Valuation of Interest Rate Caps (floors) by Simulation

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In this project, the objective is to find the intrinsic value of an interest rate cap (floor).

Interest Rate Cap

An interest rate cap is a derivative in which the buyer receives payments at the end of each period in which the interest rate exceeds the agreed strike price. An example of a cap would be an agreement to receive a payment for each month the LIBOR rate exceeds a certain level.

Example:

Time	Libor	Payment Rate	Payment
0	1.5%	NA	NA
0.5	1.7%	0%	0
1	2.2%	0.2%	1m*(0.2%/2)
1.5	2.5%	0.5%	1m*(0.5%/2)
2	2.0%	0%	0
Total			1m*0.35%

The total payment received will be 1,000,000*(0.2%/2)+1,000,000*(0.5%/2)=$3,500

If using discount method, the value should be $1{,}000{,}000*\frac{0.2\%}{2}*(1+\frac{1.7\%}{2})^{-0.5}*(1+\frac{1.5\%}{2})^{-0.5}+$

$1{,}000{,}000*\frac{0.5\%}{2}*(1+\frac{2.2\%}{2})^{-0.5}*(1+\frac{1.7\%}{2})^{-0.5}*(1+\frac{1.5\%}{2})^{-0.5}$=$3,459

Since the highly uncertainty of the payment cash flow, it is very hard to calculate the intrinsic value of the interest rate cap by analytical method. Denote V the present value of all future payments, it's a random variable, E(V) is the fair value of the cap. We can use simulation to get an estimate of E(V).

Interest Rate Model

Because V depends on the interest rate path, the first step is to simulate interest rate paths. Unlike stock price, which follows a geometric Brownian motion, interest rate process is more complicated. There are some models proposed by people, such as

- HJM model
- Hull Write model
- Rendleman and Bartter model
- Vasicek model
- Cox, Ingersoll, and Ross model (CIR)

In this project, I am using CIR model (Cox–Ingersoll–Ross model), which is a Markov process with continuous paths defined by the following stochastic differential equation

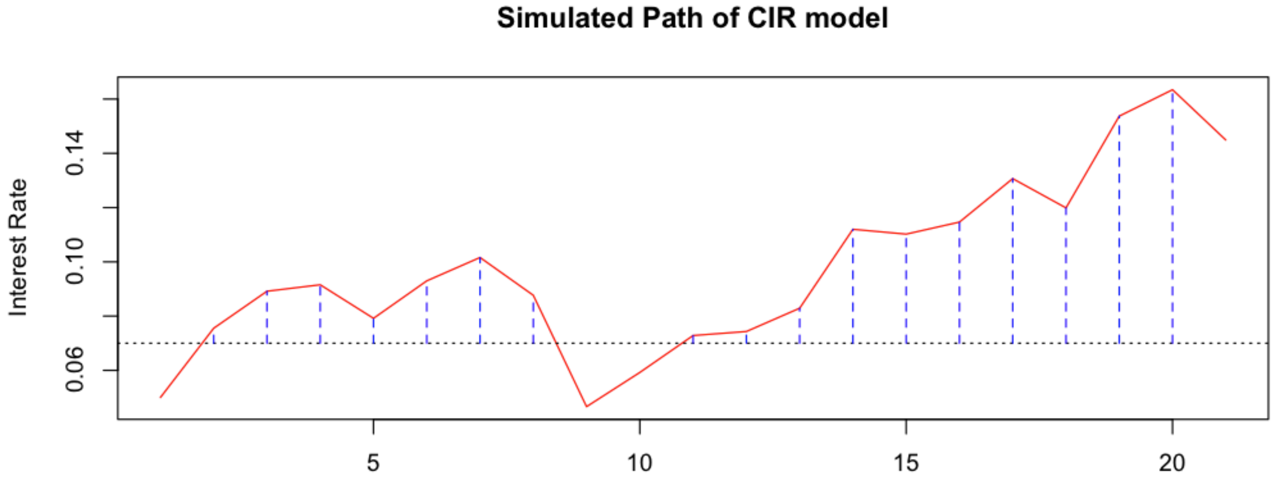
$$dr_t = \theta(\mu - r_t)dt + \sigma\sqrt{r_t}\,dW_t$$

