THE AUSTRALIAN NATIONAL UNIVERSITY

Final Examination — June 2019

Macroeconomic Theory

ECON 4422/8022

⏲ Reading Time	:	0.0 Minutes
✍ Writing Time	:	3.0 Hours
☑ Permitted Materials	:	Calculator (non programmable)
		One A4 page with notes on both sides
		Unannotated paper-based dictionary
$\langle \cdot\% \rangle$	:	Mark allocation operator

— IMPORTANT —

There are **TWO Parts** to this examination with increasing levels of difficulty:

Part I is a *partially guided* test of your ability to solve basic problems related to class material, to reason logically and to be able to interpret resulting economic insights from your analyses. This part also tests your general comprehension of basic concepts or definitions, your ability to read `Python` code and to design algorithms related an economic problem. Completion of this section would enable you to attain up to 70% of the maximal final examination mark.

- Use the **Multiple Choice Form** (MCF) to submit your answers to **Part I**.
- Pick the most accurate answer.
- Remember to include your student ID on the form.
- **Warning**: The sub-Question numbers correspond to the question numbers on the MCF. Please assign your answers to the MCF choices carefully.

Part II examines the same attributes as Part I. In addition, in this part, you must be able to apply existing knowledge and skills to problems that go beyond familiar examples. Students aiming for a superior Distinction or High Distinction grade should attempt this section.

- Answer **ONLY ONE** Question from **Part II**. If you disregard this instruction and answer both, then only the final question will be graded.
- Use the **Answer Booklet** to submit your answers to **Part II**.
- Remember to include your student ID on the cover page.
- Record the Question Number of each question you would like to have assessed on the cover sheet of the answer booklet. If you don't do so, the question will not be marked.
- Answers are expected to be succinct but complete. *Unreasonably long and irrelevant answers may be penalized.* (Recommendation: You should only require one answer booklet.)

— Part I —

Question A (10%) Time is finite and indexed by $t \in \{0, 1, ..., T\}$. Let the optimal value of a policy maker beginning with resources k_0 be given by:

$$V_0(k_0) = \max_{\{c_t, k_{t+1}\}_{t=0}^{T}} \left\{ \sum_{t=0}^{T} \beta^t U(c_t) : k_{t+1} = f(k_t) - c_t, 0 \leq c_t \leq f(k_t), k_{T+1} \geq 0 \right\}, \quad \text{(A.1)}$$

where the functions U and f are strictly concave and strictly increasing. U and f are twice-continuously differentiable. The state space $X \ni k_t$ is bounded. Assume that the Inada conditions — i.e., $\lim_{x \searrow 0} z'(x) = +\infty$ and $\lim_{x \nearrow +\infty} z'(x) = 0$ — apply for $z \in \{U, f\}$.

1. Describe precisely what we mean by a strategy in this setting. ⟨ **2%** ⟩

   (a) A strategy is a date and state contingent plan $\{g_t(k_t)\}_{t=0}^{T}$ such that $c_t = g_t(k_t)$ at each date t and state k_t.

   (b) A strategy is an optimal date and state contingent plan $\{g_t(k_t)\}_{t=0}^{T}$ such that $c_t = g_t(k_t)$ at each date t and state k_t.

   (c) A strategy is the optimal date and state contingent plan $\{g_t(k_t)\}_{t=0}^{T}$ such that $c_t = g_t(k_t)$ at each date t and state k_t.

   (d) A strategy is a policy selection $c_t = g_t(k_t)$ at each date t and state k_t.

   (e) None of the above.

2. Now re-write the sequence problem (A.1) as a recursive one. We know $V_{T+1}(f(k_T) - c_T) = 0$. At each $t \in \{0, 1, ..., T\}$, the Bellman equation is: ⟨ **2%** ⟩

   (a) $V_t(k_t) = \max_{c_t} \{U(k_t) + \beta V_{t+1}(f(k_t) - k_{t+1}) : 0 \leq k_{t+1} \leq f(k_t)\}$.

   (b) $V_t(k_t) = \max_{k_{t+1}} \{U(c_t) + \beta V_{t+1}(f(k_t) - c_t) : 0 \leq k_{t+1} \leq f(k_t)\}$.

   (c) $V_t(k_t) = \max_{c_t} \{U(c_t) + \beta V_{t+1}(f(k_t) - c_t) : 0 \leq c_t \leq f(k_t)\}$.

   (d) $V(k_t) = \max_{c_t} \{U(c_t) + \beta V(f(k_t) - c_t) : 0 \leq c_t \leq f(k_t)\}$.

   (e) There is more than one correct answer.

3. There exists a unique solution to the recursive representation of problem (A.1) because _______ [✄]. This solution is _______ [↯]. The optimizer to problem (A.1) _______ [✂].

   ⟨ **3%** ⟩

   (a) ✄ — the Bellman operator is a β-contraction
   ↯ — a unique value function
   ✂ — is unique.

   (b) ✄ — at each date, U and f are strictly concave
   ↯ — a non-singleton set of value functions
   ✂ — is unique.

   (c) ✄ — the Bellman operator is a β-contraction
   ↯ — a unique value function
   ✂ — not unique.

(d) ✂ — at each date, U and f are twice-continuously differentiable
↯ — a unique value function
✂ — is unique.

(e) None of the above.

