$x' = ax$ 只有 $x(t) = Ce^{at}$.

Proof: 设 $y(t) = e^{-at}\cdot x(t)$

若 x 为一解，则 $y'(t) = -ae^{-at}\cdot x + e^{-at}x'$

$= -ae^{-at}\cdot x + ae^{-at}x$

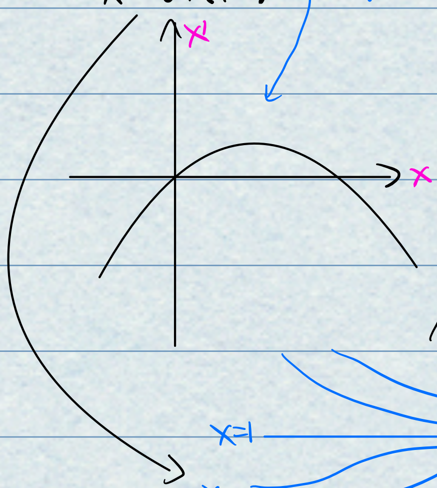
$= 0$

$\Rightarrow y'(t) = 0 \Rightarrow x = Ce^{at}$

Example:

$x' = ax\left(1-\frac{x}{N}\right)$, WLOG, $N=1$

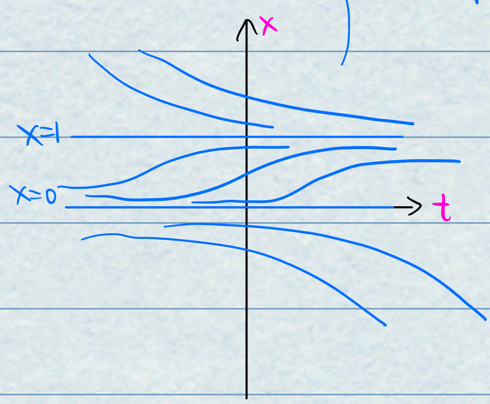
$\Rightarrow x' \overset{= f_a(x)}{=} ax(1-x)$ Non-linear.

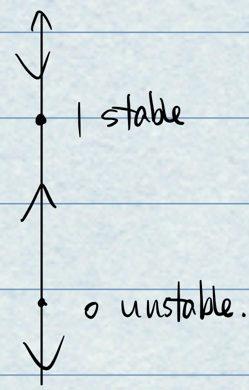


equilibrium point: $ax(1-x)=0$

$\Rightarrow x=1$ / $x=0$

从这里看出来.





Solve $x' \overset{= f_a(x)}{=} ax(1-x)$. 特解: $x(t)\equiv 1$ / $x(t)\equiv 0$

linear combination of those are not solutions. 因为 $f_a(x)$ non-linear.

"separation of variables": Review: $\frac{dx}{dt} = f(t)\,g(x)$

$g(x)=0 \Rightarrow x(t)\equiv y_0$ 为一解

① $g(x)\neq 0: \Rightarrow \frac{dx}{g(x)} = f(t)\,dt$

$\Rightarrow \int \frac{dx}{g(x)} = \int f(t)\,dt$

$\Rightarrow G(x) = F(t) + C$

$\Rightarrow x = G^{-1}(F(t)+C)$

② $\frac{dx}{dt} = f(t)g(x)$

$\Rightarrow \frac{x'(t)}{g(x(t))} = f(t)$

$\Rightarrow \int \frac{x'(t)dt}{g(x(t))} = \int f(t)dt$ $\int \frac{dx}{gx}$

$\Rightarrow G(x(t)) = F(t) + C.$ $\frac{d}{dt}G(x(t)) = \frac{x'(t)}{g(x(t))}$ 所以

$g(x_0) \neq 0 \Rightarrow g(x) \neq 0$ for small interval (C^∞)

$\Rightarrow G'(x) = \frac{1}{g(x)}$ 恒大于/小于 0 $\Rightarrow$ 可逆.

So, $x' = ax(1-x)$

$\Rightarrow \int \frac{1}{ax(1-x)}dx = t + C.$

$\Rightarrow \frac{1}{a}\int \frac{dx}{x(1-x)} = t + C.$

$\Rightarrow \int (\frac{1}{x} + \frac{1}{1-x})dx = at + C$

$\Rightarrow \ln|x| + \ln|1-x| = at + C$

$\Rightarrow \ln\left|\frac{x}{1-x}\right| = at + C$ e^C 正

$\Rightarrow \left|\frac{x}{1-x}\right| = |C_2|e^{at}$

$\Rightarrow \frac{x}{1-x} = C_2 e^{at}$

$\Rightarrow x = C_2e^{at} - xC_2e^{at}$

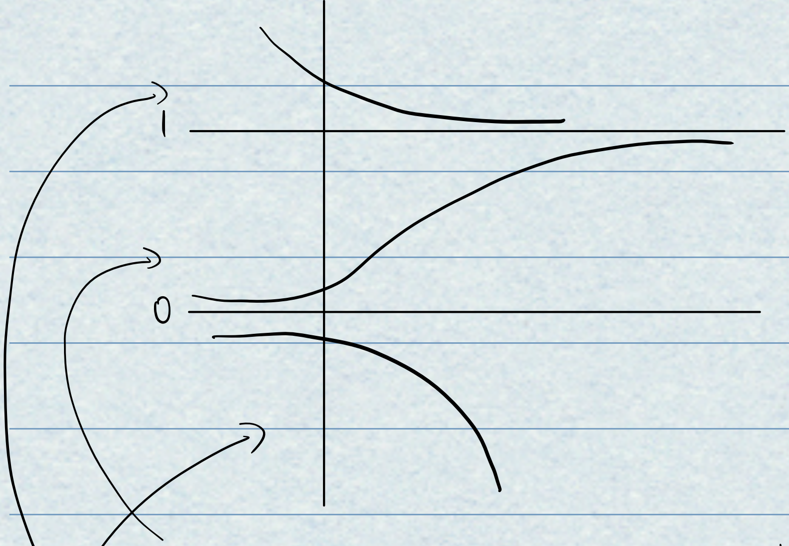
$\Rightarrow x(1 + C_2e^{at}) = C_2e^{at}$

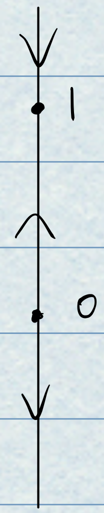
$\Rightarrow x = \frac{C_2e^{at}}{1 + C_2e^{at}}$ 之前解了 $x \equiv 1$ "$C_2 = \infty$" $x \equiv 0$ $C_2 = 0$

用初值表示 C_2:

$x(0) = \frac{C_2}{1+C_2} \to C_2 = \frac{x(0)}{1-x(0)}$

$\Rightarrow x(t) = \frac{C_2e^{at}}{1+C_2e^{at}} = \frac{x_0}{(1-x_0)e^{-at} + x_0}$



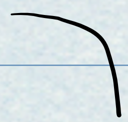


① $0<x_0<1$ $\quad t\nearrow \quad (1-x_0)e^{-at}\searrow \quad x\nearrow$

② $x_0>1$

③ $x_0<0$, $t\uparrow$, $\dfrac{x_0}{(1-x_0)e^{-at}+x_0}$ 那分母会从正变负，所以

x 会 1



渐近线

linear sys of ODEs:

$x' = A(t)x + f(t)$

$n\times n$ matrix of coefficient

inhomogeneity.

$f(t) = \begin{pmatrix} f_1(t) \\ \vdots \\ f_n(t) \end{pmatrix}$

① if A = constant $\Rightarrow$ constant coe... ODE

② $f(t)=0 \Rightarrow$ homogeneous.

($x'-Ax=f$ ($Bx=C$)也相似)

① $f=0$

if $x'=F(x)$, $\Rightarrow \exists x_0$, s.t. $F(x_0)=0$, then x_0 is an equ... of the system)

若 $F(x)=Ax$, $x=0$

若 $\det A=0$, $\Rightarrow \exists$ straight line of eqn points.

ex: $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

① diagolize $\Rightarrow \tilde{x}'=a\tilde{x}$, $\tilde{y}'=b\tilde{y}$.

$n=1$: $x'=at$ $\quad a\in\mathbb{R} \Rightarrow x(t)=e^{at}x_0$

guess: $x(t) = e^{tA} \cdot x_0 \leftarrow \mathbb{R}^n$, $t \in \mathbb{R}$.

define: e^{tA}, $A \in \mathbb{R}^{n \times n}$.

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k$$ 收敛,可导 — later.

$x(t) = e^{tA} \cdot x_0$ is a solution of $\begin{cases} x' = Ax \\ x(0) = x_0 \end{cases}$, general solution.

② $x(t) = e^{\lambda t} \cdot v$

vector scalar constant vector (ansatz)

"plug in": check $x(t) \in C$

$$x'(t) = \lambda \cdot e^{\lambda t} v$$

$$Ax = A(e^{\lambda t} v) = e^{\lambda t} Av$$

$\Rightarrow x(t) = e^{\lambda t} \cdot v$ is a solution $\Leftrightarrow e^{\lambda t} \lambda v = e^{\lambda t} A v \quad \forall t.$

$\Leftrightarrow Av = \lambda v.$

i.e. λ is a eigen value with eigen vector v.

$A = \begin{pmatrix} 2 & 3 \\ 1 & 0 \end{pmatrix} \Leftrightarrow \begin{matrix} x' = 2x + 3y \\ y' = x \end{matrix}$

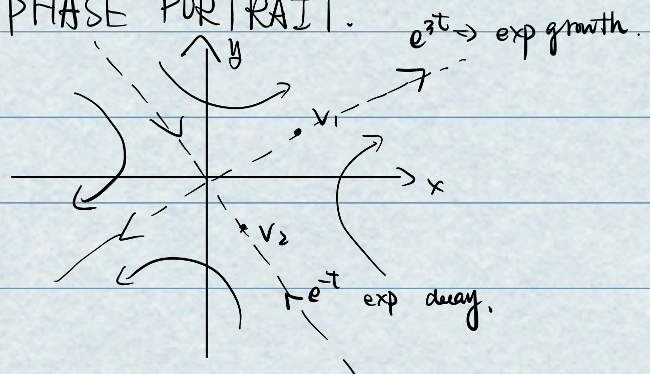
find 特征向量/值 $\lambda = -1/3$, $v_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$ $v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$x_1(t) = e^{3t} \begin{pmatrix} 3 \\ 1 \end{pmatrix} = e^{\lambda_1 t} v_1$

$x_2(t) = e^{-t} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$\Rightarrow$ to $x' = Ax$

PHASE PORTRAIT.



(linearity)

Superposition principle

若 $x_1(t)$ 为 $x'=A(t)x+f_1(t)$ 的解
$x_2(t)$ 为 $x'=A(t)x+f_2(t)$

令 $a_1,a_2\in\mathbb{R}$. 则 $x(t)=a_1x_1(t)+a_2x_2(t)$ 为 $x'=A(t)x+a_1f_1(t)+a_2f_2(t)$ 的解

$\Rightarrow$ ① $x'=A(t)x$ 的解构成向量空间

设 $X'=AX$, 且 $AV_0=\lambda V_0$, 则 $X(t)=e^{\lambda t}V_0$ 为一解

$$X'(t)=(e^{\lambda t})'V_0$$
$$=\lambda e^{\lambda t}V_0$$
$$=e^{\lambda t}(AV_0)$$
$$=A(e^{\lambda t}V_0)$$
$$=AX(t)$$

$X'=AX$; $X\in\mathbb{R}^n$, $A\in\mathbb{R}^{n\times n}$

设 $v_1,v_2,\cdots v_n$ 线性无关的特征向量. $(Av_i=\lambda_iv_i)$

则 $X_i(t)=e^{\lambda_it}v_i$ 为解

By superposition, 有 n 维 解构成的向量空间

$x(t)=a_1x_1(t)+a_2x_2(t)+\cdots+a_nx_n(t)$

general solution.

Pf (1): existence & uniqueness thm:

Pf (2): $\{v_1,\cdots v_n\}$ 为 $\mathbb{R}^n$ 的基. fix $a_1,a_2,\cdots a_n\in\mathbb{R}$, 令 $y(t)=a_1e^{\lambda_1t}v_1+\cdots+a_ne^{\lambda_nt}v_n$.

By superposition, y solves $\begin{cases}X'=AX\\ X(0)=a_1v_1+\cdots+a_nv_n\end{cases}$ $\leftarrow t=0$

下证 y 唯一.

令 $z(t)$ 为另一解 $\begin{cases} X' = AX \\ X(0) = X_0 \end{cases}$

$\{v_1, \cdots, v_n\}$ 为基 $\Rightarrow z(t) = b_1(t)v_1 + \cdots + b_n(t)v_n$. b_i 为实值函数

? 则 $b_i(0) = a_i$ 因为 $z_0 = x_0 = y(0)$

$z'(t) = b_1'(t)v_1 + \cdots + b_n'(t)v_n$.

$Az(t) = A(b_1(t)v_1 + \cdots + b_n(t)v_n)$

$= b_1(t)\lambda_1 v_1 + \cdots + b_n(t)\lambda_n v_n$.

$\begin{cases} b_i'(t) = \lambda_i b_i(t) \\ b_i(0) = a_i \end{cases} \Rightarrow b_i(t) = e^{\lambda_i t} a_i$

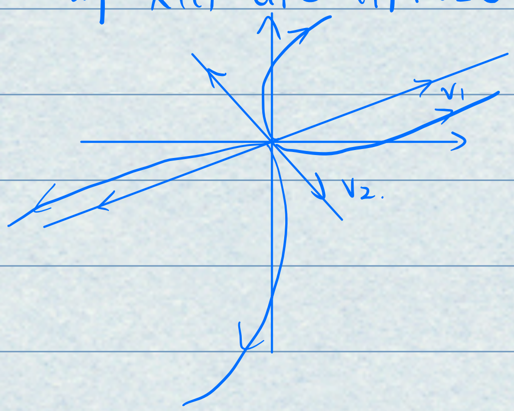
$b_i(t)$ 为 $z(t)$ 与 linear projection 的 composition, 所以可导.

不一定是 orthonormal 的.

DEF: $X' = AX$ has a saddle point @ 0 if A has a positive and negative eigenvalue.

e.g.2: $B = A + 2I = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix}$, 则 $\lambda = 5/1$, $v = \begin{pmatrix} 3 \\ 1 \end{pmatrix} / \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

则 $x(t) = a_1 e^{5t}\begin{pmatrix} 3 \\ 1 \end{pmatrix} + a_2 e^{t}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$



eg3: 若两个特征值均负

eg4: no real eigenvalues:

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \lambda = \pm i, \quad v = \begin{pmatrix} i \\ 1 \end{pmatrix} / \begin{pmatrix} -i \\ 1 \end{pmatrix}$$

$$\begin{cases} z(t) = e^{it} v_1, \\ \overline{z(t)} = e^{-it} v_2. \end{cases}$$

z is a complex solution of $X' = AX$ $\Leftrightarrow$ $z_{re} = Re(z)$, $z_{im} = Im(z)$

are solutions of $X' = AX$.

Pf: $z_{RE}'(t) + i z_{im}'(t) = z'(t) = A z(t) = A(z_{re}(t) + i z_{im}(t))$

$= A z_{RE}(t) + i A z_{im}(t)$

$F(x,y) \to F(tx,ty) = t^{\alpha} F(x,y)$

$\Rightarrow$ F is homogeneous of degree α.

* · $P(x,y)dx + Q(x,y)dy = 0$

has same

then $y = x \cdot u$, $dy = u dx + x du$

$x = y \cdot v$ $\quad dx = v dy + y dv$.

$\Rightarrow$ change * to separable variable ODE

can solve.

eg: $(x e^{\frac{y}{x}} - y \sin(\frac{y}{x}))dx + x \sin\frac{y}{x} dy = 0$

$u=\frac{y}{x}$ $(y=xu)$

$\to (xe^u \cancel{-xu\sin u})dx + \cancel{x\sin u}(xdu+\cancel{udx})=0$

$\Rightarrow xe^u dx + x^2(\sin u)du = 0$

$\Rightarrow e^u dx + x(\sin u)du=0$

$\Rightarrow -\frac{dx}{x} = \frac{\sin(u)}{e^u}du$

$\Rightarrow -\ln|x| = \frac{\sin(u)+\cos u}{e^u}(-\frac{1}{2})+C.$ 代入 y.

$J=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ $x'=Jx$

$\lambda_1=i,\ \lambda_2=-i=\bar{\lambda}_1$

$v_1=\begin{pmatrix}i\\1\end{pmatrix},\ v_2=\begin{pmatrix}-i\\1\end{pmatrix}$

So $z(t)=(c)e^{it}\begin{pmatrix}i\\1\end{pmatrix}=(\cos t+i\sin t)\left(\begin{pmatrix}0\\1\end{pmatrix}+i\begin{pmatrix}1\\0\end{pmatrix}\right)$

$=\begin{pmatrix}-\sin t\\\cos t\end{pmatrix}+i\begin{pmatrix}\cos t\\\sin t\end{pmatrix}$

$\bar{z}$ 类似

$x'=\begin{pmatrix}0&\beta\\-\beta&0\end{pmatrix}x$

note: general solution of ODE is

设 $Y(t)=\begin{pmatrix}u(t)\\v(t)\end{pmatrix}$ 为一解，则

$X(t)=a\begin{pmatrix}-\sin t\\\cos t\end{pmatrix}+b\begin{pmatrix}\cos t\\\sin t\end{pmatrix}$

$\begin{pmatrix}u'(t)\\v'(t)\end{pmatrix}=\begin{pmatrix}0&\beta\\-\beta&0\end{pmatrix}\begin{pmatrix}u(t)\\v(t)\end{pmatrix}=\begin{pmatrix}\beta v(t)\\-\beta u(t)\end{pmatrix}$

设 $y(t)=\begin{pmatrix}v(t)\\u(t)\end{pmatrix}$

$\Rightarrow f'(t)=e^{i\beta t}(u'(t)+iv'(t)+i\beta u(t)-v(t))$

let $f(t)=(u(t)+iv(t))e^{-it}$

$=0$

☆ $f'(0)\Rightarrow u(t)+iv(t)=\eta e^{it}$ $\eta\in\mathbb{C}$.

e.g. 5:

$A=\begin{pmatrix}2&-1\\1&2\end{pmatrix}$ $x'=Ax$

$\lambda=2\pm i$ $v=\begin{pmatrix}\pm i\\1\end{pmatrix}$

So $z(t)=c_1e^{\lambda_1 t}v_1+c_2e^{\lambda_2 t}v_2.$

$$x(t) = \mathrm{Re}(z(t)) = \frac{1}{2}\left(c_1 e^{\lambda_1 t} v + \bar{c} e^{\bar{\lambda}_1 t} \bar{v}\right)$$

Construct real solution explicitly:

$z = e^{\lambda t} v$, $\bar{z} = e^{\bar{\lambda} t} \bar{v}$ real solution.

$$x_1(t) = \mathrm{Re}(z(t)) = \frac{1}{2}(z + \bar{z})$$

$$x_2(t) = \mathrm{Im}(z(t)) = \frac{1}{2i}(z - \bar{z})$$

$$z(t) = e^{\lambda t} v = e^{(2+i)t}\left(i\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right)$$

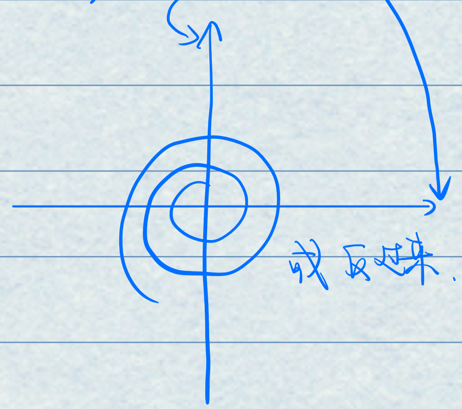
$$= e^{2t}(\cos t + i \sin t)\left(\begin{pmatrix}0\\1\end{pmatrix} + i\begin{pmatrix}1\\0\end{pmatrix}\right)$$

$$= \underbrace{e^{2t}\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix}}_{x_1(t) = \mathrm{Re}(z)} + i\underbrace{e^{2t}\begin{pmatrix}\cos t\\ \sin t\end{pmatrix}}_{\mathrm{Im}(z) = x_2(t)}$$

linear independent solution.

General solution: $x(t) = c_1 x_1(t) + c_2 x_2(t)$, $c_1, c_2 \in \mathbb{R}$.

$$x_1(t) = e^{2t}\begin{pmatrix}-\sin t\\ \cos t\end{pmatrix} \leftarrow 2\pi \text{ 周期}$$



或反过来.

$\lambda = \alpha + i\beta$ 复特征值

growth/decay ; osillation.

$$e^{(2+i)t}\left(i\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\right)$$

α. growth

β, frequency.

e.g. 6: repeated eigenvalues

$x' = Ax$, $A = \begin{pmatrix}3 & -1\\ 4 & -1\end{pmatrix}$

So $\lambda = 1$ (double) $v = \begin{pmatrix}1\\2\end{pmatrix}$

$\Rightarrow$ 1-dim space of solutions.

$x(t) = ce^{t}\binom{1}{2}$

generalized eigenvector: $(A-\lambda I)w = v \in \mathrm{Range}\,(A-\lambda I)$

i.e. $(A-\lambda I)^2 w = 0$

$w \in \ker (A-\lambda I)^2 \setminus \ker(A-\lambda I)$

$w = \binom{0}{-1} \quad v = \binom{1}{2},$

$\begin{cases} Av = \lambda v \\ Aw = \lambda w + v \end{cases}$

$x_2(t) = \alpha(t)v + \beta(t)w.$

non-0

if x C^1, then α, β C^1.

$x_2'(t) = \alpha'(t)v + \beta'(t)w$

$AX = \alpha(t)Av + \beta(t)Aw$

$= \alpha(t)\lambda v + \beta(t)(\lambda w + v)$ $\quad \swarrow \lambda = 1$

$= (\alpha(t)\lambda + \beta(t))v + (\beta(t)\lambda)w$

$\Rightarrow \begin{cases} \alpha' = \alpha\lambda + \beta \\ \beta' = \lambda\beta \end{cases}$ | triangular form.

