Contour Integral Made Easy (a.k.a. Contour Integral for Physicists)

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This note concerns a question about how to intuit the residue theorem in complex analysis with demonstrations of integrating trigonometric functions. The first demonstration is to obtain the area of a conic section. In the context of an elementary calculus course, this integral problem is frequently posed as a classic demonstration for various integration tricks for trigonometric function. Here, we show that this integral is no more than a complex contour integral and can be evaluated easily with the residue theorem. The second demonstration shows that integration over a branch cut can sometimes be converted into integrating cover poles accumulatively. We also presented a heuristic proof for the residue theorem with elementary techniques from vector calculus, which is not meant to provide a detailed discussion of residue theory whatsoever but to give an accessible derivation of the result. Specific conditions and technicalities should be referred to standard complex analysis textbooks.

I. MNEMONICS FOR THE RESIDUE THEOREM

In different formulations of the theory, the statement of the residue theorem changes slightly. Here we adopted the statement in Joseph Taylor's *Complex Variables* [1] simply to display one version of the theorem

Proposition. *For a function f which is analytic on $U \setminus E$, in which U is an open subset of $\mathbb{C}$, and E is a discrete subset of U, if γ is a null-homologous closed path on $U \setminus E$, then:*

(a) there are only finitely many points of E at which $\operatorname{Ind}_\gamma$ is non-zero;

(b) if these points are $\{z_1, \ldots, z_n\}$, then

$$\oint_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{i=1}^{n} \operatorname{Ind}_\gamma(z_i)\operatorname{Res}[f, z_i],$$

in which $\operatorname{Ind}(\cdot)$ denotes the index function (or winding number of the path γ around z_i) and $\operatorname{Res}(\cdot)$ is the residue of f at z_i. The residue is defined to be the coefficient for the $\mathcal{O}(z^{-1})$ term in the Laurent expansion of f.

Of course, the subtleties in the properties of "discrete", "homologous", "path", *etc.* are worth of a semester-long course and beyond the scope of this note. The point of showing this theorem is to demonstrate that as long as the function is decently well-behaving, an easy approach to integrate the function can be achieved. At the first glance, the $2\pi i$ and residues seem to appear out of the blue. However, if we can break the function into Laurent series without creating complications, residue theorem simply says that at a singularity $z = z_0$, the integral locally behaves like

$$\oint_{D_\epsilon(z_0)} \sum_{i=-\infty}^{\infty} c_i (z - z_0)^i \mathrm{d}z,$$

in which $D_\epsilon(z_0)$ denotes a small circular path centered at z_0. If $i \neq -1$, then by the fundamental theorem of calculus,

$$\oint (z - z_0)^i \mathrm{d}z = \left.\frac{(z - z_0)^{i+1}}{i+1}\right|_{\partial D_\epsilon(z_0)} = 0.$$

But an interesting thing happens when $i = -1$.

$$\oint (z - z_0)^{-1} \mathrm{d}z = \ln(z - z_0)\Big|_{\partial D_\epsilon(z_0)}.$$

Of course, the path $D_\epsilon(z_0)$ can be parameterized as $z = z_0 + \exp(i\theta)$, and its logarithm is just

$$\oint (z - z_0)^{-1} \mathrm{d}z = i\theta\Big|_{\partial D_\epsilon(z_0)} = 2\pi i.$$

The residue $\operatorname{Res}[f, z_0]$ is just the c_{-1} that we can to pull out before integrating over $\mathcal{O}(z^{-1})$ term.

This is a decent picture for contour integrals for practical usage. Henceforth, the note aims to connect Stokes's theorem with residue theorem and provide another heuristic argument for the residue theorem.

