

The Betti Numbers of Edge Ideals of Trees and the Regularity Offset

1 Background

1.1 Graphs and clique complexes

A simple graph G is the composition of vertex set $V(G)$ and edge set $E(G)$. We use G^c to denote the complement graph of G . If W is a subset of $V(G)$, we use $G[W]$ to denote the induced subgraph on W .

For a simple graph G whose vertex set is $[n]$, one can identify it with a square-free quadratic ideal I_G over a field k via $I_G = (x_i x_j | i, j \in E(G))$. We call I_G the edge ideal of G .

Definition 1.1. Given a graph G , where $V(G) = \{x_1, x_2, \dots, x_n\}$, the clique complex of G , denoted by ΔG , is a simplicial complex that consists of t -simplices $(x_{i_1}, x_{i_2}, \dots, x_{i_t})$ whenever $G[\{x_{i_1}, x_{i_2}, \dots, x_{i_t}\}]$ is a complete graph of t vertices.

One can observe that the edge ideal I_G is the Stanley-Reisner ideal of ΔG^c . Hence, we have a one-to-one correspondence between the family of simple graphs and the family of clique complexes. Moreover, homology of simplicial complexes ΔG^c can give Betti numbers of I_G via Hochster's formula.

Theorem 1.2. [Hochster's formula] Let I_G be the edge ideal of a graph G . Then for $i \geq 0$ and squarefree degree \mathbf{b} ,

$$\beta_{i, \mathbf{b}}(I_G) = \dim_k(\tilde{H}_{|\mathbf{b}|-i-2}(\Delta G^c[\text{supp}(\mathbf{b})], k)),$$

and $\beta_{i, \mathbf{b}}(I_G) = 0$ if \mathbf{b} is not squarefree. If we forget the multigrading, then for $j \geq 0$,

$$\beta_{i, i+j}(I_G) = \sum_{|W|=i+j} \dim_k(\tilde{H}_{j-2}(\Delta G^c[W], k)),$$

where W runs over all the subsets of $V(G)$ of size $i+j$.

More detailed information on Hochster's formula can be found in [5] and [1]. We now give the definition of Castelnuovo-Mumford regularity of an edge-ideal I_G .

Definition 1.3 (Castelnuovo-Mumford regularity). We use $\text{reg}(I_G)$ (or $\text{reg}(G)$ in short) to denote the Castelnuovo-Mumford regularity of I_G .

$$\text{reg}(I_G) = \max\{j | \beta_{i, i+j}(I_G) \neq 0\}.$$

To make the following paper more readable, for a subgraph $S \subset G$, we say the subgraph S contribute to the regularity if the multigraded betti number $\beta_{|\mathbf{s}|-reg(G), \mathbf{s}} \neq 0$, where \mathbf{s} is a grading that is all zero but only 1s on the variables that are corresponding to the vertex set $V(S)$.

By Hochster's formula, the key part of finding the regularity of an edge ideal is to find the largest number j , such that the $(j - 2)$ -nd homology group of a certain size induced subcomplex of ΔG^c is non-trivial.

1.2 The lcm-lattice

The lcm-lattice, introduced in [6], is a convenient way to compute the betti numbers and analyze the minimal free resolution of edge ideals.

Definition 1.4 ([6], Page 2). *Let $S = k[x_0, \dots, x_n]$ be the polynomial ring over a field k and $I = (m_1, \dots, m_d)$ a monomial ideal with its minimal generators. Its **lcm-lattice** L_I is the lattice with elements labeled by the least common multiples of m_1, \dots, m_d ordered by divisibility, including 1, which is regarded as the lcm of the empty set.*

It follows immediately that the atoms of L_I are precisely $\{m_1, \dots, m_d\}$.

The relabeling construction from [6] captures the adding variables construction discussed in the previous section.

Construction 1.5.

Let I and J be monomial ideals in $S = k[x_0, \dots, x_n]$, and $f : L_I \rightarrow L_J$ a map which is a bijection on atoms and preserves joins. Let \mathbf{P} be a bounded chain complex of free modules with homogeneous differentials $\partial^{\mathbf{P}}$ such that every generator of every free module \mathbf{P}_n has multidegree in L_I . The chain complex $f(\mathbf{P})$ is constructed as follows:

Let $\mathbf{P}_n = \bigoplus_{i=1}^d S[-\mathbf{a}_i]^{k_i}$, then $f(\mathbf{P})_n = \bigoplus_{i=1}^d S[-f(\mathbf{a}_i)]^{k_i}$, where we also consider f as a map $\mathbb{N}^{n+1} \rightarrow \mathbb{N}^{n+1}$ between multidegrees.

The differential $\partial_n^{\mathbf{P}}$ is homogeneous, and thus is a matrix of monomials. Suppose that its (i, j) -th entry $c\mathbf{x}^{\mathbf{t}}$ is nonzero, with $c \in k$. Suppose the i -th summand of \mathbf{P}_{n-1} has generator in degree \mathbf{a} , and the j -th summand of \mathbf{P}_n has generator in degree \mathbf{b} , then homogeneity implies $\mathbf{t} = \mathbf{b} - \mathbf{a}$. The differential $\partial_n^{f(\mathbf{P})}$ is the matrix having (i, j) -th entry $c\mathbf{x}^{f(\mathbf{b})-f(\mathbf{a})}$. This makes it the unique homogeneous differential of $f(\mathbf{P})$ having the same k -coefficients in every entry as $\partial_n^{\mathbf{P}}$.

The following theorem from [6] states that ideals with isomorphic lcm-lattice have similar minimal free resolutions.

Theorem 1.6 ([6], Theorem 3.3). *Let I and J be monomial ideals. Suppose that $f : L_I \rightarrow L_J$ is a bijection of the underlying sets and preserves joins. Let \mathbf{F} be the minimal free resolution of S/I , then $f(\mathbf{F})$ is the minimal free resolution of S/J .*

2 Edge Ideals of Trees

Trees are a type of very simple graphs whose Betti numbers are relatively simple to compute. In this section, we present a new proof to a leaf deletion theorem in [7], and use it to compute the Betti numbers of trees.

