Banach-Tarski Thm: (1924):

$\forall U, V \in \mathbb{R}^{n\geq 3}$ open, $\exists k \in \mathbb{N}$, $E_1, \cdots E_k \subset U$, $F_1, \cdots, F_k \subset V$, s.t.

$E_i \cap E_j = F_i \cap F_j = \emptyset$, $\sqcup E_i = U$, $\sqcup F_i = V$, $E_i \sim F_i$, congruent (Rotate, move, rigidly to get the same)

Let $X \neq \emptyset$, $P(X) = \{Y \mid Y \subset X\}$.

<u>Def</u>: $M \subset P(X)$ is an Algebra of sets if:

① $E \in M \Rightarrow E^c \in M$. ② $E, F \in M \Rightarrow E \cup F \in M$

an σ-Algebra if Algebra + ③ $E_i \in M, i \in \mathbb{N} \Rightarrow \cup_i E_i \in M$.

<u>Prop</u>: $\{\emptyset, X\} \in M$ on X (trivial)

$P(X)$ is a σ-Algebra.

<u>Note</u>: It is enough to check on disjoint sets.

(if E_i arbitrary, construct $E_i' = E_i \setminus \bigcup_{j<i} E_j$)

It is enough to check on increasing sequence.

Q: Are open sets an σ-Algebra? No, E^c

Let $C \subseteq P(X)$, want to define σ-Algebra generated by C,

<u>Fact</u>: Let M_α be a collection of σ-Algebras on X, $\cap_\alpha M_\alpha$ is a σ-Algebra.

<u>Def</u>: $\langle C \rangle$ the σ-Algebra generated by C, $\langle C \rangle = \cap_\alpha M_\alpha$.

$M \subset M'$, M is coorser ($\Leftarrow$ finer) than M',

Let $X \neq \emptyset$, τ be a topology on X, $\langle \tau \rangle$ is called the Borel σ-Algebra of (X, τ)

<u>Prop</u>: Let τ be the usual topology on $\mathbb{R}$, Let $B = B_\tau$, then

Then $\mathcal{B} = \langle (a,b) \rangle \overset{①}{=} \langle [a,b] \rangle = \langle [\ ) \rangle = \langle (\ ] \rangle = \langle (a,\infty) \rangle = \langle (-\infty,a) \rangle = \langle [a,\infty) \rangle = \langle (-\infty,a] \rangle$.

Let $A \subset X$, $\langle \{A\} \rangle = \{\emptyset, A, A^c, X\}$.

Lemma: Let $\mathcal{E}, \mathcal{F} \subset \mathcal{P}(X)$, if $\mathcal{E} \in \langle \mathcal{F} \rangle$, then $\langle \mathcal{E} \rangle \subseteq \langle \mathcal{F} \rangle$.

Pf: $\langle \mathcal{E} \rangle = \bigcap_{\mathcal{E} \subset \mathcal{M}} \mathcal{M}$, and $\mathcal{E} \subseteq \langle \mathcal{F} \rangle \Rightarrow \langle \mathcal{E} \rangle \subseteq \langle \mathcal{F} \rangle$

Pf of ①: $\mathcal{B} \supset \{(a,b)\}$, so $\langle \{(a,b)\} \rangle \subseteq \mathcal{B}$.

U open $\Rightarrow U = \bigcup_i (a_i, b_i)$. So open sets $\tau \subseteq \langle \{(a,b)\} \rangle \Rightarrow \mathcal{B} \subseteq \langle \{(a,b)\} \rangle$

Def: Let $X \neq \emptyset$, $\mathcal{M}$ a σ-algebra. $(X, \mathcal{M})$ is called a measurable space, $E \in \mathcal{M}$ are measurable sets in $(X, \mathcal{M})$

Def: A function $\mu: \mathcal{M} \longmapsto \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a measure if:

- $\mu(\emptyset) = 0$,
- ~~if $E \cap F = \emptyset$, then $\mu(E) + \mu(F) = \mu(E \cup F)$~~ not enough
- σ-additivity: $E_i \in \mathcal{M}$, $E_i \cap E_j = \emptyset$, $\sum_i \mu(E_i) = \mu(\bigcup_i E_i)$

E.g.: • $\forall E$, $\mu(E) = 0$

- Let $x \in X$, $\mu_x(E) = \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{otherwise.} \end{cases}$ Dirac mass.
- $\begin{cases} \text{if } X = \mathbb{R}, \ \mu(E) = \#(E \cap \mathbb{Z}) \\ \text{if } X = \mathbb{N}, \ \mu(E) = \# E, \end{cases}$ counting measure.

Def: • μ is finite if $\mu(X) < \infty$

- μ is σ-finite if $\exists E_i$, $\mu(E_i) < \infty$, $X = \bigcup_i E_i$
- μ is semi-finite if $\forall E$, s.t. $\mu(E) = \infty$, $\exists \emptyset \neq F \subseteq E$, $0 < \mu(F) < \infty$

$(X, \mathcal{M})$, $(Y, \mathcal{N})$ measurable space, a map $f: X \to Y$

<u>Def</u>: $f: X \to Y$ is $(\mathcal{M}, \mathcal{N})$ measurable, if $\forall F \in \mathcal{N}$, $f^{-1}(F) \in \mathcal{M}$.

<u>Recall</u>: $\cdot \, \forall F \subset Y$, $f^{-1}(F^c) = (f^{-1}(F))^c$

$\cdot \, \forall F_\alpha \subseteq Y$, $f^{-1}(\bigcup F_\alpha) = \bigcup f^{-1}(F_\alpha)$

Given $f: X \to \underset{(Y, \mathcal{N})}{Y}$, $f^* \mathcal{N} = \{ f^{-1}(F) \mid F \in \mathcal{N} \}$. is a σ-algebra on X.

It is the coarest σ-algebra that makes f measurable.

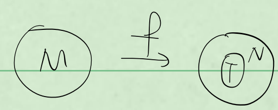
Let $(X, \mathcal{M})$, $f_* \mathcal{M} = \{ F \subset Y \mid f^{-1}(F) \in \mathcal{M} \}$. finest σ-algebra that makes f measurable.

<u>Ex</u>: Let $f: X \to Y$ constant, $\exists y \in Y$, $\forall x \in X$, $f(x) = y$

$\forall \mathcal{N} \subset \mathcal{P}(Y)$, $f^* \mathcal{N} = \{ \emptyset, X \}$, $\quad \forall F \subseteq Y$, $y \in F \Rightarrow f^{-1}(F) = X$, $y \notin F \Rightarrow f^{-1}(F) = \emptyset$

$\forall \mathcal{M} \subset \mathcal{P}(X)$, $f_* \mathcal{M} = \mathcal{P}(Y)$ $\quad \forall F$, $f^{-1}(F) = \begin{cases} X \\ \emptyset \end{cases}$. so each F must be mesurable.

<u>Lemma</u>: Let $\mathcal{N} = \langle \mathcal{F} \rangle$, f is $(\mathcal{M}, \mathcal{N})$-measurable $\Leftrightarrow$ $f^* \mathcal{F} \subset \mathcal{M}$.



<u>Pf</u>: "$\Rightarrow$" trivial

这些拉回来是 measurable 的, by

所有"拉回来是measurable"的 set 的集合.

"$\Leftarrow$" $\mathcal{F} \subseteq f_* \mathcal{M}$ by def, so $\langle \mathcal{F} \rangle \subseteq f_* \mathcal{M}$ ☆

$f^* \mathcal{F} \subset \mathcal{M} \Rightarrow \forall V \in \mathcal{F}$, $f^{-1}(V) \in \mathcal{M}$

而 $f_* \mathcal{M} = \{ V \in Y \mid f^{-1}(V) \in \mathcal{M} \}$.

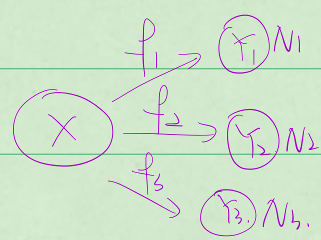
这里所有的拉回来都是 measurable 的.
所以它的子集, i.e. $\langle \mathcal{F} \rangle$, 也是.

<u>Cor</u>: Let $(X, \mathcal{B}_X)$, $(Y, \mathcal{B}_Y)$ be measurable spaces. (Borel)

then any continuous $f: X \to Y$ is $(\mathcal{B}_X, \mathcal{B}_Y)$ measurable.

Let $(Y_\alpha, \mathcal{N}_\alpha)$ be a family of mable spaces

$f_\alpha: X \to (Y_\alpha, \mathcal{N}_\alpha)$



$\langle \bigcup_\alpha f_\alpha^* \mathcal{N}_\alpha \rangle$ is coarest σ-algebra s.t. f_α is mable $\forall \alpha$.

这些使得 f_α mable.

Recall: $\prod_\alpha Y_\alpha$ is the set of all maps $\varphi: I \to \bigcup_\alpha Y_\alpha$, s.t. $\varphi(\alpha) \in Y_\alpha$.

I → Y ∋ y (small)

↓ α 是方向/坐标.

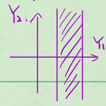
$\pi_\alpha(y) = \varphi_y(\alpha) \in Y_\alpha$, connical projection.

这里的意思是"第y个φ"，因为 $\prod_\alpha Y_\alpha = Y$ 是由很多φ构成的集合.

↓不是 countable.

这里Y在set的意义下就是 $(y_1, y_2, \cdots)$ 构成的. 我们在考虑如何 assign 一个 σ-algebra 给它.

Def: The **product σ-algebra** on $Y = \prod_\alpha Y_\alpha$ as $\bigotimes_\alpha N_\alpha = \langle \bigcup_\alpha \pi_\alpha^* N_\alpha \rangle = \langle \{\pi_\alpha^{-1}(F_\alpha) \mid F_\alpha \in N_\alpha, \alpha \in I\} \rangle$.

相当于是按坐标拉回来，所以就是  这种感觉.

$$Y = \prod_\alpha Y_\alpha \xrightarrow{\pi} Y_\alpha \ (N_\alpha) \qquad \text{So } \pi^* N_\alpha \subseteq P(Y).$$

Lemma: Let $\prod_\alpha N_\alpha = \langle \{ \pi_\alpha F_\alpha \mid F_\alpha \in M_\alpha, \alpha \in I\} \rangle$

$\bigotimes_\alpha N_\alpha$ 是被 $(F_{1k}, Y_2, Y_3, \cdots)$ generate 的

$\prod_\alpha N_\alpha$ 是被 $(F_{1k}, F_{2j}, \cdots)$ generate 的.

$\bigotimes_\alpha N_\alpha \subseteq \prod_\alpha N_\alpha$. "=" if I countable.

我们总可以取 $F_2 = Y_2$, $F_3 = Y_3$ 这种，所以 $\prod_\alpha N_\alpha$ 的 generator 包含了 $\bigotimes_\alpha N_\alpha$ 的 generator.

Pf: $\pi_\alpha^{-1} F_\alpha \in \{\prod_\alpha F_\alpha \mid F_\alpha \in N_\alpha, \alpha \in I\}$.

$\Rightarrow \underbrace{\langle \{\pi_\alpha^{-1} F_\alpha\} \rangle}_{= \bigotimes_\alpha N_\alpha} \subseteq \prod_\alpha N_\alpha$.

这是这行的意思.

↓ any generator of $\prod_\alpha N_\alpha$.

"$\supseteq$": $E \in \{\prod_\alpha F_\alpha \mid F_\alpha \in N_\alpha, \alpha \in I\} = \bigcap_\alpha \pi_\alpha^{-1}(F_\alpha)$

countable intersection is closed under σ-algebra.

$\Rightarrow \prod_\alpha N_\alpha \subseteq \bigotimes_\alpha N_\alpha$.

(X, M) mable space,

Example: $(Y_\alpha, N_\alpha) \to (\prod_\alpha Y_\alpha, \bigotimes_\alpha N_\alpha)$, $\bigotimes_\alpha N_\alpha = \langle \bigcup_\alpha \pi_\alpha^* N_\alpha \rangle$

Lemma: Let $f: X \to Y = \prod_\alpha Y_\alpha$, (X, M) mable space,

f is $(M, \bigotimes_\alpha N_\alpha)$-mable $\Leftrightarrow$ $\pi_\alpha \circ f$ is mable, $\forall \alpha$.

Pf: "$\Rightarrow$" Trivial: Since π_α mable. composition of mable maps mable.

"$\Leftarrow$" Check generating collection: $\bigcup_\alpha \pi_\alpha^* N_\alpha$

But this is $f^{-1}(\pi_\alpha^{-1}(F))$, for some α, $F \in N_\alpha$.

$\hookrightarrow = (\pi_\alpha \circ f)^{-1}(F)$ mable.

Def: A function $\mu: M \longmapsto \mathbb{R}_{\geq 0} \cup \{\infty\}$ is a measure if:

- $\mu(\emptyset) = 0$,
- σ-additivity: $E_i \in M$, $E_i \cap E_j = \emptyset$, $\sum_i \mu(E_i) = \mu(\bigcup_i E_i)$

Prop: Let (X, M, μ) mes space.

① Monotonicity: if $E \subseteq F$, $E, F \in N \Rightarrow \mu(E) \le \mu(E)$

Pf: $\mu(F) = \mu(E) \sqcup \mu(F \setminus E) = \mu(F) + \mu(F \setminus E) \geq \mu(E)$

② σ-subadditity: $\forall \{E_j\}_{j \in N} \subseteq M$, $\mu(\bigcup_i E_i) \le \sum_i \mu(E_i)$

Pf: $E_1' = E_1$, $E_j' = E_j \setminus \bigcup_1^{j-1} E_k'$

③ Upward Monotone convergence: $\forall \{E_j\}_{j \in N} \subseteq M$ s.t. $E_i \subseteq E_{i+1}$. $\mu(\bigcup E_i) = \lim_{i \to \infty} \mu(E_i)$

Pf: Let $F_1 = E_1$, $F_j = E_j \setminus \bigcup_1^{j-1} F_k$

$\mu(\bigcup_i E_i) = \mu(\bigsqcup_i F_i) = \sum \mu(F_i) = \lim_{N \to \infty} \sum_{i=1}^{N} \mu(F_i) = \lim_{N \to \infty} \mu(E_N)$

④ Downward Monotone convergence: $\forall \{E_j\}_{j \in N} \subseteq M$ s.t. $E_i \supseteq E_{i+1}$, if $\mu(E_1) < \infty$, then $\mu(\bigcap E_i) = \lim_{i \to \infty} \mu(E_i)$

Pf: Let $E_i' = E_1 \setminus E_i$, then E_i' increasing sequence.

$\mu(E_1) = \mu(E_i) + \mu(E_i')$ $\forall i$, so also ∞

$= \mu(\bigcap_i^\infty E_i) + \mu(\bigcup_i^\infty E_i')$

$\downarrow$ ③

$\mu(E_1) = \mu(\bigcap_i^\infty E_i) + \lim_{N \to \infty} \mu(E_N')$

$\Rightarrow \mu(\bigcap_i^\infty E_i) = \mu(E_1) - \lim_{N \to \infty} \mu(E_N') = \lim_{N \to \infty} (E_1 \setminus E_N') = \lim (E_N)$

Def: Let (X, M, μ), $E \in M$ is μ-null if $\mu(E) = 0$

We say a property holds μ-almost everywhere if {doesn't hold} is μ-null.

Def: M is complete w.r.t μ if $\forall N$-null, $P(N) \subset M$.

Completion scheme: Let (X, M, μ), Let $N = \{F \in M \mid \mu(F) = 0\}$.

$\bar{M} = \{E \cup F \mid E \in M, \exists K \in N, F \subset K\}$ Add every possible sets.

Prop: $\bar{M}$ is a σ-algebra. $\exists$ completion $\bar{\mu}$ on $\bar{M}$, s.t. $\forall E \in M$, $\mu(E) = \bar{\mu}(E)$.

($\bar{M}$ is complete wrt $\bar{\mu}$)

Note: $\bar{M}$ coarst wrt μ containing M.

Pf: $\cdot\,\bar{M}$ closed for countable union b/c M, N is closed wrt countable union (σ-subadditivity.)

$K \in N$, null

$\cdot\,(E\cup F)^c = \left((E\cup K)\cap(F\cup K^c)\right)^c$. Choose E, s.t. $E\cap K = \emptyset$

If $E\cap K = \emptyset$,

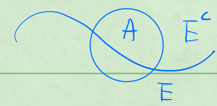
then $E\cup F = (E\setminus K)\cup(F\cup(E\cap K))$

$= \underset{\bar{M}}{\underbrace{(E\cup K)^c}} \cup \underbrace{(F\cup K^c)^c}_{= F^c\cap K \in N}$

$\cdot\,\bar{\mu}(E\cup F) = \mu(E)$

$E_1\cup F_1 = E_2\cup F_2 \Rightarrow E_1\subseteq E_2\cup K_2 \Rightarrow \mu(E_1)\le\mu(E_2)+\mu(K_2)=\mu(E_2)$

Similarly, $\mu(E_2)\le\mu(E_1) \Rightarrow \mu(E_1)=\mu(E_2) \Rightarrow \bar{\mu}$ ✓



Outer measure: $\mu^*: P(X)\longrightarrow \mathbb{R}_{\ge 0}\cup\{\infty\}$

Def: $E\in P(X)$ is μ^*-Caratheodory mable if $\forall A\in P(X)$, $\mu^*(A)=\mu^*(A\cap E)+\mu^*(A\cap E^c)$

Note: By subadditivity of μ^*, $\le$ is guaranteed, so only need to check $\ge$ for μ^*-finite set.

