Quadratic Space: A Generalized Inner Product Space

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This note provides some introduction to the notion of a quadratic space. The presentation is motivated by the need to extend the notion of dot products and metrics. This leads to the observation of the connection between quadratic spaces and symmetric bilinear forms. Then, the note shows some basic ideas behind equivalence of quadratic spaces and quadratic forms, alluding to coordinate transformations. The note conclude with a useful method to identify this quadratic equivalence by Witt's chain equivalence theorem. We attempt to find a middle ground between showing rigorous and delicate proofs and providing intuitive pictures on the topic. Some of the proofs are either intentionally omitted or placed in the appendix to avoid creating distractions or gaps in the train of logic. For curious readers who would like to learn more about quadratic forms and spaces, they should confer some advanced algebra textbooks on this matter. The argument and formulations are heavily adopted from [1, 2].

CONTENTS

I. QUADRATIC FORMS AND QUADRATIC SPACE

There have been several attempts to generalize the notion of "dot products" over a real vector space to any vector spaces over any fields. For instance, the norm over a complex vector space $\mathbf{v}^\dagger\mathbf{v}$ is a generalization of $\mathbf{v}^T\mathbf{v}$ over the reals. The first relatively successful attempt is to find a metric g over a set with

$$\begin{cases} d(x,y) \geq 0, \text{ with } d(x,y) = 0 \iff x = y, \\ d(x,y) = d(y,x), \\ d(x,y) \leq d(x,z) + d(z,y), \end{cases} \tag{1}$$

and this definition leads to various interesting metric functions over any set. But with a vector space, we have quite a lot of stringent requirements on the properties of operations over vectors[1]. Therefore, mathematicians came up with the idea of an inner product space[2]. Soon enough, this definition turns out to be a bit too strict in demanding that $\mathbf{v}\cdot\mathbf{v} > 0$ for every nonzero vector $\mathbf{v}$. In special relativity, we use a Minkowski metric $\mathrm{d}s^2 = \mathrm{d}t^2 - \mathrm{d}x^2 - \mathrm{d}y^2 - \mathrm{d}z^2$ to evaluate the "length" of a four-vector, and this "length" is a Lorentz invariant in the sense that any rotation and boosts cannot change this quantity. Identifying these invariants and vectors assists us to put down the physical law in one inertial frame, which is guaranteed to be the same in any inertial frames. However, one can easily see that the vector $(1,0,0,0)^2 = 1$ but $(0,1,0,0)^2 = -1$. So to generalize "dot products" and "lengths" to a relativistic setting requires some more relaxed assumptions. Fortunately, several mathematician pioneers

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[1] Abstractly, over any field k (which has a lot of structures already), we define an Abelian addition and a k-module multiplication over $(\oplus^n \mathsf{k}, +, \cdot)$ and call that an n-dimensional vector space.

[2] In physics, an inner product space is almost always the same as a Hilbert space, but the subtlety comes from the fact that a Hilbert space demands completeness, *i.e.* every Cauchy sequence in the space converges to some element in the space. But we really have no reason to consider the distinction if we can always perform a completion over an inner product space to form a Hilbert space.

have identified a good candidate, **quadratic space**. In this note, we will provide some basic ideas of quadratic spaces with argument and formulations heavily adopted from [1, 2]. Some other references on this topic include [3, 4]

Let's start with defining a **quadratic form**, a general notion of a vector's (squared) length.

Definition I.1. An **n-nary quadratic form over a <u>field</u>** k is an n-nary polynomial $q \in \mathsf{k}[x_1, \dots, x_n]$ that is homogeneous of degree 2, *i.e.*

$$q(x_1, \dots x_n) \triangleq \sum_{i,j}^{n} \lambda_{ij} x_i x_j, \tag{2}$$

in which at least one $\lambda_{i,j}$ is nonzero.

Of course, we can also picture an n-nary quadratic form as a matrix equation in the sense that if we identify $(x_1, \dots, x_n) \triangleq \mathbf{x} \in \mathsf{k}^n$ as an n-dimensional (k-)vector. Then, we can equivalently call that

$$q(\mathbf{x}) = \mathbf{x}^T \Lambda \mathbf{x}, \tag{3}$$

with some $n \times n$ matrix Λ representing the coefficients. This motivates the definition of a **quadratic (vector) space**.

Definition I.2. A **quadratic space** (V, q) over a field k is a k-vector space equipped with a quadratic form $q(\mathbf{v}) = q(v_1, v_2, \dots)$.

A defining feature of a quadratic form is $q(\lambda \mathbf{x}) = \lambda^2 q(\mathbf{x})$. Notice that eq. (2) is symmetric under i, j permutation, so WLOG we can demand that

$$\lambda_{ij} = \lambda_{(ij)} = \frac{1}{2}(\lambda_{ij} + \lambda_{ji}), \tag{4}$$

i.e. Λ is a symmetric matrix. If we have a symmetric matrix, then we can define a good dot product by

$$B(\mathbf{x}, \mathbf{y}) = \mathbf{x}^T \Lambda \mathbf{y} = \mathbf{x}^T \Lambda^T \mathbf{y} = B(\mathbf{y}, \mathbf{x}). \tag{5}$$

Straightforwardly from the matrix equation, we can see that $B(\mathbf{x}, \mathbf{y})$ is symmetric and bilinear (linear in either of its argument), which is precisely what we intuitively demand for an inner product $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$. This function $B(\mathbf{x}, \mathbf{y})$ is frequently referred to as a **symmetric bilinear form**. One may sense that there is a bijective correspondence between a quadratic form from V to k and a symmetric bilinear form from $V \times V$ to k, and this is true.