4. Suppose you are supplied a number $k_0 = \texttt{a} > 0$, a time horizon `T`, pre-defined function `f` representing f, and, a solution contained as a `Python` list `g` as an optimizer to problem (A.1). Which of these code snippets are logically correct? $\langle$ **3%** $\rangle$

(a) This one?

```
k = np.empty(T)
k[0] = a
y = np.empty(T-1)
for t in range(T-1):
  k[t+1] = g(k[t])
  y[t] = f(k[t])
```

(b) Or, this one?

```
k = np.empty(T)
k[0] = a
y = np.empty(T-1)
for t in range(T-1):
  k[t+1] = g[t](k[t])
  y[t] = f[t](k[t])
```

(c) Or, this one?

```
k = np.empty(T)
k[0] = a
y = np.empty(T)
for t in range(T-1):
  k[t+1] = g(k[t])
  y[t] = f(k[t])
```

(d) Or, this one?

```
k = np.empty(T)
k[0] = a
y = np.empty(T)
for t in range(T-1):
  k[t+1] = g(k[t])
  y[t] = f(k[t])
```

(e) Or, this one?

```
k = np.empty(T)
k[0] = a
y = np.empty(T-1)
for t in range(T-1):
  k[t+1] = f(k[t]) - g[t](k[t])
  y[t] = f(k[t])
```

Question B (20%) Consider a world of identical agents with population of size 1. The representative agent solves a lifetime expected utility maximization problem, given exogenous and random streams of the economy's income $\{Y_t\}_{t\in\mathbb{N}}$. The agent can demand/purchase shares $s_t \in [0,1]$ of the random income stream, and/or consume c_t given initial income Y_t each period. The agent's total expected utility is $\mathbb{E}\left\{\sum_{t=0}^{\infty}\beta^t U(c_t)|Y_0\right\}$, where $\beta \in (0,1)$. If you need, you may assume $U(c) = \ln(c)$. Individual's share and consumption demands must respect feasibility, respectively, $s_t \in [0,1]$ and $0 \leq c_t \leq Y_t$. Let $q(Y_t)$ denote the aggregate (relative) price of a share contingent on state Y_t, which is taken as given by the agent. Further, the consumer's state-contingent plan must respect her sequential budget constraint: $c_t + q(Y_t)s_{t+1} \leq [Y_t + q(Y_t)]s_t$. Assume that $\{Y_t\}_{t\in\mathbb{N}}$ is generated by an ergodic continuous-state Markov process (μ_0, P) where μ_0 is the initial unconditional distribution of Y_0 and $P(Y_t,\cdot)$ is a Markov kernel which gives the conditional probability of transiting from state Y_t to Y_{t+1}. Let $V(s_0, Y_0)$ be the agent's indirect utility from starting out at aggregate state Y_0 with share s_0. Let a variable $x := x_t$ denote its current value, and $x_+ := x_{t+1}$ its next-period value.

5. Given initial state $(s,y) \in [0,1]\times S$, the agent's Bellman equation is ⟨ **4%** ⟩

   (a) this:

$$V(s,y^i) = \max_{c\in[0,y^i],s_+\in[0,1]} \left\{ U(c) + \beta \sum_{j=1}^{N} p_{ij} V(s_+, y^j) : c + q(y^i)s_+ = [y^i + q(y^i)]s \right\}$$

   (b) this:

$$V(s,y^i) = \max_{c\in[0,y^i],s_+\in[0,1]} \left\{ U(c) + \beta \sum_{j=1}^{N} P(i,j) V(s_+, y^j) : c + q(y^i)s_+ = [y^i + q(y^i)]s \right\}$$

   (c) this:

$$V(s,y) = \max_{c\in[0,y],s_+\in[0,1]} \left\{ U(c) + \beta \int_S V(s_+, y_+)P(y, \mathrm{d}y_+) : c + q(y)s_+ = [y + q(y)]s \right\}$$

   (d) this:

$$V(s,y) = \max_{c\in[0,y],s_+\in[0,1]} \left\{ U(c) + \beta \mathbb{E}\left\{V(s_+, y_+)|y\right\} : c + q(y)s_+ = [y + q(y)]s \right\}$$

   (e) None of the above.

6. A necessary first-order (Euler) condition for utility-maximization of the agent has the form:

$$U'(c(y))q(y) = \beta \mathbb{E} \{U'(c(y_+))[q(y_+) + y_+]|y\},$$

What does this condition mean? Note: U' is the partial derivative of U with respect to its argument. ⟨ **6%** ⟩

   (a) At the optimum, the marginal utility value of a share today must equal the expected marginal utility value of future equity, which derives from future capital gains and future dividend payout.

   (b) At the optimum, the intertemporal marginal rate of substitution equals the intertemporal relative price of shares.

   (c) At the optimum, the current share price is the present value of future capital gains.

   (d) At the optimum, the current share price is the present value of future divident flows.

   (e) At the optimum, the marginal indirect utility value of capital today must equal the expected marginal indirect utility value of future capital, which derives from future share values and future dividend payout.

7. In a competitive equilibrium, we will have that $c(Y_t) = Y_t$ and $s(Y_t) = 1$ for all realizations $Y_t = y \in S$. Which statement is correct? ⟨ **2%** ⟩

   (a) There is a unique equilibrium function $q : S \to \mathbb{R}_+$ that solves the previously stated Bellman equation.

   (b) There is a unique equilibrium function $q : S \to \mathbb{R}_+$ that solves the previously stated Euler equation.

   (c) There is a unique equilibrium function $q : S \to \mathbb{R}_+$ that solves the contraction mapping problem.

   (d) There is a unique equilibrium function $V : S \to \mathbb{R}_+$ that solves the previously stated Euler equation.

   (e) There are two correct answers here.

8. The following describes the equilibrium dynamic behavior of $q : S \to \mathbb{R}_+$: ⟨ **8%** ⟩

   (a) If endowment is expected to rise over time, given the state today, the asset price must fall today: This is because of a negative correlation between future asset returns and consumption growth.

