Some notation / Review:

- Magnitude or Absolute value in $x \in \mathbb{R}$ $|x|$

  $|x+y| \le |x| + |y|$

- Notation for sequences: $(a_m)_{m\in\mathbb{N}}$, $\{a_m\}_{m\in\mathbb{N}}$

  Cauchy sequences: "(a_n) converges to b"

  if $\forall \varepsilon>0$, $\exists N$, s.t. $\forall n \ge N$, $|a_n - b| < \varepsilon$.

  call b the limit of a_n

  $b = \lim_{n\to\infty} a_n$.

Def: $\{a_n\}$ is cauchy if $\forall \varepsilon>0$, $\exists N$. s.t. $\forall n, m > N$,

$|a_n - a_m| < \varepsilon$.

Remark: If $\{a_n\}$ converges, then it's cauchy.

Pf: triangular inequality.

vise versa: Cauchy $\Rightarrow$ Convergence. Yes in $\mathbb{R}^n$

Depends on the space. Completeness

- $\mathbb{Q}$ is dense in $\mathbb{R}$

- Think about completeness: example of non-completeness.

  $\{\frac{1}{n}\}$ in $(0,1)$

- Euclidean Space $\mathbb{R}^m = \underbrace{\mathbb{R}\times\cdots\times\mathbb{R}}_{m}$

  $+$, $-$, 数乘, 点乘

  $\langle x,y\rangle = \sum x_i y_i$ $|x| = \sqrt{\langle x,x\rangle}$.

Intro / Review:

- Euclidean Space $\mathbb{R}^m$

  (Basic example for metric space, euclidean space)

  Def: Euclidean Norm:

  $x \in \mathbb{R}^m$, $\|x\| = \sqrt{\sum x_i^2} = \langle x, x \rangle$, where $\langle \cdot, \cdot \rangle$ dot product.

Inner product.

- Remark: $\langle \cdot, \cdot \rangle$ symmetric, i.e $\langle x, y \rangle = \langle y, x \rangle$

  positive definite, i.e. $\langle x, x \rangle \geq 0$, $= 0 \Leftrightarrow x = 0$

  bilinear, i.e. $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$.

  with these properties, we can define $|x| = \sqrt{\langle x, x \rangle}$

- Def: Given a vector space V (over $\mathbb{R}$), a norm on V is function

  $|\cdot| : V \to \mathbb{R}_{\geq 0}$, s.t.

  1) $\forall v$, $|v| \geq 0$, $|v| = 0 \Leftrightarrow v = 0$

  2) $|\lambda v| = |\lambda| \, |v|$,

  3) $\forall v, w$, $|v + w| \leq |v| + |w|$

- Note: A dot product always defines a norm by $|x| = \sqrt{\langle x, x \rangle}$.

  but not vise versa.

- Cauchy - Schwartz: $\langle x, y \rangle \leq |x| \cdot |y|$

- Pf: $\langle x + ty, x + ty \rangle \geq 0$, using bilinearity to expand.

  得到关于t的二次函数,算判别式.

- Observe that Cauchy-Schwartz $\Rightarrow$ triangular Ineq.

- Euclidean distance:

  Distance from x to y is $|x-y|$. satisfies $|x-y| \le |x-z| + |y-z|$

- Def: A vector space V and $\langle \cdot, \cdot \rangle$. We call $(V, \langle \cdot, \cdot \rangle)$ an

  inner product vector space. (C-S holds)

- E.g: Consider $C([a,b])$. Space of all continuous functions $f: [a,b] \to \mathbb{R}$.

  - $C([a,b])$ is a infinite dimensional vector space. Define $\langle f, g \rangle = \int_a^b f(x) g(x) dx$.

  - It follows from C-S: $\int_a^b f(x) g(x) dx = (\int_a^b f^2 dx)^{\frac{1}{2}} (\int_a^b g^2 dx)^{\frac{1}{2}}$

  Can find infinitely many linearly independent functions.

- Intervals in $\mathbb{R}$: $[a,b]$, (a,b), $(a,b]$

- Boxes in $\mathbb{R}^m$: $\prod_{i=1}^{n} [a_i, b_i]$

- Balls in $\mathbb{R}^m$: $B^m = \{x \in \mathbb{R}^m \mid |x| \le 1\}$ standard unit ball.

- Sphere: $S^{m-1} = \{x \in \mathbb{R}^m \mid |x| = 1\}$

- E is convex if $\forall x, y \in E$, $tx + (1-t)y \in E$. $\forall t \in (0,1)$

- $f: \underset{\text{domain}}{A} \to \underset{\text{codomain}}{B}$ $f(A)$ image.

- Injective, Surjective, Bijective.

Thm: $f \in C[a,b]$, f is bounded. $m \le f \le M$, then

f will take on its absolute max/min

f will take on all intermediate values

f is uniformly continuous.

METRIC SPACES

Def: A metric space is a set M, call the elements "points", with a metric d:

$\forall x, y \in M$, $d(x,y)$ is called distance from x to y.

3 properties

- $d(x,y) \geq 0$, $d(x,y) = 0 \Leftrightarrow x = y$
- $d(x,y) = d(y,x)$
- $d(x,y) \leq d(x,z) + d(y,z)$

(M, d) a metric space.

Examples: · $\mathbb{R}$, $\mathbb{R}^m$ or subsets with euclidean distance.

· $C[a,b]$ with $d_\infty(f,g) = \sup|f-g|$

$d_2(f,g) = \sqrt{\int |f-g|^2 dx}$

$d_1(f,g) = \int |f-g| dx$.

$d_p(f,g) = \left(\int_a^b |f-g|^p dx \right)^{\frac{1}{p}}$

· Discrete metric:

$$d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{otherwise.} \end{cases}$$

· $\{a_n\}$, with finite norm $(\sum |a_n|^p)^{\frac{1}{p}} < \infty$, $d_p(\{a_n\}, \{b_n\}) = (\sum |a_n - b_n|^p)^{\frac{1}{p}}$

- Sequences in (M, d)

Sequence is a list of points $p_1, \cdots, \in M$, ordered.

$\{p_n\}$ converge to a point in M, write $\lim_{n\to\infty} p_n = p$ if $\forall \varepsilon > 0$, $\exists N$, s.t. $\forall n > N$, $d(p_n, p) < \varepsilon$.

Limits are unique by triangular.

$\{q_k\}$ is a subsequence of $\{p_n\}$ if $\exists\, m_1 < m_2 < \cdots$, s.t. $q_k = p_{m_k}$.

- Continuity:

Consider fn $f: \overset{d_M}{M} \to \overset{d_N}{N}$

- Def: $f: M \to N$ continuous if it preserves sequential converge

  if $p_m \to p$ then $f(p_m) \to f(p)$

- Homeomorphism: $f: M \to N$ is a homeomorphism if

$$\begin{cases} f \text{ is bijection} \\ f \text{ is continuous} \\ f^{-1} \text{ is continuous.} \end{cases}$$

What is homeomorphism in general:

Transformation that can deform, bend, stretch,

but not puncture, tear.

- Note: f^{-1} is needed (doesn't follow from f cont bijection)

Thm: $f: M \to N$ is continuous iff it satisfies the ε-δ condition,

i.e. $\forall \varepsilon$, $\forall p \in M$, $\exists \delta$, s.t. $d_M(q,p) < \delta \Rightarrow d_N(f_q, f_p) < \varepsilon$.

Pf: "$\Longrightarrow$" Suppose f is cont. (sequential convergence)

Assume by contradiction ε-δ fails.

i.e. $\exists \delta$, $\exists p \in M$, s.t. for $\delta < \frac{1}{n}$, $\exists q_n$, s.t.

$d_M(q_n, p) < \frac{1}{n}$, $d_N(f_{q_n}, f_p) > \varepsilon$.

But $q_n \to p$, $f(q_n) \not\to f(p)$, contradiction.

"$\Longleftarrow$" Assume the ε-δ, Consider $\{p_n\}$, $p_n \to p$, WTS $f(p_n) \to f(p)$

$\forall \varepsilon > 0$, $\exists \delta$, s.t. $d_M(p_n, p) < \delta$, then $d_N(f(p_n), f(p)) < \varepsilon$.

Because $p_n \to p$, $\exists N_0$, s.t. $\forall n > N_0$, $d_M(p_n, p) < \delta$

$\Rightarrow \forall n > N_0$, $d_N(f(p_n), f(p)) < \varepsilon$.

Close / Open: (M, d) Metric space, $S \subset M$,

Def: $p \in M$ is a limit point of S if $\exists \{p_n\} \subset S$, s.t. $p_n \to p$.

Def: S is closed if it contains its all limit points.

Def: S is open if $\forall p \in S$, $\exists R > 0$, s.t. $d(p, q) < R \Rightarrow q \in S$.

Thm: Complements of open is closed and vise versa.

Note: A set may not be open or closed. It could be both. (clopen)

Def: The topology on (M, d) is the collection of all open sets, denote $\mathcal{T}$

$\mathcal{T}$ is closed under arbitrary union, under finite intersection and contains the empty set and whole set.

Remark: Passing the complements, closed set is closed under arbitrary intersection and finite union. $\emptyset$, M.

Def: $\lim S = \{p \in M \mid p \text{ a limit point of } S\}$

Def: $R > 0$, $p \in M$, define $M_R(p) = \{q \in M, \text{ s.t. } d(p, q) < R\}$ an open neighborhood of p in M.

Thm: $\lim S$ is closed, and the smallest closed set containing S.

Pf: $\{p_n\} \in \lim S$, $p_n \to p$, wts $p \in S$.

$\Downarrow$

$\exists q_{n,k} \xrightarrow{k \to \infty} p_n$

Def: $f: M \to N$, The preimage of $V \subset N$ is $f^{pre} = \{p \in M, \text{ s.t. } f(p) \in V\}$.

Thm: TFAE:

① f satisfies ε-δ

② f is continuous

③ $f^{pre}(\text{open})$ is open.

Prop: Every metric subspace N of metric M inherits its topology from M.

in sense that any $V \subset N$ open is on the form $V = N \cap U$,

where U open in M.

Pf: 走神了……

Completeness:

Def: $\{p_n\} \subseteq M$ is cauchy or satisfies cauchy condition if $\forall \varepsilon > 0, \exists N$, s.t. $\forall m, n > N$,

$d(p_m, p_n) < \varepsilon$.

Note: $\{p_n\}$ converges $\Rightarrow$ $\{p_n\}$ cauchy

But converse is not always true. (True in $\mathbb{R}^n$)

Def: (M, d) is complete if every cauchy sequence converge in M.

Remark: ① Every closed subspace of a complete metric space, then it is complete.

② Completeness is not preserved under homeomorphisms. (i.e. it's not a topological

那些在homeo下不变的东西.

perperty)

not. complete

e.g: $(-1, 1) \xleftarrow{\tanh(x)} \mathbb{R}$

Compactness: (A topological property preserved under homeomorphisms)

Def: $A \subset M$, (M, d) metric space,

the A is (sequentially) compact if any sequence $\{a_n\} \subset A$ has a convergent subsequence in A.

Remark: $\emptyset$, finite sets are CPT (compact).

Thm: Compact $\Rightarrow$ Close + Bounded. (not vise versa)

Ref: contained in $M_R(p)$ for some R, p.

Example: (Can't do $\mathbb{R}^n$, in $\mathbb{R}^n$ "$\Leftarrow$" ✓)

- $(\mathbb{N}, d)$, $d =$ discrete topology. (This is complete)
- $C([0,1])$, with $d(f,g) = \sup_{[0,1]} |f(x) - g(x)|$

Look at $A = \{f \in M \mid d(f, 0) \leq 1\}$.

Bdd by definition, closed, not compact

$f_n(x) = x^n$.

Thm: (on $\mathbb{R}/\mathbb{R}^n$):

- $[a,b]$ is compact.
- Product of 2 compact sets is compact. In particular, boxes in $\mathbb{R}^n$ are compact
- Every bdd sequence in $\mathbb{R}^n$ has convergent subsequence.
- (Heine-Borel) closed + bdd in $\mathbb{R}^n \Rightarrow$ cpt.

Thm: Given a nested sequence of sets $A_1 \supset A_2 \supset A_3 \supset \cdots$

Consider intersection $A = \bigcap_{i=1}^{\infty} A_n$

If A_i are compact, non-empty, so does A.

Thm: $f: M \to N$, $A \subset M$ compact $\Rightarrow$ $B \subset M$ compact.

in particular, compactness is a topological property.

Pf: Consider $\{b_n\} \in f(A)$, that is $b_n = f(a_n)$, $\{a_n\} \in A$.

By compactness, $\{a_{m_k}\}$ converges. $\Rightarrow \{b_{m_k}\}$ converges by continuity.

Thm: $f: M \to \mathbb{R}$, (M, d) metric space, f is continuous over A compact. It is bounded + assumes MAX/MIN.

Pf: $f(A)$ compact.

Recall: A homeomorphism is bicontinuous bijection.

Thm: If M compact, $M \overset{\text{homeomorphism}}{\cong} N$, N compact.

Moreover, if M is compact, then a continuous bijection $f: M \to N$ is a homeomorphism.

Pf: Consider $\{q_n\} \subset N$, $q_n \to q$. WTS: $f^{-1}(q_n) \to f^{-1}(q)$ (p_n)

Assume $p_n \not\to p$ $\exists \{p_{n_k}\}$. $\delta > 0$, s.t. $d(p_{n_k}, p) > \delta$. $\forall k$.