($\mu^*(A)\ge \mu^*(A\cap E)+\mu^*(A\cap E^c)$ with finite A).

Caratheodory extension thm:

Let μ^*, $M=\{E\in P(X)\,|\,E \text{ is } \mu^*\text{-mable}\}$, $\mu=\mu^*|_M$, then (X,M,μ) is complete mable space.

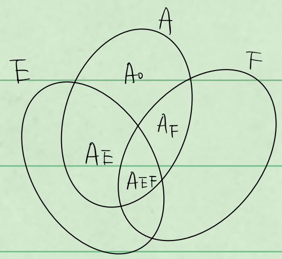
check:

① M σ-algebra ② μ is a measure ③ M is complete.

$(A\cap E)+(A\cap E^c)$

Pf: ①: Closure wrt complement is immediate

NTS closure wrt countable disjoint union.



$\mu^*(A) \overset{E\in M}{=} \mu^*(A_E\cup A_{EF})+\mu^*(A_0\cup A_F)$

$\overset{F\in M}{=} \mu^*(A_E)+\mu^*(A_{EF})+\mu^*(A_0)+\mu^*(A_F)$

☆ $\overset{F,\ E\in M}{=}$ 在 $A_E\cup A_F\cup A_{EF}$ 上分别用一次E一次F.

$= \mu^*(A_E\cup A_F\cup A_{EF})+\mu(A_0)$.

So M algebra. Let $E_i\in M$, $E_i\cap E_j=\emptyset$, $\underset{\cup\!\!\!\!\!\;M}{F_N}=\bigcup_{n=1}^{N}E_n$. $F_\infty=\bigcup_{1}^{\infty}E_n$.

$\forall A\in P(X)$, $\mu^*(A)\overset{F_N}{=}\mu^*(A\cap F_N)+\mu^*(A\cap F_N^c)$

$$\geq \mu^*(A\cap F_N)+\mu^*(A\cap F_\infty^c) \quad \text{By monotonicity of } \mu^*.$$

cutting wrt F_{N-1} $\quad \mu^*(A) \geq \mu^*(A\cap F_{N-1})+\mu^*(E_N)+\mu^*(A\cap F_\infty^c)$

$$\vdots$$

$$\mu^*(A) \geq \sum_{i=1}^{N}\mu^*(A\cap E_i)+\mu^*(A\cap F_\infty^c)$$

$\mu^*(\bigcup E_i) \leq \sum \mu^*(E_i)$

$\lim \to \infty$ $\quad \mu^*(A) \geq \sum_{i=1}^{\infty}\mu^*(A\cap E_i)+\mu^*(A\cap F_\infty^c)$

$A=F_\infty$

Sub-add $\quad \geq \mu^*(A\cap F_\infty)+\mu^*(A\cap F_\infty^c)$

② : $\mu = \mu^*|_{\mathcal{M}} \Rightarrow \mu(\emptyset)=0$

If $E_i \in \mathcal{M}$, $E_i\cap E_j=\emptyset$, then $\sum \mu(E_i) = \mu(\underset{F_\infty}{\underset{\parallel}{\bigcup E_i}})$

$\mu^*(F_\infty) \geq \sum_{i=1}^{\infty}\mu^*(E_i) \quad \Rightarrow \sigma$- additivity.

③ If N is a μ-null set $F\subset N$, $\mu^*(F) \leq \mu^*(N) = \mu(N)=0$

NTS $\mu^*(F)=0 \Rightarrow F\in\mathcal{M}$.

$\forall A\in P(X)$, $\mu^*(A) \leq \mu^*(A\cap F)+\mu^*(A\cap F^c) \leq \mu^*(A\setminus F) \leq \mu^*(A)$.

$\Rightarrow \mu^*(A)=\mu^*(A\cap F)+\mu^*(A\cap F^c) \Rightarrow F\in\mathcal{M}$.

CE] : μ^*-mble set : $E\in\mathcal{M} \Leftrightarrow \forall A\subseteq X$, $\mu^*(A) \geq \mu^*(A\cap E)+\mu^*(A\cap E^c)$

$\forall \mu^*$, $(X,\mathcal{M},\mu)$ is complete m-space.

Hahn-Kolmogorov (1933) :

Let $\mathcal{A}$ algebra on X, μ_0 pre-measure, then μ_0 can be extended to a complete measure μ on $\langle\mathcal{A}\rangle$. Moreover, if ν is another extension to $\langle\mathcal{A}\rangle$, then $\forall E\in\langle\mathcal{A}\rangle$,

$\nu(E) \le \mu(E)$ (= holds if $\mu(E) < \infty$ or if $(X, \mathcal{A}, \mu)$ is σ-finite)

Pf: Step ①: Construct μ^* on X using μ_0

Step ②: Use CET $\to (X, \mathcal{M}, \mu)$.

show that $\mathcal{M} \supset \mathcal{A}$

Step ③: Show $\mu|_{\mathcal{A}} = \mu_0$

Step ④: Uniqueness.

Pf ①: $\forall A \subset X$, $\mu^*(A) = \inf\left\{\sum_i \mu_0(E_i) \middle| E_i \in \mathcal{A}, \text{ s.t. } \cup_i E_i \supset A\right\}$

Lemma*: μ^* is an outer measure.

Pf ②: CET $\rightsquigarrow (X, \mathcal{M}, \mu)$.

WTS: $\forall E \in \mathcal{A}$, $\forall A \subseteq X$, $\mu^*(A) \ge \mu^*(A \cap E) + \mu^*(A \cap E^c)$

Fix $\varepsilon > 0$, by def of μ^*, $\exists \cup_i E_i \supset A$ s.t. $\sum_i \mu_0(E_i) \le \mu^*(A) + \varepsilon$

take $E_i \cap E \in \mathcal{A}$, cover $A \cap E \Rightarrow \mu^*(A \cap E) \le \sum_i \mu_0(E \cap E_i)$

$E_i \cap E^c \in \mathcal{A}$ - - - - - - - $\Rightarrow \mu^*(A \cap E^c) \le \sum_i \mu_0(E^c \cap E_i)$

$\Rightarrow \mu^*(A \cap E) + \mu^*(A \cap E^c) \le \sum_i \mu_0(E_i \cap E) + \sum_i \mu_0(E_i \cap E^c) = \sum_i \mu_0(E_i) \le \mu^*(A) + \varepsilon$. □

Pf ③: WTS: $\forall E \in \mathcal{A}$ $\mu(E) = \mu_0(E)$

As E covers itself, $\mu^*(E) \le \mu_0(E)$

Let E_i cover E, $\sum_i \mu_0(E_i) \ge \mu_0(E)$ (Let E_i's disjoint)

Let $\tilde{E}_i = E_i \cap E \Rightarrow \cup \tilde{E}_i = E$

so $\sum \mu_0(E_i) \ge \sum \mu_0(\tilde{E}_i) = \mu_0(E)$

Pf ④: Let ν another measure on $\langle \mathcal{A} \rangle \to \mathbb{R}_{\ge 0}$, s.t. $\nu(E) = \mu_0(E)$, $\forall E \in \mathcal{A}$.

Let $E \in \langle \mathcal{A} \rangle$, $E_i \in \mathcal{A}$ covering E

$\nu(E) \le \sum_i \nu(E_i) = \sum_i \mu_0(E_i)$

taking inf, $\nu(E) \le \mu^*(E) = \mu(E)$

Note: $\nu\left(\bigcup_i^\infty E_i\right) = \lim_{N \to \infty} \nu\left(\bigcup_{i=1}^N E_i\right) = \lim_{N \to \infty} \mu_0\left(\bigcup_{i=1}^N E_i\right) = \mu\left(\bigcup_i^\infty E_i\right)$

Fix $\varepsilon>0$, assume $\mu(E)<\infty$, then $\forall$ some E_i covering, the following holds:

$\cancel{\mu(E)}+\varepsilon \geq \sum^{\infty} \mu(E_i) \geq \mu(\bigcup E_i) \geq \cancel{\mu(E)} + \mu(\bigcup E_i \setminus E)$

$\Rightarrow \mu((\bigcup E_i)\setminus E) \leq \varepsilon.$

$\mu(E) \leq \mu(\bigcup E_i) = \nu(\bigcup E_i) = \nu(E) + \nu(\bigcup E_i \setminus E) \leq \nu(E) + \mu(\bigcup E_i \setminus E) \leq \nu(E)+\varepsilon.$

$\Rightarrow \mu(E)=\nu(E)$ if $\mu(E)$

if $X = \overset{\text{disjoint}}{\bigcup} X_i$ $\Rightarrow$ $\nu(E) = \nu(\bigcup E\cap X_i) = \sum \nu(E\cap X_i) = \sum_\mu (\quad) = \mu(E)$

finite

Construction of Lebesgue measure.

$\mathcal{E} = \{(a,b], (a,+\infty), (-\infty, b], \emptyset\}$ elementary family.

$\Rightarrow \mathcal{A} = \{$ finite disjoint union of $\mathcal{E}\}$

$\langle \mathcal{A} \rangle = \mathcal{B}_{\mathbb{R}}$

$\forall E \in \mathcal{A}$, $\mu_0(E) = \begin{cases} 0 & \text{if } \emptyset \\ \infty & \text{if contains } (-\infty,b] / (a,+\infty) \\ \sum b_i - a_i & \text{if } E = \bigcup_i (a_i,b_i] \end{cases}$

<u>Lemma:</u> μ_0 is a pre-measure on $\mathcal{A}$

<u>Note:</u> Applying H.K to $\mathcal{A}$ get Lebesgue measure

<u>Note:</u> $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \mu)$ is σ-finite, $\mathbb{R} = \bigsqcup (n, n+1]$

<u>Pf:</u> ① Check μ_0 is well-defined.

$E = \bigcup_{i=1}^N (a_i, b_i]$ if $b_i = a_j$ for some i,j, we can $(a_i,b_i]\cup(a_j,b_j] \to (a_i,b_j]$

$\mu_0((a_i,b_i]\cup(a_j,b_j]) = b_j - a_i = \mu_0((a_i,b_j])$

$\Rightarrow \mu_0$ is well-defined, and finite additivity

WTS: if $E_j \in \mathcal{A}$, $\bigcup_j E_j \in \mathcal{A}$, then $\mu_0(\bigcup E_j) = \sum \mu_0(E_j)$

Assume E_j disjoint. Assume E_j is in fact h-interval, $(a_i, b_i]$

$\bigcup_j E_j$ is finite union of h-intervals.

Case ①: finite h-intervals

Case ②: infinite h-intervals

Pf: Assume $\bigcup_j E_j$ is one finite interval

WTS $\bigcup_j E_j = (a,b] = E$, then $\sum \mu_0(E_i) = b-a$

Note: $b-a = \mu_0(\bigcup_{i=1}^{\infty} E_j)$

$$= \mu_0\left(\bigcup_{i=1}^{N} E_i \cup (E \setminus \bigcup_{i=1}^{N} E_i)\right)$$

$$= \mu_0(\bigcup_1^N E_i) + \underbrace{\mu_0(E \setminus \bigcup_{i=1}^{N} E_i)}_{\geq 0}$$

$$\geq \mu_0(\bigcup_1^N E_i) = \sum_i \mu_0(E_i)$$

So $b-a \geq \sum_1^{\infty} \mu_0(E_i)$ taking lim

WTS $b-a \leq \sum_{j=1}^{\infty} \mu_0(E_j)$

$\forall \varepsilon > 0$, Let $E' = [a+\varepsilon, b] \subset E = (a, b]$

Let $E_j' = (a_j, \underbrace{b_j + \varepsilon 2^{-j}}_{= b_j'}) \supset E_j = (a_j, b_j]$

Let E_j'' be a finite cover of E' with elements in $(E_j')_{j \in \mathbb{N}}$ $\quad E_j'' = (a_j'', b_j'')$

Assume $a_i < a_{i+1}$ $\quad b_j'' > a_{j+1}$ $\quad$ (((( )

$$\mu_0(E) = b - (a+\varepsilon) + \varepsilon$$

$$\leq b_N'' - a_1'' + \varepsilon$$

$$\leq b_N'' - a_N'' + \sum_{i=1}^{N-1} a_{j+1}'' - a_j'' + \varepsilon$$

$$\leq b_N'' - a_N'' + \sum_{i=1}^{N-1} b_j'' - a_j'' + \varepsilon$$

$$\leq \sum_{i=1}^{N} b_j'' - a_j'' + \varepsilon$$

$$\leq \sum_{j=1}^{N} \mu_0(\underbrace{E_j}_{\text{Not same } j}) + \varepsilon \Sigma 2^{-j} + \varepsilon$$

$$\leq \sum_{j=1}^{\infty} \mu_0(E_j) + \varepsilon.$$

So $\exists! \mu$ on $\mathcal{B}_{\mathbb{R}}$ s.t. $\mu(a,b] = b-a$

Regularity of Lebesgue measure.

Thm: If $E \in \mathcal{B}_{\mathbb{R}}$, then

$\mu(E) = \inf\{\mu(U) \mid \overset{\text{open}}{U} \supset E\}$

$= \sup\{\mu(K) \mid \overset{\text{compact}}{K} \subset E\}$

Recall: $\mu(E) = \inf\{\sum \mu_0(E_i) \mid E_i \in \mathcal{A},\ \cup E_i \supset E\}$

$= \inf\{\sum \mu_0((a_j, b_j]) \mid \cup (a_j, b_j] \supset E\}$

Lemma: $\forall E \in \mathcal{B}_{\mathbb{R}}$,

$\mu(E) = \inf\{\sum_i \mu(a_i, b_i) \mid \cup (a_i, b_i) \supset E\}$

Pf: Let $\nu(E) = \inf\{\sum_i \mu(a_i, b_i) \mid \cup (a_i, b_i) \supset E\}$

$(a_j, b_j) = \bigsqcup (c_{j_k}, c_{j_{k+1}}]$, $c_{j_1} = a_j$, $c_{j_k} \nearrow b_j$

So $\bigcup_{j,k} (c_{j_k}, c_{j_{k+1}}] \supset E$.

$\mu(E) \leq \sum \mu(c_{j_k}, c_{j_{k+1}}]$

$\leq \sum_j \mu(a_j, b_j) = \nu(E)$

So $\mu(E) \leq \nu(E)$.

NTS $\nu(E) \leq \mu(E)$

$\forall \varepsilon$, $\exists ((a_i, b_i])_{i \in \mathbb{N}}$, s.t.

$\mu(E) + \varepsilon \geq \sum_i \mu((a_i, b_i])$ $\qquad (a_i, b_i + \varepsilon 2^{-i}) \supset (a_i, b_i]$

$\geq \sum_i \mu((a_i, b_i + \varepsilon 2^{-i})) - \varepsilon 2^{-i}$

$\geq \sum_i \mu((a_i, b_i + \varepsilon 2^{-i})) - \varepsilon$

$\geq \nu(E) - \varepsilon$.

So $\nu(E) \leq \mu(E) + 2\varepsilon$. $\Rightarrow \nu(E) \leq \mu(E)$.

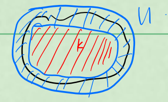
Pf: ① $\forall E \in \mathcal{B}_{\mathbb{R}}$, $\mu(E) = \inf\{\mu(U) \mid \overset{\text{open}}{U} \supset E\}$

By Lemma and fact that any open set in $\mathbb{R}$ is $\cup_i (a_i, b_i)$

② $\forall E \in \mathcal{B}_{\mathbb{R}}$, $\mu(E) = \sup\{\mu(K) \mid \overset{\text{compact}}{K} \subset E\}$

Assume first E bounded. If E is closed, then by monotonicity, $\mu(E) \geq \mu(K)$

$\forall K \subset E$, $\sup < \mu(E)$.



Assume E not closed, $\forall \varepsilon > 0$, $\exists U \overset{\text{open}}{\supset} \bar{E} \setminus E$, s.t. $\mu(U) = \mu(\bar{E} \setminus E) + \varepsilon$

Let $K^{cpt} = \bar{E} \setminus U$

$\mu(K) = \mu(E) - \mu(E \cap U)$

$= \mu(E) - (\mu(U) - \mu(U \setminus E))$

$\geq \mu(E) - \mu(U) + \mu(\bar{E} \setminus E)$

$\geq \mu(E) - \varepsilon$

Assume E is unbounded, Let $E = \bigcup (\underbrace{E \cap [j, j+1]}_{E_j})$

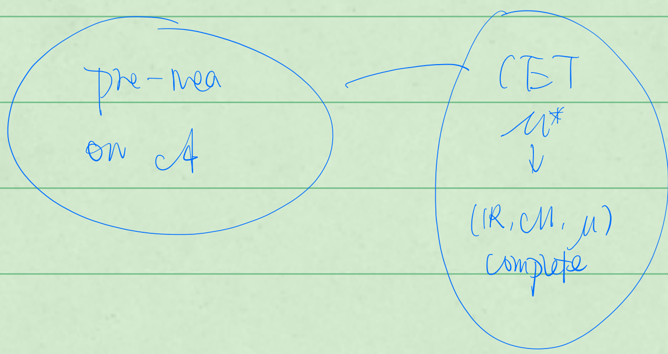
$\forall j$, $\exists K_j \subset E_j$, s.t. $\mu(K_j) \geq \mu(E_j) - \varepsilon 2^{-j}$

Let $H_N = \bigcup_{-N}^{N} K_j$ compact

$\mu(H_N) \geq \mu(\bigcup_{-N}^{N} E_j) - \sum 2^{-j} \varepsilon$

$\geq \mu(\bigcup_{-N}^{N} E_j) - 2\varepsilon$

$N \to \infty$, then we conclude.



