L^p spaces:

Let $1 \le p \le \infty$. Given a measure space (X, μ)

$f: X \to \mathbb{R}(\mathbb{C})$ measurable.

Def: $\|f\|_p = \begin{cases} (\int |f|^p d\mu)^{\frac{1}{p}} & 1 \le p < \infty \\ \operatorname{esssup} |f| & p = \infty \end{cases}$

Def: $L^p(d\mu) := \{f: X \to \mathbb{R}/\mathbb{C} \text{ mable}, \|f\|_p < \infty\}/\sim$

$f \sim g \Leftrightarrow f = g \ \mu\text{.a.e.}$

• Vector space • $\|\cdot\|_p$ a norm.

Prop: Let $f, g \in L^p(d\mu)$, then $\|f+g\|_p \le \|f\|_p + \|g\|_p$

Pf: Consider the unit ball: $B = \{f \in L^p, \|f\|_p \le 1\}$.

① B is convex$\overset{p<\infty}{.}$ Let $f, g \in B$, $\lambda \in (0,1)$, we estimate

$\|(1-t)f + tg\|_p^p = \int ((1-t)f + tg)^p d\mu \le \int ((1-t)|f| + t|g|)^p d\mu$

Let $h(y) = y^p$, convex. $\Longrightarrow \le \int (1-t)|f|^p + t|g|^p d\mu \le 1$

$p = \infty$ case similar

② $\forall f, g$, if $\|f\|_p = 0$ or $\|g\| = 0$, ✓

else: $f + g = (\|f\| + \|g\|)\Big(\underbrace{\underbrace{\frac{\|f\|}{\|f\| + \|g\|}}_{t} \frac{f}{\|f\|} + \underbrace{\frac{\|g\|}{\|f\| + \|g\|}}_{1-t} \frac{g}{\|g\|}}_{= u \in B} \Big)$

$\Rightarrow \|f+g\| = (\|f\| + \|g\|)\overset{\le 1}{\|u\|}$ ✓

Hölder's Inequality:

Given $1 \le p \le \infty$, Define p' s.t. $\frac{1}{p} + \frac{1}{p'} = 1$ $\overset{1 \leftrightarrow \infty}{}$ (dual exponent)

Then, $\forall f \in L^p$, $g \in L^{p'}$, $|\int fg \, d\mu| \le \|f\|_p \|g\|_{p'}$ $\quad \int fg \le (\int f^p)^{\frac{1}{p}} (\int g^{p'})^{\frac{1}{p'}}$

Pf: $p = 1, \infty$: $|\int fg \, d\mu| \le \int |f||g| d\mu \le \operatorname{esssup}|f| \int |g| d\mu = \|f\|_\infty \|g\|_1$

fix $p \in (1, \infty)$

<u>claim</u>: $\forall a,b>0,\ \frac{1}{p}a^p + \frac{1}{p'}b^{p'} \geq ab.$ Young's / Convexity inequality

Consider $h(x) = bx - \frac{1}{p}x^p$, concave in x

$h'(x) = b - x^{p-1}$ max when $b^{\frac{1}{p-1}}$

$h(b^{\frac{1}{p-1}}) = b^{1+\frac{1}{p-1}} - \frac{1}{p}b^{\frac{p}{p-1}} = (1-\frac{1}{p})b^{\frac{p}{p-1}} = \frac{1}{p'}b^{p'}$

$\frac{p}{p-1} = p' \Leftarrow p = p'(p-1)$

$\Leftarrow p+p' = pp'$

$\Leftarrow \frac{1}{p}+\frac{1}{p'} = 1$ ✓.

$\Rightarrow \forall x,\ h(x) \leq \frac{1}{p'}b^{p'} \overset{x=a}{\Longleftrightarrow} ba - \frac{1}{p}a^p \leq \frac{1}{p'}b^{p'}$

<u>claim</u>: $ab = \min_{t>0} \frac{1}{p}t^p a^p + \frac{1}{p'}t^{-p'}b^{p'}$ Sharp Young

$t^pa^p = t^{-p'}b^{p'}$

$t^{p+p'} = b^{p'}a^{-p}$

$(t^{p+p'})^{\frac{p}{p+p'}} = b^{\frac{pp'}{p+p'}}a^{-\frac{p^2}{p+p'}}$

$\Rightarrow t^pa^p = b\,a^{\frac{p^2+pp'}{p+p'}-\frac{p^2}{p+p'}} = ab$. , $t^{-p'}b^{p'}$ similar.

① Take $h(t) = \frac{1}{p}t^pa^p + \frac{1}{p'}t^{-p'}b^{p'}$ convex

$h'(t) = \frac{1}{t}(t^pa^p - t^{-p'}b^{p'})$, vanishes at $(ta)^p = (\frac{b}{t})^{p'}$

$|\int fg\,d\mu| \overset{①}{\leq} \int |f||g|\,d\mu \overset{\text{sharp}}{\leq} \int \inf_{t>0}\left(\frac{1}{p}t^p|f|^p + \frac{1}{p'}t^{-p'}|g|^{p'}\right)$ $\quad \int \inf(\frac{1}{p}t^p|f|^p) \leq \inf \int \frac{1}{p}t^p|f|^p$

$\overset{②}{\leq} \inf_{t>0}\left(\frac{1}{p}t^p\|f\|_p^p + \frac{1}{p'}t^{-p'}\|g\|_{p'}^{p'}\right)$ $\quad = \inf \frac{1}{p}t^p\|f\|_p^p$

$\overset{\text{sharp}}{=} \|f\|_p\|g\|_{p'}$

<u>Holder's ineq suggests the following question:</u>

Given $g \in L^{p'}$

<u>Define</u> $L: L^p \to \mathbb{R}$ by $L(f) = \int fg\,d\mu$.

- Linear
- continuous.
- $\underset{(L^p)^*}{\|L\|} = \sup_{f\neq 0}\frac{|L(f)|}{\|f\|_p} \overset{\text{claim: "="}}{\leq} \sup \frac{\|f\|_p\|g\|_{p'}}{\|f\|_p} = \|g\|_{p'}$

Given $g \in L^{p'}$, $\|g\|_{p'} > 0$, consider $f := |g|^{p'-1}\overline{\operatorname{sign}(g)}$ $\quad \operatorname{sign} = \begin{cases}+1\\0\\-1\end{cases}$ / $\operatorname{sign}(re^{i\theta}) = e^{i\theta}$

$|f|^p = |g|^{p(p'-1)} = |g|^{p'} \Rightarrow \int|f|^p = \int|g|^{p'} < \infty$

$\int fg\,d\mu = \int |g|^{p'-1}\underbrace{\overline{\operatorname{sign}(g)}g}_{|g|}\,d\mu$

$= \int|g|^{p'} = \|g\|_{p'}^{p'}$

Plug in $\frac{L(f)}{\|f\|_p} = \frac{\int fg}{\|f\|_p} = \frac{\|g\|_{p'}^{p'}}{\|g\|_{p'}^{p'/p}} = \|g\|_{p'}^{p'(1-\frac{1}{p})} = \|g\|_{p'}$

So $L^{p'} \hookrightarrow (L^p)^*$ injective.

$g \longrightarrow L_g,\quad L_g(f) = \int fg$

surjective? $\begin{cases} \text{Yes} & p<\infty \\ \text{No} & p=\infty \end{cases}$

Equality in ① : $\Leftrightarrow \exists t$, s.t. $fg = e^{it}|fg|$

Equality in $\int |f||g| d\mu \le \frac{1}{p}\|f\|_p + \frac{1}{p'}\|g\|_{p'}$

$\Leftrightarrow |f|^p = |g|^{p'}$

Equality in ② : $\exists t$ constant, $(t|f|)^p = (t^{-1}|g|)^{p'}$ a.e.

<u>Minkowski's inequality</u>:

Special case: triangle inequality.

Given (X,μ), (Y,ν) s.t. Fubini holds. $f: X\times Y \to \mathbb{R}$ measurable.

$x \xrightarrow{f(\cdot,y)} f(x,y)$ $\quad \| \int_Y f(\cdot,y) d\nu(y) \|_{L^p(d\mu)} \le \int \|f(\cdot,y)\|_{L^p(d\mu)} d\nu(y)$ $\quad$ ($x \mapsto \int f(x,y)dy$) $\quad$ $f(\cdot,y)$: $x \to f(x,y) = k_y(x)$

That is, $\left(\int_X \left| \int_Y f(x,y)\, dy \right|^p dx \right)^{\frac{1}{p}} \le \int_Y \left(\int_X |f(x,y)|^p d\mu(x) \right)^{\frac{1}{p}} dy$.

<u>Pf</u>: $f \ge 0$ Set $H(x) = \int_Y f(x,y) d\nu(y)$

$(LHS)^p = \int_X H^p(x) dx = \int_X H^{p-1}(x) \int_Y f(x,y)\, dy\, dx$

$\overset{\text{Fubini}}{=} \int_Y \int_X H^{p-1}(x) f(x,y)\, dx\, dy$

$\overset{\text{Holder}}{\le} \int_Y \|H\|_p^{p-1} \left(\int_X f(x,y)^p dx \right)^{\frac{1}{p}} dy$.

$\|H\|_p \le \int_Y \left(\int f(x,y)^p dx \right) dy$

<u>Recall</u>: Equality in Holder's ineq

$p=1$, $p'=\infty$ $\quad \left|\int fg\right| \le \int |f||g| \le \operatorname{esssup}|g| \int |f| = \|g\|_\infty \|f\|_1$

equality: $|g(x)| = \operatorname{esssup}|g|$ a.e on $\{x \mid f(x) \neq 0\}$.

<u>Minkowski's ineq</u>:

$\left(\int_X \left| \int_Y f(x,y) d\nu(y) \right|^p d\mu(x) \right)^{\frac{1}{p}} \le \int_Y \left(\int |f(x,y)|^p d\mu(x) \right)^{\frac{1}{p}} d\nu(y)$

Let $Y = \{1,2\}$, $\nu(1) = \nu(2) = \frac{1}{2}$ $\Rightarrow$ triangle ineq

Key step:

$$\|H\|_p^p = \int_X H(x)^{p-1}\int_Y f(x,y)\,d\nu(y)\,d\mu(x) \overset{\text{Fbn}}{=} \int_Y\int_X \overset{\in L^{p'}}{H(x)^{p-1}}\ \overset{\in L^p}{f(x,y)}\,d\mu(x)\,d\nu(y)$$

Apply Holder.

Equality: $H^{(p-1)p'}$ constant multiple of f $|f(x,y)|^p$

$|f(x,y)| = h(y)\, H^{(p-1)\frac{p'}{p}}$ $f(x,y) = h(x)\, l(y)$

$p=1$ case: $\int_X\int_Y f(x,y)\,d\nu(y)\,d\mu(x) \overset{\text{Fbn}}{=} \int_Y\int_X f(x,y)\,d\mu(x)\,d\nu(y)$

So $f \geq 0 \Rightarrow$ equility always hold.

Thm: Let (X,μ) measure space, $1\leq p\leq\infty$, then $L^p(\mu)$ compute.

Pf: Given a cauchy $\{x_n\}$

Key: Choose a ("fast" cauchy sequence) subsequence $\{x_{n_k}\}$

s.t. $\|x_{n_k} - x_m\| \leq 2^k$ $\forall m \geq n_k$.

Idea: Write $f_{n_k} = f_{n_1} + \sum_{i=1}^{k} (f_{n_i} - f_{n_{i-1}})$, $\sum \|f_{n_i} - f_{n_{i-1}}\|$ converges.

Consider $g_k(x) = |f_{n_1}| + \sum_{i=1}^{k} |f_{n_i} - f_{n_{i-1}}| \in L^p$

$\|g_k\|_p \leq \|f_{n_1}\|+1$
$\Rightarrow \|g\| < \infty$

So $g_k \nearrow g$ ptwise by MCT.

$\|g_k\|_p^p = \int g_k^p\,d\mu = \int |g_k(x)|^p\,dx$ $\lim g_k(x)$ exists $\forall x$ (may be ∞)

$\int |g|^p = \lim \int |g_k|^p$ by MCT

$\sqrt{}$?

Hanner's inequality: Let $1<p<\infty$

① If $1<p\leq 2$, then

$$\|f+g\|_p^p + \|f-g\|_p^p \geq (\|f\|_p + \|g\|_p)^p + |\,\|f\|_p - \|g\|_p\,|^p$$

② If $2\leq p<\infty$, ... "$\leq$" ....

In particular, $\|f+g\|_2^2 + \|f-g\|_2^2 = 2(\|f\|_2^2 + \|g\|_2^2)$

"parallelogram"

Plug in $f \to f+g$, $g \to f-g$

$$\Rightarrow 2^p(\|f\|_p^p + \|g\|_p^p) \overset{\geq}{\leq} (\|f+g\|_p + \|f-g\|_p)^p + |\,\|f+g\|_p - \|f-g\|_p\,|^p$$

Pf: $p=2$:

LHS $\|f+g\|_2^2+\|f-g\|_2^2=\int (f+g)(\bar f+\bar g)+(f-g)(\bar f-\bar g)\,d\mu$

$$=\int 2|f|^2+2|g|^2$$

$$=2(\|f\|_2^2+\|g\|_2^2)$$

$$=(\|f\|_2+\|g\|_2)^2+(\|f\|_2-\|g\|_2)^2$$

$p\geq 2$: Assume $\|f\|=1\geq R=\|g\|>0$

Construct a function $F:[0,1]\to\mathbb{R}$

$F(r)=\alpha(r)+\beta(r)R^p$ $(\alpha,\beta$ TBD$)$ $\quad \min\limits_{0\leq r\leq 1} F(r)\overset{①}{=}(1+R)^p+(1-R)^p$

Moreover: $\forall A,B\in\mathbb{C}$, will show that

$$|A+B|^p+|A-B|^p\overset{②}{\leq}\alpha(r)|A|^p+\beta(r)|B|^p$$

Given such F, we estimate

$$\|f+g\|_p^p+\|f-g\|_p^p=\int |f(x)+g(x)|^p+|f(x)-g(x)|^p$$

$$\overset{②}{\leq}\int \alpha(r)|f(x)|^p+\beta(r)|g(x)|^p \qquad (\forall r\in[0,1])$$

$$=\alpha(r)\|f\|_p^p+\beta(r)\|g\|_p^p$$

$$=\alpha(r)+\beta(r)R^p$$

$$\overset{①}{\leq}(1+R)^p+(1-R)^p$$

Now to construct F

Choose $\alpha(r)=(1+r)^{p-1}+(1-r)^{p-1}$

$$\beta(r)=r^{1-p}\left((1+r)^{p-1}-(1-r)^{p-1}\right)$$

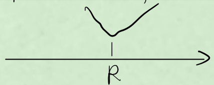
$F'(r)=\alpha'+\beta' R^p$

$$=(p-1)\Big((1+r)^{p-2}-(1-r)^{p-2}+R^p\big(-r^{-p}((1+r)^{p-1}+(1-r)^{p-1})$$
$$-r^{1-p}((1+r)^{p-2}+(1-r)^{p-2})\big)\Big)$$

$$\frac{F'}{p-1}=(1+r)^{p-2}\left(1-\left(\frac{R}{r}\right)^p(1+r-r)\right)+(1-r)^{p-2}\left(-1+\left(\frac{R}{r}\right)^p(1-r+r)\right)$$

$$=\left(1-\left(\frac{R}{r}\right)^p\right)\left((1+r)^{p-2}-(1-r)^{p-2}\right)$$

$F'(R)=0$, $\quad F'(r)\begin{cases}<0 & \text{if } 0<r<R\\ >0 & \text{if } R<r<1\end{cases}$



plug in $r=R$ $\Rightarrow$ get ①.