(2) $\Rightarrow$ $\beta = Ce^{\lambda t}$

(1): $(\lambda = 1)$

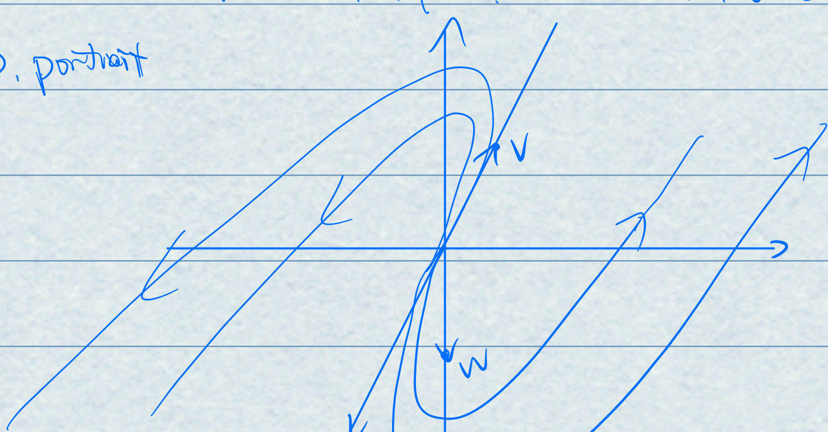
$\alpha' - \alpha = e^t$ $\leftarrow$ inhomogeneous

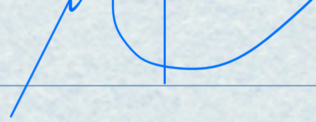
$\hookrightarrow \alpha = te^t$

So $\begin{cases} x_1(t) = e^t v \\ x_2(t) = te^t v + e^t w. \end{cases}$

$\Rightarrow x = \alpha x_1 + \beta x_2 \Rightarrow a_1 e^t v + a_2 e^t (tv + w)$

P. portrait





e.g. 7: $\lambda = -2$ double.

$$B = \begin{pmatrix} 0 & -1 \\ 4 & -4 \end{pmatrix} \qquad v = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \qquad Av = -2v$$

$$w = \begin{pmatrix} 0 \\ -1 \end{pmatrix}. \qquad Aw = -2w + v$$

General solution: $x(t) = a_1 e^{-2t} \underbrace{\begin{pmatrix} 1 \\ 2 \end{pmatrix}}_{=v} + a_2 e^{-2t}(w + tv)$

8

EXAMPLE 8: 0 EIGENVALUE:

$x' = Ax$, $A \in \mathbb{R}^{2\times 2}$,

$\lambda_1 > 0$, $\lambda_2 = 0$

GENERAL SOL: $x(t) = a_1 e^{\lambda_1 t} v_1 + a_2 v_2$

$e^{0t} = 1$

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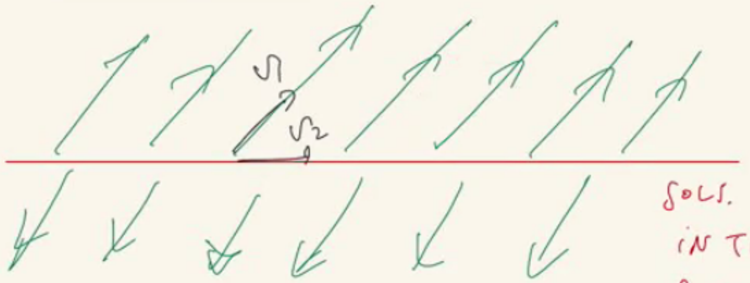


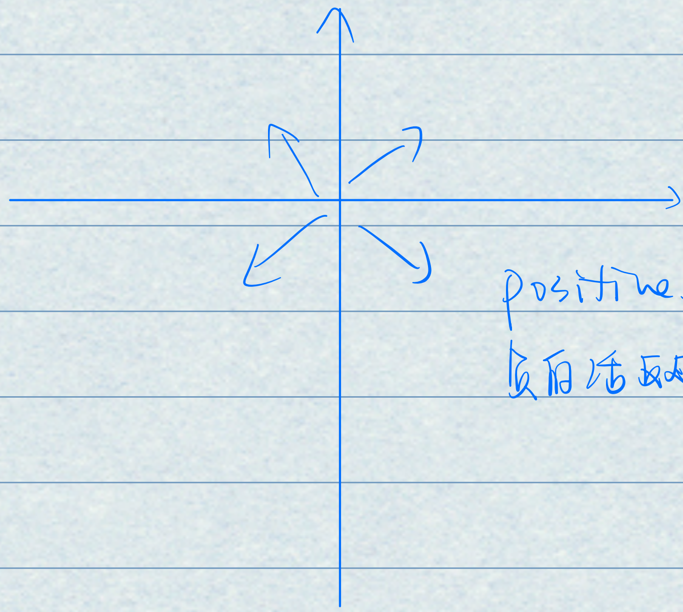
SOLS. DON'T MOVE IN THIS LINE

STEADY-STATES

EQUILIBRIA

e.g. ④: λ double, v_1, v_2 independant.

$$x(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} v_2$$

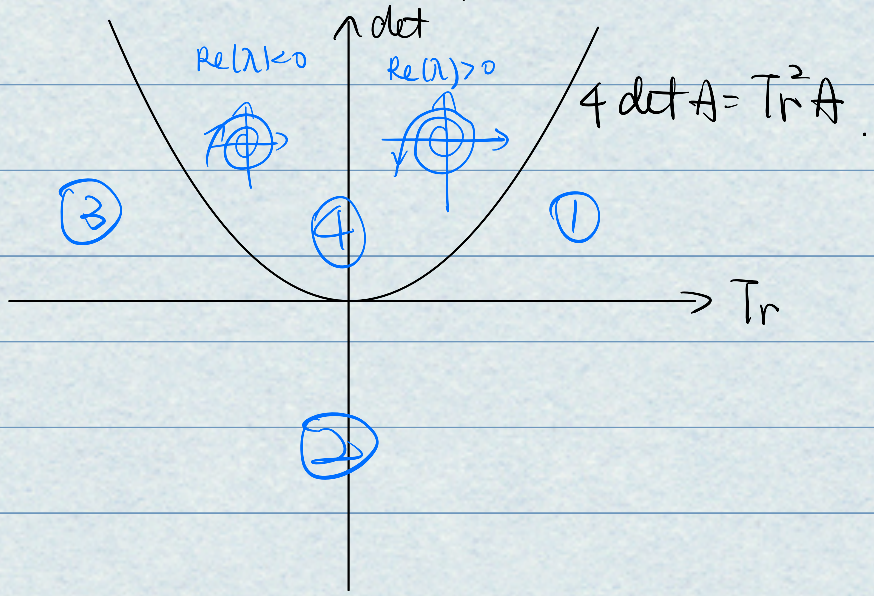


positive.
负的话反过来.

$\underline{x' = Ax}$ classification $A \in \mathbb{R}^{2\times 2}$

$$P(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A)$$

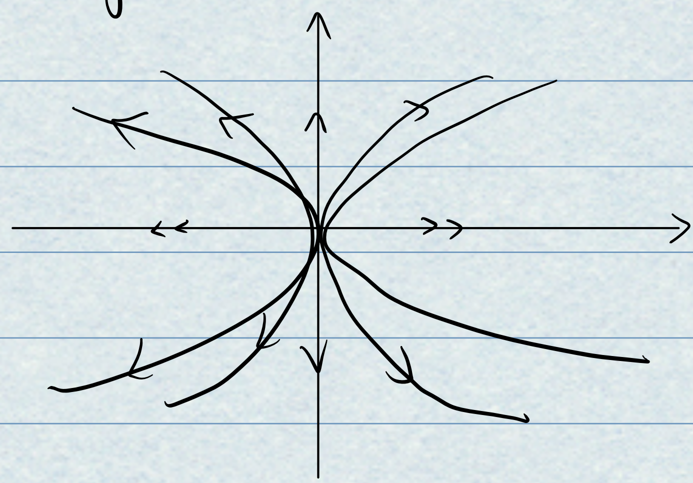
discriminant: $\mathrm{Tr}^2 - 4\det$



Case ①: $\lambda_1 > \lambda_2 > 0$

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

nondegenerate $(\lambda_1 \neq \lambda_2)$



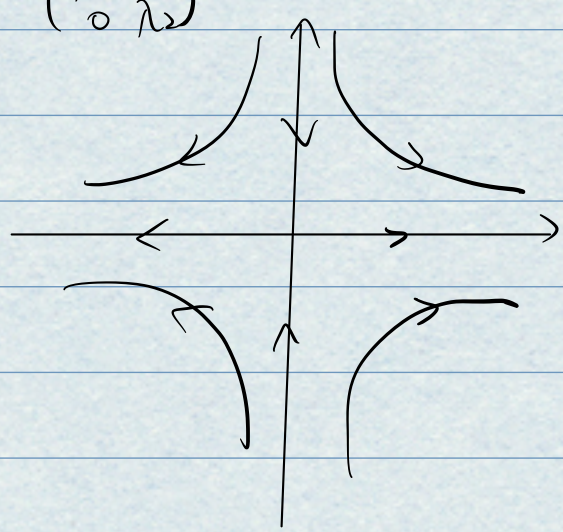
0 is an unstable node.

DETECT: $Tr^2(A) > 4\det(A)$ ([illegible])

$(Tr > 0)$

case ②: $\lambda_1 > 0 > \lambda_2$.

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$

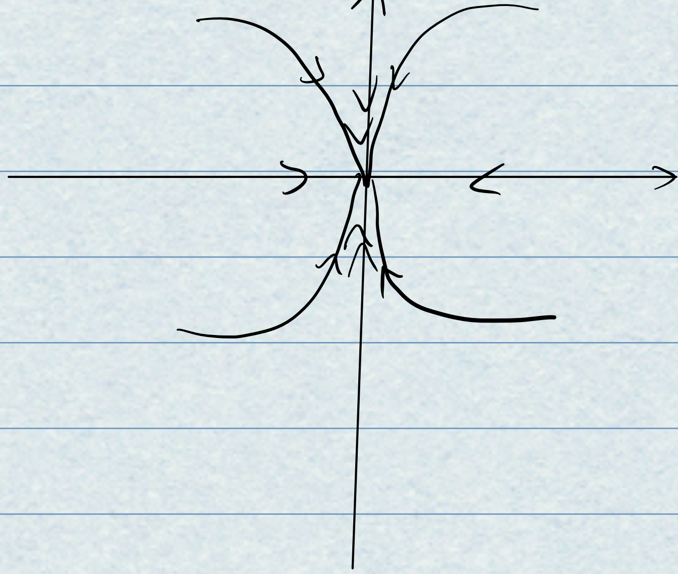


saddle
(unstable)

DETECT: $\det(A) < 0$

case ③ : $0 > \lambda_1 > \lambda_2$

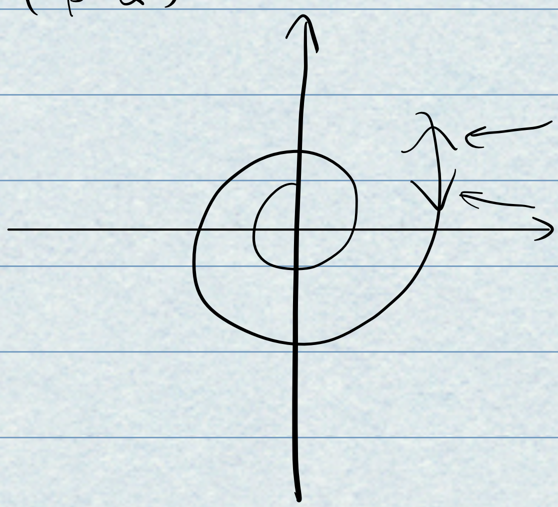
$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$$



sink
(stable)

DETECT: $\det(A) > 0$, $\operatorname{tr}(A) < 0$,
$4\det(A) < \operatorname{Tr}^2(A)$

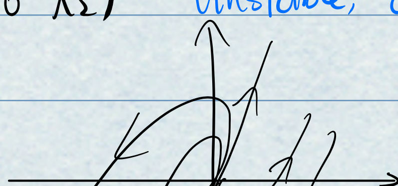
Case ④: $\lambda = \alpha \pm i\beta$
$\begin{pmatrix} \alpha & -\beta \\ \beta & \alpha \end{pmatrix}$



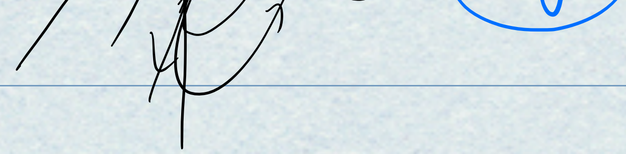
$\alpha > 0$ unstable
$\alpha < 0$ stable.

Degenerate / exceptional case:

Case ⑥: $\lambda_1 = \lambda_2 > 0$ 1 eigenvector. 负的同理.
$\begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}$ unstable, degenerate.

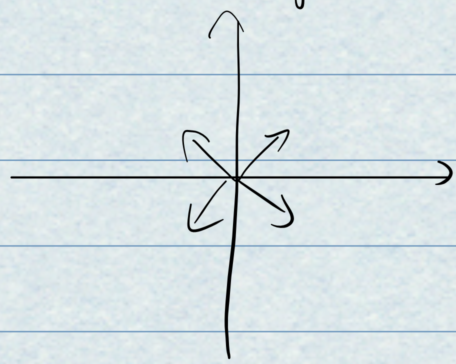


e.g. 6



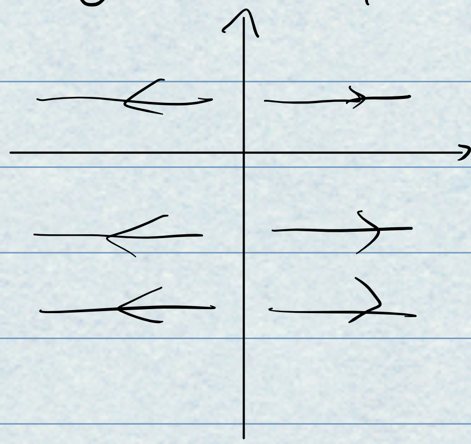
DETECT: $Tr(A) > 0$, $4\det(A) = Tr^2(A)$

case ⑦ $\lambda_1 = \lambda_2 > 0$, 2 eigenvector. with curve Σ.

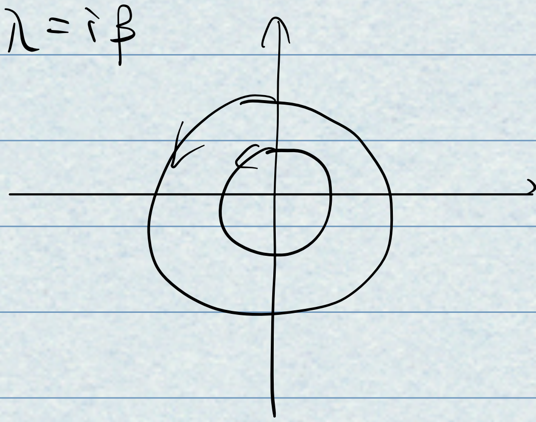


负的同理.

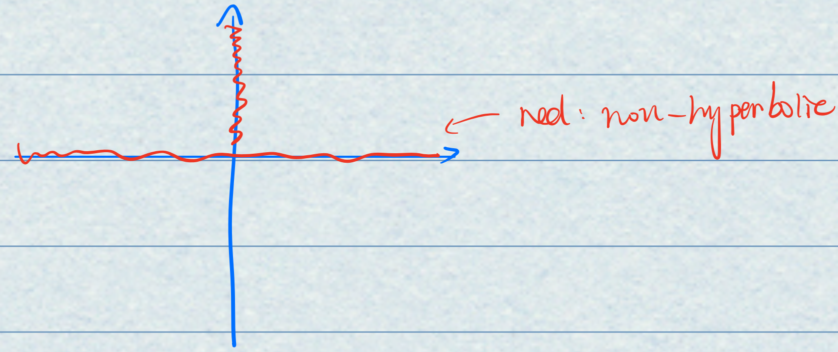
case ⑧: $\begin{pmatrix} \lambda_1 & 0 \\ 0 & 0 \end{pmatrix}$ $\lambda_1 > 0$



case ⑨: $\lambda = i\beta$



DEF: 0 is a hyperbolic equlibium for $x' = Ax$ if all eigenvalue of A have non-zero real part.



DEF: A property is called generic if it is satisfied on an open, dense subset on $\mathbb{R}^{n\times n}$.

e.g. $\det A \neq 0$, hyper...

Thm: The set of matrices in $\mathbb{R}^{n\times n}$ that have n distinct eigenvalues is generic in $\mathbb{R}^{n\times n}$ (P101)

pf: if 有 $P_t(x) = \sum_0^N a_i x^i$ $a_n(t)$. 若 a_n 为连续函数 (fact), 待证

则 $P_t(x)$ 仍然连续.

idea: $\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \ddots & \lambda_n \end{pmatrix}$

CALEY-HAMILTON Thm:

$A \in \mathbb{R}^{n\times n}$, $P_A(\lambda)$ 特征多项式, 则

$$P_A(A) = 0 \qquad (*)$$

PF: ① A 对角 — 显然.

② * 在基变换下保持不变

$A = CDC^{-1} \Leftrightarrow \left(P_A(A) = 0 \Leftrightarrow P_A(D) = 0 \right)$

??? ★ ③ 用 ---- Matrix (↑) are dense

?????? 因为 $a_n(t)$ 连续, 则 (*) stable under lim

考虑 $x' = AX$ $A \in \mathbb{R}^{n\times n}$, change of variable (basis)

$X = Ty$, $T \in \mathbb{R}^{n\times n}$, $\det(T) \neq 0$

y ?

$y' = (T^{-1}x)' = T^{-1}x' = T^{-1}Ax = T^{-1}A(Ty)$.

$\Rightarrow y' = (T^{-1}AT)\, y$ $= C$.

Goal: 化简 C.

e.g. $A = \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \in \mathbb{R}^{2\times 2}$.

$\lambda = 1, 3$, $v = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Know: $A v_i = \lambda_i v_i \Leftrightarrow A(v_1 \cdots v_n) = (v_1 \cdots v_n)\begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$

$\Rightarrow \begin{pmatrix} 1 & 0 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} = \underbrace{\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}}_{\text{"T"}} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$\Rightarrow T^{-1}AT = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

A可对角：有n个线性无关特征值.

DEF: $C = T^{-1}AT$

Solve $x' = AX$, $A \in \begin{cases} \mathbb{R}^{n\times n} \\ \mathbb{C}^{n\times n} \end{cases}$

① find $C = TAT^{-1}$

② 解 $y' = Cy$ $\quad (T = (v_1 \cdots v_n))$

③ $X(t) = Ty(t)$

Jordan Canonical form

$C = \begin{pmatrix} J_1 & \\ & \ddots \end{pmatrix}$ $\quad J_i = \begin{pmatrix} \lambda_i & 1 & \\ & \ddots \end{pmatrix}$

$$C=\begin{pmatrix} \ddots & \\ & J_n \end{pmatrix} \qquad J_1=\begin{pmatrix} \ddots & \\ 0 & \lambda_1 \end{pmatrix}$$

同样的λ可以出现在不同的J中. J与$\bar{J}$ 可以同时出现

J-C form

① find C of T:

设λ为一特征值 of algebraic multiplicity m, 找J?

e.g. m=5

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & 1 \\ & & & & \lambda \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & 1 & \\ & & & \lambda & 0 \\ & & & & \lambda \end{pmatrix} \quad etc$$

5 $\qquad$ 4+1

$$J=\lambda Id+\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \leftarrow N$$

$N^L=0, \quad N^{L-1}\neq 0$

考虑 $Null(A-\lambda)^m$ 来决定

Step ① find J

Step ② Solve $y'=Jy$

$$J=\lambda I+\underbrace{\begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & \ddots & \\ & & & 0 \end{pmatrix}}_{=L}$$

①

to simplify: $y(t)=e^{\lambda t}z(t) \Rightarrow z(t)=e^{-\lambda t}y(t)$ 消掉λI项

$$z'=-\lambda e^{-\lambda t}y+e^{-\lambda t}y'$$

$$=-\lambda e^{-\lambda t}y+e^{-\lambda t}(\lambda I+L)y$$

$$=Lz$$

So $z'=\begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 0 \end{pmatrix}z \Rightarrow \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}'=\begin{pmatrix} z_2 \\ \vdots \\ 0 \end{pmatrix}$

$\Rightarrow z_1'=z_2, \ z_2'=z_3 \ldots \ z_n'=0$

$z_n=C_1 \qquad z_{n-1}=C_1x+C_2 \quad \cdots \quad z_1=C_1x^{n-1}+\cdots+C_n$

$\Rightarrow \quad y=e^{\lambda t}z$

②

$y'=(\lambda I+N)y$

Claim: solution is $y(t) = e^{t(\lambda I + N)} y_0$ ↙ $y(0)$

$= e^{t\lambda I} e^{tN} y_0$

$e^{t\lambda I} = (e^{t\lambda}) I$ 因为 $\quad = e^{t\lambda}\left(I + tN + \frac{t^2}{2}N^2 + \cdots + \frac{t^{l-1}}{(l-1)!}N^{l-1}\right) y_0$

$e^{tA} = \sum \frac{t^k}{k!} A^k$

Assume claim, $y' = Cy$, $x' = Ax$, $C = T^{-1}AT$, $x = Ty$.

recall $y' = (T^{-1}x)' = T^{-1}Ax = T^{-1}ATy$.

$e^{tA} = \sum \frac{t^k}{k!} A^k$. 而 $A^k = TC^kT^{-1}$

$= \sum \frac{t^k}{k!} TC^kT^{-1}$

$= Te^{tC}T^{-1}$

Question: ? $E(t) = e^{tA}$ solve $E' = AE$ with $E(0) = Id$

Check: e^{tA} 的 defined / 收敛 / 可导

Define norm: $|A| = \sup\limits_{v \in \mathbb{R}^n} \frac{|Av|}{|v|} = \sup\limits_{|u|=1} |Au|$

Note: $|Av| \le |A||v|$, $|AB| \le |A||B|$

pf: $|AB| = \sup \frac{|ABv|}{|v|} \le \sup \frac{|A|\,|Bv|}{v} = |A||B|$

b/c 有 $e^{tA} = \sum$ converges absolutely:

$\sum \left|\frac{t^k}{k!} A^k\right| = \sum \frac{|t^k|}{k!} |A^k| \le \sum \frac{|t^k|}{k!} |A|^k$

$= e^{|t||A|}$

FACT: ① $e^{(s+t)A} = e^{sA} e^{tA}$

② $e^{t(A+B)} = e^{tA} \cdot e^{tB} = e^{tB} e^{tA}$

$\Updownarrow$ $\quad$ ⇑ we need

$AB = BA$

pf ①: $e^{(t+s)A} = \sum \frac{(t+s)^k}{k!} A^k$

$= \sum\limits_k \frac{\sum\limits_{l=1}^{k} \frac{k!}{l!(k-l)!} t^l s^{k-l}}{k!} A^k$

$$= \sum_{k=1} \sum_{l=1} \frac{t^l}{l!} \frac{s^{k-l}}{(k-l)!} A^l A^{k-l}$$

$$= e^{sA} e^{tA}$$

② $AB = BA \Rightarrow (A+B)^k = \sum_l \binom{k}{l} A^l B^{k-l}$

$$e^{tA} = I + tA + \sum_{k\geq 2} \frac{t^k}{k!} A^k$$

$$+ t^2 \sum_{\geq 0} \frac{t^k}{(k+2)!} A^k \leftarrow \text{bounded near } 0$$

$$e^{tA} = I + tA + o(t^2)$$

$$\Rightarrow \frac{d}{dt} e^{tA} \Big|_{t=0} = A$$

$$\Rightarrow E(s+t) = e^{(s+t)A}$$

$$= e^{sA} e^{tA}$$

$$= e^{sA} I + e^{sA} tA + o(t^2)$$

$$\Rightarrow E'(s) = A\,E(s)$$

Claim: $\begin{cases} X' = AX \\ X(0) = X_0 \end{cases}$ 的唯一解为 $X(t) = e^{tA} X_0$

Pf: existance: $y(t) = e^{tA} x_0$

$y_0 = x_0$,

$$y'(t) = \frac{d}{dt}\left(e^{tA} x_0\right) = \left(\frac{d}{dt} e^{tA}\right) x_0$$

$$= A e^{tA} x_0$$

$$= A y(t)$$

unique:

$z(t) = e^{tA} x_0$

$w(t) = e^{-tA} z(t)$

$w(0) = x_0$, $(w(t))' = \frac{d}{dt}\left(e^{-tA} z(t)\right)$

$$= -A e^{-tA} z(t) + e^{-tA} z'(t)$$

$\Rightarrow W(t)=0$

e^{tA} 是 $\begin{cases} x'=Ax \\ x(0)=y \end{cases}$ 的解. i.e. e^{tA}, by definition, the unique solution of IVP

流 映 map: $e^{tA}: (t,y) \to e^{tA}y = x(t)$

$y \to x(\cdot) = e^{\cdot A}y$.

equivalently: $e^{0A}=Id$, $\frac{d}{dt}e^{tA} = Ae^{tA}$

e.g. ① $e^{tA} = \sum \frac{t^k}{k!}A^k$

② if $A = TCT^{-1}$, (Jordan c form,

$e^{tA} = Te^{tC}T^{-1}$

③ $\begin{cases} \overset{\text{solve}}{x'} = Ax \\ x_0 = e_i = i^{th} \text{ unique vector} \end{cases}$ $\Rightarrow x_1, \dots x_n$ ↑ each i.

$e^{tA} = \begin{pmatrix} \vdots & \vdots & & \vdots \\ x_1 & x_2 & \cdots & x_n \\ \vdots & \vdots & & \vdots \end{pmatrix}$

e.g. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $A^2 = -Id$

$$A^k = \begin{cases} Id & k=4l \\ A & k=4l+1 \\ -Id & k=4l+2 \\ -A & 4l+3. \end{cases}$$

So $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k = \sum_{j=0}^{\infty} \frac{t^{2j}}{(2j)!}(-1)^j Id + \frac{t^{(2j+1)}(-1)^j}{(2j+1)!}A$

$= \cos t\, Id + \sin t\, A$

$= \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$

e.g. want $A \in \mathbb{R}^{3\times 3}$ skew-symmetric

$$A = \begin{pmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{pmatrix} \qquad e^{tA}\ ?$$

$\{e^{tA}\}$ is a subgroup (copy of SO2) in SO3.

$\uparrow$ rotations.

① Normalization: $a^2+b^2+c^2=1$

To compute e^{tA}, use eigenvalue/vector.

$\lambda_1 = 0$ $\quad$ ($A^T = -A$ $\leftarrow$ det of skew-symmetric)

$\lambda_2 = -\lambda_3$ (skew-skew symmetric

$\lambda_2 = i$, $\lambda_3 = -i$

$\det(A-\lambda I) = \det(A-\lambda I)^T = (-1)^n \det(A+\lambda I)$ ☆)

因为 $a^2+b^2+c^2=1$，展开特征多项式

$v \perp \bar{v}$, $v, \bar{v} \perp \begin{pmatrix} c \\ b \\ a \end{pmatrix}$ in $\mathbb{C}^3$

$|\langle x, y\rangle| = x_1\bar{y}_1 + x_2\bar{y}_2 + x_3\bar{y}_3$

Fact: $Re(v) \perp Im(v) \Rightarrow$ orthogonal

So canonical form $\begin{pmatrix} 0 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{pmatrix}$

real: $\left(\begin{array}{c|cc} 0 & 0 & 0 \\ \hline 0 & 0 & -1 \\ 0 & 1 & 0 \end{array}\right) = C$

$$e^{tC} = \left(\begin{array}{c|cc} 1 & 0 & 0 \\ \hline 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{array}\right)$$ blocks move seperately.