$\mathcal{A}$ = disjoint union of h-intervals.

$\langle \mathcal{A} \rangle = \mathcal{B}_{\mathbb{R}}$ $\mathcal{M}$ = Lebesgue mble sets = $\overline{\mathcal{B}_{\mathbb{R}}}$

$\mu_0(E) = \begin{cases} \infty & (\pm\infty) \\ 0 & \emptyset \\ \sum (b_i - a_i) \end{cases}$

Observe: recall $f: (X, M) \to (Y, N)$ is mble if $f^{*} N \subseteq M$

$(\mathbb{R}, M) \to (\mathbb{R}, \mathcal{B}_{\mathbb{R}})$

Def: f is Lebesgue mble if it is $(\mathcal{M}, \mathcal{B}_{\mathbb{R}})$-mble.

$g\circ f$ may not be Lebesgue-mable even if both are.

Notation: m always denote Lebesgue measure on $\mathbb{R}$ ($\mathbb{R}^d$)

Thm: Let $E \subseteq \mathcal{B}_{\mathbb{R}}$ (shown before)

$$m(E) = \inf\{m(U) \mid U^{op} \supset E\} \quad \text{outer regularity}$$

$$= \sup\{m(K) \mid K^{cpt} \subset E\} \quad \text{Inner regularity}$$

Thm: TFAE

① $E \in \mathcal{M}$

② $\exists G_\delta$ set V, m-null set N_1, s.t. $E = V \setminus N_1$

③ $\exists F_\sigma$ set W, m-null set N_2, s.t. $E = W \cup N_2$

Pf: ② $\Rightarrow$ ①, ③ $\Rightarrow$ ① automatic.

① $\Rightarrow$ ②: Let $E \in \mathcal{M}$, assume $m(E) < \infty$

$E = E' \cup F$, $F \subset N$, Borel n-set, $\overline{E} = E \cup N$

outer of $\overline{E}$

$\Rightarrow \exists U_n$ open, s.t. $m(U_n \setminus \overline{E}) < 2^{-n}$, $U_n \supset E$

$V = \bigcap U_n$ is G_δ, $\Rightarrow m(V \setminus E) = 2^{-n}$, $\forall n$.

m-null

$\Rightarrow E = V \setminus (V \setminus E)$

① $\Rightarrow$ ③: Same.

Remark: Def of m implies: $m(E) = m(\{x+c \mid x \in E\})$ $\forall c \in \mathbb{R}$

$|\lambda| m(E) = m(\{\lambda x \mid x \in E\})$, $\forall \lambda \in \mathbb{R}$

Cantor set: $\bigcap_n C_n = C$

Prop: ① C is compact, totally disconnected, nowhere dense, perfect.

② $m(C) = 0$

③ # C is uncountable.

Pf ②: $m(C) = m(\cap_n C_n)$

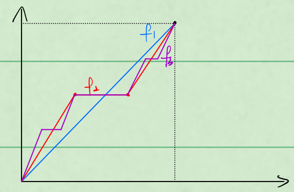
$$= \lim_{n\to\infty} m(C_n) = 0$$

$$\begin{cases} m(C_1) = 1 \\ m(C_2) = 1 - \frac{1}{3} \\ m(C_3) = 1 - \frac{1}{3} - 2\frac{1}{9} \\ m(C_4) = 1 - \frac{1}{3} - 2\frac{1}{9} - 4\frac{1}{27} \end{cases} \Rightarrow m(C_{n+1}) = 1 - \sum_{k=0}^{n-1} \frac{2^{k-1}}{3^k}$$

$$= 1 - \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 0$$

③: $|f_{n+1} - f_n| < 2^{-n}$

$f_n \xrightarrow{\text{unif}} f$



$f(C) = [0,1] \Rightarrow C$ uncountable.

Recall: • $\mathcal{B}_{\mathbb{R}^n} = \langle \{ U \in \mathbb{R}^n \mid U \text{ open} \} \rangle$

$$= \mathcal{B}_{\mathbb{R}} \otimes \mathcal{B}_{\mathbb{R}} \otimes \cdots \otimes \mathcal{B}_{\mathbb{R}}$$

• $f: X \to \prod_\alpha Y_\alpha$ mble $\Leftrightarrow$ $\pi_\alpha \circ f$ mble.

• $\mathbb{C} \cong \mathbb{R}^2$, $f: X \to \mathbb{C}$ mble if real, imaginary part both mble.

• f, g both mble $\Rightarrow$ so is $f+g$, fg.

• $\tilde{\mathbb{R}} = \{-\infty, +\infty\} \cup \mathbb{R}$.

Prop: $\{f_i\}_{i\in\mathbb{N}}$, $X \to \tilde{\mathbb{R}}$: sequence of measurable functions.

then $x \mapsto \begin{cases} \sup_i f_i(x) \\ \inf_i f_i(x) \\ \limsup_i f_i(x) \\ \liminf_i f_i(x) \end{cases}$ are all mble.

Pf: Let $E_i(a) = f_i^{-1}([a, +\infty])$, mble.

$\{x \mid \sup_i f_i(x) \geq a\} = \cap_{n=1}^\infty \cup_{i=1}^\infty E_i(a - \frac{1}{n})$, $\inf_i f_i(x) = -\sup(-f_i(x))$

Recall: $\limsup_i f_i = \inf_n \sup_{i\geq n} f_i$

$\liminf_i f_i = \sup_n \inf_{i\geq n} f_i$

Def: For $f: X \to \tilde{\mathbb{R}}$ measurable.

$f^+(x) = \max\{f(x), 0\}$, $f^-(x) = -\min\{f(x), 0\} = \max\{-f, 0\}$.

$\Rightarrow f = f^+ - f^-$, $f^+, f^- \geq 0$ both measurable.

For $f: X \to \mathbb{C}$ measurable.

define $\operatorname{sgn} z = \begin{cases} z/|z| & z \neq 0 \\ 0 & z = 0. \end{cases}$

So $f = \operatorname{sgn}(f)\,|f|$

f is simple if it is measurable, $f(X)$ is finite, $\subseteq \mathbb{R}$

Std representation of a simple function is

$$f(x) = \sum_{y \in \mathrm{Im}(f)} y \cdot \chi_{f^{-1}(y)}(x)$$

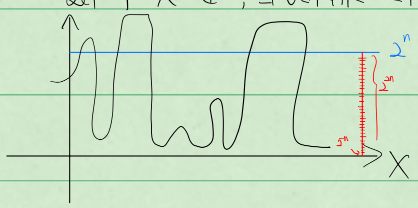
$$= \sum_{i=1}^{N} a_i \chi_{E_i}(x) \qquad \bigsqcup E_i = X$$

<u>Notation:</u> φ, ψ simple fns.

<u>Thm:</u> (Approximation with simple fns)

a) Let $f: X \to \overline{\mathbb{R}}_{\geq 0}$ mable, $\exists\ 0 \leq \varphi_1 \leq \varphi_2 \leq \cdots \leq \varphi_n \leq \cdots \leq f$, s.t. $\varphi_n \to f$ pointwise, uniformly on any set where f bounded

b) Let $f: X \to \mathbb{C}$, $\exists\ 0 \leq |\varphi_1| \leq \cdots \leq |\varphi_n| \leq \cdots \leq |f|$, same holds



<u>Pf:</u> Let $n \in \mathbb{N}$ $\quad 0 \leq l \leq 2^{2n}-1$

$E_n^l = f^{-1}([l \cdot 2^{-n}, (l+1)2^{-n}))$, $\quad R_n = f^{-1}([2^n, +\infty])$

$$\varphi_n(x) = \sum_{l=1}^{2^{2n}-1} l \cdot 2^{-n} \cdot \chi_{E_n^l}(x) + 2^n \chi_{R_n}(x)$$

- $0 \leq \varphi_n \leq \varphi_{n+1}$
- $\sup_{x \in R_n^c} |f(x) - \varphi_n(x)| \leq 2^{-n}$

For b), Let $f = (\mathrm{Re} f)^+ - (\mathrm{Re} f)^- + i(\mathrm{Im} f)^+ - i(\mathrm{Im} f)^-$

Thm: Let $(X,\mathcal{M},\mu)$ mable space, $(X,\overline{\mathcal{M}},\overline{\mu})$ its completion

If $f: X \to \overset{*}{\mathbb{R}}$, f is $\overline{\mathcal{M}}$-mable, $\exists g$ $\mathcal{M}$-mable. s.t. $f = g$ $\overline{\mu}$-a.e.

Pf: If f is $\overline{\mathcal{M}}$-mable, $\exists \varphi_n$ of $\overline{\mathcal{M}}$-simple fns, $\varphi_n \to f$ pointwise.

$$\varphi_n = \sum_{i=0}^{L-1} a_i \chi_{\underbrace{E_n^i}_{\overline{\mathcal{M}}\text{-mable.}}} \longrightarrow E_n^i \in \overline{\mathcal{M}} \Rightarrow E_n^i = \underbrace{E_n'^i}_{\in \mathcal{M}.} + \underbrace{F_n^i}_{\subset N_n^i}$$

$$\varphi_n' = \sum_{i=0}^{L-1} a_i \chi_{E_n'^i}$$

$$\{\varphi_n \neq \varphi_n'\} \subseteq \bigcup_i N_n^i = N_n \leftarrow \text{null}$$

Notice: $\varphi_n \to f$ pointwise

$$\varphi_n' = \varphi_n \text{ on } \Big(\overbrace{\bigcup_n N_n}^{\text{null}}\Big)^c = N^c$$

$$\Rightarrow \varphi_n' \to f \text{ on } N^c$$

Let $\varphi_n'' = \overset{\text{kill outside}}{\chi_{N^c}} \varphi_n'$

$\varphi_n'' \to g$ $\mathcal{M}$-mable, $g - f = 0$ on N^c

$$L^+(X,\mathcal{M},\mu) = \{f: X \to \overset{*}{\mathbb{R}}_{\geq 0} \mid f \text{ mable}\}$$

Def: If $\varphi \in L^+$ is simple, then $\varphi = \sum_{i=0}^{N-1} a_i \chi_{E_i}$

$$\int \varphi \, d\mu = \sum_{i=0}^{N-1} a_i \mu(E_i)$$

For any $A \in \mathcal{M}$, $\varphi \cdot \chi_A$ is also simple

$$\int_A \varphi \, d\mu = \sum_{i=0}^{N-1} a_i \mu(E_i \cap A)$$

Properties: If $\varphi, \psi \in L^+$

a) If $c > 0$, $\int c\varphi = c \int \varphi$

b) $\int \varphi + \psi = \int \varphi + \int \psi$

c) $\varphi \leq \psi \Rightarrow \int \varphi \leq \int \psi$

d) fix φ, $\nu = \overset{\mathcal{M}}{\underset{\Downarrow}{A}} \mapsto \int_A \varphi \, d\mu$ is a measure

Pf d): $\nu(\emptyset) = 0$

$$A = \bigsqcup_i A_i, \quad \int_A \varphi \, d\mu = \sum_{j=0}^{L-1} a_j \mu(A \cap E_j)$$

σ-additivity of μ

$= \sum_{n,j} a_j \mu(A_n \cap E_j)$

$= \sum_n \int_{A_n} \varphi \, d\mu = \sum_n \nu(A_n)$

Def: Let $f \in L^+(X, \mathcal{M}, \mu)$

$\int f \, d\mu := \sup\{ \int \varphi \, d\mu \mid \varphi \leq f \}$

Notice if $f \leq g$, then $\int f \, d\mu \leq \int g \, d\mu$, $\forall c > 0$, $\int cf = c \int f$

Thm: (Monotone Convergence Thm) MCT

Let $f_n \in L^+$, $0 \leq f_n \leq f_{n+1}$

Let $f = \sup f_n = \lim_{n\to\infty} f_n$, then

$\int \lim_{n\to\infty} f_n \, d\mu = \int f \, d\mu = \lim_{n\to\infty} \int f_n \, d\mu$

Def: If $\varphi \in L^+(X, \mathcal{M}, \mu)$ simple, $\int \varphi \, d\mu = \sum_{y \in \mathrm{Rg}(\varphi)} y \cdot \mu(\varphi^{-1}(y))$

$= \sum_i a_i \mu(E_i)$

Recall: fix φ, $A \longmapsto \int_A \varphi \, d\mu$ A measure.

Def: Let $f \in L^+$, $\int f := \sup\{ \int \varphi \, d\mu \mid 0 \leq \varphi \leq f \}$

Pf: By monotonicity, $\int f_n$ form non-decreasing sequence.

$\forall n$, $\int f_n \leq \int f \Rightarrow \lim_{n\to\infty} \int f_n \leq \int f$

Fix $\alpha \in (0,1)$, let $0 \leq \varphi \leq f$, $\forall n$, let

$E_n = \{ x \in X \mid f_n(x) \geq \alpha \varphi(x) \}$, $E_n \nearrow X$

$\int_X f_n \geq \int_{E_n} f_n \geq \alpha \int_{E_n} \varphi$

$E_n \xmapsto{\nu_\varphi} \int_{E_n} \varphi \, d\mu$ is a measure

$E_n \nearrow X \to \nu_\varphi(X) = \int_X \varphi \, d\mu = \lim_{n\to\infty} \nu_\varphi(E_n)$

$$\Rightarrow \lim_{n\to\infty}\int_X f_n \geq \lim_{n\to\infty} \alpha \int_{E_n} \varphi(x) = \alpha \int_X \varphi$$

sup

sup

$$\Rightarrow \lim_{n\to\infty}\int_X f_n \geq 1 \int_X f$$

Cor: $f, g \in L^+$, $\int f+g = \int f + \int g$

Pf: $\varphi_n \nearrow f$, $\psi_n \nearrow g$, $\varphi_n + \psi_n \nearrow f+g$

$$\Rightarrow \lim \int \varphi_n + \psi_n = \lim \int \varphi_n + \lim \int \psi_n$$

WTS: $\int \sum_i^\infty f_i = \sum_i^\infty \int f_i$

Let $F_n = \sum_i^n f_i$. then $\int F_n = \sum_{i=1}^n f_i$

$F_n \nearrow F = \sum_{i=1}^\infty f_i$ so MCT

Non-Example:

- Let $f_n = \chi_{[n,n+1]}$
- Let $f_n = n\chi_{[0,\frac{1}{n}]}$

Fatou's Lemma: If $(f_n) \subseteq L^+$

Fatou's Lemma: If $(f_n) \subseteq L^+$

$$\int \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int f_n$$

Recall: $\liminf a_n = \lim_{l\to\infty}(\inf_{n\geq l} a_n)$

Pf: $\forall l \geq 1$, $\inf_{n\geq l} f_n \leq f_j$, $\forall j \geq l$

$$\Rightarrow \int \inf_{n\geq l} f_n \leq \int f_j$$

$$\Rightarrow \int \inf_{n\geq l} f_n \leq \inf_{j\geq l} \int f_j$$

$$\int \liminf f_n \overset{MCT}{=} \lim_{l\to\infty} \int \lim_{n\geq l} f_n \leq \lim_{l\to\infty} \inf_{j\geq l} \int f_j$$

Markov inequality.

$f \in L^+$, $\forall \lambda \in (0,\infty)$, $E_\lambda = \{x \mid f(x) \geq \lambda\}$.

$\Rightarrow \mu(E_\lambda) \leq \frac{1}{\lambda}\int_X f\,d\mu.$

Pf: $\int_X f \geq \int \varphi_\lambda \geq \lambda \int_{E_\lambda} 1$

Cor: If $f \in L^+$, $\int f < \infty$, then f is a.e. finite.

Pf: $\{x \mid f(x) = \infty\} = \bigcap_n E_n$

$\mu(\quad) = \mu(\bigcap_n E_n) \leq \frac{1}{n}\int_X f \leq \frac{M}{n}$

Cor: $\int f = 0 \Rightarrow f$ a.e. 0

Pf: $Z = \{x \mid f(x) \neq 0\} = \bigcup_n E_n$

If $\mu(Z) > 0 \Rightarrow \exists\, \mu(E_{\frac{1}{n}}) > 0 \Rightarrow \frac{1}{n}\int f \geq \mu(E_{\frac{1}{n}}) > 0$

Upgrade MCT to:

Assume $f_n \nearrow f$ μ. a.e. then $\int \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int f_n$

Pf: $\exists N$ null, s.t. $\underbrace{f_n \cdot \chi_{N^c}}_{\tilde{f}_n} \nearrow \underbrace{f \cdot \chi_{N^c}}_{\tilde{f}}$

$\Rightarrow \int \lim_{n\to\infty} \tilde{f}_n \overset{MCT}{=} \lim_{n\to\infty} \int \tilde{f}_n \overset{Cor}{=} \lim_{n\to\infty} \int f_n$

$\quad \| $

$\int \tilde{f} \overset{Cor}{=} \int f$

Def: f mable ($f \Rightarrow \{\mathbb{R}, \mathbb{C}\}$) is integrable on $E \subseteq M$ if $\int_E |f| < \infty$

Remark: Set of integrable fns form a linear space.

