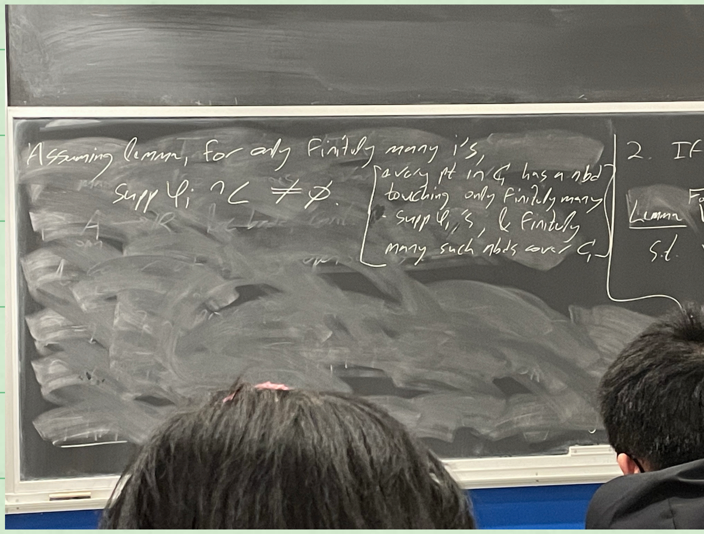
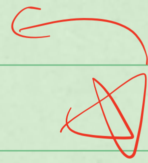
• $\forall A$, $\exists$ compact C, s.t. $vol(A\setminus C) < \varepsilon$. $(\Leftrightarrow \lim \sum\int \varphi f = \lim \int \sum \varphi f)$

任意紧集可以被包括有限个 N 覆盖



找 $N \geq i$, 则

$\sum^{N} \varphi_i = 1$ on C.



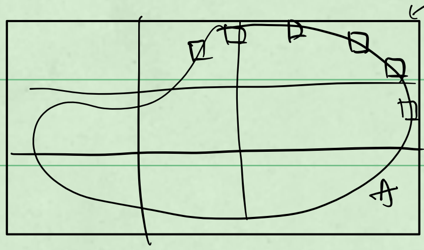
考虑

有界　　在C上是1,其它地方(0,1)

$$\left|\int_A^{old} F - \sum_{i=1}^{N}\int^{old}\varphi_i F\right| = \left|\int_A^{NT} F - \sum^{N}\varphi_i F\right| \le \int_A |F|\left(1-\sum\varphi_i\right) \le \int_A M\left(1-\sum\varphi_i\right) \le M\int_A\left(1-\sum^{N}\varphi_i\right) \le M\int_{A\setminus C} 1 \le \varepsilon.$$

Lemma? 作业　(前提是 A Jordan-measurable)

$Bd A$ 零测 $\Leftrightarrow$ χ 可积.



← 有 partition, $U - L < \varepsilon$.
$\Rightarrow$ 在边界上的 rectangle 的体积和足够小.
取 C 为里面的那些 rectangle

Facts: ① $\int$ 具有线性: $\int_A^{NT} f+g = \int_A^{NT} f + \int_A^{NT} g$

$\int_A^{NT} cf = c\int_A^{NT} f$　　$(f, g$ 可积$)$

$\int^{NT} f+g = \sum^{\infty}\int^{old}\varphi_i(f+g)$ ← old 具有线性

$= \sum^{\infty}\int^{old}\varphi_i f + \varphi_i g$

$= \sum\int\varphi_i f + \sum\int\varphi_i g = \int f + \int g$.

好像不

② (Fubini-NT)

若 $A \subset \mathbb{R}^n$ 开, $B \subset \mathbb{R}^m$ 开.　　可以更松

$f: A\times B \to \mathbb{R}$, 若 f locally bounded, / continuous except finite points.

太对

$\int_{A\times B} f(x,y)\,dxdy = \int_B\left(\int_A f(x,y)\,dx\right)dy$

前提是设 f 在 $A\times B$ 上可积.

证等号左右两 PoU 不一样?

若 U, V open cover A, B, $\Phi=\{\varphi_i\}$, $\Psi=\{\psi_j\}$ 对应的 PoU,

取 $W=\{u\times v \mid u\in U, v\in V\}$, 则 W open cover $A\times B$,

取 $\Lambda=\{\varphi_i(x)\psi_j(y)\}$ 是 W 的 PoU.

可数, sum 1, $\left(\sum_{i,j=1}^{\infty}\lambda_{ij}(x,y)=\sum\varphi_i(x)\psi_j(y)=\sum\varphi_i\sum\psi_j=1\right)$

③ If $B\subset A$ 开, $f: A\to\mathbb{R}$ 可积, 则 $f|_B$ 可积,

且 if $f\geq 0$, then $\int_B f\leq\int_A f$

Pf: NTS, 若 φ_i^B is a PoU for B, subordinate $\{U_\beta\}_{\beta\in B}$,

则 $\sum_{i=1}^{\infty}\int\varphi_i^B|f|$ 收敛.

只需证 $\sum_{i=1}^{N}\int\varphi_i^B|f|$ 有界, Indeed,

$\sum_{i=1}^{N}\int\varphi_i^B|f| = \int\left(\sum_{i=1}^{n}\varphi_i^B|f|\right) \leq \int_A\left(\sum_{i=1}^{N}\varphi_i^B\right)|f| \leq \underbrace{\int_A|f|}_{\text{上界}} < \infty$

old

$\sum^{\infty}\int\varphi_i^A|f|$

④ 设 $\{A_n\}$ a nested open set sequence. Let $A=\bigcup^{\infty}A_n$

设 $f: A\to\mathbb{R}$ 可积, 则 f 在任意 A_n 上可积, 且 $\int_{A_n}f$ 收敛至 $\int_A f$

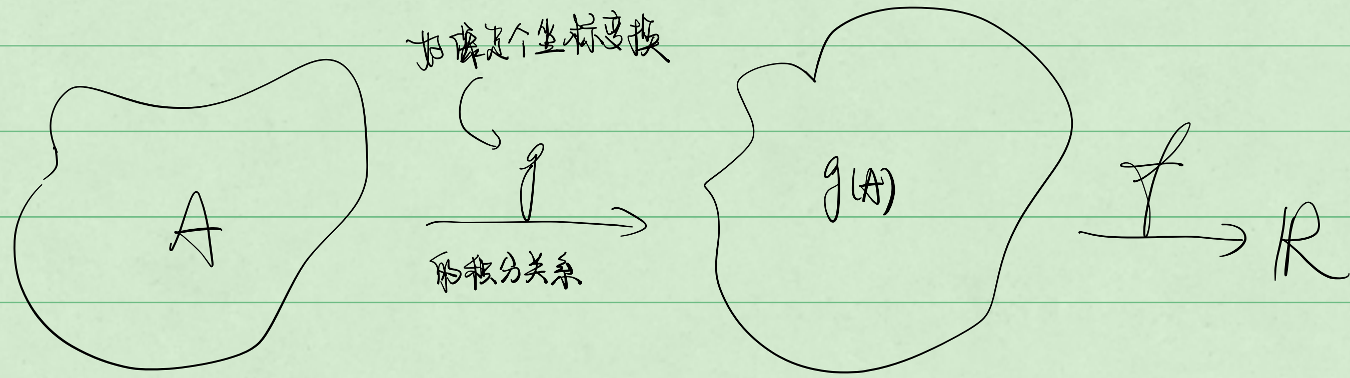
e.g. $A=(0,\infty)$, $A_n=(0,n)$, 设 $f:(0,\infty)$ 可积, 则 $\int_{(0,\infty)}^{\text{NT}}f=\lim_{n\to\infty}\int_{(0,n)}f=\lim_{n\to\infty}\int_0^n f$

$\int_A f=\sum^{\infty}\int\varphi_i^A f$ 对于 large enough N, $\int f\sim\sum^{N}\int\varphi_i^A f$ (差 ε)

$=\int\left(\sum^{N}\varphi_i^A\right)f$

$\operatorname{supp}\sum^{N}\varphi_i^A\subset\bigcup^{N}\operatorname{supp}\varphi_i^A$, compact $\Rightarrow$ A_n 的有限并 $\Rightarrow$ 包含于 A_k.

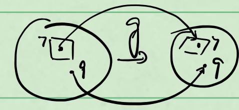
• Change of variables



$\int_{g(A)} f \overset{?}{=} \int_A f\circ g$ ×

"积分"? "体积"

所以 $\int_A f\circ g$ 是啥?



每个点都映到了同样的值.(当然!)

但这里的平四形状由 g' 决定. 但映过来后面积(体)不同.

Prop: 若 $L: \mathbb{R}^n \to \mathbb{R}^n$, R is a rectangle in $\mathbb{R}^n$,

则 $\mathrm{Vol}(L(R)) = |\det(L)|\cdot \mathrm{Vol}(R)$

(行列式值的本质就是体积)

所以在这里要乘上修正量.

$\Rightarrow \int_{g(A)} f = \int_A f\circ g \cdot |\det g'|$

Thm (C O V) (change of variables): 设 $A\subset\mathbb{R}^n$ 开, $g: A\to\mathbb{R}^n$, 为 continuously differentiable, 单, 且 $\forall a\in A$,

$g'(a)$ 在小区间内可逆. $\Rightarrow$ 这个保证了 $g(A)$ 是开的, 所以在这里可积.

若 $f: g(A)\to\mathbb{R}$ 可积, 则 $\int_{g(A)} f = \int_A f\circ g\cdot|\det g'|$

Example: "most important integral"

$I_1 = \int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}dx \qquad I_n = \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{2}}dx,$

考虑 I_2.

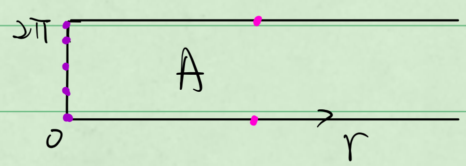
① $I_2 = \int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}dx\,dy \overset{\text{"Fubini" ①}}{=\!=\!=} \int_{\mathbb{R}_x} dx \int_{\mathbb{R}_y} dy\, e^{-\frac{x^2}{2}}\cdot e^{-\frac{y^2}{2}}$

$= \int_{\mathbb{R}_x} dx\, e^{-\frac{x^2}{2}} \int_{\mathbb{R}_y} dy\, e^{-\frac{y^2}{2}}$

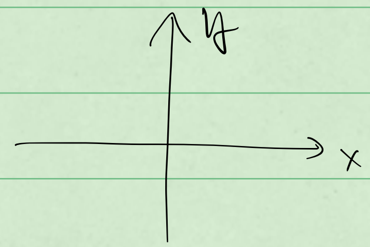
$= \left(\int_{\mathbb{R}_x} dx\, e^{-\frac{x^2}{2}}\right) I_1$

$= I_1^2$.

② use COV



$g(r,\theta)=(r\cos\theta, r\sin\theta)$



$A=[0,\infty)\times[0,2\pi]$, 但不单（因为 • •）$\Rightarrow$ 将 $A=(0,\infty)\times[0,2\pi)$ ← 重合的点没有了.

$\Downarrow$ ②

$A=(0,\infty)\times(0,2\pi)$

$$g' = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$$

$|\det g'| = r$ （回忆 f ）

$-\left(e^{-\frac{r^2}{2}}\right)' = re^{-\frac{r^2}{2}}$

$(f\circ g)(r,\theta) = e^{-\frac{r^2}{2}}$, by COV,

$$I_1 = \int_A e^{-\frac{r^2}{2}}\cdot(r)\cdot dr\,d\theta \overset{\text{Fubini ①}}{=\!=} \int_0^{2\pi} d\theta \int_0^{\infty} dr\cdot re^{-\frac{r^2}{2}}$$

$$= \int_0^{2\pi} \cdot 1$$

$$= 2\pi \Rightarrow I_1 = \sqrt{2\pi}$$

① : 无界定义域无法用 Fubini $\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}dx\,dy$:

$a_N \to \int_{[-N,N]^2} e^{-\frac{x^2+y^2}{2}}dx\,dy = \int_{[-N,N]} e^{-\frac{x^2}{2}}dx \int_{[-N,N]} e^{-\frac{y^2}{2}}dy$ 定积分, 只需证 $N\to\infty$ 时分别收敛

$\int_{\mathbb{R}^2} e^{-\frac{x^2+y^2}{2}}dx\,dy = \sum_{i=1}^{\infty}\int \varphi_i f = \lim_{p\to\infty}\sum_{i=1}^{p}\int\varphi_i f = \lim_{p\to\infty}\int\underbrace{\left(\sum_{i=1}^{p}\varphi_i\right)f}_{b_p}$

a_N, b_p 均单增, 证 $\forall N$, $\exists P$ s.t. $b_P \geq a_N$. 反之亦然, 即可证收敛至同值

固定 N, φ_i 们在 $[N,-N]$ 上和为 1, φ_i 们紧, 则有限个 φ_i 可覆盖.

反之, 找盖住所有 φ_i 的 N.

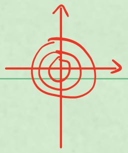
② : "You can always ignore closed sets with measure 0"

无界:

有界: 则 content $= 0$ 了.

$\sigma_n = \mathrm{Vol}(S^n)$, $S^n = \{x \in \mathbb{R}^{n+1}, |x| = 1\}$ Fubini 拆 n次

$$I_{n+1} = \int_{\mathbb{R}^{n+1}} e^{-\frac{|z|^2}{2}} dz = I_1^{\,n+1} = \sqrt{2\pi}^{\,n+1}$$



$$\to \; = \int_{\mathbb{R}^+} dr\,(S_r^n)\cdot e^{-\frac{r^2}{2}} = \sigma_n \int_{\mathbb{R}^+} r^n e^{-\frac{r^2}{2}} dr = \sigma_n I_n = \sqrt{2\pi}^{\,n+1}$$

这里的I和后面的I是这个函数,我和之前的写混了.

到原点距离

$\mathrm{Vol}(S_r^n) = r^n \cdot \underbrace{\mathrm{Vol}(S^n)}_{\sigma_n}$

$$I_{n-2} = \int_0^\infty \overset{u'}{r^{n-2}}\, \overset{v}{e^{-\frac{r^2}{2}}} dr = -\int_0^\infty \left(\frac{1}{n-1}\right) r^{n-1} \cdot \left(-re^{-\frac{r^2}{2}}\right) dr + \underbrace{uv\Big|_0^\infty}_{是0}$$

分部积分

$$= \frac{1}{n-1}\int_0^\infty r^n e^{-\frac{r^2}{2}} dr$$

$$= \frac{1}{n-1} I_n \quad (n \geq 1)$$

$$\sigma_n = 2\pi^{\frac{n+1}{2}} / I_n = \frac{2\pi \cdot 2\pi^{\frac{n-1}{2}}}{(n-1) I_{n-2}} = \frac{2\pi}{n-1}\cdot \sigma_{n-2} \Rightarrow \sigma_n = \frac{2\pi}{n-1}\sigma_{n-2}$$

$\sigma_0 = 2$, $\quad \sigma_2 = 4\pi$

$\sigma_1 = 2\pi \quad \sigma_3 = 2\pi^2$ - - - - - - -

设 $\beta_n = \mathrm{Vol}(B_1(0) \subset \mathbb{R}^n)$, 证 $\beta_n = \frac{\sigma_{n-1}}{n}$

$\beta_0 = 0 \quad \beta_2 = \pi$

$\beta_1 = 2 \quad \beta_3 = \frac{4}{3}\pi$

$\beta_n = \int_0^1 \mathrm{Vol}(S_r^{n-1})\,dr$, 而 $\mathrm{Vol}(S_r^{n-1}) = r^{n-1}\,\mathrm{Vol}(S^{n-1})$

$$= \int_0^1 r^{n-1}\,\mathrm{Vol}(S^{n-1})\,dr$$

$$= \sigma_{n-1}\int_0^1 r^{n-1}\,dr$$

$$= \sigma_{n-1}\left(\frac{r^n}{n}\right)\Big|_0^1$$

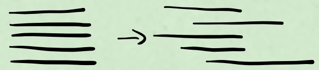
$$= \frac{\sigma_{n-1}}{n}$$

Proof of cov: WTS: 把g拆成n个"简单"的map.

Lemma: $A \xrightarrow{h} \bigcirc \xrightarrow{g} \bigcirc \xrightarrow{f} \mathbb{R}$

①?

$\mathrm{cov}(g), \mathrm{cov}(h) \Rightarrow \mathrm{cov}(g \circ h)$

layer-perserving map (l.p.):  (也可能旋转之类的)

$g: \mathbb{R}^n \to \mathbb{R}^n$, $g(x_1, x_2, \cdots, x_n) = (\cdots\cdots, x_n)$ (最后那坐标不变)

好处: 可以分步积分 ("这是哪层?" → 按层积分) ② → Fubini COV(n-1) ⇒ COV(n) for l.p. maps.

③ every g is a l.p. map? → only locally

$g: \mathbb{R}^n_{x_i} \to \mathbb{R}^n_{y_i}$, $y_i = g_i(x_1, x_2, \cdots x_n)$

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ x_n \end{pmatrix} \xrightarrow{\alpha} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-2} \\ x_{n-1} \\ y_n \end{pmatrix} \xrightarrow{\beta} \begin{pmatrix} y_1 \\ \vdots \\ \vdots \\ y_n \end{pmatrix}$$

α, β 均 l.p. (只是不同一层)

P01

④ local ⇒ global. i.e. cov(small sets) ⇒ cov(large set)

⑤ cov(1D) ⑥ R COV ⇒ COV (istricted) → 连续. ⑦ holds for swaps coordinates.

从中间走

①: Lemma: $A \xrightarrow{h} \bigcirc \xrightarrow{g} \bigcirc \xrightarrow{f} \mathbb{R}$ ($g\circ h$) 如果已经对 g 和 h 成立.

Now to show

NTS: $\int_{(g\circ h)(A)} f = \int_A (f\circ g\circ h)\,|\det(g\circ h)'|$

这里因为对 h, g COV 成立

$$\int_{(g\circ h)A} f = \int_{h(A)} \underbrace{(f\circ g)\,|\det g'|}_{\bar{f}}$$

$$= \int_A (\bar{f}\circ h)\,|\det h'|$$

代入 $\bar{f}$

$$= \int_A (f\circ g\circ h)\,|\det g'\circ h|\,|\det h'|$$

$$= \int_A (f\circ g\circ h)\,|\det((g'\circ h)\cdot h')| \quad (g'\circ h)\cdot h' = (g\circ h)'$$

$$= \int_A (f\circ g\circ h)\,|\det(g\circ h)'|$$

② Lemma: Assume COV holds for n-1 dim. 这个地方很连续.

Let $g: U \to \mathbb{R}^n$. 开, 有界 be a l.p map. s.t. $g(U)$ 仍有界.

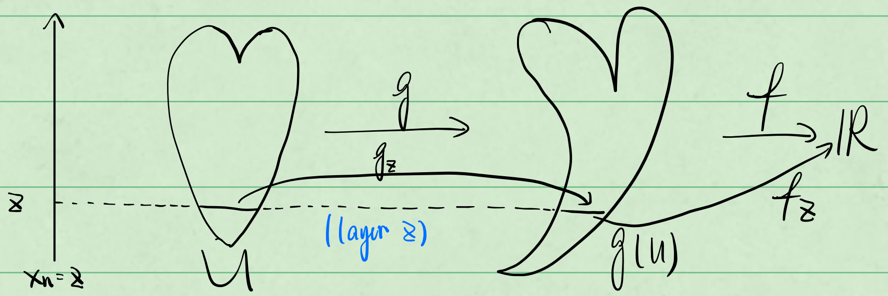
($g(x_1, x_2, \cdots, x_n) = (\cdots\cdots, x_n)$ / $g_n(x_1, x_2 \cdots x_n) = x_n$)

And, a restricted COV(g) holds: If $f: g(U) \to \mathbb{R}$

is continuous and $\operatorname{supp}(f) \subset g(U)$ $\quad U \xrightarrow{g} g(U) \xrightarrow{f} \mathbb{R}$

因为 $\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} (f \circ g)\,|\det g'|$ (同理)

$\int_{\cancel{g(U)}} \quad \int_{\cancel{U}}$



$\forall z \in \mathbb{R}$, define $g_z : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by $g_z(x) = (g_1(x,z), g_2(x,z), \cdots, g_{n-1}(x,z))$

$f_z : \mathbb{R}^{n-1} \to \mathbb{R}$ by $f_z(x) = f(x,z)$ → 这俩是"第z层的函数"

Fubini

$$\int_{\mathbb{R}^n} f \overset{\text{Fubini}}{=} \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} dx\, f(x,z) = \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} dx\, f_z(x) = \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} (f_z \circ g_z)\,|\det g_z'|$$

$|\det g_z'|$ 是什么?

$$g' = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{n-1}} & \frac{\partial g_1}{\partial z} \\ \vdots & g_z' & \vdots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_{n-1}} & \frac{\partial g_n}{\partial z} \end{pmatrix}$$

$\frac{\partial g_n}{\partial x_1} = 0, \ \cdots = 0, \ \frac{\partial g_n}{\partial x_{n-1}} = 0, \ \frac{\partial g_n}{\partial z} = 1$

最后一行是 $(0,0,\cdots,0,1)$ 因为 g l.p.

$$= \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} f_z \circ g_z\,|\det g'| \quad \underline{\underline{\text{fubini}}}$$

② : Lemma : $\forall a \in A \quad \exists\, U \ni a$ (open) s.t. on U, g is a composition of l.p. maps and coordinate swaps.

$T_{ij} : \mathbb{R}^n \to \mathbb{R}^n$, $T_{ij}(x_1, \cdots x_i, \cdots x_j, \cdots x_n) = (x_1, \cdots, x_j, \cdots, x_i, \cdots, x_n)$.

pf : $y_i = g_i(x_1, x_2, \cdots, x_n)$.

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \xrightarrow{\alpha_k} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_k \end{pmatrix} \xrightarrow{\beta_k = g \circ \alpha_k^{-1}} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

$\underbrace{\qquad\qquad\qquad\qquad}_{g}$

α invertible?

