DS-GA 3001.001 Special Topics in Data Science: Probabilistic Time Series Analysis
Homework 2

Due date: Oct 22, by 5pm

Problem 1. LDS model, 10p
Given the model parameters:
$A = \begin{bmatrix} 0.9 & 0.1 \\ 0.3 & 0.65 \end{bmatrix}$, $C = \begin{bmatrix} 1.1 & 0.2 \\ 0.1 & 0.95 \end{bmatrix}$, $Q = \sigma_Q^2 \mathbf{I}$, $R = \sigma_R^2 \mathbf{I}$ with default parameter values $\sigma_q^2 = 0.4$, $\sigma_r^2 = 0.01$ and initial condition parameters $\mu_0 = [0,\, 0]^{\mathrm{t}}$, $\Sigma_0 = \mathbf{I}$.

- draw a sequence of length $t = 20$ observations from this model.
- using this artificial data and the true model parameters, run kalman filtering and smoothing; display the data, observations, and corresponding marginal posteriors (filtering, and smoothing): true latent trajectory in blue, observations in green, red cross and ellipse for mean and cov of Kalman filter, same in yellow for the kalman smoother.
- Manipulate parameters $\sigma_{q,r}^2$ such that a) the latent dynamics are close to deterministic and b) the observation noise is very small. Re-plot inference figure for each parameter configuration and comment on results.

Problem 2. LDS prediction, 10p
Given the standard parametrization of the LDS model, and the Kalman filtering estimates $\mu_{i|i}$ and $\Sigma_{i|i}$, obtained for a dataset $\mathbf{x}_{1:t}$ write down the expressions for predicting the following 2 observations in the sequence $\mathbf{x}_{t+1}$ and $\mathbf{x}_{t+2}$.

Problem 3. Generalizations of Kalman filtering, 20p
Consider a generalization of the traditional LDS model, where both observations and latents are one-dimensional, but where the observations are generated by passing the latent state through a nonlinear function $f(\cdot)$, with additive gaussian noise: $x_t = f(z_t) + v_t$. Derive the Kalman filtering update rules for this scenario.
Hint: linearize the observation model dynamically around the prior mean $\mu_{i|i-1}$ (using a first order Taylor expansion).

Problem 4. Particle filtering, 20p
For the same LDS model used in Problem 1, implement a full step of the particle filtering algorithm: use the closed form solution for $\mathrm{P}\left(z_i | \mathbf{x}_{1:i-1}\right)$ to draw an ensemble of N particles, combine with observation $\mathbf{x}_i$ and build a sampling-based approximation for the predicting distribution $\mathrm{P}\left(z_{i+1} | \mathbf{x}_{1:i}\right)$, compare the analytical posterior mean (Kalman filtering) to the sampling based estimates, for $N = 10, 100, 1000$.

Problem 5. HMM, 10p
Consider the HMM with K=3 latent states and discrete observations $\{1, 2, 3\}$, with parameters specified by: initial distribution $\pi = [1,\, 0,\, 0]$, transition matrix $\mathbf{A} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, where $A_{ij} = \mathrm{P}(z_{t+1} = j | z_t = i)$ and likelihood $\mathrm{P}(x_t | z_t)$ described by matrix entries B_{xz}: $\mathbf{B} = \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0.5 \end{bmatrix}$.
Write down all possible state sequences consistent with observations a) 1, 2, 3 and b) 1, 3, 1.

Problem 6. HMM, 10p
Construct an HMM that generates the observation sequence $A^{k_1} C^{k_2} A^{k_3} C^{k_4}$ where A^{k_1} denotes k_1 repeats of symbol A and the number of repeats k_i are drawn from the set $\{1, 2, 3\}$ with equal probability.