Since M compact, $\exists \{p_{n_{k(i)}}\}$, s.t. $p_{n_{k(i)}} \longrightarrow p^* \in M$ $(p^* \neq p)$

Since f continuous $f(p_{n_{k(i)}}) \to f(p^*)$. but $q_{n_{k(i)}} \to q = f(p) \neq f(p^*)$. contradiction.

Uniform continuity:

Def: $f: M \to N$ is uniformly continuous if $\forall \varepsilon > 0, \exists \delta$, s.t. if $d(p,q) < \delta$

$\Rightarrow d(f(p), f(q)) < \varepsilon$.

Thm: f continuous on a compact set is uniformly continuous.

Pf:

Connectedness

Def: Let $A \subset M$ metric space, we say A is proper if $A \neq \emptyset, M$.

Def: M is disconnected if it has proper clopen subset.

M is connected if it is not disconnected.

Thm: If M is connected, $f: M \to N$ is cont., f onto N, then N is connected. (a topological property).

Pf: 设 $S \subsetneq N$ clopen, $\Rightarrow$ $f^{-1}(S) \in M$. clopen. $f^{-1}(S) \neq M$ since surjectivity $\Rightarrow$ 矛盾.

Cor: (Intermediate value)

If $f: M \to \mathbb{R}$ (f cont.), M connected, then f has intermidiate value property (attains all intermediate values)

$f(M)$ connected, if $a \in (\inf(f(M)), \sup(f(M)))$ 取不到, 则 $(\inf(f(M)), a)$ clopen.

Recall: (M, d) metric space. $S \subset M$ is disconnected if it has proper clopen subset.

is connected if not disconnected.

- Connetedness is a topolgical property (preserved by homeomorphism)
- $f: M \to \mathbb{R}$ (M connected) continuous $\Rightarrow$ has IVP

Note: $\mathbb{R}$, (a, b) is connected.

E.g:
- Can use connetedness to say when two spaces are not homeomorphic.
- $[a, b] \overset{?}{\cong} S^1$?

No. remove a point make disconnected.

Thm: Closure of connected is connected.

If S connected and $S \subset T \subset \overline{S} = \lim S$, then T is connected.

if $T = A \cup B$ clopen, $\Rightarrow$ $S = \underbrace{(A \cap S)}_{\text{clopen}} \cup \underbrace{(B \cap S)}_{\text{clopen.}}$, S connected,

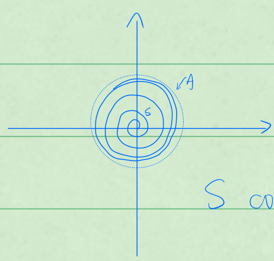
wlog, $A \cap S = \emptyset$ $\Rightarrow B \cap S = S \Rightarrow S \subseteq B \Rightarrow \overline{S} \subseteq \overline{B} \Rightarrow T \subseteq \overline{B}$

$\overline{S} \supseteq T$

但我们有 $A \cap B \overset{B \text{ close}}{=} A \cap \overline{B} = \emptyset$

$\Rightarrow A \cap T = \emptyset$.

E.g:



S connected $\Rightarrow$ $S \cup A$ connected.

Thm: The union of connected sets that have point in common is connected. Suppose $\beta \in \bigcap_\alpha S_\alpha$, $\bigcup_\alpha S_\alpha = A \cup B$ clopen, wlog $\beta \in A$.

S_α connected, $\bigcap_\alpha S_\alpha \neq \phi \Rightarrow \bigcup_\alpha S_\alpha$ connected.

A open $\Rightarrow \forall \alpha$. $A \cap S_\alpha = S_\alpha$ (因为 $A \cap S_\alpha$, $B \cap S_\alpha$ 分别 clopen, $\Rightarrow$ 有一个是 ϕ 加上 $\beta \in A$)

$\Rightarrow A \cap (\bigcup S_\alpha) = A$.

Path-connectedness

Def: A **path** joining points p and q in (M, d), is a cont fn f

$f: [a,b] \to M$, $f(a) = p$, $f(b) = q$.

Def: M is **path-connected** if every two point can be joined by path.

Remark: Path-connected $\Rightarrow$ connected.

if $S = A \cup B$ clopen, $f(0) = a$, $f(1) = b$ cannot be cont.

Exercise: $A \subset \mathbb{R}$ connected $\Rightarrow$ path connected. 笨思路是考虑 $(\inf(A), \sup(A))$, 然后区间内每个点都在 A 里不然不连通, 发现 A 是个区间后就好说明了.

↓允许取无穷

in $\mathbb{R}^{n \geq 2}$, not true. $\sin\frac{1}{x}$, spiral above.

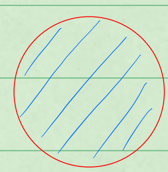
"$\Leftarrow$" $Q = \{q \mid p, q \text{ path connected}\}$ clopen.

A few other metric space concepts:

$S \subset M$, $\bar{S}$ = closure of S = lim S

$\mathring{S}$ = interior of S = largest open set contained in S.

$\partial S = \bar{S} \setminus \mathring{S}$ (topological boundary)



蓝+紫: set S

蓝+红+紫: $\bar{S}$

蓝+红: S'

Def: p is a cluster (accumulation) point of S if any nbhd ($M_r(p)$) of p contains infinitely many points of S

condensation

uncountably

open.

Note: Same as saying contains at least 1 point of S distinct from p.

Def: $S \subset M$, S' = set of cluster points of S. 显然 $S' \subset \bar{S}$, 在 $s' \in S'$ 周围取 $\{M_{\frac{1}{n}}(s)\}$ 构造数列即可。

Def: A metric space is **perfect** if $M = M'$ e.g. $\mathbb{Q}$ is perfect. 但不 complete.

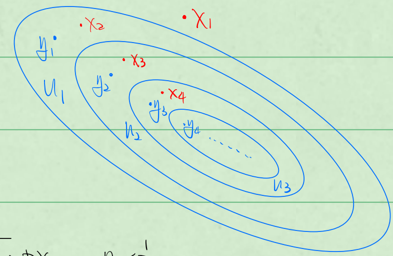
closed and no isolated points

Perfect 和 complete 有点"相反"的味道. 一个是"所有数列极限都在 set 里", 另一个是"每一个 set 里的点都是某个(非常数数列)的极限"

Thm: If M metric is perfect and complete $\Rightarrow$ uncountable.

Pf: If M is not uncountable, Then $M = \{x_1, x_2, \cdots\}$

思路是用 perfect + countable $\Rightarrow$ not complete.



We know every point is a cluster point.

Define $\overline{M_R(p)}$ be $\{q \in M \mid d(p,q) \le R\}$.

Take $y_1 \neq x_1$, $R_1 < 1$, s.t. $Y_1 = \overline{M_{R_1}(y_1)} \not\ni x_1$

M clusters at $y_1 \Rightarrow \exists y_2 \in Y_1$, $y_2 \neq x_2$ $\quad Y_2 = \overline{M_{R_2}(y_2)} \not\ni x_1$, $R_n < \frac{1}{n}$

Iterate this procedure to find $\{y_n\}$, s.t. $\overline{Y_{nR_n}(y_n)}$ is nested.

y_i 们 converge, 但任何 x_i 都不在 $\cap U_i$ 里. 矛盾.

i.e. $Y_1 \supset Y_2 \supset Y_3, \cdots$, $\quad x_n \notin Y_n$.

能持续找到每个 y_i 是因为每个 y_i 都是 cluster point.

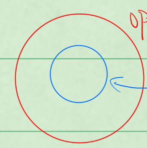
So $\{y_n\}$ is cauchy $\Rightarrow$ converges to $y \in M$.

but $y \notin M$ since $x_i \notin Y_i$, $\forall i$. contradiction.

We will use this to conclude cantor is uncountable.

Cor: Every non-empty perfect complete metric space is everywhere uncountable.

i.e. any $R > 0$ neighborhood is uncountable.



open, complete, perfect

closed, complete, uncountable

用上面的thm.

Pf: Because $M_{R/2}(p)$ is perfect. So $\overline{M_{R/2}(p)}$ is also perfect (just add more cluster points), it is a closed subset of complete, therefore also complete $\Rightarrow$ uncountable.

2 more things about topology of metric spaces.

1) Coverings of compactness

2) Generalized Heine-Borel

$\quad$ CPT $\Leftrightarrow$ closed + totally bdd.

① Coverings.

Def: A collection of subsets $\mathcal{U}$ of M **covers** A if $A \subset \bigcup U_\alpha$, $U_\alpha \in \mathcal{U}$

If $\mathcal{U}, \mathcal{V}$ both cover A, and $\forall V \in \mathcal{V} \Rightarrow V \in \mathcal{U}$, then $\mathcal{V}$ is said to be **a subcovering of $\mathcal{U}$.**

Def: If all sets in a covering $\mathcal{U}$ are open $\Rightarrow$ **open covering.**

Def: We say $A \subset M$ is **covering compact** if every open covering reduces to a finite subcovering.

Note: In def above, note that every open covering must admit a finite subcoverings.

E.g: $(0,1]$, $\{(\frac{1}{n}, 1]\}$. X.

Thm: $A \subset M$ metric space

covering CPT $\Leftrightarrow$ sequentially CPT.

Pf: "$\Rightarrow$" is easier.

Suppose A is covering CPT but not seq CPT

$\Rightarrow \exists \{p_n\} \subset A$, s.t. $\forall a \in A$, $\exists R > 0$, s.t. $M_R(a)$ contains finitely many P_n's.

(if $\forall R$, $M_R(a)$ contains infinitely many $p_n \Rightarrow$ subsequence converge)

找那个不收敛的$\{p_n\}$，则每个 $a \in M$ 附近都有个小区域只有有限个(甚至不含有) $\{p_n\}$，这些小区域有穷(compact) 但 $\{p_n\}$ 无穷，矛盾.

Then $\{M_R(a) \mid a \in A\}$ is an open covering of A

By assumption, $\downarrow$ reduces to a finite subcovering

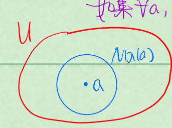
$M_R(a_1), \cdots, M_R(a_n)$ cover A, finitely many p_i's

$\Rightarrow$ Contradiction.

"$\Leftarrow$" seq CPT $\Rightarrow$ covering CPT.

Need: Def: A Lebesgue number of a covering $\mathcal{U}$ of set A is $\lambda > 0$, s.t. $\forall a \in A$, $\exists U \in \mathcal{U}$ s.t. $M_\lambda(a) \subset U$

如果$\forall a$，存在这样的U，则 λ 为 Lebesgue number.



Note: $\mathcal{U}$ depends on a but λ is independent e.g. $(0,1)$, $\{(0,1)\}$ has no Lesbegue number.

Pf: 1) <u>Lemma:</u> (Lebesgue number lemma)

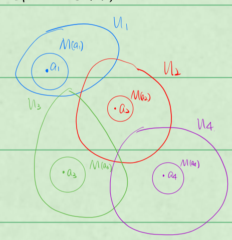
Every open covering of a seq CPT set admits positive Lebesgue number. seq. cpt sets 有 Lebesgue number.

2) Given $\lambda>0$ Lebesgue number of an open covering A, show it is covering CPT.

Pf of 2): Given Lemma,

大概意思是我们能用这个 λ + seq. compact 直接构造出那个 finite subcover.

Let $\mathcal{M}$ be an open covering of A, choose $a_1\in A$, $\exists U_1\in\mathcal{M}$, s.t. $M_\lambda(a_1)\subset U_1$.



if U_1 covers $A \Rightarrow$ done

if not, $\exists a_2\notin U_1$, $\Rightarrow \exists U_2\in\mathcal{M}$, s.t. $M_\lambda(a_2)\subset U_2$.

Suppose we can continue this procedure infinitely,

We get $\{a_n\}$, with $M_\lambda(a_n)\subset U_n$, $U_n\in\mathcal{M}$,

with $a_{n+1}\in A\setminus\bigcup_{i}^{n} U_i$

假设这个过程不会结束，我们考虑$\{a_i\}$的收敛子列 $\{a_{n_k}\}$. 取 $\varepsilon<\lambda$ 即有矛盾，因为后面的 a_{i+k} 不能在前面的 U_i 里.

By seq CPT, $\exists a_{m_k}\to p\in A$. For $k\geq N$ with large enough N,

$d(a_{m_k},p)<\lambda \Rightarrow p\in M_\lambda(a_{m_k})\subset U_{m_k}$

太神了.

We know $a_{m_l}\notin U_{m_k}$ if $l>k$

$\Rightarrow$ limit $p\notin U_{m_k}$, contradiction.

TOTAL BOUNDEDNESS

Recall: in $\mathbb{R}^n$, cpt $\Leftrightarrow$ closed and bounded.

"$\Rightarrow$" Always true.

"$\Leftarrow$" May false.

Example: · $\mathbb{Q}$, take $\mathbb{Q}\cap(-\pi,\pi)$ ($\mathbb{Q}$ is incomplete)

· $C([0,1])$ The closed unit ball is not compact ($f_n = x^n$)

bounded. $\not\Rightarrow$ totally bounded 考虑 $d_{\mathbb{R}}(x,y) = \min\{|x-y|, 1\}$ 即可.

所以这是个更强的条件 (合理)

Def: $A \subset M$ metric space is totally bounded if $\forall \varepsilon > 0$, it has covering by finitely many ε-neighborhood.

Thm: Let M be complete, $A \subset M$. A is compact $\Leftrightarrow$ closed + totally bounded.

