Notes on Optimization

Static and Dynamic Optimization

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Preface

Nothing to say here.

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1.1 Unconstrained minimizer

Definition 1.1.1 (Unconstrained optimization) *A point $x_0 \in \mathbb{R}^n$ is a minimizer of the function $f : \mathbb{R}^n \to \mathbb{R}$ if $f(x0) \leq f(x)$ for all $x \in \mathbb{R}^n$. And we write*

$$f(x_0) = \min_{x \in \mathbb{R}^n} f(x). \tag{1.1}$$

1.1.1 First variation

Theorem 1.1.1 (First variation) *Assume that $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable and x_0 is a minimizer. Then*

$$\nabla f(x_0) = 0. \tag{1.2}$$

Equivalently, we have

$$\frac{\partial f}{\partial x_i} f(x_0) = 0, \quad i = 1, \ldots, n. \tag{1.3}$$

Proof. Let us define $\phi(t) = f(x_0 + ty)$. Since x_0 is a minimizer,

$$\phi(0) = f(x_0) \leq f(x_0 + ty) = \phi(t) \tag{1.4}$$

we apply the chain rule from multivariable calculus:

$$\phi'(t) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} f(x_0 + ty) y_i. \tag{1.5}$$

For $t = 0$ and all $y \in \mathbb{R}^n$,

$$0 = \phi'(0) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} f(x_0) y_i = \nabla f(x_0)^T y. \tag{1.6}$$

Finally, by taking $y = \nabla f(x_0)$, we have $\nabla f(x_0) = 0$. □

Definition 1.1.2 (Critical point, extremal) *A point $x \in \mathbb{R}^n$ is called a critical point (or an extremal) of f if $\nabla f(x) = 0$.*

Remark 1.1.1 For a differentiable function f, if x_0 is a minimizer, then it is a critical point.

1.1.2 Second variation

Definition 1.1.3 *A symmetric $n \times n$ matrix A is nonnegative definite if for all $y \in \mathbb{R}^n$,*

$$y^T A y = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} y_i y_j \geq 0. \tag{1.7}$$

And in this case, we write

$$A \geq 0. \tag{1.8}$$

Theorem 1.1.2 (Second variation) *Assume $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ is twice differentiable and x_0 is a minimizer. Then*

$$\nabla^2 f(x_0) \geq 0. \tag{1.9}$$

Proof. Define $\phi(t) = f(x_0 + ty)$. we again apply the chain rule from multivariable calculus:

$$\phi''(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0 + ty) y_i y_j. \tag{1.10}$$

Since $t = 0$ is a minimizer of ϕ which suggests $\phi''(0) \geq 0$, we have

$$\phi''(0) = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(x_0) y_i y_j \geq 0. \tag{1.11}$$

By Definition 1.1.3, $\nabla^2 f(x_0) \geq 0$. □

1.2 Equality constraints

We turn our attention now to optimization problems with constraints. We assume $f, g_1, \ldots, g_m : \mathbb{R}^n \longrightarrow \mathbb{R}$ are twice continuously differentiable functions. Then we define $g : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ by

$$g = \begin{bmatrix} g_1 \\ \vdots \\ g_m \end{bmatrix}. \tag{1.12}$$

Then the gradient of g is

$$\nabla g = \begin{bmatrix} (\nabla g_1)^T \\ \vdots \\ (\nabla g_m)^T \end{bmatrix} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix}. \tag{1.13}$$

This is an $m \times n$ matrix-valued function.

Definition 1.2.1 (Constrained optimization) *A constrained optimization problem is to find* $x_0 \in \mathbb{R}^n$ *that*

$$\min_{x \in \mathbb{R}^n} f(x) \tag{1.14}$$

$$\text{s.t. } g(x) = 0. \tag{1.15}$$

Definition 1.2.2 *A point* $x \in \mathbb{R}^n$ *is called feasible for if* $g(x) = 0$*; that is, if* $g_k(x) = 0$ *for* $k = 1, \ldots, m$.

1.2.1 Lagrange multipliers

The method of Lagrange multipliers reveals a linear relationship between the gradients of $f, g1, \ldots, gm$ at a minimizer x_0.

1.2.2 Constrained first variation

Theorem 1.2.1 (F. John's form of the constrained first variation formula) *Suppose* x_0 *solves the constrained optimization problem defined in Definition 1.2.1. Then there exist real numbers* $\gamma_0, \lambda_0^1, \ldots, \lambda_0^m$ *(not all equal to 0) such that*

$$\gamma_0 \nabla f(x_0) + \sum_{k=1}^{n} \lambda_0^k \nabla g_k(x_0) = 0. \tag{1.16}$$

By defining $\lambda_0 = [\lambda_0^1 \ldots \lambda_0^m]^T$*, equivalently we have*

$$\gamma_0 \nabla f(x_0) + \lambda_0^T \nabla g(x_0) = 0. \tag{1.17}$$

Remark 1.2.1 When $\gamma_0 = 0$, we call it an abnormal multiplier. If $\gamma_0 \neq 0$, it is a normal multiplier, in which case we can divide and convert to the case $\gamma_0 = 1$ (for possibly new constants $\lambda_0^1, \ldots, \lambda_0^m$.)

Proof. Step 1: Fix $\beta > 0$. For each $\alpha > 0$, define

$$F^\alpha(x) = f(x) + \frac{\alpha}{2}|g(x)|^2 + \frac{\beta}{2}|x - x_0|^2 \tag{1.18}$$

We will later send $\alpha \to \infty$; this procedure is the penalty method. Let $B = \{x \in \mathbb{R}^n : |x - x_0| \leq 1\}$. The Extreme Value Theorem tells us that for continuous $F^\alpha : B \to \mathbb{R}$ and closed and bounded B, there exists x_α that minimizes F^α. Then we have $F^\alpha(x_\alpha) = \min_{x \in B} F^\alpha(x)$.

$$f(x_\alpha) + \frac{\alpha}{2}|g(x_\alpha)|^2 + \frac{\beta}{2}|x_\alpha - x_0|^2 = F^\alpha(x_\alpha) \leq F^\alpha(x_0) = f(x_0), \tag{1.19}$$

since $g(x_0) = 0$. Because $\alpha|g(x_\alpha)|^2 \leq 2(f(x_0) - f(x_\alpha)) - \beta|x_\alpha - x_0|^2$, $\{\alpha|g(x_\alpha)|^2\}_{\alpha>0}$ is bounded. Consequently,

$$\lim_{\alpha \to \infty} g(x_\alpha) = 0. \tag{1.20}$$

Step 2: Next, we use the Bolzano-Weierstrass Theorem to select a convergent subsequence $\{x_{\alpha_j}\}_{j=1}^{\infty}$ from $\{x_\alpha\} \subset B$ so that $x_{\alpha_j} \to \bar{x}$ as $\alpha_j \to \infty$, for some $\bar{x} \in B$. Equation 1.20 shows that $g(\bar{x}) = 0$ and therefore $\bar{x}$ is feasible. Hence $f(x_0) \leq f(\bar{x})$ Moreover, Equation 1.19 gives

$$f(\bar{x}) + \frac{\beta}{2}|\bar{x} - x_0|^2 \leq f(x_0) \leq f(\bar{x}), \tag{1.21}$$

which means that $|\bar{x} - x_0|^2 = 0$. So, $\bar{x} = x_0$. This is true for all convergent subsequences $x_{\alpha_j} \to \bar{x}$ and consequently

$$\lim_{\alpha\to\infty} x_\alpha = x_0. \tag{1.22}$$

Step 3: So if α is large enough, F^α has a minimum over B at point x_α. And in view of Equation 1.22, x_α does not lie on the boundary of B. It follows that[1]

$$0 = \nabla F^\alpha(x_\alpha) = \nabla f(x_\alpha) + \alpha \nabla g(x_\alpha)^T \nabla g(x_\alpha) + \beta(x_\alpha - x_0). \tag{1.23}$$

Then

$$0 = \gamma_\alpha \nabla f(x_\alpha) + \nabla g(x_\alpha)^T \lambda_\alpha + \gamma_\alpha \beta(x_\alpha - x_0), \tag{1.24}$$

for

$$\gamma_\alpha = (1 + \alpha^2 |g(x_\alpha)|^2)^{1/2} \tag{1.25}$$

$$\lambda_\alpha = \gamma_\alpha \alpha g(x_\alpha) \tag{1.26}$$

Step 4: Now $\gamma_\alpha^2 + \lambda_\alpha^2 = 1$ and therefore $\{(\gamma_\alpha, \lambda_\alpha)\}_{\alpha>0}$ is bounded. Consequently, the Bolzano-Weierstrass Theorem asserts that there is a sequence $\alpha_j \to \infty$ such that $\gamma_{\alpha_j} \to \gamma_0 \in \mathbb{R}$, $\lambda_{\alpha_j} \to \lambda_0 \in \mathbb{R}^m$, and $(\gamma_0, \lambda_0) \neq (0,0)$. Let α in Equation 1.24 behaves like $\alpha_j \to \infty$ and recall Equation 1.22 to derive Equation 1.17. We finish the proof. $\square$

1: We have

$$\nabla\left(\frac{|g(x)|^2}{2}\right) = \nabla\left(\frac{1}{2}\sum_i g_i(x)^2\right) = \begin{bmatrix} \frac{1}{2}\frac{\partial}{\partial x_1}\sum_i g_i(x)^2 \\ \vdots \\ \frac{1}{2}\frac{\partial}{\partial x_n}\sum_i g_i(x)^2 \end{bmatrix} = \begin{bmatrix} \sum_i g_i(x)\frac{\partial g_i(x)}{\partial x_1} \\ \vdots \\ \sum_i g_i(x)\frac{\partial g_i(x)}{\partial x_n} \end{bmatrix} = \nabla g(x)^T g(x).$$

Remark 1.2.2 The term β played no role, but will be used in the later proof of Theorem 1.2.4.

