# Clifford Algebras by a Physicist

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This note is on the idea of Clifford algebras, some of its fundamental properties, some important operations over a Clifford algebra, and its relation to orthogonal groups. This note leverages the concept of quadratic space and quadratic forms to show some statements over this "quadratic algebra" that are essential for understanding symmetries over the quadratic spaces. Clifford algebras show themselves more and more frequently in different topics in physics; thus, this article also connects the symmetries of the orthogonal group of the quadratic form to the symmetry of spinors, a type of quantities of great importance in the study of quantum field theory.

## CONTENTS



## I. INTUITIVE EXAMPLE: REDISCOVER COMPLEX NUMBERS

What happens to the "algebraic structure" if we have a quadratic space? What is the most general way to produce an "algebraic structure" over a quadratic space? Before diving into the beautiful proofs and theorems, let's start by building some intuition about this enticing topic. (This section's content is elicited from the argument presented in the first few chapters of [1].)

Let's consider a vector $\mathbf{v} \in \mathbb{R}^2$ with a reasonable definition that $\mathbf{v}^2 = |\mathbf{v}|^2$. Then, we can simply take the "stupid" product of $\mathbf{v}^2$ to decipher what we mean by this. Pick the standard basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$, and we can rewrite $\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$ with $v_x, v_y \in \mathbb{R}$. Then,

$$\begin{aligned} v_x^2 + v_y^2 = |\mathbf{v}|^2 = \mathbf{v}^2 &= (v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}})(v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}) \\ &= v_x^2\hat{\mathbf{x}}^2 + v_y^2\hat{\mathbf{y}}^2 + v_xv_y(\hat{\mathbf{x}}\hat{\mathbf{y}} + \hat{\mathbf{y}}\hat{\mathbf{x}}). \end{aligned} \tag{1}$$

Therefore, it is rather intuitive to demand that

$$\hat{\mathbf{x}}^2 = \hat{\mathbf{y}}^2 = 1, \quad \hat{\mathbf{x}}\hat{\mathbf{y}} + \hat{\mathbf{y}}\hat{\mathbf{x}} = 0. \tag{2}$$

Now, interestingly, we can find an odd algebraic rule on $\hat{\mathbf{x}}\hat{\mathbf{y}}$

$$(\hat{\mathbf{x}}\hat{\mathbf{y}})^2 = \hat{\mathbf{x}}(\hat{\mathbf{y}}\hat{\mathbf{x}})\hat{\mathbf{y}} = -\hat{\mathbf{x}}(\hat{\mathbf{x}}\hat{\mathbf{y}})\hat{\mathbf{y}} = -1. \tag{3}$$

This means that in this algebraic system, we naturally have $\hat{\mathbf{x}}\hat{\mathbf{y}} = i$. Not only have we found a basis of the algebra that naturally behaves like $i \in \mathbb{C}$, but also if we restrict our attention to just $\{1, \hat{\mathbf{x}}\hat{\mathbf{y}}\}$, then this subspace is closed under addition and multiplication since

$$(a + b\hat{\mathbf{x}}\hat{\mathbf{y}}) + (c + d\hat{\mathbf{x}}\hat{\mathbf{y}}) = (a + c) + (b + d)\hat{\mathbf{x}}\hat{\mathbf{y}}, \tag{4}$$

$$(a + b\hat{\mathbf{x}}\hat{\mathbf{y}}) \cdot (c + d\hat{\mathbf{x}}\hat{\mathbf{y}}) = (ac - bd) + (bc + ad)\hat{\mathbf{x}}\hat{\mathbf{y}}. \tag{5}$$

This means that this subspace itself forms an "algebraic structure" which we will define later, and even better is that this subspace is isomorphic to $\mathbb{C}$ as we just showed. But, of course, this algebraic structure also admits operations on elements with basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$, so this new "algebraic structure" consists of two vector spaces: $\mathbb{R}^2 \oplus \mathbb{C}$.

## II. ALGEBRA, TENSOR ALGEBRA, AND CLIFFORD ALGEBRA

With the goal of understanding Clifford algebras, we must make concrete definitions about what an **algebra** is and, if possible, which useful properties it has for our discussion. In this section, we will establish some definitions about algebras and Clifford algebras that contribute to our understanding of some interesting properties of the Clifford algebras in the next section.

**Definition II.1** ((associative) $\mathsf{k}$-algebra)**.** Given a field $\mathsf{k}$, a **$\mathsf{k}$-algebra** $(V, +, \cdot)$ is a $\mathsf{k}$-vector space $(V, +)$ with an (associative) bilinear multiplication $\cdot$.

*Remark* II.2. Note that the bilinearity of multiplication implies $\forall \mathbf{x}, \mathbf{y}, \mathbf{z} \in V$

$$(\mathbf{x} + \mathbf{y}) \cdot \mathbf{z} = \mathbf{x} \cdot \mathbf{z} + \mathbf{y} \cdot \mathbf{z}, \tag{6}$$

$$\mathbf{z} \cdot (\mathbf{x} + \mathbf{y}) = \mathbf{z} \cdot \mathbf{x} + \mathbf{z} \cdot \mathbf{y}, \tag{7}$$

$$a\mathbf{x} \cdot b\mathbf{y} = (ab)\mathbf{x} \cdot \mathbf{y} \tag{8}$$

This means that a $\mathsf{k}$-algebra is a ring.

**Definition II.3** (Algebra homomorphism)**.** Given two $\mathsf{k}$-algebras $A$ and $B$, an **algebra homomorphism** is a linear function $f : A \to B$ that is compatible with the multiplication, *i.e.*

$$f(\mathbf{x} \cdot_A \mathbf{y}) = f(\mathbf{x}) \cdot_B f(\mathbf{y}), \quad \forall \mathbf{x}, \mathbf{y} \in A. \tag{9}$$

*Remark* II.4. Since a $\mathsf{k}$-algebra is a ring, an algebra homomorphism is a linear function and a ring homomorphism (which is precisely what the compatibility with multiplication says).

A Clifford algebra is an algebra that emerges from a quadratic space, *i.e.* a vector space equipped with a quadratic form. Thus, it is useful to have a good command on algebras over a vector space. Fortunately, the most "general" algebra over a vector space is known and has an "explicit" form. We will introduce it with the following defineorem.