2.1 Full graphs and a leaf deletion theorem

By Hochster's formula (Theorem 1.2), the nonzero Betti numbers of a graph can appear only in squarefree multidegrees. The largest such multidegree is $\mathbf{1} = (1, \dots, 1)$, and we make the following definition.

Definition 2.1. A **full graph** is a graph G such that for some i we have $\beta_{i,1}(I_G) \neq 0$. A **full subgraph** of a graph G is an induced subgraph H that is itself a full graph.

We will see later (THEM??) that a tree is full if and only if all vertices contribute to its regularity, hence the name 'full'.

We now turn to a leaf deletion theorem, implicitly mentioned in [7], that lies at the core of the computation of tree Betti numbers. We first present an auxiliary lemma ([7], Theorem 2.1.1) and a detailed proof different from the original proof in [7].

Lemma 2.2 (Mapping cone lemma). *Let G be a simple graph and $V(G) = \{x_1, \dots, x_n\}$, with x_n a leaf and x_{n-1} its neighbor. Let $S = k[x_1, \dots, x_n]$ and*

$$J = (x_i x_j \mid \{x_i, x_j\} \in E(G), x_i x_j \neq x_{n-1} x_n),$$

$$-\mathbf{2} = (0, \dots, 0, -1, -1),$$

then there is a short exact sequence

$$0 \longrightarrow S/(J : x_{n-1} x_n)[-2] \xrightarrow{x_{n-1} x_n} S/J \longrightarrow S/I_G \longrightarrow 0. \quad (1)$$

Moreover, let $\mathbf{E}, \mathbf{F}, \mathbf{G}$ be the minimal free resolutions of the three modules in (1), from left to right respectively, then the induced short exact sequence of chain complexes

$$0 \longrightarrow \mathbf{E} \xrightarrow{\phi} \mathbf{F} \longrightarrow \mathbf{G} \longrightarrow 0$$

presents \mathbf{G} as the mapping cone of ϕ .

Proof. It is clear that (1) is an exact sequence. For the second statement, as minimal free resolutions are unique up to isomorphism, it suffices to prove that the mapping cone $\mathbf{H} = C(\phi)$ is a minimal free resolution of S/I_G . It is a complex of free modules by definition, and by the long exact sequence of homology it is a resolution of S/I_G . To prove it is minimal, we need to prove that there are no non-zero constants in the (matrices of the) differentials of \mathbf{H} . As \mathbf{E} and \mathbf{F} are both minimal, there are no nonzero constants in their differentials. By the structure of the differentials of the mapping cone, it suffices to show that there are no nonzero constants in every ϕ_i .

Consider the construction of the induced chain map ϕ .

$$\begin{array}{ccccccc} \cdots & \longrightarrow & \mathbf{E}_{t+1} & \xrightarrow{d_{t+1}^{\mathbf{E}}} & \mathbf{E}_t & \xrightarrow{d_t^{\mathbf{E}}} & \mathbf{E}_{t-1} \xrightarrow{d_{t-1}^{\mathbf{E}}} \cdots \\ & & \downarrow \phi_{t+1} & & \downarrow \phi_t & & \downarrow \phi_{t-1} \\ \cdots & \longrightarrow & \mathbf{F}_{t+1} & \xrightarrow{d_{t+1}^{\mathbf{F}}} & \mathbf{F}_t & \xrightarrow{d_t^{\mathbf{F}}} & \mathbf{F}_{t-1} \xrightarrow{d_{t-1}^{\mathbf{F}}} \cdots \end{array}$$

We abuse notation a little and set $\mathbf{E}_{-1} = S/(J : x_{n-1} x_n)$ and $\mathbf{F}_{-1} = S/J$ and $\phi_{-1} = x_{n-1} x_n$. The inductive construction of ϕ is as follows. Having constructed ϕ_t , we have $d_t^{\mathbf{F}} \phi_t d_{t+1}^{\mathbf{E}} =$

$\phi_{t-1} d_t^{\mathbf{E}} d_{t+1}^{\mathbf{E}} = 0$, so $\text{im}(\phi_t d_{t+1}^{\mathbf{E}}) \subset \ker d_t^{\mathbf{F}} = \text{im} d_{t+1}^{\mathbf{F}}$. For every generator e_i of \mathbf{E}_{t+1} , pick a $f_i \in \mathbf{F}_{t+1}$ such that $d_{t+1}(f_i) = \phi_t(d_{t+1}(e_i))$, and this selection defines the map ϕ_{t+1} .

We now prove by induction that the image of every ϕ_t lies in $x_{n-1}x_n\mathbf{F}_t$, which proves the lemma. The base case ϕ_{-1} already holds. Suppose that this is true for ϕ_t , then for every generator e_i of \mathbf{F}_{t+1} we have $\phi_t(d_{t+1}(e_i)) = x_{n-1}x_n\epsilon_i$ for some $\epsilon_i \in \mathbf{F}_t$. Now $x_{n-1}x_n\epsilon_i \in \ker d_t^{\mathbf{F}}$ implies that $\epsilon_i \in \ker d_t^{\mathbf{F}} = \text{im} d_{t+1}^{\mathbf{F}}$. Therefore there exists $f_i \in \mathbf{F}_{t+1}$ such that $d_{t+1}(f_i) = \epsilon_i$. The map ϕ_{t+1} can be taken to send e_i to $x_{n-1}x_nf_i$. \square

By Hochster's formula, the non-zero Betti numbers of graphs only appear in square-free multidegrees. Squarefree multidegrees are in one-to-one correspondence with subsets of the vertex set, so we introduce the following notation. For a graph G with $V(G) = \{x_1, \dots, x_n\}$ and a subset $H \subset V(G)$, write $\beta_i(G, H)$ for the betti number $\beta_{i, \mathbf{a}}(k[V(G)]/I_G)$, where $a_i = 1$ when $x_i \in H$ and $a_i = 0$ when $x_i \notin H$. We abbreviate $\beta_i(G, V(G))$ as $\beta_i(G)$, so full graphs are graphs with $\beta_i(G) = 1$ for some i .

