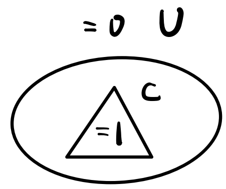
11.19 Prel 1:

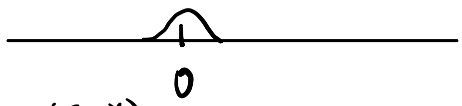
Given $C \subset U$ (compact, open) $\exists F \in C^\infty(\mathbb{R}^n)$ such that $F|_C \equiv 1$, $\operatorname{supp} F \subset U$



step 1: $\exists$ smooth 1D seashore $\sigma \in C^\infty(\mathbb{R})$ $\sigma(x) = 0 \quad x \le 0$, $\sigma(x) > 0 \quad x > 0$

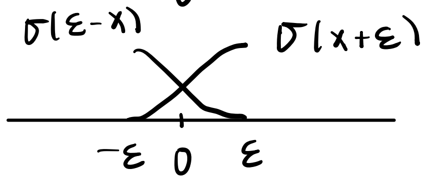
$$\sigma(x) = \begin{cases} 0, & x \le 0 \\ e^{-\frac{1}{x}}, & x > 0 \end{cases} \quad \text{(decreases expotentially)}$$

Step 2: $\exists$ smooth 1D bumps $\beta_\varepsilon \in C^\infty(\mathbb{R})$ $\beta_\varepsilon \geq 0$ everywhere



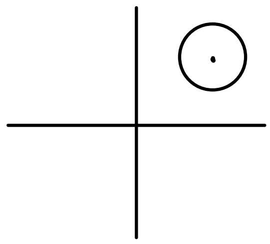
$\beta_\varepsilon(x) = 0$ if $|x-a| \geq \varepsilon$

$\beta_\varepsilon(x) > 0$ if $|x-a| < \varepsilon$



Set $\beta_\varepsilon(x) = \sigma(x+\varepsilon)\,\sigma(\varepsilon - x)$

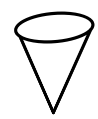
Step 3: $\exists$ smooth nD bumps, given $a \in \mathbb{R}^n$ & $\varepsilon > 0$



$\exists\ \beta_{a,\varepsilon}$ s.t. $\beta \in C^\infty(\mathbb{R}^n)$, $\beta \geq 0$

$\beta(a) > 0$, $\beta(x) = 0$ if $|x-a| \geq \varepsilon$

Wrong: $\beta_{a,\varepsilon}(x) := \beta_\varepsilon(|x-a|)$ $\quad | \ |$ function



not smooth

Correct: $\beta_{\varepsilon^2}(|x-a|^2)$

$$\downarrow$$

$$\sum_i |x_i - a_i|^2$$

$\exists$ smooth "step functions" $\theta \in C^\infty(\mathbb{R})$ $\theta: \mathbb{R} \to [0,1]$ s.t.

$$\theta(x) = \begin{cases} 0, & x \le 0 \\ 1, & x \geq 1 \end{cases}$$

one way

$$\theta_0(x) = \int_{-\infty}^{x} \beta_{\frac{1}{2},\frac{1}{2}}(t)$$

$$\theta(x) = \frac{1}{\theta_0(1)}\,\theta_0(x)$$

Proof of Prel 1:

For each $x \in C$ find $\varepsilon > 0$ s.t. $B_{\varepsilon_x}(x) \subset U$ (possible as U is open)

$\{B_{\varepsilon_x}(x)\}$ is an open cover of C, hence it has a finite subcover

$x_i,\ i = 1, \cdots, m \qquad \bigcup B_{\varepsilon_{x_i}}(x_i) \supset C$

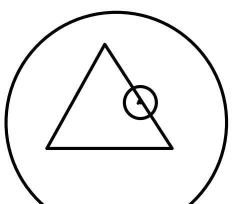
Set $F_0(x) = \sum_{i=1}^{m} \beta_{x_i, \varepsilon_{x_i}}(x)$

$F_0(x) > 0$ for $x \in C$

F_0 is a continuous function on a compact set bounded below by some $b > 0$ on C

$F(x) = \theta\left(\frac{1}{b} F_0(x)\right)$

Prel 2: Given C compact $\subset \underset{\text{open}}{U} \subset \mathbb{R}^n$, $\exists$ compact D s.t. $\underset{\text{compact}}{C} \subset \underset{\text{open}}{D^{int}} \subset \underset{\text{compact}}{D} \subset \overset{\text{open}}{U}$



