2022.9.12 Lec1

Introduction :

General form of PDE: Look for a function $u(x_1, \cdots, x_n) : \Omega \subset \mathbb{R}^n \to \mathbb{R}$

that satisfies $F(x_1, \cdots, u, \partial_x u, \cdots \partial^2_{x_i x_j} u \cdots) = 0$

" $x_1 \cdots, x_n$ " : independent variables $u$: dependent variable.

Notation: $\frac{\partial u}{\partial x_i} = \partial x_i = u_{x_i}$

Fundemental PDE:

Transport eq: $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \quad (x, t) \in \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$

$k > 0$

Diffusion equation: $\frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0$

Wave equation : $\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$

Laplace equation eq: $\Delta u = 0$

A problem is well-posted if:

① 有解

② 有唯一解

③ Stability (初值条件的小改变导致解的小改变)

Classification

① Order of the PDE. (highest)

② Linearity. $\begin{cases} \text{homogeneous} \\ \text{non-homogeneous.} \end{cases}$

判断是否是 homogeneous 的方法：

写成 $f(u \text{ terms}) = g(x, y, \cdots)$ 看g是不是0。

Write $L[u] = f(x)$ $L$ as a operator acts on $u$.

Ex: $u_t + 3u_x = x^2$ 这里 u 该们前面的系数不重要.

$\downarrow$

$L[u] = (\partial_t + 3\partial_x) u$

we say the PDE is linear if the operator is linear.

i.e. $L[\alpha_1 u_1 + \alpha_2 u_2] = \alpha_1 L[u_1] + \alpha_2 L[u_2]$

$u_t - u_{xx} = 0$ ✓ (diffusion)

$u_t + u u_x + u_{xxx} = 0$ ✗ "$u u_x$"

Homogeneous: $f(x) \equiv 0$ / inhomogeneous.

Remark: $L[u] = f(x)$ is linear, $u_1, u_2$ 2 solutions

$\Rightarrow$ $u_1 - u_2$ is a solution of $L[u] = 0$ (associated homogeneous PDE)

③ general solution:

a) general solution

E.x of O.D.E.s: $y' = y \Rightarrow y(x) = ae^x$ ← general solution.

E.x. Find all solution $u(x,y)$ $(x,y) \in \mathbb{R}^2 \to \mathbb{R}$ constant function.

Satisfying $\frac{\partial u}{\partial x}(x,y) = 0$. $\Rightarrow u(x,y) = f(y)$

Ex. Find all solution $u(x,y)$ $(x,y) \in \mathbb{R}^2 \to \mathbb{R}$

Satisfying $\frac{\partial^2 u}{\partial x^2} = 0$ $\Rightarrow u(x,y) = x f(y) + g(y)$, $f, g$ arbitrary function.

常微分中用常数作参 偏微分中用函数作参.

$\frac{\partial^2 u}{\partial x \partial y} = 0 \Rightarrow u(x,y) = f(x) + g(y)$ ( ∀ only this solution?)

$\Leftrightarrow \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = 0 \Rightarrow \frac{\partial u}{\partial y} = f(y)$

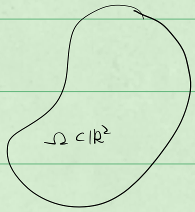
$\Rightarrow u(x,y) = \left(\int_0^y f(s)\, ds\right) + g(x)$

$= g(x) + F(y)$

若 $y(0) = 1$, 则我们令 $a = 1$.

在 PDE 中, 2 Classes of auxiliary condition

| Initial Condition | Boundary condition |
|---|---|
| Initial value prob | boundary value prob. |
| $\begin{cases} u_t - k u_{xx} = 0 & x \in \mathbb{R} \\ u(x,0) = f(x) \end{cases}$ | $u(x,y) \in \Omega \to \mathbb{R}$ <br> $\begin{cases} \Delta u = 0 \\ u/_{\partial\Omega} = f(x) \end{cases}$ |




Initial and bd value probs.

$$\begin{cases} u_t - k u_{xx} = 0 & 0 < x < l, \quad t > 0. \\ u(x,0) = f(x) \\ u(0,t) = u(l,t) = 0 & \text{(Dirichlet)} \end{cases}$$

$$\begin{cases} u_t - k u_{xx} = 0 & 0 < x < l, \quad t > 0. \\ u(x,0) = f(x) \\ \frac{\partial u}{\partial x}(0,t) = 0 & \text{(Neumann)} \\ \frac{\partial u}{\partial x}(l,t) = 0 \end{cases}$$

2nd order linear P.D.E.s of 2 variables with constant coefficient.

$a_1 u_{xx} + a_2 u_{yy} + a_3 u_{xy} + a_4 u_x + a_5 u_y + a_6 u = f(x,y)$ $a_i$ constant.

$u: (x,y) \in \Omega \subset \mathbb{R}^2 \to \mathbb{R}$

Reform linear change of variables.

$(x,y) \to (x', y')$ , $u(x,y) \to v(x', y')$

$$\begin{cases} v_{x} - v_{y'y'} = 0 & \text{parabolic} \\ v_{x'x'} + v_{y'y'} + a v_x + b v_{y'} + cv = 0 & \text{elliptic} \\ v_{x'x'} - v_{y'y'} + a v_x + b v_{y'} + cv = 0 & \text{hyperbolic.} \end{cases}$$

**Theorem 1.** By a linear transformation of the independent variables, the equation can be reduced to one of three forms, as follows.

(i) *Elliptic case:* If $a_{12}^2 < a_{11}a_{22}$, it is reducible to

$$u_{xx} + u_{yy} + \cdots = 0$$

(where $\cdots$ denotes terms of order 1 or 0).

(ii) *Hyperbolic case:* If $a_{12}^2 > a_{11}a_{22}$, it is reducible to

$$u_{xx} - u_{yy} + \cdots = 0.$$



(iii) *Parabolic case:* If $a_{12}^2 = a_{11}a_{22}$, it is reducible to

$$u_{xx} + \cdots = 0$$

(unless $a_{11} = a_{12} = a_{22} = 0$).

First order PDEs (2 variables.) and the method of characteristics.

$F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = 0$

- $a u_x + b u_y = 0$

• $a(x,y)u_x + b(x,y)u_y = 0$

• Burgers: $\partial_t u + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = 0$ / $\partial_t u + \partial_x f(u) = 0$

Conservation laws:

Remark: Surface in $\mathbb{R}^3$ $z = u(x,y)$ in $\mathbb{R}^3$

Normal to the surface $\begin{pmatrix} u_x \\ u_y \\ 1 \end{pmatrix}$

Given a PDE of first order in $\Omega = \{(x,y), y \geq 0\}$

aux--- condition $\begin{cases} F(u, x, y, u_x, u_y) = 0, & x, y \in \Omega \\ u(x,0) \text{ given.} \end{cases}$

① Find the characteristic curve (should not intersect)

在交点处会有问题.

② All character curves start from a point on the real axis.

③ Solve ODEs for curve

④ All curves cover the domain.

Notion of directional derivative.

$f(x,y)$ $\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{pmatrix}$

given a direction $\vec{v} = \begin{pmatrix} a \\ b \end{pmatrix}$,

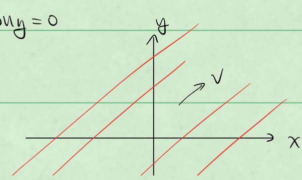
the directional derivative in the direction of $v$:

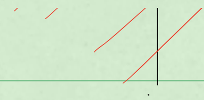
$\partial_v f = \frac{v}{\|v\|} \cdot \nabla f$

Consider the following P.D.E: $a u_x + b u_y = 0$

$u: (x,y) \in \mathbb{R}^2 \to \mathbb{R}$

$v = \begin{pmatrix} a \\ b \end{pmatrix}$ $v \cdot \nabla u = 0 \Rightarrow \partial_v u = 0$



Along each line, $u$ doesn't change. 

( but the constant is dfrt on each line)

These lines are characteristic lines.

equation of these lines: $bx-ay=c$

the solution only depends on $bx-ay$. $u(x,y)=f(bx-ay)$

Consider $\begin{cases} au_x+bu_y=0 \\ u(x,0)=x^2 \end{cases}$

$u(x,y)=f(bx-ay)$

Choose $f$ s.t. $u(x,0)=f(bx)=x^2 \Rightarrow f(x)=\frac{x^2}{b^2}$.

$\Rightarrow u(x,y)=\frac{1}{b^2}(bx-ay)^2$

to solve $au_x+bu_y=0$

analytic approach

change of variables $\begin{cases} x'=ax+by \\ y'=bx-ay. \end{cases}$

$\Rightarrow u(x,y)\to v(x',y')$ chain rule

$au_x+bu_y=(a^2+b^2)\partial_{x'}v$

$u_x=v_{x'}\frac{\partial x'}{\partial x}+v_{y'}\frac{\partial y'}{\partial x}=av_{x'}+bv_{y'}$

去重新算一下!!!

$u_y=v_{y'}\frac{\partial x'}{\partial y}+v_{y'}\frac{\partial y'}{\partial y}=bv_{x'}-av_{y'}$

$au_x+bu_y=a^2v_{x'}+abv_{y'}+b^2v_{x'}-abv_{y'}=(a^2+b^2)v_{x'}=0$

First order PDEs. (2 variables)

lit. Strauss

a) Constant coefficients
b) non-constant coefficient } linear.

a) Have seen: $au_x + bu_y = 0$

Claim: General solution: $u(x,y) = f(bx - ay)$, where $f$ is an arbitrary function of 1 variable.

Transpose equation: $u = (x,t) \to \mathbb{R}$ $\begin{cases} u_t + cu_x = 0 & x \in \mathbb{R} \\ u(x,0) = u_0(x) \end{cases}$

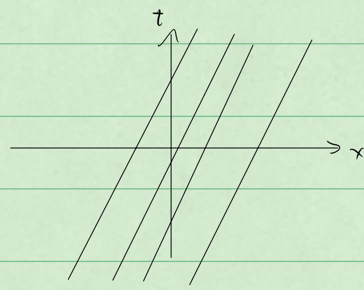
$\Rightarrow u(x,t) = f(x - ct)$

$u(x,0) = u_0(x) \Rightarrow f(x) \longrightarrow \Rightarrow f(x-ct) = u_0(x-ct) = u(x,t)$

$a(x,y)\,u_x + b(x,y)\,u_y = c_1(x,y)\,u + c_2(x,y)$

Define some curves in $(x,t)$ plane.

$\begin{cases} \dot{X}^a(t) = \dfrac{dX}{dt} = c. \\ X^a(0) = a \end{cases} \Rightarrow X^a(t) = ct + a.$




restrict $u$ to curves

$V^a(t) = u(X^a(t), t)$

这里只跟着 $\begin{cases} u_x + cu_x = 0 \\ u(x,0) = u_0 x \end{cases}$ 和

The goal of the method of characteristic is to reduce the original PDE to a family of ODE.

$\dfrac{dV^a(t)}{dt} = \left(\dfrac{\partial u}{\partial x} \cdot \underset{\substack{\| \\ c}}{\dfrac{dX^a}{dt}} + \dfrac{\partial u}{\partial t}\right)(X^a(t), t)$

$= 0$

$\Rightarrow V^a(t)$ is constant w.r.t $t$

$\Rightarrow V^a(t) = f(a)$ (与 t 无关)

$$\begin{cases} u(ct+a,t) = f(a) \\ u(a,0) = u_0(a) = f(a) \end{cases}$$

这里说明了用 chara… 的 main idea，
只不过是以常系数.

$$u(ct+a,t) = u_0(a)$$

$$\downarrow$$

$$u(x,t) = u_0(a) = u(x-ct)$$

a) Find the general solution of $u_x + y u_y = 0$ / $g(x,y)$

b) Solve $\begin{cases} u_x + y u_y = 0 \\ u(0,y) = y^2 \end{cases}$

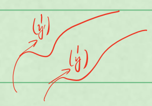
$$\begin{pmatrix} 1 \\ y \end{pmatrix} \cdot \nabla u = 0$$

$y = y(x)$ in $\mathbb{R}^2$ $\quad \begin{cases} x = x \\ y = y(x) \end{cases}$

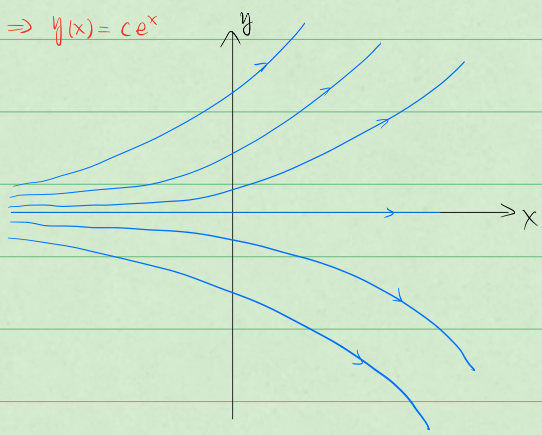
Look for a curve s.t. $\forall$ point on the curve, its tangent vector $\begin{pmatrix} 1 \\ y'(x) \end{pmatrix}$

is parallel to $\begin{pmatrix} 1 \\ y \end{pmatrix}$

$\hookrightarrow$ $y'(x) = y$




$\Rightarrow$ $y(x) = ce^x$




restrict $u$ to a characteristic curve.

$$u(x, y^{(c)}(x)) = v^{(c)}(x)$$

$$\frac{d}{dx} v^{(c)}(x) = \left( \frac{\partial u}{\partial x} + \frac{dy^c}{dx} \frac{\partial u}{\partial y} \right) (x, y^{(c)}(x))$$

$$= \left(\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)(x, y^{(c)}(x))$$

$$= 0 \,/\, g(x, y^c(x))$$

$\Rightarrow V^{(c)}(x) = f(c)$

$\Rightarrow u(x, ce^x) = f(c)$

$\Rightarrow u(x,y) = f(ye^{-x})$

2022.9.21 dec ?

① General solution of character for linear PDEs

② Wave eq in 1D.

解：

$$\begin{cases} a(x,y)\,u_x + b(x,y)\,u_y = c_1(x,y)\,u + c_2(x,y) \\ u(x,y) \text{ is given on a curve } \Gamma \text{ of } \mathbb{R}^2. \end{cases}$$

Characteristic curves in $\mathbb{R}^2$ : $\begin{cases} \dfrac{dx}{ds} = a(x(s), y(s)) & ① \\ \dfrac{dy}{ds} = b(x(s), y(s)) & ② \end{cases}$

system of 2 $1^{st}$ order ODEs.

$z(s) = u(x(s), y(s))$

$$\frac{dz}{ds} = \frac{\partial u}{\partial x}\frac{dx}{ds} + \frac{\partial u}{\partial y}\frac{dy}{ds}$$

$$= \frac{\partial u}{\partial x}(x(s),y(s))\cdot a(x(s),y(s)) + \frac{\partial u}{\partial y}(x(s),y(s))\cdot b(x(s),y(s))$$

$$= c_1(x(s),y(s))\,z(s) + c_2(x(s),y(s))$$

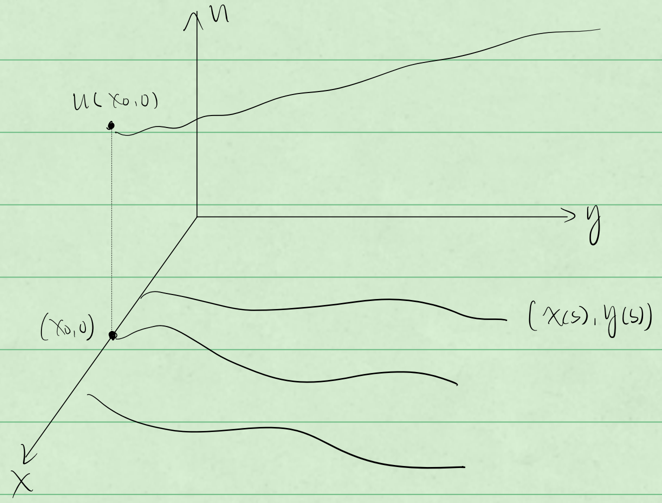
$$\Rightarrow \frac{dz}{ds} = c_1(x(s),y(s))\,z(s) + c_2(x(s),y(s)) \quad ③$$

3个式子，$x(s), y(s), z(s)$ 3个变量.

2 conditions.

Step 1: Solve ① - ② + $x(0)=x_0,\ y(0)=y_0$

Step 2: Solve the linear ODE ③ with $z(0)=u(x_0,y_0)$




Application:

① $\begin{cases} a u_x + b u_y = 0 \\ u(x,0) = x^3 \end{cases}$

② transport eq. $\begin{cases} \\ \end{cases}$

$\begin{cases} a u_x + b u_y = 0 \\ u(x,0) = x^3 \end{cases}$

$$\begin{cases} \dot{x}(s) = \dfrac{dx}{ds} = a \\ \dot{y}(s) = b \end{cases} \Rightarrow \begin{cases} x(s) = as + x_0 & ① \\ y(s) = bs + y_0. & ② \end{cases}$$

$$\frac{dz}{ds} = 0 \ ③ \Rightarrow z(s) = C \Rightarrow z(s) = u(x(s), y(s)) = C.$$

$\begin{cases} x_0 \\ y_0 \\ c \end{cases}$ 未知，只有 $u(x,0)=x^3$ 已知.

$$\Rightarrow \begin{cases} x(s) = as + x_0 \\ y(s) = as + y_0 \end{cases}$$

choose $y(0) = y_0 = 0$

$z(s) = c$ $\qquad z(0) = u(x_0, y_0) = u(x_0, 0) = x_0^3$

这里取0的意义是 characteristic curve 的起点.

Now, given $(x, y)$, find $u(x, y)$

$u(x, y) = u(as + x_0, bs)$ 在哪个 characteristic curve 上?

$$\begin{cases} x = as + x_0 \\ y = bs \end{cases} \Rightarrow s = \frac{y}{b}, \quad x_0 = \left(x - a\frac{y}{b}\right)$$

在哪个经过 $\left(\left(x - a\frac{y}{b}\right), 0\right)$ 的 characteristic curve 上

有 $z(s) = x_0^3 = u(x(s), y(s))$

又有 $u\left(\left(x - a\frac{y}{b}\right), 0\right) = \left(x - \frac{a}{b}y\right)^3$

$$u(x, y) = \left(x - \frac{a}{b}y\right)^3$$

$$u = (x, y, z, t) \longmapsto u(x, y, z, t) \in \mathbb{R}$$

$(x, y, z) \in \mathbb{R}^3, \quad t \geq 0$

$$u_t + (\vec{a} \cdot \nabla) u = f(x, y, z)$$

$$\Updownarrow$$

$$\begin{cases} u_t + a_1 \frac{\partial u}{\partial x} + a_2 \frac{\partial u}{\partial y} + a_3 \frac{\partial u}{\partial z} = f(x, y, z) \\ u(x, y, z, 0) = g(x, y, z) \end{cases}$$

$$\begin{cases} \dot{x}(t) = a_1 \\ \dot{y}(t) = a_2 \\ \dot{z}(t) = a_3 \end{cases}$$

$$w(t) = u(x(t), y(t), z(t), t)$$

$$\frac{dw}{dt} = f(x(t), y(t), z(t))$$

$$= f(a_1 t + x_0, a_2 t + y_0, a_3 t + z_0)$$

abuse notation 懒得改了

$$\Rightarrow w(t) = \int_0^t f(a_1 t + x_0, a_2 t + y_0, a_3 t + z_0)\, dt + w(0)$$

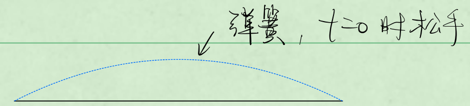
$$= \int_0^t f(a_1 t + x_0, a_2 t + y_0, a_3 t + z_0)\, dt + g(x_0, y_0, z_0)$$

$$\Rightarrow u(\overset{x}{a_1 t + x_0}, \overset{y}{a_2 t + y_0}, \overset{z}{a_3 t + z_0}, t) = \int_0^t f(a_1 s + x_0, a_2 s + y_0, a_3 s + z_0)\, ds + g(x_0, y_0, z_0)$$

$$\Rightarrow\ u(x,y,z,t)=\int_0^t f\big(x+a_1(t-s),\ y+a_2(t-s),\ z+a_3(t-s)\big)\,ds+g\big(x-a_1t,\ y-a_2t,\ z-a_3t$$

The wave equation in $\mathbb{R}$:

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=0$$




$u=(x,t)\in\mathbb{R}\times\mathbb{R}^+\to\mathbb{R}$

① General solution

② Initial conditions: $\begin{cases} u(x,0)=f(x) & \text{position}\\ \dfrac{\partial u}{\partial t}(x,0)=g(x) & \text{velocity}\end{cases}$

Solving wave equation

$$\left(\frac{\partial^2}{\partial t^2}-c^2\frac{\partial^2}{\partial x^2}\right)u=0$$

$$\Leftrightarrow\ \left(\frac{\partial}{\partial t}+c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t}-c\frac{\partial}{\partial x}\right)u=0$$

$$\Leftrightarrow\ \begin{cases}\dfrac{\partial u}{\partial t}-c\dfrac{\partial v}{\partial x}=v & ②\\[2mm] \dfrac{\partial v}{\partial t}+c\dfrac{\partial v}{\partial x}=0 & ①\end{cases}$$

①: $v(x,t)=F(x-ct)$, $F$ arbitrary

Change of variables:

$$\begin{cases}\xi=x+ct\\ \eta=x-ct\end{cases}$$

$w(\xi,\eta)=u(x,t)$

$\dfrac{\partial}{\partial t}=c\dfrac{\partial}{\partial\xi}-c\dfrac{\partial}{\partial\eta}$ $\qquad\left(\dfrac{\partial}{\partial t}\right)^2=$

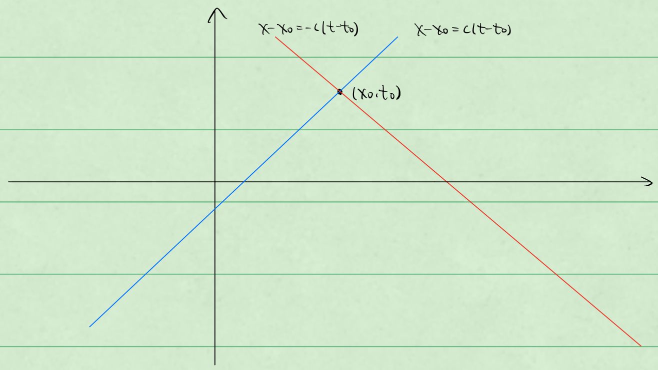
$\dfrac{\partial}{\partial x}=\dfrac{\partial}{\partial\xi}+\dfrac{\partial}{\partial\eta}$ $\qquad\left(\dfrac{\partial}{\partial x}\right)^2=$

$$\frac{\partial^2 u}{\partial t^2}-c^2\frac{\partial^2 u}{\partial x^2}=-4c^2\boxed{\frac{\partial^2 w}{\partial\xi\,\partial\eta}=0}$$

general sol: $w(\xi,\eta) = F(\xi) + G(\eta)$

Prop: The general sol of the wave equation is:

$$u(x,t) = F(x+ct) + G(x-ct)$$




$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 \\ u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

Find $F, G$ in terms of $f, g$.

$$u(x,0) = f(x) = F(x) + G(x)$$

$$u_t(x,t) = cF'(x+ct) - cG'(x-ct)$$

$$u_t(x,0) = g(x) = cF'(x) - cG'(x)$$

$$\begin{cases} f(x) = F(x) + G(x) \\ g(x) = cF'(x) - cG'(x) \end{cases}$$

$$\Rightarrow \begin{cases} f'(x) = F'(x) + G'(x) \\ g(x) = cF'(x) - cG'(x) \end{cases}$$

$$\Rightarrow \begin{cases} 2cF'(x) = cf'(x) + g(x) \\ 2cG'(x) = cf'(x) - g(x) \end{cases}$$

$$\Rightarrow \begin{cases} F(x) = \frac{1}{2}\left(f(x) + \frac{1}{c}\int_0^x g(s)\,ds\right) + A \\ G(x) = \frac{1}{2}\left(f(x) - \frac{1}{c}\int_0^x g(s)\,ds\right) + B \end{cases}$$

$$F+G = f(x) + \cancel{A+B} \quad {=0}$$

$$u(x,t) = F(x+ct) + G(x-ct)$$

$$= \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds + \cancel{A}$$

$$+ \frac{1}{2} f(x-ct) - \frac{1}{2c}\int_0^{x-ct} g(s)\,ds + \cancel{B} \quad =0$$

Prop: The solution of is:

$$u(x,t) = \frac{1}{2}\left( f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$ D'A lembert formula

2022.9.26. Lec 7

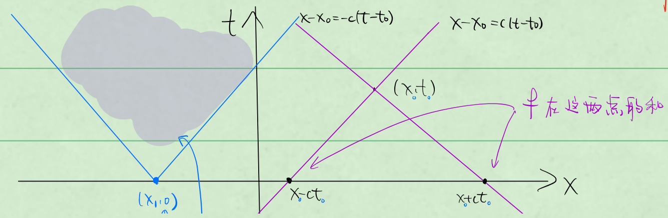
The wave equation in 1 dim

2.2

- D'A lembert formula
- Interpret { Notion of causality. / Conservation of energy.
- Diffusion equation.

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - c^2 \dfrac{\partial^2 u}{\partial x^2} = 0 & x \in \mathbb{R},\ t>0 \\ u(x,0) = f(x), & \dfrac{\partial u}{\partial t}(x,0) = g(x) \end{cases}$$

$$u(x_0,t_0) = \frac{1}{2}\left( f(x_0-ct_0) + f(x_0+ct_0)\right) + \frac{1}{2}\int_{x_0-ct_0}^{x_0+ct_0} g(s)\,ds.$$ finite speed of propagation.



g在这个区域的积分.

三维的 wave equation 的 domain of dependence 是一个 sphere.
二维 --- disk.

The domain of dependence of $u$ at point $(x_0, t_0)$ is $[x_0 - ct_0, x_0 + ct_0]$

哪些点上的解会被这个点的值影响? 这个区域上的点.

The domain of influence

$$u(x,t) = \frac{1}{2}\left(f(x-ct) + f(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

$$= \frac{\partial}{\partial t}\left(\frac{1}{2c}\int_{x-ct}^{x+ct} f(s)\,ds\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

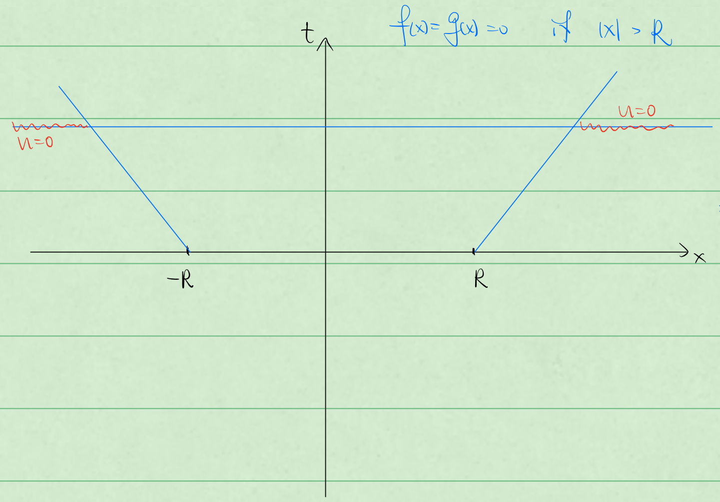
Energy: $E(t) = \frac{1}{2}\left[\int_{\mathbb{R}} |\partial_t u(x,t)|^2 dx + c^2 \int_{-\infty}^{+\infty} |\partial_x u(x,t)|^2 dx\right]$

Proposition (Conservation of energy): $E(t) = E(0)$ ← 这个东西的收敛性.