II. DELTA FUNCTIONS FROM DIFFERENTIATION

There is a well-studied trick in physics to discover delta function naturally for spaces with different dimensions. As a demonstration, we will use $\mathbb{R}^3$ as our "stage". Recall that the Gauß's theorem states that

$$\int_\Omega \boldsymbol{\nabla} \cdot \mathbf{F} \mathrm{d}\tau = \oint_{\partial\Omega} \mathbf{F} \cdot \mathrm{d}\mathbf{a},$$

in which $\mathrm{d}\tau$ denotes the volume measure, and $\mathrm{d}\mathbf{a}$ is the directional area measure (pointing outward). To parse this seemingly mysterious statement, one simply looks up the differential and integral form of Maxwell's equations

[1] American Mathematical Society (Providence, 2011), Sally series, ISBN: 9780821869017

(Gauß's Law specifically) as a reference. We are going to show the following statement

$$\delta^{(3)}(\mathbf{r}) = \frac{1}{4\pi}\boldsymbol{\nabla}\cdot\frac{\hat{\mathbf{r}}}{r^2}.$$

This statement seems wrong because if we directly apply the divergence operator in spherical coordinates, we find

$$\boldsymbol{\nabla}\cdot\frac{\hat{\mathbf{r}}}{r^2} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) = \frac{1}{r^2}\cancelto{0}{\frac{\partial}{\partial r}(1)}.$$

But the differential form of Maxwell's equations makes sense only if we parse it with integrals. The differential operators itself may simply not be able to "see" a weird stuff, here a delta function. First, we set $\mathbf{F} = r^{-2}\hat{\mathbf{r}}$ and invoke the Gauß's theorem by integrating over a sphere centered at 0 with radius R. Therefore, the measures can be just parameterized as $\mathrm{d}\tau = r^2\mathrm{d}r\mathrm{d}\Omega$ and $\mathrm{d}\mathbf{a} = \hat{\mathbf{r}}R^2\mathrm{d}\Omega$ in which $\mathrm{d}\Omega = \sin\theta\mathrm{d}\theta\mathrm{d}\phi$ is the differential solid angle.

$$\int \boldsymbol{\nabla}\cdot\frac{\hat{\mathbf{r}}}{r^2}\mathrm{d}\tau = \oint \frac{\hat{\mathbf{r}}}{R^2}\cdot\hat{\mathbf{r}}R^2\mathrm{d}\Omega = \oint \mathrm{d}\Omega = 4\pi.$$

Notice that the result is independent of the radius R. One can also perform another integral around a small sphere not containing $\mathbf{0}$ (a.k.a. Shell theorem integral) and find that once the region does not include 4π, the integral becomes 0. This can only happen if we are integrating over a delta function which is defined as

$$\int_\Omega \delta^{(n)}(\mathbf{r}-\mathbf{x})\mathrm{d}r^n = \begin{cases} 1, & \mathbf{x}\text{ inside }\Omega,\\ 0, & \mathbf{x}\text{ outside }\Omega.\end{cases}$$

The slogan which we are celebrating here is that delta function exists for an integral. Without an integral, we cannot "see" a delta function. So, actually,

$$\boldsymbol{\nabla}\cdot\frac{\hat{\mathbf{r}}}{r^2} = 4\pi\delta^{(3)}(\mathbf{r})$$

as promised. As a fun practice, the readers are encouraged to check the following identity in $\mathbb{R}^3$

$$\nabla^2\left(\frac{1}{r}\right) = -4\pi\delta^{(3)}(\mathbf{r}).$$

This is simply stating that the Green's function of ∇^2 is $-(4\pi r)^{-1}$.

Now, let's turn our focus to $\mathbb{R}^2$. The "2D Gauß's theorem" is the Green's theorem

$$\oint \mathbf{F}\cdot\mathrm{d}\mathbf{l} = \int \text{“}\boldsymbol{\nabla}\times\mathbf{F}\text{”}\mathrm{d}a = \int \det\begin{vmatrix}\partial_x & \partial_y\\ F_x & F_y\end{vmatrix}\mathrm{d}x\mathrm{d}y.$$

Now, we can switch to polar coordinates and find that the "curl" operator is just

$$\frac{1}{r}\left(\frac{\partial}{\partial r}(rF_\theta) - \frac{\partial F_r}{\partial\theta}\right).$$

