The BTL Model

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1 The Standard BTL Model

Suppose that there are K underlying aspects, the latent preference vector for each user is denoted as $u \in \mathbb{R}+^K$, and $\sum_k u_k = 1$, latent item feature vector $i, j \in \mathbb{R}+^K$.

Base on the standard BTL model:

$$p(i \succ j|u) = \frac{u^T i}{u^T i + u^T j}$$
$$p(i \prec j|u) = 1 - p(i \succ j|u)$$

We use the maximal likelihood estimation. The likelihood function can be written as follows. The model parameters are denoted as $\Theta = \{v \in V, u \in U)\}$, $\succ_d$ represents a pairwise ranking observation in a session d.

$$L(\Theta) = \Pi_d \Pi_{i \succ_d j} \frac{\sum_{k=1}^K u_k i_k}{\sum_{k=1}^K u_k i_k + \sum_{k=1}^K u_k j_k} \tag{1}$$

Thus, the log likelihood and its lower bound is:

$$\begin{aligned} l(\Theta) &= \log L(\Theta) \qquad (2)\\ &= \sum_u \sum_{d \in D_u} \sum_{i \succ_d j} \log \frac{\sum_k u_k i_k}{\sum_k u_k i_k + \sum_k u_k j_k} \\ &\geq \sum_u \sum_{d \in D_u} \sum_{i \succ_d j} [1 - \frac{\sum_k u_k i_k + \sum_k u_k j_k}{\sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t} + \log \frac{\sum_k u_k i_k}{\sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t}] \qquad (3) \end{aligned}$$

where we apply

$$\ln \frac{y}{x} \geq 1 - \frac{x}{y}$$

To maximize the lower-bound in Equ. **??**, we employ conjugate gradient method. We first fix all latent vectors for the item [1], and apply $\log \sum_k u_k i_k \geq$

[1] We use the supscript t for parameters in the t^{th} round

$\frac{\sum_k i_k \log u_k}{\sum_k i_k} + \log \sum_k i_k, \forall i_k \geq 0$. Let's compute $c^t(d,i,j) = \sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t$, $c_k^t(d,i,j) = \frac{i_k^t + j_k^t}{c^t(d,i,j)}$ and $f_k^t(d,i) = \frac{i_k^t}{\sum_k i_k^t}$ for all the pairwise ranking observations in d using the $t-$th round parameters, we have

$$\frac{\partial l}{\partial u_k} = \sum_{d \in D_u} \sum_{i \succ_d j} (\frac{f_k^t(d,i)}{u_k} - c_k^t(i,j,d)) = 0 \tag{4}$$

Here we omit the constraint that $u_k \geq 0$ and hope that this constraint will be satisfied by the solution.

$$u_k = \frac{\sum_{d \in D_u} \sum_{i \succ_d j} \frac{i_k}{\sum_k i_k}}{\sum_{d \in D_u} \sum_{i \succ_d j} \frac{i_k + j_k}{c^t(i,j,d)}} \tag{5}$$

Next we fix all user latent vectors, and again apply the Jensen's inequality $\log \sum_k u_k i_k \geq \frac{\sum_k u_k \log i_k}{\sum_k u_k} + \log \sum_k u_k, \forall u_k \geq 0$ For convenience, we use v to denote an arbitral item, and O_u is the set of all observed pairwise ranking of user u, if v is the winning item then we say $o \in W(v)$. We first compute $e_k^t(u) = \frac{u_k^t}{\sum_k u_k^t}$.

$$\frac{\partial l}{\partial v_k} = \sum_u \sum_{o \in W(v) \& \& o \in O_u} \frac{e_k^t(u)}{v_k} - \sum_u \sum_{o \in O_u, i=v \| j=v} \frac{u_k}{c^t(i,j,d)} = 0 \tag{6}$$

$$v_k = \frac{\sum_u \sum_{o \in O_u, i=v \| j=v} \frac{u_k}{c^t(i,j,d)}}{\sum_u \sum_{o \in W(v) \& \& o \in O_u} e_k^t(u)} \tag{7}$$

Again we omit the constraint that $v_k \geq 0$ because that is satisfied automatically by the solution. We have to renormalize $\forall k, \sum_v v_k = 1$.

Algorithm 1: Standard BTL Algorithm

Input: A set of pairwise rankings D, number of aspects K, maximal number of iterations $MaxIter$

Output: U, V

initialization: Randomize U, V, $\forall k, u_k \geq 0, v_k \geq 0, \sum_v v_k = 1$;

for $t = 1; t \leq MaxIter; t++$ **do**

  Randomize order of observations ;

  for $u \in U$ **do**

    $c^t(d,i,j) = \sum_k u_k^t i_k^t + \sum_k u_k^t j_k^t, f^t(d,i) = \sum_k i_k^t$;

    for $k = 1; k \leq K, k++$ **do**

      $c_k^t(d,i,j) = \frac{i_k^t + j_k^t}{c^t(d,i,j)}, f_k^t(d,i) = \frac{i_k^t}{f^t(d,i)}$;

    end

    $u_k^{t+1} = \frac{\sum_{d \in D_u} \sum_{i \succ_d j} f_k^t(d,i)}{\sum_{d \in D_u} \sum_{i \succ_d j} c_k^t(d,i,j)}$;

  end

  for $v \in U$ **do**

    $e^t(u) = \sum_k u_k^t$;

    for $k = 1; k \leq K, k++$ **do**

      $e_k^t(u) = \frac{u_k^t}{e^t(u)}$;

    end

    $v_k = \frac{\sum_u \sum_{o \in O_u, i=v \| j=v} \frac{u_k}{c^t(i,j,d)}}{\sum_u \sum_{o \in W(v) \&\& o \in O_u} e_k^t(u)}$;

  end

end