- Topological Space: A set X together with topology on X: Collection S of subsets of X, closed under finite $\cap$, arbitrary $\cup$, containing $\emptyset, X$.

  open sets

- $Z \subset X$ closed: $X \setminus Z$ is open
- $f: X \to Y$ function (mapping) between topo. spaces: f is continuous if $f^{-1}(\text{open})$ is open.
- Continuous $f: X \to Y$ is a homeomorphism if f^{-1} exists and f^{-1} continuous.
- Subspace of X: subset Z of X with the induced topology

  $= \{Z \cap U\}$, U open in X

  $Z \hookrightarrow X$ continuous.

- Quotient space: $Y = X/\sim$ set of equivalence classes, where $\sim$ equivalent relation in X.

  projection map

  $p: X \to Y$: $V \subset Y$ open if $p^{-1}(V)$ open in X.

  then p continuous (this is the smallest topology of Y to make p cont.)

  p is a quotient mapping: A surjective mapping s.t. $V \subset Y$ open $\Leftrightarrow$ $p^{-1}(V) \subset X$ open.

  Q: Is every cont. surjective map a quotient mapping?

  A: No. $X = (-\infty, 0) \cup [0, \infty)$ (disjoint union), $Y = \mathbb{R}$, $p = \mathrm{id}$

  open in X, but not a $p^{-1}(\text{open})$

- Basis of topology of X: Collection $\mathcal{B}$ of open sets of X, s.t. every open sets is a union of sets in $\mathcal{B}$.
- Neighborhood of point x in X: subset of X, which includes open set containing x.

- <u>Neighborhood basis</u> of x: Collection of nbhds of x s.t. every neighborhood of x included in the collection. 假设 $\forall$ nbhd V, 可以找到 $B \subseteq V$. $\mathcal{B} \subseteq N(x)$, s.t. $\forall V \in N(x)$, $\exists B \in \mathcal{B}$, s.t. $B \subset V$.
- Property of X hold locally on X: if there is nbhd basis of x on which property holds s.t. every nbhd of x have that property
- <u>X compact</u> (as a space): if every open cover has a finite subcover.

  <u>Subset compact</u>: Compact as a subspace topology.
- The image of a compact set through a continuous map is compact.

  Therefore continuous $f: X \to \mathbb{R}$ on compact set X takes on max / min value.
- In $\mathbb{R}^n$, compact $\Leftrightarrow$ closed and bounded.

<u>Example:</u> Metric Space: (X, d)

$d = (X \times X) \to \mathbb{R}$, satisfying:

$$\begin{cases} d(x,y) \geq 0 \quad , \quad = 0 \Leftrightarrow x = y \\ d(x,y) = d(y,x) \\ d(x,y) \leq d(x,z) + d(z,y) \end{cases}$$

Metric space has a topology with basis given by open balls: $B(x,r) = \{ y \in X \mid d(x,y) < r \}$.

- Topo. space is metrilized if its underlying set has a metric which defines the topology.
- X is <u>disconnected</u> if it is a disjoint union of 2 non-empty open / closed sets.
- Subset of X is connected if connected in the subspace topology.
- <u>Connected components</u> of X: Maximal connected subsets of X.

  They form partition of X.

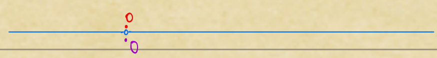
- Every connected component is closed (its closure is itself)
- Connected components not necessarily open

  e.g. $\mathbb{Q} \subset \mathbb{R}$ with subspace topology (Connected components: single points)

- Topological space is a manifold of dimension n if it locally looks like $\mathbb{R}^n$

  i.e., every point has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.

- <u>Example:</u> $\mathbb{R}$ with double 0.



  i.e., the open neighborhood of '0' consists that point and $U \setminus \{0\}$, where U open in $\mathbb{R}$.

  It's the same as $\mathbb{R} \sqcup \mathbb{R} / \sim$, where $x \sim y$ means $x = y$, $x \neq 0$ as point of $\mathbb{R}$

  disjoint union

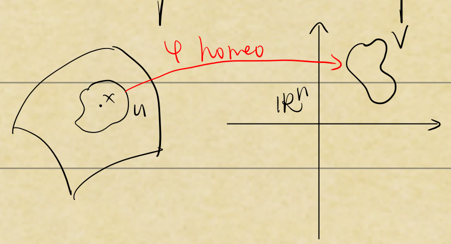
  粘在一起.

  How to avoid this? Hausdorff

- A topological space is <u>hausdorff</u> if any 2 distinct points lies in distinct open sets.

("not too big")
Second countable (定义在下面)

- <u>Def:</u> <u>n-dim topological manifold M</u> := Hausdorff topology spaces s.t. every pt $x \in M$ has an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^n$.



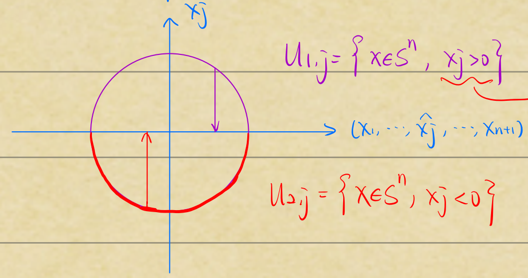
• Every manifold is locally compact and locally connected.

In particular, any connect component of a manifold is open

(Because $\mathbb{R}^n$ is)

Examples: ① S^n unit square in $\mathbb{R}^{n+1}$ $\left(\sum_1^{n+1} x_i^2 = 1\right)$

use hemisphere as coordinate charts:



$U_{1,j} = \{x \in S^n,\ x_j > 0\}$ 不能取 ≥，因为要不然不 open

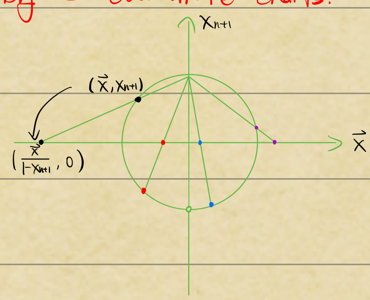
$U_{2,j} = \{x \in S^n,\ x_j < 0\}$

We just use projection to be the homeomorphism map.

We cover S^n by $2(n+1)$ charts.

• We can actually cover S^n by 2 coordinate charts.

(using sterographic projection).



(first countable: every point has countable neighborhood basis)

Topo Space <u>second countable</u> if it has countable basis.

not important. → If we don't demand second countable in def, TFAE:

(1) Every component of M is σ-compact (a countable union of compact set)

(2) Every component is 2nd countable.

(3) M metrizable.

(4) M paracompact.

Example:

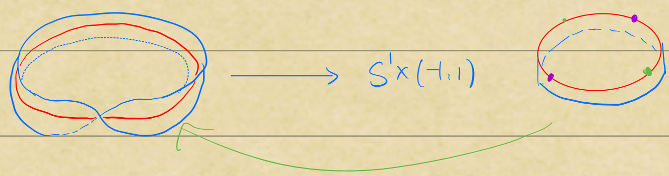
① An open subset of a manifold is a manifold.

② If M_1, M_2 are topological manifold, so is $M_1 \times M_2$ dim = $\dim M_1 + \dim M_2$.

if $(x_1, x_2) \in M_1 \times M_2$, U_1, U_2 coordinate charts, then $U_1 \times U_2$ is the coordinate chart.

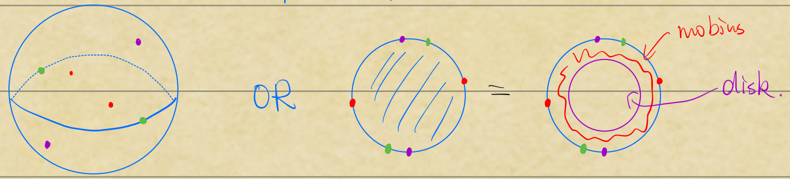
- torus $= S^1 \times S^1$ is a manifold of dim 2, submanifold of $\mathbb{R}^4 (= \mathbb{R}^2 \times \mathbb{R}^2)$, $S^1 \subset \mathbb{R}^2$. In fact, submanifold of $\mathbb{R}^3$

③ Möbius Strip

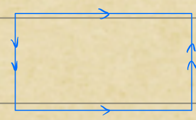


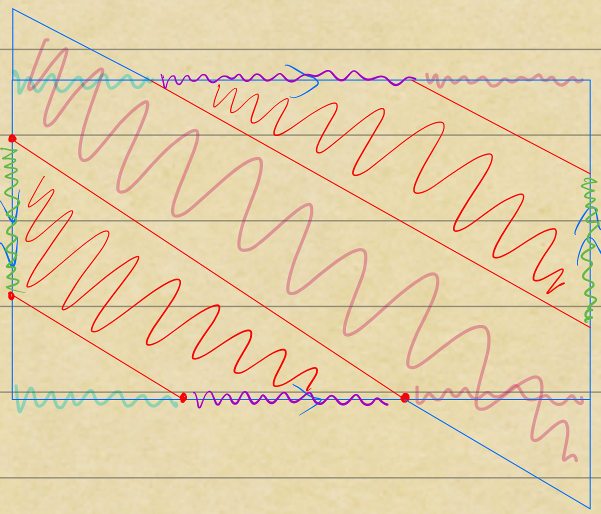
④ Real projective plane: $S^2/\sim$ $\quad \mathbb{RP}^2$

"$\sim$": identify antipodal points.



⑤ Klein bottle:





DIFFERENTIAL MANIFOLD.

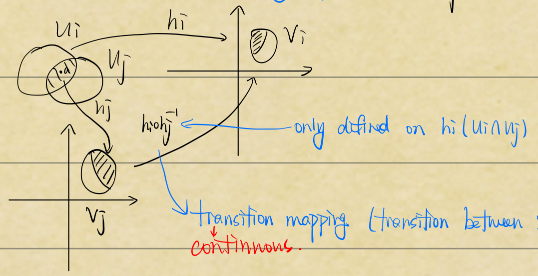
n. dim topological manifold : A second countable hausdorff space M, s.t.

$\forall a$, $\exists U \ni a$, $h(U) = V \in \mathbb{R}^n$, h homeomorphism.

open

coordinate chart

在U变化时相对应的 coordinate chart (h) 可能变化.



n. dim manifold with a smooth atlas:

C^∞ (differentiable)

一个 atlas 的意思是一组 $\{\varphi_i, U_i\}$,
其中 $\varphi_i \circ \varphi_j^{-1}$ 是 smooth 的

A 2nd countable Hausdorff space M together with a $\boxed{\text{covering}}$ M by n dim $\boxed{\text{coordinate chart}}$, which are $\boxed{C^\infty \text{ related}}$. → smooth atlas.

$M \subseteq \cup U_i$

previous h_i to $\mathbb{R}^n$

the transition mappings are C^∞

↳ 因为还没法对 h_i 求导, 所以讨论 transition mappings 的光滑性.

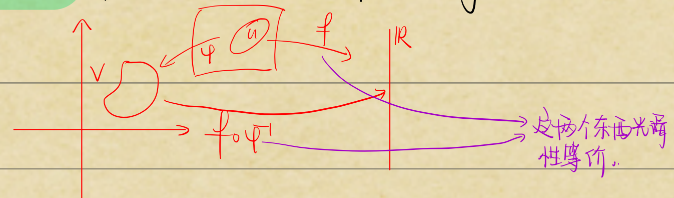
懒得改了!

$h = \varphi$

我们讨论 $M \to \mathbb{R}$ 的 function 的光滑性的时候要用 φ^{-1} 变成 $\mathbb{R}^n \to \mathbb{R}$ 的 function

M dim manifold with a smooth atlas: 然后再讨论.

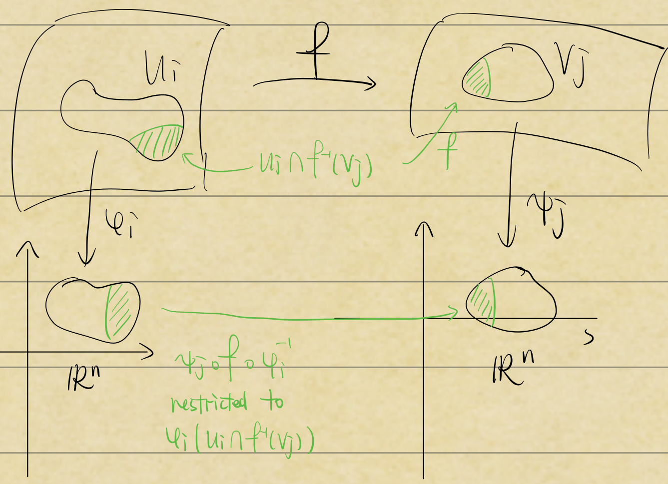
Function $f : M \to \mathbb{R}$ is smooth if $f \circ \varphi^{-1}$ is C^∞ for every coordinate chart $\varphi : U \to \mathbb{R}^n$.



mapping : $f : M^n \to N^p$ between manifolds with smooth atlases is smooth if $\psi_j \circ f \circ \varphi_i^{-1}$ is C^∞ $\forall i, j$.

n dim

和上面类似, 讨论 manifold 之间的 function 光滑性时我们移到 $\mathbb{R}^n$ 上讨论.



Diffeomorphism: Smooth mapping with a smooth inverse.

↳ 我们之前定义了 manifold 之间 function 的 smoothness, 于是我们可以讨论流形间的 diffeomorphism.

- Every smooth atlas is contained in a unique atlas.

- Add all coordinate charts that are C^∞ related in all previous ones.

为什么加进 U_1, U_2 后, U_1, U_2 C^∞ related? Chain rule.

- **Def**: An n-dim smooth manifold is an n-dim manifold with a maximal smooth atlas. 意思就是说有一个巨大无比的 atlas 盖住了所有的 manifold.

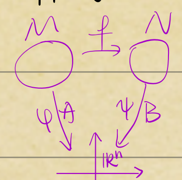
- **Lemma**: $(M, \mathcal{A})$, $(N, \mathcal{B})$ smooth manifold with maximal atlases,

  $f: M \to N$ cont. bijection mapping, TFAE:

  (1) f a diffeomorphism

  (2) $\psi \circ f \in \mathcal{A} \Leftrightarrow \psi \in \mathcal{B}$

  (3) function g on $(N, \mathcal{B})$ smooth iff $g \circ f$ is smooth on $(M, \mathcal{A})$



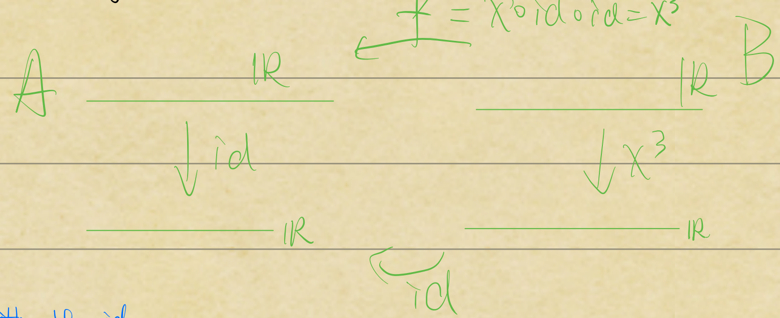
(1) ⇒ (2): f diffeo ⇔ $\psi \circ f \circ \varphi^{-1}$ $\mathbb{R}^2$ diffeo, 想证 $\psi: N \to \mathbb{R}^n$ homeo ⇔ $\psi \circ f$ homeo. compose f 即可. homeo 的 composition 仍然 homeo.

(2) ⇒ (3): compose 即可

(3) ⇒ (1): 懒了……

反正这个的意思大概就是说如果俩 manifold diffeomorphic, 我们可以随便用一个 diffeomorphism f 在它们俩之间切换.

$f = x^3 \circ id \circ id = x^3$



- Example: $M = \mathbb{R}$:

  $\mathcal{A}$ = All open sets U of $\mathbb{R}$ with $\varphi = id$

  $\mathcal{B}$ = $\varphi = \begin{matrix} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 \end{matrix}$ together with all charts that are C^∞ related to φ?

  Q: Are $(\mathbb{R}, \mathcal{A})$, $(\mathbb{R}, \mathcal{B})$ diffeomorphic? diffeomorphism between them.

  A: Yes, $f = \begin{matrix} \mathbb{R} \to \mathbb{R} \\ x \mapsto x^3 \end{matrix}$ defines diffeomorphism from $(\mathbb{R}, \mathcal{B})$ to $(\mathbb{R}, \mathcal{A})$

Q: Are there differentiable manifolds, which are homeo but not diffeo

A: No, up to dim 3. First example: Milnor exotic differentiable structures on S^7

In dim 4, there are uncountably many open subsets of $\mathbb{R}^4$, pairwise not diffeo, but all homeo to $\mathbb{R}^4$ (Donaldson, Friedman)

Say 2 smooth atlases on a topological manifold M are equivalent if the identity map of M is a diffeomorphism

manifold with a maximal atlas

Lemma: n-dim smooth manifold is a manifold with an equivalence class of smooth atlases

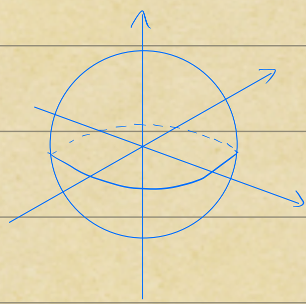
Point of this: 2 smooth atlases are equivalent iff they lie in same maximal atlas.

Q: Can a topological space have atlas of different dimensions?

A: No. Open sets in $\mathbb{R}^n$, $\mathbb{R}^p$ can't be diffeomorphic if $n \neq p$ (by Inverse Function Thm).

C^0 case: No, by invarience of domain

Example: $S^n \subset \mathbb{R}^{n+1}$: $\sum_{i=1}^{N} x_i^2 = 1$



$x_j = \pm\sqrt{1 - \sum_{j \neq i} x_i^2}$

C^∞ atlas:

$U_i^\pm = \{x \in S : x_i > 0 / < 0\}$

$\downarrow \varphi_i^\pm \qquad X = (x_1, \dots, x_{n+1})$

$\mathbb{R}^n \qquad \downarrow$

$\qquad (x_1, \dots, \hat{x}_i, \dots, x_{n+1})$

$\varphi_i^\pm : U_i^\pm \xrightarrow{\text{homeo}} \text{open unit disk}$

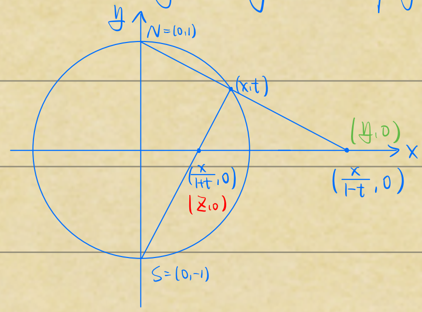
$V_i^\pm := \varphi_i^\pm(U_i^\pm)$

transition mapping from $\varphi_i^\pm$ to $\varphi_j^\pm$, $j \neq i$

looks like $(x_1, \cdots, \hat{x}_j, \cdots, x_{n+1}) \longmapsto (y_1, \cdots, \hat{y}_j, \cdots, y_{n+1})$

where $x_k = y_k$ for all $k \neq i$, and $y_i = \pm\sqrt{1 - \sum_{k \neq i} x_i^2}$

- C^∞ atlas given by stereo proj. S_n: $x_1^2 + \cdots + x_n^2 + t^2 = 1$ (notation)



transition mapping: $z = \frac{1-t}{1+t} y$ C^∞ on the overlap $\mathbb{R}^n \setminus \{0\}$.

Show these 2 atlas equivalent.

- Example of manifold structures:

(1) Open subset of a smooth manifold M has the structure of manifold.

if $\{\varphi_i : U_i \to \mathbb{R}^n\}$ is an atlas, then $\{\varphi_i|_{U \cap U_i} : U \cap U_i \to \mathbb{R}^n\}$ is also an atlas.

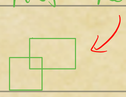
(2) $M_1 \sqcup M_2$ disjoint union of manifolds M_1, M_2.

If $\mathcal{A}_1, \mathcal{A}_2$ atlas for M_1, M_2, then $\mathcal{A}_1 \cup \mathcal{A}_2$ is an atlas for $M_1 \sqcup M_2$.