α_k, β_k 不是 l.p., 但不完全不是 (up to coord. swaps)

Let $\alpha_k(x_1, \cdots x_n) \to (x_1, \cdots x_{n-1}, g_k(x_1, \cdots x_n))$ 下证可逆.

$$\alpha_k' = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ \frac{\partial g_k}{\partial x_1} & \cdots & & & \cdots & \frac{\partial g_k}{\partial x_n} \end{pmatrix} \neq 0,$$

$\Rightarrow$ 只需 $\frac{\partial g_k}{\partial x_n}$ 不为0. 但 g 在小区间内可逆, 所以会能找到至少一个 k 使 $\frac{\partial g_k}{\partial x_n}$ 不为0.

By IFT, α_k is invertable near a.

near a, set $\beta_k = g \circ \alpha_k^{-1}$, then $g = \beta_k \circ \alpha_k = T_{kn} \circ T_{kn} \circ \beta_k \circ T_{in} \circ T_{in} \circ \alpha_k \circ T_{in} \circ T_{in}$

$$= T_{kn} \circ \underbrace{(T_{kn} \circ \beta_k)}_{\text{l.p}} \circ T_{in} \circ \underbrace{(T_{in} \circ \alpha_k \circ T_{in})}_{\text{l.p}} \circ T_{in}$$

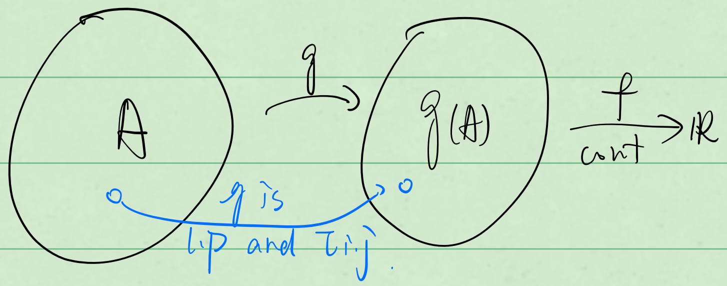
④ local COV $\Rightarrow$ global COV for continuous.

连续

$\hookrightarrow$ $\forall$ 给定 g, $\forall a \in A$, 存 $U \ni a$ 使得 COV 成立.

完成域

pf:



令 $\mathcal{V} = \left\{ \underset{\text{open}}{V} \subset g(A) : \begin{array}{l} V \text{ bounded, on } g^{-1}(V),\ g \text{ is} \\ g^{-1}(V) \text{ bounded, l.p. and T.i.j.} \end{array} \right\}$

$\mathcal{V}$ is a cover of $g(A)$. 因所有点都在.

$\forall b \in g(A)$. $g^{-1}(b) = a$ $\leftarrow g$ 单, 而 a 周围有 bdd, l.p map.

找 PoI ψ_i sub to $\mathcal{V}$

$\varphi_i = \psi_i \circ g$ is a PoI for A, sub to $\mathcal{U} = \{ g^{-1}(V), V \in \mathcal{V} \}$.

$$\int_{g(A)} f = \sum_i \int_{\substack{g(A) \\ \text{some } V \in \mathcal{V}}} \psi_i \cdot f = \sum_i \int_{\substack{A \\ \text{some } u \in \mathcal{U}}} (\psi_i \circ g)(f \circ g)\,|\det g'| = \sum_i \int_{\text{small}} \varphi_i\,(f \circ g)\,|\det g'| \overset{\text{定义}}{=\!=} \int_A f \circ g\,|\det g'|$$

⑤ COV (1D). WLOG, $A = (a,b)$.

$g : (a,b) \to \mathbb{R}$ is 1-1. 且 cont $\to$ 单调

$g(A) = g((a,b)) = \begin{cases} (g(a), g(b)) & g \uparrow \\ \ldots & \ldots \end{cases}$

$[g(b), g(a)]$ $g\downarrow$

设 $g\downarrow$:

$$\int_{g(A)} f = \int_{g(b)}^{g(a)} = \int_b^a f\circ g\cdot \underbrace{g'}_{-|\det(g')|} = -\int_a^b f\circ g\cdot(-|\det g'|)$$

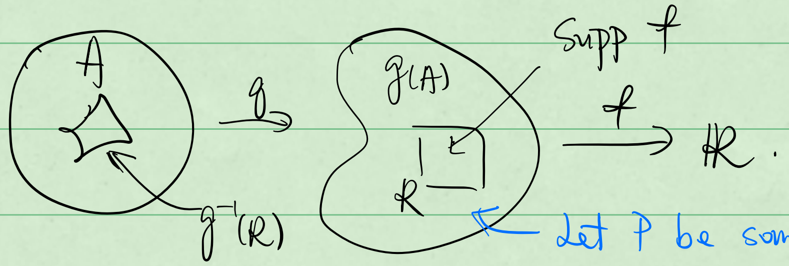
$$= \int_a^b f\circ g\cdot|\det g'|$$

⑥ 若对于连续 f 成立，则对于任意 f 成立.

$$O \xrightarrow{h} O \xrightarrow{g} O \xrightarrow[\text{constant}]{f} \mathbb{R}$$

$F = (f\circ g)|\det g'|$ ← × constant.

We'll not prove the full theorem, only local version of COV for integral functions.



Let P be some partition of R.

"$L(f,P)$ ... $L(f\circ g)$ ... $U(f\circ g)$ ... $U(f,P)$" ← 希望.

$$\underline{L(f,P)} = \sum_{S\in P} V(S)\cdot m_S(f) = \sum_{S\in P}\int_S \underbrace{m_S(f)}_{f\text{常值}} \overset{\text{COV (constant)}}{=} \sum_{S\in P} \underline{\int}_{g^{-1}(S)} \overset{常}{m_S(f)}\,|\det g'| \le \sum_{S\in P} \underline{\int}_{g^{-1}(S)} f\circ g\,|\det g'|$$

$$= \sum_{S\in P} \underline{\int}_{g^{-1}(R)} \chi_{g^{-1}(S)}\, f\circ g\,|\det g'| \le \underline{\int}_{g^{-1}(R)} \sum_S \chi_{g^{-1}(S)} (f\circ g)\,|\det g'| = \underline{\int}_{g^{-1}(R)} (f\circ g)\,|\det g'| \le \overline{\int}_{g^{-1}(R)} (f\circ g)\,|\det g'|$$

(exercise: $\underline{\int} h_1 + \underline{\int} h_2 \le \underline{\int}(h_1+h_2)$)

和前面一样，反过来. → $\le U(f,P)$

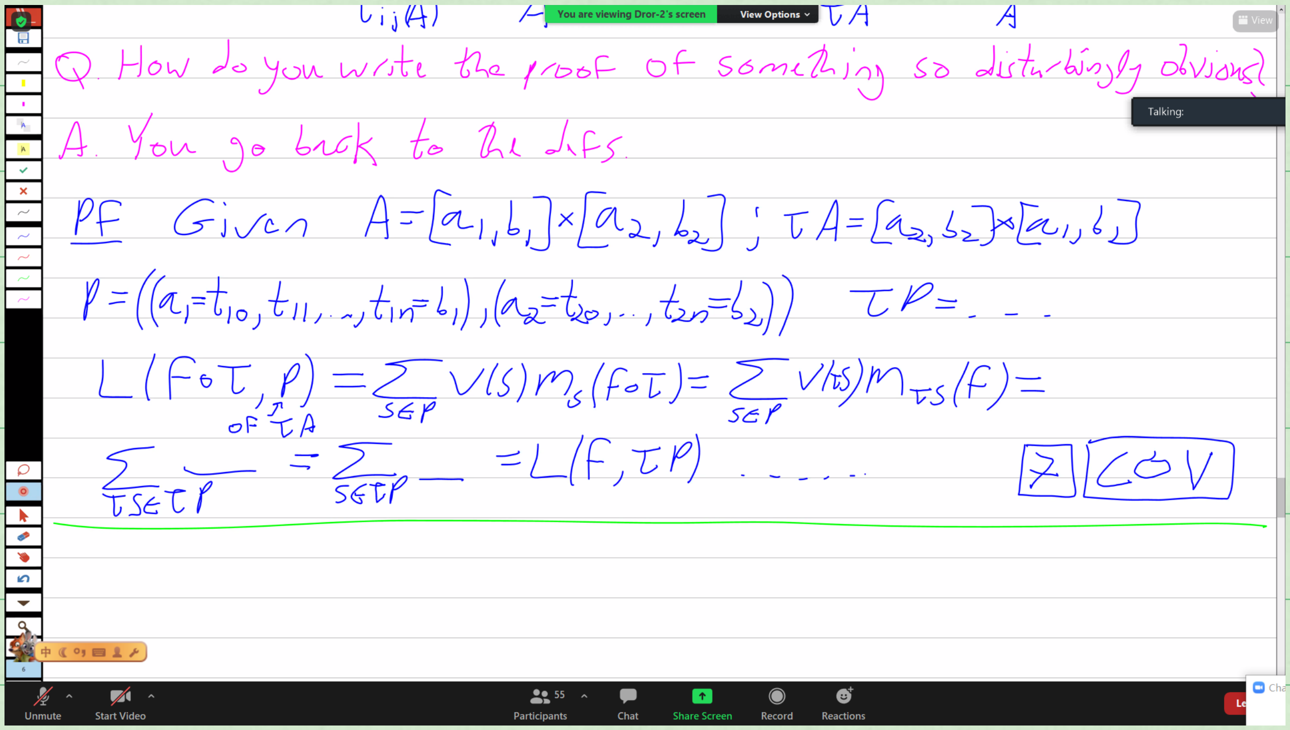
f 可积 → $L(f,P)$ 与 $U(f,P)$ 任意接近 → $f\circ g$ 可积.

⑦ COV holds for T_{ij}

NTS $\int_{T_{ij}(A)} f = \int_A f\circ T_{ij}$ ($\det = \pm 1$)

e.g. $\int_{T(A)} f\circ T = \int_A f$ (2D, $T(x,y) = (y,x)$)

过于显然 (rename)



Q. How do you write the proof of something so disturbingly obvious?

A. You go back to the defs.

Pf Given $A=[a_1,b_1]\times[a_2,b_2]$; $\tau A=[a_2,b_2]\times[a_1,b_1]$

$P=((a_1=t_{10},t_{11},\dots,t_{1n}=b_1),(a_2=t_{20},\dots,t_{2n}=b_2))$ $\quad \tau P = \dots$

$L(f\circ\tau, \underset{\text{of } \tau A}{P}) = \sum_{S\in P} v(S)\, m_S(f\circ\tau) = \sum_{S\in P} v(\tau S)\, m_{\tau S}(f) =$

$\sum_{\tau S\in\tau P} \_\_ = \sum_{S\in\tau P} \_\_ = L(f,\tau P)$ ...... $\boxed{7}$ $\boxed{COV}$

Baby Sard Thm: $A\subset\mathbb{R}^n$ 开, $g:A\to\mathbb{R}^n$, cont. diffable.

$C=\{x\in A \mid \underbrace{\det g'(x)}_{h(x)}=0\}$. Then $g(C)$ is of measure-0

"critical set of g"

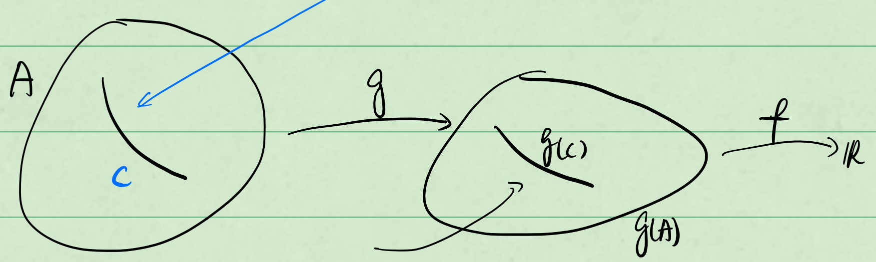
Claim: C is closed. $g'(x)$ = $n\times n$ array of cont functions.

$\Rightarrow h(x)$ is cont.

So $C=h^{-1}(0)$ $\Rightarrow$ 闭集的原像也为闭集.

Cor (of Sard): In COV Thm, can drop the condition that "g' is 1-1"

Pf:



measure-0 and also closed?

$g(A)\setminus g(C) = g(A\setminus C)$ is open (IFT holds.

open on which g' 可逆

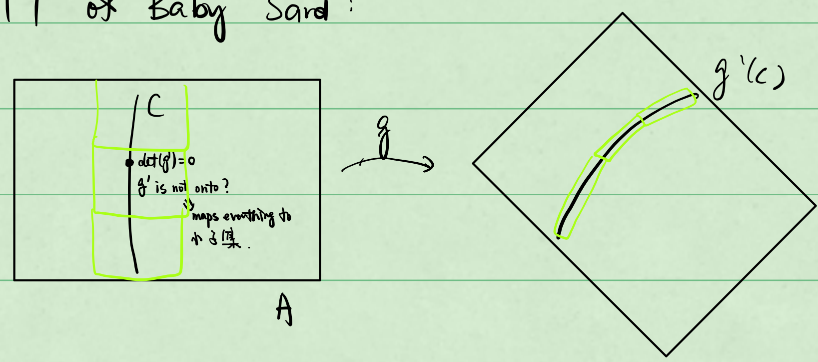
$$\int_A (f\circ g)\,|\det g'| \overset{?}{=} \int_{g(A)} f$$ ← ignore. $g(C)$ measure-0 + close = content = 0.

$$\| \qquad\qquad \|$$

在C上积分为0. $$\int_{A\setminus C} (f\circ g)\,|\det g'| \overset{\checkmark}{=} \int_{g(A)-g(C)} f$$

old COV formula.

Pf of Baby Sard:



Adult Sard thm: $g : \overset{\text{open}}{A} \subset \mathbb{R}^n \to \mathbb{R}^m$

$C = \{x \in A : \operatorname{rank}(g'(x)) < m\}$

g is k times cont diffable. $k = \max(1, n-m+1)$

Then $g(C)$ is measure-0 超难!

DEF: 令 V为 $\mathbb{R}$上向量空间

$T: V^k \longrightarrow \mathbb{R}$ is called "multilinear" or "k-linear"

if $T(u_1, \dots, \alpha u_i + \beta u_i', \dots + u_k)$

$= \alpha T(u_1, \dots u_i, \dots u_k) + \beta T(u_1, \dots, u_i', \dots, u_k)$

e.g. 内积为2-linear map

e.g. $M_{n\times n}(\mathbb{R}) \xrightarrow{\det} \mathbb{R}$

e.g. 1-linear map $\varphi: V \to \mathbb{R}$

"linear functional" $\varphi \in V^*$ dual space

e.g. 0-linear map.

$W: V^0 = \{()\} \to \mathbb{R}$

$W(()) \in \mathbb{R}$ → 0-linear $\cong$ a element in $\mathbb{R}$.

V是F上向量空间

对偶空间V^*是V到F上线性函数的集合.

DEF: $\mathcal{J}^k(V) = \{$k-linear maps on V$\}$

warning: many other sources call it $\mathcal{J}^k(V^*)$

$\langle , \rangle \in \mathcal{J}^2(V)$. $\quad \det \in \mathcal{J}^n(V) \quad \mathcal{J}^0 V \cong \mathbb{R}$

$\mathcal{J}^1 V = V^* =$ n-dim $\quad \dim \mathcal{J}^k V$?

设 $\dim(V) = n$,

设 $v_1, v_2, \cdots v_n$ 为 V 的基 $\exists!$ 基 $\varphi_1, \varphi_2 \cdots \varphi_n$ of $V^* = \mathcal{J}^1 V$,

s.t. $\varphi_i(v_j) = \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & \text{otherwise} \end{cases}$. $\{\varphi_i\}$ is called the dual basis

eg 1: dual basis of $\left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 3 \\ 4 \end{pmatrix} \right\} \in \mathbb{R}^2$?

$\mathbb{R}^2 = \left\{ \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} \right\} \Rightarrow \mathbb{R}^{2*} = \{(*, *)\}$ linear maps from $\mathbb{R}^2 \to \mathbb{R}$

$\varphi_1 = (-, -)$, $\varphi_2 = (-, -)$.

$\varphi_1(v_1) = 1$, $\varphi_1(v_2) = 0$

$\varphi_2(v_1) = 0$, $\varphi_2(v_2) = 1$

$$\Rightarrow \begin{pmatrix} - \varphi_1 - \\ - \varphi_2 - \end{pmatrix} \begin{pmatrix} | & | \\ v_1 & v_2 \\ | & | \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}^{-1} = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$$

$\Rightarrow \varphi_1 = (-2, \frac{3}{2})$, $\varphi_2 = (1, -\frac{1}{2})$

Claim: $\mathcal{J}^k(V)$ is a vector space.

$T_1, T_2 \in \mathcal{J}^k$, $(T_1 + T_2)(u_1, u_2, \cdots, u_k) = T_1(u_1, \cdots u_k) + T_2(u_1, \cdots, u_k)$

$$(\alpha T_1)(u_1,u_2,\cdots u_k) = \alpha(T_1(u_1,\cdots,u_k))$$

$$“0_T” = 0.$$

Also a map $\quad \overset{\text{“Tensor multiplication”}}{\otimes} : \mathcal{T}^k \times \mathcal{T}^l = \mathcal{T}^{k+l}$

$$: (T_1 \otimes T_2)(u_1,u_2,\cdots u_{k+l}) = T_1(u_1,\cdots u_k)\cdot T_2(u_{k+1},\cdots,u_{k+l})$$

$$T_1 \otimes T_2 = T_1 T_2 \in \mathcal{T}^{k+l}$$

$\otimes$ is associative, distributive and non-commutative

$\overset{k}{T_1}(\overset{l}{T_2}\overset{m}{T_3}) = (T_1 T_2)T_3.$ 显然

要同维度. $(T_1+T_2)T_3 = T_1T_3 + T_2T_3.$ 展开后同样显然.

$\Rightarrow$ $\otimes$ is bilinear,

T_1T_2 一般不等于 T_2T_1, 考虑 $V=\mathbb{R}^2$, $e_1\ e_2$ 为两个基, φ_1, φ_2 为 V^* 中对应的基. $(\varphi_1, \varphi_2 \in \mathcal{T}^1(V))$, 则

$$(\varphi_1\varphi_2)(e_1e_2) = \varphi_1(e_1)\varphi_2(e_2) = 1 \neq 0 = \varphi_1(e_2)\varphi_2(e_1)$$

Notation: $\underline{n} = \{1,2,\cdots,n\}$. $\underline{n}^k = \{\bar{i} = I = (i_1 i_2 \cdots i_k),\ i_\alpha \in \underline{n}\}$. $(|\underline{n}^k| = n^k)$

$(v_j)_{j=1}^n \in V^n$, $\quad J \in \underline{n}^k \quad$ “multi-index”

$$v_J = (v_{j_1}, v_{j_2}, \cdots v_{j_k})$$

这里的乘是 $\otimes$

$\varphi_i \in V^*$, $i=1,2,\cdots,n$. $I \in \underline{n}^k \quad \varphi_I = \varphi_{i_1}\cdot\varphi_{i_2}\cdot \cdots \cdot \varphi_{i_n}$

E.g. $\varphi_1\varphi_2 = \varphi_{(1,2)}$. $\quad \varphi_2\varphi_1 = \varphi_{(2,1)}$

$$\varphi_{(1,2)}(e_{(1,2)}) = 1, \quad \varphi_{(2,1)}(e_{(1,2)}) = 0.$$

若 V is a vector space with basis $v_1,\cdots,v_n$, 其 dual basis $\varphi_1,\varphi_2,\cdots,\varphi_n$.

若 $I, J \in \underline{n}^k$,

$$\mathcal{T}^k \ni \underset{\substack{\uparrow \\ \text{product of } k \text{ tensors}}}{\varphi_I}(\underbrace{v_J}_{\text{a list of } k \text{ vectors.}}) = (\varphi_{i_1}\cdots\varphi_{i_k})(v_{j_1},\cdots,v_{j_k}) = \prod_{\alpha=1}^{k}\varphi_{i_\alpha}(v_{j_\alpha}) = \prod_{\alpha=1}^{k}\underset{\substack{\uparrow \\ \begin{cases} 0 & i_\alpha \neq j_\alpha \\ 1 & i_\alpha = j_\alpha \end{cases}}}{\delta_{i_\alpha j_\alpha}} = 1 \text{ 或 } 0.$$

$=1$ 意味着 $I=J$

Thm: V with basis $v_1, \cdots, v_n$, dual basis $\varphi_1, \cdots, \varphi_n$.

Then $\{\varphi_I \mid I \in \underline{n}^k\}$ is a basis of $\mathcal{J}^k(V)$. Hence,

$\dim(\mathcal{J}^k(V)) = n^k$.

Proof: Lemma 1: If $T_1, T_2 \in \mathcal{J}^k(V)$, then $T_1 = T_2 \Leftrightarrow \forall I,\ T_1(v_I) = T_2(v_I)$

pf: "$\Rightarrow$" 显然.

"$\Leftarrow$" 设 $\forall I \in \underline{n}^k$, $T_1(v_I) = T_2(v_I)$

set $T = T_1 - T_2$, 只需证 $T = 0$.

$T(u_1, \cdots, u_k) = T(\sum a_{1j} v_j, \sum a_{2j} v_j, \cdots, \sum a_{kj} v_j)$

通过 multilinear 的性质拆成全部 v_i, 则 $T=0$

Lemma 2: $\{\varphi_I\}$ spans $\mathcal{J}^k(V)$

Given $T \in \mathcal{J}^k(V)$, 如何 $T \overset{?}{=} \sum \alpha_I \varphi_I$? 考虑代入 v_J 的值

$T(v_J) = \sum \alpha_I \varphi_I(v_J)$

$= \sum \alpha_J \delta_{IJ} = \alpha_J$.

Given $T \in \mathcal{J}^k$, set $a_I = T(v_I)$. Claim: $T = \sum a_I \varphi_I$

By lemma 1, 只需证 $T(v_J) = (\sum a_I \varphi_I)(v_J)$

$\Rightarrow a_J = a_J$, 成立.