Theorem 2.3 (Leaf deletion theorem). *Let G be a graph with $V(G) = \{x_1, \dots, x_n\}$ and $H \subset V(G)$. Suppose x_n is a leaf of G and x_{n-1} its neighbor. Then*

$$\beta_i(G, H) = \begin{cases} \beta_i(G \setminus x_n, H), & x_n \notin H, \\ \beta_{i-\#(N(x_{n-1}) \cap H)}(G_{(n)}, H \cap G_{(n)}), & x_{n-1} \in H, x_n \in H, \\ 0, & x_{n-1} \notin H, x_n \in H. \end{cases}$$

The set $N(x_{n-1})$ is the open neighborhood of x_{n-1} in G .

The graph $G_{(n)}$ is the induced subgraph of G on $\{x_i \mid \{x_i, x_{n-1}\} \notin E(G), i \neq n-1\}$. That is, it is the induced subgraph on all vertices not adjacent to x_{n-1} and not equal to it.

To deal with the case where $G_{(n)} = \emptyset$ is the empty graph, we define $\beta_i(\emptyset, \emptyset) = 1$ when $i = 0$ and $\beta_i(\emptyset, \emptyset) = 0$ otherwise.

Proof. Let \mathbf{a} be the indicator multidegree of H , and $S = k[V(G)]$. By Lemma 2.2 we have $\mathbf{G}_i = \mathbf{F}_i \oplus \mathbf{E}_{i-1}$. Therefore

$$\beta_i(G, H) = \beta_{i, \mathbf{a}}(S/I_G) = \beta_{i, \mathbf{a}}(S/J) + \beta_{i-1, \mathbf{a}}(S/(J : x_{n-1}x_n)[-2]) \quad (2)$$

$$= \beta_{i, \mathbf{a}}(S/J) + \beta_{i-1, \mathbf{a}-\mathbf{2}}(S/(J : x_{n-1}x_n)). \quad (3)$$

Observe that J is an ideal contained in the subring $k[x_1, \dots, x_{n-1}] \subset S$, therefore its minimal free resolution considered on S is the same as its minimal free resolution considered on $k[x_1, \dots, x_{n-1}]$, just with a different base ring. Using the notation of Lemma 2.2, no generator of \mathbf{F}_i has a multidegree \mathbf{a} with $a_n \neq 0$. This means that

$$\beta_{i, \mathbf{a}}(S/J) = \begin{cases} \beta_{i, (a_1, \dots, a_{n-1})}(k[x_1, \dots, x_{n-1}]/I_J) = \beta_i(G \setminus x_n, H), & a_n = 0, \\ 0, & a_n = 1. \end{cases} \quad (4)$$

On the other hand, we have

$$(J : x_{n-1}x_n) = (J : x_{n-1}) = I_{G_{(n)}} + (x_i \mid x_i \in N(x_{n-1}) \setminus x_n).$$

Write $K = (x_i \mid x_i \in N(x_{n-1}) \setminus x_n)$, then $I_{G_{(n)}}$ and K have completely disjoint generators.

Therefore by ?THM? ?? we have

$$\begin{aligned} & \beta_{i-1, \mathbf{a}-2}(S/(J : x_{n-1}x_n)) \\ &= \sum_{s+t=i-1} \beta_{s, (\mathbf{a}-2)[V(G_{(n)})]}(k[V(G_{(n)})]/I_{G_{(n)}}) \cdot \beta_{t, (\mathbf{a}-2)[N(x_{n-1}) \setminus x_n]}(k[N(x_{n-1}) \setminus x_n]/K). \end{aligned}$$

Now K is generated by single variables, so its minimal free resolution is the Koszul complex on these variables. Therefore $\beta_{t, (\mathbf{a}-2)[N(x_{n-1}) \setminus x_n]}(k[N(x_{n-1}) \setminus x_n]/K) \neq 0$ if and only if $\mathbf{a} - 2 \geq \mathbf{0}$ and t is the number of 1's in $\mathbf{a} - 2$. This is equivalent to $a_{n-1} = a_n = 1$ and t equal to the number of a_i 's such that $a_i = 1$ and $1 \leq i < n - 1$. In this case $\beta_{t, (\mathbf{a}-2)[N(x_{n-1}) \setminus x_n]}(k[N(x_{n-1}) \setminus x_n]/K) = 1$ and $s = i - 1 - t = i - \#(N(x_{n-1}) \cap H)$, so

$$\beta_{s, (\mathbf{a}-2)[V(G_{(n)})]}(k[V(G_{(n)})]/I_{G_{(n)}}) = \beta_{i-\#(N(x_{n-1}) \cap H)}(G_{(n)}, H \cap G_{(n)}).$$

Therefore

$$\beta_{i-1, \mathbf{a}-2}(S/(J : x_{n-1}x_n)) = \begin{cases} \beta_{i-\#(N(x_{n-1}) \cap H)}(G_{(n)}, H \cap G_{(n)}), & a_{n-1} = a_n = 1, \\ 0, & \text{otherwise.} \end{cases} \quad (5)$$

The above reasoning assumed $N(x_{n-1}) \setminus x_n \neq \emptyset$. If $N(x_{n-1}) \setminus x_n = \emptyset$ then $(J : x_{n-1}x_n) = I_{G_{(n)}}$ and $\#(N(x_{n-1}) \cap H) = 1$, and so formula (5) follows directly.