Preliminarys: Assume that $f, g$ has compact support, what can we say about the solution

↳ $f(x), g(x) = 0$ outside a finite interval.

$f(x) = g(x) = 0$ if $|x| > R$




fix $t = t_0$.

$\Rightarrow$ for each fixed $t_0$, $u(x,t)$ has a compact support i.e. $[-R-ct, R+ct]$

$\frac{\partial}{\partial t}u(x,0) = g(x)$, 条件.

同理.

Assume that $f, g$ are compactly supported. $g$ and $f, f'$ are continuous.

$$E(0) = \frac{1}{2}\left(\int_{\mathbb{R}} g(x)^2\,dx + c^2\int_{\mathbb{R}} |f'(x)|^2 dx\right)$$

Proof: 想证一个东西与自变量 t 无关 → 对 t 求导. Compute $E'(t)$

$$E'(t) = \int \partial_{tt}u\,\partial_t u\,dx + c^2\int \partial_t\partial_x u\;\partial_x u\,dx$$

因为 $\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} = 0$.

$$\frac{\partial}{\partial t}\int_a^b h(x,t)\,dx = \int_a^b \frac{\partial}{\partial t} h(x,t)$$

$$= \int c^2\frac{\partial^2 u}{\partial x^2}\,\partial_t u\,dx + c^2\int \partial_t\partial_x u\cdot\partial_x u\,dx$$

$$= c^2\int_{\mathbb{R}} \left(\partial_{xx}u\,\partial_t u + \partial_x u\,\partial_{tx}u\right)dx$$

$$= c^2\int_{\mathbb{R}} \partial_x\left(\partial_x u\,\partial_t u\right)dx$$

$$= c^2\;\partial_x u\,\partial_t u\Big|_{-\infty}^{+\infty}$$

$$= 0$$

给定那 t, u 在 ±∞ 都是 0.

Heat.

## Diffusion Equation in 1d

$$\begin{cases} \frac{\partial u}{\partial t} - k\frac{\partial^2 u}{\partial x^2} = 0 \quad x\in\mathbb{R},\ t>0. & (1) \\ u(x,0) = f(x) & (2) \end{cases}$$

$2^{nd}$ order P.D.E, Parabolic eq.

Goal: explicit formula for solution.

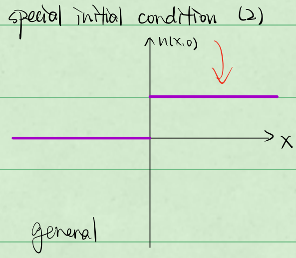
① Use fourier transform

$f(x)\ (x\in\mathbb{R}) \to \hat{f}:\ \hat{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{ikx} f(x)\,dx$

②:

Step 1: List of properties of solutions of (1)

Step 2: Solve explicitly (1) with special initial condition (2)




Step3: Solution of (1), (2) for general

$f(x)\in C'(\mathbb{R})$

Step4: Formula obtainted is valid if $f\notin C'(\mathbb{R})$

## Step 1: Properties of solutions of (1)

(a) if $u(x,t)$ sol$^n$ of (1), then let $v(x,t) = u(x-a,t)$ is also a sol$^n$

(Invariance of diffusion by translation)

(b) If $u$ is sol$^n$. $u_x$, $u_{xx}$ are also sol$^n$ of (1)

(C) If $u_1, u_2$ sol$^n$, $\alpha u_1 + \beta u_2$ sol$^n$.

(d) If $S(x,t)$ is a solution of (1), then $v(x,t) = \int_{-\infty}^{+\infty} S(x-y,t) g(y) dy = (S * g)(x)$

with arbitrary $g$ is also a function.

☆ ?

$$\frac{\partial v}{\partial t} - k\frac{\partial^2 v}{\partial x^2} = \int_{\mathbb{R}} \left(\frac{\partial S}{\partial t} - k\frac{\partial^2 S}{\partial x^2}\right)(x-y,t)\, g(y)\, dy.$$

(e) If $u(x,t)$ sol$^n$ of (1), then $v(x,t) = u(\sqrt{a}x, at)$ also a sol$^n$ of (1) for $a>0$ "Scale invariance"

## Step 2: Solve

$$\begin{cases} Q_t - kQ_{xx} = 0 \\ Q(x,0) = \begin{cases} 1 & x>0 \\ 0 & x<0 \end{cases} \end{cases}$$

Look for sol$^n$ $Q(x,t) = g\left(\frac{x}{\sqrt{t}}\right)$ "Self-similar solution"

$p = \frac{x}{\sqrt{t}}$ $Q(x,t) = g(p)$ Goal: Find $g$.

$Q_t - kQ_{xx} = 0$. $Q_t = g'(p)\left(\frac{\partial p}{\partial t}\right) = g'(p)\left(-\frac{x}{2t^{\frac{3}{2}}}\right) = g'(p)\,p\left(-\frac{1}{2t}\right)$

$Q_x = g'(p)\frac{1}{\sqrt{t}} \Rightarrow Q_{xx} = g''(p)\frac{1}{t}$

$\Rightarrow -p g'(p)\frac{1}{2t} - k\frac{1}{t}g''(p) = 0$

$\Rightarrow g''(p) + \frac{p}{2k}g'(p) = 0$

$\Rightarrow h'(p) + \frac{1}{2k}p\,h(p) = 0$ first order linear ODE

$$h' + \frac{1}{2k}ph = 0 \Rightarrow \left(h' + \frac{1}{2k}ph\right)e^{\int_0^p \frac{s}{2k}ds} = \left(h' + \frac{1}{2k}ph\right)e^{\frac{p^2}{4k}} = 0$$

$$\Rightarrow \frac{d}{dp}\left(e^{\frac{p^2}{4k}}h\right)=0$$

$$\Rightarrow h = ce^{-\frac{p^2}{4k}}$$

$$\Rightarrow g=\int h = c_1\int_0^p e^{-\frac{s^2}{4k}}ds + c_2. \qquad c_1, c_2 \text{ arbitrary.}$$

$$\Rightarrow Q(x,t) = c_1\int_0^{\frac{x}{\sqrt{t}}} e^{-\frac{s^2}{4k}}ds + c_2 \quad \text{with} \quad Q(x)=\begin{cases}1\\0\end{cases}$$

As $t$ goes to $0^+$,

$$x>0 \Rightarrow Q(x,0)=1=c_1\int_0^{\infty} e^{-\frac{s^2}{4k}}ds + c_2$$

$$x<0 \Rightarrow Q(x,0)=0=c_1\int_0^{-\infty} e^{-\frac{s^2}{4k}}ds + c_2$$

$$= -c_1\int_0^{\infty} e^{-\frac{s^2}{4k}}ds + c_2$$

$$\int_0^\infty e^{-s^2}ds = \frac{1}{2}\sqrt{\pi}$$

$$\begin{cases} 2c_2 = 1 \\ c_1 = \frac{1}{2\sqrt{\pi}} \end{cases} \rightarrow ☆.$$

$$Q(x,t) = \frac{1}{2}+\frac{1}{\sqrt{\pi}}\int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2}ds.$$

Step 3: Solving (1) with general Initial Condition $u(x,0)=f(x)$, $\lim_{x\to\pm\infty} f=0$, $f\in C^1(\mathbb{R})$

Define $S(x,t)=\frac{\partial Q}{\partial x}$ is also a sol$^n$ of (1) "(b)"

From (d), From sol$^n$ $S$, we construct a new sol$^n$

$V(x,t) = (S * f)(x)$ also a solution. This solution satisfies (1) and (2) $(V(x,0)=f(x))$

$$V(x,t)=\int_{\mathbb{R}} S(x-y,t)f(y)\,dy.$$

we need to check that $u(x,0)=f(x)$

$$(S*f)(x)=\int_{\mathbb{R}}\frac{\partial Q}{\partial x}(x-y,t)f(y)\,dy. \qquad \frac{\partial Q}{\partial x}+\frac{\partial Q}{\partial y}=0$$

$$= -\int_{\mathbb{R}}\frac{\partial Q}{\partial y}(x-y,t)f(y)\,dy \qquad \text{分部积分}$$

$$= \int_{\mathbb{R}} Q(x-y,t)\frac{\partial f}{\partial y}\,dy.$$

剩下那项是 $[Qf]_{-\infty}^{+\infty}$，$f$ 在 $\pm\infty$ 上是 0

At $t=0$, $$(S*f)(x)_{t=0} = \int_{\mathbb{R}} Q(x-y,0)\frac{\partial f}{\partial y}\,dy$$

$=0$ if $x-y<0$

$$Q(z)=\begin{cases}1 & z>0\\ 0 & z<0.\end{cases}$$

$$=\int_{-\infty}^{x}\frac{\partial f}{\partial y}\,dy = f(x)$$

We have concluded a sol$^n$ of (1)-(2) $\left( f(x)\in C^1,\ f(\pm\infty)=0 \right)$

of the form $u(x,t)=\int_{-\infty}^{+\infty} S(x-y,t)f(y)\,dy$, where $S(x,t)=\frac{\partial}{\partial x}Q=\frac{1}{\sqrt{4k\pi t}}e^{-\frac{x^2}{4kt}}$

$$\Rightarrow \quad = \frac{1}{\sqrt{4\pi kt}}\int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} f(y)\,dy$$

2022.10.3.

1) Remarks on sol$^n$ of diffusion equation on $\mathbb{R}$

2) Max/Min principle for diffusion equation.

We constructed solution to $\begin{cases} \partial_t u - k\partial_{xx}u=0 \\ u(x,0)=f(x) \end{cases}$ $k>0$.

$$u(x,t)=\frac{1}{\sqrt{4\pi kt}}\int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}} f(y)\,dy = \int_{\mathbb{R}} S(x-y,t)f(y)\,dy,$$

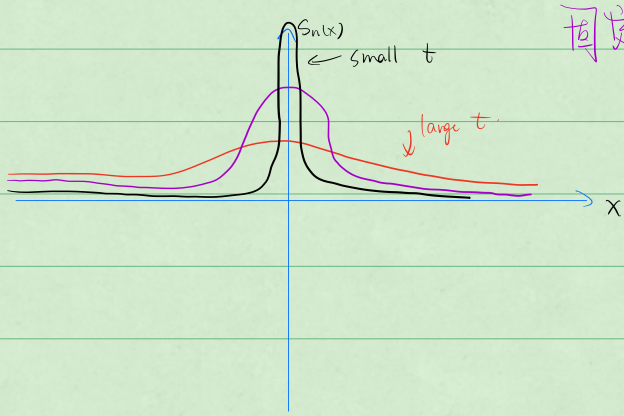
其中 $S(x,t)=\frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}$.

Properties: (1) $S(x,t)$ is $C^\infty$ $\forall x, t>0$. even function.

(2) $S(x,t)>0$, $\int_{\mathbb{R}} S(x,t)\,ds=1$

$$\int \frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}\,dx,\ \text{Let } z=\frac{x}{\sqrt{4kt}} \longrightarrow = \frac{1}{\sqrt{\pi}}\int e^{-z^2}\,dz=1$$

(3) $S_n(x)=S(x,t_n)=\frac{1}{\sqrt{4\pi kt_n}}e^{-\frac{x}{4kt_n}}$. $\qquad S(0,t)=\sqrt{\frac{1}{4\pi kt}}$




周览 n.

Fix $\delta>0$, very small.

$$\sup_{|x|>\delta} S_n(x) = \frac{1}{\sqrt{4\pi k t_n}} e^{\frac{-\delta^2}{4kt_n}} \quad \text{as } t_n \to 0^+.$$

$S_n(x)$ 在单减 取那点 $x=\delta$ 的值

$$= 0 \quad (当\ t_n \to 0\ 时)$$

取了两个极限.

(4) $S(x,t) \xrightarrow{t\to 0} \delta(x)$ (Dirac distribution).

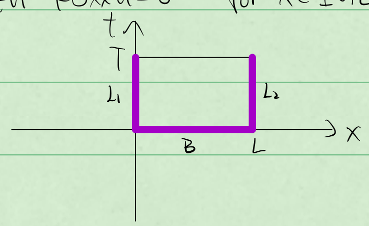
$$\begin{cases} \partial_t u - k\Delta u = 0 & x\in\mathbb{R}^n,\ t>0 \\ u(x,0) = f(x) \end{cases}$$

$$\Rightarrow u(x,t) = \int_{\mathbb{R}^n} S(x-y,t) f(y)\,dy$$

$$S(x,t) = \left(\frac{1}{\sqrt{4\pi k t}}\right)^n e^{-\frac{|x|^2}{4kt}}, \quad |x| = \sqrt{\sum x_i^2}$$

## Maximum / Min principle for sol$^n$ of diff eq.

$\partial_t u - k\partial_{xx} u = 0$ for $x\in[0,L]$ and $t=[0,T]$ (在内)




$u\in C^2$ in $x$, $C^1$ in $t$.

$R = [0,L]\times[0,T]$

The maximum and min of $u$ is obtained on the lateral sides or the bottom side of the rectangle and nowhere else. (either $x=0$ or $x=L$ or $t=0$)

We'll prove the weak version. (不加 "nowhere else")

Consider diff$^n$ eq

$$\partial_t u - \sum_{i,j=1}^n \partial_{x_i}(a_{ij}(x)\partial_{x_j} u) = 0 \quad \text{in } \Omega\subseteq\mathbb{R}^n$$

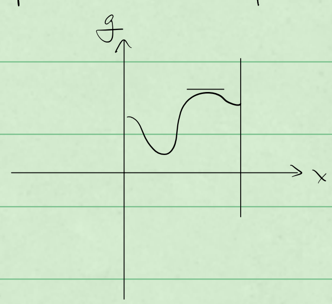
Condition on the coefficient $a_{ij}(x)$

"coexivity": $\sum^n a_{ij}(x)\xi_i\xi_j \geq c|\xi|^2 = c(\sum \xi_i^2)$ (vectors)

考虑 1维 $\partial_t u - \partial_x(a_i(x)\partial_x u) = 0$ 则 $\forall x$, $a(x)=0$

若 $a_{ij} = \begin{cases} 1 & i=j \\ 0 & i\neq j \end{cases} \Rightarrow \partial_t u - \Delta u = 0$

## Preparation for the pf




max: $g'(x) = 0$

$g''(x) \le 0$

所以若 $(x_0, t_0)$ max,

$$\Rightarrow \begin{cases} u_{xx}(x_0, t_0) \le 0 \\ u_t(x_0, t_0) = 0 \end{cases} \Rightarrow (\underbrace{\partial_t u}_{=0} - \underbrace{k\partial_{xx} u}_{\ge 0})(x_0, t_0) = 0$$

"contradiction"

但没法说明不等于，所以只是个 idea.

Let $M = \max u(x,t)$ for $(x,t) \in L_1 \cup L_2 \cup B$. ← closed.

goal: to proof $u(x,t) \le M$ for all $(x,t) \in R$.

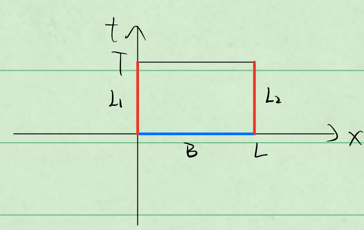
Introduce $v(x,t) = u(x,t) + \varepsilon x^2$

$\partial_t v - k\partial_{xx} v = (\partial_t u - k\partial_{xx} v) - 2k\varepsilon$.

## Thm:

$(\partial_t u - k\partial_{xx} u = 0)$

$$\begin{cases} B = \{(x,0) \mid 0 \le x \le l\} \\ L_1 = \{(0,t) \mid 0 \le t \le T\} \\ L_2 = \{(l,t) \mid 0 \le t \le T\} \end{cases}$$




- $u$ reaches its max in a closed rectangle.
- $M = \max_{L_1 \cup L_2 \cup B} u(x,t)$

Thm (weak): $\forall (x,t) \in [0,L] \cup [0,T]$, $u(x,t) \le M$.

Pf: Let $v(x,t) = u(x,t) + \varepsilon x^2$. ($\varepsilon > 0$ given)

Step 1): Prove that

$$v(x,t) \le M + \varepsilon l^2 \qquad (\forall\, x,t \in \square)$$

取 $\varepsilon \to 0$ 好像即可。

Step 2): Deduce from Step 1 that $u(x,t) \le M$

假设 1) 已知，如何证 2)?

$\Rightarrow \quad v(x,t) = u + \varepsilon x^2 \le M + \varepsilon l^2$

$\Rightarrow \quad u \le M + \varepsilon \underbrace{(l^2 - x^2)}_{\ge 0}$. 然后对 $\varepsilon$ 取极限.

Now we prove 1) 当点取在 $L_1 \cup L_2 \cup B$ 上时 1) 是显然的，只需证在内部/顶上时。

1) Inside the rectangle: $u_t - k u_{xx} = 0$

$$\Rightarrow \quad v_t - k v_{xx} = u_t - k u_{xx} - 2k\varepsilon = -2k\varepsilon < 0.$$

Assume $(x_0, t_0) \in$ inside reaches max,

then at $(x_0, t_0)$, $\underbrace{\underbrace{v_t}_{=0} - \underbrace{k v_{xx}}_{\le 0}}_{\ge 0} = \underbrace{\underbrace{-2k\varepsilon}_{<0}}_{<0}$. contradiction.

在最大值处一阶导等于零，二阶导小于零。

2) 在顶上，i.e. $(x_0, t_0)$ $x_0 \in (0, L)$, $t_0 = T$

at $(x_0, T)$, $\underbrace{v_t}_{\ge 0} - \underbrace{k v_{xx}}_{\le 0} = \underbrace{-2\varepsilon k}_{< 0}$ contradiction

因为可以在 t 增加时继续升

Q.E.D (quite easily done)

Consequences:

1) Uniqueness:

$$\text{(I)} \begin{cases} u_t - k u_{xx} = 0 \ \leftarrow f(x) \\ u(x,0) = f(x) \\ u(0,t) = g(t), \quad u(l,t) = h(t) \end{cases}$$

Theorem: The solution of (I) is unique (if exist)

Method 1: max/min principle.

Method 2: Energy method.

Pf: 设 $u_1, u_2$ 为两个 sol$^n$

则 $w = u_1 - u_2$ 为

$$\begin{cases} \partial_t u - k\partial_{xx} u = 0 \\ u(x,0) = 0 \\ u(0,T) = u(l,T) = 0 \end{cases}$$

那解，而 min, max 同时在 $L_1 \cup L_2 \cup B$ 上取到，并且边上那值

全部是零，于是 $w=0$.

Method 2: Energy method:

$$\int_0^l w(w_t - kw_{xx})\,dx = 0$$

$(ww_x)' = w_x^2 + ww_{xx}$

$\Rightarrow \quad ww_x = \int w_x^2 + \int ww_{xx}$

$$\Rightarrow \int_0^l \frac{1}{2}\frac{\partial}{\partial t}(w)^2\,dx - k\int_0^l w w_{xx}\,dx = 0$$

$w$ 在 $0, l$ 上为 $0$.

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial t}\int_0^l w^2\,dx - k\left(\left[w w_x\right]_0^l - \int_0^l w_x^2\,dx\right) = 0 \qquad \Rightarrow = 0.$$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial t}\underbrace{\int_0^l w^2(x,t)\,dx}_{:= E(t)} + \underbrace{k\int_0^l w_x^2\,dx}_{\geq 0} = 0.$$

$\Rightarrow \quad E'(t) \leq 0 \quad \Rightarrow \quad E(t) \leq E(0) = 0$, 但 $E(t) \geq 0$ (平方)

$\Rightarrow \quad E = 0 \quad \Rightarrow \quad w = 0.$

## Stability:

$$(\text{II}) \begin{cases} u_t - ku_{xx} = 0 \\ u(x,0) = f(x) \\ u(0,t) = u(l,t) = 0. \end{cases}$$

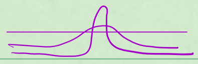
$u_1$ satisfies II with $f = f_1(x)$

$u_2$ satisfies II with $f = f_2(x)$

$\Rightarrow$ • $\int_0^l |u_1(x,t)-u_2(x,t)|^2 dx \le \int_0^l |f_1(x)-f_2(x)|^2 dx \quad \forall t \geq 0$

这个的意思是，当 $t$ 很大时，$u_1$ 与 $u_2$ 会更"靠近"？

• $\max\limits_{0\le x\le l} |u_1(x,t)-u_2(x,t)| \le \max\limits_{0\le x\le l} |f_1(x)-f_2(x)|, \quad \forall t.$

杜绝了  这种情况.

Sect 3.5 Return diffusion equsion $\underbrace{x\in\mathbb{R}^n}$.

$\partial_t u - k\partial_{xx}u = 0. \quad \underbrace{x\in\mathbb{R}^n} \to ?$

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Solution of constructed sol$^n$ of heat equation.

Thm: Let $\varphi(x)$ be a bounded continuous function for $x\in\mathbb{R}$. Let

$u(x,t)$ be defined as

$$(*) \quad u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy$$

(0) 收敛.

Then (1) $u$ satisfies $\partial_t u - k\partial_{xx}u = 0$.

(2) $u$ is infinitely differentiable $(C^\infty)$ in $x\in\mathbb{R}$, $t>0$.

smoothing property.

(3) $\lim\limits_{t\to 0^+} u(x,t) = \varphi(x)$

Prop: $f(x,y)$, $\frac{\partial}{\partial x}f(x,y)$ continuous for $x\in(a,b)$, $y\in\mathbb{R}$. Assume

that $\int_{\mathbb{R}} |f(x,y)|\,dy$ and $\int_{\mathbb{R}} |\frac{\partial}{\partial x}f(x,y)|\,dy$ uniformly convergent $\forall x\in(a,b)$

Then:

$$\frac{\partial}{\partial x}\int_{\mathbb{R}} f(x,y)\,dy = \int_{\mathbb{R}} \frac{\partial}{\partial x} f(x,y)\,dy.$$

We want to prove $\partial_t u - k\partial_{xx}u = 0$

(1) Check that integral $(*)$ is convergent.

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy$$

$$= \frac{1}{\sqrt{4\pi k t}} \int_{\mathbb{R}} e^{-\frac{y^2}{4kt}} \varphi(x-y)\,dy \qquad \text{(因为 } \int_{\mathbb{R}} f(x-y)g(y)\,dy = \int_{\mathbb{R}} f(y)g(x-y)\,dy\text{)}$$

$$= \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-p^2} \varphi(x - p\sqrt{4kt})\,dp$$ (令 $p = \frac{y}{\sqrt{4kt}}$)

$$\Rightarrow |u| = \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-p^2} |\varphi(x - p\sqrt{4kt})|\,dp$$ $\varphi$ bdd $\Rightarrow |\varphi| \leq M$

$$\leq M \frac{1}{\sqrt{\pi}} \int_{\mathbb{R}} e^{-p^2} dp = M.$$

$\Rightarrow$ 收敛.

导数们的收敛性类似，所以每个都存在.

$$\partial_t u - k\partial_{xx} u = \int_{\mathbb{R}} \underbrace{(\partial_t S - k\partial_{xx} S)}_{=0} (x-y,t)\,\varphi(y)\,dy$$

$\Downarrow$

$= 0.$

Recall: Consider $\begin{cases} \partial_t u - k\partial_{xx} u = 0 & ① \\ u(x,0) = f(x) & ② \end{cases}$

if $f$ is $C^1$, and if $f(x) \to 0$ as $x \to \pm\infty$, (比上面更强的条件)

Then the solution of ①, ② can be found: $u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} e^{-\frac{(x-y)^2}{4kt}} f(y)\,dy$.

所以要做的是试着消除这么多的限制

① $u$ given by $*$ $\quad u = c\int_{\mathbb{R}} S(x-y,t)\,\varphi(y)\,dy$

$$\Rightarrow \partial_t u - k\partial_{xx} u = \int_{\mathbb{R}} \underbrace{\left(\frac{\partial}{\partial t} - k\frac{\partial}{\partial xx}\right) S}_{=0}\, \varphi\, dy = 0.$$

? $\quad$ $S$ satisfies the heat equation for $t > 0$.

absolutely convergent，所以可以交换.

? 还没证.

$$u(x,t) = \int_{\mathbb{R}} S(x-y,t)\,\varphi(y)\,dy.$$

$$\int_{\mathbb{R}} \frac{\partial S}{\partial t}(x-y,t)\,\varphi(y)\,dy = \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} \frac{x-y}{2kt} e^{-\frac{(x-y)^2}{4kt}} \varphi(y)\,dy$$, Let $p = \frac{x-y}{\sqrt{4kt}}$.

$$= \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} \frac{\sqrt{4kt}}{\sqrt{kt}} e^{-p^2} \varphi(x - \sqrt{4kt}\,p)$$ $\quad \Rightarrow dp = -\frac{dy}{\sqrt{4kt}}$

$$= \frac{1}{\sqrt{\pi kt}} \int_{\mathbb{R}} p e^{-p^2} \varphi(x - \sqrt{4kt}\,p)\,dp$$

$$\uparrow \qquad |\quad| \leq \frac{M}{\sqrt{\pi kt}} \int_{\mathbb{R}} |p| e^{-p^2} dp$$

重复这个过程可以说明对于 $x$ $C^\infty$

下证 $\lim_{t\to 0} u(x,t) = \varphi(x)$

fix $x$, $u(x,t) - \varphi(x)$ evaluate as $t \to 0^+$

$u(x,t) - \varphi(x) = \int_{\mathbb{R}} S(x-y,t)\,\varphi(y)\,dy - 1\,\varphi(x)$

$\int_{\mathbb{R}} S(z,t)\,dz = 1$. Let $z = x-y$, $\int_{+\infty}^{-\infty} S(x-y,t)(-dy) = 1$

$\Rightarrow \int_{\mathbb{R}} S(x-y,t)\,dy = 1$.

$= \int_{\mathbb{R}} S(x-y,t)\,[\varphi(y) - \varphi(x)]\,dy$.

$= \frac{1}{\sqrt{4\pi k t}} \int e^{-\frac{(x-y)^2}{4kt}} [\varphi(y) - \varphi(x)]\,dy$. $\quad \frac{x-y}{\sqrt{4kt}} = t$.

$= \frac{1}{\sqrt{\pi}} \int e^{-p^2} [\varphi(x - \sqrt{4kt}\,p) - \varphi(x)]\,dp$.

idea: $p$ 巨大时 巨小，$p$ 不大时 巨小.

Pf: fix $x$ and $\varepsilon$. $\exists \delta > 0$, s.t.

$\max_{|x-y|\le\delta} |\varphi(x) - \varphi(y)| < \frac{\varepsilon}{2}$

$\text{Integral} = I = \underbrace{\int_{|p|<\frac{\delta}{\sqrt{4kt}}} e^{-p^2} [\varphi(x-\sqrt{4kt}\,p) - \varphi(x)]\,dp}_{I_1} + \underbrace{\int_{|p|>\frac{\delta}{\sqrt{4kt}}} e^{-p^2} [\varphi(x-\sqrt{4kt}\,p) - \varphi(x)]\,dp}_{I_2}$.