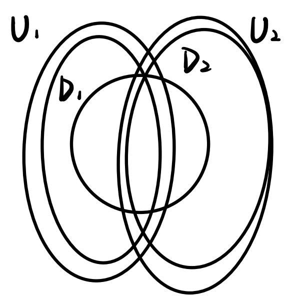
for each $x \in C$, find a open ball B_x s.t. $\overline{B_x} \subset U$. Clearly, $\{B_x\}_{x \in C}$ covers C. By compactness, it has a finite subcover $B_{x_i},\ i = 1, \cdots, p$. Take $D = \bigcup \overline{B_{x_i}}$ a finite union of closed and bounded sets. So it's closed and bounded hence compact.

It's easy to check that $D^{int} = \bigcup B_{x_i} \supset C$

Pf of POI:

Case I. A is compact

WLOG. U is finite



$\{U_i\}_{i=1,\dots,p}$

Pf: Let $E_1 = \underbrace{A \setminus \bigcup_{i=2}^{p} U_i}_{}$ exclusive zone of U_1 $\quad E_1$ is compact (because we subtract finite union of open sets from a compact set) and $E_1 \subset U_1$. By Prel 2, we can find compact D_1 s.t. $E_1 \subset D_1^{int} \subset D_1 \subset U_1$

We have that $D^{int} \cup \bigcup_{i=2}^{p} U_i \supset A$

Suppose $D_1, \dots, D_{q-1}$, where $2 \le q \le p$ have been constructed s.t. $D_i \subset U_i$

$\bigcup_{i=1}^{q-1} D_i^{int} \cup \bigcup_{i=q}^{p} U_i \supset A$

Set $E_q = A \setminus (\bigcup_{i=1}^{q-1} D_i^{int} \cup \bigcup_{i=q}^{p} U_i)$ Then E_q is compact, $E_q \subset U_q$. So use Prel 2 to find D_q compact s.t. $E_q \subset D_q^{int} \subset D_q \subset U_q$ and we still have that $\bigcup_{i=1}^{q} D_i^{int} \cup \bigcup_{i=q+1}^{p} U_i \supset A$. Continue by repeating the last step until q reaches p.

Now we can shrink U_i to a compact D_i s.t. $\{D_i^{int}\}$ still covers A. Namely, we can find compact D_i s.t. $D_i \subset U_i$ & $\bigcup D^{int} \supset A$

Assuming claim, find $\varphi: \mathbb{R}^n \to [0,1]$ that are C^∞, s.t. $\varphi_i|_{D_i} = 1$ & $\operatorname{supp} \varphi_i \subset U_i$

Define $\psi_i(x) = \begin{cases} F(x)\dfrac{\varphi_i(x)}{\sum_{j=1}^{p} \varphi(x)}, & x \in \bigcup D_i^{int} \supset A \\ 0, & x \notin \bigcup D_i^{int} \end{cases}$

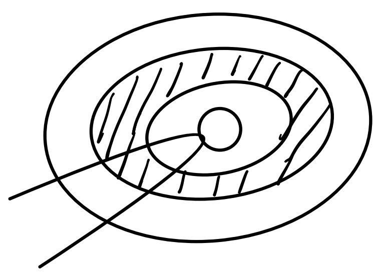
where $F(x)$ is smooth & satisfies $F|_A \equiv 1$ & $\operatorname{supp} F \subset \bigcup D_i^{int}$

Case II: $A = \bigcup_{k=0}^{\infty} A_k$ with A_k compact & $A_k \subset A_{k+1}^{int}$ & $A_0 = \emptyset$

Aside: $\bigcup A_k^{int} \subset A = \bigcup A_k = \bigcup A_{k+1}^{int} = \bigcup A_k^{int} \Rightarrow A$ is open

Let $L_k = A_{k+1} \setminus A_k^{int}$ for $k \geq 0$.

Layer k



L_k is compact.