Proposition I.3. *There is a bijective correspondence between a quadratic form $q : V \to \mathsf{k}$ and a symmetric bilinear form $B : V \times V \to \mathsf{k}$ by the following equalities*

$$B_q(\mathbf{x}, \mathbf{y}) = \frac{1}{2}[q(\mathbf{x} + \mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y})], \tag{6}$$

$$q_B(\mathbf{x}) = B(\mathbf{x}, \mathbf{x}). \tag{7}$$

These equalities can be readily verified with the matrix equations, and from now on, we will freely choose to switch between quadratic forms and symmetric bilinear forms as they are effective the same entity.

II. CHANGE OF COORDINATES REVISITED

Now, it is reasonable to ask if two quadratic spaces are the "same" (or isometric). A good starting point is to observe the transformation of linear function under coordinate transformations. From any linear algebra class, one learns that a matrix is truly a linear function expressed with respect to a given basis, and any bijective linear function corresponds to an invertible matrix with the collection of $n \times n$-invertible matrices denoted as $\mathrm{GL}_n(\mathsf{k})$. Any coordinate transformation over a vector space is specified by such an invertible linear function. Suppose we have a linear function $\mathbf{y} = f(\mathbf{x})$. Then, if we have a coordinate transformation $T \in \mathrm{GL}_n(\mathsf{k})$, we can identify the "same" linear function (in another coordinate) as

$$\begin{aligned} T(\mathbf{y}) &= f(T(\mathbf{x})) = f \circ T(\mathbf{x}) \\ \implies \mathbf{y} &= T^{-1} \circ f \circ T(\mathbf{x}). \end{aligned} \tag{8}$$

One can translate this equation into matrix language, and this precisely defines similar matrices with similarity transformation $M \mapsto T^{-1} M T$. With this notion of similarity (or conjugacy), we identify the class of equivalent linear transformations as long as we can perform some coordinate transformations to "match" them from one to another.

III. EQUIVALENCE OF QUADRATIC FORMS: FROM LINEAR FUNCTIONS TO QUADRATIC FORMS

Let's play the same game over a quadratic form or a bilinear form. To make this absolutely obvious, we will use the matrix equations. Consider evaluating a quadratic form $q\mathbf{x}$ and then switching to another coordinate with a linear function $T \in \mathrm{GL}_n(\mathsf{k})$. Then, in this new coordinate

$$q(T(\mathbf{x})) = (T(\mathbf{x}))^T \Lambda T(\mathbf{x}) = \mathbf{x}^T T^T \Lambda T \mathbf{x}. \tag{9}$$

So quite interestingly, we find another way of identifying the "same" matrix by $M \mapsto T^T M T$. This is called **congruence**. This type of equivalence is showing that Λ is not associated with a linear function $f : V \to V$ but a quadratic form (or equivalently a symmetric bilinear form) $q : V \to \mathsf{k}$ that defines the "metric" (notion of length) over a space. With this identification, we can define equivalence quadratic forms to be the same form up to coordinate transformations.

Definition III.1. Two quadratic forms q_1 and q_2 are **equivalent** (denoted as $q_1 \cong q_2$) if there exists some bijective linear function $T \in \mathrm{GL}_n(\mathsf{k})$ such that

$$q_1(\mathbf{x}) = q_2(T(\mathbf{x})),\ \forall \mathbf{x} \in V. \tag{10}$$

Similarly, two symmetric bilinear forms B_1 and B_2 are **equivalent** (denoted as $B_1 \cong B_2$) if there exists some bijective linear function $T \in \mathrm{GL}_n(\mathsf{k})$ such that

$$B_1(\mathbf{x}, \mathbf{y}) = B_2(T(\mathbf{x}), T(\mathbf{y})),\ \forall \mathbf{x}, \mathbf{y} \in V. \tag{11}$$

Note that we have observed the difference between a quadratic form and a linear function which can all be represented by a matrix in some form. But fundamentally, a bijective linear function is a function (homomorphism) from V to V whereas a quadratic form is a function from V to k. They also transform differently under coordinate transformations. A linear function follows the similarity transform $M \mapsto T^{-1}MT$, and a quadratic form follows the congruence transform $M \mapsto T^T MT$.

IV. DIRECT SUM AND DIAGONALIZATION OF QUADRATIC FORMS

Before we proceed a little deeper into operations on quadratic forms, let's first get a handle over quadratic forms. As it turns out, we can always put the quadratic form into a diagonal form, *i.e.* the matrix representing a quadratic form can be diagonal with some congruence transform. Of course, this is not too surprising a result over $\mathbb{R}$ or $\mathbb{C}$ since we can directly invoke the spectral theorem

Proposition IV.1 (Spectral Theorem)**.** *For a* ***normal matrix*** *N (i.e. $N^\dagger N = NN^\dagger$ in which $\dagger$ denotes conjugate transpose), there exists an eigendecomposition such that*

$$N = U^\dagger \Lambda U, \tag{12}$$

in which U is a ***unitary matrix*** *(i.e. $U^\dagger = U^{-1}$) and Λ is a diagonal matrix. If N is* ***Hermitian matrix*** *(i.e. $N^\dagger = N$), then Λ is necessarily a real matrix.*

Note that if we restrict to $N \in \mathbb{R}^{n\times n}$, then the spectral theorem can be reinterpreted as the diagonalizability of any symmetric bilinear form over $\mathbb{R}$. But the spectral theorem is about solving an eigenproblem, *i.e.* diagonalization with similarity transform. However, if we slightly "relax" the condition to congruence transforms, we can diagonalize any symmetric bilinear form over any field ($\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{F}_p$, *etc.*). Let's start by defining a notion of "appending vector space to another vector space" (or simply "adding two vector spaces"). This notion is called a **direct sum** of vector spaces, enlightened by the Cartesian product.

