Further Mathematics: FS1 Topic Questions

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Topic 1: Alpha

Subtopic: AlphaOne

Problem 1

a)

when $\min\left(\|\boldsymbol{x}\|_2\right)$, $\boldsymbol{x} = \boldsymbol{x}^*$ is the solution to the problem, which is $x^* = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{pmatrix}$

b) We have a matrix $\boldsymbol{A} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}$, the projection operator is

$$\boldsymbol{P} = \boldsymbol{A}\left(\boldsymbol{A}^T A\right)^{-1}\boldsymbol{A}^T = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

hence,

$$\boldsymbol{x}^* = \boldsymbol{P}\boldsymbol{v} = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{pmatrix}.$$

c) We have a matrix $\boldsymbol{A} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \\ 2 & 2 \end{pmatrix}$, the projection operator is

$$\boldsymbol{P} = \boldsymbol{A}\left(\boldsymbol{A}^T A\right)^{-1}\boldsymbol{A}^T = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

hence,

$$\boldsymbol{x}^* = \boldsymbol{P}\boldsymbol{v} = \begin{pmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ 0 \end{pmatrix}.$$

Problem 2

a) we know that:

$$\operatorname{prox}_{\varphi}(z) = \arg\min_{x\in\mathbb{R}}\left\{\frac{1}{2}\|x - z\|^2 + \phi(x - c)\right\}.$$

let $x' = x - c$

$$\operatorname{prox}_{\varphi}(z) = \arg\min_{x\in\mathbb{R}}\left\{\frac{1}{2}\|x' - (z - c)\|^2 + \phi(x' + c - c)\right\} + c = \operatorname{prox}_{\phi}(z - c) + c.$$

b) if we want to $f(x) = \frac{1}{2}\|x - z\|^2 + \phi(x)$ to be minimized, we need to find the x that makes the derivative of the function equal to zero.

we know

$$\partial f(x) = \begin{cases} x - z + \lambda \text{ when } x > 0 \\ [x - z - \lambda, x - z + \lambda] \text{ when } x = 0 \\ x - z - \lambda \text{ when } x < 0 \end{cases}$$

. Hence, let

$$\partial f(x) = 0$$

, we have

$$\text{prox}_{\phi(z)} = x^* = \begin{cases} z - \lambda \text{ when } z > \lambda \\ [z - \lambda, z + \lambda] \text{ when } z \in [-\lambda, \lambda]. \\ z + \lambda \text{ when } z < -\lambda \end{cases}$$

c) if $\varphi(x) = \lambda|x - c|$, where $c \in \mathbb{R}$ and $\lambda > 0$. Use the result from part a.

$$\text{prox}_{\varphi(z)} = \text{prox}_{\phi(z-c)} + c = \begin{cases} z - \lambda \text{ when } z > \lambda + c \\ [z - \lambda, z + \lambda] \text{ when } z \in [-\lambda + c, \lambda + c] \\ z + \lambda \text{ when } z < -\lambda + c \end{cases}$$

Subtopic: AlphaTwo

Problem 3

a) If we take the derivative of $\frac{1}{2}\left\|\boldsymbol{x} - \boldsymbol{x}^{t-1}\right\|^2 + \gamma g(\boldsymbol{x})$, we have

$$\boldsymbol{x^t} = \text{prox}_{\gamma g}(\boldsymbol{x}^{t-1}) = \boldsymbol{x}^{t-1} - \gamma \nabla g(\boldsymbol{x^t})$$

b) By the convexity of g, we know that $g(\boldsymbol{x}^t) + \nabla g(\boldsymbol{x}^t)^T(\boldsymbol{x}^{t-1} - \boldsymbol{x}^t) \leq g(\boldsymbol{x^{t-1}})$. Hence, we have

$$g(\boldsymbol{x}^t) \leq g(\boldsymbol{x^{t-1}}) - \nabla g(\boldsymbol{x}^t)^T(\boldsymbol{x}^{t-1} - \boldsymbol{x}^t) = g(\boldsymbol{x^{t-1}}) - \gamma \nabla \left\|g(\boldsymbol{x^t})\right\|_2^2$$

c) because $\boldsymbol{x^t} = \boldsymbol{x}^{t-1} - \gamma \nabla g(\boldsymbol{x^t})$ which is a gradient descent method, so

$$-\infty < g(\boldsymbol{x}^t) \leq g(\boldsymbol{x}^{t-1})$$

and we have

$$g(\boldsymbol{x}^t) \leq g(\boldsymbol{x^{t-1}}) - \gamma \nabla \left\|g(\boldsymbol{x^t})\right\|_2^2$$