But as always, there is a missing delta function, which can be found by integrating the field $\mathbf{F} = \hat{\boldsymbol{\theta}}/r$ over a circle with radius R centered at the origin. The line integral gives

$$\oint \mathbf{F}\cdot\mathrm{d}\mathbf{l} = \int \frac{\hat{\boldsymbol{\theta}}}{R}\cdot\hat{\boldsymbol{\theta}}R\mathrm{d}\theta = 2\pi.$$

Again, simply substituting $F_\theta = 1/r$ into the differential operator will not give us a delta function, but once we probe it with an integral, we immediately see a hidden delta function. So in 2D

$$\text{“}\boldsymbol{\nabla}\times\frac{\hat{\boldsymbol{\theta}}}{r}\text{”} = 2\pi\delta^{(2)}(\mathbf{r}).$$

One can check again that this is no more than simply saying that the Green's function for a 2D Laplacian is $\ln r/(2\pi)$, or equivalently,

$$\nabla^2_{(2)}(\ln r) = 2\pi\delta^{(2)}(\mathbf{r}).$$

III. A HEURISTIC ARGUMENT FOR THE RESIDUE THEOREM

In the argument, we claimed that

$$\oint f(z)\mathrm{d}z = 2\pi i\sum_{s\text{ singularities}}\mathrm{Res}[f,s],$$

but why is this true? This is not too complicated to physicist's eyes because the statement is no more than just integrating over delta functions. Consider $f(z)$ as a 2D vector field (u,v) and the differential measure as a path at (x,y). This integral is just

$$\oint (u+iv)(\mathrm{d}x + i\mathrm{d}y) = \oint (u,-v)\cdot\mathrm{d}(x,y) + i\oint (v,u)\cdot\mathrm{d}(x,y).$$

As long as we are integrating over a path which $f(z)$ is analytic (having no singularities), we can invoke the Cauchy-Riemann equation

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y},\quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

Again, this seems strange, but you can easily check that this is simply saying that we are allowed to take a well-defined integral of the (u,v) vector field because their mixed derivative vanishes, *i.e.* we can rewrite $f(z)$ as $\nabla\phi = (u,v)$ since $\boldsymbol{\nabla}\times\nabla\phi = 0$. [2] (For more discussion on

[2] Some other thought that occurred to me is that this also mirrors the symplectic (from the Greek-ization of the Latinate word "complex") nature of 1D integrable classical mechanics. Recall that an assumption of classical system is that its state is completely determined by its two initial conditions and Newton's second law. If we demand that the system is integrable, we effectively demand that it satisfies a set of Cauchy-Riemann-like equations (a.k.a. Hamilton's equations).

this and its generalization, see exact (differential) form.) Now, let's massage the expression of the contour integral into a form which we can recognize

$$\oint (u+iv)(\mathrm{d}x+i\mathrm{d}y) = \oint (v,u)\cdot \mathrm{d}(-y,x) + i\oint(-u,v)\cdot\mathrm{d}(-y,x).$$

Notice that $\mathrm{d}(-y,x)$ is just $\mathrm{d}\mathbf{l}$ (because $\mathrm{d}\theta = (-y\mathrm{d}x + x\mathrm{d}y)/r^2$), and to convert the line integral into a surface integral, we just need to use the 2D "curl" again

$$\text{“}\boldsymbol{\nabla}\times(v,u)\text{”} = \det\begin{vmatrix}\partial_x & \partial_y\\ v & u\end{vmatrix} = \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0,$$
$$\text{“}\boldsymbol{\nabla}\times(-u,v)\text{”} = \det\begin{vmatrix}\partial_x & \partial_y\\ -u & v\end{vmatrix} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.$$

Hmmm, so does complex integral over a path without singularities always vanish? No! Remember our previous conclusion? There might be a "mysterious" delta function which you can only "feel" by integration. We simply need to find the term which contributes to $\hat{\boldsymbol{\theta}}$. The first possibility is just

$$(v,u) = \frac{\hat{\boldsymbol{\theta}}}{r} = \frac{1}{r}\frac{(-y,x)}{r} = \frac{(-y,x)}{x^2+y^2} \implies (u,v) = \frac{(x,-y)}{x^2+y^2}$$
$$f(z=x+iy) = \frac{x-iy}{x^2+y^2} = \frac{\cancel{x-iy}}{(x+iy)\cancel{(x-iy)}} = \frac{1}{z},$$

and the other possibility is

$$(-u,v) = \frac{\hat{\boldsymbol{\theta}}}{r} = \frac{(-y,x)}{x^2+y^2} \implies (u,v) = \frac{(y,x)}{x^2+y^2}$$
$$f(z=x+iy) = \frac{y+ix}{x^2+y^2} = \frac{i(x-iy)}{x^2+y^2} = \frac{i}{x+iy} = \frac{i}{z}.$$