$e^{tA} = T e^{tC} T^{-1}$

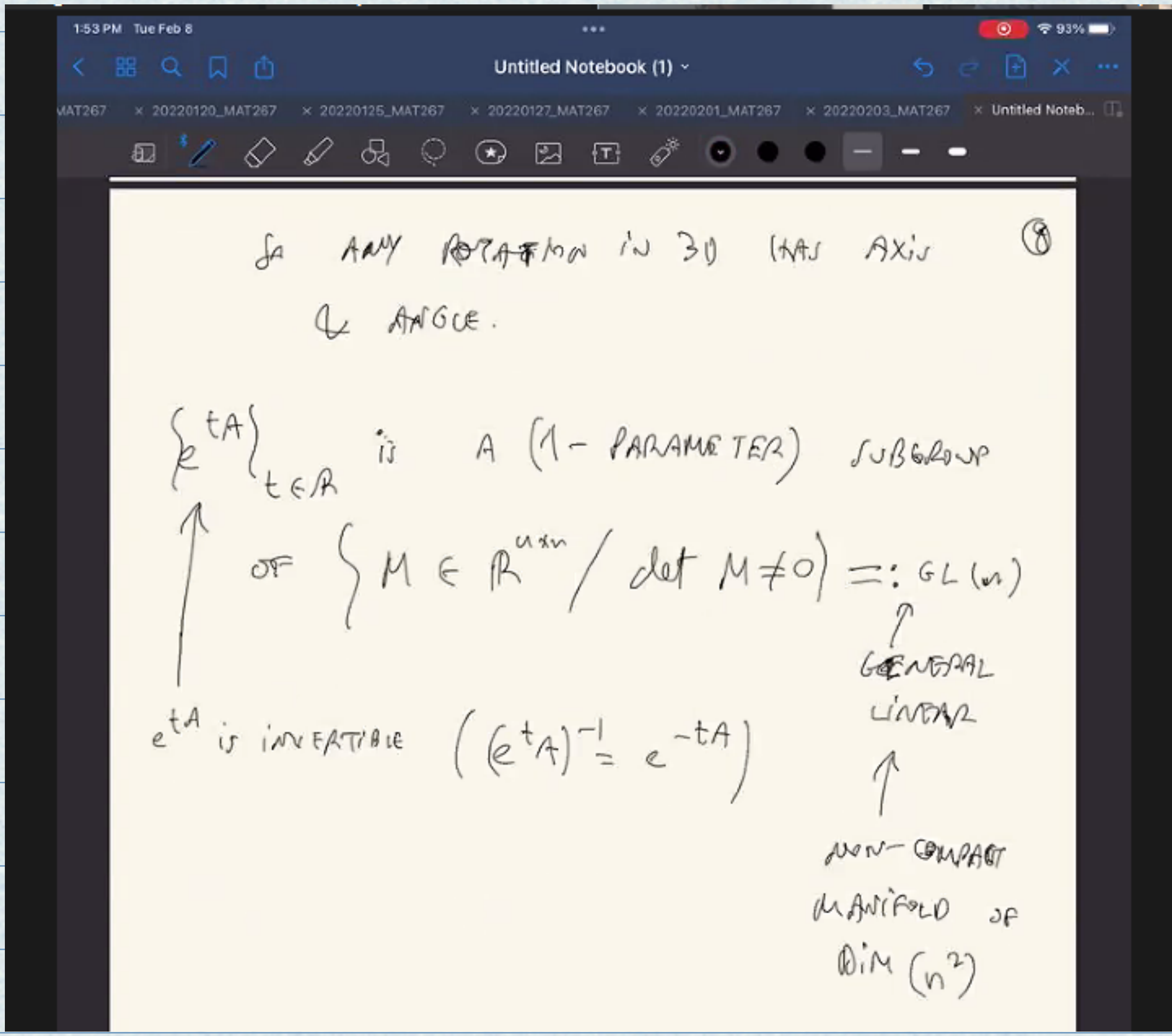
$T = \begin{pmatrix} c & | & | \\ b & Re(v) & Im(v) \\ a & | & | \end{pmatrix}$ $\Rightarrow e^{tA}$ is a rotation.

Solving $X' = AX + f(t)$

know: STRUCTURE Thm: general solution is given

$x(t) = y(t) + e^{tA} v$

特解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8

SO ANY ROTATION IN 3D HAS AXIS & ANGLE.

$\{e^{tA}\}_{t \in \mathbb{R}}$ IS A (1-PARAMETER) SUBGROUP OF $\{M \in \mathbb{R}^{n \times n} / \det M \neq 0\} =: GL(n)$

GENERAL LINEAR

NON-COMPACT MANIFOLD OF DIM (n^2)

e^{tA} IS INVERTIBLE $\left((e^{tA})^{-1} = e^{-tA}\right)$

Goal: find a specific solution of the ode

$x' = Ax + f(t)$ "Duhamel's principle"

$x(t) = y(t) + e^{tA} v$

Guess: $y(t) = e^{tA} v(t)$

$x' = Ax + f(t)$

$y' = Ae^{tA} v(t) + e^{tA} v'(t)$ LHS

$= Ae^{tA} v(t) + f(t)$ RHS

$\Rightarrow \quad e^{tA} v'(t) = f(t)$

$\Rightarrow \quad v'(t) = e^{-tA} f(t)$ 两边积分

$\Rightarrow \quad v(t) = \underset{v_0 = 0}{v_0} + \int_0^t e^{-sA} f(s)\,ds$ （特解）

Transform back: $y(t) = e^{tA} v(t)$ 代值

$= e^{tA} \int_0^t e^{-sA} f(s)\,ds$

$= \int_0^t e^{(t-s)A} f(s)\,ds$ Duhamel's formula.

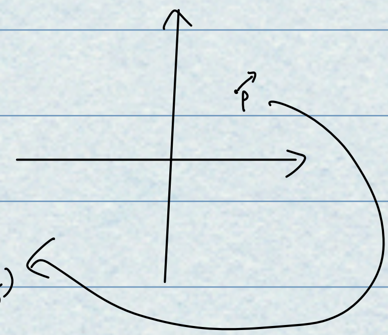
"Linearization" (overview)

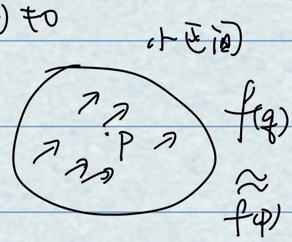
$x' = f(x)$; $f: \mathbb{R}^n \to \mathbb{R}^n$

(autonomous system)

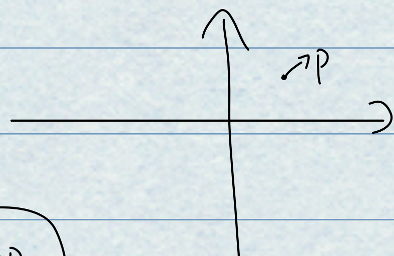
p点周围的运动

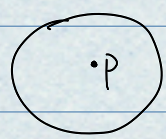
② cases ① $f(p) \neq 0$





② $f(p) = 0$





$x(t) \equiv p$ is a solution

taylor: $f(x) = f(p) + \underbrace{Df(p)}_{=A}(x-p) + \bar{o}(|x-p|)$

令 $y = x - p$，则 $y' = \underbrace{Df(p)}_{A} \cdot y + \bar{o}(|y|)$

?: 何时亦被A确定

(N) $y' = Ay + \bar{o}(|y|)$, $A = Df(p)$, $\bar{o}(|y|)$ 余项

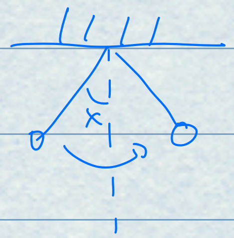
線性化 ← (L) ⇒ $y' = Ay$

Q1: Suppose the linear system has a source/sink, is the same for (N)? √

Q2: Does phase portrait look similar?

Yes if ? is hyperbolic. ($Re(\lambda) \neq 0$ ← all)

e.g. 机械钟摆



$$\underset{=1}{m}x'' + \overset{\text{friction}}{r x'} = -\underset{=1}{c}\sin(x) \quad (\text{“设”})$$

$$x'' + rx' + \sin x = 0$$

1st order system: $x_1' = x_2$

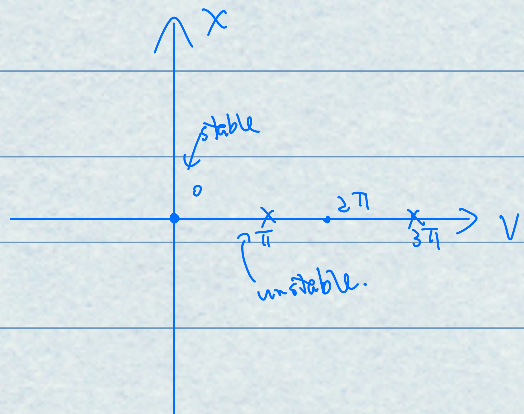
$x_2' = -rx_2 - \sin(x_1)$

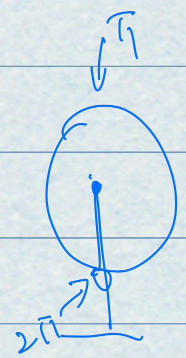
$\left(\begin{array}{l} x_1 \to x \\ x_2 = x' = v \end{array} \right)$

$$\begin{cases} x' = v \\ v' = -rv - \sin(x) \end{cases} \qquad (r > 0 \text{ constant})$$

steady state ← 稳态: $v = 0$, $x = k\pi$

(angle mod 2π is same)





f_1

$$f(x,v) = \begin{pmatrix} v \\ -rv-\sin(x) \end{pmatrix} \leftarrow f_2 \qquad \begin{pmatrix} x \\ v \end{pmatrix}' = f\begin{pmatrix} x \\ v \end{pmatrix}$$

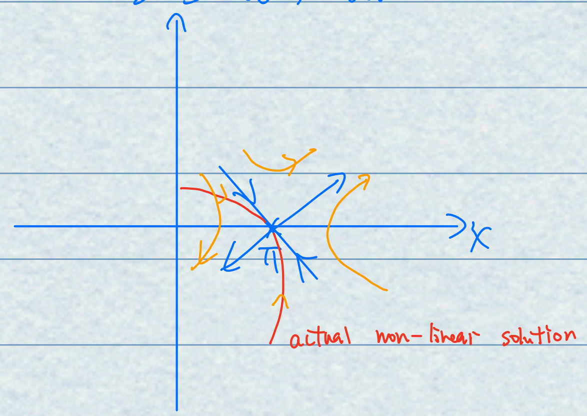
$$Df(k\pi,0) = \begin{pmatrix} 0 & 1 \\ -\cos x, & -r \end{pmatrix} \Big|_{x=k\pi}$$

$$= \begin{cases} \begin{pmatrix} 0 & 1 \\ -1 & -r \end{pmatrix} & k \text{ even} \Rightarrow \text{Top} \\ \begin{pmatrix} 0 & 1 \\ 1 & -r \end{pmatrix} & k \text{ odd} \Rightarrow \text{bot} \end{cases}$$

top: (L) $\Rightarrow \begin{pmatrix} 0 & 1 \\ 1 & -r \end{pmatrix} = A$

det $= -1 \Rightarrow \lambda$ are real, 异号

$\Rightarrow$ saddle, unstable.

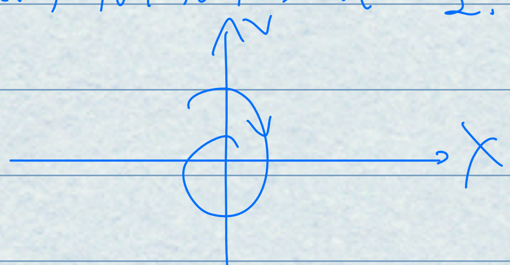


bot: $\begin{pmatrix} 0 & 1 \\ -1 & -r \end{pmatrix}$ $r \geq 0$

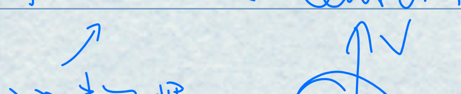
$r > 0$: linearization has a sprial

我作 ↻, N 也一样

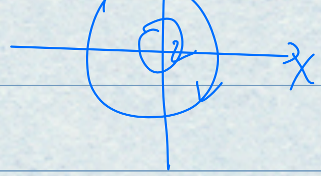
$p(\lambda) = \lambda^2 + r\lambda + 1 \Rightarrow \lambda = \frac{-r \pm \sqrt{r^2-4}}{2}$.



if $r=0$: $\Leftarrow$ center, $\lambda = \pm i$



大限来回摆



Procedure: ① find steady states
② linearize near ↓
③ tie all up in big picture.

Digerssion: [TP] lesson 9-11

Exact diff equations:

The ODE $P(x,y)\,dx + Q(x,y)\,dy = 0$ is exact if $\exists\, f(x,y)$, s.t. $\frac{\partial f}{\partial x} = P(x,y)$, $\frac{\partial f}{\partial y} = Q(x,y)$. In that case, 1 parameter family of solutions is $f(x,y) = C$.

e.g. $y\,dx + x\,dy = 0$, $f(x,y) = xy$

$\Rightarrow xy = C$ ⇐ solution.

Thm: have ODE

(ODE) $P(x,y)\,dx + Q(x,y)\,dy = 0$

P, Q, $\frac{\partial P}{\partial x}$, $\frac{\partial P}{\partial y}$, $\frac{\partial Q}{\partial x}$, $\frac{\partial Q}{\partial y}$, 存在. 连续 in simply connected region R. (open, connected, simply connected)

(a) (ODE) is exact $\Leftrightarrow$ (b) $\frac{\partial}{\partial y} P(x,y) = \frac{\partial}{\partial x} Q(x,y)$

Proof: (Also give a method of construction)

"$\Rightarrow$": $P = \frac{\partial f}{\partial x}$, $Q = \frac{\partial f}{\partial y}$.

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x \partial}$$

"$\Leftarrow$" if f exist, has to satisfy

$$\frac{\partial f}{\partial x} = P(x,y) \Rightarrow f(x,y) = \int_{x_0}^{x} P(s,y)\,ds + R(y)$$ ↙ y 的函数

$$\frac{\partial f}{\partial y} = Q(x,y) \Rightarrow$$

$$Q(x,y) = \frac{\partial}{\partial y}\left(\int_{x_0}^{x} P(s,y)\,ds + R(y)\right)$$

$$= \int_{x_0}^{x} \underbrace{\frac{\partial}{\partial y} P(s,y)}_{= \frac{\partial}{\partial x} Q(x,y)}\,ds + R'(y)$$

$$= Q(x,y) - Q(x_0,y) + R'(y)$$

$$\Rightarrow R'(y) = Q(x_0,y)$$

$$R(y) = \int_{y_0}^{y} Q(x_0,y)\,dy$$

$$f(x,y) = \int_{x_0}^{x} P(s,y)\,ds + \int_{y_0}^{y} Q(x_0,y)\,dy$$

Now check: $\frac{\partial f}{\partial x} = P$ ✓

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int_{x_0}^{x} P(s,y)\,ds + Q(x_0,y)$$

$$= \int_{x_0}^{x} \frac{\partial}{\partial y} P(s,y)\,ds + Q(x_0,y)$$

$$= \int_{x_0}^{x} \frac{\partial}{\partial s} Q(s,y)\,ds + Q(x_0,y) = Q(x,y)$$

So $f = C$ solve the ode.

e.g. $(2x + y\cos x)\,dx + (2y + \sin x - \sin y)\,dy = 0$

Exact: $\frac{\partial P}{\partial y} = \cos x = \frac{\partial Q}{\partial x}$

$$f(x,y) = \int 2x + y\cos x\,dx = x^2 + y\sin x + R(y)$$

↓导

$$\frac{\partial f}{\partial y} = 2y + \sin x - \sin y = \sin(x) + R'(y)$$

$$R'(y) = y^2 + \cos y + C$$

$$f(x,y) = x^2 + y\cos x + y^2 + \cos(y) = C.$$

If not exact:

DEF: a multiplying factor that will convert an inexact ODE $\to$ exact is called a integrating factor.

e.g. $(t^2x-t)\,dx + x\,dt = 0$

$$\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$$

if $\exists$ in... f. $f(t)$

Let $P^*(t,x) = (t^2x-t)f(t)$

$Q^*(t,x) = x f(t)$

$$\frac{\partial P^*}{\partial t} = (2tx-1)f(t) + (t^2x-t)f'(t)$$

$$\frac{\partial Q^*}{\partial x} = f(t)$$

$$\Rightarrow \frac{f'(t)}{f(t)} = \frac{2-2tx}{t^2x-t} = -\frac{2}{t}$$ 只与t有关 (luckily)

可解 $\Rightarrow f(t) = \frac{1}{t^2}$

in... factor depends on x.

· similar if $f(x)$ $f(tx)$ $f(\frac{x}{t})$ $f(\frac{t}{x})$

$h(\frac{x}{t})$

GOOD NEWS: ANY ODE $x' = f(x,t)$ WITH $f \in C^1$ IN A NEIGHBOURHOOD OF (x_0, t_0) ADMITS AN INTEGRATING FACTOR

BAD NEWS: THIS METHOD IS PRETTY USELESS IN PRACTICE. (17)

ONSTANTS

(2) INTEGRATING FACTOR:

REWRITE ODE AS:

$$\left[P(x)\, y - Q(x)\right] dx + dy = 0$$

ASSUME $\exists$ $u(x)$ INTEGRATING FACTOR (18)

WANT:

$$\left[u(x)\, P(x)\cdot y - u(x)\, Q(x)\right.$$

ASSUME $\exists$ $u(x)$ INTEGRATING FACTOR (18)

WANT:

$$\frac{\partial}{\partial y}\Big[\underbrace{u(x)\, P(x)\cdot y - u(x)\, Q(x)}_{= P^*}\Big] = \frac{\partial}{\partial x}\overbrace{u(x)}^{= Q^*}$$

$$\rightarrow u(x) \cdot P(x) = u'(x) \rightarrow u(x) = e^{\int P(x)\,dx}$$

SO NOW HAVE:

$$\underbrace{P(x)\, e^{\int P(x)\,dx}\, y\, dx + e^{\int P(x)\,dx}\, dy}_{= d\left(e^{\int P(x)\,dx}\cdot y\right)} = e^{\int P(x)\,dx}\, Q(x)\,dx$$

$$\rightarrow \left(e^{\int P(x)\,dx}\right)\cdot y = \int \left(e^{\int P(x)\,dx}\right) . Q(x) + C$$

$$\rightarrow \boxed{y = \left(e^{-\int P(x)\,dx}\right) \int \left(e^{\int P(x)\,dx}\right) Q(x) + C\, e^{-\int P(x)\,dx}}$$

$\cdot\ \frac{dy}{dx} + P(x)y = Q(x)$ ☆ 把左边凑成 $(k(x)y)'$ 形式

$\Rightarrow F\frac{dy}{dx} + F\,p(x)y = F\,Q(x)$ 其中 $F = e^{\int p(x)}$

$\Rightarrow \frac{d}{dx}(F\cdot y) = F\,Q(x)$

$\Rightarrow Fy = \int F\,Q(x)dx$

$\Rightarrow y = \frac{\int F\,Q(x)dx}{F}$

㊟ $\frac{dy}{dx} + P(x)y = Q(x)y^n$ $n=1$ ✓.

$n\neq 1$: $\downarrow$

$\underbrace{(1-n)y^{-n}\frac{dy}{dx}}_{\frac{d}{dx}(y^{1-n}) = \frac{du}{dx}} + (1-n)p(x)y^{(1-n)} = (1-n)Q(x)$

$\frac{du}{dx} + \tilde{P}(x)u = \tilde{Q}(x)$

e.g. $y' + xy = \frac{x}{y^3}$ $y\neq 0$

$4y^3 = (1-n)y^{-n}$

$4y^3y' + 4xy^4 = 4x$

$\frac{dy^4}{dx} = \frac{du}{dx}$ $u = y^4$

$\frac{du}{dx} + 4xu = 4x$, $F = e^{\int 4x}$

乘 $h(x)$

$h(x)\,du + (h(x)4xu - h(x)4x)dx = 0$

$\frac{d}{du}$

$\Rightarrow h'(x) = h(x)\,4x$

$\Rightarrow h(x) = e^{2x^2}$

$e^{2x^2}du + (4xu\,e^{2x^2} - 4x\,e^{2x^2})dx = 0$

$\int du$

$\Rightarrow f(x,u) = u\cdot e^{2x^2} + R(x)$

$\frac{d}{dx}$

$\frac{\partial f}{\partial x} = 4xu\,e^{2x^2} - 4x\,e^{2x^2} = 4xue^{2x^2} + R'(x)$

$\Rightarrow R(x) = e^{-2x^2} + C.$

$f(x,u) = ue^{2x^2} - e^{2x^2}$

So $y^4 e^{2x^2} - e^{2x^2} = C$ 为原方程的解.

Dynamic systems

$X' = AX$. $A \in \mathbb{R}^{n\times n}$

$\hookrightarrow$ $x(t) = e^{tA} V$, $X_0 = V$.

1) fix $V = x(0)$, solution $t \to x(t) = e^{tA} x_0$

2) fix t, $V \to e^{tA} V$, isomorphism on $\mathbb{R}^n$

$\{e^{tA}\}_{t\in\mathbb{R}}$ family of isomorphisms.

成 group of identity.

pf of $e^{(t+s)A} = e^{tA} e^{sA}$

$e^{(t+s)A} V$ = solution $x(t+s)$ where $x_0 = V$

$= y(t)$ where $y(0) = e^{sA} V$

DEF: A Dynamic system $(\Phi_t)_{t\in\mathbb{R}}$ is a family of maps

$\Phi_t : \mathbb{R}^n \to \mathbb{R}^n$ (CC)

$\Phi_0 = Id$

$\Phi_{s+t} = \Phi_s \circ \Phi_t$ $\forall s,t$.

Here, $\Phi_t = e^{tA}$

e.g.s. (bad)

$X' = \begin{cases} -1 & x \geq 0 \\ 1 & x < 0 \end{cases}$ 在 0 处有问题

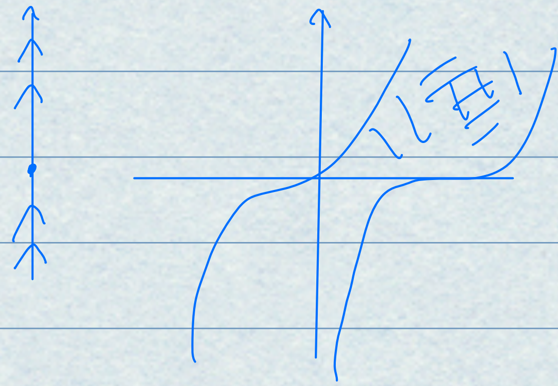
$\Rightarrow$ X cont

e.g. F. cont but bad

$x' = 3x^{\frac{2}{3}}$, seperation of variables.

$\int \frac{dx}{3x^{\frac{2}{3}}} = \int dt$

$x^{\frac{1}{3}} - x_0^{\frac{1}{3}} = t - t_0 \xrightarrow{t_0=0} x = (x_0^{\frac{1}{3}} + t)^3$



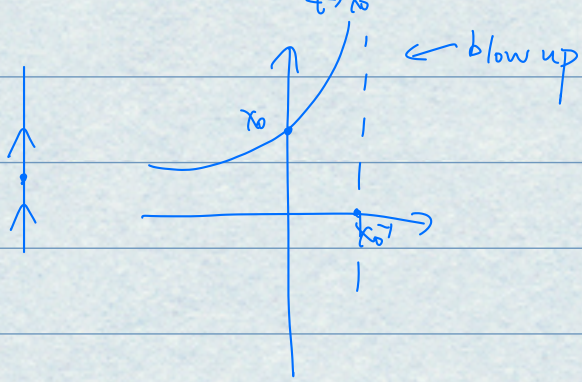
continues but not smooth $\Rightarrow$ exist ✓ uniqueness may fail.

e.g. 3. $x' = x^2$

$\int \frac{dx}{x^2} = \int dt \Rightarrow -\frac{1}{x} + \frac{1}{x_0} = t - t_0$ $\swarrow t_0 = 0$

$\Rightarrow \frac{1}{x} = \left(\frac{1}{x_0} - t\right)$

$x = \frac{x_0}{1 - x_0 t}$ $\qquad \lim_{t \to x_0^{-1}} x(t) = +\infty$



(Local) unique/exist thm

consider $x' = f(x)$, $f: \mathbb{R}^n \to \mathbb{R}^n$, C^1.

Thm: $\forall a \in \mathbb{R}^n$ $\exists I = (-\alpha, \beta) \ni (0,0)$. s.t. $\exists$ solution on $(-\alpha, \beta)$, s.t. $x(0) = a$.

Ascali_arzela thm:

① Def: $\mathcal{F} = \{f_\alpha\}_{\alpha\in A}$ $\underset{\in\mathbb{R}^n}{E} \to \mathbb{R}^n$

is uniformly bounded : $\exists M, |f(x)|<M$.

② DEF: $\mathcal{F}$ above,

$\mathcal{F}$ is equicontinuous if $\forall \varepsilon\ \exists \delta$. s.t. $\forall y \in E$,

$|x-y|<\delta \Rightarrow |f(y)-f(x)|<\varepsilon$.

e.g. 1) $\{f_\alpha(x) = ax\}_{\alpha\in\mathbb{R}}$, ① ✓ ② ×

但 $\alpha\in[3,5]$, ② ✓

2) $\{g_n(x) = x^n ; x\in[0,1]\}$ not equicont...

3) $\{f_\alpha(x)\}_{\alpha\in A}$ are lipschi? with same constant

$|f_\alpha(x) - f_\alpha(y)| < L|x-y|$

$f(x)-f(y) = f((1-t)x+ty) =$

$= \int_0^1 \nabla f(\ ) \cdot (y-x)\,dt$

$\leq \left(\int_0^1 |\nabla f|\,dt\right)|y-x|$

4) any finite family of uni... conti-- functions are equicont.