It has some nice properties, like

- Mean Reversion
- Level Dependent Volatility
- The process will never touch zero

Example:

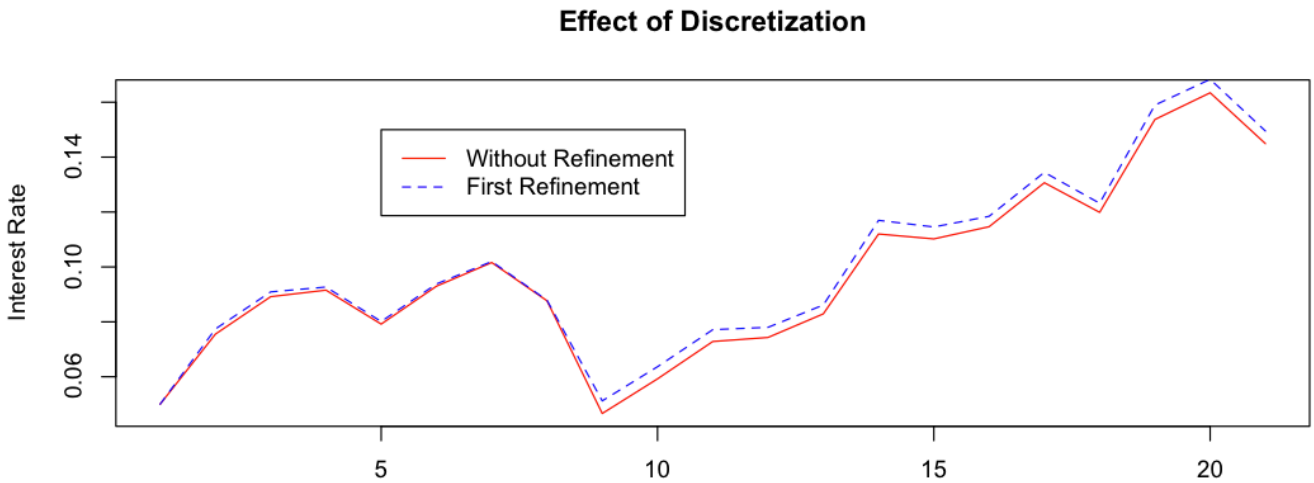
$$with\ r_o = 5\%, k = 7\%, \theta = 0.25,\ \mu = 0.1,\ \sigma = 0.1,\ T = 5,$$
$$and\ quarterly\ \ payment$$



Discretization

One step is a quarter, thus h=0.25. It's a little big such that the approximation is not so accurate, so I use first refinement to make it more accurate.

$$X(t+h) = X(t) + a(X(t))h + b(X(t))\{W(t+h) - W(t)\} + 1/2 * b'(X(t))b(X(t))[\{W(t+h) - W(t)\}^2 - h]$$



After discretization, the path changes a little.

Brownian Bridge

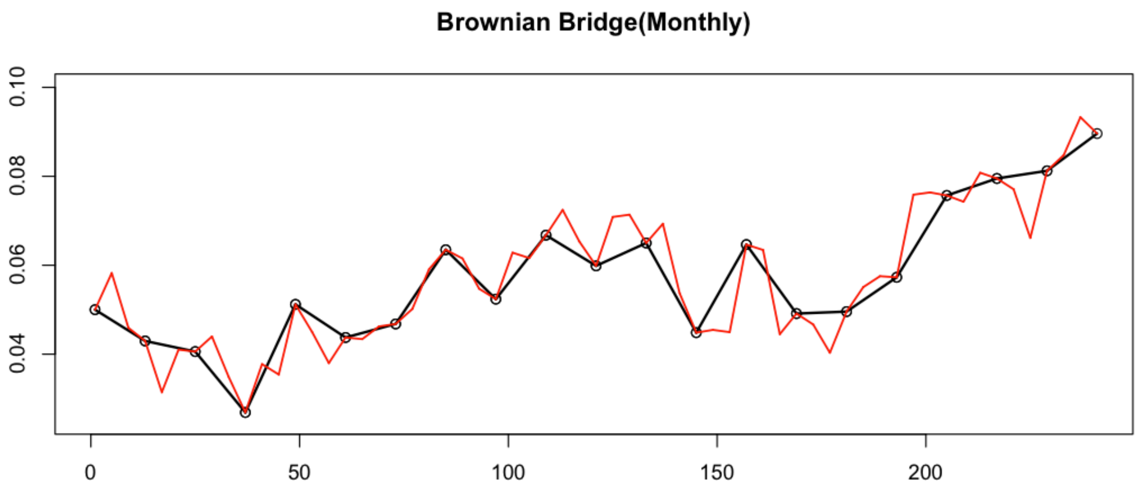
Suppose we have simulated a interest rate path with quarterly step, and we want to change it to monthly or even weekly step.