$\int f := \begin{cases} \int f^+ - \int f^- \\ \int \mathrm{Re}(f) + i \int \mathrm{Im}(f) \end{cases}$

Prop: Let f integrable, (X, M, μ), then $|\int f| \leq \int |f|$

Pf: Assume $f \to \mathbb{R}$, $|\int f| = |\int f^+ - \int f^-| \leq \int f^+ + \int f^- = \int |f|$

If $f \to \mathbb{C}$, Assume $\int f \neq 0$.

$$|\int \mathrm{Re} f| \le \int |\mathrm{Re} f| \le \int |f| \overset{|\alpha|=1}{=\!=} \int |\alpha f|$$

$|\int f| =$

$$\Rightarrow \max_{|\alpha|=1} |\int \mathrm{Re}\,\alpha f| \le \int |f| \quad . \quad \text{Let } \alpha = \overline{\mathrm{sgn} \int f} = \frac{\bar{z}}{|z|}$$

$$\int \mathrm{Re}\,\alpha f = \mathrm{Re}\,\alpha \int f = |\int f|$$

<u>Prop</u>: a) f integrable $\Rightarrow \{x \mid f(x) \neq 0\}$ is σ-finite.

b) f, g integrable

$$\forall E \in \mathcal{M},\ \int_E f = \int_E g \Leftrightarrow f = g \text{ a.e } \Leftrightarrow \int |f-g| = 0$$

<u>Pf</u>: "$\Rightarrow$" Let $E' = \{f \neq g\}$ not null

$\Rightarrow$ At least one of $\{f^\pm \neq g^\pm\}$ not null.

$\Rightarrow \int_{E^+} |f^+ - g^+| \neq 0$ contradiction.

<u>Def</u>: $f \sim g$ if $f = g$ μ.a.e.

$\hat{f} = [f] \in \{f \text{ mable}\}/\sim$, $\int [f]$ well-defined.

<u>Def</u>: $L^1(X, \mathcal{M}, \mu) = \{[f] \mid \int [f] d\mu < \infty\}$.

$f, g \in L^1$, $d(f,g) = \int |f-g|$

<u>Dominant Convergence Theorem</u>:

Let $f_n \in L^1$, $f_n \to f$ a.e. . $\exists g \in L^1 \cap L^+$, $|f_n| \le g$ a.e.

then $f \in L^1$, $\int f = \lim_{n\to\infty} \int f_n$.

<u>Cor</u>: Under the same assumptions, $f_n \xrightarrow{L^1} f$

<u>Pf</u>: $|f_n - f| \longrightarrow 0$ a.e , $|f_n - f| \le 2g$. By DCT, $\int |f_n - f| \longrightarrow 0$

Pf: Assume that f_n real (Otherwise pick $Re/Im\ f_n$)

f is mble, f integrable. observe $g+f_n,\ g-f_n \geq 0$.

Fatou: $\int g + \int f = \int \liminf (g+f_n) \leq \int g + \liminf \int f_n$

$\int g - \int f = \int \liminf (g-f_n) \leq \int g - \limsup \int f_n$ $\qquad$ $\lim = \liminf = \limsup$.

$\Rightarrow \limsup \int f_n \leq \int f \leq \liminf \int f_n$

Thm: Assume $(f_n) \subseteq L^1$. $\sum_{j=1}^{\infty} \int |f_j| < \infty$

then $\sum_{j=1}^{\infty} f_j \xrightarrow{L^1} f \in L^1$. $\int \sum_{j=1}^{\infty} f_j = \sum_{j=1}^{\infty} \int f_j$

Pf: $\int \Sigma |f_j| = \Sigma \int |f_j| < \infty$

Let $g(x) = \Sigma |f_j|$ is in L^1, so g is a.e. finite.

$h_n = \sum_{j=1}^{n} f_j \quad |h_n(x)| \leq \sum_{j=1}^{n} |f_j| \leq g$ $\quad$ a.e.

$DCT(h_n)$: $h_n \xrightarrow{L^1} h$

Thm: (Density of simple fns in L^1)

Assume $f \in L^1$, $\forall \varepsilon$, $\exists \varphi$ simple, s.t. $\int |f - \varphi| < \varepsilon$.

- If $f \in L^1(\mathbb{R}, M, m)$, then ① φ can be chosen to be "very simple",

i.e, linear combination of indicators of open intervals.

Or, ② $\exists$ continuous g, $\int |f-g| < \varepsilon$

Pf: We found φ_n, $\cdots \leq |\varphi_n| \leq \cdots \leq |f|$, $\varphi_n \xrightarrow{a.e.} f$,

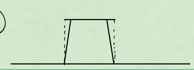
$\varphi_n \xrightarrow{L_1} f$ by DCT.

Lebesgue case: ① Given $\varphi_n = \sum_{i=1}^{N} a_i \chi_{E_i}$, $\mu(E_i) \leq |a_i|^{-1} \int_{E_i} \varphi \leq |a_i|^{-1} \int \varphi < \infty$

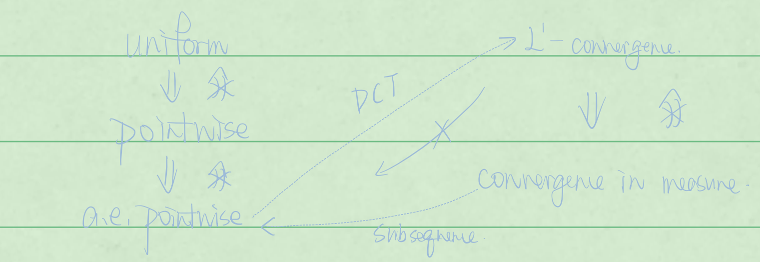
So $\exists$ union of open intervals $\tilde{E}_i \supset E_i$, $m(\tilde{E}_i \setminus E_i) < |a_i|^{-1} \frac{\varepsilon}{N}$ (Property of Lebesgue measure)

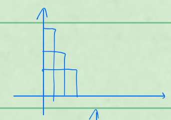
$\tilde{\varphi} = \Sigma a_i \chi_{\tilde{E}_i}$, $\int |\varphi - \tilde{\varphi}| < \varepsilon$.

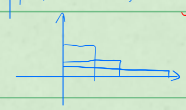
②



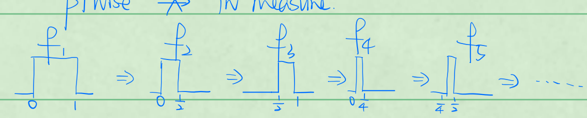
Notions of convergence:



Examples: ① (Squeeze up): $f_n = n\chi_{[0,\frac{1}{n}]}$  $\int f_n = 1$

② (Squeeze down): $f_n = n^{-1}\chi_{[0,n]}$  $\int f_n = 1$

③ $f_n = \chi_{[n,n+1]}$ ptwise $\not\Rightarrow$ in measure.

④ (Zeno's piano) 

$f_n \xrightarrow{\text{in measure}} 0$ not ptwise to anything

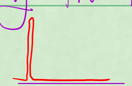
Def: f_n is cauchy in measure if: $\forall \varepsilon>0$, $i \geq j \geq n$,

$\lim_{n\to\infty} \mu(\{x \mid |f_i(x)-f_j(x)|\}) = 0$

Recall: $f_n \xrightarrow{\text{in measure}} f$ if $\lim_{n\to\infty} \mu(\{x \mid |f_n(x)-f(x)|>\varepsilon\}) = 0$.

Lemma: $f_n \xrightarrow{L^1} f \Rightarrow f_n \xrightarrow{\text{measure}} f$

Pf: $\mu\{x \mid |f_n - f| > \varepsilon\} \overset{\text{Markov ineq}}{\leq} \varepsilon^{-1}\int |f_n - f| \longrightarrow 0$

Cauchy in measure 允许  , 只要domain上差不多就行, 但 L^1 不行.

Lemma: If f_n is cauchy in measure, $f_{n_j} \longrightarrow f$ in measure,

then $f_n \longrightarrow f$ in measure.

一般来说 Cauchy ⇒ convergence. 但这里 in measure 不是个好的 metric, 所以我们需要子列收敛来帮助我们。

Lemma: If $f_n \xrightarrow{\text{in measure}} f$, $f_n \xrightarrow{\text{in measure}} g$, then $f=g$ a.e.

Pf: Fix $\varepsilon>0$, $\mu\{x \mid |f-g|>\varepsilon\} \leq \mu(\{x \mid |f-f_n| > \frac{\varepsilon}{2}\} \cup \{x \mid |g-f_n| > \frac{\varepsilon}{2}\})$
$\searrow 0$ $\searrow 0$

$\forall n$, $|f(x)-g(x)| = |f(x) - f_n(x) + f_n(x) - g(x)|$

$\leq |f(x)-f_n(x)| + |f_n(x)-g(x)|$

Pf: $\mu(\{|f-f_n|>\varepsilon\}) \leq \mu(\{|f_m - f| > \frac{\varepsilon}{2}\} \cup \{|f_m - f_n| > \frac{\varepsilon}{2}\})$, $\forall m$.

Choose $m = n_j$ $\searrow 0$ $\searrow 0$

Thm: Let f_n cauchy in measure, $\exists f$, $f_n \xrightarrow{measure} f$

and $\exists n_j$ s.t. $f_{n_j} \xrightarrow{ae\ ptwise} f$

Pf: Let g_j be the subsequence f_{n_j}, s.t.

$\mu(\underbrace{\{|g_j(x) - g_{j+1}(x)| \geq 2^{-j}\}}_{E_j}) \leq 2^{-j}$

Let $F_l = \bigcup_{j=l}^{\infty} E_j$, $\mu(F_l) < 2 \cdot 2^{-l}$

If $r \geq s \geq l$, $\forall x \notin F_l$, $|g_r(x) - g_s(x)| \leq \sum_{j=s}^{r-1} |g_{j+1}(x) - g_j(x)| \leq 2 \cdot 2^{-s}$

Let $F = \bigcap F_l$, then $\mu(F) = 0$

$\mu(\{|f - g_s| > 2 \cdot 2^{-s}\}) \leq \mu(F_s)$, so $g_s \longrightarrow f$ in measure.

Let $f(x) = \lim_{j\to\infty} g_j(x)$ for $x \in F$

Def: f_n mble on $(X, \mathcal{M}, \mu)$ converges almost uniformly to f

if $\forall \varepsilon$, $\exists E$, s.t. $\mu(E) < \varepsilon$, $f_n \xrightarrow{uni} f$ on E^c

Thm: (Egoroff): if $\mu(X) < \infty$, any a.e. converging sequence of f_n

converges almost uniformly.

Exercise: Almost uni $\Rightarrow$ a.e pt
con in measure

Pf: WLOG, $f_n \to f$ pointwise.

Fix $l, n \in \mathbb{N}$, $\bigcup_{m \geq n}^{\infty} \{x \mid |f_m(x) - f(x)| \geq l^{-1}\} = E_n(l)$

for fixed l, $E_n(l) \downarrow \emptyset$ as $n \to \infty$

Continuity from above $(\mu(X) < \infty) \Rightarrow \mu(E_n(l)) \overset{n}{\downarrow} 0$

$\Rightarrow$ Fix ε, $\exists n_l$, $\mu(E_{n_l}(l)) < \varepsilon 2^{-l}$

Let $E = \bigcup_l E_{n_l}$, $\mu(E) < \varepsilon$

If $x\in E^c$, $\forall l$, $\forall n>n_l$, $|f_n(x)-f(x)|<l^{-1}$

Product Measures:

Monotone Class Lemma: $C\subseteq P(X)$ is MC if:

$\forall E_n\subseteq E_{n+1}$, $\bigcup E_n\in C$, $\forall F_n\supseteq F_{n+1}$, $\bigcap F_n\in C$.

Let $\mathcal{E}\subseteq P(X)$ $\langle\mathcal{E}\rangle_c$ generated by $\mathcal{E}$

MCL: Let $\mathcal{A}$ be an algebra on X, $\langle\mathcal{A}\rangle_c=\langle\mathcal{A}\rangle_m$.

Pf: Since σ-algebra are MC's, $\langle\mathcal{A}\rangle_c\subseteq\langle\mathcal{A}\rangle_m$

If we show $\langle\mathcal{A}\rangle_c$ a σ-algebra, we're done. Let $C=\langle\mathcal{A}\rangle_c$.

Let $B\in C$, $C(B)=\{B'\in C$, s.t. $B'\in C$, s.t. $B'\setminus B,\ B\setminus B',\ B\cap B'\in C\}$.

Note: ① $\varnothing\in C(B)$

② $B'\in C(B)\iff B\in C(B')$

③ $C(B)$ is a MC.

$\forall E\in\mathcal{A}$, $\mathcal{A}\subseteq C(E)\Rightarrow C\subseteq C(E)$

因为 $B\in C(E)\iff E\in C(B)$

$\forall B\in C$, $\mathcal{A}\subseteq C(B)$. So if $E,F\in C$, $E\setminus F$, $E\cap F$ are also in C

$E\in C(B)\iff B\in C(E)$ $\forall B$,

$\Rightarrow C$ is an algebra.

Also we have C is a MC. So C is a σ-algebra.

$(X,\mathcal{M},\mu)$, $(Y,\mathcal{N},\nu)$

Recall: $\mathcal{M}\otimes\mathcal{N}=\langle\{E\times F\}\rangle$ $E\in\mathcal{M}$, $F\in\mathcal{N}$.

Q: $(X\times Y,\mathcal{M}\otimes\mathcal{N},?)$

Monotone Class Lemma: $\mathcal{A}$ algebra, $\langle \mathcal{A} \rangle_c = \langle \mathcal{A} \rangle$

1) Rectangles form a elementary family

$\Rightarrow$ Disjoint union of rectangles form an algebra.

$\mathcal{A} = \{ \sqcup_i E_i \times F_i \mid E_i \in \mathcal{M}, F_i \in \mathcal{N} \}$.

$\mu \times \nu (A \in \mathcal{A}) = \sum_{i=1}^{N} \mu(E_i) \nu(F_i)$

Lemma: $\downarrow$ is a pre-measure.

Pf: $\mu \times \nu(\emptyset) = 0$ ✓

Let $E \times F = \sqcup_i E_i \times F_i$

$\chi_{E \times F}(x,y) = \chi_E(x) \cdot \chi_F(y)$

$= \sum_i \chi_{E_i}(x) \chi_{F_i}(y)$

y constant $\begin{cases} \int \chi_{E\times F}(x,y) \, d\mu = \chi_F(y) \mu(E) \\ \int \sum_i \chi_{E_i}(x) \chi_{F_i}(y) \, d\mu = \int \left(\sum_i \chi_{E_i}(x) \right) \chi_{F_i}(y) \, d\mu \end{cases}$ $\qquad \int \chi_F(y) \mu(E) \, d\nu = \mu(E) \nu(F)$

$\overset{MCT}{\downarrow}$ $= \sum_i \left(\int \chi_{E_i}(x) \, d\mu \, \chi_{F_i}(y) \right)$

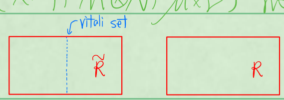
$= \sum_i \left(\mu(E_i) \chi_{F_i}(y) \right)$

$\int \sum_i \left(\mu(E_i) \chi_{F_i}(y) \right) d\nu \overset{MCT}{=} \sum \mu(E_i) \int \chi_{F_i}(y) \, d\nu = \sum \mu(E_i) \nu(F_i)$.

So we have $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ by Hahn-Kolmogorov

Notice: $\mu \times \nu$ is uniquely defined if μ, ν are σ-finite.

Remark: 1) $(X \times Y, \mathcal{M} \otimes \mathcal{N}, \mu \times \nu)$ not necessarily complete. (so we need to compute it)



$m(\longrightarrow) = \mu(0,1) \times \nu(\{y\}) = 0$.

$\mu \times \nu (R - \tilde{R}) = 0$. But $\tilde{R}$ not mble.

Prop: a) If $E \in \mathcal{M} \otimes \mathcal{N}$, $\boxed{\forall x \in X, E_x \in \mathcal{N}. \ \forall y \in Y, E^y \in \mathcal{M}}$ *

b) If f is $\mathcal{M} \otimes \mathcal{N}$, then f_x is $\mathcal{N}$-mble, $\forall x \in X$, y same.

$(E \in \mathcal{M} \otimes \mathcal{N}$, $E_x = \{ y \in Y \mid (x,y) \in E \}$ x fixed.

Pf: a) Let $\mathcal{R}$ be the collection of subsets $E \subseteq X \times Y$, s.t. * holds

WTS: 1) $\mathcal{R}$ contains rectangles
2) $\mathcal{R}$ is a σ-algebra.
$\Rightarrow \mathcal{M} \otimes \mathcal{N} \subseteq \mathcal{R}$

1) holds: $E = A \times B$, $E_x = \begin{cases} B & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}$ same for y.

2): $\mathcal{R}$ is σ-algebra: $E \in \mathcal{R} \Rightarrow E^c \in \mathcal{R}$:

$\forall x \in X$, $(E^c)_x = \{y \in Y \mid (x,y) \in E^c\} = \{y \in Y \mid (x,y) \notin E\} = \{y \notin E_x\} = E_x^c$

Countable union is the same.

b) f_x is $\mathcal{N}$-mable $\Leftrightarrow f_x^{-1}(A) \in \mathcal{N}$

$(f_x)^{-1}(A) = [\underbrace{f^{-1}(A)}_{\in \mathcal{M}\otimes\mathcal{N}}]_x \in \mathcal{N}$

Remark: Construction extends to $\mu_1 \times \mu_2 \times \cdots \times \mu_n$

In particular, Lebesgue in $\mathbb{R}^n$ (after completing)

Thm: (Fubini-Tonelli) Let $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ σ-finite.

The measure using here is before completion, which garentees g, h to be measurable.