Notation: $f_n \rightrightarrows f$ $f_n \to f$ continuous converge

$\Leftrightarrow \|f_n - f\|_\infty = 0$

$\|f\|_\infty : \{\sup|f|\}$

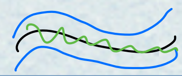
Continuous functions on $[0,1] = C[0,1]$, with $\|\cdot\|_\infty$ complete

metric space. Since

Thm: $f_k : E^{\subset \mathbb{R}^n} \to \mathbb{R}^m$ 连续.

if E 紧, $f_n \rightrightarrows f$, then $\{f_n\}$ is uniformly bounded and equicontinuous.

pf: f_k 连续, 紧, f_k bounded. 一致连续:



同理 $f(x)$ bounded. 任取 ε 后 f_k 就有限个 bound.

equi: $f_n \rightrightarrows f$

$$|f_n(x)-f_n(y)| \le \overset{\le\varepsilon}{|f_n x - f x|} + \overset{\le \varepsilon \text{ if } |x-y|<\delta}{|f x - f y|} + \underset{\le\varepsilon}{|f y - f_n y|}$$

$$\le 3\varepsilon \quad \text{if } |x-y|<\delta$$

Thm: (Ascoli-Arzela):

$\mathcal{F} = \{f_\alpha\}_{\alpha \in A > \infty}$, unif. bounded, equicont.

then $\exists \{f_n\}$ a sequence of diff functions in $\mathcal{F}$ that 在 E 上一致收敛

pf: 教授tm赶着回家做饭吗……

A set B is relatively compact if $\overline{B}$ is.

Cor: $A \subset C[K, \mathbb{R}^m]$ ($K \subset \mathbb{R}^n$ 紧)

is rela… compact in $\|\ \|_\infty \iff A$ is bounded and equicont.

3.2. Function Approximation:

Lemma: Let $f: \overline{B(0,r)} \to \mathbb{R}^m$ be cont.
$\overline{B(0,r)} \subseteq \mathbb{R}^n$

$$F(x): \mathbb{R}^n \to \mathbb{R}^n = \begin{cases} f(x) & |x| \le r \\ f\left(r\frac{x}{|x|}\right) & \end{cases}$$

f is cont and agree on $\overline{B(0,r)}$

3. Fixed point theorems:

3.1 B 啥啥啥 theorem (tut 6)

3.2. Brouwer's thm: Let $T: \overline{B(0,1)} \to \overline{B(0,1)} \subseteq \mathbb{R}^n$

T has at least 1 fix point.

$$\begin{cases} x' = f(x) \\ x_0 = v \end{cases}$$

$\nearrow$ 至少有一个解 exist

$\rightarrow$ 最多…… unique

$\searrow$ continue dependence on parameters.

① Existance

Cauchy's problem: (IVP)

IVP $\Bigg\{$ Given $f: [t_0, t_1] \in \mathbb{R}^n \to \mathbb{R}^n$, cont. $c_0 \in \mathbb{R}^n$,

find $x: [t_0, t_1] \to \mathbb{R}^n$, C^1,

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = c_0 \end{cases}$$

IEq. Given f, c_0 above

find $x: C^1$

$\forall t \in [t_0, t_1]$, $x(t) = \underbrace{c_0 + \int_{t_0}^{t} f(s, x(s))\,ds}_{= T_x(t) = U_x(t)}$

• IVT = IEq 显然

want: $T_x = x$

Cauchy-Peano. thm

(IVP): $\begin{cases} x'(t) = f(t, x(t)) \text{ in } [t_0, t_1],\ x(t) \in \mathbb{R}^n, \\ c_0 \end{cases}$

$f: [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ is cont. bdd $\left(|f(t, x(t))| \le M \right)$

then 有至少一个解

pf 1: (IEq) $x(t) = c_0 + \int_{t_0}^{t} f(s, x(s))\,ds \longrightarrow = T\,X(t)$ $(T_x(t))$

x是function.

$B = \{ x \in C([t_0, t_1]; \mathbb{R}^n) / |x(t)| \le 1,\ t \in [t_0, t_1] \}$

$|T_x(t)| \le |c_0| + \int | \overset{\le M}{\cdot} | ds \le |c_0| + M(t_1 - t_0)$

Case ①: ⓐ $|c_0| + M(t_1 - t_0) \le 1$: $T_B \subset B$

ⓑ If $x \in B$, $t, t' \in t_0, t$, $|T_x(t) - T_x(t')| \le M(t - t')$

So T_B is equicont.

$\overset{\shortparallel}{\{T_x\}_{x \in B}}$

ⓒ T is cont: 设 $\{x_k\}$, $x \in B$ and $x_k \overset{\text{uniform}}{\rightrightarrows} x$

i.e. $\|x_k - x\|_\infty \to 0$

$\|T_{x_k}(t) - T_x(t)\|_\infty \xrightarrow{?} 0.$

$|T_{x_k}(t)-T_x(t)| \le \int_{t_0}^{t} |f(s,x_k(s)) - f(s,x(s))|\,ds$

$\to 0$ 因为 $x_k \rightrightarrows x$, f is cont on $[t_0,t_1]\times[-1,1]^n$.

By Schauder – T…ff, $\exists x\in B,\ Tx = x.$

Case ②: $\underbrace{|c_0|+M(t_1-t_0)}_{H}>1$

define $g(t,x)=\frac{1}{H} f(t, Hx)$

则 g 满足以上所有条件，并入 case 1.

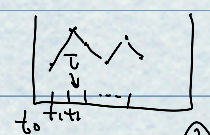
pf 2: digression: euler's poly.

$x'(t)=f(t,x(t))$

用 poly 接近.

$$\frac{x_{j+1}-x_j}{t_{j+1}-t_j}=f(t_j,x_j)$$

$$x_{j+1}=x_j+\tau f(t_j,x_j)$$



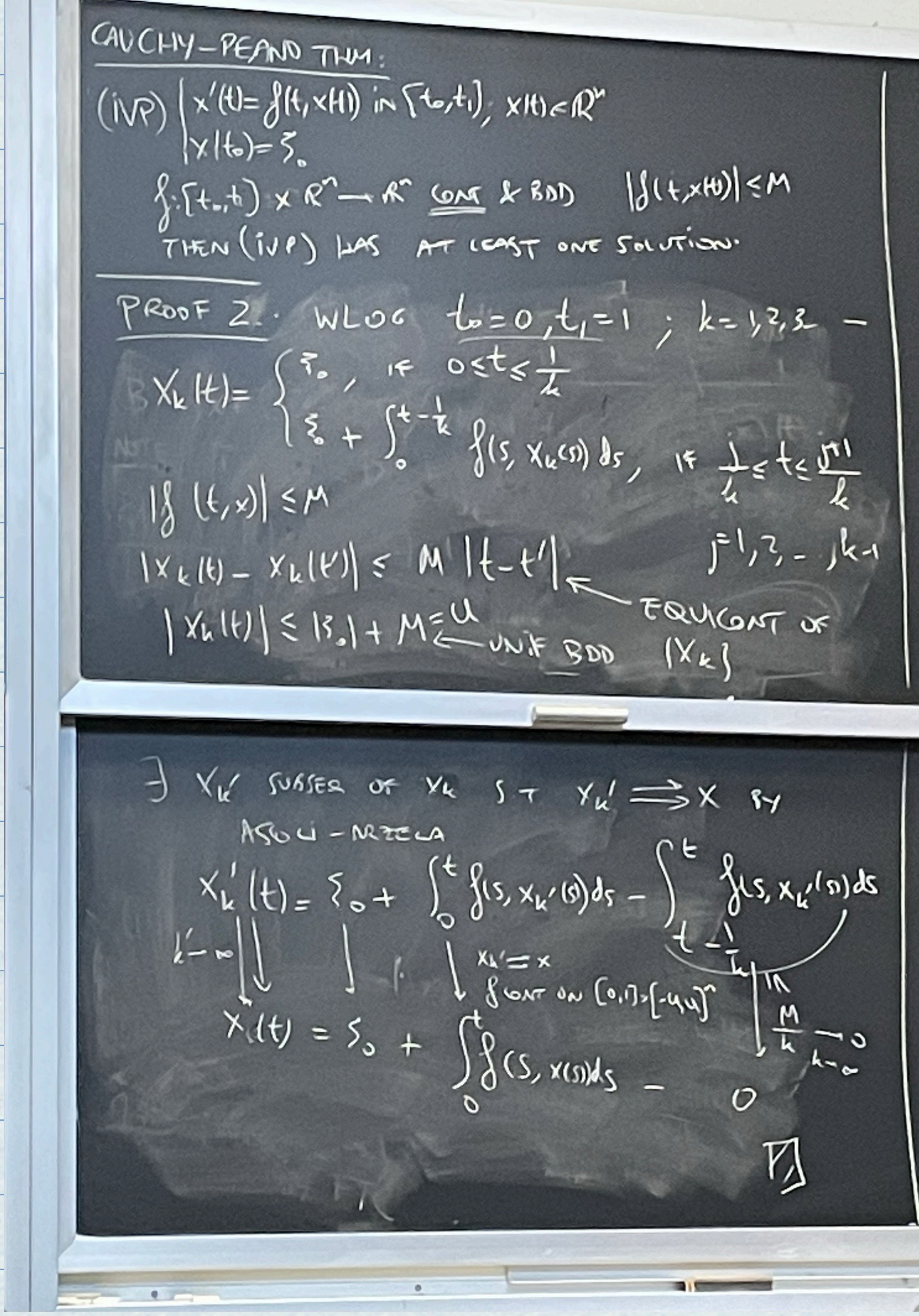
$k\in\mathbb{Z}$

WLOG: $t_0=0,\ t_1=1$; $x_k(t)=\begin{cases} c_0 & \text{if } 0\le t\le \frac{1}{k} \\ c_0+\int_0^{t-\frac{1}{k}} f(s,x_k(s))\,ds & \text{if } \frac{j}{k}\le t\le \frac{j+1}{k}\end{cases}$

$|f(t,x)|\le M$

$|x_k(t)-x_k(t')|\le M|t-t'|$

$|x_k(t)|\le |c_0|+M$



B- contraction mapping thm:

E : complete metric space, $T: E \to E$, s.t. $\exists\ 0 \le q < 1$,

s.t. $|T(x) - T(y)| \le q\, d(x,y)$

则有唯一不动点.

Consider

$$(IVP)_+ = \begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = \xi_0 \end{cases}$$

Goal: Find $I = [t_0, t_0+h]$ / $)$ $h > 0$, with $x \in C^1(I, \mathbb{R}^n)$,

s.t. $x \in IVP$ for $t \in I$.

If I closed: $x'(t_0+h) =$
left derivative.

Notation: X solves IVP_+, defined on I,

say (X, I) solves $(IVP)_+$

- The solution (X, I) can be continued to the right

if $\exists\ (\bar{X}, \bar{I})$, s.t. $\bar{I} \supset I$, $\bar{X}|_I = X$. Then $(\bar{X}, \bar{I})$ = continuation of (X, I)

Continuation is strict if $\bar{I} \supsetneq I$

Define order $(X_1, I_1) \geqslant (X_2, I_2) \Leftrightarrow I_1 \supset I_2$ and $X_2|_{I_1} = X_1$

Zorn's lemma, $\exists$ a maximal (x^*, I^*) $\left(I^* = \bigcup \bar{I} \right)$

Thm: Let $(t_0, \xi_0) \in \mathbb{R} \times \mathbb{R}^n$,

$f: (t, \xi) \in A = [t_0-h, t_0+h] \times \overline{B(\xi_0, a)} \to f(t, \xi) \in \mathbb{R}^n$.

$h > 0$, $a > 0$ f cont.

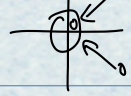
Let $M = \sup\{ |f(t,s)| : (t, s) \in A \} < \infty$.

Then ivp $\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = \xi_0 \end{cases}$

在 $I = [t_0 - \min\{h, \frac{a}{M}\}, t_0 + \min]$ 上有至少一个解

any solution to ivp defined on $J \subset I$,

J a nbhd of $a = t_0$, containing to I.



pf: pt① Lemma 2.1: $f \to \bar{f} : [t_0-h, t_0+h] \times \mathbb{R}^n \to \mathbb{R}^n$ cont.

$\bar{f}|_A = f$, $|\bar{f}(t,\xi)| \le M$.

Apply cauchy-peano to $\overline{IVP}$: $\begin{cases} x'(t) = \bar{f}(t, x(t)) \\ x(t_0) = \xi_0 \end{cases}$

$\Rightarrow$ $\bar{x}$ solution to $\overline{IVP}$

$\bar{x}$ is cont. $\bar{x}(t_0) = \xi_0 \Rightarrow \exists J$, $0<j\le h$, s.t. if $t \in J = [t_0, t_0+j]$

then $|\bar{x}(t) - \xi_0| \le a$, so $\forall t \in J$, $(t, \bar{x}(t)) \in A$. So $\bar{f}(t, \bar{x}(t)) = f(t, \bar{x}(t))$

So $x := \bar{x}|_J$ satisfy ivp for $t \in J$.

pt②: take (x, J), continue to the right as much as we can

to get $(\tilde{x}, \tilde{I})$ that cannot be strictly cnt to right.

let $t_1 = \tilde{t} = \tilde{I}$ 的右 end point. $\le t_0+h$.

claim → ②a: $\tilde{I} = [t_0, \tilde{t}] = [t_0, t_1]$ i.e. $t_1 \in \tilde{I}$

pf: if not, then $\tilde{I} = [t_0, t_1)$. $\forall t'\, t'' \in [t_0, t_1)$, $t' < t''$,

$|\tilde{x}(t'') - \tilde{x}(t')| \le \int_{t'}^{t''} |f(s, x(s))| ds \le M(t''-t')$

$\Rightarrow$ $\tilde{x}(t)$ converge as $t \to t_1$

$\tilde{x}(t) \in \overline{B(\xi, a)}$ if $t \in [t_0, t_1) \to \tilde{x}(t_1) \in \overline{B(\xi_0, a)}$

$\tilde{x}(t_1) = \xi_0 + \lim_{t \to t_1} \int_{t_0}^{t} f(s, \tilde{x}(s)) ds = \xi_0 + \int_{t_0}^{t_1} f(s, \tilde{x}(s)) ds.$

连续,有界 在 $[t_0, t_1)$.

$\Rightarrow$ $\tilde{x}$ satisfies (IVP) at $t = t_1$

② Claim: $t_1 \ge t_0 + \min\{h, \frac{a}{M}\}$.

if $t_1 = t_0 + h$. done.

WLOG, $t_1 < t_0 + h$. then $\tilde{x}(t_1) \in \partial \overline{B(\xi_0, a)}$

↳ 若非，可以继续延长.

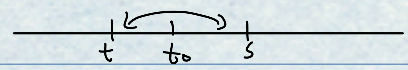
$$\Rightarrow a = |\tilde{x}(t_1) - \xi_0| = |\tilde{x}(t_1) - \tilde{x}(t_0)| \le (t_1 - t_0) \sup\{|\underbrace{\tilde{x}'(t)}_{= f(t, \tilde{x}(t))}| : t \in [t_0, t_1]\} \le M(t_1 - t_0)$$

$\Rightarrow t_1 \ge t_0 + \frac{a}{M}$.

③: to the left of t_0.

Consider $(IVP)_* \begin{cases} y'(t) = g(t, y(t)) \\ y(t_0) = \xi_0 \end{cases}$

$g(t, \xi) = -f(\overset{s}{2t_0 - t}, \xi)$



$(IVP)_*$ falls under hyp of cont to the right.

$y(t)$ solves $(IVP)_*$ in $[t_0, t_0 + h]$.

$\Rightarrow x(t) = y(2t_0 - t)$ solves IVP in $[t_0 - h, t_0]$.

Thm: ("$\exists !$")

$(IVP): \begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$

D open in $\mathbb{R}^2$ ($\mathbb{R} \times \mathbb{R}^n$)

s.t. $(x_0, y_0) \in D$,

$f: D \to \mathbb{R}^n$ cont in x and lipschitz in y, k.

then $\exists a > 0$, s.t. (IVP) has a solution on $(x_0 - a, x_0 + a)$,

unique.

pf via contraction mapping:

choose rectangle $R' \subset D$ & center (x_0, y_0).

$R' = [x_0 - A, x_0 + A] \times [y_0 - L, y_0 + L]$

f cont $\Rightarrow |f(x,y)| \le M \quad \forall (x,y) \in R'$.

~~Let~~ $a < \min\{\frac{L}{M}, A, \frac{1}{K}\}$.

Let

$R = [x_0 - a, x_0 + a] \times [y_0 - L, y_0 + L]$

$X = \{y = C[x_0 - a, x_0 + a] \text{ s.t. } \|y - y_0\|_\infty \le L.\}$

if $y \in X$, $\text{graph}(y) \subset R. \Rightarrow |f(x, y(x))| \le M. \quad \forall x \in x_0 \pm a$

$\Gamma = C([x_0 - a, x_0 + a])$ ~~S~~.

$(+)$: $\Gamma(y)(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\,ds$

if we find $y \in C([x_0 - a, x_0 + a])$

s.t. $\Gamma y = y \Rightarrow y \in (IE_f)$ so $y \in (IVP)$

Note: $y_1 \to \Gamma y_1 = y_2$

$y_2 \to \Gamma y_2 = y_3$ picard iterates.

$\vdots$

① $\Gamma: X \to X$. let $y \in X$, $\Gamma y \in C([x_0 - a, x_0 + a])$

$|\Gamma(y)(x) - y_0| = \left|\int_{x_0}^{x} f(s, y(s))\,ds\right| \le \int_{\min\{x, x_0\}}^{?} |f(s, y(s))|\,ds.$

WLOG, $x_0 < x$

$\le \int_{x_0}^{x} M\,ds = M|x - x_0| \le Ma \le L.$

$\Rightarrow \|\Gamma y - y_0\|_\infty \le L$

<1

② $\forall y, z \in X \Rightarrow \|\Gamma(y) - \Gamma(z)\|_\infty \le aK|y - z|$

$\forall x \in [x_0 - a, x_0 + a]$,

$\Gamma y(x) - \Gamma y(z) = \left|\int_{x_0}^{x} (f(s, y(s)) - f(s, z(s)))\,ds\right|$

$$\le \int_{x_0}^{x} |f(s,y(s)) - f(s,z(s))| ds$$

$$\le \int_{x_0}^{x} k|y(s) - z(s)| ds \quad \leftarrow f \text{ is lip with } K$$

$$\le k \, \|y-z\|_\infty \underbrace{|x-x_0|}^{\le a.}$$

$$\le ak \, \|y-z\|$$

③ contraction thm. (also give uniqueness)

Uniqueness by picard's method: [TP, pp. 739–740]

设 $z(x_0) = y_0$, z on $[x_0-a, x_0+a]$

$\text{graph}(z) \subset R = [x_0-a, x_0+a] \times [y_0-L, y_0+L]$

因为 $L \ge M a$, $\quad |z(x) - z(x_0)| \le \left| \int_{x}^{x_0} f(t, z(t)) dt \right|$

height, max speed, distance $\in$

$$\le |x-x_0| M \le L$$

$$|y(x) - z(x)| = \left| \int_{x_0}^{x} f(t, y(t)) - f(t, z(t)) dt \right|$$

$$\le \int_{x_0}^{x} |f - f| \underset{\text{Lip}}{\le} k \int_{x_0}^{x} |y(t) - z(t)| dt \le 2Lk(x-x_0)$$

"†"

☆ $\Rightarrow |y(x) - z(x)| \le \int_{x_0}^{x} 2Lk|t-t_0| dt = \frac{2L(k|x-x_0|)^2}{2}$

$\vdots$

$$|y(x) - z(x)| \le k \int_{x_0}^{x} \frac{2Lk^{n-1}|t-x_0|^{n-1}}{(n-1)!} dt = \frac{2L(k|x-x_0|)^n}{n!} \to 0$$

$(\forall x \in [x_0-a, x_0+a])$

So $y(x) = z(x)$

another proof:

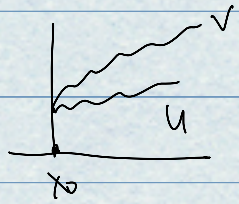
as in previous pf, "†"

$u(x) = |y(x) - z(x)|$, assume $x > x_0$.

$u(x) \le k \int_{x_0}^{x} u(t) dt$ (哈不等式)

$\Leftrightarrow \mathcal{U}'(x) \le k\,\mathcal{U}(x)$ $\qquad (\mathcal{U}(x) = \int_{x_0}^{x} u(t)\,dt)$

if had "=", $\mathcal{U}(x) = \underbrace{\mathcal{U}(x_0)\cdot e^{k(x-x_0)}}_{V}$ $\qquad \mathcal{U}(x_0)=0$



$$\left(\frac{\mathcal{U}(x)}{e^{k(x-x_0)}}\right)' = \frac{\mathcal{U}'(x) - k\,\mathcal{U}(x)}{e^{k(x-x_0)}} \le 0.$$

$$\Rightarrow \frac{\mathcal{U}(x)}{e^{k(x-x_0)}} \le \frac{\mathcal{U}(x_0)}{e^{k(x_0-x_0)}} \Rightarrow \underbrace{\mathcal{U}(x) \le \mathcal{U}(x_0)e^{k(x-x_0)}}$$

$$\mathcal{U}(x_0)=0 \quad \Rightarrow \quad \underset{\int_{x_0}^{x} \underset{\ge 0}{u}\,dt}{0 \le \mathcal{U}(x)} \le 0$$

Thm: (Gronwall's $\le$)

if $u'(x) \le k(x)\,u(x)$, k 连续. $u(x)$ 在 $x>x_0$ 可导.

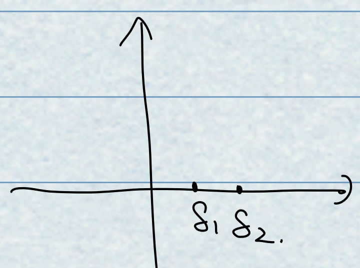
则 $u(x) \le u(x_0)\, e^{\int_{x_0}^{x} k(t)\,dt}$

↑ 通解是 $u'(x) = k(x)\,u(x)$

Remark:

IVP $\begin{cases} y'=f(x,y) \\ y(0)=y_0 \end{cases}$ $\quad \exists\ \delta_1>0$, s.t. 在 $[x_0, x_0+\delta_1]$ 上有解

δ_2 - - - - $\qquad +\delta_1$, $+\delta_1+\delta_2$.



Have seen: if 有 $f: \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}^n$

$$|f(t,x(t)) - f(t,y(t))| \le K|x(t)-y(t)|$$

and $T=a$, s.t. $KT=1$, then

$\mathcal{U}(x)(t) = \Gamma(x)(t) = \underset{x_0}{\vec{V}} + \int_{t_0}^{t} f(s,x(s))\,ds$ is a

contraction in $(C(0,T)\ \|x\|_\infty) \Rightarrow \exists!$ 定点,

Q1: how does sol depends on v. $(y_0 = y(x_0))$

. __ __ __ __ __ __ __ on parameter.

treated as same way. (para/ v)

e.g. $P_\lambda \begin{cases} x'(t) = f(t,x(t),\lambda) \\ x(t_0) = v. \end{cases}$ $\Rightarrow$ define $z(t) = \begin{pmatrix} x(t) \\ \lambda \end{pmatrix}$ $\quad g(t,z(t)) = \begin{pmatrix} f(t,x(t)) \\ 0 \end{pmatrix}$.

$z(t_0) = \begin{pmatrix} x_0 \\ \lambda \end{pmatrix} = w.$

$\Rightarrow P_\lambda \Rightarrow (\underline{P}) = \begin{cases} z'(t) = g(t,z(t)) \\ z(t_0) = \end{cases}$

Q2: contructed solution

$0 \le t \le T$ extend? can we always?

T_{max}?

Q3: if ∞ ,

OSGood's uni thm.

We have $\begin{cases} y' = f(x,y) \\ y'(x_0) = y_0 . \end{cases}$

D open $\subset \mathbb{R}^2$ $(\mathbb{R}^{n+1})$, assume $\forall (x,y_1)(x,y_2) \in D$,

$\ni$ (x_0, y_0) $\quad |f(x,y_1) - f(x,y_2)| \le \varphi(|y_1 - y_2|)$.

$\varphi : [0, \infty) \circlearrowleft$ cont,

随意上限

$\varphi(0) = 0$, $\varphi(u) > 0$ for $u > 0$. and $\int_0 \frac{dy}{\varphi(u)} = 0$

$\Rightarrow$ No more than one solution passes through (x_0, y_0)

remarks: • we don't assume f is cont in X.

(can't use peano's thm)

• e.g. $\varphi(x) = kx$. $\Rightarrow$ f is Lip in y.

$\int_0^1 \frac{du}{\varphi(u)} = \infty$ 显然,

$\Rightarrow$ OSGood aplies there, (generalization of previous one).

• If $\left|\frac{\partial f}{\partial y}\right| \le K$ on D, then

Lip cond (K) in by "MVT"

pf: 反证，假有2解. $y_1(x), y_2(x)$ on $(\alpha, \beta) \ni x_0$

$\Rightarrow y_1(x_0) = y_2(x_0) = y_0$.