   (b) If endowment is expected to rise over time, given the state today, the asset price must fall today: This is because of a positive correlation between future asset returns and consumption growth.

   (c) If endowment is expected to rise over time, given the state today, the asset price must rise today: This is because of a positive correlation between future asset returns and consumption growth.

(d) If endowment is expected to fall over time, given the state today, the asset price must rise today: This is because of expected higher consumption growth and future asset returns.

(e) None of the above.

Question C (20%)

9. Which of these statements is the most accurate? $\langle$ **5%** $\rangle$

   (a) Inflation is costly only if firms face a cost of adjusting prices.

   (b) Inflation is costly only if firms face a cost of adjusting prices and they have some market power.

   (c) Inflation is costly only if agents cannot exchange goods based on private contracts.

   (d) Inflation cannot exist in a Walrasian equilibrium because excess demand is homogeneous of degree one in prices.

   (e) None of the above.

10. In a recursive competitive equilibrium of an economy with Arrow securities, $\langle$ **5%** $\rangle$

    (a) equilibrium relative prices are equivalent to that from a Nash bargaining solution.

    (b) allocation of resources are necessarily efficient.

    (c) agents do not renegotiate the terms of their securities contracts *ex post* by assumption.

    (d) agents do not renegotiate the terms of their securities contracts *ex post* if their preferences are dynamically consistent.

    (e) agent's *ex-post* choices are always consistent with their initial optimal plans.

    (f) There may be more than one correct answer.

11. A special case of the Ramsey-Cass-Koopmans model yields an optimal growth process as $k_{t+1} = \alpha\beta k_t^{\alpha}$, where $0 < \alpha, \beta < 1$. Suppose this model represents Argentina in the 18^{th} to the early 20^{th} century. So initially, $k_0 > (\alpha\beta)^{1/(1-\alpha)}$. The half life of initial quantity k_0, according to this model is: $\langle$ **5%** $\rangle$

    (a) $T_{\text{half}} = \lceil \frac{1}{\ln(\alpha)} \ln\left[\left(\frac{1}{2}k_0 - \bar{k}\right)/\left(k_0 - \bar{k}\right)\right] \rceil$

    (b) $T_{\text{half}} = \lceil \frac{1}{\ln(\alpha)} \ln\left[\left(2k_0 - \bar{k}\right)/\left(k_0 - \bar{k}\right)\right] \rceil$

    (c) $T_{\text{half}} = \lceil \frac{1}{\ln(\alpha)} \ln\left[\left(k_0 - \bar{k}\right)/\left(\frac{1}{2}k_0 - \bar{k}\right)\right] \rceil$

    (d) $T_{\text{half}} = \lceil \frac{\ln(2)}{\ln(\alpha)} \ln\left[\left(k_0 - \bar{k}\right)/\bar{k}\right] \rceil$

    (e) None of the above.

    Hint: The half life of an initial quantity k_0 is the time it takes for that quantity to be halved. The notation $\lceil\cdot\rceil$ refers to the ceiling function.

12. Consider the following approximate recursive competitive equilibrium conditions in a Real Business Cycle model:

$$\mathbb{E}_t\left\{\hat{c}_{t+1}\right\} \approx \left[1 + (1-\alpha)\frac{\beta Y_{ss}}{K_{ss}}\right]^{-1}\left[\hat{c}_t + \beta\left(\frac{Y_{ss}}{K_{ss}}\right)\rho\hat{a}_t\right], \tag{LRCE-1}$$

and

$$\hat{k}_{t+1} \approx \left[(1-\delta) + \frac{Y_{ss}}{K_{ss}}\right]\hat{k}_t - \left[\frac{C_{ss}}{K_{ss}} + \frac{Y_{ss}}{K_{ss}}\left(\frac{1-\alpha}{\alpha}\right)\right]\hat{c}_t + \frac{Y_{ss}}{K_{ss}}\left(\frac{1}{\alpha}\right)\hat{a}_t. \quad \text{(LRCE-2)}$$

where $\hat{c}_t$ and $\hat{k}_t$, respectively, refer to consumption and capital stock expressed as percentage deviations from their respective constant reference points. All parameters are positive valued.

This implies a linear, expectational stochastic difference equation system of the form

$$\begin{bmatrix} \hat{k}_{t+1} \\ \mathbb{E}_t \hat{c}_{t+1} \end{bmatrix} = \mathbf{M} \begin{bmatrix} \hat{k}_t \\ \hat{c}_t \end{bmatrix} + \mathbf{L}\hat{a}_t,$$

given an exogenous specification for the random variable $\hat{a}_t$.

What are $\mathbf{M}$ and $\mathbf{L}$? ⟨ **5%** ⟩

(a) $\mathbf{M} = \begin{bmatrix} 1-\delta+\frac{Y_{ss}}{K_{ss}} & -\frac{C_{ss}}{K_{ss}} - \frac{Y_{ss}(1-\alpha)}{K_{ss}} \\ 0 & \frac{1}{1+\frac{Y_{ss}\beta(1-\alpha)}{K_{ss}}} \end{bmatrix}$ and $\mathbf{L} = \begin{bmatrix} \frac{Y_{ss}}{K_{ss}\alpha} \\ \frac{Y_{ss}\beta\rho}{K_{ss}\left(1+\frac{Y_{ss}\beta(1-\alpha)}{K_{ss}}\right)} \end{bmatrix}$

