2022.9.13 Lec2-3

Extend $\mathbb{C}$ to ∞:

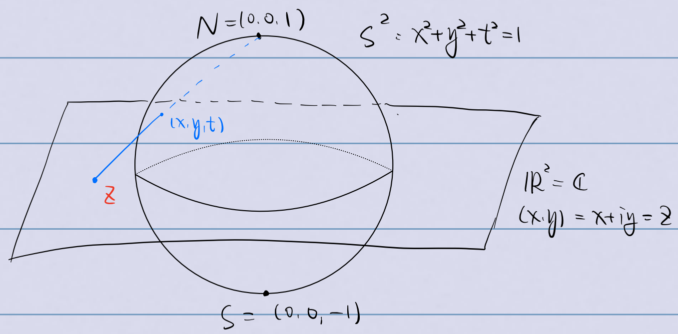
What property should ∞ have?

$a+\infty=\infty+a=\infty$

$a\cdot\infty=\infty\cdot a=\infty$ (if $a\neq 0$)

similarly, $a/\infty=0$, $a/0=\infty$

$(a\neq\infty)$ $(a\neq 0)$



Stereographic projection of (x,y,t) from N

"z"? $z=\dfrac{x+iy}{1-t}$ (三点共线) (x,y,t) $(0,0,1)$ $(0,0,t)$

Homeomorphism of $S^2\setminus\{N\}$ onto $\mathbb{C}$ $\Rightarrow$ $N=$ pt at ∞

这个对吗
检查下

Stereographic projection of (x,y,t) from S:

$z=\dfrac{x+yi}{1-t}$ (只是1变号了而已)

Also consider: complex conjugate of stereographic projection

$z'=\dfrac{x-yi}{1+t}$

For any point of $S^2\setminus\{N,S\}$

$zz'=\dfrac{x+iy}{1-t}\,\dfrac{x-iy}{1+t}=\dfrac{x^2+y^2}{1-t^2}=1$ $\Rightarrow$ $zz'=1$, $z'=\dfrac{1}{z}$

这啥?

在 $z\to\infty$ 时，$z'=0$，所以在 ∞ 附近我们用 z' (coordinate chart) 来表示 z

Under stereo proj from N:

Straight line in plane corresponding to a circle including N. "把这条直线包上去"

Any circle in S^2 correspond to a circle or straight line in $\mathbb{C}$

pf: Any circle in S^2 lies in $\mathbb{R}^3$: $ax+by+cz=d$.

$(x,y,t) \to \frac{x+yi}{1-t}$

$|z|^2 = z\bar{z} = \frac{x^2+y^2}{(1-t)^2} = \frac{1-t^2}{(1-t)^2} = \frac{1+t}{1-t}$

$\Rightarrow \quad t = \frac{|z|^2-1}{|z|^2+1} \qquad 1-t = \frac{2}{|z|^2+1}$

$\frac{1}{2}(z+\bar{z}) = \frac{x}{1-t} \Rightarrow x = \frac{1}{2}(z+\bar{z})(1-t) = \frac{z+\bar{z}}{|z|^2+1}$

$\frac{1}{2}(z-\bar{z}) = \frac{yi}{1-t} \Rightarrow y = \frac{1}{2}(z-\bar{z})(1-t) = \frac{z-\bar{z}}{i(|z|^2+1)}$

$\Downarrow$ x, y, z 在面上

$\Rightarrow \quad a(z+\bar{z}) + \frac{b}{i}(z-\bar{z}) + c(|z|^2-1) = d(|z|^2+1)$

write $z=u+vi$, 整理到 RHS

$(d-c)(u^2+v^2) - 2au - 2bv + (d+c) = 0$

$\Rightarrow \begin{cases} \text{circle if } d \neq c \\ \text{line if } d = c \end{cases}$

Complex function?

linear functions? $T: \mathbb{C} \to \mathbb{C}$

$Tz = az$ (composition of $\underset{a>0}{\text{scaling}}$ and $\underset{|a|=1}{\text{rotation}}$)

preserves angles. and orientation.

e.g. $z \mapsto \bar{z}$ real linear but not complex linear.

real linear of $\mathbb{R}^2 = \begin{pmatrix} \alpha & \beta \\ \theta & \gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

$\mathbb{C} = az = (a+bi)(x+yi)$

$= (ax-by)+(bx+ay)i \Rightarrow \begin{pmatrix} a & b \\ -b & a \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \subseteq$ Real

Exercise: find all angle-preserving $\mathbb{R}$-lin function T of $\mathbb{C}$

let S be homothetic lin trans s.t. $S^{-1}T$ fixes $(1,0)$

$S^{-1}T = (0,1) \mapsto (0,c) \quad c \neq 0$

Consider $(S^{-1}T)(1,1) = (1,c)$ + angle preserving, $c = \pm 1$

$$\begin{cases} c=1: & S^{-1}T = Id, \quad Tz = az \\ c=-1: & S^{-1}T: z \mapsto \bar{z} \Rightarrow Tz = a\bar{z} \end{cases}$$

Polynomial functions:

$f(z) = a_n x^n + \cdots + a_0$

$= a_n(z-c_1)(z-c_2)\cdots(z-c_n)$

Rational functions:

$R(z) = \frac{P(z)}{Q(z)}$

zeros of $Q(z)$ called poles of $R(z)$

$\lim_{z \to z_0} R(z) = \infty \quad (Q(z_0) = 0)$

Order of pole: order of corresponding zero of Q

Poles of $R'(z)$ same as $R(z)$

z_0 k 阶 $R(z) \Rightarrow k+1$ 阶 $R'(z)$

define $R(\infty) = \lim_{z \to \infty} R(z)$

Order of a pole at ∞?

We should think both z and $R(z)$ as points in Riemann Sphere.

Consider $R(z)$ in coordinate at ∞. Consider $R_1(z) = R(1/z)$

R has $\overset{\circ}{\text{pole}}$ at ∞ $\Leftrightarrow$ R_1 has $\overset{\circ}{\text{pole}}$ at 0.

Order of a $\begin{cases}\text{zero}\\ \text{pole}\end{cases}$ of $R(z)$ at ∞

$\Leftrightarrow$ . . ─ ─ ─ ─ ─ ─ ─ ─ ─ ─ $R_1(z)$ at 0.

e.g. $R(z) = \dfrac{a_m x^m + \cdots + a_0}{b_n x^n + \cdots + b_0}$ $(a_m, b_n \neq 0)$

$R_1(z) = R(\frac{1}{z}) = z^{n-m} \dfrac{a_0 x^m + \cdots + a_m}{b_0 x^n + \cdots + b_n}$

$\begin{cases} n > m: R(z) \text{ has a zero at } \infty \text{ of order } n-m \\ n < m: \text{- - - - - - - pole - - - - - - - - - } m-n \\ n = m: R(\infty) = \frac{a_0}{b_0} \end{cases}$

Moral: Total # of zeros of $R(z)$ in Riemann Sphere.

$= \max(m, n)$

$=$ . . - - - - - - - poles . . . . - - - - - - -

所有 Rational function 在 R.S 上有同样个数的 0/pole.

Lec 4

$R(x) = \dfrac{P(x)}{Q(x)}$ 最简

order $(R) = \max\{\deg(P), \deg(Q)\}$

If R is not constant, then R has k poles / zeros in Riemann Sphere. $R(x) = a$ also.

Ex. Rational function of order 1:

$S(z) = \dfrac{az+b}{cx+d}$ $(ad - bc \neq 0)$ 实开分母?

fractional linear / Möbius transformation.

$R(z) = w$ has exactly 1 root $\Rightarrow$ S invertible $\Rightarrow$ $w = S^{-1}(z) = \dfrac{dz-b}{-cz+a}$

Ex. $S(z) = z + a$ $\leftarrow$ ∞ fixed point

$S(z) = \frac{1}{z}$ $\leftarrow$ ± 1 fixed point.

Representation of rational functions by partial fraction:

$R(z) = \frac{P(z)}{Q(z)}$, first do long division until $\deg(P) < \deg(Q)$

$$\Rightarrow R(z) = \underbrace{G(z)}_{\text{poly without constant term}} + \frac{\tilde{H}(z)}{Q(z)}$$

$\frac{\tilde{H}(z)}{Q(z)} =: H(z)$

$H(z)$ is finite in ∞

$\deg(G)$ = the order of pole of $R(z)$ at ∞

如果 $P > Q$，则在 ∞ 处有 pole.

< - - - - - - - zero.

$G(z)$: singular part of $R(z)$ at ∞

Let $\beta_1, \cdots, \beta_t$ be distinct finite poles of $R(z)$

$z = \beta_j + 1/\zeta \Rightarrow \zeta = \frac{1}{z - \beta_j}$

$R(\beta_j + 1/\zeta)$: Rational function of ζ with pole at ∞

So we can write

$R(\beta_j + 1/\zeta) = G_j(\zeta) + H_j(\zeta)$ like before.

G_j: poly without const. H_j: finite @ ∞

$$\Rightarrow R(z) = G_j\left(\frac{1}{z-\beta_j}\right) + H_j\left(\underbrace{\frac{1}{z-\beta_j}}\right)$$

G_j: poly w/out const of $\frac{1}{z-\beta_j}$, singular part of R @ β_j

H_j: finite at $z = \beta_j$

ζ的作用是，每个"ζ"将一个对应的 β_j pole "拉到" ∞ 去，这帮助我们重新进行 这个过程.

想说明 $\sum G_j = R + G$ 因为 $G = \frac{\alpha_0}{z-\beta_j} + \frac{\alpha_1}{(z-\beta_j)^2} + \cdots$ 是想要的形式

Consider $R(z) - G(z) - \sum_{j=1}^{l} G_j\left(\frac{1}{z-\beta_j}\right) =$

Rational function, poles at most $\{\beta_j\} \cup \{\infty\}$

when $z = \beta_j$, only 2 terms might become infinity i.e. $R(z)$, $G_j\left(\frac{1}{z-\beta_j}\right)$

However, $R(z) - G_j\left(\frac{1}{z-\beta_j}\right) < \infty$

when $z = \beta_j$, same thing. $R(\infty) - G(\infty) < \infty$

这意味着 $R(z) - G(z) - \sum_{j=1}^{l} G_j\left(\frac{1}{z-\beta_j}\right)$ 是一个没有 pole 的 rational function.

$\Rightarrow R(z) - G(z) - \sum_{j=1}^{l} G_j\left(\frac{1}{z-\beta_j}\right) = c$ (constant)

we can absorb constant into $G(z)$

$\Rightarrow R(z) = G(z) + \sum_{j=1}^{l} G_j\left(\frac{1}{z-\beta_j}\right)$

What about real case? 共轭根成对出现.

$$\frac{1}{z-\beta} + \frac{1}{z-\bar{\beta}} = \frac{*}{(z-\beta)(z-\bar{\beta})} = \frac{*}{z^2 - (\underbrace{\beta+\bar{\beta}}_{\text{real}})z + \underbrace{\beta\bar{\beta}}_{\text{real}}}$$

Rational Functions:

Classification of Rational functions of order 2. to fractiona linear changes of sources and targets: 在 coordinate change 下不变.

Rational function of order 2 has 2 poles

$\Rightarrow$ Either 1 double pole or 2 distinct poles.

① Double poles.

Make fractional linear transformation to move β to ∞.

$z = \beta + \frac{1}{z}$ $\Rightarrow$ A rational function with poles at ∞

分母是0.

$\Rightarrow$ A polynomial of order 2.

$\Rightarrow w = az^2 + bz + c$

$$= a\left(z+\frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c$$

$$\underbrace{w + \frac{b^2}{4a} - c}_{w'} = a\underbrace{\left(z+\frac{b}{2a}\right)}_{z'}^2$$

w上的坐标变换.

z上的坐标变换

$\Rightarrow w = z^2$

② 2 poles.

Make fractional linear transformation to move a, b to ∞

$z \to \frac{z-b}{z-a}$? 啊?

$w = Az + B + \frac{C}{z}$ $\Rightarrow$ $w = z + \frac{1}{z}$

① Make the coefficient of $z, \frac{1}{z}$ equal:

$z' = \sqrt{\frac{A}{C}}\, z$

$\Rightarrow w = A\left(z + \frac{1}{z}\right) + B$

$\Rightarrow \frac{1}{A}(w - B) = z + \frac{1}{z}$

Rational Function of $w = S(z) = \frac{az+b}{cz+d}$ $ad - bc \neq 0$

$S(\infty) = \frac{a}{c}$, $S\left(-\frac{d}{c}\right) = \infty$

① When $c = 0$,

$S(z) = az + b$

② $c \neq 0$

$$\frac{az+b}{cz+d} = \frac{a}{c} + \frac{\frac{bc-ad}{c}}{cz+d}$$

$$= \frac{a}{c} + \frac{1}{z+\frac{d}{c}} \cdot \frac{bc-ad}{c^2}$$

a composite of translation $z_1 = z + \frac{d}{c}$ $z_2 = \frac{1}{z_1}$ $z_3 = \text{const} \cdot z_2$,

$z_4 = z_3 + \frac{a}{c}.$

Thm: FLT takes $\begin{Bmatrix} \text{circles} \\ \text{lines} \end{Bmatrix} \xrightarrow{T} \begin{Bmatrix} \text{circles} \\ \text{lines} \end{Bmatrix}$

Given any pair $\{\text{circles}/\text{lines}\}$, $\exists$ FLT take one to the other

Lemma: Given any $z_2, z_3, z_4 \in \hat{S}$, $\exists!$ FLT S, s.t. $S(z_2, z_3, z_4) = 1, 0, \infty$

$$S(z) = \frac{z-z_3}{z-z_4} \Big/ \frac{z_2-z_3}{z_2-z_4}$$

if $z_2 = \infty$: $S(z) = \dfrac{z-z_3}{z-z_4}$

if $z_3 = \infty$: $S(z) = \dfrac{z_2-z_4}{z-z_4}$

if $z_4 = \infty$: $S(z) = \dfrac{z-z_3}{z_2-z_3}$

unique ? 若 g 为另一个,

$f \circ g^{-1}$: $1, 0, \infty \mapsto 1, 0, \infty$

$\hookrightarrow \dfrac{az+b}{cz+d}$ 带进去后发现 $=1$.

Def: Cross-ratio $(z_1 : z_2 : z_3 : z_4) = S z_1$

Thm 1: if $z_1, z_2, z_3, z_4 \in \hat{S}$, distinct, T FLT,

$\Rightarrow (Tz_1, Tz_2, Tz_3, Tz_4) = (z_1, z_2, z_3, z_4)$

Thm 2: $(z_1 : z_2 : z_3 : z_4) \in \mathbb{R} \Leftrightarrow$ 4 points lies on a circle or a line.

Proof of Thm:

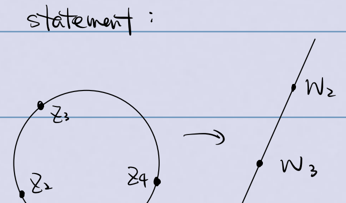
1st statement directly follows from Thm 1.2.

? $\{z_i\}$ 在 ○/| $\Rightarrow (z_i) \in \mathbb{R}$

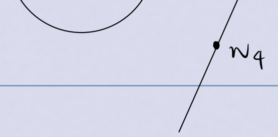
$\Rightarrow (Tz_i) \in \mathbb{R}$

$\Rightarrow \{Tz_i\}$ 在 ○/|

2nd statement:



先 pull 到 $1, 0, \infty$,
再 pull back.



$w = Tz \Leftrightarrow \underbrace{(w : w_2 : w_3 : w_4)}_{\text{关于 } w \text{ 的式子}} = \underbrace{(z ; z_2 : z_3 : z_4)}_{\text{关于 } z \text{ 的式子}}$

Proof of Thm 1:

Let $S(z) = (z : z_2 : z_3 : z_4)$

因为 $S: z_2, z_3, z_4 \mapsto 1, 0, \infty$

$\Rightarrow ST^{-1}: Tz_2, Tz_3, Tz_4 \mapsto 1, 0, \infty$

$(Tz_1, Tz_2, Tz_3, Tz_4) = G(Tz_1)$, where $G \mapsto Tz_2, Tz_3, Tz_4 \mapsto 1, 0, \infty$

这里是 Cross-r 的定义.

$\Rightarrow G = ST^{-1}$

$\Rightarrow \quad = ST^{-1}(Tz_1)$

$= S(z_1) = (z_1 : z_2 : z_3 : z_4)$

Proof of Thm 2:

(a) Image of real axis under FLT T^{-1} is $\{C/L\}$

$w = T^{-1}(z) \quad z$ real

To see: w satisfies equations of C/L

$z = T(w) = \dfrac{aw+b}{cw+d}$

$\Rightarrow Tw = \overline{Tw}$

$\Rightarrow \dfrac{aw+b}{cw+d} = \dfrac{\bar{a}\bar{w}+\bar{b}}{\bar{c}\bar{w}+\bar{d}}$

$\Rightarrow (aw+b)(\bar{c}\bar{w}+\bar{d}) = (\bar{a}\bar{w}+\bar{b})(cw+d)$

$\Rightarrow \underbrace{(a\bar{c}-c\bar{a})}_{\text{Imagnary}}|w|^2 + \underbrace{(a\bar{d}-c\bar{b})w + (b\bar{c}-d\bar{a})\bar{w}}_{\text{Im}} + \underbrace{(b\bar{d}-d\bar{b})}_{\text{Im}} = 0$

$w = x+iy$, 全部是虚的 $\Rightarrow$ 乘 i 之后是关于 x, y 的实的方程

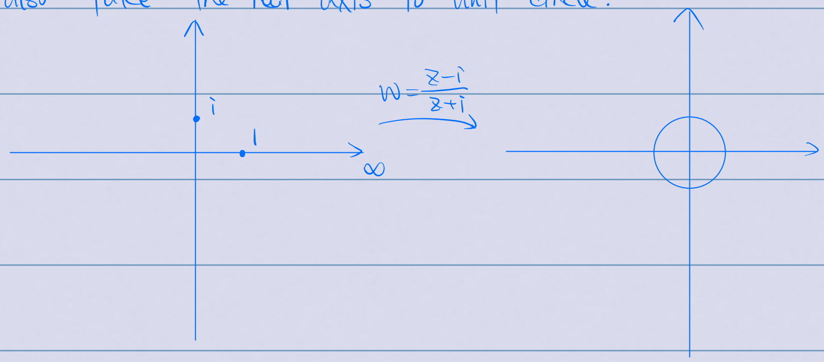
$\begin{cases} (a\bar{c}-c\bar{a}) \neq 0, & \text{满足圆的方程} \\ \qquad = 0 & \text{······ 直线方程.} \end{cases}$

b) $S z = (z : z_2 : z_3 : z_4)$ is real on image of real axis under

S^{-1} and nowhere else. 草...