Pf: "$\Rightarrow$" CPT $\Rightarrow$ closed

To prove totally bounded, take open covering of A of form $\{M_\varepsilon(p)\}_{p\in A}$, so there are finitely many such $\{M_\varepsilon(p)\}$.

"$\Leftarrow$" We'll show that A is seq. compact. (complete + close)

有点像闭区间套的证法，
再从无穷模择有穷得无穷.

(WLOG, all distinct)

$\forall \{a_n\} \subset A$, only have to find cauchy subsequence.

Set $\varepsilon_k = \frac{1}{k}$, for $k=1$, we find finitely many $M_{\varepsilon_1}(q_i)$ covering A.

找一系列半径分别为 $\frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots$ 的 nested ball，每个里取一个点即构成柯西列

So at least 1 of $\{M_{\varepsilon_1}(q_i)\}$ contains infinitely many $\{a_n\}$s. Arbitrarily pick an element to be our b_1.

Repeat, with $k=2$, cover $M_{\varepsilon_1}(q_i)$, pick b_2. 闭集内柯西列收敛.

So $\{b_n\}$ is cauchy.

$\Rightarrow$ (seq.) compact.

Cor: M metric space.

M compact $\Leftrightarrow$ complete + totally bounded.

Pf "$\Leftarrow$" M closed.

"$\Rightarrow$" compact $\Rightarrow$ complete

取个 cauchy seq. 即可

CANTOR SET:

Interesting set in $[0,1]$

It is
- Maximally disconnected. (locally every set disconnected)
- Closed
- Perfect
- Uncountable.
- Zero measure.

Construction: Start with $[0,1]$

Divide into 3 equal parts, throw away middle part.

Continue.

$C_0 = [0,1]$

$C_1 = [0,\frac{1}{3}],[\frac{2}{3},1]$

$\vdots$

$C = \bigcap_{i}^{\infty} C_i$

$\Rightarrow$ cor: C is uncountable.

Theorem: C is compact, non-empty, perfect ①, totally Ⓐ disconnected, nowhere dense Ⓑ, zero Ⓒ measure.

Pf: Compact: Closed + Bdd.
arbitrary intersection of closed set is closed.

1): Take $x \in C$, $\varepsilon > 0$, WTS $\exists \infty$ points of C in $(x-\varepsilon, x+\varepsilon)$

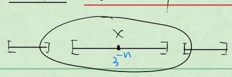
$x \in C \Rightarrow x \in C_n$ for all n. Take n large, s.t. $3^{-n} < \varepsilon$.

So that $x \in I$, where I one of intervals of C_n (length 3^{-n})

$I \subset (x-\varepsilon, x+\varepsilon)$ perfect: cluster points 等于说 $\Leftrightarrow$ 找一个身边的点即可.

"E"

Note: All endpoints of intervals in C_n lies in C.

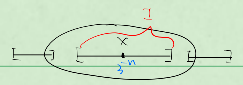


有很多"断点"。用断点分开即有 clopen set.

Ⓐ: Def: C is totally disconnected if each $p \in M$ has arbitrarily small clopen neighborhood

$\forall \varepsilon,\ \forall p \in M,\ \Rightarrow$ clopen U, $p \in U$, s.t. $U \subseteq M_\varepsilon(p)$

e.g. $\mathbb{Q}$, discrete set.

Pf:  $3^{-n} < \varepsilon$.

$x \in I$, I closed in C_n, therefore in C.

Also, $J = C_n \setminus I$ is finite union of closed intervals.

$\Rightarrow J$ closed. I, J clopen. $\Rightarrow I \cap C$, $J \cap C$ clopen

Ⓑ Def: $S \subset M$ metric space is dense if $\bar{S} = M$

$S \subset U$ is somewhere dense if $\exists U$ open, s.t.

$\overline{S \cap U} \supset U$ 这个的意思是S"填满了"某个开集(在这里:开区间)

$S \subset U$ is nowhere dense if not somewhere dense.

Pf: Prove first, C contains no interval. 这个很显然的,会被足够大的 C_n 切断.

Suppose $(a,b) \in C$, take n, s.t. $3^{-n} < b-a$,

can't have $(a,b) \in C_n$.

NTS: nowhere dense.

If not, then $\exists U$, s.t. $\overline{C \cap U} \supset U \supset (a,b)$ for some a,b.

But $\overline{C \cap U} \subset \bar{C} = C$, contradiction. 但C是闭的,想"填满"某个集合必须是本身就包含它,而这是不可能的.

Ⓒ Def: Zero measure / Zero outer measure

(In $\mathbb{R}$), S is a set of zero outer measure, if $\forall \varepsilon$, $\exists$ a covering by

open intervals $\{(a_i, b_i)\}$, s.t. $\sum_{i=1}^{\infty} (b_i - a_i) < \varepsilon$.

Pf: C is a zero set: 因为 C_n 的长度去0了,顶多是端点们盖不住,而端点又可数,即零测。

$C \subset C_n$. $\forall n$, C_n is union of 2^n intervals of length 3^{-n}

Can cover C_n by 2^n open intervals (a_i, b_i), s.t. $b_i - a_i < (2.9)^{-n}$.

$\Rightarrow \sum_{i=1}^{2^n} (b_i - a_i) = 2^n (2.9)^{-n} < \varepsilon$

FUNCTION SPACES: (in particular, continuous)

Recall: $C([a,b])$: space of cont. functions $[a,b] \to \mathbb{R}/\mathbb{C}$

Uniform convergence:

$f_n: [a,b] \to \mathbb{R}$ sequence, f_n converges uniformly to f if $\forall \varepsilon$,

$\exists N$, s.t. $\forall n > N$, $|f_n(x) - f(x)| < \varepsilon$, $\forall x \in [a,b]$

Notation: $f_n \rightrightarrows f$ / $\text{unif} \lim_{n\to\infty} f_n = f$ / $f_n \to f$ unif.

Intuition: ε-neighborhood of f eventually.

non-) Example: $x^n \to \begin{cases} 0 \\ 1 \end{cases}$ in $[0,1]$ not uniformly.

Question: What does unif. convergence preserve?

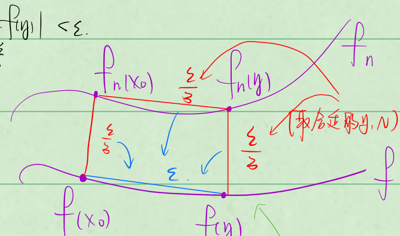
A: ① Continuity. ② Integrability (Riemann) ③ Differentiability.

Theorem: f_n is continuous at x_0, and $f_n \rightrightarrows f$, then f is continuous at x_0

Pf: We know that $\forall \varepsilon > 0$, $\exists N$, s.t. if $n > N$, then $|f_n(x) - f(x)| < \frac{\varepsilon}{3}$

f_n cont $\Rightarrow \exists \delta$, s.t. $|y - x_0| < \delta \Rightarrow |f_n(y) - f_n(x_0)| < \frac{\varepsilon}{3}$

then $|f(y) - f(x_0)| \le \underbrace{|f_n(y) - f_n(x_0)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_n(x_0) - f(x_0)|}_{<\frac{\varepsilon}{3}} + \underbrace{|f_n(y) - f(y)|}_{<\frac{\varepsilon}{3}} < \varepsilon.$



这里说的都是与轴上的距离

Note: $\{x^n\}$ continuity may not be preserved if not uniform.

Q: If f_n cont, $f_n \xrightarrow{\text{pointwise?}} f$, f cont, does $f_n \rightrightarrows f$?

A: No. wlog $f=0$

Thm: $C([a,b])$ with sup metric is complete.

/ ·In this case, $C([a,b])$ is a vector space & induces a norm.

- A complete normed vector space is called the Banach Space.
- If the norm is induced by an inner product, it's called the Hilbert Space.

Pf: First look at bound, not necessarily continuously functions. call it C_b (vector space with sup metric)

$C([a,b]) \subset C_b([a,b])$

Second: Show that $C_b([a,b])$ with sup metric is complete.

Third: We get that $C([a,b])$ is complete because it's closed.

Because unif lim for cont fns are cont. (Shown above)

Thm: $C_b([a,b])$ with $d(f,g) = \sup|f-g|$ is complete.

Pf: Take $\{f_n\} \subset C_b$ cauchy, that is, $\forall \varepsilon, \exists N$, s.t. $\forall m,n > N$, $d(f_m, f_n) < \frac{\varepsilon}{2}$

Fix $x \in [a,b]$, $f_n(x)$ is cauchy in $\mathbb{R}$.

So it converges, can define pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$

Fix $x \in [a,b]$, can find $\overset{\text{depends on } x}{m} \geq N$, s.t. $|f_m(x) - f(x)| < \frac{\varepsilon}{2}$

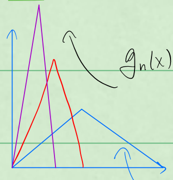
So $\forall n \geq N$, $|f_n(x) - f(x)| \leq |f_n(x) - f_m(x)| + |f_m(x) - f(x)|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

$\Rightarrow \sup|f_n - f| < \varepsilon$, $\forall > N$.

Easy to show f bdd ($f \leq M + \varepsilon$, $\forall \varepsilon$)

Remind: Ex from last time.



$g_n(x)$ continuous, $g_n \to 0$ pointwise, not uniform.

bounded invariant $f_n = \frac{1}{2n} g_n$, $f_n \to 0$ pointwise, not uniform.

Note: $\int_0^1 g_n(x)\,dx = 1 \neq \int_0^1 \lim g_n(x)\,dx$.

Thm: If f_n a sequence of Riemann integrable functions and $f_n \rightrightarrows f$

then f is Riemann integrable and $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$

Note: Let $\mathbb{Q} \cap [0,1] = \{q_1, q_2, \cdots\}$

$f_n(x) = \chi_{\{q_1, \cdots, q_n\}} = \begin{cases} 1 & x = q_i \\ 0 & \text{otherwise.} \end{cases}$

$\Rightarrow f_n \in R$ (riemann integrable)

($g \in R \Leftrightarrow$ bdd + discont. in zero set)

okay for Lebesgue integrable.

$f_n(x) \xrightarrow{n\to\infty} \chi_{\mathbb{Q}}$ pointwise. not $\in R$

Lemma: Countable union of zero set is zero set.

From lemma, $f_n \rightrightarrows f \Rightarrow f \in R$.

Pf of Thm: (Exchanging integrals)

$\forall \varepsilon > 0$, $\sup |f_n - f| < \frac{\varepsilon}{b-a}$

$\Rightarrow \left| \int_a^b f_n(x)\,dx - \int_a^b f(x)dx \right|$

$= \left| \int_a^b (f_n(x) - f(x))dx \right|$

$\leq \int_a^b |f_n(x) - f(x)|dx$

$< \int_a^b \frac{\varepsilon}{b-a} dx = \varepsilon.$

- Unif. convergence is simplest criterion for exchanging lim and $\int$.
- Lebesgue theory will give us better criteria.

Prop: $\{f_n\}$ sequence of differentiable functions, $f_n \rightrightarrows f$ uniformly

$+ f_n' \rightrightarrows g$, f' differentiable and $f_n' \rightrightarrows g = f'$

?

non-e.g: Take $f_n = \sqrt{x^2 + \frac{1}{n}}$, $f_n \rightrightarrows f = |x|$ $f_n' = \frac{x}{\sqrt{x^2+\frac{1}{n}}}$

(Something much worse than can happen. Can have f_n diff (smth)

$f_n \rightrightarrows f$, f may not be differentiable at any point)

Pf: Note: If we know a little more about $f_n \in C'$,

with $f_n \rightrightarrows f$, $f_n' \rightrightarrows g$ unif

Pf is easy using F.T.C + previous result of $\int$

$$f_n(x) = f_n(a) + \int_a^x f_n'(a)$$

$$\downarrow \quad \downarrow \quad \downarrow$$

$$f(x) \quad f(a) \quad \int_a^x g = f'$$

If we just know f diff,

fix $x \in [a,b]$. let $\varphi_n(t) = \begin{cases} \frac{f_n(t)-f_n(x)}{t-x} & t \neq x \\ f_n'(x) & t = x \end{cases}$

$\Rightarrow \varphi(t) = \begin{cases} \frac{f(t)-f(x)}{t-x} & t \neq x \\ g(x) & t = x \end{cases}$, $\varphi_n \to \varphi$ pointwise.

WTS: uniform. Then we're done because $\varphi_n \in C[a,b] \Rightarrow g = f'$

To show $\varphi_n \rightrightarrows \varphi$, suffice to show $\{\varphi_n\}$ is cauchy is do.

$$\varphi_n(t) - \varphi_m(t) = \frac{f_n(t) - f_n(x) - f_m(t) + f_m(x)}{t-x} = \frac{(\overbrace{f_n(t) - f_m(t)}^{h(t)}) - (\overbrace{f_n(x) - f_m(x)}^{h(x)})}{t-x}$$

mean value thm $\overset{}{=} \frac{h'(\theta)\cancel{(t-x)}}{\cancel{t-x}} = h'(\theta)$ for some $\theta \in [t,x]$.

草 这里是 f_n 还是 f_n'?

Since $h' = f_n' - f_m'$, as $n, m \to \infty$, $h' \to 0$ (since $f_n' \to g$)

$\Rightarrow \varphi_n - \varphi_m \to 0$ unif.

COMPACTNESS in C

We have generalized Heine-Borel for general metric spaces.