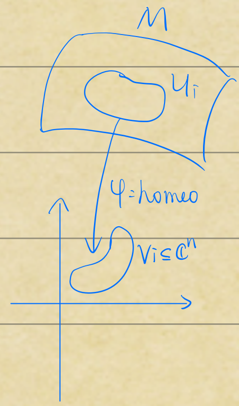
(3) Product $M^n \times N^p$ of manifolds of dim n, p, is a manifold of dim $n+p$.

Let $\varphi : U \to \mathbb{R}^n$, $\psi : V \to \mathbb{R}^p$, $\varphi \times \psi : U \times V \longrightarrow \varphi(U) \times \psi(V) \subseteq \mathbb{R}^n \times \mathbb{R}^p \cong \mathbb{R}^{n+p}$.

not necessarily maximal even though both atlas are maximal.



(4) We can also define complex manifold: i.e. manifold with complex atlas



the transition mappings that are complex analytic.

(5): Real projective space $\mathbb{RP}^n$.

<u>1st Def:</u> $S^n/\sim$ antipodal points. i.e. $(x_0,\cdots,x_n) \sim (-x_0,\cdots,-x_n)$

unit sphere $\subset \mathbb{R}^{n+1}$

Standard atlas for S^n induces atlas for $\mathbb{RP}^n$.

equivalence class

$U_i = \{[(x_0,\cdots,x_n)] \in \mathbb{RP}^n,\ x_i \neq 0\}$

$\downarrow \qquad\qquad \downarrow$

$\mathbb{R}^n \quad (\operatorname{sign} x_i)(x_0,\cdots,\hat{x}_i,\cdots,x_n)$ 这个项为了让 equivalence class 里那两个元素投影到一起.

<u>2nd Def:</u> $\mathbb{RP}^n = \{\text{lines through } 0 \text{ in } \mathbb{R}^{n+1}\}$

$= \mathbb{R}^n \setminus \{0\} / \sim.$

where $(x_0,\cdots,x_n) \sim (x_0',\cdots,x_n')$ if $(x_0,\cdots,x_n) = \lambda(x_0',\cdots,x_n')$

we use $[x_0,\cdots,x_n]$ to denote equivalence class.

<u>Atlas:</u> $U_i = \{[x_0,\cdots,x_n] \in \mathbb{RP}^n : x_i \neq 0\}$

$\downarrow \varphi_i \qquad\qquad \downarrow$

$\mathbb{R}^n \quad \left(\frac{x_0}{x_i},\cdots,\widehat{\frac{x_i}{x_i}},\cdots,\frac{x_n}{x_i}\right)$ Image $= \mathbb{R}^n$.

inverse:

所有 equivalence class.

$(y_1,\cdots,y_n) \longmapsto [y_1,\cdots,y_i,1,y_{i+1},\cdots,y_n]$.

<u>Examples of manifold structures:</u>

$\mathbb{RP}^n$: (1) $S^n/\sim$ antipodal

"2 different atlas"

(2) $\mathbb{R}^{n+1}\setminus\{0\}/\sim$. same line.

These 2 smooth manifold structures are <u>equivalent</u>.

① id is a diffeo

② same max atlas.

Complex projective space: $\mathbb{CP}^n$

Space of complex lines through 0 in $\mathbb{C}^{n+1}$, $\mathbb{C}^{n+1}/\sim$ ← complex lines through 0. identify points on the same

$[x_0, \cdots, x_n]$ equivalence class of $(x_0, \cdots, x_n) \in \mathbb{C}^n \setminus \{0\}$.

Covering by coordinate chart: $U_i = \{[x_0, \cdots, x_n] : x_i \neq 0\}$

$$\begin{array}{ccc} U_i & & [x_0,\cdots,x_n] \\ \text{homeo} \downarrow \varphi_i & & \downarrow \\ \mathbb{C}^n & & \left(\frac{x_0}{x_i}, \cdots, \widehat{\frac{x_i}{x_i}}, \cdots, \frac{x_n}{x_i}\right) \end{array}$$

Complex manifold structure

Transition mappings complex analytic. or, even better, rational (for this example)

$\mathbb{CP}^n$ has complex dim n, real dim $2n$. (complex-algebraic manifold)

Remark: $\mathbb{RP}^{2n} \ncong \mathbb{CP}^n$ ← always have orientation. (Just introduction).

$\mathbb{RP}^2$ = Disk + Mobius ← not orientable.

E.g: $\mathbb{CP}^1 = \mathbb{C}^2 \setminus \{0\}/\sim$.

Coordinate chart: $U_0 = \{[x_0, x_1] : x_0 \neq 0\}$, $U_1 = \{[x_0, x_1] : x_1 \neq 0\}$

$$\begin{array}{cccc} \downarrow \varphi_0 & \downarrow & \downarrow \varphi_1 & \downarrow \\ \mathbb{C} & x_1/x_0 = z & \mathbb{C} & x_0/x_1 = w \end{array}$$

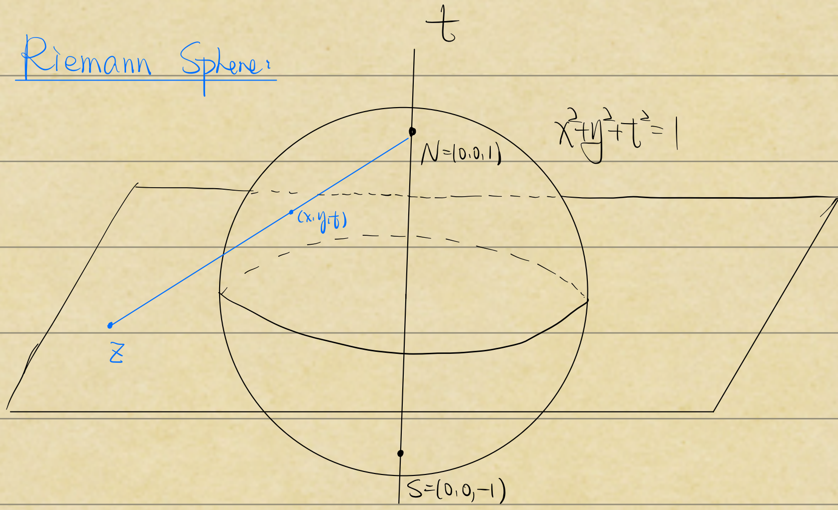
故有 $\{[1,0]\}$ 不在 U_0 里

$z = \frac{1}{w}$ ← coordinate around ∞

$\Rightarrow \mathbb{CP}^1 = \mathbb{C} \cup \{[1,0]\}$ Riemann Sphere!!!

!

Riemann Sphere:



Sterographic proj from N: $z = \frac{x+iy}{1-t}$

Sterographic proj from S: $\cancel{z = \frac{x+iy}{1+t}}$

$\downarrow$

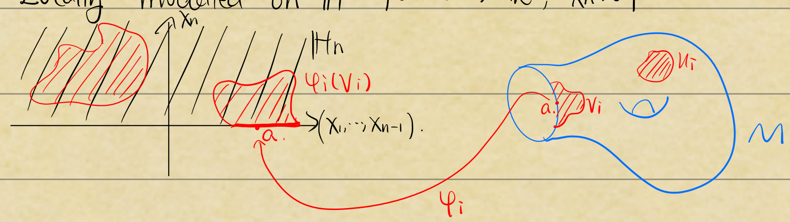
Complex conjugate of → $w = \frac{x-iy}{1+t}$

$zw = \frac{x^2+y^2}{1-t^2} = 1 \Rightarrow w = \frac{1}{z}$

Smooth manifolds: locally modelled $\mathbb{R}^n$.

Smooth manifolds with boundary:

Locally modelled on $\mathbb{H}^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n,\ x_n \geq 0\}$.



i.e. second countable Haus space covered by coordinate charts

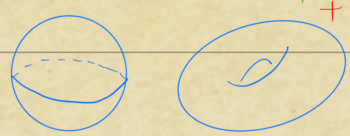
$U_i \xrightarrow[\text{homeo}]{\varphi_i}$ onto open subset of $\mathbb{H}^n$, with C^∞ transition mappings.

Q: What is C^∞ function on $\mathbb{H}^n$?

A: ① Restriction of C^∞ fn on $\mathbb{R}^n$

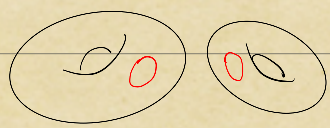
② C^∞ fn on $\{x_n > 0\}$ s.t. all partial derivatives extends continuously to bdy.

Classification of compact orientable 2-manifolds.



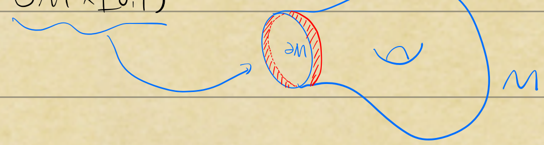
+ n-hole torus
$\Downarrow$
connected sum of n torus

Connected sum: (of 2 manifolds)



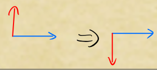
cut a hole and glue them together.

In general, ∂M of M smooth has open neighborhood U diffeo to $\partial M \times [0,1)$



Orientation:

Smooth manifold M is orientable if it has smooth atlas s.t. all transition mappings are "orientation preserving", i.e, Jacobian Determinants >0. 不会有类似 $\begin{cases} x \mapsto x \\ y \mapsto -y \end{cases}$ 的映射

 不可能.

Smooth functions and smooth mappings:

A smooth manifolds has lots of smooth functions.

Bump function lemma:

M smooth manifold,

$\underset{\text{compact}}{C} \subset \underset{\text{open}}{U} \subset M$.

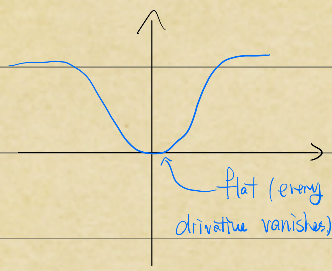
There exist C^∞ function $f: M \to [0,1]$, s.t.

$f = 1$ on C, $f = 0$ on $M \setminus U$.

Later: (1) also true C closed.

(2) Any close set $C \subset M$ is the zero set of a C^∞ fn.

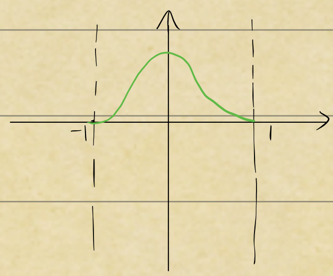
Pf: (1)



$h: \mathbb{R} \to \mathbb{R}$ C^∞

$$h(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

(2)



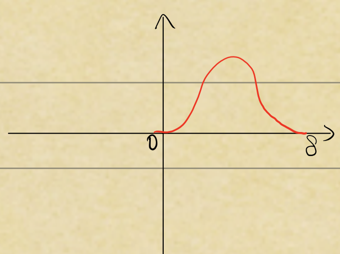
$$j(x) = \begin{cases} e^{-\frac{1}{(x+1)^2}} e^{-\frac{1}{(x-1)^2}} & x \in (-1,1) \\ 0 & \text{otherwise.} \end{cases}$$

$h(x+1) \cdot h(x-1)$

在 $(-1,1)$ 上, >0

在 $\{-1,1\}$ 上 $=0$

(3)

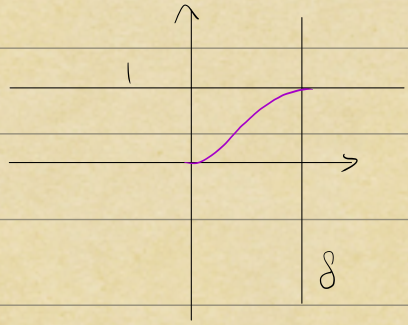


$k(x) > 0$ on $(0, \delta)$

$= 0$ otherwhere.

Deform $j(x)$

(4)

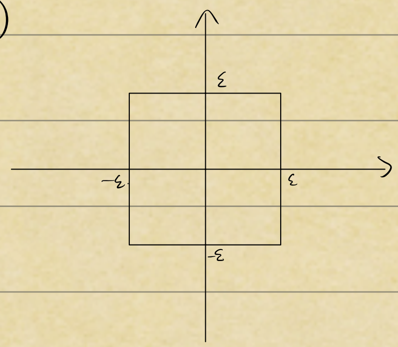


$l(x) = \int_0^x k \Big/ \int_0^\delta k$

对 $j(x)$ 积分后 normalize.

以上所有 function 都是 $\mathbb{R}\to\mathbb{R}$ 的.

(5)

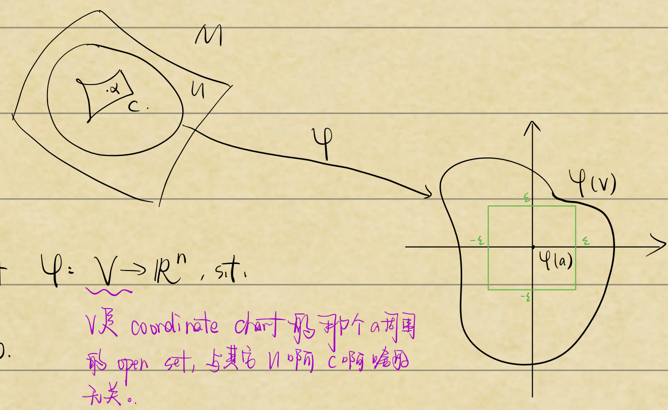


$g: \mathbb{R}^n \to \mathbb{R}$ C^∞, $\begin{cases} >0 & \text{on } (-\varepsilon,\varepsilon)^n \\ =0 & \text{otherwise.} \end{cases}$

$g(x_1,\cdots,x_n) = j\left(\frac{x_1}{\varepsilon}\right)\cdots j\left(\frac{x_n}{\varepsilon}\right)$ 只要有一个 $x_k=\pm\varepsilon$, 则 $g(x)=j(\frac{x_k}{\varepsilon})=0$

所以在边界上是0.

在里面每个 j 都 >0, 乘起来 >0.



Proof of Lemma:

Given $a \in C$,

choose coordinate chart $\varphi: V \to \mathbb{R}^n$, s.t.

V是 coordinate chart 的那个a周围的 open set, 与其它 U啊 C啊啥的无关。

$a\in V$, $\overline{V}\subset U$, $\varphi(a)=0$.

Choose $\varepsilon>0$, s.t. $(-\varepsilon,\varepsilon)^n \subset \varphi(V)$, apply g. 这个 f_a 是依赖于选点 a 的。

因为边界是0

$\Rightarrow g\circ\varphi$ extends to C^∞ fn f_a on M, which is

preimage. 也是我们要用的 open cover.

>0 in some open nbhd W_a of a where closure lies in U. compact.

There are finitely many points $a_1,\cdots,a_k$ s.t. $\{W_{a_k}\}$ covers C

Consider $f_{a_1}+f_{a_2}+\cdots+f_{a_k}$, 在C上>0, 在U外<0

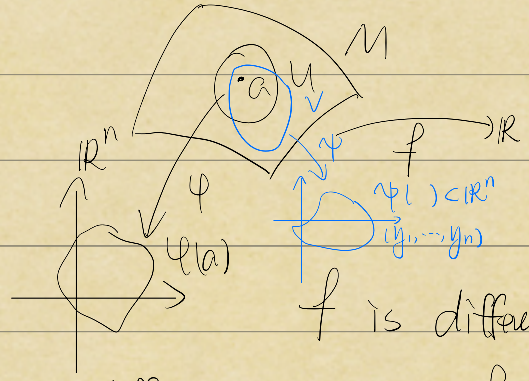
this lies in U, >0 on C, so $\geq\delta$ on C for some $\delta>0$.

take $f = l\circ(f_{a_1}+f_{a_2}+\cdots+f_{a_k})$. 要除以δ吧?

l 的图如是, <0时是0, >1时是1. ∎

$(f_{a_1}+f_{a_2}+\cdots+f_{a_k})/\delta$ smooth, 是个很基本的函数, C上1 U外0, 所以 apply l 后正好..

Differentation in local coordinates:



f is differentiable at a if $f \circ \varphi^{-1}$ is differentiable at $\varphi(a)$ for one (因为 transition map smooth. $\Leftrightarrow$ every) coor. chart @ a.

We consider coordinates x_i of $\mathbb{R}^n$ as function on U. $x_i \circ \varphi = \varphi_i$

We define $\dfrac{\partial f}{\partial x_i}(a) := \dfrac{\partial (f\circ\varphi^{-1})}{\partial x_i}(\varphi(a))$

Q: ψ?

$$\frac{\partial f}{\partial y_i}(a) = \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a)\cdot\frac{\partial x_j}{\partial y_i}(a)$$

just defined

functions on M

$$= \frac{\partial (f\circ\psi^{-1})}{\partial y_i}(\psi(a)) \qquad = \frac{\partial (f\circ\varphi^{-1})}{\partial x_j}(\varphi(a))$$

$\mathbb{R}^n \to \mathbb{R}^n$ $\quad \mathbb{R}^n \to \mathbb{R}$ chain Rule!

$$= \frac{\partial (f\circ\varphi^{-1} \circ \varphi\circ\psi^{-1})}{\partial y_i}(\psi(a))$$

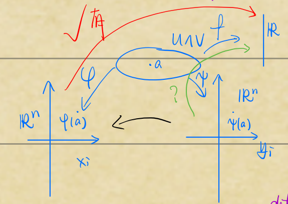
$x_j\circ\psi^{-1}$ $\quad \Rightarrow \dfrac{\partial (x_j\circ\psi^{-1})}{\partial y_i}(\psi(a)) = \dfrac{\partial x_j}{\partial y_i}(a)$ def!

$$= \sum_{j=1}^n \left(\frac{\partial (f\circ\varphi^{-1})}{\partial x_j}(\varphi(a))\right)\times\left(\boxed{\frac{\partial (\varphi_j\circ\psi^{-1})}{\partial y_i}}(\psi(a))\right)$$

$$= \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a)\cdot\frac{\partial x_j}{\partial y_i}(a)$$

我们在这里定义了f在(某个特定) coor. chart 下的偏导, 方式是用相对应的 transition map pull back 回 coor. chart 后对第i个方向求导.

我们现在需要解决的问题是, 如何在两个相互重叠的 coor. chart 间定义 chain rule.



绿色=红色∘黑色

好处在于, 红色我们有, 红色 + 黑色我们可以用$\mathbb{R}^n$里的 chain rule.

def, 是我们要求的东西.

$$\Rightarrow \frac{\partial f}{\partial y_i}(a) = \frac{\partial (f\circ\psi^{-1})}{\partial y_i}(\psi(a))$$

$$= \frac{\partial (f\circ\varphi^{-1}\circ\varphi\circ\psi^{-1})}{\partial y_i}(\psi(a))$$

$$= \sum_{j=1}^n \frac{\partial (f\circ\varphi^{-1})}{\partial x_j}(\varphi(a))\cdot\frac{\partial (\varphi\circ\psi^{-1})_j}{\partial y_i}(\psi(a))$$

def $\quad$ def.

$$= \sum_{j=1}^n \frac{\partial f}{\partial x_j}(a)\cdot\frac{\partial x_j}{\partial y_i}(a)$$

这里是 $(\varphi\circ\psi^{-1})(\psi(a))$的 jth coord

$= \varphi(a)$的 jth coord.

$= (x_j\circ\varphi)\, a$

$=$ "x_j" (a) 这里的 x_j 是 $M\to\mathbb{R}$ 的 function

$$\Rightarrow \frac{\partial}{\partial y_i} = \sum_{j=1}^n \frac{\partial x_j}{\partial y_i}\frac{\partial}{\partial x_j}$$

$\left.\dfrac{\partial}{\partial y_i}\right|_a$ is a derivation ∂ at a.