Let $z(x) = y_1(x) - y_2(x)$, 满足 $\begin{cases} z'(x) = f(x, y_1(x)) - f(x, y_2(x)) \\ z(x_0) = 0. \end{cases}$

$(y_1 \ne y_2)$,

☆ 设 $z(x_1) \ne 0$,

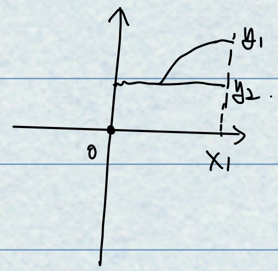
因为 $z(x) > 0$, 加上 $\varphi(u) > 0$ for $u > 0$.

$$\frac{dz}{dx} = f(x, y_1(x)) - f(x, y_2(x)) \le \varphi(|z(x)|) < 2\varphi(|z(x)|)$$

ⓑ comparison argument:

since $y_1(x) \neq y_2(x)$, $\exists\, x_1 \in (\alpha, \beta)$, s.t. $y_1(x_1) \neq y_2(x_1)$

Case ①: if $x_1 > x_0$, $y_1(x_1) > y_2(x_1)$



Let v be the solution of

$$(ivp)_v = \begin{cases} \frac{dv}{dx} = \lambda \varphi(v) \\ v(x_1) = z(x_1) = z_1 > 0. \end{cases}$$

saparable. $\Rightarrow$ solution: $\Phi(v) = \int_v^{z_1} \frac{du}{\varphi(u)}$

then $\Phi(v(x)) = \int_{v(x)}^{z} \frac{dv}{\varphi(v)} = \int_x^{x_1} \lambda\, dx = \lambda(x_1 - x)$

well-defined, 因为 $\varphi > 0$.

© this solution is defined clearly in an ( ) containing 区间

x_1.

- Claim: it's actually defined $\forall x$, $x < x_1$

pf: φ cont, >0 on $(0, +\infty)$,

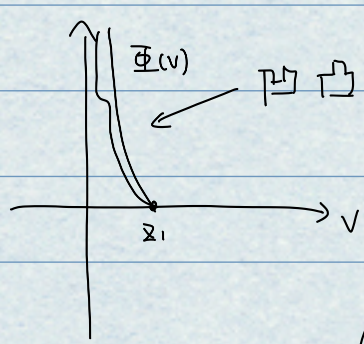
$\Rightarrow \Phi(z_1) = 0$.

$\int_0^1 \frac{du}{\varphi(u)} = \infty \Rightarrow \Phi(v) \nearrow +\infty$ as $v \searrow 0$

$\Rightarrow \Phi : (0, z_1] \to [0, +\infty)$, $\Phi'(v) = \frac{-1}{\varphi(v)} < 0$,

$\Rightarrow \Phi \searrow \Rightarrow \Phi$ invertible.

$\Rightarrow \Phi^{-1} : [0, +\infty) \to (0, z_1]$.



凹凸性未知.

$v(x) = \Phi^{-1}(\lambda(x_1 - x))$

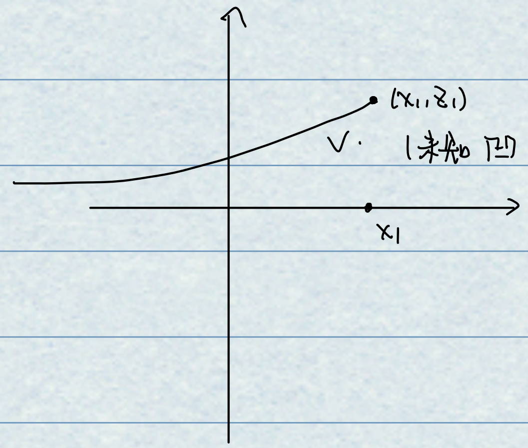
domain is $[0,+\infty)$

$\Rightarrow V(x)$ is defined for $x_1 - x \geq 0$ i.e. $x \leq x_1$. □

range $(\Phi^{-1}) > 0$

- V is ↗, positive function on $(-\infty, x_1]$

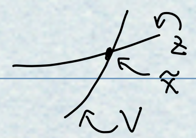
因为 Φ^{-1} ↗, $(x_1 - x)$ ↘



- Compare $V(x)$, $z(x)$: both 过 (x_1, z_1)

by ☆, if $\exists\, \tilde{x}$, s.t. $V(\tilde{x}) = z(\tilde{x})$ 　　必须要检验ψ

then $z'(\tilde{x}) < V'(\tilde{x})$, 因为 $\frac{dz}{dx} \leq \varphi(|z|) < 2\varphi(|z|) = \frac{dV}{dx}$

so → 

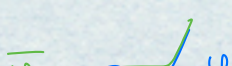
这意味着只能有这一个交点.

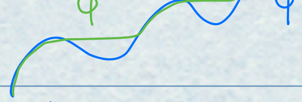
$\Rightarrow$ graph(V) and z 在 (α, x_1) 上无交点,

So by construction, $z(x_1) = V(x_1)$, z is above $V^{\text{(graph)}}$ on $\{x < x_1\}$

but $z(x_0) = 0 < V(x_0)$, 意味着有交点. 所以矛盾.
↑ positive

Note: given φ, $\overline{\varphi}(u) = \sup\limits_{\tilde{u} < u} \varphi(\tilde{u})$





不一定

if $\int_0 \frac{du}{\varphi(u)} = \infty$, is $\int_0 \frac{du}{\varphi(u)} = \infty$?

• if $\varphi(0)=0$, $\varphi(u)>0$, $\varphi'(0)$ 存在, 则 $\int_0^1 \frac{du}{\varphi(u)} = \infty$

Had seen: if $D \subset \mathbb{R}^n$ $f: D \to \mathbb{R}^n$ Locally Lip. cont

($\forall K$ compact $\subset D$,

$\exists L = L(K)$, s.t. $|f(x)-f(y)| \leq L|x-y|$

$\forall x, y \in K$)

then $\exists u \in D$, $\exists T > 0$ time,

s.t. ivp = $\begin{cases} x' = f(x) \\ x(0) = v \end{cases}$ has unique sol $x: [-T, T] \to D$

depends cont on v.

Lemma: $D \subset \mathbb{R}^n$, $f: D \to \mathbb{R}^n$, C^1, then it's localy Lipschipz.

[$\forall K$ 紧致, $L = \sup_K \|Df(x)\|$ Lip cont on K.

Gloable uni, $\exists$, thm.

$\mathcal{D} \subset \mathbb{R}^n$ open, connected.

then $f: \mathcal{D} \to \mathbb{R}^n$ Locally Lip vector field.

$v \in \mathcal{D}$, $\exists!$ maximal interval of existence $I_{max} = (\underline{T}\ \overline{T}) \ni 0$,

s.t. ① $\begin{cases} x' = f(x) \\ x(0) = v \end{cases}$ 在 I 上有唯一解

② if $\overline{T} < 0$, then $\lim = |x(t)| + \frac{1}{dist(x(t), \partial D)} = +\infty$

$t \to T$

$t < T$

$\to$ either $|x(t)|$ goes to ∞,

or approaches the ∂.

pf: by contrapositive, 设 ① $x(t)$ exists up to T (at least)

② $|x(t)| \not\to \infty$ at T,

③ $x(t) \not\to \partial D$ at T.

by ①, $\exists\ t_j \nearrow T$, s.t.

$$\begin{cases} ① \ M := \sup_j |x(t_j)| < \infty \\ ② \ \delta := \inf_j \operatorname{dist}(x(t_j), \partial D) > 0 \end{cases}$$

Let $K \subset D$, $K = \{ z \in D \,/\, |z| \le M,\ d(z, \partial D) \ge \delta \}$ 紧.

By lemma, f is Lip on K, with constant $L = L(K)$

$\Rightarrow \exists$ fixed time $\tau > 0$, s.t. $\forall w \in K$, ivp $\begin{cases} x' = f(x) \\ x_0 = w \end{cases}$

has a solution on $[-\tau, \tau]$.

Take $(\text{ivp})_j : \begin{cases} y' = f(y) \\ y(0) = x(t_j) = w_j. \end{cases} \Rightarrow$ sol: y_j

look @ $y_j(t - t_j)$ defined on $[t_j - \tau, t_j + \tau]$

agrees with $x(t)$ @ t_j.

since $x(t)$ maximal solution, then

$\Rightarrow x(t)$ is defined up to $t_j + \tau$

if $t_j \nearrow \infty$, $t_j \to T$.

So $t_j+\tau>T$. so $\underset{T_{max}}{\bar{T}} > T+\frac{\tau}{2}$, so T is not max.

Collary in [HSD, p399]

if $\exists\, [0,\beta)$, $\forall \beta>0$.

Uniform contractions:

Given $f: \underset{\mathbb{R}^n}{D_1}\times \underset{\mathbb{R}^m}{D_2} \to \underset{\mathbb{R}^n}{D}$.

设 f is a uniform contraction, $0\le q<1$ fixed

$|f(x_1,v)-f(x_2,v)|\le q|x_1-x_2|$, $\forall v\in D_2$, $\forall x_1,x_2\in D_1$.

设 f is C^1 in (x,v)

Solve $x=f(x,v)$ $\leftarrow$ x is fixed point.

contracting mapping

$\underset{\text{CMT,}}{\uparrow}$ $\Longrightarrow$ $\exists\, x=x(v)$, solution.

$\underbrace{x-f(x,v)}_{C_1}=0 \overset{}{\underset{\text{IFT}}{\Longrightarrow}}$ check $D_x(x-f(x,v)) = \overbrace{Id-D_xF}^{\text{invertible.}}$

but $\|D_xF\|\le q<1$

Lemma: $A\in\mathbb{R}^{n\times n}$ $|A|=q<1$

$\left(\|A\| = \sup\limits_{x\neq 0}\frac{|Ax|}{|x|}\right) \Rightarrow Id-A$ invertible.

pf: $(Id-A)^{-1}=\sum\limits_{k=0}^{\infty}A^k$ $\left(\frac{1}{1-a}=\sum a^i\right)$

① absolutely convergent.

② $(Id-A)(\sum A^k)=Id$. $\Longrightarrow$ $\square$

Solving $\underbrace{X - f(x,v)}_{Id - f} = 0$, $D_x(X - f(x,v)) > 0$, $f(x,v)$ invertible.

$$(IVP)_v = \begin{cases} x' = f(t,x) \\ x(0) = v. \end{cases} \to X_v(t)$$

continuous dependence

know: if $v = v_k \to w$,

$$\Rightarrow X_{v_k}(t) \to X_w(t)$$

dont know how fast.

Gronwall's lemma (generalized version)

$f : [a,b] \to \mathbb{R}$ cont.

$g : [a,b] \to \mathbb{R}^+$

$y : [a,b] \to \mathbb{R}$ cont s.t. $\forall t \in [a,b]$,

$$(\dagger) \quad y(t) \le f(t) + \int_a^t g(s) y(s)\, ds$$

then $\forall t \in [a,b]$

"LHS"

$$y(t) \le f(t) + \int_a^t f(s) g(s)\, e^{\int_s^t g(u)\,du}\, ds$$

in particular: $f \equiv k$. ? 并不显然.

$$\Rightarrow y(t) \le k\, e^{\int_a^t g(s)\,ds}$$

plug in $f = k$.

RHS

$$LHS = k + k\int_a^t g(s) \left(e^{\int_s^t g(u)\,du}\right) ds \overset{?}{\le} k e^{\int_a^t g(s)\,ds}$$

$$\left(e^{\int_s^t g(u)\,du}\right) = e^{\int_a^t} \cdot e^{-\int_a^s}$$

与s无关.

$LHS|_{t=a} = RHS|_{t=a}$

$$\Rightarrow \underbrace{e^{-\int_a^t g(u)\,du} + \int_a^t g(s)\left(e^{-\int_a^s g(u)\,du}\right) ds}_{\text{求导后}=0} = 1$$

同除 $ke^{\int_a^t}$

pf 0: HW3

pf 1: 设 $f(t)=k$.

WLOG: $|y(t)|\le 1$

$$y(t)\le k+\int_a^t g(t_1)\left(k+\int_a^{t_1} g(t_2)\left(k+\int_a^{t_2} g(t_3)y(t_3)\,dt_3\right)dt_2\right)dt_1$$

$$=k+k\int_a^t g(t_1)\,dt_1+k\int_a^t g(t_1)\int_a^{t_2} g(t_2)\,dt_2\,dt_1+$$

$$\int_0^t g(t_1)\int_a^{t_1} g(t_2)\int_a^{t_2} g(t_3)y(t_3)\,dt_3\,dt_2\,dt_1$$

Let $I=\int_a^t g(t_1)\int_a^{t_2} g(t_2)\,dt_2\,dt_1 = \int_a^t M'(t_1)\cdot M(t_1)\,dt_1$

$M(u)=\int_a^u g(s)\,ds$

$M'(u)=g(u)$, $M(a)=0$

$$=\frac{1}{2}(M(t))^2=\frac{1}{2}\left(\int_a^t g(s)\,ds\right)^2$$

Similarly, in same way get

$$y(t)\le k+k\frac{\int_a^t g(t_1)\,dt_1}{1!}+k\frac{\left(\int_a^t g(t_1)\,dt_1\right)^2}{2!}+\cdots$$

$$+k\frac{\int_a^t g(t_1)\,dt_1}{n!}+\underbrace{\int_a^t g(t_1)\int_a^{t_1} g(t_2)\cdots\int_a^{t_n} g(t_{n+1})y(t_{n+1})\,dt_{n+1}\cdots dt_1}_{=R}$$

$$R\le\frac{\left(\int_a^t g(t_1)\,dt_1\right)^{n+1}}{(n+1)!}$$ 因为 $g\ge 0$, $|y(t)|\le 1$

$$\Rightarrow y(t)\le k e^{\int_a^t g(s)\,ds}+\varepsilon,$$ 令 $\varepsilon\to 0$.

$$\Rightarrow y(t)\le k e^{\int_a^t g(s)\,ds}$$

hypotsis

pf 2: 令 $z(t)=\int_a^t g(s)y(s)\,ds$.

"$\frac{dy}{dx}+py$".

$\Rightarrow$ "H" $\times g(t)\Rightarrow z'(t)-g(t)z(t)\le f(t)g(t)$

define $w(t)=z(t)e^{-\int_a^t g(s)\,ds}$

integrading factor.

$$w'(t) \le f(t)g(t)e^{-\int_a^t g(s)ds}$$

积分 有 $w(a) = z(a) = 0$

$$w(t) \le \int_a^t f(s_1)g(s_1)e^{-\int_a^t g(s)ds} ds_1$$

$$\Rightarrow z(t) \le \int_a^t f(s_1)g(s_1)e^{\int_{s_1}^t g(s)ds} ds_1.$$

注意到 $y(t) \le f(t) + z(t) \le f(t) + \int_a^t f(s_1)g(s_1)e^{\int_{s_1}^t g(s)ds} ds_1$

$$y(t) \le f(t) + \int_a^t f(s_1)g(s_1)e^{\int_{s_1}^t g(s)ds} ds_1$$

即得证

Thm: 令 $x_1 : [a,b] \to \mathbb{R}^n$ be differable, PP.394 [HSD]

$x_2 : [a,b] \to \mathbb{R}^n$,

s.t. $|x_1(a) - x_2(a)| \le \delta$.

令 $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ be a function that's Lip in second variable. $|f(t_1, \xi_1) - f(t_2, \xi_2)| \le L|\xi_1 - \xi_2|$

设 $|x_1'(t) - f(t, x_1(t))| \le \varepsilon_1$, $(t \in (a,b))$

$|x_2'(t) - f(t, x_2(t))| \le \varepsilon_2$

then $|x_1(t) - x_2(t)| \le \delta e^{L(t-a)} + (\varepsilon_1 + \varepsilon_2)\dfrac{e^{L(t-a)} - 1}{L}$

pf: Let $\varepsilon = \varepsilon_1 + \varepsilon_2$. $g(t) = x_1(t) - x_2(t)$.

$(+)$: $|g'(t)| = |x_1'(t) - x_2'(t)|$

$\le |f(t, x_1(t)) - f(t, x_2(t))| + \varepsilon$ ← 三角不等式

$\le L|g(t)| + \varepsilon$ ← Lip. f with L.

$\Rightarrow |g(t)| = \left|\int_a^t g'(s)ds + g(a)\right|$

$\le \int_a^t |g'(s)|ds + \delta$

$\le \delta + \varepsilon(t-a) + \int_a^t L|g(s)|ds$ ← "(+)"

用 gronwall

$$\Rightarrow |g(t)| \le \delta + \varepsilon(t-a) + \int_a^t L(\delta+\varepsilon(s-a))e^{L(t-s)}\,ds$$

$$= \delta e^{L(t-a)} + \frac{\varepsilon}{L}(e^{L(t-a)}-1) \longleftarrow$$ 自己算一下!

Cor: [HSD] p 401

令 $A(t) \in \mathbb{R}^{n\times n}$ cont.

IVP: $\begin{cases} X'(t) = A(t)X & t_0 \in I \\ X(t_0) = X_0 \end{cases}$

有唯一解 on all I.

$X(t) = X_0\, e^{\int_{t_0}^t A(s)\,ds}$ ("the" solution)

Section 17.6 [HSD]

Differentiablity of the flow:

$X' = f(x)$, f is C^1, $X \in \mathbb{R}^n$

Solution: $\Phi(t,x) = \Phi_t(x)$ is C^1 in t and X.

time derivative. $\longrightarrow$ $\frac{d}{dt}(\Phi(t,x)) = x'(t) = f(x(t))$

space derivative. $\longrightarrow$ $\frac{d}{dx}(\Phi_t(x))$?

Consider $\begin{cases} x' = f(x) \\ x(0) = x_0 \in \mathbb{R}^n \end{cases}$ $t \in J \ni 0$. ← 時

$\forall t \in J$, $A(t) = Df_{x(t)} = Df(x(t)) \Rightarrow$ Jacob matrix of f @ $x(t)$

$f \in C^1 \Rightarrow A(t)$ cont

Define "variational equation" along the solution $x(t)$

$u' = A(t)u$

$\uparrow$ 有 unique solution on J, $\forall u(0)=u_0$ (cor $\uparrow$)

($\dagger$) As in autonomous case, solution of this system 是线性的.

$\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$ $\qquad$ $\begin{cases} u' = A(t)u \\ u(0) = u_0 \end{cases}$ $\qquad$ $A(t) = Df(x(t))$

Meaning: 若 u_0 small, then $t \to x(t) + u(t)$ is a good approximation to the solution of $\begin{cases} x' = f(x) \\ x(0) = x_0 + u_0 \end{cases}$

prop: $0 \in J$ closed $[\,]$,

$x(t)$ solves $\begin{cases} x' = f(x) \\ x(0) = x_0 \end{cases}$ for $t \in J$,

then $u(t,\xi)$ solves

$$\begin{cases} u' = A(t)u \\ u(0,\xi) = \xi \end{cases} \qquad A(t) = Df(x(t))$$

Assume ξ, $x_0 + \xi \in D$ (domain of f)

$f: D \to \mathbb{R}^n$ is C^1, $D \in \mathbb{R}^n$

Assume $y(t)$ solves $\begin{cases} y' = f(y) \\ y_0 = x_0 + \xi, \end{cases}$

then $\lim\limits_{\xi \to 0} \dfrac{|y(t,\xi) - x(t) - u(t,\xi)|}{|\xi|}$ uniformly converges to 0.

i.e. $\forall \varepsilon, \exists \delta$ s.t. $|\xi| < \varepsilon \Rightarrow$

$$|y(t,\xi) - x(t) - u(t,\xi)| \le \varepsilon |\xi| \quad \forall t \in J.$$

Note: $u(t,\xi)$ is linear in ξ (by "($\dagger$)")

Thm:

$\Phi(t,x)$, flow of $x' = f(x)$, is a C^1 function that is AC^1, i.e. $\frac{\partial \Phi}{\partial t}$, $\frac{\partial \Phi}{\partial x}$ $\exists$, cont, in t, x.

pf: $\frac{\partial \Phi}{\partial t} = f(\Phi_t(x)) = f(\Phi(t,x))$ cont.

$\frac{\partial \Phi}{\partial x}$: if ξ 小, then $\Phi(t, x_0+\xi) - \Phi(t, x_0) = y(t,\xi) - x(t)$

prop $\Rightarrow$ $\frac{\partial \Phi(t,x_0)}{\partial x}$ is a linear map

$\xi \to u(t,\xi)$ cont.

note: $\frac{\partial \Phi(t,x)}{\partial t} = D\Phi_t(x)$

Special Case: (important)

设 $x(t) \equiv a$ a equilibrium solution of $x' = f(x)$,

then $A(t) = Df(x(t)) = Df(a)$ $\to$ constant.

get: $\begin{cases} \frac{d}{dt} D\Phi_t(a) = A(D\Phi_t(a)) \\ \Phi_0(a) = Id. \end{cases}$

Solution: $D\Phi_t(a) = e^{tA}$.

In a neighborhood of the equilibrium point, the flow is approximate linear.

Proof of prop:

$$x(t) = x_0 + \int_0^t f(x(s))\,ds$$

$$y(t,\xi) = x_0 + \xi + \int_0^t f(y(s,\xi))\,ds$$

$$u(t,\xi) = \xi + \int_0^t \underbrace{[Df(x(s))]}_{= Df_{x(s)}}(u(s,\xi))\,ds$$

$$|y(t,\xi) - x(t) - u(t,\xi)| \leq \int_0^t |f(y(s,\xi)) - f(x(s)) - Df_{x(s)}(u(s,\xi))|\,ds$$

$$f(y) = f(z) + Df_z(y-z) + R(z, y-z)$$

Linear. $\qquad \lim_{y\to z} \frac{R(z,y-z)}{|y-z|} = 0$

$$\underbrace{|y(t,\xi) - x(t) - u(t,\xi)|}_{=g(t)}$$

$$\leq \int_0^t |Df_{x(s)}(y(s,\xi) - x(s) - u(s,\xi))|\,ds + \int_0^t |R(x(s), y(s,\xi) - x(s))|\,ds$$

$$N := \max\{Df_{x(t)} : t \in J\} < \infty.$$

Then $g(t) \leq N\int_0^t g(s)\,ds + \int_0^t \underbrace{|R(x(s), y(s,\xi) - x(s))|}_{\leq \varepsilon\cdot|y(s,\xi)-x(s)|}\,ds.$

Fix $\varepsilon > 0$, by Taylor,

$\exists\, \delta_0 > 0$ / $|y(s,\xi) - x(s)| \leq \delta_0$, $s \in J$

from the thm following gronwall, $\exists\, \delta, k > 0$.

s.t. $|y(s,\xi) - x(s)| \leq |\xi| e^{ks} \leq \delta_0$ if $|s| \leq \delta_0$ and $s \in J$.

If $|\xi| < \delta_1$, $t \in J$.

$$g(t) \leq N\int_0^t g(s)\,ds + \int_0^t \varepsilon|\xi| e^{ks}\,ds$$

$$= N\int_0^t g(s)\,ds + c\cdot\varepsilon\cdot|\xi| \qquad \int_0^t e^{ks}\,ds.$$

某常数, 与 K, $\mathrm{Length}(J)$ 有关.

Gronwall:

$g(t) < C\varepsilon e^{Nt} \cdot |\xi|$, $t \in J$, $|\xi| \le \delta_1$

$\Rightarrow \dfrac{g(t)}{|\xi|} \xrightarrow[\xi \to 0]{} 0$

Variational EQ:

Formal Calculation.

$u(t) = D_v \Phi_t(v)$

derivative w.r.t v.

$\Phi_t(v)$ solves $\begin{cases} x' = f(x) \\ x(0) = v. \end{cases}$

$\frac{d}{dt} u(t) = \frac{d}{dt} D_v \Phi_t(v)$

formally, $= D_v\left(\frac{d}{dt} \Phi_t(v)\right)$

$= D_v\left(f(\Phi_t(v))\right)$

$= Df(x(t)) \cdot u(t)$ ⇐ chain rule.

$\Rightarrow \begin{cases} u' = Df(x(t)) \cdot u \\ u(0) = Id. \end{cases}$

equivilently, "linearization about x,"

$\begin{cases} z(t) = Df(x(t))\, z \\ z(0) = w \end{cases} \rightarrow z(t) = u(t)\, w$

$= \left(D_v(\Phi_t(v))\right) w$

$\Rightarrow x(t) + z(t)$ solves $\begin{cases} y' = f(y) \\ y_0 = v + w \end{cases}$

If $x(t) = a$ is a steady state.

$\Rightarrow D(f(x(t))) = Df(a)$ 常数.