Ex: FLT that takes the upper half plane H^+ to unit disk D

also take the real axis to unit circle.



$0 \mapsto -1$
$\infty \mapsto 1$
$1 \mapsto -i$

2. HOLOMORPHIC Functions.

$f(z)$ complex valued function in open set $\Omega \subseteq \mathbb{C}$

$f(z)$ is holomorphic at $z \in \Omega$ if $f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists.

i.e, $f(z+h) - f(z) = ch + \varphi(h)h$ where $\lim_{h\to 0} \varphi(h) = 0$

$z = x+iy$, $f = u+iv$, $c = a+ib$, $h = \xi + i\eta$

Derivative at z: $h \mapsto ch$

对 c 有要求.

$$\begin{pmatrix} \xi \\ \eta \end{pmatrix} \mapsto \begin{pmatrix} \underbrace{\begin{matrix} a \\ b \end{matrix}}_{\frac{\partial f}{\partial x}} & \underbrace{\begin{matrix} -b \\ a \end{matrix}}_{\frac{\partial f}{\partial y}} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

在 $\mathbb{C}$ 中，对 z 求导
$\Updownarrow$
在 $\mathbb{R}^2$ 中，对 (x,y) 求导

f is holomorphic at z

$\Leftrightarrow$ differentiable at z (as function (u,v) of (x,y))

在 z 点上的导数应该是 $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ 的形式

and partial derivative at z satisfies $\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \end{cases}$ or $\frac{\partial f}{\partial x} + i \frac{\partial f}{\partial y} = 0$

Cauchy - Riemann equations.

Jacobian determinant : $\det\left(\frac{\partial(u,v)}{\partial(x,y)}\right) = |f'(z)|^2$

Consider $f(x,y)$, differentiable complex valued function

<u>Differential</u> $df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$

Example: 关于x,y的函数

$z = x+iy$ $\qquad\qquad dz = dx+idy$

$\Rightarrow$

$\bar{z} = x-iy$ $\qquad\qquad d\bar{z} = dx-idy$

$$dx = \frac{dz+d\bar{z}}{2}$$

$$dy = \frac{dz-d\bar{z}}{2i}$$

$$\Rightarrow \quad df = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)dz + \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)d\bar{z}$$

So we define $\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

为什么要这么定义？

Thus we can write $df = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$???

∂x, dy span tangent space of $\mathbb{R}^2$, $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$ span cotangent space of $\mathbb{R}^2$.

∂z $\partial\bar{z}$ - - — — —— — — $\mathbb{C}$, $\frac{\partial}{\partial z}$, $\frac{\partial}{\partial \bar{z}}$ - - - - - - - — —— — - — -

$\Leftrightarrow$ $\frac{\partial f}{\partial \bar{z}} = 0$ ($\Rightarrow$ 是一个只与z有关的式子 (与$\bar{z}$无关))

$\Leftrightarrow$ holomorphic.

$$\frac{\partial^2 f}{\partial z \partial \bar{z}} = 0$$

$\Updownarrow$

$f(x,y)$ is <u>harmonic</u> if $f \in C^2$ and $\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ Laplace's equation

$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ Laplacian operator.

We'll see holomorphic function is harmonic (if f C^2)

$\Rightarrow$ So real and imaginary part of holomorphic function are harmonic

Remark: $\frac{\partial f}{\partial \bar z} = 0 \iff \frac{\partial \bar f}{\partial z} = 0$ $\qquad f = u + iv \Rightarrow \bar f = u - iv$

$$\frac{\partial f}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

$$\Rightarrow \overline{\frac{\partial f}{\partial \bar z}} = \frac{1}{2}\left(\overline{\frac{\partial f}{\partial x}} - i\overline{\frac{\partial f}{\partial y}}\right)$$

$$= \frac{\partial \bar f}{\partial z}$$

Lemma: if $f(z)$ holomorphic (in connect open set) and $f'(z) = 0$ (identically)

then f is a constant.

Pf: $\underbrace{df}_{=0} = \underbrace{\frac{\partial f}{\partial z}}_{=0 \text{ hypothesis}} dz + \underbrace{\frac{\partial f}{\partial \bar z}}_{=0 \text{ holom}} d\bar z$

可以把 df 写成 dx, dy 的形式,然后是0.

Basic properties of holomorphic functions.

if $f(z)$ holomorphic (in connect open set) and $f'(z) = 0$ (identically) then f is a constant.

Prop: Given f holomorphic in connected open Ω

(1) If $|f|$ constant $\Rightarrow f(z)$ constant

(2) If $|Re(f)|$ constant $\Rightarrow f(z)$ constant.

Pf: (1): $|f(z)|^2 = f \cdot \bar f$

$$0 = \frac{\partial}{\partial z}|f(z)|^2 = \frac{\partial f}{\partial z}\bar f + f \cdot \frac{\partial \bar f}{\partial z} = \overline{\frac{\partial f}{\partial \bar z}} = 0, \text{ holomorphic.}$$

$$\Rightarrow 0 = \frac{\partial f}{\partial z}\bar f = \left\{\begin{matrix} \frac{\partial f}{\partial z} = 0 \\ \bar f = 0 \end{matrix}\right\} \Rightarrow f \text{ constant.}$$

但这里只考虑了某个点的情况.

(2) $\mathrm{Re}(f) = \dfrac{f+\bar{f}}{2}$

应该要在 nbhd上成立才行.

所以:

$|f(z)|$ const $\begin{cases} \text{either } 0 \\ \text{not } 0. \end{cases}$

$$\frac{d}{dz}\mathrm{Re}(f) = \frac{1}{2}\left(\frac{df}{dz} + \frac{d\bar{f}}{dz}\right) = 0$$

① 若 $|f(z)|=0$. 即成立

$$\Rightarrow \frac{\partial f}{\partial z}dz + \underbrace{\frac{\partial f}{\partial \bar{z}}d\bar{z} + \frac{\partial \bar{f}}{\partial z}dz}_{=0} + \frac{\partial \bar{f}}{\partial \bar{z}}d\bar{z} = 0$$

② 若 $|f(z)| \neq 0 \Rightarrow \bar{f} \neq 0$ everywhere.

$\Rightarrow \dfrac{\partial f}{\partial \bar{z}} = 0$

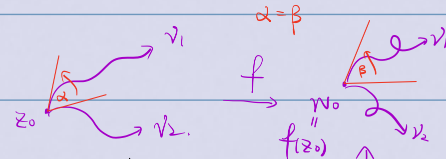
$$\Rightarrow \frac{\partial f}{\partial z}dz + \frac{\partial \bar{f}}{\partial \bar{z}}d\bar{z} = 0 \Rightarrow \frac{\partial f}{\partial z} = \frac{\partial \bar{f}}{\partial \bar{z}} = 0.$$

Mapping Properties

$f(z)$ holom at some points. z_0

Tangent mapping f at z_0:

导数给出的线性映射 $w = c\cdot z$, $c = f'(z_0)$



If $c \neq 0$, then the tangent mapping perserves angle and orientation.

We say that holo f is conformal at any pt z_0 if $f'(z_0) \neq 0$

Lemma: A $\mathbb{R}$-linear transformation from $\mathbb{C} \to \mathbb{C}$ is whether of form $w = cz$ or $w = c\bar{z}$

$\hookrightarrow \dfrac{\partial f}{\partial \bar{z}} = 0$ $\qquad \hookrightarrow \dfrac{\partial f}{\partial z} = 0.$

Consider $w = f(z)$ in a connected open set Ω, assume f is continuously differentiable, and Jac det $\neq 0$ at every point.

If f preserves angles at every pt of Ω, then either $\frac{\partial f}{\partial z} = 0$ or $\frac{\partial f}{\partial \bar{z}} = 0$ at every points of Ω.

They can't be both 0, since $\partial z, \partial \bar{z}$ can be written as $\partial x, \partial y$, violates that Joc det $= 0$

Since $\frac{\partial f}{\partial z}$, $\frac{\partial f}{\partial \bar{z}}$ continuous, $\Rightarrow \{\frac{\partial f}{\partial z} = 0\}$, $\{\frac{\partial f}{\partial \bar{z}} = 0\}$ disjoint closed set, So one of them must be empty.

的意思是 tangent map preserves angle.

Therefore, if f preserves angles at every point, then either

$$\begin{cases} \dfrac{\partial f}{\partial \bar{z}} = 0 \Rightarrow f \text{ holomorphic} \\ \dfrac{\partial f}{\partial z} = 0 \Rightarrow f \text{ anti-holomorphic.} \end{cases}$$

这是当且仅当.

$$\begin{cases} \text{保角保 orientation} \Rightarrow \text{holom} \\ \text{只保角: holo / anti-holo} \end{cases}$$

Inverse function theorem

Suppose f holomorphic around z_0 and $f'(z_0) \neq 0$

Then $\exists\, U \ni z_0$, $V \ni f(z_0)$, s.t. f maps U onto V with an inverse.

$z = g(w)$, which is also holomorphic.

follows that $g'(w) = \dfrac{1}{f'(z)}$ ← chain rule

Pf: To be completed later. : we'll use the fact that partial derivatives of holo f is continuous. 之后可以从实数上的 IFT.

然后 $f'(z) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \Rightarrow g'(w) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}^{-1}$, 也是相同的形式.

Complex power series.

$\sum_{n=0}^{\infty} a_n (z-z_0)^n \quad a_n \in \mathbb{C}. \qquad (z_0 \in \mathbb{C}$, 不妨设 $z_0 = 0)$

Thm: $\sum_{n=0}^{\infty} a_n z^n$, $\exists\, 0 \leq R \leq \infty$, s.t.

① $\forall\, r < R$, series $\sum a_n z^n$ converges uniformly and absolutely in disk $|z| \leq r$

② $\forall\, |z| > R$, the series diverges. Terms of series unbounded.

R: radius of convergence.

○: circle of convergence.

③ Derived series: $\sum n a_n z^{n-1}$ has same radius of convergence R

and let $f(z) = \sum a_n z^n \quad (|z|<R)$. Then $f(z)$ is holomorphic and

$f'(z) = \sum n a_n z^{n-1}$

Example:

$\sum n! z^n$, radius of convergence $= 0$.

$\sum \frac{z^n}{n!}$, radius of convergence $= \infty$

$\sum z^n$, radius of convergence $= 1$

$\sum \frac{z^n}{n}$, radius of convergence $= 1$

$\sum \frac{z^n}{n^2}$, radius of convergence $= 1$

Exponential and logrithmic function.

Def: $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ $\qquad \infty$

$\frac{d}{dz} e^z = \sum \frac{z^n}{n!} = e^z$

$e^{z+w} = e^z e^w$ ① 展开

②: Let $g(z) = e^z e^{c-z}$

$g'(z) = e^z e^{c-z} + (-e^z e^{c-z}) = 0$ product rule.

$\Rightarrow g(z)$ constant. $g(0) = e^c$.

$\Rightarrow \quad e^z e^w = e^z e^{(z+w)-z} = e^{z+w}$.

Let $z = x+iy$, then $e^{x+iy} = e^x e^{iy}$

$e^{iy} = \cos y + i \sin y$. 带到定义里。偶数次项会给出 cos，奇数次项提出 i 之后是 sin.

$\Rightarrow e^{i\pi} = -1$

$|e^{iy}| = 1 \in$ unit circle. $= S^1$

$|e^z| = e^x$

Mapping : $y \mapsto e^{iy}$

$\mathbb{R} \to S^1 \leq \mathbb{C}$ $\longrightarrow$ group homomorphism.

group under addition

group under multiplication

kernel $= \{2k\pi \mid k \in \mathbb{N}\}$

$\Rightarrow$ $\mathbb{R}/\{2k\pi\} \cong S^1$ homeomorphism, where LHS is the quotient topology.

why homeo? Both compact Hausdorff space.

(continuous bijective mapping between compact hausdorff space is homeo)

$$\mathbb{R}/\{2k\pi\} \xrightarrow{\cong} S^1$$

called "arg"

$$\arg(z) \longleftarrow z$$

only defined up to $2k\pi$.

for any non-zero $z \in \mathbb{C}$, we define $\arg(z) = \arg\left(\frac{z}{|z|}\right)$

$\Rightarrow$ $z = |z| e^{i\arg(z)}$

Trig functions:

Def $\begin{cases} \cos(z) = \dfrac{e^{iz} + e^{-iz}}{2} \\ \sin(z) = \dfrac{e^{iz} - e^{-iz}}{2i} \end{cases}$

$\Rightarrow$ $e^{iz} = \cos z + i \sin z$

$\Rightarrow$ $\cos^2 z + \sin^2 z = 1$ $\quad (e^{iz} \overline{e^{iz}} = 1)$

$\cos' z = -\sin z$

$\sin' z = \cos z$

$\begin{cases} \cos(z+w) \cdots \\ \sin(z+w) \cdots \end{cases}$

Complex log

Want to solve $e^w = z$, when $z \neq 0$.

$$z = |z| e^{i \arg z}$$
$$= e^{\log|z|} e^{i \arg z}$$
$$= e^{\log|z| + i \arg z}$$

$w = \log|z| + i \arg z$

Def: $\log z = \log|z| + i \arg z$, Defined up to integral multiples of $2k\pi$

$e^{\log z} = z$

if $z = x > 0$, then we get the classical $\log x$ if we allow only value 0 for $\arg z$.

if $z, z' \neq 0$, $\log(zz') = \log(z) + \log(z')$?

this make sense mod $2k\pi$.

Branches of $\log z$:

Let $f(z)$ be a continuous function in a connected open set Ω, we say $f(z)$ is a branch of $\log z$ if $\forall z \in \Omega$, $e^{f(z)} = z$

We'll study: what condition must Ω satisfy for a branch of log to exist?

Lemma: suppose there is a branch $f(z)$ of $\log z$ in a connected set Ω, then any other branches has form

$$f(z) + 2k\pi i, \quad \text{for } k \in \mathbb{Z}$$

conversely, $\forall k \in \mathbb{Z}$, $f(z) + 2k\pi i$ is a branch.

Pf: Suppose $f(z)$, $g(z)$ are both branches of $\log z$,

let $h(z) = \dfrac{f(z) - g(z)}{2\pi i}$ $\to$ continuous, values integers. Since Ω connect,

h is constant.

We can likewise define a branch of $\arg(z)$ in an connect open set (not containing 0)

Any branch of $\operatorname{Arg} z$ defines a branch of $\log z$, and vise versa.

Prop: If $f(z)$ is a branch of $\log z$ in a conneted open set Ω,

then $f(z)$ is holomorphic, $f'(z) = \frac{1}{z}$

Pf: $\lim\limits_{h \to 0} \dfrac{f(z+h) - f(z)}{h}$ have to see this exists and $= \frac{1}{z}$

$$= \lim_{h \to 0} \frac{f(z+h) - f(z)}{z+h-z}$$

$$= \lim_{h \to 0} \frac{\underbrace{f(z+h)}_{=w} - f(z)}{e^{f(z+h)} - e^{f(z)}}$$

$\lim\limits_{h \to 0} w \to f(z)$

$$= \lim_{w \to f(z)} \frac{w - f(z)}{e^w - e^{f(z)}}$$

$$= \lim_{w \to f(z)} \frac{1}{\frac{e^w - e^{f(z)}}{w - f(z)}}$$

$= 1/(e^w)'$ 在 $f(z)$ 的值

$= 1/e^{f(z)} = \frac{1}{z}$.

Power series operation

Complex power series $f(w) = \sum_{n=0}^{\infty} a_n w^n$ $\quad g(w) = \sum_{n=0}^{\infty} b_n w^n$

Does composite $f(g(z))$ make sense? $\to$

是个无穷级数和.

$f(g(z)) = a_0 + a_1(b_0 + b_1 z + \cdots) + a_2(b_0 + b_1 z + \cdots)^2 + \cdots$ coefficient of z^n?

当 $b_0=0$ 时, k足够大时 $(b_0+b_1z+\cdots)^k$ 不会提供 ($k=n+1$) ↳ Yes, if $b_0=0$

z^n 项, 所以 z^n 前系数是有限和.

e.g. Infinite taylor series of $f(g(z))$ at $z=z_0$

$w_0=g(z_0)$, Taylor series of f at w_0 $\sum a_n(\underbrace{w-w_0}_{=b_0})^n$

然后将 g 在 z_0 处展开. $\sum b_p(z-z_0)^n$.

为什么 ↗ 总是可以形? 好像因为 反正因为没有常数项.

Formal Inverse Function thm

Given formal power series $f(w)=\sum_{n=0}^{\infty}a_nw^n$, then $\exists\ g(z)=\sum_{p=0}^{n}b_pz^p$,

s.t. $b_0=0$ and $f\circ g=id$ ($id: f(z)=z$)

iff $f(0)=0$, $f'(0)\neq 0$ $\Leftrightarrow$ $a_0=0$, $a_1\neq 0$. 需要f导数不为0.

($f=\sum a_nz^n \Rightarrow f'=\sum na_nz^{n-1}$)

In this case, g is unique and $g\circ f=id$

Proof by method of undetermined coefficient.

$$a_0+a_1(b_1z+b_2z^2+\cdots)+a_2(b_1z+b_2z^2+\cdots)+\cdots=z.$$

$\Rightarrow a_0=0,\ a_1b_1=1 \Rightarrow f(0)=0 \neq f'(0)$ necessary. (同时可确定g的唯一性)

coefficient of z^n in LHS same as coefficient of z^n in

$a_0+a_1g(z)+\cdots+a_ng(z)^n$ (更大的项不提供 z^n 项)

$a_1b_n+\underbrace{P(a_2,\cdots,a_n,b_1,\cdots,b_{n-1})}_{\text{poly}}$, $b_1=1/a_1$, 之后可以归纳地算.

正整数系数, 对 a_i, b_i 线性.

Since $g(0)=0$, $g'(0)\neq 0$, $\Rightarrow g\circ f_1=id$

$f_1=id\circ f_1=(f\circ g)\circ f_1=f\circ(g\circ f_1)=f$

$f(w)=\sum_{n=0}^{\infty} a_n w^n$, $\quad g(z)=\sum_{p=1}^{n} b_p z^p$

<u>Prop</u>: f, g convergent $\Rightarrow$ $f\circ g$ convergent

无常数项，所以可以做到

Take $r>0$, s.t. $\sum_{1}^{\infty}|b_p|r^p < R(f)$ $\leftarrow$ f的收敛半径

Then (1) $R(f\circ g)\geq r$

and if $|z|<r$, then (2) $|g(z)|<R(f)$

(3) $\underbrace{f(g(z))}_{\text{substituting}} = \underbrace{(f\circ g)(z)}_{\text{power series 意义下的.}}$

Pf of (1): $\underbrace{\sum_{n=0}^{\infty}|a_n|\left(\sum_{p=1}^{\infty}|b_p|r^p\right)^n}_{\sum_{k=0}^{\infty}\nu_k r^k} < \infty$

Say $(f\circ g)(z) = \sum_{k=0}^{\infty} c_k z^k$, 则有 $|c_k|\leq \nu_k$,

三角不等式. c_k里的每一项取norm后加起来是ν_k.