**Defineorem II.5** (Free algebra over a vector space $V$)**.** Given a vector space $V$, there exists a $\mathsf{k}$-algebra $F(V)$ and a function $V \to F(V)$ such that for any $\mathsf{k}$-algebra, $A$, and any linear function, $f : V \to A$, there is a unique algebra homomorphism $\tilde{f} : F(V) \to A$ such that the diagram

$$\begin{tikzcd}
V \arrow[r, "f"] \arrow[d, "i"'] & A \\
F(V) \arrow[ur, dashed, "\exists! \tilde{f}"'] &
\end{tikzcd}$$

commutes. This free algebra over $V$ is the **tensor algebra**

$$F(V) = \bigoplus_{m=0}^{\infty} V^{\otimes m} = \mathsf{k} \oplus V \oplus (V \otimes V) \oplus \dots. \tag{10}$$

The slogan of our defineorem is that the tensor algebra is the free algebra over a vector space $V$ with base field $\mathsf{k}$, or that this tensor algebra is the most general algebra containing $V$. This result may not be too surprising since we have seen the "free bilinear space" $(V \otimes W)$ and the "free commutative ring" $(\mathbb{Z}[S])$, and drawing connections from the two ideas yields the tensor algebra.

Now, we have both an explicit expression for the free algebra and a quite handy universal property which will be a powerful theoretical gadget for proving general and abstract theorems. We are ready to set up a quadratic structure on top of a vector space and introduce Clifford algebras.

**Definition II.6** (Clifford Algebra)**.** Given a quadratic space $(V, q)$, its **Clifford algebra** is

$$C(V, q) \triangleq F(V)/(\mathbf{v} \otimes \mathbf{v} - q(\mathbf{v}) \cdot 1), \tag{11}$$

in which $(\mathbf{v} \otimes \mathbf{v} - q(\mathbf{v}) \cdot 1)$ denotes the (two-sided) ideal generated by elements of the form $\mathbf{v} \otimes \mathbf{v} - q(\mathbf{v}) \cdot 1$ for all $\mathbf{v} \in V$.

**Example II.7.** The quaternion algebra is a Clifford algebra over a 2-dimensional vector space $V$ over $\mathsf{k}$.

Consider the quadratic space $(V, q)$ with $q = \langle a, b \rangle$ with respect to the standard basis (conveniently named as $\{i, j\}$). Then, it is trivial to note that $i^2 = a$ and $j^2 = b$ are satisfied by modulo-ing out $(\mathbf{x} \otimes \mathbf{x} - q(\mathbf{x}) \cdot 1)$. But we also naturally obtain the $k = ij$ by notice that the anticommutation relation is satisfied since

$$\begin{aligned}
a + b &= q(i + j) = (i + j) \otimes (i + j) \\
&= \cancelto{a}{(i \otimes i)} + \cancelto{b}{(j \otimes j)} + i \otimes j + j \otimes i \\
&\implies (i \otimes j) = -(j \otimes i).
\end{aligned} \tag{12}$$

## III. FUNDAMENTAL PROPERTIES AND OPERATIONS

With the previous discussions building on our intuitions and theoretical tools, we can finally formalize some of our observations into theorems. In this section, we are going to survey some basic properties of Clifford algebra and some useful operations on it. Most of the proofs are credited to or enlightened by [2].

### A. Universal Property and $C(V, q)$ as a Vector Space

**Theorem III.1** (Universal property of $C(V, q)$)**.** *Given a quadratic space $(V, q)$ and a $\mathsf{k}$-algebra $A$, if a linear function $f : V \to A$ satisfies $f(\mathbf{v})^2 - q(\mathbf{v}) \cdot 1 = 0 \in A$, $\forall \mathbf{v} \in V$, then there exists a unique algebra homomorphism $\tilde{f} : C(V, q) \to A$ such that the diagram*

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "i"'] & A \\ C(V,q) \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

*commutes.*

*Proof.* Applying the universal property of the tensor algebra on $V$, we know that there is a unique algebra homomorphism $f_F : F(V) \to A$ extended from $f$. Observe that

$$f_F(\mathbf{x} \otimes \mathbf{x} - q(\mathbf{x}) \cdot 1) = f(\mathbf{x})^2 - q(\mathbf{x}) \cdot 1 = 0. \quad (13)$$

Therefore, $(\mathbf{x} \otimes \mathbf{x} - q(\mathbf{x}) \cdot 1) \subset \ker(f_F)$. This induces an algebra homomorphism $\tilde{f} : C(V, q) \to A$ with the desired commutativity under canonical inclusion $\tilde{f}(\mathbf{x}) = f(\mathbf{x})$. Since $C(V, q)$ is generated by $V$, the mapping $\tilde{f}$ is unique. $\square$

*Remark* III.2. In hindsight, our observation in example II.7 that a quaternion algebra is a Clifford algebra is sensible. Or, at least, a quaternion algebra should emerge naturally as a subalgebra from a Clifford algebra since Clifford algebra is the most general algebra over a quadratic space, and all quaternion demands are some quadratic relation among its basis.

**Corollary III.3.** *For a quadratic space $V$ with two equivalent*[1] *quadratic forms $q \cong q'$, their Clifford algebra $C(V, q) \cong C(V, q')$.*

*Proof.* See problem 4. $\square$

**Theorem III.4** (Basis (Poincaré-Birkhoff-Witt))**.** *For an $n$-dimensional quadratic space $V$ with its basis $\mathcal{B}_V = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$, its Clifford algebra $C(V, q)$ is spanned by*

$$\mathcal{B}_{C(V,q)} = \{1\} \cup \{\mathbf{v}_{i_1} \mathbf{v}_{i_2} \ldots \mathbf{v}_{i_k} \mid 1 \leq i_1 < \ldots < i_k \leq n\}. \quad (14)$$

[1] Two quadratic forms are equivalent if the two are the same up to an invertible linear function. For more information, see, for example, my notes on quadratic spaces.

*Remark* III.5. Just to make crystal-clear about my notation, let's consider $C(\mathbb{R}^3)$. Then, it is spanned by $\{1, \hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}, \hat{\mathbf{x}}\hat{\mathbf{y}}, \hat{\mathbf{x}}\hat{\mathbf{z}}, \hat{\mathbf{y}}\hat{\mathbf{z}}, \hat{\mathbf{x}}\hat{\mathbf{y}}\hat{\mathbf{z}}\}$.

*Proof.* Since we can always diagonalize $q$, we can pick a basis $\mathcal{B}_V$ such that $q(\mathbf{v}_j) = q_j$ in which $q_j$ is the $j^{\text{th}}$ entry of the diagonalization. Notice that $F(V)$ is spanned by vectors of the form $\mathbf{v}_i \otimes \mathbf{v}_j \otimes \ldots$. However, the ideal generated by $\mathbf{x} \otimes \mathbf{x} - q(\mathbf{x}) \cdot 1$ demands that $\mathbf{v}_i \ldots \mathbf{v}_j \mathbf{v}_j \ldots \mathbf{v}_k = q_j \big(\mathbf{v}_i \ldots \cancelto{1}{\mathbf{v}_j \mathbf{v}_j} \ldots \mathbf{v}_k\big) \in C(V, q)$. Then, as discussed in example II.7,

$$q_i + q_j = (\mathbf{v}_i + \mathbf{v}_j) \otimes (\mathbf{v}_i + \mathbf{v}_j) \implies \mathbf{v}_i \otimes \mathbf{v}_j + \mathbf{v}_j \otimes \mathbf{v}_i = 0, \quad (15)$$

*i.e.* $\mathbf{v}_i \mathbf{v}_j = -\mathbf{v}_j \mathbf{v}_i \in C(V, q)$. Therefore, we can always put the monomial $\mathbf{v}_i \ldots \mathbf{v}_k$ into a standard form $\mathbf{v}_{i_1} \mathbf{v}_{i_2} \ldots \mathbf{v}_{i_k}$ in which $1 \leq i_1 < \ldots < i_k \leq n$. $\square$

**Corollary III.6.** $\dim C(V, q) \leq 2^{\dim V}$.

*Proof.* See problem 5. $\square$

*Remark* III.7. In fact, just like what we intuitively expected, $\dim C(V, q) = 2^{\dim V}$. One can make an argument by introducing the notion of gradation, graded product, and filtration, which we will not do here. One can find some enlightening discussions and proofs in Section V.1 of [3].