So in fact, any linear combination of z^{-1} and iz^{-1} yields a delta function. Thus, the term of interest is just all cz^{-1} for some $c\in\mathbb{C}$. A careful reader at this point will question: where goes the i in the factor of $2\pi i$ then? This i is reflected by the "nature" of the surface integral on $\mathbb{C}$. You may think that $\mathrm{d}\tilde{a} \stackrel{?}{=} \mathrm{d}x\mathrm{d}y$, but recall that $\mathrm{d}z = \mathrm{d}(x+iy)$, so in fact the differential area $\mathrm{d}\tilde{a}$ on $\mathbb{C}$ is $\mathrm{d}\tilde{a} = \underline{\underline{i}}\mathrm{d}x\mathrm{d}y = i\mathrm{d}a$. So, by Green's theorem,

$$\oint f(z)\mathrm{d}z = \int \text{“}\boldsymbol{\nabla}\times f\text{”}\, i\mathrm{d}a = 2\pi i\,\mathrm{Res}\,.$$

And of course, if there are other singularities with a expansion containing a $\mathcal{O}(z)^{-1}$ term, it also contributes a delta function. If the path wrap around the region twice, the area should be integrated over twice. So

$$\oint_\gamma f(z)\mathrm{d}z = 2\pi i \sum_{s\text{ singularities}} \mathrm{Ind}_\gamma(s)\,\mathrm{Res}[f,s].$$

IV. MORE DISCUSSIONS AND REMARKS

Here we would like to put forth a small remark on the "curl" operator. One can notice that at least in 2D, defining this curl operator in terms of determinant is redundant if not cumbersome. After all, this determinant enforces nothing but simply a negative sign. So we can write the operator as

$$\boldsymbol{\nabla}\times(u,v) = \nabla_{(y,-x)}(u,v) = \nabla_{(x,y)}(v,-u).$$

So with this in mind, we can rewrite the two Cauchy-Riemann equations as

$$\begin{aligned} 0 &\stackrel{!}{=} \boldsymbol{\nabla}\times(v,u) + i\boldsymbol{\nabla}\times(-u,v)\\ &= \nabla_{(x,y)}(u,-v) + i\nabla_{(x,y)}(v,u)\\ &= \nabla_{(x,y)}(u+iv, i(u+iv)) = \nabla_{(x,y)}(f, if).\end{aligned}$$

Now, we can define a conjugate derivative operator which replace $\nabla_{(x,y)}$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right).$$

(Why 1/2? It is because $x = \mathrm{Re}(z) = (z+\bar{z})/2$ and $y = \mathrm{Im}(z) = (z-\bar{z})/(2i)$.) Thus, in some other formulation of the Riemann-Cauchy equation, one can also compact the expression into

$$\frac{\partial f}{\partial \bar{z}} = 0.$$

V. 1ST WORKING EXAMPLE: AREA OF CONIC SECTION

The area of a conic section is an interesting problem. Recall that any conic section can be parameterized in polar coordinates as

$$r(\theta) = \frac{l}{1+e\cos\theta}.$$

Therefore, the area enclosed by a conic section is

$$A = \int_0^{2\pi} \mathrm{d}\theta\, \frac{1}{2}\left(\frac{l}{1+e\cos\theta}\right)^2.$$

If we consider the following change of variable $z = \exp(i\theta)$ with $\mathrm{d}z = iz\mathrm{d}\theta$ (a.k.a. Weierstrass substitution), then the integral becomes

$$\begin{aligned} A &= \oint \frac{\mathrm{d}z}{iz}\frac{1}{2}\left(\frac{l}{1+e(z+z^{-1})/2}\right)^2\\ &= \frac{2l^2}{i}\oint \frac{z\mathrm{d}z}{(2z+ez^2+e)^2}.\end{aligned}$$

