Walrasian Heterogeneous Agent Models with Asset Market Incompleteness

Homework 0. Go back and use the `Bewley-Aiyagari` bespoke class we developed for last lecture. Play with the partial equilibrium problem of a household (i.e., its dynamic program) taking the relative price r as fixed.

1. Run two cases of this problem with different levels of borrowing constraint: $\bar{a} = 0$ and $\bar{a} = -A$, where A is some positive constant. After solving for each experiment's optimal savings function $(a, e) \mapsto g(a, e)$, compute a sample path for consumption in each case. What do you see?

2. Take the baseline setting as in the lecture. Now, do as in the lecture: Simulate an agent's stochastic outcome for a_t for a very, very long time period, save this and plot the histogram of the sample individual time series. Also, compute some moments of this ditribution (e.g., mean, variance, kurtosis).

3. Repeat the last simulation, but beginning from different initial conditions $a_0 \in X := [\underline{a}, \overline{a}]$. I suggest trying for, say $N = 5{,}000$ or $10{,}000$ initial conditions (individuals). Effectively, you are simulating a panel data of individuals $\{a_{t+1} = g^t(a_0, \epsilon_t) : \forall a_0 \in X\}_{t=0}^{T}$ as you vary across the initial cross-section of people indexed by a_0. Make T as long as possible without stalling your PC! Construct the pooled statistics of this panel (mean, variance and etc.). Compare with your result from the last question.

   Comment on your comparative results.

Homework 1. Consider a variation of Aiyagari's model with idiosyncratic shocks to employment status.[1]

Agents are identified by their *individual* employment status $e \in E := \{0, 1\}$ (0 for unemployed and 1 for employed) and their asset position a. If they are employed, they earn a gross wage income of $w(f)$. They earn/pay before-tax interest $r(f)$ on their positive/negative asset balance. In a steady-state/stationary and along a dynamic equilibrium, aggregate relative prices (r, w) and the government-budget-balancing tax rate (τ) would depend on the equilibrium's current distribution of agents f. (See market clearing conditions below for why.) As a result, the current (unconditional) aggregate distribution, given by a p.d.f. f over (a, e) outcomes, is also a relevant state variable. Let Q denote the Markov matrix of an individual's employment status, i.e.,

$$Q(e'|e) = \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix},$$

where $p = \Pr\{e_{t+1} = 0 | e_t = 0\}$ and $q = \Pr\{e_{t+1} = 1 | e_t = 1\}$.

Agents derive utility from consumption but not leisure/labor. (In equilibrium, each employed agent will supply a unit of labor time inelastically.) An individual household named $(a, e; f)$ has valuation

$$\begin{aligned} V(a, e; f) = \max_{c, a'} \Bigg\{ \frac{c^{1-\eta}}{1-\eta} + \beta \sum_{e' \in E} Q(e'|e) V(a', e'; f') : \\ a' &= (1 - \tau(f)) w(f) e + b(1-e) + [1 + (1 - \tau(f)) r(f)]\, a - c, \\ a &\geq a_{min}, \\ f'(a', e') &= \sum_{e \in E} Q(e'|e) f(g^{-1}(a', e; f), e), \\ Q(e'|e) &= \begin{bmatrix} p & 1-p \\ 1-q & q \end{bmatrix} \Bigg\}. \end{aligned}$$

The expression $g^{-1}(a', e, f)$ denotes the current-period asset position a of an individual who realized shock e and has best-responded by choosing next-period asset position as $a' = g(a, e; f)$. In any competitive equilibrium (whether stationary or in transition), we will also have factor relative prices being equal to their respective marginal products:

$$r(f) = \alpha \left(\frac{K}{N} \right)^{\alpha - 1} - \delta,$$

and,

$$w(f) = \alpha \left(\frac{K}{N} \right)^{\alpha}.$$

(This is derived from firm profit maximization.)

Market clearing will require aggregate capital demanded by firms to equal the net supply of assets by households,

$$K = \sum_{e \in E} \int_{a_{min}}^{\infty} a f(a, e) \mathrm{d}a,$$

[1] See Gary Hansen and Ayse İmhoroğlu (1992) "The Role of Unemployment Insurance in an Economy with Liquidity Constraints", *J. Pol. Econ., vol. 100(1), 118-142*.

and also labor demand equals supply:

$$N = \int_{a_{min}}^{\infty} 1 \cdot f(a,1) \mathrm{da}.$$

Government budget balance requires that

$$\tau(f)\left[w(f)N + r(f)K\right] = \int_{0}^{\infty} bf(a,0)\mathrm{d}a, \qquad b = \rho\,(1-\tau(f))w(f).$$

Think of this as a government that finances unemployment-benefit transfers to the agents in state $e = 0$ (i.e., the unemployed) by levying a flat-rate, income tax from owners of capital and labor. The unemployment benefit b is paid out as a "replacement ratio" (ρ), which is a fraction of after-tax income of the employed.