The existence theory for Lagrange multipliers for inequality constraints of the form $h_j(x) \leq 0$ for $j = 1, \dots, p$ is more complicated and will be discussed later.

1.2.3 Regular points

Definition 1.2.3 (Regular point) *A point x_0 is regular if the vectors $\{\nabla g_k(x_0)\}_{k=1,\dots,m}$ are linearly independent in $\mathbb{R}^n$.*

Theorem 1.2.2 (Constrained first variation formula) *Suppose that x_0 solves the constrained optimization problem defined in Definition 1.2.1 and furthermore that x_0 is regular. Then there exist real numbers $\lambda_0^1, \dots, \lambda_0^m$ such that*

$$\nabla f(x_0) + \sum_{k=1}^{n} \lambda_0^k \nabla g_k(x_0) = 0. \tag{1.27}$$

By defining $\lambda_0 = [\lambda_0^1 \dots \lambda_0^m]^T$, equivalently we have

$$\nabla f(x_0) + \lambda_0^T \nabla g(x_0) = 0. \tag{1.28}$$

Proof. From Theorem 1.2.1, we know that

$$\gamma_0 \nabla f(x_0) + \lambda_0^T \nabla g(x_0) = 0 \tag{1.29}$$

for constants $\gamma_0, \lambda_0^1, \ldots, \lambda_0^m$ are not all zero.

We claim that if x_0 is regular, then $\gamma_0 \neq 0$. To see this, suppose instead that $\gamma_0 = 0$, then

$$\lambda_0^T \nabla g(x_0) = 0 \tag{1.30}$$

and $\lambda_0 \neq 0$. But this is impossible as $\nabla g(x_0)$ are independent. Thus we divide both sides of Equation 1.17 by γ_0 to obtain an expression of the form in Equation 1.28. □

1.2.4 Constrained second variation

We discuss next how to compute second variations when we have regular equality constraints.

Lemma 1.2.3 *If x_0 is regular, then the $m \times m$ matrix $\nabla g(x_0) \nabla g(x_0)^T$ is nonsingular.*

Proof. Suppose that $y \in \mathbb{R}^m$ and $\nabla g(x_0) \nabla g(x_0)^T y = 0$. If a nonzero y satisfies this, then $\nabla g(x_0) \nabla g(x_0)^T$ is singular. However, if not (only $y = 0$ satisfies this), then matrix $\nabla g(x_0) \nabla g(x_0)^T$ is nonsingular.

We multiply both sides by y^T.

$$0 = y^T \nabla g(x_0) \nabla g(x_0)^T y = |\nabla g(x_0)^T y|^2 \tag{1.31}$$

Therefore,

$$0 = \nabla g(x_0)^T y = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_1}{\partial x_n} & \cdots & \frac{\partial g_m}{\partial x_n} \end{bmatrix} = \sum_{k=1}^{m} y_k \nabla g_k(x_0) \tag{1.32}$$

Since x_0 is regular, the vectors $\{\nabla g_k(x_0)\}_{k=1,\ldots,m}$ are linearly independent in $\mathbb{R}^n$ so that we must have $y = 0$. The $m \times m$ matrix $\nabla g(x_0) \nabla g(x_0)^T$ is nonsingular. □

Theorem 1.2.4 (Constrained second variation formula) *Suppose that x0 solves the constrained optimization problem defined in Definition 1.2.1 and that x_0 is regular. Let $\lambda_0^1, \ldots, \lambda_0^m$ be corresponding Lagrange multipliers, satisfying the first variation formula, Equation 1.28. Then*

$$y^T \left(\nabla^2 f(x_0) + \sum_{k=1}^{m} \lambda_0^k \nabla^2 g_k(x_0) \right) y \geq 0 \tag{1.33}$$

for all $y \in \mathbb{R}^n$ such that $\nabla g(x_0) y = 0$.

Proof. Step 1: We return to the proof of Theorem 1.2.1 and extract more detailed information. Since x_0 is regular, we know from Theorem 1.2.2 that

$$\gamma_\alpha = (1+\alpha^2|g(x_\alpha)|^2)^{1/2} \longrightarrow \gamma_0 \neq 0. \tag{1.34}$$

Hence $\{\alpha|g(x_\alpha)|\}_{\alpha>0}$ is bounded. We can assume that $\alpha_j g(x_{\alpha_j}) \longrightarrow \lambda_0$.

Step 2: Since, x_α lies within the interior of ball B for large α, we have $z^T\nabla^2 F^\alpha(x_\alpha)z \geq 0$ for all $z \in \mathbb{R}^n$ where

$$\nabla^2 F^\alpha(x) = \nabla^2 f(x) + \alpha\nabla g(x)^T\nabla g(x) + \alpha\sum_{k=1}^m g_k(x)\nabla^2 g_k(x) + \beta I. \tag{1.35}$$

Now,

$$z^T\left(\nabla^2 f(x_\alpha) + \alpha\nabla g(x_\alpha)^T\nabla g(x_\alpha) + \alpha\sum_{k=1}^m g_k(x_\alpha)\nabla^2 g_k(x_\alpha) + \beta I\right)z \geq 0 \tag{1.36}$$

Remember from the Lemma that $\nabla g(x_0)\nabla g(x_0)^T$ is nonsingular; consequently $\nabla g(x_\alpha)\nabla g(x_\alpha)^T$ is invertible for a large α. Given $y \in \mathbb{R}^m$ with $\nabla g(x_0)y = 0$, we define

$$z_\alpha = y - \nabla g(x_\alpha)^T\left(\nabla g(x_\alpha)\nabla g(x_\alpha)^T\right)^{-1}\nabla g(x_\alpha)y. \tag{1.37}$$

Then $\nabla g(x_\alpha)y = 0$. Observe that $z_\alpha \longrightarrow y$ as $\alpha \longrightarrow \infty$. This follows since $x_\alpha \longrightarrow x_0$ and $\nabla g(x_0)y = 0$.

Step 3: Take $z = z_\alpha$ in Equation 1.36, then

$$(z_\alpha)^T\left(\nabla^2 f(x_\alpha) + \alpha\nabla g(x_\alpha)^T\nabla g(x_\alpha) + \alpha\sum_{k=1}^m g_k(x_\alpha)\nabla^2 g_k(x_\alpha) + \beta I\right)(z_\alpha) \geq 0, \tag{1.38}$$

where $(z_\alpha)^T\alpha\nabla g(x_\alpha)^T\nabla g(x_\alpha)(z_\alpha) = 0$. Let $\alpha \longrightarrow \infty$ and recall that $\alpha_j g(x_{\alpha_j}) \longrightarrow \lambda_0$,

$$y^T\left(\nabla^2 F^\alpha(x_\alpha) = \nabla^2 f(x_\alpha) + \sum_{k=1}^m \lambda_0^k g_k(x_\alpha)\nabla^2 g_k(x_\alpha) + \beta I\right)y \geq 0, \tag{1.39}$$

for $y \in \mathbb{R}^m$ that $\nabla g(x_0)y = 0$. To conclude send $\beta \longrightarrow 0$. □

1.3 Applications

1.3.1 Least squares

Let A denote an $m \times n$ matrix and assume $b \in \mathbb{R}^m$ is given. If the linear system $Ax = b$ has no solution, we can nevertheless build an approximate solutions by finding $x_0 \in \mathbb{R}^n$ that solves the minimization problem

$$\min_{x\in\mathbb{R}^n} |Ax - b|^2 \tag{1.40}$$

Theorem 1.3.1 *If $A^T A$ is invertible, then the solution of Equation 1.40 is*

$$x_0 = (A^T A)^{-1} A^T b \tag{1.41}$$

Proof. Step 1: Let $f(x) = |Ax - b|^2 = |Ax|^2 - 2(Ax)^T b + |b|^2$. We have

$$\nabla f(x) = 2A^T Ax - 2A^T b. \tag{1.42}$$

So that $\nabla f(x_0) = 0$ implies $x_0 = (A^T A)^{-1} A^T b$.

Step 2: For uniqueness, we must show that $f(x_0) \leq f(x_0 + y)$ for all $y \neq 0$.

$$\begin{aligned} f(x_0 + y) - f(x_0) &= |A(x_0 + y) - b|^2 - |Ax_0 - b|^2 \qquad (1.43)\\ &= 2(Ax_0 - b)^T Ay + |Ay|^2 \\ &= |Ay|^2 \geq 0. \end{aligned}$$

□

1.3.2 Roots of polynomials

A novel application of Lagrange multipliers by [1] shows that the existence of a root for a complex polynomial of degree $n \geq 1$:

[1]: Jong (2009), "Lagrange Multipliers and the Fundamental Theorem of Algebra"

$$f(z) = z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \tag{1.44}$$

The coefficients $a_0, \ldots, a_{n-1}$ here are complex numbers.

If we substitute $z = x + iy$, then we can write f in terms of real and imaginary parts:

$$f(z) = u(x, y) + iv(x, y), \tag{1.45}$$

where $u, v : \mathbb{R}^2 \to \mathbb{R}$ are polynomials.

Lemma 1.3.2 *The functions u, v solve the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \tag{1.46}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{1.47}$$

Proof. We first apply induction to $f_n(z) = z^n$. When $n = 1$, $f_1(z) = z = x + iy = u_1(x, y) + iv_1(x, y)$ It is trivial that $u_1 = x, v_1 = y$ solve the Cauchy-Riemann equations.

Then suppose that for $n = k - 1$, u_{k-1}, v_{k-1} solve the Cauchy-Riemann equations.