这里放大了更多

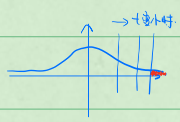
$|I_1| \le \frac{\varepsilon}{2} \underbrace{\frac{1}{\sqrt{\pi}} \int_{-\infty}^{+\infty} e^{-p^2}\,dp}_{=1} \le \frac{\varepsilon}{2}$

if $|p| \le \frac{\delta}{\sqrt{4kt}}$, then $\max |\varphi(x-\sqrt{4kt}\,p) - \varphi(x)| < \frac{\varepsilon}{2}$

$|\varphi - \varphi| \le 2M$

$|I_2| \le 2M \frac{1}{\sqrt{\pi}} \int_{|p|>\frac{\delta}{\sqrt{4kt}}} e^{-p^2}\,dp \le \frac{\varepsilon}{2}$

这里，当t足够小时，可以让积分巨小.




即得证 (Quiet Easily Done)

$\longrightarrow$ Solve wave / diff $x \in$ half plane.

Solve inhomogeneous equation : $\partial_t u - k\partial_{xx} u = f(x,t)$

Wave equation: No smoothing effect D'Alembert

$u(x,t) = \frac{1}{2}(f(x+ct)+f(x-ct))$ $(g=0)$

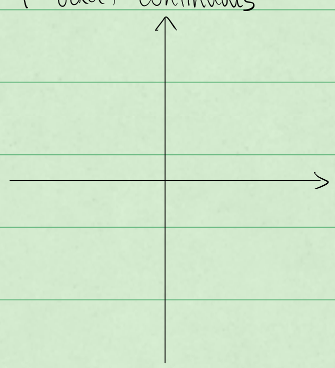
## Chp 3.1

Solving diff eq on a half-line.

$$\begin{cases} \partial_t u - k\partial_{xx} u = 0 & x,t>0 \\ u(x,0) = \varphi(x) \\ u(0,t) = 0 \end{cases}$$

(homogeneous) Dirichlet Boundary Condition.

$\varphi$ odd, continuous $-\varphi(x) = \varphi(-x)$, $\varphi(0)=0$.



若$\varphi$在$x_0$处不连续，则

$u(x,t) \to \frac{1}{2}[\varphi(x_0^+)+\varphi(x_0^-)]$

$$\varphi_{odd}(x) = \begin{cases} \varphi(x) & x>0 \\ 0 & x=0 \\ -\varphi(-x) & x<0 \end{cases}$$

$\hookrightarrow$ "odd extension of $\varphi$"

$$\begin{cases} \partial_t v - k\partial_{xx} v = 0 \\ v(x,0) = \varphi_{odd}(x) \end{cases} \Rightarrow v \text{ also odd in } x\,?$$

# Chap 3.1 Solving the diffusion eq on $\frac{1}{2}$ Real line

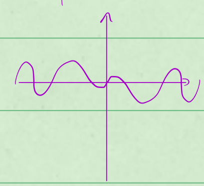
homogeneous, Dirichlet B.C.

$$
\text{(I)}\quad
\begin{cases}
\text{a)}\ \partial_t v - k\partial_{xx} v = 0 & t, x>0. \\
\text{b)}\ v(x,0) = \varphi(x) & x>0 \quad (t=0) \qquad \text{bdd. cont.} \\
\text{c)}\ v(0,t) = 0 & t>0 \quad (x=0)
\end{cases}
$$

## Method of reflection.

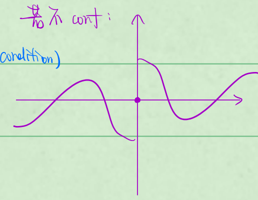
Step 1: Extend $\varphi(x)$ to the whole line (odd extension) why odd? if $f$ odd. cont. (on $\mathbb{R}$.)

to $\varphi_{odd}(x) = \begin{cases} \varphi(x) & x>0 \\ 0 & x=0 \\ -\varphi(-x) & x<0 \end{cases}$



没懂，好像跟之后算的v有关

若不 cont:



Later: same problem with dfrt B.C. (Naimem condition) ?

则我们用 even extension $\longrightarrow \frac{\partial v}{\partial x}(0,t) = 0.$

Robin B.C. $\quad c\,v(0,t) + \frac{\partial v}{\partial x}(0,t) = 0$

Step 2: Consider the problem on $\mathbb{R}$.

$$
\text{(II)}\quad
\begin{cases}
\partial_t u - k\partial_{xx} u = 0 & x\in\mathbb{R}. \\
u(x,0) = \varphi_{odd}(x) & \text{bdd, cont for } x\neq 0,
\end{cases}
$$

Step 3: Solve II:

$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y)\,\varphi_{odd}(y)\,dy.$$

$u(x,t)$ is continuous ($C^\infty$) $\forall x, t>0.$

$u(x,t)$ is a odd condition (wrt $x$) 解的奇偶性与B.C相关.

$$u(-x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} S(-x-y)\,\varphi_o(y)\,dy.$$

$\downarrow$ S even

$$= \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} S(x+y)\,\varphi_o(y)\,dy. \qquad y=-z$$

$$= \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} S(x-z,t)\,\varphi_o(-z)\,dz$$

负号与上下界交换时的负号抵消了.

$$= \frac{1}{\sqrt{4\pi kt}} \int_{\mathbb{R}} -S(x-z,t)\,\varphi_o(z)\,dz$$

$$= -u(x,t)$$

$$u(0,t) \overset{(t>0)}{=} \int_{\mathbb{R}} S\, \varphi_{odd} = 0$$

even, odd

$\lim\limits_{t\to 0^+} u(0,t) = 0$，因为若 $\varphi$ 连续，则 $\lim\limits_{t\to 0^+} u(x,t) = \varphi(x)$

平均值.

但如果 $\varphi$ 不连续，$\lim\limits_{t\to 0^+} u(x_0,t) = \frac{1}{2}\left(\lim\limits_{x\to x_0^+} \varphi(x) + \lim\limits_{x\to x_0^-} \varphi(x)\right)$

## Step 4: Find $v$.

$v(x,t)$ = separation of $u(x,t)$ to $x \geq 0$

$v(x,t) = u(x,t)$ for $x \geq 0$. 验证一下.

a) $v$ sol$^n$ of dif. eq 显然

b) $v(x,0) = \varphi(x)$, $\forall x > 0$, $t > 0$.

Because $\varphi_{odd} = \varphi$ $\forall x > 0$.

c) $v(0,t) = 0$

$\Rightarrow$ $v$ 满足 (a), (b), (c).

## Step 5: Write formula wrt $\varphi$.

$$v(x,t) = \int_{-\infty}^{0} S(x-y,t)\, \varphi_{odd}(y)\, dy + \int_0^{\infty} S(x-y,t)\, \varphi_{\cancel{odd}}(y)\, dy$$

$$\parallel$$

$$-\int_{-\infty}^{0} S(x-y,t)\, \varphi_{odd}(-y)\, dy \qquad (\varphi_{odd}(-x) = -\varphi_{odd}(x)) \qquad -y = z$$

$$\parallel$$

$$-\int_{+\infty}^{0} S(x+z,t)\, \varphi_{odd}(z)\, (-dz)$$

三个负号

$$\parallel$$

$$-\int_0^{\infty} S(x+z,t)\, \varphi(z)\, dz$$

$$v(x,t) = \int_0^{\infty} \left(S(x-y,t) - S(x+y,t)\right) \varphi(y)\, dy$$

## Wave eq:

$$\begin{cases} \partial_{tt} v - c^2 \partial_{xx} v = 0 & x,t > 0 \\ v(x,0) = f(x), \quad v_t(x,0) = g(x) & x > 0. \\ v(0,t) = 0 & t > 0 \end{cases}$$

1) Extend $f, g$ by odd $f_{odd}$, $g_{odd}$.

2) Solve $V$ with I.C. $f_{odd}$, $g_{odd}$. ($f_{odd} \leftrightarrow f_0$)

Statement: $u$ odd.

$$u(x,t) = \frac{1}{2}\left(f_0(x+ct) + f_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_0(s)\,ds.$$

restrict $u$ to $x \geq 0$. 只需考虑 $x=0$ 时的情况.

$$V(0,t) = \frac{1}{2}\left(f_0(ct) + f_0(-ct)\right) + \frac{1}{2c}\int_{-ct}^{ct} g_0 = 0 \quad \text{odd.}$$

① $f, t > 0$.

## Section 3.2. Waves on $\frac{1}{2}$ line

$$\begin{cases} V_{tt} - c^2 V_{xx} = 0 & t>0, \quad x>0 \\ V(x,0) = f(x), \quad V_t(x,0) = g(x) & x>0. \\ V(0,t) = 0, \quad t>0. & \text{(Dirichlet)} \end{cases}$$

/ ( $\frac{\partial v}{\partial x}(0,t) = 0$, Niemann condition)

Extend $f \to f_{odd}$, $g \to g_{odd}$ on $\mathbb{R}$,

Solve $u_{tt} - c^2 u_{xx} = 0$ with $f_{odd}$, $g_{odd}$,

$\Rightarrow$ find $u$ $\Rightarrow$ $V$ is the restriction of $u$.

$$u(x,t) = \frac{1}{2}\left(f_0(x+ct) + f_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_0(s)\,ds.$$

$V(x,t) = u(x,t)$ with $x>0$.

$$= \frac{1}{2}\left(f_0(x+ct) + f_0(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_0(s)\,ds.$$

$x, t > 0$, $x+ct > 0 \Rightarrow f_0(x+ct) = f(x+ct)$

if $x-ct>0$, $f_0(x-ct) = f(x-ct)$

if $x-ct<0$, $f(x-ct) = -f(ct-x)$

① $x-ct, x+ct > 0$: $u$ the same as $V$.

② $x-ct<0,\ x+ct>0.$

$$V(x,t)=\frac{1}{2}\Big(f(x+ct)-\underbrace{f(ct-x)}_{=+f_{odd}(x-ct)}\Big)+\frac{1}{2c}\int_{\underbrace{ct-x}}^{x+ct}g(s)\,ds$$

$\int_{x-ct}^{ct-x}$ 部分被抵消了. 因为 $g_{odd}$ is odd.

## Section 3.3 Diffusion with source.

$$\begin{cases}\partial_t u-k\partial_{xx}u=f(x,t) & x\in\mathbb{R},\ t>0 \quad (1.1)\\ u(x,0)=g(x) & x\in\mathbb{R}. \quad (1.2)\end{cases}$$

$f(x,t)$ given, "source" (of heat)

**Step 0:** $\begin{cases}y'+ay=f(t)\\ y(0)=y_0\end{cases}$

① 特解 + 通解

② integrating factor (同乘 $e^{at}$)

$\Rightarrow\ e^{-at}\frac{d}{dt}(e^{at}y)=f(t)$

$\Rightarrow\ e^{at}y-y_0=\int_0^t e^{as}f(s)\,ds.$

$\Rightarrow\ y=e^{-at}\left(\int_0^t e^{as}f(s)\,ds+y_0\right)$

$$y(t)=e^{-at}y_0+\int_0^t e^{-a(t-s)}f(s)\,ds.$$

$$=\mathcal{J}(t)y_0+\int_0^t\mathcal{J}(t-s)f(s)\,ds.$$

$\mathcal{J}(t)y_0$ 为 homogeneous 的通解

**Step 1:** Guessing

$\mathcal{J}(t)$ 是一个 operator, s.t. $(\mathcal{J}(t)g)(x)=(S*g)(x)$

$$u(x,t)=\mathcal{J}(t)g(x)+\int_0^t\mathcal{J}(t-s)f(s)\,ds.$$

$$=\int_{\mathbb{R}}S(x-y,t)g(y)\,dy+\int_0^t\int_{\mathbb{R}}S(x-y,t-s)f(y,s)\,dy\,ds.$$

**Step 2** Check (1.1), (1.2).

(1.2) $t=0$, $\int_0^t\int_{\mathbb{R}}S(x-y,t-s)f(y,s)\,dy\,ds=0$

$$\lim_{t\to 0}\int_{\mathbb{R}} S(x-y,t)\,g(y)\,dy = g(x) \quad \text{(why?)}$$

$$(1.1) = \partial_t u - k\partial_{xx}u = \int_{\mathbb{R}} (\partial_t S - k\partial_{xx}S)(x-y,t)\,g(y)\,dy.$$

$$+\underbrace{(\partial_t - k\partial_{xx})\left(\int_0^t\int_{\mathbb{R}} S(x-y,t-s)\,f(y,s)\,dy\,ds.\right)}$$

$$= \int_0^t \underbrace{\int_{\mathbb{R}} (\partial_t S - k\partial_{xx}S)(x-y,t-s)\,f(y,s)\,dy}_{=0}\,ds.$$

chain rule, $\frac{d}{dt}\int_0^t g(s)ds = g(t)$

咋导的?

$$+\underbrace{\lim_{\varepsilon\to 0}\int_{\mathbb{R}} S(x-y,\varepsilon)\,f(y,t)\,dy}_{=f(x,t)}$$

## Inhomogeneous wave eq.

$$\begin{cases} \partial_{tt}u - c^2\partial_{xx}u = f(x,t) \\ u(x,0) = \varphi(x), \quad u_t(x,0) = \psi(x) \end{cases}$$

① solve $\begin{cases} \partial_{tt}u_1 - c^2\partial_{xx}u_1 = f(x,t) \\ u_1(x,0) = u_{1t}(x,0) = 0 \end{cases}$

② Solve $\begin{cases} \partial_{tt}u_2 - c^2\partial_{xx}u_2 = 0 \\ u_2(x,0) = \varphi(x), \quad u_{2t}(x,0) = \psi(x) \end{cases}$

$\Rightarrow u = u_1 + u_2$. $u_2$解过了,考虑$u_1$.

$$\begin{cases} \partial_{tt}u - c^2\partial_{xx}u = f(x,t) \\ u(x,0) = u_t(x,0) = 0 \end{cases}$$

3 methods

we use: Factorization of the wave eq.

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u}_{v} = f(x,t)$$

$$\Longleftrightarrow \begin{cases} \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)v = f & ; \quad v(x,0) = \underbrace{u_t(x,0)}_{=0} - \underbrace{c u_x(x,0)}_{=0,\ \text{因为 } u(x,0)=0} = 0 \\ \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = v & ; \quad u(x,0) = 0 \end{cases}$$

转换成了两个 1st order Inhomogeneous eq.

• Solving $\begin{cases} v_t + c v_x = f(x,t) \end{cases}$

$v(x,0)=0.$

char. line: $\begin{cases} \dfrac{dx}{dt} = c \\ x(0) = a \end{cases} \Rightarrow x^{(a)}(t) = a+ct$

$\Rightarrow w^{(a)}(t) = v(x^{(a)}(t), t)$

$\Rightarrow \dfrac{dw^{(a)}}{dt} = \dfrac{\partial v}{\partial x}\underbrace{\dfrac{dx}{dt}}_{=c} + \dfrac{\partial v}{\partial t}\dfrac{dt}{dt} = f(x^{(a)}(t), t)$

积分

$\Rightarrow w^{(a)}(t) - w^{(a)}(0) = v(x^{(a)}(t),t) - v(x^{(a)}(0),0) = \int_0^t f(x^{(a)}(s),s)\,ds.$

$v(a+ct,t) - \underbrace{v(a,0)}_{=0} = \int_0^t f(a+cs,s)\,ds.$

$\Rightarrow v(a+ct,t) = \int_0^t f(a+cs,s)\,ds$

$\Rightarrow v(x,t) = \int_0^t f(x-ct+cs, s)\,ds$   令 $x=a+ct$ 后换元

只是换了notation.

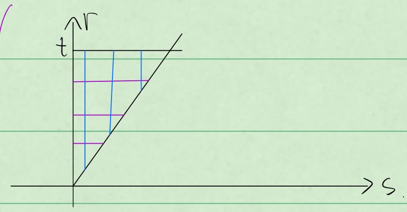
$\Rightarrow u(x,t) = \int_0^t v(x+ct-cr, r)\,dr$   把所有 $c$ 换成 $-c$，$f$ 换成 $v$.

$\Rightarrow v(y,r) = \int_0^r f(y-cr+cs, s)\,ds$

$\Rightarrow u(x,t) = \int_0^t \int_0^r f(x+ct-cr-cr+cs, s)\,ds\,dr$

$= \int_0^t \int_0^r f(x+ct-2cr+cs, s)\,ds\,dr.$

交换积分顺序



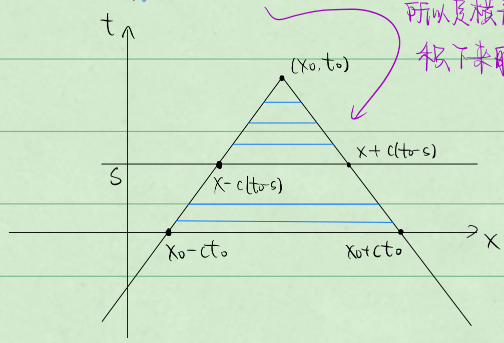

$= \int_0^t \int_s^t f(x+ct-2cr+cs, s)\,dr\,ds$

$p = x+ct-2cr+cs \Rightarrow dp = -2c\,dr \Rightarrow \begin{cases} r=s \Rightarrow p = x+ct-cs \\ r=t \Rightarrow p = x-ct+cs \end{cases}$

$= -\dfrac{1}{2c}\int_0^t \int_{x+c(t-s)}^{x-c(t-s)} f(p,s)\,dp\,ds$

$= \dfrac{1}{2c}\int_0^{t_0} \int_{x-c(t_0-s)}^{x+c(t_0-s)} f(p,s)\,dp\,ds$

所以是横着一条条积下来的.

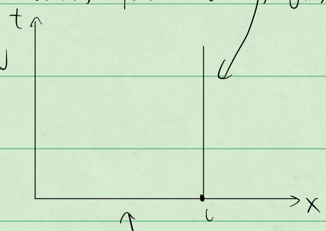


## Diff / Wave eq on an interval.

(1) $\begin{cases} \partial_{tt} u - c^2 \partial_{xx} u = 0 & 0 < x < l. \quad t > 0 \\ u(0,t) = u(l,t) = 0 & \text{Dirichlet} \quad t > 0 \\ u(x,0) = f(x) \quad \partial_t u(x,0) = g(x) & 0 < x < l. \end{cases}$

(2)

(3)



Eigenvalue problems.

matrix $A v = \lambda v$ (linear algebra)

operator ☆ $\begin{cases} A \varphi_{fun} = \lambda \varphi \\ \text{boundary cond.} \end{cases}$ O.D.E.

Fourier Series (Harmonic analysis)

Method of separation of variables.

① Look for solutions $u(x,t)$ for (1). (2) in the form of

$u(x,t) = X(x)\,T(t)$

② Satisfies (3)

①: Substitude into (1)

$$X(x)\,T''(t) - c^2 X''(x)\,T(t) = 0$$

$$\Rightarrow \quad c^2 \frac{T''(t)}{T(t)} = \frac{X''(x)}{X(x)} = \underset{\sim\sim\sim}{\text{constant}} = -\lambda$$

因为所有的 x, t, 都成立

$$\Rightarrow \begin{cases} T''(t) + c^2 \lambda T(t) = 0 \\ X''(x) + \lambda X(x) = 0 \end{cases}$$

→ satisfies (1)

$0 < x < l$

Condition (2) $\Rightarrow \begin{cases} u(0,t) = X(0)\,T(t) = 0 & \Rightarrow X(0) = 0 \\ u(l,t) = X(l)\,T(t) = 0 & \Rightarrow X(l) = 0 \end{cases}$

(EVP) $\begin{cases} T''(t) + c^2 \lambda T(t) = 0 \\ X''(x) + \lambda X(x) = 0 \\ X(0) = X(l) = 0 \end{cases}$

Look for non-trivial solution.

Consider $\begin{cases} X''+X=0 \\ X(0)=X(1)=0 \end{cases}$

$\Rightarrow X(x)=A\cos(x)+B\sin(x)$

$\begin{cases} X(0)=A=0 \\ X(1)=B\sin(1)=0 \Rightarrow B=0 \end{cases}$

The only solution is trivial solution.

$\Rightarrow$ Find $\lambda$ carefully!

Eigenvalue function, $(X,\lambda)$ eigenvalue/function are unknown

Solving the eigenvalue function. $\rightarrow$ w.r.t $L=-\frac{\partial^2}{\partial x^2}$, $LX=\lambda X$.

① Is $\lambda=0$ a eigenvalue? No. $X''=0 \Rightarrow X=Ax+B \Rightarrow A=B=0$.

② Is $\lambda<0$ a eigenvalue? $\lambda=-\alpha^2$ No!

$\begin{cases} X''-\alpha^2X=0 \\ X(0)=X(l)=0. \end{cases} \Rightarrow X(x)=Ae^{\alpha x}+Be^{-\alpha x}$

$\left.\begin{cases} X(0)=A+B=0 \\ X(l)=Ae^{\alpha l}+Be^{-\alpha l}=0 \end{cases}\right\} \Rightarrow A=B=0$

③ Are there $\lambda>0$ eigenvalue? $\lambda=\alpha^2$

$\begin{cases} X''+\alpha^2X=0 \\ X(0)=X(l)=0 \end{cases}$

$X(x)=A\cos(\alpha x)+B\sin(\alpha x)$

$X(0)=A=0$

$X(l)=B\sin(\alpha l)$

$\sin(\alpha l) \Rightarrow \alpha l=n\pi \Rightarrow \alpha=\frac{n\pi}{l}$ $n\in\mathbb{Z}^+$ (不能等于0)

$\Rightarrow$ We found $\infty$ # of e.v and e.f of the form

$\begin{cases} \lambda_n=\frac{n^2\pi^2}{l^2} \\ X_n(x)=B\sin\left(\frac{n\pi x}{l}\right) \end{cases}$

any constant

For each $\lambda_n$, Solve $T_n''(t) + c^2\lambda_n T_n(t) = 0$

$$\Rightarrow T_n(t) = A\cos\left(\frac{cn\pi t}{L}\right) + B\sin\left(\frac{cn\pi t}{L}\right)$$

$\Rightarrow$ We found sol$^n$ of (1)-(2)

$$U_n(x,t) = \sin\left(\frac{n\pi x}{L}\right)\left(A\cos\left(\frac{cn\pi t}{L}\right) + B\sin\left(\frac{cn\pi t}{L}\right)\right) \quad n=1,2,3,\cdots$$

Any finite linear combination of $U_n$ is the solution of (1)-(2)

Is it possible to find coefficients $\{A_n, B_n\}_{n=1}^{N}$

s.t. $\begin{cases} \sum_{n=1}^{N} U_n(x,0) = f(x) \Rightarrow f(x) = \sum_{n=1}^{N} A_n \sin\left(\frac{n\pi x}{L}\right) \\ \sum_{n=1}^{N} \frac{\partial}{\partial t} U_n(x,0) = g(x) \Rightarrow g(x) = \sum_{n=1}^{N} B_n\left(\frac{cn\pi}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \end{cases}$

No if $N$ finite. Yes if $N$ infinite.

Wave on an interval

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(0,t) = u(L,t) = 0 \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases}$$

$$\Rightarrow u(x,t) = \sum\left(A_n\cos\left(\frac{n\pi c}{L}t\right) + B_n\sin\left(\frac{n\pi c}{L}t\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

Question: find $A_n$, $B_n$ to satisfies the I.C.

$f(x) = \sum A_n \sin\frac{n\pi x}{L}$ $\leftarrow$ "Convergence of Fourier Series"

We will show: $A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$

Thm: (Pointwise convergence)

Assume $f(x): [0,L] \to \mathbb{R}$, continuous, $f'(x)$ is piecewise continuous.

Define $A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$. Then we have: sine-Fourier series.

$$\forall x \in (0,L), \quad \sum_{n=1}^{N} A_n \sin\frac{n\pi x}{L} \longrightarrow f(x) \quad \text{as } N \to \infty$$

在 $x=0/x=l$ 时为 0

Diffusion:

$$\begin{cases} \partial_t u - k\partial_{xx} u = 0 \\ u(0,t) = u(l,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

$\Rightarrow \begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$ , $\quad T' - k\lambda T = 0$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{l}\right)^2$ , $X_n(x) = \sin\left(\frac{n\pi}{l}x\right)$ $\quad \Rightarrow T_n(t) = e^{-k\left(\frac{n\pi}{l}\right)^2 t}$

$u_n(x,t) = X_n(x)T_n(t)$ , "Sine - Fourier series"

$u(x,t) = \sum^{\infty} A_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} \sin\left(\frac{n\pi}{l}x\right)$

## Diffusion with Neumann B.C.

$$\begin{cases} (1)\ \partial_t u - k\partial_{xx} u = 0 \\ (2)\ \partial_x u(0,t) = \partial_x u(l,t) = 0 \\ (3)\ u(x,0) = f(x) \end{cases}$$

$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = X'(l) = 0 \end{cases}$ $\quad \partial_x u(0,t) = X'(0)T(t)$

(a) Is $\lambda$ an eigenvalue ? Yes!

$X'' = 0 \Rightarrow X(x) = Ax + B \Rightarrow A = 0$ $\quad X(x) = B$

(2) Are there $\lambda_n < 0$ e.v? $\quad \lambda_n = -\alpha^2$

$X_n(x) = Ae^{\alpha_n x} + Be^{-\alpha_n x}$ 代入后 $a=b=0$.

(3) $\lambda_n > 0$ ?

$\{ X'' + \alpha^2 X = 0$

$X'(0) = X'(l) = 0$

$\Rightarrow X(x) = A\cos\alpha x + B\sin\alpha x$

$X'(x) = \alpha(-A\sin\alpha x + B\cos\alpha x)$

$X'(0) = \alpha B = 0 \Rightarrow B = 0$

$X'(l) = -A\sin(\alpha l) = 0 \Rightarrow \sin(\alpha l) = 0 \Rightarrow \alpha = \frac{n\pi}{l} \quad n = 1, 2, \cdots$

$\Rightarrow$ Eigenvalues: $\lambda_0 = 0, \quad \lambda_n = \left(\frac{n\pi}{l}\right)^2$

$\downarrow \qquad \downarrow$

$X_0 = B \quad X_n = \cos\left(\frac{n\pi x}{l}\right)$

For each $\lambda_n$, $T_n' + k\lambda_n T_n = 0$

$$\begin{cases} n = 0, \quad T_0' = 0, \quad \lambda_0 = 0, \Rightarrow T_0(t) = \text{constant} \\ n = 1, 2, \cdots : \quad T_n' + k\left(\frac{n\pi}{l}\right)^2 T_n = 0 \end{cases}$$

$\Rightarrow T_n(t) = e^{-k\left(\frac{n\pi}{l}\right)^2 t}$

$Sol^n$ for (1)-(2):

$X_n(x)\,T_n(t) \qquad n = 0, 1, 2, \cdots$

$U(x,t) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{l}\right)^2 t} \cos\left(\frac{n\pi x}{l}\right)$

$U(x,0) = f(x) = \frac{B_0}{2} + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi x}{l}\right)$ 除以2是为了对上这里的B0.