Now the theorem follows from the formulas (3) (4) (5) when $G_{(n)} \neq \emptyset$. When $G_{(n)} = \emptyset$, G is a star with center x_{n-1} , and we assume $x_{n-1}, x_n \in H$ because this is always true when $G_{(n)}$ appears in the theorem. For convenience, relabel the vertices of G as $\{y_0, y_1, \dots, y_t\}$ with the center being y_0 and such that $y_0, y_1 \in H$, so $I_G = (y_0y_1, \dots, y_0y_t)$. Let $S = k[y_0, \dots, y_t]$ then the minimal free resolution of S/I_G is almost the Koszul complex on $\{y_1, \dots, y_t\}$. The only difference is the differential in degree 0: instead of (y_1, \dots, y_t) it is (y_0y_1, \dots, y_0y_t) . We are assuming $y_0, y_1 \in H$, and in this case the Betti number $\beta_i(G, H) = 1$ if and only if $i = \#(H \cap \{y_1, \dots, y_t\}) = \#(N(y_0) \cap H)$. Therefore the theorem holds precisely when $\beta_i(\emptyset, \emptyset)$ is defined the way it is defined in the statement of the theorem. \square

2.2 Computing Betti numbers

Now we turn to the computation of $\beta_i(G, H)$ for forests. Theorem 2.3 provides us with a powerful tool, letting us delete leaves and reduce a forest inductively into trivial forms. We first take a look at the trivial case.

Proposition 2.4. *Let G be an edgeless graph with $V(G) = \{x_1, \dots, x_t\}$ ($t \geq 0$) and $H \subset V(G)$, then $\beta_i(G, H) = 1$ when $H = \emptyset$ and $i = 0$, and $\beta_i(G, H) = 0$ otherwise.*

Proof. The ideal $I_G = 0$, and its ambient ring is $S = k[x_1, \dots, x_t]$. Its minimal free resolution is $0 \rightarrow S \rightarrow 0$, so its only nonzero Betti number is 1, lying in the multidegree $\mathbf{0}$ and homological degree $i = 0$. \square

Theorem 2.5. *Let T be a forest and $H \subset V(T)$, then $\beta_i(T, H)$ can only be 0 or 1, and we have*

$$\beta_i(T, H) = \beta_i(T[H], H).$$

Equivalently, $\beta_i(T, H) = 1$ when the induced subforest $T[H]$ is a full graph, and $\beta_i(T, H) = 0$ otherwise.

Proof. We provide an algorithmic procedure of deleting parts of the forest T until only an edgeless graph remains. Initialize the 6-tuple as:

$$(i_0, A_0, B_0, j_0, C_0, D_0) := (i, T, H, i, T[H], H).$$

From $(i_t, A_t, B_t, j_t, C_t, D_t)$, pick a leaf x_n and its neighbor x_{n-1} in the graph A_t and construct $(i_{t+1}, A_{t+1}, B_{t+1}, j_{t+1}, C_{t+1}, D_{t+1})$ as follows.

Case 1: $x_n \notin B_t$. Set

$$\begin{aligned} i_{t+1} &:= i_t, \\ A_{t+1} &:= A_t \setminus x_n, \\ B_{t+1} &:= B_t, \\ j_{t+1} &:= j_t, \\ C_{t+1} &:= C_t, \\ D_{t+1} &:= D_t. \end{aligned}$$

Case 2: $x_{n-1} \in B_t, x_n \in B_t$. Set

$$\begin{aligned} i_{t+1} &:= i_t - \#(N_{A_t}(x_{n-1}) \cap B_t), \\ A_{t+1} &:= A_t[V(A_t) \setminus N_{A_t}(x_{n-1}) \cup \{x_{n-1}\}], \\ B_{t+1} &:= B_t \cap V(A_{t+1}), \\ j_{t+1} &:= j_t - \#(N_{C_t}(x_{n-1}) \cap D_t), \\ C_{t+1} &:= C_t[V(C_t) \setminus N_{C_t}(x_{n-1}) \cup \{x_{n-1}\}], \\ D_{t+1} &:= D_t \cap V(C_{t+1}). \end{aligned}$$

Case 3: $x_{n-1} \notin B_t, x_n \in B_t$. Terminate.

Case 4: A_t is an edgeless graph. Terminate.

There are a few things to prove. First, it is clear from Theorem 2.3 that $\beta_{i_t}(A_t, B_t) = \beta_{i_{t+1}}(A_{t+1}, B_{t+1})$ and $\beta_{j_t}(C_t, D_t) = \beta_{j_{t+1}}(C_{t+1}, D_{t+1})$. Therefore $\beta_{i_t}(A_t, B_t) = \beta_i(T, H)$ and $\beta_{j_t}(C_t, D_t) = \beta_i(T[H], H)$. Second, we have the following relations.

$$\begin{aligned} C_t &= A_t[B_t], \\ D_t &= B_t, \\ j_t &= i_t. \end{aligned}$$

For Case 1 they are obvious. For Case 2, we prove them inductively on t . Suppose that they hold for t , then:

- (1) C_{t+1} consists of the vertices in $C_t = A_t[B_t]$ that are not adjacent to x_{n-1} and not equal to x_{n-1} , and A_{t+1} consists of the vertices in A_t that are not adjacent to x_{n-1} and not equal to x_{n-1} . Therefore, the vertices of C_{t+1} are precisely the vertices of A_{t+1} that lie in B_t , so $C_{t+1} = A_{t+1}[B_t \cap V(A_{t+1})] = A_{t+1}[B_{t+1}]$.
- (2) $D_{t+1} = D_t \cap V(C_{t+1}) = B_t \cap V(A_{t+1}[B_{t+1}]) = B_{t+1}$.

(3) $V(C_t) = V(A_t[B_t]) = B_t = D_t$, so $N_{C_t}(x_{n-1}) \cap D_t = N_{C_t}(x_{n-1}) = N_{A_t[B_t]}(x_{n-1}) = N_{A_t}(x_{n-1}) \cap B_t$. Therefore $j_t = i_t$ implies $j_{t+1} = i_{t+1}$.