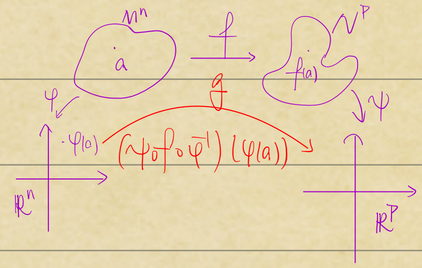
i.e. $\partial : C^\infty(M) \to \mathbb{R}$ linear

s.t. $\partial(fg)(a) = f(a)\partial g + g(a)\partial f$

$\left(\dfrac{\partial x_i}{\partial y_j}(a)\right)$ is the Jacobian matrix of $\varphi\circ\psi^{-1}$ at $\psi(a)$

$\left(\dfrac{\partial x_i}{\partial y_j}(a)\right)^{-1} = \left(\dfrac{\partial y_i}{\partial x_j}(a)\right)$

Mapping between manifolds:



Given $f: M^n \to N^p$ smooth, $a \in M$:

$\varphi: U \to \mathbb{R}^n_{(x_1,\dots,x_n)}$ coordinate system for M at a.

$\psi: V \to \mathbb{R}^p_{(y_1,\dots,y_p)}$ coordinate system for N at $f(a)$

$\left(\frac{\partial(y_j\circ f)}{\partial x_i}(a)\right)$ is the Jacobian matrix of $\psi\circ f\circ\varphi^{-1}$ at $\varphi(a)$. 我们把 Df 的 rank 定义为 Dg 的 rank.

$p\times n$ matrix.

rank of this matrix doesn't depend on the choice of coordinate. We call it **rank of f** at a

We say a is a **critical point** if rank of f at $a < p$.

regular point otherwise.

We say $b \in N$ is a **critical value** of f if $b = f(a)$ for some critical point a. / **regular value** otherwise.

Example: $f: \mathbb{R}^n \to \mathbb{R}$ C^1, all points critical

$\Rightarrow f$ constant.

Set of critical value should be small measure-0.

$A \subset \mathbb{R}^n$ measure 0 if every $\varepsilon > 0$, A can be covered by countably many open (or closed) rectangles of total value $< \varepsilon$.

Lemma: $A \subset \mathbb{R}^n$ rectangle, $f: A \to \mathbb{R}^n$ C^1, $f = (f_1, \dots, f_n)$

Assume $\left|\frac{\partial f_i}{\partial y_j}\right| \le M$ on A, then $\forall x, y \in A$,

$|f(x) - f(y)| \le n^2 M |x-y|$

Lemma: If $f: \mathbb{R}^n \to \mathbb{R}^n$ is C^1, and $A \subset \mathbb{R}^n$ has measure 0

$\Rightarrow f(A)$ has measure 0.

Sard's Theorem and the Rank Theorem

Lemma: If $f: \mathbb{R}^n \to \mathbb{R}^n$ C^1, if $A \subset \mathbb{R}^n$ measure 0,

$\Rightarrow f(A)$ measure 0.

M smooth manifold. $A \subset M$ has measure 0 can be covered by countably many coor. chart $\varphi_i : U_i \to \mathbb{R}^n$, s.t. $\varphi_i(A \cap U_i)$ has measure 0 for all i.

$\Downarrow$ $\varphi(A \cap U)$ = union of $\varphi_i(A \cap U_i)$

$\Leftrightarrow$

$\Uparrow$ since M is secondly countable.

if, for every coor. chart $\varphi: U \to \mathbb{R}^n$, $\varphi(A \cap U)$ has measure 0

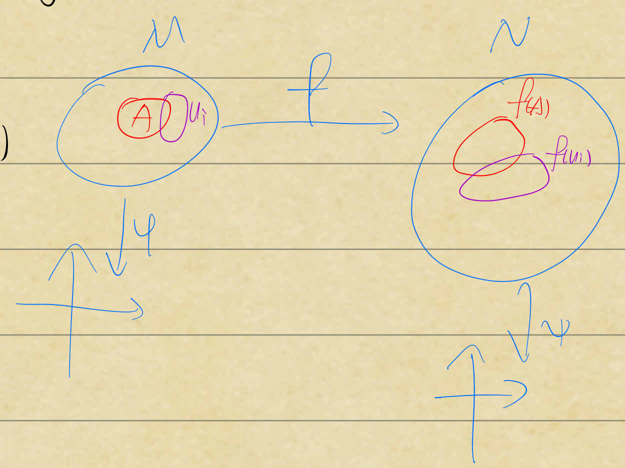
Cor: If $f: M \to N$ C^1 map of manifolds of same dim, and $A \cap M$ has measure 0, then $f(A)$ has measure 0.

Pf: WTS: $\psi(f(A) \cap V)$ has measure 0 for every coordinate chart for N.

Let $\varphi_i : U_i \to \mathbb{R}^n$ be countable cover of M by coord. charts.

$\Rightarrow f(A) \cap V = \bigcup_i \left(f(A \cap U_i) \cap V\right)$

$\psi(f(A \cap U_i) \cap V) = \left(\psi \circ f \circ \varphi_i^{-1}\big|_{(f \circ \varphi_i^{-1})^{-1}(V)}\right)\left(\varphi_i(A \cap U_i)\right)$



SARD'S THEOREM:

If $f: M \to N$ C^1 map between manifolds with same dim n,

Then set of critical values has measure 0.

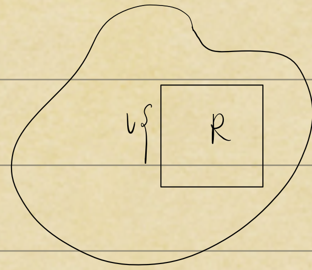
(Stronger version: if $f: M^n \to N^p$ C^r mapping, then the set of critical values has measure 0 if $r \geq 1 + \max\{n-p, 0\}$)

Pf: Enough to consider $\overset{\mathbb{R}^n \text{ open}}{\overset{\cup}{U}} \to \mathbb{R}^n$ (because we can restrict to coord. chart).

Let $A = \{x \in U \mid \det f'(x) = 0\}$ critical points.

Enough to show $f(A \cap \overset{\text{closed cube in } U}{R})$ has measure 0.

Subdivide R into N^n subcube of side length $\frac{L}{N}$. denote S.



Given $\varepsilon > 0$, we can choose N large enough that, if $a \in S$, then $\forall y \in S$,

$|f(y) - f(x) - Df(x)(x-y)| < \varepsilon |x-y| \leq \varepsilon \sqrt{n} \frac{L}{N}$ 对角线长度

If $S \cap A \neq \emptyset$, take $x \in S \cap A$, then $\{Df(x)(y-x) : y \in S\} \subset V$. (not full ranked.)

V: (most) $(n-1)$ dim. linear subspace of $\mathbb{R}^n$.

$\Rightarrow \{f(y) \mid y \in S\}$ lies within $\overset{\text{距离}}{\varepsilon \sqrt{n} \frac{L}{N}}$ of hyperplane $f(x) + V$.

$\Rightarrow \exists$ constant M. s.t. $|f(x) - f(y)| \leq M|x-y|$ (Lemma last time)

So if $x \in S \cap A$, then $\{f(y) \mid y \in S\}$ lies in cylinder of height $\varepsilon \sqrt{n} \frac{L}{N}$, where base is ball in $f(x) + V$, of radius $M\sqrt{n}\frac{L}{N}$.

This cylinder has volume $\overset{\text{constant.}}{C}\varepsilon (\frac{L}{N})^n$

R is covered by N^n cubes. So $f(R \cap A)$ lies in the set of volume $N^n \cdot C\varepsilon(\frac{L}{N})^n = \overset{\text{arbitrarily small.}}{C\varepsilon L^n}$

RANK THEOREM: $f: M^n \to N^p$ C^∞

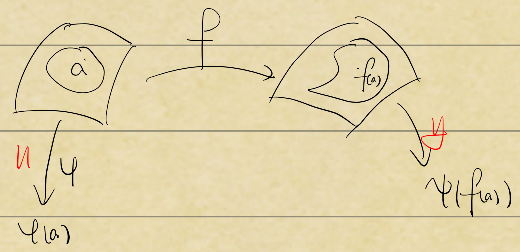
(1) If f has rank k at $a \in M$,

then there are coordinate system $(x_1, \cdots, x_n)$ at a, $(y_1, \cdots, y_p)$ at $f(a)$ in which f has the form $(y_1, \cdots, y_p) = (x_1, \cdots, x_k, f_{k+1}(x), \cdots, f_p(x))$

(2) If f has rank k (around a) (nbhd of a)

Then we can choose coord. system in which

$(y_1, \cdots, y_p) = (x_1, \cdots, x_k, 0, 0, \cdots, 0)$



Pf of 1) : Consider coord. system $(u_1, \cdots, u_n)$ @ a, $(y_1, \cdots, y_n)$ @ $f(a)$

After permuting coordinates, we may assume the first k block of $\left(\frac{\partial y_i}{\partial u_j}\right)$ has rank k. i.e. $\det\left(\frac{\partial y_i}{\partial u_j}\right)_{i,j\in\{1,\dots,k\}} \neq 0$

$$\text{Define } x_i = \begin{cases} \overset{(\psi_i \circ f \circ \varphi^{-1})}{y_i \circ f} \quad \underset{y_i(u)}{} & i = 1, 2, \cdots, k \\ u_i & i = k+1, \cdots, p. \end{cases}$$

Why coor. chart?

$$\text{Consider } \left(\frac{\partial x_i}{\partial u_j}(a)\right) = \begin{pmatrix} \frac{\partial y_i}{\partial u_j}(a) & * \\ 0 & \mathrm{Id} \end{pmatrix}$$

By the inverse function theorem, $x = x(u)$ is a coord. change at a.

"$y = x$"

and $(y_1, \cdots, y_p) = (x_1, \cdots, x_k, f_{p+1}(x), \cdots, f_p(x)) = f(x_1, \cdots, x_n)$

Pf of 2) : By (1), we can choose $\overset{a}{\underset{f(a)}{}}$ $\overset{x}{\underset{v}{}}$ at $\overset{a}{f(a)}$ in which f has form

$$(v_1, \cdots, v_p) = (x_1, \cdots, x_k, f_{p+1}(x), \cdots, f_p(x))$$

$$\left(\frac{\partial v_i}{\partial x_j}\right) = \begin{pmatrix} \mathrm{Id} & 0 \\ * & \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial f_{k+1}}{\partial x_{k+1}} & \cdots & \frac{\partial f_{k+1}}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_p}{\partial x_{k+1}} & \cdots & \frac{\partial f_p}{\partial x_n} \end{pmatrix} = 0 \quad \text{since rank} = p.$$

only depends on other variables.

$\Rightarrow$ So $\forall i = \{k+1, \cdots, p\}$, $f_i(x) = f_i(x_1, \cdots, x_k)$

$$\text{Define } y_i = \begin{cases} v_i & i = 1, \cdots, k \\ v_i - f_i(v_1, \cdots, v_k) & i = k+1, \cdots, p. \end{cases}$$

To show it's a coord. change:

$$\left(\frac{\partial y_i}{\partial v_i}\right) = \begin{pmatrix} Id & 0 \\ * & Id \end{pmatrix}$$

In these coordinates: f has form:

$$\begin{cases} y_i = v_i = x_i & i \in \{1, \cdots, k\} \\ y_i = v_i - f_i(v_1, \cdots, v_k) = 0 . & i \in \{k+1, \cdots p\} \end{cases}$$

$f_i(x_1, \cdots, x_k)$ $\qquad$ $f_i(x_1, \cdots, x_k)$

<u>Cor</u>: (1) <u>Submersion theorem</u>:

If $p \le n$ and f $M^n \to N^p$ has rank p at a then for any coord. system $(y_1, \cdots, y_p)$ at $f(a)$, we can find a coordi. system $(x_1, \cdots, x_n)$ at a in which $f: (y_1, \cdots, y_p) = (x_1, \cdots, x_p)$ (simply the statement of the rank theorem) (no permutation of the y coordinate needed).

(2) <u>Immersion thm</u>:

If $n \le p$ and f has rank n at a, then, for any coord. system $(x_1, \cdots, x_n)$ at a, we can choose the coord. system $(y_1, \cdots, y_p)$ at $f(a)$, in which $f: (y_1, \cdots, y_p) = (x_1, \cdots, x_n, 0, \cdots 0)$

<u>Pf</u>: Rank thm gives x, y in which

$$(y_1, \cdots, y_p) = (x_1, \cdots, x_n, 0, \cdots, 0)$$

But given a coord. system $(u_1, \cdots, u_n)$ for M at a, we may need coordinate change $x = X(u)$

<u>Claim</u>: We can write $(z_1, \cdots, z_p) = (u_1, \cdots, u_n, 0, \cdots, 0)$ after coord. change $z = z(y)$

We have: $(y_1, \cdots, y_p) = (x_1(u), x_2(u), \cdots, x_n(u), 0, \cdots, 0)$

where $X = X(u)$ is a coordinate change. (invertible)

Define $(z_1(y), \cdots, z_p(y))$ by:

$$(z_1, \cdots, z_n) = X^{-1}(y_1, \cdots, y_n)$$

$$(z_{n+1}, \cdots, z_p) = (y_{n+1}, \cdots, y_p)$$

This is a coordinate change because

$$\text{Jac. matrix} = \left(\begin{array}{c|c} X^{-1} & 0 \\ \hline 0 & Id \end{array}\right)$$

In these coord. f given by $(u, 0)$

Smooth mapping $f: M^n \to N^p$ is an <u>immersion</u> if rank $f = n$ at every point of M.

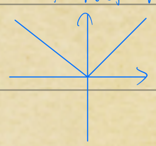
Immersion is locally 1-1, but not necessarily 1-1

and smooth 1-1 map is not necessarily an inversion.

All 1-1 but not inersion.

<u>Example:</u> 1) $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^3$ 1-1 but not rank=1 @ 0.

2) $h: \mathbb{R} \to \mathbb{R}^2$ with graph

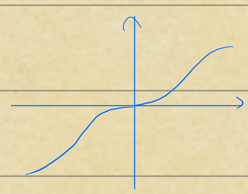


$y = |x|$

smooth, 1-1

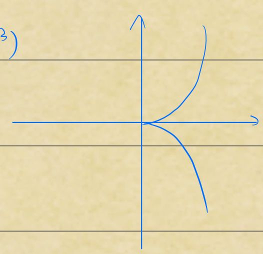
$h(t) = (f(t), g(t))$

$$f(t) = \begin{cases} e^{-\frac{1}{t^2}} & t>0 \\ 0 & t=0 \\ -e^{-\frac{1}{t^2}} & t<0 \end{cases}$$

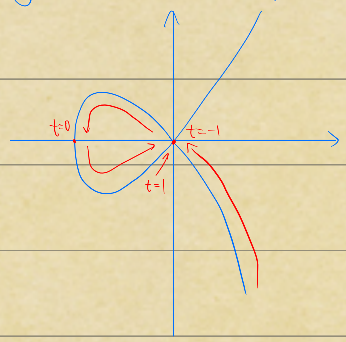


$$g(t) = \begin{cases} e^{-\frac{1}{t^2}} & t \neq 0 \\ 0 & t=0 \end{cases}$$

3) $(x, y) = (t^2, t^3)$



Immersions: 4) $y^2 = x^2(x+1)$ $f : \mathbb{R} \to \mathbb{R}^2$



$f|_{(-\infty,-1)}$  immersion but not homeomorphism.

Image of $\begin{cases} x = t^2-1 \\ y = t(t^2-1) \end{cases}$

∞ : image of $\theta \mapsto (\sin\theta, \sin 2\theta)$ immersion. $\iff$ $S^1 \to \mathbb{R}^2$ $(x,y) \overset{f}{\mapsto} (y, 2xy)$

$\Downarrow$

Also image of $f|_{S^1 \setminus \{(1,0)\}}$

 $\Rightarrow$ ∞

$P \subset S^1 \times S^1$

image of $\theta \mapsto (e^{i\theta}, e^{i\alpha\theta})$ irrational

immersion, but P is dense in $S^1 \times S^1$

Immersions and embeddings:

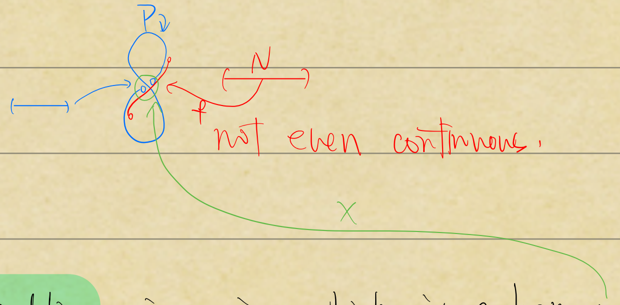
M smooth manifold.

$P \subset M$ an immersed submanifold if P is image of an injective immersion.

i.e. P can be given structure of smooth manifold s.t. inclusion $P \hookrightarrow M$ is an immersion.

Suppose $P \subset M$ an immersed manifold, $f : N \to M$ is smooth map s.t. $f(N) \subset P$. Is f smooth as map $N \to P$?

E.g:



Embedding: immersion which is a homeomorphism onto its image (with subspace topology)

(Smooth) Submanifold P of M means the image of an embedding.

OR, the immersed submanifold $P \subset M$ is a submanifold when the inclusion $P \hookrightarrow M$ is an embedding.

Prop: $f: M^n \to N^p$ with constant rank k in some neighborhood of a fiber $f^{-1}(b)$, $b \in N$, then

① $f^{-1}(b)$ is a closed submanifold of M of dim $n-k$.

(Exercise using rank theorem)

② In particular, if f is a submersion, then $f^{-1}(b)$ closed submanifold of dim $n-p$. (submersion thm)

PARTITION OF UNITY

Lemma 1: M smooth manifold (or top. manifold), $\mathcal{O}$ open cover of M.

Then there is an open cover $\mathcal{O}'$ which is locally finite and refines $\mathcal{O}$ i.e. M is paracompact.

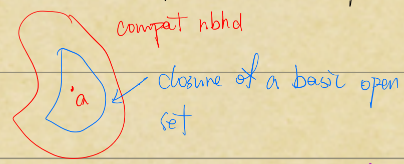
$\forall$ point, $\exists$ nbhd $\cap$ finite many $U \in \mathcal{O}'$.

(And every $U \in \mathcal{O}'$ is diffeo (homeo) to $\mathbb{R}^n$).

Proof: We can assume M connected

M is a countable union of compact sets (σ-compact). (Say, $C_1, C_2 \cdots$)

(Because each point has compact neighborhood)



+ M has countable basis.

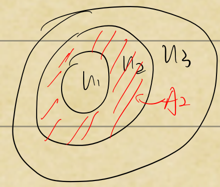
在O里找C_1的 open cover 即可

☆ C_1 has open neighborhood U_1 with compact closure in M.

$\overline{U_1} \cup C_2$ has open neighborhood U_2 with compact closure in M

$\overline{U_2} \cup C_3$ has open neighborhood U_3 with compact closure in M

M covered by open $\{U_i\}$ s.t. $\overline{U_i}$ compact $\overline{U_i} \subset U_{i+1}$



M union of compact annuli $A_i = \overline{U_i} \setminus U_{i-1}$

<u>Claim:</u> We can cover A_i by finite number of open sets V_i, each in some element of O, and each in $U_{i+1} \setminus \overline{U_{i-2}}$ 只接触周围的那两个

This gives us O'. Locally finite since U_i intersects only finitely many of the V_j.

Note: A locally finite open cover is countable.

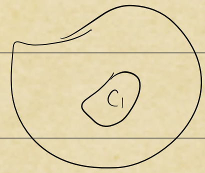
<u>Lemma 2:</u> Given a locally finite open cover O of M, then for every $U \in O$, there is open $U' \subset \overline{U'} \subset U$, s.t. the U' still form open cover.

<u>Proof:</u> We can assume M connected.

$O = \{U_1, U_2, \cdots\}$

Let $C_1 = U_1 \setminus U_2 \cup U_3 \cup \cdots$ is closed and in U_1

So there exist U_1', s.t. $C_1 \subset U_1' \subset \overline{U_1'} \subset U_1$



U_1 exercise

Let $C_2 = U_2 \setminus U_1' \cup U_3 \cup \cdots$

Do the same thing, we can find U_2'.

$C_3 = U_3 \setminus U_1' \cup U_2' \cup U_4 \cup \cdots$

$\mathcal{O}' = \{U_1', U_2', \cdots\}$ is obviously locally finite.