So $\sum|c_k|r^k<\infty \Rightarrow R(f\circ g)\geq r$.

<u>Reciprocal of a power series</u>:

If $f=\sum_{n=0}^{\infty}a_n z^n$, $a_0\neq 0$, then $\exists\, g(z)$, s.t. $f(z)g(z)=1$

if $R(f)>0$, so does $R(g)>0$.

<u>Pf</u>: We can assume $a_0=1$ let $f(z)=1-h(z)$, then $h(0)=0$

$(1-h(z))^{-1}$, composite of $\underbrace{1+\sum_{n=1}^{n} w^n}_{\frac{1}{1-w}}$ with $w=h(z)$

?怎么就证完了?

<u>Inverse function thm for convergent power series</u>

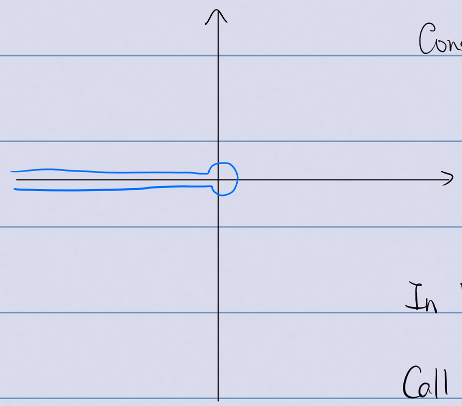
In the previous statement, $R(f)>0 \Rightarrow R(g)>0$.

By direct estimate

or follow from inverse fn thm for holomorphic fns.

once we know holomorphic fn has infinite Taylor series, that converges and represents the fn.

Principal branch of $\log z$



Consider $\mathbb{C} \setminus (-\infty, 0]$

In here, there is a unique value of $\arg z \in (-\pi, \pi)$

Call it $\operatorname{Arg} z$, principal branch of $\arg(z)$

Enough to show it's continuous on $S^1 \setminus \{-1\}$

$$\begin{tikzcd} S^1 \setminus \{1\} \arrow[r, bend right, "\operatorname{Arg}(z)"'] & (-\pi, \pi) \arrow[l, bend right, "z = e^{i\theta}"'] \end{tikzcd}$$

principal branch of $\log z$: $\log|z| + i\operatorname{Arg} z$, continuous on $\mathbb{C} \setminus (-\infty, 0]$

在 $\mathbb{R}_{>0}$ 上与 $\log$ 重合.

Prop: Power series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^n}{n}$ converges if $|z| < 1$

and sum = principal branch of $\log(1+z)$

Pf: The power series $f(z)$ and $g(w) = \underbrace{\sum_{n=1}^{\infty} \frac{w^n}{n!}}_{e^w - 1}$ are inverses.

pf by 1st year calculus

我可没学过!

$g(f(z)) = z \Rightarrow e^{f(z)} = 1+z \Rightarrow f$ is a branch of $\log(z+1)$

Principal ? 只需检查 $z=0$ 时取值即可.

Def: $\alpha \in \mathbb{C}$, $z \neq 0$.

$z^{\alpha} = e^{\alpha \log z}$ for fixed α, z^{α} many valued fn of z.

Has a branch in any domain Ω where log has a branch.
con - open - set

Any branch of $\log z$ defines branch of z^{α}.

e.g. binomial series $(1+z)^{\alpha} = e^{\alpha \log(1+z)}$

power series expansion in $|z|<1$ is $\sum \binom{\alpha}{n} z^n$

W4 LEC 2

<u>Mapping properties of holomorphic functions</u>

$w = z^{\alpha}$, α real positive

conformal

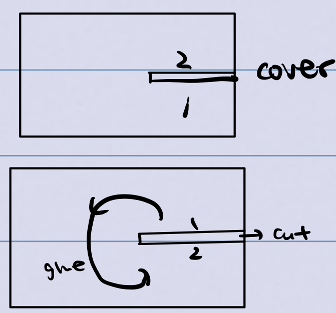
z^{α} not 1-1 if $\alpha \neq 1$, multivalued if α fractional.

Holomorphic in wedge $\frac{dw}{dz} = \frac{\alpha w}{z}$ $\quad z^{\alpha} = e^{\alpha \log z}$

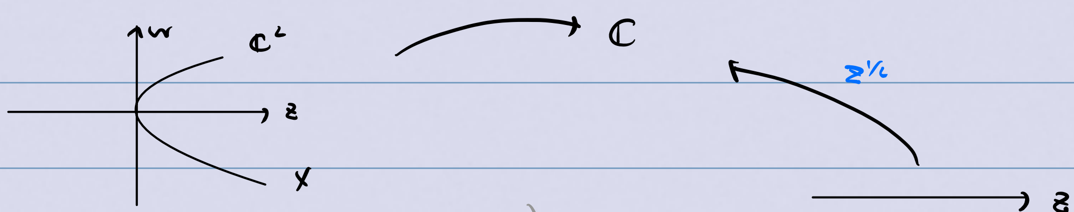
e.g $w = z^{1/2}$

$$\begin{tikzcd} X \arrow[dr, "\text{single value of function}"] \arrow[d, "\text{branched covering}"'] & \\ \mathbb{C} \arrow[r, "z^{1/2}"'] & \mathbb{C} \end{tikzcd}$$

multi value of function



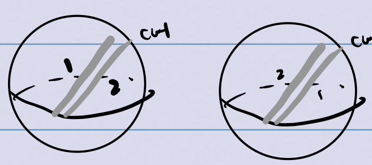
$z = w^2$, $X: \{z = w^2\} \subset \mathbb{C}^2$ $\quad (z,w) \to w$



(manifold as a graph of a function,
local coordinate : (z,w))

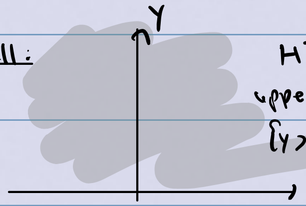
function on the manifold : $(z,w) \mapsto w$

Multivalued function
$w = z^{1/2}$ lift to
single valued function
$(w,z) \to w$ on covering
surface X
↓
<u>Riemann Surface</u>



⇓ gives another sphere

Recall:

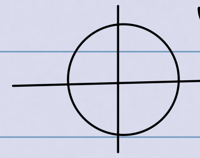


H^+
upper hemi-sphere
$\{y>0\}$

$z = x+iy$

fractional linear transformation

$w = \dfrac{z-i}{z+i}$, $i \to 0$



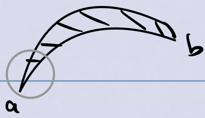
unit disk

$D = \{|z|<1\}$

check three points : $0 \to -1$
$\infty \to 1$
($+$ or $-i$) $1 \to -i$

since fractional lin. transformations are orientation perserving

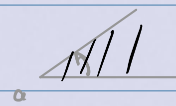
conforming mapping of circular wedge onto D on H^+



$\to$ conform: straighten angle by using power of z

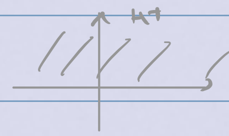
① $a \to 0$, $b \to \infty$ by $\xi = \dfrac{z-a}{z-b}$

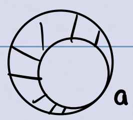
(frac. lin. tr. takes line $\leftrightarrow$ circle)



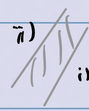
OR

② $w = \xi^{\alpha}$ (after rotation)



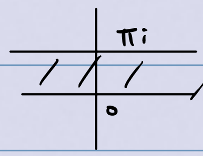


$\to$ ① $a \to \infty$ $\xi = \dfrac{1}{z-a}$

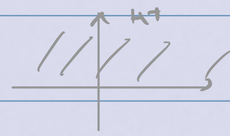


② i) $\to$ positive x-axis
ii) $\to$ negative x-axis

by

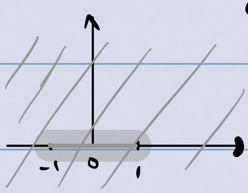


exponential function $\longrightarrow$

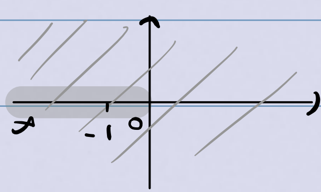


Exercise

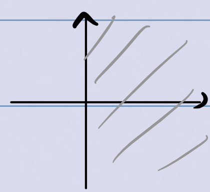
conformal mapping of complement of line segment to interior (or exterior) of unit disk



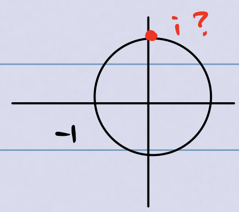
$-1 \to 0$
$1 \to \infty$ by $z_1 = \dfrac{z+1}{z-1}$
$0 \to -1$



$\xrightarrow{z_2 = z_1^{1/2}}$



$$w = \frac{z_2 - 1}{z_2 + 1}$$



$\therefore$ we get $w = z - \sqrt{z^2 - 1}$

check: which square root ?

$$(z - \sqrt{z^2-1})(z + \sqrt{z^2-1}) = 1$$

we know $|w| < 1$

show: $z = \frac{1}{2}\left(w + \frac{1}{w}\right)$

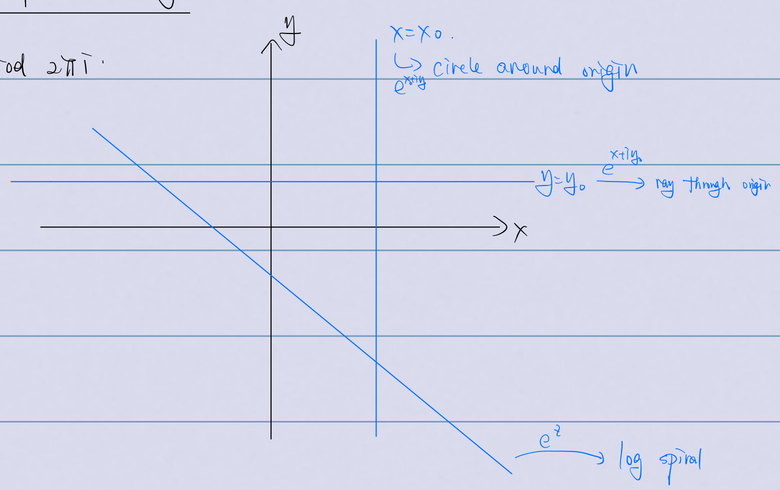
Mapping properties of exp and log

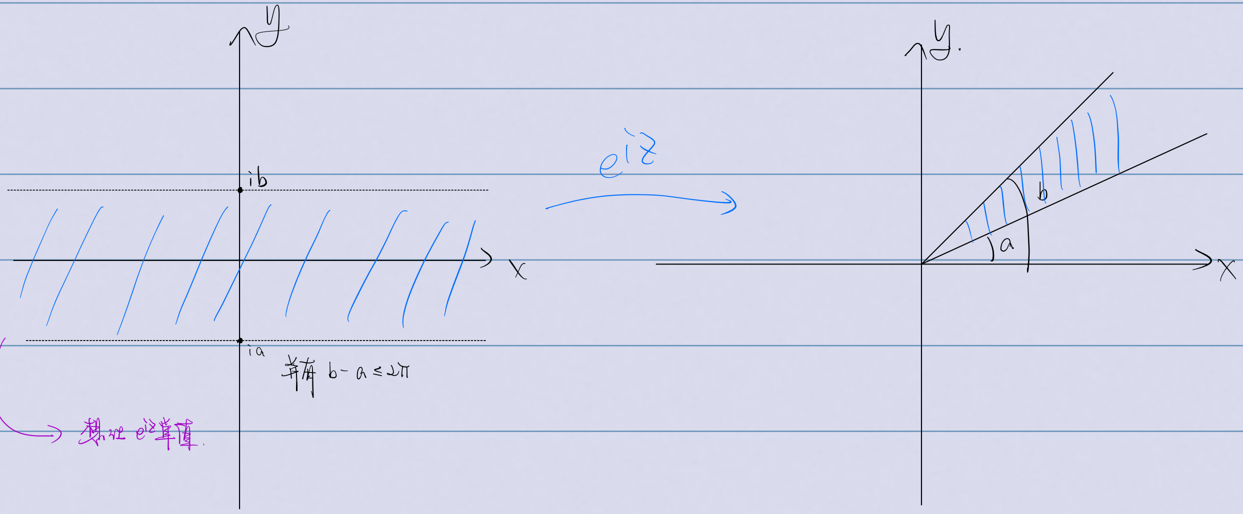
$w = e^z$: periodic. period $2\pi i$.

$= e^{x+iy}$

$= e^x e^{iy}$

$= e^x(\cos y + i \sin y)$





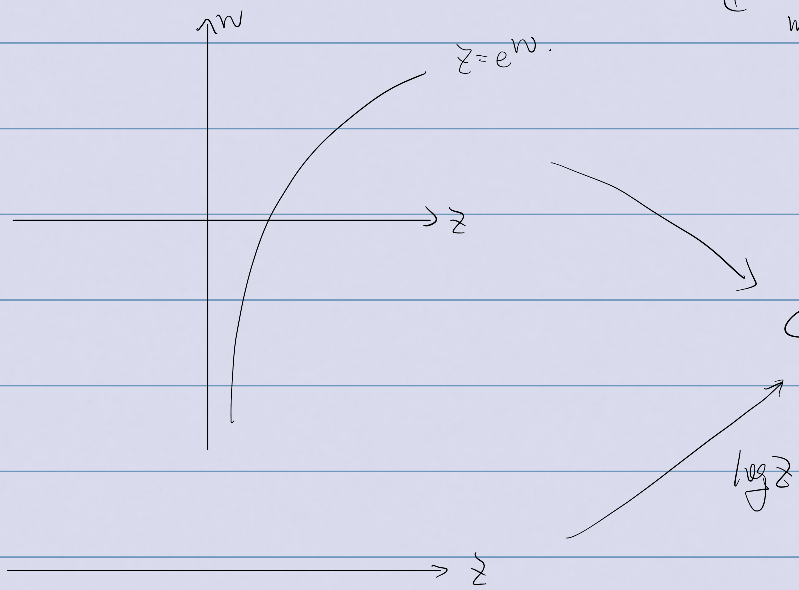
并有 $b - a \leq 2\pi$

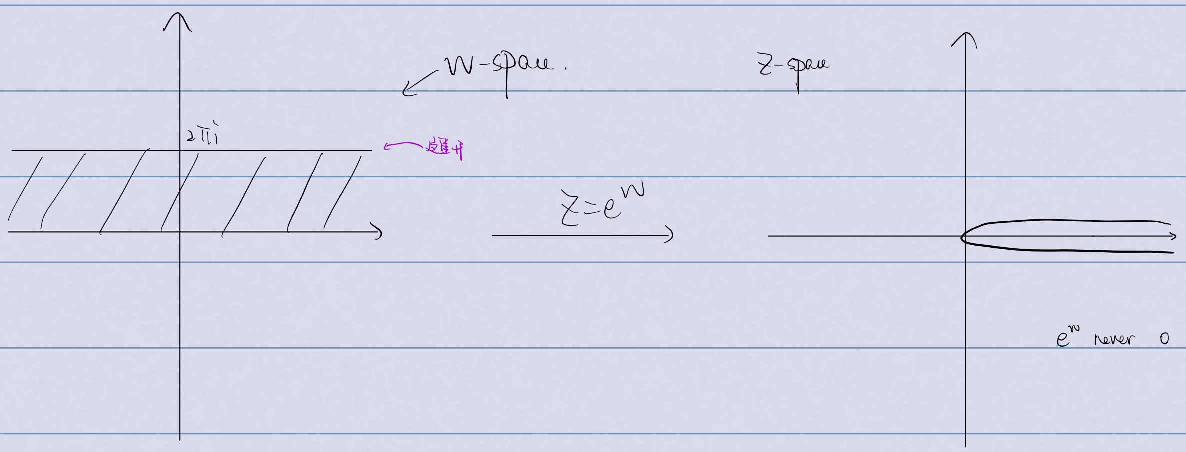
想让 e^{iz} 单值.

Riemann surface of $w=\log z$

$$\begin{tikzcd} X \subset \mathbb{C}^2 \arrow[r, "\text{single-valued}"] \arrow[d, "\text{covering space}"'] & \mathbb{C} \\ \mathbb{C} \arrow[ur, "\log z"'] & \end{tikzcd}$$

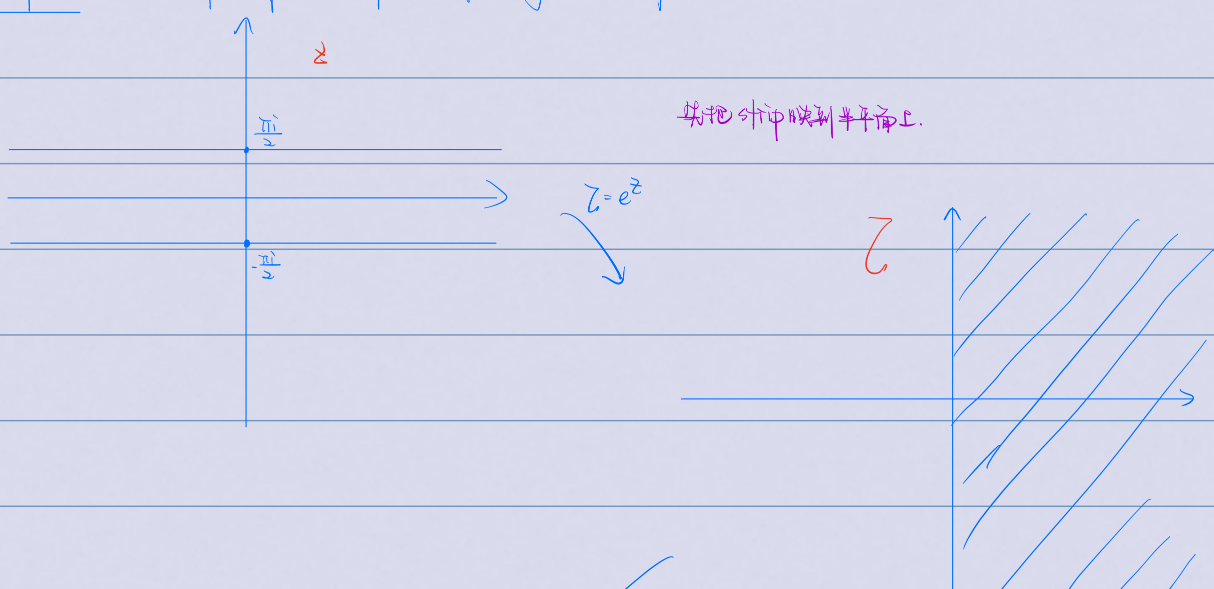
multi-valued.