These results show that

$$\begin{aligned}\beta_i(T, H) &= \beta_{i_t}(A_t, B_t), \\ \beta_i(T[H], H) &= \beta_{i_t}(A_t[B_t], B_t).\end{aligned}$$

The operations of Case 1 and Case 2 are only valid when both A_t and C_t have at least one edge. If A_t is edgeless then we reach Case 3 which terminates the algorithm. If C_t is edgeless but A_t is not, then $C_t = A_t[B_t]$ is a union of isolated points of A_t , and so the connected components that contain edges are disjoint from the vertex set B_t , so we can choose a leaf from these components and this will fall into Case 1. This shows that this algorithm can indeed run until it terminates at Case 3 or Case 4.

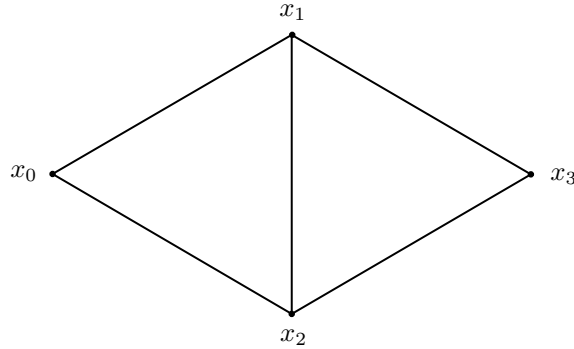
If the algorithm terminates at Case 3, then $\beta_{i_t}(A_t, B_t) = 0$ by Theorem 2.3. The subforest C_t has an isolated point, so $\beta_{i_t}(C_t) = 0$. Therefore $\beta_i(T, H) = \beta_i(T[H], H) = 0$.

If the algorithm terminates at Case 4, then A_t is an edgeless graph. By Proposition 2.4, $\beta_{i_t}(A_t, B_t) = 1$ precisely when $i_t = 0$ and $B_t = \emptyset$, and this is also precisely when $\beta_{i_t}(A_t[B_t], B_t) = 1$. Therefore $\beta_i(T, H) = \beta_i(T[H], H)$. \square

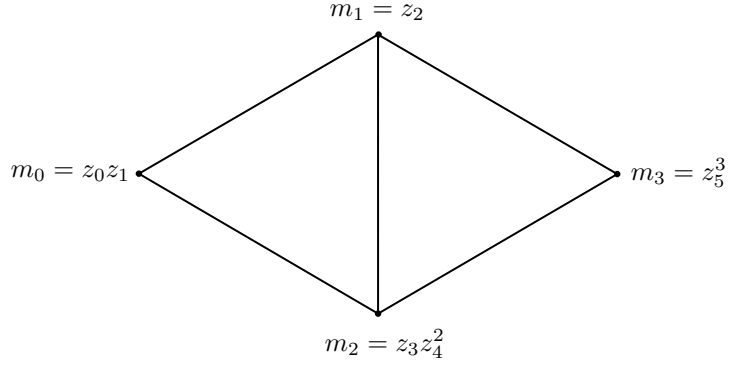
2.3 Classification of full forests

3 Adding Variables and Adding Degrees

The edge ideal of a graph is obtained by assigning a unique variable to each vertex. A natural generalization is assigning a monomial to each vertex such that no two vertices share a variable. We call this the **generalized edge ideal**. We can also think of it as starting from the ordinary edge ideal and adding extra variables to each vertex.



$$I = (x_0x_1, x_0x_2, x_1x_2, x_1x_3, x_2x_3)$$



$$J = (z_0 z_1 z_2, z_0 z_1 z_3 z_4^2, z_2 z_3 z_4^2, z_2 z_5^3, z_3 z_4^2 z_5^3)$$

We would like to analyze the resolution and some other related properties of the generalized edge ideal.

Let G be a graph with $n + 1$ vertices and $S = k[x_0, \dots, x_n]$. Let I_G be its ordinary edge ideal and J_G be its generalized edge ideal with monomials $m_0, \dots, m_n \in k[z_0, \dots, z_t]$. As the m_i consist of disjoint sets of variables, the lcm-lattices L_I and L_J are isomorphic via the bijection $f(x_i) = m_i$. Therefore, by Theorem 1.6, the corresponding free modules in the free resolution of S/I_G and the free resolution of S/J_G have the same number of generators, and the generator of degree \mathbf{a} in the former corresponds to the generator of degree $f(\mathbf{a})$ in the latter. In other words, we have

Proposition 3.1. *With the previous notation, the \mathbb{Z}^{n+1} -graded Betti numbers satisfy*

$$\beta_{i,\mathbf{a}}(S/I_G) = \beta_{i,f(\mathbf{a})}(S/J_G).$$

Let $d_i = \deg m_i$ and $\mathbf{d} = (d_0, \dots, d_n)$, then $|f(\mathbf{a})| = \mathbf{a} \cdot \mathbf{d}$. Now suppose that, when we are dealing with S/I_G , instead of the ordinary \mathbb{Z} -grading of S where each variable x_i had degree 1, we worked with a \mathbb{Z} -grading of S where x_i had degree d_i . It follows that the degree of $\mathbf{x}^{\mathbf{a}}$ would be $\mathbf{a} \cdot \mathbf{d} = |f(\mathbf{a})|$ instead of $|\mathbf{a}|$. That is, the degree of every generator of every free module in the resolution of S/I_G , under the grading $\deg x_i = d_i$, is the same as the degree of every generator of every free module in the resolution of S/J_G , under the grading $\deg x_i = 1$. This means that the regularity and \mathbb{Z} -graded Betti numbers of the generalized edge ideal is equivalent to the regularity and \mathbb{Z} -graded Betti numbers of the ordinary edge ideal, under another \mathbb{Z} -grading. Therefore, in the rest of this paper, we will only focus on the ordinary edge ideal under other \mathbb{Z} -gradings.