For ②, NTS if $A \cdot B > 0$, then can rescale from ①.

$A = |A| e^{i\theta}$, $B = |B| e^{i\varphi}$, rotate s.t. $\varphi = 0$.

$$|A+B|^p + |A-B|^p = \left(|A|^2 + |B|^2 + 2\cos\theta |A||B|\right)^{\frac{p}{2}} + \left(|A|^2 + |B|^2 - 2\cos\theta |A||B|\right)^{\frac{p}{2}}$$

$$= 2\left(\left(|A|^2 + |B|^2 + 2\cos\theta |A||B|\right)^{\frac{p}{2}} + \left(|A|^2 + |B|^2 - 2\cos\theta |A||B|\right)^{\frac{p}{2}}\right)/2$$

$t \to t^{\frac{p}{2}}$ convex

get maximum when $\cos\theta = 1 \Rightarrow \theta = 0$

Thm: (Projection onto closed convex sets in L^p)

Let (X, μ) measure space, $1 < p < \infty$,

$C \subseteq L^p$ convex closed.

(Important special case: C is closed linear subspace)

Let $f \in L^p \setminus C$, let $d := \inf_{g \in C} \|f - g\| > 0$

Thm: $\exists! \, g_0 \in C$, s.t. $\|f - g_0\| = d$. "projection" of f.

Pf: $2 \le p < \infty$

Step 1: Construct a minimizing sequence $\{g_k\}$ in C,

i.e. $\lim_{k \to \infty} \|f - g_k\|_p = d$

Step 2: Get a converged subsequence $g_{k_j} \longrightarrow g_0$

Step 3: Show $\|f - g_0\|_p = d$

In our case, also to show uniqueness.

Given $f \in L^p$, by translating C to $\{g - f \mid g \in C\}$,

we can take $f = 0$.

Consider $\{g_k\}$ minimizing sequence $\lim_{k \to \infty} \|g_k\| = d$

By H ineq, $\forall k, l \in \mathbb{N}$,

$$\|g_k + g_l\|_p^p + \|g_k - g_l\|_p^p \le \left(\|g_k\|_p + \|g_l\|_p\right)^p + \left(\|g_k\|_p - \|g_l\|_p\right)^p$$

As $k, l \to \infty$ $\quad 2^p \left\|\frac{g_k + g_l}{2}\right\|_p^p \ge 2^p d^p \quad (2d)^p \quad 0$ $\quad \frac{g_k+g_l}{2} \in C$

So this term sufficiently small $\Rightarrow \{g_k\}$ Cauchy.

$\Rightarrow \{g_k\}$ converges, $\|g_0\|_p = \lim\limits_{\infty} \|g_k\|_p = d$.

What can we say about the projection of f onto C?

We know $\|f-g\|_p^p \geq \|f-g_0\|_p^p$, $\forall g \in C$.

In particular, $\|f-((1-t)g_0+tg)\|_p^p \geq \|f-g_0\|_p^p$ for $0 \leq t \leq 1$

$0 \leq \frac{d}{dt}\|f-((1-t)g_0+tg)\|_p^p\Big|_{t=0} = \frac{d}{dt}\int |f-g_0+t(g-g_0)|(\ )$

Differentiate the integrand

$$\frac{d}{dt}\left(|f-g_0+t(g-g_0)|^2\right)^{\frac{p}{2}}\Big|_{t=0} = \frac{p}{2}|f-g_0|^{p-2}\frac{d}{dt}\left(f-g_0+t(g-g_0)\right)\overline{\left(f-g_0+t(g-g_0)\right)}\Big|_{t=0}$$

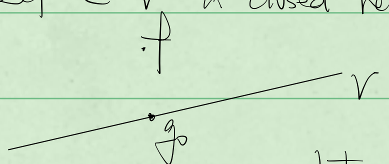
$$= \frac{p}{2}|f-g_0|^{p-2}\, 2\,\mathrm{Re}(g-g_0)\overline{(f-g_0)}$$

$$= p\,\mathrm{Re}\left(|f-g_0|^{p-2}\overline{f-g_0}\right)(g-g_0)$$

New condition:

$$\mathrm{Re}\int\left(|f-g_0|^{p-2}\overline{f-g_0}\right)(g_0-g)\,d\mu \geq 0. \quad \forall g \in C.$$ Necessary condition for g_0.

Let $C = V$ a closed vector subspace of L^p, $f \notin V$.



$$\mathrm{Re}\int\left(|f-g_0|^{p-2}\overline{f-g_0}\right)(g_0-g)\,d\mu \geq 0, \quad \forall g$$

Let $g \to g+g_0$ $\quad \mathrm{Re}\int\left(|f-g_0|^{p-2}\overline{f-g_0}\right)g\,d\mu \geq 0$

$g \to -g$ $\quad \mathrm{Re}\int\left(|f-g_0|^{p-2}\overline{f-g_0}\right)g\,d\mu = 0$

$g \to ig$ $\quad \int\left(|f-g_0|^{p-2}\overline{f-g_0}\right)g\,d\mu = 0$

Riesz Representation Thm:

(X,μ), Let $1 \leq p < \infty$. If $p=1$, assume μ is σ-finite.

Then $(L^p)^* = L^{p'}$

The canonical embedding $L^{p'} \hookrightarrow (L^p)^*$, $g \longmapsto T_g$

where $T_g(f) = \int fg\,d\mu$.

Pf: Let $T \in (L^p)^*$, i.e. $T: L^p \to \mathbb{C}$, linear, cont.

Consider $N = \mathrm{null}(T) = \{h \in L^p \mid T(h) = 0\}$. (N is closed since T cont.)

If $N = L^p$, then $T \equiv 0$.

Otherwise, pick $f_0 \notin N$. Take $h_0 \in N$, s.t.

$d(f_0, N) = \|f_0 - h_0\|$. By rescaling and translation we may

WLOG assume $h_0 = 0$, $\|f_0\|_p = 1$. I.e. the projection of f_0 on N is 0.

We proved that $\int |f_0|^{p-1} \overline{\operatorname{sign} f_0}\, h \, d\mu = 0$, $\forall h \in N$.

$$\text{Set } g = \underset{\text{constant}}{C} |f_0|^{p-1} \overline{\operatorname{sign} f_0}$$

Then $\int g h = 0$, $\forall h \in N$. $\int g f_0 = C \int |f_0|^{p-1} \overline{\operatorname{sign} f_0}\, |f_0| \operatorname{sign} f = C \int |f_0|^p = C$.

Set $C = T f_0$.

$$\text{So } \forall f \in L^p, \quad f = \overbrace{f - \frac{Tf}{Tf_0} f_0}^{\in N(T)} + \overbrace{\frac{Tf}{Tf_0} f_0}^{\in \operatorname{span}\{f_0\}}$$

$$\int fg = \underbrace{\int g\left(f - \frac{Tf}{Tf_0} f_0\right)}_{0} + \underbrace{\int g\left(\frac{Tf}{Tf_0} f_0\right)}_{Tf}$$

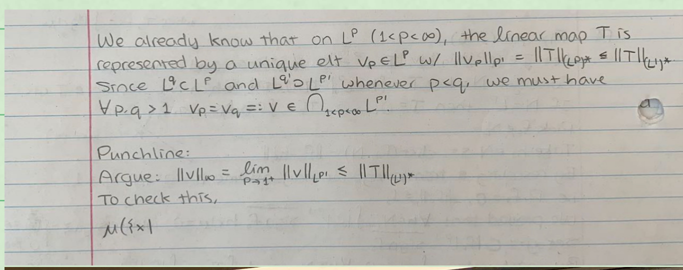
$$= Tf.$$

So $\exists g$, s.t. $T = T_g$. ✓

For $p = 1$, firstly consider the case where X finite measure.

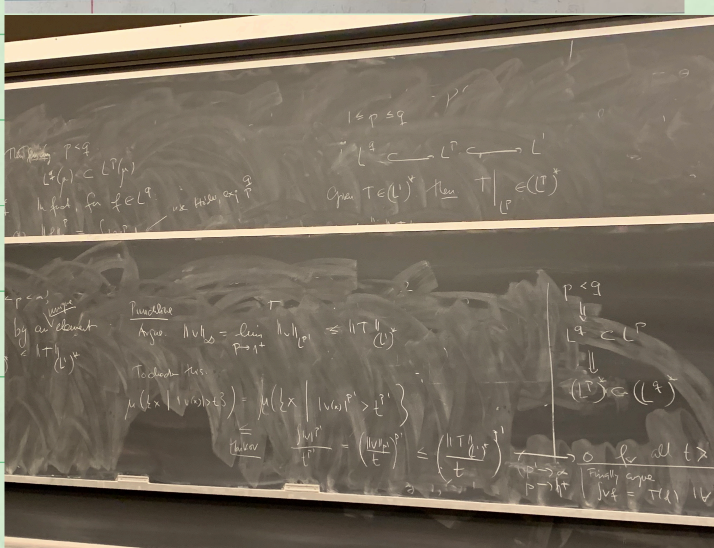
WLOG $\mu(X) = 1$.

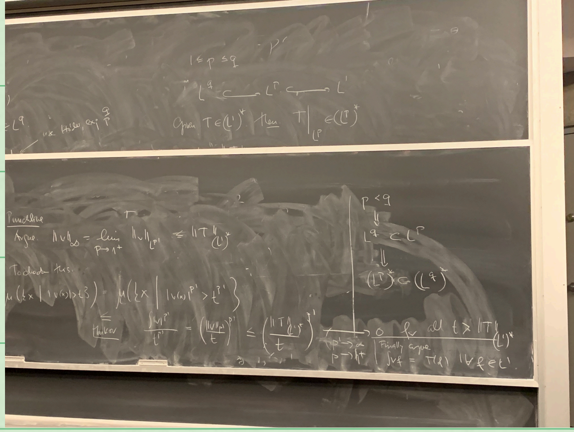
$$\left(p < q \Rightarrow L^q \subseteq L^p \Rightarrow (L^p)^* \subseteq (L^q)^*\right.$$



We already know that on L^p $(1<p<\infty)$, the linear map T is represented by a unique elt $v_p \in L^{p'}$ w/ $\|v_p\|_{p'} = \|T\|_{(L^p)^*} \leq \|T\|_{(L^1)^*}$.
Since $L^q \subset L^p$ and $L^{q'} \supset L^{p'}$ whenever $p<q$, we must have
$\forall p, q > 1$ $v_p = v_q =: v \in \bigcap_{1<p<\infty} L^{p'}$.

Punchline:
Argue: $\|v\|_\infty = \lim_{p \to 1^+} \|v\|_{L^{p'}} \leq \|T\|_{(L^1)^*}$
To check this,
$\mu(\{x |$



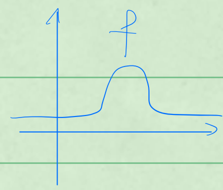


• $\|f\|_p = \sup_{\substack{\|g\|_{p'}=1 \\ g\in L^{p'}}} \left|\int fg\right| = \sup \operatorname{Re}\int fg \qquad (1<p<\infty)$

$g = \frac{|f|^{p-1}\,\overline{\operatorname{sign} f}}{\|f\|_p^{p-1}}$ → actually max.

• $p=1$ case: $\|f\|_1 = \int |f|\, d\mu = \int f\, \overline{\operatorname{sign} f}\, d\mu$.

$p=\infty$: $\|f\|_\infty = \sup_{\|g\|=1} \left|\int fg\, d\mu\right|$



$g_t = \frac{\chi_{\{f\ge t\}}\, \overline{\operatorname{sign} f}}{\mu\{f\ge t\}}$

Consider $1\le p_0<p_1<\infty$, $p\in(p_0,p_1)$

Lemma: $L^{p_0}\cap L^{p_1} \subseteq L^p \subseteq L^{p_0}+L^{p_1}$ (Continuous embeddings)

Def: $L^{p_0}+L^{p_1} = \{f \mid f=f_0+f_1,\ f_0\in L^{p_0},\ f_1\in L^{p_1}\}$.

$\|f\|_{L^{p_0}+L^{p_1}} = \inf \|f_0\|_{p_0}+\|f_1\|_{p_1}$

Pf: $L^{p_0}\cap L^{p_1}\subseteq L^p$

$p_0<p<p_1 \iff \frac{1}{p_1}<\frac{1}{p}<\frac{1}{p_0}$

Find $t\in(0,1)$ s.t. $\frac{1}{p} = \frac{(1-t)}{p_0}+\frac{t}{p_1} \Rightarrow 1=(1-t)\frac{p}{p_0}+t\frac{p}{p_1}$

Let $f\in L^{p_0}\cap L^{p_1}$.