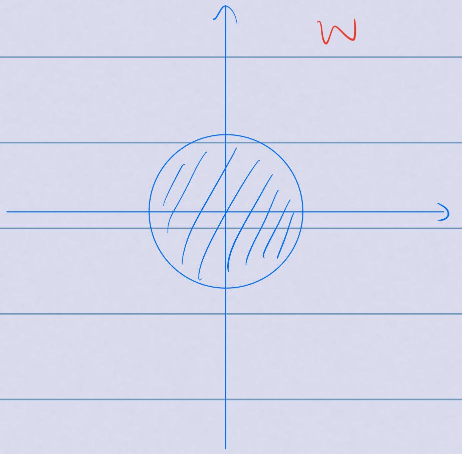




Example: Map open strip conformally onto open unit disk.

先把 strip 映到半平面上.

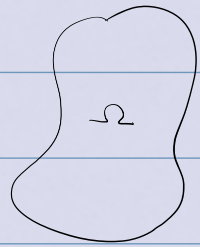




$w = \frac{z-1}{z+1}$

$\Rightarrow \quad w = \frac{e^z - 1}{e^z + 1}$

Analytic functions



$f(z)$ analytic in open Ω if it has convergent series representation at every point $z_0 \in \Omega$

指的是 power series 在 z_0 附近开集收敛到 f

If $f(z)$ has convergent series representation at z_0, then it is a convergent power series $g(z)$ at z_0, s.t. $g'(z) = f(z)$ in same disk $|z - z_0| < r$

$$f(z) = \sum a_n (z-z_0)^n$$

$$g(z) = \underbrace{\sum \frac{a_n}{n+1} (z-z_0)^{n+1}}_{\text{分项积分}}$$

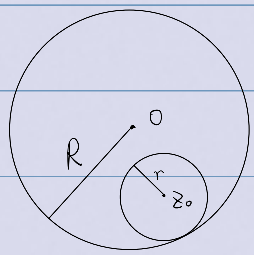
收敛域相同

Primitive uniquely determined up to a constant.

这tm是 antiderivative.

Does convergent series define an analytic function?

Prop: If $f(z)=\sum a_n z^n$, $R(f)=R$, then $f(z)$ analytic in $|z|<R$



i.e., for any $|z_0|<R$, f 在 z_0 处有 convergent series representation

We'll show there's a g, s.t. g converges uniformly and absolutely in $|z-z_0|\le r$

for any $r<R-|z_0|$

Pf: $f(z)=\sum a_n\left(z_0+(z-z_0)\right)^n$

$$=\sum a_n\left(\sum \binom{n}{k} z_0^{n-k}(z-z_0)^k\right)$$

$$\sum |a_n|\left(|z_0|+|z-z_0|\right)^n=\sum |a_n|\sum\binom{n}{k}|z_0|^{n-k}|z-z_0|^k$$ 这个东西收敛,所以可以 rearrange.

$$\Rightarrow f(z)=\sum_{k=0}^{\infty}\left(\underbrace{\sum_{n=0}^{\infty} a_n\binom{n}{k} z_0^{n-k}}_{}\right)(z-z_0)^k$$

$= \frac{1}{k!}f^{(k)}(z)$ $\leftarrow$?? nb.

Principle of analytic continuation:

Thm: $f(z)$ analytic in domain Ω, $z_0\in\Omega$,

TFAE: 1) $f^{(n)}(z_0)=0$, $n=0,1,\cdots$

2) f identically zero in nbhd of z_0

3) $f\equiv 0$ in Ω

Pf: 3) $\Rightarrow$ 1) 显然

1) $\Rightarrow$ 2) 因为 power series 中的每个系数都是 0.

2) $\Rightarrow$ 3):

Let $\Omega'=\{z\in\Omega: f$ is identitally zero in a neiborhood of z in $\Omega\}$

$z_0 \in \Omega' \Rightarrow \Omega' \neq \emptyset$

Ω' open by definition

Ω' closed?

Take $z \in \overline{\Omega'}$ w/ $f^{(n)}(z) = 0$ $n = 0, 1, 2, \cdots$ (by continuity)

Since 1) $\Rightarrow$ 2), we know that $z \in \Omega'$, so

Cor: If f, g analytic in Ω, if $f = g$ in a neighborhood of some point, then $f = g$ in Ω

Corollary: Ring $A(\Omega)$ of analytic functions on Ω, then $A(\Omega)$ is an integral domain.

i.e. $fg = 0$ then $f \equiv 0$ or $g \equiv 0$.

if $f \not\equiv 0$, then $\exists z_0$ s.t. $f \neq 0$ around z_0, then $g = 0$ in this nbhd, so $g \equiv 0$.

Zeros and poles:

Consider f analytic around z_0. $f(z) = \sum a_n (z - z_0)^n$ $|z - z_0| < \delta$.

Suppose $f(z_0) = 0$, but $f \not\equiv 0$.

Let k be the smallest integer s.t. $f^{(k)}(z_0) \neq 0$ $\Leftrightarrow a_k \neq 0$

Then $f(z) = (z - z_0)^k g(z)$, where g analytic and $g(z_0) \neq 0$

$\Rightarrow g(z) = \sum_{n=k}^{\infty} a_n (z - z_0)^{n-k}$ (k: the order / multiplicity of zero z_0 of f)

(Characterized by $f^{(j)} = 0$, $f^{(k)} \neq 0$

This shows that $f(z) \neq 0$, for $0 < |z - z_0| < \varepsilon$ for some ε

isolated 0.

If we make a coordinate change near z_0

$$\zeta = (z-z_0)\, g(z)^{\frac{1}{k}}$$

then $f(z(\zeta)) = \zeta^k$ i.e, $w=f(z)$ becomes $w=\zeta^k$

Quotient of a function

$\frac{f(z)}{g(z)}$, where $g(z) \neq 0$ around z_0.

↳ well-defined, analytic in nbhd of any z_0 s.t. $g(z_0) \neq 0$.

What if $g(z_0) = 0$? $\begin{cases} f(z) = (z-z_0)^k f_1(z) \\ g(z) = (z-z_0)^l g_1(z) \end{cases}$ where $f_1(z_0), g_1(z_0) \neq 0$.

$$\Rightarrow \frac{f(z)}{g(z)} = (z-z_0)^{k-l} \frac{f_1(z)}{g_1(z)}$$ → analytic, z_0附近≠0.

$\begin{cases} ① \ k \geq l, \text{ then } f/g \text{ extends to be analytic at } z_0 \\ ② \ k < l, \ z_0 \text{ is a pole of } f/g, \text{ order } l-k. \end{cases}$

i.e, $\left|\frac{f}{g}\right| \to \infty$ as $z \to z_0$. 在 Riemann Sphere 上有意义.

Meromorphic function: in open Ω.

fn which is well-defined, analytic in complement of discrete set.

and expressible in nbhd of any point of Ω. (比如局部才形 f/g)

Mero fns in Ω form a field.

Exercise: if $f(z)$ mero in Ω, then $f'(z)$ is also mero.

same poles.

Cauchy's Integral Formula

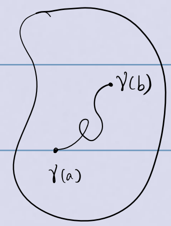
Integration over curves

$\Omega \subset \mathbb{R}^2$ open

Curve in Ω : $\gamma : [a,b] \to \Omega$

usually C^1

$\gamma(t) = (x(t), y(t))$



Differential (1-) form : $\omega = P\,dx + Q\,dy$, where P, Q are continuous functions on Ω, ($\mathbb{R}$- or $\mathbb{C}$- valued)

Def : $\int_\gamma \omega = \int_a^b F(t)\,dt$, where $F(t) = P(x(t), y(t))\,x'(t) + Q(x(t), y(t))\,y'(t)$

$\left(\int_\gamma w = \int_a^b \gamma^* w \right.$, $\gamma^*(P) = P \circ \gamma$, $\gamma^*(dx) = d(\gamma^* x) = d(x \circ \gamma)$ $(x \circ \gamma)(t) = x(\gamma(t)) = x(t)$ $\left. = x'(t) \right)$

Change of parameter : $\delta(u) = \gamma(t(u))$, where $t : [c,d] \to [a,b]$,

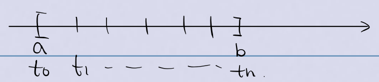
$t(c,d) = (a,b)$, $t'(u) > 0$

$\delta^* w = (\gamma \circ t)^* w = t^* \circ \gamma^* w = F(t(u))\,t'(u)\,du$.

$\Rightarrow \int_\delta w = \int_\gamma w$ by integration by substitution So $\int_\gamma w$ depend on w, γ, not on choice of parametrion.

oriented curve

If $t(c,d) = (b,a)$, $t'(u) < 0$, then $\int_\gamma w = -\int_\delta w$



then $\int_\gamma \omega = \sum_{i=1}^{n} \int_{\gamma_i} \omega$, where $\gamma_i = \gamma|_{[t_{i-1}, t_i]}$

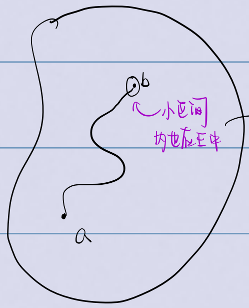
Independent of choice of partition.

So piecewise C^1 is enough.

closed curve : $\gamma(a) = \gamma(b)$. $\int_\gamma$ 与起点终点无关.

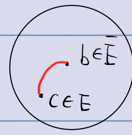
Lemma: Any 2 points of connect open set $\Omega \subset \mathbb{R}^2$ can be connected by piecewise C^1 curve

Pf: Fix $a \in \Omega$, Let $E = \{b \in \Omega \mid a, b$ can be connected by piecewise C^1 curve$\}$



$\Rightarrow E$ open, not empty. 只需证 E closed.

$\forall b \in \bar{E}$



$\Rightarrow b \in E$

$\Rightarrow E$ clopen $\Rightarrow E = \Omega$.

Primitive of ω:

C^1 function F on Ω, s.t. $\omega = dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy$.

$\int_\gamma dF = \int_a^b (F \circ \gamma)'(t)\, dt = F(\gamma(b)) - F(\gamma(a))$

e.g. if Ω connected, $dF = 0 \Rightarrow F$ is a constant.

Q: Given w, can we find primitive?

Prop: w has a primitive iff $\int_\gamma w = 0$ for every piecewise C^1 closed curve.

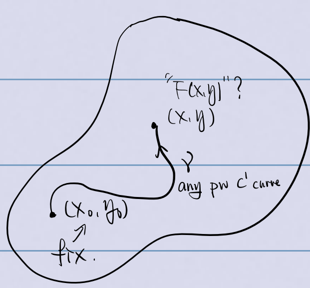
Proof: "$\Rightarrow$" is simple, since $\int_\gamma w = \int_\gamma dF = F(\gamma(b)) - F(\gamma(a)) = 0$.

"$\Leftarrow$": Define $F(x,y) = \int_\gamma w$. $\leftarrow$ why well-defined?

要证与路径选择无关,
而这是题设.

因为 $\int_{\gamma_1} w + \int_{-\gamma_2} w = 0$

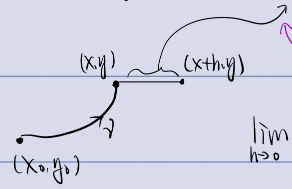
$\Rightarrow \int_{\gamma_1} w = \int_{\gamma_2} w$.



只需证偏导存在且是我们想要的.

$$F(x+h,y)-F(x,y)=\int_x^{x+h} P(t,y)\,dt$$

h足够小时



$$\lim_{h\to 0}\frac{F(x+h,y)-F(x,y)}{h}=\lim_{h\to 0}\frac{1}{h}\int_x^{x+h}P(t,y)\,dt=P(x,y)$$

Q同理.

Is this still true if we change the thm to "whenever γ is a boundary of a retangle"

Cauchy's Thm

$w = P\,dx + Q\,dy$ diff form in $\Omega \subseteq \mathbb{R}^2$. P, Q cont.

ω has primitive iff $\int_\gamma \omega = 0$ $\forall$ piecewise continuous close γ is Ω.

parallel to axis.

Can we replace this by "γ a bd of a retangle?"

Yes, if Ω is an open disk. 可以把前面证明中的 x_0 取成圆心. 这样的"retangle bds" 都会在 disk 中.

ω is closed if $\int_\gamma \omega = 0$, whenever γ is bd of some small enough retangle in Ω.

可以把这个视作 closed 的定义.

$\Leftarrow$ 显然. "$\Rightarrow$"? 可以把大的retangle切成很多小的retangle

$\Leftrightarrow$ whenever γ is bd of any retangle.

$\Leftrightarrow$ w locally has a primitive.

在disk中,这两个是等价的.

$\Rightarrow$ In a disk, a closed differential form ω has a primitive.

Closed differential form in a domain Ω needn't have a primitive.

connected open set

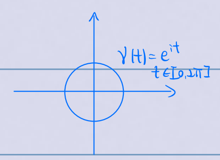
Example: $\Omega = \mathbb{C}\setminus\{0\}$, $\omega = \frac{dz}{z}$

a branch of log.

w is closed because locally, $d\log z = \frac{dz}{z}$

No global primitive: find a curve γ, $\int_\gamma \omega$ is not 0.

$$\int_\gamma w\,dz$$



$z = e^{it}$

$dz = ie^{it}dt$

$$\Rightarrow \int_0^{2\pi}\frac{dz}{z}=\int_0^{2\pi} i\,dt = 2\pi i.$$

$$\int_\gamma w = \int_0^{2\pi} w(\gamma(t))\gamma'(t)\,dt$$

$$=\int_0^{2\pi} w(e^{it})\,\gamma'(t)\,dt$$

$$=\int_0^{2\pi}\frac{1}{\cancel{e^{it}}}\,i\cancel{e^{it}}\,dt$$

$$\int_\gamma \eta = 2\pi.$$

$$\int_\gamma \frac{1}{\sin(z)}\,dz$$

$$=\int_0^{2\pi}\frac{\gamma'(t)}{\sin(2\cos 2t - i2\sin 2t)}\,dt$$

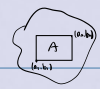
needn't be complex: $\frac{dz}{z}=\frac{d(x+iy)}{x+iy}=\frac{x\,dx+y\,dy}{x^2+y^2}+i\left(\frac{x\,dx-y\,dy}{x^2+y^2}\right)=\eta$

$\cdot(x-iy)$

$\eta = dt$, where $t=\arctan y/x$

Cauchy's thm: If $f(z)$ holomorphic function in open $\Omega \subset \mathbb{C}$, then $f(z)dz$ is closed. (locally, $\exists F(z)$, s.t. $F'(z) = f(z)$)

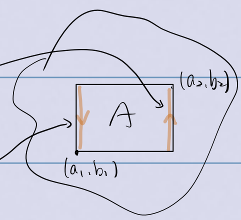
无关!

Aside: Green's thm: Assume P, Q continuous, with continuous partials: $\frac{\partial P}{\partial y}, \frac{\partial Q}{\partial x}$ in some neiberhood of closed rectangle A  γ the oriented bd

$$\Rightarrow \int_\gamma \underbrace{P\,dx + Q\,dy}_{\omega} = \iint_A \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy}_{d\omega}.$$

pf: $\iint_A \frac{\partial Q}{\partial x} = \int_{b_1}^{b_2} \left(\int_{a_1}^{a_2} \frac{\partial Q}{\partial x} dx \right) dy.$

$= \int_{b_1}^{b_2} \left(Q(a_2, y) - Q(a_1, y) \right) dy$



$= \int_\gamma Q\,dy.$

Green's Formula tell us: If $\frac{\partial P}{\partial x}, \frac{\partial Q}{\partial y}$ exist and continuous,

then $\int_\gamma \omega = 0 \Leftrightarrow \iint_A \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$ for small enough rect

$\Leftrightarrow \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 0$

the definition of "closed"!

With additional assumption. ($\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ exist, continuous).

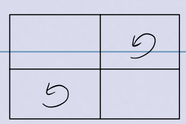
Pf: $f(z)dz = \underbrace{f(z)}_{P}dx + \underbrace{i f(z)}_{Q}dy.$

By Green's formula, it's enough to show $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \Leftrightarrow \underbrace{\frac{\partial f}{\partial y} = i \frac{\partial f}{\partial x}}$

Cauchy-Riemann equation!

Pf (with no additional ass):

Enough to show $\underbrace{\int_\gamma f(z)dz}_{\mu(R)} = 0 \quad \forall \gamma$ bd of any R in Ω.

R  divide R into 4 equal parts R_i, each with oriented bd γ_i:

$\int_\gamma f(z)dz = \sum_{i=1}^{4} \int_{\gamma_i} f(z)dz$

pigeon hole.

So for at least 1 i, we know $\left|\int_{\gamma_i} f(z)dz\right| \geq \frac{1}{4} \left|\int_\gamma f(z)dz\right|$

Let the corresponding $R_i = R^{(1)}$, $\gamma_i = \gamma^{(1)}$ $\Rightarrow |\mu(R^{(1)})| \geq \frac{1}{4} |\mu(R)|$

Continue to subdivide. $R \supset R^{(1)} \supset R^{(2)} \supset \cdots$

$\Rightarrow |\mu(R^{(k)})| = \left| \int_{\gamma^{(k)}} f(z)\,dz \right| \geq \frac{1}{4^k} \left| \int_{\gamma} f(z)\,dz \right| = \frac{1}{4^k} |\mu(R)|$

$\Rightarrow \exists\, z_0 \in \bigcap_k R^{(k)}$

Since f holomorphic at z_0, $f(z) = f(z_0) + f'(z_0)(z-z_0) + \varphi(z_0)(z-z_0)$, where $\lim_{x \to x_0} \varphi = 0$

$$\int_{\gamma^{(k)}} f(z)\,dz = \underbrace{\int_{\gamma^{(k)}} f(z_0)\,dz}_{0} + \int_{\gamma^{(k)}} f'(z_0)(z-z_0)\,dz + \int_{\gamma^{(k)}} \varphi(z)\,|z-z_0|\,dz$$

$= f'(z_0) \int_{\gamma^k} (z-z_0)\,dz = f'(z_0) \left| \frac{1}{2}(z-z_0)^2 \right|_{\gamma^{(1)}(1)}^{\gamma^{(k)}(0)} = 0$

定义.