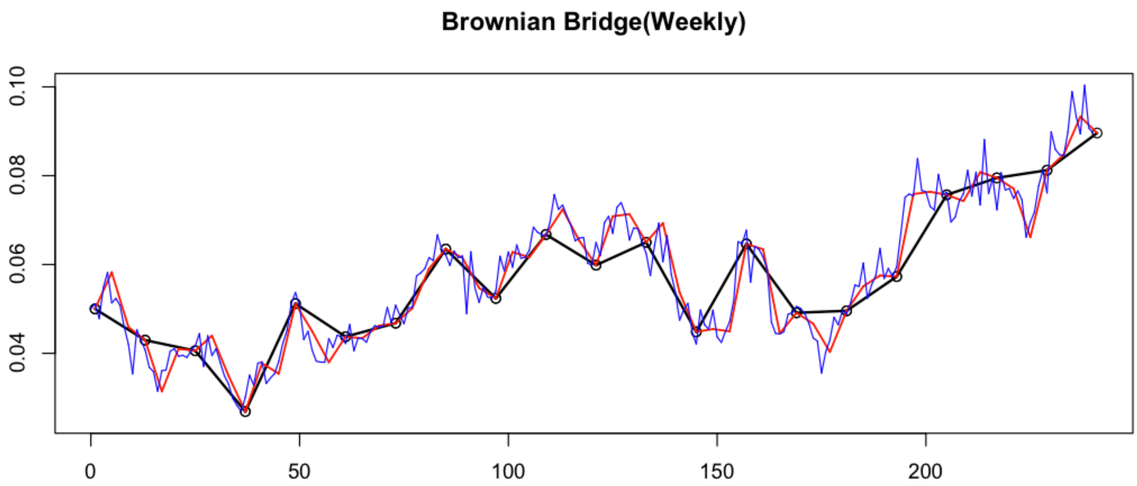
The new knot should also be normal distributed, with

$$E(r(s) \mid r(u), r(t)) = \frac{(t-s)*r(u) + (s-u)*r(t)}{t-u}$$

and

$$Var(r(s) \mid r(u), r(t)) = \frac{(s-u)(t-s)}{t-u}\sqrt{Var(r(u))*Var(r(t))}$$





Jump Diffusion

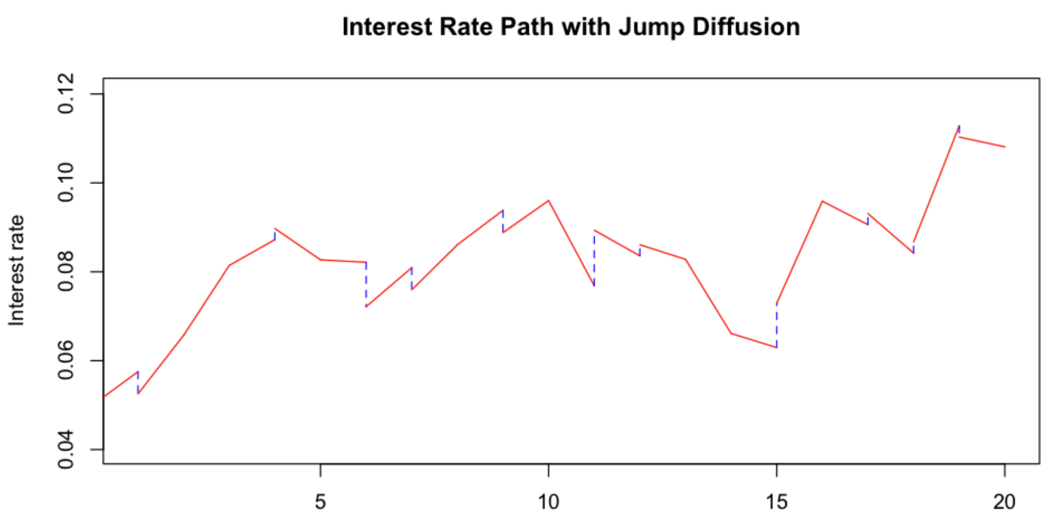
Federal Reserve occasionally changes the Federal Funds rate, and thus influences the underlying interest rate. To capture this kind of event, we can add a jump diffusion term into our model.