Definition IV.2. Given two Abelian groups $(G_1, +_1)$ and $(G_2, +_2)$, the **direct sum**[3] of the two groups is defined as a new Abelian group $(G_1 \oplus G_2, +)$ equipped with the coordinate-wise addition

$$(g_{11}, g_{21}) + (g_{12}, g_{22}) \triangleq (g_{11} +_1 g_{21}, g_{12} +_2 g_{22}),\\ \forall g_{11}, g_{12} \in G_1,\ g_{21}, g_{22} \in G_2. \tag{13}$$

Then, a direct sum of two vector spaces means the direct sum of their underlying addition (Abelian) group. If the two (or finitely many) vector spaces have the same base field k, then their direct sum looks just like the usual Cartesian product.

Example IV.3. $\mathbb{R}^m \oplus \mathbb{R}^n = \mathbb{R}^{m+n}$, and we can identify any $(\mathbf{v}, \mathbf{w}) \in \mathbb{R}^m \oplus \mathbb{R}^n$ as $\mathbf{v} + \mathbf{w} \in \mathbb{R}^{m+n}$.

Accompanied with the notion of a direct sum of vector spaces, we can also "add" two quadratic forms or symmetric bilinear form by an **orthogonal sum**.

Definition IV.4. Given two symmetric bilinear forms B_1 (over V_1) and B_2 (over V_2), their **orthogonal sum** is a new symmetric bilinear form $B_1 \perp B_2 : (V_1 \oplus V_2) \times (V_1 \oplus V_2) \longrightarrow \mathsf{k}$ defined as $\forall \mathbf{v}_{11}, \mathbf{v}_{12} \in V_1$ and $\mathbf{v}_{21}, \mathbf{v}_{22} \in V_2$

$$\begin{aligned}&(B_1 \perp B_2)(\mathbf{v}_{11} + \mathbf{v}_{21}, \mathbf{v}_{12} + \mathbf{v}_{22})\\ \triangleq &B_1(\mathbf{v}_{11}, \mathbf{v}_{12}) + B_2(\mathbf{v}_{21}, \mathbf{v}_{22}).\end{aligned} \tag{14}$$

Now, we are ready to investigate the diagonalizability of quadratic (or symmetric bilinear) forms. First, let's define the set of represented elements (or the set of diagonal elements)

Definition IV.5. Given a k-quadratic space (q, V), an element $\lambda \in \mathsf{k}$ is **represented** by q if there exists some vector $\mathbf{v} \in V$ such that $q(\mathbf{v}) = \lambda$. The set of represented elements then is defined as

$$D_{\mathsf{k}}(q) = D(q) = \{q(\mathbf{v}) \mid \mathbf{v} \in V\} \setminus \{0\}. \tag{15}$$

Just like the notation suggests, these elements are actually potential candidates for the diagonal entries

[3] In fact, over any group, one can define a notion of direct product in a similar fashion. The fact that we are using a plus sign is entirely due to the commutativity of the group operation.

of the quadratic form. Then, we will also introduce a shorthand for a diagonal form. A quadratic form q associated with a diagonal matrix

$$\Lambda = \begin{pmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{pmatrix} \tag{16}$$

has a shorthand notation $q = \langle \lambda_1, \lambda_2, \ldots, \lambda_n \rangle$.

Perfect! Let's start with a lemma with a symmetric bilinear form (so that the notion of a "perpendicular subspace" or an orthogonal complement is clearer, see appendix A).

Lemma IV.6 (Representation Criterion)**.** *For a symmetric bilinear form B over V and $\lambda \in \mathsf{k}^\times$, $\lambda \in D(B) \iff B \cong \langle \lambda \rangle \perp B'$ for some other symmetric bilinear form B' over V'.*

Proof. See appendix A. □

Theorem IV.7 (Diagonalizability of Quadratic Forms and Symmetric Bilinear Forms)**.** *For any symmetric bilinear form B over a k vector space V, there exists $\lambda_1, \ldots, \lambda_n \in \mathsf{k}$ such that*

$$B \cong \langle \lambda_1, \ldots, \lambda_n \rangle\,. \tag{17}$$

Proof. This is a straightforward application of lemma IV.6. If $D(B) = \emptyset$, we can simply write $B \cong \langle 0, \ldots, 0 \rangle$. If $\exists \lambda \in D(B)$, then $B \cong \langle \lambda \rangle \perp B'$ by lemma IV.6. And the rest of the argument proceeds by induction. □

With this theorem, we may safely assume that every quadratic form can be expressed in a diagonal form, and from now on, we will freely assume at our convenience that a quadratic form $q = \langle \lambda_1, \ldots, \lambda_n \rangle$ for some $\lambda_n \in \mathsf{k}$. Given a diagonal matrix, some matrix invariant becomes relatively easy to evaluate. Consider an n-dimensional form. Then, we can calculate its determinant (also called the discriminant)

$$\det \Lambda = \det \begin{vmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{vmatrix} = \prod_{k=1}^{n} \lambda_k. \tag{18}$$

Why would we be interested in calculating the determinant? Observe the following interesting phenomena consider the form

$$q(\mathbf{v}) = v_x^2 + 2 v_x v_y + v_y^2 = (v_x + v_y)^2. \tag{19}$$

This form can be represented by the following symmetric matrix

$$\Lambda_q = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \implies \det \Lambda_q = 0. \tag{20}$$

What does it mean? This zero determinant hints us that instead of looking like a 2-dimensional quadratic form, we can actually just observe the span of $\{\hat{\mathbf{x}} + \hat{\mathbf{y}}\}$ and find that this quadratic form is in fact only one-dimensional, *i.e.* this form is actually **degenerate**[4]. Conveniently, in a diagonal form, this simply means that one of the diagonal entry is 0, killing off the product.