Given $\varepsilon > 0$, if $|z-z_0| < \delta$, then $\left| \int_{\gamma^{(k)}} \varphi(z)\,|z-z_0|\,dz \right| \leq \varepsilon \left| \int_{\gamma^{(k)}} |z-z_0|\,dz \right|$

$$\leq \varepsilon \cdot \underbrace{\operatorname{diam}(R^{(k)})}_{\frac{1}{2^k}\operatorname{diam} R} \cdot \underbrace{\operatorname{param}(R^{(k)})}_{\frac{1}{2^k}\operatorname{param}(R)} = \varepsilon \frac{1}{4^k} \operatorname{diam}(R)\operatorname{param}(R)$$

$$|\mu(R)| \leq 4^k \left| \int_{\gamma^{(k)}} f(z)\,dz \right| \leq \varepsilon \operatorname{diam} R \operatorname{param} R$$

$\forall \varepsilon$!

$\Rightarrow \mu(R) = 0$

Corollary: Holomorphic function f in $\Omega \subseteq \mathbb{C}$ locally has a primitive which is also holomorphic

Pf: Consider local primitive $F(z)$, $f(z)\,dz = dF(z) = \frac{\partial F}{\partial z}\,dz + \frac{\partial F}{\partial \bar{z}}\,d\bar{z}$

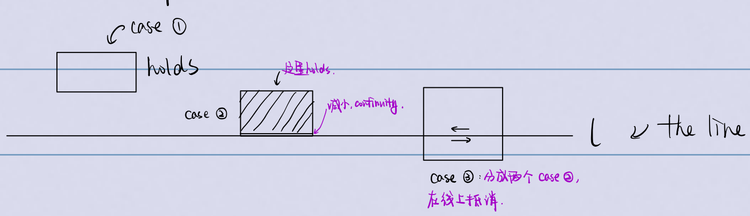
系数分别相等. $\Rightarrow \frac{\partial F}{\partial \bar{z}} = 0$ linearly independent!

$\Rightarrow F$ holomorphic

Corollary: (Generalization of Cauchy's thm)

In Cauchy's thm, it's enough to assume f continuous in Ω, and holomorphic outside a line // to x-axis.

pf:



We'll prove:

Thm: Closed differential form in a simply-connected open subset Ω of $\mathbb{R}^2$ has primitive (global)

Next time: A closed differential form $\omega = Pdx + Qdy$ in open $\Omega \subset \mathbb{R}^2$ always has a primitive along the curve $\gamma(t)$ $t \in [a,b]$

i.e, a continuous function $f(t)$, s.t. $\forall t_0 \in [a,b]$, $\exists$ primitive F of ω, s.t. $f(t) = F(\gamma(t))$, to t sufficiently close to t_0.

Homotopy:

Primitive of a closed diff form?

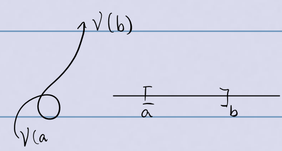
Closed differential form ω in open $\Omega \subset \mathbb{R}^2$.

(doesn't necessarily have a global primitive)

but always has primitive along a curve.

Prop: $\Omega \subseteq \mathbb{R}^2$ open, ω closed diff form in Ω,

Let $\gamma: [a,b] \to \Omega$ continuous curve.



then there is a continuous fn $f(t)$ on $[a,b]$

s.t. $\forall t_0 \in [a,b]$, there is a primitive F of ω

in a nbhd of $\gamma(t_0)$, s.t. $f(t) = F(\gamma(t))$ for t near t_0.

f is uniquely determined up to a constant.

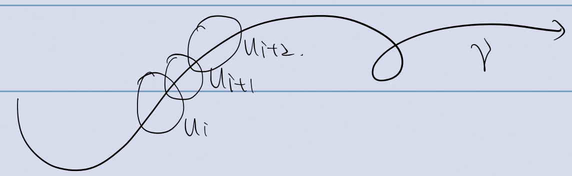
Pf: Uniqueness: Suppose f_1, f_2 are primitives of ω along γ,

then in nbhd of t_0, $f_1(t) - f_2(t) = F_1(\gamma(t)) - F_2(\gamma(t))$

locally constant + continuous $\Rightarrow$ constant $\leftarrow$ 2 local primitives near the same point differs by a constant.

Existance: There is a partition $a = t_0 < t_1 < \cdots < t_n = b$ of $[a,b]$,

s.t. $\gamma([t_{i-1}, t_i])$ lies in a open disk U_i in which w has a primitive F_i 因为 close 说明 locally 有, $[a,b]$ compact finite.



U_i, U_{i+1} disks, so $U_i \cap U_{i+1}$ connected. So $F_{i+1} - F_i = \text{constant}$.

$\Rightarrow$ adjust F_i one by one to make continuous.

Define $f(t)$ as $F_i(\gamma(t))$, $t \in [t_{i-1}, t_i]$.

If γ is piecewise C^1 and f is a primitive along γ, then

$$\int_\gamma w = f(b) - f(a)$$

Why? Consider partitions in Prop, $\gamma_i = \gamma|_{[t_i, t_{i+1}]}$.

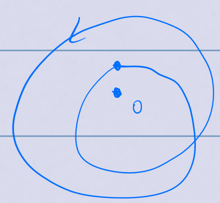
$$\Rightarrow \int_\gamma \omega = \sum \int_{\gamma_i} w = \sum \left(F_i(\gamma(t_i)) - F_i(\gamma(t_{i-1})) \right)$$

$$= \sum \left(f(t_i) - f(t_{i-1}) \right)$$

$$= f(b) - f(a)$$

So we can define $\int_\gamma w$ for any continuous curve as $f(b) - f(a)$

Example:



γ closed $(\gamma(a) = \gamma(b))$ curve not containing 0.

$$\int_\gamma \frac{dz}{z} = \underbrace{f(b) - f(a)}_{} = 2i\pi n.$$

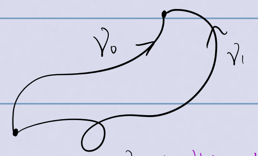
difference between 2 branchs of log at $\gamma(a) = \gamma(b)$.

primitive of this.

local

$\int_{\gamma} \frac{x\,dy - y\,dx}{x^2+y^2} = 2\pi n$ same reason.

"the variation of the argument along γ"

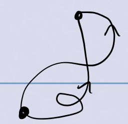


$\gamma_0(t)$, 关于s,t 的二元函数, $s=1$ 时 $\gamma_1(t)$, $s=0$ 时 $\gamma_0(t)$

Homotopy:

$\gamma_0, \gamma_1 : [0,1] \to \Omega$, s.t. $\gamma_0(0) = \gamma_1(0)$, $\gamma_0(1) = \gamma_1(1)$

are homotopic in Ω (with fixed endpoint)



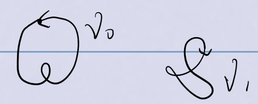
If there is a $\gamma : I \times I \to \Omega$,

s.t. $\gamma(0,t) = \gamma_0(t)$ $\gamma(1,t) = \gamma_1(t)$

$\gamma(s,0) = \gamma_0(0) = \gamma_1(0)$, $\gamma(s,1) = \gamma_0(1) = \gamma_1(1)$

$\gamma_0, \gamma_1 : [0,1] \to \Omega$ continuous closed curves,

$\gamma_1(1) = \gamma_1(0)$, $\gamma_0(1) = \gamma_0(0)$



are homotopic in Ω (as closed curves)

γ_0 homotopic to a point if $\gamma_1 = pt.$

If there is a $\gamma : I \times I \to \Omega$,

s.t. $\gamma(0,t) = \gamma_0(t)$ $\gamma(1,t) = \gamma_1(t)$, $\gamma(s,0) = \gamma(s,1)$

Thm: If ω is a closed differential form in Ω and $\gamma_0, \gamma_1 : [0,1]$ cont curves in Ω, homotopic either (1) with fixed endpoint or (2) as closed curves, then $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$.

Homotopy

Thm: If ω is a closed differential form in Ω and $\gamma_0, \gamma_1 : [0,1]$ cont curves in Ω, homotopic either (1) with fixed endpoint or (2) as closed curves, then $\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$.

Lemma: ω closed form in Ω, $\gamma : [a,b] \times [c,d] \to \Omega$ cont.

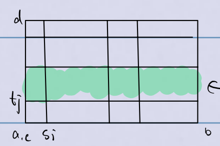
Then there is cont fn $f : [a,b] \times [c,d] \to \mathbb{C}$.

s.t. $\forall (s_0,t_0) \in [a,b] \times [c,d]$, $\exists F$ primitive of ω defined in a nbhd of $\gamma(s_0,t_0)$, s.t. $f(s,t) = F(\gamma(s_0,t_0))$ in a nbhd of (s_0,t_0)

F is unique up to a constant.

Pf: Choose partitions $\{s_i\}, \{t_j\}$ of $[a,b]$, $[c,d]$, s.t. γ maps

$[s_{i-1}, s_i] \times [t_{j-1}, t_j] \longrightarrow \underbrace{U_{ij}}_{\text{open disk}}$, where ω has a local primitive F_{ij}.



for fixed j, there is a primitive f_j along $\gamma|_{[a,b]\times[t_j, t_{j+1}]}$ just like before (F_{ij}, $F_{i+1,j}$ differ by a constant in $U_{ij} \cap U_{i+1,j}$)

↳ 可以一个一个调整.

So we define f_j as $F_{ij} \circ \gamma$ in $[s_{i-1}, s_i] \times [t_{j-1}, t_j]$

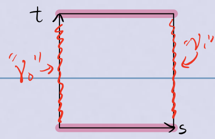
For each j, f_j and f_{j+1} differ by constant in $[a,b] \times \{t_j\}$

Proof of Thm:

(1): We have homotopy $\gamma : [0,1] \times [0,1] \to \Omega$, s.t.:

$\gamma(0,t) = \gamma_0(t)$, $\gamma(1,t) = \gamma_1(t)$, $\gamma(s,0) = \gamma_0(0) = \gamma_1(0)$, $\gamma(s,1) = \gamma_0(1) = \gamma_1(1)$

Let f be primitive of ω along γ,



f is constant on the 2 horizontal sides. $f = F \circ \gamma$.

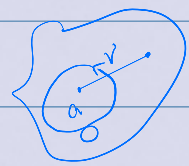
i.e, $f(0,0) = f(1,0)$, $f(0,1) = f(1,1)$.

$$\int_{\gamma_0} w = f(0,1) - f(0,0) = f(1,1) - f(1,0) = \int_{\gamma_1} w$$

Ω is simply-connected if every closed curve in Ω is null-homotopic.

<u>Cor</u>: In simply connected open set, any closed form has a primitive.

<u>Example</u>: (1) Star-shaped



$\gamma(s,t) = a + s(\gamma(t) - a)$

(2) $w = \frac{dz}{z}$

has a primitive has a primitive in any simply connected Ω

not containing 0.

i.e, $\log(z)$ has branch in any simply connected Ω not containing 0.

Let $\log z = w_0 + \int_{z_0}^{z} \frac{dz}{z}$, where $e^{w_0} = z_0$

(3) $\mathbb{C} \setminus \{0\}$ not simply connected, since $\int_{S^1} \frac{dz}{z} = 2\pi i \neq 0$

<u>Cauchy's integral formula</u>:

γ closed curve in $\mathbb{C}$

pt not on γ

<u>Winding number</u> of γ wrt a:

$$w(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} \in \mathbb{Z}.$$

<u>Properties</u>: (1) Fix a, Then $w(\gamma, a)$ invariant under homotopy of γ not passing through a. (by early thm today)

(2) In particular, if γ lies in a simply connected set not containing a, then $w(\gamma, a) = 0$

(3) Fix γ, Then $w(\gamma, a)$ constant on connected component

of complement of γ (Enough to show locally constant)

因为固定 γ 移动 a 等价于固定 a 移动 γ.

$\swarrow w(\gamma, \text{center}) = 1$

(4) If γ is a circle described in positive sense:

Then $w(\gamma, a) = \begin{cases} 0 & a \text{ outside } \gamma \\ 1 & \cdots \text{ inside } \cdots \end{cases}$

Cauchy's integral formula:

$\Omega \subset \mathbb{C}$ open, $f(z)$ holomorphic in Ω

γ closed curve in Ω not containing a, homotopic to a point.

Then $\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz = w(\gamma, a) \cdot f(a)$

Cor: If $f(z)$ holomorphic in nbhd of closed disk D,

and γ bd of disk (in pointwise sense)

then $\frac{1}{2\pi i} \int_\gamma \frac{f(z)}{z-a} dz = \begin{cases} f(a) & \text{if } a \in \text{circle} \\ 0 & \text{if } \& \end{cases}$

Pf: Let $g(z) = \begin{cases} \frac{f(z) - f(a)}{z-a} & z \neq a \\ f'(a) & z = a \end{cases}$

$\Rightarrow g(z)$ is continuous, holomorphic when $z \neq a$.

柯西定理课证的

$\Rightarrow g(z) dz$ is closed

$\Rightarrow 0 = \int_\gamma g(z) dz = \int_\gamma \frac{f(z) - f(a)}{z-a} dz$

$\Rightarrow \int_\gamma \frac{f(z)}{z-a} dz = f(a) \int_\gamma \frac{1}{z-a} dz = f(a)\, 2\pi i\, w(\gamma, a)$

Cauchy's theorem wrap-up

Consider $f(z)$ continuous in Ω,

TFAE: (1) $f(z)$ is holomorphic in Ω

(2) $f(z)dz$ closed

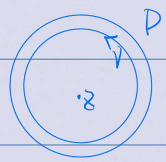
(3) $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$ when $z \in$ interior of closed disk in

positively

Ω, with oriented bd γ.

Remark: Holomorphic $f(z)$ in open disk D is infinitely differential

in D.



z inside γ, $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z}d\zeta$ by Cauchy's integral formula.

So $f'(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^2}$. $\forall n$, $f^{(n)} = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta - z)^n}$

Pf: (1) ⇒ (2): Cauchy's thm.

(1) ⇒ (3): Cauchy's Integral formula.

(3) ⇒ (1)

(2) ⇒ (1): ("Morera's thm")

(2) ⇒ $f(z)$ locally has a primitive $g(z)$, which is holomorphic.

⇒ $f(z) = g'(z)$ holomorphic.

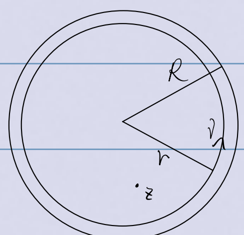
Cor: Continuous fn which is holomorphic except a line

"finitely union of lines"

is holomorphic everywhere.

Application of Cauchy's formula.

$f(z)$ holom in $|z| < R$.



$0 < r < R$

If $|z|<r$, then $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$

For all n $f^{(n)} = \frac{n!}{2\pi i}\int_\gamma \frac{f(\zeta)d\zeta}{(\zeta-z)^n}$

Taylor expansion of f at 0:

$f = \sum a_n z^n$, where $a_n = \frac{f^{(n)}(0)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(\zeta)}{\zeta^{n+1}} d\zeta$

Thm: $f(z)$ has convergent powseries expansion in $|z|<R$.

Pf: $\frac{1}{\zeta - z} = \frac{1}{\zeta}\left(1-\frac{z}{\zeta}\right)^{-1} = \frac{1}{\zeta}\left(1+\frac{z^2}{\zeta^2}+\frac{z^3}{\zeta^3}+\cdots\right)$ convergent for $|z|<|\zeta|$

$\Rightarrow f(z) = \frac{1}{2\pi i}\int_\gamma \sum_{n=0}^{\infty} \frac{z^n f(\zeta)}{\zeta^{n+1}} d\zeta$ for fixed z, $|z|<r$, this is convergent

commute. for $|\zeta|=r$. (By comparison with geometric series)

这里要全收敛

$\Rightarrow$ QED.

Cor: Every holom fn is analytic.

$f(re^{i\theta}) = \sum_{m=0}^{\infty} a_m r^m e^{im\theta}$ So $a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\theta}) e^{-in\theta} d\theta$

乘 $e^{in\theta}$ 后积分, $m\neq n$ 那项为0

Integral formula gives upper bound for Taylor coeff a_n:

Let $M(r) = \sup_\theta |f(re^{i\theta})|$ $|a_n r^n| \leq M(r) \Rightarrow |a_n| \leq \frac{M(r)}{r^n}$

Liouville's Thm: A bounded holomorphic fn in $\mathbb{C}$ is constant.

Pf: $M(r) \leq M$ for all r and some f.

$\Rightarrow |a_n| \leq \frac{M}{r^n}$ for all $r>0$

$\Rightarrow a_n = 0$.

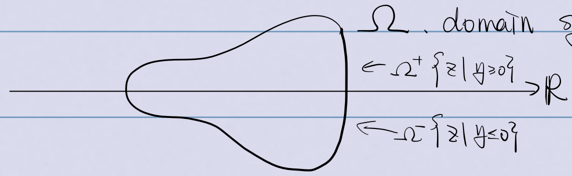
Cor: Fundamental thm of algebra:

Pf: Suppose $P(z)$ has no roots.

$\Rightarrow \frac{1}{P(z)}$ is a holomorphic fn in $\mathbb{C}$, bounded 稍微讨论一下即可

$\Rightarrow \frac{1}{P(z)} = C \Rightarrow P$ constant.

Schwartz's reflection principle



$f(z)$ continuous on Ω^+, real in $\Omega \cap \mathbb{R}$, holom on $\Omega^+ \cap \{y>0\}$

Then $f(z)$ extends to a holomorphic function in Ω (唯一)

By reflection

Pf: Define $g(z) = \begin{cases} f(z) & z \in \Omega^+ \\ \overline{f(\bar{z})} & z \in \Omega^- \end{cases}$

Then $g(z)$ holomorphic in both $\Omega^+ \cap \{y>0\}$ and $\Omega^- \cap \{y<0\}$ 验证

and g cont, holom except a line, then g holomorphic

$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad r>0$ small enough, $a_n r^n = \frac{1}{2\pi} \int_0^{2\pi} e^{in\theta} f(re^{i\theta}) d\theta$

$\xleftarrow{n=0}$

$f(0) = a_0 = \frac{1}{2\pi} \int_0^{2\pi} f(re^{i\theta}) d\theta$

$\hookrightarrow$ mean value of $f(z)$ on $|z|=r$

Mean value property:

If $f(z)$ holomorphic in $\Omega \subset \mathbb{C}$, then f has MVP, i.e., $\forall$ disk in Ω

$f(\text{center}) =$ mean value on bd.

Real and Im part of f also satisfies this.