Claim: φ_I are linear independent:

设 $a_I \in \mathbb{R}$, s.t. $\sum a_I \varphi_I = 0$. $\forall J \in \underline{n}^k$,

$\underbrace{(\sum a_I \varphi_I)}_{= \sum a_I \delta_{IJ} = a_J}(v_J) = 0(v_J) = 0$. $\Rightarrow \forall J,\ a_J = 0$.

由以上证毕

若 $V \xrightarrow{L} W$ 线性. 如何联系 $\mathcal{T}^k(V)$ 与 $\mathcal{T}^k(W)$? $\mathcal{T}^k(V) \xleftarrow{L^*} \mathcal{T}^k(W)$

$T \in \mathcal{T}^k(W)$.

$\longmapsto (L^*T)(u_1, \cdots, u_k) = T(Lu_1, \cdots, Lu_k)$ $\leftarrow u_i \in V$.

Claim 1. If $T \in \mathcal{T}^k(W)$, then $L^*T \in \mathcal{T}^k(V)$

(L^*T is multilinear)

Pf: $(L^*T)(u_1, \cdots, \alpha u_i' + \beta u_i'', \cdots u_k)$

$= T(Lu_1, \cdots, L(\alpha u_i' + \beta u_i''), \cdots Lu_k)$

$\underset{\text{linear}}{\overset{L \text{ is}}{=\!=}} T(Lu_1, \cdots, \alpha Lu_i' + \beta Lu_i'', \cdots, Lu_k)$

$\underset{\text{mul-lin}}{\overset{T \text{ is}}{=\!=}} \alpha T(Lu_1 + \cdots + Lu_k) + \beta T(Lu_2, \cdots, Lu_k)$

Claim 2: $L^*: \mathcal{T}^k W \to \mathcal{T}^k V$ is linear.

NTS: 若 $T_1, T_2 \in \mathcal{T}^k(W)$, 则 $L^*(\alpha T_1 + \beta T_2) = \alpha L^* T_1 + \beta L^* T_2$.

Claim 3: L^* is compatible with $\otimes$.

i.e. if $T_1 \in \mathcal{T}^k(W)$, $T_2 = \mathcal{T}^l(W)$, then $\underbrace{L^*(\overset{\overset{\text{k+l tensor (W)}}{\downarrow}}{T_1 T_2})}_{\text{k+l tensor(V)}} = L^*(T_1) L^*(T_2)$

DEF: $T \in \mathcal{T}^k(V)$ is "alternating"

if $T = (\cdots u, \cdots, w, \cdots) = -T(\cdots, w, \cdots, u, \cdots)$

$\Lambda^k(V) := \{T \in \mathcal{T}^k(V) \mid T \text{ is alternating}\}$ $(\Lambda^k(V^*))$

$\Lambda^k(V)$ is a subspace of $\mathcal{T}^k(V)$

Prop: $T \in \mathcal{T}^k$ Then T kills repetition $\Leftrightarrow$ T is alternating

Pf: T is alternating, 令 $u = w$ 即可 "$\Leftarrow$"

"$\Rightarrow$":

$0 = T(\cdots u+w, \cdots, u+w) = T(\cdots u \cdots u \cdots) + T(\cdots u \cdots w \cdots)$

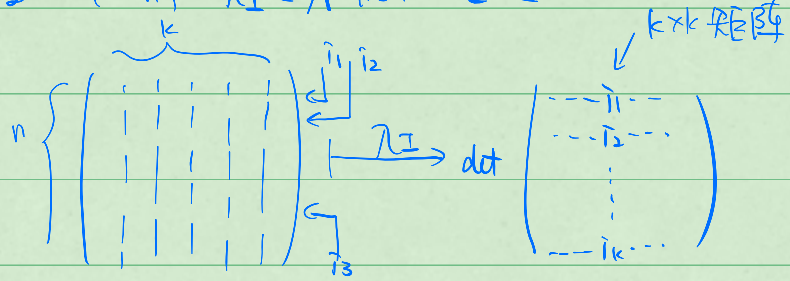
$+T(\cdots w \cdots u \cdots) + T(\cdots w \cdots w \cdots)$

$\Rightarrow \qquad 0 = T(\cdots w \cdots u \cdots) + T(\cdots u \cdots w \cdots)$

eg1: $\det(M_{n\times n}(\mathbb{R})) \in \mathcal{J}^n(\mathbb{R}^n)$ — 每一列为一元素

$\qquad\qquad\qquad\qquad \cap\ \Lambda^n(\mathbb{R}^n)$

eg2: $k \le n$, $\lambda_I \in \Lambda^k \mathbb{R}^n$. $I \in \underline{n}^k$



只需考虑 $\{I \in \underline{n}^k : i_1 < \cdots < i_k\} = \underline{n}_a^k = \binom{n}{k}$

$|\underline{n}_a^k| = C_n^k$

DEF: A permutation of order k $\sigma: \underline{k} \to \underline{k}$ 双射

$S_k = \{\sigma\}$ $\qquad |S_k| = k!$

ι: identity

$\mathrm{sign}(\sigma) = \prod_{i<j} \mathrm{sign}(\sigma_i - \sigma_j)$

Claim: 若 $T \in \Lambda^k$, $\sigma \in S_k$, $\qquad T: V^k \to \mathbb{R}$

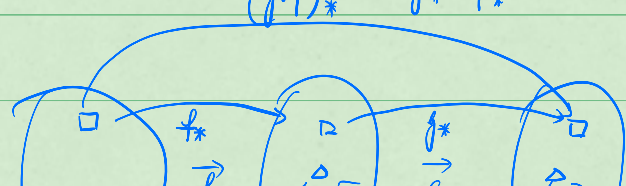
则 $T \circ \sigma^* = (-1)^\sigma T$.

$\sigma^*: V^k \to V^k$. $\quad \sigma^*(v_1, \cdots, v_k) = (v_{\sigma(1)}, \cdots, v_{\sigma(k)})$

Aside①: V^k (事实上) $= \{\text{function}: \underline{k} \to V\}$.

So $\quad \underline{k} \xrightarrow{\sigma} \underline{k} \to V$

Aside②: $\qquad (g\circ f)_* = g_* \circ f_*$



$(g\circ f)^* = f^*\circ g^*$

特别地, $(\sigma\circ\tau)^* = \tau^*\circ\sigma^*$

Pf of Claim: write $\sigma = \tau_1\circ\cdots\circ\tau_l$ transpositions.

$$T\circ\sigma^* = T\circ(\tau_1\circ\cdots\circ\tau_l)^*$$ 这里是 pull back 所以反过来 why?

$$= T\circ\tau_l^*\circ\cdots\circ\tau_1^*$$

$$= (T\circ\tau_l^*)\circ\cdots\circ\tau_1^*$$ T alternating.

$$= -T\circ\tau_{l-1}^*\circ\cdots\tau_1^*$$

$$= (-1)^\sigma T.$$

Basis for $\Lambda^k(V) = \{\omega_I : I\in \underline{n}_a^k\}$ ascending sequences.

DEF: 若 $I\in\underline{n}^k$ (especially if $I\in\underline{n}_a^k$)

$$\omega_I = \sum_{\sigma\in S_k}(-1)^\sigma \varphi_I\circ\sigma^*$$ "anti-symmetrization"

本质一样

e.g. $F:\mathbb{R}^2\to\mathbb{R}$ is anti-symmetric if $F(x,y) = -F(y,x)$

若 $g:\mathbb{R}^2\to\mathbb{R}$ is any function:

Let $F(x,y) = g(x,y) - g(y,x)$

Claim: ω_I is alternating.

Pf: $\omega_I\circ\tau^* = \left(\sum_\sigma(-1)^\sigma\varphi_I\circ\sigma^*\right)\circ\tau^*$ (τ: transposition)

$$= \sum_\sigma(-1)^\sigma\varphi_I\circ\sigma^*\circ\tau^*$$

$$= \sum_\sigma(-1)^\sigma\varphi_I\circ(\tau\circ\sigma)^*$$

$$= (-1)\sum_\sigma(-1)^{\sigma\tau}\varphi_I\circ(\tau\sigma)^*$$

$$= (-1)\sum_{\sigma\tau}(-1)^{\sigma\tau}\varphi_I\circ(\tau\sigma)^*$$

$$= (-1)\,\omega_I$$

重点在于如果I,J中元素一样, 则通过重排是线性相关的

Thm: $\{w_I \mid I \in \underline{n}_a^k\}$ is a basis of $\Lambda^k V$. So $\dim \Lambda^k V = C_n^k$

pf: 1. $w_I(v_J) = \delta_{IJ}$ $(I, J \in \underline{n}_a^k)$ (V v.s. $v_1, \cdots v_n$ 基, $\varphi_1, \cdots \varphi_n$ 对偶基)

pf: $w_I(v_J) = \sum_{\sigma \in S_k} (-1)^\sigma \varphi_I(\sigma^* v_J)$

$= \sum_{\sigma \in S_k} (-1)^\sigma (\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})(v_{j_{\sigma_1}}, v_{j_{\sigma_2}}, \cdots, v_{j_{\sigma_k}})$

$= \sum (-1)^\sigma (\varphi_{i_1}(v_{j_{\sigma_1}}))(\varphi_{i_2}(v_{j_{\sigma_2}})) \cdots (\varphi_{i_k}(v_{j_{\sigma_k}}))$ $\quad$ $\sigma \neq$ identity $\Rightarrow$ 这里不 ascending.

$= \sum (-1)^\sigma \cdot \begin{cases} 1 & \text{if } i_1 = j_{\sigma_1}, i_2 = j_{\sigma_2} \cdots i_k = j_{\sigma_k} \Leftrightarrow (i_1 \cdots i_k) = (j_{\sigma_1} \cdots j_{\sigma_k}) \\ 0 & \text{otherwise} \end{cases}$

$= \sum_{\sigma = \mathrm{Id}} (-1)^\sigma \delta_{I,J} = \delta_{IJ}$

2: 设 $\lambda_1, \lambda_2 \in \Lambda^k V$, $\lambda_1 = \lambda_2 \Leftrightarrow \forall I \in \underline{n}_a^k$, $\lambda_1(v_I) = \lambda_2(v_I)$

"$\Rightarrow$" 显然

"$\Leftarrow$" NTS $(\lambda_1 - \lambda_2)(u_1, \cdots, u_k) = 0$

$\Leftarrow$ $(\lambda_1 - \lambda_2)(v_{i_1} \cdots v_{i_k}) = 0$ $\forall i_1 \cdots i_k$.

$\Leftarrow$ $(\lambda_1 - \lambda_2)(v_{i_1} \cdots v_{i_k}) = 0$ $\forall (i_1, \cdots, i_k) \in \underline{n}_a^k$

$\Leftrightarrow$ $\lambda_1(v_I) = \lambda_2(v_I)$ $\forall I \in \underline{n}_a^k$ 即得证

3. Span: given $\lambda \in \Lambda^k$, find a_I s.t. $\lambda = \sum a_I w_I$

Let $a_I = \lambda(v_I)$ NTS $\lambda = \sum a_I w_I$.

只需证 $\lambda(v_J) = (\sum a_I w_I)(v_J)$ $\forall I, J \in \underline{n}_a^k$

$= \sum a_I \delta_{IJ}$

$= a_J$

4 线性无关

$\sum b_I w_I = 0$

$\sum b_I w_I(v_J) = 0$

$\Rightarrow b_J = 0$ $\forall J \in \underline{n}_a^k$

e.g. $V = \mathbb{R}^3$, $v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$. $\varphi_1 = x$, $\varphi_2 = y$, $\varphi_3 = z$
$(1,0,0)$ $(0,1,0)$ $(0,0,1)$

$\mathbb{R}^3 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

basis for $\Lambda^0 \mathbb{R}^3$, $\Lambda^1 \mathbb{R}^3$, $\Lambda^2 \mathbb{R}^3$, $\Lambda^3 \mathbb{R}^3$ ....

$C_3^0 \quad \Lambda^0\mathbb{R}^3: w_0$

$C_3^1 \quad \Lambda^1\mathbb{R}^3: w_1, w_2, w_3$

$C_3^2 \quad \Lambda^2\mathbb{R}^3: w_{12}, w_{23}, w_{13}$

$C_3^3 \quad \Lambda^3\mathbb{R}^3: w_{123}$

Aside: $\underline{n}_a^k = \{(i_1,\cdots i_k) \mid 1\le i_1 < i_2 < \cdots < i_k \le n\}$, $\quad |\underline{n}_a^k| = \binom{n}{k}$

"$\{(i_1,\cdots,i_k) \mid 1\le i_1 \le i_2 \le \cdots \le i_k \le n\}$" $= \underline{n}_{nd}^k \leftarrow$ non-decreasing

$|\underline{n}_{nd}^k| = \binom{n+k-1}{k} \leftarrow$ 插板 $\quad \{(1\,\boxed{\star 2 \star\star 3 \star 4\, 5 \star\star})\} \leftarrow (122355)$

Thm: $\exists!\ (\lambda, \eta) \mapsto (\lambda \wedge \eta)$ "wedge product"

$\lambda \in \Lambda^k,\ \eta \in \Lambda^l,\ \lambda\wedge\eta \in \Lambda^{k+l}$ s.t.

0. bilinear $\quad (\alpha\lambda_1 + \beta\lambda_2 \wedge \eta) = \alpha\lambda_1\wedge\eta + \beta\lambda_2\wedge\eta$

1. associative $\quad (\lambda\wedge\eta)\wedge\varphi = \lambda\wedge(\eta\wedge\varphi)$

2. super-commutative (graded commutative)

$$\lambda\wedge\eta = (-1)^{k\cdot l}\,\eta\wedge\lambda \qquad \lambda\in\Lambda^k,\ \eta\in\Lambda^l$$

均奇时才负

3. $w_I = \varphi_{i_1}\wedge\varphi_{i_2}\wedge\cdots\wedge\varphi_{i_k}$ if $I \in \underline{n}_a^k$

Pf uniqueness: if w, η given, 则 $w\wedge\eta$ 被以上性质唯一确定

$$\varphi_i\wedge\varphi_j = \begin{cases} w_{(ij)} \ ③ & \text{if } i<j \quad (ij)\in\underline{n}_a^k. \\ -w_{(ji)} \ ② & \text{if } j<i \\ 0 & \text{if } i=j \end{cases}$$

$\varphi_i\wedge\varphi_i = -\varphi_i\wedge\varphi_i = 0$

若 $I\in\underline{n}_a^k$, $J\in\underline{n}_a^l$

$$\omega_I \wedge \omega_J \overset{③}{=} (\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}) \wedge (\varphi_{j_1} \wedge \cdots \wedge \varphi_{j_l})$$

$$\overset{①}{=} \varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k} \wedge \varphi_{j_1} \wedge \cdots \wedge \varphi_{j_l}$$

$$= \begin{cases} 0 & \text{if } I \cap J \neq \emptyset \\ \pm\, \omega_{I \cup J} \ (\text{同时 sort}) & \text{if } I \cap J = \emptyset \end{cases}$$

$\omega_{23} \wedge \omega_{14} = (-1)^2\, \omega_{1234}$

$\omega_{13} \wedge \omega_{24} = (-1)^1\, \omega_{1234}$

$\omega \wedge \eta \qquad \omega = \sum a_I \omega_I$

$\eta = \sum b_J \omega_J$

$$\Rightarrow \omega \wedge \eta = \left(\sum a_I \omega_I\right) \wedge \left(\sum b_J \omega_J\right)$$

$$= \sum a_I b_J (\omega_I \wedge \eta_J)$$ ← know how to do.

仍然需要证在基($\{\omega_I\}$)变换下保持不变 (well-defined)

Existence: need a basis-independent formula for $\lambda \wedge \eta$ ($\lambda \in \Lambda^k$, $\eta \in \Lambda^l$).

DEF: $$(\lambda \wedge \eta)(u_1, \cdots u_{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma \lambda(u_{\sigma(1)}, \cdots, u_{\sigma(k)})\, \eta(u_{\sigma(k+1)}, \cdots, u_{\sigma(k+l)})$$

$$= \sum_{\substack{\sigma \in S_{k+l} \\ \sigma_1 < \cdots < \sigma_k \\ \sigma_{k+1} < \cdots < \sigma_{k+l}}} (-1)^\sigma \lambda(u_{\sigma_1} \cdots u_{\sigma_k})\, \eta(u_{\sigma_{k+1}}, \cdots, u_{\sigma(k+l)})$$

有 $k!$ ways 来 permute $\lambda(u_{\sigma(1)}, \cdots, u_{\sigma(k)})$, 每一项的值都一样 所以只选了一个.

e.g., $\lambda \in \Lambda^3$, $\eta \in \Lambda^2$

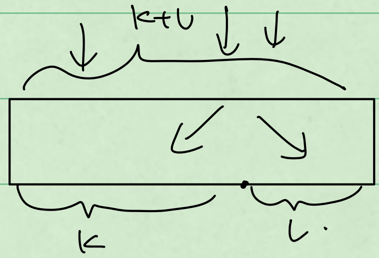
$(\lambda \wedge \eta)(u_1, u_2, u_3, u_4, u_5) = \sum$ 选 2 个给 η C_5^2

NTS: properties 0–3 holds. ε (alternating)

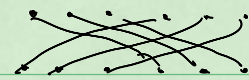
ε: same as $\omega_I \in \Lambda^k$

0: trivial

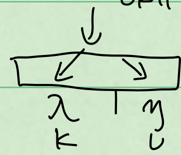
1 ~ 3:



$= \sum_{\substack{\sigma\in S_{k+l}\\ \sigma_1<\cdots<\sigma_k\\ \sigma_{k+1}<\cdots<\sigma_{k+l}}} (-1)^{\sigma}$

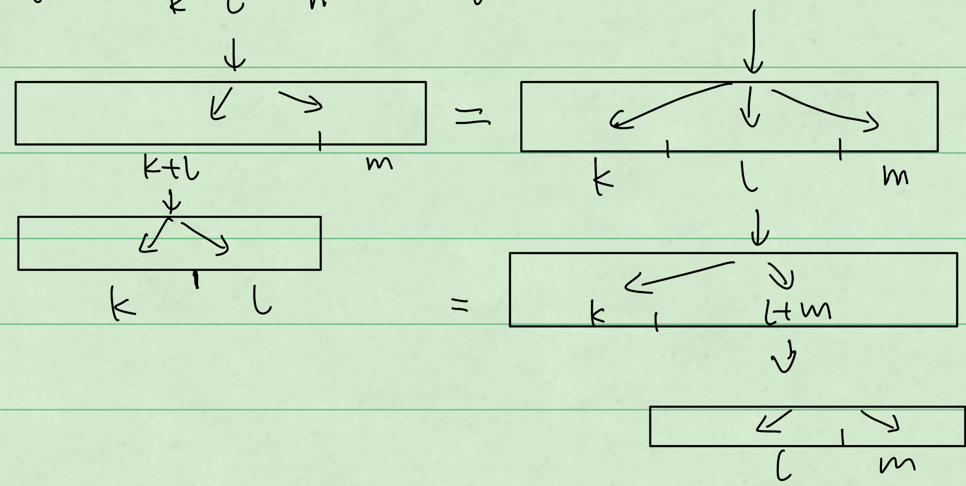


$(\lambda\wedge\eta)(u_1,\dots,u_{k+l}) =$



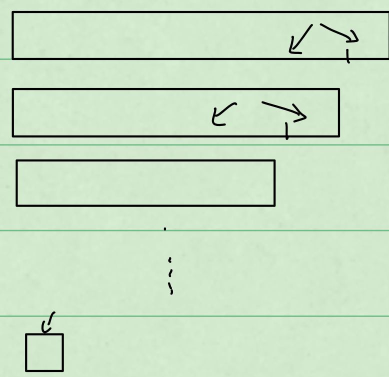
1: (Assoicativity) $\underset{k}{(\lambda}\wedge\underset{l}{\eta)}\wedge\underset{n}{\varphi} = \lambda\wedge(\eta\wedge\varphi)$

LHS =

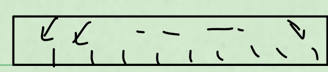


3. $\omega_I = ((\varphi_{i_1}\wedge\varphi_{i_2})\wedge\varphi_{i_3})\wedge\cdots\cdots\wedge\varphi_{i_k})$

$(\varphi_{i_1}\wedge\varphi_{i_2}\wedge\cdots\wedge\varphi_{i_k})(u_1,\dots,u_k) =$



$=$



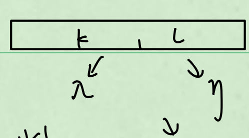
permutation

$= \sum(-1)^{\sigma}(\varphi_{i_1}\cdots\varphi_{i_k})(u_{\sigma_1}\cdots u_{\sigma_k})$

$= \omega_I$

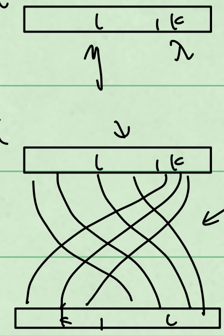
2. $\lambda\wedge\eta = (-1)^{kl}\eta\wedge\lambda$

LHS =



RHS $= (-1)^{kl}$

$= (-1)^{kl}$



有 kl 个交点, 所以是 $(-1)^{kl}$.

Pullbacks $V \xrightarrow{L} W$ $\quad \Lambda^k V \xleftarrow{L^*} \Lambda^k W$

e.g. $\dim V = n$, $\Lambda^n V = \Lambda^{top} V$

1-d space, $(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n)$

Thm: $L: V \to V$. $\quad L^*: \Lambda^{top} V \to \Lambda^{top} V$. is multiplication of $\det(L)$

If $w \in \Lambda^{top}$, then $L^* w = \det(L) w$.