$\|f\|_p^p = \int |f|^p\, d\mu = \int |f|^{(1-t)p}\,|f|^{tp}$

$\overset{\text{Holder}}{\le} \left(\int|f|^{p_0}\right)^{\frac{(1-t)}{p_0}p} \left(\int |f|^{p_1}\right)^{\frac{t}{p_1}p}$

$= \|f\|_{p_0}^{(1-t)p}\,\|f\|_{p_1}^{tp}$

$\Rightarrow \|f\|_p \le \|f\|_{p_0}^{(1-t)}\,\|f\|_{p_1}^{t}$

NTS $L^p\subseteq L^{p_0}+L^{p_1}$. Given $f\in L^p$, need to decompose $f=f_0+f_1$

Take $f_0=(|f|-1)_+\operatorname{sign}f \qquad f_1=\min(|f|,1)\operatorname{sign} f$

Markov's ineq: $\mu\{x\mid |f(x)|\ge t\} \overset{☆}{\le} \frac{\int|f|^p}{t^p} = \left(\frac{\|f\|_p}{t}\right)^p$

$\mu\{x\mid |f(x)|\ge t\} = \mu\{|f|^p\ge t^p\} \le \frac{\int|f|^p}{t^p}$

We estimate $\|f_0\|_{p_0}^{p_0} = \int_0^\infty \mu\{x \mid |f_0(x)|>t\}\, p_0 t^{p_0-1}\,dt$

$$= \int_0^\infty \mu\{x \mid |f(x)|>t+1\}\, p_0 t^{p_0-1}\,dt$$

$$\le \int_0^\infty \|f\|_p^p\, p_0 \frac{t^{p_0-1}}{(t+1)^p}\,dt$$

$$\le \int_0^\infty \|f\|_p^p\, p_0\, t^{-1-(p-p_0)}\,dt < \infty$$

p_0, p_1. Determine p_t by $\frac{1}{p_t} = \frac{1-t}{p_0} + \frac{t}{p_1}$

Know: $\|f\|_{p_t} \le \|f\|_{p_0}^{1-t}\|f\|_{p_1}^{t}$

$T: L^{p_0}_{d\mu} + L^{p_1} \to L^{q_0}_{d\nu} + L^{q_1}$, s.t. $\|Tf\|_{q_0} \le \|T\|_{p_0\to q_0}\|f\|_{p_0}$

$$\|Tf\|_{q_1} \le \|T\|_{p_1\to q_1}\|f\|_{p_1}$$

Riesz-Thorin Theorem: (X,μ), (Y,ν) semi-finite

Let $p_0, p_1, q_0, q_1 \in [1,\infty]$ Fix $t\in(0,1)$ let $p(t), q(t)$ be

$$\frac{1}{p(t)} = \frac{(1-t)}{p_0} + \frac{t}{p_1}, \quad \frac{1}{q(t)} = \frac{(1-t)}{q_0} + \frac{t}{q_1}$$

Then $\|T\|_{L^{p(t)}\to L^{q(t)}} \le \|T\|^{1-t}_{L^{p_0}\to L^{q_0}}\, \|T\|^{t}_{L^{p_1}\to L^{q_1}}$

$\|T\|_{p_t\to q_t} = \sup \frac{\|T(f)\|_{q(t)}}{\|f\|_{p(t)}}$ $\qquad$ good: $\|T(f)\|_{q(t)} \le \|Tf\|_{q_0}^{1-t}\, \|Tf\|_{q_1}^{t}$

bad: $\|f\|_{p(t)} \le \|f\|_{p_0}^{1-t}\, \|f\|_{p_1}^{t}$

Key observation: $t \longrightarrow \left(\int |f|^{p(t)}\right)^{\frac{1}{p(t)}}$, $\quad t \longrightarrow a^t = e^{t\log a}$

analytic, holomorphic over $\mathbb{C}$

$p(t) = \left(\frac{1-t}{p_0} + \frac{t}{p_1}\right)^{-1}$

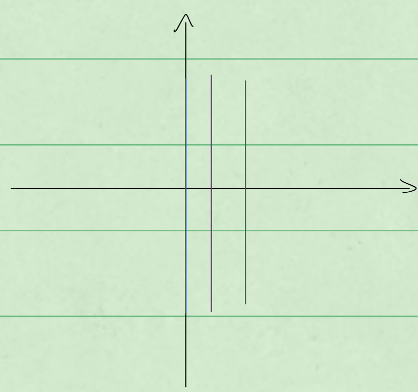
3-line lemma: Let $S = \{z\in\mathbb{C} \mid 0 < \mathrm{Re}(z) < 1\}$

$\varphi: S \to \mathbb{C}$ holomorphic, cont on $\bar{S}$

Assume that $|\varphi(z)| \le M_0$, if $\mathrm{Re}(z) = 0$

$|\varphi(z)| \le M_1$ if $\mathrm{Re}(z) = 1$

Then $\sup_{\mathrm{Re}(z)=t} |\varphi(z)| \le M_0^{1-t} M_1^{t}$



Pf: WLOG $M_0 = M_1 = 1$, otherwise consider $\psi(z) = \frac{\varphi(z)}{M_0^{1-z} M_1^{z}}$

Consider $\varphi_\varepsilon(z) = \varphi(z)\cdot \exp(\varepsilon(z^2-1))$

Note $|\tilde{\varphi}(z)| \le |\varphi(z)|$ holomorphic, $|\tilde{\varphi}(z)| \to 0$ as $|z| \to \infty$ on S

$\tilde{\varphi}(z) \xrightarrow{\varepsilon \to 0} \varphi(z)$ ptwise.

By maximum principal, $\sup_z |\tilde{\varphi}(z)| \le 1$

Pf: Given $1 < p_0, p_1, q_0, q_1 < \infty$. T.

$T|_{L^{p_0}} : L^{p_0} \to L^{q_0}$, $T|_{L^{p_1}} : L^{p_1} \to L^{q_1}$, bounded linear.

WTS $\|T\|_{p(t)\to q(t)} \le \|T\|^{1-t}_{p_0 \to q_0} \|T\|^{t}_{p_1 \to q_1}$

$\|T\|_{p(t)\to q(t)} = \sup_{\substack{f \in L^{p(t)} \\ \|f\|_{p(t)}=1}} \|Tf\|_{q(t)}$

$\overset{\text{dual repre.}}{=} \sup_{\|f\|_{p(t)}=1} \sup_{\|g\|_{q'(t)}=1} \left| \int T(f) g \, d\mu \right|$ $\quad \frac{1}{q} + \frac{1}{q'} = 1$

WLoG $f = \sum_i^n a_i \chi_{E_i}$, $g = \sum_j^m b_j \chi_{F_j}$

Interpolation: For $0 \le \operatorname{Re} z \le 1$, $\frac{1}{p(z)} = \frac{1-z}{p_0} + \frac{z}{p_1}$

$f_z := |f|^{\frac{p(t)}{p(z)}} \operatorname{sign} f$, $g_z = |g|^{\frac{q(t)}{q(z)}} \operatorname{sign} g$ $\quad (g_t = g,\ f_t = f)$

Clearly: $z \longrightarrow \int T(f_z) g_z$ is holomorphic on $S = \{0 \le \operatorname{Re}(z) \le 1\}$,

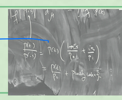
continuous on $\bar{S}$.

Write $\int T(f) g \, d\mu = \int T(\sum_i^m a_i \chi_{E_i})(\sum b_j \chi_{F_j}) \, d\mu$

$= \sum_i \sum_j a_i b_j \int T(\chi_{E_i}) (\chi_{F_j}) \, d\mu$

Note: $f_t = f$, $\|f_0\|_{p_0}^{p_0} = \int |f|^{\frac{p(t)}{p_0} p_0} = \|f\|_{p_t}^{p_t} = 1$

Similarly, $\|f_1\|_{p_1} = \|g_0\|_{q_0'} = \|g_1\|_{q_1'} = 1$

$\|f_{is}\|_{p_0}^{p_0} = \int \left| |f|^{\frac{p(t)}{p_{is}}} \right|^{p_0} d\mu = 1$ 

For $z = is$, $\int T(f_{is}) g_{is} \, d\mu \le \|T\|_{L^{p_0} \to L^{q_0}} \|f_{is}\|_{p_0} \|g\|_{q_0'}$

$\varphi(z) = \int T(f_z) g_z \, d\mu$, $\varphi(1+is) = \left| \int T(f_{is}) g_{is} \, d\mu \right| \le \|T\|_{L^{p_1} \to L^{q_1}} \|f_{1+is}\|_{p_1} \|g_{1+is}\|_{q_1'}$

3-line lemma: $|\varphi(t)| \le \|T\|^{1-t}_{L^{p_0} \to L^{q_0}} \|T\|^{t}_{L^{p_1} \to L^{q_1}}$

Let f = μ-measurable function, complex valued.

The level set of $|f|$ at height α is given by

$E_f(\alpha) = \{x \in X \mid |f| > \alpha\}$,

Distribution function of f:

$\lambda_f(\alpha) = \mu(E_f(\alpha))$

Recall: $\|f\|_p^p = \int_0^\infty \lambda_f(\alpha) p\alpha^{p-1} d\alpha$

Markov ineq: $\mu(E_\alpha) \le \left(\frac{\|f\|_p}{\alpha}\right)^p$

Def: $[f]_p := \inf\{C \in \mathbb{R} \mid \lambda_f(\alpha) \le (\frac{C}{\alpha})^p\} = \sup \alpha (\lambda_f(\alpha))^{\frac{1}{p}}$

weak-$L^p = \{f$ meble, $[f]_p < \infty\}/\mu$.a.e.

By Markov ineq, weak-$L^p \supseteq L^p$

Note: $[cf]_p = |c|\, [f]_p$, but $[\cdot]$ is not a norm.

$\lambda_{f+g}(2\alpha) = \mu(\{x \mid f(x)+g(x) > 2\alpha\}) \le \mu\{x \mid |f(x)| > \alpha$ or $|g(x)| > \alpha\} \le \lambda_f(\alpha) + \lambda_g(\alpha)$

$[f+g]_p \le 2^p([f]_p + [g]_p)$

Def: $f_n \to f$ in weak-L^p $\Leftrightarrow$ $\lim [f_n - f]_p = 0$

Lemma: If $1 \le p_0 < p < p_1 < \infty$, then $(\text{weak-}L^{p_0}) \cap (\text{weak-}L^{p_1}) \subseteq L^p$

Example: $f \in L^1 \Rightarrow$ Hardy-Littlewood $\Rightarrow$ max fn lies in weak-L^1

Pf: Assume C_0, C_1 constant, s.t. $\lambda_f(\alpha) \le (\frac{C_0}{\alpha})^{p_0}$, $\lambda_f(\alpha) \le (\frac{C_1}{\alpha})^{p_1}$, $\forall \alpha > 0$

Write $\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}$, take $a > 0$

We estimate $\|f\|_p^p = \int_0^\infty \lambda_f(\alpha) p\alpha^{p-1} d\alpha$

$$= \int_0^a \lambda_f(\alpha) p\alpha^{p-1} d\alpha + \int_a^\infty \lambda_f(\alpha) p\alpha^{p-1} d\alpha$$

$$\le \int_0^a \left(\frac{C_0}{\alpha}\right)^{p_0} p\alpha^{p-1} d\alpha + \int \left(\frac{C_1}{\alpha}\right)^{p_1} p\alpha^{p-1} d\alpha$$

$$= C_0^{p_0} \frac{p}{p-p_0} a^{p-p_0} + C_1^{p_1} \frac{p}{p_1-p} a^{-(p_1-p)}$$

$$\overset{\min}{\underset{\alpha}{=}} \frac{C_0^{1-t} C_1^t}{(1-t)^{1-t} t^t}$$

Consider a map $T: L^p \to \text{weak-}L^q$

We say T is sublinear if

(1) $|T(f+g)| \le |Tf| + |Tg|$ (pointwise)

(2) $|T(cf)| = |c|\,|T(f)|$ (pointwise)

We say T is strong-type (p,q) if $\|Tf\|_q \le \text{constant}\, \|f\|_p$

- - - - - - - - - - weak-type (p,q) if $[Tf]_q \le \text{constant}\, \|f\|_p$

Thm: (M real-interpretation)

Let $1 \le p_0, p_1, q_0, q_1 < \infty$, $q_0 \ne q_1$, $p_0 < q_0$, $p_1 < q_1$, $0 < t < 1$

Let $L^{p_0} + L^{p_1} \longrightarrow (\text{weak } L^{q_0}) + (\text{weak } L^{q_1})$ be sublinear.