Cover M because:

Let $a \in M$. $\exists$ biggest k, s.t. $a \in U_k$ (locally finiteness)

So $a \in \underbrace{U_1' \cup \cdots \cup U_k' \cup \boxed{U_{k+1} \cup \cdots}}_{= M \text{ 因为 } k \text{ finite}}$ 与a无关.

$\Rightarrow \forall a,\ a \in \bigcup_{i=1}^{\infty} U_i'$

<u>Theorem</u>: (Partition of unity)

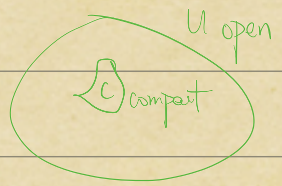
Given locally finite open cover of smooth manifold M, then $\forall U \in \mathcal{O}$,

there is C^∞ function $\psi_U: M \to [0,1]$

s.t. ① $\operatorname{Supp} \psi_U \subset U$

② $\sum_{U \in \mathcal{O}} \psi_U(\ ) = 1 \quad \forall a$.

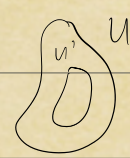
<u>Lemma</u>: (from before):



$\exists\ C^\infty\ \varphi$, s.t.
$\begin{cases} \varphi = 1 \text{ on } C \\ \operatorname{supp} \varphi \subset U. \end{cases}$

<u>Proof</u>: I. First assume every $U \in \mathcal{O}$ has compact closure.

Choose $U' \subset U$ as in Lemma 2



$\overline{U'} \subset U$ compact.

By Lemma, $\exists\ C^\infty$ fn

$\varphi_U: M \to [0,1]$, $\varphi_U = 1$ on $\overline{U'}$ compact, $\operatorname{supp} \varphi_U \subset U$.

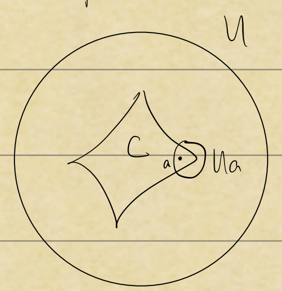
$\Sigma \psi_u > 0$ everywhere because $\mathcal{U}'$ covers M.

Define $\varphi_u = \frac{\psi_u}{\Sigma \psi_u}$

II : In general, same argument will work if we have

Lemma for C closed.

Now prove this



$\forall a \in C$, take open nbhd U_a

with compact closure in U.

Also cover $M \setminus C$ by open

sets V_β with compact closure.

$\Rightarrow \{U_a, V_\beta\}$ open cover of M

So it has locally finite refinement $\mathcal{O}$

Apply case I to $\mathcal{O}$: φ_u $u \in \mathcal{O}$ from case I.

Let $f = \sum_{w \in \mathcal{O}'} \varphi_w$, where $\mathcal{O}' = \{U_a\}$

C^∞ since Σ is finite.

1 on C (because $\Sigma \varphi_u = 1$ everywhere, but φ_u vanishes on

☆? C when $U \subset V_\beta$)

$\operatorname{supp} f \subset U$

Cor: Let $\mathcal{O}$ be open cover of M, there is collection of

C^∞ fns, $\varphi_i : M \to [0,1]$ (countable)

Def of C^∞ partition of unity.

(1) Each point a of M has open nbhd on which only

finitely many φ_i nonzero.

(2) $\sum_i \varphi_i(a) = 1$ $\forall a \in M$.

(3) $\forall i$, $\exists U \in \mathcal{O}$, s.t. $\operatorname{supp} \varphi_i \subset U$. (Def of subordinate to $\mathcal{O}$)

Proof: Lemma 1 of the thm.

Whitney embedding theorem: (Weak version):

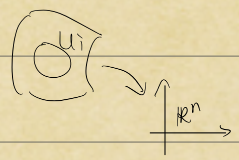
Every compact smooth manifold has an embedding in $\mathbb{R}^N$, for some N.

Later: we can take $N = 2n+1$, $n = \dim M$.

Best: $2n$.

Proof: We can cover M by finitely many coordinate charts:

$\eta_i : U_i \longrightarrow \mathbb{R}^n$, $i \in \{1, 2, \cdots k\}$



with coordinates $(x_{i1}, \cdots, x_{in})$

We can shrink $U_i \Rightarrow U_i'$, still forms a covering.

Choose $\varphi_i : M \longrightarrow [0,1]$ C^∞, s.t. $\varphi_i = 1$ on U_i', $\operatorname{supp} \varphi_i \subset \overline{U_i}$.

乘起来.

for 1-1 perpose.

Take $f: M \to \mathbb{R}^{kn+k}$, $f = (\underbrace{\varphi_1\eta_1, \cdots, \varphi_k\eta_k}_{nk}, \underbrace{\varphi_1, \cdots, \varphi_k}_{k})$

① f is an immersion: i.e. rank n at every point

$a \in U_i'$ for some i, on U_i', $\varphi_i = 1$. So

$f(a) = (\cdots, \overset{k\text{-dim}}{\eta_i}, \cdots)$

On U_i', Jac. matrix of f includes Jac. matrix of η_i

as submatrix $\left(\frac{\partial x_{ij}}{\partial x_{ik}}\right) = Id$

② f is 1-1: Suppose $f(a) = f(b)$

$a \in U_i'$ for some i, $\Rightarrow \varphi_i(a) = 1$ $\Rightarrow \varphi_i(b) = 1$

$\Rightarrow b \in U_i$

$\Rightarrow \eta_i(a) = \varphi_i(a)\eta_i(a) = \varphi_i(b)\eta_i(b) = \eta_i(b)$

$\Rightarrow a=b$ (η_i homeo)

③ f is homeomorphic. onto its image.

because bijective map $g: X \to Y$ of hausdorff space,

where X compact, is a homeomorphism.

Tangent bundle:

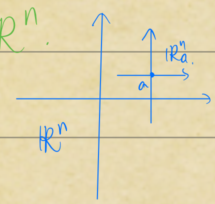
Tangent space TM_a at point $a \in M$?

Given smooth $f: M^n \to N^p$, we want Linear map

$$f_{*a}: TM_a \to TN_{f(a)} \quad \text{(derivative)}$$

Recall: If $a \in U \overset{\text{open}}{<} \mathbb{R}^n$

$TU_a \sim T\mathbb{R}^n_a \sim \mathbb{R}^n_a$ just a copy of $\mathbb{R}^n$.



$V \in \mathbb{R}^n_a$, denoted V_a or (a, V)

$f: \mathbb{R}^n \to \mathbb{R}^p$, $f_{*a}: T\mathbb{R}^n_a \to T\mathbb{R}^p_{f(a)}$

$$V_a \longmapsto (Df(a)V)_{f(a)}$$

Tangent bundle of $U < \mathbb{R}^n$

$TU = U \times \mathbb{R}^n$, pairs (x, V),

$$\begin{array}{ccc} TU & & (x,V) \\ \pi \downarrow & & \downarrow \\ U & & x. \end{array}$$

TU_a means fiber $\pi^{-1}(a)$

$f_*: TU \to T\mathbb{R}^p$, $f_*(x, V) = (f(x), Df(x)V)$

$U \xrightarrow{f} \mathbb{R}^p \xrightarrow{g} \mathbb{R}^q$

WTS: $(g \circ f)_* = g_* \circ f_*$

$(g \circ f)_*(x, V)$

$= (g\circ f(x),\ D(g\circ f)(x)(v))$

$= (g(f(x)),\ (Dg(f(x))(Df(x))(v)))$

$= g_*(\overset{f_*(x,v)}{f(x),\ Df(x)v})$

$= g_*\circ f_*(x,v)$

Tangent vectors v_a operate on differentiable fns:

$v_a(f) = D_v f(a) = Df(a)(v)$ directional dirivative.

$v_a = e_{i,a}$? $D_{e_i} f(a) = \frac{\partial f}{\partial x_i}(a) = \frac{\partial}{\partial x_i}\Big|_a (f)$

Tangent Bundle

$T\mathbb{R}^n_a$. Tangent vectors $v_a \in T\mathbb{R}^n_a$ operate on differentiable functions

$v_a(f) = D_v f(a) = Df(a)\cdot v$.

e.g. $e_{i,a}(f) = D_{e_i} f(a) = \frac{\partial f}{\partial x_i}(a) = \frac{\partial}{\partial x_i}\Big|_a (f)$

So we identify $\frac{\partial}{\partial x_i}\Big|_a$ with tangent vector $e_{i,a}$.

$f: \mathbb{R}^n \to \mathbb{R}$:

$f_{*a}\left(\frac{\partial}{\partial x_i}\Big|_a\right) = f_{*a}(e_{i,a}) = (Df(a)e_{i,a})_{f(a)} = \left(\frac{\partial f}{\partial x_i}(a)\right)_a$

$f: \mathbb{R}^n \to \mathbb{R}^p$

$f_{*a}: T\mathbb{R}^n_a \to T\mathbb{R}^p_{f(a)}$

$f_{*a}\left(\frac{\partial}{\partial x_i}\Big|_a\right) = \sum_{j=1}^{p} \frac{\partial f_j}{\partial x_i}(a) \frac{\partial}{\partial y_j}\Big|_{f(a)}$ (why?)

LHS $= f_{*a}(e_{i,a}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(a) & \cdots & \frac{\partial f_1}{\partial x_n}(a) \\ \vdots & & \vdots \\ \cdots & & \frac{\partial f_p}{\partial x_n}(a) \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th} = \begin{pmatrix} \frac{\partial f_1}{\partial x_i} \\ \vdots \\ \frac{\partial f_p}{\partial x_i} \end{pmatrix}_{f(a)} =$ RHS.

$\frac{\partial}{\partial x_i}\Big|_a$ is a derivation at a, i.e. linear mapping $l: \underset{C^\infty_{\mathbb{R}^n_a}}{\underbrace{\overset{C^\infty \text{ fns in}}{\text{a nbhd of } a}}} \to \mathbb{R}$

s.t. $l(f\cdot g) = l(f)\cdot g(a) + g(a)\, l(f)$.

C^∞_a or $C^\infty_{\mathbb{R}^n_a}$ ring of germs of C^∞ fn at a

germ of C^∞ fns at a: equivalence class of pairs (U, f), U open nbhd of a, f: C^∞ fn on U.

where $(U,f) \sim (V,g)$ if $f=g$ on a neighborhood W of a, where $W \subset U \cap V$

Prop: Space $\mathrm{Der}(C^\infty_{\mathbb{R}^n_a})$ of linear derivations at a is n-dim vector space spanned by $\frac{\partial}{\partial x_i}\big|_a$.

"Pf": Hadamard's lemma: If $f(x_1, \cdots, x_n)$ is C^r, then

$$f(x) - f(0) = \sum_{i=1}^{n} x_i g_i(x), \text{ where } g_i \ C^{r-1} \qquad g_i(0) = \frac{\partial f}{\partial x_i}(0)$$

(if replace $f(0)$ by $f(a)$

$f(x) - f(a) = \sum_{i=1}^{n} (x_i - a_i) g_i(x)$, $g_i(a) = \frac{\partial f}{\partial x_i}(a)$)

Pf: $f(x) - f(0) = \int_0^1 \frac{\partial f(tx)}{\partial t} dt = \int_0^1 \sum \frac{\partial f}{\partial x_i}(tx) x_i$

$$= \sum x_i \underbrace{\int_0^1 \frac{\partial f}{\partial x_i}(tx) dt}_{g_i(x)}$$

Pf: Let $l \in \mathrm{Der}(C^\infty_{\mathbb{R}^n_a})$

$l(1) = l(1 \cdot 1) = 1 l(1) + l(1) \cdot 1 = 2 l(1) = 0$

$l(c) = c l(1) = 0$.

$l(f) = l(f - \underset{\text{constant}}{f(a)})$

$= l\left(\sum_{i=1}^{n} (x_i - a_i) g_i(x)\right)$

product

$= \sum_{i=1}^{n} \left((a_i - a_i) l(g_i) + g_i(a) l(x_i - a_i) \right)$

$= \sum_{i=1}^{n} l(x_i - a_i) \underset{= \frac{\partial f}{\partial x_i}(a)}{g_i(a)}$

$= \sum_{i=1}^{n} l(x_i - a_i) \frac{\partial}{\partial x_i}\big|_a (f)$

So $\frac{\partial}{\partial x_i}\big|_a$ ~~span~~ $\mathrm{Der}(C^\infty_{\mathbb{R}^n_a})$ spans.

$\frac{\partial}{\partial x_i}\big|_a$ linear independent?

$\sum c_i \frac{\partial}{\partial x_i}\big|_a = 0$ apply with $f = x_j \Rightarrow c_j = 0$

So $\mathrm{Der}(C^\infty_{\mathbb{R}^n_a}) \cong T\mathbb{R}^n_a$.

How to define <u>tangent space</u> TM_a to a smooth manifold M at a point a?

$C^\infty_{M,a}$ 是一系列在 a 附近光滑的函数（准确地说，equivalent class，要模掉那些在 a 点附近长得一模一样但在外面不一样的函数。）然后 derivative 可以理解为这里的 derivation？毕竟它只满足 product rule. $Der(C^\infty_{M,a})$ 是这些 linear functional（因为 derivation 把 function 送到 function.）构成的空间，它的 basis 是 $\partial_i|_a$（不好懂），因为之前 identify 了 $e_{i,a}$ 和 $\partial_i|_a$（在 $\mathbb{R}^n$ 中），所以可以把 $\partial_i|_a$ 们理解为"切方向"，换也就构成了 tangent space 的 basis.

<u>Def</u>: Tangent Space TM_a is the linear space $Der(C^\infty_{M,a})$ of derivations of the ring $C^\infty_{M,a}$ of germs of C^∞ fns on M at a.

$f: M \to N$ smooth, $a \in M$ induces $f^*_a: \begin{array}{l} C^\infty_{N,f(a)} \to C^\infty_{M,a} \\ h \mapsto h \circ f \end{array}$ ring homomorphism

f^*_a induces $f_{*a}: \underset{\substack{\| \\ Der(C^\infty_{M,a})}}{TM_a} \longrightarrow \underset{\substack{\| \\ Der(C^\infty_{N,f(a)})}}{TN_{f(a)}}$

f_{*a} 其实就是 f 在 a 点的"导数"。我们一方面可以把它理解为一个把 TM_a 里向量送去 $TN_{f(a)}$ 里向量的 linear map（只从直观上可以这么理解），而严格地定义则是定义为把 derivation $\in C^\infty_{M,a}$ 送去 derivation $\in C^\infty_{N,f(a)}$ 的 linear functional. 本质上是因为（在 $\mathbb{R}^n$ 的 case 中）"tangent vector acts on fns"

注意：f_{*a} 方向与 f 相同，f^*_a 相反.

因为我们知道"tangent vector acts on fns"（dire. deri.）

$$\underbrace{f_{*a}}_{\text{linear fnl}}\underbrace{(\lambda)}_{\to Der(C^\infty_{M,a}) \in TM_a}\underbrace{(h)}_{C^\infty_{N,f(a)}} : \quad f_{*a}(\lambda) 是"导数"$$
$$= \lambda(f^*_a h)$$
$$= \lambda(h \circ f)$$

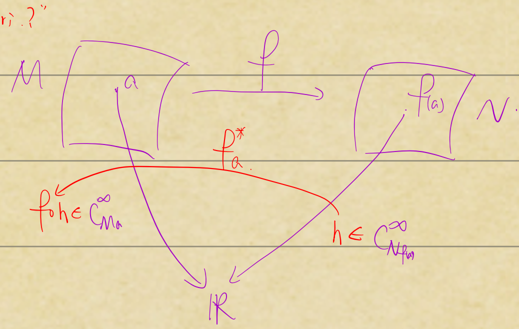
这个算 def？

"如何用 f^* induce f_*"？

$$\underset{\substack{\swarrow \\ \in Der(C^\infty_{M,a}) \\ \text{tangent space 里的点}}}{f_{*a}(\lambda)}\underset{\substack{\searrow \\ \in C^\infty_{N,f(a)}}}{(h)} := \lambda(f^*_a(h)) = \lambda(h \circ f)$$

是否可以理解为"被 f induce 的，在 λ 方向上的 dire. deri.？"

Why is $f_{*a}(\lambda)$ a derivation at $f(a)$?



$$f_{*a}(\lambda)(g \cdot h) = \lambda((g \cdot h) \circ f) \quad // (g \circ f) \cdot (h \circ f)$$
$$= g(f(a)) \cdot \lambda(h \circ f) + h(f(a)) \cdot \lambda(g \circ f)$$
$$= g(f(a)) \cdot f_{*a}(\lambda)(h) + h(f(a)) \cdot f_{*a}(\lambda)(g)$$

■

<u>Lemma</u>: (1) f_{*a} is a linear mapping.

(2) If $M \xrightarrow{f} N \xrightarrow{g} P$, then

$$(g \circ f)_{*a} = g_{*f(a)} \cdot f_{*a} \qquad \text{chain rule.}$$

$$TM_a \xrightarrow{f_{*a}} TN_{f(a)} \xrightarrow{g_{*f(a)}} TP_{g(f(a))} \quad ☆.$$

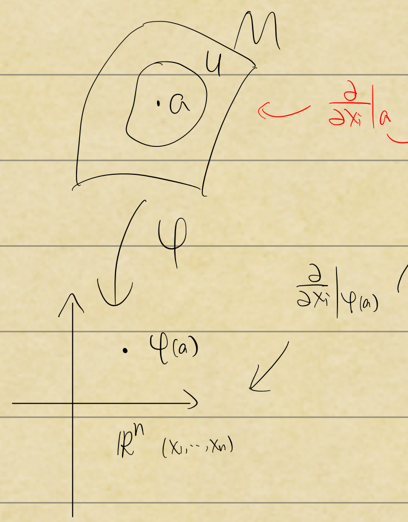
<u>Pf</u>: $(g \circ f)_{*a}\underset{\swarrow TM_a}{(\lambda)}\underset{\swarrow \text{in } C^\infty_{P,g(f(a))}}{(h)} = \lambda(h \circ g \circ f)$

$$= \underbrace{\overbrace{(f_{*a}(\lambda))}^{\text{derivation of } f(a)}}_{\lambda'}(h \circ g)$$

这里是 $g_{*f(a)}(\lambda')(h) = \lambda'(g^*_{f(a)} h) = \lambda'(h \circ g)$

$$= g_{*f(a)} \circ \underbrace{f_{*a}(\lambda)}_{\lambda'}(h)$$

In particular, if f diffeo, then f_{*a} isom.



