Probabilistic Time Series Analysis: Lab 11

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Conceptual Review of Spectral Density Kernels

Let's look at X_t a *discrete* centered stationary gaussian process, over $t \in \mathbb{Z}$, with covariance $\gamma(k)$. Let's write the covariance matrix of $X_{[-N,N]} = (X_t)_{t=-N}^{N}$:

$$\Sigma_N = \begin{bmatrix} \gamma(0) & \gamma(1) & \cdots & \gamma(2N) \\ \gamma(1) & \gamma(0) & \cdots & \gamma(2N-1) \\ \vdots & \vdots & \ddots & \vdots \\ \gamma(2N) & \gamma(2N-1) & \cdots & \gamma(0) \end{bmatrix}.$$

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How would you find the eigenvectors and eigenvalues of this matrix?

$$\lambda_j(\Sigma_N) = \sum_{k=-N}^{N} \gamma(k) e^{-2\pi i \frac{jk}{2N+1}} \approx \sum_{k=-N}^{N} \gamma(k) e^{-\pi i \frac{jk}{N}}.$$

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Let's look at a frequency scaling with N, $j = \omega N$ (as opposed to, say, $j = 5$ fixed as $N \to \infty$). Then,

$$\lim_{N \to \infty} \lambda_{\omega N}(\Sigma_N) = \sum_{k=-\infty}^{\infty} \gamma(k) e^{-\pi i \omega k} = \hat{\gamma}(\omega),$$

nothing but the *discrete-time Fourier transform* of γ (not to be confused with the discrete Fourier transform!).

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So, by specifying $\hat{\gamma}(\omega)$, we directly specify how the limiting spectrum (eigenvalues) of Σ_N will look.

What property of X ensures that we don't need to worry about specifying the eigenvectors?

What Makes $\hat{\gamma}$ a "Density"?

What can we say about the eigenvalues of Σ_N?

1. They are real.
2. They are non-negative.
3. Their sum is roughly $2N\gamma(0)$.

What do these facts translate to for properties of $\hat{\gamma}$? Can you draw a picture of this function?

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A nice property: if $\gamma(0) = 1$, then $\hat{\gamma}$ just looks like a probability distribution on $[-1, 1]$!

How Things Change for Continuous Processes

The key ideas are the same; the main difference is that $\hat{y}$ may extend to the entire real line instead of just an interval.

The idea of the original paper [Adams, Wilson] was to model $\hat{y}$ with a *mixture of gaussians*. **What do you think were the motivations for this choice?**

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1. Recovering y with an inverse Fourier transform is tractable. (If you don't believe me, what's the inverse Fourier transform of a gaussian?)
2. Any distribution can be approximated as a mixture of gaussians. When do you think such a representation can be especially accurate with just a few gaussians? What does such a covariance look like?