Assume T is weak-type (p_0, q_0) and weak-type (p_1, q_1)

Define $\frac{1}{p} = \frac{1-t}{p_0} + \frac{t}{p_1}$, $\frac{1}{q} = \frac{1-t}{q_0} + \frac{t}{q_1}$

Then T is strong-type (p,q)

Pf: Assume $p_0 < p_1$, ($p_0 = p_1$ case equivalent to the lemma)

For $a > 0$, write $f = f^a + f_a$, where $f^a = (|f| - a)_+ \operatorname{sign} f$, $f_a = \min(|f|, a) \operatorname{sign} f$

$\lambda_{f^a}(\alpha) = \lambda_f(a+\alpha) \le \lambda_f(\alpha)$, $\lambda_{f_a}(\alpha) = \begin{cases} \lambda_f(\alpha) & \alpha \le a \\ 0 & \alpha > a. \end{cases}$

$\lambda_{Tf}(2\alpha) \le \lambda_{Tf^a}(\alpha) + \lambda_{Tf_a}(\alpha)$

We write $\|Tf\|_q^q = \int_0^\infty \lambda_{Tf}(\alpha)\, q\alpha^{q-1}\, d\alpha$

$$\le 2^q \int_0^\infty \left(\lambda_{Tf^a}(\alpha) + \lambda_{Tf_a}(\alpha)\right) q\alpha^{q-1}\, d\alpha$$

$$\le 2^q \int_0^\infty \left(\left(\frac{[Tf^a]}{\alpha}\right)^{q_0} + \left(\frac{Tf_a}{\alpha}\right)^{q_1} \right) q\alpha^{q-1}\, d\alpha$$

$$\le 2^q \int_0^\infty \left(\left(\frac{C_0 \|f\|_{p_0}}{\alpha}\right)^{q_0} + \left(\frac{C_1 \|f\|_{p_1}}{\alpha}\right)^{q_1} \right) q\alpha^{q-1}\, d\alpha \qquad \text{Since } [f]_{q_j} \le C_j \|f\|_{p_j}$$

? Markov

$$\le 2^q q C_0^{q_0} \int_0^\infty \left(\int_{\cancel{0}\, a}^\infty \lambda_{f^{\cancel{a}}}(\beta)\, \beta^{p_0 - 1} p_0\, d\beta \right)^{\frac{q_0}{p_0}} \alpha^{q - q_0 - 1}\, d\alpha$$

$$+ 2^q q C_1^{q_1} \int_0^\infty \left(\int_0^{\cancel{\infty}\, a} \lambda_{f_{\cancel{a}}}(\beta)\, \beta^{p_1 - 1} p_1\, d\beta \right)^{\frac{q_1}{p_1}} \alpha^{q - q_1 - 1}\, d\alpha$$

Key idea: we can take a to depend on α.

$a = \alpha^\sigma$ (① integral: $0 \le \beta \le \alpha^\sigma$; ② - - — — $\alpha^\sigma \le \beta < \infty$) σ, TBD.

First integral: Apply Minkowski's inequality with $r_0 = \frac{p_0}{q_0}$ and integrand

教授挂黑板了....

Important example of real interpolation.

Consider $L^1(\mathbb{R}, m)$

Lebesgue differential thm

Assume $f: \Omega \to \mathbb{R}$ integrable. Then

$$\fint_{B_r(x)} f(y)\,dy = \frac{1}{m(B_r(x))}\int_{B_r(x)} f$$

$$\lim_{r\to 0} \fint_{B_r(x)} f(y)\,dy = f(x) \text{ for a.e. } x.$$

Proof using Hardy-Littlewood maximal fn.

$$Hf(x) = \sup_{r>0} \fint_{B_r(x)} f(y)\,dy.$$

Today's Thm: If $f \in L^p(\mathbb{R}^n)$, then $Hf \in L^p(\mathbb{R}^n)$ for $1<p<\infty$

And $\|Hf\|_p \leq C_{p,n} \|f\|_p$.

Proof: Interpolation between $p=1$, $p=\infty$.

$p=\infty$: $\|Hf\|_\infty = \sup_x \sup_r \fint_{B_r(x)} f(y)\,dy \leq \|f\|_\infty$

$p=1$: Hardy-Littlewood maximum thm: $\exists$ constant $C_n \leq 3^n$,

s.t. $\forall \alpha>0$, $m(\underbrace{\{x \mid Hf(x)>\alpha\}}_{\lambda_{Hf}(\alpha)}) \leq \frac{C_n \|f\|_{1}}{\alpha}$

So Hf is weak-type $(1,1)$, strong-type (∞,∞) therefore weak.

Is H sublinear?

$$H_{cf}(x) = \sup_r \fint_{B_r(x)} |cf|\,dy = c\,Hf(x)$$

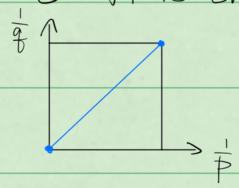
$$H_{f+g}(x) = \sup_r \fint_{B_r(x)} |f(y)+g(y)|\,dy$$

$$\leq \sup_r \fint_{B_r(x)} |f(y)|+|g(y)|\,dy$$

$$\leq \sup_r \fint_{B_r(x)} |f(y)|\,dy + \sup_r \fint_{B_r(x)} |g(y)|\,dy$$

$= Hf(x) + Hg(x)$

So H is sublinear, weak-type $(1,1)$, (∞,∞)



$p_0 = q_0 = 1$

$p_1 = q_1 = \infty$

Hilbert Space

Def: A **Hilbert Space** is a vector space with a norm given by inner product. the vector space is assumed to be complete with that norm.

$L^2(d\mu)$, $\langle f, g\rangle = \int f\bar{g}\, d\mu$

Let V be a vector space

Def: An **inner product** on V is a map: $\langle \cdot, \cdot \rangle$ from $V \times V$ to $\mathbb{C}$ such that

① $\langle x, x\rangle \geq 0$ with "=" $\Leftrightarrow x = 0$, positivity

② $\langle \alpha x + \beta y, z\rangle = \alpha\langle x, z\rangle + \beta\langle y, z\rangle$, linearity

③ $\langle x, y\rangle = \overline{\langle y, x\rangle}$, hermitian.

Norm is $\|x\| = \sqrt{\langle x, x\rangle}$

key inequality: $|\langle x, y\rangle| \leq \frac{1}{2}(\|x\|^2 + \|y\|^2)$, $|\langle x, y\rangle| \leq \|x\|\,\|y\|$.

Triangular inequality $\|x+y\| \leq \|x\| + \|y\|$

Parallelgram Identity:

$\|x+y\|^2 + \|x-y\|^2 = 2(\|x\|^2 + \|y\|^2)$

Consequences: If H a Hilbert Space, $C \subseteq H$ closed, convex,

$x \notin C$, then $\exists! y_0 \in C$ s.t. $\|x-y_0\| = \inf_{y \in C} \|x-y\|$ $y_0 = P_C(x)$

necessary condition on y_0: $\mathrm{Re}\langle x-y_0, y-y_0 \rangle \geq 0$, $\forall y \in C$.

If C subspace, $\langle x-y_0, y-y_0 \rangle = 0$, $\forall y \in C$.

Def: x, y are orthogonal if $\langle x, y \rangle = 0$.

The angle between x, y are $\cos\alpha = \frac{\langle x, y\rangle}{\|x\|\|y\|}$

An orthonormal sequence in H is a sequence $\{u_k\}_{k \geq 1}$ s.t. $\langle u_k, u_j \rangle = \delta_{ij}$.

Remarks: · If $x_1, \cdots, x_n$ mutually orthogonal, then

$\|x_1 + \cdots + x_n\|^2 = \sum \|x_i\|^2$.

· Let $V \subseteq H$ closed subspace. $x \notin V$, $\exists y_0 \in V$, $\|x-y_0\| = \inf_V \|x-y\|$

$\Rightarrow (x-y_0) \perp y$, $\forall y \in V$.

Is the converse true? Yes.

Assume $\langle x-y_0, y \rangle = 0$, $\forall y \in V$.

$\|x-y\|^2 = \|(x-y_0)-(y-y_0)\|^2 = \|x-y_0\|^2 + \|y-y_0\|^2 \geq \|x-y_0\|^2$.

Def: $V \subseteq H$ linear subspace, its orthogonal complement is

$V^\perp = \{x \in H \mid \langle x, y \rangle = 0, \forall y \in V\}$.

Fact: $V^\perp$ closed $(V^\perp)^\perp = \overline{V}$.

Excursion: Infinite sum.

Let N be a set, $\{a_\lambda\}_{\lambda \in N}$ a family of non-negative numbers.

Def: $\sum_{\lambda\in N} a_\lambda = \sup_{F\subseteq N \text{ finite}} \sum_{\lambda\in F} a_\lambda$

Suppose $\sum_{\lambda\in N} a_\lambda < \infty$

Claim: then $\{\lambda\in N \mid a_\lambda>0\}$ is countable.

Bessel's inequality

Assume $(u_\lambda)_{\lambda\in N}$ orthonormal family, then $\forall x\in H$, we have

$\sum_{\lambda\in N} |\langle x,u_\lambda\rangle|^2 \le \|x\|^2$

Consider first a finite sequence $u_1,\cdots,u_n$ orthonormal.

Given x, consider $x-\sum_{j=1}^{n}\langle x,u_j\rangle u_j$

$V=\text{span}\{u_1,\cdots,u_n\}$. Claim: $\sum_{j=1}^{n}\langle x,u_j\rangle u_j = P_V x$, since $\langle x-\sum_{j=1}^{n}\langle x,u_j\rangle u_j, u_k\rangle = 0$

We conclude $\|x\|^2=\|x-\sum_{j=1}^{n}\langle x,u_j\rangle u_j\|^2+\sum_{j=1}^{n}|\langle x,u_j\rangle|^2 \ge \sum_{j=1}^{n}|\langle x,u_j\rangle|^2$ (equality $\Leftrightarrow x\in V$)

$\Rightarrow \sum_{\lambda\in N}|\langle x,u_\lambda\rangle|^2 \le \sum_{\text{finite}}|\langle x,u_\lambda\rangle|^2 \le \|x\|^2$

Thm: Let H be an infinite-dim separable Hilbert space, and let

$(u_n)_{n\in\mathbb{N}}$ orthonormal sequence. TFAE

① Density: $\text{Span}\{u_n \mid n\in\mathbb{N}\}$ dense in H

② Completeness: $\forall x\in H$, if $\langle x,u_n\rangle=0$ $\forall n$, then $x=0$

③ Equality in Bessel: $\forall x\in H$, $\|x\|^2=\sum_{n=1}^{\infty}|\langle x,u_n\rangle|^2$

④ Representation: $\forall x\in H$, $\exists (a_n)_{n\in\mathbb{N}}\subseteq\mathbb{C}$, s.t.

$$x=\lim_{N\to\infty}\sum_{n=1}^{N}a_nu_n=\sum_{n=1}^{\infty}a_nu_n.$$

Pf: ① $\Rightarrow$ ② $\Rightarrow$ ③ $\Rightarrow$ ④ $\Rightarrow$ ①

① $\Rightarrow$ ②: Assume span is dense,

Let $x\in H$ be given, If $\langle x,u_n\rangle=0$, $\forall n\in\mathbb{N}$,

then $\langle x,y\rangle=0$, $\forall y\in\text{span} \Rightarrow \forall y\in\overline{\text{span}}$ since $\langle\ \rangle$ conti.

$\Rightarrow \forall y \in \text{span} = H$ (dense) $\Rightarrow \langle x, x\rangle = 0 \Rightarrow x = 0$

② ⇒ ③ Consider $S_N = \lim_{N\to\infty} \sum_{n=1}^{N} \langle x, u_n\rangle u_n$. Claim: S_N cauchy in H.

$\|S_N - S_{N+k}\|^2 = \|\sum_{n=N+1}^{N+k} \langle x, u_n\rangle u_n\|^2 = \sum_{N+1}^{N+k} |\langle x, u_n\rangle|^2 \xrightarrow{N\to\infty} 0$ Since $\sum_1^\infty |\langle x, u_n\rangle|^2 < \infty$

So $S := \sum_{n=1}^\infty \langle x, u_n\rangle u_n$ converges in H

Check $\langle x - S, u_k\rangle = \langle x - \sum_1^\infty \langle x, u_n\rangle u_n, u_k\rangle$

$= \langle x, u_k\rangle - \sum_1^\infty \langle x, u_n\rangle\langle u_n, u_k\rangle$

$= 0 \Rightarrow x = S$ by ②

$\Rightarrow x = \sum_{n=1}^\infty \langle x, u_n\rangle u_n \Rightarrow \|x\|^2 = \sum \langle x, u_n\rangle$

③ ⇒ ④: Set $a_n = \langle x, u_n\rangle$. $S := \sum_{n=1}^\infty \langle x, u_n\rangle u_n$ (Absolute convergent in H by Bessel)

$\|x - S\|^2 = \lim_{N\to\infty} \|x - \sum_{n=1}^N \langle x, u_n\rangle u_n\|^2 = \lim_{N\to\infty} (\|x\|^2 - \|\sum_{n=1}^N \langle x, u_k\rangle u_k\|^2) \overset{③}{=} 0$

Example: $H = L^2(-\pi, \pi)$, measure $\frac{1}{2\pi} dx$

Standard Fourier basis: $\{e_n = e^{inx}\}$

$\langle e_n, e_m\rangle = \int_0^{2\pi} e^{inx} e^{-imx} = \int_0^{2\pi} e^{i(n-m)x} = \delta_{mn}$

Fourier coefficient: given $f \in L^2(-\pi, \pi)$, $a_n = \hat{f}(n) = \int_{-\pi}^{\pi} f(x) e^{-inx} dx$

is $\{e_n\}$ an orthonormal basis? Yes, by showing density.

by using trig polynomial. $\sum a_n e^{inx}$

The standard Fourier basis

Thm: $\{e_n(x)\}$ is an orthonormal basis of H.

Pf: NTS span$\{e_n\}$ is dense.

Consider for the moment, $f: \mathbb{R} \to \mathbb{R}$, 2π-periodic, continuous.

Clearly, $f|_{(-\pi,\pi)} \in H$, Define $\hat{f}(k) = \langle f, e_k\rangle = \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$

Consider $F_r(x) = \sum_{k=-\infty}^{\infty} |r|^k \hat{f}(k) e^{ikx}$ $r<1$, $x \in \mathbb{R}$

$$= \sum_{k=-\infty}^{\infty} |r|^k \int_{-\pi}^{\pi} e^{ik(x-y)} dy$$

$$= \int_{-\pi}^{\pi} f(y) \left(\sum_{k=-\infty}^{\infty} r^{|k|} e^{ik(x-y)} \right) dy$$

Compute $P_r(x) := \sum_{k=-\infty}^{\infty} r^{|k|} e^{ikx}$

$$= \sum_{k\geq 0} (re^{ix})^k + \sum_{k\leq 0} (re^{-ix})^k - 1$$

$$= \frac{1}{1-re^{ix}} + \frac{1}{1-re^{-ix}} - 1$$

$$= \frac{1-re^{ix}+1-re^{-ix}}{(1-r\cos x)^2+(r\sin x)^2} - 1$$

So $\frac{1}{2\pi} P_r(x) = \frac{2-2r\cos x - (1+r^2-2r\cos x)}{1+r^2-2r\cos x}$

$$= \frac{1-r^2}{1+r^2-2r\cos x} \cdot \frac{1}{2\pi}$$

So $F_r(x) = \int_{-\pi}^{\pi} f(y) P_r(x-y) dy$ Convolution!