For $n = k$, $f_k(z) = z f_{k-1}(z) = (x + iy)(u_{k-1} + iv_{k-1})$, so that $u_k =$

$xu_{k-1} - yv_{k-1}, v_k = yu_{k-1} + xv_{k-1}$. And it is easy to find

$$\frac{\partial u_k}{\partial x} = u_{k-1} + x\frac{\partial u_{k-1}}{\partial x} - y\frac{\partial v_{k-1}}{\partial x} \tag{1.48}$$

$$\frac{\partial v_k}{\partial y} = x\frac{\partial v_{k-1}}{\partial y} + u_{k-1} + y\frac{\partial u_{k-1}}{\partial y} \tag{1.49}$$

$$\frac{\partial u_k}{\partial y} = x\frac{\partial u_{k-1}}{\partial y} - v_{k-1} - y\frac{\partial v_{k-1}}{\partial y} \tag{1.50}$$

$$\frac{\partial v_k}{\partial x} = v_{k-1} + x\frac{\partial v_{k-1}}{\partial x} + y\frac{\partial u_{k-1}}{\partial x}. \tag{1.51}$$

Since u_{k-1}, v_{k-1} solve the Cauchy-Riemann equations, u_k, v_k also solve the Cauchy-Riemann equations.

Moreover, constant terms go into u, v do not matter. The proof for a general general polynomials follows by linearity. When $f(z) = a_0 + \sum_{k\geq 1} a_k f_k(z)$, we have $u = a_0 + \sum_{k\geq 1} a_k u_k$ and $v = \sum_{k\geq 1} a_k v_k$. The functions u, v solve the Cauchy-Riemann equations. □

Theorem 1.3.3 (Fundamental Theorem of Algebra) *There exists a point $z_0 \in \mathbb{C}$ which $f(z_0) = 0$.*

Proof. Step 1: We introduce the level sets $L_c = \{(x, y) \in \mathbb{R}^2 : u(x, y) = c\}$ and $M_c = \{(x, y) \in \mathbb{R}^2 : v(x, y) = c\}$, where $c \in \mathbb{R}$. Since $u(x, 0) = x^n$ + lower order terms, the function $u(x, 0)$ takes on infinitely many values. It follows that the sets L_c are nonempty for infinitely many values of the parameter c. We observe next that except for finitely many values of c, we have $\nabla u \neq 0$ on L_c and $\nabla v \neq 0$ on M_c. To see this, note that f' is a polynomial, and consequently has at most finitely many zeros. But $f' = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$[2], according to Cauchy-Riemann equations, and therefore $\nabla u \neq 0$ except at finitely many points.

2: The derivatives for complex variable functions.

Step 2: Select a value of the parameter c so that $L_c \neq \emptyset$ and $\nabla u \neq 0$ on L_c. We introduce the constrained optimization problem

$$\min v^2 \tag{1.52}$$

$$\text{s.t. } u = c. \tag{1.53}$$

Since $u^2 + v^2 = |f|^2 \longrightarrow \infty$ as $|z| \longrightarrow \infty$, we can use the Extreme Value Theorem to show that there exists a point $(x_c, y_c) \in L_c$ solving the constrained optimization problem. We claim that $v(x_c, y_c) = 0$. To see this, note that since $\nabla u \neq 0$ on L_c, (x_c, y_c) is a regular point. Hence Theorem 1.2.2 asserts that there exists a Lagrange multiplier λ such that $2v(x_c, y_c)\nabla v(x_c, y_c) + \lambda \nabla u(x_c, y_c) = 0$. But the Cauchy-Riemann equations imply $\nabla u \cdot \nabla v = 0$ and $|\nabla u| = |\nabla v|$. Since $\nabla u(x_c, y_c) \neq 0$, it follows that $\lambda = 0$ and $\nabla v(x_c, y_c) = 0$.

Step 3: In view of $v(x_c, y_c) = 0$, we see that $M_0 \neq \emptyset$. Select a sequence $c_k \longrightarrow 0$, such that $M_{c_k} \neq \emptyset$ and $\nabla v \neq 0$ on M_{c_k}. Then the argument above (with the roles of u and v reversed) shows that there exist points $(x_k, y_k) \in M_{c_k}$ for which $u(x_k, y_k) = 0$. The sequence $\{(x_k, y_k)\}_{k=1}^{\infty}$ is bounded, and so, passing if necessary to a subsequence, we may assume $(x_k, y_k) \longrightarrow (x_0, y_0) \in M_0$. Then $u(x_0, y_0) = v(x_0, y_0) = 0$, and therefore $f(z_0) = 0$ for $z_0 = x_0 + iy_0$. □

Linear Optimization | 2

Linear optimization theory, most commonly known as linear programming, concerns the minimization of linear functions, subject to affine equality and inequality constraints.

2.1 Theory

2.1.1 Basic concepts

Before start, we introduce some notations. If $x = [x_1 \dots x_n]^T \in \mathbb{R}^n$, we write $x \geq 0$ to mean that $x_i \geq 0$ for all $i = 1, \dots, n$. Similarly, $x > 0$ means that $x_i > 0$ for all $i = 1, \dots, n$.

We have $x \geq y$ if $x_i \geq y_i$ for all $i = 1, \dots, n$ and $x > y$ if $x_i > y_i$ for all $i = 1, \dots, n$.

Definition 2.1.1 (Canonical primal linear programming problem) *Let $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$ and assume A is an $m \times n$ matrix. The canonical primal linear programming problem is to find $x_0 \in \mathbb{R}^n$ to*

$$\begin{aligned} &\min c^T x \\ &\text{s.t. } Ax = b \\ &\quad x \geq 0 \end{aligned} \tag{P}$$

Definition 2.1.2 *We say $x \in \mathbb{R}^n$ is feasible if $Ax = b$, $x \geq 0$, that is, if x satisfies the constraints in (P). We will often call a feasible x a feasible solution.*

Definition 2.1.3 (Canonical dual linear programming problem) *A canonical dual problem is to find $y_0 \in \mathbb{R}^m$ to*

$$\begin{aligned} &\max b^T y \\ &\text{s.t. } A^T y \leq c \end{aligned} \tag{D}$$

Definition 2.1.4 *We say $y \in \mathbb{R}^m$ is feasible for (D) if $A^T y \leq c$.*

The most important fact of linear programming is that the primal and dual problems contain information about each other.

Theorem 2.1.1 (Duality and optimality) *If x is feasible for (P) and y is feasible for (D), then*

$$b^T y \leq c^T x. \tag{2.1}$$

If x_0 is feasible for (P) and y_0 is feasible for (D), and if $b^T y_0 = c^T x_0$ then x_0 solves (P) and y_0 solves (D).

Proof. Let x, y be feasible, $Ax = b, x \geq 0$ and $A^T y \leq c$. We have

$$b^T y = (Ax)^T y = x^T A^T y \leq x^T c = c^T x. \tag{2.2}$$

Suppose x_0, y_0 are feasible and $b^T y_0 = c^T x_0$. By (2.2), $b^T y0 \leq c^T x$ for all feasible x for (P). So $c^T x0 \leq c^T x$ for all feasible x, and thus x_0 is optimal for (P). A similar argument works for y_0. □

Then we introduce some other forms of linear programming problems.

Definition 2.1.5 (Standard linear programming problem) *The standard linear programming problem is to find $x_0 \in \mathbb{R}^n$ to*

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax \geq b \\ & x \geq 0 \end{aligned} \tag{P^*}$$

Definition 2.1.6 (Dual standard linear programming problem) *The dual standard linear programming problem is to find $y_0 \in \mathbb{R}^m$ to*

$$\begin{aligned} \max\ & b^T y \\ \text{s.t. } & A^T y \leq c \\ & y \geq 0 \end{aligned} \tag{D^*}$$

Remark 2.1.1 Note carefully that we now have an inequality $Ax \geq b$ and an additional sign constraint $y \geq 0$.

Remark 2.1.2 We could check that the duality and optimality theorem (Theorem 2.1.1) applies to the standard forms (P^*) and (D^*).

Definition 2.1.7 (General linear programming problem) *The general linear programming problem is to find $x_0 \in \mathbb{R}^n$ to*

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & \sum_{j=1}^{n} a_{ij} x_j \geq b_i, && i \in I_1 \\ & \sum_{j=1}^{n} a_{ij} x_j = b_i, && i \in I_2 \\ & x_j \geq 0, && j \in J_1 \end{aligned} \tag{P^o}$$

where $I_1 \cup I_2 = I$, $I_1 \cap I_2 = \emptyset$, $J_1 \subseteq J$, $I = \{1, \ldots, m\}$, and $J = \{1, \ldots, n\}$. Here I_1 and J_1 are the indices of the inequality constraints.

Definition 2.1.8 (Dual general linear programming problem) *The dual* *general linear programming problem is to find* $y_0 \in \mathbb{R}^m$ *to*

$$\begin{aligned} &\max b^T y && && (D^o) \\ &\text{s.t.} \sum_{i=1}^{m} y_i a_{ij} \leq c_j, && j \in J_1 \\ &\sum_{i=1}^{m} y_i a_{ij} = c_j, && j \in J_2 \\ &y_i \geq 0, && i \in I_1 \end{aligned}$$

where $J_1 \cup J_2 = J$*,* $J_1 \cap J_2 = \emptyset$*,* $I_1 \subseteq I$*,* $I = \{1, \ldots, m\}$*, and* $J = \{1, \ldots, n\}$*. And* I_1 *and* J_1 *are the indices of the inequality constraints.*

Remark 2.1.3 We write $[A, b, c, I_1, J_1]$ to display the relevant information determining in a general linear programming problem (P^o) defined in Definition 2.1.7.

The canonical problem (P) and its dual (D) correspond to $J_1 = J, J_2 = \emptyset, I_1 = \emptyset, I_2 = I$.

The standard problem (P^*) and its dual (D^*) correspond to $J_1 = J, J_2 = \emptyset, I_1 = I, I_2 = \emptyset$.

Remark 2.1.4 We could check that the duality and optimality theorem (Theorem 2.1.1) applies to the general forms (P^o) and (D^o).