$$X(t+h) = X(t) + a(X(t))h + b(X(t))\{W(t+h) - W(t)\} + 1/2 * b'(X(t))b(X(t))[\{W(t+h) - W(t)\}^2 - h] + \sum_{k=1}^{N} Y_k$$

where $N \sim Poisson(\lambda h), \lambda = 4, h = 1/4$

and Y follows the following discrete distribution

Change	+75 bp	+50 bp	+25 bp	-25 bp	-50 bp	-75 bp
Prob	0.05	0.1	0.35	0.35	0.1	0.05

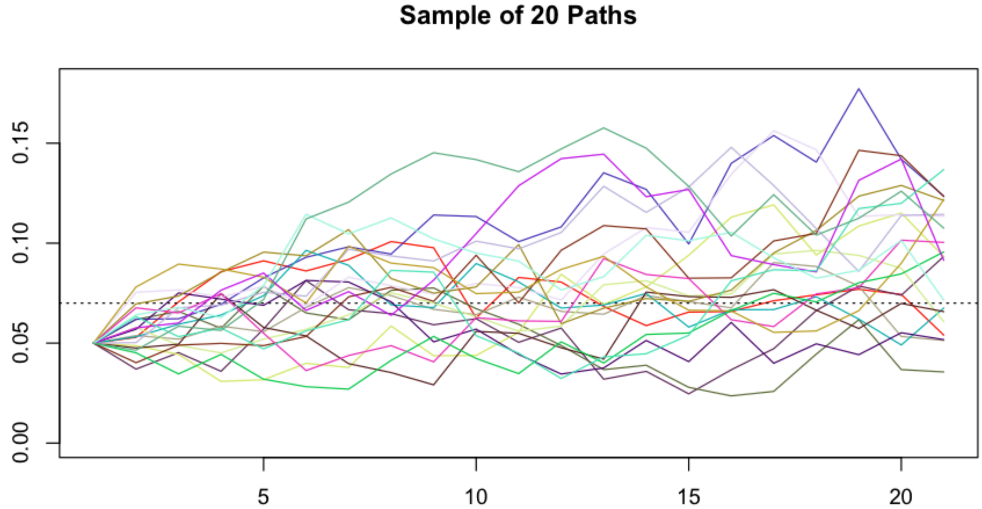


Valuation

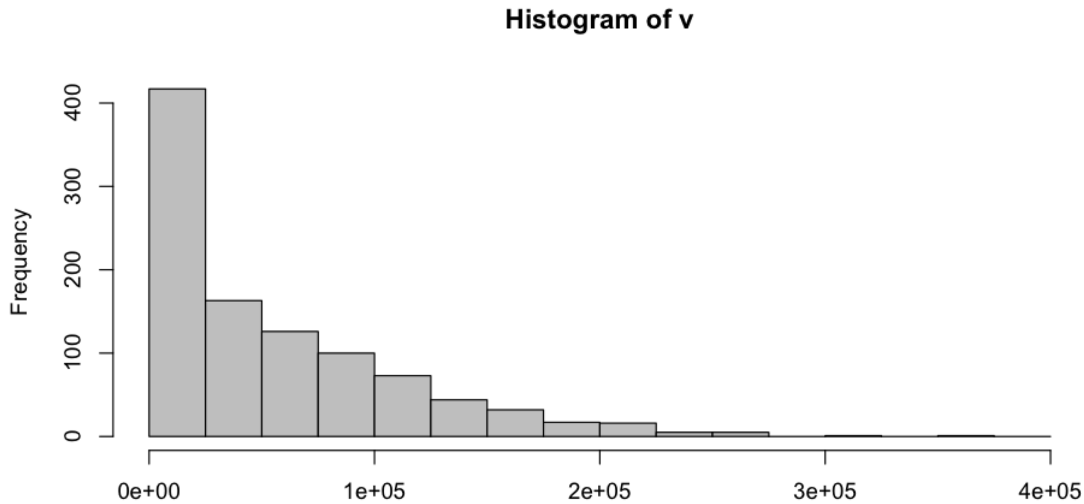
Once we simulate a path, V can be calculated as $V = \sum_{i=1}^{20} \max(0, r_i - 0.07) * 1{,}000{,}000 / 4 * \prod_{j=1}^{i} (1 + \frac{r_j}{4})^{-1}$