(compact $\Leftrightarrow$ totally bounded + closed)

Look for a more "practical" criterion.

To say $\{f_n\} \subset C$ has a convergent subsequence.

Def: $\{f_n\} \subset C$ is equicontinuous (uniform-continuous) if $\forall \varepsilon, \exists \delta$, s.t. $|t-s| < \delta$

$\Rightarrow |f_n(t) - f_n(s)| < \varepsilon, \ \forall n.$

($\delta = \delta(\varepsilon)$ is the same for all f_n).

Thm: (ARZELA-ASCOLI)

Every $\overset{\text{bdd}}{\vee}$ equicontinuous sequence $f_n \subset C$ admits $\overset{\text{unif.}}{\vee}$ convergent subsequence.

Pf: $[a,b]$ has dense countable subset, say, the rationals, call $D = \{d_1, d_2, \cdots\}$

f_n bounded, i.e. $|f_n(x)| \le M$, $\forall x \in [a,b]$.

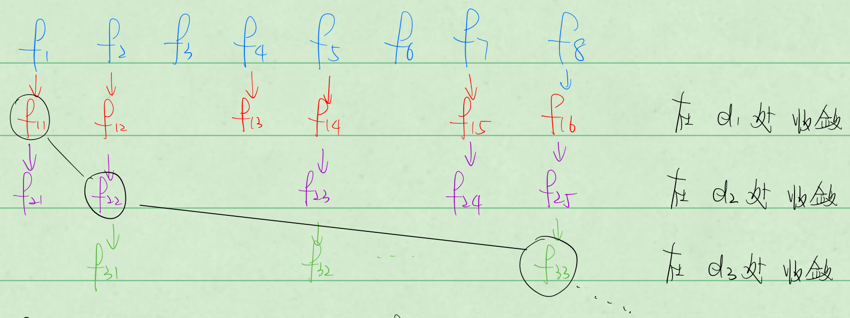
In particular, $f_n(q_i)$ is bdd sequence in $\mathbb{R}$, $\forall i$. (in particular, $i=1$)

Then we can extract subsequence converging $f_{1,k}(d_1) \to y_1 \in \mathbb{R}$

Also, $f_{1,k}(d_2)$ bdd $\Rightarrow \exists$ subsequence $f_{2,k}(d_2) \to y_2 \in \mathbb{R}$. (also $f_{2,k}(d_1) \to y_1$)

Iterating, get $f_{m,k}$ s.t. $f_{m,k}$ is a subsequence of $f_{m-1,k}$,

and $f_{m,k}(d_i) \to y_i$, $\forall i \le m$.



Consider diagonal sequence $g_m = f_{m,m}$.

NTS: g_m conve. unif.

First: Show pointwise convergent $\forall d_j$.

$$g_m(d_j) = f_{m,m}(d_j) = f_{m-1,k_1}(d_j) = f_{m-2,k_2}(d_j) = \cdots = f_{j,k_{m-j}}(d_j) \xrightarrow{m\to\infty} y_j$$

Now we have $\{g_n\}$ pointwise converging in D, and g_n is equicont.

NTS: pointwise converging + equicont. $\Rightarrow$ unif. conv. in D.

① $\forall \varepsilon > 0$, $\exists \delta$ s.t. $\overset{\forall n}{\vee}$ $|s-t| < \delta \Rightarrow |g_n(t) - g_n(s)| < \frac{\varepsilon}{3}$

② Choose J large enough, s.t. $[a,b]$ is covered by $\{N_\delta(d_i)\}_{i=1,\cdots,J}$

Look at $g_m(d_i) \xrightarrow{m\to\infty} y_i$ $\forall i = 1, \cdots, J$

③ In particular, $|g_m(d_i) - g_l(d_i)| < \frac{\varepsilon}{3}$ for m, l sufficiently large

Proof now that g_m is cauchy:

$$|g_m(x) - g_l(x)| \le \underbrace{|g_m(x) - g_m(d_i)|}_{<\frac{\varepsilon}{3}\ \text{(equi)}\ ①} + \underbrace{|g_m(d_i) - g_l(d_i)|}_{<\frac{\varepsilon}{3}\ ③} + \underbrace{|g_l(d_i) - g_l(x)|}_{<\frac{\varepsilon}{3}\ \text{(equi)}\ ①}$$

Pick d_i s.t. $|d_i - x| < \delta$

Think: $F \subset C$ equicont + bdd $\Leftrightarrow$ totally bdd, closed

E.g.: of equicont fns.

① On $[a,b]$, $\alpha + \beta x$

② Lipschitz functions with constant $L \le L_0$

($\exists L$, s.t. $|f(x) - f(y)| < L|x-y|$)

③ seq of f_n with $|f_n'(x)| < M$.

④ Holder cont fns of bounded Holder constant.

Def: $f \in C^\alpha$, $0 < \alpha < 1$ (α-Holder) iff $\sup_{x \ne y} \frac{|f(x) - f(y)|}{|x-y|^\alpha} < \infty$

Recap of spaces:

$$\underset{\text{bdd}}{C_b} \supsetneq \underset{\text{riemann-}}{R} \supsetneq C^\alpha \supsetneq \overset{\alpha' > \alpha}{C^{\alpha'}} \supsetneq Lip \supsetneq C' \supsetneq C^\infty \supsetneq \overset{e^{-\frac{1}{x^2}}}{\text{analytic}}$$

Theorem: Polynomials are dense in $C([a,b])$

i.e. $\forall$ cont. f, ε, we can find a $p(x)$ poly

s.t. $\|f - p\|_\infty < \varepsilon$.

Later: Thm: (STONE-WEIERSTRASS)

An algebra $A \subset C([a,b])$ $\overset{\text{+ some assumptions}}{\vee}$ is dense in $C([a,b])$.

↳ closed under addition / mult. ($p(x)$, sin, cos(x))

Pf: f is given, want to find p.

step 0: Reduce to $C([0,1])$

WLOG, $f(0)=f(1)=0$ (减掉一个 linear 即可).

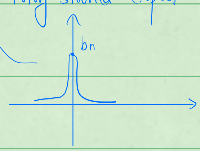
Now we extend our p to 0 outside $[0,1]$.

step 1: $\beta_n(t)=(1-t^2)^n b_n$. ← constants to estimate. for $|t|\le 1$, 0 otherwise.

Look at convolution of f and β_n.

$$P_n(x)=\int_{-1}^{1} f(x+t)\beta_n(t)\,dt \quad \text{← } x \text{ 附近 } f \text{ 的加权平均} \xrightarrow{n\to\infty} f(x+t)\big|_{t=0}=f(x)$$

Why should expect this to approximate f as $n\to\infty$?



About convolution: Def: given f and g, $f*g(x)=\int_{-\infty}^{\infty} f(x-t)g(t)\,dt$

Idea: $f*g$ gets the best regularity of f, g.

Step 2: Show P_n is a polynomial

$$P_n(x)=\int_{-1}^{1} f(\underbrace{x+t}_{u})\beta_n(t)\,dt$$

$$=\int_{x-1}^{x+1} f(u)\beta_n(u-x)\,du \quad (x\in[0,1])$$

$\to (1-(u-x)^2)^n b_n$ 可以展开成系数关于 n 的 poly. Say, $Q_n(x)=\sum c_i(n)x^i$

$$=\int_0^1 f(u)Q_n(x)\,du$$

$$=\sum\left(\int_0^1 f(u)c_i(u)\,du\right)x^i$$

Step 3: Show $P_n \rightrightarrows f$ on $[0,1]$

How do we choose b_n? We choose b_n s.t. $\int_{-1}^{1}\beta_n(x)=1$, $\forall n$.

i.e. $b_n=\dfrac{1}{\int_{-1}^{1}(1-x^2)^n dx}$

check: $b_n\le C\sqrt{n}$.

$$1=b_n\int_{-1}^{1}(1-t^2)^n \ge b_n\int_{-\frac{1}{\sqrt n}}^{\frac{1}{\sqrt n}}(1-t^2)^n dt \ge b_n\int_{-\frac{1}{\sqrt n}}^{\frac{1}{\sqrt n}}\left(1-\frac{1}{\sqrt n}^2\right)^n dt$$

$$\Rightarrow \quad 1 \ge b_n\left(1-\frac{1}{n}\right)^n\frac{2}{\sqrt n}$$

$$\Rightarrow \quad b_n\le \frac{\sqrt n}{2}\underbrace{\left(1-\frac{1}{n}\right)^{-n}}_{\to e}\le C\sqrt{n}$$

$\underline{NTS}$: $P_n(x) \rightrightarrows f(x)$

$$|P_n(x) - f(x)| = \left| \int_{-1}^{1} \beta_n(t) f(x-t)\,dt - f(x) \right|$$

$$= \left| \int_{-1}^{1} \beta_n(t) \left(f(x-t) - f(x) \right) dt \right|$$

Idea: if $|t| > \delta$, $\beta_n \rightrightarrows 0$, whole $\int \to 0$

if $|t| - \delta$, $f(x-t) - f(x)$ small.

$\forall \varepsilon > 0$, $\exists \delta$, s.t. $|t| < \delta \Rightarrow |f(x-t) - f(x)| < \frac{\varepsilon}{2}$

$\forall n$ large enough, $|\beta_n(t)| \le \frac{\varepsilon}{4 \sup f}$

$$|P_n(x) - f(x)| \le \int_{-1}^{1} \beta_n(t) |f(x-t) - f(x)|\,dt$$

$$= \int_{-\delta}^{\delta} \beta_n(t) \underbrace{|f(x-t) - f(x)|}_{B}\,dt + \int_{\delta \le |t| \le 1} \beta_n(t) \underbrace{|f(x-t) - f(x)|}_{A}\,dt$$

$$\le \int_{-\delta}^{\delta} \beta_n(t) \cdot \frac{\varepsilon}{2} + \int_{\delta \le |t| \le 1} \beta_n(t)\, 2 \sup_{[0,1]} f\,dt$$

$$< \frac{\varepsilon}{2} + 2 \sup_{[0,1]} |f| \underbrace{\int_{\delta \le |t| \le 1} \beta_n(t)\,dt}_{\le \frac{\varepsilon}{4 \sup f}}$$

$\le \varepsilon$ for n large enough.

STONE-WEIERSTRASS

Extend Weierstrass to general M metric space and function algebra $A < C(M, \mathbb{R})$

Def: A subset of $C(M)$ is a function algebra if $\forall f, g \in A$,

$f + g \in A$, $fg \in A$, $cf \in A$ $(c \in \mathbb{R})$

Def: A algebra separate points if $\forall p_1, p_2 \in M$, $\exists f \in A$, s.t. $f(p_1) \ne f(p_2)$

Def: A algebra vanishes nowhere if $\forall p \in M$, $\exists f \in A$, s.t. $f(p) \ne 0$.

Thm: If A function algebra $\subset C(M)$ M compact, and A separates points and vanishes nowhere, then A is dense in $C(M)$

Pf: Use some supporting lemmas (later)

step 1: A fn algebra $f, g \in \bar{A}$

$\Rightarrow \max(f, g)$, $\min(f, g)$ also in $\bar{A}$

$\max(f,g) = \frac{f+g}{2} + \frac{|f-g|}{2}$ ($\bar{\mathcal{A}}$)

Lemma 1: $\mathcal{A}$ fn algebra $\Rightarrow$ $\bar{\mathcal{A}}$ fn algebra

(By prop of limits)

Lemma 2: $f \in \mathcal{A} \Rightarrow |f| \in \bar{\mathcal{A}}$

Can approximate fn $g(y) = |y|$ by $P(y)$ 常数项会有问题，但 $g(y)=|y|$ 常数项0，可以去掉.

$\Rightarrow$ - - - - - - - - - - - - $|f(x)|$ by $P(f(x)) \in \mathcal{A}$. $\Rightarrow |f(x)| \in \bar{\mathcal{A}}$

$\max(f,g) \in \bar{\mathcal{A}}$, $\min(f,g) = \frac{f+g}{2} - \frac{|f-g|}{2} \in \bar{\mathcal{A}}$

Step 2: Want to find $G \in \mathcal{A}$. s.t. $F(x) - \varepsilon < G(x) < F(x) + \varepsilon$ $\quad (F(x) \in C(M))$

Lemma 3: If $\mathcal{A}$ separates points and vanishes nowhere

Then $\exists p_1 \neq p_2 \in M$, $c_1, c_2 \in \mathbb{R}$, $f \in \mathcal{A}$. s.t. $f(p_i) = c_i$

Fix $p \neq q$, find H_{pq}, s.t. $H_{pq}(p) = F(p)$, $H_{pq}(q) = F(q)$

Keep p fixed, vary q, each q has nbhd U_q, s.t.

$\forall x \in U_q$, $F(x) - \varepsilon < H_{pq}(x)$

By compactness of M, extract finite subcover $U_{q_1}, \cdots U_{q_n}$ (H_{pq_1}, H_{pq_n})

Take $G_p(x) = \max\{H_{pq_1}(x), \cdots, H_{pq_n}(x)\}$

Have $F(p) = G_p(p)$, and $F(x) - \varepsilon < G_p(x)$

Step 3: Know $\forall p$ have G_p s.t. $G_p(p) = F(p)$, each p has U_p s.t.

$G_p(x) < F(x) + \varepsilon$, extract $U_{p_1}, \cdots, U_{p_m}$

Let $G(x) = \min\{G_{p_1}(x), \cdots, G_{p_m}(x)\} \in \bar{\mathcal{A}}$ have $G(x) < F(x) + \varepsilon$.