WTS $\lim_{r\to 1} F_r(x) = f(x)$, $\forall x$.

① $\int_{-\pi}^{\pi} P_r = 1$ ② $P_r \geq 0$

③ $\lim_{r\to 1} P_r(x) = 0$ uniformly on $(-\pi,\pi)\setminus(-\delta,\delta)$ $\forall \delta$.

Let $\delta > 0$ TBD.

We estimate

$$|F_r(x) - f(x)| = \left| \int_{-\pi}^{\pi} P_r(x-y) f(y) dy - f(x) \right|$$

$$= \left| \int_{-\pi}^{\pi} P_r(x-y) (f(y) - f(x)) dy \right|$$

$$\leq \int_{-\pi}^{\pi} P_r(x-y) (|f(x)| + |f(y)|) \chi_{|x-y|\in(\delta,\pi)} dy + \int \chi_{|x-y|\notin(\delta,\pi)} P_r(x-y) |f(x) - f(y)| dy.$$

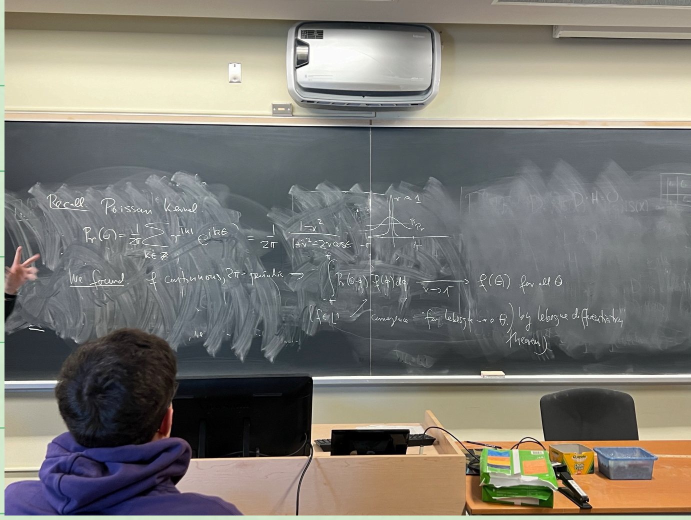
Given $\varepsilon > 0$, pick δ s.t. $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\varepsilon}{3}$

pick r close to 1 that $|P_r(x-y)| < \frac{\varepsilon}{3}$ ☆☆☆

$$\underbrace{\int_{-\pi}^{\pi} P_r(x-y) (|f(x)| + |f(y)|) \chi_{|x-y|\in(\delta,\pi)} dy}_{\leq \frac{\varepsilon}{3}} + \underbrace{\int \chi_{|x-y|\notin(\delta,\pi)} \underbrace{P_r(x-y)}_{\leq 1} \underbrace{|f(x) - f(y)|}_{\leq \frac{\varepsilon}{3}} dy}_{\leq \frac{\varepsilon}{3}} \leq \frac{2}{3}\varepsilon.$$

Fix r s.t. above estimation holds.

$F_r(x) = \sum_{k\in\mathbb{Z}} r^{|k|} \hat{f}(k) e^{ikx}$. Choose N. s.t. $\left| F_r(x) - \sum_{|k|\leq N} r^{|k|} \hat{f}(k) e^{ikx} \right| < \frac{\varepsilon}{3}$ ✓



① . Consider f as a function $D \to \mathbb{C}$, Assume f is real.

<u>Define</u>: $u(r\cos\theta, r\sin\theta) = \int_{-\pi}^{\pi} P_r(\theta-\varphi) f(\varphi)\, d\varphi$. in unit disk.

$r^k e^{ik\theta} = (re^{i\theta})^k = z^k$ holomorphic

$k<0 = \quad r^{|k|} e^{ik\theta} = (re^{-i\theta})^{|k|} = (\bar{z})^{|k|}$

<u>Equivilantly</u>, $u(z) = \sum_{k>0} \hat{f}(k) z^k + \sum_{k>0} \hat{f}(-k)(\bar{z})^k + \hat{f}(0)$

extending from S^1 to D.

<u>Fact</u>: A real-valued fn g $D \to \mathbb{R}^2$ is harmonic $(\Delta g = 0)$

$\Leftrightarrow g = \operatorname{Re} \Phi$, Φ holomorphic.

u solves the boundary-value problem $\begin{cases} \Delta u = 0 \text{ on } D \\ u|_{\partial D} = f \end{cases}$

<u>We proved</u>: $f \in L^2(-\pi,\pi) \Rightarrow \int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{k\in\mathbb{Z}} |\hat{f}(k)|^2$. (i.e. $\{\hat{f}(k)\} \in l^2$)

injective ✓. Surjective? Yes.

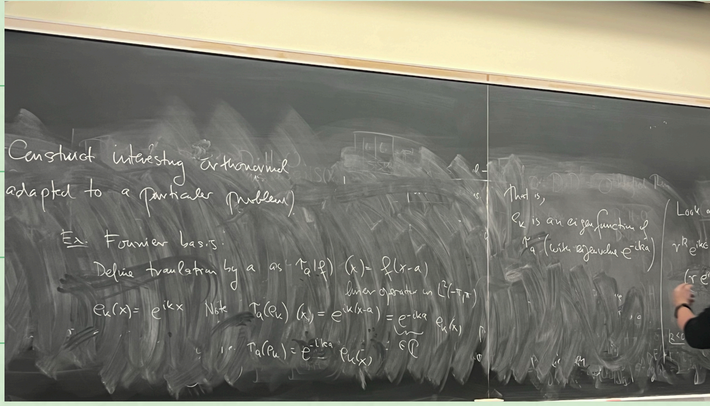
Q: Given a sequence $(a_k)_{k\in\mathbb{Z}}$, square summable.

<u>Define</u>: $f(x) := \sum_k a_k e^{ikx}$. converges in L^2. pointwise convergence?

Consider partial sum $S_N(x) = \sum_{-N}^{N} a_k e^{ikx}$

General theorem: ($f_n \to f$ in L^2) $\Rightarrow f_{n_k} \to f$ ptwise a.e.

<u>Goal</u>: Construct interesting orthonormal basis



<u>Finite dim case:</u> If $A \in \mathbb{C}^{n\times n}$ is Hermitian $(\bar{A}^T = A)$, then $\mathbb{C}^n$ has a basis of orthonormal eigenvectors

<u>Ex: no eigenvector:</u>

$T: l^2 \to l^2$, $T(a_1, a_2, a_3, \cdots) = (0, a_1, a_2, \cdots)$

<u>Def:</u> Let H_1, H_2 Hilbert Spaces,

$T: H_1 \to H_2$ a bounded linear operator

The adjoint of T is the operator $T^*: H_2 \to H_1$, s.t.

$\langle Tx, y\rangle_{H_2} = \langle x, T^*y\rangle_{H_1}$.

We say $T: H \to H$ self-adjoint if $T^* = T$

Does $*$ uniquely determine T^*?

Given y, consider the map $L_y: x \mapsto \langle Tx, y\rangle \in \mathbb{C}$

$L_y \in (H_1)^* \quad \left(\frac{|Lx|}{\|x\|_{H_1}} \le \frac{\|Tx\|_{H_2}\|y\|_{H_2}}{\|x\|_{H_1}} \le \|T\|_{H_1\to H_2} \|y\|_{H_2} \right)$

$L_y(x) = \langle x, z\rangle$ for some $z \in \mathbb{C}$, $T^*y = z$.

Unique? $\langle Tx, y\rangle = \langle x, Ay\rangle = \langle x, By\rangle$

$\Rightarrow \langle x, (A-B)y\rangle = 0, \ \forall x, y. \ \Rightarrow A-B=0$

Norm of T^*? $\|T^*y\|_{H_1} = \sup_{\|x\|=1} \mathrm{Re}(\langle x, T^*y\rangle)$

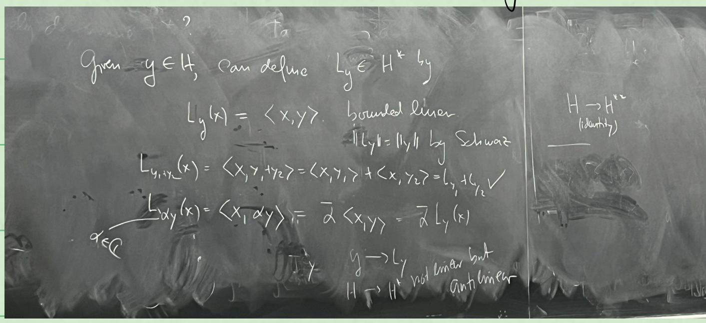
$\|T^*\| = \sup \frac{\mathrm{Re}\langle x, T^*y\rangle}{\|x\|_{H_1}\|y\|_{H_2}} = \sup \frac{\mathrm{Re}\langle Tx, y\rangle}{\|x\|_{H_1}\|y\|_{H_2}} = \|T\|$

Remark: $(T^*)^* = T$

$$\langle Tx, y\rangle = \langle x, T^*y\rangle = \overline{\langle T^*y, x\rangle} = \overline{\langle y, T^{**}x\rangle}$$

Excension: $H^* = H$?

Given $y \in H$, can define $L_y \in H$ s.t. $L_y(x) = \langle x, y\rangle$



Spectral Thm:

Def: Let X, Y Banach space, $L: X \to Y$ bounded, linear.

L is compact if every bounded sequence (x_n) in X

the sequence (Lx_n) has a convergent subsequence.

Thm: Let H be separable Hilbert Space, $A: H \to H$ compact,

self-adjoint. then $\exists (u_n)$ orthonormal basis, (λ_n) in $\mathbb{R}$,

s.t. $Au_n = \lambda_n u_n$

Strategy: ① Construct one eigenvalue / eigenvector pair.

Case 1: $\lambda = \|A\|$ an eigenvalue.

$\lambda = \cancel{\sup}^{\max} \langle x, Ax\rangle$

Case 2: $\lambda = -\|A\|$, replace A by $-A$

② Induction: Assume we have $\{u_1, \cdots, u_n\}$, $\{\lambda_1, \cdots, \lambda_n\}$

orthonormal eigenvectors.

Restrict A to $\{u_1, \cdots, u_n\}^\perp$

③ Error estimate $\| P_{\{u_1,\cdots,u_n\}^\perp} A P_{\{u_1,\cdots,u_n\}^\perp} \| \le \| \lambda_{n+1} \|$

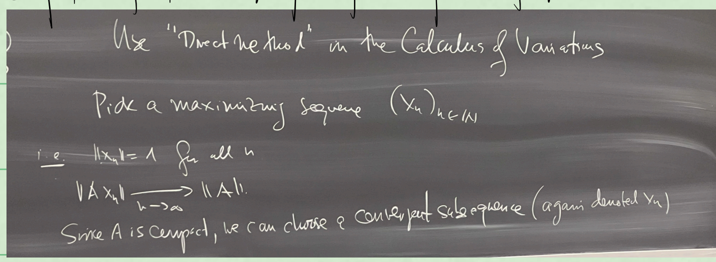
<u>Pf</u>: Step 1: $\|A\| = \sup\limits_{\|x\|=1} \|Ax\| = \sup\limits_{\|y\|=1} \sup\limits_{\|x\|=1} |\langle Ax, y\rangle| \overset{\text{polarization}}{\le} \frac{1}{4} \mathrm{Re}\big(\langle A(x+y), x+y\rangle - \langle A(x-y), x-y\rangle + \underbrace{i\langle A(x+iy), x+iy\rangle - i\langle A(x-iy), x-iy\rangle}_{\text{purely imaginary}} \big)$

$\le \frac{1}{4} \left| \|A\| \left((\|x\| + \|y\|)^2 + (\|x\| - \|y\|)^2 \right) \right| = \frac{1}{2} \|A\| (\|x\|^2 + \|y\|^2)$

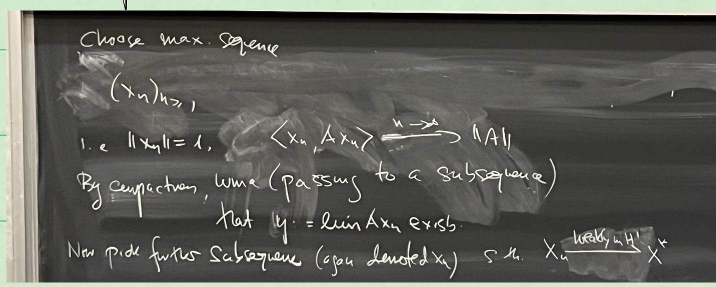
<u>Claim</u>: $\|A\| = \sup\limits_{\|x\|=1} |\langle x, Ax\rangle|$

"$\ge$" is clear by schwartz "$\le$" later

$\sup |\langle x, Ax\rangle| = \max\{ \sup \langle x, Ax\rangle, \sup \langle x, -Ax\rangle \}$



By compactness, $\lim\limits_{n\to\infty} Ax_n = y \in H$ exists.



<u>Def</u>: For $f \in \mathcal{L}^1(\mathbb{R}^n)$, $\hat{f}(k) : \mathbb{R}^n \to \mathbb{C}$:

$$\hat{f}(k) = \int_{\mathbb{R}^n} e^{-2\pi i k \cdot x} f(x)\,dx \quad , \quad kx = k_1 x_1 + \cdots + k_n x_n$$

By Def: $\mathcal{F} : f \to \hat{f}$ linear.

$$|\hat{f}(k)| = \left| \int_{\mathbb{R}^n} e^{-2\pi i k \cdot x} f(x)\,dx \right| \le \|f\|_1$$

So $\mathcal{F} : \mathcal{L}^1 \to \mathcal{L}^\infty$ bounded.

Goal: Define $\mathcal{F} : \mathcal{L}^2 \to \mathcal{L}^2$ (Note $C_c^\infty(\mathbb{R}^n) \subseteq \mathcal{L}^1 \cap \mathcal{L}^2 \subseteq \mathcal{L}^2$ dense)

Interaction of $\mathcal{F}$ with geometric operations.