Maximum modulus principle : If f is a cont. complex valued fn with MVP in $\Omega \subset \mathbb{C}$, and $|f|$ has a local maximum at $a \in \Omega$, then f constant around a.

Pf: ① If $f(a) = 0$, ✓

② $f(a) \neq 0$: then we can assume $f(a)$ real, >0 ← 乘个常数即可.

for $r>0$ small enough, $M(r) = \sup_{\theta \in [0,2\pi]} |f(a+re^{i\theta})| \le f(a) \le M(r)$ ← 因为在a周围积分是f(a). MVP.

$\Rightarrow f(a) = M(r)$

Let $g(z) = \mathrm{Re}(f(a) - f(z)) = \underset{\sup_{|z-a|=r}|f(z)|}{f(a)} - \mathrm{Re}(f(z))$ $\Rightarrow = \sqrt{\mathrm{Re}^2+\mathrm{Im}^2} \ge \mathrm{Re}$

$\Rightarrow g(z) \ge 0$ on $|z-a| = r$

$g(z) = 0 \iff f(z) = f(a)$

MV of $g(a)$ is $\underbrace{\text{MV of } f(a)}_{=f(a)} - \underbrace{\text{MV of } \mathrm{Re} f(z)}_{= \mathrm{Re} f(a) = f(a)} = 0$. MVP

Since $g(z)$ cont, ≥ 0 on $|z-a| = r$, and MV $g(z) = 0 \Rightarrow g(z) = 0$ on $|z-a| = r$.

Quite Easily Done.

Cor: Suppose Ω bounded in $\mathbb{C}$.

Let $f(z)$ be continuous, complex valued fn on $\overline{\Omega}$, with MVP in Ω.

$M = \sup |f(z)|$ on frontier $\overline{\Omega}$ ← $\overline{\Omega} \setminus \Omega$

Then $|f(z)| \le M$ $z \in \overline{\Omega}$. if $\exists z_0 \in \Omega$, $|f(z_0)| = M \Rightarrow f$ constant.

Pf: Let $M' = \max |f(z)|$

$z\in\Omega$

M' attained at point $a\in\bar{\Omega}$.

① If $a\in$ frontier : ✓ 直接就上面那个定理.

$f^{-1}(f(a))$

② If $a\in\Omega$: $\{z\in\Omega \mid f(a)=f(z)\}$ open by MVP, closed

Schwartz's lemma :

$f(z)$ is holom in $|z|<1$, $|f(z)|<1$. if $|z|<1$. $f(0)=0$

then ① $|f(z)|\le|z|$ for $|z|<1$

② if $|f(z_0)|=|z_0|$, for some $0<|z_0|<1$,

then $f(z)=\lambda z$, $|\lambda|=1$.

Pf: Since $f(0)=0$, $f(z)=\sum_{n=1}^{\infty}a_nz^n$ ← 没有 a_0.

$\Rightarrow \frac{f(z)}{z}$ holomorphic.

$\Rightarrow \left|\frac{f(z)}{z}\right|\le\frac{1}{r}$ on $|z|=r$. where $r<1$

$\Rightarrow \left|\frac{f(z)}{z}\right|\le\frac{1}{r}$ on $|z|\le r$. by

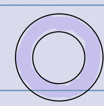
$\Rightarrow$ for fixed z, $\left|\frac{f(z)}{z}\right|<\frac{1}{r}$ for all $|z|\le r<1$.

Let $r\to1$. $|f(z)|\le|z|$. ✓

if $|f(z_0)|=|z_0|$ for $z\ne0$, then $\frac{f(z)}{z}$ attains its max

at a interior point $\Rightarrow \frac{f(z)}{z}$ is a constant.

Laurent expansion : Holomorphic f in annulus $0\le R_2<|z|<R_1\le\infty$

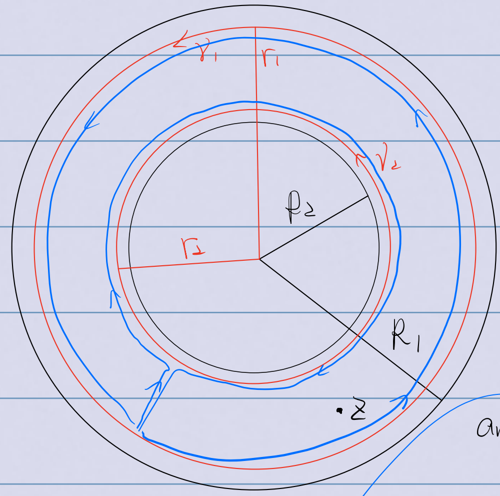


has a convergent Laurent expansion in annulus.

i.e. $\sum_{n=-\infty}^{\infty}a_nz^n=\underbrace{\sum_{n<0}a_nz^n}_{\substack{\text{convergent}\\ \text{for } |z|>R_2}}+\underbrace{\sum_{n\ge0}a_nz^n}_{\substack{\text{convergent}\\ \text{for } |z|<R_1}}$.

Let $\zeta=\frac{1}{z}$, $\sum_{n<0}a_nz^n=\underline{\sum_{n>0}a_{-n}\zeta^{-n}}$ → convergent if $|\zeta|<\frac{1}{R_2}$

Pf:



By Cauchy's thm.

$$f(z) = \frac{1}{2\pi i}\int_{\gamma_1} \frac{f(\zeta)}{\zeta - z}\, d\zeta - \frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

因为蓝色部分与 Z_1 homotopy.

$\Rightarrow \sum_{n=0}^{\infty} a_n z^n$ where

$$a_n = \begin{cases} \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_1} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta & \text{if } n \geq 0 \\ \dfrac{1}{2\pi i}\displaystyle\int_{\gamma_2} \frac{f(\zeta)}{\zeta^{n+1}}\, d\zeta & \text{if } n < 0 \end{cases}$$

$$(\zeta - z)^{-1} = \frac{1}{\zeta}\left(1 - \frac{z}{\zeta}\right)^{-1}$$

这个前面处理过.

$$-\frac{1}{2\pi i}\int_{\gamma_2} \frac{f(\zeta)}{\zeta - z}\, d\zeta$$

两负号消了.

$$\frac{1}{\zeta - z} = -\frac{1}{z}\frac{1}{1 - \zeta/z} = -\sum_{n=0}^{\infty} \frac{\zeta^n}{z^{n+1}} = -\sum_{n<0}^{-\infty} \frac{z^n}{\zeta^{n+1}}$$

平移了角标.

uniformly and absolutely convergent on $|\zeta| = r_2$.

Holom fn in punctured disk e.g. $0 < |z| < r$

has a isolated singularity at $z = 0$ if f cannot be extended to a holom fn in $|z| < r$.

重点在于负项有多少个.

$\sum_{-\infty}^{\infty} a_n z^n$ Laurent expansion of $f(z)$ in $0 < |z| < r$.

poles or essential singularity

↳ finitely many $n < 0$ ↳ infinitely many $n < 0$.

这里最负的一项

Pole: 可以乘一个足够大的 z^n 使得 $f(z)$ holom

$f(z) = \frac{g(z)}{z^n}$ meromorphic in $|z| < r$

去掉有限个点后 holom, 在去掉的点附近可以写成 $\frac{f}{g}$ 的形式.

在 Riemann sphere 意义下是 holom 的

Isolated singularity and residue

Holom fn $f(z)$ in a punctured disk $0<|z|<R$

has isolated singularity at 0 if f can be extended

to be holom in $|z|<R$.

poles or essential singularity

劳伦展开有限负项   infinitely many.

Extension is possible iff f is bounded around 0.

Why?

乘 $e^{-in\theta}$ 后积分.

$f(re^{i\theta}) = \sum_{n=-\infty}^{\infty} a_n r^n e^{in\theta} \Rightarrow a_n r^n = \frac{1}{2\pi}\int_0^{2\pi} e^{-in\theta} f(re^{i\theta})\, d\theta$

☆ $e^{in\theta}$ 呢? $M(r)$代进来

Let $M(r)=\sup_{|z|=r}|f(z)| \Rightarrow |a_n|r^n \le M(r) \Rightarrow |a_n| \le \frac{M(r)}{r^n} \quad \forall n\in\mathbb{Z}$.

$\Rightarrow f$ bounded $\Leftrightarrow M(r)\le M$ constant.

if $n<0$, then $|a_n| \le M r^{\underbrace{-n}_{>0}} \to 0$ as $r\to 0$

So $\forall n<0$, $a_n=0 \Rightarrow$ 这里没有 singularity (这里在干啥)

If 0 pole, then $\lim_{z\to 0} f(z)=\infty$

A meromorphic fn is a holom fn with values in S^2.

?

Weierstrass Thm

If 0 is an essential singularity of f, then $\forall \varepsilon>0$, $f(0<|z|<\varepsilon)$ dense in $\mathbb{C}$.

Pf: Otherwise, $\exists\, \delta>0$, $a\in\mathbb{C}$ s.t. $|f(z)-a|>\delta$ if $0<|z|<\varepsilon$.

Let $g(z)=\frac{1}{f(z)-a} \Rightarrow \begin{cases} \text{holom in } 0<|z|<\varepsilon \\ \text{bounded by } \frac{1}{\delta} \end{cases} \Rightarrow$ holom in $|z|<\varepsilon$.   Contradiction.

quotient of holom fn

$\Rightarrow f(z) = a + \frac{1}{g(z)}$ meromorphic, so 0 is a pole.

In fact, $f(0<|z|<\varepsilon) = \mathbb{C}$, omits at most 1 value

Picard's big thm   比如 $e^{\frac{1}{z}} \neq 0$

At infinity: Suppose f holom in $|z|>R$,

f is holomorphic at ∞ if $f(\frac{1}{z})$ holomorphic in $|z|<\frac{1}{R}$

f has a pole at ∞ if $f(\frac{1}{z})$ has a pole at 0

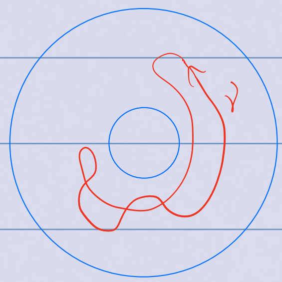
-- essential singularity @ ∞ -------- essential singularity @ 0

$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$ poles at 0: finitely many $n>0$, s.t. $a_n \neq 0$.

Laurent expansion

$f(\frac{1}{z}) = \sum_{-\infty}^{\infty} a_{-n} z^n$, 之后相同.

Exercise: $f(z)$ holom in annulus, $R_2<|z|<R_1$.



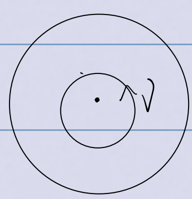
$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = ?$

$f(z) = \sum_{-\infty}^{\infty} a_n z^n = \frac{a_{-1}}{z} + \sum_{n\neq -1} a_n z^n \quad {=0}$

$= \frac{1}{2\pi i}\int_\gamma \frac{a_{-1}}{z}\,dz$

$= a_{-1} \cdot w(\gamma, 0)$

In particular, if f holom in punctured disk, $0<|z|<\varepsilon$,



$\frac{1}{2\pi i}\int_\gamma f(z)\,dz = a_{-1}$.

↳ residue of $w = f(z)\,dz$ at 0

Residue at ∞ of $f(z)$ holom in $|z|>R$.

is $\frac{1}{2\pi i}\int_\gamma f(z)\,dz$, where γ big circle (around ∞),

positively oriented wrt ∞ / negative oriented wrt 0?)

Let $z = \frac{1}{z'} \Rightarrow f(z)dz = -\frac{1}{z'^2} f(\frac{1}{z'})dz'$

$= -\frac{1}{2\pi i}\int_{\gamma'} -\frac{1}{z'^2} f(\frac{1}{z'})dz'$

small circle in positive sense.

The Residue Theorem

$f(z)$ holomorphic fn in punctured nbhd of pt a.

Residue of $f(z)dz$ at a, $\text{res}(f(z)dz, a) = \frac{1}{2\pi i}\int_\gamma f(z)dz$

γ circle around a, in positive sense

"residue of $f(z)$ at a" "$\text{res}(f(z), a)$" $= a_{-1}$, where $\sum a_n z^n$ is the Laurent expan

Residue of $f(z)dz$ at ∞, $\frac{1}{2\pi i}\int_\gamma f(z)dz$

small circle around ∞, in + sense wrt ∞

在此之外 holom

i.e., a big circle in $\mathbb{C}$, oriented in negative sense.

In coordinate at ∞, $z' = \frac{1}{z}$, $\int f(z)dz = \int -\frac{1}{z'^2} f(\frac{1}{z'})dz'$

变换后的 corr. system.

$\text{res}(f(z)dz, \infty) = \frac{1}{2\pi i}\int_{\gamma'} -\frac{1}{z'^2} f(\frac{1}{z'})dz'$ small circle around $z' = 0$

$\text{res}(f, \infty) = -a_{-1}$

Use Laurent expansion of $f(z) = \sum a_n z^n$

$= \cdots + a_{-2}z^{-2} + a_{-1}z^{-1} + a_0 + a_1 z + a_2 z^2 + \cdots$

$z = \frac{1}{z'}$

$= + a_{-2}z'^2 + a_{-1}z' + a_0 + a_1\frac{1}{z'} + a_2\frac{1}{z'^2} + \cdots$

$\div (z')^2$

$= + a_{-2} + a_{-1}\frac{1}{z'} + a_0\frac{1}{z'^2} + a_1\frac{1}{z'^3} + \cdots$

别忘了负号

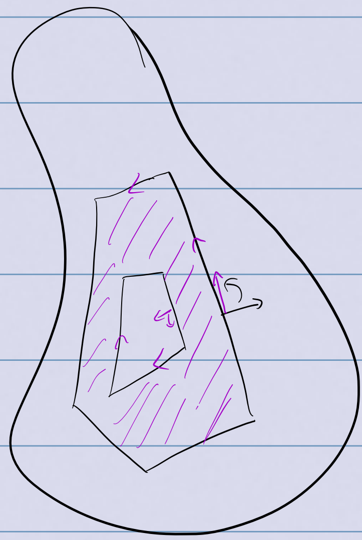
Residue Theorem:

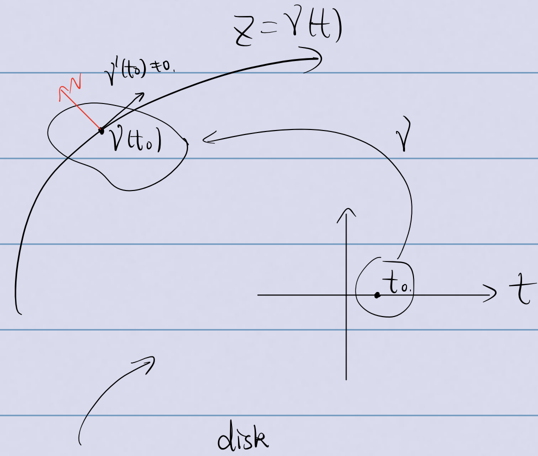
Ω open in S^2, $f(z)$ holom in Ω, except maybe at isolated point.

K compact set with piecewise C^1 oriented boundary Γ in Ω, where Γ contain no singular pt or ∞

Then $\frac{1}{2\pi i}\int_\Gamma f(z)\,dz = \sum_{\substack{\text{singularities}\\ z_k}} \operatorname{res}(f, z_k)$

Γ disjoint union of closed piecewise C^1 curve, γ, positively oriented wrt K.





There is a open disk U centered at t_0, open nbhd V of $\gamma(t_0)$, s.t. $\gamma(t)$ extend to a C^1 map, $\varphi: U \to V$, with C^1 inverse, s.t. $\varphi(t,0)=\gamma(t)$, $\det\varphi'>0$ why?

Pick $N \perp \gamma'(t_0)$. Let $\varphi(t,u) = \gamma(t) + uN$. If doesn't work, use $-N$.

γ is positively oriented wrt K if upper half disk mapped into the interior of K.

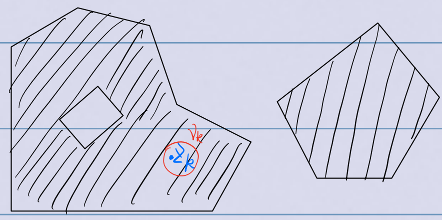
Green's Thm:

If P, Q are C^1 in nbhd of K, then
$$\int_\Gamma P\,dx + Q\,dy = \iint_K \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$

In particular, if $P\,dx + Q\,dy$ closed, then $\int_\Gamma P\,dx + Q\,dy = 0$

Pf of Residue Theorem.

① $\infty \notin K$.



z_k singular pt.

γ_k bd of $D_k \ni z_k$, small disk.

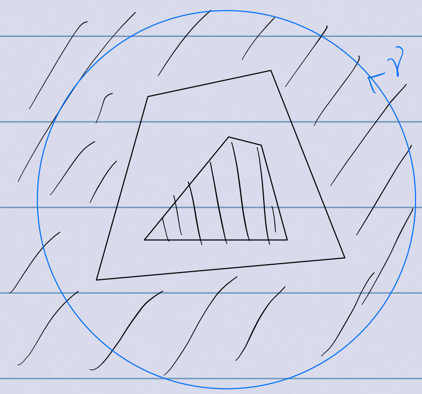
Let $K' = K \setminus \bigcup_k \mathrm{Int}\, D_k$, $f(z)$ holom in nbhd of K'

By Green's Thm, $\int_{\mathrm{Bd}\,K'} f(z)\,dz = 0$

$\Rightarrow \int_{\mathrm{Bd}(K \setminus \bigcup_k \mathrm{Int}\, D_k)} f(z)\,dz = 0$

$\Rightarrow \int_\Gamma f(z)\,dz = \int_{\cup \gamma_k} f(z)\,dz = 2\pi i \sum_k \mathrm{res}(f, z_k)$

② $\infty \in K$.



Choose r, s.t.

$\{|z| \geq r\} \subset \mathrm{Int}\, K$. and

($f(z)$ holom on $\{|z| > r\}$

except at ∞)

Let $K'' = K \setminus \{|z| > r\} \Rightarrow \mathrm{Bd}\, K'' = \Gamma \cup \gamma$.

By ①, $\int_\Gamma f(z)\,dz + \underbrace{\int_\gamma f(z)\,dz}_{= -\mathrm{res}(f, \infty)} = \sum \mathrm{res}(f, z_k)$

$\Rightarrow \int_\Gamma f = \sum \mathrm{res}$.

Example:

① $K = S^2$ ($\Gamma = \emptyset$) $\Rightarrow \sum_k \mathrm{res}(f, z_k) = 0$

$\Rightarrow$ sum of residue of any rational function is 0.