From before, $G(x) > F(x) - \varepsilon$.

ODEs and PICARD's THM on $\exists!$

(M, d) metric space.

Def: $f: M \to M$, p is a fixed point of f if $f(p) = p$.

Def: $f: M \to M$ contraction if $\forall x, y \in M$, $d(f(x), f(y)) \leq k\, d(x,y)$ for $k < 1$

Thm: (Banach fixed point)

$f: M \to M$, f contraction, M complete $\Rightarrow \exists!$ fixed point of f.

Pf: idea: pick x_0, consider $\{f^n(x_0)\}$.

$d(f^n(x_0), f^{n-1}(x_0)) \leq k^{n-1} d(f(x_0), x_0) \Rightarrow$ cauchy. $\Rightarrow x_n \xrightarrow{n\to\infty} p$

$x_{n+1} \to p$

p is fixed point because $f(x_n) \to f(p)$

left to show uniqueness. 显然.

Note: More general thm about $\exists$ of fixed points

(e.g. Brower $f: B \to B$ closed ball $\subset \mathbb{R}^n$ f cont $\Rightarrow \exists$ fixed pt)

ODE: System $U \in \mathbb{R}^n$ open

$t \in I \subset \mathbb{R}$

seek $(x_1(t), x_2(t), \cdots, x_n(t)) = \vec{x}(t)$

solving $x_1'(t) = f_1(\vec{x}(t))$, $x_2'(t) = f_2(\vec{x}(t))$, $\cdots$, $x_n(t) = f_n(\vec{x}(t))$

$f_i : U \to \mathbb{R}$

Contraction and ODEs:

Def: $f: M \to M$ $f(p) = p$ is a fixed point.

Def: $f: M \to M$ is a contraction if for some $k < 1$,

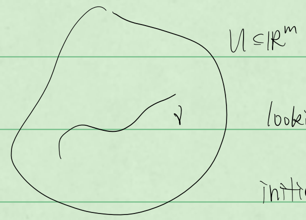
$\forall p, q \in M$, $d(f(p), f(q)) \leq k\, d(p, q)$

Thm: (Banach fixed point) M complete metric space,

f contraction $\Rightarrow \exists!$ fixed point.

Pf: By iteration of the map f.

ODE:



looking for $\vec{\gamma}(t)$ for given velocity and some initial contial.

$$* \begin{cases} \vec{\gamma}'(t) = \vec{F}(\vec{\gamma}(t)) & \vec{F}: U \to \mathbb{R}^m, \text{ vector field.} \\ \vec{\gamma}(t_0) = p & \text{WLOG, } t_0 = 0 \end{cases}$$

Thm: (Picard)

The ode $\begin{cases} \vec{\gamma}'(t) = \vec{F}(\vec{\gamma}(t)) \\ \vec{\gamma}(t_0) = p \end{cases}$ $\vec{F}: U \to \mathbb{R}^m$, vector field has unique solution locally near p, provided $\vec{F}$ Lipschitz.

$\vec{F} = (F_1, \cdots, F_n)$ $\quad |\vec{F}(x) - \vec{F}(y)| \le L|x-y|$, $L>$

Note: "Locally near p" we'll find a small piece of the curve

Pf: Step 1:

Quantifiers:

$\vec{F}$ cont, $\exists N = \overline{M_R(p)}$, s.t. $|\vec{F}(x)| \le M$, $\forall x \in N$.

We'll consider "times" τ, s.t. $\tau M < R$, $\tau L < 1$

write $*$ in integral form

integral of $\left(\int_0^t F_1, \cdots, \int_0^t F_n \right)$

$$* \Rightarrow \vec{\gamma}(t) = p + \int_0^t \vec{F}(\gamma(s))ds.$$

Look at $\Phi: \vec{\gamma} \longmapsto p + \int_0^t \vec{F}(\vec{\gamma}(s))ds$

Solution $\Leftrightarrow$ fixed point of Φ.

Take the metric space to be $C = C([-\tau, \tau], N)$

Need to show $\begin{cases} ① \ \Phi: C \to C \\ ② \ \Phi \text{ contraction.} \end{cases}$

①: $\vec{\gamma} \in C$, $|\Phi(\vec{\gamma}) - p| \overset{?}{\le} R$.

$$\Leftarrow \left| \int_0^t \vec{F}(\gamma(s))ds \right| \le \int_0^t |\vec{F}(\gamma(s))| ds.$$

$(t>0)$

$$\le \int_0^t M ds \le \tau M \le R.$$

② $\vec{\gamma}, \vec{\sigma} \in C$, $|\Phi(\gamma) - \Phi(\sigma)|$

$$= \left| p + \int_0^t \vec{F}(\gamma(s))\,ds - \left(p + \int_0^t \vec{F}(\sigma(s))\,ds\right) \right|$$

$$= \left| \int_0^t \left(\vec{F}(\gamma(s)) - \vec{F}(\sigma(s))\right) ds \right|$$

$$\leq \int_0^t \left|\vec{F}(\gamma(s)) - \vec{F}(\sigma(s))\right| ds$$

$$\leq \int_0^t L \left|\vec{\gamma}(s) - \vec{\sigma}(s)\right| ds.$$

$$\leq L \int_0^t \sup_{[-\tau,\tau]} |\vec{\gamma} - \vec{\sigma}|\, ds$$

$$= L\, d(\vec{\gamma}, \vec{\sigma}) \cdot \tau$$

≤ 1

Remarks on Picard's thm:

Recall: Consider $x: \underset{\mathbb{R}}{I} \to \underset{\mathbb{R}^n}{U}$ $t \mapsto x(t)$ $F: \underset{\mathbb{R}^n}{U} \to \mathbb{R}^n$.

ODE $\begin{cases} \dot{x} = F(x) \\ x(0) = p \end{cases}$ ← initial condition

initial time

Assuming F Lipschitz with constant L, then $\exists!$ solution of ODE, this solution belongs to $C([-\tau,\tau], M_R(p))$ with $\tau L < 1$, $\tau M < R$

$M = \sup_{M_R(p)} |F(x)|$

Pf: Write $\dot{x} = F(x)$ as fixed pt of map on $C([-\tau,\tau], M_R(p))$ and prove it's a contraction.

Rk: Thm gives us a local-in-time solution. one is interested in what happens after? 在边界处取初始值，再用一遍thm.

Ex: $\begin{cases} \dot{x} = x^2 \\ x(0) = x_0 > 0 \end{cases}$ $x: [-\tau,\tau] \to \mathbb{R}$

$F(x) = x^2$ is Lip locally.

$x(t) = \dfrac{x_0}{1 - t x_0}$ nice smooth as $t < \frac{1}{x_0}$

Rk: Uniqueness may fail if F not Lip.

$\dot{x} = x^{\frac{1}{3}}$, $F = x^{\frac{1}{3}}$ not Lip at 0

① $x \equiv 0$ solution

② $x(t) = ct^{\frac{3}{2}}$, $c = \left(\frac{2}{3}\right)^{\frac{3}{2}}$

Also, $x(t) = \begin{cases} 0 & 0 < t < t_0 \\ c(t-t_0)^{\frac{3}{2}} & \end{cases}$

Thm: (Peano) consider solution as above, with F cont, then $\exists$ sol^n

(not necessarily unique)

Pf: Use Arzelà-Ascoli.

LEBESGUE THEOREM:

- Build theory of measuring / integrating (more general / useful than Riemann)
- Theory is at core of analysis, funtional analysis, math physics.
- Build a notion of integral ("Area under graph") (will mostly work on $\mathbb{R}^2$)
- More general than Riemann:

Sets that we can measure (According to notion of Lebesgue measure)

$\chi_{\mathbb{Q}\cap[0,1]}$: $\int \chi_{\mathbb{Q}\cap[0,1]} = \text{Area}(\mathbb{Q}\cap[0,1]\times[0,1])$

$= \text{length}(\mathbb{Q}\cap[0,1]) \times 1 = 0.$

OUTER MEASURE (Not a measure, from it we'll construct Lebesgue measure)

Def: length of $I = (a,b)$ $|I| = b-a$

Lebesgue outer measure of a set $A \subset \mathbb{R}$ is:

$m^*(A) = \inf\{\sum |I_k| : I_k \text{ intervals covering } A\}$

Note: (details later)

$\exists A, B \subset \mathbb{R},\ A \cap B = \emptyset,\ m^*(A \cup B) \neq m^*(A) + m^*(B)$

$\{I_k\}$ countable open covering

- If $\forall \{I_k\}$, $\sum |I_k| = \infty$, $\Rightarrow m^*(A) = \infty$
- So we can define m^*, $\forall A \subset \mathbb{R}$.

<u>Thm</u>: (Basic properties of m^*)

a) $m^*(\emptyset) = 0$

b) $A \subset B \Rightarrow m^*(A) \le m^*(B)$

c) If $A = \bigcup_{i=1}^{\infty} A_i \Rightarrow m^*(A) \le \sum_{i=1}^{\infty} m^*(A_i)$

<u>Pf</u>: a) ✓

b) If $\{I_k\}$ covers $B \Rightarrow \{I_k\}$ covers A

$\Rightarrow$ Coverings of $A \Rightarrow$ Coverings of B.

c) $\{I_{i,k}\}_k$ covering of A_i, then $\bigcup_{i=1}^{\infty}\bigcup_{k=1}^{\infty} I_{i,k}$ covering of A

$m^*(A) \le \sum_{i,k} |I_{i,k}|$

$\forall \varepsilon > 0$ choose covering $\{I_{i,k}\}$ of A_i, s.t. $m^*(A_i) + \frac{\varepsilon}{2^i} \ge \sum_{k=1} |I_{i,k}|$

$m^*(A) \le \sum_{i,k} |I_{i,k}| \le \sum_{i\ge1} m^*(A_i) + \frac{\varepsilon}{2^i} = \left(\sum m^*(A_i)\right) + \varepsilon$.

<u>Note</u>: "b)": Monotonicity

"c)": Countable subadditivity.

These (a), (b), (c) in general for some $\mu^*: 2^X \longrightarrow [0, \infty]$

defines an abstract outer measure on X.

In $\mathbb{R}^2$:

<u>Def</u>: Area of $(a,b)\times(c,d) = R$ $\quad |R| = (d-c)(b-a)$

Planar outer measure of $A \subset \mathbb{R}^2$ is $m^*(A) = \inf\{\sum_k |R_k|,\ R_k \text{ covers } A\}$.

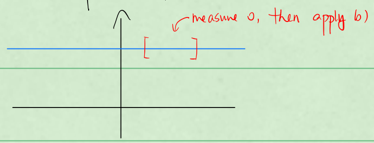
In $\mathbb{R}^n$, do the same (Denote m_n^* ← dim)

• Recall: Z zero set $\Leftrightarrow m^*(Z)=0$

Prop: • Subset of Z is zero set

• Countable union of zero set is zero set

• Each plane $\{x_i=a\}$ in $\mathbb{R}^n$ $n\geq 1$, is a zero set.



Thm: • Linear outer measure of $[a,b]$ is $b-a$

• Planar outer measure of $[a,b]\times[c,d]$ is $(b-a)(d-c)$

• n-dim outer measure of $\prod_{i=1}^{n}[a_i,b_i]$ is $\prod_{i=1}^{n}(b_i-a_i)$

Pf: • WTS $m^*([a,b])=b-a$

$\forall \varepsilon$, $(a-\frac{\varepsilon}{2}, b+\frac{\varepsilon}{2})$ covers $[a,b] \Rightarrow m^*([a,b]) \leq b-a+\varepsilon$.

Left to show $m^*([a,b]) \geq b-a$

Given $\{I_k\}$ open covering, WTS $\sum |I_k| \geq b-a$

$[a,b]$ compact $\Rightarrow \{I_k\}_{k=1}^{N}$

Say $I_k=(a_k,b_k)$

$N=1 \Rightarrow [a,b] \subset (a_1,b_1) \Rightarrow b_1-a_1 > b-a$

By induction: Say that any covering of $[c,d]$ by N intervals $\{I_k\}_{k=1}^{N}$ satisfies $\sum_{k=1}^{N}|I_k| \geq d-c$

Show this holds for $N+1$.

Consider $\{I_i\}_{i=1}^{N+1}$ $I_i=(a_i,b_i)$ cover of $[a,b]$

say, $I_1 \ni a$.

1) If $b_1 \geq b \Rightarrow (a_1,b_1) \supset [a,b]$

2) If $b_1 < b$, then $[a,b]=[a,b_1) \cup [b_1,b]$

$\sum_{i=2}^{N+1}|I_i| \geq b-b_1$ by induction

$\Rightarrow \sum_{i=1}^{N+1}|I_i| \geq b-b_1+b_1-a = b-a$ ■

Note: now know $m^*([a,b]) = b-a$.

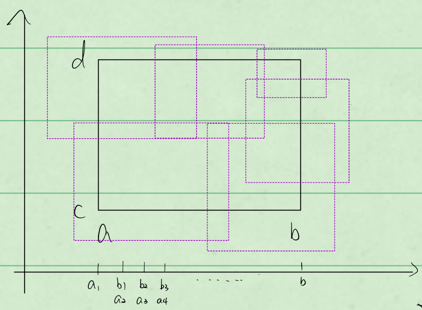
then $I = (a,b] = [a,b) = (a,b)$

$m^*(I) = b-a$ since $[a,b] \supset I \supset [a+\varepsilon, b-\varepsilon]$, monotonicity.