第一种方法 to get linerization:

$f(x) = f(a) + Df(a)(x-a) + \underbrace{\text{Error}}_{= o(|x-a|)}$

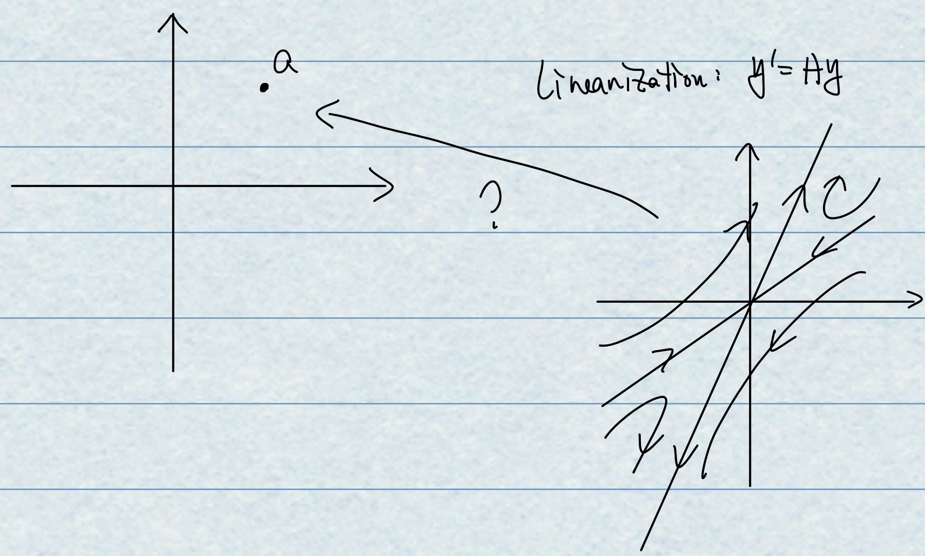
Look the dynamtics near steady state.

write solution as

$$x(t) = a + y(t)$$

then $0 + y' = f(x) = \underbrace{0}_{f(a)} + Ay + \underbrace{\text{Error}}_{o(|y|)}$.

$y' = Ay$ tell us? about 原问题.



know: if Φ_t generated by $x' = f(x)$.

then $\underset{\substack{\uparrow \\ \text{initial value}}}{D_v} \Phi_t\underset{\substack{\uparrow \\ \text{steady state.}}}{(a)} = e^{tA}$ for t fixed.

e.g. [8.3]

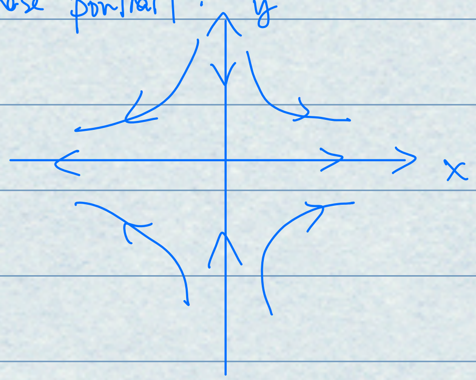
$\begin{cases} x' = x + y^2 \end{cases} \Rightarrow \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x+y^2 \end{pmatrix} = 0$

$\left\{ y' = -y \right. \iff \begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \cdots \\ -y \end{pmatrix} = f(x,y)$

$\Rightarrow a = (0,0)$

$Df(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = A. \to e^{tA} = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$

So phase portrait:



linearization.

Solve 原方程:

$y' = -y \Rightarrow y(t) = y_0 e^{-t}.$

$\hookrightarrow x'(t) = x(t) + y_0^2 e^{-2t}$

$\Rightarrow x(t) = \underbrace{x_0 \cdot x(t)}_{x' = x(t)} + x^*(t)$

$x' = x(t) + y^2 e^{-2t}.$ particular solution

guess $x^* = b \cdot e^{-2t} \Rightarrow -2b = b + y_0^2 \Rightarrow b = -\frac{y_0^2}{3}$

So $x(t) = \left(x_0 + \frac{y_0^2}{3}\right)e^t - \frac{y_0^2}{3} e^{-2t}$

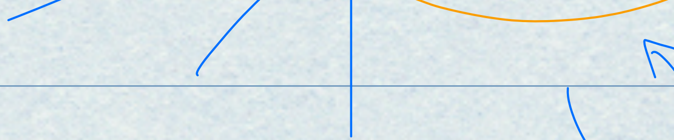
初值检验

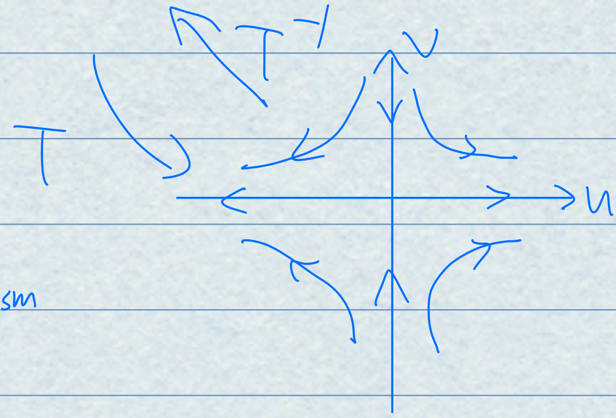
$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \left(x_0 + \frac{y_0^2}{3}\right)e^t - \frac{y_0^2}{3} e^{-2t} \\ y_0 e^{-2t} \end{pmatrix}$

$\Rightarrow$

这两个是分开的部分

(exist/∃ thm, 0 is steady).





hope: T is a diffeomorphism

$$\begin{cases} u = x + \frac{1}{3}y^2 \\ v = y \end{cases}$$

dynamic $\overset{\text{diffeo, steady point}}{\underset{\text{locally}}{\longleftrightarrow}}$ linear

Note: generally, T is just homeo and local.

Conjucacy.

for matrices: $A = TBT^{-1}$

normal form for a pblm, make A 简单.

for group: $a \Leftrightarrow gbg^{-1}$

for our: $\Phi = h \circ \varphi \circ h^{-1}$

e.g. 2.

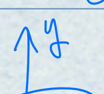
$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} - \frac{1}{2}(x^2+y^2)\begin{pmatrix} x \\ y \end{pmatrix}$$

$a = (0,0)$

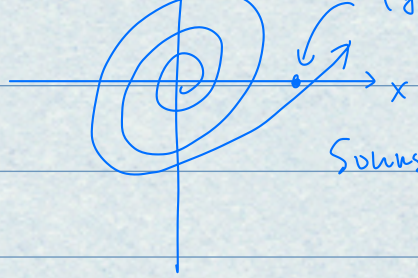
Linearization:

$$Df(0,0) = \begin{pmatrix} \frac{1}{2} & -1 \\ 1 & \frac{1}{2} \end{pmatrix}, \quad \lambda = \frac{1}{2} \pm i$$

$$e^{tA} = e^{\frac{t}{2}} \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$



$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad A\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$



Sounse (unstable)

Solve: $x = r\cos\theta$, $y = r\sin\theta$.

$$\Rightarrow \begin{cases} x' = r'\cos\theta - r\sin\theta(\theta') & ① \\ y' = r'\sin\theta + r\cos\theta(\theta') & ② \end{cases}$$

$xx' + yy' = rr' \Leftarrow ①x + ②y$ ③

$-x'y + y'x = r^2\theta'$ ④

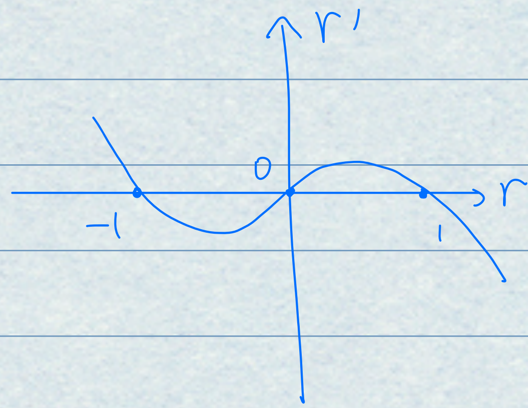
$$③ \Rightarrow r' = \frac{xx'+yy'}{r}$$

$$= \frac{1}{2}r - \frac{1}{2}r^3 \leftarrow \text{"ODE"}$$

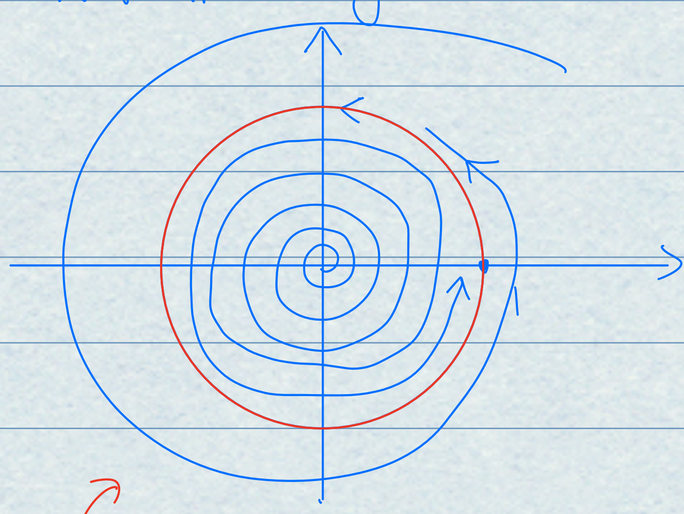
$$④ \Rightarrow \theta' = \frac{-x'y+y'x}{r^2}$$

$$= 1 \leftarrow \text{"ODE"}$$

r:

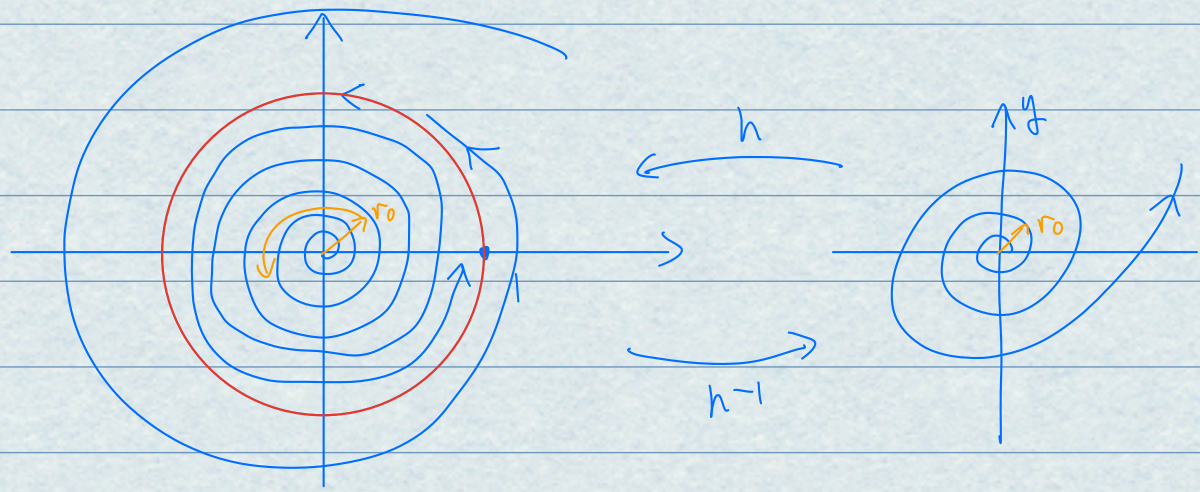


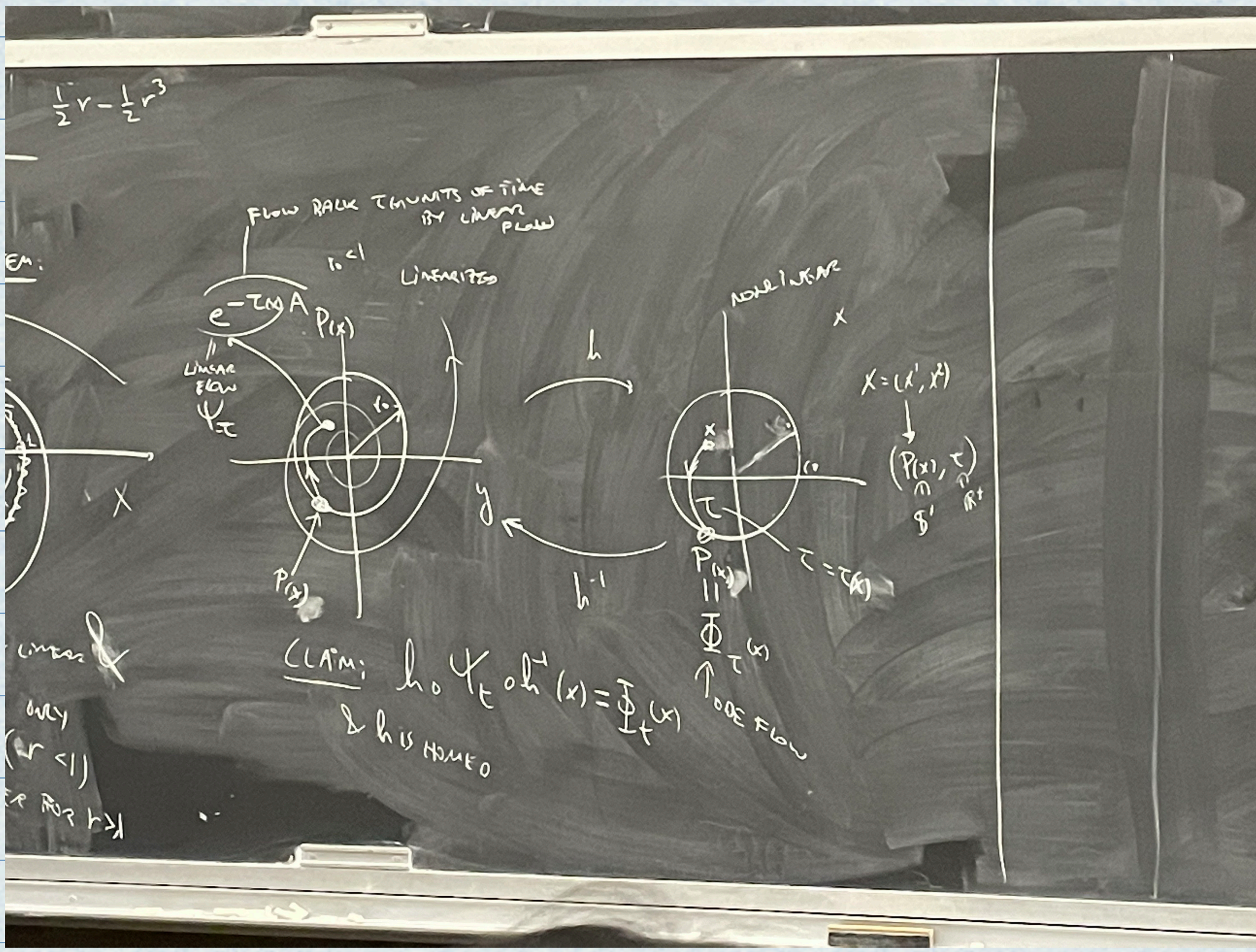
So Non-linear system:



only locally conjugate. (linear and non-linear)
$(r<1)$

how to build such conjugacy?





e.g. 3: $x' = -y + \varepsilon x(x^2+y^2)$

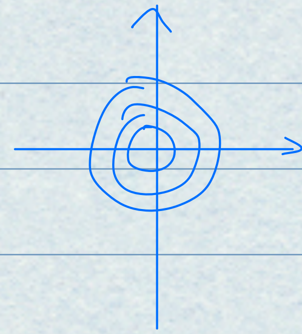
$y' = x + \varepsilon y(x^2+y^2)$

linearization:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Rightarrow e^{tA} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

$\lambda = \pm i$. $Re(\lambda) = 0$. $\longrightarrow$

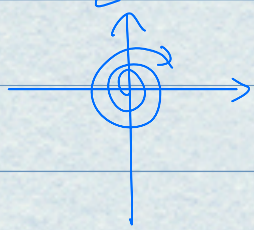


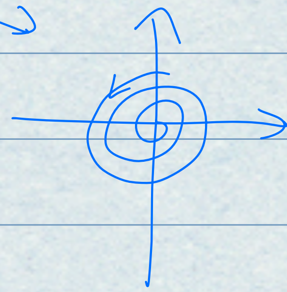
极坐标

$\theta' = 1$

$r' = \varepsilon r^3$

$\varepsilon > 0$ $\quad$ $\varepsilon < 0$





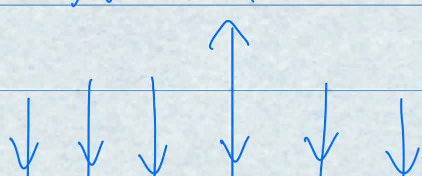
e.g. 4: $x' = x^2$

$y' = -y$. $\quad f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x^2 \\ -y \end{pmatrix}$

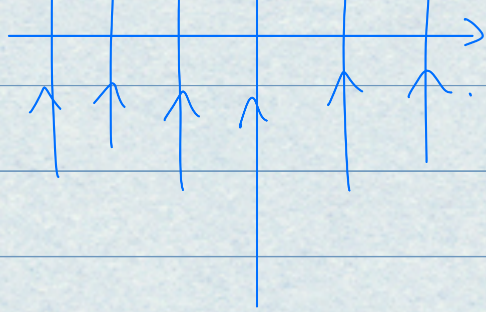
linearization:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 2x & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \text{ @ } (0,0)$$

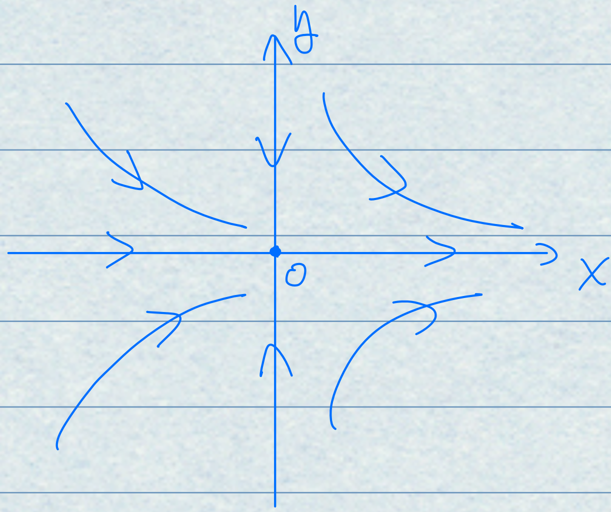
$\Rightarrow \lambda = 0, -1.$



stable, x sink.



non-linear:



unstable

$\Rightarrow$ no conjugacy.

e.g. 5: $\begin{pmatrix}x\\y\end{pmatrix}' = f(x,y)$, s.t. $f(0)=0$.

($\begin{pmatrix}x\\y\end{pmatrix}' = (x^2+y^2)\, g(x,y)$)

same geometry with $\begin{pmatrix}x\\y\end{pmatrix}' = g(x,y)$

Stability: (In the sense of Lyapunov)

Def: $(\Phi_t)_{t\geq 0}$ a dynamic system in $\Omega \subset \mathbb{R}^n$.

i.e. $\Phi_0 = \mathrm{Id}$, $\Phi_{s+t} = \Phi_s \circ \Phi_t$

· Let "a" a steady state / equilibrium / fixed point.

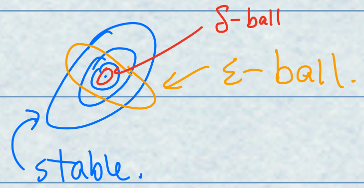
i.e. $\Phi_t(a) = a \quad \forall t$,

We say "a" is stable if

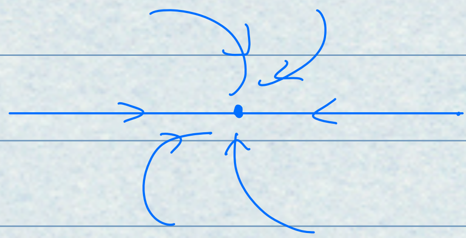
☆ $\forall \varepsilon > 0, \ \exists \delta > 0$, s.t. $|x-a| < \delta \Rightarrow$

$$\sup_{t \geq 0} |\Phi_t(x) - a| < \varepsilon$$

e.g. center



e.g.



$\varepsilon = \delta$. 即可.

The steady state is unstable if $\exists \delta > 0$, and $x_j \to a$,

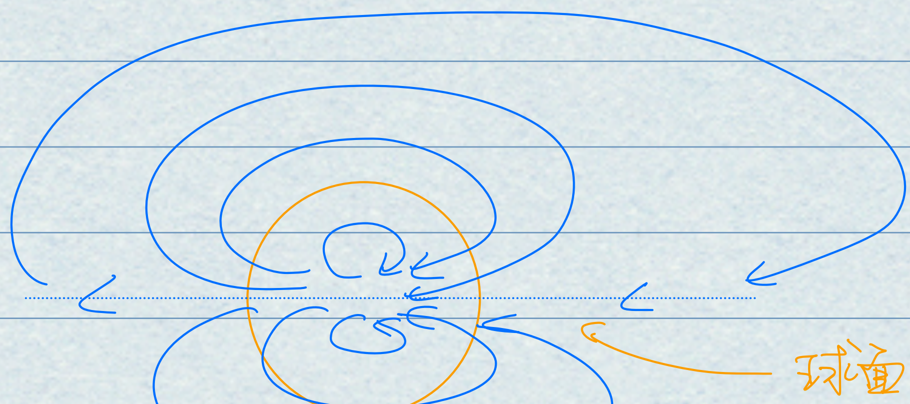
s.t. $\sup_{t \geq 0} |\Phi_t(x_j) - a| > \varepsilon, \quad \forall j = 1, 2, \cdots, n, \cdots$

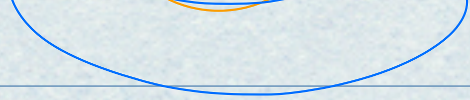
The equilibrium point is "asymptotacally stable" if

① stable.

② $\exists \delta > 0$, s.t. $\lim_{t \to \infty} |\Phi_t(x) - a| = 0$, $\forall x$ with $|x-a| < \delta$.

e.g. ② $\not\Rightarrow$ ① on S^2.





Thm: (Groeman-hartman).

(Topological conjugacy). $\leftarrow$ homeo, not iso

Consider $x' = f(x)$, $f \in C^1$,

$f(a) = 0$.

$A = Df(a)$.

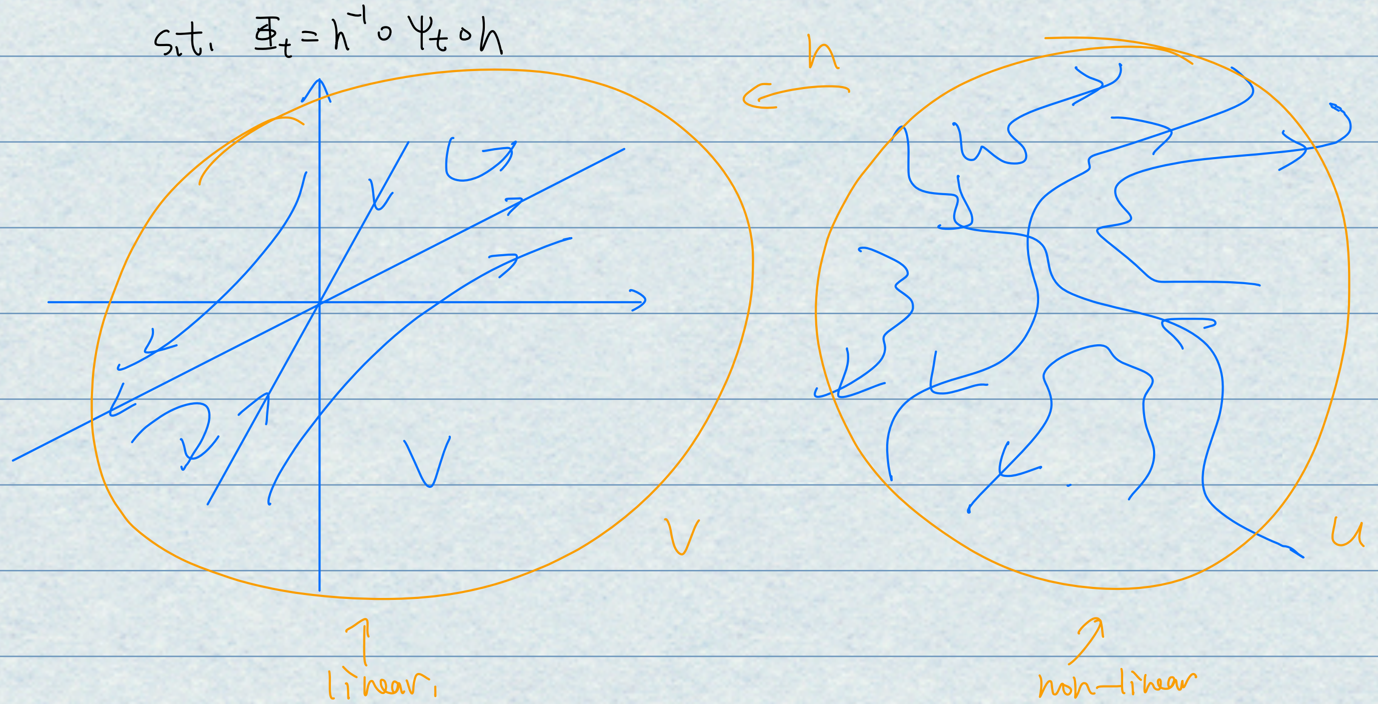
Assume A is hyperbolic, $(Re(\lambda) \neq 0)$

Let Φ_t = Dynamical sys of

$\psi_t = e^{tA}$

then $\exists$ nbhds $U \ni a$, $V \ni 0$, $\exists\, h : U \to V$, homeo,

s.t. $\Phi_t = h^{-1} \circ \psi_t \circ h$



- What if we want h to be diffeo?

need: non resonance condition.

eigenvalues $\lambda_1, \cdots, \lambda_n$

linear combination: $\lambda_j = \sum_{k \neq j} a_k \lambda_k$.