Pf: WLOG, $w = w_{123\cdots n} = \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$, where φ_i's are dual basis for v_i's $\leftarrow$ basis of V

With LOG: $n = 3$. $\to$ 可以推广

$L = V \to V = A \in M^{3\times 3} \to \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

是线性的, 所以只需验证一个 vector.

$$L^* w_{123}(v_{123}) = L^* w_{123}(v_1, v_2, v_3)$$

$$= w_{123}(Lv_1, Lv_2, Lv_3)$$

$$= \varphi_1 \wedge \varphi_2 \wedge \varphi_3 (Lv_1, Lv_2, Lv_3)$$

拿定义展开就是行列式

$$= \varphi_1 \wedge \varphi_2 \wedge \varphi_3 \left(\begin{matrix} a_{11}v_1 \\ + \\ a_{12}v_2 \\ + \\ a_{13}v_3 \end{matrix}, \begin{matrix} a_{21}v_1 \\ + \\ a_{22}v_2 \\ + \\ a_{23}v_3 \end{matrix}, \begin{matrix} a_{31}v_1 \\ + \\ a_{32}v_2 \\ + \\ a_{33}v_3 \end{matrix} \right)$$

$$= \det(L)\, w_{123}.$$

DEF: An "orientation" of a dimension-n vector space is a choice $(v_1, \cdots, v_n)$ of an ordered basis, two such choices are considered the same if they charge of basis matrix between them has determinent >0

1. if $(v_1, \cdots, v_n)$ is a choice, $\overset{A}{\sim} (v_1', \cdots, v_n') \overset{B}{\sim} (v_1'', \cdots, v_n'')$

same orientation

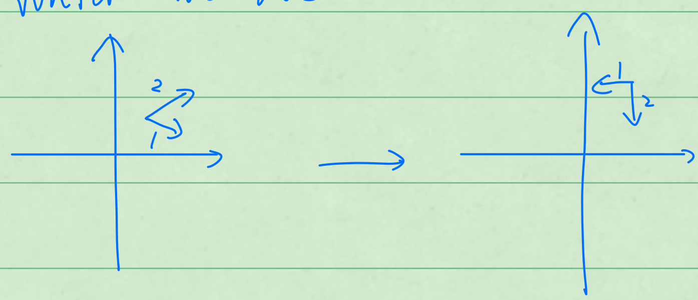
$\det A > 0$, $\det B > 0 \Rightarrow \det AB > 0$, so 1, 3 same.

2. Symmetric, $(v_1 \cdots v_n) \sim (v_1', \cdots, v_n') \Rightarrow (v_1' \cdots v_n') \sim (v_1, \cdots v_n)$

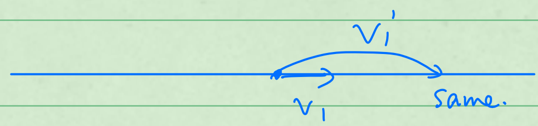
3. $(v_1, \cdots, v_n) \sim (v_1, \cdots, v_n)$

e.g. Right (left) hand represent well-defined orientations in $\mathbb{R}^3$ in which we live

e.g.



e.g. Orientation of $\mathbb{R}^1$



(正向/负向)

every finite-dim vector space has exactly 2 orientation.

DEF: An "orientation'" of the same vector space is a choice of $\omega \in \Lambda^{top} V$, $\omega \neq 0$.

ω_1, ω_2 is considered equivalent when $\omega_1 = \alpha \omega_2$ $(\alpha > 0)$

Thm: The two def are equivalent.

i.e. $\forall$ vector space V, $\{\text{orientation}\} = \{\text{orientation}'\}$

We say that an orientation' $\omega \in \Lambda^{top} V$ agrees with orientation $= (v_1, \cdots, v_n)$ if $\omega(v_1, \cdots, v_n) > 0$.

$V \to W$ push / pull? neither in general and both if L is invertible.

Tangent space: (nearly just language)

$p \in \mathbb{R}^n$, $(p, \vec{v})$ — point in $\mathbb{R}^n$; vector (direction) in $\mathbb{R}^n$.

$\xi = (p, \vec{v})$ → tangent vector to $\mathbb{R}^n$ at p.

given p, $\{(p, v) \mid v \in \mathbb{R}^n\} = T_p\mathbb{R}^n = \mathbb{R}^n_p$ ⇒ the tangent space to $\mathbb{R}^n$ at p.

$T_p\mathbb{R}^n$ is a vector space. $(\alpha(p, v_1) + \beta(p, v_2) = (p, \alpha v_1 + \beta v_2))$

$\mathbb{R}^n_p$ has an inner product $\langle (p, v_1), (p, v_2) \rangle = \langle v_1, v_2 \rangle$

- - - — — — — norm: $\|(p, v)\| = \|v\|$

push / pull? $\mathbb{R}^n \xrightarrow{F} \mathbb{R}^m$, $\xi = (p, v) \xrightarrow{F_*} (F(p), F'(p)\, v)$

可导

Claims: 1. $F_* : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$ is a linear map. (显然，- p 是定值)

2. $\mathbb{R}^n \xrightarrow{F} \mathbb{R}^m \xrightarrow{G} \mathbb{R}^k$

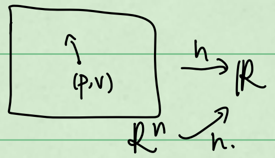
$$(G \circ F)_* : T_p\mathbb{R}^n \to T_{G(F(p))}\mathbb{R}^k.$$
$$\overset{\checkmark}{=}\ G_* \circ F_*$$

pf: $\xi = (p, v)$. $(G \circ F)_*(\xi) = (g(f(p)), (g \circ f)'(p)\, v)$

$= (g(f(p)), g'(f(p)) \cdot f'(p) \cdot v)$

$= g(f(\xi))$

$T_p\mathbb{R}^n \Leftrightarrow$ "directional derivatives at p"



$D_{(p,v)}h = D_\xi(h) =$ "directional derivative... of h in the direction of ξ"

$$= \frac{d\, h(p+tv)}{dt}\Big|_{t=0}$$

if $p=(x_1,\cdots,x_n)$, $V=(V_1,\cdots,V_n)$, ↳ $\frac{d}{dt} h(x_1+tv_1,\cdots,x_n+tv_n)\big|_{t=0}$

$$= \frac{dh}{dx_1}(p)\cdot V_1 + \frac{dh}{dx_2}(p)\cdot V_2 + \cdots + \frac{dh}{dx_n}(p)\cdot V_n$$

$$= h'(p)\cdot V$$

the instructors' answer, *where instructors collectively construct a single answer*

We view the function in 2 steps:

$t \mapsto (x_1+tv_1,\cdots,x_n+tv_n) \mapsto h(x_1+tv_1,\cdots,x_n+tv_n)$

The first map has derivative $\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ and the second map is h, so it has derivative $Dh = \left(\frac{\partial h}{\partial x_1},\cdots,\frac{\partial h}{\partial x_n}\right)$

So using the chain rule, the derivative of the whole things is $Dh\cdot\begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} = \frac{\partial h}{\partial x_1}v_1+\cdots+\frac{\partial h}{\partial x_n}v_n$

thanks! | 1

followup discussions *for lingering questions and comments*

Claims: 1. "bilinear" $D_{\alpha\xi+\beta\eta}h = \alpha D_\xi h + \beta D_\eta h$.

⇒ "D" 的本质是把 $h'(p)$ 作用在V上，而 p 是同个.

$D_\xi(h) = D_{(v,p)}h$

$= h'(p)V$.

所以线性

$D_\xi(\alpha h_1+\beta h_2) = \alpha D_\xi h_1 + \beta D_\xi h_2$.

2. Leibniz : $D_\xi(f\cdot g) = f(p)D_\xi g + D_\xi(f)\, g(p)$

$\xi=(p,v)$ 前导后不导 + 后导前不导

$$= \Big(\underbrace{\sum v_i \frac{\partial}{\partial x_i}}_{\xi} h\Big)\Big|_p$$

$\left(p, \begin{pmatrix}2\\3\end{pmatrix}\right) \sim 2\frac{\partial}{\partial x} + 3\frac{\partial}{\partial y}$

$\sim 2\partial x + 3\partial y$

$\sim 2\partial_1 + 3\partial_2.$

Claim: pushing vectors is compatible with pulling functions.

$\xi \in \mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{h} \mathbb{R}$ all 可导

$\xi \in T_p\mathbb{R}^n$, $h: \mathbb{R}^m \to \mathbb{R}$ $f: \mathbb{R}^n \to \mathbb{R}^m$

$D_{f_*\xi} h = D_\xi (f^* \circ h)$

pf: $D_{f_*\xi}(h) = D_{(f(p), f'(p)v)} h$ $\quad D_{(p,v)} h = h'(p) v$

$= h'(f(p)) \cdot (f'(p) \cdot v)$

$= (h \circ f)'(p) \cdot v$

$= D_\xi (h \circ f)$

$= D_\xi (f^* \circ h)$

DEF: a vector field on $\mathbb{R}^n$ is

$F: \mathbb{R}^n \to \bigcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n$, s.t. $F(p) \in T_p\mathbb{R}^n$

$F(p) = (p, \sum F^i(p) e_i)$, $F^i(p): \mathbb{R}^n \to \mathbb{R}$, the component functions of F.

$= \sum F^i(p) \frac{\partial}{\partial x_i}$

F is continuous / diffable $\Rightarrow \forall i$, F^i is continuous / diffable

$(D_F h)(p) = D_{F(p)} h$

向量场

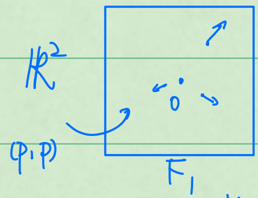
可以乘向量场(标量或函数). i.e. $g: \mathbb{R}_p^n \to \mathbb{R}$, $F \to gF = (gp, v)$.

可以计算两个向量场的内积：在点 p 算 v_1, v_2 的内积，就得到了 $\mathbb{R}^n \to \mathbb{R}$ 的 map

诶这个好奇怪，再想想。

$$\begin{array}{c} T_p\mathbb{R}^n \\ \cup \\ F(p) \end{array} = \sum_{i=1}^{n} F^i(p)\,(p, e_i) = \sum F^i(p)\,\partial_i$$ （本质是 v.F.，是 $\mathbb{R}^n \to \mathbb{R}^n$ 的函数）

e.g



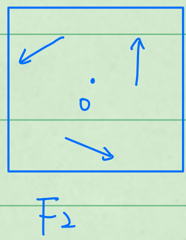
at $p = \begin{pmatrix} x \\ y \end{pmatrix}$, $v = \begin{pmatrix} x \\ y \end{pmatrix} = x\begin{pmatrix} 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$= x\partial_x + y\partial_y$

the radial v.F.

$$D_{F_1}(x^2+y^2) = x\partial_x(x^2+y^2) + y\partial_y(x^2+y^2) = 2x^2+2y^2$$

在每一点求导会是个值，所以是function.

e.g.



垂直，相等

$p = \begin{pmatrix} x \\ y \end{pmatrix}$ $V = \begin{pmatrix} \cos 90^\circ, -\sin 90^\circ \\ \sin 90^\circ, \cos 90^\circ \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$

$$F_2 = -y\partial_x + x\partial_y$$

$$D_{F_2}(x^2+y^2) = -y\,2x + x\,2y = 0.$$

DEF: A k-form on $\mathbb{R}^n$ is a function $\omega : \mathbb{R}^n \to \bigcup_{p\in\mathbb{R}^n} \Lambda^k(T_p\mathbb{R}^n)$ ($\Lambda^k(T_p\mathbb{R}^n) \cong \mathbb{R}^n$)

s.t. $\omega(p) \in \Lambda^k(T_p\mathbb{R}^n)$

同构!

就那个

$\omega_I(p)((p,v_1),\cdots,(p,v_k)) = \omega_I(v_1,\cdots,v_k)$

因为依赖于p的位置

Now, given a k-form, ω, $\omega(p) = \sum_{I\in\Delta_n^k} \lambda_I(p)\,\omega_I(p)$

$\lambda_I : \mathbb{R}^n \to \mathbb{R}$, coefficient functions.

DEF: ω is cont./diffable/C^r if $\forall I$, $\lambda_I(p)$ is

DEF: $\Omega^k(\mathbb{R}^n) = \left\{ \begin{matrix} \text{all} & C^\infty & k\text{-forms} \\ \text{in} & \mathbb{R}^n & \end{matrix} \right\}$ = "diffable k-forms"

Technolitties: $+$, $\times s$ (sclar), $\times g$ (function), $\wedge$: $\Omega^k(\mathbb{R}^n) \times \Omega^l(\mathbb{R}^n) \to \Omega^{k+l}(\mathbb{R}^n)$

加 乘

$$\omega \quad \eta \to \omega\wedge\eta$$

$$\omega\wedge\eta(p) = \omega(p)\wedge\eta(p) \quad (p \in \mathbb{R}^n)$$

这意味着"乘"是被包含在^里的.

diffable forms pull. $F: \mathbb{R}^n \to \mathbb{R}^m$, $\Rightarrow F^*: \Omega^k(\mathbb{R}^m) \to \Omega^k(\mathbb{R}^n)$

$$\omega \quad ? \quad F^*\omega$$

$$\underset{F^*(\omega)}{\mathbb{R}^n} \xrightarrow{F} \underset{\omega}{\mathbb{R}^m}$$

$$f^*(\omega)(p)(\xi_1, \xi_2, \cdots, \xi_k) = \omega(f(p))(f_*(\xi_1), \cdots, f_*(\xi_k)) \qquad \left(f_*(v_p) = (Df(p)(v))_{f(p)} \right)$$

ξ 的 p 相同

$F^*(\omega)$是 $\mathbb{R}^n \to \Lambda^k$ 的, 所以 $F^*(\omega)(p)$ 才是个 Λ^k, 从而才能喂$(\xi_1, \cdots, \xi_k)$

Thm: pulling diff forms is compatible with everything (all the operations above)

- $F^*(\omega_1 + \omega_2) = F^*(\omega_1) + F^*(\omega_2)$
- $F^*(7\omega) = 7F^*(\omega)$
- if $g: \mathbb{R}^n \to \mathbb{R}$, $\quad F^*(g\omega) = F^*(g) \cdot F^*(\omega)$
- $F^*(\omega\wedge\eta) = F^*(\omega) \wedge F^*(\eta)$
- contra-variance.

$$\underbrace{\square \xrightarrow{f} \square \xrightarrow{h}}_{h\circ f} \underset{\omega}{\square}$$

$$(h\circ f)^*(\omega) = f^*(h^*(\omega))$$

If $f \in \Omega^0 \mathbb{R}^n = C^\infty(\mathbb{R}^n)$, 这个数是$f$和$\omega$的加起来.

$\omega \in \Omega^k \mathbb{R}^n$, then $f \wedge \omega \in \Omega^k \mathbb{R}^n$ $= f \cdot \omega$

就是相当于 ω 和一个 function 的 compose

Claim: If $\omega \in \Omega^k \mathbb{R}^n$, $\eta \in \Omega^l \mathbb{R}^m$, $g: \mathbb{R}^n \to \mathbb{R}^n$ smooth.

则 $g^*(\omega \wedge \eta) = g^*(\omega) \wedge g^*(\eta)$ $\in \Omega^{k+l}(\mathbb{R}^n)$

pf: By def: $g^*(\omega \wedge \eta)(\xi_1, \cdots, \xi_{k+l})$ $\quad \xi_i \in T_p\mathbb{R}^n$

$= \omega \wedge \eta (g\xi_1, \cdots, g\xi_{k+l})$

$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma \omega(g\xi_{\sigma_1}, \cdots g\xi_k) \cdot \eta(g\xi_{\sigma(k+1)} \cdots g\xi_{\sigma(k+l)})$

相等.

RHS $= (g^*\omega) \wedge (g^*\eta)(\xi_1, \cdots, \xi_{k+l})$

$= \frac{1}{k!\,l!} \sum_\sigma (-1)^\sigma g^*\omega(\xi_{\sigma(1)}, \cdots, \xi_{\sigma(k)}) \cdot g^*\eta(\xi_{\sigma(k+1)}, \cdots, \xi_{\sigma(k+l)})$

$d: \underbrace{\Omega^0(\mathbb{R}^n)}_{\text{functions}} \to \underbrace{\Omega^1(\mathbb{R}^n)}_{\text{吃一个 tangent vector, 吐一个数.}}$ $\quad$ (later: $\Omega^k \to \Omega^{k+1}$)

DEF: $(dF)(\underset{\in T_p\mathbb{R}^n}{\xi}) = D_\xi F$ $\quad F(p') \cdot V$ 在 p 点的向量 V 的 directional derivative.

$= \sum v_i \frac{\partial F}{\partial x_i}$

$= \sum_{I \in \mathcal{I}_1} \lambda_I(p) \omega_I = \sum_{i=1}^n \lambda_i(p) \varphi_i(v) = \sum_{i=1}^n \lambda_i(p) v_i$

$\lambda_i = \frac{\partial F}{\partial x_i}$

$dF = \sum \frac{\partial F}{\partial x_i} \cdot \varphi_i$ $\quad$ (projection map), $\mathbb{R}^n \to \mathbb{R}$. $p \mapsto x_j$

$F = x_j, \Rightarrow dF = \varphi_j.$

$$dF = \sum \frac{\partial F}{\partial x_i} dx_i$$

- d is linear. $d(\alpha f + \beta g) = \alpha dF + \beta dg$
- Leibnitz Rule: $d(f \cdot g) = (df)g + f(dg)$
- Compatible with pull-backs : $d(g^* f) = g^*(df)$

$$\xi \in \mathbb{R}^n \xrightarrow{g} \mathbb{R}^r \xrightarrow{f} \mathbb{R}$$

$$d(g^* f)(\xi) = D_\xi (g^* f) \qquad \text{LHS}$$

$$\| =$$

前面证过这里的等号了.

$$g^* df(\xi) = df(g(\xi)) = D_{g(\xi)}(f) \qquad \text{RHS}$$

e.g.

$$\mathbb{R}^2_{r,\theta} \xrightarrow{g : \binom{r}{\theta} \to \binom{r\cos\theta}{r\sin\theta}} \mathbb{R}^2_{x,y}, \quad \omega = dx \wedge dy$$

$$\begin{cases} dx, dy \text{ are 1-forms.} \\ = \varphi_1 \wedge \varphi_2 = \omega_{12}. \end{cases}$$

Can we compute $g^*(dx \wedge dy)$?

$$g^*(dx \wedge dy) = g^*(dx) \wedge g^*(dy)$$

$$= d(g^* x) \wedge d(g^* y)$$

$$= d(r\cos\theta) \wedge d(r\sin\theta)$$

$$= \left(\frac{\partial r\cos\theta}{\partial r} dr + \frac{\partial r\cos\theta}{\partial \theta} d\theta\right) \wedge \left(\frac{\partial r\sin\theta}{\partial r} dr + \frac{\partial r\sin\theta}{\partial \theta} d\theta\right)$$

$$= (\cos\theta dr - r\sin\theta d\theta) \wedge (\sin\theta dr + r\cos\theta d\theta)$$

$$= r\cos^2\theta \, dr \wedge d\theta - r\sin^2\theta \, d\theta \wedge dr.$$

$$= r \, dr \wedge d\theta$$

anti-symmetric for odd

$$d : \Omega^k(\mathbb{R}^n) \Rightarrow \Omega^{k+1}(\mathbb{R}^n)$$

这里的顺序重要，上面因为 commute 所以顺序无所谓.

$df = \sum \frac{\partial f}{\partial x_i} dx_i = \sum dx_i \wedge \frac{\partial f}{\partial x_i}$

DEF: $d\omega = \sum_{i=1}^{n} dx_i \wedge \frac{\partial \omega}{\partial x_i}$ ← 和之前的 agree.

if $\omega = \sum_{I \in \mathcal{N}^k} \lambda_I dx_I$, then $\frac{\partial \omega}{\partial x_i} = \sum_{I \in \mathcal{N}^k} \frac{\partial \lambda_I}{\partial x_i} dx_I$

e.g.: $\theta = \frac{y}{x^2+y^2} dx - \frac{x}{x^2+y^2} dy \in \Omega^1(\mathbb{R}^2_{x,y})$

$$d\theta = dx \wedge \frac{\partial \theta}{\partial x} + dy \wedge \frac{\partial \theta}{\partial y}$$

$$= dx \wedge \Big(\underbrace{\cancel{\_\_\ dx}}_{dx \wedge dx = 0} - \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} dy \Big) + dy \wedge \Big(\frac{x^2+y^2-2y^2}{(x^2+y^2)^2} dx \Big)$$

另一项也没了

$$= \frac{x^2-y^2}{(x^2+y^2)^2} dx \wedge dy - \frac{x^2-y^2}{(x^2+y^2)^2} dx \wedge dy$$

$$= 0$$

$$d(\overset{h}{\overbrace{f dx + g dy}}) = dx \frac{\partial h}{\partial x} + dy \frac{\partial h}{\partial y}$$

$$= dx \frac{\partial g dy}{\partial x} + dy \frac{\partial f dx}{\partial y}$$

$$= \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) dx \wedge dy$$

exterior derivite.

Thm: d 有如下性质

① linear. $d(a\omega - b\eta) = a\,d\omega - b\,d\eta$.

② Leibnitz $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^{\omega} \omega \wedge d\eta$

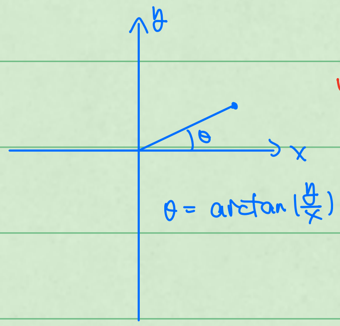
③ $d^2 \equiv 0$ (really: $d_k = \Omega^k \to \Omega^{k+1}$)

$$\Omega^k \xrightarrow{d_k} \Omega^{k+1} \xrightarrow{d_{k+1}} \Omega^{k+2}$$

so

$$d_{k+1} \circ d_k \equiv 0$$

e.g.