Calculation:

Example:

① $f(z) = \dfrac{g(z)}{h(z)}$, a point where $g(a) \neq 0$, h has simple zero

$$\frac{(z-a)\,g(z)}{h(z)} = \sum_{k=0}^{\infty} b_k (z-a)^k$$ holom @ a

$$\Rightarrow \frac{g(z)}{h(z)} = \sum_{k=0}^{\infty} b_k (z-a)^{k-1}$$ only need to find b_0

$$b_0 = \lim_{z\to a} \frac{(z-a)g(z)}{h(z)} = g(a) \frac{1}{\lim_{z\to a} \frac{h(z)-h(a)}{z-a}} = \frac{g(a)}{h'(a)}$$

$\operatorname{res}(f, a) = \dfrac{g(a)}{h'(a)}$ 只有这一项贡献了 $\frac{1}{z}$ 项.

(准确地说, $\frac{1}{z-a}$)

$g(z) = g(a) + g'(a)(z-a) + \cdots$

$0 = h(a) +$

$h(z) = h'(a)(z-a)\,(1 + \cdots)$

② $f(z) = \dfrac{e^{iz}}{z(z^2+1)^2}$ $\operatorname{res}(f, i) = -\dfrac{3}{4e}$ double pole.

$\because (z+i)(z-i)$

Let $z = i + \zeta$

$$f(i+\zeta) = \frac{e^{i(i+\zeta)}}{(i+\zeta)\,\zeta^2(2i+\zeta)^2}$$

① $e^{i(i+\zeta)} = e^{i^2} e^{i\zeta} = e^{-1}(1 + i\zeta + \cdots)$

② $(i+\zeta)^{-1} = -i(1 - i\zeta)^{-1} = -i(1 + i\zeta + \cdots)$

③ $(2i+\zeta)^{-2} = -\frac{1}{4}(1 - \frac{i}{2}\zeta)^{-2} = -\frac{1}{4}(1 + i\zeta + \cdots)$

① × ② × ③ × ζ^{-2}

$= \dfrac{i}{4e}(1 + 3i\zeta + \cdots) \times \zeta^{-2}$

$= \dfrac{i}{4e}(\zeta^{-2} + 3i\zeta^{-1} + \cdots) \Rightarrow a_{-1} = -\dfrac{3}{4e}$

③ $f(z)$ mero in nbhd of $z=a$ ($f(z) = (z-a)^k g(z)$, where g holom in nbhd of a, $g(a) \neq 0$) pole if $k<0$

$\operatorname{res}\left(\dfrac{f'}{f}, a\right) = k$

$f'(z) = k(z-a)^{k-1} g + (z-a)^k g'$

$\dfrac{f'}{f} = \dfrac{k}{z-a} + \dfrac{g'}{g} \Rightarrow \dfrac{f'}{f}$ has simple pole at a, with residue k.

Argument Principle:

$f(z)$ non-constant mero fn in open Ω

K compact with oriented Bd Γ in Ω

Suppose f doesn't take value a on Γ, has no poles on Γ,

Then $\displaystyle \frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)-a}\,dz = \underbrace{D}_{\#\text{ of zeros of } f(z)=a} - \underbrace{P}_{\#\text{ poles of } f(z) \text{ in } K,\ \text{with multiplicity}}$

和 $f(z)$ 的 pole 是一样的.

再听一遍

Ex: $f(z)$ mero in nbhd of a,

$\text{res}\left(\frac{f'}{f}, a\right) = ?$

$f(z) = (z-a)^k g(z) \to$ holom, $g(a) \neq 0$

$k<0$ pole

$k>0$ zero

$\frac{f'}{f} = \frac{k}{z-a} + \underbrace{\frac{g'}{g}}$ $\to$ holomorphic, res$=0$

$\text{res}\left(\frac{f'}{f}, a\right) = \text{res}\left(\frac{k}{z-a}, a\right) = k$

Argument Principle

$f(z)$ non-constant mero fn in Ω, K compact set with oriented Bd Γ.

Given a, assume no zeros of $f(z)-a$, no poles of $f(z)$ on Γ.

Then $\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)-a}\,dz = \underbrace{D}_{\#\text{ of zeros of } f(z)=a} - \underbrace{P}_{\#\text{ poles of } f(z) \text{ in } K,\ \text{with multiplicity.}}$

和f(z)的pole是一样的.

Pf: From the above example. and residue thm.

Residue thm: $\frac{1}{2\pi i}\int_\Gamma \frac{f'(z)}{f(z)-a}\,dz = \sum \text{res}\left(\frac{f'(z)}{f(z)-a}, z_k\right)$

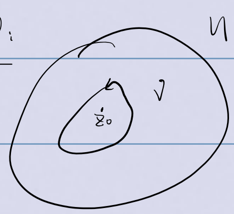
Thm: $f(z)$ non-constant holom fn in nbhd of $z=z_0$ where z_0

root of order k of $f(z)-a$, $a \in \mathbb{C}$

For every sufficient small nbhd U of z_0, $\forall$ b suff close to a, $b \neq a$,

$f(z)-b$ has k simple (order 1) roots in U.

Proof:



U small enough s.t. $f(z)-a$ has no zero but z_0

in closed disk.

And $f'(z) \neq 0$ in the closed disk except z_0.

$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-b}\,dz$ constant for b in connected component

of $f\circ\gamma$ by integration by substitution ? ☆

$\Rightarrow \quad = \frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)-a}\,dz = k$ for b close enough

By the argument principle. $f(z)-b$ has k roots inside γ, All simple because $f'(z)\neq 0$.

Rouche's Theorem:

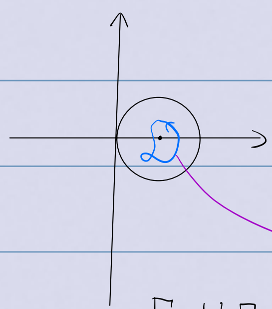
$f(z), g(z)$ holomorphic in open Ω, K compact set with oriented Bd Γ in Ω.

no zero in Γ

If $|f(z)-g(z)| < |f(z)|$ on Γ, then $f(z), g(z)$ have same number of zeros (with multiplicity) in K.

Proof: $\left|1-\frac{g(z)}{f(z)}\right| < 1$ on Γ

so values of $F(z) = \frac{g(z)}{f(z)}$ on Γ lie in open disk centered 1, radius 1.



$$\frac{1}{2\pi i}\int_\Gamma \frac{F'(z)}{F(z)} = \#\text{ zeros of } g - \#\text{ of zeros of } f \text{ in } K.$$

$\Gamma = \bigcup_i \Gamma_i$. Sum of winding numbers of $F\circ\Gamma_i$ at 0 值 $F\circ\Gamma_i$ 在这里，所以都是0.

Evaluation of definite integrals by residue calculus.

① $\int_0^{2\pi} R(\cos\theta, \sin\theta)\,d\theta$, R rational function with no poles on unit circle.

$z = e^{i\theta}$ $\quad \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$, $\sin\theta = \frac{1}{2i}\left(z-\frac{1}{z}\right)$

$dz = ie^{i\theta}d\theta, \quad d\theta = -i\frac{dz}{z}$

$$= -i\int_{|z|=1} R\left(\frac{1}{2}\left(z+\frac{1}{z}\right), \frac{1}{2i}\left(z-\frac{1}{z}\right)\right)\frac{1}{z}\,dz$$

$$= 2\pi \sum_{|z|<1} \operatorname{Res}(\qquad)$$

Example: $\int_0^{\pi} \frac{d\theta}{a+\cos\theta}$ $\quad a>1 \quad z=e^{i\theta}.$

(even)

$$= \frac{1}{2}\int_0^{2\pi}\frac{d\theta}{a+\cos\theta} \qquad \frac{d\theta}{a+\cos\theta} = -i\frac{dz}{z}\cdot\frac{1}{a+\frac{1}{2}\left(z+\frac{1}{z}\right)}$$

$$= -i\int_{C}\frac{dz}{z^2+2az+1} \qquad z^2+2az+1 = (z-\alpha_1)(z-\alpha_2) \quad \alpha_i = -a\pm\sqrt{a^2-1}$$ ，只有 $-a+\sqrt{a^2-1}$ 在 ⊕ 里

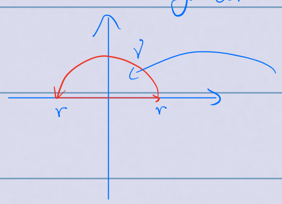
residue at this point is: $\left.\frac{1}{2(z+a)}\right|_{z=-a+\sqrt{a^2-1}} = \frac{1}{2\sqrt{a^2-1}}$

Why? 教授解释了，我还是没懂

$\int_{-\infty}^{\infty} R(x)\,dx$ $\quad R$ rational with no real poles.

when does it converge? ($\int_0^{\infty}$, $\int_{-\infty}^{0}$ 都 converge)

$\Longleftrightarrow$ deg deno $\geq$ deg num $+2$. $\Longleftrightarrow$ $\lim_{x\to\pm\infty} xR(x) = 0$



large enough to enclose all poles in $y>0$.

$$\int_{-r}^{r} R(x)\,dx + \int_{\text{half-circle}} R(z)\,dz = 2\pi i\sum_{y>0}\operatorname{Res} R(z)$$

$\leq M(r)\cdot\pi r \to 0$ as $r\to\infty$ $\qquad M(r) = \sup_{|z|=r}|R(z)|$

$$\int_{-\infty}^{\infty} R(x)\,dx = 2\pi i\sum_{y>0}\operatorname{Res} R(z) \quad \left(\Longleftrightarrow \int_{-\infty}^{\infty} = -2\pi i\sum_{y<0}\operatorname{Res} R(z)\right)$$

Example:

① $f(z) = \frac{g(z)}{h(z)}$, a point where $g(a)\neq 0$, h has simple zero

$\operatorname{res}(f, a) = \frac{g(a)}{h'(a)}$, 只有这一项贡献了 $\frac{1}{z}$ 项.

(准确地说, $\frac{1}{z-a}$)

$g(z) = g(a) + g'(a)(z-a) + \cdots$

$0 = h(a) +$

$h(z) = h'(a)(z-a)(1+\cdots)$

$\int_0^{+\infty}\frac{dx}{1+x^6}$ $\qquad \frac{1}{1+z^6}$ has all 6 poles in unit circle

In $y>0$, $e^{\frac{\pi i}{6}}$, $e^{\frac{\pi i}{2}}$, $e^{\frac{5\pi i}{6}}$

At each pole α, residue is $\frac{1}{6\alpha^5} = -\frac{\alpha}{6}$ $\qquad$ 这里到底为啥啊.

$$= \frac{1}{2}\int_{-\infty}^{\infty}\frac{dx}{1+x^6}$$

$$= -\frac{\pi i}{6}\left(e^{\frac{\pi i}{6}} + e^{\frac{\pi i}{2}} + e^{\frac{5\pi i}{6}}\right)$$

$$= \frac{\pi i}{6}\left(2\sin\frac{\pi}{6}+1\right)$$

$$\int_{-\infty}^{\infty} R(x)e^{ix}dx = \int_{-\infty}^{\infty} R(x)\cos x\,dx + i\int_{-\infty}^{\infty} R(x)\sin x\,dx$$

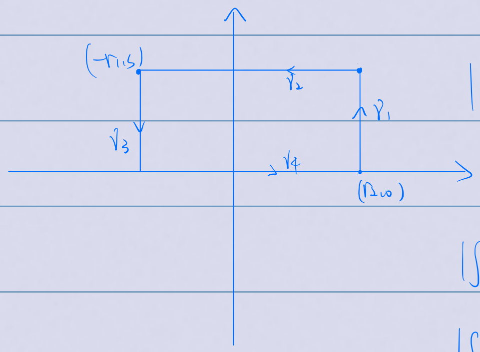
if $R(z)$ has a zero of order ≥ 2 at ∞,

then $\int_{-\infty}^{\infty} R(x)e^{ix}dx = 2\pi i \sum_{y>0} \mathrm{Res}(R(z)e^{iz})$

because $|e^{iz}| = e^{-y}$ bounded in $y>0$.

Enough here that $R(z)$ has zero of order ≥ 1 at ∞.

i.e., $|zR(z)|$ bounded. $|R(z)| < \frac{\text{const}}{|z|}$



$$\left|\int_{\gamma_1}\right| \leq \text{const}\int_0^s e^{-y}\frac{dy}{|z|}$$

$$\leq \frac{\text{const}}{r_2}\underbrace{\int_0^\infty e^{-y}dy}_{=1}$$

$$\left|\int_{\gamma_3}\right| \leq \frac{\text{const.}}{r_1}$$

$$\left|\int_{\gamma_2}\right| \leq \text{const}\,\frac{e^{-s}}{s}(r_1+r_2) \to 0 \text{ as } s\to\infty \text{ for fixed } r_1, r_2.$$

$$\Rightarrow \left|\int_{-r_1}^{r_2} R(x)e^{ix}dx - 2\pi i\sum_{y>0}\mathrm{Res}(R(z)e^{iz})\right| \leq \text{const}\left(\frac{1}{r_1}+\frac{1}{r_2}\right)$$

$\Rightarrow \int_{-\infty}^{\infty} R(x)e^{-ix}dx$ 用下半平面

Likewise for integrals involving e^{inx}, $\cos mx$, $\sin nx$, $\cos^m x$, $\sin^n x$.

$$\int_0^\infty \frac{\cos mx}{x^2+1}dx = \frac{1}{2}\int_{\mathbb{R}} = \frac{1}{2}\,\mathrm{Re}\left(\int_{-\infty}^{\infty}\frac{e^{imx}}{x^2+1}dx\right)$$

$$\int_{-\infty}^{\infty}\frac{e^{imx}}{x^2+1}dx = \frac{2\pi i}{m}\sum_{y>0}\mathrm{Res}\left(\frac{e^{iz}}{\frac{z^2}{m^2}+1}\right)$$

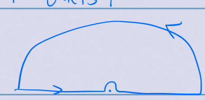
$$= \pi e^{-m}$$

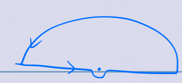
$\frac{m^2e^{iz}}{m^2+z^2}$, only poles in $y>0$ is $z=im$

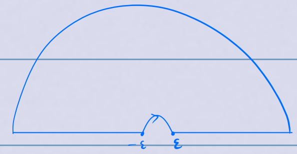
Residue is $\frac{m^2e^{-m}}{2im}$

$$= \frac{\pi e^{-m}}{2}$$

What if $R(z)$ has poles on real axis?

Simple pole: choose contour to bypass 

or enclose  差了个 residue



if 0 simple pole of $f(z)$

$\lim_{\varepsilon\to0}\int_{\gamma(\varepsilon)} f(z)dz = \pi i \operatorname{Res}(f,0)$ 若 $f=\frac{a}{z}+g(z)$ holom.

positive oriented semi-circle. 因为 $\int_{\gamma(\varepsilon)} f(z)dz = \underbrace{\int_{\gamma(\varepsilon)}\frac{a}{z}dz}_{\pi i a} + \underbrace{\int_{\gamma(\varepsilon)} g(z)dz}_{0}$

$$\int_0^{\infty}\frac{\sin x}{x}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\sin x}{x}dx$$

$$= \frac{1}{2}\operatorname{Im}\left(\lim_{\varepsilon\to0}\left(\int_{-\infty}^{-\varepsilon}\frac{e^{ix}}{x}dx+\int_{\varepsilon}^{\infty}\frac{e^{ix}}{x}dx\right)\right)$$

$$= \frac{1}{2}\operatorname{Im}\Big(\underbrace{\lim_{\varepsilon\to0}\int_{\gamma(\varepsilon)}\frac{e^{iz}}{z}dz}_{=\pi i(\operatorname{Res}(\frac{e^{iz}}{z},0))=\pi i}\Big)$$

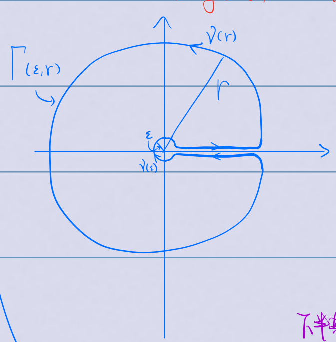
$$= \frac{\pi}{2}$$

④ $\int_0^{+\infty}\frac{R(x)}{x^\alpha}dx$ ⑤ $\int_0^{+\infty}R(x)\log x\,dx$

• $\int_0^{+\infty}\frac{R(x)}{x^\alpha}dx$ ← rational with no poles on $[0,\infty)$ $0<\alpha<1$

converges at 0, ✓

converges at ∞ iff $R(x)$ has zero of order ≥ 1 at ∞, i.e. $xR(x)$ bounded.



Take $\arg(z)$ in $[0,2\pi)$

ε small enough.

$\int_{\Gamma(\varepsilon,r)}\frac{R(z)}{z^\alpha}dz = 2\pi i\sum_{\mathbb{C}\setminus[0,\infty)}\operatorname{res}(R(z)/z^\alpha)$ if r large enough

$$\text{LHS} = \underbrace{\int_{\gamma(r)}\frac{R(z)}{z^\alpha}dz}_{r\to\infty,\ \to 0} + \underbrace{\int_{\gamma(\varepsilon)}\frac{R(z)}{z^\alpha}dz}_{\varepsilon\to0,\ \to0} + \underbrace{\int_\varepsilon^r\frac{R(x)}{x^\alpha}dx}_{\text{上半条线}} - e^{-2\pi i\alpha}\int_\varepsilon^r\frac{R(x)}{x^\alpha}dx$$

下半条线呢? $\arg z=2\pi i \Rightarrow z^\alpha = e^{2\pi i\alpha}|z|^\alpha$. ?

$$(1-e^{-2\pi i\alpha})\int_0^{+\infty}\frac{R(x)}{x^\alpha}dx = 2\pi i\sum_{\mathbb{C}\setminus[0,\infty)}\operatorname{res}(R(z)/z^\alpha)$$

• $\int_0^{\infty}\frac{dx}{x^\alpha(1+x)}$ $0<\alpha<1$

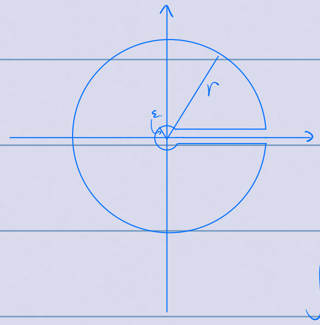
$$(1-e^{-2\pi i\alpha})\int_0^{\infty}\frac{dx}{x^\alpha(1+x)} = 2\pi i\operatorname{Res}\left(\frac{1}{z^\alpha(1+z)},-1\right)$$

$$= \frac{1}{e^{\pi i\alpha}}$$

$$\Rightarrow \int_0^{\infty}\frac{dx}{x^\alpha(1+x)} = \frac{2\pi i}{e^{\pi i\alpha}(1-e^{-2\pi i\alpha})}$$

- $\int_0^\infty R(x)\log x\,dx$ ← rational with no poles on $[0,\infty)$

need $xR(x) \xrightarrow{x\to\infty} 0$, then converge.