Now suppose we start with a graph G with $V(G) = \{x_0, \dots, x_n\}$, then $S = k[x_0, \dots, x_n]$. If we add t_i extra variables to the vertex x_i , then x_i would have $1 + t_i$ variables. In light of the discussion in the previous paragraph, this is equivalent to the having a grading on S with $\deg x_i = 1 + t_i$. We call this “adding t_i degrees to the vertex x_i .” Let $N = t_0 + \dots + t_n$, then we also say we “added N degrees to the graph.” Now by Hochster’s formula (Theorem 1.2), all non-zero Betti numbers of S/I_G lie in square-free multidegrees. For a square-free multidegree \mathbf{a} , we obviously have $|\mathbf{a}| \leq |f(\mathbf{a})| \leq |\mathbf{a}| + N$. That is, after adding N degrees to the graph, the \mathbb{Z} -degree of every generator in the minimal free resolution of S/I_G increases by at most N . By the definition of regularity we have

Proposition 3.2. *Let I_G be the ordinary edge ideal of a graph G and J_G be its edge ideal after adding N degrees. Then*

$$\operatorname{reg}(I_G) \leq \operatorname{reg}(J_G) \leq \operatorname{reg}(I_G) + N.$$

4 The Regularity Offset

With Proposition 3.2, we have the upper and lower bounds of the regularity. When we were doing examples, we found the regularity tends to get closer to the upper bound rather than the lower bound. It is very intuitive, cause when we adding degrees, the regularity normally tends to grow. So, the regularity keeping the same after adding degrees is actually an abnormal phenomenon.

Definition 4.1 (Regularity offset). *Given a graph G , for a certain grading $\mathbf{g} \in \mathbb{N}^n$, where $\sum_{i=1}^n (\deg x_i - 1) = N$, let I_G be the ordinary edge ideal and J_G be the edge ideal with grading \mathbf{g} , we define the regularity offset of G under grading \mathbf{g} as $ro(G, \mathbf{g}) = \operatorname{reg}(I_G) + N - \operatorname{reg}(J_G)$. And we say that a graph G has regularity offset if there exists a grading such that the regularity offset under that grading is non-zero.*

By the definitions of resolution and regularity and the conclusion before, the regularity offset of a certain edge ideal will exist, if and only if a vertex or some vertices doesn't occur in any subgraph that contributes to the regularity or don't occur in them simultaneously.

Lemma 4.2. *A graph G doesn't have regularity offset if and only if $ro(G, \mathbf{t}) = 0$ where \mathbf{t} is a grading that is all 2.*

Proof. If a graph doesn't have regularity offset, then by definition, $ro(G, \mathbf{t}) = 0$.

Now, for the other direction, we can think $\operatorname{reg}(J_G)$ under grading \mathbf{t} as adding one degree to a vertex and doing this process to each vertex. And, by Proposition 3.2, if we add one degree, then the regularity will go up at most by 1. Since then, the statement $ro(G, \mathbf{t}) = 0$ becomes, for any permutation of vertices, if we add one degree to each vertex under that permutation, the regularity will always go up by 1 each time. Hence, it is not possible that a vertex or some vertices doesn't occur in any subgraph that contribute to the regularity or don't occur in them simultaneously, because if then, when we add degree to just that vertex or those vertices (which is possible, as we add degree under any permutation), the regularity offset will occur, and then $ro(G, \mathbf{t}) \neq 0$. So that, the graph G doesn't have regularity offset. \square

With this lemma, we can have a topological conclusion for the existence of the regularity offset.

Theorem 4.3. *A graph G doesn't have regularity offset if and only if the whole graph contributes to the regularity, which means $\tilde{H}_{\operatorname{reg}(G)-2}(\Delta G^c, k)$ is non-trivial.*

Proof. In Lemma 4.2, we have already shown the equivalence between the nonexistence of regularity offset and $ro(G, \mathbf{t}) = 0$. So, to prove this theorem, we just need to show that $ro(G, \mathbf{t}) = 0$ if and only if the whole graph contributes to the regularity.

One direction is immediate. If the whole graph contributes to the regularity, then the monomial corresponding to the whole graph will occur in the resolution on a right position.

Hence, all the added degrees will be collected in this monomial, which means the graph G can not have any regularity offset, then, for sure, $ro(G, \mathbf{t}) = 0$.

For the other direction, we are going to prove its contrapositive. Suppose the whole graph doesn't contribute to the regularity, we denote the largest subgraphs that contribute to the regularity as $\{S_i\}$, and $|V(S_i)| = m < n = |V(G)|$ for all i . Under grading \mathbf{t} , the degree of monomials that contribute to the regularity will double. So the regularity goes up by m , because this is the largest degree of those monomials. Since the number of added degrees in grading \mathbf{t} is n , then $ro(G, \mathbf{t}) = n - m > 0$. In consequence, we prove that if $ro(G, \mathbf{t}) = 0$, then the whole graph contributes to the regularity. \square

After understanding the existence of regularity offset, we are naturally interested in where the regularity offset comes from, or in other words, which are the tricky vertices?

Definition 4.4. *Given a graph G , offset vertices are the vertices that either don't occur in any subgraph that contribute to the regularity or don't occur in those subgraphs simultaneously with some other offset vertices. We use $V_o(G)$ to denote all the offset vertices in the graph G .*

In the following paper, we are going to discuss the regularity offset numerically and find the offset vertices for some types of graphs.

4.1 Co-Chordal graphs

Chordal graphs are graphs such that they have no induced cycles with length more than 3. We call a graph is co-chordal if its complement graph is chordal. In the study of edge ideals, if the graph is co-chordal then it will have a relatively simple resolution.

Theorem 4.5 ([2], Fröberg's Theorem). *The edge ideal of a graph G has a linear resolution if and only if the complement graph of G is chordal.*

By this theorem, we can have a useful corollary immediately.