$\frac{\partial}{\partial x_i}\Big|_a$ $\quad \frac{\partial f}{\partial x_i}(a)$ means $\frac{\partial (f\circ\varphi^{-1})}{\partial x_i}(\varphi(a))$

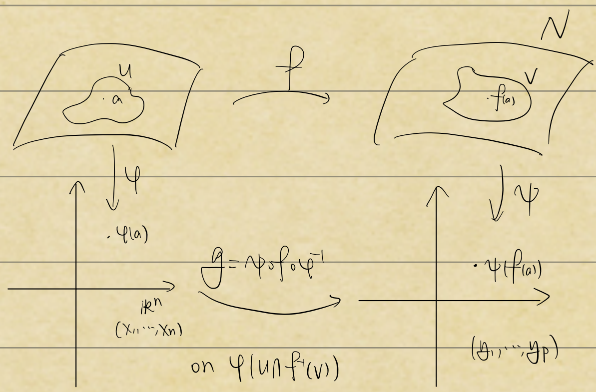
$\Rightarrow \frac{\partial}{\partial x_i}\Big|_a(f) \longleftrightarrow \frac{\partial}{\partial x_i}\Big|_{\varphi(a)}(f\circ\varphi^{-1})$

$(\varphi^{-1})_{*\varphi(a)}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(a)}\right) = \frac{\partial}{\partial x_i}\Big|_a.$

$\varphi_{*a}\left(\frac{\partial}{\partial x_i}\Big|_a\right) = \frac{\partial}{\partial x_i}\Big|_{\varphi(a)}$

$\Rightarrow (\varphi^{-1})_{*\varphi(a)} = (\varphi_{*a})^{-1}$

<u>Lemma</u>: TM_a is a n-dim vector space with basis $\frac{\partial}{\partial x_i}\Big|_a$ (from a given coordinate chart).



$$\begin{tikzcd} TM_a \arrow[r, "f_{*a}"] \arrow[d, "\varphi_{*a}"] & TN_{f(a)} \arrow[d, "\psi_{*f(a)}"] \\ T\mathbb{R}^n_{\varphi(a)} \arrow[r, "g_{*\varphi(a)}"] & T\mathbb{R}^p_{\psi(f(a))} \end{tikzcd}$$

WTS: $g_{*\varphi(a)} = Dg(\varphi(a))$

derivation at $g(\varphi(a)) = \psi(f(a))$

$$g_{*\varphi(a)}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(a)}\right)(h) = \frac{\partial}{\partial x_i}\Big|_{\varphi(a)}(h\circ g)$$

$$= \frac{\partial(h\circ g)}{\partial x_i}(\varphi(a))$$

chain rule. $$= \underbrace{\sum_{j=1}^{n}\frac{\partial h}{\partial y_j}(g(\varphi(a)))}_{\frac{\partial}{\partial y_j}\big|_{\psi(f(a))}(h)}\frac{\partial g_j}{\partial x_i}\varphi(a)$$

$$\Rightarrow g_{*\varphi(a)}\left(\frac{\partial}{\partial x_i}\Big|_{\varphi(a)}\right) = \sum_{j=1}^{n}\frac{\partial g_j}{\partial x_i}(\varphi(a))\frac{\partial}{\partial y_j}\Big|_{\psi(f(a))}$$

$$= Dg(\varphi(a))\frac{\partial}{\partial x_i}\Big|_{\varphi(a)}$$

Tangent bundle TM of a smooth manifold M.

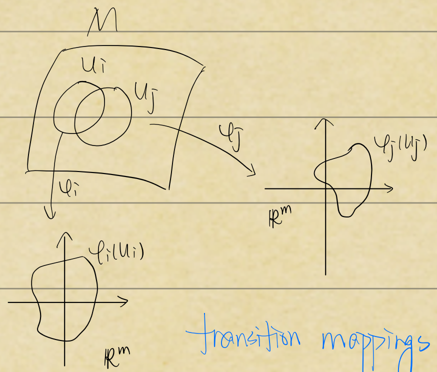
Recall: If U an open subset of $\mathbb{R}^m$, Define $TU = U \times \mathbb{R}^m$ $\left((a,v) = v_a\right)$

"trivial bundle"

$$U \xrightarrow{f} \mathbb{R}^n \quad C^\infty \qquad TU \xrightarrow{f_*} T\mathbb{R}^n$$

$$(x,v) \longmapsto (f(x), \underbrace{Df(x)v}_{f_{*x}(v)})$$

As a set, TM will be disjoint union of all TM_x, $x \in M$



$$\varphi_j \circ \varphi_i^{-1} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

$$T\varphi_i(U_i \cap U_j) \xrightarrow{(\varphi_j \circ \varphi_i^{-1})_*} T\varphi_j(U_i \cap U_j)$$

$$\varphi_i(U_i \cap U_j) \times \mathbb{R}^m \qquad \qquad \varphi_j(U_i \cap U_j) \times \mathbb{R}^m$$

transition mappings!

$$(x, v) \longmapsto \left((\varphi_j \circ \varphi_i^{-1})(x),\ D(\varphi_j \circ \varphi_i^{-1})(x)(v) \right)$$

Given transition data, use this to construct manifold.

Transition mappings define equivalence relations on disjoint union of all $\varphi_i(U_i) \times \mathbb{R}^n$ $(T\varphi_i(U_i))$

where $(x, v) \in \varphi_i(U_i) \times \mathbb{R}^m$, $(y, w) \in \varphi_j(U_j) \times \mathbb{R}^m$,

$(x,v) \sim (y,w)$ if $x \in \varphi_i(U_i \cap U_j)$ and $(y,w) = \left((\varphi_j \circ \varphi_i^{-1})(x),\ D(\varphi_j \circ \varphi_i^{-1})(x)(v)\right)$

Quotient of disjoint union by equivalence relationship with the quotient topology defines manifold TM.

Because the quotient topology is hausdorff and second countable.

Atlas of a special form

$$x \longmapsto D(\varphi_j \circ \varphi_i^{-1})(x)$$

$$\varphi_i(U_i \cap U_j) \longrightarrow L(\mathbb{R}^m, \mathbb{R}^m), \quad C^\infty$$

This structure defines "bundle atlas"

TM a manifold of dim $2m$

$\downarrow \pi_M$

M smooth (projection on the first coordinate)

$\pi_M^{-1}(a) = TM_a$

C^∞ section: $s: M \to TM$: C^∞ mapping $s: M \to TM$. s.t. $\pi_M \circ s = id_M$.

"vector field"

$f: M \to N$ $\quad TM_x \xrightarrow{f_{*x}} TN_{f(x)}$

$TM \xrightarrow{f_*} TN$?

TANGENT BUNDLE : TM of smooth M.

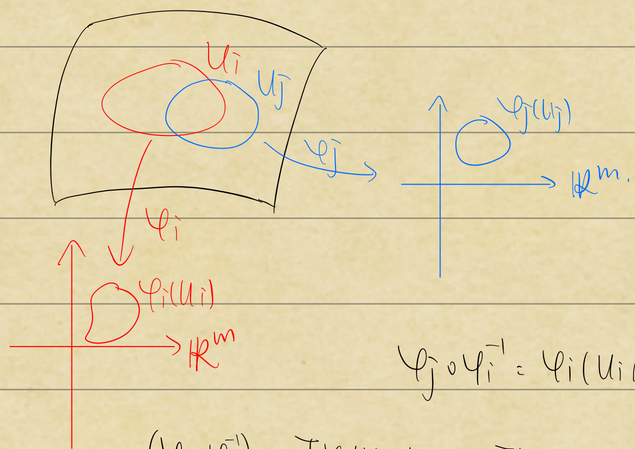
Recall: For $U \overset{open}{\subset} \mathbb{R}^m$, $TU = U \times \mathbb{R}^m$ $\quad (x, v) \in TU_x$

$f: U \to \mathbb{R}^n$ C^∞ induces $f_*: TU \xrightarrow{f_*} T\mathbb{R}^n$.

$(x, v) \longmapsto (f(x), Df(x)(v))$

$f_{*x}: \mathbb{R}^m_x \to \mathbb{R}^n_{f(x)}$

TM disjoint union of TM_x $\quad x \in M$.



$\varphi_j \circ \varphi_i^{-1}: \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$

$(\varphi_j \circ \varphi_i^{-1})_*: T\varphi_i(U_i \cap U_j) \to T\varphi_j(U_i \cap U_j)$

$\varphi_i(U_i \cap U_j) \times \mathbb{R}^m \to \varphi_j(U_i \cap U_j) \times \mathbb{R}^m$

$(x, v) \longmapsto ((\varphi_j \circ \varphi_i^{-1})x, \underbrace{(\varphi_j \circ \varphi_i^{-1})_{*x}}(v))$

$= D(\varphi_j \circ \varphi_i^{-1})(x)$

If consider $x \longmapsto (\varphi_j \circ \varphi_i^{-1})_{*x}$

$\varphi_i(U_i \cap U_j) \longrightarrow L(\mathbb{R}^m, \mathbb{R}^m)$

This structure defines "bundle atlas". Transition mappings define an equivalence relationship on disjoint union of $T\varphi_i(U_i)$:

$(x, v) \in T\varphi_i(U_i)$, $(y, w) \in T\varphi_j(U_j)$ are equivalent if

$x \in \varphi_i(U_i \cap U_j)$ and $(y,w) = \left((\varphi_j \circ \varphi_i^{-1})(x),\ D(\varphi_j \circ \varphi_i^{-1})(x)(v) \right)$.

Quotient of $\coprod T\varphi_i(U_i)$ by the equivalence relation defines C^∞ manifold TM because quotient space Hausdorff and second countable.

$$\begin{array}{l} TM \quad \text{smooth manifold of dim } 2m \\ \downarrow \pi = \pi_M \quad \text{submersion.} \\ M \end{array}$$

each fiber, $\pi_M^{-1}(a) = TM_a$ a vector space.

C^∞ section of tangent bundle TM:

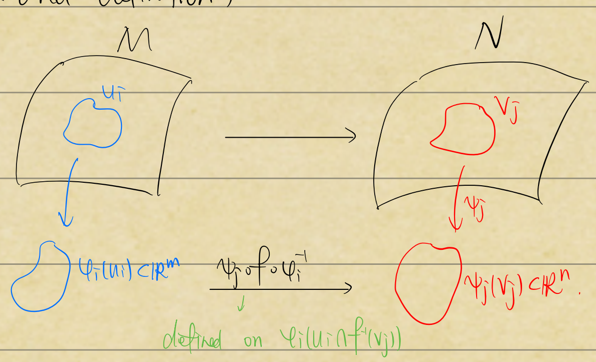
Smooth map $X : M \to TM$, s.t. $\pi_M \circ X = id_M$. 把每个M的点,送到fiber上

If $a \in$ coord. chart U_i with smooth coord. $(x_1, \cdots, x_m)$ 的某个点.

then $X(a) = \sum c_i(a) \frac{\partial}{\partial x_i}\Big|_a$ $\quad X$ smooth $\Leftrightarrow$ all c_i smooth

<u>Lemma:</u> If $f : M^m \to N^n$ smooth, then there is an induced "bundle mapping", $f_* : TM \to TN$ which restricts to f_{*x} on TM_x, with every $x \in M$.

<u>Pf:</u> (and definition)



$$\begin{tikzcd} U_i \cap f^{-1}(V_j) \arrow[r, "f"] \arrow[d, "\varphi_i"'] & V_j \arrow[d, "\psi_j"] \\ \varphi_i(U_i \cap f^{-1}(V_j)) \arrow[r, "\psi_j \circ f \circ \varphi_i^{-1}"] & \psi_j(V_j) \end{tikzcd}$$

Over this, we have

$$\begin{array}{ccc}
TM & & TN \\
\cup & & \cup \\
\pi_M^{-1}(U_i \cap f^{-1}(V_j)) & & \pi_N^{-1}(V_j) \\
\downarrow & & \downarrow \\
\varphi_i(U_i \cap f^{-1}(V_j)) \times \mathbb{R}^m & \longrightarrow & \psi_j(V_j) \times \mathbb{R}^n
\end{array}$$

$$(x, v) \longmapsto \left((\psi_j \circ f \circ \varphi_i^{-1})(x), (\psi_j \circ f \circ \varphi_i^{-1})_{*x}(v) \right)$$

这里有个 example.

Def: A C^∞ vector bundle E of fibre dim k over a C^∞ manifold M is a submersion $\pi: E \to M$ of corank k (i.e. $\dim E - \dim M = k$) s.t. each fiber $E_a = \pi^{-1}(a)$ is a vector space of dim k, together with a bundle atlas. $\{(\pi^{-1}(U_i), \Phi_i)\}$, which makes the following diagram commute:

$$\begin{array}{ccc} \pi^{-1}(U_i) & \xrightarrow{\text{diffeo } \Phi_i} & \varphi_i(U_i) \times \mathbb{R}^k \\ \downarrow \pi & & \downarrow \text{proj} \\ U_i & \xrightarrow{\varphi_i} & \varphi_i(U_j) \subseteq \mathbb{R}^m \end{array}$$

- Moreover, $\forall i, j$, we have

$$\begin{array}{ccc} & U_i \cap U_j & \\ \swarrow \varphi_i & & \searrow \varphi_j \\ \varphi_i(U_i \cap U_j) & \xrightarrow{\varphi_j \circ \varphi_i^{-1}} & \varphi_j(U_i \cap U_j) \end{array}$$

and over this diagram, we must have:

$$\begin{array}{ccc} & \pi^{-1}(U_i \cap U_j) & \\ \swarrow \Phi_i & & \searrow \Phi_j \\ \varphi_i(U_i \cap U_j) \times \mathbb{R}^k & \xrightarrow{?} & \varphi_j(U_i \cap U_j) \times \mathbb{R}^k \end{array}$$

The "?" mapping must be:

$$\varphi_i(U_i \cap \varphi_j) \times \mathbb{R}^k \longrightarrow \varphi_j(U_i \cap U_j) \times \mathbb{R}^k$$

$$(x, v) \longmapsto \left((\varphi_j \circ \varphi_i^{-1})(x),\ \Lambda_{ij}(x)(v)\right),$$

where $\Lambda_{ij}: \varphi_i(U_i \cap U_j) \longrightarrow GL(k, \mathbb{R})$

This is what defines a bundle atlas.

Def: A bundle mapping F from (E_1, π_1, M_1) (fiber dim k) to (E_2, π_2, M_2) (k) is a map:

$$\begin{array}{ccc} E_1 & \xrightarrow{F} & E_2 \\ \downarrow \pi_1 & & \downarrow \pi_2 \\ M_1 & \xrightarrow{f} & M_2 \end{array}$$

s.t. F is linear on each fiber, and such that there are bundle atlases as above, over atlases $\{(U_i,\varphi_i)\}$ for M_1 and $\{(V_j,\psi_j)\}$ for M_2, s.t. $\forall i,j$, we get:

$$\begin{tikzcd}
E_1|_{U_i \cap f^{-1}(V_j)} \arrow[r, "F"] \arrow[d, "\Phi_i"'] & E_2|_{V_j} \arrow[d, "\Psi_j"] \\
\varphi_i(U_i \cap f^{-1}(V_j)) \times \mathbb{R}^k \arrow[r, "(*)"] & \psi_j(V_j) \times \mathbb{R}^l
\end{tikzcd}$$

s.t. $(*)$: $(x,v) \longmapsto \left((\psi_j \circ f \circ \varphi_i^{-1})(x),\ \Lambda_{ij}(x)(v)\right)$

where $\Lambda_{ij} : \varphi_i(U_i \cap f^{-1}(V_j)) \longrightarrow L(\mathbb{R}^k, \mathbb{R}^l)$

C^∞ vector bundle isomorphism:

C^∞ vector bundle mapping with inverse which is also C^∞ vector bundle mapping

Remark: Every C^∞ vector bundle is C^∞ locally trivial.

$$E \xrightarrow{\pi} M.$$

Every pt of M has open nbhd s.t.

$$\begin{tikzcd} \pi^{-1}(U) = E|_U \arrow[r, "\cong"] \arrow[d, "\pi"] & U \times \mathbb{R}^k \arrow[d, "proj"] \\ U \arrow[r, equal] & U \end{tikzcd}$$

} trivial v.b.

Example: (1) M^n smooth manifold

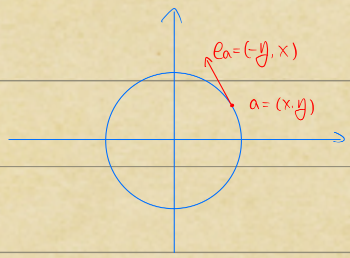
$A: M \to M(p,n)$ lin. space of $p \times n$ matrices.

$$\begin{tikzcd} M \times \mathbb{R}^n \arrow[r] \arrow[d] & M \times \mathbb{R}^p \arrow[d] \\ M \arrow[r, equal] & M \end{tikzcd} : \quad (x,v) \longmapsto (x, A(x)v)$$

C^∞ v.b map iff A C^∞

C^∞ v.b. isom iff $p=n$ and A has image in $GL_n(\mathbb{R})$

(2) TS^1 is trivial, i.e. TS^1 looks like $S^1 \times \mathbb{R}$



$$\begin{tikzcd} TS^1 \arrow[rr, "\cong"] \arrow[dr] & & S^1 \times \mathbb{R} \arrow[dl] \\ & S^1 & \end{tikzcd}$$

$$\begin{array}{ccc} TS^1 & & S^1 \times \mathbb{R} \\ te_a & \longleftarrow & (a, t) \end{array}$$

TS^2 not trivial — 毛球定理.

(3) Tautological line bundle over $\mathbb{RP}^n$ (or $\mathbb{CP}^n$)

$\mathbb{RP}^n$. space of lines through 0 in $\mathbb{R}^{n+1}$

$$\begin{array}{c} E \\ \downarrow \\ \mathbb{RP}^n \end{array} \qquad E_\lambda \quad \text{lines in } \mathbb{R}^{n+1} \text{ corresponding to } \lambda.$$

$\lambda = [\lambda_0, \cdots, \lambda_n]$

$$E = \{(\lambda, x) \in \mathbb{RP}^n \times \mathbb{R}^{n+1} : \underline{x \in \lambda}\}$$

$$\begin{array}{ccc} E & & (\lambda, x) \ni (x_0, \cdots, x_n) \\ \downarrow & & \downarrow \\ \mathbb{RP}^n & & \lambda = [\lambda_0, \cdots, \lambda_n] \end{array}$$

$x_i \lambda_j = x_j \lambda_i \ , \forall ij.$

$\mathbb{RP}^n$ is covered by coord. charts: $U_i : i = 0, \cdots, n.$

$$U_i \xrightarrow{\varphi_i} \mathbb{R}^n$$

$$\{\lambda = [\lambda_0, \cdots, \lambda_n] : \lambda_i \neq 0\} \longmapsto \left(\frac{\lambda_0}{\lambda_i}, \cdots, \widehat{\frac{\lambda_i}{\lambda_i}}, \cdots, \frac{\lambda_n}{\lambda_i}\right)$$

coordinates on U_i

$E|_{U_i} \ = \ x_j = x_i \frac{\lambda_j}{\lambda_i}$ "A graph of a function"

$$E|_{U_i} \longrightarrow \mathbb{R}^n \times \mathbb{R}$$

$$(\lambda, x) \longmapsto \left(\left(\frac{\lambda_0}{\lambda_i}, \cdots, \widehat{\frac{\lambda_i}{\lambda_i}}, \cdots, \frac{\lambda_n}{\lambda_i}\right), \ ?\right) \quad x_i$$

$$([y_0, \cdots, 1, \cdots, y_n], x) \longleftarrow ((y_0, \cdots, \hat{y}_i, \cdots, y_n), t)$$

where x is : $\begin{cases} x_i = t \\ x_j = t y_j = x_i \frac{\lambda_j}{\lambda_i} \end{cases}$

What about $\begin{array}{c} E \\ \downarrow p \\ \mathbb{R}^{n+1} \end{array} \quad \begin{array}{c} (\lambda, x) \\ \downarrow \\ x \end{array} \quad p^{-1}(x) = \begin{cases} \text{single pt} & x \neq 0 \\ \mathbb{RP}^n & x = 0. \end{cases}$

Vector Bundle examples:

Grassmannian:

$Gr(k,n)$: Space of k dim linear subspaces of $\mathbb{R}^n$ $(\mathbb{C}^n)$

manifold of dim ? $k(n-k)$

Tautological bundle (corank k)

$$\begin{array}{ccc} E & = & \{(\lambda, x) \in Gr(k,n) \times \mathbb{R}^n, \ x \in \lambda\} \\ \downarrow \pi & & \downarrow \\ Gr(k,n) & & \lambda \end{array}$$

$Gr(1,n) = \mathbb{RP}^{n-1}.$

Given $\lambda \in Gr(k,n)$, choose basis $w_1, \cdots, w_k$ of λ.

$W = \begin{pmatrix} \text{column} \\ \text{vectors} \\ w_j \end{pmatrix}$ w.r.t standard basis of $\mathbb{R}^n$.

↳ "homogeneous coord."

W and W' determines the same λ if $W' = W \cdot A$, where $A \in GL_k(\mathbb{R})$

For example, suppose first k rows of W are linear independent

By elementary column operations, can transform W to $\begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ \hline a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{n-k,1} & \cdots & a_{n-k,k} \end{pmatrix}$ uniquely determined.

In general, if rows $i_1, \cdots, i_k$ lin. independent,

Let $W_{i_1 \cdots i_k}$ be corresponding submatrix,

$\Rightarrow W \cdot W_{i_1 \cdots i_k}^{-1}$ reduces such submatrix to Id.