① <u>translation</u>: $\forall a \in \mathbb{R}^n$, $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$, $\tau_a(x) = x + a$.

$$\forall f : \mathbb{R}^n \to \mathbb{C}, \quad (\tau_a f)(x) = f(\tau_a^{-1} x) = f(x-a)$$

$$\widehat{\tau_a f}(k) = \int e^{-2\pi i k x} f(x-a)\,dx = \int e^{-2\pi i k y} e^{-2\pi i k a} f(y)\,dy = e^{-2\pi i k a} \hat{f}(k)$$

② <u>derivative</u>: $\widehat{\partial_{x_j} f}(k)$:

$$\partial_{x_j} f(x) = \lim_{h \to 0} \frac{\tau_{he_j} f(x) - f(x)}{h}$$

$$\Rightarrow \widehat{\partial_{x_j} f}(k) = \lim_{h \to 0} \left(\frac{e^{2\pi i k_j h} - 1}{h} \right) \hat{f}(k) = 2\pi i k_j \hat{f}(k)$$

③ <u>dilation</u>: $\forall \lambda > 0$, $\sigma_\lambda(x) = \lambda x$, $\sigma_\lambda f(x) = f(\lambda^{-1} x)$

$$\widehat{\sigma_\lambda f}(k) = \int e^{-2\pi i k x} f(\lambda^{-1} x)\,dy \qquad x = \lambda y, \quad dx = \lambda^n dy$$

$$= \int \lambda^n e^{-2\pi i k \lambda y} f(y)\,dy$$

$$= \lambda^n \hat{f}(\lambda k)$$

④ <u>L-linear transform</u> ($\det L = 1$)

$$\widehat{f \circ L^{-1}}(k) = \int e^{-2\pi i (Ly) k} f(y)\,dy = \hat{f}(L^* y)$$

⑤ <u>convolution</u>:

$$\widehat{f * g}(k) = \int e^{-2\pi i k x} \int f(x-y) g(y)\,dy\,dx = \hat{f}(k)\,\hat{g}(k) \quad \text{clear.}$$

<u>Lemma</u>: Let $g_\lambda(x) = e^{-\pi |x|^2 \lambda}$ be a gaussian on $\mathbb{R}^n$

then $\hat{g}(k) = \lambda^{-\frac{n}{2}} e^{-\frac{\pi |k|^2}{\lambda}}$

In particular, $g_1=\hat{g}_1$

Pf: suffice to compute $\lambda=1$

$$\hat{g}_1(k)=\int_{\mathbb{R}^n} e^{-2\pi i k\cdot x}e^{-\pi|x|^2}=\int_{\mathbb{R}^n} e^{-\pi(x+ik)(x+ik)}dx\cdot \underbrace{e^{-\pi|k|^2}}_{\hat{g}_1(k)}$$

Claim: $\int_{-\infty}^{\infty} e^{-\pi(x+ik)^2}dx=\int_{-\infty}^{\infty}e^{-\pi x^2}dx=1.$

$\mathcal{F}: \mathcal{L}_1\to\mathcal{L}_\infty$ bounded linear. range? complicated.

Will prove: $\forall f,g\in\mathcal{L}^1\cap\mathcal{L}^2$, $\langle\hat{f},\hat{g}\rangle=\langle f,g\rangle \Rightarrow \|\hat{f}\|_2=\|f\|_2$

We proved: $\sigma_\lambda f(x)=f(\lambda^{-1}x)$ for $f\in\mathcal{L}^1$, $\widehat{\sigma_\lambda f}(k)=\lambda^n\hat{f}(\lambda k)$

$$\Rightarrow \widehat{(\lambda^{-\frac{n}{2}}\sigma_\lambda f)}(k)=\lambda^{\frac{n}{2}}\hat{f}(\lambda k)$$

$$\|\sigma_\lambda f\|_2^2=\int|f(\lambda^{-1}x)|^2dx=\lambda^{-n}\|f\|_2^2$$

$$\|\widehat{\sigma_\lambda f}\|_2^2=\int|\lambda^n\hat{f}(\lambda k)|^2dk=\lambda^n\|\hat{f}\|_2^2$$

We also compute $g_\lambda(x)=e^{-\pi\lambda|x|^2}$, $\hat{g}_\lambda(x)=\lambda^{-\frac{n}{2}}e^{-\pi|x|^2/\lambda}$

In particular, $\hat{g}_1=g_1$

Claim: $\lim_{\lambda\to 0}g_\lambda=1$

$$\lim_{\lambda\to 0}\hat{g}_\lambda*\varphi=\varphi \quad (\mathcal{L}^1)$$

Thm: If $f\in\mathcal{L}^1\cap\mathcal{L}^2(\mathbb{R}^n)$, then $\|f\|_2=\|\hat{f}\|_2$

Pf: $\iiint \overbrace{e^{-\pi\lambda|k|^2}}^{g_\lambda(k)}e^{-2\pi ikx}f(x)e^{2\pi iky}\overline{f(y)}\,dx\,dy\,dk$

$=\iint\hat{g}_\lambda(x-y)f(x)\overline{f(y)}\,dx\,dy$

$=\int(\hat{g}_\lambda*f)(x)\overline{f}(x)\,dx\xrightarrow{\lambda\to 0}\|f\|_2^2.$

$$\iiint \overbrace{e^{-\pi\lambda|k|^2}}^{g_\lambda(k)}e^{-2\pi ikx}f(x)e^{2\pi iky}\overline{f(y)}\,dx\,dy\,dk=\int e^{-\pi\lambda|k|^2}|\hat{f}(k)|dk\xrightarrow[\lambda\to 0]{MCT}\int|\hat{f}(k)|^2dk \quad \checkmark$$

For $f\in\mathcal{L}^2$, we define $\mathcal{F}(f)$ by: pick any $f_l\to f$ in $\mathcal{L}^1\cap\mathcal{L}^2$,

Set $\hat{f}:=\lim_{l\to\infty}\hat{f}_l$

$\{f_l\}$ cauchy in $L^2 \Rightarrow \{\hat{f}_l\}$ cauchy $\Rightarrow f \in L^2$ exists.

<u>Example:</u> $f(x) = \frac{1}{1+|x|} \in L^2 \qquad \int |f(x)|^2 = 2\frac{-1}{1+|x|}\Big|_0^\infty < \infty$

$\hat{f}_l = \widehat{\chi_{[-l,l]} f} = \int_{-l}^{l} e^{-2\pi i k x} \frac{1}{1+|x|} dx$

For $f, h \in L^1 \cap L^2$,

$$\langle \hat{f}, h\rangle = \iint e^{-2\pi i k x} f(x)\,dx\,\overline{h(k)}\,dk.$$

$$= \int f(x) \int \overline{h(k) e^{2\pi i k x}}\,dk\,dx = \langle f, \check{h}\rangle,$$

where $\check{h}(x) = \int e^{2\pi i k x} h(k)\,dk.$

So $\check{\mathcal{F}}(h) = \check{h}$, $\check{h}(x) = \hat{h}(-x)$

<u>Prop:</u> $\forall f \in L^2(\mathbb{R}^n)$, $\check{\hat{f}} = \hat{\check{f}} = f$

<u>Pf:</u> Consider $\hat{g}_\lambda * f(x) = \int \hat{g}_\lambda(x-y) f(y)\,dy$

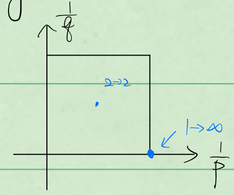
$$= \iint e^{-2\pi i k(x-y)} e^{-\pi\lambda|k|^2} f(y)\,dy\,dk$$

$$= \int e^{-2\pi i k x} e^{-\lambda\pi|k|^2} \check{f}(k)\,dk \xrightarrow{\lambda \to 0} \hat{\check{f}}$$

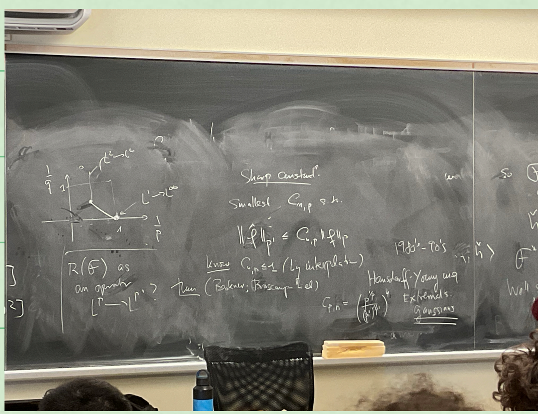
<u>Conclusion</u>: $\mathcal{F}: L^2 \to L^2$ unitary

$\|\mathcal{F}\|_{1\to\infty} \le 1$

$\|\mathcal{F}\|_{2\to 2} = 1$



$\mathcal{F}: L^p \to L^{p'} \quad \forall p \in [1,2]$, $\|\mathcal{F}\| \le 1$



<u>Observe</u>: f has compact support $\Rightarrow \hat{f}$ smooth.

$$\hat{f}(k) = \int_{\Omega} e^{-2\pi i k x} f(x)\,dx$$

$$f \in C_c^{\infty},\ \widehat{\partial_{x_j} f} = \int e^{-2\pi i k x} \partial_{x_j} f(x)\,dx = \int 2\pi i k_j\, e^{-2\pi i k x} f(x)\,dx = (2\pi i k_j)\hat{f}(k)$$

<u>Def</u>: Schwartz Space:

$$S := \{ f \in \mathbb{R}^n \to \mathbb{C} \mid x^{\beta} D^{\alpha} f \ \text{ bounded} \}. \quad \alpha = (\alpha_1, \cdots, \alpha_n) \quad \beta = (\beta_1, \cdots \beta_n)$$

$$D^{\alpha} f = \prod (\partial_{x_j})^{\alpha_j} f, \quad x^{\beta} = \prod (x_j)^{\beta_j}$$

<u>Fact</u>: $\mathcal{F}: S \to S$ continuous bij.

<u>Application</u>: Consider Poisson problem : $-\Delta u = f$, $\Delta u = \sum \partial_{x_j}^2$

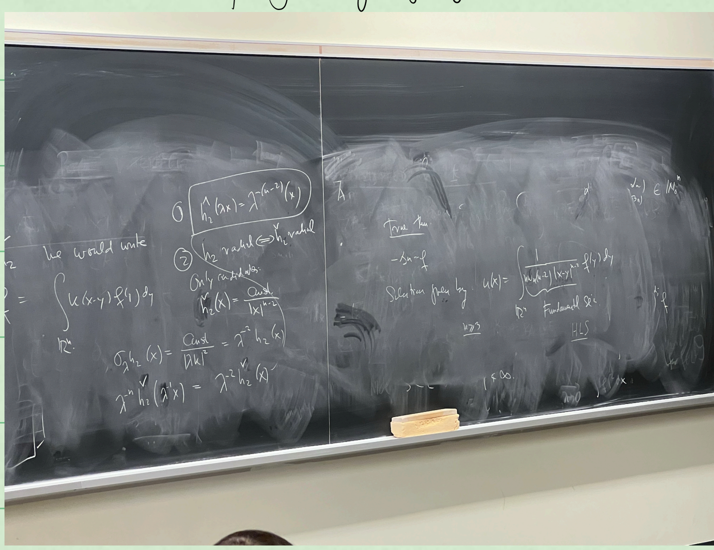
$$-\widehat{\Delta u}(k) = (2\pi)^2 |k|^2 \hat{u}(k)$$

$$\Rightarrow (2\pi)^2 |k|^2 \hat{u}(k) = \hat{f}(k)$$

$$\Rightarrow \hat{u}(k) = \frac{1}{(2\pi)^2 |k|^2} \hat{f}(k)$$

Let $\quad h_2(x) = \dfrac{1}{(2\pi)^2 |k|^2}$

$$u(x) = \check{h}_2 * f = \int u(x-y) f(y)\,dy$$



<u>Distributions</u>: (LL Chapter 6, Folland 8, 9).

Distributions = generalized fns.

In $\mathbb{R}^n$, $C_c^\infty \subseteq \underset{\text{Shwartz}}{S} \subseteq L^2 \subseteq L^1_{loc} \subseteq \overset{\text{dual of } S}{S'} \subseteq \mathcal{D}'$ distributions.

Let $\emptyset \neq \Omega \subseteq \mathbb{R}^n$ open connected.

$$\mathcal{D} = C_c^\infty = \{\varphi : \Omega \to \mathbb{C} \mid \varphi \text{ smooth, } \operatorname{supp}\varphi \overset{\text{compact}}{\subseteq} \Omega\}$$

Elements of $\mathcal{D}$ are called test functions.

Topology: $\varphi_n \to \varphi$ in $\mathcal{D}$ if:

① $\exists K \overset{\text{cpt}}{\subseteq} \Omega$, s.t. $\forall n$, $\operatorname{supp}\varphi_n \subseteq K$.

② $D^\alpha \varphi_n \to D^\alpha \varphi$ $\forall$ multi-index α.

Def: A multi-index is a vector $\alpha = (\alpha_1, \cdots, \alpha_n)$ of ≥ 0 integers. n = length. $|\alpha| = \sum \alpha_i$ order

For $x \in \mathbb{C}^n$, $x^\alpha = \prod x_i^{\alpha_i}$, $D^\alpha f = \sum (\partial x_i)^{\alpha_i} f$

$\mathcal{D}' = \{T : \mathcal{D} \to \mathbb{C} \text{ linear cont.}\}$.