So is the determinant invariant under congruence? Not quite. Let's take a look at the unary form $q = \langle \lambda \rangle$ with $\lambda \in \mathsf{k}^\times$. Notice that $q = \lambda x^2 \implies D(q) = \lambda \mathsf{k}^{\boxtimes}$ in which $\mathsf{k}^{\boxtimes} \triangleq \{x^2 | x \in \mathsf{k}^\times\}$. One can extend this observation to an arbitrary n-nary form by noticing that $\lambda \in D(q) \implies \lambda^{-1} \in D(q)$. This shows that $D(q)$ is a union of different sets of elements of the form $\lambda \mathsf{k}^{\boxtimes}$ for some $\lambda \in \mathsf{k}^\times$. These sets of elements form cosets called **square classes** $\mathsf{k}^\times / \mathsf{k}^{\boxtimes}$. So when we find the determinant, it was not the value of the determinant that matters but the square class of the determinant since each diagonal element actually represents a square class.

Definition IV.8. The **determinant** or the **discriminant** of a quadratic form is the determinant of the matrix associated with the form up to square classes. For a form $q = \langle \lambda_1, \ldots, \lambda_n \rangle$,

$$\det q = \prod_{k=1}^{n} \lambda_k \cdot \mathsf{k}^{\boxtimes}. \tag{21}$$

Another way to yield the same conclusion is to note that under congruence,

$$\begin{aligned} \det f &= \det\!\left(T^T \Lambda T\right) \\ &= \det\!\left(T^T\right) \det(\Lambda) \det(T) = \det(T)^2 \det(\Lambda). \end{aligned} \tag{22}$$

So only if we identify the determinant up to square classes can the determinant of a quadratic form be an invariant under equivalence.

V. WITT'S CHAIN EQUIVALENCE

As it turns out, the equivalence of quadratic forms is very controlled that we can see if two forms are equivalence to performing a stepwise equivalence check over binary forms. Each step consists of a check on **simple equivalence**.

Definition V.1. Two quadratic forms are **simply equivalent** or of **simple equivalence** if the two

[4] To see more discussions about degenerate forms, read appendix A.

forms' diagonalizations differ at most by a binary piece and binary pieces are equivalent. More precisely, let $f = \langle \lambda_1, \dots, \lambda_n \rangle$ and $g = \langle \mu_1, \dots, \mu_n \rangle$. Then, f is simply equivalent to g if $\exists\ 1 \le i \le j \le n$ such that $\lambda_k = \mu_k\ \forall k \in [1, n] \setminus \{i, j\}$ and $\langle \lambda_i, \lambda_j \rangle \cong \langle \mu_i, \mu_j \rangle$.

Note that if $i = j$, we interpret as an equivalence of the unary form $\langle \lambda_i \rangle = \langle \mu_i \rangle$. The main reason why we are interested binary equivalence is that they are "well-controlled". How? See the following proposition.

Proposition V.2. *For two binary quadratic forms f and g, $f \cong g \iff$ both $\det f = \det g$ and $D(f) \cap D(g) \neq \emptyset$.*

Proof. The forward direction is trivial. For the backward direction, suppose that $f = \langle a, b \rangle$ and $g = \langle c, d \rangle$. Let $\lambda \in D(f) \cap D(g)$. Then, by lemma IV.6, we know that $f \cong \langle \lambda, x \rangle$ and $g \cong \langle \lambda, y \rangle$. Recall that the determinant is an invariant under congruence. To ensure that the new diagonalization yields the same determinant, we must conclude that $f \cong \langle \lambda, \lambda ab \rangle$ and $g \cong \langle \lambda, \lambda cd \rangle$. But since $\det f = \det g$, we know that $ab\mathsf{k}^{\boxtimes} = cd\mathsf{k}^{\boxtimes}$, whence $\lambda ab\mathsf{k}^{\boxtimes} = \lambda cd\mathsf{k}^{\boxtimes}$. Thus, the "second part" of the two quadratic forms represents effectively the same thing, and $f \cong g$. □

Abstractly, the binary equivalence condition is simple, but why do we insist on that $D(f) \cap D(g) \neq \emptyset$? Why would the determinant condition be insufficient? Let's consider a slightly stupid example. Consider two quadratic forms over $\mathbb{R}^2$, $f(x, y) = \langle 1, 0 \rangle (x, y) = x^2$ and $g = \langle -1, 0 \rangle (x, y) = -x^2$. Both of them have zero determinant, but, surely, they are not equivalent. Consider any bijective linear transformation[5] $T(x) = kx$ for some $k \in \mathsf{k}^{\times}$. There is obvious no such k that yields $-1 \cdot k^2 = k^2$ over the reals.

Of course, simply equivalent forms are equivalent. But not all equivalent forms are simply equivalent because the transitivity does not hold for simple equivalence. (See for example that $\langle 1, 1, 1 \rangle \cong \langle 1, 2, 2 \rangle \cong \langle 2, 2, 2 \rangle \in \mathbb{R}$, but $\langle 1, 1, 1 \rangle$ and $\langle 2, 2, 2 \rangle$ are not simply equivalent.) Therefore, we need to define a transitive equivalence relation.

Theorem V.3 (Witt's Chain Equivalence Theorem)**.** *Two forms are* ***chain equivalence*** *if there is a sequence of simple equivalences of diagonal forms that one can perform from one form to the other. Two quadratic forms are equivalent $\iff$ two quadratic forms are chain equivalent.*

[5] We can ignore the y component since it is always mapped to a null form $\langle 0 \rangle$.