Note that given an exogenous unemployment benefit transfer per person b, the tax rate $\tau(f)$ is endogenously determined by the government budget constraint.

1. Define precisely a recursive competitive equilibrium for this economy.
2. Design an algorithm for finding a stationary (steady-state) recursive competitive equilbrium. Hint: You will need to find both relative prices $(w(f), r(f))$ and tax policy $\tau(f)$ that satisfies equilibrium.

Homework 2. Continue from the last problem. Assume that $a_{min} = -2$, $\alpha = 0.36$, $\beta = 0.995$, $\eta = 2$, $\delta = 0.005$, $\rho = 0.25$, $p = 0.5$ and $q = 0.9565$.

You can modify from our given Bewley-Aiyagari model class. Do so and compute a stationary equilibrium under this parametric setting.

Think about what are important statistics or results to present. Write up a concise and clear report describing the equilibrium you compute.

What is the ex-ante welfare associated with the given government policy?

Homework 3. Suppose all else equal from the last question, the government now decides to increase the policy on unemployment benefit permanently. Specifically, ρ is increased by one-hundred percent from the previous setting. Compute the new steady-state equilibrium.

Contrast the results here with the previous one. Comment.

What is the ex-ante welfare associated with this new government policy? You can calculate welfare in terms of a compensating variation measure in units of the initial steady-state consumption.

Homework 4. (Advanced, optional) Consider now the task of calculating transition dynamics in term of K_t and f_t between the initial stationary equilibrium and the new, permanent stationary equilibrium.

Tell us how you would design an algorithm for calculating the dynamic equilibrium's transition.

Go forth and compute.

What is the total welfare along the transition path?

Homework 5. (Reprise) This question modifies a version of the Bewley-Aiyagari heterogeneous agent model (used in the study of wealth inequality) in two ways. First, individuals (i.e., households) are finitely-lived overlapping-generations agents. Intergenerational connection is through one's primitive preference over one's children (through educational knowledge transfer). Second, effective labor supply is endogenized through human capital accumulation, where there is a non-degenerate distribution of initial human capital across households.

As in a version of Bewley-Aiyagari, the laissez-faire (no-government) economy below has a stark notion of asset-market incompleteness. Agents cannot borrow to compensate for low idiosyncratic human capital risk (determined one period ahead) and thus cannot perfectly insure against idiosycratic income. The only partial source of consumption smoothing is through saving by owning capital stock, but this trades off with a motive of investing in one's children through privately funded education. For simplicity, we have no private asset market to borrow from to privately finance education.

This modified environment admits analytical solutions. In this highly disciplined story, we can derive the dynamics of aggregate growth and inequality using pencil-and-paper plus a bit of high-school calculus/algebra brain.

We see how the private funding of education versus public funding of education affect the rate of growth, and, the size of inequality in income and wealth, dynamically and towards the long run.

Households. Assume in each period there is a constant population of young people of size one. For each household cohort born at date t, their lifetime payoff depends on their own consumption when young and old, respectively, c_t and d_{t+1}, and some expenditure on their children's education e_t:

$$U(c_t) + \beta U(d_{t+1}) + \gamma U(e_t), \qquad \beta \in (0,1), \gamma > 0. \tag{5.1}$$

The parameters, β and γ, respectively, capture household impatience and the degree of altruism towards one's own children. Assume $U(x) = \ln(x)$.

The date t household faces these budget constraints over their lifetime:

$$c_t + s_t + e_t = w_t h_t, \tag{5.2}$$

and,

$$d_{t+1} = R_{t+1} s_t, \tag{5.3}$$

where s_t is household saving, w_t is a competitive real wage rate, R_{t+1} is a competitive gross return on investment in capital goods. Note that each household has one unit of labor time, but their effective labor supply is $h_t \times 1$ units of time.

It takes time for current educational investment e_t to be embodied in next-period children. Also, a child's human capital (h_{t+1}) will also inherit from her parents' human capital (h_t). Suppose the evolution (or production) of within-household human capital follows the non-homogeneous difference equation:

$$h_{t+1} = H(h_t, e_t) \equiv A e_t^{\theta} h_t^{1-\theta}, \qquad \theta \in (0,1), A > 0. \tag{5.4}$$

