Positivity of polynomials in the symbolic square

1 Introduction

We denote as $\mathbb{P}^n$ the n-dimensional projective space over $\mathbb{C}$. We write $S = \mathbb{C}[x_0, \cdots, x_n]$ and $\mathfrak{m} = (x_0, \cdots, x_n) \subset S$.

Definition 1.1. *Let $I \subseteq S$ be a homogeneous ideal. The* ***saturation*** *of I is defined to be*

$$I^{\mathrm{sat}} = \{ \ f \in S \mid f\mathfrak{m}^k \subseteq I \text{ for some } k \geqslant 0 \ \}.$$

Theorem 1.2 ([1])**.** *Let $X \subseteq \mathbb{P}^n$ be a smooth variety and $I \subseteq S$ be its homogeneous ideal. Then $I^{(d)} = (I^d)^{\mathrm{sat}}$ for all d.*

Theorem 1.3. *Let $X \subseteq \mathbb{P}^n$ be a smooth irreducible variety and $I \subseteq S$ be its homogeneous ideal. Let $P \in \mathbb{P}^n$ be a real point of X, then I^2 and $I^{(2)}$ coincide when localized at P.*

Proof. We only need to prove $I^{(2)} \subseteq I$ when localized at P. Let $I = (g_1, \cdots, g_m)$ and d be the maximal degree of the g_i. As I is prime, I^2 and $I^{(2)}$ agree when localized at I. Take any form $f \in I^{(2)}$, then by Theorem 1.3 we have $f \in (I^2)^{\mathrm{sat}}$, and so $f\mathfrak{m}^k \subset I^2$ for some k. In particular, $(x_0^k + \cdots + x_n^k)f \in I^2$ for some k. Take k to be even, then $x_0^k + \cdots + x_n^k \neq 0$ at P, so $f \in I^{(2)}$ when localized at P. ☐

Theorem 1.4. *Let $X \subseteq \mathbb{P}^n$ be a smooth irreducible variety and $I \subseteq S$ be its homogeneous ideal.*

References

[1] Robert Lazarsfeld Lawrence Ein, Huy Tai Ha. Saturation bounds for smooth varieties. *Algebra & Number Theory*, 16:1531–1546, 2022.