$n = 2+$: $m^*(R) \leq (b-a)(d-c)$ 显然

for converse, $R = \bigcup_{k=1}^{N} R_k \Rightarrow$ have positive lebesgue number $\lambda > 0$.

$\Rightarrow \forall p \in R, \exists R_k$. s.t. $M_\lambda(p) \subset R_k$.



partion $[a,b]$ and $[c,d]$

into (a_i, b_i), (c_i, d_i), say that

$S_{ij} = (a_i, b_i) \times (c_j, d_j)$ of diam $< \lambda$.

$\Rightarrow \sum_{i,j} |S_{ij}| = |R| = (b-a)(d-c)$

Union of S_{ij} contained in R_k, form a smaller rect R_k'

$\Rightarrow \sum_{S_{ij} \subset R_k} |S_{ij}| \leq |R_k|$

$(b-a)(d-c) = |R| \overset{*}{=} \sum_{i,j} |S_{ij}| \leq \sum_k \sum_{\substack{i,j \\ S_{ij} \subset R_k}} |S_{ij}|$

Note: $m^*(R) = (b-a)(d-c)$

any rect. open / close / half

Here, we want : $\omega(\bigcup A_i) = \sum \omega(A_i)$ (countable additivity)

not "sub"

MEASURABILITY

Def: A set $E \subset \mathbb{R}^n$ is (lebesgue) measurable if $\forall X \in \mathbb{R}^n$ $m^*(X) = m^*(X \cap E) + m^*(X \cap E^c)$

(Any test set splits nicely) $(\omega(X) = \omega(X \cap E) + \omega(X \cap E^c)$ for more general setting$)$

Denote $\mathcal{M}(\mathbb{R}^n) = \mathcal{M}$: collection of all measurable sets.

Thm: Collection of measurable sets $\mathcal{M}$ w.r.t any abstract outer measure is a σ-algebra ($\phi \in \mathcal{M}$. $\mathcal{M}$ closed under complements and countable unions), and is countably additive (i.e., if $E_i \in \mathcal{M}$, $E_i \cap E_j = \phi, i \neq j$, then $w(\sqcup E_i) = \sum w(E_i)$)

① 取补集 ② 可数并
③ disjoint union

Also, zero sets are measurable.

Pf: • $\mathcal{M}$ is σ-algebra:

• $\phi \in \mathcal{M}$ ✓

• $E^c \in \mathcal{M}$ ✓ $(X \cap E^c \to X \cap E)$

Also, zero sets are measurable:

$w(X) \le \cancel{w(X \cap Z)}^{0} + w(X \cap Z^c) = w(X \cap Z^c) \le w(X)$

• M is closed under countable unions:

① M is closed under differences: $(A, B \in M \Rightarrow A \cap B^c \in M)$

② M is closed under finite unions.

③ M is finitely additive on M. (E_i finite, disjoint. $w^*(\sqcup E_i) = \sum w^*(E_i)$)

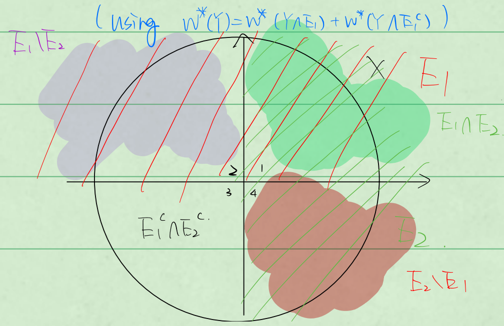
④ w^* Satisfies a countable addition formula.

pf: ①: $E_1, E_2 \in M$, X test set.

WTS $E_1 \setminus E_2 = E_1 \cap E_2^c \in M$

i.e, $w^*(X) = w^*(X \cap (E_1 \setminus E_2)) + w^*(X \cap (E_1 \setminus E_2)^c)$

(using $w^*(Y) = w^*(Y \cap E_1) + w^*(Y \cap E_1^c)$)



Associate # to w^* of sets.

$2 = w^*(\ )$, $1 = w^*(\ )$

$12 = w^*(\ )$

$w^*(X) = w^*(X \cap (E_1 \setminus E_2)) + w^*(X \cap (E_1 \setminus E_2)^c)$

$1234 = 2 \quad 134$

$134 = 1 + 34$ (E_1 is measurable)

i.e. $w^*(X \cap (E_1 \setminus E_2)^c) = w^*(\underbrace{X \cap (E_1 \setminus E_2)^c \cap E_1}_{E_1^c \cup E_2 \cap E_1}) + w^*(\underbrace{X \cap (E_1 \setminus E_2)^c \cap E_1^c}_{E_1^c \cup E_2 \cap E_1^c})$

$= w^*(X \cap (E_1 \cap E_2))$

$34 = 3 + 4$ similar.

So $2 + 134 = 1 + 2 + 3 + 4$

Now, $1234 = 12 + 34$ ($E_1 \in \mathcal{M}$)

$= 1 + 2 + 3 + 4$ ($E_2 \in \mathcal{M}$)

So LHS = RHS.

② WTS: $E_i \in \mathcal{M}$, $\bigcup_{i=1}^{N} E_i \in \mathcal{M}$

Suffice to show $E_1 \cup E_2 \in \mathcal{M}$

$(E_1 \cup E_2)^c = E_1^c \cap E_2^c = \underset{\mathcal{M}}{E_1^c} \setminus \underset{\mathcal{M}}{E_2} \in \mathcal{M}$ by a)

③ finite additivity: $E_1 \cap E_2 = \emptyset$

$w^*(E_1 \cup E_2) = w^*(E_1 \cup E_2 \cap E_1) + w^*(E_1 \cup E_2 \cap E_1^c)$

$= w^*(E_1) + w^*(E_2)$

④: WTS: $\mathcal{M}$ closed under countable union, and outer measure m^* is countably additive on $\mathcal{M}$.

Notation: In $\mathbb{R}^n$, define m^* the Lebesgue outer measure.

call m Lebesgue measure

(Just the restriction of m^* to measure sets)

Proof of claim:

test set $\to X \in 2^X$, $\{E_i\}_{i=1}^{\infty}$ disjoint, $E = \bigcup E_i$, then

$w(X) = \sum w(X \cap E_i) + w(X \cap E^c)$

We know "$\leq$" by subadditivity.

Show: $\geq$: $F := \bigsqcup_{i=1}^{k} E_i \in M$ $\quad F \subset E$

$w(X) = w(X \cap F) + w(X \cap F^c)$

$\geq w(X \cap F) + w(X \cap E^c)$

finite additivity $= \sum_{i=1}^{k} w(X \cap E_i) + w(X \cap E^c)$

Take $\lim_{k \to \infty}$ to get $w(X) = \sum_{1}^{\infty} w(X \cap E_i) + w(X \cap E^c)$

Using claim with $X = E = \bigsqcup E_i$

Get $w(E) = \sum \underbrace{w(E \cap E_i)}_{E_i} + \cancel{w(E \cap E^c)}$

$= \sum w(E_i)$

Left to show if $E_i \in M \Rightarrow \bigcup E_i \in M$ (not disjoint)

First, we rewrite $E = \bigcup E_i = \bigsqcup E_i'$ $\quad (E_i' = E_i \setminus \bigcup_{j=1}^{i-1} E_j)$ (call E_i' by E_i)

Apply again to $X \cap E$, then

$w(X \cap E) = \sum^{\infty} w(X \cap E \cap E_i) + \cancel{w(X \cap E \cup E_i^c)}$

$= \sum^{\infty} w(X \cap E_i)$

Earlier: $w(X) = \sum w(X \cap E_i) + w(X \cap E^c)$

$= w(X \cap E) + w(X \cap E^c)$

Examples of outer measures / measures

- Counter measures: $w : 2^X \to [0, +\infty]$

  $\forall S \in 2^X$, $w(S) = \# S$ or ∞ (if 不是 finite)

- $0/\infty$ measure: $w(\emptyset) = 0$, $\quad w(S \neq \emptyset) = \infty$

- m^* outer Lebesgue measure.

- δ measure at $x \in X$,

$$\delta(S) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$$

- Weighted Lebesgue outer measure:

replace $|I|$ by $f(c)|I|$, f nice $\nearrow$ fn, c midpoint.

- Outer Jordan measure

Note: When know how to measure sets, then

$$\int f \, dm = m(\text{under graph of } f)$$

<u>Thm</u>: (Measure continuity):

$\{E_k\}$, $\{F_k\}$ measurable

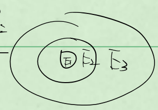
Assume $E_k \nearrow E$, i.e., $E_1 \subset E_2 \subset \cdots \subset E_i \cdots \subset E = \bigcup E_i$

$F_k \searrow F$, $F_1 \supset F_2 \supset \cdots \supset F_i \supset \cdots \supset F = \bigcap F_i$

Then $\underset{\nearrow}{w(E_k) \xrightarrow{k\to\infty} w(E)}$, $\underset{\searrow}{w(F_k) \xrightarrow{k\to\infty} w(F)}$ $(w(F_1) < \infty)$

Note: $F_k = [k, \infty)$ $F_k \searrow \emptyset$

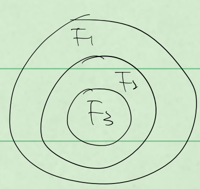
<u>Pf</u>:



Same trick, $E = \bigcup E_i = \bigsqcup E_i'$

$$\Rightarrow w(E) = \sum_{i=1}^{\infty} w(E_i') = \lim_{k\to\infty}\left(\sum_{i=1}^{k} w(E_i')\right) = \lim_{k\to\infty} w(E_k)$$

- $F_k \searrow F$, $F_1 = \left(\bigsqcup_{i=1}^{\infty} F_i'\right) \sqcup F$ $\quad F_i' = F_i \setminus F_{i+1}$



$$\underbrace{w(F_1)}_{<\infty} = \underbrace{\sum_{i\geq 1} w(F_i')}_{\text{converge}} + w(F) \Rightarrow$$

$$w(F_k) = \left(\sum_{i \geq k} w(F_i')\right) + w(F)$$

$$\lim_{k\to\infty} w(F_k) = \lim_{k\to\infty} \cancel{\sum_{i\geq k} w(F_i')} + w(F) = w(F)$$

- <u>Example of non-measurable sets</u>

Section 3:

Def of measure space $(X, \mathcal{M}, \mu)$

Def names of transformations depending on how they transform $\mathcal{M}$ or μ.

Section 4: Now

Relate measurability (w.r.t m = lebesgue measure on $\mathbb{R}^n$)

Thm: open and closed sets are measurable.

Prop: Half space $[0,\infty) \times \mathbb{R}^{n-1}$ $(0,\infty)$ are measurable.

From prop get all boxes

Write any open / closed set of countable union of open boxes.

Def: G_δ set is countable $\bigcap$ of open

F_σ - - - - - - - - - $\bigcup$ of close

Measurable set are G_δ or F_σ + a 0 set

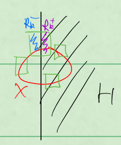
Pf: $n=2$, wlog (up to rotation)

$H = (a,\infty) \times \mathbb{R}$, WTS $\forall X \subset \mathbb{R}^n$,

$m^*(X) = m^*(X \cap H) + m^*(X \cap H^c)$

$\forall \varepsilon > 0$, $\exists$ covering $\{R_k\}$ by open rectangles.

s.t. $\sum |R_k| < m^*(X) + \varepsilon$



$X^- = \{(x,y) \in X, x < a\}$

$X^+ = \{(x,y) \in X, x > a\}$.

call $\mathcal{R}^\pm$ collection of rectangles $R_k^\pm$

$R \in \mathcal{R}$, $R^\pm = \{(x,y) \in R, \pm(x-a) > 0\}$. $\mathcal{R}^+$ covers X^+, $\mathcal{R}^-$ covers X^-

$$m^*(X) \leq m^*(\overbrace{X \cap H}^{X^+}) + m^*(\overbrace{X \cap H^c}^{X^-})$$

$$\leq \sum_{R \in \mathcal{R}^+} |R| + \sum_{R \in \mathcal{R}^-} |R|$$

$$= \sum_{R \in \mathcal{R}} |R| < m^*(X) + \varepsilon$$

■

<u>Thm</u>: Lebesgue measure is regular, i.e. each $E \in \mathcal{M}$

can be sandwiched, i.e. $\exists F, G$,

$F \subset E \subset G$, $F \in \mathcal{F}_\sigma$, $G \in \mathcal{G}_\delta$, and $m(G \setminus F) = 0$

(not $m(F) = m(G)$)

Converse, if $\exists F, G$ sandwich as above then $E \in \mathcal{M}$

<u>Prop</u>: (Regularity)

If $E \in \mathcal{M}$, then $\exists F \in \mathcal{F}_\sigma$, $G \in \mathcal{G}_\delta$ s.t. $F \subset E \subset G$ with

$m(G \setminus F) = 0$ (converse easily true)

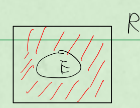
<u>Pf</u>: Let's look at the case bdd $E \subset R$ (rectangle)

Take a decreasing sequence of open $U_n \supset E$, s.t. $m(U_n \setminus E) \longrightarrow 0$

Can take U_n union rectangle in def of m^*, s.t. $U_n = \bigcup R_k$, with

$m(\bigcup R_k) \leq m(E) + \frac{1}{n}$, Let $U_n = \bigcup R_k \cap U_{n-1}$ (为了保 decreasing)

So we have $m(U_n) \to m(E)$



Can do the same thing with $E^c := R \setminus E$

We find $V_n \supset E^c$ decreasing with $m(V_n) \to m(E^c)$

Take $K_n = R \setminus V_n$, increasing and closed.