Let $\mathcal{U}_k = \{ U \cap A_{k+2}^{int} \cap A_{k-1}^{c} : U \in \mathcal{U} \}$

$\mathcal{U}_k$ is an open cover of L_k. So use step 1 to find a PDI $\Psi_k = \{\psi_{ki}\}$ for L_k subordinate to $\mathcal{U}_k$. Let $\Psi = \bigcup_{k=0}^{\infty} \Psi_k = \{\psi_i\}$ it is still a countable collection of C^∞ functions.

extend these to $A = \bigcup A_k$ by 0

Set $\psi_i = \dfrac{\varphi_i(x)}{\sum_{j=1}^{\infty} \varphi_j(x)}$ And clearly $\{\psi_i\}$ is the desired POI For $A = \bigcup A_k$

Case III: A is an arbitrary open set. $A = \bigcup_{k \geq 1} A_k$

$A_k = \{ x \in A \ \& \ |x| \leq k \ \& \ \mathrm{dist}(x, A^c) \leq \frac{1}{k} \}$

Claim: $A_k \subset A_{k+1}^{\mathrm{int}}$, A_k is compact, $A = \bigcup A_k$.

Step 4: Find a POI for $A' = \bigcup_{u \in U} u$ It works for A, too.

Prel 1: $\int_R^{leg} (f+g) = \int_R^{leg} f + \int_R^{leg} g$ assuming f & g are integrable.

$\int_R^{leg} cf = c \int_R^{leg} f$

Prel 2: $f \leq g$ and both are integrable

$\int f \leq \int g$

Warning:

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \log 2$

$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots = \frac{\pi}{4}$

Riemann rearrangement can rearrange the series to get any sum you want.

Good	Evil
$1 \ \frac{1}{3} \ \frac{1}{5} \ \frac{1}{7}$	$-\frac{1}{2} \ -\frac{1}{4} \ -\frac{1}{6}$

For a sum, get from good until it exceeds the sum, and get from evil until it goes slightly below the sum.

sum works whenever $\sum |a_n| = \infty$ $\sum a_i$ converges

If $\sum |a_n| < \infty$, sums are the same under rearrangement.

Let A be an open set (not necessarily bounded), $F: A \to \mathbb{R}$ locally bounded (meaning $\forall x \in A$ $\exists$ open $V \ni x$ s.t. F is bounded on V)

Aside: $F(x) = \frac{1}{x}$ on $(0,1)$ locally bounded and F is continuous except on a measure 0 set.

Let $\mathcal{U} = \{U\}$ be an open cover A by bounded open sets of A on which f is bounded

Let $\Phi = \{\varphi_i\}$ be an POI for A subordinate to $\mathcal{U}$.

Say that F is $(\mathcal{U}, \Phi)$ - integrable if $\sum_i \int \varphi_i |F| < \infty$

In that case, define

$$\int_A^{(\mathcal{U},\Phi)} F = \sum_i \int \varphi_i F$$

rectangle bigger than support φ_i

Aside: $\int F \leq \int |F| \qquad -\int F \leq \int |F|$

$\left| \int F \right| \leq \int |F|$

$$\sum \left| \varphi_i F \right| \leq \sum \int |\varphi_i F| = \sum \int \varphi_i |F| < \infty$$

The series is absolutely convergent.

Thm: If A & F are as before, then

1. If $\mathcal{U}$ & $\mathcal{U}'$, Φ & Φ' are open covers & POI as before (Φ subordinates to $\mathcal{U}$ and Φ' subordinates to $\mathcal{U}'$)

Then F is $(\mathcal{U}, \Phi)$-integrable $\Leftrightarrow$ F is $(\mathcal{U}', \Phi')$ - integrable

(We can say in either case that F is "NT- integrable")

In that case $\int_A^{(\mathcal{U},\Phi)} F = \int_A^{(\mathcal{U}',\Phi')} F =: \int_A^{NT} F$

Pf: $$\int_A^{(\mathcal{U},\Phi)} g \underset{(1)}{=} \sum_i \int \varphi_i g \underset{(2)}{=} \sum_i \int \Big(\sum_j \varphi_j'\Big) \varphi_i g \underset{(3)}{=} \sum_i \sum_j \int \varphi_j' \varphi_i g$$

$$(4) \parallel$$

$$\int_A^{(\mathcal{U}',\Phi')} g \underset{(1)}{=} \sum_j \int \varphi_j' g \underset{(2)}{=} \sum_j \int \Big(\sum_i \varphi_i\Big) \varphi_j' g \underset{(3)}{=} \sum_j \sum_i \int \varphi_i \varphi_j' g$$