全整, then resonance.

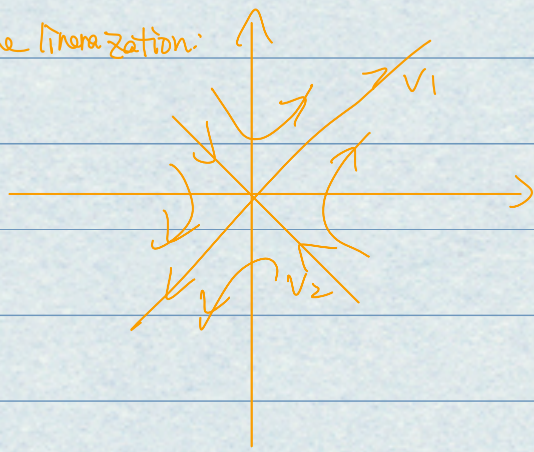
追導 non-resonance.

- What does "topological conjugacy mean?"
  - preserve stability, asymptotic stability.
  - But, all sinks are topological conjugate.

- Other consequences?

2-D case: $A = \begin{pmatrix} + & 0 \\ 0 & - \end{pmatrix}$ $\lambda_1 \lambda_2 < 0$ ($\lambda_1 > 0$, $\lambda_2 < 0$)

then the linearization:



general principle: if the linear sys is "structuratcy stable"; (hyperbolic)

then non-linear looks like linear. (locally)

Case ①

pf: $Df(a)$ has distinct real negative eigenvalues.

WLOG, $a=0$, $A = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$.

$\lambda_1, \cdots, \lambda_n < 0$.

$$X' = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix} X + \bar{o}(|X|)$$ 高阶项.

lemma: if $\lambda_1, \cdots, \lambda_n$ are real, negative,

then 0 is asymptotacalcy stable.

i.e. 0 is a sink.

pf: Consider

$$\frac{d}{dt}\frac{1}{2}\left(y_1^2+y_2^2+\cdots+y_n^2\right)$$

$$L(x)=\frac{1}{2}|x|^2.$$

if $y(t)$ solves the linear system, then

$$\frac{d}{dt}L(y(t))=\frac{d}{dt}\frac{1}{2}|y^2|=y_1y_1'+\cdots+y_ny_n'$$

$$=y^Ty'=y^TAy.$$

$$=\lambda_1y_1^2+\cdots+\lambda_ny_n^2$$

$$\leq -\min\{|\lambda_1|,\cdots,|\lambda_n|\}\,|y^2|$$

$$=-cL(y) \longrightarrow \text{微分不等式}$$

So $L(y(t))\leq L(y(0))\,e^{-ct}$, $\forall t>0$ gronwall.

Now, for non-linear, $x'=f(x)$

repeat,

$$\begin{pmatrix} x_1\\ \vdots \\ x_n\end{pmatrix}$$

$$\frac{d}{dt}L(x(t))=x^T\cdot x'(t)$$

$$=x^T\cdot\left(\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}x+\bar{o}(|x|)\right)$$

$$\leq -c|x^2|+\bar{o}(|x|^2)$$

$$|x^T\bar{o}(x)|\leq\left|\begin{pmatrix} x_1\\ \vdots \\ x_n\end{pmatrix}k\sqrt{x_1^2+\cdots x_n^2}\right|$$

$$=k\sqrt{x_1^2(x_1^2+\cdots+x_n^2)+\cdots+x_n^2(x_1^2+\cdots+x_n^2)}$$

$$=k|x|^2$$

So choose nbhd $V\ni 0$, s.t. on V,

$$\sup_{x\in V}\frac{\left|f(x)-\begin{pmatrix}\lambda_1 & & \\ & \ddots & \\ & & \lambda_n\end{pmatrix}x\right|}{|x|^2}\leq\frac{c}{2}$$

taylor.

then $\frac{d}{dt}L(x(t))\leq -\frac{1}{2}c|x|^2$ when $x(t)\in V$.

Choose $V=\{x\,|\,|x|<\rho\}$. for solutions $x'=f(x)$ on V,

we have $\frac{d}{dt}L(x(t))\leq -\tilde{c}\,L(x(t))$.

$$\Rightarrow \text{ on } V,\ L(x(t))\leq L(x(0))\cdot e^{-\tilde{c}t}\xrightarrow[t\to\infty]{}0$$

$\Rightarrow \quad |x(t)| \leq \sqrt{2e^{-\tilde{c}t}}\, |x(0)| \xrightarrow[t\to\infty]{} 0$

$\hookrightarrow \quad x(t) \in V, \quad \forall t, \quad \text{and} \quad x(t) \xrightarrow[t\to\infty]{} 0$

Def: a function $L: \underset{\subset \mathbb{R}^n}{U} \to \mathbb{R}$ is called a

Lyaponov function (energy)

for a system $x' = f(x)$ on U,

if L is non-increasing along solutions.

Sufficient: $\frac{d}{dt} L(x(t)) \leq 0$.

"Strict": $\frac{d}{dt} L(x(t)) < 0$

Prop: Under same hypo, $\lambda < 0$ ← linear

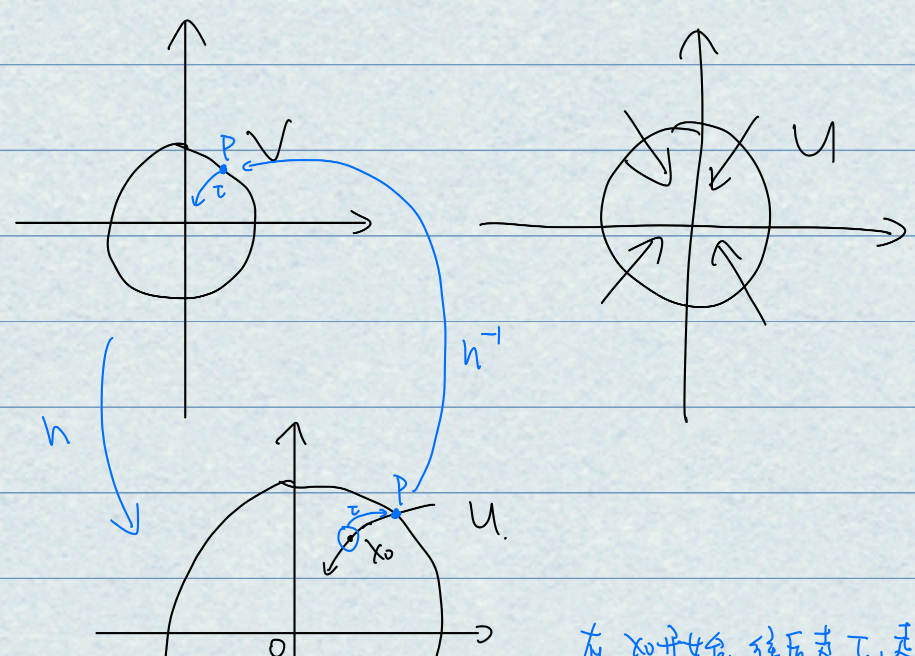
$\exists\, h: V \to U$, s.t. $\Phi_t = h \cdot e^{tA} \cdot h^{-1}$

open nbrd of 0 ; open nbrd of a. ← $\Phi_t(x) = x(t)$, $x(0) = x_0$.

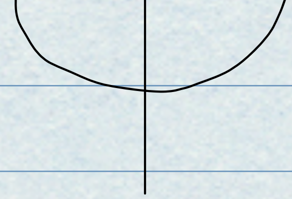
pf: WLOG, $a = 0$, $A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$.

Constructed $U = B_\rho(0)$, ρ small, s.t. if $x_0 \in U$,

then $x(t) \in U$, $\forall t > 0$. $|x(t)| \xrightarrow[t\to\infty]{} 0$



在 x_0开始，往后走 τ，走到 ∂，映到



V上对应的点，往前走τ.

p点，存在足够为 U 内没有其它 steady point.

pf: (formally)

$\forall x \in B_\rho(0) = U$

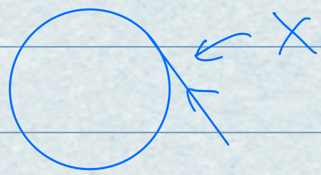
define $\tau(x) = \sup\{t<0,\ \Phi_t(x) \notin U\}$.

p点

fact: τ is differentiable for $x \in U \setminus \{0\}$.

pf: $t \to \Phi_t(x)$ is C^1 intersecting ∂U.

transversally (Derivative vector @ 交点 is not tangent to the surface).



然后用 implicit function thm.

So $\tau \to -\infty$ if $x \to 0$

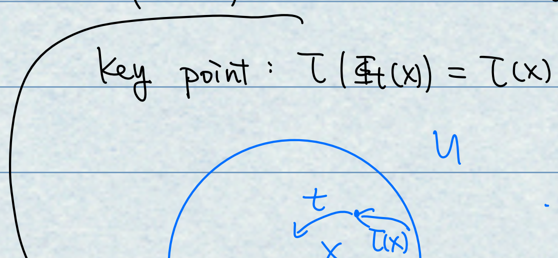
$$h^{-1}(x) = \underbrace{e^{-\tau(x)A}}_{\text{往前 flow } \tau(x) \text{ time.}} \circ \overbrace{\Phi_{\tau(x)}(x)}^{\text{"p"}}$$

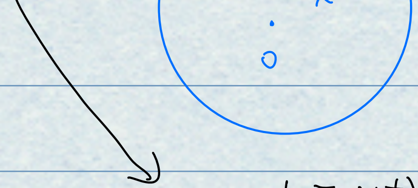
negative

Want to check: $h^{-1} \circ \Phi_t = e^{tA} \circ h^{-1}$, $\forall t>0$.

$h^{-1}(\Phi_t(x)) = e^{-\tau(\Phi_t(x)) \cdot A} \circ \Phi_{\tau(\Phi_t(x))}(\Phi_t(x))$ 只是代入.

Key point: $\tau(\Phi_t(x)) = \tau(x) - t$.





$$S_0 = e^{(-\tau(x)+t)A} \circ \underbrace{\Phi_{\tau(x)-t}(\Phi_t(x))}_{\text{x开始的，往前 } t\text{，往后 } \tau(x)-t\text{，} = \rho.}$$

$// \ \Phi_{\tau(x)}(x)$

$(k\cdot f(x))\circ g(x)$

$\Rightarrow k\cdot f(g(x))$

$$= (e^{tA}\cdot e^{\tau(x)A}) \circ \Phi_{\tau(x)}(x)$$

constant? $\; = e^{tA}\cdot \left(e^{\tau(x)A} \circ \Phi_{\tau(x)}(x)\right)$

$$= e^{tA}\circ h^{-1}(x)$$

Check: h homeo. 只需 check cont @ 0.

Let $y = e^{-\tau(x)A}\underbrace{\Phi_{\tau(x)}(x)}_{|\cdot|=\rho} \xrightarrow[x\to 0]{} 0.$ exponentially.

$-\tau(x)A$ 每一项都负，所以负.

In fact, enough to assume $\mathrm{Re}(\lambda)<0$. ($\forall\, Df(a)$ 的 λ)

2×2 case: $\lambda = -\alpha + i\beta$.

real Jordan form: $\begin{pmatrix} -\alpha & -\beta \\ \beta & -\alpha \end{pmatrix}$, the ODE: $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} -\alpha & -\beta \\ \beta & -\alpha \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$.

Same Lip function,

$$\frac{d}{dt}\frac{1}{2}|x|^2 = (x_1, x_2)\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$$

代λ.

$$= -\alpha|x|^2$$

↳ no β.

Grobman–Hartman thm:

若 $Df(a)$ hyperbolic,

then $\exists\, h: U\to V$. homeomorphism,

$0 \to a$.

s.t. $\Phi_t = h \circ e^{tA} \circ h^{-1}$

Solved: • if $\lambda < 0$, and A 可对角化. ✓

• If $\mathrm{real}(\lambda) < 0$, 2×2 case. 共轭 → 消掉.

• if Jordan block: $\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$ $\lambda < 0$,

$$y' = \begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix} y.$$

$$\frac{d}{dt}\left(\tfrac{1}{2}|y|^2\right) = (y_1, y_2)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}' = (y_1, y_2)\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \lambda(y_1^2 + y_2^2) + y_1 y_2$$

$ab \leq \frac{a^2+b^2}{2}$.

所以若λ足够大, ✓.

$\Rightarrow$ if $\varepsilon < \frac{|\lambda|}{3}$, $|y_1|$ <0

Claim: $\begin{pmatrix} \lambda & 1 \\ & \lambda \end{pmatrix}$ is conjugate to $\begin{pmatrix} \lambda & \varepsilon \\ & \lambda \end{pmatrix}$ 或需.

$$\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} = \begin{pmatrix} \lambda & \varepsilon \\ & \lambda \end{pmatrix}$$

q.e.d.

Summary:

if $\mathrm{Re}(\lambda) < 0$,

$\Downarrow$ Jordan form

symme, positive,

Lyap func, $L(x) = \frac{1}{2} x^T M x$

w.r.t "t?"

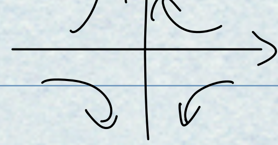
$L(x) \downarrow$, so a sink. Asympt. stable.

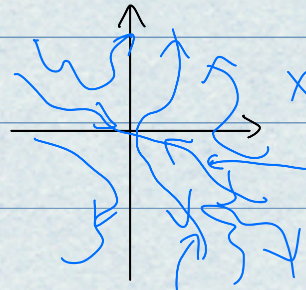
So, conjugate to the Linear system.

Saddle. $x' = f(x) = Ax + \bar{o}(|x|)$

linear is saddle. (λ 有正有负)

normal form: $\lambda_1 \uparrow$ $y' = Ay$,





$x'=f(x)$

stable.

↘ 收敛.

unstable manifold. → 发散

- These two manifolds are target at $x=a$. to the corresponding Linear manifold in linear system

Section 9.1 – 9.3

nullclines:

$x'=f(x)$ on $D\subset\mathbb{R}^n$. $f(a)=0$.

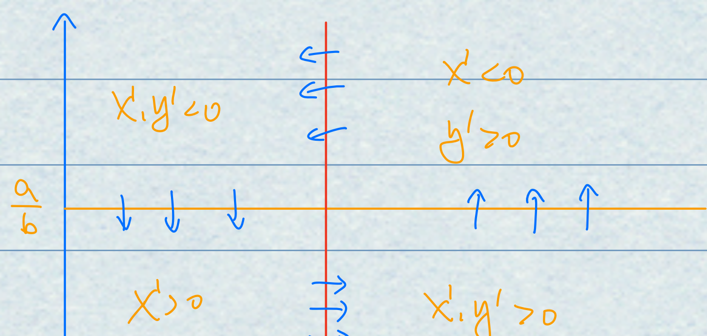
Def: j-th nullcline $=\{x\in D \mid f_j(x)=0\}$.

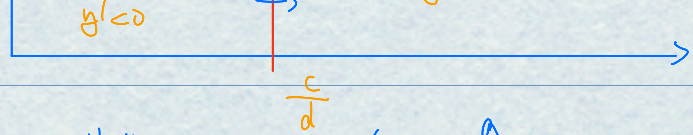
partition D into subdomains. 符号确定.

在 a 处相交.

e.g. $x'=x(a-by)$ $\quad$ $D>0$.

$y'=y(-c+dx)$





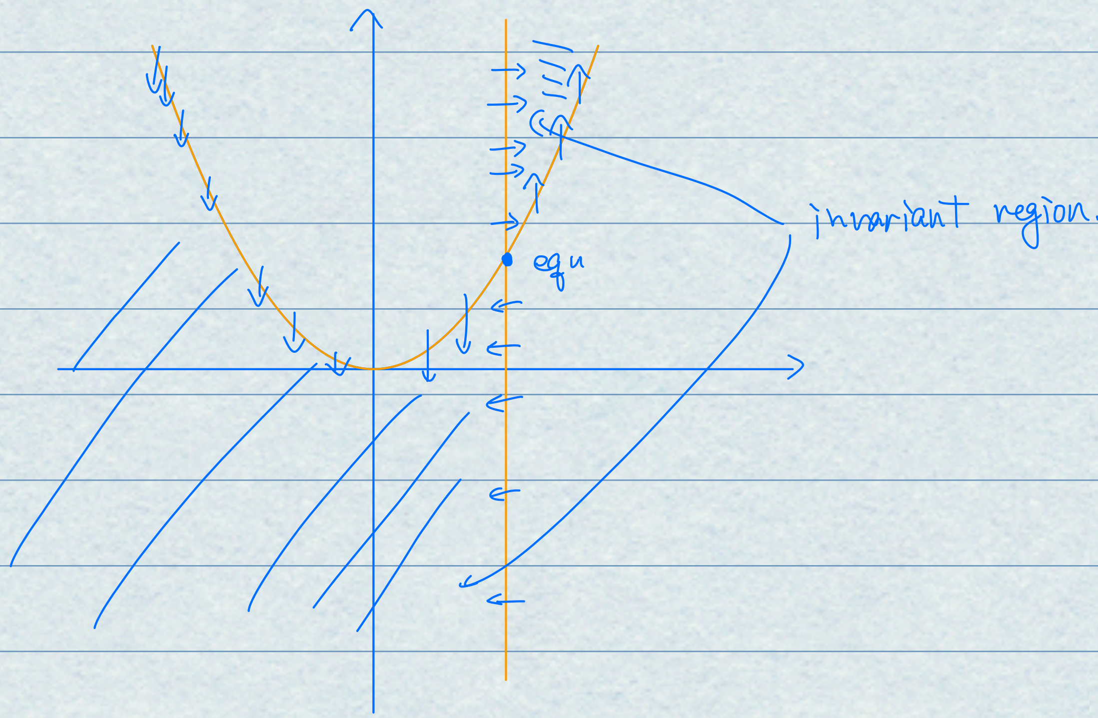
X-nullcline: $X=0$ / $y=\frac{a}{b}$

y- - - - -. $y=0$ / $X=\frac{c}{d}$.

e.g. $X'=y-X^2$

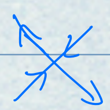
$y'=X-2$.

X-ncline: $y=X^2$ y... $X=2$.



linearize at (2,4)

$Df(a)=\begin{pmatrix}-4 & 1\\ 1 & 0\end{pmatrix}$, saddle (det<0)

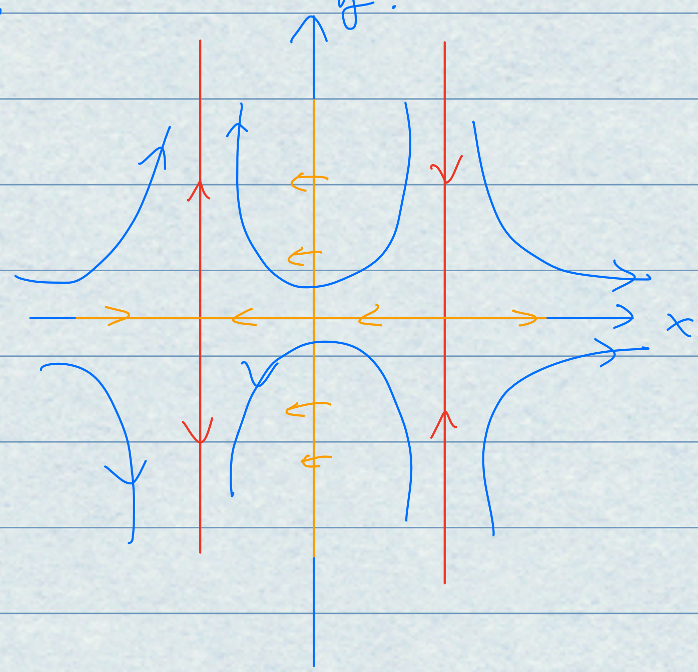


e.g. 3

$X'=X^2-1 \rightarrow X=\pm 1$

$y'=-Xy+a(X^2-1) \rightarrow Xy=a(X^2-1)$ a 相关

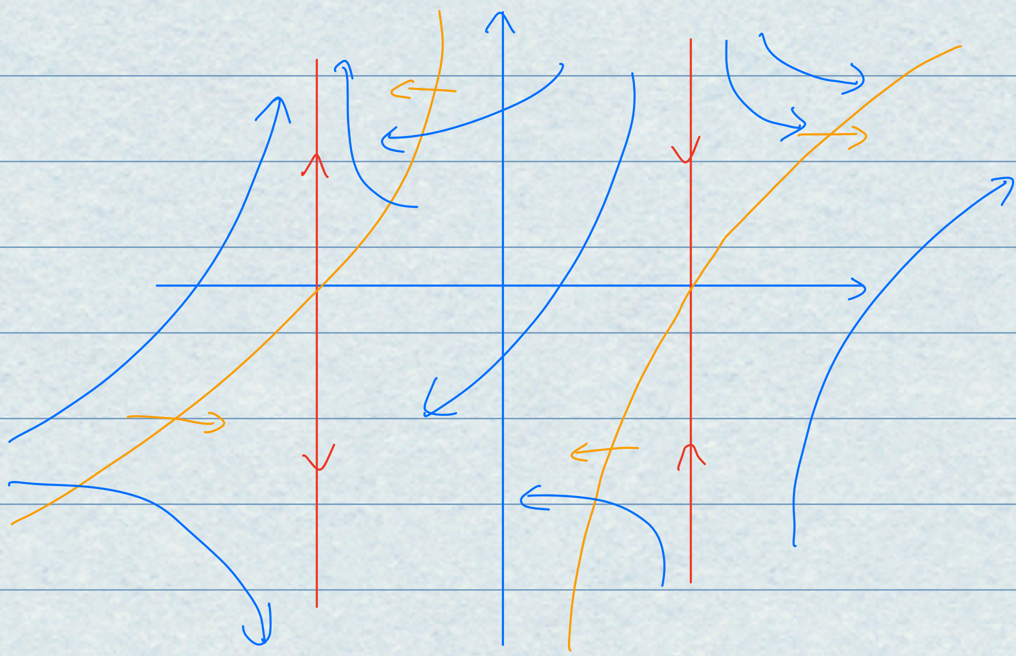
① $a=0$,



② $a>0$.

$x' = x^2 - 1$

$y' = -xy + a(x^2-1) \quad y = \dfrac{a(x^2-1)}{x} = ax - \dfrac{a}{x}$



Thm: 设 $x_1 : [a,b] \to \mathbb{R}^n$ be differable, PP. 394 [HSD]

$x_2 : [a,b] \to \mathbb{R}^n$, (recall)

s.t. $|x_1(a) - x_2(a)| \leq \delta$.

设 $f : [a,b] \times \mathbb{R}^n \to \mathbb{R}^n$ be a function that's Lip in second variable. $|f(t_1,\xi_1)-f(t_2,\xi_2)|\le L|\xi_1-\xi_2|$

设 $|x_1'(t)-f(t,x_1(t))|\le\varepsilon_1$, $(t\in(a,b))$

$|x_2'(t)-f(t,x_2(t))|\le\varepsilon_2$

then $|x_1(t)-x_2(t)|\le\delta e^{L(t-a)}+(\varepsilon_1+\varepsilon_2)\dfrac{e^{L(t-a)}-1}{L}$

Lyapunov's Thm:

$L: D\subset\mathbb{R}^n\to\mathbb{R}$, differentiable.

That has isolated strict local min at $x^*\in D$.

and $\frac{d}{dt}L(x(t))\le 0$, $\forall x$ solve $x'=f(x)$ in D

then x^* is stable equilibrium for C^1.

if, in addition, $\frac{d}{dt}(L(x(t)))<0$, for every non-constant x, then x^* is a sink (Asmp stable).

pf: ① stability: given x^*, given $U\ni x^*$, what find $U\ni x^*$, s.t. $x(0)\in U\Rightarrow x(t)\in V$, $\forall t>0$.

first of all, shrink U. WLOG, x^* is the only critical point in U.

$$\begin{cases} L(x^*)=\min\limits_{x\in U}L(x)\\ U \text{ is bounded.}\end{cases}$$

Let $m:=\min\limits_{x\in\partial U}L(x)>L(x^*)$

$V:\{x\mid L(x)<m\}$

So $x(0)\in V\Rightarrow x(t)\in V\subset U$. ✓.

② Asmp. stability.

Assuming $\frac{d}{dt}L(x(t)) < 0$, $\forall x$. non-constant.

$L(x(t)) \downarrow$, bonded. below

So $\lim\limits_{t\to\infty} L(x(t)) \geq L(x^*)$ ⇐ want : " = "

fix $x_0 \in V$, let $x(t)$ is the solution, $x(0) = x_0$.

then $\overline{V}$ is compact $\Rightarrow \exists (t_j)_{j\geq 1} \nearrow \infty$. s.t.