Notation: $\langle T, \varphi \rangle = T(\varphi)$

i.e. if $\varphi_n \to \varphi$ in $\mathcal{D}$, then $T(\varphi_n) \to T(\varphi)$

Eg: • Let $f \in L^1_{loc}(\Omega)$

Define $\langle T, \varphi \rangle = \int_\Omega f(x)\varphi(x)dx$

$|\langle T, \varphi_n \rangle - \langle T, \varphi \rangle| = \left|\int_\Omega f(x)(\varphi_n - \varphi)\right| \leq \sup_{x \in K} |\varphi_n - \varphi| \int_\Omega |f|$

• $T_n(\varphi) = \int \varphi$

• $\delta_\alpha(\varphi) := \varphi(\alpha)$

Let $T \in \mathcal{D}'$, $\psi : \mathbb{R}^n \to \mathbb{C}$

• Define ψT by:

$$\langle \psi T, f\rangle := \langle T, \psi f\rangle$$

• Let $L \in GL(n, \mathbb{R})$ $(\det L \neq 0)$

$$\langle T_{f\circ L}, \varphi\rangle = \int f(Lx)\varphi(x)\,dx = \int f(y)\varphi(L^{-1}y)\frac{1}{\det L}\,dy = \langle T_f, (\varphi\circ L^{-1})\frac{1}{\det L}\rangle$$

Define $T\circ L$ by: $\langle T\circ L, \varphi\rangle = \langle T, \frac{1}{|\det L|}\varphi\circ L^{-1}\rangle$

How to define $\hat{T}$?

$$\langle T_{\hat{f}}, \varphi\rangle = \int \hat{f}(x)\varphi(x)\,dx$$
$$= \int f(x)\hat{\varphi}(x)\,dx$$

So $\langle \hat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle$

• Derivative: $T \in \mathcal{D}'$, $\alpha = (\alpha_1, \cdots, \alpha_n)$ multi-index.

$$\langle T_{\partial x_i f}, \varphi\rangle = \int \partial x_i f\,\varphi = -\int f\,\partial x_i\varphi \quad \text{(Integration by part)}$$

So $\langle D^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\varphi\rangle$

• Let φ be a test fn, define f' by $\quad f = |x|$

$$\langle f', \varphi\rangle = -\langle f, \varphi'\rangle = -\int_{\mathbb{R}} |x|\varphi'(x)\,dx$$
$$= -\int_0^\infty x\varphi'(x) + \int_{-\infty}^0 x\varphi'(x)$$
$$\overset{\text{IBP}}{=} -x\varphi(x)\Big|_0^\infty + \int_0^\infty \varphi(x)\,dx + x\varphi(x)\Big|_{-\infty}^0 - \int_{-\infty}^0 \varphi(x)\,dx$$
$$= \int_{-\infty}^\infty \operatorname{sgn}x\,\varphi(x)\,dx$$

$$\langle f'', \varphi\rangle = \langle \operatorname{sgn}x', \varphi\rangle = -\int_{\mathbb{R}} \operatorname{sgn}x\,\varphi'(x)\,dx$$
$$= -\int_0^\infty \varphi' + \int_{-\infty}^0 \varphi'$$
$$= -\varphi\Big|_0^\infty + \varphi\Big|_{-\infty}^0$$
$$= 2\varphi(0)$$

We define $\langle D^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\varphi\rangle$

Linear ✓ Continuity: say $\varphi_k \to \varphi$ in $\mathcal{D}$

$\forall\beta.$ $D^\beta(\varphi_k - \varphi) \longrightarrow 0$

Then $\langle D^\alpha T, \varphi_n\rangle = (-1)^{|\alpha|}\langle T, D^\alpha\varphi_n\rangle = T(D^\alpha\varphi_n)$

Thm: Let $T \in \mathcal{D}'$ distribution. Assume T is positive, i.e.

$\langle T, \varphi \rangle \geq 0$, $\forall \varphi \in \mathcal{D}$ with $\varphi \geq 0$

Then T "is" a measure. $\exists \mu$ s.t. $\langle T, \varphi \rangle = \int \varphi(x) d\mu$.

$\mu(A) = \sup\{T(\varphi) \mid \operatorname{supp} \varphi \subset A\}$.

Pf: Define $\psi * T$ by $\langle \psi * T, \varphi \rangle = \langle T, \varphi * \tilde{\psi} \rangle$, where $\tilde{\psi}(x) = \psi(x)$

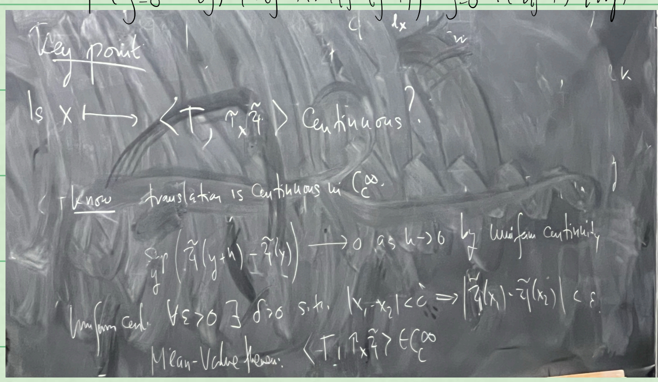
$\langle T_{\psi * f}, \varphi \rangle = \iint \psi(x-y) f(y) dy\, \varphi(x) dx$ $= \int \langle T, \tau_x \tilde{\psi} \rangle \varphi(x) dx$

$= \iint \varphi(x) \psi(x-y) dx\, f(y) dy$

Write $\varphi * \tilde{\psi}(x) = \int \varphi(y) \tilde{\psi}(y-x) dy$. approximate by Riemann sum

$\sum_{j=0}^{N} \varphi(t_j) \overbrace{\psi(t_j - x)}^{\tau_{t_j}(\tilde{\psi})(x)} (t_j - t_{j-1})$

$\Rightarrow T\left(\sum_{j=0}^{N} \varphi(t_j) \psi(t_j - x)(t_j - t_{j-1})\right) = \sum_{j=0}^{N} T(\tau_{t_j}(\tilde{\psi})) \varphi(y_j)$ $=$



Def: (Sobolev Spaces)

For $k \in \mathbb{N} \cup \{0\}$, $p \in [1, \infty]$, define

$W^{k,p}(\Omega) = \{T \in \mathcal{D}'(\Omega) \mid T$ is represented by a fn $f \in L^p$, $\forall |\alpha| \leq k$ multi-index, $D^\alpha T$ is represented by a fn $g_\alpha \in L^p\} \subseteq L^p(\Omega)$

$\Omega = \mathbb{R}^n$, $p = 2$:

$W^{1,2} = H^1 = \{f \in L^2 \mid (1 + |k|^2)^{\frac{1}{2}} f \in L^2\}$

Norm on $W^{k,p}$: $\sum_{|\alpha|\le k} \|g_\alpha\|_p = \|f\|_{W^{k,p}}$

<u>Example</u>: $W^{1,p}(\Omega) = \{ f \in L^p(\Omega) \mid \exists g_1, \cdots, g_n \in L^p(\Omega),\ s.t.\ -\int_\Omega f \partial_{x_j}\varphi = \int_\Omega g_j \varphi \in C_c^\infty(\Omega), \}$

i.e. $g_j = \partial_j f$ in sense of distributions

$\|f\|_{W^{1,p}} = \|f\|_{L^p} + \sum_{j=1}^n \|g_j\|_{L^p}$

<u>Prop</u>: $W^{1,p}(\Omega)$ is complete.

<u>Pf</u>: Let $\{f_n\}$ be a cauchy sequence in $W^{1,p}$

i.e. $\{f_n\}$ is cauchy in L^p, also the distributional derivatives $\{(g_j)_k\}$.

Completeness of $L^p \Rightarrow f, g_1, \cdots, g_n \in L^p$,

$f_k \to f$, $(g_j)_k \to g_j$. NTS $g_j = \partial_j f$ as distributions.

Let φ be a test function

$$\langle T_{\partial_j f}, \varphi \rangle = -\int f(x) \partial_j \varphi(x) dx$$

$$= -\lim_{k\to\infty} \int \overset{\in W^{1,p}}{f_k(x)} \partial_j \varphi(x) dx$$

$$= \lim_{k\to\infty} \int (g_j)_k(x) \varphi(x) dx$$

$$= \int g_j(x) \varphi(x) dx$$

<u>Sobolev inequlities</u>:

Consider $W^{1,p}(\mathbb{R}^n)$, $1 \le p < n$, set $p^* = \frac{np}{n-p}$

$\exists C = C(n,p)$, s.t. $\|u\|_{p^*} \le C(n,p) \|\nabla u\|_p$. $= \sum_{j=1}^n \|\partial_j u\|_p$

LHS $= \left(\int |u|^{p^*} dx\right)^{\frac{1}{p^*}} \sim m^{\frac{n}{p^*}}$

RHS $= \left(\int |\nabla u|^p dx\right)^{\frac{1}{p}} \sim m^{\frac{n-p}{p}}$ $\Rightarrow \frac{n}{p^*} = \frac{n-p}{p}$

Thm (Hahn-Banach):

Let X be a $\mathbb{R}$-vector space, p a sublinear functional on X, $M \subseteq X$ a subspace, f a linear functional on M s.t. $f(x) \le p(x)$, $\forall x \in M$.

Then $\exists F: X \to \mathbb{R}$ linear s.t. $F(x) \le p(x)$, $\forall x \in X$, $F|_M = f$

Cor: X a normed vector space,

If $M \subseteq X$, $f: M \to \mathbb{R}$ bounded linear, then

$\exists$ a bdd linear $F: X \to \mathbb{R}$, s.t. $\|F\|_X = \|f\|_M$

Pf: $p(x) := \|f\| \|x\|$

then $F(x) \le \|f\| \|x\|$

Remark: X is a locally convex topological V.S. (TVS)

$\Rightarrow$ cor holds for cont. extension

e.g. $L^p[0,1]$ when $0 \le p \le 1$, there are no cont. linear functionals on L^p. the only open convex sets are $\{0\}$ and L^p

Convexity: X a TVS

Def: The gauge function of $K \subseteq X$ with $0 \in K$ is

$p_K(x) = \inf\{r > 0 : x \in rK\}$.

E.g.: $\cdot$ X has a norm $\|\cdot\|$, $K = \{\|x\| \le 1\}$. $p_K(x) = \|x\|$

$\cdot$ X is $\mathbb{R}^n$ with $\|\cdot\|_p$, $K = \{\|x\|_q \le 1\}$. $p_K(x) = \|x\|_q$

$\cdot$ $f: X \to \mathbb{R}$, $K = \{x \mid |f(x)| \le 1\}$. $p_K(x)$ almost a norm.

Assume X is a NVS

Lemma: $C \subseteq X$ open, convex, and $0 \in C$, $p = p_C$.

① p is sublinear.

② $\exists M>0$, s.t. $p(x) \le M \cdot \|x\|$, $\forall x$.

③ $C = \{x : p(x) = 1\}$.

Pf: ②: $\exists r>0$, s.t. $B(r,0) \subseteq C$

$\Rightarrow p(x) \le \frac{1}{r}\|x\|$, $M = \frac{1}{r}$

③ If $x \in C \Rightarrow \exists \varepsilon>0$, s.t. $(1+\varepsilon)x \in C$.

$\Rightarrow p(x) \le \frac{1}{1+\varepsilon} < 1$

If $p(x)<1 \Rightarrow \exists r<1$ s.t. $\frac{1}{r}x \in C$.

Convexity: $r(\frac{1}{r}x) + (1-r)(0) = x \in C$.

①: If $\lambda>0$,

$p(\lambda x) = \inf\{r>0 \mid \lambda x \in rC\}$

$= \lambda \inf\{\lambda^{-1} r > 0 \mid x \in \lambda^{-1} r \cdot C\}$

$= \lambda p(x)$

$p(x+y) \le p(x) + p(y) + \varepsilon$, $\forall \varepsilon > 0$.

Remark: If C is open, convex, symmetric $(-C = C)$

and bounded w.r.t $\|\cdot\|$

$\Rightarrow p$ is a norm equiv to $\|\cdot\|$

Def: A Hyperplane $H \subseteq X$ is

$H = \{x : f(x) = \alpha\}$ for some linear functional f and $\alpha \in \mathbb{R}$.

Lemma: (Brezis)

$H = [f = \alpha]$ is closed iff f is continuous.

Def: $A, B \subseteq X$, $H = [f = \alpha]$

H separates A and B if $f(a) \leq \alpha \leq f(b)$ $\forall a \in A, b \in B$.

H strictly separates A and B if $\exists \varepsilon > 0$, $f(a) + \varepsilon \leq \alpha \leq f(b) - \varepsilon$.

Lemma: If $C \subseteq X$ open convex, $x_0 \notin C$, then $\exists [f = f(x_0)]$ separates x_0 and C.

Pf: Assume $0 \in C$. $M = \mathrm{span}\{x_0\}$.

$f : M \to \mathbb{R}$, $f(tx_0) = t$, p is the gauge function of C.

Lemma. since $C = \{p(x) < 1\}$.

$1 = f(x_0) \leq p(x_0)$

$\forall t \in \mathbb{R}$, $f(tx_0) = t f(x_0) \leq |t| p(x_0) = p(tx_0)$.

$\Rightarrow f$ dominated by M.

$\Rightarrow \exists$ An ext^n $F : X \to \mathbb{R}$, $\|F\|_X = \|f\|_M$.

$x \in C$ $F(x) \leq p(x) < 1$

$F(x_0) = 1$

$H = [F = 1]$

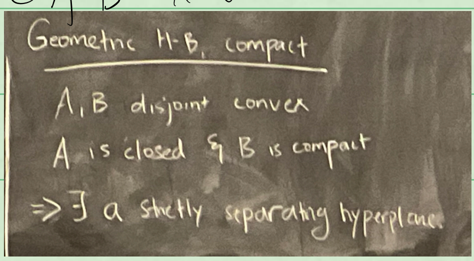
Key idea: separating A from B $\Leftrightarrow$ separating $A - B$ from 0.

Geometric H-B for open sets:

A, B disjoint, non-empty convex sets and A is open

$\Rightarrow \exists$ a separating hyperplane.

Pf: $C = A - B$. $x_0 = 0$



$\underline{Pf}$: $C = A - B$ closed, convex, $0 \notin C$

$\Rightarrow \exists B_r(0)$ disjoint from C

$\Rightarrow \exists [f = \alpha]$ separates $B_r(0)$, C

i.e. $f(a-b) \le \alpha \le f(v)$, $\forall a \in A, b \in B, v \in B_r(0)$

$f(a-b) \le -r \cdot \|f\|$ (take inf of v)

$\Rightarrow f(a) + \frac{r\|f\|}{2} \le f(b) - \frac{r\|f\|}{2}$

pick β: $f(a) + \frac{r\|f\|}{2} \le \beta \le f(b) - \frac{r\|f\|}{2}$

$\Rightarrow$ strickly separate.