Go through the above first with $g = |F|$

(1) idc irrelevant

(2) sum=1 in POI

(3) NTS $\int (\sum_j \varphi_j') \cdot h = \sum_j \int \varphi_j' \cdot h$

have support $h \subset$ support $\varphi_i \subset U \in \mathcal{U}$ so support h is bounded & closed hence compact

By compactness of support h & local finiteness of Φ', only finitely many i's have support $\varphi_i' \cap$ support $h \neq \phi$

Hence, (3) holds by finite linearity.

(4) $|g| \geq 0$, $\varphi_i \geq 0$, $\varphi_j' \geq 0$, so all terms of the sums are non-negative

Exercise: A sum of non-negative terms is convergent iff every rearrangement is convergent.

We know the absolute convergence since F is $(\mathcal{U}, \Phi)$-integrable.

Go through the above again with $g = F$.

(4) By absolute convergence.

Suppose A & F are as above.

Thm 1: If A & F are bounded, then F is NT-integrable on A.

2. If in addition A is "Jordan measurable" (meaning A is bounded & $\text{Bd}\, A$ is measure-0 $\Leftrightarrow \chi_A$ is continuous except measure 0)

Then $\int_A^{NT} F = \int_A^{old} F$ $\qquad \int_A^{old} F = \int_{\text{big } R}^{old} \chi_A F$

Pf of 1: Assume $|F| < M$ & $A \subset \underset{\text{rectangle}}{R}$

$$\sum_{i=1}^{N} \int_R \varphi_i |F| = \int_R \sum_{i=1}^{N} \varphi_i |F| \leq \int_R M = M \text{vol}(R)$$

Pf of 2: Lemma: We can find a compact $C \subset A$ s.t $\underset{\displaystyle \int \chi_{A \setminus C}}{\overset{\shortparallel}{\text{vol}(A \setminus C)}} < \varepsilon$

Assuming Lemma, for only finitely many i's, support $\varphi_i \cap C \neq \emptyset$ (every point in C has a neighbourhood touching only finitely many supp φ_i's and finitely many such neighbourhoods cover C)

Find N bigger that all of those i's. Then $\sum_{i=1}^{N} \varphi_i = 1$ on C.

$$\left| \int_A^{old} F - \sum_{i=1}^{N} \int^{old} \varphi_i F \right| = \left| \int_A F - \sum_{i=1}^{N} \varphi_i F \right| \leq \int_A |F| \left(1 - \sum_{i=1}^{N} \varphi_i\right) \leq M \int_A \left(1 - \sum_{i=1}^{N} \varphi_i\right)$$

$$\leq M \int_{A \setminus C} 1 = M \operatorname{vol}(A \setminus C) < M\varepsilon$$

Facts: 1. $\int^{NT}$ is linear. $\int_A^{NT} F+g = \int_A^{NT} F + \int_A^{NT} g \qquad \int_A^{NT} cf = c \int_A^{NT} f$

assuming F & G are NT-integrable.

Pf: $\int_A^{NT} F+g = \sum_{i=1}^{\infty} \int \varphi_i (F+g) = \sum_{i=1}^{\infty} \int \varphi_i F + \int \varphi_i g = \sum_{i=1}^{\infty} \int \varphi_i F + \sum_{i=1}^{\infty} \int \varphi_i g = \int_A^{NT} F + \int_A^{NT} g$

2. (Fubini-NT) If $A \subset \mathbb{R}^n$ is open, $B \subset \mathbb{R}^m$ is open, $F\ A \times B \to \mathbb{R}$ is locally bounded & continuous except on finitely many points.

Then $\int_{A \times B} F(x,y)\,dx\,dy = \int_A dx \int_B dy\, F(x,y)$

assuming F is integrable on $A \times B$.

If U is an open cover of A, V is an open cover of B. $\Phi = (\varphi_i)$ is a POI for A subordinate to U, $\Psi = (\psi_i)$ is a POI for B surbordinate to V.