$\arg(z)\in[0,2\pi)$

Integrate $R(z)\log^2(z)$

$\arg z = 2\pi i \Rightarrow \log z = \log|z| + 2\pi i$

$\int_{\gamma(r)}, \int_{\gamma(\varepsilon)} \xrightarrow[\varepsilon\to 0]{r\to\infty} 0$

$$\int_0^{+\infty} R(x)\log^2 x\,dx - \int_0^\infty R(x)(\log x + 2\pi i)^2 dx = 2\pi i \sum_{\mathbb{C}\setminus[0,\infty)} \operatorname{res}(R(z)\log^2 z)$$

$$= -2\int_0^\infty R(x)\log x\,dx - 2\pi i\int_0^\infty R(x)dx = \sum_{\mathbb{C}\setminus[0,\infty)} \operatorname{res}(R(z)\log^2 z)$$

- $\int_0^\infty \frac{\log x}{(1+x)^3}dx$

Residue of $\frac{\log z}{(1+z)^3}$ at $z=1$ $\quad z=-1+z_1$ $\quad 1-\pi i$

$\log(-1+z_1) = \log(-1) + \log(1-z_1) = \pi i + \log(1-z_1)$

$\log^2 z = \left(\pi i + \left(-z_1 - \frac{z_1^2}{2} - \cdots\right)\right)^2$

$= (1-\pi i)z_1^2$

<u>Harmonic Functions:</u> Derichlet problem and mean value property

$f(x,y)$ fn of 2 real variables in open subset of $\mathbb{R}^2(\mathbb{C})$ with real / complex value is harmonic if C^2 and $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$ Laplace

$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$, $\frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$ $\xleftrightarrow{\text{duals of}}$ $(dz = dx + idy,\quad d\bar z = dx - idy)$

$\Rightarrow \Delta = 4\frac{\partial^2}{\partial z\,\partial \bar z}$ (holomorphic fn is harmonic)

Real and Imagerary part of harmonic fns are harmonic.

A fn has MVP iff its Re, Im has.

A real valued harmonic fn $g(x,y)$ is locally the real part of a holomorphic function $f(z)$, which is uniquely determined up to a constant.

Why? $\frac{\partial^2 g}{\partial z \partial \bar{z}} = 0$, so $\frac{\partial g}{\partial z}$ holomorphic; therefore,

$\frac{\partial g}{\partial z} dz$ locally has a primitive $f(z)$

$df = \frac{\partial g}{\partial z} dz$ $\qquad$ $d\bar{f} = \frac{\partial g}{\partial \bar{z}} d\bar{z}$ $\qquad$ 对所有取共轭, g real.

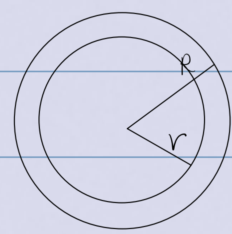
$\hookrightarrow = \frac{\partial f}{\partial z} dz + \cancel{\frac{\partial f}{\partial \bar{z}} d\bar{z}}$ $\qquad$ $\bar{f}(\bar{z}) = \overline{f(z)}$ $\qquad$ $RHS = \frac{1}{2}\overline{\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) g} \cdot \overline{(dx + idy)}$

$= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) g \cdot (dx - idy)$

$$\Rightarrow d(f + \bar{f}) = dg$$

$$\Rightarrow g = 2\,\mathrm{Re}(f) + \text{constant}$$

Real-valued harmonic



$g(x,y) =$ real part of holom fn f

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $|z| < R$

Assume $a_0 \in \mathbb{R}$ $\qquad$ $\mathrm{Re}(a_n \overset{z^n}{r^n e^{in\theta}}) = r^n \frac{a_n e^{in\theta} + \overline{a_n e^{in\theta}}}{2}$

x, $y \Rightarrow z = re^{i\theta}$

$$g(r\cos\theta, r\sin\theta) = a_0 + \sum_{n=1}^{\infty} r^n \left(\frac{a_n e^{in\theta} + \bar{a}_n e^{-in\theta}}{2}\right)$$

$a_0 = \frac{1}{2\pi} \int_0^{2\pi} g(r\cos\theta, r\sin\theta)\, d\theta$

$\int_0^{2\pi} g(r\cos\theta, r\sin\theta) e^{-in\theta} d\theta = \frac{1}{2} 2\pi r^n a_n$ $\qquad$ 因为 $\int_0^{2\pi} e^{in\theta} d\theta = \int_0^{2\pi} \underbrace{\cos n\theta}_{0} + i \underbrace{\sin n\theta}_{0}\, d\theta$

$\Rightarrow a_n = \frac{1}{\pi} \int_0^{2\pi} g(r\cos\theta, r\sin\theta) \frac{1}{r^n e^{in\theta}} d\theta$ $\qquad$ rational fn of z $= \frac{re^{i\theta} + z}{re^{i\theta} - z}$

$$\Rightarrow f(z) = \frac{1}{2\pi} \int_0^{2\pi} g(r\cos\theta, r\sin\theta) \left(1 + 2\underbrace{\sum_{n=1}^{\infty} \left(\frac{z}{re^{i\theta}}\right)^n}_{\text{Converge: } |z| < r}\right) d\theta$$

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} g(r\cos\theta, r\sin\theta) \frac{re^{i\theta} + z}{re^{i\theta} - z}\, dz$$

Express holom fn $f(z)$ in terms of real part of bd.

real part: $\frac{re^{i\theta} + z}{re^{i\theta} - z} \cdot \frac{re^{-i\theta} - \bar{z}}{re^{-i\theta} - \bar{z}}$

$$\Rightarrow g(x,y) = \frac{1}{2\pi} \int_0^{2\pi} g(r\cos\theta, r\sin\theta) \underbrace{\frac{r^2 - |z|^2}{|re^{i\theta} - z|^2}}_{\text{poisson kernel}} d\theta$$

$|z| < r$, g real valued harmonic.

用了"g是 holom fn 的 real part"

Dirichlet problem for a disk

Given continuous fn $f(\theta)$ real, on circle centric 0, radius r,

can we find fn $F(z)$, cont. continuous in $|z|\le r$,

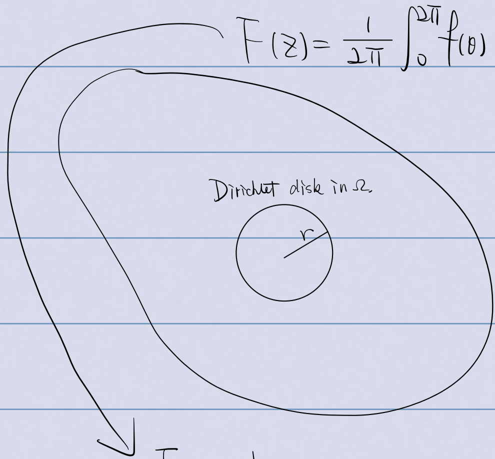
harmonic in $|z|<r$, s.t. $F(re^{i\theta})=f(\theta)$

Thm: Yes, the solution is unique.

Pf: Uniqueness: By max mod principle.

Existance: Define

$$F(z)=\frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta \quad , \quad |z|<r$$



Ω $\quad f|_{\partial D}$ cont.

So $\exists$ F cont on D, harmonic on int D,

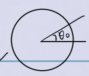
$F|_{\partial D}=f|_{\partial D}$. $F-f|_{\partial D}=0 \Rightarrow F-f=0$

$F(z)$ harmonic in $|z|<r$ because it's real part of a holomorphic fn.

We have to show that

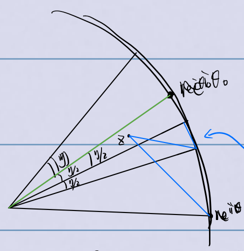
$$\lim_{z\to re^{i\theta_0}} F(z)=f(\theta_0)$$

Note: $\frac{1}{2\pi}\int_0^{2\pi}\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta=1$ (Take $g=1$) ?



Lemma: $\forall \eta>0$, $\frac{1}{2\pi}\int_{|\theta-\theta_0|>\eta}\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta \xrightarrow[z\to re^{i\theta_0}]{} 0$

Pf:



$z=\rho e^{i\alpha}$, $|\alpha-\theta_0|<\frac{\eta}{2}$

Then $|\alpha-\theta|\ge\frac{\eta}{2}$, $\forall\ |\theta-\theta_0|>\eta$.

$\Rightarrow |re^{i\theta}-z|>r\sin\frac{\eta}{2}$

$$\Rightarrow \frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\le\frac{r^2-\rho^2}{r^2\sin^2\frac{\eta}{2}} \xrightarrow[\substack{\rho\to r}]{\text{as } z\to re^{i\theta_0}} 0 \quad \checkmark$$

$$F(z)-f(\theta_0)=\frac{1}{2\pi}\int_0^{2\pi}\left(f(\theta)-f(\theta_0)\right)\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta$$

$$=\frac{1}{2\pi}\int_{|\theta-\theta_0|\le\eta}+\frac{1}{2\pi}\int_{|\theta-\theta_0|>\eta}$$

Dirichlet Problem for a disk

Given continuous $f(\theta)$, periodic with period 2π,

To find $F(z)$ continuous on disk $\{|z| \le r\}$, harmonic in $\{|z|<r\}$

s.t. $F(re^{i\theta}) = f(\theta)$

Sol^n: For $|z|<r$, let

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,\frac{r^2-|z|^2}{|re^{i\theta}-z|^2}\,d\theta$$

harmonic because it's the real part of a holomorphic fn.

To show: $\lim\limits_{z\to re^{i\theta_0}} F(z) = f(\theta_0)$

Lemma: If $\eta>0$, then $\displaystyle\int_{|\theta-\theta_0|\ge\eta} \frac{r^2-|z|^2}{|re^{i\theta}-z|}\,d\theta \xrightarrow{z\to re^{i\theta_0}} 0$ 上面

$$\Rightarrow F(z)-f(\theta_0) = \frac{1}{2\pi}\int_0^{2\pi}\left(f(\theta)-f(\theta_0)\right)\frac{r^2-|z|^2}{|re^{i\theta}-z|}\,d\theta$$

$$= \frac{1}{2\pi}\int_{|\theta-\theta_0|<\eta} ① + \frac{1}{2\pi}\int_{|\theta-\theta_0|\ge\eta} ②$$

因为这个积分总小于1.

Let $\varepsilon>0$. $① \le \sup\limits_{|\theta-\theta_0|<\eta}|f(\theta)-f(\theta_0)|$, choose η, s.t. $① \le \frac{\varepsilon}{2}$

$② \le \sup\limits_{\text{Disk}}|f(\theta)-f(\theta_0)|\cdot\frac{1}{2\pi}\displaystyle\int_{|\theta-\theta_0|\ge\eta}\frac{r^2-|z|^2}{|re^{i\theta}-z|} < \frac{\varepsilon}{2}$ for $z\to re^{i\theta_0}$

Runge's approximation theorem:

Can a holomorphic function be approximated uniformly by polynomial, on a given compact set?

① holom fn in open disk has convergent power series expansion.

So the partial sum $\longrightarrow$ fn

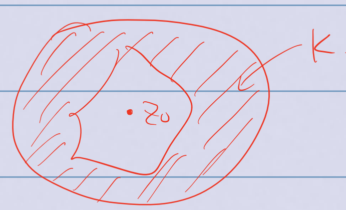
② $1/z$ cannot be uniformly approximated by polynomials on S^1

$$\begin{cases}\displaystyle\int_{S^1}\frac{dz}{z} = 2\pi i \\ \displaystyle\int_{S^1} \text{poly}\,dz = 0\end{cases}$$

K simply connected?

the compact set

It's necessary that $\mathbb{C}\setminus K$ is connected.



Otherwise $\mathbb{C}\setminus K$ has bounded component.

Suppose $\frac{1}{z-z_0}$ can be uniformly approximated by polys:

Then, choose poly $p(z)$ s.t., 假设找到了这样的poly.

$$\left|\frac{1}{z-z_0} - p(z)\right| < \frac{1}{c}, \quad \text{where } |z-z_0| < c \text{ on } K.$$

$\Rightarrow |(z-z_0)p(z) - 1| < 1$ on K.

then $|(z-z_0)p(z)-1| < 1$ on the bdd component of $\mathbb{C}\setminus K$ (maximal module principle)

$\Rightarrow$ Contradiction. Let $z = z_0$.

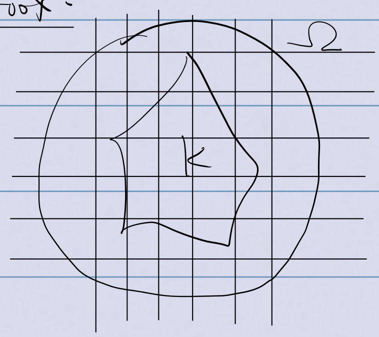
Runge's theorem:

$K \subset \Omega \subset \mathbb{C}$ (compact, open), $f(z)$ holomorphic function on Ω.

Then ① f can be approximated on K uniformly by rational functions with poles on $\mathbb{C}\setminus K$

② if $\mathbb{C}\setminus K$ connected, then f can be approximated by polys.

Proof:



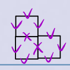
Grid of squares of side lenth

$d < \frac{1}{\sqrt{2}} d(K, \mathbb{C}\setminus\Omega)$

So any square that intersect K will lie inside Ω.

Let $Q = \{Q_1, Q_2, \cdots, Q_m\}$ squares intersect K, bdries positively oriented.

$\gamma_1, \gamma_2, \cdots \gamma_n$ bdry segment of Q_j which aren't bdry of 2 adjacent squares of Q.



So each γ_k doesn't intersect K.

If $z \in K$, then $f(z) = \sum_{i=1}^{n} \frac{1}{2\pi i} \int_{\gamma_i} \frac{f(\zeta)}{\zeta - z} d\zeta \qquad z \in K$.

Using Cauchy's thm: Consider $z \in Q = Q_1 \cup \cdots \cup Q_m$, not on bd of any Q_j.

If $z \in Q_j$, then $\frac{1}{2\pi i} \int_{\partial Q_m} \frac{f(\zeta)}{\zeta - z} d\zeta = \begin{cases} f(z) & m = j \\ 0 & \text{otherwise} \end{cases}$

$\Rightarrow f(z) = \frac{1}{2\pi i} \sum_{m=1}^{M} \int_{\partial Q_m} \frac{f(\zeta)}{\zeta - z} d\zeta = \frac{1}{2\pi i} \sum_{i=1}^{n} \int_{\gamma_i} \frac{f(\zeta)}{\zeta - z}$ 内部共用那些那积分被消掉了.

Then this also true for any $z \in Q$ by continuity 之前说不能在边上.

(1) Enough to prove:

<u>Lemma:</u> γ line segment on $\Omega \setminus K$.

Then $\int_\gamma \frac{f(\zeta)}{\zeta - z} d\zeta$ can be approximated on K by rational fns with poles on γ.

<u>Proof:</u> $\gamma : [0,1] \to \Omega$

$$= \int_0^1 \frac{f(\gamma(t))}{\gamma(t) - z} \gamma'(t)\, dt$$

Integral $F(z,t)$, continuous fn on $K \times [0,1]$, so uniformly cont.

So $\forall \varepsilon > 0$, $\exists \delta > 0$, s.t.

$$\sup_{z \in K} |F(z,t_1) - F(z,t_2)| < \varepsilon, \text{ where } |t_1 - t_2| < \delta.$$

It follows that Riemann sums of $\int_0^1 F(z,t)\,dt$ approximate $\int$ uniformly on K.

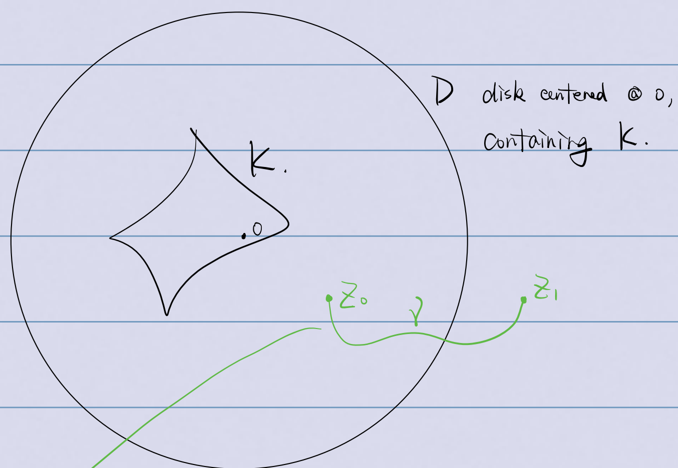
Riemann sums are rational fns with poles on γ.

带每个t_0进去的时候这里总是一个常数，变成 $\frac{b}{a-z}$ 的形式，唯一在变的量只有z（在给定partition时），所以是 Rational fn.

(2) Enough to proof: (By partial fraction decomposition)

<u>Lemma:</u> If $\mathbb{C} \setminus K$ connected and $z_0 \notin K$, then $\frac{1}{z - z_0}$ can be approximated uniformally on K by polynomials.

<u>Pf:</u>



D disk centered @ 0, containing K.

<u>Case 1:</u> $z_0 \notin D$.

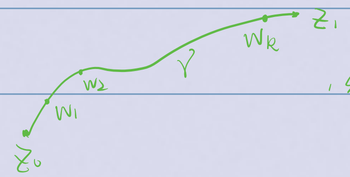
$$\frac{1}{z - z_0} = -\frac{1}{z_0} \frac{1}{1 - \frac{z}{z_0}} = -\sum_{n=0}^{\infty} \frac{z^n}{z_0^{n+1}} \quad \text{power series expansion @ 0.}$$

Converges uniformly on K. Take the partial sum

<u>Case 2:</u> $z_0 \in D$, Take $z_1 \notin D$.

Show $\frac{1}{z-z_0}$ can be approximated uniformly on K

by polys in $\frac{1}{z-z_1}$



, s.t. $|w_{i+1} - w_i| < \frac{1}{2} d(\gamma, K) = \delta$, say.

<u>Claim:</u> if $w \in \gamma$ and $|w - w'| < \delta$, then $\frac{1}{z-w}$

can be approx unif on K by polys in $\frac{1}{z-w'}$

<u>Pf:</u> $\frac{1}{z-w} = \frac{1}{(z-w')-(w-w')} = \frac{1}{z-w'} \cdot \frac{1}{1-\frac{w-w'}{z-w'}} = \sum_{0}^{\infty} \frac{(w-w')^n}{(z-w')^{n+1}}$

Converges uniformly on K.

Use partial sum.

Apply this step by step from w_1 to w_2, etc.