A, B disjoint convex $\neq \emptyset$

finite dim $\Rightarrow$ always separate

A open $\Rightarrow$ separate

A closed, B compact $\Rightarrow$ strictly separate

A, B closed $\Rightarrow$ might not separate.

$\underline{Recall}$: C open, convex, $0 \in C$, p gauge

$\Rightarrow C = \{x \mid p(x) < 1\}$

$\underline{Cor}$ $\bar{C} = \{x \mid p(x) \le 1\}$

$\varphi: X \to [0, +\infty]$ 'nice' convex fn

$K = \{x : \varphi(x) \le 1\}$ closed convex

$K = \{x \mid p_K(x) \le 1\}$

For $\lambda > 0$

$\{x : \varphi(x) < \lambda\} = \{x : p_K(x) < \lambda\}$

Convex fns $\varphi : X \to (-\infty, +\infty]$

<u>Claim</u>: $\{x : \varphi(x) \le \lambda\}$, $\forall \lambda \in \mathbb{R}$

are closed $\iff$ φ is lower semi-continuous (l.s.c)

if $x_n \to x$, $\liminf \varphi(x_n) \ge \varphi(x)$

<u>Claim</u>: φ is convex $\Rightarrow$ $\{x \mid \varphi(x) \le \lambda\}$ is convex.

<u>Def</u>: $\mathrm{epi}(\varphi) = \{(x, \lambda) : \varphi(x) \le \lambda\} \subseteq X \times \mathbb{R}$.

<u>Claim</u>: ① φ is l.s.c $\iff$ $\mathrm{epi}(\varphi)$ closed

② φ is convex $\iff$ $\mathrm{epi}(\varphi)$ convex

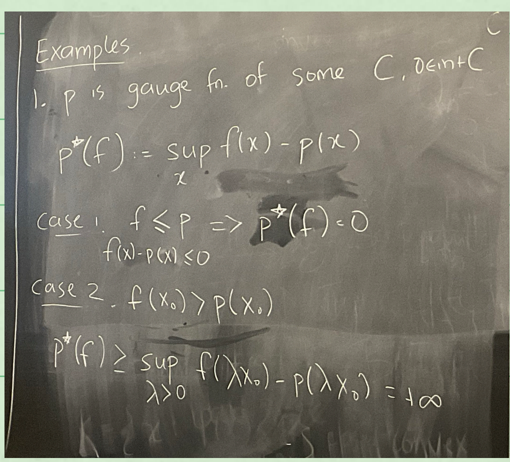
③ φ is proper ($\varphi \not\equiv \infty$) $\iff$ $\mathrm{epi}(\varphi) \ne \emptyset$

H-B: p is gauge, want $f \in X^*$, $f \le p$.

<u>Def</u>: $p^* : X \to (-\infty, \infty]$ indicator fn

$$p^*(f) = \begin{cases} 0 & f \le p \\ +\infty & \text{otherwise.} \end{cases}$$

<u>Def</u>: φ is proper $\Rightarrow$ $\varphi^*(f) = \sup_x f(x) - \varphi(x)$



2: $\varphi: \mathbb{R} \to \mathbb{R}$, $p \geq 1$

$\varphi(a) = \frac{1}{p} a^p$, identify $\mathbb{R}^* = \mathbb{R}$

if $b \in \mathbb{R}$, $\varphi^*(b) = \sup_a ab - \frac{1}{p} a^p = \frac{1}{p'} b^{p'}$

Young's ineq: $\varphi: X \to (-\infty, +\infty]$, $\forall x \in X$, $f \in X^*$

$f(x) \leq \varphi(x) + \varphi^*(f)$

Prop: if $\varphi: X \to (-\infty, \infty]$ is convex, l.s.c, proper

$\Rightarrow$ so is φ^*

Pf: proof

$$\text{epi}(\varphi^*) = \{(f, \lambda) : \lambda \geq \sup_x (f(x) - \varphi(x))\}$$

$$= \bigcap_{x \in X} \{(f, \lambda) : \lambda \geq \underbrace{f(x) - \varphi(x)}_{\text{convex lsc on } X^*}\}$$

$$\Rightarrow \text{epi}(\varphi^*) = \bigcap \text{closed + convex} = \text{closed + convex}$$

Wait? $\text{epi}(\varphi^*) \neq \emptyset$

need $f \in X^*$ and $K \in \mathbb{R}$ s.t. $f(x) - \varphi(x) \leq K$ $\forall x \in X$

rearrange: $\varphi(x) \geq f(x) - K$ $\forall x \in X$

Choose $x_0 \in X$ s.t. $\varphi(x_0) < +\infty$

and $\lambda_0 \in \mathbb{R}$ s.t. $\lambda_0 < \varphi(x_0) \Rightarrow (x_0, \lambda_0) \notin \text{epi}(\varphi)$

$\Rightarrow \exists$ a strictly sep $H = [F = \alpha] \subseteq X \times \mathbb{R}$

$\exists f \in X^*$ and $t \in \mathbb{R}$ s.t. $F(x, \lambda) = f(x) + t\lambda$.

If $\varphi(x) < +\infty$

$f(x_0) + t\lambda_0 < \alpha < f(x) + t\,\varphi(x)$

Set $x = x_0$

$f(x_0) + t\lambda_0 < f(x_0) + t\,\varphi(x_0)$

$\Rightarrow t(\varphi(x_0) - \lambda_0) > 0 \Rightarrow t > 0$

so $f(x) + t\varphi(x) > \alpha$ $\forall x \in X$

$\varphi(x) \geq \left(\frac{-f}{t}\right)(x) + \frac{\alpha}{t}$

$\varphi^*\left(\frac{-f}{t}\right) < \frac{-\alpha}{t} < +\infty$

Example: Fix $\varphi = X \to (-\infty, +\infty]$.

1. Translate by constant $\alpha \in \mathbb{R}$

$(\varphi+\alpha)(x) = \varphi(x) + \alpha$.

$(\varphi+\alpha)^*(x) = \sup_x f(x) - \varphi(x) - \alpha$

$= \varphi^*(f) - \alpha$.

2. Scale: $\lambda \in \mathbb{R}$ $(\lambda\varphi) = \lambda(\varphi x)$

$(\lambda\varphi)^*(x) = \sup f(x) - \lambda\varphi(x)$

$= \lambda \sup \lambda^{-1} f - \varphi$

$= \lambda \varphi^*(\lambda^{-1} f)$

3. $(\varphi+g)$, $g \in X^*$

$(\varphi+g)^*(f) = \sup f - \varphi - g$

$= \varphi^*(f-g)$

Claim: $\varphi \le \psi \Rightarrow \varphi^* \ge \psi^*$

Avoid $X \ne X^{**}$

Def: $\varphi : X \to (-\infty, +\infty]$

$\varphi^{**} : X \to (-\infty, +\infty]$

$\varphi^{**}(x) = \sup_{f \in X^*} f(x) - \varphi^*(f)$

Thm: (Fenchel–Moreau):

If φ convex, proper, l.s.c.

then $\varphi = \varphi^{**}$

Pf: Case 1: $\varphi \ge 0$

$f(x) - \varphi^*(f) \le \varphi(x)$, $\forall f, x$

$\Rightarrow \varphi^{**}(x) \le \varphi(x)$

Say $\varphi^{**}(x_0) < \varphi(x_0)$

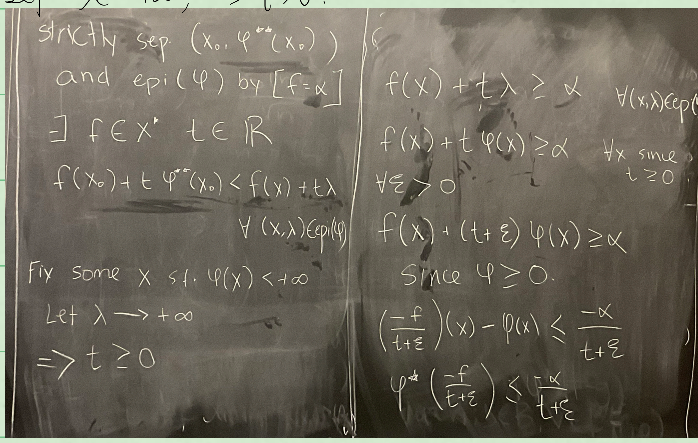
Strictly sep. $(x_0, \varphi^{**}x_0)$ and $\text{epi}(\varphi)$

$\exists f \in X^*,\ t \in \mathbb{R}$

$f(x_0) + t\varphi^{**}(x_0) < f(x) + t\lambda \quad , \forall (x,\lambda) \in \text{epi}(\varphi)$

Fix some x, s.t. $\varphi(x) < +\infty$

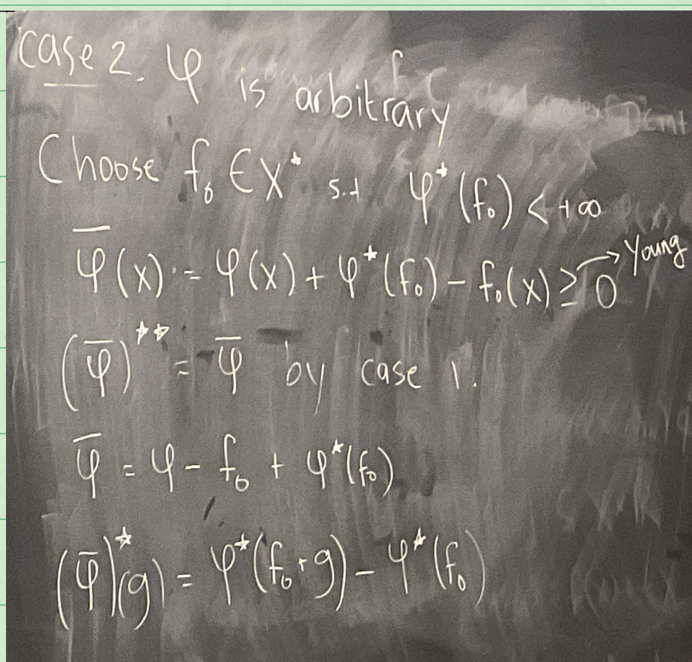
Let $\lambda \to +\infty$, $\Rightarrow t \geq 0$.



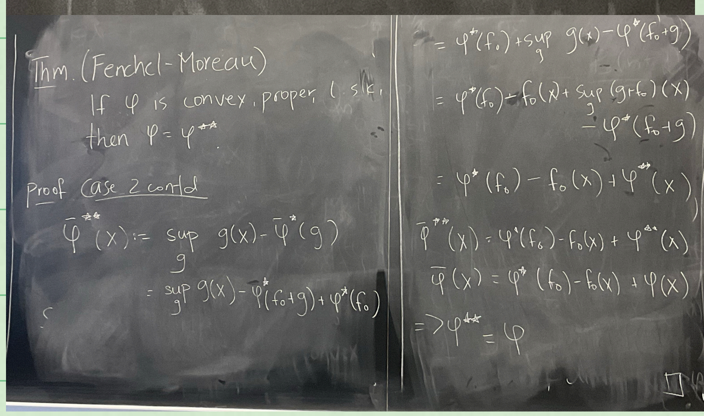
strictly sep. $(x_0, \varphi^{**}(x_0))$
and $\text{epi}(\varphi)$ by $[f = \alpha]$
$\exists f \in X^*$ $t \in \mathbb{R}$
$f(x_0) + t\varphi^{**}(x_0) < f(x) + t\lambda$
$\forall (x,\lambda) \in \text{epi}(\varphi)$
Fix some x s.t. $\varphi(x) < +\infty$
Let $\lambda \longrightarrow +\infty$
$\Rightarrow t \geq 0$

$f(x) + t\lambda \geq \alpha \quad \forall (x,\lambda) \in \text{epi}(\varphi)$
$f(x) + t\varphi(x) \geq \alpha \quad \forall x$ since $t \geq 0$
$\forall \varepsilon > 0$
$f(x) + (t+\varepsilon)\varphi(x) \geq \alpha$
since $\varphi \geq 0$.
$\left(\frac{-f}{t+\varepsilon}\right)(x) - \varphi(x) \leq \frac{-\alpha}{t+\varepsilon}$
$\varphi^*\left(\frac{-f}{t+\varepsilon}\right) \leq \frac{-\alpha}{t+\varepsilon}$

Case 2:



case 2. φ is arbitrary
Choose $f_0 \in X^*$ s.t. $\varphi^*(f_0) < +\infty$
$\overline{\varphi}(x) := \varphi(x) + \varphi^*(f_0) - f_0(x) \geq 0 \rightarrow$ Young
$(\overline{\varphi})^{**} = \overline{\varphi}$ by case 1.
$\overline{\varphi} = \varphi - f_0 + \varphi^*(f_0)$
$(\overline{\varphi})^*(g) = \varphi^*(f_0 + g) - \varphi^*(f_0)$



Thm. (Fenchel-Moreau)
If φ is convex, proper, l.s.c.,
then $\varphi = \varphi^{**}$.

Proof Case 2 cont'd
$\overline{\varphi}^{**}(x) := \sup_g g(x) - \overline{\varphi}^*(g)$
$= \sup_g g(x) - \varphi^*(f_0+g) + \varphi^*(f_0)$
$= \varphi^*(f_0) + \sup_g g(x) - \varphi^*(f_0+g)$
$= \varphi^*(f_0) - f_0(x) + \sup_g (g+f_0)(x) - \varphi^*(f_0+g)$
$= \varphi^*(f_0) - f_0(x) + \varphi^{**}(x)$
$\overline{\varphi}^{**}(x) = \varphi^*(f_0) - f_0(x) + \varphi^{**}(x)$
$\overline{\varphi}(x) = \varphi^*(f_0) - f_0(x) + \varphi(x)$
$\Rightarrow \varphi^{**} = \varphi$
$\square$