$\mathrm{Im}(d_k) = \mathrm{Ker}(d_{k+1})$

$\omega = d\lambda$

{exact forms} {closed forms} ($d\omega = 0$)

exact forms are closed

$$d\theta = dx \frac{\partial \theta}{\partial x} + dy \frac{\partial \theta}{\partial y}$$

$$= \quad \vdots \quad \left(\frac{1}{1+\frac{y^2}{x^2}}\right) \frac{1}{x}\, dy$$

$$= -\frac{y}{x^2+y^2}\, dx + \frac{x}{x^2+y^2}\, dy$$ 和上面一样.

④ $g^* d(w) = d(g^* w)$

pf of ②:

$$d(w \wedge \eta) = \sum dx_i \wedge \left(\frac{\partial\, w\wedge\eta}{\partial x_i}\right) \quad ?$$

$$= \sum dx_i \wedge \left(\frac{\partial w}{\partial x_i} \wedge \eta + w \wedge \frac{\partial \eta}{\partial x_i}\right)$$

$$= \sum \left(\underbrace{dx_i \wedge \frac{\partial w}{\partial x_i}}_{dw}\right) \wedge \eta + (-1)^{w}\, w \wedge \underbrace{dx_i \wedge \frac{\partial \eta}{\partial x_i}}_{d\eta}$$

$$= (dw) \wedge \eta + (-1)^{w}\, w \wedge d\eta$$

claim: $\frac{\partial (w\wedge\eta)}{\partial x_i} = \left(\frac{\partial w}{\partial x_i}\right) \wedge \eta + w \wedge \left(\frac{\partial \eta}{\partial x_i}\right)$

WLOG: $w = f dx_i$, $\eta = g\, dx_j$

$$w \wedge \eta = fg\, dx_i \wedge dx_j$$

$$\frac{\partial\, w\wedge\eta}{\partial x_i} = \left(\frac{\partial fg}{\partial x_i}\right) dx_i \wedge dx_j$$

$$= (f_{x_i} g + f g_{x_i})\, dx_i \wedge dx_j$$

pf of 3: $d(dw) \overset{?}{=} 0$

$$d(dw) = d\left(\sum dx_i \wedge \frac{\partial w}{\partial x_i}\right) \quad \text{线性.}$$

$$\overset{①}{=} \sum d\left(dx_i \wedge \frac{\partial w}{\partial x_i}\right)$$

Leibniz $$\overset{②}{=} \sum \underset{0}{d(dx_i)} \wedge \frac{\partial w}{\partial x_i} - dx_i \wedge d\left(\frac{\partial w}{\partial x_i}\right)$$

$$= -\sum dx_i \wedge d\frac{\partial w}{\partial x_i} \quad \text{展开}$$

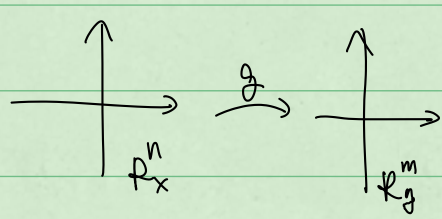
$$= -\sum dx_i \wedge \left(\sum dx_j \wedge \frac{\partial^2 w}{\partial x_i \partial x_j}\right)$$

$$= -\sum dx_i \wedge dx_j \wedge \left(\frac{\partial^2 w}{\partial x_i \partial x_j}\right)$$

$$= 0$$ i,j, 成对出现

$d(dx_i) = 0$, 因为 dx_i 为常系数

pf of 4: WLOG: $w = f \wedge dy_I$



$$g^*(dw) = g^*\left((df) \wedge dy_I + f \wedge d(dy_I)\right)$$

$$= g^*\left((df) \wedge dy_I\right)$$

$$= g^*(df) \wedge g^*(dy_I) \quad \text{function.}$$

$$= d(g^* f) \wedge d(g^* y_{i_1}) \wedge \cdots \wedge d(g^* y_{i_k})$$

$dy_I = dy_{i_1} \wedge \cdots \wedge dy_{i_k}$

$d(dy_I) = \sum (-1)^{?} dy_{i_1} \wedge \cdots \wedge d(dy_{i_\alpha}) \wedge \cdots \wedge dy_{i_k}$

=0

=0

另一边:

$$d(g^*w) = d(g^*(f \wedge dy_I))$$

$$= d(g^*f \wedge g^*dy_{i_1} \cdots \wedge g^*dy_{i_k})$$

$$= d(g^*f) \wedge g^*dy_{i_1} \cdots \wedge g^*dy_{i_k}$$

Leibniz, 其它项都是0.

k+1 form

$$dw\,|\xi_1, \cdots \xi_{k+1}| = \sum_{i=1}^{k+1} \pm W_p(\xi_1, \cdots, \hat{\xi}_i, \cdots \xi_k) \pm W_{p+v_i}(\xi_1, \cdots, \hat{\xi}_i, \cdots \xi_k)$$

dont use

k-form

k+1个基点相同 构成个平四

在这个平四里积分

$$\text{“}\int_{P(\xi_1, \cdots, \xi_{k+1})} dw\text{”} = \int_{\partial P}$$

Thm: with $\xi_i = (p, v_i)$, $w \in \Omega^k(\mathbb{R}^n)$

硬 omit.

$$(dw)(\xi_1, \cdots, \xi_{k+1}) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{k+1}} \sum_{i=1}^{k+1} (-1)^{i-1} \Big(w(p+\varepsilon v_i)(\varepsilon v_1, \cdots, \widehat{\varepsilon v_i}, \cdots, \varepsilon v_{k+1}) - w(p)(\varepsilon v_1, \cdots, \widehat{\varepsilon v_i}, \cdots, \varepsilon v_{k+1}) \Big)$$

pf: WLOG, $w = f \wedge \lambda$, λ has 常系数.

$$dw = d(f \wedge w) = df \wedge w + f \wedge dw$$

$$= (df) \wedge w$$

$$(dw)(\xi_1, \cdots, \xi_{k+1}) = (df \wedge \lambda)(\xi_1, \cdots, \xi_{k+1}) \quad \text{(LHS)}$$

$$df(\;),\ \lambda(\cdots\cdots)$$

交叉点的个数

$$= \sum_{i=1}^{k+1} (-1)^{i-1} df(\xi_i) \cdot \lambda(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_{k+1})$$

$$= \sum_{i=1}^{k+1} (-1)^{i-1} D_{\xi_i}(f)\, \lambda(\xi_1, \cdots, \hat{\xi}_i, \cdots, \xi_{k+1})$$

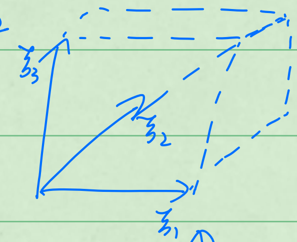
用这些 ε 消掉前面

$$RHS = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^{k+1}} \sum_{i=1}^{k+1} (-1)^{i-1} \Big(f(p+\varepsilon v_i)\cdot \lambda(\varepsilon v_1, \cdots, \varepsilon \hat{v}_i, \cdots, \varepsilon v_{k+1}) - f(p)\cdot \lambda(\varepsilon v_1, \cdots, \varepsilon \hat{v}_i, \cdots, \varepsilon v_{k+1}) \Big)$$

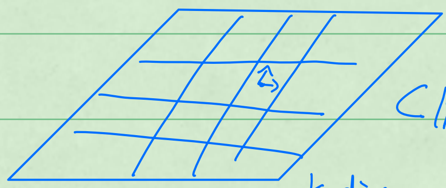
$$= \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \sum_{i=1}^{k+1} (-1)^{i-1} \big(f(p+\varepsilon v_i) - f(p) \big) \big(\lambda(v_1, \cdots, \hat{v}_i, \cdots, v_{k+1}) \big)$$

$$= \sum_{i=1}^{k+1} (-1)^{i-1} \Big(\lim_{\varepsilon \to 0} \frac{f(p+\varepsilon v_i) - f(p)}{\varepsilon} = D_{\xi_i} f \Big) \lambda(v_1, \cdots, \hat{v}_i, \cdots, v_{k+1}) = LHS.$$

$k=2$



$P(\xi_1, \cdots; \xi_k)$



$\subset \mathbb{R}^n$

k-dim

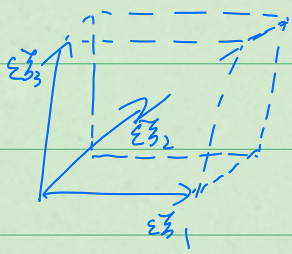
$\omega \in \Omega^k(\mathbb{R}^n)$

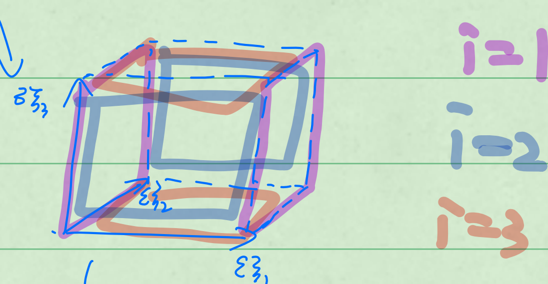
$$LHS: \cancel{\lim_{\varepsilon \to 0}} \cancel{\frac{1}{\varepsilon^{k+1}}} (d\omega)(\varepsilon\xi_1, \cdots, \varepsilon\xi_{k+1})$$

$$\cancel{\lim_{\varepsilon \to 0}} \cancel{\frac{1}{\varepsilon^{k+1}}} \sum_{i=1}^{k+1} (-1)^{i-1} \Big(\omega(p+\varepsilon v_i)(\varepsilon v_1, \cdots, \varepsilon \hat{v}_i, \cdots, \varepsilon v_{k+1}) - \omega(p)(\varepsilon v_1, \cdots, \varepsilon \hat{v}_i, \cdots, \varepsilon v_{k+1}) \Big)$$

Small ε.

$(k+1)\times 2$ 个 prllgram

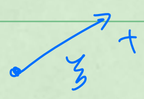




$$\int_{P(\varepsilon\xi_1, \cdots, \varepsilon\xi_{k+1})} d\omega = LHS$$

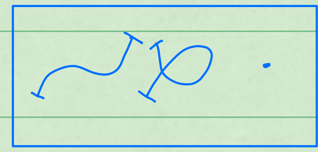
$$\int_{\partial P(\xi_1, \cdots, \xi_k)} \omega$$

$$df(\xi) = \lim \frac{1}{\varepsilon} \big(f(p+\varepsilon v) - f(p) \big)$$



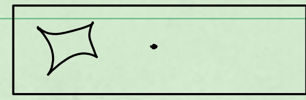
DEF: A singular k-cube in $A \subset \mathbb{R}^n$ open is a cont function $I^k=[0,1]^k \to \mathbb{R}^n$.

3 1-cube in $\mathbb{R}^2$:



0-cube: a point

2-cube:



$$C_k(A) = \begin{pmatrix}\text{space of}\\ \text{k-chains}\\ \text{in } A \subseteq \mathbb{R}^n\end{pmatrix} = \begin{pmatrix}\text{The free abelian group}\\ \text{generated by All k-cubes in } A\end{pmatrix}$$

$$= \left\{ \sum_{i=1}^{p} \alpha_i c_i \;\middle|\; \alpha_i \in \mathbb{Z},\ c_i : \overset{\text{cube}}{F} : I^k \to A \right\}$$ (順序不重要/can drop 0's/分配律)

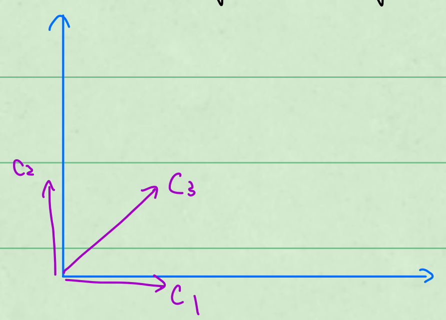
• 可加, $C_k(A)$ 有 "+", "×" (常整數.), 有 "0"

$c_i : [0,1] \xrightarrow{(t)} \mathbb{R}^2$

$c_1 = (t, 0)$ $\quad$ $\cancel{c_1 + c_2 - c_3 = 0}$

$c_2 = (0, t)$

$c_3 = (t, t)$



in $C_1(\mathbb{R}^2)$

goal

$\partial : C_k(A) \to C_{k-1}(A)$

phase maps?

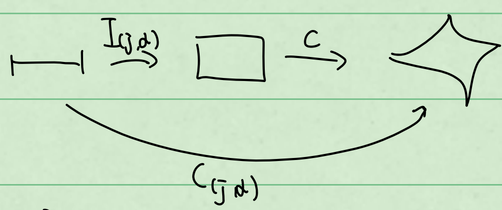
DEF: $I^k_{(j,\alpha)} \in C_{k-1}(I^k)$ $\qquad j \in \underline{k}^{\{1,\cdots,k\}}$ $\quad \alpha \in \{0,1\}$

$\hookrightarrow I^{k-1}_{y_1,\cdots,y_{k-1}} \to I^k_{x_1,\cdots,x_k}$

$$I^k_{(j,\alpha)}(y_1,\cdots,y_{k-1}) = (\underbrace{y_1,\cdots,y_{j-1},\alpha_j,y_j,\cdots y_{k-1}}_{k})$$

if $c : I^k \to A$, $c_{(j,\alpha)} = c \circ I_{(j,\alpha)}$

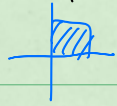
C^1_k (可導)



DEF: $\partial c = \sum_{j \in \underline{k}} (-1)^{j+\alpha} c_{(j,\alpha)} \in C_{k-1}(A)$

$\alpha\in\{0,1\}$

Extend linearly to $C_k(A)$: $\partial\left(\sum a_i c_i\right)=\sum a_i \partial c_i$

$c\in C_2(\mathbb{R}^2)$  $c(x,y)=(x,y)$

$\partial c = -c_{(1,0)}+c_{(1,1)}+c_{(2,0)}-c_{(2,1)}$

$= -(0,t)+(1,t)+(t,0)-(t,1)$ 在第2个位置放1. 剩下保持

$C_0(\mathbb{R}^2)\ni\partial(\partial c) = -\partial(0,t)+\partial(1,t)+\partial(t,0)-\partial(t,1)$

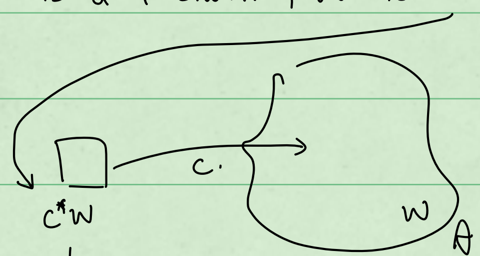
$= -\left(\cancel{(0,1)}-\cancel{(0,0)}\right)+\left(\cancel{(1,1)}-\cancel{(1,0)}\right)+\left(\cancel{(1,0)}-\cancel{(0,0)}\right)-\left(\cancel{(1,1)}-\cancel{(0,1)}\right)$

$=0$

Thm: $\partial^2=0$ $\quad \left(\underbrace{C_k(A)\xrightarrow{\partial}C_{k-1}(A)\xrightarrow{\partial}C_{k-2}(A)}_{\partial^2} \right)$

Integration: $\int_c w$

c is a k-chain, w is a k-form.



$= f\,dx_1\wedge dx_2\wedge\cdots\wedge dx_n.$

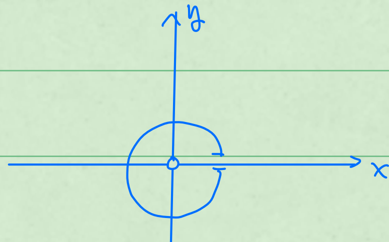
DEF: $\int_{I^k_{x_1,\dots,x_k}} f\,dx_1\wedge\cdots\wedge dx_k = \int_{I^k} f$

$\int_c w = \int_{I^k} c^* w \qquad (c^k: I^k\to A. \quad w\in\Omega^k(A))$

$\int_{\sum a_i c_i} w = \sum a_i \int_{c_i} w$

e.g. 1: $k=1$, $n=2$. $\quad A=\mathbb{R}^2-\{0\}$

$w=\frac{-y\,dx}{x^2+y^2}+\frac{x\,dy}{x^2+y^2}$



$c\in C_1^1(A) \qquad c: I\to A=\mathbb{R}^2\setminus\{0\}$

$c(t)=\begin{pmatrix}\cos 2\pi t\\ \sin 2\pi t\end{pmatrix}$

$$\int_c w = \int_{I=[0,1]} c^*w = \int_0^1 \frac{-\sin 2\pi t \, d\cos 2\pi t}{1} + \frac{\cos 2\pi t \, d\sin 2\pi t}{1}$$

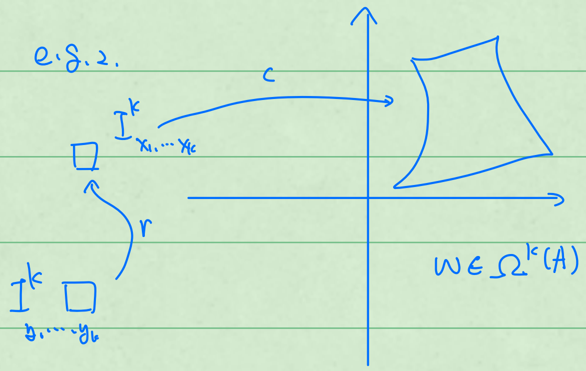
$$= \int_0^1 2\pi(+\sin^2 2\pi t + \cos^2 2\pi t)\,dt$$

$$= \int_0^1 2\pi\,dt = \int_0^1 2\pi(1) = 2\pi$$

e.g. 0: $k=0$, n anything $w=f$ $\underset{\text{"}(dw)\text{'}}{\sqcup}$

$$\int_{\sum a_i p_i} f = \sum a_i f(p_i)$$

e.g. 2.



i) c 不重要, image (c) 重要

ii) $\int_c w = \int_{c\circ r} w$ provided $r: I^k \to I^k$, which is C', 1-1 and onto,

$\det(r')>0$ (r is orientation-perserving).

pf: $\int_{c\circ r} w = \int_{I^k} (c\circ r)^* w = \int_{I^k} r^*(c^*w) = \int_{I^k} r^*(f\,dx_1\wedge\cdots\wedge dx_k)$

write $c^*w = f\,dx_1\wedge\cdots\wedge dx_k$ $\quad = \int_{I^k} (f\circ r)\cdot r^*(dx_1\wedge\cdots\wedge dx_k)$

$$= \int_{I^k} (f\circ r)\cdot|\det(r')|\cdot dy_1\wedge\cdots\wedge dy_k$$

cov $\quad = \int_{r(I^k)} f\,dy_1\wedge\cdots\wedge dy_k$

$$= \int_{I^k} f\,dy_1\wedge\cdots\wedge dy_k$$

$$= \int_{I^k} c^*w$$

$$= \int_c w$$

$\mathbb{R}^3$

$$\Omega^0 \longrightarrow \Omega^1 \longrightarrow \Omega^2 \longrightarrow \Omega^3$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$\{f\} \xrightarrow{\text{grad}} \left\{\begin{matrix} f_1 dx_1 \\ + f_2 dx_2 \\ + f_3 dx_3 \end{matrix}\right\} \xrightarrow{\text{curl}} \left\{\begin{matrix} f_1 dx_2 dx_3 \\ + f_2 dx_3 dx_1 \\ + f_3 dx_1 dx_2 \end{matrix}\right\} \xrightarrow{\text{div}} \{g\, dx_1 dx_2 dx_3\}$$

vector space

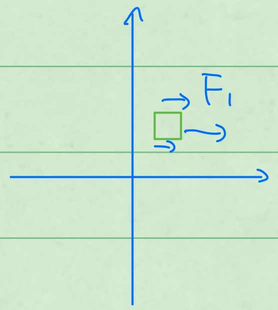
$$\operatorname{div}(G) = \partial_1 G_1 + \partial_2 G_2 + \partial_3 G_3$$

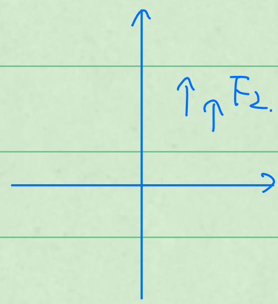
$$\operatorname{curl}(F) = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \partial_3 F_1 - \partial_1 F_3 \\ \partial_1 F_2 - \partial_2 F_1 \end{pmatrix}$$

$$\operatorname{grad} F = \begin{pmatrix} \partial_1 F \\ \partial_2 F \\ \partial_3 F \end{pmatrix}$$

↳ direction of fastest increasing

div: $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, $\operatorname{div}(F) = \partial_1 F_1 + \partial_2 F_2$

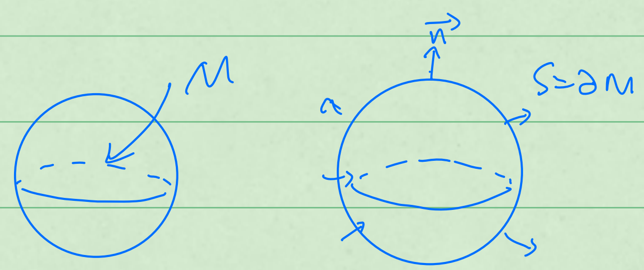




(在某一点"多"了多少)

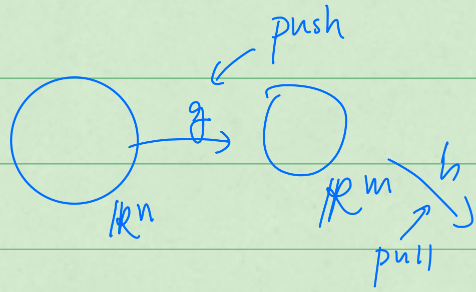
curl: $(\operatorname{curl} F)_3 = \partial_1 F_2 - \partial_2 F_1$

$\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ measures spins.



div G

$$\int_{\substack{M \\ (3D)}} \operatorname{div} G = \int_{\substack{\partial M = S \\ (2D)}} G \cdot \vec{n}\, dA \qquad \text{(Gauss's thm)}$$

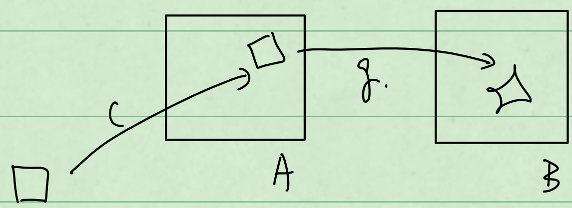


Cubes / Chains push.