Let $U_{i_1 \cdots i_k}$ denote the equivalence class of matrices W, s.t.

$W_{i_1 \cdots i_k}$ is invertible: corresponds to n.dim linear subspace λ

of $\mathbb{R}^n$ whose orthogonal projection to subspace $\mathbb{R}^{i_1, \cdots, i_k}$ ($\cong \mathbb{R}^k$ spanned by $i_1 \cdots i_k$) of $\mathbb{R}^n$

given by $x_i = 0$ if $i \neq i_1, \cdots, i_k$. is invertible.

If $W \in U_{i_1, \cdots, i_k}$, let $A_{i_1 \cdots i_k} = W \cdot W_{i_1, \cdots, i_k}^{-1}$

Submatrix $B_{i_1, \cdots, i_k}$ given by all rows given by all rows except

rows $i_1, \cdots, i_k$, define homeomorphism

$U_{i_1, \cdots, i_k} \longrightarrow \mathbb{R}^{k(n-k)} = M(n-k, k)$

Transition mappings bet. coord. charts $U_{i_1, \cdots, i_k}, U_{j_1, \cdots, j_k}$:

If $W \in U_{i_1, \cdots, i_k} \cap U_{j_1, \cdots, j_k}$, Then: $A_{i_1 \cdots i_k} W_{i_1 \cdots i_k} = A_{j_1 \cdots j_k} W_{j_1 \cdots j_k}$ $(= W)$

So transition maps are rational functions.

Vector bundle charts: (i.e. trivialization of $E|_{U_{i_1 \cdots i_k}}$).

$E|_{U_{i_1 \cdots i_k}} \longrightarrow M(n-k, k) \times \mathbb{R}^{i_1 \cdots i_k}$

$(\lambda, x) \longmapsto (B_{i_1 \cdots i_k}, P_{i_1 \cdots i_k}(x))$ orthogonal projection onto $\mathbb{R}^{i_1 \cdots i_k}$

Inverse:

$(A_{i_1, \cdots, i_k}, \text{pt of } \lambda \text{ projecting to } t) \longleftarrow (B_{i_1 \cdots i_k}, t)$ $\quad t = (t_{i_1}, \cdots, t_{i_k})$

?

Point $(x_1, \cdots, x_n)$, s.t. $x_{i_j} = t_{i_j}$, $j = 1, \cdots, k$. $(x_1, \cdots \hat{x}_{i_j}, \cdots, x_n) = B_{i_1, \cdots, i_k} \cdot t$.

Orientation of a manifold or vector bundle.

$T: V \to V$: isom of finitely dimensional vector space to itself,

is orientation-perserving if $\det T > 0$ with given basis!

- - - - - - reversing if $\det T < 0$.

Say 2 ordered basis $\{v_i\}$ $\{w_j\}$ of V equivalent if $\det \begin{pmatrix} \text{basis} \\ \text{change} \\ \text{matrix} \end{pmatrix} > 0$.

Orientation of V: equiv. class $\mu = [v_1, \cdots, v_n]$ of ordered basis.

Given orientated vector spaces (V, μ), (W, ν)

say lin. isom $T: V \to W$ orientation preserving if

$[Tv_1, \cdots, Tv_n] = \nu$, whenever $[v_1, \cdots, v_n] = \mu$.

Given trivial bundle $\begin{matrix} M \times \mathbb{R}^k \\ \downarrow \\ M \end{matrix}$,

we can put standard orientation of $\mathbb{R}^n$ on every fiber,

If $f: M \times \mathbb{R}^k \to M \times \mathbb{R}^k$ is isom of this bundle to itself,

then f is either orientation-perserving or orientation-reversing on

all fibers provided that M connected. ($\det A(x)$ cannot change sign)

$\begin{matrix} E \\ \downarrow \pi \\ M \end{matrix}$ smooth vector bundle

Orientation μ means collection of orientation μ_x of fiber E_x

which are compatiable: if $E|_U \xrightarrow{\Phi} U \times \mathbb{R}^k$ is a local connected

trivialization and $\mathbb{R}^k$ has standard orientation

Then $\exists$ either orientation-preserving or orientation-reversing on all fibers.

(If this is true for one trivialization over U, then it's true for any other).

If E has orientation μ, then it has another $-\mu = \{-\mu_x\}$.

But not true that every v.b has orientation.

Bundle orientable if it has an orientation

Oriented bundle: bdl together with orientation

<u>Orientation of TM</u> called orientation of M.

<u>Orientated manifold</u>: $(M, \underset{\uparrow\,\text{orientation}}{\mu})$

<u>Prop</u>: M orientable iff there is atlas $\{(U_i, \varphi_i)\}$, s.t. $\forall i, j$,

$\det D(\varphi_i \circ \varphi_j^{-1})(x) > 0, \quad \forall x \in \varphi_i(U_i \cap U_j)$

<u>Pf</u>: Given orientation μ of M, we can take all charts (U, φ)

s.t. $TM|_U \to \varphi(U) \times \underbrace{\mathbb{R}^n}_{\text{standard orien.}}$ orien. preser.

Conversely:

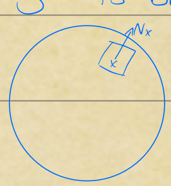
Given such atlas, we have oriented fibers of TM, s.t. each φ_* is

orien. pres.

<u>Examples</u>:

(1) $S^{n-1} \subset \mathbb{R}^n$ orientable.

S^{n-1} is bdry of B^n, you can always orient boundary



given $v_1, \cdots, v_{n-1} \in TM_x$, define μ_x as $[v_1, \cdots, v_{n-1}]$

provided $[n_x, v_1, \cdots, v_{n-1}]$ is standard orientation of $\mathbb{R}^n$.

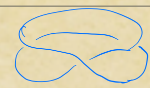
(2) (open) Mobious strip is a vector bundle E of fiber dim 1 over S^1.

E is quotient of $\mathbb{R}\times\mathbb{R}$ by equi. rela $(x,t)\sim(x+1,-t)$

$$\begin{array}{ccc} E & & (x,t) \\ \downarrow & & \downarrow \\ S^1 & & (\cos 2\pi x, \sin 2\pi x) \end{array}$$

E not orientable (as a vector bundle, or as manifold)

E has no non-vanishing section



But can pick 2 pt in each fiber to get $A = \mathbb{R}\times\{-1,1\}/\sim$

If E has an orientation $\{\mu_p\}$, $p\in S^1$, we could define non-vanishing section $s(p)$ by taking $s(p)$ as the unique point of $A\cap E_p$, s.t. $[s(p)]=\mu_p$

(3) $\mathbb{RP}^{2n}$ ✗ , $\mathbb{RP}^{2n+1}$ ✓ in general.

Antipital point mapping is orien. rere. for $2n$

---- pre for $2n+1$.

Orientation example:

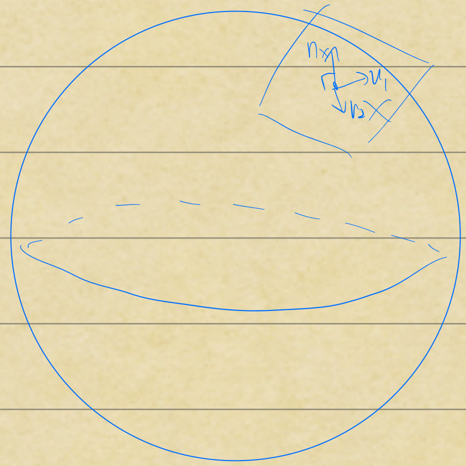
$\mathbb{RP}^2$ is not orientable.

Consider $A: S^2\to S^2$, antipodal map (induced by antipodal map from $\mathbb{R}^3$)
$x\mapsto -x$

$\mathbb{RP}^2 = S^2/\sim$ where $x\sim A(x)$

$A_{*x}: TS^2_x \longrightarrow TS^2_{-x=A(x)}$

$(x,v)\longrightarrow (A(x), A(v))$



u_1, u_2 basis of TS_x

$(n_x, u_1, u_2) \xrightarrow{A_{*_x}} (-n_x, -u_1, -u_2)$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad\; \shortparallel$
$\quad\quad\quad\quad\quad\quad\quad\quad\quad n_{-x}$

$\Rightarrow$ Orientation, reversing.

$S^2 \xrightarrow{g} \mathbb{RP}^2$ Suppose $\mathbb{RP}^2$ has orientation $\{\mu_y\}_{y \in \mathbb{RP}^2}$.
$x \longmapsto [x]$ Then we can define orien $\{\nu_x\}_{x \in S^2}$
by require g to be orien. preserving
Then antipodal point map preserves this orientation,
contradiction.

- $A: S^3 \to S^3$ orientation preserving
  $g: S^3 \to \mathbb{RP}^3$ Can define orientation for $\mathbb{RP}^3$ by requiring
  $x \longmapsto [x]$ orien. preserving

$\Rightarrow \mathbb{RP}^3$ orientable.

So $\mathbb{RP}^n$ is orientable if n odd, non-orientable if n even.

STRONG(er) WHITNEY EMBEDDING THM

Every compact C^∞ manifold M with dim n has an embedding in $\mathbb{R}^{2n+1}$

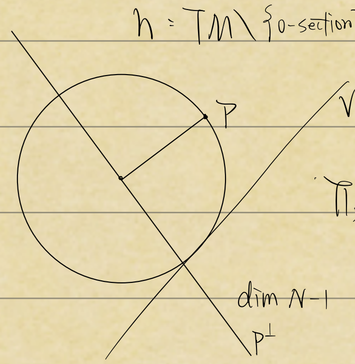
- Enough to show there is injective immersion into $\mathbb{R}^{2n+1}$
(Because compact)

<u>Lemma</u>: If M has injective immersion onto $\mathbb{R}^N$, then it has injective immersion into $\mathbb{R}^{N-1}$, where $N \geq 2n+2$.

<u>Proof</u>: Consider $g: M \times M \setminus \overset{\text{diagonal}}{\Delta} \longrightarrow S^{N-1}$

$$(x,y) \longmapsto \frac{i(x)-i(y)}{|i(x)-i(y)|}$$

$$h: TM \setminus \{0\text{-section}\} \longrightarrow S^{N-1}$$



$$v \longmapsto \frac{i_*(v)}{|i_*(v)|}$$

$$\pi_p: \mathbb{R}^N \to p^\perp$$

$p \notin \operatorname{im}(g) \iff \pi_p \circ i$ injective.

$p \in \operatorname{im}(g) \iff i(x)-i(y)$ multiple of p

$\Rightarrow \pi_p \circ i(x) - \pi_p \circ i(y) = 0$

$i(x), i(y)$ projects to the same point

$p \notin \operatorname{Im}(h) \iff (\pi_p \circ i)_*$ injective.

$p \in \operatorname{Im}(h) \iff \underbrace{\pi_{p*}}_{\text{linear map}} \circ \underbrace{i_*(v)}_{\neq 0} = 0$ some v.

$\Rightarrow p \notin \operatorname{Im} g \cup \operatorname{Im} h$

$\iff \pi_p \circ i: M \longrightarrow p^\perp \cong \mathbb{R}^{N-1}$ an injective immersion

$(\pi_p \circ i)_*$ isom ?

So we only have to find such p.

$M \times M$, TM manifold of dim $2n$, $N-1 > 2n$. so p disjoint from $\operatorname{Im} g \cup \operatorname{Im} h$ iff p is a regular value for g, h.

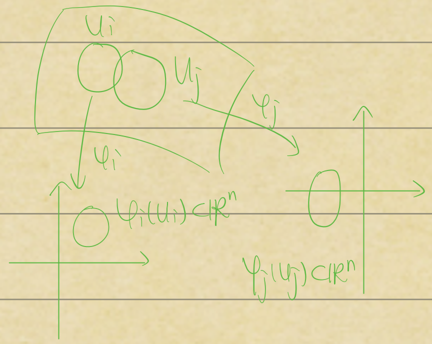
By Sard's thm, regular value is dense.

Vector Field and Differential Forms:

Cotangent bundle T^*M : bundle dual to TM.

fibers T^*M_x are dual spaces to TM_x.

Recall: defn of TM by bundle charts



$$(\varphi_j\circ\varphi_i^{-1}) : \varphi_i(U_i\cap U_j) \to \varphi_j(U_i\cap U_j)$$

$$(\varphi_j\circ\varphi_i^{-1})_* : T\varphi_i(U_i\cap U_j) \to T\varphi_j(U_i\cap U_j)$$

$$\quad\quad \| \quad\quad\quad\quad\quad\quad \|$$

$$\varphi_i(U_i\cap U_j)\times\mathbb{R}^n \quad\quad \varphi_j(U_i\cap U_j)\times\mathbb{R}^n$$

$$(x, v) \longmapsto ((\varphi_i\circ\varphi_j^{-1})x,\ (\varphi_i\circ\varphi_j^{-1})_{*x}(v))$$

$$\downarrow$$

$$= D(\varphi_i\circ\varphi_j^{-1})(x)v.$$

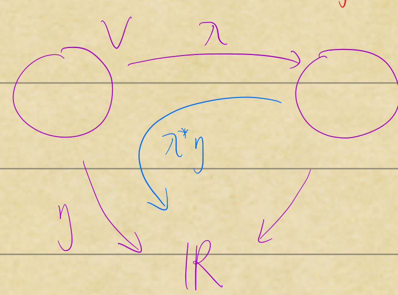
$\varphi_i(U_i\cap U_j) \to L(\mathbb{R}^n,\mathbb{R}^n)$ C^∞

In particular, $x \longmapsto (\varphi_j\circ\varphi_i^{-1})_{*x}$

A linear mapping of vector spaces

$\lambda : V \to W$ induces $\lambda^* = W^* \to V^*$

$\lambda^*(\eta)(\underset{\in V}{v}) = \eta(\lambda(v))$



要"绕"去 v 一趟.

$(\varphi_j\circ\varphi_i^{-1})_{*x} : \underset{V}{\mathbb{R}^n} \to \underset{W}{\mathbb{R}^n}$ induces $(\varphi_j\circ\varphi_i^{-1})^*_x : \underset{W^*}{(\mathbb{R}^n)^*} \to \underset{V^*}{(\mathbb{R}^n)^*}$

T^*M has bundle with overlap mappings:

$$(\varphi_j\circ\varphi_i^{-1})^* : T^*\varphi_j(U_i\cap U_j) \to T^*\varphi_i(U_i\cap U_j)$$

$$\|\text{def} \quad\quad\quad\quad \|$$

$$\varphi_j(U_i\cap U_j)\times(\mathbb{R}^n)^* \quad\quad \varphi_i(U_i\cap U_j)\times(\mathbb{R}^n)^*$$

$$(y, \eta) \longmapsto (\underbrace{(\varphi_i\circ\varphi_j^{-1})(y)}_{x},\ (\varphi_j\circ\varphi_i^{-1})^*_x(\eta))$$

$$\eta \longmapsto (\varphi_j\circ\varphi_i^{-1})^*_{\underset{(\varphi_i\circ\varphi_j^{-1})(y)}{x}}$$

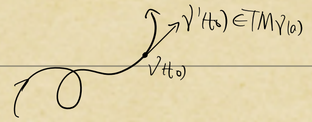
C^∞ function on $\varphi_i(U_i\cap U_j)$ with values in $L(\overset{W^*}{(\mathbb{R}^n)^*}, \overset{V^*}{(\mathbb{R}^n)^*})$

Vector Fields:

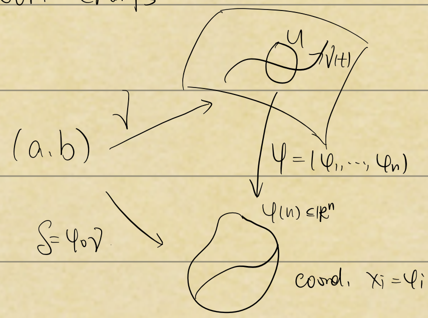
Tangent vector to a smooth curve $\gamma : (a,b) \to M$

at $t_0 \in (a,b)$ is defined as $\gamma_{*t_0}(e_{1,t_0})$, denote this as $\gamma'(t_0)$

"$\frac{d}{dt}\big|_{t_0}$"



In local coor. charts:



$\delta = (\delta_1, \cdots, \delta_n)$, $\delta_i = \varphi_i \circ \gamma$

$= x_i \circ \gamma$

$= \gamma_i$

Tangent vectors act on smooth functions.

$X \in TM_a$, $f \in C^\infty(M)$

$X(f) = f_{*a}(X)$, in coordinates: $X_a = \sum_{i=1}^{n} \xi_i \frac{\partial}{\partial x_i}\big|_a$ $\in \mathbb{R}$

$= \sum_{i=1}^{n} \xi_i \frac{\partial f}{\partial x_i}(a)$

"$\frac{\partial(f \circ \varphi_i^{-1})}{\partial x_i}\Big|_a$"

这里基本上只是def.

但其实问题是 $\gamma'(t_0)$ 到底是啥.

是一个 $TM_{\gamma(t_0)}$ 里的向量 (因为 $\gamma \to M$)。

理解成 $\gamma:(a,b) \to M$ 被 lower star push 到 γ_{t_0} 来.

In particular: $\gamma'(t_0)(f) = f_{*\gamma(t_0)}\, \gamma_*(t_0)\, (e_{1,t_0})$

$\in TM_{\gamma(t_0)}$

这是个tangent vector 它act on f, 就像 这样,

$g_{*f(a)} \circ f_{*a} = (g \circ f)_{*a}$.

这是 $N = (a,b)$, TN_{t_0} 那一个 vector.

$= (f \circ \gamma)_{*t_0}(e_{1,t_0})$

$= \frac{d(f \circ \gamma)}{dt}(t_0)$

$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\gamma(t_0)) \frac{d\gamma_i}{dt}(t_0)$

$= \sum_{i=1}^{n} \left(\frac{\partial \gamma_i}{\partial t}(t_0) \frac{\partial}{\partial x_i}\Big|_{\gamma(t_0)} \right)(f)$

C^∞ 1-form ω on M: C^∞ section of T^*M.

$\begin{cases} \omega & C^\infty \text{ 1-form on } M \\ X & C^\infty \text{ vector field on } M. \end{cases}$

We get C^∞ function on M:

$$\omega(X)(x) = \underbrace{\omega(x)}_{\in TM_x^*} \underbrace{X(x)}_{\in TM_x}$$

是它的dual, 所以是个数.

If $f \in C^\infty(M)$ we define C^∞ 1-form df (differential of f)

by $\boxed{df(x)(\underbrace{X}_{\in TM_x}) = f_{*x}(X) = X(f)}$

Eg. If $\varphi: U \to \mathbb{R}^n$ coord. chart for M, then

$dx_i = d\varphi_i$ section of $T^*M|_U$.

$dx_i(x)\left(\frac{\partial}{\partial x_j}\Big|_x\right) = \delta_{ij}$

$\Rightarrow$ $\boxed{dx_i(x) \text{ form basis of } T^*M_x = (TM_x)^*}$

$\boxed{\text{dual basis to } \frac{\partial}{\partial x_i}\Big|_x}$

In particular, C^∞ 1-form ω can be expressed as

$\omega = \sum \underbrace{\omega_i}_{\text{functions}} dx_i$ in local coordinates charts U.

$\omega(x) = \sum \omega_i(x)\, dx_i(x)$

ω is C^∞ 1-form $\Leftrightarrow$ ω_i are C^∞ fns.

Prop: If $f \in C^\infty(M)$, then in local coord. charts $\varphi: U \to \mathbb{R}$

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

Pf: Let $X \in TM_x$, $X = \sum \xi_i \frac{\partial}{\partial x_i}\Big|_x$

So $\xi_i = X(x_i) = dx_i(x)(X)$

$df(x)(X) = X(f)$

$= \sum \xi_i \frac{\partial f}{\partial x_i}(x)$

$= \left(\sum \frac{\partial f}{\partial x_i}(x)\, dx_i(x)\right)(X)$

=

Important Remark:

$f: M \longrightarrow N$ C^∞, induces bundle map $f_*: TM \to TN$:

In particular, $\forall x$, $f_{*x}: TM_x \to TN_{f(x)}$ linear mapping.