Cosine-Fourier series: $B_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

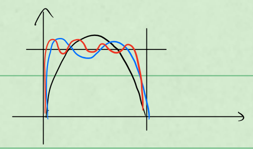
## Chapter 4:

$A_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx \qquad B_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

if $l = \pi$, $\quad A_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)\,dx, \quad B_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos(nx)\,dx$

$f(x) = 1 = \lim_{N\to\infty}\left(A_1\sin(x) + \cdots + A_N\sin(Nx)\right)$

$$A_n = \frac{2}{\pi}\int_0^{\pi}\sin(nx)\,dx = -\frac{2}{\pi}\left[\cos nx\right]_0^{\pi} = \begin{cases} 0 & n \text{ even} \\ \frac{4}{n\pi} & n \text{ odd.} \end{cases}$$



Eigenvalue pb. with Robin B.C.

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < l \\ \begin{cases} X'(0) - a_0 X(0) = 0, \\ X'(l) + a_l X(l) = 0 \end{cases} & a_0,\ a_l \text{ constant.} \end{cases}$$

$$\begin{cases} u_t - k u_{xx} = 0. \\ \partial_x u(0,t) - a_0 u(0,t) = 0 \\ \partial_x u(l,t) + a_l u(l,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

a) $\lambda = 0$?

$X'' = 0$, $X(x) = Ax + B$. $X' = A$

$$\begin{cases} A - a_0 B = 0 \\ A + a_l(Al + B) = 0 \end{cases} \Rightarrow \begin{cases} A - a_0 B = 0 \\ (l + a_l)A + a_l B = 0 \end{cases} \qquad A = a_0 B$$

$$\Rightarrow ((1 + a_l l)a_0 + a_l)B = 0.$$

Yes, if $(1 + a_l l)a_0 + a_l = 0$, then $X = a_0 Bx + B$.

$\hookrightarrow a_0 + a_l = -a_0 a_l l.$

No, if $(1 + a_l l)a_0 + a_l \neq 0$

b) $\lambda = \beta^2 > 0$?

$X(x) = A\cos(\beta x) + B\sin(\beta x)$

$X'(x) = -\beta A \sin(\beta x) + \beta B\cos(\beta x)$

$$\Rightarrow \begin{cases} X'(0) - a_0 X(0) = \beta B - a_0 A = 0 \\ X'(l) + a_l X(l) = -\beta A\sin(\beta l) + \beta B\cos(\beta l) + a_l(A\cos(\beta l) + B\sin(\beta l)) = 0 \end{cases}$$

A, B的线性组合.

$$\Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix} = 0 \quad \text{for non-trivial } \begin{pmatrix} A \\ B \end{pmatrix}$$

$\det = 0$

↳是一个关于β的式子。解出满足要求的β.

$$\begin{cases} (-a_0)A + (\beta)B = 0 \\ (-\beta\sin(\beta l) + a_l\cos(\beta l))A + (\beta\cos(\beta l) + a_l\sin(\beta l))B = 0 \end{cases}$$

$$\Rightarrow a_0\beta\cos\beta l + a_0 a_1 \sin\beta l - \beta^2\sin(\beta l) + \beta a_1\cos(\beta l) = 0$$

$$\Rightarrow a_0\beta + a_0 a_1 \tan\beta l - \beta^2\tan\beta l + \beta a_1 = 0$$

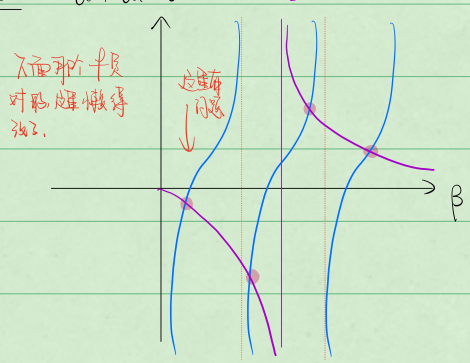
$$\Rightarrow (a_0+a_1)\beta = (\beta^2 - a_0 a_1)\tan\beta l.$$

$$\Rightarrow \tan(\beta l) = \frac{(a_0+a_1)\beta}{\beta^2 - a_0 a_1}.$$

$$\Rightarrow \begin{cases} f(\beta) = \tan(\beta l) & \beta > 0 \\ g(\beta) = \dfrac{(a_0+a_1)\beta}{\beta^2 - a_0 a_1}. \end{cases}$$

Draw the curves and find their intersection

<u>Case 1</u>: $a_0, a_1 > 0$




Infinitely many intersections.

$$\Rightarrow \lambda_n = \beta_n^2 \quad \text{交点}$$

$$\lim_{n\to\infty} \lambda_n = 0. \quad \left(\frac{(2n+1)\pi}{2l}\right)^2 < \lambda_n < \left(\frac{(2n+3)\pi}{2l}\right)^2.$$

Robin B.C.

$$\begin{cases} X'' + \lambda X = 0 & 0 < x < l \\ X'(0) - a_0 X(0) = 0 & X'(l) + a_1 X(l) = 0 \end{cases}$$

(1) e.v $= 0$ 必须有 $a_0 + a_1 = -a_0 a_1 l$

(2) e.v $\lambda = \beta^2$ $X(x) = A\cos(\beta x) + B\sin(\beta x)$

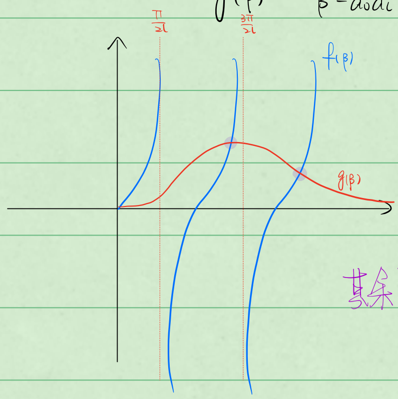
这里有问题.

for each $\beta_n$, $X_n(x) = \cos(\beta_n x) + \frac{a_0}{\beta_n}\sin(\beta_n x)$, $\lambda_n = \beta_n^2$

To exclude the trivial solution，令行列式为零。

反之最后得出 $\tan(\beta l) = \dfrac{(a_0+a_1)\beta}{\beta^2 - a_0 a_1}.$

case (2): $\begin{cases} f(\beta) = \tan(\beta l) \\ g(\beta) = \dfrac{(a_0+a_1)\beta}{\beta^2 - a_0 a_1}. \end{cases}$ $\quad a_0 < 0, \quad a_1 > 0.$




其余同理，总会有无穷个交点。

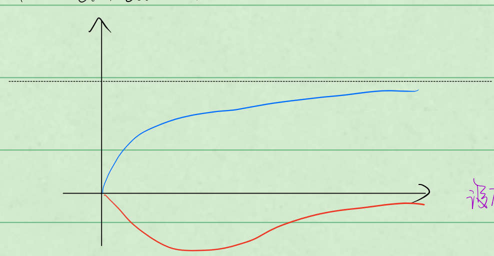
C) $\lambda = -\gamma^2 < 0$

$\Rightarrow X(x) = A\cosh(\gamma x) + B\sinh(\gamma x)$

substitute the boundary condition and get a system like $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

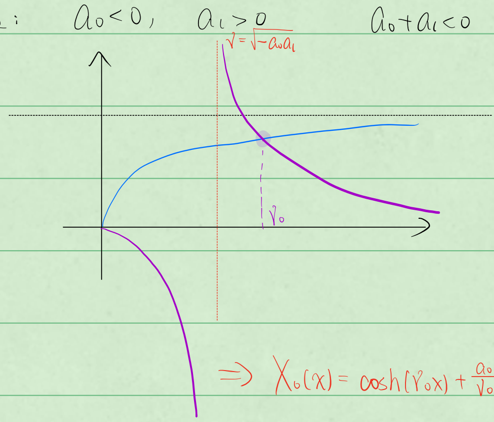
$\Rightarrow \tanh(\gamma x) = -\dfrac{(a_0+a_1)\gamma}{\gamma^2 + a_0 a_1}.$

Case 1: $a_0, a_1 > 0.$



没有交点 → 无解.

Case 2: $a_0 < 0, \quad a_1 > 0 \qquad a_0 + a_1 < 0$




$\Rightarrow X_0(x) = \cosh(\gamma_0 x) + \frac{a_0}{\gamma_0}\sinh(\gamma_0 x)$

$$\begin{cases} u_t - k u_{xx} = 0 \\ u(x,0) = \varphi(x) \\ u(0,t) - a_0 u(0,t) = 0, \quad u(l,t) + a_1 u(l,t) = 0 \end{cases} \qquad \begin{cases} a_0 < 0, \ a_1 > 0 \\ a_0 + a_1 < 0 \end{cases}$$

Solution:

$$u(x,t) = A_0 e^{\gamma_0^2 kt}\left(\cosh(\gamma_0 x) + \frac{a_0}{\gamma_0}\sinh(\gamma_0 x)\right) + \sum_{n=1}^{\infty} A_n e^{-\beta_n^2 kt}\left(\cos(\beta_n x) + \frac{a_0}{\beta_n}\sin(\beta_n x)\right)$$

Initial Condition:

$$u(x,0) = \varphi(x) = A_0\left(\cosh(\gamma_0 x) + \frac{a_0}{\gamma_0}\sinh(\gamma_0 x)\right) + \sum_{n=1}^{\infty} A_n\left(\cos(\beta_n x) + \frac{a_0}{\beta_n}\sin(\beta_n x)\right)$$

Boundary Condition:

- Dirichlet:

  $u(0,t) = u(l,t) = 0$

- Neumann

  $\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(l,t) = 0$

- Mixed

  Dir at $x=0$, Neu at $x=l$

- Robin

- periodic $\begin{cases} u_t - k u_{xx} = 0 & -l < x < l \\ u(-l,t) = u(l,t) \\ \frac{\partial u}{\partial x}(-l,t) = \frac{\partial u}{\partial x}(l,t) \end{cases}$

How to compute coefficient in a fourier series?

$f(x) = \sum A_n \sin\left(\frac{n\pi x}{l}\right)$

Lemma: Assume $\{X_n, \lambda_n\}_1^\infty$ are e.f and eigenvalues of $\begin{cases} X'' + \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$

$\Rightarrow$ If $\lambda_m \neq \lambda_n$, then $\int_0^l X_m(x) X_n(x)\,dx = 0$

Pf 1: 代入 嗯算积分. $\int_0^l \sin\frac{n\pi x}{l} \sin\frac{m\pi x}{l}\,dx.$

Pf 2: directly from the system.

## Fourier Series. (5.11–5.13)

- Orthogorality (scaler product in $L^2$)

$$L^2(a,b) = \{ f : (a,b) \to \mathbb{R}/\mathbb{C} \ \text{ s.t. } \|f\|_{L^2}^2 = \int_a^b |f(x)|^2 dx \}$$

$$(f, g) = \int_a^b f(x)\overline{g(x)}\,dx.$$

$$\|f\|_{L^2}^2 = \int_a^b |f(x)|^2 dx$$

Lemma: Consider $\begin{cases} X'' + \lambda X = 0 & (0, L) \\ X(0) = X(L) = 0 \end{cases}$

2 e.f $X_m(x)$, $X_n(x)$ with $\lambda_m \neq \lambda_n$ are orthogonal.

Pf: ① $X_m'' + \lambda_m X_m = 0 \qquad X_m(L) = X_m(0) = 0$

② $X_n'' + \lambda_n X_n = 0 \qquad X_n(L) = X_n(0) = 0$

Goal: $\int_0^L X_m(x) X_n(x) = 0$

$\int$ ① $\times X_n$ − ② $\times X_m$

$$\Rightarrow \int X_n X_m'' + \lambda_m X_m X_n - X_n'' X_m - \lambda_n X_n X_m dx = 0$$

$$= \int \underbrace{(X_m'' X_n - X_n'' X_m)}_{\text{分部积分}} + \underbrace{(\lambda_m - \lambda_n)}_{\neq 0} X_m X_n dx = 0$$

消了两项

$$\Rightarrow \left[ X_m' X_n - X_n' X_m \right]_0^L = -(\lambda_m - \lambda_n) \int X_m X_n dx$$

$\Downarrow \quad \to = 0$

所以在这里我们其实不需要Direchlet这么强的条件，满足这式子即可.

Consequence: E.V.P with Neumann → orthog 显然.

E.V.P with Robin ?

$$\begin{cases} X'(0) - a_0 X(0) = 0 \\ X'(L) + a_1 X(L) = 0 \end{cases}$$

Computing $\left[ X_m' X_n - X_n' X_m \right]_0^L$

$$= X_m'(L) X_n(L) - X_n'(L) X_m(L) - X_m'(0) X_n(0) + X_n'(0) X_m(0).$$

$-a_1 X_m(l)$ $\quad -a_1 X_m(l)$ $\quad a_0 X_m(0)$ $\quad a_0 X_n(0)$

$=0$ 消掉了.

Consider $X''+\lambda X=0$ in $(0,l)$, assuming general boundary condition of the form

$$\begin{cases} \alpha_1 X(0)+\beta_1 X'(0)+\gamma_1 X(l)+\delta_1 X'(l)=0 \\ \alpha_2 X(0)+\beta_2 X'(0)+\gamma_2 X(l)+\delta_2 X'(l)=0 \end{cases}$$

We say this is a symmetric B.C. if $f, g$ satisfying these B.C. s.t. $[f'g-g'f]_0^l=0$

Exercise: Assume $f:[0,l]\to\mathbb{R}$ and has a convergent Fourier sin series, i.e. $f(x)=\sum_{n=1}^{\infty} A_n \sin\frac{n\pi x}{l}$

同乘后积分即可.

More generally,

本质没区别.

$$\begin{cases} X''+\lambda X=0 \\ \text{Sym B.C.} \\ f(x)=\sum_{1}^{\infty} A_n X_n(x) \end{cases}$$

$$\Rightarrow A_n=\frac{1}{\|X_n\|^2}\int_0^l X_n(x) f(x)\,dx.$$

Solving wave / diff in periodic geometry.

$$\Rightarrow \begin{cases} X''+\lambda X=0 & -l<x<l \\ X(-l)=X(l) \\ X'(-l)=X'(l) \end{cases}$$

- Is $\lambda=0$ an e.v.? $\quad\Rightarrow X(l)=B$

$X(x)=Ax+B$

$X(-l)=X(l)\Rightarrow -Al+B=Al+B \quad\Rightarrow A=0.$

$X'(-l) = X'(l) \Rightarrow \quad A = A$

- $\lambda = \beta^2 > 0$.

$X(x) = A\sin(\beta x) + B\cos(\beta x)$

$A\sin(\beta l) + B\cos(\beta l) = -A\sin(\beta l) + B\cos(\beta l) \Rightarrow 2A\sin(\beta l) = 0$

$-A\beta\cos(\beta l) - B\beta\sin(\beta l) = -A\beta\cos(\beta l) + B\beta\sin(\beta l) \overset{\beta\neq 0}{\Rightarrow} 2B\sin(\beta l) = 0$

$\left.\begin{array}{l} A\sin(\beta l) = 0 \\ B\sin(\beta l) = 0 \end{array}\right\} \Rightarrow \left\{\begin{array}{l} A = B = 0 \quad \times \\ \beta l = n\pi. \Rightarrow \beta = \frac{n\pi}{l} \end{array}\right.$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{l}\right)^2 \qquad X_n(x) = A_n\sin\left(\frac{n\pi x}{l}\right) + B_n\cos\left(\frac{n\pi x}{l}\right)$

$f: (-l, l) \to \mathbb{R}$

$f(x) =$ full fourier series $= \frac{B_0}{2} + \sum_{n=1}^{\infty} A_n\sin\left(\frac{n\pi x}{l}\right) + B_n\cos\left(\frac{n\pi x}{l}\right)$

Where $A_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx$

$B_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx.$

From full fourier (→ cos, sin) to complex fourier series

$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$

$\Rightarrow f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{\frac{in\pi x}{l}}$, $C_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{-\frac{in\pi x}{l}}dx.$

$f: \mathbb{R} \to \mathbb{C}$ $\quad \left\{\begin{array}{l} f \xrightarrow{\text{direct}} \hat{f} \\ \hat{f} \xrightarrow{\text{inverse}} f \end{array}\right.$ $\quad \hat{f}(k) = (\mathcal{F}f)(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} f(x)dx$

$f(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}\hat{f}(x)dx$

3 types of Fourier series:

① $f: [0, l] \to \mathbb{R}$, $f(x) = \left\{\begin{array}{l} \sum^{\infty} A_n\sin\frac{n\pi x}{l} \\ \frac{B_0}{2} + \sum_{1}^{\infty} B_n\cos\frac{n\pi x}{l} \end{array}\right.$

② $f: [-l, l] \to \mathbb{R}$ $\quad f(x) = \frac{B_0}{2} + \sum_{1}^{\infty}\left(A_n\sin\frac{n\pi x}{l} + B_n\cos\frac{n\pi x}{l}\right)$

Consider e.v. pb with symmetric b.c.

$$\begin{cases} X''+\lambda X=0 \\ X'(0)=X'(l)=0. \end{cases}$$

Lemma: All e.v.s are real. $(X_0, \lambda)$, NTS: $\lambda=\bar{\lambda}$

pf: ① $X''+\lambda X=0$

② $\bar{X}''+\bar{\lambda}\bar{X}=0$

$① \cdot \bar{X} - ② \cdot X$

$$\Rightarrow \underbrace{\int_0^l (X''\bar{X}-\bar{X}''X)\,dx}_{=0} + (\lambda-\bar{\lambda})\int_0^l |X(x)|^2 dx = 0$$

=0 ，因为 symmetric B.C.

$$\Rightarrow 0=(\lambda-\bar{\lambda})\int_0^l \underbrace{|X(x)|^2}_{\text{non-trivial}} dx \Rightarrow \lambda=\bar{\lambda}$$

Knowing that $\lambda\in\mathbb{R}$, we can always choose $X$ to be real valued.

$$\begin{cases} X''+\lambda X=0 \\ \bar{X}''+\lambda \bar{X}=0 \end{cases} \Rightarrow (X+\bar{X})''+\lambda(X+\bar{X})=0$$

<u>3 notions of consequence</u>

1) We say that the series $\sum_{n=0}^{\infty} f_n(x)$ converges to $f(x)$ pointwise in $(a,b)$ if $\forall x_0\in(a,b)$, $S_N(x)=\sum_{n=1}^{N} f_n(x)$, $\lim_{N\to\infty}|f(x_0)-S_N(x_0)|=0$

Example: $f_n(x)=(1-x)x^{n-1}$ $\quad (0,1)\to\mathbb{R}$, $\sum_{n=1}^{\infty} f_n(x)$ converge?

$$S_N(x)=\sum_{n=1}^{N}(x^{n-1}-x^n)=1-x^N$$

$x\in(0,1)$, $S_N(x)\to 1$.

2) We say that the series $\sum_{n=0} f_n(x)$ converges to $f(x)$ uniformly in $(a,b)$ if $\lim_{N\to\infty}\max_{a\le x\le b}|f(x)-S_N(x)|=0$

这个就不 uniformly converge 了.

3) We say that the series $\sum_{n=0} f_n(x)$ converges to $f(x)$ in $L^2$ in $(a,b)$ if $\lim_{N\to\infty} \int_0^l |f(x) - S_N(x)|^2 dx = 0$

## Convergence of Fourier Series 5.3–5.4

3 THMs

① Pointwise convergence:

Assume $f(x): [a,b] \to \mathbb{R}/\mathbb{C}$, continuous, $f(x)$ (piecewise) cont. on $[a,b]$. Then it has Fourier series (sin / cos / full / complex),

have $\{X_n\}_0^\infty$ set of e.f associated to an e.v. pb with symmetric b.c.

Consider its Fourier coefficient. $A_n$ (expansion in either ),

Then $\forall x \in (a,b)$, $\sum_0^N A_n X_n(x) \xrightarrow{N\to\infty} f(x)$

$$A_n = \frac{\int_a^b f(x) X_n(x) dx}{\|X_n\|_{L^2}^2} = \frac{(f, X_n)}{\|X_n\|_{L^2}^2}$$

② Uniform convergence:

Assume $f, f', f''$ continuous on $[a,b]$, and $f$ satisfies the same b.c. as the set $\{X_n\}$,

Then $\max_{a\le x\le b} |f(x) - \sum_0^N A_n X_n(x)| \xrightarrow{N\to\infty} 0$

③ $L^2$ - convergence:

Assume that $\int_a^b |f(x)|^2 dx < \infty$. Define $A_n = \frac{(f, X_n)}{\|X_n\|}$ be fourier coef. why finite?

Then we have $\int_a^b |f(x) - \sum_0^N A_n X_n(x)| dx \xrightarrow{N\to\infty} 0$

Remark: Assume $\int_a^b |f(x)|^2 dx$ is finite, we have

$(f, X_n) = \int_a^b f(x) \overline{X_n(x)} dx$.

柯西不等式: $(f, X_n) \le \underbrace{\|f\|}_{<\infty} \cdot \underbrace{\|X_n(x)\|}_{<\infty}$

Goal: Prove thm ①.

"The $L^2$-theory"

$$\begin{cases} X'' + \lambda X = 0 & \text{in } (a,b) \\ + \text{symmetric b.c.} \end{cases}$$ ⇒ 使 $X_n$ 正交…吧?

**Prop 1**: "Least-Square approximation"

Given $f:[a,b]\to \mathbb{R}/\mathbb{C}$, Assume $\|f\|<\infty$, Define $A_n = \dfrac{(f,X_n)}{\|X_n\|^2}$

Fix $N$, consider all possible choices of $N$ constants:

$(c_1,\cdots,c_N)$. Define

$I(c_1,\cdots,c_N) = \|f - \sum_1^N c_n X_n(x)\|^2$, 想找合适的$\{c_i\}$使$I$尽可能小. $C_n = A_n$时

**Pf**: $E_N = I(c_1,\cdots,c_N) = \|f - \sum_1^N c_n X_n(x)\|^2$

$$= \int_a^b \Big(f(x) - \sum_1^N c_n X_n(x)\Big)\Big(\overline{f(x)} - \sum_1^N \bar{c}_n \overline{X_n(x)}\Big)\,dx$$

$$= \|f\|^2 - \Big(\sum_1^N c_n \overline{(f,X_n)} + \sum_1^N \bar{c}_n (f,X_n)\Big) + \sum_1^N |c_n|^2 \|X_n\|^2$$

$$= \left(\sum_1^N \|X_n\|^2 \Big(c_n - \frac{(f,X_n)}{\|X_n\|^2}\Big)\Big(\bar{c}_n - \overline{\frac{(f,X_n)}{\|X_n\|^2}}\Big)\right) + \|f\|^2 - \sum_1^N \frac{|(f,X_n)|^2}{\|X_n\|^2}$$

在凑后面的平方　　如果在实里, 这里是平方

$$= \sum_1^N \|X_n\|^2 \left| c_n - \frac{(f,X_n)}{\|X_n\|^2} \right|^2 + \|f\|^2 - \sum_1^N \frac{|(f,X_n)|^2}{\|X_n\|^2}$$

$C_n$ 只与这里有关, ⇒ $C_n = A_n$.

$$E_N \geq \|f\|^2 - \sum_1^N \frac{|(f,X_n)|^2}{\|X_n\|^2}$$

**Cor**: "Bessel inequality"　　$A_n = \dfrac{(f,X_n)}{\|X_n\|^2}$

$$0 \leq E_N = \|f\|^2 - \sum_1^N \frac{|(f,X_n)|^2}{\|X_n\|^2} = \|f\| - \sum_1^N |A_n|^2 \|X_n\|^2$$

$$\Rightarrow \sum_1^N |A_n|^2 \|X_n\|^2 \leq \|f\|$$

Let $N\to\infty$, $\sum_1^\infty |A_n|^2 \|X_n\|^2 \leq \|f\|$

## $L^2$ Theory

Let $\{X_n\}_0^\infty$ orthogonal set of function $X_n : [-\pi,\pi] \to \mathbb{R}/\mathbb{C}$

least square approximation:

fix $N$, $f \in L^2$, $E_N = \|f - \sum^N C_n X_n\|_{L^2}^2$

Then let $C_n = \frac{(f, X_n)}{\|X_n\|^2} = A_n$, minimize $E_N$.

$$E_N \le \|f\|_{L^2}^2 - \sum_{n=1}^N |A_n|^2 \|X_n\|^2$$

Consequences:

① If $f : [-\pi,\pi] \to \mathbb{R}/\mathbb{C}$, with $\|f\|^2$ bounded

We have the Besell inequality:

$$\sum_{n=1}^N |A_n|^2 \|X_n\|^2 \le \|f\|^2$$

$$\Rightarrow \underbrace{\sum_{n=1}^\infty |A_n|^2 \|X_n\|^2}_{\text{收敛级数}} \le \|f\|^2$$

② Passeval identity

Assume in addition that if $\sum_{n=1}^N A_n X_n \xrightarrow{N\to\infty} f$ in $L^2$,

(i.e, $\|f - \sum A_n X_n\| \to 0$ as $N \to \infty$) 但这只需 f $L^2$ 可积

then $\sum_{n=1}^\infty |A_n|^2 \|X_n\|^2 = \|f\|^2$

这里的 $E_N$ 里的系数是 $A_n$, 所以这里是等号

Pf: $0 \le E_N = \|f(x) - \sum_1^N A_n X_n\|^2 = \|f\|^2 - \sum_1^N |A_n|^2 \|X_n\|^2$

We know that $\|f(x) - \sum_1^N A_n X_n\|^2 \xrightarrow{N\to\infty} 0$

$\Rightarrow \|f\|^2 - \sum_1^N |A_n|^2 \|X_n\|^2 \xrightarrow{N\to\infty} 0$

$\Rightarrow \sum_{n=1}^\infty |A_n|^2 \|X_n\|^2 = \|f\|^2$

## Application of Parseval id

$f(x) = 1$, $x \in [0, \pi]$

$$\begin{cases} X'' + \lambda X = 0 & 0<x<\pi \\ X(0) = X(\pi) = 0 \end{cases}$$

$\lambda_n = n^2, \quad X_n(x) = \sin(nx)$

$\int_0^\pi |f(x)|^2 dx = \pi$

$\|X_n\|^2 = \int_0^\pi \sin^2(nx)\,dx = \int_0^\pi \frac{1-\cos(2nx)}{2}\,dx = \frac{\pi}{2}$

$A_n = \frac{1}{\|X_n\|^2} \int_0^\pi f(x) X_n(x)\,dx$

$= \frac{2}{\pi} \int_0^\pi \sin(nx)$

$= -\frac{2}{\pi} \cdot \frac{1}{n} \cdot [\cos(nx)]_0^n$

$= \frac{2}{\pi n}(1-(-1)^n) = \begin{cases} \frac{4}{\pi n} & \text{odd} \\ 0 & \text{even} \end{cases}$

$\Rightarrow \quad \sum_{n \text{ odd}} \left(\frac{4}{\pi n}\right)^2 \frac{\pi}{2} = \pi$

$\Rightarrow \quad \sum_{n \text{ odd}} \frac{8}{n^2 \pi^2} = 1$

$\Rightarrow \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

We use $X_n(x) = e^{inx}$

**Thm:** $f: [-\pi,\pi] \to \mathbb{R}/\mathbb{C}$, continuous on $[-\pi,\pi]$, $f'$ continuous on $[-\pi,\pi]$

Define Fourier coef: $C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{inx} dx$

$\forall x \in (-\pi,\pi)$, $S_N = \sum_{n=-N}^{N} C_n e^{inx} \to f(x)$ as $N \to \infty$. pointwisely.