It is straightforward to see that chain equivalence $\implies$ equivalence, but an equivalence $\implies$ the existence of chain may be a bit surprising. We will omit the proof here, but curious readers are encouraged to dive into textbooks on quadratic forms and symmetric bilinear forms for more detailed discussions. As a warm up, we will demonstrate an equivalence among several quaternary quadratic forms.

$$\langle 1, 1, 1, 1 \rangle \cong \langle 1, 1, 2, 2 \rangle = \langle 1, 2, 1, 2 \rangle \cong \langle 1, 2, 3, 6 \rangle . \tag{23}$$

Some of the steps may seem mysterious, but the key to the secret lies in the following proposition

Proposition V.4.

$$\langle \lambda, \mu \rangle \cong \langle \lambda + \mu, (\lambda + \mu)\lambda\mu \rangle \quad \text{for } \lambda, \mu \in \mathsf{k}. \tag{24}$$

Proof. This is a direct consequence of simple binary equivalence. □

VI. TENSOR PRODUCT: A MULTIPLICATION OF QUADRATIC FORMS

As discussed in section IV, we can introduce an orthogonal sum that "adds" quadratic forms together. With a notion of addition, we would like to introduce a notion of multiplication of quadratic forms. But what does it mean to multiply vector spaces?

A. Tensor Product of Vector Spaces

Tensor product is something that we intuitive know as a way to form higher dimensional matrices with various tensor basis that may look like $\hat{\mathbf{x}} \otimes \hat{\mathbf{y}} \otimes \hat{\mathbf{z}}$. Also, we know that $\hat{\mathbf{x}} \otimes \hat{\mathbf{y}} \neq \hat{\mathbf{x}} \otimes \hat{\mathbf{y}}$, but somehow $(2\hat{\mathbf{x}}) \otimes \hat{\mathbf{y}} = 2\hat{\mathbf{x}} \otimes \hat{\mathbf{y}} = \hat{\mathbf{x}} \otimes (2\hat{\mathbf{y}})$. To give this concept a bit more taste of physics, we actually know that it looks like some kind of indexed vector "glued together". For example, $v_i w_j = (\mathbf{v} \otimes \mathbf{w})_{ij}$, or $p^\mu p^\nu = (p \otimes p)^{\mu\nu}$. These intuitions seem to point to some kind of multilinearity conditions. But what is it?

Defineorem VI.1 (Tensor Product)**.** Given two k-vector spaces V and W, there exists a new k-vector space $V \otimes W$ such that any k-bilinear function $f : V \times W \to U$ induces a unique linear function $\tilde{f} : V \otimes W \to U$ that makes the following diagram commutes

$$\begin{tikzcd}
V \times W \arrow[r, "f"] \arrow[d, "i"'] & U \\
V \otimes W \arrow[ur, dashed, "\exists! \tilde{f}"'] &
\end{tikzcd}$$

This new vector space is the **tensor product** of V and W. Specifically, $V \otimes W = \mathsf{k}^{\oplus V\times W}/R$ in which $\mathsf{k}^{\oplus V\times W}$ denotes the direct sum over the underlying set of the vector space $V \times W$ and R is the subspace generated by elements of the form

$$\begin{cases}(\mathbf{v}_1+\mathbf{v}_2,\mathbf{w}_1)-(\mathbf{v}_1,\mathbf{w}_1)-(\mathbf{v}_2,\mathbf{w}_1),\\ (\mathbf{v}_1,\mathbf{w}_1+\mathbf{w}_2)-(\mathbf{v}_1,\mathbf{w}_1)-(\mathbf{v}_1,\mathbf{w}_2),\\ (\lambda\mathbf{v}_1,\mathbf{w}_1)-\lambda(\mathbf{v}_1,\mathbf{w}_1),\\ (\mathbf{v}_1,\lambda\mathbf{w}_1)-\lambda(\mathbf{v}_1,\mathbf{w}_1),\end{cases} \tag{25}$$

$\forall \mathbf{v}_i \in V, \mathbf{w}_i \in W, \lambda \in \mathsf{k}$.

This structure looks complicated but modulo out this subspace is effectively saying that we will demand these elements to be equivalent over the new vector space. As one intuitively expect, $\dim(V\otimes W) = \dim V \cdot \dim W$ compared to $\dim(V\oplus W) = \dim V + \dim W$.

B. Hom Space and Dualities

At this point, we have seen a linear function over and over playing different roles in our theoretical development. But what does the set of all linear functions look like?

Definition VI.2. The **Hom space** is the vector space formed by all linear functions with a pointwise coordinatewise addition and the obvious scalar multiplication. Specifically, given two k-vector spaces V and W

$$\mathrm{Hom}(V,W) \triangleq \{f : V \to W \mid f \text{ a linear function}\}. \tag{26}$$

A familiar Hom space is the dual space consisting of all dual vectors, $V^* \triangleq \mathrm{Hom}(V,\mathsf{k})$. Given a basis of V called $\mathcal{B}_V = \{\mathbf{v}_1,\dots,\mathbf{v}_n\}$, the dual space has a canonical dual basis that looks like

$$\mathbf{v}_i^*(\mathbf{v}) \triangleq \begin{cases}0, & \forall \mathbf{v}\in\mathcal{B}_V\setminus\{\mathbf{v}_i\},\\ 1, & \mathbf{v}=\mathbf{v}_i.\end{cases} \tag{27}$$

Or in short, $v_i^*(\mathbf{v}_j) = \delta_{ij}$ for $i,j\in[1,n]$.