### B. Involution and $\mathbb{Z}/2\mathbb{Z}$-Gradation

Recall that in our intuitive example, we constructed a vector space isomorphic to $\mathbb{R}^2 \oplus \mathbb{C}$ and discovered that the $\mathbb{C}$ part is closed under both addition and multiplication. We can recognize that this intuitive construction is precisely how one constructs a Clifford algebra $C(\mathbb{R}^2, \langle 1, 1 \rangle)$; however, why do we observe a subalgebra inside this Clifford algebra? Do we always get a subalgebra for free while constructing a Clifford algebra? These puzzles can be solved by introducing a canonical involution.

**Definition III.8** (Canonical involution)**.** For a non-characteristic-2 field $\mathsf{k}$ and a $\mathsf{k}$-vector space $V$, a **canonical involution** over $C(V, q)$ is

$$\alpha : C(V, q) \to C(V, q), \quad \mathbf{v} \mapsto -\mathbf{v}. \quad (16)$$

*Remark* III.9. To clarify the notation here, the negation on $\mathbf{v}$ here means that for any element in $C(V, q)$ of the form

$$\mathbf{x} = \sum_i a_i \left( \prod_r^m \mathbf{v}_{i_r} \right), \quad (17)$$

the canonical involution yields

$$\alpha(\mathbf{x}) = \sum_i a_i \left( \prod_r^m -\mathbf{v}_{i_r} \right) = \sum_i (-1)^m a_i \left( \prod_r^m \mathbf{v}_{i_r} \right). \tag{18}$$

So for instance, $\alpha(1) = 1$, $\alpha(\hat{\mathbf{x}}) = -\hat{\mathbf{x}}$, $\alpha(\hat{\mathbf{x}}\hat{\mathbf{y}}) = \hat{\mathbf{x}}\hat{\mathbf{y}}$.

**Proposition III.10.** *The canonical involution is an automorphism (an isomorphic endomorphism) over $C(V, q)$.*

*Proof.* See problem 6. □

**Theorem III.11** ($\mathbb{Z}/2\mathbb{Z}$-grading of Clifford Algebra)**.** *Any Clifford algebra (over a non-characteristic-2 field $\mathsf{k}$) can be decomposed into*

$$C(V, q) = C^+(V, q) \oplus C^-(V, q), \tag{19}$$

*such that*

$$\begin{cases} C^+(V,q)C^+(V,q) = C^-(V,q)C^-(V,q) = C^+(V,q), \\ C^-(V,q)C^+(V,q) = C^+(V,q)C^-(V,q) = C^-(V,q). \end{cases} \tag{20}$$

*Proof.* Consider the canonical involution $\alpha$. As shown in the previous remark,

$$\alpha\left( \sum_i a_i \left( \prod_r^m \mathbf{v}_{i_r} \right) \right) = \sum_i (-1)^m a_i \left( \prod_r^m \mathbf{v}_{i_r} \right). \tag{21}$$

Since $(-1)^m = 1$ if $2 \mid m$ and $(-1)^m = -1$ if $2 \nmid m$, we know that we can rewrite it as

$$\begin{aligned} \alpha\left( \sum_i a_i \left( \prod_r^m \mathbf{v}_{i_r} \right) \right) = & \sum_i a_i \left( \prod_r^{\text{even } m} \mathbf{v}_{i_r} \right) \\ & - \sum_i a_i \left( \prod_r^{\text{odd } m} \mathbf{v}_{i_r} \right). \end{aligned} \tag{22}$$

From this, we can infer that if $C^+(V, q) \triangleq \{\mathbf{x} \in C(V, q) \mid \alpha(\mathbf{x}) = \mathbf{x}\}$ and $C^-(V, q) \triangleq \{\mathbf{x} \in C(V, q) \mid \alpha(\mathbf{x}) = -\mathbf{x}\}$, then $C(V, q) = C^+(V, q) \oplus C^-(V, q)$. Since $\alpha$ is an isomorphism on $C(V, q)$, $\alpha(C^+(V, q)) = C^+(V, q)$ and $\alpha(C^-(V, q)) = -C^-(V, q)$. Thus,

$$\begin{aligned} &\begin{cases} \alpha(C^+(V,q)C^+(V,q)) = C^+(V,q)C^+(V,q), \\ \alpha(C^-(V,q)C^-(V,q)) = C^-(V,q)C^-(V,q), \\ \alpha(C^+(V,q)C^-(V,q)) = -C^+(V,q)C^-(V,q), \\ \alpha(C^-(V,q)C^+(V,q)) = -C^-(V,q)C^+(V,q) \end{cases} \\ &\implies \text{eq. (20)}. \end{aligned} \tag{23}$$

□

*Remark* III.12. Another way to show eq. (20) is by noticing that $C^{\pm}(V, q)$ consists only of sums of even/odd-"dimensional" simple tensors[2]. The multiplication adds the dimension since

$$\left( \bigotimes_i^n \mathbf{v}_i \right) \otimes \left( \bigotimes_j^m \mathbf{u}_j \right) = \bigotimes_k^{m+n} \mathbf{w}_k, \tag{24}$$

in which $\mathbf{w}_k = \mathbf{v}_i$ if $k \leq m$ and $\mathbf{w}_k = \mathbf{u}_j$ if $k > m$. Leveraging this equation, one can show both directions of the inclusion of eq. (20) straightforwardly.