(b) $\mathbf{M} = \begin{bmatrix} 1-\delta+\frac{Y_{ss}}{K_{ss}} & -\frac{C_{ss}}{K_{ss}} - \frac{Y_{ss}(1-\alpha)}{K_{ss}\alpha} \\ 0 & \frac{1}{1+\frac{Y_{ss}\beta(1-\alpha)}{K_{ss}}} \end{bmatrix}$ and $\mathbf{L} = \begin{bmatrix} \frac{Y_{ss}}{K_{ss}\alpha} \\ \frac{Y_{ss}\beta\rho}{K_{ss}\left(1+\frac{Y_{ss}\beta(1-\alpha)}{K_{ss}}\right)} \end{bmatrix}$

(c) $\mathbf{M} = \begin{bmatrix} 1-\delta\frac{Y_{ss}}{K_{ss}} & -\frac{C_{ss}}{K_{ss}} - \frac{Y_{ss}(1-\alpha)}{K_{ss}\alpha} \\ 0 & \frac{1}{1+\frac{Y_{ss}\beta(1-\alpha)}{K_{ss}}} \end{bmatrix}$ and $\mathbf{L} = \begin{bmatrix} \frac{Y_{ss}}{K_{ss}\alpha} \\ \frac{Y_{ss}\beta\rho}{K_{ss}\left(1+\frac{Y_{ss}\beta(1-\alpha)}{K_{ss}}\right)} \end{bmatrix}$

(d) There are two correct answers

(e) None of the above

Question D (20%) *WoShiCaiJingGaoShou Pty Ltd.* is a company that produces a product called `iChEat`. Assume units of `iChEat`s are divisible goods, and are representable as non-negative real numbers. To produce y units of `iChEat` requires capital input $k \in \mathbb{R}_+$. The production technology $f : \mathbb{R}_+ \to \mathbb{R}_+$ is such that $y = f(k)$.

WoShiCaiJingGaoShou Pty Ltd. is a price taker in both the capital market and in the market for its generic product. Let the price of a unit of `iChEat` be 1, so that q is the price of a unit of capital in terms of an `iChEat`. *WoShiCaiJingGaoShou Pty Ltd.* discounts future profit flows at a constant discount factor $\beta \in (0,1)$. Time is denumerable, $t \in \mathbb{N}$. Suppose that capital must be purchased one period ahead for production next period, and during production, it depreciates at rate $\delta \in (0,1)$ per period. Denote the per-period new investment demand of *WoShiCaiJingGaoShou Pty Ltd.* as i_t, and,

$$i_t = k_{t+1} - (1-\delta)k_t, \tag{D.1}$$

where k_t is predetermined by the end of $t-1$. Given initial capital stock k_0, *WoShiCai-JingGaoShou Pty Ltd.* has a value given by

$$\pi(k_0) = \max_{(i_0,k_1,i_1,k_2,\ldots)\in\mathbb{R}_+^\infty} \sum_{t=0}^{\infty} \beta^t \left[f(k_t) - q i_t\right]$$

subject to (D.1) for every $t \in \mathbb{N}$. Assume that $f(k) = \sqrt{k}$. Let a variable $x := x_t$ denote its current value, and $x_+ := x_{t+1}$ its next-period value.

13. Show that the state space containing k_t can be restricted to a set $X \subset \mathbb{R}_+$ which is compact. What is X, explicitly? ⟨ **4%** ⟩

    (a) $X = \left[0, \bar{k}\right]$ where $\bar{k} = \max\{(\delta q)^{-1}, k_0\}$.
    (b) $X = \left[0, \bar{k}\right]$ where $\bar{k} = \min\{(\delta q)^{-1}, k_0\}$.
    (c) $X = \left[0, \bar{k}\right]$ where $\bar{k} = \max\{(\delta q)^{2}, k_0\}$.
    (d) $X = \left[0, \bar{k}\right]$ where $\bar{k} = \max\{(\delta q)^{-2}, k_0\}$.
    (e) $X = \left[0, \bar{k}\right]$ where $\bar{k} = \max\{\sqrt{\delta q}, \sqrt{k_0}\}$.

14. The Bellman equation is

$$\pi(k) = \max_{k_+ \in \Gamma(k)} \left\{ f(k) - q[k_+ - (1-\delta)k] + \beta\pi(k_+) \right\},$$

where $\Gamma : X \rightrightarrows 2^X$, where $\Gamma(k) := \{k_+ \in X : k_+ \in [0, q^{-1}f(k) + (1-\delta)k\}$ for every $k \in X$, describe the firm's feasible choice correspondence.

From Theorem 3 in the Appendix, since f is continuously differentiable, then the value function is also differentiable. Moreover there is a unique optimal strategy and the Inada conditions ensure that it is always in the interior of $X^\infty = X \times X \times \cdots$.

The marginal value of capital to *WoShiCaiJingGaoShou Pty Ltd.* is __________ ⟨ **6%** ⟩

    (a) $\pi'(k) = f'(k) + q\delta$.
    (b) $\pi'(k) = f'(k) + q(1-\delta)$.

(c) $q = \beta\pi'(k_+)$.

(d) $q = \beta[f'(k_+) + q(1-\delta)]$.

(e) $f(k) - q[k_+ - (1-\delta)k$.

15. The optimal stationary investment policy is such that k_{t+1} equals __________ [⚒]. The firm's value at k_0 is __________ [🏭].