Another interesting Hom space is the vector space V itself $V \cong \mathrm{Hom}(\mathsf{k},V)$. (What is the isomorphism here that maps a vector in V to a linear function?) So with this setup, we can identify some interesting isomorphisms between linear functions and vector spaces. Here are a few interesting propositions.

Proposition VI.3 (Hom-Tensor Duality)**.** $\mathrm{Hom}(V\otimes W,U) \overset{\phi}{\cong} \mathrm{Hom}(V,\mathrm{Hom}(W,U))$ *with*

$$\begin{aligned}\phi : \mathrm{Hom}(V\otimes W,U) &\longrightarrow \mathrm{Hom}(V,\mathrm{Hom}(W,U))\\ f &\longmapsto (v\mapsto(w\mapsto f(v\otimes w)))\end{aligned} \tag{28}$$

Proposition VI.4 (Dual-Hom Duality)**.** *For a finite-dimensional V, $V^*\otimes W \overset{\phi}{\cong} \mathrm{Hom}(V,W)$ with*

$$\begin{aligned}\phi : V^*\otimes W &\longrightarrow \mathrm{Hom}(V,W),\\ f\otimes W &\longmapsto (v\mapsto f(v)w).\end{aligned} \tag{29}$$

Proposition VI.5 (Dual of Tensor Product)**.** $V^*\otimes W^* \overset{\phi}{\cong} (V\otimes W)^*$ *with*

$$\begin{aligned}\phi : V^*\otimes W^* &\longrightarrow (V\otimes W)^*,\\ f\otimes g &\longmapsto ((\mathbf{v}\otimes\mathbf{w})\mapsto f(\mathbf{v})g(\mathbf{w})).\end{aligned} \tag{30}$$

These are almost direct consequence of the definition of Hom space or tensor product, so we will leave the verification to the readers.

C. Tensor Product of Quadratic Forms

Notice that a symmetric bilinear form is a function $B : V\times V \to \mathsf{k}$. We can then put this function into a tensor product space by identifying a corresponding $\tilde{B} : V\otimes V \to \mathsf{k}$. Now, $\tilde{B} \in \mathrm{Hom}(V\otimes V,\mathsf{k}) \cong \mathrm{Hom}(V,\mathrm{Hom}(V,\mathsf{k})) = \mathrm{Hom}(V,V^*)$. If we tensor two symmetric bilinear forms together, say (V_1,B_1) and (V_2,B_2), we should tensor V_1 and V_2 together, and this new function $\tilde{B}_1\otimes\tilde{B}_2$ must lies in $\mathrm{Hom}(V_1\otimes V_2, V_1^*\otimes V_2^*) \cong \mathrm{Hom}(V_1\otimes V_2,(V_1\otimes V_2)^*)$. So we start with "separating" the two arguments of B_i to form $\tilde{B}_i$ as shown in proposition VI.3, and then, we identify the $\tilde{B}_1\otimes\tilde{B}_2$, and at last, we filter the result through the isomorphism described by proposition VI.5 to identify the image function on $(V_1\otimes V_2)^*$. This sounds complicated, but the key step is using the isomorphism shown in proposition VI.5. With that step, we can find that

Definition VI.6. The **tensor product of two symmetric bilinear forms** (V_1,B_1) and (V_2,B_2) follows

$$\begin{aligned}&(B_1\otimes B_2)(\mathbf{v}_{11}\otimes\mathbf{v}_{21},\mathbf{v}_{12}\otimes\mathbf{v}_{22})\\ &\quad= B_1(\mathbf{v}_{11},\mathbf{v}_{12})B_2(\mathbf{v}_{21},\mathbf{v}_{22}).\end{aligned} \tag{31}$$

Similarly, the **tensor production of two quadratic spaces** (V_1,q_1) and (V_2,q_2) follows

$$(q_1\otimes q_2)(\mathbf{v}_1\otimes\mathbf{v}_2) = q_1(\mathbf{v}_1)q_2(\mathbf{v}_2).^6 \tag{32}$$

With this definition, one can easily check the following properties of the orthogonal sum and tensor products

[6] Conveniently, with two diagonal forms, $\langle\lambda_1,\dots,\lambda_n\rangle \otimes \langle\mu_1,\dots,\mu_m\rangle = \langle\lambda_1\mu_1,\lambda_1\mu_2,\dots,\lambda_1\mu_m,\lambda_2\mu_1,\dots,\lambda_n\mu_m\rangle$.

Proposition VI.7. *Given three quadratic spaces* (V_1, q_1)*,* (V_2, q_2)*, and* (V_3, q_3)*, the following relations hold*

$$q_1 \perp q_2 \cong q_2 \perp q_1, \tag{33}$$

$$q_1 \otimes q_2 \cong q_2 \otimes q_1, \tag{34}$$

$$(q_1 \perp q_2) \perp q_3 \cong q_1 \perp (q_2 \perp q_3), \tag{35}$$

$$(q_1 \otimes q_2) \otimes q_3 \cong q_1 \otimes (q_2 \otimes q_3), \tag{36}$$

$$q_1 \otimes (q_2 \perp q_3) \cong q_1 \otimes q_2 \perp q_1 \otimes q_3. \tag{37}$$

These properties hint that we are handling a very promising candidate that can form a commutative ring. There is just one severe drawback: we have no candidate for a subtraction! After all, there is no such thing as an inverse to the orthogonal sum, or is there? A careful reader may realize that we have consistently used a cancellation-like condition in our argument for equivalent forms. We have never formally introduced it, but it is so intuitive that we just assume that it is true. Let's put it down.