The name of the game now changes to evaluating this contour integral. By residue theorem, we know that

$$\oint f(z)\mathrm{d}z = 2\pi i \sum_{s\,\text{singularities}} \mathrm{Res}[f,s],$$

and the residue is simply the -1^{st} Laurent coefficient at s. First, the pole appears when

$$ez^2+2z+e=0 \implies r_\pm \triangleq -\frac{2\pm\sqrt{4-4e^2}}{2e} = \frac{\mp\sqrt{1-e^2}-1}{e},$$

in which $r_\pm$ denote the root with the same/opposite sign for the quadratic equation. Notice that the only singularity that is within the contour is $r_- = \left(\sqrt{1-e^2}-1\right)/e$. So we just need to find

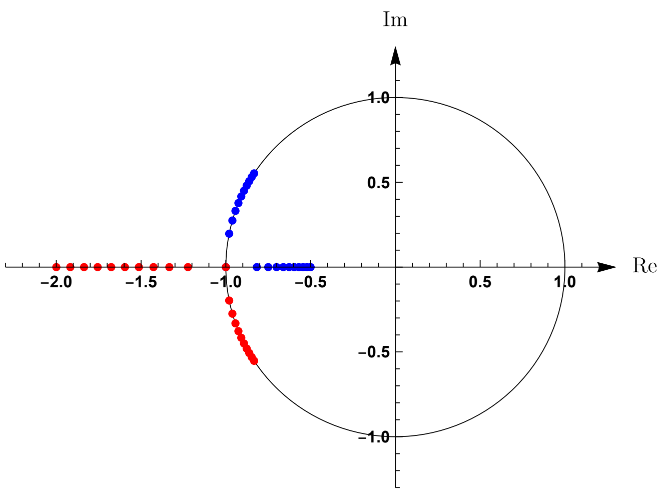

FIG. 1. Singularities on $\mathbb{C}$ as e changes

$$\text{Res}\left[g \triangleq \frac{z}{\left(2z+ez^2+e\right)^2}, r_-\right].$$

But this is not too complicated because we can separate the function into two parts.

$$g(z) = \frac{z}{e^2(z-r_-)^2(z-r_+)^2} = \frac{1}{(z-r_-)^2}\frac{z}{e^2(z-r_+)^2}.$$

The first term has a double pole at $z = r_-$, but the second term is analytic at r_-. This analyticity ensures a Taylor expansion at r_-. So we know that the residue is just

$$\begin{aligned}\text{Res}[g, r_-] &= \frac{\mathrm{d}}{\mathrm{d}z}\frac{z}{e^2(z-r_+)^2}\bigg|_{r_-} \\ &= \frac{(r_- - r_+)^2 - 2r_-(r_- - r_+)}{e^2(r_- - r_+)^4} \\ = -\frac{r_- + r_+}{e^2(r_- - r_+)^3} &= \frac{2}{e}\cdot\frac{e^3}{8e^2(1-e^2)^{3/2}} = \frac{1}{4(1-e^2)^{3/2}}.\end{aligned}$$

Therefore,

$$A = \frac{2l^2}{i}\oint g(z)\mathrm{d}z = \cancel{2}\pi\cancel{i}\times\frac{\cancel{2}l^2}{\cancel{i}}\frac{1}{\cancel{4}(1-e^2)^{3/2}} = \frac{\pi l^2}{(1-e^2)^{3/2}}.$$

Note that the divergence of the area of conic sections [3] can be observed from the behavior of the singularities as shown in FIG. 1. As e increases, the two singularities move from the real line to the point $z = -1$ and then traces along the integration contour, resulting in a divergent integral.