where 1million is the notional value, and $\prod_{j=1}^{i} (1 + \frac{r_j}{4})^{-1}$ is the discount factor.

Then, simulate more paths.



In the next step, I simulated 1,000 paths, and get 1,000 values, the histogram plot as below

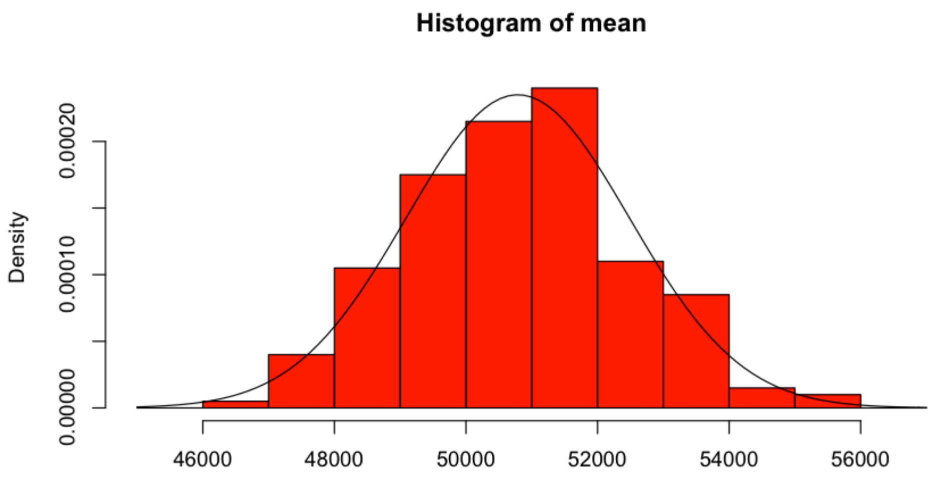


Min.	1st Qu.	Median	Mean	3rd Qu.	Max.
0.0	8393	35470	54560	84970	367200

$E(V) = \bar{V} = 54560$, should be the fair value of the cap.

Variance Reduction

In the previous example, I used the average of 1,000 paths to estimate E(V). It should be noted that the calculation process involve many steps and iterations, so increasing the number of paths is sometimes slow. Thus, the estimator we achieve may have large variation. In this section, I use variance reduction method to improve the efficiency.



Use the previous method to obtain 200 estimators, and plot them together. It can be seen that the span is as large as 10,000, that is roughly 20% of V.

☙ Control Variate

The first method to overcome this problem is to use the R_T, the interest rate at maturity as control variate. It is easy to imagine that when R_T is high, V should also be high. So the correlation between R_T and V is positive, thus control variate method can reduce the variance.

Use $\bar{V}(b) = \bar{V} - b(\bar{r}_T - E(r_T))$ instead of $\bar{V}$

$\bar{V}$ and $\bar{r}$ can be obtained from simulation, and b is the slope coefficient by regression V against R_T.