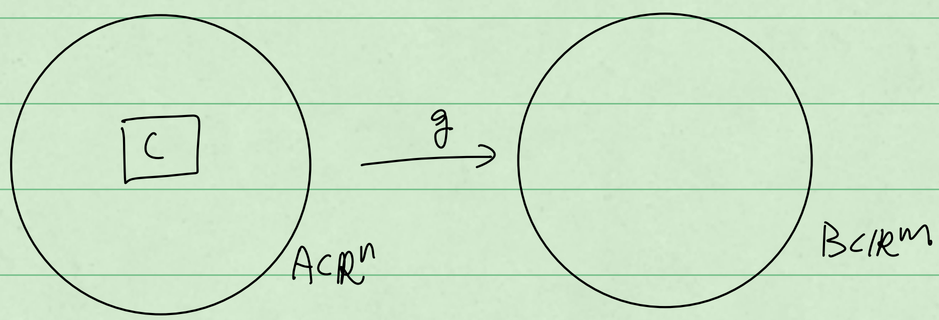
$$g: \underset{\cap\,\mathbb{R}^n}{A} \to \underset{\cap\,\mathbb{R}^m}{B} \implies g_*: C_k(A) \to C_k(B)$$

g_* is compatible with "+" "×7"

. . . . . – – – – – – – "∂" i.e. $\partial(g_* c) = g_*(\partial c)$



每个面都被映到了面上.

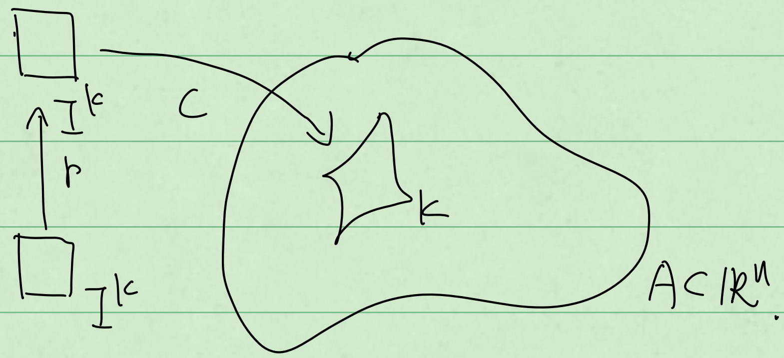


$c \in C_k(A)$ $\qquad$ $w \in \Omega^k(B)$

prop: $\int_c g^* w = \int_{g_* c} w$

pf: $LHS = \int_{I^k} c^*(g^* w)$ "="

$RHS = \int_{I^k} (g_* c)^* w$



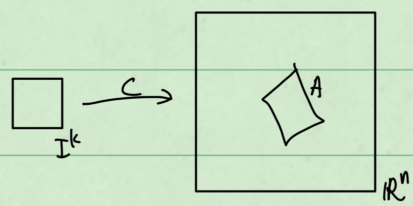
$c: I^k \to A$, $w \in \Omega^k(A)$, $r: I^k \to I^k$. 1-1, onto, $\det r^* > 0$

$$\int_c w = \int_{c \circ r} w$$

prop: 設 $c: I^n \to \mathbb{R}^n_{(x_1,\dots,x_n)}$ one to one. $\det c' > 0$.

Let $A = c(I^n)$. Suppose also $f: \mathbb{R}^n \to \mathbb{R}$ is given,

Then $\int_A f = \int_c f \, dx_1 \wedge \cdots \wedge dx_n$.



pf: $\Leftarrow \int_A f \overset{?}{=} \int_{I^n} (f \circ c) \cdot \det(c')$, COV.

Thm: Given $c \in C_k^1(A \subset \mathbb{R}^n)$ & $w = \Omega^{k-1}(A)$, then $\int_c dw = \int_{\partial c} w$

pf: WLOG, c is a single cube.

$$\int_c dw = \int_{I^k} c^*(dw) = \int_{I^k} d(c^* w) \overset{?}{=} \int_{\partial I^k} c^* w = \int_{c_*(\partial I^k)} w$$

$$= \int_{\partial(c_* I^k)} w = \int_{\partial c} w$$

So we only have to show "?"

Assume $w \in \Omega^{k-1}(I^k)$, WLOG, $w = f \cdot dx_1 \wedge \cdots \wedge \hat{dx_i} \wedge \cdots \wedge dx_k$

$= f \cdot dx_{no\,i}$ (notation)

$$\text{LHS} = \int_{I^k} dw = \int_{I^k} (-1)^{i-1} \frac{\partial f}{\partial x_i} (dx_1 \wedge \cdots \wedge dx_k) = (-1)^{i-1} \int_{I^k} \frac{\partial f}{\partial x_i} \ \underset{\text{FTC}}{\overset{\text{Fubini}}{=}}$$

Fundamental thm cal

$$= (-1)^{i-1} \int_{I^{k-1}_{y_1, \dots, y_{k-1}}} [F(y_1, \dots, \overset{i}{\tilde{1}}, \dots y_{k-1}) - F(y_1, \dots \overset{i}{\tilde{0}}, \dots y_{k-1})]$$

$$\text{RHS} = \int_{\partial I^k} f dx_{no\,i} = \sum_{j=1}^{k} \sum_{\alpha \in \{0,1\}} (-1)^{j+\alpha} \int_{I_{(j,k)}} f \, dx_{no\,i} =$$

$$\sum_{j,\alpha} (-1)^{j+\alpha} \int_{I^{k-1}_{y_1, \dots, y_{k-1}}} (f \circ I_{(j,\alpha)}) \cdot d(x_1 \circ I_{j\alpha}) \wedge \cdots \wedge \widehat{d(x_i \circ I_{(j,\alpha)})} \wedge \cdots \wedge d(x_k \circ I_{j\alpha})$$

the function x_j is constant on the image of $I_{(j,\alpha)}$

So $(x_j \circ I_{(j,\alpha)})$, $\to$ constant. $\to$ $d(x_j \circ I_{(j,\alpha)}) = 0$

$\to j$ 只能 $= i$

$$= \sum_{\alpha} (-1)^{i+\alpha} \int_{I^k_{y_1, \dots, y_{k-1}}} f(y_1, \dots, \overset{i}{\tilde{\alpha}}, \dots y_{k-1}) \, dy_1 \wedge \cdots \wedge dy_{k-1}$$

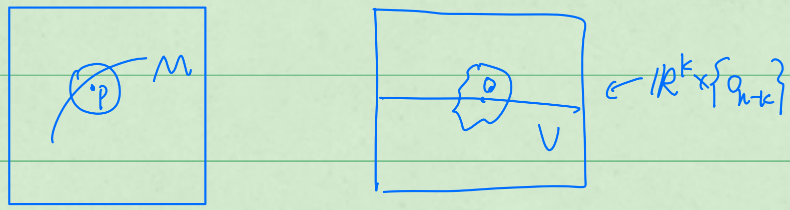
$$= \sum_{\alpha} (-1)^{i+\alpha}$$

Thm: given $k \le n$, $M \subset \mathbb{R}^n$, $p \in M$, TFAE

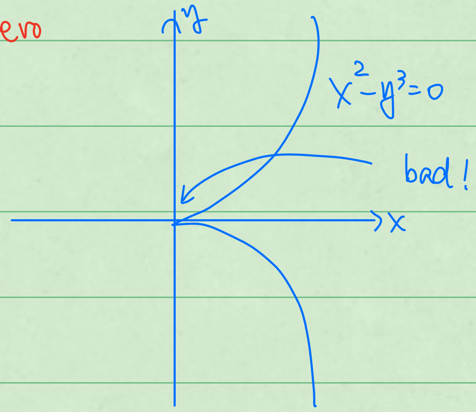
(M): $\exists$ open $U \ni p$, open $V \subset \mathbb{R}^n$, $h: U \to V$. 局部和"标准"欧式空间类似

h: 1-1, onto, smooth, h^{-1} smooth,

$h(U) \cap M = V \cap (\mathbb{R}^k \times \{0_{n-k}\})$ (near p, space is uniform)



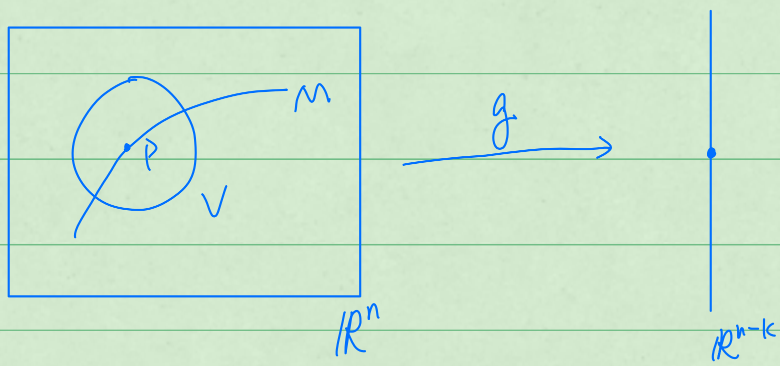
(Z): ero



M是g的kernel的一部分.

dim(M) = k ⇒ dim(image(M)) = n-k.

$y = x$

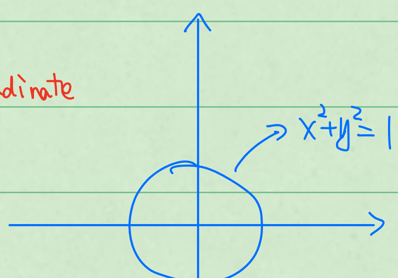


"好"的函数

$\exists$ open $U \ni p$, smooth $g: U \to \mathbb{R}^{n-k}$, s.t.

$U \cap M = U \cap g^{-1}(0)$, and rank $g' = n-k$ at p.

(C)oordinate

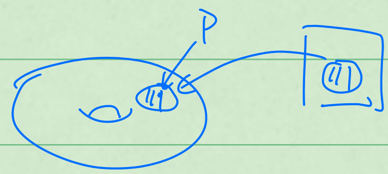


"kernel"

在p旁边有好的

"coordinate system"

$\hookrightarrow \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$ is "image."



$\exists$ open $U \ni p$, open $W \subset \mathbb{R}^k$, smooth, 1-1 f

$f: W \to \mathbb{R}^n$, s.t.

① $f(W) = U \cap M$.

② $f^{-1}: M \cap U \to W$ cont

③ $\forall a \in W$, $\operatorname{rank}(f'(a)) = k$.

DEF: $M \subset \mathbb{R}^n$ is a k-manifold $(k \le n)$ if $(M\,z\,c)$ holds.

e.g.: ① $S^2 \subseteq \mathbb{R}^3$ is a 2-manifold

$\{x^2+y^2+z^2-1=0\} \Rightarrow$ "z"

Satisfies "c" (经纬度)

$\operatorname{rank}(g') = 1$?

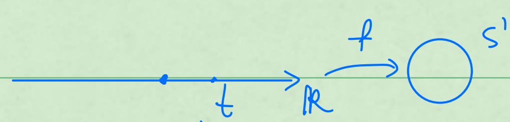
$g'(p) = (2x \quad 2y \quad 2z) \neq 0$.

e.g. $S^1 \subseteq \mathbb{R}^2$

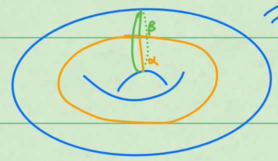
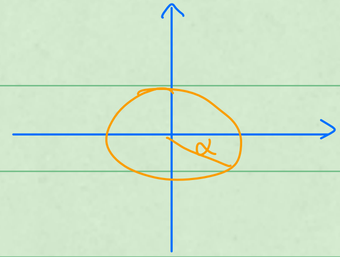
$\{x^2+y^2-1=0\}$

"z" $\to$ ✓

"c":



$f(t) = \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$

e.g.  $= T^2 \subseteq \mathbb{R}^3$ 

"(C)" $F(\alpha,\beta) = \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} + \frac{1}{3}\left(\cos\beta \begin{pmatrix} \cos\alpha \\ \sin\alpha \\ 0 \end{pmatrix} + \sin\beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right)$

$$= \begin{pmatrix} \cos\alpha + \frac{1}{3}\cos\beta\cos\alpha \\ \sin\alpha + \frac{1}{3}\cos\beta\sin\alpha \\ \frac{1}{3}\sin\beta \end{pmatrix}$$

3D-manifold (3个方向 rotate)

e.g. $SO(3) \subseteq M_{3\times3}(\mathbb{R}) = \mathbb{R}^9$

$\parallel$

sets of rotations of $\mathbb{R}^3$

perserving orientation.

"(Z)": $A \in SO(3) \Leftrightarrow A^TA = I,\ \det(A) = 1$

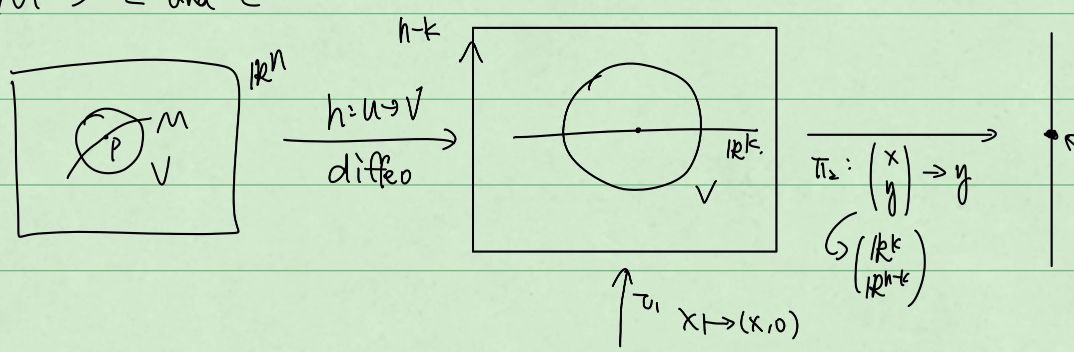
$$\begin{cases} A^TA - I = 0 \\ \det A - 1 = 0 \end{cases}$$

$\in$ symmetric matrices of $\mathbb{R}^{3\times3}$. so have 6 dimension.

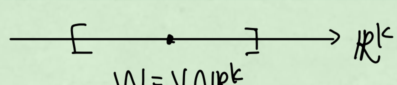
可以推广到 $SO(n)$

pf:

① $M \Rightarrow$ "Z" and "C"

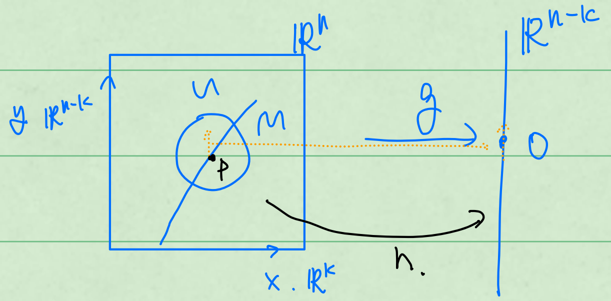


"Z": $g = (\pi_2 \circ h)$



"c": $f = (h^{-1} \circ \tau_1)$

② "2 ⇒ M"



know: rank $g' = n-k$ ⇒ $n-k$ 个线性无关的

Name the variables in $\mathbb{R}^n$: $\{x_1, \cdots, x_k, y_1, \cdots, y_{n-k}\}$ in some order

s.t. $\frac{\partial g}{\partial y}$ is non-singular

$M_{(n-k)(n-k)}$

Define: $h(x, y) = (x, g(x, y))$

$$h'(p) = \begin{pmatrix} \frac{\partial x}{\partial x} & \frac{\partial x}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix} = \begin{pmatrix} I & 0 \\ ? & \frac{\partial g}{\partial y} \end{pmatrix}$$

不知道, 无所谓 non-singular.

I, $\frac{\partial g}{\partial y}$ invertible ⇒ $h'(p)$ invertible.

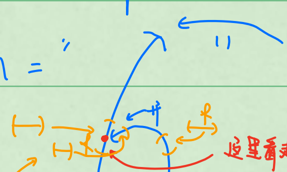
$\overset{\text{IFT}}{\Longrightarrow}$ $\exists\, U \ni p$ on which h is invertible. with smooth

inverse (因为 h smooth)

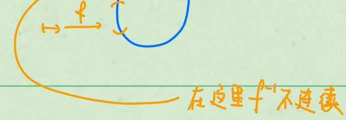
$h: U \to V = h(U)$ is diffeomorphism

③ i) C − ② ⇏ M

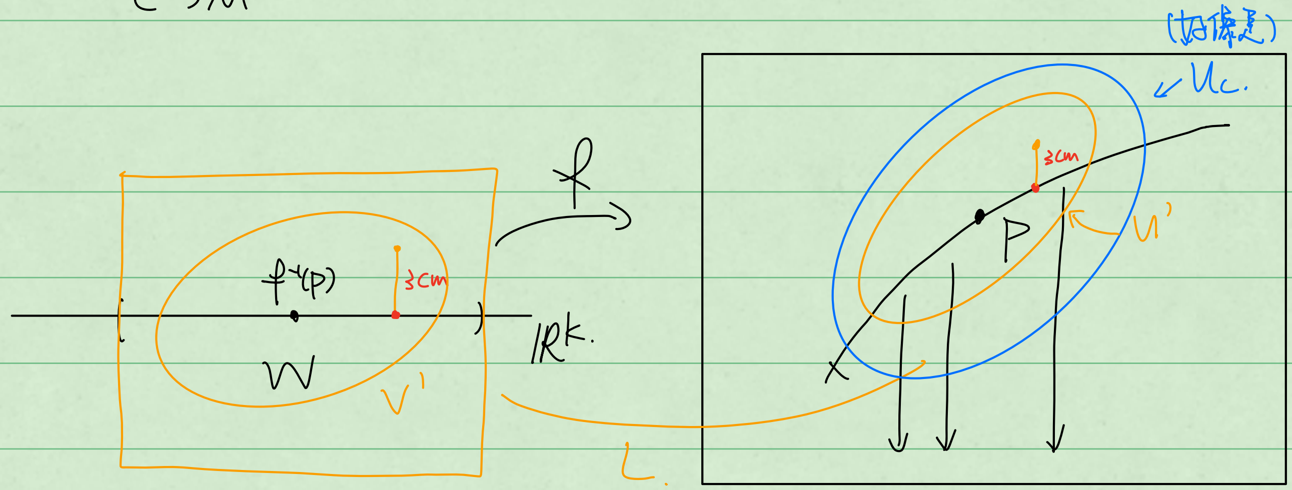
Counterexample of: $k=1$, $n=2$.

M = "  " 开,若闭, 不符合

这里看起来像 $\perp$, ‡ —



"C ⇒ M"



extend W to $\square$ to $\bigcirc$

WLOG, $(\pi_{x_1,\dots,x_k}\circ f)' = (\pi_1\circ f)'$ is of rank k, so invertible.

(找到那个 rank 为 k 的 $x_1,\dots,x_k$)

Define $L: \underbrace{W}_{x_1,\dots,x_k}\times\underbrace{\mathbb{R}^{n-k}}_{y_1,\dots,y_{n-k}} \longrightarrow \mathbb{R}^n$. by $L(x,y) = \overbrace{f(x)}^{\in\mathbb{R}^n} + \begin{pmatrix}0\\y\end{pmatrix}$

$$L' = \begin{pmatrix} \frac{\partial L_1}{\partial x} & \frac{\partial L_1}{\partial y} \\ \frac{\partial L_2}{\partial x} & \frac{\partial L_2}{\partial y} \end{pmatrix} = \begin{pmatrix} \frac{\partial F}{\partial x} & \sim \\ 0 & I \end{pmatrix} \Rightarrow \text{invertible.}$$

L' is invertible and so is L in some $V' \in W\times\mathbb{R}^k$.

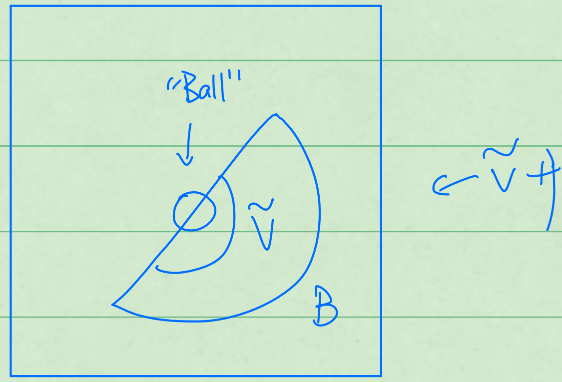
set $U' = L(V')$

Now to prove: $L(V\cap(\mathbb{R}^k\times\{0\})) = U\cap M$

$L(V''\cap\mathbb{R}^k) = U''\cap M \Leftrightarrow f(V''\cap\mathbb{R}^k) = U''\cap M$

Remainder: $g: \overset{\in\mathbb{R}^n}{A} \to \overset{\in\mathbb{R}^m}{B}$ is cont

$\Leftrightarrow$ $g^{-1}(\tilde{V})$ is open in A whenever $\tilde{V} \in \mathbb{R}^m$ is open in B.



$\Leftrightarrow$ " $\tilde{V}$ is open in B " $\Leftrightarrow$ $\exists$ open $V \in \mathbb{R}^m$, s.t. $\tilde{V} = B \cap V$

$g^{-1}(V)$ is of the form $A \cap U$, U open $\in \mathbb{R}^n$, whenever $V \subset \mathbb{R}^m$ open.

Up above, f^{-1} is cont

$\Rightarrow$ f is "open" $\Rightarrow$ f carries open sets to open sets.