VI. 2$^{\text{ND}}$ WORKING EXAMPLE: METHOD OF ACCUMULATING POLES

In most cases, residue theorem suffices to evaluate complex integrals. However, there are other cases that we encounter a branch cut. A branch cut appears when an implicit logarithm function is present in the integral. Recall that in complex analysis, we define most power functions by logarithms, for instance, $z^{1/2} = e^{1/2\ln z}$, $z^{\sqrt{2}} = e^{\sqrt{2}\ln z}$. But evaluating a contour integral over logarithms requires careful treatments, specifically the contour should stay on the same sheet without crossing the branch cut. Normally, this process is done by reparameterize the contour in the complex plane and analyze the integral by taking some limits. However, there are occasion in which we integrate over certain types of branch cut that we can use a sneaky trick, called method of accumulating poles.

The method of accumulating poles uses the idea that a logarithmic branch cut can be handled as a collection of simple poles; therefore, we can first perform a contour integral over the simple pole and systematically accumulate back the poles by an integration over some other parameter.

Here is an interesting demonstration of this method. Consider the integral

$$I = \int_0^{2\pi}\mathrm{d}\phi\, 2\cot\phi\arctan\left(\frac{c\sin\phi}{1-c\cos\phi}\right) - \ln\left(1+c^2-2c\cos\phi\right),$$

in which $c \in (0,1)$. Because the integral has dependence on trigonometric functions of ϕ, we can use the same trick on integrating over a unit circle with a change of variable $z = e^{i\phi}$. Then, the integral becomes

$$\begin{aligned} I =& \frac{2}{i}\oint\frac{\mathrm{d}z}{z}i\cdot\frac{z+z^{-1}}{z-z^{-1}}\arctan\left(\frac{z-z^{-1}}{2i/c-iz-iz^{-1}}\right) \\ &-\frac{1}{i}\oint\frac{\mathrm{d}z}{z}\ln\left(1+c^2-cz-cz^{-1}\right) \\ =& -\frac{2}{i}\oint\mathrm{d}z\frac{z^2+1}{z(z+1)(z-1)}\,\text{artanh}\left(\frac{z^2-1}{z^2+1-2z/c}\right) \\ &-\frac{1}{i}\oint\frac{\mathrm{d}z}{z}\ln\left(\frac{z+c^2z-cz^2-c}{z}\right), \end{aligned}$$

[3] A good animation for this divergence can be found on Wikipedia at `https://en.wikipedia.org/wiki/Conic_section#Polar_coordinates`

in which artanh denotes the inverse hyperbolic tangent, *viz. area tangens hyperbolicus*. Notice that we can express $\operatorname{artanh}(x)$ in terms of logarithms by solving the equation

$$\begin{aligned}\tanh y = \frac{e^y - e^{-y}}{e^y + e^{-y}} &= \frac{e^{2y}-1}{e^{2y}+1} \\ \implies e^{2y} &= \frac{1+\tanh y}{1-\tanh y} \\ \implies \operatorname{artanh}(x) &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right).\end{aligned}$$

Therefore, in terms of explicit logarithms, the integral is

$$\begin{aligned}I =& -\frac{1}{i}\oint \mathrm{d}z \frac{z^2+1}{z(z+1)(z-1)}\ln\left(\frac{z^2+1-2z/c+z^2-1}{z^2+1-2z/c-z^2+1}\right) \\ &- \frac{1}{i}\oint \frac{\mathrm{d}z}{z}\ln\left(\frac{-c\left[z^2-(c+1/c)z+1\right]}{z}\right) \\ =& -\frac{1}{i}\oint \mathrm{d}z \frac{z^2+1}{z(z+1)(z-1)}\ln\left(\frac{z(z-1/c)}{1/c(c-z)}\right) \\ &- \frac{1}{i}\oint \frac{\mathrm{d}z}{z}\ln\left(\frac{-c(z-c)(z-1/c)}{z}\right).\end{aligned}$$

Now, we take the principle branch of the logarithm and set a branch cut along the negative real axis. Given $c \in (0,1)$, the argument of the two logarithms are negative if

$$\begin{aligned}&\begin{cases}\dfrac{-c(z-c)(z-1/c)}{z} \le 0 \\ \dfrac{z(z-1/c)}{1/c(c-z)} \le 0\end{cases} \\ &\implies z(z-1/c)(z-c) \ge 0, \\ &\therefore z \in [0,c] \cup [1/c, +\infty),\end{aligned}$$

which will be the branch cut which we will use. Since there is no pole inside the unit circle, the contour integral around the unit circle is the same as the path near the branch cut as shown in FIG. 2.