**Corollary III.13.** *The even part of the Clifford algebra $C^+(V, q)$ is a subalgebra of $C(V, q)$.*

*Proof.* Since $C^+(V, q)C^+(V, q) = C^+(V, q)$, *i.e.* $C^+(V, q)$ is closed under multiplication, $C^+(V, q)$ is a subalgebra of $C(V, q)$. □

**Definition III.14** (Transpose and conjugation)**.** Inherited from the tensor algebra, the Clifford algebra also possesses an anti-automorphism, *viz.* the **transpose**[3]

$$\begin{aligned} T: \quad & F(V) \longrightarrow F(V), \\ & \mathbf{v}_1 \otimes \mathbf{v}_2 \otimes \ldots \otimes \mathbf{v}_n \longmapsto \mathbf{v}_n \otimes \mathbf{v}_{n-1} \otimes \ldots \otimes \mathbf{v}_1. \end{aligned} \tag{25}$$

Then, the **conjugation** (now an involutive anti-automorphism) is defined as

$$\dagger \triangleq \alpha \circ T = T \circ \alpha, \tag{26}$$

Notationally, we will use $\mathbf{v}^T$ to denote $T(\mathbf{v})$ and $\mathbf{v}^\dagger$ for $\dagger(\mathbf{v})$.

To reiterate the properties of these operations, from some $\mathbf{x}, \mathbf{y} \in C(V, q)$

$$\begin{aligned} \alpha(\mathbf{x}\mathbf{y}) &= \alpha(\mathbf{x})\alpha(\mathbf{y}), \\ (\mathbf{x}\mathbf{y})^T &= \mathbf{y}^T \mathbf{x}^T, \\ (\mathbf{x}\mathbf{y})^\dagger &= \mathbf{y}^\dagger \mathbf{x}^\dagger. \end{aligned} \tag{27}$$

## IV. SYMMETRIES OF CLIFFORD ALGEBRAS, ORTHOGONAL GROUPS, AND SPINORS

With these interesting operations, we are ready to extend our observations one step further. Over a quadratic space, isometries are interesting mappings which "preserve" the quadratic form. For a given quadratic form, there is an associated orthogonal group that entails all the isometries. So is there a similar notion of isometry over a Clifford algebra? We will make this concrete in the following section.

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[2] This notion of dimension should really be the order or degree of the tensor.

[3] This operation is also called reverse and is related to the construction of an opposite algebra.

### A. Reflection, Orthogonal Group, and Clifford-Lipschitz Group

From theorem III.1, we know that Clifford algebra is the most general algebra we can possibly build out of a quadratic space. It is natural to ask what kind of mappings is going to preserve the "length" of a vector $\mathbf{v} \in V$ on which we build our Clifford algebra, *i.e.* what are the elements $\mathbf{x} \in C(V, q)$ such that $\mathbf{x} \cdot \mathbf{v} \in V$ and $q(\mathbf{x} \cdot \mathbf{v}) = q(\mathbf{v})$? Over the quadratic space, these are just the orthogonal group $O(q)$; thus, hyperplane reflections may be a good starting point, but how can we reflect a vector $\mathbf{v} \in V$ with respect to a (normal) vector $\mathbf{x} \in V$?

**Lemma IV.1.** *A reflection of any vector* $\mathbf{v}$ *with respect to* $\mathbf{x} \in C(V, q)^\times \cap V$ *over the Clifford algebra is*

$$\rho_x : C(V, q) \cap V \mapsto C(V, q) \cap V, \quad \mathbf{v} \mapsto -\mathbf{x}\mathbf{v}\mathbf{x}^{-1}. \tag{28}$$

*Remark* IV.2. Nota bene: Since $C(V, q)$ is in general not a division ring, $C(V, q)^\times \neq C(V, q) \setminus \{0\}$. Instead, $C(V, q)^\times$ means the invertible elements in $C(V, q)$.

*Proof.* Recall that a reflection with respect to a hyperplane over a quadratic space is defined as

$$\begin{aligned} \rho_x(\mathbf{v}) =& \mathbf{v} - \frac{2B(\mathbf{x}, \mathbf{v})\mathbf{x}}{q(\mathbf{x})} \\ =& \mathbf{v} - [q(\mathbf{x} + \mathbf{v}) - q(\mathbf{x}) - q(\mathbf{v})] \frac{\mathbf{x}}{q(\mathbf{x})}. \end{aligned} \tag{29}$$

Over $C(V, q)$, this is just

$$\rho_x(\mathbf{v}) = \mathbf{v} - \left[(\mathbf{x} + \mathbf{v})^2 - \mathbf{x}^2 - \mathbf{v}^2\right] \frac{\mathbf{x}}{q(\mathbf{x})}. \tag{30}$$

Now, this is a physicist's dream because indeed over $C(V, q)$, it is straightforward to note that $\mathbf{x}/q(\mathbf{x}) = \mathbf{x}^{-1}$ (given that $\mathbf{x} \in C(V, q)^\times \cap V$). Then,

$$\rho_x(\mathbf{v}) = \mathbf{v} - (\mathbf{x}\mathbf{v} + \mathbf{v}\mathbf{x})\mathbf{x}^{-1} = -\mathbf{x}\mathbf{v}\mathbf{x}^{-1}. \tag{31}$$

□

*Remark* IV.3. A keen reader may already notice that in this proof, I was confident in putting down $q(\mathbf{x}) = \mathbf{x}^2$ in the numerator but not $q(\mathbf{x})^{-1}$ as $1/(\mathbf{x}\mathbf{x})$ in the denominator. When $\mathbf{x} \in V$, both notations are reasonably clear; however, the subtlety comes from what we mean by division over a potentially non-division ring. This is not too difficult as long as we find the center of the ring so that this division is unambiguous. Indeed, $\mathbf{Z}(C(V, q))$ is another interesting (and much more complicated) topic on its own.

Now, we will move onto a grand construction of an exciting group that is a "good" analogy of the orthogonal group in Clifford algebra.

**Definition IV.4** (Clifford-Lipschitz group)**.** The **Clifford-Lipschitz group** is defined as

$$\Gamma \triangleq \{\mathbf{x} \in C(V, q)^\times \mid \alpha(\mathbf{x})\mathbf{v}\mathbf{x}^{-1} \in V, \forall \mathbf{v} \in V\}$$

Here, we have defined a "twisted adjoint action (conjugation)"

$$\widetilde{\mathrm{Ad}}_x : C(V, q) \mapsto C(V, q), \quad \mathbf{v} \mapsto \alpha(\mathbf{x})\mathbf{v}\mathbf{x}^{-1}. \tag{32}$$

This action we designed in definition IV.4 seems weird. But this is just to include all reflections if $\mathbf{x} \in C(V, q)^\times \cap V$. Along with this action, we also find an nice mapping which later will be quite useful

**Definition IV.5** (Spinor norm)**.** The spinor norm is defined as

$$N : C(V, q) \to C(V, q), \quad \mathbf{x} \mapsto \mathbf{x}^\dagger \mathbf{x}. \tag{33}$$

*Remark* IV.6. Notice that for any $\mathbf{v} \in V$, $N(\mathbf{v}) = \mathbf{v}^\dagger \mathbf{v} = -\mathbf{v}^2 = -q(\mathbf{v})$.

### B. Short Exact Sequence Theorem: A Long Proof

Finally, we are at a point to understand why we have constructed this strange Clifford-Lipschitz group, and it all connects smoothly to $O(q)$, the orthogonal group.