**Pf of convergence theorem:**

$S_N(x) = \frac{1}{2\pi} \sum_{-N}^{N} \left( \int_{-\pi}^{\pi} f(y) e^{-iny} dy \right) e^{inx}$

$= \frac{1}{2\pi} \sum_{-N}^{N} \int_{-\pi}^{\pi} f(y) e^{in(x-y)} dy$

$= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) \sum_{-N}^{N} e^{in(x-y)} dy$

Define $K_n(\theta) = \sum_{-N}^{N} e^{in\theta}$ $\quad e^{i\theta} = s$

$= \sum_{-N}^{N} s^n$

$= \sum_{n=1}^{N} \left(\frac{1}{s}\right)^n + 1 + \sum_{n=1}^{N} s^n \qquad 1 + \cdots + s^N = \frac{1-s^{N+1}}{1-s}$

$$= 1+\sum_{n=1}^{N}\left(\frac{1}{s}\right)^n + 1 + \sum_{n=1}^{N} s^n - 1$$

$$= \frac{1-e^{-i(N+1)\theta}}{1-e^{-i\theta}} + \frac{1-e^{i(N+1)\theta}}{1-e^{i\theta}} - 1$$

$$= \frac{e^{i(N+\frac{1}{2})\theta} - e^{-i(N+\frac{1}{2})\theta}}{e^{i\frac{\theta}{2}} - e^{-i\frac{\theta}{2}}}$$

$$K_N(\theta) = \frac{\sin(N+\frac{1}{2})\theta}{\sin(\frac{\theta}{2})}$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) K_N(x-y)\, dy.$$

Properties of $K_N$: 
$$\begin{cases} 周期\ 2\pi \\ \int_{-\pi}^{\pi} K_N(\theta) = 2\pi \\ K_N(\theta) = \dfrac{\sin(N+\frac{1}{2})\theta}{\sin(\frac{\theta}{2})} \end{cases}$$

$$S_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(y) \frac{\sin\left(\left(\frac{N+1}{2}\right)(x-y)\right)}{\sin\left(\frac{x-y}{2}\right)}\, dy$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x+\theta) \frac{\sin\left(\frac{N+1}{2}\theta\right)}{\sin\frac{\theta}{2}}\, d\theta \qquad y-x=\theta$$

这里设所谓

$$S_N(x) - f(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} (f(x+\theta) - f(x)) \frac{\sin\left(\frac{N+1}{2}\theta\right)}{\sin\frac{\theta}{2}}\, d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x+\theta)-f(x)}{\sin(\frac{\theta}{2})} \sin\left(\frac{N+1}{2}\theta\right) d\theta$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\cdot \sin\left(\frac{N+1}{2}\theta\right) d\theta \quad = g_N$$

Lemma: 1) $g$ cont, bdd for $\theta \in [-\pi, \pi]$

2) $\|g\| < +\infty$

3) set $\{\sin(N+\frac{1}{2})\theta\}_{N=1}^{\infty}$ orthogonal set in $[-\pi, \pi]$
$= \{Y_N\}$

$g_N(\theta)$ = Fourier coef of $g$ for set $\{Y_N\}_1^{\infty}$

Bessel: $\sum_{N=1}^{\infty} |g_N|^2 \underbrace{\|Y_N\|^2}_{\text{finite}} \le \|g\|^2 = \text{finite}.$

因为 这个东西 $N\to\infty$ 时不是0.

$\Rightarrow g_N \longrightarrow 0 \quad \text{as } N \to \infty$

$\Rightarrow g_N = S_N - f \xrightarrow{N\to\infty} 0$ ■

Pf of the lemma:

1) when $\theta \to 0$, $g(\theta) \sim \frac{f(x+\theta)-f(x)}{\theta} \frac{\theta}{\sin\frac{\theta}{2}}$ bdd.

2) $-\pi < x < \pi$, $\int_{-\pi}^{\pi} |g(\theta)|^2 d\theta$ finite

3) $\begin{cases} Y'' + \lambda Y = 0 \\ Y(0)=0,\ Y'(\pi)=0 \end{cases} \Rightarrow Y_n = \sin(N+\frac{1}{2})Y$

## Convergence of Fourier series

$\begin{cases} \text{Uniform convergence} \\ L^2 \text{ convergence.} \end{cases}$

## Uniform convergence theorem:

$\{X_n\}$ set of orthogonal functions.

Given a fn $f: [-\pi,\pi] \to \mathbb{R}$, cont, $f'$ cont, $f''$ cont on $[-\pi,\pi]$.

$X_n(x) = e^{inx}$, $x \in [-\pi,\pi]$ $X'' + \lambda X = 0$, periodic bd condition: $X(\pi) = X(-\pi)$, $X'(\pi) = X'(-\pi)$

Assume also that $f$ satisfies the same B.C. as $X_n(x)$

Ptwise: $(-\pi,\pi)$. Uniform cont: $[-\pi,\pi]$

It's Fourier series converges uniformly to $f(x)$:

$C_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$, $S_N(x) = \sum_{n=-N}^{N} C_n e^{inx}$, $\max|f(x) - S_N(x)| \to 0$ as $N \to \infty$

Pf: $f(x) = \sum_{-\infty}^{\infty} C_n e^{inx}$ (pointwise convergent)

$|f(x) - S_N(x)| = \left|\sum_{|n|=N+1}^{\infty} C_n e^{inx}\right| \le \sum_{|n|=N+1}^{\infty} |C_n|$ ← 三角 + $|e^{inx}| \le 1$

$\max_{[-\pi,\pi]} |f(x) - S_N(x)| \le \sum_{|n|=N+1}^{\infty} |C_n|$

Goal: prove that as $N \to \infty$, $\sum_{|n|=N+1}^{\infty} |C_n| \to 0$. (the tail of $\sum_{-\infty}^{\infty} C_n$)

$\Leftarrow$ $\sum_{-\infty}^{\infty} |C_n|$ is convergent.

$f''(x) = \sum \tilde{C}_n e^{inx}$ ← $f''$ cont.

$$\tilde{c}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f'(x)e^{-inx}dx$$

$$= -\frac{1}{2\pi}(-in)\int_{-\pi}^{\pi} f(x)e^{-inx}dx + \cancel{\frac{1}{2\pi}\left[f(x)e^{inx}\right]_{-\pi}^{\pi}}$$

$$= in\, c_n$$

$$\Rightarrow \sum_{|n|=N+1}^{\infty}|c_n| = \sum_{|n|=N+1}^{\infty}\frac{1}{|n|}|\tilde{c}_n|$$

$$\sum_{-\infty}^{\infty}|c_n| = \sum_{\substack{-\infty \\ \neq 0}}^{\infty}\frac{1}{|n|}|\tilde{c}_n|$$

<u>Lemma</u>: Cauchy-Schwartz ineq for finite sum

$$\left|\sum a_n b_n\right| \le \sqrt{\left(\sum a_n^2\right)\left(\sum b_n^2\right)}$$

$$\sum_{n=1}^{N}\frac{1}{|n|}|\tilde{c}_n| \le \underbrace{\left(\sum_{n=1}^{N}\frac{1}{n^2}\right)^{\frac{1}{2}}}_{\text{convergent as } N\to\infty}\underbrace{\left(\sum_{n=1}^{N}|\tilde{c}_n|^2\right)^{\frac{1}{2}}}_{?}$$

convergent as $N\to\infty$ finite

↳ Bessel inequality: $\sum_{1}^{\infty}|\tilde{c}_n|^2\|e^{inx}\| = 2\pi\sum_{1}^{\infty}|\tilde{c}_n| \le \|f'\|$

$$\Rightarrow \sum_{-\infty}^{\infty}|c_n| \text{ convergent} \Rightarrow \sum_{|n|=N}^{\infty}|c_n| \to 0$$

$\blacksquare$

<u>Thm 3</u>: $f\in L^2[-\pi,\pi]$, define $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$

$$\left|\int_{-\pi}^{\pi} f(x)e^{-inx}dx\right| \le \underbrace{\left(\int_{-\pi}^{\pi}(e^{-inx})^2dx\right)^{\frac{1}{2}}}_{\sqrt{2\pi}}\underbrace{\left(\int_{-\pi}^{\pi}|f(x)|^2dx\right)}_{\|f\|} \Rightarrow c_n \text{ bounded.}$$

$$\Rightarrow \int_{-\pi}^{\pi}|f(x)-S_N(x)|^2dx \longrightarrow 0$$

Consequence: if $f\in L^2$, Bessel ineq becomes eq: $\Rightarrow 2\pi\sum_{-\infty}^{\infty}|c_n|^2 = \|f\|^2$

Given $f$, $S_N = \sum_{-N}^{N} c_n e^{inx}$

$$\int_{-\pi}^{\pi}|S_N(x)|^2 = \int_{-\pi}^{\pi}\left(\sum_n c_n e^{inx}\right)\left(\sum_m \overline{c_m}\,\overline{e^{imx}}\right)dx$$

消消乐

$$= 2\pi\sum_{-N}^{N}|c_n|^2$$

<u>Bessel</u>: $\sum_{-N}^{N}|c_n|^2 \le \|f\|_{L^2} \Rightarrow \sum_{-\infty}^{\infty}|c_n|^2$ convergent.

↳ $\sum_{-N}^{N}|c_n|^2\|x_n\|^2 \le \|f\|^2.$

<u>Prop</u>: Assume $\{d_m\}_{-\infty}^{\infty}$ coef, s.t. $\sum_{-\infty}^{\infty}|d_m|^2 < \infty$,

Then $\exists\, g$ of period $2\pi$, $g\in L^2[-\pi,\pi]$,

s.t. $g(x) = \sum_{-\infty}^{\infty} d_n x_n$ in $L^2$

From prop applied to $\{c_n\}$, $\exists g \in L^2$, s.t. $g = \sum c_n x_n$ in $L^2$.

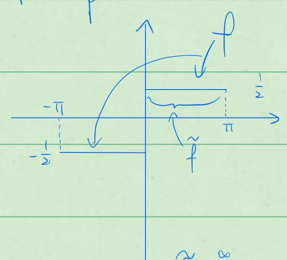
By uniqueness of Fourier coef, $g = f$ a.e.

Convergence of series pointwisely:

1) $f \in [-\pi, \pi]$, $f, f'$ cont, then $S_N \xrightarrow{N\to\infty} f$ pointwise

2) if $f$ discontinuous @ $x_0$, $X_N(x_0) \xrightarrow{N\to\infty} \frac{1}{2}\left(f(x_0^+) + f(x_0^-)\right)$

Example:




$\{x_m\} = \frac{\pi}{m}$ $\qquad S_N(x_m)$

$f(x) = \sum_{-\infty}^{\infty} c_n e^{inx}$ $\qquad c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-inx} dx$

$S_N(x) = \sum_{\substack{1 \\ \text{odd}}}^{N} \frac{2}{n\pi} \sin(nx)$

$\tilde{f} = \sum_{n=1}^{\infty} d_n \sin(nx)$, $d_n = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\sin(nx)dx = \begin{cases} \frac{2}{\pi n} & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

odd, so extend it to $[-\pi,\pi]$

$\Rightarrow \tilde{f}(x) = \sum_{\text{odd}}^{\infty} \frac{2}{n\pi} \sin(nx)$

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x-y) f(y)\, dy \qquad K_N(\theta) = \frac{\sin(N+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}$$

$$= \int_0^{\pi} \frac{\sin\left((N+\frac{1}{2})(x-y)\right)}{\sin\left(\frac{x-y}{2}\right)} \frac{1}{4\pi} dy - \int_{-\pi}^{0} \frac{\sin\left((N+\frac{1}{2})(x-y)\right)}{\sin\left(\frac{x-y}{2}\right)} \frac{1}{4\pi} dy$$

Let $M = N + \frac{1}{2}$, $\quad M(x-y) = \theta$

$$= \int_{M(x-\pi)}^{Mx} \frac{\sin\theta}{M\sin\left(\frac{\theta}{2M}\right)} \frac{1}{4\pi} d\theta - \int_{-M(x+\pi)}^{-Mx} \frac{\sin\theta}{M\sin\left(\frac{\theta}{2M}\right)} \frac{1}{4\pi} d\theta$$

$$= \int_{-Mx}^{Mx} \frac{\sin\theta}{M\sin\left(\frac{\theta}{2M}\right)} \frac{1}{2\pi} d\theta - \int_{M\pi-Mx}^{M\pi+Mx} \frac{\sin\theta}{M\sin\left(\frac{\theta}{2M}\right)} \frac{1}{2\pi} d\theta$$

Choose $x = \frac{\pi}{M}$

$S_N\left(\frac{\pi}{x}\right) = I_1 + I_2$, $\quad I_1 = \int_{-\pi}^{\pi} \frac{\sin\theta}{M\sin\left(\frac{\theta}{2M}\right)} \frac{1}{2\pi} d\theta$, $\quad I_2 = \int_{(M-1)\pi}^{(M+1)\pi} \frac{\sin\theta}{M\sin\left(\frac{\theta}{2M}\right)} \frac{1}{2\pi} d\theta$

$\frac{(M-1)}{2M}\pi \le \frac{\theta}{2M} \le \frac{(M+1)}{2M}\pi \Rightarrow \left(1-\frac{1}{M}\right)\frac{\pi}{2} \le \frac{\theta}{2M} \le \left(1+\frac{1}{M}\right)\frac{\pi}{2}$

$\hookrightarrow \approx \frac{\pi}{2}$

$\Rightarrow I_2 \xrightarrow{M\to\infty} 0$

$I_1 = \int_{-\pi}^{\pi} \frac{\sin\theta}{2M\sin\left(\frac{\theta}{2M}\right)} \frac{1}{2\pi} d\theta$. $\qquad 2M\sin\frac{\theta}{2M} \xrightarrow{M\to\infty} \theta$

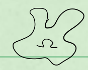
$= \int_{-\pi}^{\pi} \frac{\sin\theta}{\theta} \frac{1}{2\pi} d\theta$

$= 2\int_0^{\pi} \frac{\sin\theta\, d\theta}{2\pi\theta}$

Laplace equation : $\Delta u = 0$

1d: $\frac{d^2u}{dx^2} = 0$

2d: $\begin{cases} \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \\ u|_{\partial\Omega} = 0 \end{cases}$




Complex analysis

$f(z) = f(x+iy) = u(x+iy) + iv(x+iy)$

if $f$ analytic: Cauchy-Riemann relation: $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$

Max/Min for solution of the Laplace eq.

Thm: $D$ connected, bdd, open of $\mathbb{R}^2$ or $\mathbb{R}^3$, Assume $u \in C^2(D) \cup C^1(\bar{D})$

and $\Delta u = 0$

$u$ reaches its max and min at $\partial D$

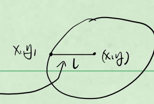
Pf: $u$ reaches its max at $(x_0, y_0) \in \bar{D}$,

$v(x,y) = u(x,y) + \varepsilon(x^2+y^2)$, $\Delta v = 4\varepsilon > 0$.

Assume $v$ reaches its max at $(x_1,y_1) \in \mathrm{int}(D)$, then $\frac{\partial^2 v}{\partial x^2}(x_1,y_1)$, $\frac{\partial^2 v}{\partial y^2}(x_1,y_1) \le 0$

But , $\Rightarrow$ $v$ reaches its max at a boundary pt.

$\forall (x,y) \in D$, $u(x,y) \le v(x,y) \le \underbrace{v(x_1,y_1)}_{\in \partial D} = u(x_1,y_1) + \varepsilon l^2$




$\le \max_{\partial D} u + \varepsilon l^2$

let $\varepsilon \to 0$, $\forall x,y \in D$, $u(x,y) \le \max_{\partial D} u$

Finding particular sol$^n$ of $\Delta u = 0$, $(x,y) \in \mathbb{R}^2$

Rewrite $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ in polar coordinate.

$u(x,y) = v(r,\theta)$ , $x, y = r\cos\theta,\ r\sin\theta$

$$\Rightarrow \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}} = 0$$

$$\Rightarrow \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = 0$$

$$\Rightarrow w' + \frac{w}{r} = 0 \Rightarrow (rw)' = 0 \Rightarrow rw(r) = C_1$$

$$\Rightarrow w(r) = \frac{C_1}{r} \Rightarrow v(r) = C_1 \ln r + C_2$$

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# Chapter 6. Harmonic functions.

(Solutions of $\Delta u = 0$ in $D \subseteq \mathbb{R}^2/\mathbb{R}^3$

In particular: 1) the min/max principle.

2) Special sol$^n$ of $\Delta u = 0$ in $\mathbb{R}^2/\mathbb{R}^3$.

3) Solve $\begin{cases} \Delta u = 0 \\ \text{B.C.} \end{cases}$ in "simple domain"  

4) Solve $\begin{cases} \Delta u = 0 \\ u|_{\partial D} = f \end{cases}$ in Disk  Poisson

5) From Poisson formula

a) Mean value thm

b) Min/Max principle (strong version in any dimension)

c) Gain of smoothness.

## 2. a: Special Sol$^n$ of $\Delta u = 0$

$u(x,y) \in \mathbb{R}^2 \to \mathbb{R}$, having "radial" symmetry.

"radial" symmetry = $u(x,y) = v(r,\theta)$ (Cartesian $\Leftrightarrow$ Polar.)

Writing $\Delta u = 0$ in polar coordinate $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$

$$\Rightarrow \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \cancel{\frac{1}{r^2}\frac{\partial^2 v}{\partial \theta^2}} = 0$$ Look for solution only depends on r.

$$\Rightarrow \frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} = 0$$ All harmonic, radically symmetric function.

$\Rightarrow \; w' + \frac{w}{r} = 0 \;\Rightarrow\; (rw)' = 0 \;\Rightarrow\; rw(r) = C_1$

$\Rightarrow \; w(r) = \frac{C_1}{r} \;\Rightarrow\; V(r) = C_1 \ln r + C_2 = \underbrace{C_1 \ln(x^2+y^2) + C_2}_{\text{only } r>0}$

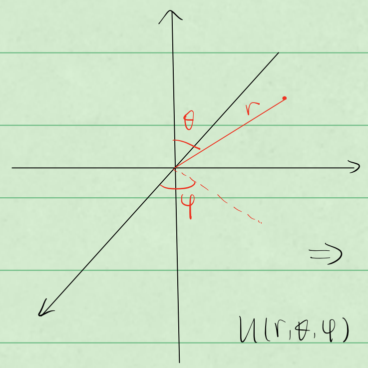
$\Delta(\ln r) = 0 \quad \forall r \neq 0.$ $\quad \underbrace{\Delta(\ln r)}_{r\text{有}\,w=0} = -2\pi\delta$

$\downarrow$

$\delta(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \qquad \delta(x,t) \xrightarrow{t \to 0^+} \delta(x)$

· 3D: $\quad \Delta u = 0, \quad u: \mathbb{R}^3 \to \mathbb{R}$

Transform to spherical coordinate



$$\Rightarrow \begin{cases} x = r\cos\varphi\sin\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\theta \end{cases}$$

$u_{xx} + u_{yy} + u_{zz} = 0$

$$\Rightarrow \frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2} = 0$$

$u(r,\theta,\varphi)$ that do not depend on $\theta, \varphi$,

$u'' + \frac{2}{r}u' = 0 \;\Rightarrow\; V' + \frac{2}{r}V = 0$

$\Rightarrow \; r^2 V' + 2rV = 0$

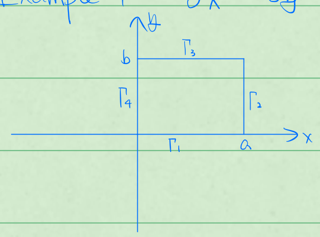
$\Rightarrow \; r^2 V = C_1$

$\Rightarrow \; V(r) = \frac{C_1}{r^2} \;\Rightarrow\; u(r) = \frac{C_1}{r} + C_2.$

$\Delta\left(\frac{1}{r}\right) = 0, \; \forall r \neq 0. \quad V(r) = \frac{1}{4\pi r}, \quad -\Delta\left(\frac{1}{4\pi r}\right) = \delta_0$

In $\mathbb{R}^n$: $\frac{\partial^2 u}{\partial r^2} + \frac{n-1}{r}\frac{\partial u}{\partial r} = 0 \;\Rightarrow\; u(r) = \frac{C_1}{r^{n-2}} + C_2$

Example 1: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad (x,y) \in (0,a) \times (0,b)$



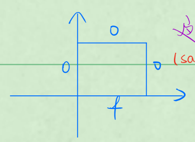

$\Gamma_1$: $u(x,0) = f(x) \qquad 0<x<a.$

$\Gamma_2$: $\frac{\partial u}{\partial x}(a,y) = g(y) \qquad 0<y<b.$

$\Gamma_3$: $\left(\frac{\partial u}{\partial y} + u\right)(x,b) = h(x) \qquad 0<x<a.$

$\Gamma_4$: $u(0,y) = k(y) \qquad 0<y<b.$

Pb1: $u_1$:




这里的"0"是让 $g, h, k = 0$，但相对应的边界条件（比如 $\Gamma_2$ 的 $\frac{\partial u}{\partial x}$，$\Gamma_3$ 的 $\frac{\partial u}{\partial y} + u$）是不变的.
(same kind of b.c.)

$u_{2,3,4}$

Pb 2, 3, 4: 分别令 $\Gamma_2, \Gamma_3, \Gamma_4 \neq 0$.

$\Rightarrow U_{original} = U_1 + U_2 + U_3 + U_4$.

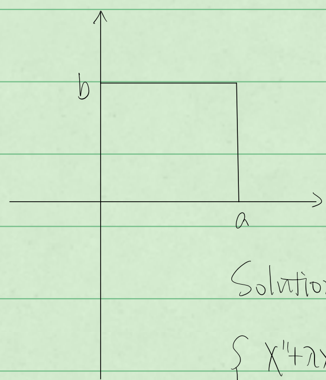
$Pb_1$: $U_1(x,y) = X(x)Y(y)$

$\Rightarrow \frac{X''}{X} + \frac{Y''}{Y} = 0$

$\Rightarrow \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$

$$\Rightarrow \begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda X = 0 \\ X(0) = X'(a) = 0 \end{cases}$$

$\Rightarrow \lambda_n = \left(\frac{n\pi}{a}\right)^2, \quad X_n = \cos\left(\frac{n\pi x}{a}\right)$




$$\begin{cases} \Delta U = 0 \quad \text{in } (a,0)\times(0,b) = D \\ U(x,0) = f(x) \\ U_x(a,y) = U_x(0,y) = 0 \\ \left(\frac{\partial u}{\partial y} + u\right)(x,b) = 0 \end{cases}$$

Solution of form $U(x,y) = X(x)Y(y)$

$$\begin{cases} X'' + \lambda X = 0 \\ Y'' - \lambda Y = 0 \end{cases} \Rightarrow \begin{matrix} X_0(x) = 1, \quad \lambda_0 = 0 \\ X_n(x) = \cos\left(\frac{n\pi}{a}x\right), \quad \lambda_n = \left(\frac{n\pi}{a}\right)^2 \end{matrix}$$

For each $\lambda = \lambda_n$, solve $Y_n'' - \lambda_n Y_n = 0$

$$\begin{cases} n = 0 \Rightarrow Y_0 = A_0 y + B \\ n \neq 0 \Rightarrow Y_n = A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y} \end{cases}$$

Construct soln: $U(x,y) = \left(\frac{A_0}{2}y + \frac{B_0}{2}\right) + \sum_{1}^{\infty}\left(A_n e^{\frac{n\pi}{a}y} + B_n e^{-\frac{n\pi}{a}y}\right)\cos\frac{n\pi}{a}x$

On $y = 0$, $f(x) = U(x,y) = \frac{B_0}{2} + \sum_{n=1}^{\infty}(A_n + B_n)\cos\frac{n\pi x}{a}$

$\Rightarrow B_0 = \frac{2}{a}\int_0^a f(x)\,dx$, $\quad A_n + B_n = \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx$ 这只是条件之一，还需另一个条件.

$\frac{\partial u}{\partial y}(x,b) = \frac{A_0}{2} + \sum_{n=1}^{\infty}\frac{n\pi}{a}\left(A_n e^{\frac{n\pi}{a}b} - B_n e^{-\frac{n\pi}{a}b}\right)\cos\frac{n\pi x}{a} + \frac{A_0 b}{2} + \frac{B_0}{2} + \sum_{n=1}^{\infty}\left(A_n e^{\frac{n\pi}{a}b} + B_n e^{-\frac{n\pi}{a}b}\right)\cos\frac{n\pi x}{a}$

$= 0$

$\Rightarrow A_0 + A_0 b + B_0 = 0 \Rightarrow A_0, B_0$ ✓

$$A_n e^{\frac{n\pi}{a}b}\left(\frac{n\pi}{a}+1\right)+B_n e^{-\frac{n\pi}{a}b}\left(-\frac{n\pi}{a}+1\right)=0$$

Laplace eq in $D=\{r<a\}$

$$\begin{cases}\Delta u=0\\ u|_{r=a}=f\end{cases}$$

$u(r,\theta)=$

$$\begin{cases}\dfrac{\partial^2 u}{\partial r^2}+\dfrac{1}{r}\dfrac{\partial u}{\partial r}+\dfrac{1}{r^2}\dfrac{\partial^2 u}{\partial \theta^2}=0\\ \text{boundary condition.}\\ u(a,\theta)=f(\theta),\quad \theta\in[0,2\pi)\end{cases}$$

设解为 $R(r)P(\theta)$

$2\pi$ periodic.

$u(r,\theta)=R \Rightarrow R''P+\frac{1}{r}R'P+\frac{1}{r^2}RP''=0$

$\Rightarrow r^2\left(\frac{R''}{R}+\frac{R'}{rR}\right)+\frac{P''}{P}=0$

$$\Rightarrow \begin{cases}\dfrac{P''}{P}=-\lambda \iff P''+\lambda P=0 & ①\\ r^2R''+rR'-\lambda R=0 & ②\end{cases}$$

$P(0)=P(2\pi)\Rightarrow \begin{cases}P''+\lambda P=0\\ P(0)=P(2\pi),\quad P'(0)=P'(2\pi)\end{cases}$

$\Rightarrow \lambda_0=0,\quad \lambda_n=n^2$

$P_0=\frac{A_0}{2},\quad P_n=A_n\cos n\theta+B_n\sin n\theta$

For each $\lambda_n$, solve ②

$n=0:\ r^2R_0''+rR_0'=0$ $\Rightarrow ty'+y=0 \Rightarrow y=\frac{1}{t} \Rightarrow s=R_0'=\frac{1}{r}$

$\Rightarrow R_0(r)=C_1\ln r+C_2$. Let $C_1=0$ to avoid singularity.