$\lim\limits_{j\to\infty} x(t_j) = y_0$

if $y_0 = x^*$, ✓ $\left(\Rightarrow \lim L(x(t_j)) = L(y_0) = L(x^*) \right)$

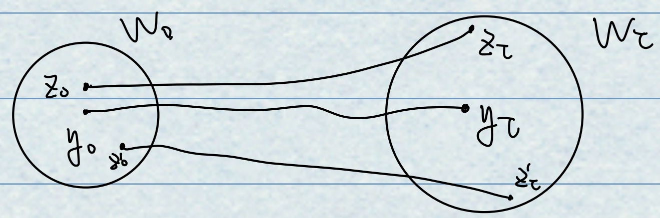
Call $y(t)$ solution with $y(0) = y_0$.

$\frac{d}{dt} L(y(t)) < 0$, $\Rightarrow L(y(t)) < L(y(0))$, fix $\tau > 0$. $\Rightarrow L(y(\tau)) < L(y_0)$

$\exists W_\tau$ nbhd of $y(\tau)$. s.t. $\forall w \in W_\tau$, $L(w) < L(y_0)$

$\xRightarrow[\text{continuous dependence of initial}]{} \exists W_0$ s.t. $y_0 \in W_0$, $\forall z_0 \in W_0$,

$\begin{cases} z_0 = z(0) \\ z' = f(z) \end{cases} \Rightarrow z(\tau) \in W_\tau$

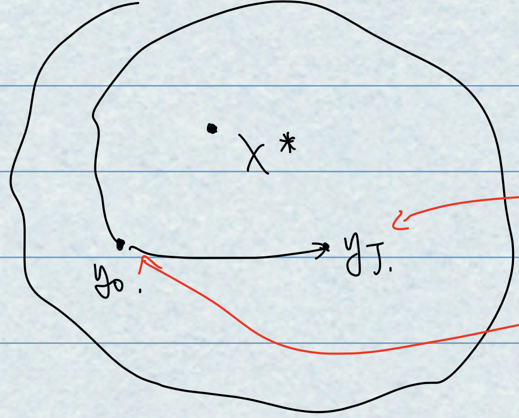


Set $z_0 = x(t_j)$ $(j \geq 1)$, 则 $z_0 \in W_0$,

$\Rightarrow x(t_j + \tau) \in W_\tau$.

$\Rightarrow L(x(t_j + \tau)) < L(y_0)$ 但 因为 $\frac{d}{dt} L < 0$,

又有 $\lim L(x(t_j)) = L(y_0)$, 一直 ↓, 但 这里 ↑ 了, 矛盾



这里的L值小于这里的L值.

与

"$\lim L(x(t_j)) = L(y_0)$，一直 $\downarrow$，" 矛盾

Fact : if x^* is a sink. for $x' = f(x)$, $\Rightarrow \exists L(x)$ lyp function, on $U \ni x^*$, s.t. $\frac{d}{dt}(L(x(t))) < 0$, $\forall x$ non-constant.

Def: Limit Sets.

$x' = f(x)$ on $D \subset \mathbb{R}^n$

$x(t)$ is a solution, the ω-limit set of x

$= \{ y \in D \mid \exists t_j \nearrow +\infty, \text{ s.t. } \lim (t_j) = y \}$

$= \bigcap_{\substack{s>0 \\ s \in \mathbb{R} \\ s = 1, 2, \cdots}} \overline{\{ x(t) \mid t \geq s \}}$

the α-limit set:

$= \{ y \in D \mid \exists t_j \searrow -\infty, \text{ s.t. } \lim (t_j) = y \}$

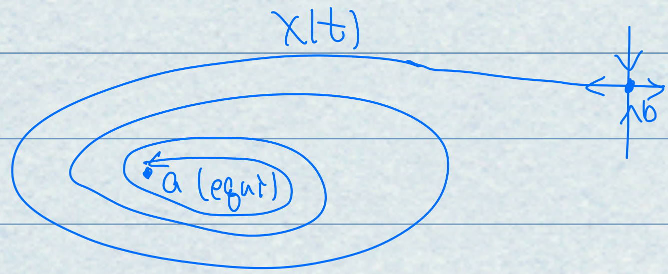
$= \bigcap_{\substack{s<0 \\ s \in \mathbb{R} \\ s = -1, -2, \cdots}} \overline{\{ x(t) \mid t \leq s \}}$

Observation: limits could be $\emptyset$.

Like: $x' = 1$

e.g.s:

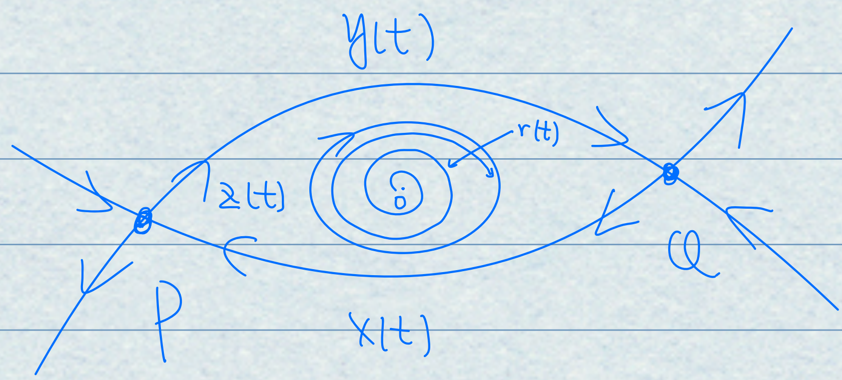
①



$w(x) = a$

$\alpha(x) = b$

②



x, y 连接两个 $\overset{\text{diff}}{\vee}$ equi，所以 x, y 叫 heteroclinic

- - - - same - - - - - homoclinic

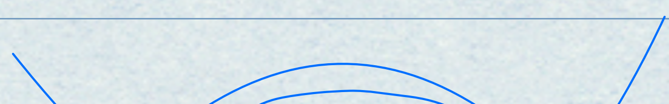
$w(x) = P = \alpha(y)$

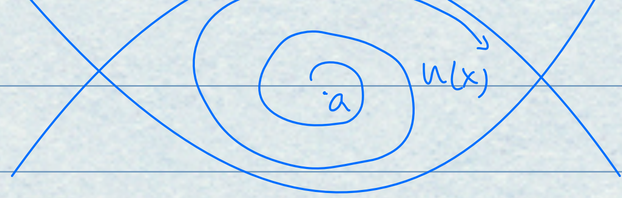
$\alpha(x) = Q = w(y)$

$w(z) = \alpha(z) = \{z(t)\} \ (t \in \mathbb{R})$

$\alpha(r) = a. \quad w(a) = \{z(t)\} \ (t \in \mathbb{R})$

③





$\alpha(u) = a,$

$\omega(u) = \{P\} \cup \{Q\} \cup \{x(t)\} \cup \{y(t)\}.$

Limit sets could be:

equi, connecting orbit, cycles.

Assume now that $x(t)$ is a bounded solution.

$x(t) \subset D \quad , \forall t \in \mathbb{R}.$

then $\omega(x) = \bigcap_s \overline{\{x(t) / t \geq s\}}$ is chain of non-empty compact sets.

Cantor's intersection thm:

① $A_1 \supset A_2 \supset A_3 \supset \cdots\cdots$, sequence of compact set,

if $\bigcap_j A_j = \emptyset$, then $\exists j_0$ s.t. $A_{j_0} = \emptyset$.

② if $\{C_j\}_{j \geq 1}$ is compact, then $\bigcap_j C_j = \emptyset$

$\Longleftrightarrow \exists N,$ s.t. $C_1 \cap C_2 \cap \cdots \cap C_N = \emptyset.$

① $\Longleftrightarrow$ ②

Properties:

① $\omega(x) \neq \emptyset$ (因为紧)

② if $y \in \omega(x)$, then [the entire orbit of y]

$= \{ \Phi_t(y) \mid t \in \mathbb{R} \} \subset W(x)$

③ $W(x)$ is conneited.

pf of ②: given $y \in W(x)$, $t \in \mathbb{R}$.

$\exists\, t_j \nearrow +\infty$, $\lim_j x(t_j) = y$.

$$\Phi_t(y) = \lim_j \Phi_t(x(t_j)) = \lim_j (t_j + t) \in W(x)$$

cont

因为 $\rightarrow \{t_j\} \nearrow +\infty \Rightarrow \{t_j + t\} \nearrow +\infty$ (constant)

2^{nd} pf:
$$\begin{aligned}
\Phi_t(W(x)) &= \Phi_t\left(\bigcap_s \overline{\{x(r) \mid r \geq s\}}\right) \\
&\subseteq \bigcap_s \Phi_t \overline{\{x(r) \mid r \geq s\}} \\
&\subseteq \bigcap_s \overline{\Phi_t(\{x(r) \mid r \geq s\})} \\
&= \bigcap_s \overline{(\{x(t+r) \mid r \geq s\})} \\
&= \bigcap_s \overline{(\{x(r) \mid r \geq s+t\})} = W(x)
\end{aligned}$$

Def: A closed set $C \subseteq \mathbb{R}^n$ is conneited

if $\forall$ partition $C = C_1 \cup C_2$, $C_1 \cap C_2 = \emptyset$,

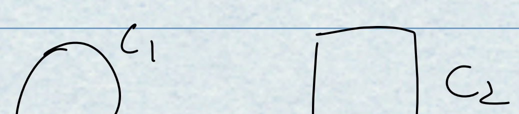
C_1, C_2 closed, we have either $C_1 = \emptyset$ or $C_2 = \emptyset$.

pathwise connected: $\forall x, y \in C$, $\exists$ path $\gamma_{x,y}$

conneting x, y in C.

proof of ③: if not, then $W(x) = C_1 \cup C_2$,

$C_1 \cap C_2 = \emptyset$. $y_1 \in C_1$, $y_2 \in C_2$.



$\exists d$, s.t. $|C_1 - C_2| \geq d$

Hamilton system:

$x'' = -\nabla V(x)$

$$\begin{cases} x' = y \\ y' = -\nabla V(x) \end{cases}$$ 2n functions.

Lip function: $L(x,y) = \frac{1}{2}|y|^2 + V(x)$ $\to$ energy.

$$\frac{d}{dt}L(x,y) = y\cdot y' + \nabla V(x)\cdot x'$$
$$= 0$$

所以能量守恒.

equivalently: $\exists H: \mathbb{R}^2 \to \mathbb{R}$, smooth. s.t. $x' = \frac{\partial H}{\partial y}(x,y)$

$y' = -\frac{\partial H}{\partial x}(x,y)$

$$\begin{pmatrix} 0 & 1 \\ * & 0 \end{pmatrix} \qquad \begin{pmatrix} \frac{\partial^2 H}{\partial x \partial y} & \\ & -\frac{\partial^2 H}{\partial x \partial y} \end{pmatrix}$$

So $\lambda^2 + 0\lambda + * = 0$.

Limit set – continued.

Properties:

① $w(x) \neq \emptyset$ (因为紧)

② if $y \in w(x)$, then [the entire orbit of y]

$= \{\Phi_t(y) \mid t \in \mathbb{R}\} \subset w(x)$

③ $w(x)$ is connected.

Jordan curve thm:

γ is simple closed curve in $\mathbb{R}^2$.

i.e. $\gamma: [a,b] \to \mathbb{R}^2$. $\gamma(a) = \gamma(b)$

$\gamma(s) \neq \gamma(t)$

Then $\mathbb{R}^2 \setminus \gamma$ has 2 connected component.

1 is Bdd, homeomorphic to $[x^2+y^2<1] = D$

1 is unBdd, homeo to $\mathbb{R}^2 \setminus \bar{D}$

Def: ACD domain.

$x' = f(x)$ on D. A is positively invariant

if $\begin{cases} \Phi_t(A) \subset A, \ \forall t>0 \\ x \in A \Rightarrow x(t) \in A. \end{cases}$

negative ---- same.

· A is invariant if it's both positively / nega--- invariant.

Prove that Limit sets are invariant.

Poincare Bendixson Thm:

$x' = f(x) \subset D \subset \mathbb{R}^2$.

f is C^1.

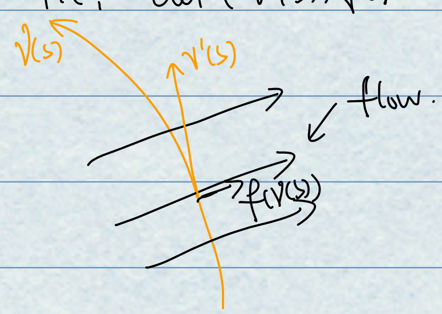
Assume $x \subset D$, assume $W(x)$ contains no equlibrium point, and contained in compact set, then $W(x)$ is a periodic orbit

i.e. $\exists\ y(t) = y(t+T)$, $W(x) = \{y(t) \mid 0 \le t \le T\}$

Def: A section (γ) for the flow $*\ x' = f(x)$ is a curve segment s.t. it intersect the vector field transversally.

i.e. $\det(\gamma'(s), f(\gamma(s))) \neq 0 \quad (\forall\ \gamma)$

WLOG ≥ 0.



$\exists$ small neighborhood of γ on which flows are almost 平行的

By IFT: $\exists$ section near any x_0, s.t. $f(x_0) \neq 0$.

Key idea: Suppose a solution $x(t)$ intesect A section $\gamma(s)$

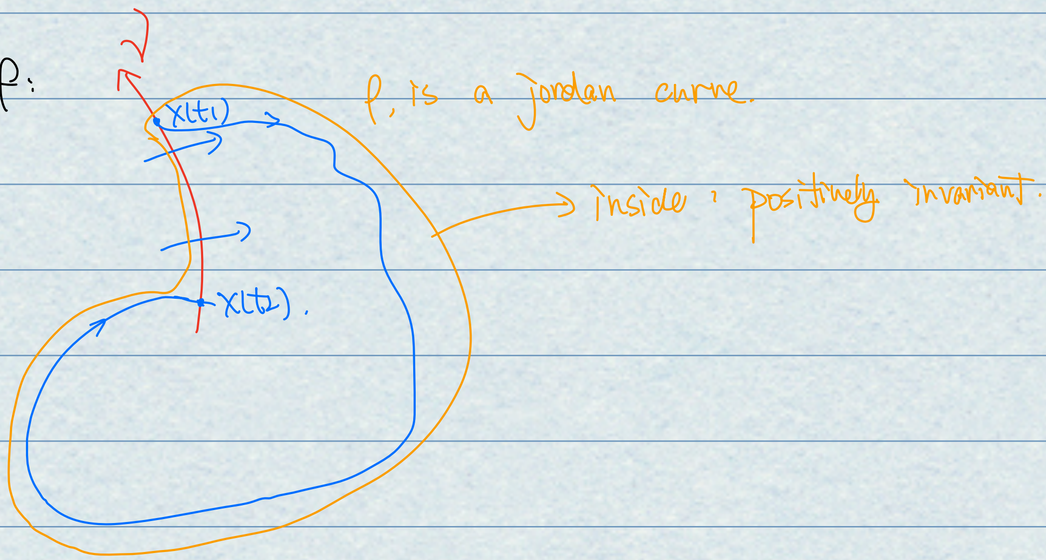
in multiple of times, on

$$x(t_1) = \gamma(s_1)$$
$$x(t_2) = \gamma(s_2)$$
$$x(t_3) = \gamma(s_3)$$
$$\vdots$$

Claim: If $t_1 < t_2 \cdots <$ is monotone, then s_j is also

monotone $(s_1 > s_2 > \cdots)$ or $(s_1 < s_2 < \cdots)$

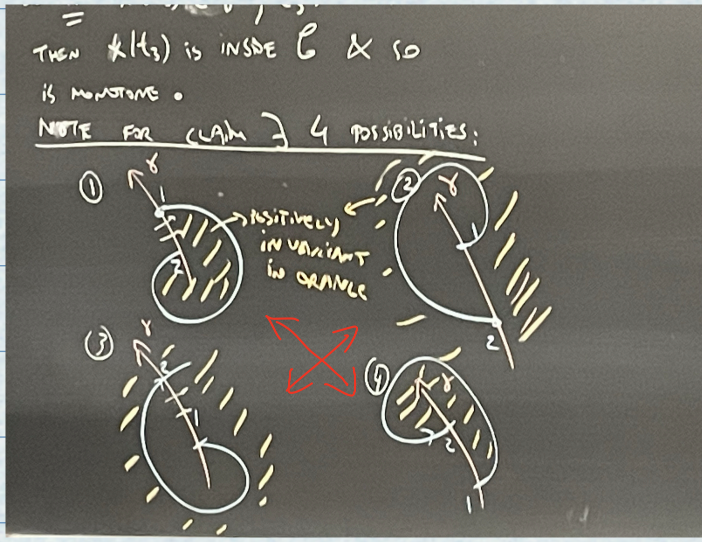
pf:



if $x(t_3) \in \gamma$, $t_3 > t_2 > t_1$ then $x(t_3)$ inside ρ, so

it's monotone.

Note: for the claim: there are 4 possibilities. (正负组合)

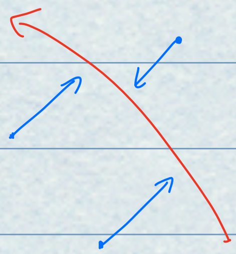


same.

Note: can take $x(t_j) \in \gamma$.

$y \in \omega(x) \Rightarrow \exists \ \underline{x(\tau_j)} \to y.$

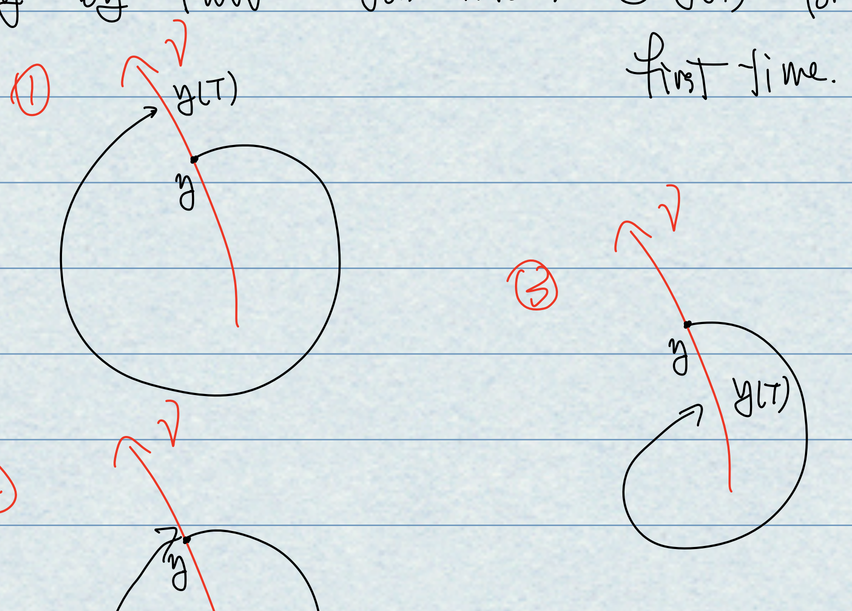
need not belong to γ.

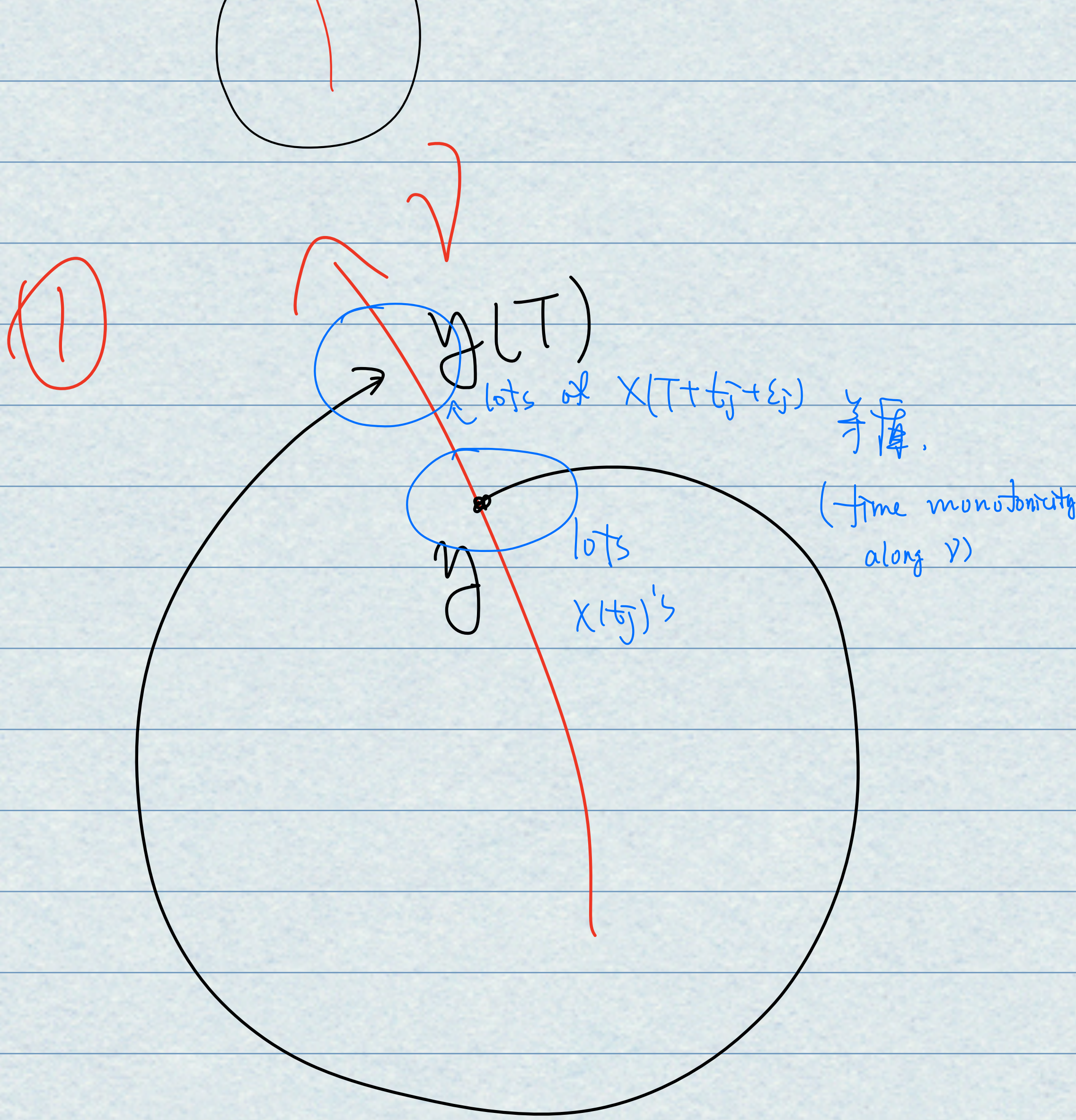


把点"拉"到γ上.

By IFT, if γ is smooth, $x \to x(t_1)$ is smooth.

Evolve y by flow: $\to y(t)$ hits γ @ $y(T)$ for first time.



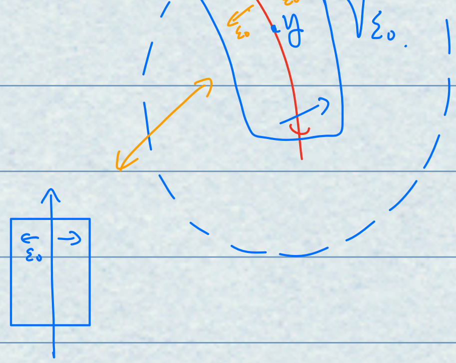


Solved so far: $y \in \gamma(x) \Rightarrow \Gamma = \{y(t)\}$ is periodic,

$\Gamma \subset \omega(x)$

Want: if Γ is a closed orbit in $\omega(x)$ then $\omega(x) = \Gamma$

只需证 $\text{dist}(\underset{\substack{\uparrow \\ \Phi_t(x)}}{x(t)}, \Gamma) \xrightarrow[t\to+\infty]{} 0$



Claim: $\exists\, t_0 < t_1 < \cdots < t_n <,$ s.t.

① $\Phi_{t_n}(x) \in \tilde{\gamma}$

② $\Phi_{t_n}(x) \to y$

③ $\Phi_t(x) \notin \tilde{\gamma}$ if $t_n < t < t_{n+1}$

$x_n = \Phi_{t_n}(x)$

fact: $\exists\, \bar{T} > 0,\ |\ t_{n+1} - t_n \le \bar{T}$ for $n \ge n_0$

pf: $\exists\, T > 0 \ /\ y(T) = y.$

if x_n close to y, then $\Phi_T(x_n) \in V_{\varepsilon_0}$

So $\Phi_{T+t}(x_n) \in \tilde{\gamma}$, for some $t \in (-\varepsilon_0, \varepsilon_0)$

So $t_{n+1} - t_n \le T + \varepsilon_0 = \bar{T}$

Take $\beta > 0$

Continuous dependence on initial $\Rightarrow \exists\, \delta > 0$, s.t.

$|z - y| < \delta,\ |t| \le \bar{T} \Rightarrow |\Phi_t(z) - \Phi_t(y)| < \beta.$

if $n > n_0$, $|x_n - y| < \delta \Rightarrow |\Phi_t(x_n) - \Phi_t(y)| < \beta$ if $n \ge n_0$, $|t| \le \bar{T}$.

$\underbrace{\mathrm{dist}(\Phi_t(x), \Gamma)}_{t_n < t < t_{n+1}} \le |\Phi_t(x) - \Phi_{t-t_n}(y)| = |\Phi_{t-t_n}(x_n) - \Phi_{t-t_n}(y)| < \beta$

因为 $t - t_n < \beta$