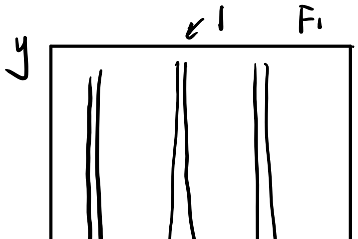
take $\mathcal{W} = \left\{ U \times V : \begin{matrix} U \in \mathcal{U} \\ V \in \mathcal{V} \end{matrix} \right\}$ is an open cover of $A \times B$, and $\Lambda = \{ \lambda_{ij} = \varphi_i(x) \psi_j(y) \}$ is a POI for $A \times B$.

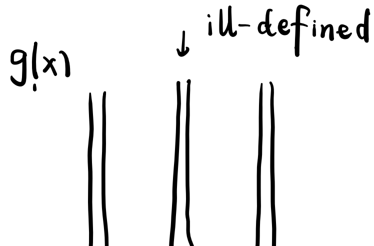
① countable

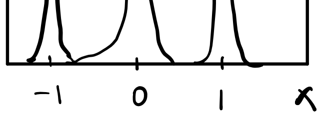
② $\sum_{i,j=1}^{\infty} \varphi_i(x) \psi_j(y) = 1 \times 1 = 1$

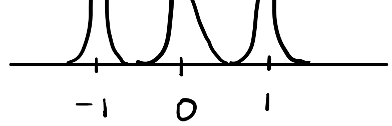
False: If F is integrable on $A \times B$ Then $g(x) = \int_B F(x,y)\,dy$ & $\int_A g(x) = \int_{A \times B} F$

Fails even for continuous F on $\mathbb{R}^2$









$F(x) = \sum_{n=1}^{\infty} \frac{1}{2^n} F_1(nx)$ $F(x)$ is bounded by 1.

$g(x) = \int F(x,y)\,dy$ is not defined at least when $x \in \mathbb{Q}$.

Fact 3: If $B \subset A$ are open sets, $F: A \to \mathbb{R}$ integrable, then $F|_B$ is also integrable. Furthermore, if $F \geq 0$, $\int_A F \geq \int_B F$

Pf: NTS if φ_i^B is a POI for B subordinate to some U^B, then $\sum_{i=1}^{\infty} \varphi_i^B |F|$ is convergent. NTS $\sum_{i=1}^{N} \int \varphi_i^B |F|$ is bounded (independent of N) Indeed,

$\sum_{i=1}^{N} \int \varphi_i^B |F| = \int^{old} (\sum_{i=1}^{N} \varphi_i^B) |F| = \int_A (\sum_{i=1}^{N} \varphi_i^B) |F| \leq \int_A |F| = \sum \int \varphi_i^A |F| < \infty$

Fact 4. Assume A_n is open for each n, $A_1 \subset A_2 \subset A_3 \cdots$ and let $A = \bigcup A_n$.

Assume $F: A \to \mathbb{R}$ is integrable. Then $\int_{A_n} F \longrightarrow \int_A F$

Example: $A = (0,\infty)$ $A_n = (0,n)$ $F: (0,\infty) \to \mathbb{R}$ integrable, then

$\int_{(0,\infty)}^{NT} F = \lim_{n\to\infty} \int_{(0,n)} F = \lim_{n\to\infty} \int_0^n F$

$\int_A F = \sum_{i=1}^{\infty} \int \varphi_i^A F$ for large N. $\int F \underset{\text{up to } \epsilon}{\sim} \sum_{i=1}^{N} \int \varphi_i^A F = \int (\sum \varphi_i^A) F$

support $\sum_{i=1}^{N} \varphi_i^A \subset \bigcup_{i=1}^{N} \operatorname{supp} \varphi_i^A$ a compact set is covered by A_n's so it is contained in some single A_k $\int (\sum \varphi_i^A) F = \int_{A_k} (\sum \varphi_i^A) F \sim \int_{A_k} F$

need to show they're close.

Eg $\int_{[0,\infty)} r e^{-\frac{r^2}{2}} dr = \lim_{R\to\infty} \int_0^R r e^{-\frac{r^2}{2}} dr = \lim_{R\to\infty} \left[-e^{-\frac{r^2}{2}} \right]_0^R$

$= \lim_{R\to\infty} (-e^{-\frac{R^2}{2}}) - (-1)$

$= 1$