1) Tonelli: If $f \in L^+(X \times Y)$, then $g(x) = \int f_x \, d\nu$, $h(y) = \int f^y \, d\mu$ are in $L^+(X)$, $L^+(Y)$, respectively. And $\int f \, d(\mu \times \nu) = \int g(x) \, d\mu = \int h(y) \, d\nu$

or, $\boxed{\int f \, d(\mu \times \nu) = \int \left(\int f(x,y) \, d\nu \right) d\mu = \int \left(\int f(x,y) \, d\mu \right) d\nu}$ **

2) Fubini: If $f \in L^1(X \times Y)$, then for almost every x, $f_x \in L^1(\nu)$

for a.e. y, $f^y \in L^1(\mu)$, and ** holds.

Thm: (Baby Tonellino)

Suppose $(X, \mathcal{M}, \mu)$, $(Y, \mathcal{N}, \nu)$ are σ-finite, let $E \in \mathcal{M} \otimes \mathcal{N}$, then

$x \overset{g}{\longmapsto} \nu(E_x)$, $y \overset{h}{\longmapsto} \mu(E^y)$ are measurable fns, and

$\mu \times \nu(E) = \int \mu(E^y) \, d\nu = \int \nu(E_x) \, d\mu$

Pf: Prove it first if X, Y are finite measure spaces.

Let $C \subseteq \mathcal{M}\otimes\mathcal{N}$ be collection of E s.t. the statement holds.

NTS: C is a monotone class, contains disjoint union of rectangles.

($\Rightarrow \mathcal{M}\otimes\mathcal{N} \subseteq C$ by MCT)

<u>Claim:</u> $E = \underset{\mathcal{M}}{A} \times \underset{\mathcal{N}}{B} \in C$

$\nu(E_x) = \nu(B)\chi_A(x)$. $\mu(E^y) = \mu(A)\chi_B(y)$

$\mu(A)\nu(B) = \mu\times\nu(E) = \int \mu(A)\chi_B(y)\,d\nu = \int \nu(B)\chi_A(x)\,d\mu$

<u>Claim:</u> C is a MC. Let $E_n \in C$, $E_n \subseteq E_{n+1}$

$E = \bigcup_n E_n$. Let $h_n(y) = \mu((E_n)^y)$, $g_n(x) = \nu((E_n)_x)$

Notice $h_n(x)$ is increasing, mable, $h_n \longrightarrow h$ ptwise, so h mable.

By MCT, $\int h = \lim \int h_n d\nu = \lim \int \mu(E_n^y)\,d\nu = \lim \mu\times\nu(E_n) = \mu\times\nu(E)$

Same for g.

<u>Prove:</u> C is closed by decreasing sequence using DCT, using

the fact that $g(x) \le \nu(Y) < \infty$, $\forall x$.

$\Rightarrow g(x)$ integrable b/c $\mu(X) < \infty$

$\rightsquigarrow$ + DCT + convergence from above $\Rightarrow C$ is a MC.

NTS Tonelli: Tonellimo $\Rightarrow$ Tonelli for $f = \chi_E$, and

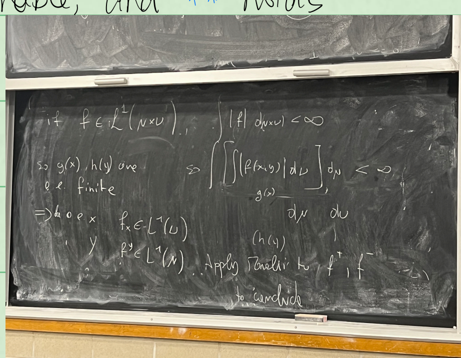
for simple functions.

Let $\varphi_n \nearrow f$. Let $g_n(x) = \int (\varphi_n)_x d\nu$, $h_n(y) = \int (\varphi_n)^y d\mu$

then by MCT, $g_n \nearrow g$, $h_n \nearrow h$ ptwise.

so g, h a mable, and ** holds

If $f \in L^1$,



$\mathcal{L}^1(\mu)$: $f \longmapsto \int |f| d\mu$ is a norm.

Linear Space: $v+w \in X$, $\lambda v \in X$ $\lambda \in K$ ($\mathbb{R}/\mathbb{C}$)

semi-norm: $p: X \to \mathbb{R}_{\geq 0}$ homogeneous, trig ineq.

is a norm iff $p(x)=0 \Leftrightarrow x=0$

We denote norms by $\|\cdot\|$

$(X, \|\cdot\|)$ is Banach if it's complete wrt metric induced by $\|\cdot\|$

Prop: $(X, \|\cdot\|)$ is Banach iff every absolute convergent series is convergent

Recall: We showed that in $\mathcal{L}^1$ holds $\Rightarrow$ $\mathcal{L}^1$ banach.

If $(X, \|\cdot\|_X)$, $(Y, \|\cdot\|_Y)$ are Banach spaces, then $(X \times Y, \|\cdot\|_{X\times Y})$ is normed.

say $\|(x,y)\|_{X\times Y} = \max\{\|x\|, \|y\|\}$.

Linear operator: $T(\lambda x + \mu y) = \lambda T(x) + \mu T(y)$ $\{T: X \to Y\} = \mathcal{L}(X,Y)$.

$T \in \mathcal{L}(X,Y)$ is bounded if $\exists C>0$, s.t. $\forall x \in X$, $\|Tx\|_Y \leq C\|x\|_X$

equivalently

Prop: T is bounded if ① $\sup\limits_{\|x\|=1} \|Tx\|_Y < \infty$ ② $\sup\limits_{\|x\|\leq 1} \|Tx\|_Y < \infty$ ③ $\sup\limits_{x \in X} \dfrac{\|Tx\|_Y}{\|x\|_X} < \infty$

Prop: TFAE:

① T is continuous (wrt d_X, d_Y)

② T is continuous at 0

③ T is bounded.

Graph of T: $T: X \to Y$

$$X \times Y \supseteq \Gamma_T = \{(x,y).\ s.t.\ y = Tx\}.$$

Note: $L(X,Y)$ is itself a linear space. $T, T' \in L(X,Y)$, $\lambda \in K$

$$(T+T')(x) = Tx + T'x \qquad (\lambda T)x = \lambda(Tx)$$

We can define $L(X,Y)$ with $\|\cdot\|_{L(X,Y)}$ as follows:

$$\|T\|_{L(X,Y)} = \sup_{\|x\| \le 1} \|Tx\|_Y$$

Lemma: On bounded $B(X,Y) \subseteq L(X,Y)$. $\|\cdot\|_L$ is a norm.

Prop: If $(Y, \|\cdot\|_Y)$ Banach, then $(B(X,Y), \|\cdot\|)$ is Banach.

Pf: If T_n is Cauchy, then $T_n \to T \in B$.

$\exists T_{n_j}$, s.t. $\|T_{n_{j+1}} - T_{n_j}\| < 2^{-j}$

$\forall x$, $\overset{*}{T_{n_0}x + \sum_{j=0}^{N-1} (T_{n_{j+1}} - T_{n_j})x}$ is absolutely convergent (in Y)

$$Tx = \lim_{N\to\infty} *$$

Show $\|T\| = \lim_{n\to\infty} \|T_n\|$

Given X, let $B(X,K) = X^*$ be the space of Linear functionals called the dual space to X.

In L^1, $\int \in B(L^1, K)$ $f \mapsto Tf = \int f$. $\|Tf\| = |\int f| \le \int |f| = \|f\|$ So $\int \in B$

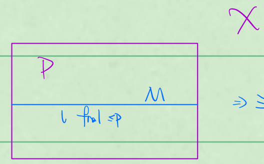
$\|T\| = 1$.

Observe: X^* is always Banach

Def: Let X be a vector space, $p: X \to \mathbb{R}$ is a sublinear functional on X

if $\forall x, x' \in X$, $\lambda \ge 0$,

$p(x+x') \le p(x) + p(x')$, and $p(\lambda x) = \lambda p(x)$



$\Rightarrow \exists\, \tilde{l}$ on whole X $\le p$.

Thm: (Hahn-Banach):

Let X be a $\mathbb{R}$-vector space, p a sublinear functional on X, $M \subseteq X$ a subspace, l a linear functional on M s.t. $l(x) \le p(x)$, $\forall x \in M$.

Then $\exists\, \tilde{l}: X \to \mathbb{R}$ linear s.t. $\tilde{l}(x) \le p(x)$, $\forall x \in X$, $\tilde{l}|_M = l$. $*$

Idea: Extend l to some bigger space.

Zorn's Lemma: Let S be partially ordered, s.t. every totally ordered subset has an upper bound

(i.e, $S' \subset S$ totally ordered, $\exists\, y \in S$, s.t. $\forall x \in S'$, $y \ge x$)

then $\exists$ maximal element in S

(i.e. $\exists\, \tilde{y} \in S$, s.t. if $y \ge \tilde{y} \Rightarrow y = y'$)

Pf: Let $M \subsetneqq X$ subspace, let $x \in X \setminus M$, then $\exists\, l'$ extension of l to $M + \mathbb{R}x = M'$, s.t. $*$ holds.

$S = \{$ family of all linear functionals defined on some subspace $N \supset M$, which satisfies $*$ $\}$

Recall : given $l \in S$, $\Gamma_l = \{(x, l(x)) \mid x \in N\} \subseteq N \times \mathbb{R} \subseteq X \times \mathbb{R}$.

So $l_1 \le l_2 \overset{def}{:=} \Gamma_{l_1} \subseteq \Gamma_{l_2}$

If $(\Gamma_\alpha)_{\alpha \in A} \nearrow$, $\bigcup_{\alpha \in A} \Gamma_\alpha$ is its upper bound.

Proof of the claim: Let $x \in X \setminus M$, define $\tilde{l}$ on $M + \mathbb{R}x \ni z = y + \lambda x$

Note: if $y_1, y_2 \in M$, $l(y_1) + l(y_2) = l(y_1 + y_2) \le p(y_1 + y_2) \le p(y_1 - x + x + y_2) \le p(y_1 - x) + p(y_2 + x)$

$\Rightarrow l(y_1)-p(y_1-x)\le p(x+y_2)-l(y_2)$

$\forall x\in X,\quad \sup_{y\in M}(l(y)-p(y-x))\le \inf_{y\in M}(p(x+y)-l(y))$

$\Rightarrow \exists\alpha,\quad \sup_{y}(l(y)-p(y-x))\le\alpha\le\inf_{y}(p(x+y)-l(y))$

$\tilde{l}(z=y+\lambda x):=l(y)+\lambda\alpha$ $\quad \tilde{l}$ is linear. $\tilde{l}|_M=l$

NTS $\tilde{l}(x)\le p(x)$ $\qquad \lambda=0$ ✓

① if $\lambda>0$, $\tilde{l}(y+\lambda x)=l(y)+\lambda\alpha=\lambda(l(\frac{y}{\lambda})+\alpha)$

$$\le\lambda(l(\tfrac{y}{\lambda})+p(x+\tfrac{y}{\lambda})-l(\tfrac{y}{\lambda}))$$

$$=\lambda p(x+\tfrac{y}{\lambda})=p(y+\lambda x).$$

② $\lambda<0$ same.

<u>$\mathbb{C}$ Hahn-Banach:</u> X a $\mathbb{C}$-space, p is a semi-norm.

<u>Prop:</u> Let $(X,\|\cdot\|)$ be a normed vector space. Then:

i) if $M\subseteq X$ closed subspace, $x\in X\setminus M$, $\exists l\in X^*$ s.t.
$l|_M=0$, $\quad l(x)\ne0$. (In fact let $\delta=\inf_y\|x-y\|$, then
$\exists l,\ l(x)=\delta$)

ii) if $x\ne0$, $\exists l\in X^*$ s.t. $\|l\|=1$, $l(x)=\|x\|$

iii) $\forall x,y\in X$, $\exists l\in X^*$, s.t. $l(x)\ne l(y)$

iv) if $x\in X$, let $\hat{x}:X^*\to\mathbb{C}$. $\hat{x}(l)=l(x)$

then $x\mapsto\hat{x}$ is a linear isometry from X to X^{**}

i.e. $\|\hat{x}-\hat{y}\|_{X^{**}}=\|x-y\|_X$

Let $\hat{X}=\{\hat{x}, \text{ s.t. } x\in X\}\subseteq X^{**}$. Let $\overline{\hat{X}}\subseteq X^{**}$

then $\overline{\hat{X}}$ is called the completion of X.

If X is Banach, then $\overline{\hat{X}}=\hat{X}$, but still $\hat{X}\subsetneq X^{**}$

equality holds for finite dim space.

if X Banach is s.t. $\hat{X} = X^{**}$ X is called Reflexive.

Def: Let X metric space, $S \subseteq X$ is nowhere dense if $\bar{S}$ does not contain any ball in X

S is meager if it is countable union of nowhere dense sets.

Thm: (Baire Category Thm)

If X is complete, $S \subseteq X$ meager, then S does not contain any ball

Thm: (Uniform boundedness principle)

Let X, Y normed vector space. let $A \subseteq B(X,Y)$

a) If $S \subseteq X$ non-meager, s.t. $\sup_{T\in A} \|Tx\| < \infty$, $\forall x \in S$, then $\sup_{T\in A} \|T\| < \infty$

b) If X Banach, if $\sup_{T\in A} \|Tx\| < \infty$, $\forall x \in X$, then $\sup_{T\in A} \|T\| < \infty$

Pf: a) $\xRightarrow{BCT}$ b) so NTS a)

Define $E_n = \{x \in X \mid \sup_A \|Tx\| \le n\}$

$$= \bigcap_{T\in A} \{x \in X \mid \|Tx\| \le n\} = \bigcap_{T\in A} T^{-1}(\overline{B(0,n)}) \quad \text{closed in } X.$$

$\bigcup_n E_n \supseteq S$ non-meager.

$\Rightarrow \exists n$. s.t. E_n is somewhere dense.

So $E_n \supseteq \overline{B(x_0, r_0)}$ (E_n closed)

$\Rightarrow \forall T \in A$, $T(\overline{B(x_0,r_0)}) \subseteq \overline{B(0,n)} = \{y \in Y, \|y\| \le n\}$.

want to show $\sup_{T\in A} \|T\| < \infty$

Claim: $\overline{B(0,r_0)} \subseteq E_{2n}$ $\left(\Rightarrow T(\overline{B(0,r_0)}) \subseteq \overline{B(0,2n)} \Rightarrow T(\overline{B(0,1)}) \subseteq \overline{B(0,\tfrac{2n}{r_0})} \right)$

Pf: Let $\|x\| < r_0$, $\forall T \in A$, $\|Tx\| = \|T(x + x_0 - x_0)\| \le \|T(x+x_0)\| + \|Tx_0\| \le n + n = 2n$

E.g: $C_{00} = \{(a_n) \subset \mathbb{C}$, s.t. only finitely many non-zeros$\}$

$\|(a_n)\| = \sup|a_n|$ $\qquad a_n^1 = (1,0,0,\cdots)$

$\qquad\qquad a_n^2 = (1,\frac{1}{2},0,\cdots)$

$\Rightarrow (C_{00}, \|\cdot\|)$ not banach $\quad a_n^3 = (1,\frac{1}{2},\frac{1}{4},\cdots)$

If it is C_0, then T not surjective.

$T: C_{00} \circlearrowleft \quad [T(a_n)] = (\frac{1}{n}a_n)$

$\ker T = 0$. $\quad$ Range $T = C_{00}$ $\quad (T^{-1} = (na_n))$

T bounded since $\sup|\frac{1}{n}a_n| \le \sup|a_n|$. $\boxed{\text{But } T^{-1} \text{ unbounded}}$

Thm: (Open mapping thm):

Let X, Y Banach, $T \in B(X,Y)$ surjective, then T is open

Cor: If $T \in B(X,Y)$ bijective, T^{-1} is bounded

Remark: T is open iff $T(B(0,1))$ contains a ball centered at 0.

$B_r = B(0,r)$

Pf: Let $X = \bigcup_{n=1}^{\infty} B_n$. Since T surjective, $Y = \bigcup_{n=1}^{\infty} TB_n$

Y is Banach, By BCT, $\exists n$, TB_n not nowhere dense.

$TB_n = nTB_1 \Rightarrow TB_1$ not nowhere dense

So $\exists y_0 \in Y$, $r_0 > 0$, s.t. $\overline{TB_1} \supseteq B(y_0, r_0)$

1) Recover the ball centered @ 0

Let $y_1 = Tx_1$, s.t. $\|y_1 - y_0\| < \frac{r_0}{2}$, so that $B(y_1, \frac{r_0}{2}) \subset \overline{TB_1}$

So if $\|y\| < \frac{r_0}{2}$, $\|y\| \le \underbrace{\|y+y_0\|}_{\substack{\in\overline{TB_1}\\ \exists x_n' \to x_1}} + \underbrace{\|y_0\|}_{\substack{\in\overline{TB_1}\\ \exists x_n^2 \to x_2}} \Rightarrow y \in \overline{TB_2}$

So $x_n'+x_n^2 \to x_1+x_2 \in \overline{B_2}$ $\Rightarrow y$

$\Rightarrow \overline{TB_2} \supseteq B_{\frac{r_0}{2}} \Rightarrow \overline{TB_1} \supseteq B_{r_0/4}$

2) Show $TB_1 \supset B_{r'}$ $\quad$ Let $\rho = \frac{r_0}{4}$, then $\forall n$, $\overline{TB_{2^{-n}}} \supseteq B_{\rho 2^{-n}}$

Let $\|y\| < \frac{\rho}{2}$, WTS $\exists x \in B_1$, s.t. $Tx = y \Rightarrow TB_1 \supseteq B_{\frac{\rho}{2}}$

$\exists x_1 \in B(0,\frac{1}{2})$, s.t. $\|y - Tx_1\| < \frac{\rho}{4}$

$\exists x_2 \in B(0,\frac{1}{4})$, s.t. $\|(y-Tx_1) - Tx_2\| < \frac{\rho}{8}$

$\exists x_n \in B(0,2^{-n})$, s.t. $\|y - \sum^{n} Tx_i\| < \rho 2^{-n-1}$

$\|x_n\| < 2^{-n} \Rightarrow \sum x_i$ converges to $x \in B_1$

$\Rightarrow Tx = y.$

So $\forall y \in B_{\rho/3}$, $\exists x \in B_1$, s.t. $y = Tx$

$\Rightarrow TB_1 \supseteq B_{\rho/3}$

Given $T \in L(X,Y)$ $\Gamma_T = \{(x,y) \mid y = Tx\}$, graph of T.