$m(K_n) = m(R) - m(V_n) \to m(R) - m(E^c) = m(E)$. Take $F = \bigcup K_n$, $m(F) = m(E)$

Similarly let $G = \bigcap U_n \in \mathcal{G}_\delta$. Know $m(U_n) \downarrow m(E) = m(G)$ — measure continuity.

■

First Application: (of unproven property)

Cor: A Lipeomorphism (bi-lip homo) $h : \mathbb{R}^n \to \mathbb{R}^n$ is a meseomorphism (bijection $h/h^{-1} : \mathcal{M} \to \mathcal{M}$)

Pf: Homeo send G_δs to G_δs and F_σ to F_σ. NTS h^{Lip} send 0 set to 0 set.

Affine motions:

T invertible linear transformation + translation ($T\vec{x} = M\vec{x} + \vec{v}$ ← ignore)

We know already $T : \mathcal{M} \to \mathcal{M}$ because its a Lipeomorphism with Lip constant $\|T\|$

Thm: An affine motion $T : \mathbb{R}^n \to \mathbb{R}^n$ is a meseomorphism, which multiplies measure by $|\det T|$

$\boxed{\begin{array}{l}\text{Sec 3 — reading} \\ \text{Sec 5 — some in class, some reading.}\end{array}}$

Thm: T ($\mathbb{R}^n \to \mathbb{R}^n$) linear invertible transformation (no constant)

T is a meseomorphism, and $m(TE) = \det(T)\, m(E)$, $\forall E \in \mathcal{M}$

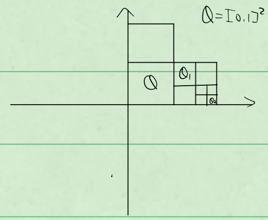
Pf: Need 2 Lemmas:

Lemma1: Every open set in $\mathbb{R}^n$ is countable union of disjoint cubes plus a zero set.

Lemma2: Every open set in $\mathbb{R}^n$ is countable union of disjoint open balls plus a zero set.

Pf: (of Lemma 1):

Use dyadic cubes:



$\mathcal{Q}_0 = \{Q + \mathbb{Z}^2\}$

$\mathcal{Q}_1 = \{4^{-1}Q + 4^{-1}\mathbb{Z}^2\} = 4^{-1}\mathcal{Q}_0$

$\mathcal{Q}_2 = 4^{-2}\mathcal{Q}_0$

O open set in $\mathbb{R}^2$, use elements in $\mathcal{Q}_0, \mathcal{Q}_1, \mathcal{Q}_2, \cdots$

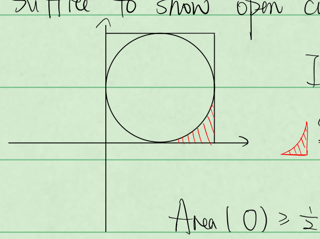
to cover O

Iterate process of bisecting and accepting open cubes to fully cover O.

Eventually, up to a zero set, given by countable union of lines, will cover O by countable union of dyadic cubes.

Pf: of Lemma 2:

Suffice to show open cubes be covered by balls:



In $n=2$, $S=[0,1]^2$, Δ disc.

$U_1 = S \setminus \bar{\Delta}$, by Lemma, a countable union of cubes.

$\text{Area}(O) \geq \frac{1}{2}\text{Area}(\square)$

Step 1: $U_1 = S \setminus \bar{\Delta} = \underbrace{\bigcup S_n}_{\text{dyadic squares}} + \underbrace{Z}_{\text{zero set}}$

Finitely many of $\underbrace{S_i}_{\text{say, } i=1,\cdots,n}$ will cover more than $\frac{1}{2}$ area of U_1.

Step 2: Call $U_2 = U_1 \setminus \bigcup_{i}^{n} S_i$, do the same thing

$m(U_1) \leq \frac{1}{2}m(S)$, $m(U_2) \leq \frac{1}{2}m(U_1)$

Iterate this, $m(U_k) \leq \frac{1}{2}m(U_{k-1}) \leq \cdots \leq \frac{1}{2^k}m(S) \longrightarrow 0$

Left with zero set $S \setminus \bigcup$ disks at each stage.

Pf of thm:

NTS $m(TE) = \det T \, m(E)$

First: Assume T diagonal, $T = \operatorname{diag}(\lambda_1, \cdots, \lambda_n)$

$\underbrace{TR}_{R \text{ rectangle}} = R' \text{ rectangle}, \quad |TR| = \underbrace{|\lambda_1 \cdots \lambda_n|}_{\det T} |R|$

E measurable, $\exists$ covering $\{R_k\}$, s.t. $\sum |R_k| \le m^*(E) + \varepsilon$

$\Rightarrow \{TR_k\}$ covering of TE

$m(TE) \le \sum |TR_k| = \det(T) \sum |R_k| \le \det(T)(m(E) + \varepsilon)$

Also we can apply T^{-1}

$m(E) = m(T^{-1}TE) \le |\det(T^{-1})| \, m(TE)$

$\Rightarrow m(TE) = \det T \, m(E)$

Second: Assume T is orthogonal, $m(TE) = m(E)$

Take $B = B_c(R) \Rightarrow TB = B_{Tc}(R) \Rightarrow m(TB) = m(B)$

Since we know $m(TB) = m(B)$, and that every $\underset{\text{open set}}{R}$ is countable

union of B_i + Z (zero set)

$\Rightarrow m(TE) = m(E) \quad \forall E \in \mathcal{M}$ by countable additivity.

Last step: Every T can be written as $T = \overbrace{O_1}^{\text{orthogonal}} \underbrace{D}_{\text{diagonal}} \overbrace{O_2}^{\text{orthogonal}}$

NEXT Important / Natural thm:

A, B measurable sets, $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^k$

$m_{n+k}(A \times B) = m_n(A) \times m_k(B)$ (In particular, $0 \times \infty = 0$)

We need notion of Hull and Kernel

(Application of regularity)

Hull and Kernels:

Recall: $A \subset \mathbb{R}^n$ bdd,

$m^*(A) = \inf\{$ measure over all open sets containing $A\}$.

$m^*(A)$ achieved by a G_δ set. Call this H_A. $m(H_A) = m^*(A)$

intersect sets above.

Dually: (Looking from inside)

can define inner measure,

$m_*(A) = \sup\{$ measure of all closed set contained in $A\}$

$m_*(A)$ achieved by a F_σ set. Call K_A.

Remark: If A unbdd, need extra minimality / maximality property.

- $m_*(A) = m(K_A) \le m^*(A) = m(H_A)$

Analogy: $H_A \longrightarrow$ Closure

$K_A \longrightarrow$ Interior.

Define measure theoretic boundary:

$\partial m(A) = H_A \setminus K_A$.

Thm: A bdd measurable $\Leftrightarrow m_*(A) = m^*(A) \Leftrightarrow m(\partial m(A)) = 0$

Previous thm on regularity sandwich $(A \in \mathcal{M}, \overset{K_A}{F} \subset A \subset \overset{H_A}{G}, m(G \setminus F) = 0$ $= m(G) - m(F)$

Thm: $A \subset \mathbb{R}^n$, $A \subset B$, B box,

$A \in \mathcal{M} \Leftrightarrow A$ divides cleanly B

$$m(B) = m^*(B \cap A) + m^*(B \cap A^c)$$

Thm: A, B measurable sets, $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^k$

$m_{n+k}(A \times B) = m_n(A) \times m_k(B)$ (In particular, $0 \times \infty = 0$)

Pf: Steps:

0): A, B boxes ✓

1): A/B zero sets ✓

2): U, V open $\Rightarrow U\times V$ measurable, $m(U\times V) = m(U)m(V)$

idea: Use $U = \sqcup F_i \cup Z$. ~~And~~ use countable additivity.

open rectangles

3): To conclude, use any measurable is a $G_\delta + Z$

$U = \bigcap_{n=1}^{\infty} U_n \cup Z_n$. U_n open, $U_n \downarrow$

+ use measure continuity downward.

Make idea more precise: Suppose $A, B \subset [0,1]$ dim=1 wlog

<u>Observation:</u> $H_A \subset A \subset K_A$, $H_B \subset B \subset K_B$ are measurable sandwichs

$m(H_A) = m(A) = m(K_A)$, same for B.

Then $K_A \times K_B \subset A \times B \subset H_A \times H_B$ is a regularity sandwich

4) $U_n \downarrow H_A$, $V_n \downarrow H_B$, U_n, V_n open

$$m(U_n \times V_n) = m(U_n)m(V_n) \longrightarrow m(H_A)m(H_B) \longrightarrow m(A)m(B)$$

$\downarrow$

$$m(H_A \times H_B) \overset{③}{=} m(A\times B)$$

<u>Slices:</u>

<u>Def:</u> The slice of $E \subset \mathbb{R}^n \times \mathbb{R}^k$ at $x \subset \mathbb{R}^n$ is

$$E_x = \{y \in \mathbb{R}^k \mid (x,y) \in E\} \subset \mathbb{R}^k$$

<u>Thm:</u> If $E \subset \mathbb{R}^n \times \mathbb{R}^k$ mesurable,

E zero set $\Leftrightarrow$ E_x zero set almost everywhere. up to a zero set

<u>Lebesgue integral:</u>

$f: \mathbb{R} \to [0,\infty]$ Assuming non-negative. In general case need some adjustment

Def: The undergraph of f is:

$$\mathcal{U}(f) = \{(x,y) \in \mathbb{R} \times [0,\infty) : y < f(x)\}$$

Def: f is measurable if $\mathcal{U}(f) \in \mathcal{M}$

Def: If f is measurable, then $\int_{D} f = m(\mathcal{U}(f))$ (no dx) (domain) . $\int f = \infty$ allowed.

Def: A measurable f is integrable if $\int f < \infty$

Note: what if f negative: $f = f_+ - f_-$

$f_+ = \begin{cases} f & f > 0 \\ 0 & f < 0 \end{cases}, \quad f_- = \begin{cases} -f & f < 0 \\ 0 & f > 0 \end{cases}$

f is measurable if both f_+, f_- are.

f integrable if f_+, f_- both, i.e. $|f|$ integrable.

$f : \mathbb{R} \to [0, +\infty)$, measurable.

Def: L^p space: $\int |f|^p < \infty$. L^2 is special: a Hilbert space

All L^p space are normed vector space w.r.t. $\|f\|_{L^p} = (\int |f|^p)^{\frac{1}{p}}$

Def: $f_n \longrightarrow f$ almost every x : $f_n(x) \longrightarrow f(x)$, $\forall x \in Z^c$, Z zero set.

Thm: (Monotone convergence):

Assume $f_n \nearrow f$ A.E.x. $\Rightarrow$ f measurable and $\lim_{n\to\infty} \int f_n = \int f$ (possibly ∞)

(outside Z, $f_n x \le f_{n+1} x$, $f_n \nearrow f$)

Note, whenever write $\int g$, g always measurable, not necessarily integrable.

Pf: f measurable $\Leftrightarrow$ $\mathcal{U}(f) \in \mathcal{M}$

we have $\mathcal{U}(f) = \bigcup \mathcal{U}(f_n)$ up to zero set.

$\int f_n = m(\mathcal{U}(f_n)) \nearrow m(\mathcal{U}(f)) = \int f$ by upward measure continuity.

Note: without $\nearrow$, this fails.

non e.g.: $f_n = n\chi_{[0,\frac{1}{n}]}$, $\int f_n = 1$. but $f_n \to f$ a.e. x.

or $f_n = -\chi_{[n,+\infty)}$

☆ $f_n = \chi_{[n,+\infty)}$, $f_n \downarrow 0$, 所以↓不行。

$f_n = \frac{1}{n} \to 0$

$\mathcal{U}(f_n) \to \{y \le 0\} \neq \emptyset = \mathcal{U}(0)$

$\mathbb{R}\times\{0\}$

Thm: If $f_n \downarrow f$, f_n integrable, $\Rightarrow \int f_n \downarrow \int f$

To get proof, need to define "Complete undergraph" $= \hat{\mathcal{U}}(f) = \{(x,y) \mid 0 \le y \le f(x)\}$.

Prop: $\hat{\mathcal{U}}(f) \in \mathcal{M} \Leftrightarrow \mathcal{U}(f) \in \mathcal{M}$. And if so $m(\mathcal{U}(f)) = m(\hat{\mathcal{U}}(f))$

Pf: for $n \in \mathbb{N}$, let $T_{1n} = \begin{pmatrix} 1 & 0 \\ 0 & 1+\frac{1}{n} \end{pmatrix} : \mathbb{R}^2 \to \mathbb{R}^2$, sends $(x,y) \to (x, y(1+\frac{1}{n}))$

T_{1n} is meseo, change measures by $1+\frac{1}{n} = \det T_{1n}$

Write $\underset{(y\le f(x))}{\hat{\mathcal{U}}(f)}$ " $=$ " $\bigcap_{n\ge1} T_n(\mathcal{U}(f))$ up to these pnts.