$n\neq 0:\ r^2R_n''+rR_n'-n^2R_n=0$

$R_n(r)=r^\alpha \hookrightarrow \alpha(\alpha-1)+\alpha-n^2=0$

$\Rightarrow \alpha=\pm n$

$R_n(r)=A_nr^n+\cancel{B_nr^{-n}}$ Avoid singularity.

$$\Rightarrow \begin{cases}u_0(r,\theta)=\frac{A_0}{2}\\ u_n(r,\theta)=r^n(A_n\cos n\theta+B_n\sin n\theta)\end{cases}$$

Find all $A_n, B_n$ s.t. $u(a,\theta)=f(\theta)$

$$u(a,\theta)=\frac{A_0}{2}+\sum_{n=1}^{\infty}a^n(A_n\cos n\theta+B_n\sin n\theta)$$

$$=f(\theta)$$

$$\Rightarrow A_0=\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)d\theta$$

$$A_n=\frac{1}{a^n}\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\cos(n\theta)d\theta,\qquad B_n=\frac{1}{a^n}\frac{1}{\pi}\int_{-\pi}^{\pi}f(\theta)\sin(n\theta)d\theta,$$

$\theta\to\varphi$

**Prop:** The sol$^n$ of $\begin{cases}\Delta u=0\\ u(a,\theta)=f(\theta)\end{cases}$ is given

$$u(r,\theta)=\frac{a^2-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(\varphi)}{a^2-2ar\cos(\theta-\varphi)+r^2}d\varphi$$

**Pf:** $$u(r,\theta)=\frac{1}{2\pi}\left(\int_{-\pi}^{\pi}f(\varphi)d\varphi+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\int_{-\pi}^{\pi}f(\varphi)\left[\underbrace{\cos n\varphi\cos n\theta+\sin n\varphi\sin n\varphi}_{\cos n(\theta-\varphi)}\right]d\varphi\right)$$

$$=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(\varphi)\left(1+2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos n(\theta-\varphi)\right)d\varphi$$

$$\theta-\varphi=\beta\Rightarrow 1+\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n(e^{in\beta}+e^{-in\beta})$$

$\beta\to$ 换回 $\varphi$

$$\sum_{1}^{\infty}x^n=\frac{x}{1-x}$$

$$=1+\sum_{n=1}^{\infty}\left(\left(\frac{r}{a}\right)e^{i\varphi}\right)^n+\sum_{n=1}^{\infty}\left(\left(\frac{r}{a}\right)e^{-i\varphi}\right)^n$$

$$=1+\frac{\frac{r}{a}e^{i\varphi}}{1-\frac{r}{a}e^{i\varphi}}+\frac{\frac{r}{a}e^{-i\varphi}}{1-\frac{r}{a}e^{-i\varphi}}$$

$$=1+\frac{re^{i\varphi}}{a-re^{i\varphi}}+\frac{re^{-i\varphi}}{a-re^{-i\varphi}}$$

$$=\frac{a^2-r^2}{a^2-2ar\cos\varphi+r^2}$$

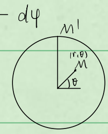
$$=\frac{a^2-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(\varphi)}{a^2-2ar\cos(\theta-\varphi)+r^2}d\varphi$$

Rewrite poisson in $x,y$ coordinate

$$u(r,\theta)=\frac{a^2-r^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(\varphi)}{a^2-2ar\cos(\theta-\varphi)+r^2}d\varphi$$

$r<a,\quad -\pi\le\theta\le\pi$

$$\begin{cases}x=r\cos\theta\\ y=r\sin\theta\end{cases}\qquad\begin{cases}x'=a\cos\theta'\\ y'=a\sin\theta'\end{cases}$$




$$|M-M'|=(x-x')^2+(y-y')^2$$

$$=x^2+y^2+x'^2+y'^2-2xx'-2yy'$$

$$=a^2+r^2-2ar\cos(\theta-\theta')$$

$$\vec{x}=(x,y)$$

$$u(x,y)=\frac{a^2-|\vec{x}|^2}{2\pi a}\int_{|\vec{x}|=a}\frac{f(\vec{x}')}{|\vec{x}-\vec{x}'|^2}ds'\qquad ds'=a\,d\theta'$$

Poisson formula

Mean value thm

**Mean value theorem:**

$$u(0)=\frac{a^2}{2\pi a}\int_{|\vec{x}'|=a}\frac{f(\vec{x}')}{a^2}ds'$$

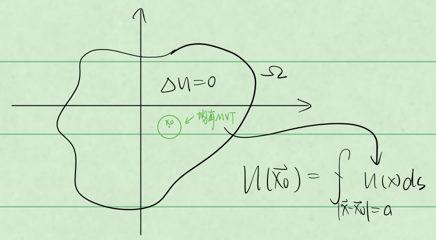
$$= \frac{1}{2\pi a}\int_{|\vec{x}|=a} f(\vec{x})\,ds'$$

$$= \frac{1}{2\pi a}\int_{|\vec{x}|=a} u(\vec{x})\,ds'$$

Strong max/min

Smoothness

Thm




**Prop:** Assume $u\in C^2(\Omega)\cap C(\bar{\Omega})$, satisfies $\Delta u=0$

Then $\forall\ \vec{x_0}\in\Omega$, we have

$$u(\vec{x_0}) = \frac{1}{2\pi a}\int_{|\vec{x}-\vec{x_0}|=a} u(\vec{x})\,ds.$$

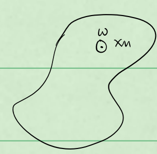
## The min/max principle in the strong form:

Assume $u\in C^2(\Omega)\cap C(\bar{\Omega})$, satisfies $\Delta u=0$

Then the max and min value of $u$

Then max/min reached on $\partial\Omega$ and nowhere else. except $u$ constant.

**pf:** We know that $u$ reaches is max in $\partial D$, but we haven't exclude that $\exists\ \vec{x_m}\in D$, s.t. $u(x_m)=M$.




By contradiction, assume $u(x_m)=M$

$$\Rightarrow M = u(x_m) = \frac{1}{2\pi a}\int_{|x-x_m|=a} u(x)\,ds(\vec{x})$$

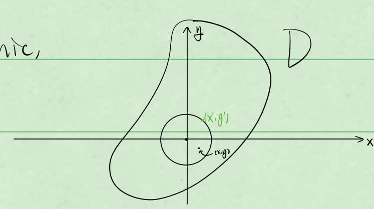
Since $M$ maximum, and $M=$ average of $u$ on $|x-x_m|=a$

$\Rightarrow u$ is constant in $|x-x_m|\le a$, where $a$ arbitrary.

By picking different $x_m$, we can conclude that $u$ is constant.

· **Smoothness:** Assume $u\in C^2(D)\cap C(\bar{D})$ harmonic,

Then $\forall x\in D$, $u\in C^\infty(D)$




pf: $$u(x,y) = \frac{a^2-(x^2+y^2)}{2\pi a}\int \frac{u(x',y')}{(x-x')^2+(y-y')^2}\,ds(x',y')$$

$$\frac{\partial u}{\partial x}(x,y) = \frac{1}{2\pi a}(-2x)\int_{x'^2+y'^2=a^2} \frac{u(x',y')}{(x-x')^2+(y-y')^2}\,ds(x',y')$$

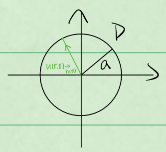
在圆上积

$$+\frac{a^2-(x^2+y^2)}{2\pi a}\int_{x'^2+y'^2=a^2}\frac{-u(x',y')\,2(x-x')}{\left((x-x')^2+(y-y')^2\right)^2}\,ds.$$

Recall:

Diffusion: $u(x,t)=\frac{1}{\sqrt{4\pi kt}}\int e^{-\frac{(x-y)^2}{4kt}}\varphi(y)\,dy \xrightarrow{t\to 0^+} \varphi(x)$

· Thm: Let $h:[0,2\pi)\longrightarrow\mathbb{R}$ a continuous $2\pi$ periodic function on $\partial D$




Let $u(r,\theta)=\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{h(\varphi)}{r^2-2r\cos(\theta-\varphi)+a^2}\,d\varphi$

1) $u$ is harmonic in $D$

2) $u$ continuous in $\overline{D}$, and $u(r,\theta)\xrightarrow{r\to a} h(\theta)$

3) $u$ is the unique sol$^n$ of $\begin{cases}\Delta u=0\\ u|_{\partial D}=\varphi\end{cases}$ (By max/min)

· Pf: Let $P(r,\theta)=\frac{a^2-r^2}{r^2-2ar\cos\theta+a^2}$

$\Rightarrow u(r,\theta)=\frac{1}{2\pi}\int_0^{2\pi}P(r,\theta-\varphi)\cdot h(\varphi)\,d\varphi$

Properties: 1) $P(r,\theta)>0$ （因为分母是 $(r,\theta)$ 到 $(a,\varphi)$ 的距离）

2) $\frac{1}{2\pi}\int P(r,\theta)\,d\theta=1$ $\left(P(r,\theta)=1+\sum_1^\infty\left(\frac{r}{a}\right)^n\cos n\theta \Rightarrow \int P=\int 1\right)$ 这些是0

3) $\Delta P=0$ $\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{r}{a}\right)^n\cos(n\theta)=0$

· Pf of $u(r,\theta)\xrightarrow{r\to a}h(\theta)$:

$u(r,\theta_0)-h(\theta_0)=\frac{1}{2\pi}\int_0^{2\pi}P(r,\theta_0-\varphi)h(\varphi)\,d\varphi-h(\theta_0)\cdot 1 = \frac{1}{2\pi}h(\theta_0)\int_0^{2\pi}\underbrace{P(r,\varphi-\theta_0)}_{=P(r,\theta_0-\varphi)}\,d\varphi$ 因为是cos

$=\frac{1}{2\pi}\int_0^{2\pi}P(r,\theta_0-\varphi)\left(h(\varphi)-h(\theta_0)\right)d\varphi$

$=\frac{1}{2\pi}\int_{|\theta_0-\varphi|<\delta}\underset{<\frac{\varepsilon}{2}}{\mathrm{I}}+\frac{1}{2\pi}\int_{|\theta_0-\varphi|>\delta}\underset{<\frac{\varepsilon}{2}}{\mathrm{II}}$

当 $r\to a$ 时, $\frac{a^2-r^2}{r^2-2ar\cos\theta+a^2}\to\frac{a^2-r^2}{(a-r)^2}\to\frac{a+r}{a-r}$ 可能会出问题

$\mathrm{I}\le\frac{\varepsilon}{2}$:

Choose $\delta$, s.t. $|\varphi-\theta_0|<\delta \rightarrow |h(\varphi)-h(\theta_0)|<\frac{\varepsilon}{2}$

$|\mathrm{I}|\le\frac{\varepsilon}{2}\frac{1}{2\pi}\int_{|\cdot|<\delta}P(r,\theta_0-\varphi)\,d\varphi\le\frac{\varepsilon}{2}\underbrace{\frac{1}{2\pi}\int_0^{2\pi}P(r,\theta_0-\varphi)\,d\varphi}_{=1}=\frac{\varepsilon}{2}$

$\mathrm{II}\le\frac{\varepsilon}{2}$:

$$\mathrm{II} \le \frac{2}{2\pi} \sup|h| \int_{|\varphi-\theta|>\delta} \frac{a^2-r^2}{r^2-2ar\cos(\theta-\varphi)+a^2}\, d\varphi$$

$$= \frac{2}{2\pi} \sup|h| \int_{|\theta|>\delta} \frac{a^2-r^2}{r^2-2ar\cos\theta+a^2}\, d\theta$$

$r^2-2ar+a^2+2ar(1-\cos\theta)$

$= (a-r)^2+2ar\,2\sin^2\frac{\theta}{2}$

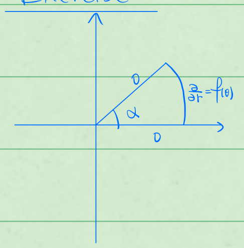
$$\le \frac{1}{\pi} \sup|h| \int_{|\theta|>\delta} \frac{a^2-r^2}{(a-r)^2+4ar\sin^2\frac{\theta}{2}}\, d\theta$$

$|\theta| > \delta \Rightarrow |\sin\frac{\theta}{2}| \geq \sin^2\frac{\delta}{2}$

$$\le \text{-------} \cdot \frac{(a-r)(a+r)}{\underbrace{4ar\sin^2\frac{\delta}{2}}_{\text{fixed}}} \le \frac{\varepsilon}{2}$$

$\square$

Exercise:




$D = \{(r,\theta),\ 0 \le r < a,\ 0 < \theta < \alpha\}$

$$\begin{cases} \Delta u = 0 \\ \frac{\partial u}{\partial r}(a,\theta) = f(\theta) \\ u(r,0) = u(r,\alpha) = 0 \end{cases}$$

$u(r,\theta) = R(r)P(\theta)$

$$\Rightarrow \begin{cases} (1)\ P'' + \lambda P = 0, \quad P(0) = P(\alpha) = 0 \\ (2)\ R'' + rR' - \lambda R = 0 \end{cases}$$

$(1) \Rightarrow \lambda_n = \left(\frac{n\pi}{\alpha}\right)^2, \quad P_n(\theta) = \sin\left(\frac{n\pi\theta}{\alpha}\right)$

For each $\lambda_n$, solve

$r^2 R_n'' + rR_n' - \frac{n^2\pi^2}{\alpha^2} R = 0$

$R_n(r) = r^s \Rightarrow s(s-1) + s - \frac{n^2\pi^2}{\alpha^2} = 0$

$\Rightarrow s = \pm\frac{n\pi}{\alpha}$

$\Rightarrow R_n(r) = Ar^{\frac{n\pi}{\alpha}} + \cancel{Br^{-\frac{n\pi}{\alpha}}}$ singular @ 0

$\Rightarrow u(r,\theta) = \sum A_n r^{\frac{n\pi}{\alpha}} \sin\frac{n\pi\theta}{\alpha}$

Then find $A_n$ by using $\frac{\partial u}{\partial r}(a,\theta) = f(\theta)$

$$\frac{\partial u}{\partial r}(r,\theta) = \sum A_n \frac{n\pi}{\alpha} a^{\frac{n\pi}{\alpha}-1} \sin\frac{n\pi\theta}{\alpha} = f(\theta)$$

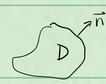
# Chapter 7

## Divergence Theorem:

$$\iiint_D \operatorname{div} F \, d\vec{x} = \iint_{\partial D} \vec{F}\cdot\vec{n}\, d\sigma.$$

$$\nabla f = \begin{pmatrix} \partial_{x_1} f \\ \partial_{x_2} f \\ \partial_{x_3} f \end{pmatrix}$$

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix}$$



$$\nabla f = \begin{pmatrix} f_x \\ f_y \\ f_z \end{pmatrix}$$ 这个是gradient, 这里f是 $\mathbb{R}^3 \to \mathbb{R}$ 形

$$\nabla F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$ vector field. $F = (F_1, F_2, F_3)$

Consequences:
$$\iiint_D v \Delta u \, d\vec{x} = -\iiint_D \nabla v \nabla u \, d\vec{x} + \iint_{\partial D} v \frac{\partial u}{\partial n} d\sigma$$ ☆

Pf: $\underset{\text{divergence}}{\nabla} \cdot (v \underset{\text{gradient}}{\nabla} u) = \sum_{i=1}^{3} \partial_{x_i}(v \partial_{x_i} u) = \sum_{i=1}^{3} \partial_{x_i} v \partial_{x_i} u + v \partial^2_{x_i} u = \nabla v \cdot \nabla u + v \Delta u$

Apply div. thm for $F = v \nabla u = \begin{pmatrix} v\partial_{x_1} u \\ v\partial_{x_2} u \\ v\partial_{x_3} u \end{pmatrix}$ ?

$(v u_x)_x = v_x u_x + v u_{xx}$

$(v u_y)_y = v_y u_y + v u_{yy}$

这他妈是 $v(x,y,z)$, $u_x(x,y,z)$ 两函数乘起来!!! 草!

$(v u_z)_z = \underbrace{v_z u_z}_{\substack{\Sigma \\ \nabla v \nabla u \\ \text{点乘}}} + \underbrace{v u_{zz}}_{\substack{\Sigma \\ v \Delta u \\ \text{普通乘法}}}$

$$\nabla\left( v \nabla u \right) = \nabla \begin{pmatrix} v u_x \\ v u_y \\ v u_z \end{pmatrix} = \ldots + \ldots$$

$$\iiint_D \nabla \cdot (v \nabla u) \, d\vec{x} = \iint_{\partial D} v (\nabla u \cdot \vec{n}) \, d\sigma.$$

$$\Rightarrow \iiint_D \nabla v \cdot \nabla u + v \Delta u \, d\vec{x} = \iint_{\partial D} v \big(\underbrace{\nabla u \cdot \vec{n}}_{= \frac{\partial u}{\partial n}}\big) \, d\sigma.$$

$$\Rightarrow \iiint_D v \Delta u \, d\vec{x} = -\iiint_D \nabla v \nabla u \, d\vec{x} + \iint_{\partial D} v \frac{\partial u}{\partial n} d\sigma \quad ①$$

$u \longleftrightarrow v$

$$\Rightarrow \iiint_D u \Delta v \, d\vec{x} = -\iiint_D \nabla u \nabla v \, d\vec{x} + \iint_{\partial D} u \frac{\partial v}{\partial n} d\sigma \quad ②$$

① − ② =

$$\iiint_D v \Delta u - u \Delta v \, d\vec{x} = \iint_{\partial D} v \frac{\partial u}{\partial n} - u \frac{\partial v}{\partial n} d\sigma$$ ☆

- In 3D domain, in $D \subseteq \mathbb{R}^3$, $\Delta\left(\frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}}\right) = 0$
- A consequence for divergence theorem is the mean value theorem for harmonic functions in $\mathbb{R}^3$.

Thm: $D$: ball of $\mathbb{R}^3 = \{ |\vec{x}| < a \}$

$\partial D$ = Sphere of $\mathbb{R}^3$ $= \{ |\vec{x}| = a \} = S$

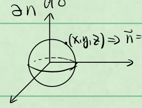
If $u \in C^2(D) \cap C(\bar{D})$ harmonic in $D$, then

$$u(0) = \frac{1}{\text{Area}(S)} \iint_S u(x)\, dS(x)$$

$$= \frac{1}{4\pi \cancel{a^2}} \int_0^{2\pi}\int_0^{\pi} u(a,\theta,\varphi)\, \cancel{a^2} \sin\theta\, d\theta\, d\varphi .$$

Pf: $\Delta u = 0$ in $D$.

$$0 = \iiint_D \Delta u\, dx = \iint_S \frac{\partial u}{\partial n}\, d\sigma$$



$(x,y,z) \Rightarrow \vec{n} = \left(\frac{x}{r}, \frac{y}{r}, \frac{z}{r}\right)$

$$\frac{\partial u}{\partial n} = \nabla u \cdot \vec{n}$$

$$= \frac{x}{r} u_x + \frac{y}{r} u_y + \frac{z}{r} u_z$$

$r^2 = x^2 + y^2 + z^2$

$$\frac{\partial u}{\partial r} = \sum_{cyc} \frac{\partial u}{\partial x} \frac{\partial x}{\partial r}$$

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \Rightarrow \frac{x}{r} = \frac{\partial x}{\partial r}$$

$$= \frac{\partial u}{\partial r}$$

同除$a^2$, 消掉.

$$= \int_0^{2\pi}\int_0^{\pi} \frac{\partial u}{\partial r}(a,\theta,\varphi)\, \cancel{a^2} \sin\theta\, d\theta\, d\varphi .$$

$$= \frac{\partial}{\partial r} \int_0^{2\pi}\int_0^{\pi} u(r,\theta,\varphi) \sin\theta\, d\theta\, d\varphi$$

$$0 = \frac{\partial}{\partial r}\left( \frac{1}{4\pi} \int_0^{2\pi}\int_0^{\pi} u(r,\theta,\varphi) \sin\theta\, d\theta\, d\varphi \right)$$

$$\Rightarrow \forall r,\ \frac{1}{4\pi r^2} \int_0^{2\pi}\int_0^{\pi} u(r,\theta,\varphi)\, r^2 \sin\theta\, d\theta\, d\varphi = \text{constant}$$ (doesn't depend on r)

Take $r \to 0$, constant $= u(0)$

MVT $\Rightarrow$ max/min

Dirichlet Principle:

Let $D \subseteq \mathbb{R}^n$ open bounded, with smooth boundary.

Fix $g: \partial D \to \mathbb{R}$. Let

$\Phi = \{ w: D \to \mathbb{R} \mid w \in C^2(D) \cap C(\bar{D}),\ w|_{\partial D} = g \}$

$$E(w)=\frac{1}{2}\int_D |\nabla w(x)|^2 dx.$$

Minimization Problem:

Find $w\in\Phi$, s.t. $E[w]$ is minimum

The function in $\Phi$ that minimize $E[w]$ is

$u$, s.t. $\begin{cases}\Delta u=0\\ u|_{\partial D}=g\end{cases}$ "Euler-Lagrange Equation associate to the min problem"

Pf: WTS $\forall w\in\Phi$, $E(u)\le E(w)$

Let $v=u-w$, then $v|_{\partial D}=0$

$|\nabla(u-v)|^2=\left|\begin{pmatrix}u_x-v_x\\u_y-v_y\\u_z-v_z\end{pmatrix}\right|^2=\sum u_x^2+v_x^2-2u_xv_x$

$$E(w)=\frac{1}{2}\int_D|\nabla(u-v)|^2dx$$

$$=\frac{1}{2}\int_D|\nabla u|^2+\frac{1}{2}\int_D|\nabla v|^2-\int_D\nabla u\cdot\nabla v.$$

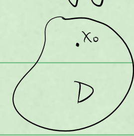
$\iiint_D v\Delta u\, d\vec{x}=-\iiint_D \nabla v\nabla u\, d\vec{x}+\iint_{\partial D} v\frac{\partial u}{\partial n}d\sigma$

$$=E(u)+E(v)+\int_D v\underbrace{\Delta u}_{=0\text{ on }\Delta u}dx-\int_{\partial D}\underbrace{v}_{=0\text{ on }\partial D}\frac{\partial u}{\partial n}dx$$

$$=E(u)+E(v)\ge E[u]$$

Prop: Let $D\subset\mathbb{R}^3$ bdd, open, $u\in C^2(D)\cap C(\bar{D})$, satisfying

$\Delta u=0$, then $\forall\vec{x}_0=(x_0,y_0,z_0)$,

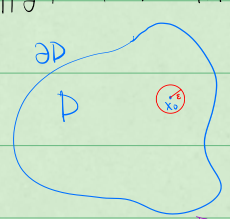



$\frac{\partial}{\partial n}=(\vec{n}\cdot\nabla)$ directional directive

$$u(\vec{x}_0)=\frac{1}{4\pi}\int_{\partial D}\left(-u(\vec{x})\frac{\partial}{\partial n}\left(\frac{1}{|\vec{x}-\vec{x}_0|}\right)+\frac{\partial u}{\partial n}(x)\frac{1}{|\vec{x}-\vec{x}_0|}\right)d\sigma(x)$$

Pf: $\int_D(\Delta u\, v-u\Delta v)dx=\int_{\partial D}\left(\frac{\partial u}{\partial n}v-u\frac{\partial v}{\partial n}\right)d\sigma$

apply this to $\begin{cases}u=u\\ v=\frac{1}{4\pi|\vec{x}-\vec{x}_0|}\end{cases}$ but this singular @ $x=x_0$. remove the neighborhood.




↓ Apply this to the domain $D_\varepsilon=D\setminus B(x_0,\varepsilon)$

因为 V harmonic in $D_\varepsilon$.

$$\int_{D_\varepsilon}(\underbrace{\Delta u}_{=0}v-u\underbrace{\Delta v}_{=0})dx=\int_{\partial D}\left(\frac{\partial u}{\partial n}v-u\frac{\partial v}{\partial n}\right)d\sigma+\int_{\partial B}\left(\frac{\partial u}{\partial n}v-u\frac{\partial v}{\partial n}\right)d\sigma$$

$$=\frac{1}{4\pi}\int_{\partial D}\left(-u(\vec{x})\frac{\partial}{\partial n}\left(\frac{1}{|\vec{x}-\vec{x}_0|}\right)+\frac{\partial u}{\partial n}(x)\frac{1}{|\vec{x}-\vec{x}_0|}\right)d\sigma(x)$$

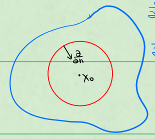
LHS = 0

So we only need to prove

$$\int_{\partial B}\left(\frac{\partial u}{\partial n}v-u\frac{\partial v}{\partial n}\right)d\sigma=-u(x_0)$$

$$\int_{\partial B}\left(\frac{\partial u}{\partial n}v-u\frac{\partial v}{\partial n}\right)d\sigma=\int_{|x-x_0|=\varepsilon}-\frac{\partial u}{\partial r}\frac{1}{4\pi\underbrace{|x-x_0|}_{\varepsilon}}+u\frac{1}{4\pi\varepsilon^2}\,d\sigma$$

$r=|x-x_0|$



$\frac{\partial}{\partial n}=-\frac{\partial}{\partial r}$ ?

$\frac{\partial}{\partial n}v=\frac{1}{4\pi}\left(-\left(\frac{\partial}{\partial r}\right)\frac{1}{r}\right)$

$=\frac{1}{4\pi r^2}\Big|_{r=\varepsilon}$

$=\frac{1}{4\pi\varepsilon^2}$

$\frac{\partial}{\partial n}(v)=\vec{n}\cdot\nabla v$, $\vec{n}=\begin{pmatrix}\frac{x}{r}\\ \frac{y}{r}\\ \frac{z}{r}\end{pmatrix}$

$=\begin{pmatrix}x/r\\ y/r\\ z/r\end{pmatrix}\cdot\begin{pmatrix}v_x\\ v_y\\ v_z\end{pmatrix}$

$=\frac{xv_x+yv_y+zv_z}{r}=\frac{\partial v}{\partial r}$ in polar coordinates.

$$=\int_{|x-x_0|=\varepsilon}\left[-\frac{\partial u(x)}{\partial r}\frac{1}{4\pi\varepsilon}-u(x)\frac{1}{4\pi\varepsilon^2}\right]\underbrace{\varepsilon^2\sin\theta\,d\theta\,d\varphi}_{=d\sigma}$$

as $\varepsilon\to 0$, $\displaystyle\int_{B_\varepsilon}-\underbrace{\frac{\partial u}{\partial r}}_{\text{bounded.}}\frac{\varepsilon}{4\pi}\sin\theta\,d\theta\,d\varphi\longrightarrow 0$

$$=\frac{-1}{4\pi\varepsilon^2}\int_{|x-x_0|=\varepsilon}u(x)\,d\sigma = \text{average value of } u \text{ in } \partial B_\varepsilon.$$

$$=-u(x_0)$$

$\varphi\in C^2(\mathbb{R}^3)$, $\varphi=0$ outside a ball on radius $R$.

$$\varphi(0)=-\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{1}{|x|}\Delta\varphi\,dx$$

## 7.3: The notion of Green's function:

$D\subset\mathbb{R}^3$

<u>Def</u>: The green's function of $\Delta$ associated with domain $D$ with Dirichlet Bd condition is defined as follows:

Fix $x_0\in D$, let $G(x,x_0)$, as a function of $x$,

$$\begin{cases}\Delta G=0 & \text{in } D\setminus\{x_0\}\\ G(x,x_0)=0 & \text{if } x\in\partial D.\\ G \text{ is } C^2 \text{ function except } x=x_0\\ H(x,x_0)=G(x,x_0)+\frac{1}{4\pi|x-x_0|} & \text{is } C^2(D) \text{ and } \Delta H=0 \text{ in } D.\end{cases}$$

<u>equivalently</u>: Let $H\in C^2(D)$ satisfy

$$\begin{cases}\Delta H=0 & \text{in } D\\ H|_{\partial D}=\frac{1}{4\pi|x-x_0|}\end{cases}$$

Define $G(x,x_0)=-\frac{1}{4\pi|x-x_0|}+H(x,x_0)$

The solution of $\Delta u = 0$, $u|_{\partial D} = f$ is:

$$u(x_0) = \int_{\partial D} f(x) \frac{\partial G}{\partial n}(x, x_0)\, d\sigma.$$

Thm: If $u \in C^2(D) \cap C(\bar{D})$ solution of $\begin{cases} \Delta u = 0 \\ u|_{\partial D} = f \end{cases}$ it is given by:

$$\forall x_0, \quad u(x_0) = \int_{\partial D} f(x) \frac{\partial}{\partial n} G(x, x_0)\, d\sigma$$

Pf: Let $H(x, x_0) \in C^2(D) \cap C(\bar{D})$ satisfying

$$\begin{cases} \Delta H = 0 \quad \text{in } D \\ H|_{\partial D} = \frac{1}{4\pi |x - x_0|} \end{cases}$$

$$G(x, x_0) = -\frac{1}{4\pi |x - x_0|} + H(x, x_0)$$

Properties of $G$:

$G(x, x_0) = 0$ if $x \in \partial D$.

$$\int_D (\Delta u \overset{H}{\ddot{v}} - u \Delta v)\, dx = \int_{\partial D} \left(\frac{\partial u}{\partial n} \overset{H}{v} - u \frac{\partial \overset{H}{v}}{\partial n}\right) d\sigma$$

$$\Rightarrow \qquad 0 = \cancel{-\int_{\partial D} \frac{\partial u}{\partial n} H} + \int_{\partial D} \frac{\partial H}{\partial n} f\, d\sigma \qquad ①$$

$$u(\vec{x_0}) = \cancel{\frac{1}{4\pi} \int \frac{1}{|x - x_0|} \frac{\partial u}{\partial n}} - \frac{1}{4\pi} \int \frac{\partial}{\partial n} \left(\frac{1}{|x - x_0|}\right) f(x)\, dx. \quad ②$$

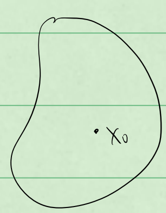
①+② $\qquad u(x_0) = \int_{\partial D} f(x) \frac{\partial}{\partial n} (G(x, x_0))\, d\sigma \qquad f$ is the same.