Def: T is closed if Γ_T closed. $\|\ \|_{X\times Y} = \max\{\|\cdot\|_X, \|\cdot\|_Y\}$.

Thm: If X, Y banach, T closed $\Rightarrow$ T bounded.

Pf: π_X, π_Y projections. Notice: $\pi_X \in B(X\times Y, X)$ π_Y same

Since X, Y Banach, so is $X \times Y$, since Γ_T closed, then

Γ_T is banach. $\pi_X : \Gamma_T \to X$ bijective. By OMT, π_X^{-1} bounded.

$\Rightarrow T = \pi_Y \circ \pi_X^{-1} \in B(X,Y)$ ∎

Remark: If E nowhere dense, $(\bar{E})^c$ is dense and open.

Pf of BCT:

$S = \bigcup_n E_n$ (now dense), Assume by contradiction that $\exists B(x_0, r_0) \subseteq S$.

But E_1 is nowhere dense, so $\exists x_1 \in B(x_0, r_0/10)$, r_1 s.t. $B(x_1, r_1) \subseteq (\bar{E}_1)^c$

Assume that $B(x_1, r_1) \subseteq B(x_0, r_0/10)$ disjoint from E_1

$\exists x_2, r_2$, s.t. $B(x_2, r_2) \subseteq B(x_1, r_1/10)$, disjoint from E_1, E_2.

..... $\exists$ sequence x_m, r_m, s.t. $r_m \le r_{m-1}/10$, $d(x_m, x_{m+1}) \le 2r_{m-1}/10$ $\Rightarrow x_m$ cauchy.

So $x_m \to x \in X$, $x \in B(x_0, r_0)$. But x disjoint from every E_n. ∎

Let X^* dual of X, X^{**} double dual $\supset X$.

Def: $x_n \xrightarrow{\text{weakly}} x$ if $\forall l \in X^*$, $l(x_n) \to l(x)$

$l_n \xrightarrow{\text{weakly}} l$ if $\forall \varphi \in X^{**}$, $\varphi(l_n) \to \varphi(l)$

$l_n \xrightarrow{\text{weak-}*} l$ if $\forall\, x \in X \subseteq X^{**}$, $x(l_n) = l_n(x) \to l(x) = x(l)$

Thm: (Banach–Alaoglu)

If X is normed, the unit ball in X^* is relatively compact wrt weak-$*$ convergence.

Recall: $l \in X^*$, $l : \begin{matrix} X \\ \cup \\ X \end{matrix} \to \mathbb{R} \setminus \mathbb{C}$

$l(f) = \int f \, d\mu$. $f \in L^+(\mu)$

If $f \geq 0$, $l(f) \geq 0$

Cone $\subset X$, s.t. if $x, y \in C$, $x + y \in C$, $\lambda x \in C$, $\lambda \geq 0$

Positive functional if $\forall x \in C$, $l(x) \geq 0$

Def: Signed measures: $\nu : \mathcal{M} \to \mathbb{R} \cup \{-\infty, +\infty\}$, s.t.

1) $\nu(\emptyset) = 0$

2) ν may assume either $+\infty$ or $-\infty$ but not both.

3) If E_j mble, disjoint, then

$\nu(\bigcup_j E_j) = \sum_j \nu(E_j)$. If LHS $< \infty$, then RHS converges absolutely.

Examples: • Let μ_1, μ_2 finite measures on $(X, \mathcal{M})$, then $\mu_1 - \mu_2 = \nu$ is a signed measure.

• Let μ be a measure, $f \in L^1(\mu)$

$E \mapsto \int_E f \, d\mu$ is a signed measure.

Fact: Continuity from above and below work as expected, i.e.

if $E_j \nearrow E$, then $\nu(\bigcup E_i) = \lim \nu(E_i)$

if $E_j \searrow E$, $|\nu(E_1)| < \infty$, then $\nu(\bigcap E_i) = \lim \nu(E_i)$

Def: $E \subseteq \mathcal{M}$ is positive wrt ν if $\forall F \overset{\mathcal{M}}{\subseteq} E$, $\nu(F) \geq 0$.

$E \subseteq \mathcal{M}$ is negative wrt ν if $\forall F \overset{\mathcal{M}}{\subseteq} E$, $\nu(F) \leq 0$.

} closed under countable unions.

$E \subseteq \mathcal{M}$ is null if both positive and negative. $\cancel{\Leftrightarrow}$ $\nu(E) = 0$

Hahn Decomposition Thm:

ν signed measure, $\exists P, N$, $P \cap N = \emptyset$, $P \sqcup N = X$.

P positive, N negative. If $P' N'$ are also as above,

then $P \triangle P'$, $N \triangle N'$ null.

(greedy)

Pf: Assume $\nu \neq +\infty$ (otherwise reverse)

Let $m = \sup_{P \text{ positive}} \nu(P) < \infty$, $\exists P_n$ positive s.t. $\nu(P_n) \to m$

$P = \bigcup_n P_n$, cont from below $\Rightarrow \nu(P) = m$. P is positive.

Let $N = X \setminus P$. NTS N negative.

Claim: N negative.

Fact 1: N does not contain any non-null positive set. (i.e., $S \subseteq N$ positive $\Rightarrow$ null)

Pf 1: Let $A \subseteq N$ positive, non-null.

$A \cap P = \emptyset$, $\nu(A) > 0$, $\nu(A \cup P) = \nu(A) + \nu(P) > \nu(P) = m$ contradiction.

Fact 2: If $A \subseteq N$, $\nu(A) > 0$, then $\exists B \subseteq A$, s.t. $\nu(B) > \nu(A)$

Pf 2: $\exists A' \subseteq A$, $\nu(A') < 0$. Take $B = A \setminus A'$

If $\exists A \subseteq N$ positive, then prove by contradiction.

Want to construct $\{n_j\}$, $\{A_j\}$ as follows:

Let n_0 smallest integer s.t. $\exists A_0$, $\nu(A_0) > n_0^{-1} > 0$

... n_1 ------------ $\exists A_1 \subseteq A_0$, $\nu(A_1) > \nu(A_0) + n_1^{-1}$

$\cdots\ n_k \cdots\cdots\ \exists A_k \subseteq A_{k-1},\ \nu(A_k) > \nu(A_{k-1}) + n_k^{-1}$

Let $A = \cap A_k$. $\infty > \nu(A) = \lim \nu(A_j) > \sum n_k^{-1} \Rightarrow n_k \longrightarrow \infty$

So $\nu(A) > 0$, $\exists B \subseteq A$, s.t. $\nu(B) > \nu(A) + n^{-1}$

So $\exists n_k > n$. Then $B \subset A \subset A_{k-1}$

$\nu(B) > \nu(A) + n^{-1} > \nu(A_{k-1}) + n$. But $n < n_k$, contradiction.

$P \setminus P' \begin{cases} \subset N' \\ \subset P \end{cases} \Rightarrow$ null. So does $P \triangle P'$, $N \triangle N'$

Jordan Decomposition thm:

Let ν signed measure, $\exists!$ positive $\nu^\pm$, s.t. $\nu^+ \perp \nu^-$, $\nu = \nu^+ - \nu^-$

Recall: $\nu \perp \mu$ if $\exists E \cup F = X$, $E \cap F = \emptyset$, $\nu(E) = \mu(F) = 0$

Pf: Let $X = P \sqcup N$ be Hahn decomposition

Let $\nu^+(A) = \nu(A \cap P)$, $\nu^-(A) = -\nu(A \cap N)$

$\nu^+(N) = \nu^-(P) = 0$

Let $\tilde{\nu}^\pm$ be another pair of positive measures,

$\exists E^+ \sqcup E^- = X$, $\tilde{\nu}^+(E^-) = \tilde{\nu}^-(E^+) = 0$.

Since $\nu = \tilde{\nu}^+ - \tilde{\nu}^-$, if $A \subseteq E^+$, $\nu(A) = \tilde{\nu}^+(A) - \tilde{\nu}^-(A) \geq 0$, $B \subseteq E^- \Rightarrow \nu(B) \geq 0$

So E^+ positive, E^- negative. $E^\pm$ form a Hahn decomposition. which is

unique (up to null) $\Rightarrow \nu^\pm = \tilde{\nu}^\pm$ $(E^\pm = P/N)$

$\nu(A \cap E^\pm) = \tilde{\nu}^\pm(A)$

Example: • ν positive, $f \in L^1(\nu)$, $E \mapsto \int_E f\, d\nu$

• Def: $L^1_{ext}(\nu) = \{ f$ mble, either $\int f^+$ or $\int f^- < \infty \}$

If $\nu = \int f\, d\mu$, $|\nu| = \int |f|\, d\mu$.

Def: $|\nu| = \nu^+ + \nu^-$

Let ν signed, μ positive $\nu \ll \mu$ if μ-null $\Rightarrow$ ν-null.

Note: $\nu \ll \mu \Leftrightarrow |\nu| \ll \mu \Leftrightarrow \begin{cases} \nu^+ \ll \mu \\ \nu^- \ll \mu. \end{cases}$

<u>Def</u>: Let $E \in M$, $\mu < \nu$ on $E \Leftrightarrow E$ is a positive set of $\nu - \mu$.

Let ν signed, $L^1(\nu) = L^1(\nu^+) \cap L^1(\nu^-)$

<u>Def</u>: $\int f d\nu = \int f d\nu^+ - \int f d\nu^-$

<u>Fact</u>: Let ν signed, μ positive, Assume $\nu \ll \mu$, $\nu \perp \mu$

then $\nu = 0$ ($\exists E \sqcup F = X$, $\nu(E) = 0 = \mu(F) \Rightarrow \nu(F) = 0$)

<u>Lemma</u>: Let ν a finite signed measure, μ positive.

$\nu \ll \mu \Leftrightarrow \forall \varepsilon > 0, \exists \delta$, s.t. if $\mu(E) < \delta$, $|\nu(E)| < \varepsilon$.

<u>Pf</u>: Notice $\forall E$, $|\nu(E)| = |\nu^+(E) - \nu^-(E)| \leq \nu^+(E) + \nu^-(E) = |\nu|(E)$

We can assume ν is positive

"$\Leftarrow$" is immediate.

"$\Rightarrow$": Assume $\exists \varepsilon > 0$, $\forall n \in \mathbb{N}$, $\exists E_n$, $\nu(E_n) \leq 2^{-n}$, $\nu(E_n) \geq \varepsilon$.

Let $F_i = \bigcup_i^\infty E_n$ $\mu(F_i) \leq 2^{1-i}$, $\nu(F_i) \geq \varepsilon$

$F = \bigcap F_i$, $\mu(F) = 0$, $\nu(F) \geq \varepsilon$. ν finite, contradiction.

<u>Thm</u>: (Lebesgne-Redon-Nikodym): Let ν be σ-finite signed measure

μ σ-finite positive, $\exists$ unique σ-finite signed measure λ, ρ

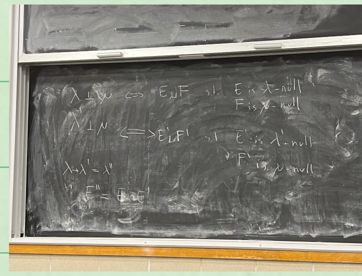
$\lambda \perp \mu$, $\rho \ll \mu$, $\nu = \rho + \lambda$

$\exists f \in L^1_{ext}(\mu)$, s.t. $\rho(E) = \int_E f d\mu$

$\frac{d\rho}{d\mu} := f$ R-N derivative of ρ wrt μ

Lemma: Let ν, μ finite positive. Either $\nu \perp \mu$, or $\exists \varepsilon > 0$, $E \in \mathcal{M}$,

$\mu(E) > 0$, s.t. $\nu \geq \varepsilon \mu$ on E. $**$



Pf: Uniqueness: $\nu = \rho + \lambda = \rho' + \lambda' \Rightarrow \overset{\ll \mu}{\rho - \rho'} = \underbrace{\lambda - \lambda'}_{\perp \mu} = 0$

Step 1: Assume ν, μ finite, prove separately for $\nu^{\pm}$

$0 \in \mathcal{F} = \{f : X \to [0, \infty],\ \int_E f\,d\mu \leq \nu(E)\}$

$\mathcal{F} \neq \emptyset$. Notice if $f, g \in \mathcal{F}$, then $\max\{f, g\} \in \mathcal{F}$

in fact, let $h = \max(f, g)$. Let $A = \{x \mid f > g\}$.

$\int_E h = \int_{E \cap A} h + \int_{E \setminus A} h = \int_{E \cap A} f + \int_{E \setminus A} g \leq \nu(E \cap A) + \nu(E \setminus A) = \nu(E)$

Let $a = \sup_{f \in \mathcal{F}} \int_X f\,d\mu$. Note: $a \leq \nu(X) < \infty$

Let $f_n \in \mathcal{F}$, s.t. $\int_X f_n\,d\mu \to a$.

Let $g_n = \max(f_1, \cdots, f_n)$ $g_n \nearrow$. $g_n \in \mathcal{F}$.

Let $f = \sup_n f_n$ $g_n \nearrow f$, $a \geq \int g_n \geq \int f_n \to a$.

By MCT. $\int f\,d\mu = \lim \int g_n = a$. So

f is our candidate density. $d\rho = f\,d\mu$

NTS $\nu - \rho \perp \mu$. But by $**$, if $\underbrace{\nu - \rho}_{= \lambda} \not\perp \mu$.

$\exists \varepsilon, E, \mu(E) > 0$, s.t. $(\nu - \rho) \geq \varepsilon \mu$ on E.

so $\varepsilon \chi_E\,d\mu \leq d\lambda = d\nu - f\,d\mu \Rightarrow f + \varepsilon \chi_E\,d\mu \leq d\nu$

LRN's Thm: $\forall \nu$ signed, μ positive, both σ-finite.

$\exists! \lambda, \rho$, $\lambda \perp \rho$, $\rho \ll \mu$, $\nu = \lambda + \rho$

$\exists! f \in L^1_{ext}(\mu)$, s.t. $\rho(A) = \int_A f\,d\mu$. $\left(f = \frac{d\rho}{d\mu}\right)$

Lemma: Let μ, ν positive, finite, then either $\nu \perp \mu$ or

$\exists \varepsilon>0$, $\exists E\in\mathcal{M}$, $\mu(E)\neq 0$, s.t. $\nu\geq\varepsilon\mu$ on E.

Pf: Let $P_n \sqcup N_n$ a Hahn decomposition of $\nu-\frac{1}{n}\mu$.

Let $P=\bigcup P_n$, $N=\bigcap N_n$

Then N is negative set for every $\nu-\frac{1}{n}\mu$.

In particular, $\forall N'\subseteq N$, $0\leq\nu(N')\leq\frac{1}{n}\mu(N')$, $\forall n$. $<\infty$

So $\forall N'$, $\nu(N')=0$, N is ν-null.

① If $\mu(P)=0$, then $\mu\perp\nu$

② If $\mu(P)>0$, $\exists\mu(P_n)>0$.

Then $(\nu-\frac{1}{n}\mu)(P_n)>0 \Rightarrow \nu(P_n)>\frac{1}{n}\mu(P_n)$

Lemma: If ν is σ-finite, signed, μ,λ positive,

$\nu\ll\mu$, $\mu\ll\lambda$ $(\nu\ll\lambda)$

a) if $g\in\mathcal{L}^1(\mu)$, then $g\cdot\frac{d\nu}{d\mu}\in\mathcal{L}^1(\nu)$, $\int g\,d\nu=\int g\frac{d\nu}{d\mu}\,d\mu$

b) $\frac{d\nu}{d\lambda}=\frac{d\nu}{d\mu}\frac{d\mu}{d\lambda}$ λ a.e.

Density $=\frac{\text{mass}}{\text{volume}}$ Let $\overset{m}{\ddot{\mu}}$ Lebesgue, $X=\mathbb{R}^n$ $=\frac{\text{mass}(E)}{\text{Leb}(E)}$

Let $\nu\ll m$, $\forall B(x,r)\in\mathcal{L}$, $\lim_{r\to 0}\frac{\nu(B(x,r))}{m(B(x,r))}\overset{?}{=}\frac{d\nu}{dm}$

Def: $L^1_{loc}(m)=\{f:\mathbb{R}^n\to\mathbb{C}\mid\forall U\in\mathcal{L}$ bounded, $\int_U|f|<\infty\}$

Def: Let $f\in L^1_{loc}$, $\forall x\in\mathbb{R}^n$, $r>0$

$(A_rf)(x)=\frac{1}{m(B(x,r))}\int_{B(x,r)}f\,dm.$

Lemma: $(x,r)\longmapsto(A_rf)(x)$ is continuous.