Corollary 4.6. *If G^c is chordal, then $reg(I_G) = 2$.*

Proposition 4.7. *If G^c is chordal, then G doesn't have regularity offset if and only if G^c is not connected.*

Proof. Since $reg(I_G) = 2$, then by Theorem 4.3, G doesn't have regularity offset if and only if $\tilde{H}_0(\Delta^{G^c}, k)$ is non-trivial. By homology theory, $dim_k(\tilde{H}_0(\Delta^{G^c}, k)) = \# \text{ connected components} - 1$. Hence, G doesn't have regularity offset if and only if G^c is not connected. \square

Corollary 4.8. *Complete graph doesn't have regularity offset.*

Proof. Complete graph's complement is just a bunch of isolated vertices. Then, it is co-chordal and doesn't have regularity offset. \square

For now, we have a brief introduction on how the regularity will behave if it has some special properties besides co-chordality. Next, we are going to introduce a general theorem for all the co-chordal graphs.

Theorem 4.9. *Given a co-chordal graph G , for a grading \mathbf{g} , we use $t_i = \deg x_i - 1$ to denote the number of added degrees on each vertex. Then we have*

$$ro(G, \mathbf{g}) = \min_j \left\{ \sum_{i=1}^{k_j} t_{(j_i)} \right\} \quad \text{where } \{x_{j_1}, x_{j_2}, \dots, x_{j_{k_j}}\} \text{ forms a minimal cut set.}$$

Proof. By multigraded Hochster's formula (Theorem 1.2), a certain monomial would occur in the resolution if and only if the corresponding homology group is non-trivial. For the co-chordal case, we just need to analyze \tilde{H}_0 . If we translate this to graph theory language, we just need to find the disconnected induced subgraph or, in another direction, find the cut sets.

Since co-chordal graph has a linear resolution, every disconnected induced subgraph would contribute to the regularity. Therefore, all the degrees in the cut sets can not be collected in that monomial corresponding to this subgraph. So that

$$ro(G, \mathbf{g}) = \min_j \left\{ \sum_{i=1}^{k_j} t_{(j_i)} \right\} \quad \text{where } \{x_{j_1}, x_{j_2}, \dots, x_{j_{k_j}}\} \text{ forms a cut set.}$$

As all the variables' degree are strictly positive, we can simplify this formula a little bit.

$$ro(G, \mathbf{g}) = \min_j \left\{ \sum_{i=1}^{k_j} t_{(j_i)} \right\} \quad \text{where } \{x_{j_1}, x_{j_2}, \dots, x_{j_{k_j}}\} \text{ forms a minimal cut set.}$$

□

Corollary 4.10. *For a co-chordal graph G , $V_o(G)$ is the union of all the minimal cut sets.*

4.2 Paths and cycles

Paths and cycles are two other families of graphs that have relatively simple structure. We will discuss the behavior of regularity offset of them in this chapter.

First of all, we need to understand the regularity of edge ideal of paths and cycles.

Proposition 4.11 ([8], Proposition 10). $reg(I_{C_n}) = reg(I_{P_n}) = \lfloor (n+1)/3 \rfloor + 1$.

To start with a relatively easy one, we would like to discuss cycles first. By Theorem 4.3, to understand cycles' regularity offset, we just need to figure out the homotopy type of ΔC_n^c . Thanks to Prof. Kozlov, in the Proposition 5.2 of [4], he has shown that

$$\Delta C_n^c \simeq \begin{cases} S^{k-1} \vee S^{k-1}, & \text{if } n = 3k; \\ S^{k-1}, & \text{if } n = 3k + 1; \\ S^k, & \text{if } n = 3k + 2. \end{cases}$$

We can check that these homotopy types fit in the regularity exactly. Hence, we can give the conclusion for cycles.

Proposition 4.12. *All the cycles don't have the regularity offset.*

Now, let's move on to the paths. Similarly, we need to introduce a topological conclusion at the beginning.

Proposition 4.13 ([4], Proposition 4.6).

$$\Delta P_n^c \simeq \begin{cases} S^{k-1}, & \text{if } n = 3k; \\ a \text{ point}, & \text{if } n = 3k + 1; \\ S^k, & \text{if } n = 3k + 2. \end{cases}$$

Therefore, we have the following corollary immediately.

Corollary 4.14. P_n doesn't have regularity offset if and only if $n \equiv 0, 2 \pmod{3}$.

For the case that $n \equiv 1 \pmod{3}$, we have the following theorem.

Theorem 4.15. For P_n , where $n \equiv 1 \pmod{3}$, we label the vertices from one end to the other as x_1, x_2, \dots, x_n . Then, for a grading \mathbf{g} , we have $V_o(P_n) = \{x_1, x_4, \dots, x_{n-3}, x_n\}$ and $ro(P_n, \mathbf{g}) = \min\{t_1, t_4, \dots, t_{n-3}, t_n\}$, where $t_i = \deg x_i - 1$.

Proof. Let $n = 3k + 1$. By Proposition 4.13, we can see that the whole P_n can not contribute to the regularity. We then think about the subgraph with $n - 1$ vertices.

For the subgraph $P_n[V(P_n) \setminus \{x_1\}]$ and $P_n[V(P_n) \setminus \{x_n\}]$, they are paths with $n - 1$ vertices, then with Proposition 4.13, we can see that their Stanley-Reisner complexes are both homotopy equivalent to S^{k-1} . And this is exactly what we need for the regularity. Therefore, these two subgraphs contribute to the regularity.

For the subgraph $P_n[V(P_n) \setminus \{x_{3s+1}\}]$, where $0 < s < k$, all these subgraphs are the unions of two connected components, and one component is P_{3s} , the other is $P_{3(k-s)}$. By Corollary 20.6 in [3], the Stanley-Reisner complex of these subgraphs is $\Delta P_{3s}^c * \Delta P_{3(k-s)}^c$. Then, by Proposition 4.13, $\Delta P_{3s}^c * \Delta P_{3(k-s)}^c \simeq S^{s-1} * S^{k-s-1} \simeq S^{(s-1)+(k-s-1)+1} = S^{k-1}$. So, as before, these subgraphs contribute to the regularity.