⟨ **6%** ⟩

(a) ⚒ $k^* \equiv [q(1/\beta - 1 + \delta)]^2$ per period

🏭 $\pi(k_0) = (k_0)^{1/2} - q[k^* - (1-\delta)k_0] + \frac{\beta}{1-\beta}\left[(k^*)^{1/2} - q\delta k^*\right]$

(b) ⚒ $k^* \equiv \sqrt{[q(1/\beta - 1 + \delta)]}$ per period

🏭 $\pi(k_0) = (k_0)^{1/2} - q[k^* - (1-\delta)k_0] + \frac{\beta}{1-\beta}\left[(k^*)^{1/2} - q\delta k^*\right]$

(c) ⚒ $k^* \equiv [q(\beta - 1 + \delta)]$ per period

🏭 $\pi(k_0) = (k_0)^2 - q[k^* - (1-\delta)k_0] + \frac{\beta}{1-\beta}\left[(k^*)^{1/2} - q\delta k^*\right]$

(d) ⚒ $k^* \equiv [(1/\beta - 1 + \delta)]^2$ per period

🏭 $\pi(k_0) = (k_0)^{1/2} - q[k^* - (1-\delta)k_0] + \frac{\beta}{1-\beta}\left[(k^*)^{1/2} - q\delta k^*\right]$

(e) None of the above.

16. From the last question, we deduced that from any initial state $k_0 \in X$, the firm's optimal investment strategy is to install capital for next period at a constant level k^*, and the same for every period thereafter. Why? ⟨ **4%** ⟩

(a) For $t \geq 0$, the marginal cost for investing into an additional unit of capital productive in the next period is constant at q. There is no additional cost from adjusting the stock immediately to k^*.

(b) The firm has perfect foresight. Therefore maximizing its long run capital stock is optimal.

(c) For $t \geq 0$, the marginal profit for investing into an additional unit of capital productive in the next period is constant at q. The marginal cost is zero. Therefore it is optimal to invest k^* each period.

(d) There is no risk. As such the optimal investment strategy is to maximizing its flow per period.

(e) None of the above.

— Part II —

Answer only Question E or Question F.
See further instructions at the start of this document.

Question E (30%) Suppose there is a continuum of identical consumers uniformly distributed on the set $[0,1]$. Each consumer derives utility from an infinite stream of consumption according to the criterion $\sum_{t=0}^{\infty} \beta^t u(C_t)$. Assume $u(c) = c$ and the discount factor $\beta \in (0,1)$. Consumption equals wage each period w_t if the consumer is employed. Otherwise, it equals some unemployment benefit b.

There is a continuum of firms, whose population is also of measure one. Each firm produces output $y = A$ by hiring one worker at a time. The profit of a productive firm is thus $A - w_t$. Each period, each firm faces a probability ρ that its employer-employee match gets dissolved. Assume that firms also discount future payoffs using β. Also, assume that $A > b$.

The labor market is decentralized: There exists a constant-returns-to-scale aggregate matching function M such that $m_t = M(u_t, v_t)$ measures the total measure of matches in the economy, given a measure of the unemployed, u_t, and a measure of advertised job vacancies, v_t. If a firm and a consumer get matched, the wage rate is determined by a (generalized) Nash bargaining solution. Let $\phi \in (0,1)$ and $1 - \phi$, respectively, denote a worker and a firm's bargaining power with regard to the (axiomatic) Nash bargaining objective function.

The aggregate state of the economy is measured by the aggregate unemployment rate, u_t. Define market tightness as the aggregate buyer-to-seller ratio, $\theta_t = v_t/u_t$.

Post-match Bargaining. The Nash bargaining solution is of the following form: If $S_t := S(u_t)$ is the total match surplus, then the worker's match surplus, is

$$\phi S_t := W^e(u_t) - W^u(u_t), \tag{E.1}$$

and the firm's surplus is

$$(1-\phi) S_t := J^f(u_t) - J^v(u_t). \tag{E.2}$$

The list (W^e, W^u, J^f, J^v) comprises the value functions of (i) an employed consumer, (ii) an unemployed consumer, (iii) a firm's whose vancancy is just filled, and (iv) a firm that remains vacant.

1. **Matching process.** By linear homogeneity of M, we have $q(\theta_t) = M(u_t, v_t)/v_t$. Also we have $p(\theta_t) = M(u_t, v_t)/u_t = \theta_t q(\theta_t)$. ⟨ **4%** ⟩

   (a) What is the interpretation of the functions q and p?

   (b) Also, M is such that $\mathrm{d}q(\theta)/\mathrm{d}\theta < 0$ and $\mathrm{d}\ln[q(\theta)]/\mathrm{d}\ln(\theta) \in (-1, 0)$. Explain in words what these two properties say about the decentralized matching market or process.

2. **Evolution of aggregate unmeployment.** Aggregate accounting of labor flows means that we have:

$$u_{t+1} = [1 - \rho - \theta_t q(\theta_t)]\, u_t + \rho. \tag{E.3}$$

Explain, with the aid of plain English, how you would arrive at this expression above. ⟨ **4%** ⟩

3. **Firms.** Given u_t, a currently productive firm has valuation

$$J^f(u_t) = A - w_t + \beta \left[\rho J^v(u_{t+1}) + (1-\rho) J^f(u_{t+1})\right]. \tag{E.4}$$

Let κ denote a fixed job-advertising cost (measured in units of the final consumption good). A firm with job vacancy has the value

$$J^v(u_t) = -\kappa + \beta \left[q(\theta_t) J^f(u_{t+1}) + (1 - q(\theta_t))\, J^v(u_{t+1})\right]. \tag{E.5}$$

There is free entry to advertising for jobs.

(a) Explain in plain English what these two recursions above say. ⟨ **2%** ⟩

(b) Show that competition or free entry in terms of vacancy posting results in the condition:

$$J^f(u_{t+1}) = \frac{\kappa}{\beta q(\theta_t)}. \tag{E.6}$$

Then, show that you can arrive at the following dynamic Job Creation condition:

$$\frac{\kappa}{q(\theta_t)} = \beta \left[A - w_{t+1} + \frac{(1-\rho)\kappa}{q(\theta_{t+1})}\right]. \tag{E.7}$$

Interpret this "asset pricing" or Euler equation in plain English.