Note: $m(B(x,r))=C_nr^n$, $C_n=m(B(0,1))$

$m(\partial B(x,r))=0$

Pf: ① as $r \to r_0$, $\chi_{B(x,r)} \longrightarrow \chi_{B(x,r_0)}$ converges pointwise everywhere

except (maybe) on $\partial B(x, r_0)$

$B(x,r) \subseteq B(x_0, r_0+1)$

② $\forall \|x - x_0\| < \frac{1}{2}$, $\forall |r - r_0| < \frac{1}{2}$, $|\chi_{B(x,r)}| < \chi_{B(x_0, r_0+1)}$

as $\substack{x_n \to x_0 \\ r_n \to r_0}$, eventually $\chi_{B(x_n, r_n)}$ is dominated by

WTS $(A_{r_n} f)(x_n) \longrightarrow (A_{r_0} f)(x_0)$

$\Updownarrow$

$\int_{B(x_n,r_n)} f \, dm \longrightarrow \int_{B(x_0,r_0)} f \, dm$.

$\| \qquad$ DCT

$\underbrace{\int f \cdot \chi_{B(x_n, r_n)}}_{= f_n \text{ is dominated by } |f| \chi_{B(x_0, r_0+1)}}$

Def: Let $f \in L^1_{loc}(m)$, we define the Hardy-Littlewood maximum function

$[Hf](x) = \sup_{r>0} (A_r f)(x)$

Notice: $[Hf]$ is measurable.

Thm: (Hardy-Littlewood maximal inequality) / (Markov inequality)

$\exists c > 0$, s.t. $\forall f \in L^1$, $\alpha > 0$,

$m\{x : [Hf](x) > \alpha\} \leq \frac{c}{\alpha} \int |f| \, dm$

Vitaly Covering Lemma:

X metric space, $\{B_j\}$ finite collection of open balls

$\exists$ subcollection of disjoint balls $\{B_{j_k}\}$

s.t. $\bigcup_k 3B_{j_k} \supset \bigcup B_j$ (radius)

Pf: (Greedy): Let $A_1 \in \{B_j\}$ largest, A_2 is the largest ball

disjoint from A_1, A_3 largest, disjoint from $A_1, A_2, \cdots$

So we get a subcollection. WTS $\bigcup 3A_i \supset \bigcup B_j$

① If $\{A_i\} = \{B_i\}$ ✓

② If $\exists B \notin \{A_j\}$, let j smallest s.t. $A_j \cap B \neq \emptyset$

then $\text{rad}(B) \le \text{rad}(A_j)$, so $B \subseteq 3A_j$

Cor: Let C be a collection of open balls in $\mathbb{R}^n$, $U = \bigcup_{B \in C} B$.

let $c < m(U)$, $\exists$ finite disjoint subcollection $\{B_j\}$ s.t. $\sum m(B_j) > 3^{-n} c$.

Pf: $\exists K^{\text{cpt}} \subset U$, s.t. $m(K) > c$, extract a finite cover of K, apply Vitaly.

Def: $f \in L^1_{loc}(m)$,

$[Hf](x) = \sup_{r>0} [A_r|f|](x)$ $\quad A_r g(x) = \frac{1}{m(B(x,r))} \int_B g \, dm$

Thm: $\forall f \in L^1$, $\alpha > 0$,

$m(\{x \mid [Hf](x) > \alpha\}) \le \frac{3^n}{\alpha} \int |f| dm$

Pf: (of HLMI)

Let $E_\alpha = \{x \mid [Hf](x) > \alpha\}$.

$\forall x \in E_\alpha$, $\exists r_x > 0$, $A_{r_x}|f|(x) > \alpha$.

$\Leftrightarrow$ $\alpha < A_{r_x}|f|(x) = \frac{1}{m(B(x,r))} \int_B |f| dm$ $\Leftrightarrow$ $\alpha \, m(B(x,r)) < \int_{B_j} |f| dm$.

$\bigcup_x B(x, r_x) \supset E_\alpha$, for any $c < m(E_\alpha)$, $\exists \{x_1, \dots, x_N\} \subseteq E_\alpha$,

s.t. $B_j = B(x_j, r_{x_j})$ are disjoint, $\sum_j m(B_j) > 3^{-n} c$.

$c < 3^n \sum_j m(B_j) < \frac{3^n}{\alpha} \sum_j \int_{B_j} |f| dm < \frac{3^n}{\alpha} \int_{\mathbb{R}^n} |f| dm$

sup.

$\Leftrightarrow$ $m(E_\alpha) = m(\{x \mid [Hf](x) > \alpha\}) \le \frac{3^n}{\alpha} \int |f| dm$

Thm: Let $f \in L^1_{loc}$, then $\lim_{r \to 0} A_r f(x) = f(x)$ a.e.

Pf: If we fix arbitrary $L > 0$ and show the statement holds for $\|x\| < L$, then we are done.

we also can assume $r<1$. So we can replace f with $f\chi_{B(0,L+1)} \in L^1$

By density of cont fns in L^1, $\exists g$ continuous $\in L^1$, s.t. $\int_{\mathbb{R}} |f-g|\, dm < \varepsilon$.

g cont $\Rightarrow \forall x \in \mathbb{R}^n, \forall \delta, \exists r$, s.t. $|y-x|<r \Rightarrow |g(y)-g(x)|<\delta$.

$$|A_r g(x) - g(x)| = \frac{1}{m(B(x,r))} \left|\int_B g(y)-g(x)\,dy\right|$$
$$\leq \frac{1}{m(B(x,r))} \int_B |g(y)-g(x)|\,dy < \delta.$$

So $A_r g(x) \xrightarrow{r\to 0} g(x)$

Now for f:

$$\limsup_{r\to 0} |A_r f(x) - f(x)| = \limsup_{r\to 0} \left| [A_r(f-g)](x) + (A_r g - g)(x) + (g-f)(x) \right|$$
$$\leq \limsup_{r\to 0} \left|[A_r(f-g)](x)\right| + 0 + |f-g|(x)$$
$$\leq \limsup A_r |f-g|(x) + |f-g|(x)$$
$$\overset{\text{HLMI}}{\leq} H|f-g|(x) + |f-g|(x)$$

Let $E_\alpha = \{x \mid \limsup_{r\to 0} |A_r f - f|(x) > \alpha\} \subseteq \{x \mid H|f-g|(x) > \frac{\alpha}{2}\} \cup \{x \mid |f-g|(x) > \frac{\alpha}{2}\}$.

$$m(E_\alpha) \leq \frac{3^n}{\frac{\alpha}{2}} \int |f-g| + \frac{1}{\frac{\alpha}{2}} \int |f-g|$$ Markov ineq.
$$\leq \frac{2(3^n+1)}{\alpha}\varepsilon$$

So $m(E_\alpha) = 0$.

$m\{\limsup_{r\to 0} |A_r f - f|(x) > 0\} = \bigcup_n E_{\frac{1}{n}} = 0$.

So for m.a.e, $\lim_{r\to 0} (A_r f - f)(x) = \frac{1}{m(B)} \int_B f(y) - f(x)\, dy = 0$

<u>Def</u>: Let $f \in L^1_{loc}$. The Lebesgue set of f

$$L_f = \left\{x \mid \lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)-f(x)|\,dy = 0\right\}$$

<u>Thm</u>: If $f \in L^1_{loc}$, then $m(L_f^c) = 0$

In particular, let $E \in \mathcal{B}$, $\chi_E \in L^1_{loc}$

Thm: (Lebesgue Differentiation Thm)

$f \in L^1_{loc}$, $\forall x \in L_f$, $\forall E_r$ shrink to x nicely.

$$\lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y)-f(x)|\,dy = 0 \quad \left(\Rightarrow \lim_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} f(y)\,dy = f(x) \right)$$

Pf: $0 \le \limsup_{r\to 0} \frac{1}{m(E_r)} \int_{E_r} |f(y)-f(x)| \le \frac{1}{m(E_r)} \int_{B(x,r)} |f(y)-f(x)|\,dy \le \frac{1}{\alpha B(x,r)} \int_B |f(y)-f(x)|$ 这行抄错了

Recall: $\lim_{r\to 0} A_r f(x) = f(x)$ a.e.

Pf: Let $c \in \mathbb{R}$, $g_c(x) = |f(x)-c|$. Apply to g_c.

$\exists E_c$ m-null, s.t. if $x \notin E_c$ $A_r g_c(x) \xrightarrow{r\to 0} g_c(x)$

$\forall c \in \mathbb{Q}$, $\bigcup_{\mathbb{Q}} E_c = E$ m-null.

If $x \notin E$, $\forall \varepsilon > 0$, $\exists c$ s.t. $|f(x)-c| < \varepsilon$.

$|f(y)-c+c-f(x)| \le |f(y)-c| + |f(x)-c|$

$$\limsup_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)-f(x)|\,dy$$

$$\le \frac{1}{m(B(x,r))} \int_{B(x,r)} \underbrace{|f(y)-c|}^{g_c(y)}\,dy + \varepsilon$$

$\le 2\varepsilon$.

$$\Rightarrow \lim_{r\to 0} \frac{1}{m(B(x,r))} \int_{B(x,r)} |f(y)-f(x)|\,dy = 0.$$

Def: Borel measure ν is regular if

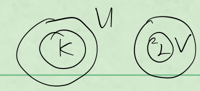
- $\forall K^{cpt}$, $\nu(K) < \infty$
- $\forall E \subseteq M$, $\nu(E) = \inf\{\nu(U) \mid U^{open} \supseteq E\}$

Def: A signed ν is regular if $|\nu|$ is regular.

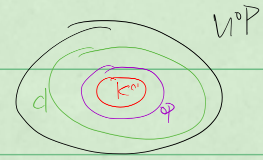
Thm: Let ν signed regular, Let $\nu = \overset{\text{singular}}{\overset{\uparrow}{\lambda}} + \rho_{\ll m}$ (LRN), $\exists! f$ s.t.

$\rho(E)=\int_E f\,dm$, then $\forall E_r$ shrink nicely, m-a.e. x,
$\lim_{r\to 0}\frac{\nu(E_r)}{m(E_r)}=f$

Def: Topological space X is normal if $\forall K^{cl}, L^{cl}$ disjoint, $\exists U^{op}\ V^{op}$ s.t. $K\subseteq U$, $L\subseteq V$



Lemma: (Urysohn): TFAE



① X normal

② $\forall K^{cl}\ U^{op}\supset K$, $\exists V^{op}, L^{cl}$, s.t. $U\supseteq L\supseteq V\supseteq K$.

③ $\forall K^{cl}, L^{cl}$, $K\cap L=\emptyset$, $\exists f\in C(X,[0,1])$, s.t. $f|_K\equiv 1$, $f|_L\equiv 0$

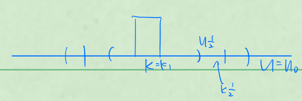
④ If $K^{cl}\subseteq U^{op}$, $\exists f\in C(X,[0,1])$, s.t. $\chi_K\le f\le \chi_U$

Pf: ① ⟺ ② : (choose $L=U^c$)

③ ⟺ ④ (choose $L=U^c$)

③ ⇒ ① : Let $U=f^{-1}(\frac{2}{3},1]$, $V=f^{-1}[0,\frac{1}{3})$

NTS ② ⇒ ④



Let $K_1=K$, $U_0=U$.

By ②, $\exists K_{\frac12}, U_{\frac12}$, s.t. $K_1\subseteq U_{\frac12}\subseteq K_{\frac12}\subseteq U_0$

$\Rightarrow \exists K_1\subseteq U_{\frac34}\subseteq K_{\frac34}\subseteq U_{\frac12}\subseteq K_{\frac12}\subseteq U_{\frac14}\subseteq K_{\frac14}\subseteq U_0$

By induction, $\forall q=\frac{a}{2^N}\in[0,1]$, $U_q\subseteq K_q$.

Moreover: if $0<q'<q<1$, $K_q\subseteq U_{q'}$

Define $f(x)=\sup\{q\mid x\in U_q\}$ $(=\inf\{q\mid x\notin K_q\})$ $\left(\begin{matrix}\inf\emptyset=0\\ \sup\emptyset=1\end{matrix}\right)$

NTS f cont.

$\Leftrightarrow \forall \alpha, f^{-1}([0,\alpha)), \ f^{-1}((\beta,1])$ open

$f^{-1}([0,\alpha)) = \bigcup_{q<\alpha} U_q$ open

$f^{-1}((\beta,1]) = \bigcup_{q>\beta} X \setminus K_q$ is open.

Hausdorff: "Normal, just for points"

Fact: Compact + Hausdorff $\Rightarrow$ Normal

Step 1: If $F^{cpt} \subseteq X^{Haus}$, $\forall x, \exists\, U^{op} \ni x, F \subseteq V^{op}$.

pf: $\forall y \in F, \exists U_y \ni x, V_y \ni y$.

$\bigcup_{y\in F} V_y$ covers F $\Rightarrow$ $\exists$ finite subcover $\bigcup_i V_{y_i} = V^{op}$

$V^{op} \supseteq F$, $\bigcap_i U_{y_i} = U^{op} \ni x$

Step 2: Do the same thing

Def: X is locally compact if $\forall x \in X$ $\exists$ compact neighborhood

$X \supseteq N \ni x$

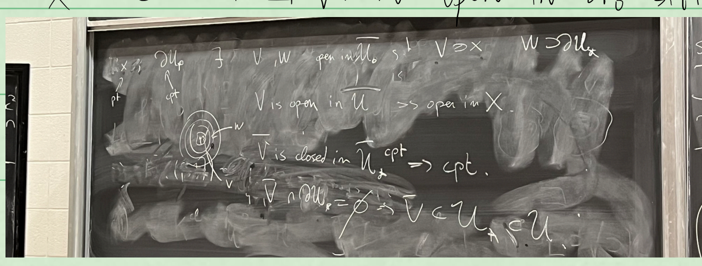
Def: LCH-space is topo X which is locally compact + Hausdorff.

Lemma: $U^{op} \subseteq X^{LCH}$, $\forall x \in U$, $\exists N^{cpt}_{nbhd}$ $N \ni x$, $N \subseteq U$.

Pf: $\exists x \in F_0^{cpt}$ nbhd, Let U_* be $U \cap \operatorname{int} F_0 \ni x$

$\overline{U_*} \subseteq F_0 \Rightarrow \overline{U_*}$ compact

$\overset{pt}{x}$ $\overset{cpt}{\partial U_*}$, $\exists V, W$ open in $\overline{U}_0$ s.t. $V \ni x$, $W \supseteq \partial U_*$



Def: LCH space: Locally compact + Hausdorff

Recall: Compact + Hausdorff $\Rightarrow$ Normal.

Urysohn's Lemma: X normal $\Leftrightarrow$ if $K^{cl} \subseteq U^{open}$, $\exists f \in C^0(X, [0,1])$,

$$\chi_K \le f \le \chi_U.$$

Lemma*: $U^{op} \subseteq X^{LCH}$, $\forall x \in U$, $\exists N^{cpt} \ni x$, $N \subseteq U$.

$\surd$ Def

Lemma: Let $K^{cpt} \subseteq U^{op} \subseteq X^{LCH}$. $\exists V$ open, precompact (closure is cpt), s.t. $K^{cpt} \subseteq V \subseteq \overline{V} \subseteq U$.

Pf: By *, $\forall x \in K$, $\exists x \in N_x^{cpt} \subseteq U$. So $\mathring{N}_x$ open $\Rightarrow \bigcup_{x \in K} \mathring{N}_x \supseteq K$.

Let $V = \bigcup_{i=1}^{n} \mathring{N}_{x_i}$ is open. $\overline{V} = \bigcup_{i=1}^{n} N_{x_i}$ compact, $\overline{V} = \bigcup_{i=1}^{n} N_{x_i} \subseteq U$.

LCH-Urysohn's Lemma: Let $K^{cpt} \subseteq U^{op} \subseteq X^{LCH}$

$\exists f \in C^0(X, [0,1])$, s.t. $f|_K = 1$, f is 0 outside some cpt subset of U.

Pf: By Lemma above, $\exists V^{op}$ s.t. $K^{cpt} \subseteq V \subseteq \overline{V} \subseteq U^{op} \subseteq X^{LCH}$

$\overline{V}$ is normal (cpt + Hausdorff)

$\xRightarrow[\text{Lemma}]{\text{U's}} \exists f \in C^0(\overline{V}, [0,1])$, s.t. $f|_K = 1$, $f|_{\partial V} = 0$.

Extend f to X by setting $f|_{\overline{V}^c} = 0$

Let $E^{cl} \in [0,1]$, if $E \not\ni 0$, $f^{-1}(E) = (f|_{\overline{V}})^{-1}(E)$ closed.

otherwise, $0 \in E$, $f^{-1}(E) = \underbrace{(f|_{\overline{V}})^{-1}(E)}_{\text{closed}} \cup \underbrace{V^c}_{\text{closed}}$

LCH space will be "nice"

Def: Let X be LCH, Borel, μ is Radon if

① $\forall K^{cpt}$, $\mu(K) < \infty$

② $\mu(E) = \inf\{\mu(U) \mid U^{op} \supseteq E \text{ mble}\}$ Outer regularity.

③ $\mu(E) = \sup\{\mu(K) \mid K^{cpt} \subseteq E \text{ open}\}$ Inner regularity.

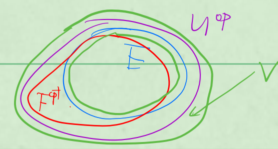
Def: If ③ holds $\forall E$ msble, then μ is regular.

weaker.

(σ-cpt)

Lemma: A Radon measure is inner regular on any σ-finite set.

Def: A set E is σ-compact if countable union of compact sets.



Pf: Let E σ-finite. Assume first E is finite measure.