**Theorem IV.7** (Short exact sequence theorem)**.** *For a Clifford algebra $C(V, q)$ over a regular quadratic space $(V, q)$ with finite dimension, the group homomorphism*

$$\widetilde{\mathrm{Ad}} : \Gamma \to O(q), \quad \mathbf{x} \mapsto \widetilde{\mathrm{Ad}}_x. \tag{34}$$

*is surjective, and its kernel is* $\mathsf{k}^\times$*, i.e. we have the following short exact sequence*

$$1 \longrightarrow \mathsf{k}^\times \longrightarrow \Gamma \longrightarrow O(q) \longrightarrow 1. \tag{35}$$

*Proof.* This proof is presented in [4]. Note that this will be a length proof with two lemmata. Also, we will invoke the **Cartan-Dieudonné theorem**: Given an $n$-dimensional regular quadratic space $(V, q)$, every element $\sigma \in O(q)$ is a composition of at most $n$ hyperplane reflections. A detailed proof can be found in [3, 4].

**Step 1:** Let's first show that the kernel of $\widetilde{\mathrm{Ad}}$ is $\mathsf{k}^\times$.

**Lemma IV.8.**

$$\ker(\widetilde{\mathrm{Ad}}) = \{\mathbf{x} \in \Gamma \mid \widetilde{\mathrm{Ad}}_x = id_V\} = \mathsf{k}^\times. \tag{36}$$

*Proof of lemma IV.8.* Pick an element $\mathbf{x} \in \ker(\widetilde{\mathrm{Ad}})$. Then, it satisfies

$$\alpha(\mathbf{x})\mathbf{v}\mathbf{x}^{-1} = \mathbf{v} \implies \alpha(\mathbf{x})\mathbf{v} = \mathbf{v}\mathbf{x}, \quad \forall \mathbf{v} \in V. \tag{37}$$

Since $C(V,q) = C^+(V,q) \oplus C^-(V,q)$, we can decompose $\mathbf{x} = \mathbf{x}^+ + \mathbf{x}^-$ in which $\mathbf{x}^+ \in C^+(V,q)$ and $\mathbf{x}^- \in C^-(V,q)$. Then, we can obtain

$$\mathbf{x}^+\mathbf{v} = \mathbf{v}\mathbf{x}^+, \quad -\mathbf{x}^-\mathbf{v} = \mathbf{v}\mathbf{x}^- \tag{38}$$

Now, we would like to show that $\mathbf{x}^+ \in \mathsf{k}^\times$, or equivalently, for any $\hat{\mathbf{y}} \in V$, $\mathbf{x}^+$ must contain an even power of $\hat{\mathbf{y}}$. Since $C^+(V,q)$ is a vector space, we can decompose $\mathbf{x}^+ = \mathbf{a} + \mathbf{b}\hat{\mathbf{y}}$ uniquely in which $\mathbf{a} \in C^+(V,q)$ and $\mathbf{b} \in C^-(V,q)$ contains no power of $\hat{\mathbf{y}}$. Let $\mathbf{v} = \hat{\mathbf{y}}$. Then, we find

$$\mathbf{a}\hat{\mathbf{y}} + \mathbf{b}q(\hat{\mathbf{y}}) = \hat{\mathbf{y}}\mathbf{a} + \hat{\mathbf{y}}\mathbf{b}\hat{\mathbf{y}} = \mathbf{a}\hat{\mathbf{y}} - \mathbf{b}q(\hat{\mathbf{y}}) \tag{39}$$

This shows that $\mathbf{b} = 0$. As for $\mathbf{a}$, notice that eq. (38) holds for all $\mathbf{v}$; therefore, $\mathbf{a}\hat{\mathbf{y}} = \hat{\mathbf{y}}\mathbf{a} \implies \mathbf{a} = a \in \mathsf{k}^\times$. An almost identical argument applies for $\mathbf{x}^-$ to conclude that $\mathbf{x}^- = 0$. Thus, $\ker(\widetilde{\mathrm{Ad}}) = \mathsf{k}^\times$. □

**Step 2:** For the sake of a well-defined group function, we need $\widetilde{\mathrm{Ad}}(\Gamma) \subset O(q)$. Although we claimed that $\widetilde{\mathrm{Ad}}$ is a group homomorphism, we are not sure that $\widetilde{\mathrm{Ad}}(\Gamma)$ is just $O(q)$ (instead of some other wild mapping on $\mathbf{v} \in V$). Let's see if this is actually true.

**Lemma IV.9.** *The restricted spinor norm*

$$N : \Gamma \mapsto \mathsf{k}^\times, \quad \mathbf{x} \mapsto \mathbf{x}^\dagger\mathbf{x} \tag{40}$$

*is a group homomorphism.*

*Proof of lemma IV.9.* The main concern about $N$ being a group homomorphism is whether the image $N(\Gamma) \subset \mathsf{k}^\times$. Pick any $\mathbf{v} \in V$ and $\mathbf{x} \in \Gamma$. Then, we know that $\widetilde{\mathrm{Ad}}_x(\mathbf{v}) = \mathbf{u}$ for some $\mathbf{u} \in V$. Now, we can apply transpose (an anti-automorphism) on both sides of the equation and find that $\left[\widetilde{\mathrm{Ad}}_x(\mathbf{v})\right]^T = \mathbf{u}$. This yields

$$\begin{aligned} \alpha(\mathbf{x})\mathbf{v}\mathbf{x}^{-1} &= (\mathbf{x}^{-1})^T\mathbf{v}\alpha(\mathbf{x})^T = (\mathbf{x}^T)^{-1}\mathbf{v}\mathbf{x}^\dagger \\ \implies \mathbf{v} &= \mathbf{x}^T\alpha(\mathbf{x})\mathbf{v}\mathbf{x}^{-1}\left(\mathbf{x}^\dagger\right)^{-1} = \alpha(\mathbf{x}^\dagger\mathbf{x})\mathbf{v}\left(\mathbf{x}^\dagger\mathbf{x}\right)^{-1}. \end{aligned} \tag{41}$$

Since this statement is true for arbitrary $\mathbf{v} \in V$, we know that $\mathbf{x}^\dagger\mathbf{x} \in \ker(\widetilde{\mathrm{Ad}}) = \mathsf{k}^\times$ by lemma IV.8. Thus, the image of $N$ is $\mathsf{k}^\times$ as expected. Now, indeed, $N(\mathbf{x}\mathbf{y}) = \mathbf{y}^\dagger\mathbf{x}^\dagger\mathbf{x}\mathbf{y} = N(\mathbf{x})N(\mathbf{y})$ as expected. □

Now, notice that $\forall\ \mathbf{v} \in V$ and $\mathbf{x} \in \Gamma$, $N(\widetilde{\mathrm{Ad}}_x(\mathbf{v})) = N(\alpha(\mathbf{x})\mathbf{v}\mathbf{x}^{-1}) = -q(\mathbf{v})N(\alpha(\mathbf{x}))N(\mathbf{x}^{-1}) = -q(\mathbf{v}) = N(\mathbf{v})$. Then, since this is true for all $\mathbf{v} \in V$, $\widetilde{\mathrm{Ad}}_x|_V \in O(q)$. Extending to all $\mathbf{x} \in \Gamma$, we know that $\widetilde{\mathrm{Ad}}(\Gamma) \subset O(q)$.