hence

$$0 \leq \gamma \nabla \left\|g(\boldsymbol{x^t})\right\|_2^2 \leq 0$$

if

$$t \to +\infty$$

Topic 2: Beta

Problem 4

a) because

$$\partial f(\boldsymbol{x}) = \{\boldsymbol{v} \in \mathbb{R}^n : f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n\}$$

if $g(\boldsymbol{x}) = \theta f(\boldsymbol{x})$,

$$\partial g(\boldsymbol{x}) = \{\boldsymbol{v} \in \mathbb{R}^n : g(\boldsymbol{y}) \geq g(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n\}$$

$$\partial g(\boldsymbol{x}) = \{\boldsymbol{v} \in \mathbb{R}^n : \theta f(\boldsymbol{y}) \geq \theta f(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n\}$$

$$\partial g(\boldsymbol{x}) = \left\{\boldsymbol{v} \in \mathbb{R}^n : f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \frac{\boldsymbol{v}^T}{\theta}(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n\right\}$$

$$\partial g(\boldsymbol{x}) = \theta\{\boldsymbol{v} \in \mathbb{R}^n : f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n\} = \theta \partial f(\boldsymbol{x})$$

b)

$$\partial h(\boldsymbol{x}) = \{\boldsymbol{v} \in \mathbb{R}^n : f(\boldsymbol{y}) + g(\boldsymbol{y}) \geq f(\boldsymbol{x}) + g(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n\}$$

all of the elements that satisfy

$$f(\boldsymbol{y}) \geq f(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n$$

and

$$g(\boldsymbol{y}) \geq g(\boldsymbol{x}) + \boldsymbol{v}^T(\boldsymbol{y} - \boldsymbol{x}), \forall \boldsymbol{y} \in \mathbb{R}^n$$

are in the set

$$\partial h(\boldsymbol{x})$$

hence

$$\partial f(\boldsymbol{x}) + \partial g(\boldsymbol{x}) \subseteq \partial h(\boldsymbol{x})$$

c) we know that

$$\partial \|x\|_1 = \begin{cases} 1 \text{ when } x > 0 \\ [-1, 1] \text{ when } x = 0 \\ -1 \text{ when } x < 0 \end{cases}$$

.

hence $\operatorname{sgn}(x) \in \partial \|x\|_1$.

Problem 5

b) Not differentiable at $\boldsymbol{x} = \mathbf{0}$, and h is convex.

c)

$$\nabla\left[\frac{1}{2}\|\boldsymbol{x}-\boldsymbol{y}\|_2^2 + \gamma\lambda\|x\|_1\right] = \begin{cases} x - y + \gamma\lambda \text{ when } x > 0 \\ [x-y-\gamma\lambda, x-y+\gamma\lambda] \text{ when } x = 0 \\ x - y - \gamma\lambda \text{ when } x < 0 \end{cases}$$

.

let it be 0, we have

$$\mathrm{prox}_{\gamma g(y)} = x^* = \begin{cases} y - \gamma\lambda \text{ when } y > \lambda \\ [y-\gamma\lambda, y+\gamma\lambda] \text{ when } y \in [-\gamma\lambda, \gamma\lambda]. \\ y + \gamma\lambda \text{ when } y < -\gamma\lambda \end{cases}$$

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% load the variables of the optimization problem
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

load('dataset.mat');

[p, n] = size(A);

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% set up the function and its gradient
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

evaluate_f = @(x) (1/n)*sum(log(1+exp(-b.*(A'*x))));
evaluate_gradf = @(x) (1/n)*A*(-b.*exp(-b.*(A'*x))./(1+exp(-b.*(A'*x))));

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% parameters of the gradient method
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

xInit = zeros(p, 1); % zero initialization
stepSize = 1; % step-size of the gradient method
maxIter = 1000; % maximum number of iterations

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% optimize
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% initialize
x = xInit;

% keep track of cost function values
objVals = zeros(maxIter, 1);

% iterate
for iter = 1:maxIter
```

```
    % update
    xNext = x - stepSize*evaluate_gradf(x);

    % evaluate the objective
    funcNext = evaluate_f(xNext);

    % store the objective and the classification error
    objVals(iter) = funcNext;

    fprintf('[%d/%d] [step: %.1e] [objective: %.1e]\n',...
        iter, maxIter, stepSize, objVals(iter));

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % begin visualize data
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    % plot the evolution
    figure(1);
    set(gcf, 'Color', 'w');
    semilogy(1:iter, objVals(1:iter), 'b-',...
        iter, objVals(iter), 'b*', 'LineWidth', 2);
    grid on;
    axis tight;
    xlabel('iteration');
    ylabel('objective');
    title(sprintf('GM (f = %.2e)', objVals(iter)));
    xlim([1 maxIter]);
    set(gca, 'FontSize', 16);
    drawnow;

    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
    % end visualize data
    %%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

    % update w
    x = xNext;
end
```