By ②. $\forall \varepsilon, \exists U^{op} \supseteq E$ s.t. $\mu(U) \leq \mu(E) + \varepsilon$

By ③, $\exists F^{cpt} \subseteq U^{op}$ s.t. $\mu(F) > \mu(U) - \varepsilon$.

$\mu(U \setminus E) < \varepsilon$, so $\exists V^{op} \supseteq U \setminus E$, $\mu(V) < \varepsilon$, $K = F^{cpt} \setminus V^{op}$ is cpt, $K^{cpt} \subseteq E$.

$$\mu(K) = \mu(F) - \mu(F \cap V)$$
$$> \mu(U) - \varepsilon - \mu(F \cap V)$$
$$> \mu(U) - \varepsilon - \varepsilon = \mu(U) - 2\varepsilon.$$

If $\mu(E) = \infty$, $E = \bigcup E_i$, s.t. $\mu(E_i) > i$,

$\exists K_i^{cpt} \subseteq E$ s.t. $\mu(K_i) > i$

If X is σ-compact, then every Radon measure is regular.

Prop: If μ is σ-finite Radon measure on X, E borel.

- $\forall \varepsilon > 0$, $\exists F^{cl} \subseteq E \subseteq U^{op}$ s.t. $\mu(U \setminus F) < \varepsilon$.
- $\exists A^{F_\sigma}$, B^{G_δ}, s.t. $A \subseteq E \subseteq B$, $\mu(B \setminus A) = 0$.

Thm: Let X be LCH, s.t. every open set is σ-compact

then any borel measure s.t. $\forall K^{cpt}$, $\mu(K)<\infty$ is regular.

Remark: Let X be LCH, let μ be borel measure with property ①.

then $\overset{\text{cpt supported cont. fn.}}{\downarrow}$ $C_{cpt}(X) \subseteq L^1(\mu)$. $\int_X f = \int_{K^{cpt}} f \le \int \chi_K \|f\|_\infty$

Lemma: Let l be a positive functional on $C_{cpt}(X^{LCH})$, then

$\forall K^{cpt} \subseteq X$, $\exists C_K$. s.t. $l(f) \le C_K \|f\|_\infty$, $\forall f$ s.t. $\operatorname{supp} f \subseteq K$.

Pf: Let l be $\mathbb{R}$-valued, for $K^{cpt} \subseteq X$, let $U^{op} \supset K$, φ be

as in LCH-Urysohn s.t. $\varphi|_K = 1$, $\varphi|_{V^c} = 0$ for some $V \subseteq U$.

If $\operatorname{supp} f \subseteq K^{cpt}$, $f \le \chi_K \|f\|_\infty \le \varphi \|f\|_\infty$

So $\varphi(x)\|f\|_\infty - f \ge 0$.

So $0 \le l(\varphi(x)\|f\|_\infty - f) = \|f\| \underbrace{l(\varphi)}_{C_K} - l(f)$

$l(f) \le C_K \|f\|$

R-Markov-Kakutani

Thm: (Riesz Representation thm):

Let l be a positive functional on $C_{cpt}(X^{LCH})$,

then $\exists!$ Radon measure X s.t. $l(f) = \int f \, d\mu$.

Moreover, $\mu(U) = \sup\{ l(f) \mid f \in C_{cpt}(X),\ f \prec U\}$ $\forall U^{op} \subseteq X$

$\mu(K) = \inf\{ l(f) \mid f \in C_{cpt}(X),\ f \ge \chi_K\}$ $\forall K^{cpt} \subseteq X$

Def: $f \prec U^{op}$ if $0 \le f \le 1$, $\operatorname{supp}(f) \subseteq U^{op}$.

(bit stronger than $0 \le f \le \chi_U$ ($\Rightarrow \operatorname{supp} f \subseteq \overline{U}$))

Thm: Let X^{LCH} that $\forall$ open $U \subseteq X$ is σ-compact

Then every Borel measure ν finite on cpt sets is regular.

<u>Pf:</u> $C_{cpt}(X) \subseteq L^1(\nu)$

So $l(f) := \int f d\nu$ is positive linear functional.

By RMK, $\exists!$ Radon measure μ s.t. $l(f) = \int f d\mu$.

$\forall U^{op}$, $U = \bigcup_i K_i^{cpt}$ Let $f_1 \in C_{cpt}(X)$ s.t. $f_1 \prec U$.

$f_1 \equiv 1$ on K_1

$f_2 \prec U$. s.t. $f_2 \equiv 1$ on $K_1 \cup K_2$

$f_l \prec U$ s.t. $f_l \equiv 1$ on $\bigcup_{i=1}^{l} K_i$

So $f_l \nearrow \chi_U$ as $l \nearrow \infty$

By MCT, $\mu(U) = \int \lim f_n d\mu = \lim \int f_n d\mu \overset{RMK}{=\!=} \lim \int f_n d\nu = \nu(U)$

So $\nu(U) = \mu(U)$

Let E borel, $\exists F^{cl} \subseteq E \subseteq V^{op}$ s.t. $\nu(V \setminus F), \mu(V \setminus F) < \varepsilon$.

$\Rightarrow \mu(V) \leq \nu(E) + \varepsilon$. ($\nu$ is outer regular)

$\nu(F) \geq \mu(E) - \varepsilon$, F is σ-compact, so $\exists K_j \uparrow F$,

$\nu(K_j) \to \nu(F)$, ν is inner-regularity.

<u>Def:</u> $C_0(X) = \{ f \in C^0(X) \mid \forall \varepsilon, \{|f| \geq \varepsilon\} \text{ is compact} \}$

Completion of C_{cpt} wrt $\|\cdot\|_{sup}$.

<u>Observe:</u> If μ Radon on X, $l(f) = \int f d\mu$ extends to C_0 iff μ is finite.

($\mu(X) = \sup \{ l(f) \mid f \in C_{cpt}(X), f \prec X \}$.)

If l is positive + bounded,

$\Rightarrow \mu$ s.t. $\int f d\mu = l(f)$ is finite Radon.

Thm: (RMK for bdd functionals) X^{LCH}, $l \in (C_0(X,\mathbb{R}))^*$

$\exists! \mu$ Radon finite signed measure s.t. $l(f) = \int f \, d\mu$

$\exists$ injective $\mathcal{R}$ $C_0(X,\mathbb{R})^* \xrightarrow{\text{isometry}} (\text{Radon measures})$

Def: $\mu = \mu^+ - \mu^-$ is Radon $\Leftrightarrow$ $\mu^\pm$ are Radon $\Leftrightarrow$ $|\mu|$ Radon

μ finite Radon signed measures form linear space $\|\mu\| = |\mu|(X)$

Lemma: if $l \in C_0(X,\mathbb{R})^*$, $\exists l^\pm \in C_0(X,\mathbb{R})^*$ positive that $l = l^+ - l^-$

Pf: for $f \geq 0$,

$l^+(f) = \sup\{ l(g) \mid g \in C_0(X,\mathbb{R}),\ 0 \leq g \leq f \}$

$l(g) \leq \|l\| \|g\|_\infty \leq \|l\| \|f\|_\infty$, $l^+(f) \geq 0$

$\Rightarrow$ $0 \leq l^+(f) \leq \|l\| \|f\|_\infty$

Claim: l^+ is linear

$l^+(cf) = c\, l^+(f)$ $\forall c > 0$ from def

Let $0 \leq g_1 \leq f_1$, $0 \leq g_2 \leq f_2$

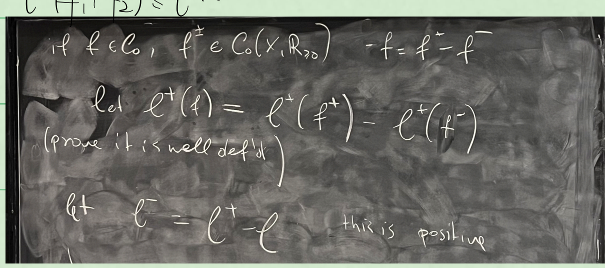
$\Rightarrow$ $0 \leq g_1 + g_2 \leq f_1 + f_2$

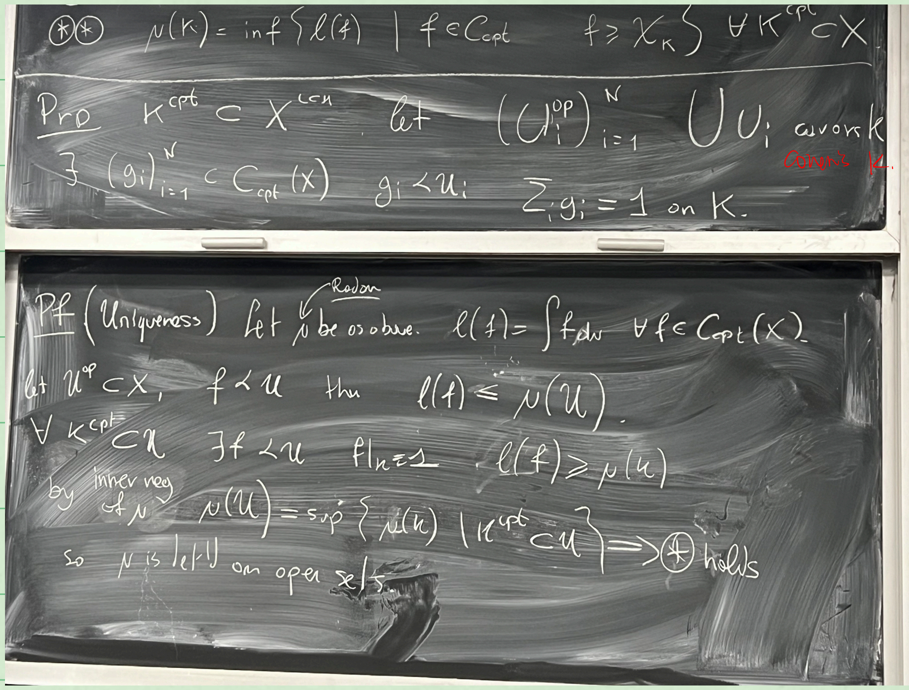
$l^+(f_1 + f_2) \geq l^+(g_1) + l^+(g_2)$

If $0 \leq g \leq f_1 + f_2$ Let $g_1 = \min(g, f_1)$, $g_2 = g - g_1 \leq f_2$

so $l(g) \leq l(g_1) + l(g_2) \leq l^+(f_1) + l^+(f_2)$

$l^+(f_1 + f_2) \leq l^+ \cdots\cdots$





P01.

Outer reg of $\mu \Rightarrow$ determined for any Borel set

Pf: (Existance)

Recall if $\Sigma \subseteq P(X)$ $\rho: \Sigma \to \mathbb{R}_{\geq 0}$ s.t. $\rho(\emptyset) = 0$,

$\mu^*(E) = \inf\{\sum \rho(E_i) \mid E_i \in \Sigma,\ \cup E_i \supset E\}$ is an outer measure

Let $\rho(U) = \sup\{l(f),\ f \prec U\}$, μ^* above. Σ here: open sets.

NTS i) μ^* outer measure

ii) open sets are μ^*-measurable.

iii) μ satisfies **

iv) $\forall f \in C_{cpt}(X)$, $\int f d\mu = l(f)$

i)+ii) $\Rightarrow$ Caratheodory thm $\Rightarrow$ Borel measure μ.

By construction, μ is outer-regular.

iii) $\Rightarrow \mu(K^{cpt}) < \infty$

Let $U^{op} \subseteq X$, let $\alpha < \mu(U)$, choose $f \prec U$, s.t. $l(f) > \alpha$

Let $K^{cpt} = \operatorname{supp} f \subseteq U$.

If $g \in C_{cpt}$, $g \geq \chi_K \geq f$ $\Rightarrow l(g) \geq l(f) > \alpha$

By (**) $\mu(K) > \alpha \Rightarrow$ inner reg of μ.

Pf i): Let $U^{op} = \cup U_i$, Assume $\rho(U) \leq \sum_i \rho(U_i)$, then $\mu^*(E) = \inf\{\sum \rho(U_i)$ s.t. $U_i^{op}, \bigcup U_i \supset E\}$.

$f \in C_{cpt}(X)$, $f \prec U$, $K^{cpt} = \mathrm{supp} f$, then $K \subseteq \bigcup_{i=1}^{N} U_i$

$\exists g_i \prec U_{i_i}$ $\sum g_i = 1$ on K, $\sum g_i f = f$ on K

$g_i f \prec U_{i_i}$

$\ell(f) = \sum_i \ell(g_i f) \leq \sum_i \rho(U_{i_i}) \leq \sum \rho(U_i)$

Recall U is μ^*-mble if $\forall E, \mu^*(E) \geq \mu^*(E \cap F) + \mu^*(E \setminus F)$

Pf ii): i.e. $\forall U^{op}$, $\infty > \mu^*(E) \geq \mu^*(E \cap U) + \mu^*(E \setminus U)$

Assume that E open, so $E \cap U$ open, $\forall \varepsilon$, $\exists f \prec E \cap U$ s.t.

$\ell(f) > \mu^*(E \cap U) - \varepsilon$, $U \setminus \mathrm{supp} f$ is open, $\exists g \prec U \setminus \mathrm{supp} f$

s.t. $\ell(g) \geq \mu^*(E \setminus \mathrm{supp} f) - \varepsilon$, $f + g \prec E$

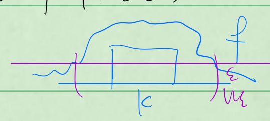
$\mu^*(E) \geq \ell(f+g) = \ell(f) + \ell(g) > \mu^*(E \cap U) + \mu^*(E \setminus U) - \varepsilon - \varepsilon$.

For arbitrary E, $\exists V^{op} \supset E$, s.t. $\mu^*(V) < \mu^*(E) + \varepsilon$

$\mu^*(E) + \varepsilon > \mu^*(V) \geq \mu^*(V \cap U) + \mu^*(V \setminus U)$

$\geq \mu^*(E \cap U) + \mu^*(E \setminus U)$

Pf iii): $K^{cpt} \subset X$, $f \in C_{cpt}$, $f \geq \chi_K$, $\forall \varepsilon$, $U_\varepsilon = f^{-1}(1-\varepsilon, \infty)$



U_ε is open: if $g \prec U_\varepsilon$,

$(1-\varepsilon)^{-1} f \geq g$.

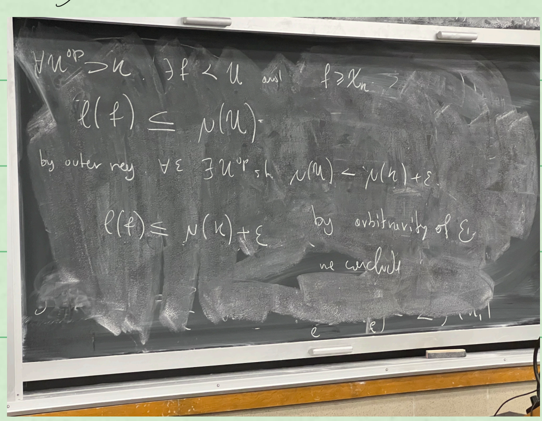
$\ell((1-\varepsilon)^{-1} f) \geq \ell(g)$

$(1-\varepsilon)^{-1} \ell(f) \geq \ell(g)$

$\mu(K) \leq \mu(U_\varepsilon) \leq \ell(f)(1+2\varepsilon)$, which shows $\mu(K) \leq \inf\{\quad\}$.

$\forall U^{op} \supseteq K$, $\exists f \prec U$ and $f \geq \chi_K$,

$\ell(f) \leq \mu(U)$



Pf iv): If $f \neq 0$, assume $\|f\|_\infty = 1$. Given $N > 0$,

for any $1 \le j \le N$, $K_j = \{x \mid f(x) \ge j N^{-1}\}$

$K_0 = \operatorname{supp} f$, $K_0 \supseteq K_1 \supseteq \cdots \supseteq K_N$.

Let $f_1, \cdots, f_N \in C_{cpt}(X)$.

$$f_j = \begin{cases} 0 & \text{outside } K_{j-1} \\ f(x) - (j-1)N^{-1} & \text{on } K_{j-1} \setminus K_j \\ \frac{1}{N} & \text{on } K_j \end{cases}$$

?

$f = \sum f_j$

Observe: $\chi_{K_j} N^{-1} \le f_j \le \chi_{K_{j-1}} N^{-1}$

$$\frac{1}{N}\mu(K_j) \le \int f_j \, d\mu \le \frac{1}{N}\mu(K_{j-1})$$

Notice: if $U^{\varphi} \supset K_{j-1} \Rightarrow N \cdot f_j \prec U \Rightarrow \ell(f_j) \le \frac{\mu(U)}{N}$

$$\frac{1}{N}\mu(K_j) \le \ell(f_j) \le \frac{\mu(K_{j-1})}{N}$$

$$\frac{1}{N}\sum_{j=1}^{N}\mu(K_j) \le \int f \, d\mu \le \frac{1}{N}\sum_{j=0}^{N-1}\mu(K_j)$$

$$\frac{1}{N}\sum_{j=1}^{N}\mu(K_j) \le \ell(f) \le \frac{1}{N}\sum_{j=0}^{N-1}\mu(K_j)$$

$$\Rightarrow \left|\int f \, d\mu - \ell(f)\right| \le \frac{1}{N}\left(\sum_{j=0}^{N-1}\mu(K_j) - \sum_{j=1}^{N}\mu(K_j)\right)$$

$$\le \frac{1}{N}(\mu(K_0) - \mu(K_N))$$

$$\le \frac{1}{N}\mu(\operatorname{supp} f)$$