Theorem VI.8 (Witt's Cancellation Theorem)**.** *For any three quadratic forms* q_1*,* q_2*, and* q_3*,*

$$q_1 \perp q_3 \cong q_2 \perp q_3 \implies q_1 \cong q_2. \tag{38}$$

This cancellative property is profound but also quite intuitive since we can always observe the subform of a quadratic form by inputing some vector $(\mathbf{v}, \mathbf{0})$. Just for practice, let's use both the cancellation theorem and the chain equivalence theorem to show an interesting statement about tensor products.

Proposition VI.9. *Let* λ *be an element of* $\mathsf{k}^\times$*, and let* q *be a quadratic form over* k *with* $\dim q = 2m$*. Then,* $q \cong \langle\lambda\rangle \otimes q \implies q \cong q_1 \perp \cdots \perp q_m$*, where each* q_i *is a binary form such that* $q_i \cong \langle\lambda\rangle \otimes q_i$*.*

Proof. We will show this by induction on the dimension (with an induction step size of 2). For the base case when $\dim q = 2$, the equality is trivially established.

Suppose that for $\dim q' = 2(m-1)$, $q' \cong \langle\lambda\rangle \otimes q' \implies q' \cong q_1 \perp \ldots \perp q_{m-1}$ in which $q_i \cong \langle\lambda\rangle \otimes q_i$ are binary forms $\forall i < m-1$. Let's find the case when $\dim q = 2m$. Pick some diagonalization so that $q = \langle\mu_1, \ldots, \mu_{2m}\rangle$ and $\langle\lambda\rangle \otimes q = \langle\lambda\mu_1, \ldots, \lambda\mu_{2m}\rangle$. If two quadratic forms $q \cong \langle\lambda\rangle \otimes q$, then, by Witt's chain equivalence theorem, $\exists$ a binary subform. f of q and a binary subform g of $\langle\lambda\rangle \otimes q$ such that $f \cong g$. This follows from the existence of chain equivalence and the definition of simple equivalence. This means that $\exists i, j, k, l \in \mathbb{N}$ such that $\langle\mu_i, \mu_j\rangle \cong \langle\lambda\mu_l, \lambda\mu_k\rangle$. However, we are allowed to permute the diagonal elements to different positions while leaving the form isometric. Hence, WLOG, we can demand that $i = l = 2m-1$ and $j = k = 2m$, *i.e.* $\langle\lambda\rangle \otimes q_m \cong q_m$ in which $q_m = \langle\mu_{2m-1}, \mu_{2m}\rangle$. Then, by the Witt's cancellation theorem, we can cancel out this part and find that $\left\langle\mu_1, \ldots, \mu_{2(m-1)}\right\rangle \cong \left\langle\lambda\mu_1, \ldots, \lambda\mu_{2(m-1)}\right\rangle$. Invoking the induction hypothesis, we can infer that the proposition holds for the $\dim q = 2m$ case. $\square$

Now, with the cancellation theorem, we can identify a well-suited candidate for the additive inverse, except a subtlety here. If we denote orthogonal sum as $+$ and cancellation as $-$. Then

$$\langle a, b, c\rangle - \langle c\rangle = \langle a, b\rangle + \langle c\rangle - \langle c\rangle = \langle a, b\rangle \tag{39}$$

is a known quadratic form, but what is

$$\langle a, b, c\rangle - \langle d\rangle? \tag{40}$$

And perhaps even more fundamentally, what is $-\langle 1\rangle$? Canceling something out of nothing?

VII. GROTHENDIECK-WITT RING: A TEASER

The crying leftover puzzles from the conclusion part of the last section turns out to be just a matter of choice, similar to the argument that negative numbers do not exist. Indeed, it may be bizarre to ask what a negative apple is. Instead, this negation can just carry the information that we can taking a formal cancellation or subtraction by the Witt's cancellation theorem, and we are simply taking the same step as extending the natural numbers to the integers[7]. With this extension of additive inverses over all quadratic forms (more precisely, the isometry classes of regular quadratic forms), one can obtain a commutative ring with operation $+ = \perp$ and $\cdot = \otimes$.

Definition VII.1. The **Grothendieck-Witt** ring of a field k, denoted as $\mathrm{GW}(\mathsf{k})$, is the commutative ring over the Grothendieck group construction of isometry classes of regular quadratic forms with $\perp$ being the ring addition and $\otimes$ being the ring multiplication.

Another way to putting down this ring is to use a free construction similar to defineorem VI.1. As we discussed, each entry in the diagonal form is determined up to square classes. So $\mathrm{GW}(\mathsf{k})$ can be seen as a commutative algebra over the square classes. The free commutative ring over a set S is the polynomial ring $\mathbb{Z}[S]$.

[7] This construction is formalized as a Grothendieck group construction that extends an Abelian monoid to an Abelian group.

Theorem VII.2. *Grothendieck-Witt ring of a field* k *can be expressed as* $\mathbb{Z}[\langle \mathsf{k}^\times/\mathsf{k}^\boxtimes\rangle]/R$ *with* R *being the ideal generated by elements of the form*

$$\begin{cases} \langle 1\rangle - 1, \\ \langle \lambda\mu\rangle - \langle\lambda\rangle\cdot\langle\mu\rangle\,, \\ \langle\lambda,\mu\rangle - \langle \lambda+\mu, (\lambda+\mu)\lambda\mu\rangle\,. \end{cases} \tag{41}$$

Similarly, one can modulo out the hyperbolic form $h \triangleq \langle 1, -1\rangle$ in $\mathrm{GW}(\mathsf{k})$ to obtain another ring, namely the Witt ring

Definition VII.3. The Witt ring of a field k, denoted as $\mathrm{W}(\mathsf{k})$, is defined as

$$\mathrm{W}(\mathsf{k}) \triangleq \mathrm{GW}(\mathsf{k})/\mathbb{Z}h = \mathrm{GW}(\mathsf{k})/\mathbb{Z}\cdot\langle 1,-1\rangle\,. \tag{42}$$

These rings can significantly simplify the process of identifying equivalent quadratic forms and are essential for understanding rings of vector bundles over a topological space, known as k-theories.