For all the other subgraphs, they are the disjoint union of P_{3p+1} and P_{3q+2} , where $p+q = k - 1$. So, the Stanley-Reisner complex is $\Delta P_{3p+1}^c * \Delta P_{3q+2}^c \simeq \{\bullet\} * S^k$. As this is a cone, so it is homotopy equivalent to a point. Hence, they don't contribute to the regularity.

We now can conclude that only $P_n[V(P_n) \setminus \{x_{3t+1}\}]$, where $0 \leq t \leq k$, are the only $(n-1)$ -vertex subgraphs that contribute to the regularity. So that, $\{x_1, x_4, \dots, x_{n-3}, x_n\} \subset V_o(P_n)$ (to make it simple, we now call them the $1 \equiv \pmod{3}$ vertices).

To show the other direction, we need to show that any subgraph that includes all the $1 \equiv \pmod{3}$ vertices in the vertex set cannot contribute to the regularity. In the previous reasoning, the bad part that let us not delete vertex other than $1 \equiv \pmod{3}$ vertices is it will produce a P_{3p+1} , which is homotopy equivalent to a point. To kill that "point", we need to delete a vertex in P_{3p+1} . If we still don't delete the $1 \equiv \pmod{3}$ vertices, it actually drags us back to the original problem, producing a "point". Therefore, we can show that if we never delete the $1 \equiv \pmod{3}$ vertices, its Stanley-Reisner complex will always homotopy equivalent to a point. Hence, there cannot be a subgraph that includes all the $1 \equiv \pmod{3}$ vertices and still contributes to the regularity. So, $V_o(P_n) = \{x_1, x_4, \dots, x_{n-3}, x_n\}$.

As we have shown before, all the added degrees on $1 \equiv \pmod{3}$ vertices cannot be collected in the subgraphs that contribute to the regularity simultaneously. So, for a grading \mathbf{g} , $ro(P_n, \mathbf{g}) = \min\{t_1, t_4, \dots, t_{n-3}, t_n\}$, where $t_i = \deg x_i - 1$. \square

5 General View

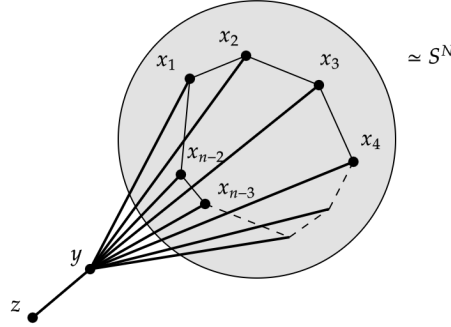
Now, we understand what the regularity offset would be like for some special types of graphs. We may be able to say that we understand the regularity offset to some sense. Hence, we can go back to deal with some general questions before.

5.1 Revisit Proposition 3.2

As we mentioned at the beginning of Section 3, the $\text{reg}(J_G)$ intuitively tends to be closer to $\text{reg}(I_G) + N$. One may think that the inequality we introduce in Proposition 3.2 is not good enough, as it seems to be possible to have a nicer lower bound. However, in this chapter, we are going to show that it is actually a sharp lower bound for any given N .

Proposition 5.1. *Given an integer $N \geq 0$, there is always a graph G such that $\text{reg}(J_G) = \text{reg}(I_G)$ after adding N degrees.*

Proof. To prove this, we are going to introduce a family of graphs and show that the regularity of this graph remains the same after adding a certain number of degrees.



The graph above is that special family of graphs, where $V(G) = \{x_1, x_2, x_3, \dots, x_{n-3}, x_{n-2}, y, z\}$ and $\Delta G^c[\{x_1, x_2, x_3, \dots, x_{n-3}, x_{n-2}\}] \simeq S^N$. We let y be adjacent to all the other vertices and z only be adjacent to y .

We claim that the $\text{reg}(I_G) = N + 2$. Since $\Delta G^c[\{x_1, x_2, x_3, \dots, x_{n-3}, x_{n-2}\}] \simeq S^N$, by Hochster's formula, $\text{reg}(I_G) \geq N + 2$. To show the other direction, we need to analyze the Stanley-Reisner complex of I_G . Let $M > N$. As y is adjacent to all the other vertices, it is an isolated vertex in the Stanley-Reisner complex. Hence, if there is a subcomplex of ΔG^c whose M -dimensional homology is non-trivial, it doesn't need to include y , because y can help nothing here. Since z is not adjacent to any x_i in G , then the subcomplex on any subset of $\{x_1, x_2, x_3, \dots, x_{n-3}, x_{n-2}\}$ together with z would be a cone, hence contractible. So, if there is a subcomplex of ΔG^c whose M -dimensional homology is non-trivial, it has to be a subcomplex of $\Delta G^c[\{x_1, x_2, x_3, \dots, x_{n-3}, x_{n-2}\}]$. However, this cannot be possible. Consequently, $\text{reg}(I_G) = N + 2$.

By the reasoning before, we can also see that, for any subcomplex, if z is in this subcomplex, then this subcomplex would always be contractible unless this subcomplex also involves y . Moreover, even we have both y and z simultaneously, we can only get a subcomplex whose 0-dimensional homology is non-trivial. So, to some sense, the vertex z is the least possible vertex that can contribute to the regularity. If we add N degrees only to vertex z , since adding degree is equivalent to adding dimension in homology in this case,

we will shift the homology dimension by adding N , therefore, any subcomplex involving y and z would be same as $\Delta G^c[\{x_1, x_2, x_3, \dots, x_{n-3}, x_{n-2}\}]$ homologically. So, no subcomplex breaks the homology red line N , then the regularity will remain the same. \square

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