Also, according to the property of CIR model,

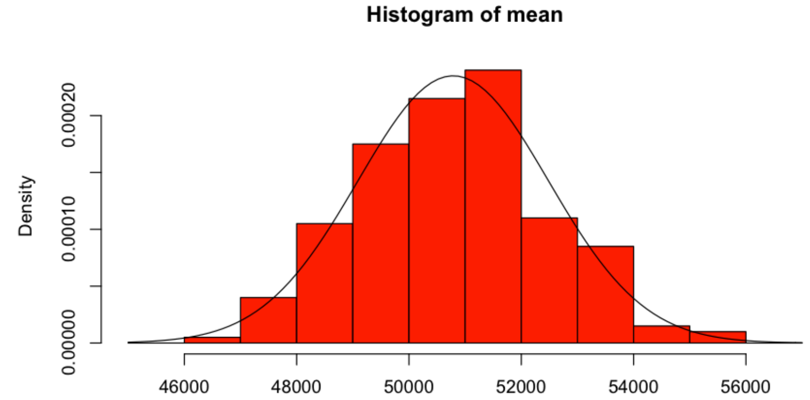
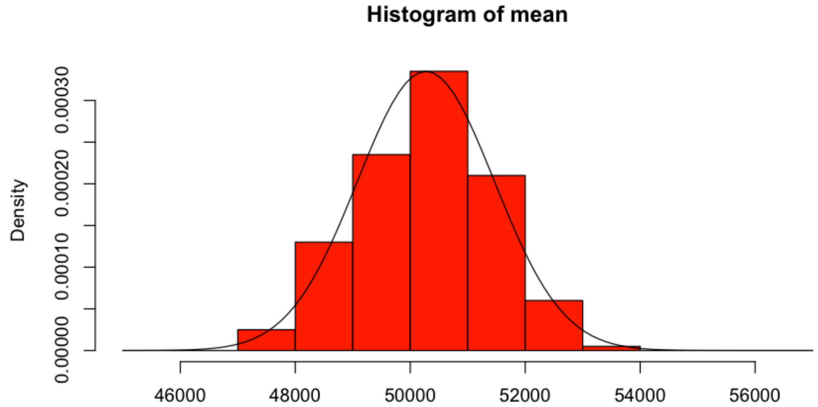
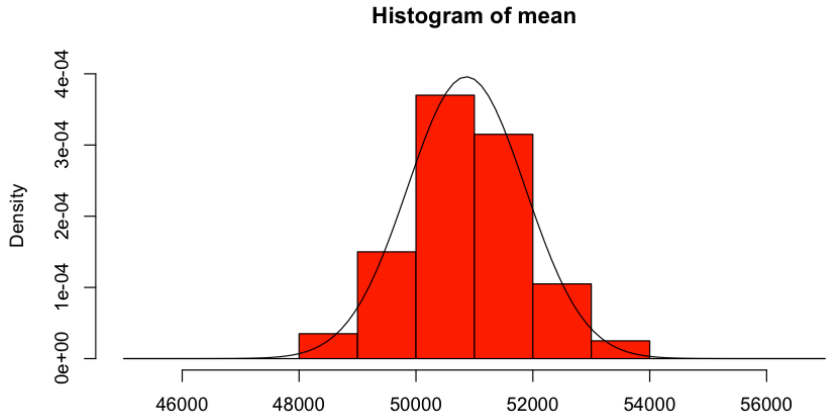
$E[r_t \mid r_0] = r_0 e^{-\theta t} + \mu(1 - e^{-\theta t})$

All parameters are known, so $E[r_T \mid r_0 = 5\%]$=8.57%

☙ Antithetic Variates

Another method is to use antithetic variates. In each step, when W(t) is simulated, I use both W(t) and –W(t) to construct two paths at one time.

↬ Comparison

<table>
<tr><th></th><th>Histogram Plots</th><th>Mean</th><th>S.D</th></tr>
<tr><td>Without Variance Reduction</td><td>
</td><td>50787.63</td><td>1697.624</td></tr>
<tr><td>Control Variate</td><td>
</td><td>50272.27</td><td>1191.321</td></tr>
<tr><td>Antithetic Variates</td><td>
</td><td>50862.46</td><td>1007.721</td></tr>
</table>

By looking at the histogram plots, the estimators of the later two are more concentrated to the middle, and the range is narrowed than the first one. The mean value are very close, while the standard deviations are significantly smaller. Meanwhile, Antithetic Variates method seems to be a little better than Control Variate.