Hint: On the left-hand-side of (E.7), $\frac{\kappa}{q(\theta_t)}$ measures the expected employee-search cost of the firm. On the right, $\beta(1-\rho)$ is the effective discount factor to the firm since there is a per-period probability ρ that a match gets destroyed. ⟨ **4%** ⟩

4. **Workers.** Given u_t, a currently employed worker-consumer has valuation

$$W^e(u_t) = w_t + \beta \left[\rho W^u(u_{t+1}) + (1-\rho) W^e(u_{t+1})\right]. \tag{E.8}$$

A currently unemployed worker-consumer has valuation

$$W^u(u_t) = b + \beta \left[\theta_t q(\theta_t) W^e(u_{t+1}) + (1 - \theta_t q(\theta_t)) W^u(u_{t+1})\right]. \tag{E.9}$$

If we combine the Nash bargaining solution, (E.1), and (E.2), with the competitive advertising result, (E.6), and with the worker's valuations (E.8) and (E.9), then we can derive the total surplus as

$$S(u_t) = A - b + \frac{\kappa\left[1 - \rho - \phi \theta_t q(\theta_t)\right]}{(1-\phi) q(\theta_t)}. \tag{E.10}$$

Further algebra will give us the following Wage Setting curve:

$$w_t = \phi(A + \kappa\theta_t) + (1-\phi) b. \tag{E.11}$$

Interpret in words what this Wage Setting curve says.

⟨ **4%** ⟩

5. **Recursive decentralized equilibrium (RDE).** *Hint*: A RDE will be given by a sequence $\{\theta_t, u_t\}_{t=0}^{\infty}$ beginning from some fixed initial state u_0 that must satisfy individually rational behavior and aggregate consistency of actions.

   Write down a precise and compact definition of RDE using all the information given to you above. $\langle$ **4%** $\rangle$

6. In the lead-up to the federal elections, the Prime Minister claimed that if his opponents (the 'Working Australians for a New Kapitalism' party) win the election, the bargaining power will shift too much towards workers. The Prime Minister was quoted as saying: "This will, in the long run, create more unemployment and businesses will have to pay higher wage rates".

   From the perspective of this model, would this claim be correct? Use the model to explain your conclusion.

   Hint: Consider a steady state of the RDE. $\langle$ **8%** $\rangle$

Question F (30%) **Households.** Assume in each period there is a constant population of young people of size one. For each household cohort born at date t, their lifetime payoff depends on their own consumption when young and old, respectively, c_t and d_{t+1}, and some expenditure on their children's education e_t:

$$U(c_t) + \beta U(d_{t+1}) + \gamma U(e_t), \qquad \beta \in (0,1), \gamma > 0. \tag{F.1}$$

The parameters, β and γ, respectively, capture household impatience and the degree of altruism towards one's own children. Assume $U(x) = \ln(x)$.

The date t household faces these budget constraints over their lifetime:

$$c_t + s_t + e_t = w_t h_t, \tag{F.2}$$

and,

$$d_{t+1} = R_{t+1} s_t, \tag{F.3}$$

where s_t is household saving, w_t is a competitive real wage rate, R_{t+1} is a competitive gross return on investment in capital goods. Note that each household has one unit of labor time, but their effective labor supply is $h_t \times 1$ units of time.

It takes time for current educational investment e_t to be embodied in next-period children. Also, a child's human capital (h_{t+1}) will also inherit from her parents' human capital (h_t). Suppose the evolution (or production) of within-household human capital follows the non-homogeneous difference equation:

$$h_{t+1} = A e_t^{\theta} h_t^{1-\theta}, \qquad \theta \in (0,1), A > 0. \tag{F.4}$$

Firm. A representative firm produces a final good using the technology:

$$f(k_t) := Z k_t^{\alpha}, \qquad \alpha \in (0,1), Z > 0. \tag{F.5}$$

The variable $k_t := K_t/H_t$ is the aggregate capital-to-efficient-labor ratio. Profit maximization gives the firm's optimal demand for effective labor and capital, respectively, as

$$w_t = f(k_t) - k_t f'(k_t), \tag{F.6}$$

and,

$$R_t = f'(k_t). \tag{F.7}$$

Aggregation and market clearing. Let the cumulative probability distribution function over human capital across households at date t be $M_t : \mathbb{R}_{++} \mapsto [0,1]$. The function M_t is non-decreasing, with $M_t(0^+) = 0$ and $M_t(+\infty) = 1$, where $0^+ := \lim_{h \searrow 0} h$. Let the distribution of the logarithm of h_t be given by the cumulative probability distribution function μ_t. There is a bijection such that

$$\mu_t(\ln(h_t)) = M_t(h_t). \tag{F.8}$$

Assume that the initial distribution of agents, μ_0, has variance $\sigma_0 > 0$.

Thus, we can account for the aggregate level of human capital through labor market clearing:

$$H_t := \int e^{\ln(h_t)} \mathrm{d}\mu_t(\ln(h_t)). \tag{F.9}$$

Capital market clearing is summarized by

$$K_{t+1} = \int e^{\ln(s_t)} \mathrm{d}\mu_t(\ln(h_t)). \tag{F.10}$$

Self-funding of education. Assume that education is privately funded within each household.

1. Derive the optimal saving supply and education demand by each current household.

   Hint: Write these down as explicit functions of their current income.