Appendix A: Proof of the Representation Criterion

In this appendix, we will prove the representation criterion (lemma IV.6).

Proof. $\Longleftarrow$) This direction is straightforward. Suppose that $B \cong \langle\lambda\rangle \perp B'$; then, simply evaluating the vector $(1,0,0,\ldots)$ over the orthogonal sum, we can find that

$$B((1,\mathbf{0}),(1,\mathbf{0})) = \langle\lambda\rangle(1,1) + B'(\mathbf{0},\mathbf{0}) = \lambda. \tag{A1}$$

Therefore, $\lambda \in D(\langle\lambda\rangle \perp B')$. So how are $D(\langle\lambda\rangle \perp B')$ related to $D(B)$? Since $B \cong \langle\lambda\rangle \perp B'$, we know that $\exists\, T \in \mathrm{GL}_n(\mathsf{k})$ such that $B(T(\mathbf{x}),T(\mathbf{y})) = (\langle\lambda\rangle \perp B')(\mathbf{x},\mathbf{y})$. But T induces a bijection from V to V itself, *i.e.* a vector $\mathbf{x}\in V$ is identified with another unique element $T(\mathbf{x}) \in V$. Thus, $D(B) = D(\langle\lambda\rangle \perp B')$. ⊠ ($\Longrightarrow$ not shown yet)

$\Longrightarrow$) This direction is slightly subtle because it is possible for a diagonal bilinear form to look like

$$B = \langle 0, 1, \ldots\rangle\,. \tag{A2}$$

Then, the first entry $\langle 0\rangle$ is more or less like a "placeholder" that takes one dimension but contributes nothing to $D(B)$ since that part can only represent 0 which is excluded from the definition of $D(B)$. We are going to separate these parts out and take care of them by the end via padding $\langle 0,\ldots\rangle$ to B. The "good" nonzero parts are defined to be **regular** or **nonsingular**, and the zero parts are **singular** or **degenerate** or **totally isotropic**.

Definition A.1. B is a **regular symmetric bilinear form** if either one of the following equivalent statement holds:

(a) The matrix associated with the form $\Lambda \in \mathrm{GL}_n(\mathsf{k})$, *i.e.* $\det\Lambda \neq 0$.

(b) For a given $\mathbf{x}\in V$, $B(\mathbf{x},\mathbf{y}) = 0$, $\forall \mathbf{y}\in V \implies \mathbf{x} = 0$.

(c) $\phi : \mathbf{x} \mapsto B(\cdot,\mathbf{x})$ defines an isomorphism $V \overset{\phi}{\cong} V^*$ in which V^* denotes all linear functions mapping V to k (*i.e.* V^* is the space of all linear functionals or dual vectors).

Any non-regular form is **singular** or **degenerate** or **totally isotropic**.

Now, we are ready to make some progress in understanding how to circumvent this subtlety.

Lemma A.2. *Over a regular quadratic space* V *equipped with the regular symmetric bilinear form* B*, if* U *is a subspace of* V*, then* $\dim U + \dim U^\perp = \dim V$ *with* $U^\perp = \{\mathbf{v}\in V \mid B(\mathbf{v},U) = 0\}$*.*

Proof. Since B is regular,

$$\phi : V \to V^*, \quad \mathbf{v}\mapsto B(\cdot,\mathbf{v}) \tag{A3}$$

is an isomorphism. (This also means that $V^\perp = \{\mathbf{0}\} \iff V$ is regular.) Note that over the dual space V^*, we have a canonical surjective map via restriction $r : V^* \twoheadrightarrow U^*$ with $U^* \cong U$. It is straightforward to see that $r\circ\phi$ is a linear function. Then, $\dim U = \dim U^* = \dim(r\circ\phi)(V)$ due to surjectivity of $r\circ\phi$. But $\ker(r\circ\phi) = \{\mathbf{v}\in V \mid (r\circ\phi)(\mathbf{v}) = 0\} = \{\mathbf{v}\in V \mid B(U,\mathbf{v}) = 0\} = U^\perp$. Therefore, by the rank-nullity theorem, we can infer that

$$\begin{aligned} \dim V &= \dim V^* = \dim(r\circ\phi)(V) + \dim\ker(r\circ\phi) \\ &= \dim U + \dim U^\perp. \end{aligned} \tag{A4}$$

□

Now, we are ready to make claims about the $\Longrightarrow$ direction of lemma IV.6.

Proof. $\Longrightarrow$) Pick $\lambda\in D(B)$. If V is not regular, we can find a subspace W of V such that $V = V^\perp \perp W$ and $D(V) = D(W)$. Since $W^\perp = 0$, we know that W is regular. So any degenerate bilinear form B with $D(B) \neq \emptyset$ will have a regular subform. Therefore, we may assume that V is regular. Since $\lambda \in D(B)$, we can pick a vector $\mathbf{v}\in V$ such that $B(\mathbf{v},\mathbf{v}) = \lambda$. Then, the subspace $U \triangleq \mathrm{span}\{\mathbf{v}\}$ is $\langle\lambda\rangle$. By lemma A.2, we know that $\dim U + \dim U^\perp = \dim V$. Since $U\cap U^\perp = \{\mathbf{0}\}$ and $\dim U + \dim U^\perp = \dim V$, we can infer that $V = U \perp U^\perp$ or that $B = \langle\lambda\rangle \perp B\,|_{U^\perp}$. □

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