Dual $(f_{*x})^* = f_x^*: T^*N_{f(x)} \longrightarrow T^*M_x$

Do we have bundle map f^* $T^*N \to T^*M$?

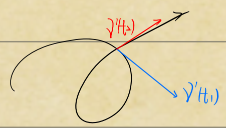
But, given C^∞ section ω of T^*N (i.e, C^∞ 1-form on N)

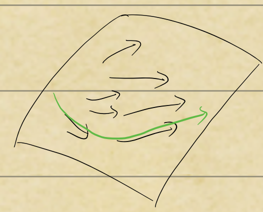
We do get a C^∞ 1-form on M by pull back.

i.e, "$f^*\omega$"

Def: $(f^*\omega)(x)(X) = \omega(f(x))(f_{*x}(X))$ ($x \in M$, $X \in TM_x$)

f_* doesn't induce map from vector field on M → vector field on N.





Given C^∞ vector field X on M,

and a point $a \in M$, is there C^∞

curve $\gamma: (-\varepsilon, \varepsilon) \to M$, s.t. $\gamma(0) = a$ and $\gamma'(t) = X(\gamma(t))$

γ is an integral curve of X with initial $\gamma(0) = a$

E.g. in $\mathbb{R}^n$, in local coord. chart $\varphi: U \longrightarrow \mathbb{R}^n$

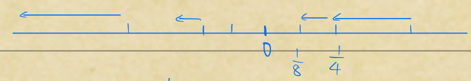
$\gamma_i'(t) = f_i(\gamma_1(t), \cdots, \gamma_n(t))$

system of ODEs with initial condition.

Example: ① find $\gamma: \mathbb{R} \to \mathbb{R}$

s.t. $\gamma'(t) = -\gamma(t)^2$, We want to integrate vector field

On $\mathbb{R}$, $f(a) = -a^2$



$\frac{dy}{dx} = -y^2 \Rightarrow -\frac{dy}{y^2} = dx \Rightarrow \frac{1}{y} = x + c \Rightarrow y = \frac{1}{x+c}$

$\gamma(0) = a \Rightarrow c = \frac{1}{a} \Rightarrow y = \frac{1}{x + \frac{1}{a}}$

Also $y \equiv 0$ for $y(0) = 0$

No integral curve can be defined for all t ithe X defined everywhere

Curves escapes to ∞ when $x \to -\frac{1}{a}$

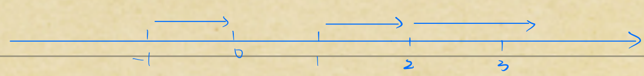
Fundemental existance and uniqueness thm

X C^∞ v.f. on open $U \subseteq \mathbb{R}^n$, $K \subseteq U$ compact

Then $\forall \varepsilon > 0$, there is a nbhd U' of K in U and unique C^∞ mapping

$g: U' \times (-\varepsilon, \varepsilon) \to U$ s.t. $\frac{\partial g(x,t)}{\partial t} = X(g(x,t))$, $g(x,0) = x$.

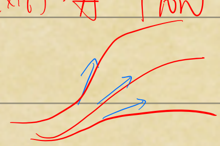
E.g. On $\mathbb{R}$, $X(a) = a^{\frac{2}{3}} \frac{\partial}{\partial t}\big|_a$



$\frac{dy}{dx} = y^{\frac{2}{3}}$ ≥ 2 sol^n with $\gamma(0) = 0$, $\begin{cases} \gamma(t) = 0 \\ \gamma(t) = \frac{1}{27} t^3 \end{cases}$

不 unique 因为 $X(a)$ 不 C^1

$g(x,t)$: A flow. $\varphi_t(x) = g(x,t)$



Corollary: $\varphi_0 = id$

$\varphi_{s+t}(x) = \varphi_s \circ \varphi_t(x)$ (when both side are defined) i.e. $|s|, |t|, |s+t| < \varepsilon$, $x, \varphi_t(x) \in U'$

$f(x,s) = g(x, t+s)$

$\frac{\partial f}{\partial s} = X(f(x,s))$

$f(x,0) = g(x,t)$

<u>Pf</u>: $\frac{d}{ds} g(x, s+t) = X(g(x, s+t))$

$\varphi_{s+t}(x) = g(x, s+t)$

$= g(\overbrace{g(x,t)}^{y}, s)$

$= \varphi_s(\varphi_t(x))$

$\frac{\partial g(y,s)}{\partial s} = X(g(y,s))$

$g(y,0) = y$.

$y = g(x,t)$ satisfies this

$h(x,s) := g(g(x,t), s)$

$\frac{\partial h}{\partial s} = X(g(\boxed{g(x,t)}, s))$ fixed.

$= X(h(x,s))$

$h(x,0) = g(x,t)$

$\Rightarrow f = h$ by unique.

<u>THM</u>: X C^∞ v.f on manifold M,

Given $a \in M$, there is open nbhd V of a, $\varepsilon > 0$, and a unique collection of diffeos

$$\varphi_t : V \longrightarrow \varphi_t(V) \subset M, \quad |t| < \varepsilon$$

s.t.

(1) $\varphi : V \times (-\varepsilon, \varepsilon) \to M$

$$(x,t) \longmapsto \varphi_t(x) \qquad C^\infty$$

(2) $\varphi_{s+t}(x) = (\varphi_s \circ \varphi_t)(x) \qquad |s|, |t|, |s+t| < \varepsilon$, $x, \varphi_t(x) \in V$

(3) $\forall x \in V$, $X(x)$ is tangent vector at $t=0$ of curve $t \longmapsto \varphi_t(x)$

<u>Support</u> of X is closure of $\{x \in M, X(x) \neq 0\}$

<u>Thm</u>: If X has compact support, (e.g. if M compact) then there are diffeos.

$\varphi_t(x)$, $t \in \mathbb{R}$, satisfies (1) – (3).

parameter group of diffeos of M.

X C^∞ vector field on M: $\quad x \longrightarrow X(x) = X_x \in TM_x$

($\mathfrak{X}(M)$: space of C^∞ vector field on M.)

Given $a \in M$, can we find an integral curve γ of X, with

initial condition $\gamma(0) = a$, $\gamma'(t) = X(\gamma(t)) = X(\gamma(t)) = X_{\gamma(t)}$

e.g. on $\mathbb{R}$: $X = -x^2 \frac{\partial}{\partial x}$,

Looking for $\gamma(t)$, s.t. $\gamma(0) = a$, $\gamma'(t) = X(\gamma(t))$

Let $x = \gamma(t)$, then $\gamma'(t) = \frac{dx}{dt} = -x^2$

$\gamma'(t) \in TM_{\gamma(t)}$ spanned by $\frac{d}{dx}\Big|_{\gamma(t)}$ $\quad$ so actually: $\gamma'(t) = \frac{dx}{dt}\frac{d}{dx}\Big|_{\gamma(t)} = -x^2\frac{\partial}{\partial x}\Big|_{\gamma(t)}$.

$$-\frac{dx}{x^2} = dt \Rightarrow \frac{1}{x} = t + c \Rightarrow x = \frac{1}{t+c} \xRightarrow[c=\frac{1}{a}]{} x = \frac{1}{t+\frac{1}{a}}$$

Fund thm: Given vector field on open $U \subseteq \mathbb{R}^n$, locally, there exist

$g(x,t)$, $x \in U$, $|t| < \varepsilon$, with values in $\mathbb{R}^n$, s.t.

$$\frac{\partial g(x,t)}{\partial t} = X(g(x,t))$$

$$g(x,0) = x$$

下面的 x 是上面的 a，
起到的是 initial condition 的作用
我们也想考虑 I.C 变化时 g 的变化.

In this example, $g(x,t) = \frac{1}{t+\frac{1}{x}}$

Also denote $\varphi_t(x) = g(x,t)$

Consider $g(x, \underset{\text{fixed}}{s+t})$

Then $\frac{\partial}{\partial s} g(x, s+t) = X(g(x,s+t))$, then g is the unique solution with

initial condition $g(x,s+t)|_{s=0} = g(x,t)$

Then consider $g(g(x,t),s)$. $g(y,s)$ is the unique solution with $g(y,0) = y$,

take $y = g(x,t)$, then $g(g(x,t),s)$ is the unique solution with initial

condition $g(x,s+t)|_{s=0} = g(x,t)$

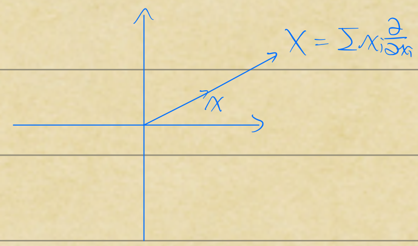
So $g(x,s+t) = g(g(x,t),s) \Leftrightarrow \varphi_{s+t}(x) = \varphi_s(\varphi_t(x)) \Rightarrow \varphi_{s+t} = \varphi_s \circ \varphi_t$

Thm: Given $X \in \mathfrak{X}(M)$, for all $a \in M$, $\exists$ open nbhd V of a, $\varepsilon > 0$, unique collection of diffeos $\varphi_t : V \to \varphi_t(V) \subset M$, $|t| < \varepsilon$, s.t.

(1) $$\begin{aligned}(-\varepsilon, \varepsilon) \times M &\longrightarrow M \quad \text{is } C^\infty \\ (t, x) &\longmapsto \varphi_t(x)\end{aligned}$$

(2) $\varphi_{s+t}(x) = \varphi_s(\varphi_t(x))$ when $|s|, |t|, |s+t| < \varepsilon$, $x, \varphi_t(x) \in V$

(3) If $x \in V$, then X_x is tangent vector at $t=0$ of curve $t \mapsto \varphi_t(x)$



Example: Radical vector field on $\mathbb{R}^n$

Find $\gamma(t)$ s.t. $\gamma'(t) = \gamma(t)$

$\gamma = x = \gamma(t)$, $\frac{dx_i}{dt} = x_i$

with initial condition $\gamma(0) = a$, $x_i = a_i e^t$

$\varphi_t(x) = e^t x$

$\mathrm{supp}(X) = \overline{\{x \mid X_x \neq 0\}}$

Thm: If X has compact support (in particular, if M compact) then there are diffeos $\varphi_t(x)$ satisfying conditions of previous thm, defined for all $t \in \mathbb{R}$.

Pf: Cover $\mathrm{supp}\, X$ by finitely many $V_1, \cdots V_p$, $\varepsilon_1, \cdots, \varepsilon_p$, $\varphi_t^1, \cdots, \varphi_t^p$.

Let $\varepsilon = \min \varepsilon_i$

If $x \in V_i, \cdots, V_j$, then $\varphi_t^i(x) = \varphi_t^j(x)$ by uniqueness.

So let's define

$$\varphi_t(x) = \begin{cases} \varphi_t^i(x) & x \in V_i \\ x & x \notin \mathrm{Supp}\, X \end{cases}$$

Then $\varphi: (-\varepsilon,\varepsilon)\times M \to M$, $(t,x)\mapsto \varphi_t(x)$ C^∞, $\varphi_{s+t}(x)=\varphi_s(\varphi_t(x))$ $|s|, |t|, |s+t|<\varepsilon$,

φ_t diffeo

To define φ_t when $|t|>\varepsilon$:

Write $t = k\frac{\varepsilon}{2}+r$, k integer, $|r|<\frac{\varepsilon}{2}$

Define $\varphi_t = \begin{cases} \underbrace{\varphi_{\frac{\varepsilon}{2}}\circ\varphi_{\frac{\varepsilon}{2}}\circ\cdots\circ\varphi_{\frac{\varepsilon}{2}}}_{k \text{ times}}\circ\varphi_r & k>0 \\ \varphi_{-\frac{\varepsilon}{2}}\circ\varphi_{-\frac{\varepsilon}{2}}\circ\cdots\circ\varphi_{-\frac{\varepsilon}{2}}\circ\varphi_r & k<0 \end{cases}$

Flow-box theorem:

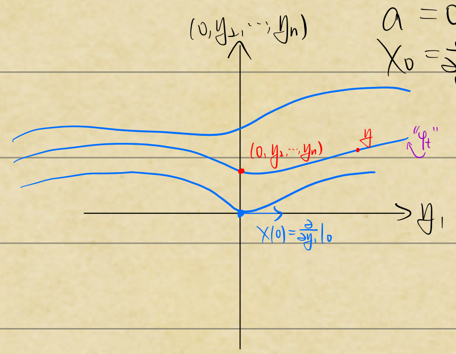
X C^∞ v.f on M, $X(a)\neq 0$, then $\exists$ coord. system at a,

$\psi: U\to \mathbb{R}^n_{(x_1,\cdots,x_n)}$, s.t. $X=\frac{\partial}{\partial x_1}$ on U, i.e., $\psi_* X = \frac{\partial}{\partial x_1}$

"choose a coord. chart"

Proof: We can assume $M=\mathbb{R}^n$ (with coord. $(y_1,\cdots,y_n)$)

$a=0$, $X_0=\frac{\partial}{\partial y_1}\Big|_0$ } 这里都是 WLOG.



Let φ_t be our flow generated by X.

Consider the map defined in the nbhd of 0 in $\mathbb{R}^n$, with values in $\mathbb{R}^n=M$

$\xi(x_1,\cdots,x_n)=\varphi_{x_1}(0,x_2,\cdots,x_n)$ WTS: ξ_* take $\frac{\partial}{\partial x_1}$ to our v.f.

Recall: Action of tangent vector of curve γ at t on a function f:

$\frac{d\gamma}{dt}(f)(t_0)=\frac{d(f\circ\gamma(t))}{dt}(t_0)=(f\circ\gamma)'(t_0)$

X_x is tangent of $t\mapsto\varphi_t(x)$ at $t=0$

$(Xf)(x)=X_x(f)=\frac{d}{dt}\Big|_{t=0}(f\circ\varphi_t)(x)=\lim_{h\to 0}\frac{f(\varphi_h(x))-f(x)}{h}$

$$\xi_{*x}\left(\frac{\partial}{\partial x_1}\Big|_x\right)(f) = \frac{\partial}{\partial x_1}\Big|_x (f\circ\xi)$$

$$= \lim_{h\to 0}\frac{1}{h}\Big(f(\xi(\underbrace{x_1+h, x_2,\dots,x_n}))-f(\xi(x_1,\dots,x_n))\Big)$$

$$= \varphi_{x_1+h}(0,x_2,\dots,x_n)$$

$$= \varphi_h(\underbrace{\varphi_{x_1}(0,x_2,\dots,x_n)}_{\to\ \xi(x)})$$

$$= \lim_{h\to 0}\frac{1}{h}\Big(f(\varphi_h(\xi(x_1,\dots,x_n)))-f(\xi(x))\Big)$$

$$= (Xf)(\xi(x)) = X_{\xi(x)}f$$

$$\Rightarrow \xi_{*x}\left(\frac{\partial}{\partial x_1}\Big|_x\right) = X_{\xi(x)} = X(\xi(x))$$

$$\Rightarrow \xi_*\left(\frac{\partial}{\partial x_1}\right) = X\circ\xi$$

Want to find: $\psi : \underset{\underset{M=\mathbb{R}^n}{\cup}}{U} \xrightarrow{\psi} \mathbb{R}^n$, s.t. $\psi_* X = \frac{\partial}{\partial x_1}$ (ξ: $\mathbb{R}^n \to U$)

So $\psi = \xi^{-1}$

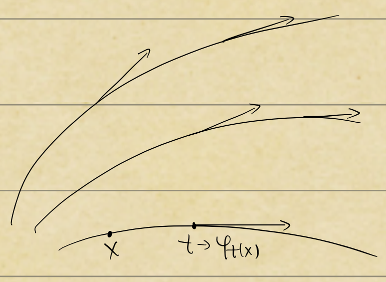
For $i\geq 1$, $\psi_{*0}\left(\frac{\partial}{\partial x_i}\Big|_0\right)(f) = \frac{\partial}{\partial x_i}\Big|_0 (f\circ\xi)$

$$= \lim_{h\to 0}\frac{1}{h}\Big(f(\xi(\underbrace{0,\dots,h,\dots,0}_{(0,0,\dots,h,0,0)}))-\overbrace{f(\xi(0))}^{f(0)}\Big)$$

$$= \lim_{h\to 0}\frac{1}{h}\Big(f(0,\dots,h,\dots,0)-f(0)\Big)$$

$$= \frac{\partial f}{\partial y_i}(0)$$

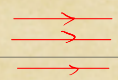
$\Rightarrow \xi_{*0}\left(\frac{\partial}{\partial x_i}\Big|_0\right) = \frac{\partial}{\partial y_i}\Big|_0 \quad \Rightarrow \xi_{*0} = id \quad \Rightarrow \xi$ is invertible by inverse fn thm.



If X generates the flow $\varphi_t(x)$

Then $(Xf)(a) = \frac{d}{dt}\Big|_{t=0}(\varphi_t^* f)(a)$

$$= \lim_{h\to 0}\frac{f(\varphi_h(a))-f(a)}{h}$$



look like this

Flow-Box thm : We can choose coords $(x_1,\dots,x_n)$ in which

$$X = \frac{\partial}{\partial x_1}$$

(1) Suppose we have 2nd v.f Y, that linearly independent of X at every point

Q: Can we find coord system, s.t. $X = \frac{\partial}{\partial x_1}$, $Y = \frac{\partial}{\partial x_2}$?

(2) What if $X_1, \cdots, X_k$, linearly independent ?

Is there analogous action of v.f.s on other objects?

$(Xf)(a) \longrightarrow (L_X f)(a)$, Lie derivative.

<u>Lie-derivative of a differential 1-form ω</u>

<u>Def</u>: $(L_X \omega)(a) = \frac{d}{dt}\Big|_{t=0} (\varphi_t^* \omega)(a)$

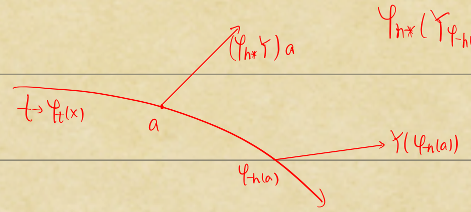
$= \lim_{h\to 0} \frac{1}{h}\left((\varphi_h^* \omega)(a) - \omega(a)\right)$ limit is in T_a^*M.

Lie derivative of vector field Y:

$(L_X Y)(a) = \lim_{h\to 0} \frac{1}{h}\left(Y(a) - (\varphi_{h*} Y)(a)\right)$

$\varphi_{h*}(Y_{\varphi_{-h}(a)})$

φ_a is a diffeo, so φ_*, φ^* make sense.



<u>Lemma</u>: $L_X Y = \lim_{h\to 0} \frac{(\varphi_h^* Y)(a) - Y(a)}{h}$

<u>Remark</u>: If $g: M \to N$ diffeo and $Y \in \mathfrak{X}(N)$, then we can define

$g^* Y \in \mathfrak{X}(M)$

$(g^* Y)(a) = (g^{-1})_* Y(g(a))$

$g^* Y = (g^{-1})_* Y$

<u>Pf</u>: RHS $= \lim_{h\to 0} \frac{Y(a) - (\varphi_h^* Y)(a)}{-h}$

$= \lim_{h\to 0} \frac{Y(a) - (\varphi_{-h}^* Y)(a)}{h}$

$= \lim_{h\to 0} \frac{Y(a) - (\varphi_{h*} Y)(a)}{h} = L_X Y$

Composite of v.f.s: $XY: \ f \longmapsto X(Yf)$

not a v.f in general e.g. $X = \frac{d}{dt}$, $X^2 = \frac{d^2}{dt^2}$ not a derivation

Lemma: Commutator $[X,Y] = XY - YX$ is derivation.