$\left( u_{x,y} \Leftrightarrow u_{r,\theta},\right.$ 说明一下 $\left.\frac{\partial u}{\partial n}\Big|_{\partial D} = \frac{\partial u}{\partial r}\Big|_{\partial D\,(a=r)} \right)$

$$\vec{n} = \begin{pmatrix} \frac{x}{r} \\ \frac{y}{r} \end{pmatrix} \quad \nabla u = \begin{pmatrix} \partial_x u \\ \partial_y u \end{pmatrix} \Rightarrow \ \hookrightarrow = \frac{x}{r} \partial_x u + \frac{y}{r} \partial_y u$$

$$\downarrow x = r\cos\theta \Rightarrow \frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r}$$

$$= \frac{\partial x}{\partial r} \frac{\partial u}{\partial x} + \frac{\partial y}{\partial r} \frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}.$$




$D \in \mathbb{R}^3$

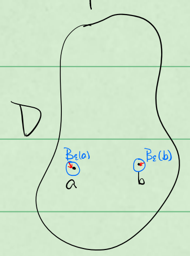
Fix $x_0 \in D$.

$G(x, x_0)$ (with $x \neq x_0$) has the properties $G(x, x_0) = -\frac{1}{4\pi |x - x_0|} + H(x, x_0)$

where $\begin{cases} H(x, x_0) \in C^2(D) \cap C(\bar{D}) \\ \Delta H = 0 \quad \forall x \in D \\ H(x, x_0) = \frac{1}{4\pi} \frac{1}{|x - x_0|} \end{cases}$

**Thm:** $\begin{cases} \Delta u = 0 \\ u|_{\partial D} = g \end{cases} \Rightarrow u(x_0) = \int_{\partial D} g(x) \frac{\partial G}{\partial n}(x, x_0)\, d\sigma$

**Prop:** $G(a,b) = G(b,a)$




**Pf:** 2nd Green's identity

$$\int_{D_\varepsilon} (f \underbrace{\Delta g}_{=0} - g \underbrace{\Delta f}_{=0})\, dx = \int_{\partial D_\varepsilon} f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n}\, d\sigma$$

g在∂D上是0.

Apply this to $D_\varepsilon = D \setminus B_\varepsilon(a) \cup B_\varepsilon(b)$.

Define $u(x) = G(x,a)$, $x \neq a$

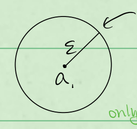
$v(x) = G(x,b)$, $x \neq b$

So LHS = 0

$$\text{RHS} = \underbrace{\cancel{\int_{\partial D}}}_{=0} + \overset{\mathrm{I}}{\int_{\partial B_\varepsilon(a)}} + \overset{\mathrm{II}}{\int_{\partial B_\varepsilon(b)}}$$

$$\int_{\partial B_\varepsilon(a)} G(x,a) \frac{\partial G}{\partial n}(x,b) - G(x,b) \frac{\partial G}{\partial n} G(x,a)\, d\sigma$$

这个地方该拆了 $G(x,a)$
因为 $\varepsilon = |x-a|$, 拆 $G(x,b)$ 会出 $\frac{1}{x-b}$




$d\sigma_2$

$$\mathrm{I} = -\int_\partial \Big( \Big( \frac{1}{4\pi \underbrace{|x-x_0|}_{=\varepsilon}} + H(x,a) \Big) \frac{\partial}{\partial r} G(x,b)\, \varepsilon^2 \sin\theta\, d\theta\, d\varphi = 0$$

H bounded

only left

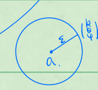
$$+ \int_\partial G(x,b) \Big( \frac{\partial}{\partial r} \Big[ -\frac{1}{4\pi|x-a|} + H(x,a) \Big] \Big) \varepsilon^2 \sin\theta\, d\theta\, d\varphi$$

$r = |x-a|$ $\frac{\partial}{\partial r}\frac{1}{r} = -\frac{1}{r^2}$

$$= \int_{\partial B_\varepsilon(a)} G(x,b) \frac{1}{4\pi\varepsilon^2} \varepsilon^2 \sin\theta\, d\theta\, d\varphi$$

$$\frac{1}{4\pi} \int_{\partial B_\varepsilon(a)} G(x,b) \sin\theta\, d\theta\, d\varphi$$

$$= \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} G(x,b) \sin\theta\, d\theta\, d\varphi.$$




$\binom{k}{\theta\varphi}$ $x \to a$

$$= \frac{1}{4\pi} G(a,b)\, \text{surf}(\text{unit sphere}) = G(a,b)$$

$$= \frac{1}{4\pi\varepsilon^2} \int_{\partial B_\varepsilon(a)} G(x,b)\, \varepsilon^2 \sin\theta\, d\theta\, d\varphi$$

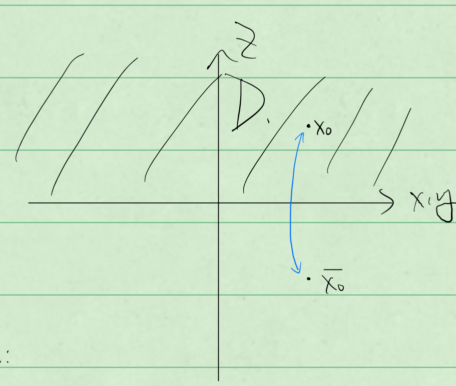
$G(x,b)$ Harmonic.

$= \text{mean value of } G(x,b) \text{ on } \partial B_\varepsilon(a) = G(a,b)$

$\mathrm{II} = -G(b,a)$

$\Rightarrow \mathrm{I} + \mathrm{II} = 0 \Rightarrow G(a,b) = G(b,a)$

$$\begin{cases} \Delta u = 0 & \text{in } \{z \geq 0\}. \\ u(x,y,0) = f(x,y) \end{cases}$$




- Calculate Green's function for harmonics:

$\vec{x} = (x,y,z)$, $G(x,x_0) = \frac{-1}{4\pi|x-x_0|} + \underbrace{H(x,x_0)}_{\text{look for this}}$

$G(x,x_0) = 0$ for $x \in \partial D$.

$V(x)=\frac{1}{|x|}$, $\Delta=0$. $V'(x)=\frac{c}{|x-a|}+c'$ 也滿 $\Delta=0$

- $G(x,x_0)=-\frac{1}{4\pi|x-x_0|}+\frac{c}{4\pi|x-\bar{x}|}$, choose $\bar{x}$, s.t. $G(x,x_0)=0$ if $x\in\partial D$.

Take $\vec{x}=(x,y,0)$, $G(x,x_0)=-\left(\frac{-1}{4\pi\sqrt{(x-x_0)^2+(y-y_0)^2+z_0^2}}-\frac{1}{4\pi\sqrt{(x-\bar{x})^2+(y-\bar{y})^2+\bar{z}^2}}\right)=0$

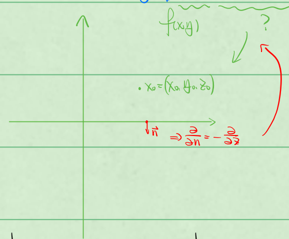
Then $\bar{x}=(x_0,y_0,-z_0):=x_0^*$

$$\Rightarrow u(\vec{x})=\frac{1}{4\pi}\int_{\mathbb{R}^2}f(x,y)\frac{2z_0}{((x-x_0)^2+(y-y_0)^2+z_0^2)^{\frac{3}{2}}}\,dxdy$$

$\vec{x}=(x,y,z)$

$(x_0,y_0,z_0)$ 和 ? $\left(u(x)=\int_{\partial D}\underbrace{f(x)}_{f(x,y)}\underbrace{\frac{\partial G}{\partial n}(x,x_0)}_{?}\,\underbrace{d\sigma}_{dxdy}\right)$

plug in $z=0$




$$\frac{\partial}{\partial z}\left(\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}}-\frac{1}{\sqrt{(x-x_0)^2+(y-y_0)^2+(z+z_0)^2}}\right)=-\left(\frac{z-z_0}{((x-x_0)^2+(y-y_0)^2+(z-z_0)^2)^{\frac{3}{2}}}-\frac{z+z_0}{((x-x_0)^2+(y-y_0)^2+(z+z_0)^2)^{\frac{3}{2}}}\right)$$

$$\Rightarrow u(x,y,z)=\frac{z_0}{2\pi}\int_{\mathbb{R}^2}\frac{f(x,y)}{((x-x_0)^2+(y-y_0)^2+z_0^2)^{\frac{3}{2}}}\,dxdy$$

$$u(x_0,y_0,z_0)\xrightarrow{z_0\to 0}f(x_0,y_0)$$

Pf: cylindrical coordinate: $(x,y,z)\longrightarrow(r,\theta,z)$, $x-x_0=r\cos\theta$, $y-y_0=r\sin\theta$

$$u(x_0,y_0,z_0)=\frac{z_0}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{f(x_0+r\cos\theta,y_0+r\sin\theta)}{(r^2+z_0^2)^{\frac{3}{2}}}\,r\,dr\,d\theta$$

$s=\frac{r}{z_0}\Rightarrow ds=\frac{dr}{z_0}$

$$=\frac{\cancel{z_0}}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{f(x_0+sz_0\cos\theta,y_0+sz_0\sin\theta)}{\cancel{z_0^3}(s^2+1)^{\frac{3}{2}}}\cancel{z_0^2}s\,ds\,d\theta$$

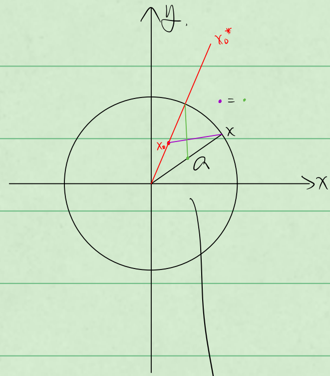
$$=\frac{1}{2\pi}\int_0^{2\pi}\int_0^{\infty}\frac{f(x_0+sz_0\cos\theta,y_0+sz_0\sin\theta)}{(s^2+1)^{\frac{3}{2}}}\,s\,ds\,d\theta$$

$z_0\to 0$

$s^2=t\Rightarrow 2s\,ds=dt\Rightarrow\int_0^{\infty}\frac{s\,ds}{(s^2+1)^{\frac{3}{2}}}=\frac{1}{2}\int_0^{\infty}\frac{dt}{(t+1)^{\frac{3}{2}}}=\frac{2}{2}\left[\frac{1}{(t+1)^{\frac{1}{2}}}\right]_0^{\infty}=1$

$$=\frac{1}{2\pi}\int_0^{2\pi}f(x_0,y_0)\underbrace{\int_0^{\infty}\frac{s\,ds}{(s^2+1)^{\frac{3}{2}}}}_{=1}\,d\theta$$

$$=f(x_0,y_0)$$




$D = \{(x,y) \in \mathbb{R}^2,\ x^2+y^2<a^2\}$

$G(x,x_0) = \frac{1}{2\pi}\ln|x-x_0| + H(x,x_0)$

Look for $H(x,x_0) = c\ln|x-x_0| + D$.

$(x_0,y_0) \longleftrightarrow x_0+iy_0$

in $\mathbb{C}$, $z_0 \to z_0^* = \frac{a^2}{\bar{z}_0} = \frac{a^2 z_0}{|z_0|}$

Goal: $G(x,x_0)=0$ if $x \in \partial D$.

$x \in \partial D$, $|\vec{x}-\vec{x}_0| = \left|\frac{x_0}{|x_0|}a - \frac{x}{a}|x_0|\right| = \frac{|x_0|}{a}\left|\frac{x_0}{|x_0|^2}a^2 - \vec{x}\right|$

$= \frac{|x_0|}{a}|x - x_0^*|$

harmonic, not singular.

$\Rightarrow \ln|x-x_0| = \ln\frac{|x_0|}{a} + \ln|x-x_0^*|$

Let $G(x,x_0) = \frac{1}{2\pi}\ln|x-x_0| - \frac{1}{2\pi}\ln\frac{|x_0|}{a}|x-x_0^*|$, $x, x_0 \in D$.

If $x \in \partial D$, then $G(x,x_0)=0$

$u(x_0) = \int_{x=a} \frac{\partial G}{\partial n}(x,x_0) f(x)\, d\sigma$

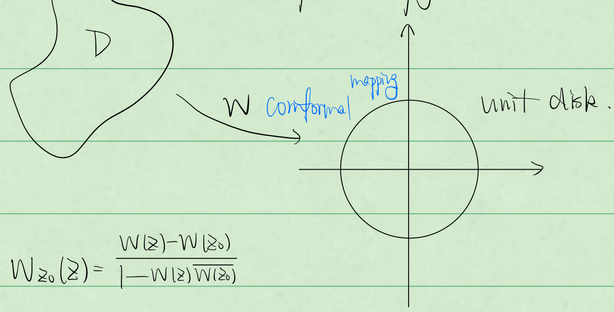
$\nabla G = \begin{pmatrix} \frac{\partial G}{\partial x} \\ \frac{\partial G}{\partial y} \end{pmatrix}$ $\quad \vec{n} = \frac{\vec{x}}{a} = \begin{pmatrix} \frac{x}{a} \\ \frac{y}{a} \end{pmatrix}$

$\Rightarrow \nabla G \cdot \vec{n} =$

$u(x_0) = \frac{a^2-|x_0|^2}{2\pi a}\int_{x=a} \frac{f(x)}{|x-x_0|^2}\, d\sigma.$

Green's function $\longleftrightarrow$ Conformal mapping.

D open, simply connect.




$W_{z_0}(z) = \frac{W(z)-W(z_0)}{1-W(z)\overline{W(z_0)}}$

**Prop**: The function $w_{z_0}(z)$ maps $D$ conformly $D$ to unit disk $\{|z|<1\}$.

(1) $w_{z_0}(\partial D) = \partial$ unit disk.

(2) $w_{z_0}(z_0) = 0$

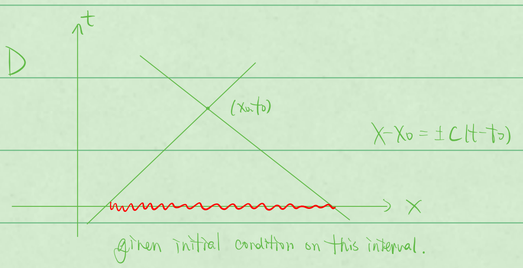
The green's function of $\Delta$ in $D$ is:

$$G(z,z_0) = \frac{1}{2\pi} \ln |w_{z_0}(z)|$$

# CHAPTER 9.1, Wave function in $\mathbb{R}^3$.

$$\begin{cases} \partial_{tt} u - c^2 \Delta u = 0 & x \in \mathbb{R}^3. \\ u(x,0) = f(x), \quad u_t(x,0) = g(x) \end{cases}$$

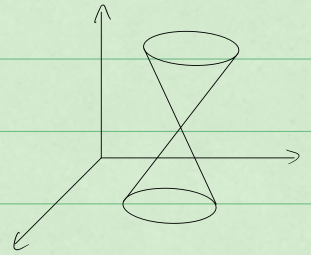
Recall: 1D




**Light cone**: Given $(\vec{x}_0, t_0) \in \mathbb{R}^3 \times \mathbb{R}$,

$G = \{ (\vec{x}, t) \in \mathbb{R}^4, \ |\vec{x} - \vec{x}_0| = c|t - t_0| \}$ "hypersurface"

inside of solid cone: $= \{ (\vec{x}, t), \ |x - x_0| \leq c|t - t_0| \}$.



# Conservation of energy:

$$\partial_{tt} u - c^2 \Delta u = 0$$

$$\Rightarrow \underbrace{\partial_t u\, \partial_{tt} u}_{=\frac{1}{2}\partial_t(\partial_t u)^2} - c^2 \partial_t u \Delta u = 0$$

$$\Rightarrow \frac{1}{2}\partial_t(\partial_t u)^2 - c^2 \underbrace{\partial_t u \Delta u}_{u_t(u_{xx}+u_{yy}+u_{zz})} = 0$$

$$u_t u_{xx} = \partial_x(\partial_t u \partial_x u) - \partial_{tx} u \partial_x u$$

$$= \partial_x(\partial_t u \partial_x u) - \frac{1}{2}\partial_t(\partial_x u)^2$$

$$\Rightarrow \frac{1}{2}\partial_t(\partial_t u)^2 - c^2\partial_x(\partial_t u\partial_x u) - c^2\partial_y(\partial_t u \partial_y u) - c^2\partial_z(\partial_t u\partial_z u) + \frac{1}{2}c^2\partial_t\left((\partial_x u)^2 + (\partial_y u)^2 + (\partial_z u)^2\right) = 0$$

$$\Rightarrow \frac{1}{2}\frac{\partial}{\partial t}\left(u_t^2 + c^2|\nabla u|^2\right) - c^2\partial_x(u_t u_x) - c^2\partial_y(u_t u_y) - c^2\partial_z(u_t u_z) = 0$$

integrate over $\mathbb{R}^3$

$$\Rightarrow \frac{d}{dt}E(t) = 0, \quad E(t) = E(0),$$

where Energy: $E(t) = \frac{1}{2}\int_{\mathbb{R}^3} |u_t|^2 + c^2|\Delta u|^2\, dx$.

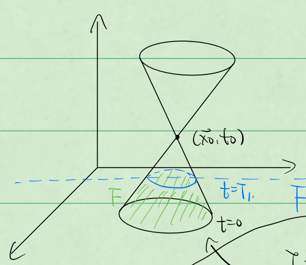
☆ 补一下n的证.

$$n = \begin{pmatrix} \frac{x-x_0}{|x-x_0|\sqrt{c^2+1}} \\ y\cdots \\ z\cdots \\ \frac{c}{\sqrt{c^2+1}}\frac{t_0-t}{|t_0-t|} \end{pmatrix}$$

$$\frac{1}{2}\frac{\partial}{\partial t}\left(u_t^2 + c^2|\nabla u|^2\right) - c^2\partial_x(u_t u_x) - c^2\partial_y(u_t u_y) - c^2\partial_z(u_t u_z) = 0 \quad *$$

light cone: $\{(\vec{x},t) \quad |\vec{x}-\vec{x_0}| = |t-t_0|\}$ 3-D set

solid cone: $\{(\vec{x},t) \quad |\vec{x}-\vec{x_0}| \le |t-t_0|\}$ 4-D set

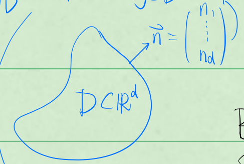



Intersect solid cone by hyperplane $t=T$.

F: inside the solid cone between $t=0$, $t=T$.

Integrate over F. $\Rightarrow \iiiint_F \{*\}\, dx\,dy\,dz\,dt = 0$

$\int_D \partial_{x_i} f\, dx = \int_{\partial D} f n_i$ (stoke's thm)




Bd F?

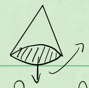
$\text{Bd}F = \underset{\text{Bottom}}{B} + \underset{\text{Top}}{T} + \underset{\text{Lateral}}{L}$

$\begin{cases} B: |\vec{x}-\vec{x_0}| \le ct_0 \\ T: |\vec{x}-\vec{x_0}| \le c(t_0-T_1) \end{cases}$

each piece, we apply stoke's thm.

$$= \iiint_B + \iiint_T + \iiint_L = 0$$

I.C. $u(x,0) = f$, $u_t(x,0) = g$



normal: $\begin{pmatrix}0\\0\\0\\-1\end{pmatrix} \leftarrow t$

$$\int_B = -\int_B \frac{1}{2}\left(u_t^2 + c^2|\nabla u|^2\right)\Big|_{t=0} d\vec{x} + 0$$

因为 $\partial_{x,y,z}$ 在 B上 的 $n_{x,y,z}=0$.

$$= -\int_B \frac{1}{2}\left(g^2 + c^2|\nabla f|^2\right) d\vec{x}$$

$$\int_T = \int_T \frac{1}{2}\left(|u_t|^2 + c^2|\nabla u|^2\right) d\vec{x}$$

Now we prove $\int_L \geq 0$, which $\Rightarrow \int_T \leq \int_B$

$$\int_L = \int_L \frac{1}{2}\left(u_t^2 + c^2|\nabla u|\right)\underbrace{\frac{c}{\sqrt{c^2+1}}}_{n_t} - \sum_{cyc} c^2(u_t u_x)\underbrace{\frac{x-x_0}{c|\vec{x}-\vec{x}_0|\sqrt{c^2+1}}}_{n_{x,y,z}} d\sigma$$

$$= \frac{c}{\sqrt{c^2+1}}\int_L \frac{1}{2}\left(u_t^2 + c^2|\nabla u|\right) - u_t\left(u_x\frac{x-x_0}{|\vec{x}-\vec{x}_0|} + u_y\frac{y-y_0}{|\vec{x}-\vec{x}_0|} + u_z\frac{z-z_0}{|\vec{x}-\vec{x}_0|}\right) d\sigma$$

$$= \nabla u \cdot \begin{pmatrix}\frac{x-x_0}{|\vec{x}-\vec{x}_0|}\\ \frac{y-y_0}{|\vec{x}-\vec{x}_0|}\\ \frac{z-z_0}{|\vec{x}-\vec{x}_0|}\end{pmatrix} = \vec{r}$$ unit vector in direction $\vec{x}-\vec{x}_0$

为啥没c?

$$= \frac{c}{\sqrt{c^2+1}}\int_L \frac{1}{2}\left(u_t^2 + c^2|\nabla u|\right) - c\,u_t(\nabla u\cdot\vec{r})\,d\sigma \quad **$$

$$= \frac{c}{\sqrt{c^2+1}}\int_L \frac{1}{2}\left(u_t - \nabla u\cdot\vec{r}\right)^2 \underbrace{-\frac{1}{2}c^2(\nabla u\cdot\vec{r})^2 + c^2|\nabla u|^2}$$

$$\underbrace{+c^2\left(|\nabla u|^2 - \frac{1}{2}(\nabla u\cdot\vec{r})^2\right)}_{\geq 0}$$

Claim: $|\nabla u|^2 - (\nabla u\cdot\vec{r})^2 = |\nabla u - (\nabla u\cdot\vec{r})\vec{r}|^2$

$$|\nabla u - (\nabla u\cdot\vec{r})\vec{r}|^2 = (\nabla u)^2 - 2(\nabla u\cdot\vec{r})(\nabla u\cdot\vec{r}) + (\nabla u\cdot\vec{r})^2(\vec{r})^2$$

$$= \nabla u^2 - (\nabla u\cdot\vec{r})^2$$

$\geq 0$

We proved: $\int_T \frac{1}{2}\left(u_t^2 + |\nabla u|^2\right)(\vec{x},T_1)\,dx \leq \int_B \frac{1}{2}\left(g^2 + c^2|\nabla f|^2\right) dx.$

If $f=g=0 \Rightarrow u_t^2 + |\nabla u|^2 = 0 \quad \forall T_1. \Rightarrow u = \text{constant} \Rightarrow u = 0.$

## §9.2: Kirchoff formula for sol$^n$ $\begin{cases}\partial_{tt}u - c^2\Delta u = 0 & x\in\mathbb{R}^3\\ u(x,0)=f(x), \quad u_t(x,0)=g(x)\end{cases}$

$$u(\vec{x}_0,t_0) = \frac{1}{4\pi c^2 t_0}\int_S g(x)\,dx + \frac{\partial}{\partial t_0}\left(\frac{1}{4\pi c^2 t_0}\int_S f(x)\,dx\right), \quad \text{where } S=\{x\in\mathbb{R}^3,\ |x-\vec{x}_0| = ct_0\},$$

a sphere of $\mathbb{R}^3$.

Spherical means:

$f(\vec{x})$, define its spherical mean

$$\bar{f}(r) = \frac{1}{4\pi r^2}\int_{|\vec{x}|=r} f(\vec{x})\,ds$$

$$= \frac{1}{4\pi}\int_0^{\pi}\int_0^{2\pi} f(r,\theta,\varphi)\sin\theta\, d\theta\, d\varphi$$

Note: The Laplacian in sphere coordinate

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$

Step 1: Define $\bar{u}(r,t) = \frac{1}{4\pi r^2}\int_{|\vec{x}|=r} u(x,t)\,ds$.

Step 2: If $u$ satisfies the wave equation, then

$$\frac{\partial^2}{\partial t^2}\bar{u} - \underbrace{c^2\left(\frac{\partial^2\bar{u}}{\partial r^2} + \frac{2}{r}\frac{\partial\bar{u}}{\partial r}\right)}_{\text{radical laplacian.}} = 0$$

Step 3: Define $v(r) = r\bar{u}(r)$, $v$ satisfies

$$\begin{cases} \partial_{tt}v - c^2\partial_{rr}v = 0 \\ v(0,t) = 0, \quad v(r,0) = r\bar{f}(r), \quad v_t(r,0) = r\bar{g}(r) \end{cases}$$

Step 4: $\bar{u}(0,t) = \lim_{r\to 0}\frac{v(r,t)}{r} = u(0,t)$

Step 5: Invariance of the wave equation by translation.