Theorem 2.1.2 (Linear programming duality) *The dual of* (D^o) *is* (P^o)*.*

Proof. The problem (D^o) is equivalent to

$$\min\, (-b)^T y \tag{2.3}$$

$$\text{s.t.} \sum_{i=1}^{m} (-a_{ij}) y_i \geq c_j, \qquad j \in J_1 \tag{2.4}$$

$$\sum_{i=1}^{m} (-a_{ij}) y_i = c_j, \qquad j \in J_2 \tag{2.5}$$

$$y_i \geq 0 \qquad i \in I_1 \tag{2.6}$$

This is $[-A^T, -c, -b, J_1, I_1]$. So duality converts

$$\underbrace{[A, b, c, I_1, J_1]}_{P^o} \rightarrow \underbrace{[-A^T, -c, -b, J_1, I_1]}_{D^o}. \tag{2.7}$$

Hence the dual of (D^o) is

$$[-(-A^T)^T, -(-b), -(-c), I_1, J_1] = [A, b, c, I_1, J_1]. \tag{2.8}$$

□

Remark 2.1.5 By adding new slack variables, we can in fact convert a general linear programming problem (P^o) into the canonical form (P).

As a consequence of these observations, when we study the theory of linear programming, it is enough to consider the canonical problem.

2.1.2 Equilibrium equations

Theorem 2.1.3 (Equilibrium equations) *Suppose x is feasible for (P) and y is feasible for (D). Then x and y are optimal iff they satisfy the equilibrium equations*

$$\sum_{i=1}^{m} y_i a_{ij} = c_j \text{ if } x_j > 0, j = 1, \ldots, n. \tag{E}$$

And equivalently,

$$\sum_{i=1}^{m} y_i a_{ij} < c_j \text{ if } x_j = 0, j = 1, \ldots, n. \tag{E'}$$

Proof. As before, we compute $b^T y = (Ax)^T y = x^T A^T y \leq x^T c$. Our question is when do we have equality in the last inequality? Note that

$$x^T(A^T y - c) = \sum_{j=1}^{n} x_j \left((A^T y)_j - c_j\right) \begin{cases} = 0, & \text{if (E) holds} \\ < 0, & \text{if (E) fails} \end{cases} \tag{2.9}$$

where $(A^T y)_j = \sum_{i=1}^{m} y_i a_{ij}$. Thus we have $b^T y = c^T x$ if (E) holds. By Theorem 2.1.1, (E) implies the optimality for both x and y. □

2.1.3 Basic solutions

We introduce next the concept of basic solutions to linear programming problems. These are solutions with the largest numbers of zero entries, which are consequently the easiest to study.

A $m \times n$ matrix A can be written as $[a^1 a^2 \cdots a^n]$, where a^j is the j-th column vector of A, and so $a^j \in \mathbb{R}^m$ for $j = 1, \ldots, n$.

Lemma 2.1.4 *If $Ax = b$, then b is a linear combination of the columns of A.*

Proof. We can write $Ax = b$ as $\sum_{j=1}^{n} x_j a^j = b$, and this shows b to be a linear combination of the column vectors of A. □

Definition 2.1.9 *We say that $x \in \mathbb{R}^n$ is a basic solution of $Ax = b$ if the columns $\{a^j : x_j \neq 0, j = 1, \ldots, n\}$ are linearly independent in $\mathbb{R}^m$. Also, we say $x = 0$ is basic.*

Theorem 2.1.5 *For each $b \in \mathbb{R}^m$ the linear system of equations $Ax = b$ has at most finitely many basic solutions.*

Proof. Look at columns $\{a^1, \ldots, a^n\}$ of A. There are only finitely many subsets $\{a^{j_1}, \ldots, a^{j_l}\} \subseteq \{a^1, \ldots, a^n\}$ which are independent in $\mathbb{R}^m$. We claim there is at most one solution of $Ax = b$ having the form $x = [0 x_{j_1} \ldots x_{j_l} 0]^T$. To see this, let $\hat{x} = [0 \hat{x}_{j_1} \ldots \hat{x}_{j_l} 0]^T$ also solve $A\hat{x} = b$. Then $A(x - \hat{x}) = b - b = 0$ and therefore $\sum_{k=1}^{l} (x_{j_k} - \hat{x}_{j_k}) a^{j_k} = 0$. Since the columns $\{a^{j_1}, \ldots, a^{j_l}\}$ are linearly independent, it follows that $x_{j_k} = \hat{x}_{j_k}$. Hence there is at most one basic solution of $Ax = b$ corresponding to each independent collection of columns. □

Theorem 2.1.6 (Basic solutions) *If there exists a feasible solution of (P), then there exists a basic feasible solution.*

If there exists an optimal solution of (P), then there exists a basic optimal solution.

Proof. Step 1: Select a feasible solution x with fewest number of nonzero components. And then let us show it is a basic feasible solution.

If $x = 0$, we are done. If not, then suppose that x has nonzero elements $x_{j_1}, x_{j_2}, \ldots, x_{j_l} > 0$ and

$$Ax = \sum_{k=1}^{l} x_{j_k} a^{j_k} = b. \tag{2.10}$$

Then suppose that x is not basic. By Definition 2.1.9, $\{a^{j_1}, \ldots, a^{j_l}\}$ are not linearly independent. There exists $\theta_{j_1}, \theta_{j_2}, \ldots, \theta_{j_l}$ not all equal to zero, such that

$$\sum_{k=1}^{l} \theta_{j_k} a^{j_k} = 0. \tag{2.11}$$

This means that $A\theta = 0$. Then Equation 2.10 and Equation 2.11 simply imply for any λ that

$$\sum_{k=1}^{l} (x_{j_k} - \lambda \theta_{j_k}) a^{j_k} = b. \tag{2.12}$$

We may assume $\theta_{j_p} > 0$ for some index j_p (if not, multiply θ by -1). Increase λ from 0 to the first $\lambda^* > 0$ for which at least one of the values $x_{j_1} - \lambda^* \theta_{j_1}, \ldots, x_{j_l} - \lambda^* \theta_{j_l}$ equals zero while others remain to be greater than zero. Since $\theta_{j_p} > 0$, this must happen at finite value of λ^*. Then let $x^* = x - \lambda^* \theta \geq 0$ and $Ax^* = 0$. But x^* then has at least one fewer nonzero entry than x. This is a contradiction, and therefore x is indeed a basic feasible solution.

Step 2: Now let x_0 be an optimal solution with the fewest number of nonzero components. We will show that x_0 is a basic optimal solution.

Suppose x_0 is not. Then as above, suppose x_0 has nonzero elements $x_{j_1}, x_{j_2}, \ldots, x_{j_l} > 0$ and $\sum_{k=1}^{l} x_{j_k} a^{j_k} = b$ and $\sum_{k=1}^{l} \theta_{j_k} a^{j_k} = 0$ for appropriate $\theta_{j_1}, \ldots, \theta_{j_l}$ not all zero. Select λ^* as before and write $x_0^* = x_0 - \lambda^* \theta$. Then $Ax_0^* = b$, $x_0^* \geq 0$, and x_0^* has fewer nonzero components than x_0 which leads to contradiction of that x_0 has the fewest number of nonzero components.

Step 3: We now claim

$$c^T x_0^* = c^T x_0 = \min\{c^T x : Ax = b, x \geq 0\}. \tag{2.13}$$

To prove this, observe first that

$$c^T \theta = 0; \tag{2.14}$$

for otherwise, we could select a small value of λ so that $c^T(x_0 - \lambda\theta) < c^T x_0$ and x_0 is not optimal consequently. Thus Equation 2.14 must hold and therefore Equation 2.13 holds. Thus x_0^* is optimal for (P), but has fewer nonzero components than x_0. And this is a contradiction: x_0 is a basic optimal solution. □

Remark 2.1.6 Our discussion of basic solutions leads to the very interesting realization that although linear programming problems are finite dimensional, with infinitely many feasible solutions, they are in effect finite optimization problems, with only finitely many basic solutions to consider and only finitely many optimal basic solutions.

The simplex algorithm, discussed next, builds upon this observation.

2.2 Simplex algorithm

The simplex algorithm comprises two procedures:

Phase I: Find a basic feasible solution of $Ax = b$, $x \geq 0$ (or show that none exists).

Phase II: Given a basic feasible solution, find a basic optimal solution (or show that none exists).

2.2.1 Nondegeneracy

Assumption 2.2.1 (Nondegeneracy) *The nondegeneracy assumptions are that*

1. $n > m$;
2. *the rows of A are linearly independent (and thus A has m columns which are independent);*
3. *b cannot be written as a linear combination of fewer than m columns of A.*

Assumption 1 implies that there are more unknowns $(x_1, \ldots, x_n)$ than the m linear equality constraints in the linear system $Ax = b$.

Assumption 2 means that $\text{rank}(A) = \dim(\text{column space}) = \dim(\text{row space}) = m$.

Assumption 3 says that if $Ax = b$, then x has at least m nonzero entries.

Remark 2.2.1 Under the nondegeneracy assumptions, any basic, feasible solution of $Ax = b, x \geq 0$ has precisely m nonzero entries. The next assertion shows that the converse is true as well

Lemma 2.2.1 (On nondegeneracy) *Assume the nondegeneracy assumptions 1-3 hold. If $Ax = b, x \geq 0$ and x has precisely m non-zero entries, then x is a basic feasible solution.*

Proof. Let $x = [0 x_{j_1} 0 \ldots x_{j_m} \ldots 0]^T$, where $x_{j_1}, \ldots x_{j_m} > 0$, and write $B = \{j_1, \ldots, j_m\}$. To show that x is a basic feasible solution, We must prove that the columns $\{a^{j_1}, \ldots, a^{j_m}\}$ are independent.