**Step 3:** $\widetilde{\mathrm{Ad}}$ is surjective: If $\mathbf{x} \in V \cap \Gamma$, $\rho_x(\mathbf{v}) = \widetilde{\mathrm{Ad}}_x(\mathbf{v})$ for any $\mathbf{v} \in V$ as shown in lemma IV.1. Then, by Cartan-Dieudonné theorem, we know that a finite compositions of $\widetilde{\mathrm{Ad}}_x$ yields any element in $O(q)$. Note that $\widetilde{\mathrm{Ad}}_y \circ \widetilde{\mathrm{Ad}}_x = \widetilde{\mathrm{Ad}}_{yx}$, thus $\widetilde{\mathrm{Ad}}$ is a surjective homomorphism. □

Excellent! We have found the best group over $C(V,q)$ that is an analogy to $O(q)$ over $V$. What is even better is that we now are convinced that the appropriate group action should be the twisted adjoint action.

### C. Spin Group: A Teaser to Enthusiasts

While mathematicians constructed these interesting Clifford algebras and their symmetries, physicists in the early to mid 20th century noticed a type of quantity called **spinors**. These quantities emerged from the spin of a particle, an intrinsic angular momentum independent of the particle's rotation. These 4-dimensional quantities, albeit similar to four-vectors, do not transform like a four-vector. This had confused physicists for a decade. The breakthrough happened in 1928 by a then-26-year-old prodigy Paul Dirac. In his proposal of the **Dirac equation**, Dirac observed that the spin and its associated symmetries can naturally appear from a differential equation equipped with matrix-like coefficients [5, 6]. For more contexts on spinors from the perspective of physicists, readers can refer to appendix A for more detailed and anecdotal discussion. This later turns out to be a Clifford algebra over the flat spacetime manifold. With this bright observation, a sequence of discoveries made the notion of symmetries over spinors precise by finding a subgroup in the Clifford-Lipschitz group, *viz.* the spin group.

This is a profound but intriguing topic with very active research. Therefore, we will only take a peek of the beauty of spin groups. But hopefully, by the end of this section, the readers will be convinced that there are interesting connections from the notion of spin groups to other fields of physics and mathematics. For those enthusiasts, this teaser section may provide more motivations for a deep dive into this interesting topic.

**Definition IV.10** (Spin group and Pin group)**.** The spin group and (poorly punned) pin group is defined as

$$\begin{aligned} \mathrm{Spin}(V,q) &= \{\mathbf{x} \in \Gamma \cap C^+(V,q) \mid \mathbf{x}^\dagger\mathbf{x} = \pm 1\}, \\ \mathrm{Pin}(V,q) &= \{\mathbf{x} \in \Gamma \mid \mathbf{x}^\dagger\mathbf{x} = \pm 1\}. \end{aligned} \tag{42}$$

These groups over $\mathbb{R}^{p,m}$ (*i.e.* $\mathbb{R}$-vector space with a quadratic form $q = p\langle 1\rangle + m\langle -1\rangle$, and we will keep the convention as $(p, m)$) are particularly enticing as discussed in [7]. We here provide a list of these spin groups as a teaser

$$\begin{aligned}
&\mathrm{Spin}(1,1) = \mathbb{R}_{>0},\ \mathrm{Spin}(2) = \mathrm{U}(1),\\
&\mathrm{Spin}(2,1) = \mathrm{SL}(2,\mathbb{R}),\ \mathrm{Spin}(3) = \mathrm{SU}(2),\\
&\mathrm{Spin}(2,2) = \mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R}),\\
&\mathrm{Spin}(3,1) = \mathrm{SL}(2,\mathbb{C}),\ \mathrm{Spin}(4) = \mathrm{SU}(2) \times \mathrm{SU}(2),\\
&\mathrm{Spin}(3,2) = \mathrm{Sp}(4,\mathbb{R}),\ \mathrm{Spin}(4,1) = \mathrm{Sp}(1,1;\mathbb{H}),\\
&\mathrm{Spin}(5) = \mathrm{Sp}(2,\mathbb{H}),\ \mathrm{Spin}(3,3) = \mathrm{SL}(4,\mathbb{R}),\\
&\mathrm{Spin}(4,2) = \mathrm{SU}(2,2),\ \mathrm{Spin}(5,1) = \mathrm{SL}(2,\mathbb{H}),\\
&\mathrm{Spin}(6) = \mathrm{SU}(4).
\end{aligned}$$

Definitely, these are interesting groups. But on top of these Clifford algebra, we can further identify a (maximal semisimple) Lie algebra, an enticing structure that gives us a notion of the elements that are "infinitesimally close" to the group identity and provides a complete description of the local structure of the group. These local behaviors are very significant for physicists, especially field theorists, working over different real-valued or complex-valued manifold to explore its local properties. Again, we compile a list of some interesting Lie algebra over the Clifford algebra for your reference. By convention, a Lie algebra associated with a Lie group is denoted with small fraktur letters.

$$\begin{aligned}
&\mathfrak{cl}(2) = \mathfrak{sl}(2,\mathbb{R}) = \mathfrak{cl}(1,1),\\
&\mathfrak{cl}(3) = \mathfrak{spin}(3,1) \simeq \mathfrak{sl}(2,\mathbb{C}) = \mathfrak{cl}(1,2),\\
&\mathfrak{cl}(2,1) = \mathfrak{spin}(2,1) \oplus \mathfrak{spin}(2,1) \simeq \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{sl}(2,\mathbb{R}),\\
&\mathfrak{cl}(4) = \mathfrak{spin}(5,1) \simeq \mathfrak{sl}(2,\mathbb{H}) = \mathfrak{cl}(1,3),\\
&\mathfrak{cl}(3,1) = \mathfrak{spin}(3,3) \simeq \mathfrak{sl}(4,\mathbb{R}),\\
&\mathfrak{cl}(5) = \mathfrak{spin}(5,1) \oplus \mathfrak{spin}(5,1) \simeq \mathfrak{sl}(2,\mathbb{H}) \oplus \mathfrak{sl}(2,\mathbb{H}),\\
&\mathfrak{cl}(4,1) = \mathfrak{sl}(4,\mathbb{C}) = \mathfrak{cl}(2,3),\\
&\mathfrak{cl}(3,2) = \mathfrak{sl}(4,\mathbb{R}) \oplus \mathfrak{sl}(4,\mathbb{R}),\\
&\mathfrak{cl}(6) = \mathfrak{su}(6,2) = \mathfrak{cl}(5,1),\ \mathfrak{cl}(7) = \mathfrak{sl}(8,\mathbb{C}),\\
&\mathfrak{cl}(6,1) = \mathfrak{sl}(4,\mathbb{H}),\\
&\mathfrak{cl}(8) = \mathfrak{sl}(16,\mathbb{R}),\ \mathfrak{cl}(7,1) = \mathfrak{sl}(8,\mathbb{H}),\\
&\mathfrak{cl}(8,1) = \mathfrak{sl}(16,\mathbb{R}),\\
&\mathfrak{cl}(9) \simeq \mathfrak{cl}^{+}(9,1) = \mathfrak{sl}(16,\mathbb{R}) \oplus \mathfrak{sl}(16,\mathbb{R}).
\end{aligned}$$