Instead of parameterizing the path near the branch cut, we will "accumulate poles" to form the branch cut. Notice that

$$\begin{aligned}\frac{\partial}{\partial c}\ln\left(-\frac{z(z-1/c)}{1/c(z-c)}\right) &= \frac{z^2-1}{c(z-c)(z-1/c)}, \\ \frac{\partial}{\partial c}\ln\left(-\frac{c(z-c)(z-1/c)}{z}\right) &= \frac{z^2-2cz+1}{c(z-c)(z-1/c)}.\end{aligned}$$

Thus, the original integral can be represented as an integral over c (here parameterized as x)

$$\begin{aligned}I =& \int_0^c \mathrm{d}x\ -\frac{1}{i}\oint \mathrm{d}z\,\frac{(z^2+1)}{z\cancel{(z^2-1)}}\frac{\cancel{z^2-1}}{x(z-x)(z-1/x)} \\ &+ \int_0^c \mathrm{d}x\ -\frac{1}{i}\oint \mathrm{d}z\,\frac{1}{z}\frac{z^2-2xz+1}{x(z-x)(z-1/x)} \\ =& \int_0^c \mathrm{d}x\ -\frac{1}{i}\oint \mathrm{d}z\,\underbrace{\frac{2z^2-2xz+2}{xz(z-x)(z-1/x)}}_{f(z)}\end{aligned}$$

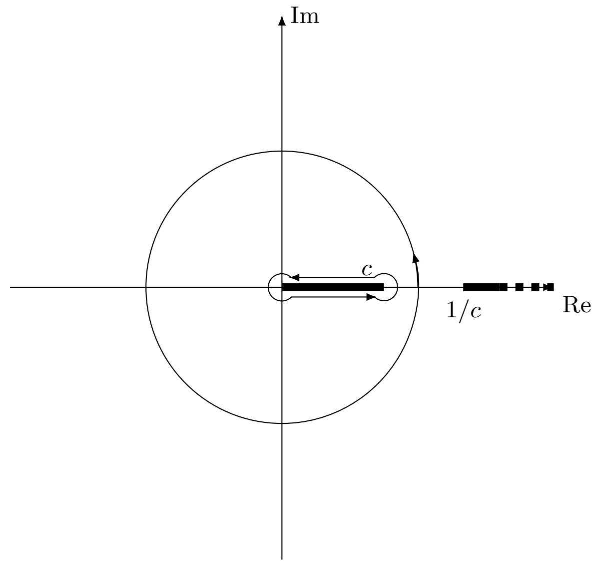

FIG. 2. Contour and the branch cut used in section VI

This function is easy to analyze because all terms are rational and its analytic properties inside the contour are conspicuous. There are only two simple poles inside the contour, and their residues are

$$\operatorname{Res}[f(z), z=0] = \frac{2}{x}, \qquad \operatorname{Res}[f(z), z=x] = \frac{2}{x(x^2-1)}.$$

Therefore, by the residue theorem, we can reduce the integral into an real integral and find the result as

$$\begin{aligned}I =& -2\pi\int_0^c \mathrm{d}x\,\frac{2x}{x^2-1} = -2\pi\int\frac{\mathrm{d}\left(x^2-1\right)}{x^2-1} \\ =& -2\pi\ln\left(\frac{c^2-1}{-1}\right) = -2\pi\ln\left(1-c^2\right).\end{aligned}$$

ACKNOWLEDGMENTS

The problem on the area of a conic section is motivated by Dr. Ying Xu's note on elementary integral tricks whereas the argument for the residue theorem is heavily adapted from Prof. Oliver Knill's note [4] on Green's theorem and Cauchy's theorem. The author would like to thank Prof. Kyle Ormsby and Prof. Andrew Larkoski's valuable discussions on this topic.

[4] http://www.math.harvard.edu/~knill/teaching/residues_1996/residue.pdf