We know that $\sum_{j \in B} x_j a^j = b$. If the columns $\{a^{j_1}, \ldots, a^{j_m}\}$ are dependent, we could write some column as a linear combination of the others. This is, for some index j_k we have $a^{j_k} = \sum_{j \in B, j \neq j_k} y_j a^j$. Then

$$b = x_{j_k} a^{j_k} + \sum_{j \in B, j \neq j_k} x_j a^j = x_{j_k} \sum_{j \in B, j \neq j_k} y_j a^j + \sum_{j \in B, j \neq j_k} x_j a^j = \sum_{j \in B, j \neq j_k} (x_{j_k} y_j + x_j) a^j. \tag{2.15}$$

Thus b is a linear combination of few than m columns of A, a contradiction to the nondegeneracy requirement 3. $\square$

2.2.2 Phase II

We discuss Phase II before Phase I. The goal of Phase II is, given a basic feasible solution x, to find a basic optimal solution x_0, or show none exists. For this, we assume the nondegeneracy condition 1, 2, 3.

So we are given $x = [0\ x_{j_1}\ 0 \ldots x_{j_m} \ldots 0]^T$ where $x_{j_1}, \ldots x_{j_m} > 0$ are the m nonzero entries of x. We also have $Ax = b$.

Step 1: Use the dual problem to check for optimality

We have a basic feasible solution x, and need to check if it is optimal or not.

Definition 2.2.1 *Suppose we have a basic feasible solution x. Let $B = \{j : x_j > 0\} = \{j_1, \ldots, j_m\}$. We call $\{a^j : j \in B\}$ the basis corresponding to x.*

The $m \times m$ matrix $M = [a^{j_1}\ a^{j_2} \cdots a^{j_m}]_{m \times m}$, is called the corresponding basis matrix.

If $c = [c_1 \ldots c_n]^T$, define $\hat{c} = [c_{j_1} \ldots c_{j_m}]^T \in \mathbb{R}^m$.

Remark 2.2.2 The matrix M is invertible since its columns are independent. Thus there exist a unique $y \in \mathbb{R}^m$ solving $M^T y = \hat{c}$ and $y = (M^{-1})^T \hat{c}$.

Recall next that y is feasible for (D) if $A^T y \leq c$. This may or may not hold. But if it holds, we are done.

Lemma 2.2.2 *If $A^T y \leq c$ holds, then x is optimal for (P).*

Proof. The equilibrium equations (E) say

$$(a^j)^T y = \sum_{i=1}^{m} y_i a_{ij} = c_j \text{ if } x_j > 0 (\text{or} j \in B). \tag{2.16}$$

Now $M = [a^{j_1} \ldots a^{j_m}]$, and so

$$M^T = \begin{bmatrix} (a^{j_1})^T \\ \vdots \\ (a^{j_m})^T \end{bmatrix} \tag{2.17}$$

But since there exists a y such that $M^T y = \hat{c}$, which means that $(a^j)^T y = c_j$ for $j \in B$. These are precisely the equilibrium question (E). So if y is feasible for (D), it follows that x is optimal for (P) and y is optimal for (D). □

Therefore we have two possibilities:

A1: $y = (M^{-1})^T \hat{c}$ satisfies $A^T y \leq c$. Then stop: x is optimal for Equation P and y is optimal for (D).
A2: $y = (M^{-1})^T \hat{c}$ does not satisfy $A^T y \leq c$. Go to step 2.

Step 2: Use a wrong way inequality to improve x

When $A^T y \leq c$ fails, there exists some index $s \in (\{1, \ldots, n\} - B)$ such that

$$\underbrace{(a^s)^T y > c_s.}_{\text{wrong way inequality}} \tag{2.18}$$

The key idea of the simplex algorithm is use this fact to change the basis $\{a^j : j \in B\}$, thereby constructing a new basic feasible solution x^* with a lower cost $c^T x^*$. To do this, we first find $t = [t_{j_1} \ldots t_{j_m}] \in \mathbb{R}^m$ so that $Mt = a^s$. Since M is invertible, we have a unique solution t. Then $a^s = \sum_{j \in B} t_j a^j$, and consequently for all λ we have $\lambda a^s + \sum_{j \in B}(x_j - \lambda t_j) a^j = b$. We then define

$$\hat{x} = \begin{bmatrix} 0 \\ x_{j_1} - \lambda t_{j_1} \\ \vdots \\ x_{j_2} - \lambda t_{j_2} \\ \vdots \\ \lambda \\ \vdots \\ x_{j_m} - \lambda t_{j_m} \\ 0 \end{bmatrix} \in \mathbb{R}^n. \tag{2.19}$$

Here λ is in the s-th slot. And for a small $\lambda > 0$, we have $A\hat{x} = b$. Thus $\hat{x}$ is feasible for (P) small $\lambda > 0$.

However, we must note that $\hat{x}$ has $m+1$ nonzero entries, and consequently is not basic.

How does replacing x by $\hat{x}$ affect the cost? We would be pleased if $c^T \hat{x} < c^T x$ after replacing x by $\hat{x}$.

$$\begin{aligned} c^T \hat{x} - c^T x &= \lambda c_s + \sum_{j \in B} (x_j - \lambda t_j) c_j - \sum_{j \in B} x_j c_j && (2.20) \\ &= \lambda c_s - \lambda \underbrace{\sum_{j \in B} t_j c_j}_{z_s} && (2.21) \\ &= \lambda (c_s - z_s). && (2.22) \end{aligned}$$

And $z_s = \sum_{j \in B} t_j c_j = \hat{c}^T t = \hat{c}^T (M^- 1 a^s) = y^T a^s$. Therefore (2.18) is equivalent to $z_s > c_s$. It follows that $c^T \hat{x} < c^T x$. Consequently, we lower the cost by shifting to $\hat{x}$ from x.

Step 3: Change the basis, lower the cost

There are two possibilities as to how much we can lower the cost by increasing λ:

B1: $t_j \leq 0$ for all $j \in B$. Then $\hat{x}_j = x_j - \lambda t_j \geq x_j > 0$ for $j \in B$ and so $\hat{x}$ is feasible for all $\lambda > 0$. Thus says

$$c^T \hat{x} = c^T x + \lambda \underbrace{(c_s - z_s)}_{<0} \longrightarrow -\infty \tag{2.23}$$

as $\lambda \longrightarrow \infty$. So we have learned that $\inf\{c^T x : Ax = b, x \geq 0\} = -\infty$ and therefore stop and concluded that (P) has no solution.

B2: $t_j > 0$ for at least one index $j \in B$. We increase λ, starting at 0 and stopping when $\lambda = \lambda^* > 0$ and $\hat{x}_{j_k} = x_{j_k} - \lambda^* t_{j_k} = 0$ for some index $j_k \in B$. Define that

$$x^* = \begin{bmatrix} 0 \\ x_{j_1} - \lambda^* t_{j_1} \\ \vdots \\ x_{j_k} - \lambda^* t_{j_k} (= 0) \\ \vdots \\ \lambda^* \\ \vdots \\ x_{j_m} - \lambda^* t_{j_m} \\ 0 \end{bmatrix}. \tag{2.24}$$

Then x^* has no more than m nonzero entries; and, since $Ax^* = b$, the nondegeneracy condition say that x^* has precisely m nonzero entries. According to Lemma 2.2.1, x^* is therefore a basic feasible solution. Furthermore, $c^T x^* < c^T x$.
Now define the new basis

$$B^* = (\underbrace{\{j_1, \dots, j_m\}}_{B} - \{j_k\}) \cup \{s\}, \tag{2.25}$$

by removing the index j_k and adding the index s. Then go to step 1 again with x^* replacing x and B^* replacing B.

Each time we cycle through step 1 to step 3, the cost strictly decreases. Thus the same collection of basis vectors B will never repeat. Hence the simplex algorithm terminates in a finite number of step. This can only occur when A1 happens which suggests we have reached an optimal solution or B1 happens which means that none exists.

Theorem 2.2.3 (Simplex algorithm finds optimal solutions) *Assume the nondegeneracy conditions 1, 2, 3 hold, and that there exist feasible x for (P), feasible y for (D). Then the simplex algorithm terminates in finitely many steps, and produces a basic optimal x_0 for (P) and an basic optimal y_0 for (D).*

Proof. If y is feasible for (D), $\inf\ \{c^T x : Ax = b, x \geq 0\} \geq b^T y > -\infty$, and so B1 cannot occur. Consequently, the simplex algorithm terminates at an optimal x_0 for (P) and y_0 for (D). □

2.2.3 Phase I

We now come back and explain how to carry Phase I of the simplex algorithm. This major focus of Phase I is to find a feasible solution of $Ax = b, x \geq 0$. The following discussions are trying to convert Phase II to Phase I. And in this way, we will not include other unproved definitions or theorems. However, alternatively we can do Gauss elimination.

Remark 2.2.3 To make a connection between Phase I and II, we need to modify the third nondegeneracy condition, to become

3′. b cannot be written as a linear combination of fewer than m columns of $\tilde{A} = [A\ I_m]$, where I_m is $m \times m$ identity matrix.

Note that 3′ indicates $b_i \neq 0$ for all $i = 1, \dots, m$.

We assume for this section that the nondegeneracy conditions 1, 2, and 3′ hold.

The goal of Phase I is to find $x \geq 0$ solving $Ax = b$, that is,

$$\sum_{j=1}^{n} a_{ij} x_j = b_i, (i = 1, \dots, m). \tag{2.26}$$

We may assume that $b_i > 0$ for otherwise we can multiply i-th constraint by -1.

Now let z be a vector of m elements and consider the modified problem:

$$\begin{aligned} \min\ & z \\ \text{s.t. } & Ax + I_m z = b \\ & x \geq 0 \\ & z \geq 0. \end{aligned} \tag{$\tilde{P}$}$$

This has the form

$$\begin{aligned} \min\ & \tilde{c}^T \tilde{x} \\ \text{s.t. } & \tilde{A}\tilde{x} = \tilde{b} \\ & \tilde{x} \geq 0 \end{aligned}$$

where $\tilde{x} = [x\ z]^T$, $\tilde{A} = [A\ I_m]$, $\tilde{b} = b$, and $\tilde{c} = [\underbrace{0 \dots 0}_{n}\ \underbrace{1 \dots 1}_{m}]^T$.