   ⟨ **4%** ⟩

2. Show that in an equilibrium growth in individual human capital is identical across individuals:

$$\frac{h_{t+1}}{h_t} = A\left[\frac{\gamma}{1+\beta+\gamma} w_t\right]^{\theta}.$$

   ⟨ **4%** ⟩

3. The last result implies that the growth rate of average human capital is:

$$\frac{H_{t+1}}{H_t} = \kappa w_t^{\theta}, \qquad \kappa := A\left(\frac{\gamma}{1+\beta+\gamma}\right)^{\theta}. \tag{F.11}$$

   Now show that capital per efficient units of workers follows this recursion:

$$k_{t+1} = \left(\frac{\beta}{(1+\beta+\gamma)\kappa}\right)\left[(1-\alpha)Zk_t^{\alpha}\right]^{1-\theta}, \qquad \kappa := A\left(\frac{\gamma}{1+\beta+\gamma}\right)^{\theta}. \tag{F.12}$$

   Then argue that there is a unique steady state equilibrium in terms of the relative prices (w, R) and allocation (in terms of the K/H ratio) k.

   Hint: Capital market clearing (F.10) implies $K_{t+1} = \left(\frac{\beta}{1+\beta+\gamma}\right) w_t H_t$.

   ⟨ **6%** ⟩

4. *Hint*: Consider a log-linear process:

$$x_{t+1} = \rho_t x_t + v_t.$$

   where $x_t := \ln(y_t)$ is distributed according to the distribution function μ_t, and, v_t is a given process that is independent of x_t.

   Assume μ_t is characterized by its first two moments: mean and variance. Then the mean and variance of x_t, respectively $\bar{x}_t$ and σ_t^2, will evolve according to

$$\bar{x}_t = \rho_t \bar{x}_t + v_t, \qquad \text{and,} \qquad \sigma_{t+1}^2 = \rho_t^2 \sigma_t^2.$$

Let $l_t := \ln(h_t)$, and $\bar{l}_t := \int l_t \mathrm{d}\mu_t(l_t)$. Derive the dynamics of distribution of the logarithm of human capital in terms of its mean and variance dynamics. What happens to labor-income and capital-income inequality (in terms of variance) over time, as $t \to \infty$?

More hints: The dynamics of k_t is monotone and converges to the unique steady state from any initial state, as $t \to \infty$.) Utilize your workings from sub-question 2 earlier.

⟨ **6%** ⟩

Public funding of education. Consider now, education being provided publicly and financed through taxation of aggregate wage income:

$$e_t = \tau_t w_t H_t. \tag{F.13}$$

That is, e_t is no longer a choice variable for individual households. Equation (F.2) becomes

$$c_t + s_t = (1 - \tau_t) w_t h_t. \tag{F.14}$$

5. Answer these questions:

   (a) Show that the optimal saving function of a household is now

   $$s_t = \left(\frac{\beta}{1+\beta}\right)(1 - \tau_t) w_t h_t. \tag{F.15}$$

   ⟨ **2%** ⟩

   (b) Using this result, derive what is each household's equilibrium preferred tax rate. *Hint*: Imagine if the household were a dictator, what tax rate would it choose to maximize its own welfare criterion? ⟨ **2%** ⟩

   (c) By the median voter theorem, the voting equilibrium implements the policy preferred by a median voter/household. In this model, the median-preferred policy is $\tau_t = \gamma/(1 + \beta + \gamma)$. Explain why.

   ⟨ **2%** ⟩

6. Using the last result from the last sub-question:

   (a) Show that the logarithm of individual human capital evolves according to

   $$h_{t+1} = \kappa w_t^\theta H_t^\theta h_t^{1-\theta}. \tag{F.16}$$

   ⟨ **1%** ⟩

   (b) Derive the equilibrium evolution of the distribution of log human capital (again, in terms of mean and variance dynamics).

   ⟨ **1%** ⟩

   (c) Compare your result here with that of the earlier economy without government financing of education as a public good. What can you say specifically about inequality of labor and capital incomes here?

   For technical reasons, assume that the initial support of μ_0 is a compact set $[l_0^{min}, l_0^{max}] \subset (0, \infty)$.

   ⟨ **2%** ⟩

Appendix: Useful definitions and results

Neumann expansion. Let A be a stable square matrix, L the lag operator, and I an identity matrix conformable to A:

$$(I - AL)^{-1} = I + AL + AL^2 + ...$$

Integration by parts.

$$\int_a^b udv = uv|_a^b - \int_a^b vdu$$

Leibniz' rule. Let $\phi(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$ for $t \in [c,d]$ and f and f_t are continuous and a, b are differentiable on $[c,d]$. Then $\phi(t)$ is differentiable on $[c,d]$ and

$$\phi(t) = f(b(t), t)\, b'(t) - f(a(t), t)\, a'(t) + \int_{a(t)}^{b(t)} f_t(x,t)\,dx.$$

Independent random variable and geometric distribution. Let N be a geometrically distributed random waiting time until the arrival of a desired signal. Let $\lambda = \int_0^{\overline{x}} dF(x)$ be the probability that the desired signal x is not observed in one period. Then,

$$\Pr\{N = 1\} = 1 - \lambda$$

$$\Pr\{N = j\} = (1-\lambda)\,\lambda^{j-1}.$$

The mean waiting time is then $\overline{N} = (1-\lambda)^{-1}$.