$\hat{\mathcal{U}}(f) \in \mathcal{M}$ if $\mathcal{U}(f) \in \mathcal{M}$

$m(\hat{\mathcal{U}}(f)) = \lim m(T_n(\mathcal{U}(f))) = \lim (1+\frac{1}{n}) m(\mathcal{U}(f)) = m(\mathcal{U}(f))$

Pf: $\infty > \int f_n = m(\hat{\mathcal{U}}(f_n)) \downarrow \overset{m(\hat{\mathcal{U}}(f))=}{m(\mathcal{U}(f))} = \int f$

因为虽然 $m\mathcal{U}f_n \downarrow m\mathcal{U}f$，但并没有 $\mathcal{U}f_n \downarrow \mathcal{U}f$.

用 $\hat{\mathcal{U}}$ 倒一下.

Def: If $f_n : \mathbb{R} \to [0,\infty]$ seq, define lower / upper envelope sequence.

$\underline{f_n}(x) = \inf\{f_k(x) \mid k \ge n\} \nearrow \liminf f_n$

$\overline{f_n}(x) = \sup\{f_k(x) \mid k \ge n\} \searrow \limsup f_n$

Prop: $\mathcal{U}(\overline{f_n}) = \bigcup_{k\ge n} \mathcal{U}(f_k)$

$\hat{\mathcal{U}}(\underline{f_n}) = \bigcap_{k\ge n} \hat{\mathcal{U}}(f_k)$

In particular, $\underline{f_n}, \overline{f_n}$ measurable if f is.

Thm: (Dominated convergence)

$f_n : \mathbb{R} \to [0,\infty]$ measurable, $f_n \to f$ a.e. x.

Assume $\exists g: \mathbb{R} \to [0,\infty]$, s.t. $|f_n| < g$ a.e. x, g integrable,

Then $\lim \int f_n = \int f$

Pf: By prop above, $\underline{f_n}$, $\overline{f_n}$ measurable, they are also dominated by g.

Using MCT (mono - conve - thm)

$$\int \underline{f_n} \le \int f_n \le \int \overline{f_n} \le \int g < \infty$$

$$\downarrow \text{MCT} \qquad \downarrow$$

$\liminf f_n = f = \limsup f_n$ $\qquad \int f \qquad \int f$

Cor: The pointwise a.e. limit of seq of measurable fn is measurable.

(If $f_n \to f$ a.e. $\mathcal{U}(\underline{f_n}) \nearrow \mathcal{U}(f)$)

Fatou's lemma:

f_n $\mathbb{R} \to [0,\infty]$ mea,

$$\int \liminf f_n \le \liminf \int f_n$$

e.g. $f_n = n\chi_{[0,\frac{1}{n}]} \longrightarrow 0$ a.e.

then $\int 0 = 0 < 1 = \int f_n$

Pf: Call $f = \liminf f_n = \lim_{n\to\infty} \underline{f_n}$, $\underline{f_n} \nearrow f$

$\Rightarrow \int \underline{f_n} \nearrow \int f = \int \liminf f_n$

Have $\int \underline{f_n} \le \int f_n$

$\Rightarrow \int \liminf f_n \le \liminf \int f_n$

Remark: "Standard" def of integral is different, a bit more involved.

Given f, look at simple fns $\sum e_i \chi_{E_i}$, E_i disjoint mea.

If $\varphi = \sum e_i \chi_{E_i}$, $\int \varphi = \sum e_i m(E_i)$

"Standard def of measurability" f

Pre-image measurability:

f measurable if $f^{-1}([a,\infty)) = \{x \mid f(x) > a\}$.

Thm: (Properties of integral)

f, g $\mathbb{R} \to [0,\infty)$, mea

a) $\int f \leq \int g$ if $f \leq g$

$m(\mathcal{U}(f) \cap X_k \times \mathbb{R})$

b) $\mathbb{R} = \bigsqcup X_k$, $X_k \in \mathcal{M}$, $\int f = \sum_k \int_{X_k} f$

c) $X \in \mathcal{M}$, $m(X) = \int \chi_X$

d) If $m(X) = 0 \Rightarrow \int_X f = 0$

e) If $f = g$ a.e $\Rightarrow \int f = \int g$

f) If $c > 0 \Rightarrow \int cf = c\int f$

g) $\int f = 0 \Leftrightarrow f = 0$ a.e.

h) $\int f + g = \int f + \int g$.

Fourier series:

$f : [0,1] \to \mathbb{R}$, $a_n = \int_0^1 f(x) e^{-2\pi i n x} dx$. (Lebesgue)

$f(x) = \sum a_n e^{2\pi i n x}$

$e^{i\theta} = \cos\theta + i\sin\theta$ $\quad f = u + iv$, $\int f = \int u + i\int v$

Fourier transform:

$f : \mathbb{R}^d \to \begin{cases} \mathbb{R} \\ \mathbb{C} \end{cases}$

Def: For $f \in L^1(\mathbb{R}^d)$ (Space of Lebesgue integrable fn. $\int |f| < \infty$)

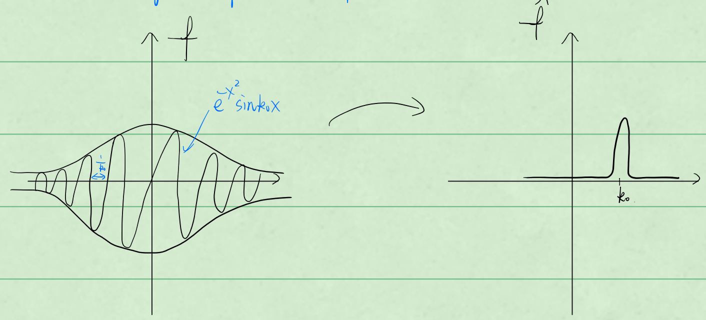
Define fourier transformation

$\hat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i x \cdot \xi} dx$ $\quad x \cdot \xi = \sum x_i \xi_i$

$\hat{f}: \mathbb{R}^d \to \begin{cases}\mathbb{R} \\ \mathbb{C}\end{cases}$, $\xi \mapsto \hat{f}(\xi)$

$\begin{cases} x \text{ variable: physical / real space} \\ \xi \text{ variable: frequency / fourier space} \end{cases}$

Idea: infer things of f from $\hat{f}$



Lemma: $f \in L^1$, $\hat{f}(\xi) = \int f(x) e^{-2\pi i x\cdot\xi} dx$

$\hat{f}$ bounded, cont. $\hat{f} \xrightarrow{|\xi|\to\infty} 0$

Pf: $|\hat{f}(\xi)| = |\int f(x) e^{-2\pi i x\cdot\xi} dx|$

$\leq \int |f(x)| \cancel{|e^{-2\pi i x\cdot\xi}|} dx$

• WTS: $\xi \to \xi_0$, then $\hat{f}(\xi) \to \hat{f}(\xi_0)$?

$\int f(x) e^{-2\pi i x\cdot\xi} dx \to \int f(x) e^{-2\pi i x\cdot\xi_0} dx$?

Yes, by dominant convergence thm.

No guarantee that $\hat{f} \in L^1$, and can make sense of operation
$\int \hat{f}(\xi) e^{2\pi i x\xi} d\xi$

To start "r" nemending this introduce Schwartz Space (class):

Def: Schwartz Space $\mathcal{S}$: fn decay faster than any poly.
$\mathcal{S} = \{ f \in C^\infty \mid \sup_{x\in\mathbb{R}^d} |x^\beta D^\alpha f(x)| < \infty \}$

e.g. $f(x) = e^{-x^2}$

Prop: (Properties of $\mathcal{F}$). If $f \in \mathcal{S}$:

	f	$\mathcal{F}f$
$x \in \mathbb{R}^d$	① $f(x+h)$	$e^{2\pi i h\cdot\xi}\hat{f}(\xi)$
	② $f(x)e^{-2\pi i x\cdot h}$	$\hat{f}(\xi+h)$
$\delta>0$	③ $f(\delta\cdot x)$	$\frac{1}{\delta^d}\hat{f}(\frac{\xi}{\delta})$
	④ $\frac{\partial}{\partial x_k} f(x)$	$2\pi i \xi_k \hat{f}(\xi)$
	⑤ $-2\pi i x_k f(x)$	$\frac{\partial}{\partial \xi_k}\hat{f}(\xi)$

Pf: ④ : $\widehat{\frac{\partial}{\partial x_k} f(x)} = \int \left(\frac{\partial}{\partial x_k} f(x)\right) e^{-2\pi i x\cdot\xi}\,dx$

$= -\int f(x)\, (-2\pi i \xi_k)\, e^{-2\pi i x\cdot \xi}\,dx$ Integration by parts.

$= 2\pi i \xi_k \hat{f}(\xi)$

⑤ WTS $\frac{\partial}{\partial \xi}\hat{f} = \widehat{-2\pi i x f}$

$\frac{\partial}{\partial \xi}\int f(x) e^{-2\pi i x\cdot\xi}\,dx = \lim_{h\to 0}\frac{\hat{f}(\xi+h)-\hat{f}(\xi)}{h}$

$= \lim_{h\to 0}\int f(x)\left(\frac{e^{-2\pi i x(\xi+h)} - e^{-2\pi i x\xi}}{h}\right)$

use dominant convergent thm

$\left|\frac{e^{-2\pi i x(\xi+h)} - e^{-2\pi i x\xi}}{h}\right| \le c(|x|+1)$ for constant c, small h.

$= \int f(x) \lim_{h\to 0}(\quad)$

$= \int -2\pi i x f(x) e^{-2\pi i x\cdot\xi}\,dx$

$= \widehat{-2\pi i x f}$

④ if $\partial_k f \in L^1 \Rightarrow |\xi \hat{f}| < \infty$

$\Rightarrow \partial^\alpha f \in L^1 \Rightarrow |\xi^\alpha \hat{f}| < \infty$

smoothness $\Rightarrow$ decay as $|\xi| \to \infty$

(localized)

⑤ localized $\Rightarrow$ smooth

Prop: $f \in \mathcal{S} \Leftrightarrow \hat{f} \in \mathcal{S}$

Pf: ④ + ⑤

Note: If $xf \in L^1$, $\Rightarrow \partial_{\xi_k} \hat{f}$ is bdd

Inversion formula: (" $f(x) = \int \hat{f}(\xi) e^{2\pi i \xi x} d\xi$)

Lemma: $f(x) = e^{-\pi x^2}$ $\quad (\int f = 1)$

then $\hat{f} = f$

Pf: $\begin{cases} f' = (-1) \cdot 2i\pi x f(x) \\ f(0) = 1 \end{cases} \xrightarrow{\mathcal{F}} \begin{cases} 2\pi i \xi \hat{f} = -i(\hat{f}(\xi))' \\ \hat{f}(0) = 1 \end{cases}$

$f, \hat{f}$ satisfies the same ODE $\Rightarrow$ same by uniqueness.

Cor: $K_\delta(x) = \frac{1}{\sqrt{\delta}} e^{-\frac{\pi x^2}{\delta}}$ $\quad$ then $\hat{K}_\delta(\xi) = e^{-\delta\pi\xi^2}$

Prop: $f \in \mathcal{S} \Rightarrow K_\delta * f \xrightarrow{\delta \to 0} f$ uniformly.

Prop: $\int f\hat{g} = \int \hat{f} g$ $\quad (f, g \in L^1)$ $\quad *$

(f, g annotated: L^1, bdd)

Pf: LHS = RHS = $\iint_{\mathbb{R}^2} f(x) g(y) e^{-2\pi i xy} dx dy$.

But why can we exchange integrals? by Fubini.

- If $f = f(x,y)$ $\Rightarrow \int \left(\int f(x,y) dx \right) dy = \int \left(\int f(x,y) dy \right) dx$
- If $f \in L^1(\mathbb{R} \times \mathbb{R})$ $\left(\iint |f| < \infty\right)$ $\Rightarrow \int \left(\int f(x,y) dx \right) dy = \int \left(\int f(x,y) dy \right) dx$

Thm: $f(x) = \int \hat{f}(\xi) e^{2\pi i x \xi} d\xi$. if $f \in \mathcal{S}$

Pf: first pf at $x = 0$: $f(0) = \int \hat{f}(\xi) d\xi$

Use $G_\delta(x) = e^{-\delta\pi x^2}$

$K_\delta = \hat{G}_\delta(\xi) = \frac{1}{\sqrt{\delta}} e^{-\frac{\pi x^2}{\delta}}$ in $*$

$LHS = \int \hat{f}\, G_\delta = \int f\, \overset{K_\delta}{\overset{\shortparallel}{\hat{G}_\delta}} = RHS$ Let $\delta \to 0$

$RHS = \int f K_\delta = \underset{\text{at } x=0}{\overset{= \int f(y) K_\delta(y-x)\,dx}{(f * K_\delta)(0)}} \xrightarrow{\delta \to 0} f(0)$

$LHS = \int \hat{f}(\xi) e^{-\delta\pi\xi^2} d\xi \xrightarrow{\delta \to 0} \int \hat{f}(\xi)$ by DCT.

<u>Second:</u> $F(y) = f(x+y)$ fn of y, x fixed.

We know $\underset{f(x)}{\underset{\shortparallel}{F(0)}} = \int \hat{F}(\xi)\,d\xi = \int \widehat{f(\cdot + x)}\,d\xi \overset{①}{=} \int e^{2\pi i x\cdot\xi} \hat{f}(\xi)\,d\xi$

<u>Remark:</u> $f \xrightarrow{\mathcal{F}} \hat{f} \xrightarrow{\mathcal{F}^{-1}} f$

We can show $\mathcal{F}^{-1} f = \mathcal{F}(-f)$ (反吗) change sign.

<u>Next:</u> Show that if $f \in \mathcal{S}$,

$\int |f|^2 = \int |\hat{f}|^2$ as isometry.