Since each $b_i > 0$, a basic feasible solution of $(\tilde{P})$ is $\tilde{x} = [0 \dots 0\ b_1 \dots b_m]^T$. Now apply Phase II to $(\tilde{P})$: we either produce a basic optimal solution $\tilde{x}_0$ of $(\tilde{P})$ or learn that none exists. Since $\tilde{c}^T \tilde{x} \geq 0$ for all feasible $\tilde{x}$, the latter cannot occur, as we will later see from . Hence Phase II provides use with a basic optimal $\tilde{x}_0 = [x\ z]^T$ for $(\tilde{P})$.

There are now two possibilities to consider:

C1: $\sum_{i=1}^{m} z_i = 0$. Then $z = 0$, and therefore $\tilde{A}\tilde{x} = \tilde{b}$ implies $Ax = b, x \geq 0$. So we have a basic feasible solution x for (P).

C2: $\sum_{i=1}^{m} z_i > 0$. In this situation, (P) does not have any feasible solution x. This is so, since if $Ax = b, x \geq 0$, then $\tilde{x}_0 = [x\ 0]^T$ would be optimal for $(\tilde{P})$, giving the cost $\tilde{c}^T \tilde{x} = \sum_{i=1}^{m} z_i = 0$.

2.3 Duality Theorem

Next we return to theory and provide an analysis of the solvability of linear programming problems in standard form (P^*) and (D^*).

We no longer need the nondegeneracy conditions from the previous section, but we do require this important assertion:

Theorem 2.3.1 (Variant of Farkas alternative) *Either*

1. $Ax \leq b, x \geq 0$ *has a solution; or*
2. $A^T y \geq 0, b^T y < 0, y \geq 0$ *has a solution,*

but not both.

The formal proof of this theorem is postponed to Theorem 3.1.7 on page 29.

Theorem 2.3.2 (Duality Theorem for standard form problems) *Precisely one of the following occurs:*

1. *Both (P^*) and (D^*) have feasible solutions. In this case, both (P^*) and (D^*) have optimal solutions and*

$$\min \{c^T x : Ax \geq b, x \geq 0\} = \max \{b^T y : A^T y \leq c, y \geq 0\}. \tag{2.27}$$

2. *There are feasible solutions for (D^*), but not for (P^*). Then*

$$\sup \{b^T y : A^T y \leq c, y \geq 0\} = \infty. \tag{2.28}$$

3. *There are feasible solutions for (P^*), but not for (D^*). Then*

$$\inf \{c^T x : Ax \geq b, x \geq 0\} = -\infty. \tag{2.29}$$

4. *Neither (P^*) nor (D^*) has feasible solutions.*

Proof. Step 1: We introduce the compound matrix

$$\hat{A} = \begin{bmatrix} -A & 0 \\ 0 & A^T \\ c^T & -b^T \end{bmatrix}_{(m+n+1)\times(n+m)} \tag{2.30}$$

$$\hat{x} = \begin{bmatrix} x \\ y \end{bmatrix} \tag{2.31}$$

$$\hat{b} = \begin{bmatrix} -b \\ c \\ 0 \end{bmatrix} \tag{2.32}$$

$$\hat{y} = \begin{bmatrix} v \\ u \\ \lambda \end{bmatrix}. \tag{2.33}$$

Then Theorem 2.3.1 states that either

1. $\hat{A}\hat{x} \leq \hat{b}, \hat{x} \geq 0$ has a solution; or
2. $\hat{A}^T \hat{y} \geq 0, \hat{b}^T \hat{y} < 0, \hat{y} \geq 0$ has a solution

but not both.

Step 2: If 1 holds, which means $\hat{A}\hat{x} \le \hat{b}, \hat{x} \ge 0$ has a solution, then there exist $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ so that $x \ge 0, y \ge 0$ and

$$\begin{bmatrix} -A & 0 \\ 0 & A^T \\ c^T & -b^T \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \le \begin{bmatrix} -b \\ c \\ 0 \end{bmatrix}. \tag{2.34}$$

Thus

$$-Ax \le -b \tag{2.35}$$
$$A^T y \le c \tag{2.36}$$
$$c^T x - b^T y \le 0 \tag{2.37}$$
$$x \ge 0 \tag{2.38}$$
$$y \ge 0 \tag{2.39}$$

Consequently, we have $c^T x \le b^T y \le (Ax)^T y = x^T A^T y \le x^T c$ which implies $c^T x = b^T y$. Hence

$$Ax \ge b \tag{2.40}$$
$$A^T y \le c \tag{2.41}$$
$$c^T x = b^T y \tag{2.42}$$
$$x \ge 0 \tag{2.43}$$
$$y \ge 0 \tag{2.44}$$

Therefore, if x is feasible for (P^*) and y is feasible for (D^*), then according to Theorem 2.1.1 and Remark 2.1.2, $c^T x = b^T y$ implies that x is optimal for (P^*) and y is optimal for (D^*). This is the statement 1 of Duality Theorem.

Step 3: If 2 holds, which means $\hat{A}^T \hat{y} \ge 0, \hat{b}^T \hat{y} < 0, \hat{y} \ge 0$ has a solution, then there exist $v \in \mathbb{R}^m, u \in \mathbb{R}^n, \lambda \in \mathbb{R}$ with $v, u, \lambda \ge 0$,

$$\begin{bmatrix} -A^T & 0 & c \\ 0 & A & -b \end{bmatrix} \begin{bmatrix} v \\ u \\ \lambda \end{bmatrix} \ge \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{2.45}$$

and

$$\hat{b}^T \hat{y} = [-b^T c^T 0] \begin{bmatrix} v \\ u \\ \lambda \end{bmatrix} = -b^T v + c^T u \le 0. \tag{2.46}$$

That is

$$-A^T v + c\lambda \ge 0 \tag{2.47}$$
$$Au - b\lambda \ge 0 \tag{2.48}$$
$$-b^T v + c^T u < 0 \tag{2.49}$$
$$v \ge 0 \tag{2.50}$$
$$u \ge 0 \tag{2.51}$$
$$\lambda \ge 0 \tag{2.52}$$

Next, we assert that $\lambda = 0$. Observe that $\lambda(b^T v) \le u^T A^T v \le u^T (c\lambda) =$

$\lambda(c^T u)$, so $\lambda(-b^T v + c^T u) \geq 0$. But as $\lambda \geq 0$, it contradicts $-b^T v + c^T u < 0$ if $\lambda > 0$. So we have $\lambda = 0$ and

$$A^T v \leq 0 \tag{2.53}$$
$$Au \geq 0 \tag{2.54}$$
$$c^T u \leq b^T v \tag{2.55}$$
$$v \geq 0 \tag{2.56}$$
$$u \geq 0. \tag{2.57}$$

The existence of u, v satisfying 2 leads us to statements 2-4 of Duality Theorem. The next step is to discuss the positiveness of $c^T u$.

Step 4: If $c^T u < 0$, (D^*) has no feasible solution. For $A^T y \leq c, y \geq 0$, 2 implies $0 \leq y^T A u = (A^T y)^T u \leq c^T u < 0$, which is a contradiction.

If in addition (P^*) has no feasible solution, we have statement 4 of the Duality Theorem. If, on the other hand, (P^*) has a feasible solution x solving $Ax \geq b, x \geq 0$ then for all $\mu \geq 0$, we have

$$A(x + \mu u) = Ax + \mu A u \geq b \tag{2.58}$$
$$x + \mu u \geq 0 \tag{2.59}$$

So $x + \mu u$ is also feasible for (P^*). But then since $c^T u < 0$, $c^T(x + \mu u) = c^T x + \mu c^T u \longrightarrow -\infty$ as $\mu \longrightarrow \infty$. This is statement 3 of the Duality Theorem.

Step 5: If $c^T u \geq 0$, (P^*) has not feasible solution. Assume $Ax \geq b, x \geq 0$. Following 2, then $0 \geq x^T A^T v = (Ax)^T v \geq b^T v > c^T u \geq 0$, which is a contradiction.

Also if (D^*) has no feasible solution, we have statement 4 of the Duality Theorem. If (D^*) has a feasible solution y solving $A^T y \leq c, y \geq 0$, then for all $\mu \geq 0$, we have

$$A^T(y + \mu v) = A^T y + \mu A^T v \leq c \tag{2.60}$$
$$y + \mu v \geq 0 \tag{2.61}$$

So $y + \mu v$ is also feasible for (D^*). But then since $b^T v > c^T u \geq 0$, $b^T(y + \mu v) = b^T y + \mu b^T v \longrightarrow \infty$ as $\mu \longrightarrow \infty$. This is the statement 2 of the Duality Theorem. □

Let us now return to the canonical forms of our primal and dual problems: (P) and (D). Since we can use slack and surplus variables to convert between the canonical and the standard form problems (P^*) and (D^*), we like wise have a duality assertion for the canonical problems.

Theorem 2.3.3 (Duality Theorem for canonical form problems) *Precisely one of the following occurs:*

1. *Both (P) and (D) have feasible solutions. In this case, both (P) and (D) have optimal solutions and*

$$\min\{c^T x : Ax = b, x \geq 0\} = \max\{b^T y : A^T y \leq c\}. \tag{2.62}$$

2. *There are feasible solutions for (D), but not for (P). Then*

$$\sup\{b^T y : A^T y \leq c\} = \infty. \tag{2.63}$$

3. *There are feasible solutions for (P), but not for (D). Then*

$$\inf\{c^T x : Ax = b, x \geq 0\} = -\infty. \tag{2.64}$$

4. *Neither (P) nor (D) has feasible solutions.*

Remark 2.3.1 What about general forms?