$w(x,t) = u(x+x_0,t)$, $\begin{cases} w_{tt} - c^2\Delta w = 0 \\ w(x,0) = f(x+x_0) \\ w_t(x,0) = g(x+x_0) \end{cases}$

$w(0,t) = u(x_0,t)$

Proof of Kirchoff formula for solution of wave eq in $\mathbb{R}^3$

$$\begin{cases} u_{tt} - c^2\Delta u = 0 & x\in\mathbb{R}^3,\ t>0 \\ u(x,0) = f(x) & u_t(x,0) = g(x) \end{cases}$$

Method of sperical mean:

$$\bar{u}(r,t) = \frac{1}{4\pi r^2}\int_{|x|=r} u(x,r)\,ds$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u(r,\theta,\varphi,t)\sin\theta\, d\theta\, d\varphi$$

$$\overline{\left(\frac{\partial u}{\partial t}\right)}(r,t) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} \frac{\partial u}{\partial t}(r,\theta,\varphi,t)\sin\theta\, d\theta\, d\varphi$$

$$= \frac{\partial}{\partial t}\bar{u}(r,t)$$

Similarly, $\overline{\left(\frac{\partial u}{\partial r}\right)} = \frac{\partial}{\partial r}\bar{u}$

Step 2: Lemma 1: If $u$ satisfies the wave eq. then

$$(\bar{u})_{tt} - c^2\left((\bar{u})_{rr} + \frac{2}{r}(\bar{u})_r\right) = 0$$

Pf: $\overline{\Delta u} = \overline{\frac{\partial^2 u}{\partial r^2}} + \overline{\frac{2}{r}\frac{\partial u}{\partial r}} + \underbrace{\overline{\frac{1}{r^2}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)}}_{①} + \underbrace{\overline{\frac{1}{r^2}\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}}}_{②}$

$$① = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right)\sin\theta\, d\theta\, d\varphi$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) d\theta\, d\varphi$$

$$= \frac{1}{4\pi}\int_0^{2\pi}\sin\theta\frac{\partial u}{\partial\theta}\Big|_0^{\pi} d\varphi$$

$$= 0$$

$$② = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{1}{\sin^2\theta}\frac{\partial^2 u}{\partial\varphi^2}\sin\theta\, d\theta\, d\varphi$$

$$= \frac{1}{4\pi}\int_0^{\pi}\Big(\underbrace{\int_0^{2\pi}\frac{\partial^2 u}{\partial\varphi^2}d\varphi}_{=\frac{\partial u}{\partial\varphi}\big|_0^{2\pi}=0}\Big)\sin\theta\, d\theta$$

$$= 0$$

So $u_{tt} - c^2\Delta u = 0 \Rightarrow \bar{u}_{tt} - c^2\Delta\bar{u} = 0$

Step 3: Change of variable: $v(r,t) = r\bar{u}(r,t)$

$v$ satisfies
$$\begin{cases} v_{tt} - c^2 v_{rr} = 0 & r>0 \\ v(0,t) = 0 \\ v(r,0) = r\bar{f}(r), \quad v_t(r,0) = r\bar{g}(r) \end{cases}$$

D'Alembert formula:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0 \end{cases}$$

$$\begin{cases} u(0,t) = 0 \\ u(x,0) = F(x), \quad u_t(x,0) = G(x) \end{cases}$$

Then on $0 < x < ct$,

$$u(x,t) = \frac{1}{2}\left(F(x+ct) - F(x-ct)\right) + \frac{1}{2c}\int_{ct-x}^{x+ct} G(s)\,ds.$$

$$= \frac{1}{2c}\frac{\partial}{\partial t}\int_{ct-x}^{x+ct} F(s)\,ds + \frac{1}{2c}\int_{ct-x}^{x+ct} G(s)\,ds.$$

$$V(r,t) = \frac{1}{2c}\int_{ct-r}^{ct+r} s\bar{g}(s)\,ds + \frac{1}{2c}\frac{\partial}{\partial t}\int_{ct-r}^{ct+r} s\bar{f}(s)\,ds$$

$r < ct$: $$V(r,t) = \frac{1}{2c}\int_{ct-r}^{ct+r} s\bar{g}(s)\,ds + \frac{1}{2}\left((r+ct)\bar{f}(r+ct) - (r-ct)\bar{f}(r-ct)\right)$$

$$= \frac{1}{2c}\int_{ct-r}^{ct+r} s\bar{g}(s)\,ds + \frac{1}{2c}\frac{\partial}{\partial t}\left(\int_{ct-r}^{ct+r} s\bar{f}(s)\,ds\right)$$

$\bar{u}(0,t) = u(0,t)$ (def of $\bar{u}$)

$$\bar{u}(0,t) = \lim_{r\to 0}\frac{V(r,t) - V(0,t)}{r}, \quad V(0,t) = 0$$

把V带进来对r求导

$$= \frac{\partial V}{\partial r}(0,t)$$

$$= \frac{1}{2c}\left(2ct\,\bar{g}(ct)\right) + \frac{\partial}{\partial t}\left(\frac{1}{2c}2ct\bar{f}(ct)\right)$$

$$= t\,\bar{g}(ct) + \frac{\partial}{\partial t}\left(t\bar{f}(t)\right)$$

$$\Rightarrow u(0,t) = \frac{\cancel{t}}{4\pi c^2 t^{\cancel{2}}}\int_{|\vec{x}|=ct} g(\vec{x})\,dS_{(x)} + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{|\vec{x}|=ct} f(\vec{x})\,dS_{(x)}\right)$$

From $u(0,t) \longrightarrow u(x_0,t)$

Let $w(x,t) = u(x+x_0,t)$, $w$ satisfies

$$\begin{cases} \partial_{tt} w - c^2\Delta w = 0 \\ w(x,0) = f(x+x_0) \\ w_t(x,0) = g(x+x_0) \end{cases}$$

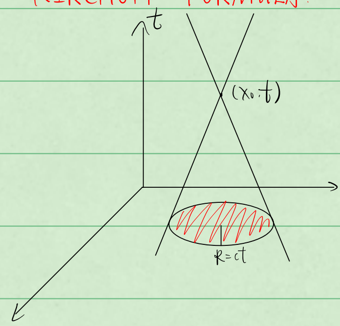
$\Rightarrow$ We can calculate $w(0,t)$, which is just $u(x_0,t)$

$$w(0,t) = \frac{1}{4\pi c^2 t}\int_{|x|=ct} g(x+x_0)\,dS_{(x)} + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{|x|=ct} f(x+x_0)\,dS_{(x)}\right)$$

Let $\vec{x}' = \vec{x} + \vec{x}_0 \Rightarrow |x' - x_0| = ct$

$$= \frac{1}{4\pi c^2 t}\int_{|x'-x_0|=ct} g(x')\,ds + \frac{\partial}{\partial t}\left(\frac{1}{4\pi c^2 t}\int_{|x'-x|=ct} f(x')\,ds\right)$$

KIRCHOFF FORMULA




Wave equation in $\mathbb{R}^2$

$$\begin{cases} \partial_{tt} u - c^2(\partial_{xx} u + \partial_{yy} u) = 0 & \vec{x} \in \mathbb{R}^2 \\ u(x,y,0) = f(x,y) \\ u_t(x,y,0) = g(x,y) \end{cases}$$

相当于在第三个维度上 extend 了所有 function, 让每条 vertical line 上的值一样.

$$u(0,0,t) = \frac{1}{4\pi c^2 t}\int_{x^2+y^2+z^2=c^2t^2} g(x,y)\,ds + \cdots$$

arbitrary

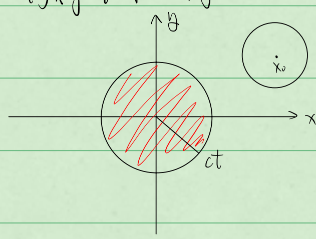
$$= 2\,\frac{1}{4\pi c^2 t}\iint_{\text{upper hemisphere}} g(x,y)\,ds + \cdots$$

$$= \frac{1}{2\pi c^2 t}\iint_{x^2+y^2\le 1} g(x,y)\,\frac{ct}{\sqrt{c^2t^2-x^2-y^2}}\,dxdy + \cdots$$

$z = \sqrt{c^2t^2 - x^2 - y^2} = h(x,y)$

$$dS = \sqrt{1 + h_x^2 + h_y^2}\,dxdy = \sqrt{1 + \frac{x^2}{c^2t^2-x^2-y^2} + \frac{y^2}{c^2t^2-x^2-y^2}}\,dxdy = \sqrt{\frac{c^2t^2}{c^2t^2-x^2-y^2}}\,dxdy.$$

$$\Rightarrow u(0,0,t) = \frac{1}{2\pi c}\int_{x^2+y^2\le c^2t^2} \frac{g(x,y)}{\sqrt{c^2t^2-x^2-y^2}}\,dxdy + \frac{1}{2\pi c}\frac{\partial}{\partial t}\int_{x^2+y^2\le c^2t^2} \frac{f(x,y)}{\sqrt{c^2t^2-x^2-y^2}}\,dxdy$$




Wave eq in $\mathbb{R}^n$:

n odd: spherical mean

$$\bar{u}(r,t) = \frac{1}{\omega_{n-1} r^{n-1}} \int_{|x|=r} u(x,t)\, dS(x)$$

to show ($\omega_n$: n 维球面面积)

$$\Rightarrow \bar{u}_{tt} - c^2 \left( \frac{\partial^2 \bar{u}}{\partial r^2} + \frac{n-1}{r} \frac{\partial \bar{u}}{\partial r} \right) = 0$$

$$\bar{u}(r,t) \xrightarrow{3D} v(r,t) = r\bar{u}(r,t)$$

$$\xrightarrow{\dim = n = 2k+1} v(r,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1} \left( r^{2k-1} \bar{u}(r,t) \right)$$

$$\dim 5 = \frac{1}{r}\frac{\partial}{\partial r} \left( r^3 \bar{u}(r,t) \right)$$

## Inhom wave in $\mathbb{R}^3$

$$\begin{cases} u_{tt} - c^2 \Delta u = h \in C^2(\mathbb{R}^3) \\ u(x,0) = u_t(x,0) = 0 \end{cases}$$

$$u(x,t) = \int_0^t \mathcal{U}(x,t,s)\, ds$$

Claim: For fixed $s$, $\mathcal{U}$ as a function of $x, t$, satisfies

$$☆ \quad \begin{cases} \partial_{tt} \mathcal{U} - c^2 \Delta \mathcal{U} = 0 \\ \mathcal{U}(x,s,s) = 0 \\ \mathcal{U}_t(x,s,s) = h(x,s) \end{cases}$$

$$u_t = \overset{0}{\mathcal{U}(x,t,t)} + \int_0^t \frac{\partial \mathcal{U}}{\partial t}(x,t,s)\, ds$$

$$u_{tt} = \frac{\partial \mathcal{U}}{\partial t}(x,t,t) + \int_0^t \frac{\partial^2 \mathcal{U}}{\partial t^2}(x,t,s)\, ds$$

$$= h(x,t) + c^2 \int_0^t \Delta \mathcal{U}(x,t,s)\, ds$$

$$\Delta u = \int_0^t \Delta \mathcal{U}(x,t,s)\, ds.$$

$$u(x,t) = \int_0^t \bar{U}(x,t,s)\, ds. \quad \Bigg| \quad \text{From } ☆, \quad \begin{cases} \partial_{tt} V - c^2 \Delta u = 0 \\ V(x,0,s) = 0 \\ V_t(x,0,s) = h(x,s) \end{cases}$$

Prop: Use Kirchoff to write $\bar{U}(x,t,s)$

Fix $s$, $V(x,t,s) = \bar{U}(x, t+s, s)$

$$V(x, t-s, s) = \bar{U}(x, t, s)$$

$\partial_{tt} V = \partial_{tt} \bar{U}$ , $\Delta V = \Delta U$

Kirchoff:

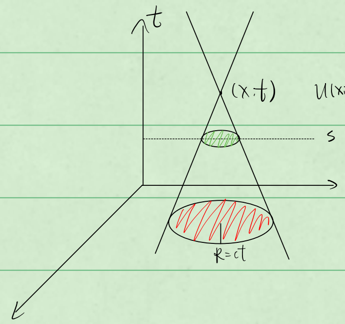
$$V(x,t,s) = \frac{1}{4\pi c^2 t}\int_{|x'-x|=ct} h(x',s)\, dS(x')$$

$$U(x,t,s) = V(x,t-s,s)$$

$$u(x,t) = \int_0^t V(x,t-s,s)\, ds \qquad s = t - \frac{|x-x'|}{c}$$

$$= \int_0^t \frac{ds}{4\pi c^2 (t-s)} \int_{|x'-x|=c(t-s)} h(x',s)\, dS.$$

double integral




$$u(x,t) = \frac{1}{4\pi c}\iint \frac{c\, h(x', t-\frac{|x-x'|}{c})}{|x-x'|}\, dS(x)\, ds$$

$$= \frac{1}{4\pi c^2}\iint_{|x-x'|\le ct} \frac{h(x', t-\frac{|x-x'|}{c})}{|x-x'|}\, dx'$$

$$\begin{cases} u_{tt} - c^2\Delta u = h \in C^2(\mathbb{R}^3) \\ u(x,0) = f \\ u_t(x,0) = g \end{cases}$$

$$u(x,t) = \mathcal{S}(t)g + \frac{\partial}{\partial t}\left(\mathcal{S}(t)f\right) + \int_0^t \mathcal{S}(t-s)h(x,s)\, ds.$$

$$\begin{cases} \partial_t u - k\Delta u = 0 \qquad x \in \mathbb{R}^3 \\ \end{cases}$$

$\begin{cases} \text{diffusion} \\ \text{Schrodinger.} \end{cases}$

# Diffusion in $\mathbb{R}^2$, $\mathbb{R}^3$.

$$\begin{cases} (1)\ \partial_t u - k\Delta u = 0 \qquad x \in \mathbb{R}^3 \\ (2)\ u(x,0) = f(x) \end{cases}$$

Recall: in 1D: $S(x,t) = \frac{1}{\sqrt{4\pi k t}} e^{-\frac{x^2}{4kt}}$ , $\partial_t S - k\partial_{xx} S = 0$

$S_3(x,y,z,t) = \prod_{cyc} S(x,t)$

$\partial_t S_3 - k\Delta f = 0$

每一行做出来反覆.

where $\partial_t S_3 = \partial_t S(x,t) \cdot S(y,t) \cdot S(z,t)$ , $\Delta S_3 = \partial_{xx} S(x,t) \cdot S(y,t) \cdot S(z,t)$

$+ S(x,t) \cdot \partial_t S(y,t) \cdot S(z,t)$ $\qquad + S(x,t) \cdot \partial_{yy} S(y,t) S(z,t)$

$+ S(x,t) \cdot S(y,t) \cdot \partial_t S(z,t)$ $\qquad + S(x,t) \cdot S(y,t) \cdot \partial_{zz} S(z,t)$

Claim: Solution of 1), 2) is:

$$u(\vec{x},t) = \int_{\mathbb{R}^3} S_3(x-x', y-y', z-z', t) f(x',y',z')\, dx'dy'dz'$$

$$= \frac{1}{(4\pi kt)^{\frac{3}{2}}} \int_{\mathbb{R}^3} e^{-\frac{(\vec{x}-\vec{x}')^2}{4kt}} f(\vec{x}')\, d\vec{x}$$

b/c $S_3(x,y,z,t) = \frac{1}{(4\pi kt)^{3/2}} \cdot e^{-x^2+y^2+z^2/4kt}$

$$\partial_t u - k\Delta u = \int_{\mathbb{R}^3} (\partial_t - k\Delta) S(\vec{x}-\vec{x}', t) f(\vec{x}')\, d\vec{x}'$$

$\downarrow$ assumed $f$ is continuous , $u \in C^2$

check $\lim_{t\to 0^+} u(\vec{x},t) = f(\vec{x})$

Solution to Schrödinger eqn'

Given $\frac{1}{i} u_t - \Delta u + V(x) u = 0$ , $V(\vec{x})$ potential

$$\begin{cases} \frac{1}{i} \partial_t u - \frac{1}{2}\Delta u = 0, & x \in \mathbb{R}^3 \\ u(x,0) = f(x) \end{cases}$$

in $\mathbb{R}^1$ : $S(x,t) = \frac{1}{\sqrt{2\pi i t}} e^{-x^2/2it}$

$$u(x,t) = \frac{1}{(2\pi t i)^{3/2}} \int_{\mathbb{R}^3} e^{-|\vec{x}-\vec{x}'|^2/2it} f(x')\, dx'$$

Let $i = i+\varepsilon$ to correspond to diffusion eqn)

$\Rightarrow \partial_t u - \frac{(i+\varepsilon)}{2} \Delta u = 0 \longrightarrow$ formula from heat eqn)

then $u(x,t) = \frac{1}{(2\pi t(i+\varepsilon))^{3/2}} \int_{\mathbb{R}^3} e^{-|\vec{x}-\vec{x}'|^2/2(\varepsilon+i)t} f(\vec{x}')\, d\vec{x}'$ as $\varepsilon \to 0^+$

where $\sqrt{i+\varepsilon}$ = complex $z$ s.t $z^2 = i+\varepsilon$ and $\mathrm{Re}(z) > 0$

$\hookrightarrow \sqrt{i} = \lim \sqrt{i+\varepsilon} = \frac{1+i}{\sqrt{2}}$

Solving harmonic oscillation:

$\mathbb{R}_n$ separation of variables , $u(x,t) = T(t)\, v(x)$

$$\begin{cases} \frac{1}{i}\partial_t u - \partial_{xx} u + x^2 u = 0 & x \in \mathbb{R} \\ u(x,0) = f(x) \end{cases}$$

$$\frac{1}{TV}\left(\frac{1}{i}T'V - TV'' + x^2 TV\right) = 0$$

$$\Rightarrow \quad \frac{1}{i}\frac{T'}{T} - \frac{V''}{V} + x^2 = 0$$

$$\Rightarrow \quad -i\frac{T'}{T} = \frac{V'' - x^2 V}{V} = -\lambda$$

$$\Rightarrow \quad \begin{cases} V'' + (\lambda - x^2)V = 0 & ; \quad T' + i\lambda T = 0 \\ V(x) \to 0 \quad \text{as} \quad x \to \pm\infty \end{cases}$$

change of variable : let $v(x) = e^{-\frac{x^2}{2}} w(x)$

$\Rightarrow$ New eq for $w(x)$ : $w'' - 2xw' + (\lambda - 1)w = 0$

" Hermite diff. eqn' "

By Fuch's theory , look for sol. in the form of taylor series near $x=0$

$w(x) = \sum_{n=0}^{\infty} a_n x^n \Rightarrow w'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$w''(x) = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - 2\sum_{n=1}^{\infty} n a_n x^n + (\lambda - 1)\sum_{n=0}^{\infty} a_n x^n = 0$$

<u>End of Chap 9</u>

① $\frac{1}{i} u_t = u_{xx} - x^2 u \quad x \in \mathbb{R}.$

② Hydrogenatom.

- Solving

$$\begin{cases} \frac{1}{i}\partial_t u = \partial_{xx} u - x^2 u & x \in \mathbb{R} \\ u \xrightarrow{x \to \pm\infty} 0 \\ u(x,0) = f(x) \end{cases}$$

$$\Rightarrow \begin{cases} V'' + (\lambda - x^2) V = 0 \quad ; \quad T' + i\lambda T = 0 \\ V(x) \to 0 \quad \text{as} \quad x \to \pm\infty \end{cases}$$

Consider

$$\begin{cases} V'' + (\lambda - x^2) V = 0 \\ V(x) \to 0 \quad \text{as} \quad x \to \pm\infty \end{cases}$$

$$V = e^{-\frac{x^2}{2}} w$$

$$\Rightarrow \quad w'' - 2xw' + (\lambda - 1) w = 0$$

Look for solutions in the form of Taylor series near $x=0$

$$w(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$\Rightarrow \sum_{k=2}^{\infty} a_k k(k-1) x^{k-2} - 2x \sum_{k=1}^{\infty} a_k k x^{k-1} + (\lambda - 1) \sum_{k=0}^{\infty} a_k x^k = 0$$

$k = 2 + k'$

$$\Rightarrow \sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k + \sum_{k=0}^{\infty} (-2k + \lambda - 1) a_k x^k = 0$$

$$\Rightarrow \sum_{k=0}^{\infty} ( * ) x^k = 0 \quad \Rightarrow \quad * = 0$$

$$\Rightarrow \forall k, \quad (k+2)(k+1) a_{k+2} + (-2k + \lambda - 1) a_k = 0$$

Recurrence relation. $a_0 \to a_2 \to a_4 \to \cdots$
$a_1 \to a_3 \to a_5 \to \cdots$

$$2a_2 = (1 - \lambda) a_0$$

$$6a_3 = (3 - \lambda) a_1$$

$$\vdots$$

$$(k+2)(k+1) a_{k+2} = (2k + 1 - \lambda) a_k.$$

$\lambda = 1, \quad \begin{cases} a_0 = 1 \\ a_1 = 0 \end{cases} \Rightarrow a_2, a_3, \cdots = 0$

$\lambda_1 = 1 \; \cdots\cdots \; w_0(x) = 1$ (we can always take $a_0 = 1$)

$\lambda = 3 \quad \begin{cases} a_0 = 0 \\ a_1 = 1 \end{cases} \quad \begin{matrix} a_{2k} = 0 \\ a_{2k+1} = 0 \end{matrix}$ 因为W满足的是二阶微分方程，所以有两个自由度.

$\lambda_2 = 3 \; \cdots\cdots \; w_1(x) = 2x \quad = H_1(x)$

$\lambda_3 = 5 \; \cdots\cdots \; w_2(x) = 4x^2 - 2 \quad = H_2(x)$

$\lambda_4 = 7 \; \cdots\cdots \; w_3(x) = 8x^3 - 12x \quad = H_3(x)$

$$\vdots$$

$W_k(x) = H_k(x) \longrightarrow v_k(x) = e^{-\frac{x^2}{2}} w_k(x)$

$\lambda_k = 2k+1$ , solve $T_k' + i\lambda_k T_k = 0$

$\Rightarrow$ We found solution $u_k(x,t) = e^{-\frac{x^2}{2}} H_k(x) \cdot e^{-i(2k+1)t}$

$u(x,t) = \sum_{k=0}^{\infty} C_k e^{-\frac{x^2}{2}} H_k(x) \cdot e^{-i(2k+1)t}$

$u(x,0) = f(x) = \sum_{k=0}^{\infty} C_k e^{-\frac{x^2}{2}} H_k(x)$

Prop: (*) $\int_{\mathbb{R}} H_m(x) H_n(x) e^{-x^2} dx = 0$

$\int_{\mathbb{R}} H_m^2(x) e^{-x^2} dx = \sqrt{\pi}\, 2^n n!$

Assume (*) is true. then

$\left(H_m(x) e^{-\frac{x^2}{2}}\right) f(x) = \left(\sum_{k=0}^{\infty} C_k e^{-\frac{x^2}{2}} H_k(x)\right) H_m(x) e^{-\frac{x^2}{2}}$

$\Rightarrow C_m = \frac{1}{\sqrt{\pi}\, 2^n n!} \int_{\mathbb{R}} H_m(x) e^{-\frac{x^2}{2}} f(x)\, dx$

Pf: Use $w'' - 2xw' + (\lambda-1)w = 0$ ($H_k$ satisfies this)

"Sturm - Liouville pb"

$(p(x)u')' + q(x)u = \lambda m(x)$

$e^{-x^2}\left(w'' - 2xw' + (\lambda-1)w = 0\right)$

$\Rightarrow \left(w' e^{-x^2}\right)' + (\lambda-1) w e^{-x^2} = 0$

$\begin{cases} w_k \Leftrightarrow \lambda = \lambda_k \\ w_l \Leftrightarrow \lambda = \lambda_l \end{cases}$

① $w_l \cdot \left(w_k' e^{-x^2}\right)' + (\lambda_k - 1) w_k e^{-x^2} = 0$

② $w_k \cdot \left(w_l' e^{-x^2}\right)' + (\lambda_l - 1) w_l e^{-x^2} = 0$

看一下.

$\int$①$-$②: $\int \left[-\cancel{w_l' w_k' e^{-x^2}} + \cancel{w_l' w_k' e^{-x^2}} + (\lambda_{k-1} - \lambda_{l+1}) w_k w_l e^{-x^2}\right] dx = 0$

$\Rightarrow \underbrace{\left(\int_{\mathbb{R}} w_l(x) w_k(x) e^{-x^2} dx\right)}_{=0} \underbrace{(\lambda_k - \lambda_l)}_{\neq 0} = 0$

- $u_t - k\Delta u = 0$, $x \in D \subseteq \mathbb{R}^{2/3}$ bounded
- $u_{tt} - c^2 \Delta u = 0$

+ Boundary conditions : either $u|_{\partial D}=0$ or Neumann $\frac{\partial u}{\partial n}|_{\partial D}=0$

+ Initial condition.

## General properties:

$$\begin{cases} u_t - k\Delta u = 0 & x \in D \subset \mathbb{R}^n \\ u|_{\partial D} = 0 \\ u(x,0) = f(x) \end{cases}$$

Separation of variables: $u = T(t)\,V(x)$

$\Rightarrow \frac{T'}{kT} = \frac{\Delta V}{V} = -\lambda$

$\Rightarrow T' + k\lambda T = 0$

EVP $\begin{cases} -\Delta V = \lambda V \\ V|_{\partial D} = 0 \end{cases}$ e.v. problem.

**Prop**: Assume $(\lambda, v)$ sol$^n$ of (EVP)

$\int_D v(\Delta v + \lambda v)\,dx = 0$

(Divergence thm: $\int_D f \Delta g\,dx = -\int_D \nabla f \nabla g + \int_{\partial D} f \frac{\partial g}{\partial n}\,ds$

$\Rightarrow -\int_D |\nabla v|^2 + \lambda \int_D |v|^2\,dx = 0$

$\Rightarrow \lambda \int_D |v|^2\,dx = \int |\nabla v|^2 \geq 0$

① $\lambda \geq 0$. $\lambda > 0$.

② Is $\lambda = 0$ e.v.?

No. $\lambda = 0 \Rightarrow \nabla V = 0 \Rightarrow V = c^{st} = 0$ $(\partial D)$

No

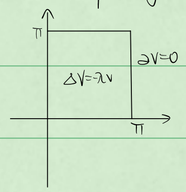
③ 2 e.f with $\neq$ e.v are orthogonal

(1): $-\Delta V_1 = \lambda_1 V_1$

(2): $-\Delta V_2 = \lambda_2 V_2$

$\int (1)V_2 - (2)V_1 = 0 \Rightarrow \int V_1 V_2 = 0$

④ Multiplicity of e.v. (in general, multiplicity can be >1)




$D = (0,\pi)\times(0,\pi)$

$$\begin{cases} -\Delta V = \lambda V \\ V = 0 \quad \text{on bd} \end{cases} \qquad V_{xx}+V_{yy}+\lambda u = 0$$

$V(x,y) = X(x)\,Y(y)$

$\Rightarrow X''Y + Y''X + \lambda XY = 0 \quad \Rightarrow \dfrac{X''}{X} + \dfrac{Y''}{Y} = -\lambda$

$\Rightarrow \dfrac{X''}{X} = -\mu, \quad \dfrac{Y''}{Y} = -\nu, \quad \mu+\nu = \lambda.$

$$\Rightarrow \begin{cases} X'' + \mu X = 0 \\ X(0) = X(\pi) = 0 \end{cases} \qquad \begin{cases} Y'' + \nu Y \\ Y(0) = Y(\pi) = 0 \end{cases}$$

$\mu_n = n^2 \quad X_n(x) = \sin(nx) \qquad \nu_m = m^2 \quad Y_m(y) = \sin(my) \qquad (m, n \in N^+)$

$V_{m,n}(x,y) = \sin(nx)\sin(my)$

associated to e.v. $\lambda_{m,n} = n^2 + m^2$