Pf: $[X,Y](fg) = ?$

① $(XY)(fg) = X(Y(fg))$

operation on functions.

$$= X(f\,Y(g) - g\,Y(f))$$

$$= f\,((XY)(g)) + X(f)\,Y(g) + g((XY)f) + X(g)\,Y(f)$$

② $(YX)(fg) = \cdots\cdots$

① − ② $\longrightarrow = f((XY - YX)(g)) + g((XY - YX)f)$

E.g. $X = \frac{\partial}{\partial x_1}$, $Y = (1+x_1^2)\frac{\partial}{\partial x_2}$ XY 不是乘起来! 是 compose!

$$[X,Y] = 2x_1\frac{\partial}{\partial x_2} + \cancel{(1+x_1^2)\frac{\partial^2}{\partial x \partial x_1}} - \cancel{(1+x_1^2)\frac{\partial^2}{\partial x \partial x_1}}$$

$$= 2x_1\frac{\partial}{\partial x_2}$$

In local coord: $X = \sum \xi_i \frac{\partial}{\partial x_i}$, $Y = \sum \eta_i \frac{\partial}{\partial x_i}$

$$X(Yf) = \sum \xi_i \frac{\partial}{\partial x_i}\left(\sum \eta_j \frac{\partial f}{\partial x_j}\right)$$

$$= \sum \xi_i \left(\frac{\partial \eta_j}{\partial x_i}\frac{\partial f}{\partial x_j} + \eta_j \frac{\partial^2 f}{\partial x_i \partial x_j}\right)$$

$$= \sum \xi_i \left(\frac{\partial \eta_j}{\partial x_i}\frac{\partial f}{\partial x_j}\right) + \sum_{ij} \xi_i \eta_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$Y(Xf) = \sum \eta_i \left(\frac{\partial \xi_j}{\partial x_i}\frac{\partial f}{\partial x_j}\right) + \sum_{ij} \xi_i \eta_j \frac{\partial^2 f}{\partial x_i \partial x_j}$$

Thm: $L_X Y = [X,Y]$

Lie derivative and Lie brackets:

Lie deriv of a v.f. Y wrt v.f X.

$$(L_X Y)(a) = \lim_{h\to 0} \frac{1}{h}\big(Y(a) - (\varphi_{h*}Y)(a)\big), \quad \varphi_t \text{ flow of } X.$$

$$= \lim_{h\to 0}\frac{1}{h}\big((\varphi_h^* Y)a - Y(a)\big)$$

If g diffeo, g_* make sense. Define g^*Y as $(g^{-1})_*Y$.

Thm: $L_X Y = [X, Y]$

We'll use Hadamard's Lemma:

$f: (-\varepsilon, \varepsilon)\times M \to \mathbb{R}$ C^∞, $f(0,x) = 0$, $\forall x$.

Then $\exists\, C^\infty$ $g: (-\varepsilon,\varepsilon)\times M \to \mathbb{R}$, s.t.

$$\begin{cases} f(t,x) = t\, g(t,x) \\ \frac{\partial f}{\partial t}(0,x) = g(0,x) \end{cases}$$

Pf: $$f(t,x) - f(0,x) = \int_0^1 \frac{\partial f(st,x)}{\partial s}\,ds$$
$$= t\underbrace{\int_0^1 \frac{\partial f}{\partial s}(st,x)\,ds}_{g(t,s)}$$

(Given f $C^\infty(M)$)

Pf: Let φ_t be the flow of X, $|t| < \varepsilon$,

$f\circ\varphi_t - f = t g_t$, $g_t(x)$ C^∞. $g_0 = \frac{\partial}{\partial t}\Big|_{t=0}\varphi_t^* f = X(f)$

☆ 第3题

$$(\varphi_{h*}Y)_a(f) = \varphi_{h*}(Y_{\varphi_{-h}(a)})f$$

$$= Y_{\varphi_{-h}(a)}(f\circ\varphi_h)$$

$$= Y_{\varphi_{-h}(a)}(f + h g_h) = Y_\varphi(f) + Y_\varphi(h g_h) \quad \text{since linear}$$

$$(L_X Y)_a(f) = \lim_{h\to 0}\frac{1}{h}\Big(\underbrace{Y(f)(a)}_{☆\, = Y(a)f} - (\varphi_{h*}Y)_a(f)\Big)$$

h提出去和$\frac{1}{h}$消掉.

$$= \lim_{h\to 0}\frac{1}{h}\big(Y(f)(a) - Y(f)(\varphi_{-h}(a))\big) - \lim_{h\to 0}\big(Y(g_h)(\varphi_{-h}(a))\big)$$

$$= L_X Y(f)(a) - Y(g_0)(a)$$

$$= X_a(Y(f)) - Y_a(X(f))$$

$$= [X, Y]f$$

So $L_X Y = -L_Y X$, $L_X X = 0$

Recall: Flow-box thm: $X_a \neq 0 \Rightarrow \exists x = (x_1, \cdots, x_n)$ s.t. $X = \frac{\partial}{\partial x_1}$

If Y, X linearly independent, is there coord sys s.t. $X = \frac{\partial}{\partial x_1}$, $Y = \frac{\partial}{\partial x_2}$.

No. $[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}] = 0$, so no hope unless $[X, Y] = 0$ (e.g. $X = \frac{\partial}{\partial x_1}$, $Y = (1+x^2)\frac{\partial}{\partial x_2}$, $[X, Y] = 2x_1\frac{\partial}{\partial x_2}$)

Remark: If $g: M \to N$ is a diffeo, then $g_*[X, Y] = [g_* X, g_* Y]$

Thm: Given $X_1, \cdots, X_k$ linearly independent v.f in nbhd of a, s.t. $[X_i, X_j] = 0$ $\forall i, j$.

Then there exists coord. chart (U, φ) around a. s.t. $X_i = \frac{\partial}{\partial x_i}$ on U.

Pf: As in proof of FBT, we can assume $M = \mathbb{R}^n_{(y_1, \cdots, y_n)}$, $a = 0$, $X_i(0) = \frac{\partial}{\partial y_i}\big|_0$ (By linear change of vectors)

Define $\varphi^{-1} = \xi(x_1, \cdots, x_n) = \varphi^1_{x_1}(\varphi^2_{x_2}(\cdots(\varphi^k_{x_k}(0, \cdots, 0, x_{k+1}, \cdots, x_n))\cdots))$

As before, we compute

$$\xi_*\left(\frac{\partial}{\partial x_i}\Big|_0\right) = \begin{cases} X_i(0) = \frac{\partial}{\partial y_i}\Big|_0 & , i = 1, \cdots, k. \\ \frac{\partial}{\partial y_i}\Big|_0 & i = k+1, \cdots, n \end{cases}$$

As before, we also see $X_1 = \frac{\partial}{\partial x_1}$

下一步：把每个φ_j换到前面. $X_iX_j = X_jX_i$允许我们这么做.

We need:

① Lemma: $g: M \to N$ diffeo, X. v.f. on M, φ_t its flow,

then $g_* X$ generates the flow $g \circ \varphi_t \circ g^{-1}$

Cor: $g: M \to M$ diffeo, then $g_* X = X \Leftrightarrow g \circ \varphi_t = \varphi_t \circ g$.

② Lemma: If $\begin{matrix} X \\ Y \end{matrix}$ generates flow $\begin{matrix} \varphi_t \\ \psi_s \end{matrix}$,

then $[X, Y] = 0$ iff $\varphi_t \circ \psi_s = \psi_s \circ \varphi_t$, $\forall s, t$.

Last Lemma shows that for each $i = 1, \cdots, k$, $\xi(x_1, \cdots, x_n) = \varphi^i_{x_i}(\varphi^1_{x_1}(\cdots))$

So $X_i = \frac{\partial}{\partial x_i}$

① Pf: $(g_* X)_b(f) = g_*(X_{g^{-1}(b)})(f)$ $\quad (Xf)(a) = \frac{d}{dt}\Big|_{t=0} (\varphi_t^* f)(a)$

$= X_{g^{-1}(b)}(f \circ g)$

$= \lim_{h\to 0} \frac{1}{h}\big(\underbrace{\varphi_h^*(f\circ g)(g^{-1}(b))}_{(f\circ g)(\varphi_h(g^{-1}(b)))} - (f\circ g)(g^{-1}(b))\big)$

$= \lim_{h\to 0} \frac{1}{h}\big(f\circ(g\circ\varphi_h\circ g^{-1})(b) - f(b)\big)$

$= \frac{d}{dt}\Big|_{t=0} (g\circ\varphi_h\circ g^{-1})^* f(b)$

② Pf: If $\varphi_t \circ \varphi_s = \varphi_s \circ \varphi_t$, $\forall s$.

then $\varphi_{t*} Y = Y$, by Cor. $g = \varphi_t$, $Y = X$.

If it holds for all t, then $L_X Y = 0 \iff [X, Y] = 0$. ✓

Suppose $[X,Y] = 0$, i.e., $\lim_{h\to 0} \frac{1}{h}(Y_b - (\varphi_{h*}Y)_b) = 0$ for all b.

Given $a \in M$, consider curve $\gamma: (-\varepsilon, \varepsilon) \longrightarrow M$, $\gamma(t) = (\varphi_{t*}Y)_a$.

$\gamma'(t) = \lim_{h\to 0} \frac{1}{h}(\gamma(t+h) - \gamma(t))$

$= \lim_{h\to 0} \frac{1}{h}\big((\varphi_{(t+h)*}Y)_a - (\varphi_{t*}Y)_a\big)$

$= \lim_{h\to 0} \frac{1}{h}\big(\underbrace{(\varphi_{t*}(\varphi_{h*}Y))_a}_{\varphi_{t*}((\varphi_{h*}Y)_{\varphi_t^{-1}(a)})} - \varphi_{t*}(Y_{\varphi_t^{-1}(a)})\big)$

$= \varphi_{t*}\big(\lim_{h\to 0} \frac{1}{h}((\varphi_{h*}Y)_{\varphi_t^{-1}(a)} - Y_{\varphi_t^{-1}(a)})\big) = 0$

So $\gamma'(t) = 0 \Rightarrow \gamma(t) = \gamma(0)$

$\Rightarrow \varphi_{t*} Y = Y$

$\Rightarrow \varphi_t \circ \varphi_s = \varphi_s \circ \varphi_t$ ✓

Frobenius integrability theorem

k-dim distribution Δ on M

$\Delta: a \longmapsto \Delta_a$, k-dim vector space of TM_a.

Δ C^∞ if every point of M has nbhd U on which there are

C^∞ vector fields $X_1, \cdots, X_k$, s.t. $X_i|_a$ form basis of Δ_a, for all $a \in U$.

k-dim submanifold N of M is an integral submanifold of Δ if $\forall a \in N$,

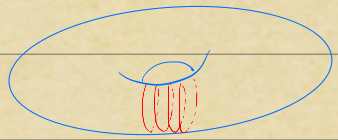
$i_{*a} TN_a = \Delta_a$, where $i: N \hookrightarrow M$.

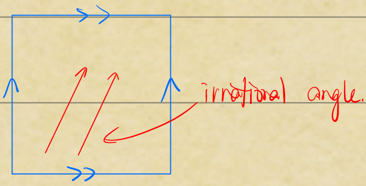
N need not exist even locally.

e.g. $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial y} + x \frac{\partial}{\partial z}$ in $\mathbb{R}^3$

But $[X, Y] = \frac{\partial}{\partial z}$, linearly inde of X, Y.

We should allow <u>immersed submanifolds</u> N as integral submanifold of Δ





If N integral manifold of distribution Δ on M, then $[X_i, X_j]_a \in TN_a$ whenever X_i's are as above (Span Δ locally)

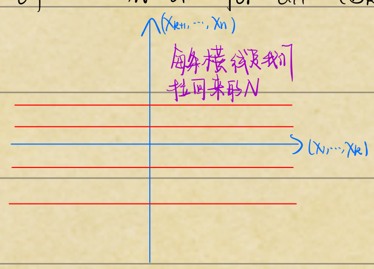
Say Δ integrable if $[X, Y]$ belongs to Δ whenever X, Y both belong to Δ

<u>Local integrability thm:</u>

Δ integrable k-dim distribution on M, Then:

(1) For all $a \in M$, there is coord. chart $U \to (-\varepsilon, \varepsilon)^n \in \mathbb{R}^n$, $\varepsilon > 0$,

s.t. $\varphi(a) = 0$, $\{x_i = b_i \mid i = k+1, \cdots n\}$ are integral submanifolds of Δ in U for all $(b_{k+1}, \cdots b_n) \in (-\varepsilon, \varepsilon)^{n-k}$.



(2) Any connected integral manifold of $\Delta|_U$ is contained in one of these leaves.

Vector fields related by a C^∞ function $f: M \to N$,

$X \in \mathfrak{X}(M)$, $Y \in \mathfrak{X}(N)$, s.t. $\overset{(*)}{f_{*a} X_a = Y_{f(a)}}$ $\forall a \in M$.

(*) Means: $\forall g \in C^\infty(N)$, $(f_{*a} X_a)(g) = Y_{f(a)}(g)$

$X_a(g \circ f) = X(g \circ f)(a) = Y(g)(f(a))$

$\Leftrightarrow X(g \circ f) = Y(g) \circ f$ for all $g \in C^\infty(N)$

Example: (1) If f is a diffeomorphism, then $\forall X \in \mathfrak{X}(M)$,

X is f-related to $f_* X$.

(2) If f is an immersion, $Y \in \mathfrak{X}(N)$, and $Y_{f(a)} \in f_{*a} TM_a$

for all $a \in M$, then $\exists!$ vector field $X \in \mathfrak{X}(M)$, s.t.

X, Y f-related (By immersion thm)

Lemma: If X_i, Y_i f-related, $i = 1, 2$, then $[X_1, X_2]$, $[Y_1, Y_2]$

are f-related.

Pf: $\forall g \in C^\infty(N)$, $Y_i(g) \circ f = X_i(g \circ f)$

$$\begin{aligned}
[Y_1, Y_2](g) \circ f &= Y_1(Y_2(g)) \circ f - Y_2(Y_1(g)) \circ f \\
&= X_1(Y_2(g) \circ f) - X_2(Y_1(g) \circ f) \\
&= X_1(X_2(g \circ f)) - X_2(X_1(g \circ f)) \\
&= [X_1, X_2](g \circ f)
\end{aligned}$$

Lemma: Suppose $X_1, \cdots, X_k$ span Δ in some neighborhood U of $a \in M$

Then Δ integrable on U $\Leftrightarrow$ $[X_i, X_j] = \sum_{l=1}^{n} c_{ij}^l X_l$, where $c_{ij}^l \in C^\infty(U)$

Pf: " $\Rightarrow$ " We have to show that if X belongs to Δ on U then

$X = \sum_{i=1}^{k} c_i X_i$ where $c_i \in C^\infty(U)$

We can complete $X_1, \cdots, X_k$ to $X_1, \cdots, X_n$ in local

~~☆~~ 没抄下来.

"$\Leftarrow$" to show, if $X, Y \in \Delta$, then $[X, Y] \in \Delta$

$X = \sum f_i X_i$, $Y = \sum g_i Y_i$, $f_i, g_i \in C^\infty$

Enough to show $[f_i X_i, g_j Y_j] \in \Delta$, $\forall i, j$.

In general, $[fX, gY] = fg\underbrace{\overset{[X_i, X_j]}{[X, Y]}}_{\in \Delta} + fX(g)\underbrace{Y}_{\in \Delta} - gY(f)\underbrace{X}_{\in \Delta}$

So LHS $\in \Delta$ if $X, Y, [X, Y] \in \Delta$.

Pf: (1) We can assume $M = \mathbb{R}^n_{(y_1, \cdots, y_n)}$, $a = 0$

$\Delta_0 \in T\mathbb{R}^n_0$, spanned by $\frac{\partial}{\partial y_i}\Big|_0$, $i = 1, 2, \cdots, k$.

$\pi = \mathbb{R}^n \to \mathbb{R}^k$ $\quad$ π_* injective on Δ_a near 0

$y \mapsto (y_1, \cdots, y_k)$ $\quad$ (by continuity)

We can choose $X_1, \cdots, X_k$ belonging to Δ near 0

s.t. $\pi_*(X_i(a)) = \frac{\partial}{\partial y_i}\Big|_{\pi(a)}$, a near 0.

Why?: We can choose $X_1, \cdots, X_k$, s.t.

$X_i = \sum_{j=1}^{k} a_{ij} \frac{\partial}{\partial y_j} + \sum_{j=k+1}^{n} b_{ij} \frac{\partial}{\partial y_j}$, where $(a_{ij}(0)) = Id$, $(b_{ij}(0)) = 0$

Define X_j' by $X_i = \sum a_{ij} X_j'$, $i = 1, 2, \cdots, k$

i.e. X_i, $\frac{\partial}{\partial y_i}$ π-related, $i = 1, \cdots, k$

$\pi_*[X_i, X_j]_a = [\frac{\partial}{\partial y_i}, \frac{\partial}{\partial y_j}]_{\pi(a)}$, so $[X_i, X_j] = 0$ since π_* injective.

By flow-box theorem, we can choose local coords $(x_1, \cdots, x_n)$

s.t. $X_i = \frac{\partial}{\partial x_i}$, $i = 1, 2, \cdots, k$.

Leaves as in (1) are integral manifolds because tangent space

spanned by $\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_k}$.

(2) Any connected integral manifold of $\Delta|_U$ is contained in one of these leaves.

Pf (2): $i: N \hookrightarrow U$

$x_j \circ i$, $j = k+1, \cdots, n$ $\quad$ WTS it's a constant

$\Leftrightarrow d(x_j \circ i) = 0$

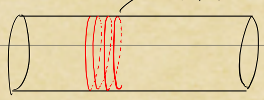
$$d(x_j \circ i)(\underbrace{X_b}_{\in TN_b}) = X_b(x_j \circ i)$$

$$= \underbrace{i_{*b} X_b}_{\in \Delta_b,\ \text{spanned by } \{\frac{\partial}{\partial x^i}\}_{i \le k}}(x_j) \quad j > k,$$

k-dim Foliation: of M: k-dim immersed submanifold N s.t. every point of M lies in some component of N and every pt has coord. nbhd U in which components of $N \cap U$ are as in (1).

Components (connected component) of N called leaves of foliation

Could look like this:

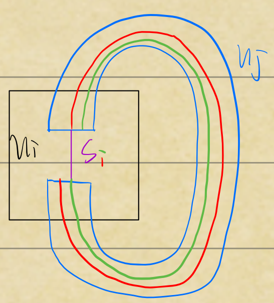


Global Integrability thm:

If Δ integrable, C^∞ k-dim distribution on M, then M is foliated by integral submanifolds of Δ.

Pf: M is covered by countably many coord. charts (U_i, φ_i) as in (1), $x^i = (x^i_1, \cdots, x^i_n)$ coords in (1).

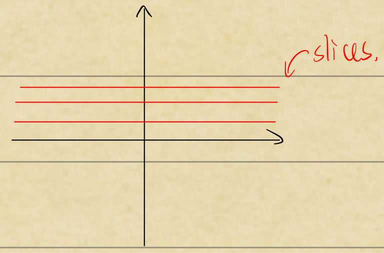
Slices (S) of U_i may intersect U_j on more than 1 slice of U_j.



$S \cap U_j$ has only countably many components, each in single slice of U_j, by (2)

So $S \cap U_j$ contained in at most countably many slices of U_j.

Given $a \in M$, choose i_0 s.t. $a \in U_{i_0}$, let S_{i_0} be slice of U_{i_0} containing a.



is joined to S_{i_0}, can joined to a.

Say slice S of U_i, for some i, is given to S_{i_0}

if $\exists\ i_0, i_1, \cdots, i_q = i$, s.t. $S_{i_j} \cap S_{i_{j+1}} \neq \emptyset$

$\Updownarrow$ $\qquad j = 0, \cdots, q-1$

slice $S_{i_0}, S_{i_1}, \cdots, S_{i_q} = S$.

For each sequence $i_0, \cdots, i_q = i$, there's only countably many chains

$S_{i_0}, \cdots, S_{i_q} = S$, but also only countably many $i_0, \cdots, i_q = i$.

So there's only countably many chains $S_{i_0}, \cdots, S_{i_q} = S$.

So there's only countably many slices joint to a.

Union of these slices is a immersed submanifold of M.

Given $b \neq a$, the corresponding union is either equals to a, or is disjoint.

M is foliated by the disjoint union of all these submanifolds.