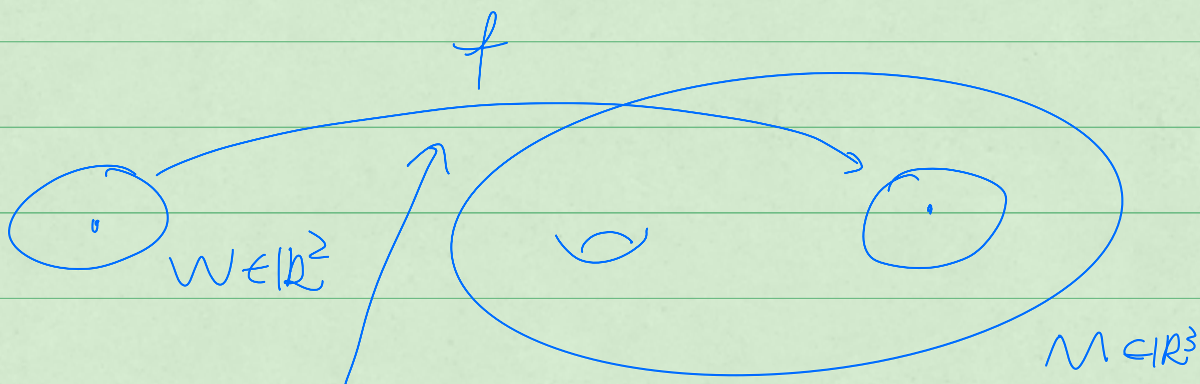
meaning in our context,

$f(V' \cap \mathbb{R}^k)$ must be open in $M \cap U_c$.

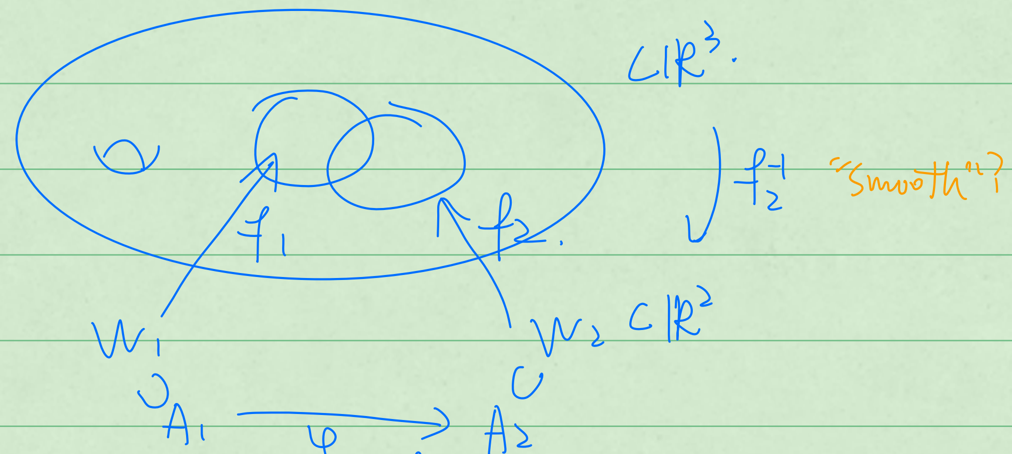
$\Rightarrow$ $\exists$ open $U'' \subset \mathbb{R}^n$, s.t. $f(V' \cap \mathbb{R}^k) = U'' \cap M \cap U_c = M \cap \underbrace{(U_c \cap U'')}_{U'''}$

$= M \cap U'''$

Now, set $V''' = f^{-1}(U''')$, done.



Coordinate patch.



$\varphi = f_2^{-1} \circ f_1$ (transition function)

A_1, A_2: Inverse image of $\text{image}(f_1) \cap \text{image}(f_2)$

φ is a diffeomorphism.

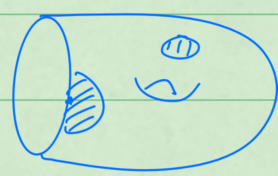
DEF: $\mathbb{R}^k_+ = \{ x \in \mathbb{R}^k ;\ x_k \geq 0 \}$

DEF: A "k manifold with boundary." is a subset $M \subset \mathbb{R}^n$,

s.t. $\forall p \in M$, $\exists \underset{\text{open}}{W} \in \mathbb{R}^k_+$,

and an open $U \in \mathbb{R}^n$. a

smooth, 1-1 function $f: W \to \mathbb{R}^n$, s.t.

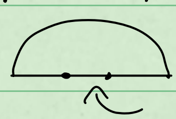


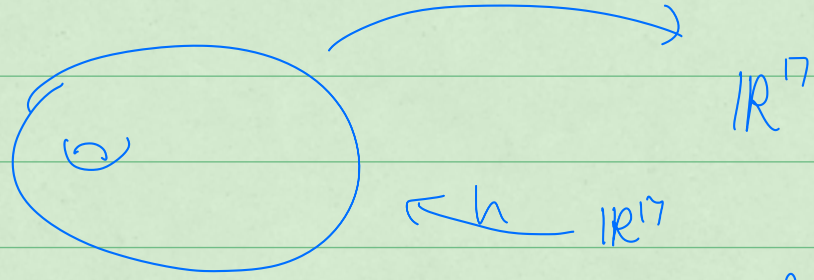
① $f(W) = M \cap U$,

②

(As above)

DEF, if M is "k manifold with boundary."

then $\partial M = \{ p \in M$, s.t.  $\}$



g is smooth if $\forall$ coordinate chart, $f : \mathbb{R}^k \supset W \to M$

$g \circ f : W \to \mathbb{R}^{17}$ is smooth

h is smooth if $\forall$ coordinate chart $f : W \to M$

$f^{-1} \circ h|_{h^{-1}(f(W))} : h^{-1}(f(W)) \to W$ is smooth.

Suppose M^k, N^l ← manifolds of deg k/l.

$\varphi : M \to N$ is smooth if

$$\begin{tikzcd} M \arrow[r, "\varphi"] & N \arrow[d, shift left, "f_N^{-1}"] \\ \mathbb{R}^k \arrow[u, "f_M", "c.c."'] & \mathbb{R}^l \arrow[u, shift left, "f_N"] \end{tikzcd}$$

for $\forall f_M, f_N$, the partially defined function,
$f_N^{-1} \circ \varphi \circ f_M$ is smooth.

$\forall p \in M$, $\exists$open $W \subset \mathbb{R}^k_+ = \{x_k \geq 0\}$, open $p \in U \subset \mathbb{R}^n$,
smooth 1-1 function $f : W \to U$ s.t. ① ② ③
then M is "manifold with boundary."

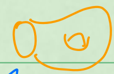
Thm: If M is a manifold with boundary, then,
$\forall p \in M$, either one of the 2 following is true.
① if $f : W \to M$ is coordinate chart s.t. $f(a) = p$,
then $a_k > 0$
② if $f : W \to M$ is coordinate chart s.t. $f(a) = p$,
then $a_k = 0$

↑

Def: $\partial M = \{p \in M,\ s.t.\ ② \text{ holds}\}$.

Warning:

① $\partial \neq$ bd.

$M =$ 

∂M.   bd$M = M$.

② Every manifold is a "manifold with boundary"

(可能 $\partial M = \emptyset$)

但反之不成立.

Thm: If M is a manifold with ∂, then ∂M is a manifold. M deg (k), ∂M deg $(k-1)$.

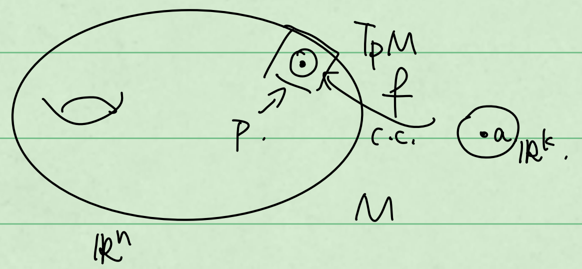
大概 X_k.

$\Rightarrow \partial^2 = 0$.

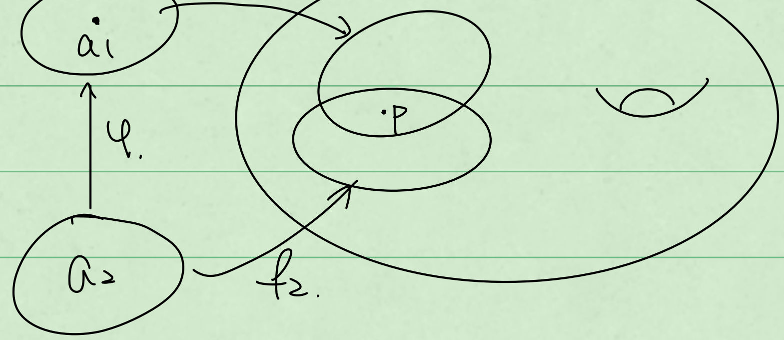
Let M be a k-dim manifold, (with ∂)

Let $p \in M$, if $f: W \to M$ is a coordinate chart for m with $f(a) = p$, then the tangent space at p of M

$T_pM = f_* T_a(\mathbb{R}^k) \subset T_p\mathbb{R}^n$.



1. this is well-defined.



$$f_{1*}T_{a_1}\mathbb{R}^k \overset{?}{=} f_{2*}T_{a_2}\mathbb{R}^k.$$

$$f_2 = f_1 \circ \varphi.$$

$$f_{2*}T_{a_2}\mathbb{R}^k = (f_1\circ\varphi)_* T_{a_2}\mathbb{R}^k.$$

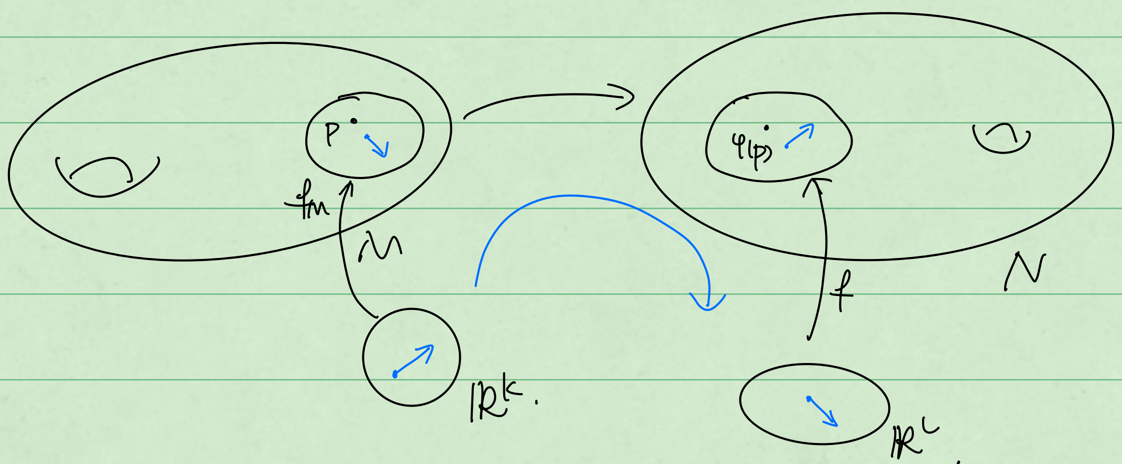
$$= f_{1*}\left(\varphi_* T_{a_2}\mathbb{R}^k\right)$$

$$= f_{1*}\left(T_{a_1}\mathbb{R}^k\right)$$ (φ_* is invertible, as φ' is. as φ is)

2. $\dim T_pM = \operatorname{rank}(f') = k$.

Suppose $\varphi: M^k \to N^l$, smooth, $p \in M$.

Claim: $\exists\ \varphi_* : T_pM \to T_{\varphi(p)}N$.



Def: A vector field on M is
$f: M \longrightarrow \bigcup_{p\in M} T_pM$,

$f: \mathbb{R}^n \to \bigcup_{p\in\mathbb{R}^n} T_p\mathbb{R}^n$,
$f(p) \in T_p\mathbb{R}^n$.

s.t. $f(p) \in T_pM$.

这里每一点的 $T_p\mathbb{R}^n$ 一样

这里不一样

Def: Such f is smooth if

同上.

Def: $\Omega^l(M^k) \ni w$ if

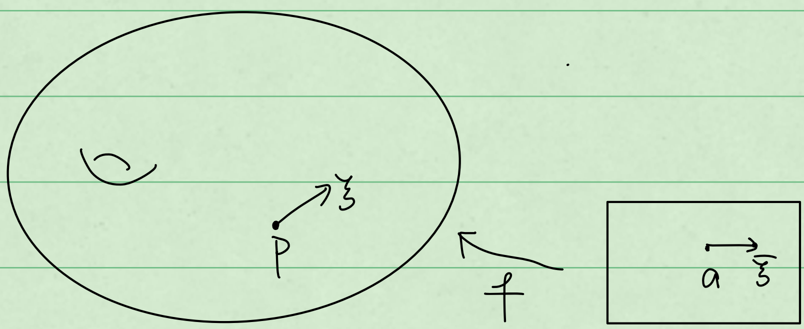
$w : M \longrightarrow \bigcup_{p\in M} \Lambda^l T_pM$.

s.t. $w(p) \in \Lambda^l T_pM$, smooth

Theme: a fish on a manifold is a fish on every coordinate chart, provided they agree.

$\xi \in T_pM$, $g: M \to \mathbb{R}$ smooth

$D_\xi g \in \mathbb{R}$



$\bar{g}$ on W, f^*g.

$\bar{\xi}$ on $T_a\mathbb{R}^n$

Def: $D_\xi g = D_{\bar{\xi}} \bar{g}$

also cubes, chains,

$\varphi : M^k \to N^l$

$\Rightarrow \varphi_* \; C_p(M) \to C_p(N)$

Compatible with ∂.

Stoke's thm (cube version) holds.

On $\Omega(M)$, we have

$+$, $\wedge$, $d : \Omega^l(M) \to \Omega^{l+1}(M)$, φ^*, φ_*

Then obey all rules from $\mathbb{R}^n$, except $W=\sum_I f_I dx_I$

e.g. on $S^2=[x^2+y^2+z^2=1]$

$W_1 = x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy \in \Omega^2(S^2)$

y is a function on S^2, $\mathbb{R}^3$, therefore 0-form. so make sence.

但两两线性相关 (S^2上的), 所以非标准形式, 冗余

但可以局部. 比如 $z=1$ 附近, 可以用 $dxdy$ 表示.

$W_2 = x\,dx + y\,dy + z\,dz \in \Omega^1(S^2)$

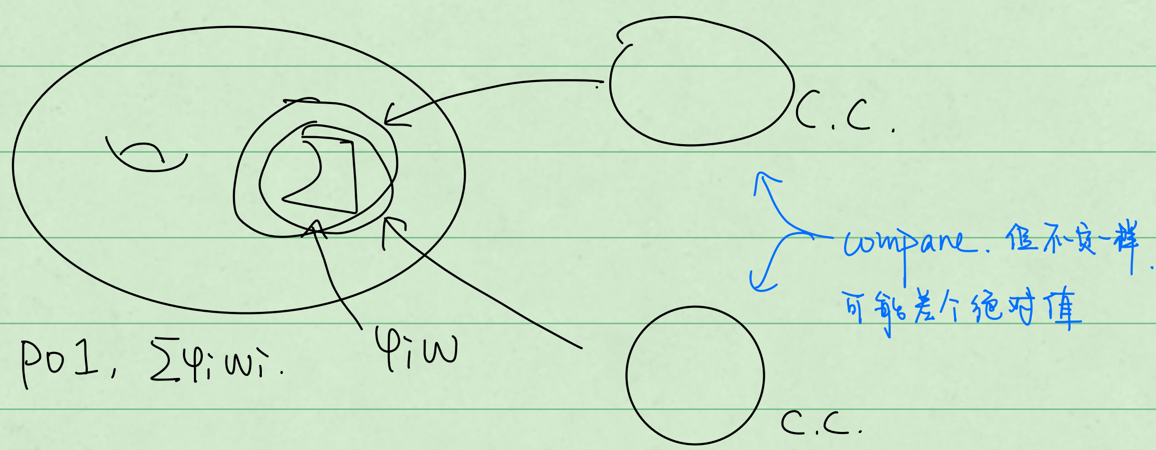
$=0$ (as a form on S^2)

Consider $g = x^2+y^2+z^2 \equiv 1$ (on S^2)

$0 = dg = 2x\,dx + 2y\,dy + 2z\,dz$

So, 0 form.

$$\int_M dw = \int_{\partial M} w$$



Loose Def: An orientable manifold is a manifold

s.t. an atlas can be chosen, s.t. all

sys of coordinate

transition functions are orientation-perserving

i.e. $\forall \varphi$, $\det(\varphi') > 0$

vector space

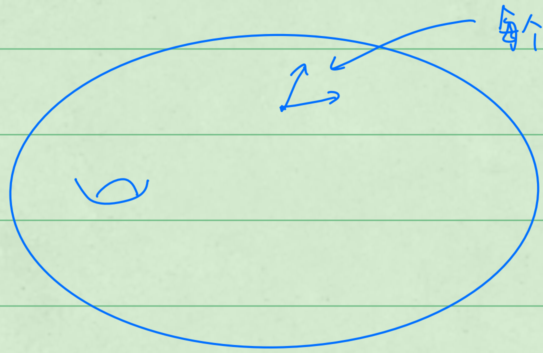
Remainder: Orientation of V^k: either an

① ordered basis 模掉正的坐标变换.

② $0 \neq \eta \in \Lambda^k(V)$ 模掉"乘正实数倍"

Def: An orientation on M^k is a continuously varying choice of orientation η_p for T_pM for each $p \in M$

每个 tangent space choose 一个 basis



Namely: it is a choice of $\eta \in \Omega^k(M)$, s.t. $\eta(p) \neq 0$ $\forall p$.

$\eta_1 \sim \eta_2$ if $\eta_1 = f\eta_2$. $f \neq 0$, f cont.

或者: {nowhere 0 top forms on M}.

在 ordered basis 视角下:

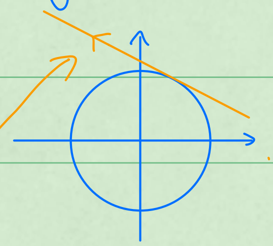
A choice O_p of an ordered basis of T_pM (modullo positive det change of basis)

(如何刻画 cont?)

s.t. $\forall p \in M$, There is a nbhd $U \ni p$, and vector fields $X_1, \cdots, X_k$, defined on U, s.t. $O_{p'} = (X_1(p'), X_2(p'), \cdots, X_n(p'))$

for every $p' \in U$.

e.g. 1: $S^1 = [x^2+y^2=1]$

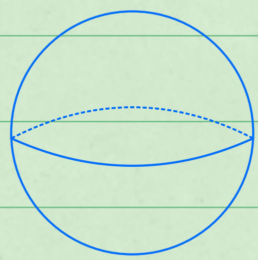


either $\begin{cases} \text{1-form, nowhere 0.} \longrightarrow \eta = \text{“}d\theta\text{”} = x\,dy - y\,dx \quad (x^2+y^2=1) \\ \text{choice of a basis of } T_pM, \forall p. \end{cases}$

$X_1\begin{pmatrix} x \\ y \end{pmatrix} = \left(\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} -y \\ x \end{pmatrix} \right)$

在上 check 这个，发现是正数即可

e.g. 2: $S^2 = [x^2+y^2+z^2=1]$.

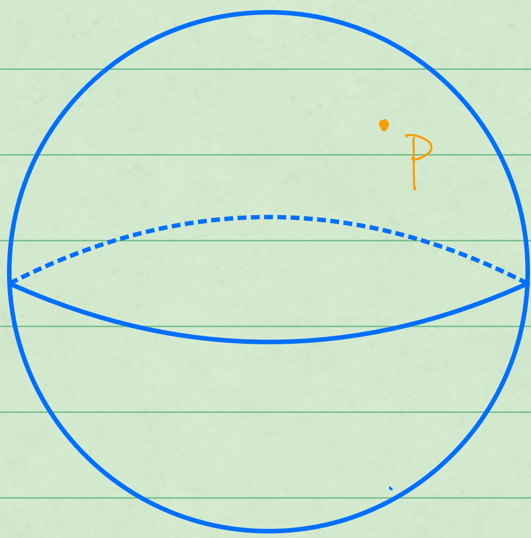


① $\eta = \sum_{cyc} x\,dy \wedge dz$

② find 2 vector field, 构成 T_pM 的基.

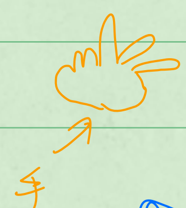
但不可能. 毛球定理.

$\nexists$ a nowhere 0 vector field on S^2. 但只要找 locally 即可



半径.

大拇指向外指，食，中指构成 basis



e.g. 0: a pt $p \in \mathbb{R}^0$

In terms of top forms, namely, function.

a function on a point is just a scalar, y.

$y_1 \sim y_2 \Leftrightarrow \frac{y_1}{y_2} > 0$

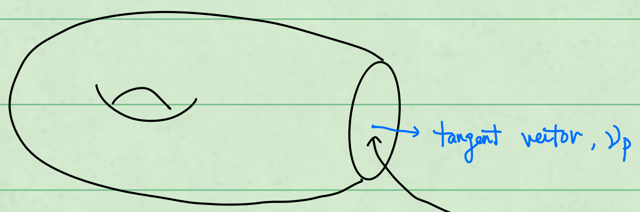
e.g. m: Möb，莫比乌斯环，转一圈会转成反方向

我tm画不出来莫比乌斯环啊啊啊啊啊啊啊啊啊啊啊啊

$\Rightarrow$ 无法 oriente.

Def: M is oriented if it is given with choice of a orientation. — Orientatible.

Given a oriented M, there is a standard way to oriente ∂M.



A orientation $O_p^{\partial M}$ of ∂M at p is:

if you prepend to it the outward pointing normal vector to ∂M,

then get the orientation. O_p^M at p.

i.e. $O_p^M = (\nu_p, O_p^{\partial M})$

e.g. $S^2 = \partial D^3$ $\quad D^3 = [x^2+y^2+z^2 \le 1]$

$O^M = (\partial x, \partial y, \partial z)$. 则在 S^2 上，同上，把拇指向外.

∂Möb is oriented.