2.4 Applications

We discuss in the subsequent sections several interesting applications and extensions of linear programming.

2.4.1 Two-person, zero-sum matrix games

In a two-person, zero-sum matrix game, we have two participants: player I (who wants to maximize some payoff) and player II (who wants to minimize this payoff). Each player selects his/her strategy without knowing what the other will do.

For a matrix game, the payoff is determined by a given $m \times n$ payoff matrix:

$$\begin{bmatrix} a_{11} & \cdots & & & a_{1n} \\ \vdots & \ddots & & & \\ & & a_{ij} & & \\ & & & \ddots & \vdots \\ a_{m1} & & & \cdots & a_{mn} \end{bmatrix} \tag{2.65}$$

Player I selects a row index $i \in \{1, \ldots, m\}$, and player II selects a column index $j \in \{1, \ldots, n\}$. The payoff to player I is a_{ij} and the loss to player II is a_{ij}.

What are optimal strategies for the players?

Definition 2.4.1 (Saddle point) *The (k, l)-th entry a_{kl} of the matrix A is a saddle point if*

$$\max_{1\leq i\leq m} a_{il} = a_{kl} = \min_{1\leq j\leq n} a_{kj}. \tag{2.66}$$

Equivalently, a_{kl} is a saddle point if

$$a_{il} \leq a_{kl} \leq a_{kj} \tag{2.67}$$

for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Theorem 2.4.1 (Minmax and saddle points) *The matrix A has a saddle point iff the minimax condition*

$$\min_{1\leq j\leq n} \max_{1\leq i\leq m} a_{ij} = \max_{1\leq i\leq m} \min_{1\leq j\leq n} a_{ij} \tag{2.68}$$

holds.

Example 2.4.1 For the matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, we have $\min_j \max_i a_{ij} = 1$ and $\max_i \min_j a_{ij} = -1$. Therefore the minimax condition (Theorem 2.4.1) fails. So A does not have a saddle points.

2.4.2 Network flows

TODO

2.4.3 Transportation problem

TODO

Convexity 3

3.1 Convex geometry

3.1.1 Convex sets

Definition 3.1.1 *A set $C \subseteq \mathbb{R}^n$ is convex if for all $a, b \in C$ and $0 \le \theta \le 1$, we have $\theta a + (1-\theta)b \in C$.*

If two points belong to a convex set, then all points that are on the line between these two points belong to this convex set.

Definition 3.1.2 (Convex combination) *Let $\{a^1, \dots, a^p\} \subset \mathbb{R}^n$. A convex combination of $a^1, \dots a^p$ is $\sum_{k=1}^p \theta_k a^k$ for some $\theta_k \ge 0$ and $\sum_{k=1}^p \theta_k = 1$.*

The convex combination can be regarded as a linear combination with restraints on coefficients θ_k.

Definition 3.1.3 (Convex polytope) *The convex polytope generated by $a^1, \dots, a^p$ is $\langle a^1, \dots, a^p \rangle = \left\{\sum_{k=1}^p \theta_k a^k : \theta_k \ge 0, \sum_{k=1}^p \theta_k = 1\right\}$*

A convex polytope is a set of all possible convex combinations.

Theorem 3.1.1 (Caratheodory's Theorem) *Let b belong to the convex polytope $\langle a^1, \dots, a^p \rangle \subset \mathbb{R}^n$. Then we can write $b = \sum_{k=1}^{n+1} \theta_k a^{j_k}$ where $1 \le j_1 < \cdots < j_{n+1} \le p$, $\theta_k \ge 0$, $\sum_{k=1}^{n+1} \theta_k = 1$.*

Proof. Since $b \in \langle a^1, \dots, a^p \rangle$, there exists a solution $x \in \mathbb{R}^p$ of

$$Ax = \begin{bmatrix} b \\ 1 \end{bmatrix}, x \ge 0 \tag{3.1}$$

for the $(n+1) \times p$ matrix

$$A = \begin{bmatrix} a^1 & a^2 & \cdots & a^p \\ 1 & 1 & \cdots & 1 \end{bmatrix}. \tag{3.2}$$

And according to Theorem 2.1.6, there exists a basic solution $x^* \in \mathbb{R}^p$ of

$$Ax^* = \begin{bmatrix} b \\ 1 \end{bmatrix}, x^* \ge 0. \tag{3.3}$$

This means that x^* has at most $n+1$ nonzero entries $\{x_{j_1}, \dots, x_{j_{n+1}}\}$ corresponding to the independent rows of A. Then relabel $\theta_k = x_{j_k}$ for $k = 1, \dots, n+1$. □

Any element belongs to n-dimensional convex polytope can be represented by a convex combination of atmost $n+1$ vectors.

3.1.2 Separating hyperplanes

We discuss now the geometry of convex sets and of hyperplanes.

Lemma 3.1.2 *Let C be a nonempty, closed, convex, subset of $\mathbb{R}^n$ and suppose $0 \notin C$. Then there exists a unique point $x_0 \in C$ such that $|x_0| = \min\ \{|x| : x \in C\}$. Furthermore, $0 \leq x_0^T(x - x_0)$ for all $x \in C$.*

For a convex set C that does not contain the origin, we can find a point x_0 that is closest to the origin. Then, draw a line perpendicular to the line connects the origin and x_0 at x_0. And we find out that the convex set C is (not strictly) on one side of the perpendicular line.

Proof. Step 1: Let $\delta = \inf\ \{|x| : x \in C\} \geq 0$. Select $\{x^k\}_{k=1}^{\infty} \subset C$ with $\delta = \lim_{k\to\infty} |x^k|$. According to Bolzano-Weierstrass Theorem, there exists a convergent subsequence $\lim_{j\to\infty} x^{k_j} = x_0$. Consequently, $|x_0| = \lim_{j\to\infty} x^{k_j} = \delta$. Since C is closed, $x_0 \in C$. And since $0 \notin C$, $\delta > 0$.

Step 2: We claim that x_0 is a unique point in C with $|x_0| = \delta$. Too see this, suppose $x_1 \in C$ also satisfies $|x_1| = \delta$. Then $\frac{x_0+x_1}{2} \in C$ and $\left|\frac{x_0+x_1}{2}\right| \geq \delta$. However,

$$\underbrace{|x_0 - x_1|^2}_{\geq 0} + \underbrace{|x_0 + x_1|^2}_{\geq 4\delta^2} = \underbrace{2(|x_0|^2 + |x_1|^2)}_{=4\delta^2}. \tag{3.4}$$

Thus we must have $x_0 = x_1$.

Step 3: Let x be any point in C. Then for $0 < \theta < 1$, $(1-\theta)x_0 + \theta x \in C$. Therefore,

$$|x_0|^2 \leq |(1-\theta)x_0 + \theta x|^2 \tag{3.5}$$
$$= |x_0 + \theta(x - x_0)|^2 \tag{3.6}$$
$$= |x_0|^2 + 2\theta x_0^T(x - x_0) + \theta^2|x - x_0|^2 \tag{3.7}$$
$$0 \leq 2x_0^T(x - x_0) + \theta|x - x_0|^2 \tag{3.8}$$

Sending $\theta \to 0$, we have $0 \leq x_0^T(x - x_0)$. □

Definition 3.1.4 (Hyperplane) *Let $a \in \mathbb{R}^n$, $b \in \mathbb{R}$. An expression of the form $a^T x + b = 0$ determines a hyperplane in $\mathbb{R}^n$. The hyperplane is $\{x \in \mathbb{R}^n : a^T x + b = 0\}$.*

It is an $(n-1)$-dimensional affine subspace and passes through the origin if and only if $b = 0$.

Definition 3.1.5 *Let S_1, S_2 be two subsets of $\mathbb{R}^n$. The hyperplane $a^T x + b = 0$ separates S_1 and S_2 if $a^T x + b \geq 0$ for all $x \in S_1$ and $a^T x + b \leq 0$ for all $x \in S_2$. The hyperplane $a^T x + b = 0$ strictly separates S_1 and S_2 if $a^T x + b > 0$ for all $x \in S_1$ and $a^T x + b < 0$ for all $x \in S_2$.*

Theorem 3.1.3 (Separating Hyperplane Theorem) *Let C be convex, closed and non-empty, and suppose $e \notin C$. Then there exists a hyperplane $a^T x + b$ that strictly separates C and $\{e\}$.*

Remark 3.1.1 It is important for subsequent applications that in Separating Hyperplane Theorem, we do not require that C be bounded.

Proof. Step 1: Upon shifting the coordinates if necessary, we may assume $e = 0$. According to Lemma 3.1.2, there exists $x_0 \in C$ such that $0 < \delta = |x_0| = \min\{|x| : x \in C\}$. We construct the separating hyperplane by finding a perpendicular bisector of the segment $\overline{ex_0}$. Let $m = x_0/2$ and $a = x_0/\delta$, then $a^T(x - m) = 0$ is the separating hyperplane. Or equivalently, $a^T x + b = 0$ where $b = -a^T m$.

Step 2: For $e = 0$, $a^T(0 - m) = -\frac{|x_0|^2}{2\delta} < 0$.

For C, taking $x \in C$, we have

$$a^T(x - m) = \frac{1}{\delta} x_0^T (x - x_0/2) \tag{3.9}$$

$$= \frac{1}{\delta}\left(\underbrace{x_0^T(x - x_0)}_{\geq 0} + \underbrace{|x_0|^2/2}_{>0} \right) \tag{3.10}$$

$$> 0 \tag{3.11}$$

Thus the hyperplane $a^T x + b$ strictly separates $\{e\}$ and C. □

3.1.3 Dual convex sets

As a first application of separating hyperplanes, we discuss next a geometric form of convex duality.