There are too many ideas unexplored and too many connections not addressed on this topic. However, what matters the most is that with these initial attempts to understand Clifford algebra, we may agree that this is a fascinating topic waiting for some grand quest of exploration.

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## V. EXERCISES

Practice makes perfect. Readers may opt to complete as many problems as possible. The difficulty of a problem is indicated by the number of stars in the parenthesis by the problem title. Solutions (or partial solutions) to all of the problems are available for readers to check their understandings.

*Problem* 1 ($\star$). As shown in section I, there is a hidden subalgebra $\mathbb{C}$ inside $C(\mathbb{R}^2, |v|^2)$. But actually, $\mathbb{C}$ itself is a Clifford algebra. Find the quadratic form that generates the "honest" complex number. What is the quadratic space $V$ in this case? The exterior algebra $\bigwedge(V)$ over an $n$-dimensional vector space $V$ is an anticommutative algebra such that for any two basis vectors $\mathbf{e}_i$, $\mathbf{e}_j$, we have

$$0 = \{\mathbf{e}_i, \mathbf{e}_j\} = \mathbf{e}_i \wedge \mathbf{e}_j + \mathbf{e}_j \wedge \mathbf{e}_i, \tag{43}$$

in which $\wedge$ denotes the bilinear multiplication over the algebra. Given an $n$-dimensional vector space $V = \mathsf{k}^n$, argue that its exterior algebra $\bigwedge(V)$ is a Clifford algebra by finding its corresponding quadratic form.

*Problem* 2 ($\star$). Find $C(\mathbb{R}^3, |\mathbf{v}|^2)$ with the standard Euclidean metric $|\cdot|^2 = \langle 1,1,1\rangle$. What is its even subalgebra $C^+(\mathbb{R}^3, |\mathbf{v}|^2)$?

*Problem* 3 ($\star\star$). Given a quadratic space $(V, q)$, we know that by polarizing $q$, we obtain a (bijective) symmetric bilinear space $(V, B)$. Therefore, it is quite natural to realize that there is an equivalent definition of the Clifford algebra over a symmetric bilinear space $(V, B)$. How would you modify definition II.6 to make it compatible over $(V, B)$? *Hint*: Some hinted structure from eq. (12) may be helpful.

*Problem* 4 ($\star\star$). Show corollary III.3. *Hint*: One may find the universal property shown in theorem III.1 very handy, and also to find an isomorphism is to find an inverse for a known homomorphism.

*Problem* 5 ($\star$). Show corollary III.6. *Hint*: Recall that $2^n = \sum_{k=0}^{n} \binom{n}{k}$.

*Problem* 6 ($\star$). Show proposition III.10. *Hint*: Corollary III.3 may be helpful.

## Appendix A: More Contexts on Spinors in Physics

As a physicist, I am more interested in applying mathematical structures to understand the phenomena around us. As many of my mathematician friends concluded, physicists are horrible mathematicians who can only make arguments over $\mathbb{R}$. (This is true because physicists always group those colleagues who frequently use discrete lattice structure instead of the standard Taylor expansion as "applied mathematicians".) Nonetheless, there is a sound reason for physicists to focus on $\mathbb{R}$ vector spaces due to its analytic structure which admits a field theory naturally. In the following two sections, I will provide a quick narrative for the motivation of understanding spin, spinors, and Clifford algebra in physics for readers' references.

### 1. Four-Vectors

Many of you may have seen Lorentz transformations

$$\begin{cases} x' = \dfrac{1}{\sqrt{1-v^2/c^2}}(x-vt), \\ t' = \dfrac{1}{\sqrt{1-v^2/c^2}}\Big(t-\dfrac{vx}{c^2}\Big). \end{cases} \tag{A1}$$

However, those of you who are interested in trigonometric functions may recognize the structure $\big(1-v^2/c^2\big)^{-1/2}$ as something suspiciously like $\big(1+\tan^2\theta\big)^{-1/2}$ in many trigonometric identities. In fact, if we make a substitution that $v/c = \tanh\eta$, then the entire Lorentz transformation becomes just

$$\begin{pmatrix} x' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\eta & \sinh\eta \\ \sinh\eta & \cosh\eta \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}, \tag{A2}$$

which functions like a rotation but in some hyperbolic space. This discovery in the early $20^{\text{th}}$ century leads to a revolutionary idea about physics on a manifold consisting of a 3-dimensional Euclidean part along with a temporal-spatial mixing with a hyperbolic part. This leads to a flat spacetime metric $g = \langle 1,-1,-1,-1\rangle$ which acts like the norm over the flat spacetime manifold[4]. Fortunately, the manifold is simple enough that we can identify it as a quadratic space $(\mathbb{R}^4, g)$. Then, there is an orthogonal group of $g$ denoted as $O(1,3)$ and its special subgroup $SO(1,3)$, which physicists call the group of Lorentz transformations[5] or the **Lorentz group**.

With this construction, we naturally obtain an invariant under Lorentz transformation, the length of a four-dimensional object (the **four-vector**). This is quite shocking because, unlike the vector (or three-vector, *e.g.* velocity, momentum, position), a four-vector (*e.g.* four-position, four-velocity) has the same length even under boost from one inertial frame to another moving inertial frame. Just like the advantage of using vectors in classical mechanics, physicists use a four-component object called the four-vector while putting down expressions in relativistic theories. Then, a natural scalar to form with these four-vectors is the dot product $X \cdot Y = X_\mu Y^\mu = B(X,Y)$ with the polarized symmetric bilinear form $B$ from $g$.

### 2. Dirac Equation: Taking the Square Root of Differential Operators

However, there was something from quantum mechanics that bothered physicists in the early to mid $20^{\text{th}}$ century. According to quantum mechanics, a physical system can carry some angular momentum in its wavefunction. Usually, the angular momentum is induced by rotation; however, particles such as electrons seem to possess some intrinsic angular momentum, *viz.* **spin**, which was not explained by the theory. Specifically, this spin can be added to the system's angular momentum; however, this spin cannot be induced by rotation.