Firm. A representative firm produces a final good using the technology:

$$f(k_t) := Z k_t^{\alpha}, \qquad \alpha \in (0,1), Z > 0. \tag{5.5}$$

The variable $k_t := K_t/H_t$ is the aggregate capital-to-efficient-labor ratio. Profit maximization gives the firm's optimal demand for effective labor and capital, respectively, as

$$w_t = f(k_t) - k_t f'(k_t), \tag{5.6}$$

and,

$$R_t = f'(k_t). \tag{5.7}$$

Aggregation and market clearing. Let the cumulative probability distribution function over human capital across households at date t be $M_t : \mathbb{R}_{++} \mapsto [0,1]$. The function M_t is non-decreasing, with $M_t(0^+) = 0$ and $M_t(+\infty) = 1$, where $0^+ := \lim_{h \searrow 0} h$. Let the distribution of the logarithm of h_t be given by the cumulative probability distribution function μ_t. There is a bijection such that

$$\mu_t(\ln(h_t)) = M_t(h_t). \tag{5.8}$$

Assume that the initial distribution of agents, μ_0, has variance $\sigma_0 > 0$.

Thus, we can account for the aggregate level of human capital through labor market clearing:

$$H_t := \int e^{\ln(h_t)} \mathrm{d}\mu_t(\ln(h_t)). \tag{5.9}$$

Capital market clearing is summarized by

$$K_{t+1} = \int e^{\ln(s_t)} \mathrm{d}\mu_t(\ln(h_t)). \tag{5.10}$$

Self-funding of education. Assume that education is privately funded within each household.

1. Derive the optimal saving supply and education demand by each current household.

   Hint: Write these down as explicit functions of their current income.

2. Using the last step, now show that in an equilibrium, growth in aggregate human capital stock is:

$$\frac{H_{t+1}}{H_t} = \kappa w_t^{\theta}, \qquad \kappa := A\left(\frac{\gamma}{1+\beta+\gamma}\right)^{\theta}. \tag{5.11}$$

3. Using the last few steps, show that in an equilibrium, capital per efficient units of workers follows this recursion:

$$k_{t+1} = \left(\frac{\gamma}{(1+\beta+\gamma)\kappa}\right)\left[(1-\alpha)Zk_t^{\alpha}\right]^{1-\theta}, \qquad \kappa := A\left(\frac{\gamma}{1+\beta+\gamma}\right)^{\theta}. \tag{5.12}$$

   Then argue that there is a unique steady state equilibrium in terms of the relative prices (w, R) and allocation (in terms of the K/H ratio) k.

   Hint: Capital market clearing (5.10) implies $K_{t+1} = \left(\frac{\beta}{1+\beta+\gamma}\right) w_t H_t$.

4. Consider a log-linear process:

$$x_{t+1} = \rho_t x_t + v_t.$$

where $x_t := \ln(y_t)$ is distributed according to the distribution function μ_t, and, v_t is a given process that is independent of x_t.

Assume μ_t is characterized by its first two moments: mean and variance. Then the mean and variance of x_t, respectively $\bar{x}_t$ and σ_t^2, will evolve according to

$$\bar{x}_t = \rho_t \bar{x}_t + v_t, \qquad \text{and,} \qquad \sigma_{t+1}^2 = \rho_t^2 \sigma_t^2.$$

Let $l_t := \ln(h_t)$, and $\bar{l}_t := \int l_t \mathrm{d}\mu_t(l_t)$. Derive the dynamics of distribution of the logarithm of human capital in terms of its mean and variance dynamics.

What happens to labor-income and capital-income inequality (in terms of variance) over time, as $t \to \infty$?

Public funding of education. Consider now, education being provided publicly and financed through taxation of aggregate wage income:

$$e_t = \tau_t w_t H_t. \tag{5.13}$$

That is, e_t is no longer a choice variable for individual households. Equation (5.2) becomes

$$c_t + s_t = (1 - \tau_t) w_t h_t. \tag{5.14}$$

5. Answer these questions:

   (a) Show that the optimal saving function of a household is now

   $$s_t = \left(\frac{\beta}{1+\beta}\right)(1-\tau_t) w_t h_t. \tag{5.15}$$

   (b) Using this result, derive what is each household's equilibrium preferred tax rate. *Hint*: Imagine if the household were a dictator, what tax rate would it choose to maximize its own welfare criterion?

   (c) By the median voter theorem, the voting equilibrium implements the policy preferred by a median voter/household. In this model, the median-preferred policy is $\tau_t = \gamma/(1+\beta+\gamma)$. Explain why.

6. Using the last result from the last sub-question:

   (a) Show that the logarithm of individual human capital evolves according to

   $$h_{t+1} = \kappa w_t^\theta H_t^\theta h_t^{1-\theta}. \tag{5.16}$$

   (b) Derive the equilibrium evolution of the distribution of log human capital (again, in terms of mean and variance dynamics).

   (c) Compare your result here with that of the earlier economy without government financing of education as a public good. What can you say specifically about inequality of labor and capital incomes here?

   For technical reasons, assume that the initial support of μ_0 is a compact set $[l_0^{min}, l_0^{max}] \subset (0, \infty)$.