法向量

iota

$\eta_{\partial M}(p) = i^*_{\partial M \to M} \circ \iota_{\nu} \circ \eta_M(p)$

inner multiplication. inclusion map.

If X is a vector field on M, (locally)

$\iota_X : \Omega^l(M) \to \Omega^{l-1}(M)$. "inner multiplication by X" as follows:

$w \in \Omega^l(M)$, $\xi_i \in T_p(M)$

$\iota_X(w)(\xi_1, \cdots, \xi_{l-1}) = w(X(p), \xi_1, \cdots, \xi_{l-1})$

Claim: The two def agree.

Assume $p \in \partial M$, T_pM is oriented, say it is given by the outward pointing normal. (ν_p).

$(\nu_p, \xi_1, \cdots, \xi_{k-1}) \Rightarrow \xi_i \in T_p\partial M$.

Assume also that this orientation of T_pM is given by $\eta_M \in \Omega^k(M)$

$\Rightarrow \eta_M(\nu_p, \xi_1, \cdots, \xi_{k-1}) > 0$.

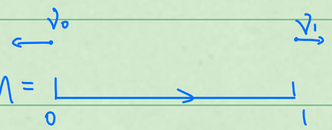
The orientation of ∂M is given by $\underset{\text{def } 1}{(\xi_1, \cdots, \xi_{k-1})} \overset{?}{\sim} \underset{\text{def } 2}{i^*_{\partial M \to M} \iota_\nu \eta_M}$.

i.e, $\eta_{\partial M}(\xi_1, \cdots, \xi_{k-1}) \overset{?}{>} 0$

$= i^*_{\partial M \to M} \circ \iota_\nu \circ \eta_M(\xi_1, \cdots, \xi_m)$

$= \iota_\nu \circ \eta_M(\xi_1, \cdots, \xi_m)$

$= \eta_M(\nu, \xi_1, \cdots, \xi_m)$

e.g. $M =$ 

$\partial M = \{0, 1\}$. as an oriented manifold?

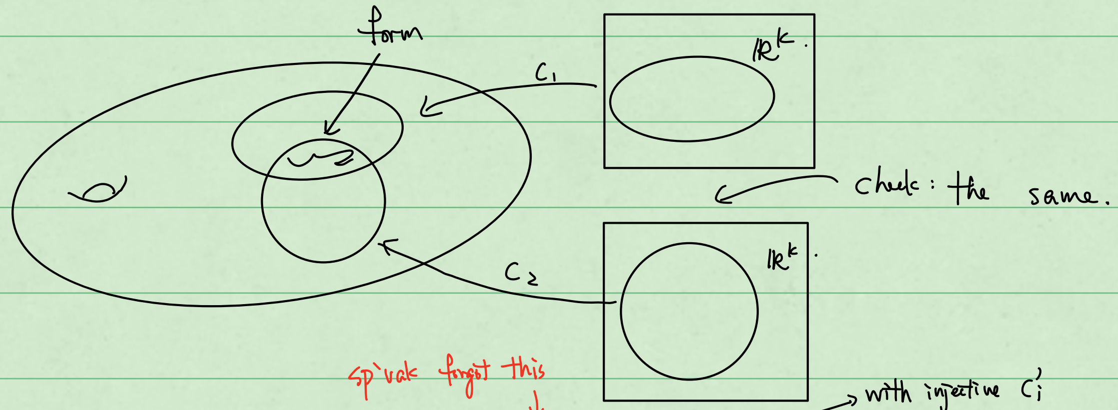
orientation at $\begin{cases} 0 : (+/-) \\ 1 : (+/-) \end{cases}$

$\eta = dx$

$p = 0$: $\eta_{\partial M}(0) = i^*_{\partial M \to M} \iota_{\nu_0} dx = \iota_{-\partial x} dx = dx(-\partial x) = -1$

Similarly: $p = 1 \Rightarrow$ "+"

Integration on an oriented manifold.



spivak forgot this

with injective c_i

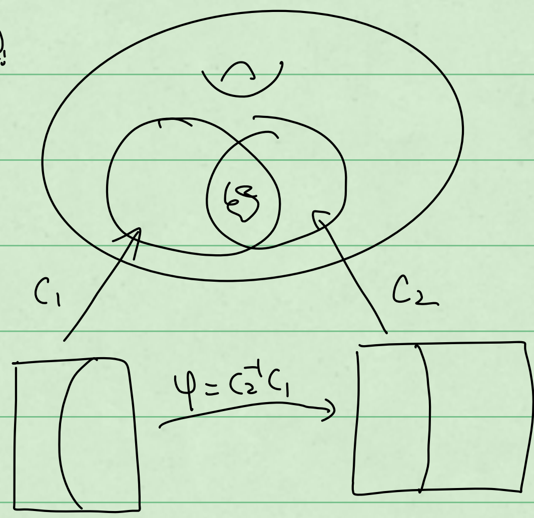
Prop: Let c_i $(i = 1,2)$ be smooth, injective, orientation preserving k-cubes in an oriented manifold M^k. and assume $w \in \Omega^k(M)$, s.t. $\operatorname{supp} w \subset \operatorname{img}(c_1) \cap \operatorname{img}(c_2)$

then. $\displaystyle\int_{c_1} w = \int_{c_2} w$

$\displaystyle \int_{c_1} w = \int_{I^k} c_i^* w$

This means we can define $\displaystyle\int_{c_1} w = \int_{c_2} w = \int_M w.$

pf:



$$\int_{I^k} c_i^* w = \int_{I^k} (c_2 \circ \varphi)^* w = \int_{I^k} \varphi^* \circ c_2^* w \overset{\text{cov}}{=\!=} \int_{I^k} c_2^* w$$

φ is orientation preserving. $\Leftrightarrow$ $\det(\varphi') > 0$

Suppose $w \in \Omega^k(M)$, choose POI, φ_i subordinate. to open sets that can be covered by good cubes as above.

Define $\int_M w = \sum_i \int_M \varphi_i w$

Comment:

① if M compact, then $\{\varphi_i\}$ could be finite, so finite sum,

no issue

② In general, first define "integrable forms"

Integrable: $\sum_i \int_M \varphi_i |w| < \infty$.

$C^*(\varphi_i w) \in \Omega^k(I^k)$

$\int_{I^k} |F| \, dx_1 \cdots dx_k$

Important: This is independent of PoU.

$$\int_M^{(\varphi_i)} w \sim \int_M^{(\psi_j)} w$$

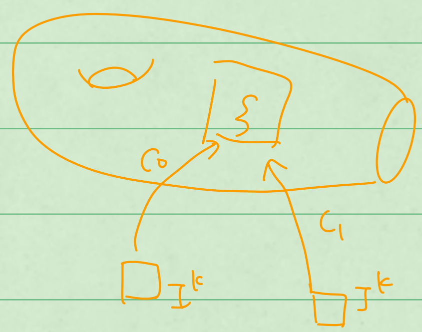
$$\int_M^{(\varphi_i)} w = \sum_i \int_M 1 \cdot \varphi_i w = \sum_{i,j} \int_M \varphi_i \psi_j w = \sum_j \int_M 1 \cdot \psi_j w = \int_M^{(\psi_j)} w$$

$1 = \sum_j \psi_j$ $\qquad$ $1 = \sum_i \varphi_i$

Notes: 1. $\int_M w$ is linear in w.

2. $\int_{-M} w = -\int_M w$

"$-M$" same manifold, reverse orientation.



$C_0(x_1, \cdots, x_k) = C_1(1-x_1, \cdots, x_k)$

Stoke's thm:

If M^k is compact / oriented, $w \in \Omega^{k-1}(M)$,

then $\int_M dw = \int_{\partial M} w$

pf: Suppose we know the thm on w's with "small" support

$\mathrm{Sup}(w) \subset \mathrm{Im}(c)$ for some "good" c.

边界的 sup 变大.

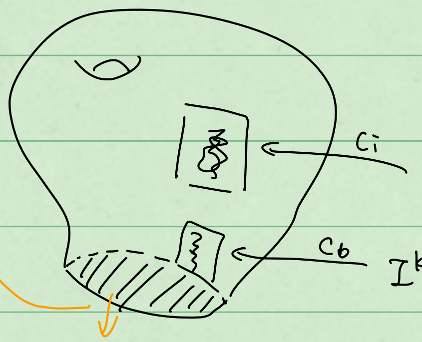
$$\int_{\partial M} w = \sum_i \int_{\partial M} \varphi_i w \xlongequal{\varphi_i w \text{ 小}} \sum_i \int_M d\varphi_i w = \sum_i \int_M d\varphi_i \cdot w + \underbrace{\sum_i \int_M \varphi_i dw}_{= \int_M dw} .$$

take an PoI for M, is also for ∂M

$$\sum_i \int_M d\varphi_i \cdot w = \int_M d(\sum_i \varphi_i) w = \int_M (d1) w = 0 .$$

finite sum

Now to prove "small" case



I^k (inside, case I)

I^k (boundary, Case B)

边界就是$(C_b)_{(1,0)}$的反方向.

Case I: $\mathrm{supp}(w) \subset \mathrm{int}(M)$, can be covered by C_i.

Case B: $\mathrm{Supp}(w) \cap \partial M \neq 0$, 〃 — — — —— —— —

Case I: $\int_{\partial M} w = 0$, 因为在内部

$$\int_M dw = \int_{C_i} dw = \int_{I^k} c_i^*(dw) = \int_{I^k} d(c_i^* w)$$

$$= \int_{\partial I^k} c_i^* w = \int_{C_{i*} I^k} w = 0 \quad (\text{因为 } C_i > w)$$

Case B: $\int_M dw = \int_{\partial I^k} c_b^* w = \int_{\partial c_b} w$

Choose C_b, such that only $C_b(0, \underbrace{x_2, \cdots, x_k}_{y_1, \cdots, y_{k-1}})$ intersects ∂M.

$$= -\int_{(C_b)_{(1,0)}} w = -\int_{\underbrace{((C_b)_{(1,0)}}_{y_1,\dots,y_{k-1}}} w = \int_{\partial M} w.$$

每个面有自己的符号.

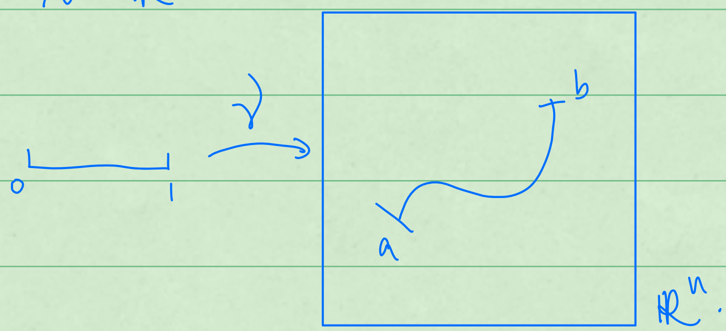
this cube is orientation reversing.

$M^1 \subset \mathbb{R}^1$

e.g.1. $M^1 = [a, b]_x \quad w = f. \quad \partial M = \{b\} \cup (-\{a\}) \quad dw = f' dw.$

$$\int_a^b f' dx = \int_M dw = \int_{\partial M} w = f(b) - f(a)$$

e.g.2: $M^1 \subseteq \mathbb{R}^n$.



$M = \gamma([0,1]) \sim \gamma.$

$w = f : \mathbb{R}^n \to \mathbb{R}.$

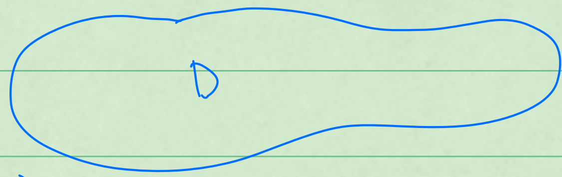
$$f(b) - f(a) = \int_{\partial M} w = \int_{\gamma([0,1])} dw = \int_{\gamma([0,1])} \sum \frac{\partial f}{\partial x_i} dx_i$$

$$= \int_{[0,1]} \sum \frac{\partial f}{\partial x_i}(\gamma(t)) \cdot \gamma_i'(t) \cdot dt = \int_{[0,1]} (\operatorname{grad} f) \cdot \dot{\gamma}(t)\, dt.$$

$$x_i = \gamma_i(t) \Rightarrow dx_i = \gamma_i'(t)\, dt$$

$$\dot{\gamma}(t) = \begin{pmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_n'(t) \end{pmatrix}$$

e.g.3. $\underset{D}{\overset{\shortparallel}{M^2}} \subseteq \mathbb{R}^2_{x,y}$.



$\partial D = \gamma : [0,1] \to \mathbb{R}^n.$

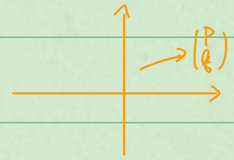
$w = P\,dx + Q\,dy. \quad dw = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$$\int \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy. = \int_D \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \int_M dw \quad) \text{ e.g.2.}$$

$$= \int_{\partial M} w = \int_\gamma P\,dx + Q\,dy = \int_{[0,1]} \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \dot\gamma\,dt$$

two interpertations:

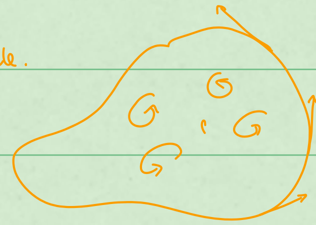
1. $F = \begin{pmatrix} P \\ Q \end{pmatrix}$



$\frac{\partial Q}{\partial x}$: ↑

$\frac{\partial P}{\partial y}$: →

$\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)$: how much the flow want to spin a particle.



← 在边界处的旋转.

2. $Q \to P \quad P \to -Q$.

则 $\int_D \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}\right) = \int_{[0,1]} \begin{pmatrix} -Q \\ P \end{pmatrix} \dot\gamma\,dt$ ← $\begin{pmatrix} P \\ Q \end{pmatrix}$ 转了 $90°$.

$= \int_{[0,1]} \begin{pmatrix} P \\ Q \end{pmatrix} (\dot\gamma$ 顺时针转 $90°)$ ← outward pointing normal.

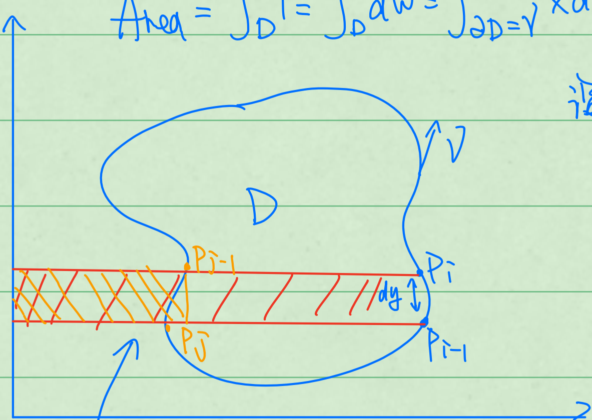
$= \underbrace{\int_{[0,1]} \begin{pmatrix} P \\ Q \end{pmatrix} \cdot \vec{n}}_{\text{how much flow comes out of the domain.}}$ (因为 $\dot\gamma(t)$ 是切向量)

$\operatorname{div}\begin{pmatrix} P \\ Q \end{pmatrix}$

how much flow is created @ here.

Sub. e.g. $w = x\,dy$. $dw = dx \wedge dy$

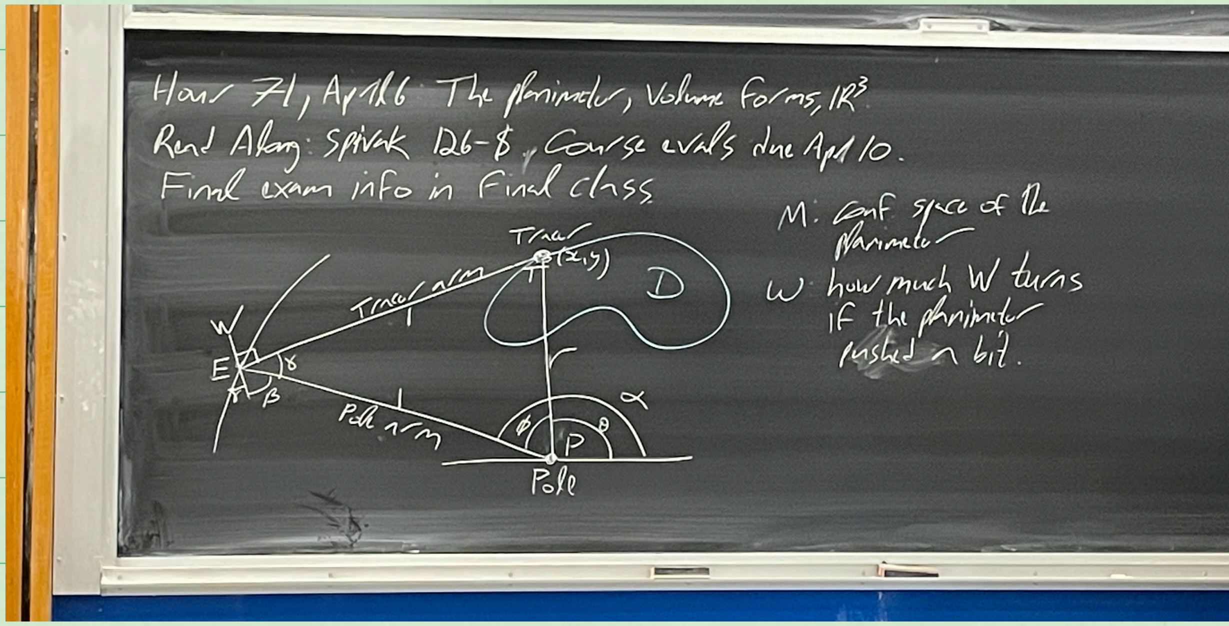
$$\text{Area} = \int_D 1 = \int_D dw = \int_{\partial D = \gamma} x\,dy \sim \sum_{i=1}^{n} x_i (y_i - y_{i-1})$$



红-橙.

沿着 γ 的时间

$$\begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ \vdots & \vdots \\ x_n & y_n \end{pmatrix}$$



$\alpha, \beta, \gamma, \varphi, \theta, r, x, y : M \to \mathbb{R}.$

$\omega \in \Omega^1(M)$

$$\int_{\partial D} \omega = \int_D d\omega \qquad \text{“}\omega\text{”?}$$

$\omega = (d\alpha)\cos\gamma \qquad \gamma = \pi - 2\varphi$

W与α变化同步

$$= \cos(\pi - 2\varphi)\, d(\theta + \varphi)$$

$$= -\cos(2\varphi)\cdot d(\theta+\varphi)$$

$$d\omega = 2\sin(2\varphi)\, d\varphi \wedge d\theta.$$

$$= \underbrace{2\cos\varphi}_{r}\ \underbrace{2\sin\varphi\, d\varphi}_{-dr}\, d\theta.$$

$$= -r\,dr\,d\theta = -dx\,dy.$$

$$= \int_D -dx \wedge dy = -\text{Area}.$$

The $\mathbb{R}^3$ thm:

$$\int_{M^3} \operatorname{div} F \underset{\text{volume form}}{dV} = \int_{\partial M^3} F\cdot \overset{\text{norm}}{n}\, \overset{\text{area form}}{dA} \qquad \text{“Gauss”}$$

$$\int_{M^2} (\operatorname{curl} F)\, n\, dA = \int_{\partial M^2} F\cdot \overset{\text{tangent}}{T}\, ds. \qquad \text{“Stokes”}$$

M^k oriented manifold in $\mathbb{R}^n$, with orientation given by

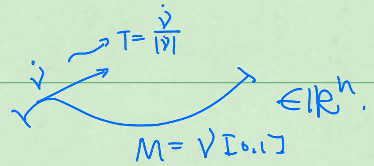
$\eta = \Omega^k(M)$. $\quad \eta \neq 0$.

Single symble. { dV : "The volume form on M. , is that form for which

$dV(\xi_1, \xi_2, \cdots \xi_k) = 1$ if $\xi_1, \cdots \xi_k$ make a positive orthonormal

basis of T_pM.

e.g 1 :



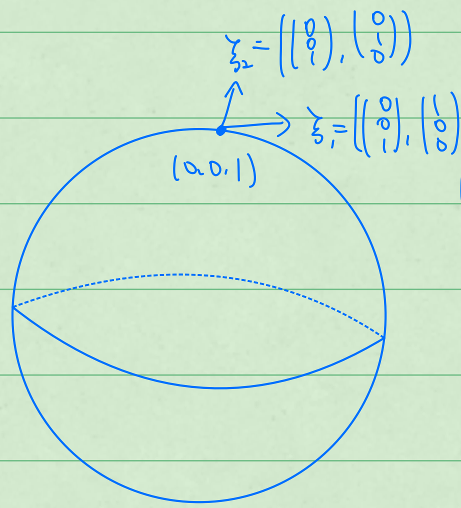
dV? $\qquad dV \in \Omega^1(M)$

$\qquad\qquad\quad \| $

$\qquad\qquad\quad dl = ds$

$ds(T) = 1$

e.g.



On $S^2 \subseteq \mathbb{R}^3_{xyz}$.

$$dV(\xi_1, \xi_2) = dA(\xi_1, \xi_2) \overset{def}{=} 1$$

$$dx \wedge dy(\xi_1, \xi_2) = 1$$

$\Rightarrow$ at $N = (0,0,1)$, $dA = dx \wedge dy$

$M^2 \subset \mathbb{R}^3$ oriented, Let $n(x)$ $x \in M$, be the positive unit normal to M.

Def: $n: M \to \bigcup_{x \in M} T_x\mathbb{R}^3$

s.t. • $n(x) \in T_x\mathbb{R}^3$

• $n(x) \perp T_x M$

• $|n(x)| = 1$

• if u, v are tangent to M at x, s.t. (u, v) is positive relative to the orientition of M, Then n, u, v is a positive basis of $\mathbb{R}^3$.

dA : volume form on M.

$$dA(u,v) = \begin{vmatrix} - & u & - \\ - & v & - \\ - & n & - \end{vmatrix}$$

$u, v \in T_x M$