Fortunately, Wolfgang Pauli introduced a set of three $2\times 2$ complex matrices that capture the interactions between the usual angular momentum and spin. The introduction of Pauli matrices was a huge phenomenological triumph, but physicists still did not fully comprehend the origin of these matrices. The breakthrough happened in 1928 by a then-26-year-old prodigy Paul Dirac. His treatment was to factorize the second-order relativistic equation for quantum mechanics into two linear parts [5, 6]. The $2^{\text{nd}}$-order equation, known as the Klein-Gorden equation, reads

$$\Big(m^2 + \frac{\partial^2}{\partial t^2} - \nabla^2\Big)\psi = 0,^{6} \tag{A3}$$

[4] Note that some authors use the "GR convention" or the "mostly-plus convention", *i.e.* $g = \langle -1,1,1,1\rangle$. We here will stick with the "HEP convention" or the "mostly-minus convention".

[5] Caveat lector: Technically, since determinant of a linear transformation does not preserve the direction of the temporal coordinate, physicists also frequently restrict the definition of the Lorentz group to $SO^+(1,3)$, *i.e.* the proper ($\det\Lambda = 1$) orthochronous ($\Lambda(t) = |k|t+\beta$) transformations

[6] Again, we are using the HEP convention by demanding that

in which $m$ denotes the mass of the system, $\nabla^2$ is the three-dimensional Laplacian, and $\psi$ denotes the wavefunction of the system. Dirac proposed a formal linear factorization of the differential operator into the form

$$\begin{aligned}0 &= \left[\left(i\gamma^0\frac{\partial}{\partial t} - i\gamma^1\frac{\partial}{\partial x} - i\gamma^2\frac{\partial}{\partial y} - i\gamma^3\frac{\partial}{\partial z}\right)^2 - m^2\right]\psi \\ &= \left(i\partial\!\!\!/ + m\right)\left(i\partial\!\!\!/ - m\right)\psi,\end{aligned} \tag{A4}$$

in which $\partial\!\!\!/ \triangleq \gamma^0\partial_t - \gamma^1\partial_x - \gamma^2\partial_y - \gamma^3\partial_z$. Dirac realized that these gamma objects must satisfy a relation that

$$\begin{cases}\gamma^i\gamma^j + \gamma^j\gamma^i = 0, \quad i \neq j, \\ \gamma^i\gamma^i = g(X^i).\end{cases} \tag{A5}$$

He then explicitly constructed several $4 \times 4$ matrices that satisfy the multiplication rules, named as gamma matrices or Dirac matrices. To parse the new equation, Dirac demanded that the wavefunction of the system is a 4-dimensional object, called a **spinor**[7]. As discussed in problem 7, there are many representations of the gamma matrices that satisfy the same rules as eq. (A5), referred to as different bases. Noticeably, one of the bases, Weyl basis, demonstrates clearly the relation between Pauli matrices and the gamma matrices. This opens a new chapter in physics research to understand the spin of a particle as a natural relativistic quantum effect.

However, many readers at this stage may start chuckle or even laugh at the ingenious physicist's idea. After all, the relations shown in eq. (A5) are precisely the equivalence relation for a Clifford algebra. Indeed, physicists are more and more aware of the connection between Clifford algebra and spin and using Clifford algebra to understand spins and spinors. It seems that it is strange not to call all 4-dimensional objects four-vectors, but there is a reason that we separate the notion a four-vector (or just a vector) and a spinor. Although the two are both four-dimensional objects, vectors follow the transformation rule as stated in eq. (A1) while spinors seems to transform in an entirely different way. It happens to be that the spinor's transformation group "looks like" rotations in 3-dimensional space that enlightened physicists to consider the interaction between spin and the usual angular momentum. This connection between spinor and Clifford algebras is through the notion of the Clifford-Lipschitz group

*Problem* 7 ($\star$). Check the following 4-dimension matrix representation of the Dirac matrix is valid, *i.e.* check their squares and anticommutation relation.

(a) Dirac basis:

$$\gamma^0 = \begin{pmatrix}1&&&\\&1&&\\&&-1&\\&&&-1\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}&&&1\\&&1&\\&-1&&\\-1&&&\end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix}&&&-i\\&&i&\\&i&&\\-i&&&\end{pmatrix}, \quad \gamma^3 = \begin{pmatrix}&&1&\\&&&-1\\1&&&\\&-1&&\end{pmatrix}.$$

(b) Majorana basis:

$$\gamma^0 = \begin{pmatrix}&&&-i\\&&i&\\&-i&&\\i&&&\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}i&&&\\&-i&&\\&&i&\\&&&-i\end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix}&&&i\\&&-i&\\&-i&&\\i&&&\end{pmatrix}, \quad \gamma^3 = \begin{pmatrix}&-i&&\\-i&&&\\&&&-i\\&&-i&\end{pmatrix}.$$

(c) Weyl basis:

$$\gamma^0 = \begin{pmatrix}&&1&\\&&&1\\1&&&\\&1&&\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}&&&1\\&&1&\\&-1&&\\-1&&&\end{pmatrix},$$

$$\gamma^2 = \begin{pmatrix}&&&-i\\&&i&\\&i&&\\-i&&&\end{pmatrix}, \quad \gamma^3 = \begin{pmatrix}&&1&\\&&&-1\\-1&&&\\&1&&\end{pmatrix}.$$

[Notice that this basis can be expressed as

$$\gamma^0 = \begin{pmatrix}&\mathbb{1}\\\mathbb{1}&\end{pmatrix}, \quad \gamma^i = \begin{pmatrix}&\sigma_i\\-\sigma_i&\end{pmatrix},$$

in which $\sigma_i$ denotes the $2 \times 2$ Pauli matrices. ]

as discussed in section IV A.

$\hbar = c = 1$.

[7] For readers interested in quantum field theory, this object is actually the **spinor field**. It is only the non-propagating coefficient that is the actual spinor.

[1] P. Lounesto, *Clifford Algebras and Spinors*, London Mathematical Society Lecture Note Series No. 286 (Cambridge University Press, 2001).
[2] D. Cheung, *Classification of Quadratic Forms with Clifford Algebras*, Ph.D. thesis, The University of New South Wales (2015).
[3] T.-Y. Lam, *Introduction to Quadratic Forms over Fields*, Vol. 67 (American Mathematical Soc., 2005).
[4] B. Casselman, “Clifford algebras and spinors,”.
[5] M. E. Peskin and D. V. Schroeder, *An Introduction to Quantum Field Theory* (Addison-Wesley, Reading, USA, 1995).
[6] M. D. Schwartz, *Quantum Field Theory and the Standard Model* (Cambridge University Press, 2014).
[7] I. Todorov, Bulg. J. Phys. **38**, 3 (2011), arXiv:1106.3197 [math-ph].