Definition 1 *A correspondence* $\Gamma : X \rightrightarrows Y$ *is* lower semi-continuous (lsc) at x *if for every open set* V *that meets* $\Gamma(x)$ *– i.e.* $V \cap \Gamma(x) \neq \emptyset$ *– there is an open set* $U(x) \ni x$ *such that if* $x' \in U(x)$*, then* V *also meets* $\Gamma(x')$ *or* $\Gamma(x') \cap V \neq \emptyset$*. The correspondence* Γ *is said to be lsc if it is lsc at every* $x \in X$*.*

Definition 2 *A correspondence* $\Gamma : X \rightrightarrows Y$ *is* upper semi-continuous (usc) at x *if for every open set* $V \supset \Gamma(x)$*, there is an open set* $U(x) \ni x$ *such that if* $x' \in U(x)$*, then* $V \supset \Gamma(x')$*. A correspondence is said to be upper semi-continuous and compact-valued if it is usc at every* $x \in X$*.*

Definition 3 *A correspondence* $\Gamma : X \rightrightarrows Y$ *is* continuous at x *if it is both usc and lsc at* x*. Then we say* Γ *is* continuous *if it is both usc and lsc (i.e. usc and lsc at every* $x \in X$*).*

Definition 4 *Let* (S,d) *be a metric space and the map* $T : S \to S$*. Let* $T(w) := Tw$ *be the value of* T *at* $w \in S$*.* T *is a contraction with modulus* $0 \leq \beta < 1$ *if* $d(Tw, Tv) \leq \beta d(w,v)$ *for all* $w, v \in S$*.*

Theorem 1 (Banach Fixed Point Theorem) *If* (S,d) *is a complete metric space and* $T : S \to S$ *is a contraction, then there is a fixed point for* T *and it is unique.*

Theorem 2 *The following metric spaces are complete:*

1. $(\mathbb{R}, |\cdot|)$.
2. $(C_b(X), d_\infty)$*, where* $C_b(X)$ *is the set of continuous and bounded functions on* X*.*
3. $(C'_b(X), d_\infty)$*, where* $C'_b(X)$ *is the set of continuous, bounded and nondecreasing functions on* X*.*
4. $(C''_b(X), d_\infty)$*, where* $C''_b(X)$ *is the set of continuous, bounded and strictly increasing functions on* X*.*

Furthermore, $C''_b(X) \subset C'_b(X) \subset C_b(X)$*, are closed subsets relative to* $C_b(X)$*.*

Theorem 3 *Suppose* (S,d) *is a complete metric space, and* $T : S \to S$ *is a* $\beta < 1$ *contraction mapping with fixed point* $v \in S$*. If* $S' \subset S$ *is closed and* $T(S') \subseteq S'$*, then* $v \in S'$*. Furthermore, if* $T(S') \subseteq S'' \subseteq S'$*, then* $v \in S''$*.*

Theorem 4 (Blackwell's sufficient conditions for a contraction) *Let $M : S \to S$ be any map satisfying*

1. *Monotonicity: For any $v, w \in S$ such that $w \geq v \Rightarrow Mw \geq Mv$.*
2. *Discounting: There exists a $0 \leq \beta < 1$ such that $M(w+c) = Mw + \beta c$, for all $w \in S$ and $c \in \mathbb{R}$. (Define $(f+c)(x) = f(x) + c$.)*

Then M is a contraction with modulus β.

Let (P, λ_0) be a Markov chain on a finite state space S.

Theorem 5 *If $P_{ij}^{(\tau)} > 0$ for all $i, j = 1, ..., n$, then there exists a unique invariant distribution $\lambda^* = \lim_{t\to\infty} \lambda_0 P^t$ satisfying $\lambda^* = \lambda^* P$.*

Theorem 6 *Let $h : S \to \mathbb{R}$. If $\{\varepsilon_t\}$ is a Markov chain (P, λ_0) on the finite set $S = \{s_1, ..., s_n\}$ such that it is asymptotically stable with stationary distribution λ^*, then as $T \to \infty$,*

$$\frac{1}{T}\sum_{t=0}^{T} h(\varepsilon_t) \to \sum_{j=1}^{n} h(s_j)\lambda^*(s_j)$$

with probability one.

Theorem 7 *The characteristic polynomial of a (2×2) matrix $\mathbf{F}$ is*

$$P(\lambda) = \lambda^2 - \text{trace}(\mathbf{F})\lambda + \det(\mathbf{F}).$$

The (at most two distinct) eigenvalues, λ, solve $P(\lambda) = 0$.

Theorem 8 *Let $\mathbf{A}$ be a $n \times n$ matrix with eigenvalues $\lambda_1, ..., \lambda_n$. Then*

1. *$\lambda_1 + \lambda_2 + ... + \lambda_n = trace(\mathbf{A})$, and*
2. *$\lambda_1 \cdot \lambda_2 \cdot ... \cdot \lambda_n = \det(\mathbf{A})$.*

Theorem 9 *$\mathbf{F}$ is a stable matrix, or all of its eigenvalues are such that $|\lambda_i| < 1$, if and only if*

1. *$|\det(\mathbf{F})| < 1$, and*
2. *$|-trace(\mathbf{F})| - \det(\mathbf{F}) < 1$.*

Note: The trace of a matrix $\mathbf{A}$ is the sum of its diagonal elements. The determinant of a (2×2) matrix $\mathbf{F}$ is given by $f_{11}f_{22} - f_{21}f_{12}$, where f_{ij} is the row-i and column-j element of the matrix.

Implicit differentiation (bivariate example). Consider a smooth, bivariate function $(x, y) \mapsto R(x, y)$. If $R(x, y) = 0$, the derivative of the implicit function $x \mapsto f(x) \equiv y$ is $\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{\partial R/\partial x}{\partial R/\partial y} = -\frac{R_x}{R_y}$. (This formula is obtained from the generalized chain rule to obtain the total derivative with respect to x.)

——— *End of Examination* ———