Definition 3.1.6 (Polar dual) *Let $C \subset \mathbb{R}^n$ be closed and convex, with $0 \in C$. Its polar dual is the set $C^0 = \{y \in \mathbb{R}^n : x^T y \leq 1, \forall x \in C\}$.*

Theorem 3.1.4 (Dual convex sets) *Under Definition 3.1.6:*

1. *C^0 is closed, convex, with $0 \in C^0$.*
2. *We have the duality assertion $(C^0)^0 = C$.*

Proof. Step 1: To see C^0 is closed, we can equivalently show that $(C^0)^c = \{y \in \mathbb{R}^n : x^T y > 1, \exists x \in C\}$ is open. To check openness, we must show that for any $y_0 \in (C^0)^c$, there exists a r-ball such that $B_r(y_0) \subset (C^0)^c$. Let us take a fixed $y_0 \in (C^0)^c$, there exists $x_0 \in C$ with $x_0^T y_0 = \delta > 1$. Let $r > 0$ satisfy $|x_0| r < \delta - 1$, $y_r \in B_r(y_0)$. And denote $\Delta y_r = y_r - y_0$. Clearly, $|\Delta y_r| \leq r$ and

$$x_0^T y_r = x_0^T y_0 + x_0^T \Delta y_r \tag{3.12}$$

$$\geq \delta - |x_0^T \Delta y_r| \tag{3.13}$$

$$\geq \delta - |x_0||\Delta y_r| \tag{3.14}$$

$$\geq \delta - |x_0| r > 1. \tag{3.15}$$

So that for all $y_0 \in (C^0)^c$, there exists a r-ball such that $B_r(y_0) \subset (C^0)^c$. Thus $(C^0)^c$ is open and C^0 is closed.

Step 2: C^0 is convex. For any $a, b \in C^0$, for all $x \in C$, we have $x^T a \leq 1$ and $x^T b \leq 1$. Consequently, for all $0 \leq \theta \leq 1$, $x^T(\theta a + (1-\theta)b) \leq 1$. Thus $\theta a + (1-\theta)b \in C^0$ as well.

Step 3: For any $x \in C$, $x^T 0 = 0 \leq 1$. Therefore $0 \in C^0$.

Step 4: Note that $(C^0)^0 = \{z \in \mathbb{R}^n : y^T z \leq 1, \forall y \in C^0\}$. Let $x \in C$. Then $y^T x \leq 1$ for all $y \in C^0$ and thus $x \in (C^0)^0$. Consequently $C \subseteq (C^0)^0$.

If $z \in (C^0)^0 - C$, then since C is closed, Theorem 3.1.3 says there exists $a \in \mathbb{R}^n, b \in \mathbb{R}$ such that

1. $a^T z + b < 0$,
2. $a^T x + b > 0$ for all $x \in C$.

Since $0 \in C$, 2 implies $b > 0$. So if we write $y = -\frac{a}{b}$, 2 says also that $y^T x < 1$ for all $x \in C$. Hence $y \in C^0$. Since $z \in (C^0)^0$, it follows that $y^T z \leq 1$; and therefore $a^T z + b \geq 0$. But this contradicts 1. Thus $(C^0)^0 - C$ is empty and hence $(C^0)^0 = C$. □

3.1.4 Farkas alternative

Our next goal is the Farkas alternative, a statement about solving vector inequalities. This turns out to have a surprising geometric interpretation involving separating hyperplanes for certain convex cones.

Definition 3.1.7 (Finite cone) *Let $\{a^1, \ldots, a^n\} \subset \mathbb{R}^m$. The finite cone generated by $\{a^1, \ldots, a^n\}$ is the set $C = \{\sum_{i=1}^n x_i a^i : x_i \geq 0\}$.*

The difference between a convex polytope under Definition 3.1.3 and a finite cone is that a finite cone does not require $\sum_{i=1}^n x_i = 1$.

Observe that $b \in C$ precisely when we can solve $Ax = b, x \geq 0$, when $A = [a^1 \ldots a^n]$ is the $m \times n$ matrix whose columns are $\{a^1, \ldots, a^n\}$.

Definition 3.1.8 (Basic cone) *If $\{a^1, \ldots, a^k\}$ are independent, we call the finite cone they generate a basic cone.*

For a basic cone, the solution for $Ax = b, x \geq 0$ is unique where again $A = [a^1 \ldots a^n]$.

Lemma 3.1.5 *Suppose $\{a^1, \ldots, a^n\}$ generate the finite cone C. Let $C_1, \ldots, C_q$ be the basic cones generated by all linearly independent subsets of $\{a^1, \ldots, a^n\}$. Then*

$$C = \bigcup_{i=1}^{q} C_i. \tag{3.16}$$

Proof. Obviously, $C_i \subseteq C$, so that $\cup_{i=1}^q C_i \subseteq C$.

Now select $b \in C = \{Ax : x \geq 0\}$. There exists a solution of $Ax = b, x \geq 0$. And according to Theorem 2.1.6, there in fact exists a basic solution x^* such that $Ax^* = b, x^* \geq 0$. This means that the columns $\{a^{j_1}, \ldots, a^{j_m}\}$ of A corresponding to the nonzero entries of x^* are independent. So b belongs to the basic cone generated by $\{a^{j_1}, \ldots, a^{j_m}\}$ and thus $b \in \cup_{i=1}^q C_i$. This is true for all $b \in C$, consequently $C \subseteq \cup_{i=1}^q C_i$. □

Theorem 3.1.6 *Let C be the finite cone generated by $\{a^1, \ldots, a^n\} \subset \mathbb{R}^m$. Then C is convex and closed.*

Proof. Step 1: We need to show that C is convex. Let $b^1, b^2 \in C, 0 \leq \theta \leq 1$. Then there exist x^1, x^2 such that $b^1 = Ax^1, x^1 \geq 0$ and $b^2 = Ax^2, x^2 \geq 0$. Therefore $(1-\theta)b^1 + \theta b^2 = A\left((1-\theta)x^1 + \theta x^2\right)$ for $(1-\theta)x^1 + \theta x^2 \geq 0$. Thus $(1-\theta)b^1 + \theta b^2 \in C$, and so C is convex.

Step 2: Let C_i be a basic cone generated by an independent set$\{a^{j_1}, \dots, a^{j_l}\} \subseteq \{a^1, \dots, a^n\}$. Assume $\{b^k\}_{k=1}^{\infty} \subset C_i$, with $\lim_{k\to\infty} b^k = b^0$. We claim that $b^0 \in C_i$ and this will show that C_i is closed.

First, let us write $B = \{j_1, \dots, j_l\}$. Since the vectors $\{a^j : j \in B\}$ are independent, if $u = [u_{j_1} \dots u_{j_l}]^T \in \mathbb{R}^l$ and $\sum_{j\in B} u_j a^j = 0$, it follows that $u = 0$. Therefore for all $u \in \mathbb{R}^l$ with $|u| = 1$, $\sum_{j\in B} u_j a^j \neq 0$. Hence the Extreme Value Theorem implies that there exists $\epsilon > 0$ such that $\min\left\{|\sum_{j\in B} u_j a^j| : |u| = 1\right\} = \epsilon > 0$. Thus if $v \in \mathbb{R}^l$,

$$|\sum_{j\in B} v_j a^j| \geq \epsilon |v|. \tag{3.17}$$

We now turn to the proof of $b^0 \in C_i$. Observe that we can write $b^k = Ax^k$, where $x^k \geq 0, x^k = [0\, x^k_{j_1}\, 0 \dots x^k_{j_l}\, 0]^T$. Then $b^k = \sum_{j\in B} x^k_j a^j$, and therefore (3.17) implies

$$|x^k| \leq \frac{1}{\epsilon}|b^k| \tag{3.18}$$

for $k = 1, \dots$ The sequence $\{x^k\}_{k=1}^{\infty}$ is therefore bounded, and so we can apply the Bolzano-Weierstrass Theorem to extract a convergent subsequence:

$$\lim_{j\to\infty} x^{k_j} = x^0. \tag{3.19}$$

Then $x^0 \geq 0$ and $Ax^0 = \lim_{j\to\infty} Ax^{k_j} = \lim_{j\to\infty} b^{k_j} = b^0$. Furthermore $x^0_j = 0$ except possibly for the indices $j \in B$. Hence $b^0 \in C_i$.

Step 3: So each basic cone C_i is closed. The finite union of closed sets is closed, and hence $C = \cup_{i=1}^q C_i$ is closed. □

In view of the previous theorem, we can apply the Separating Hyperplane Theorem to a finite cone. This has the following major payoff:

Theorem 3.1.7 (Farkas alternative) *Let A be an $m \times n$ matrix, $b \in \mathbb{R}^m$. Then either*

1. $Ax = b, x \geq 0$ *has a solution* $x \in \mathbb{R}^n$*; or*
2. $A^T y \geq 0, b^T y < 0$ *has a solution* $y \in \mathbb{R}^m$*,*

but not both.

Proof. Step 1: Assume x solves 1, y solves 2. Then $0 \leq x^T(A^T y) = (Ax)^T y = b^T y < 0$ which is a contradiction. So 1 and 2 cannot both be true.

Step 2: Suppose 1 fails. We will show that then 2 must hold. The failure of 1 means $b \notin C = \{Ax : x \geq 0\}$. Since C is a finite cone, Theorem 3.1.6

tells us that C is closed and convex. Then the Separating Hyperplane Theorem (Theorem 3.1.3) asserts that there exist $a \in \mathbb{R}^m, c \in \mathbb{R}$ such that

$$a^T z + c > 0 \quad (z \in C) \tag{3.20}$$

and

$$a^T b + c < 0. \tag{3.21}$$

Let $x \geq 0, \mu \geq 0$. Set $z = \mu A x = A(\mu x) \in C$. Then according to (3.20), $a^T(\mu A x) + C > 0$. Dividing by $\mu > 0$ and letting $\mu \to \infty$, we see that $a^T A x \geq 0$. So $(A^T a)^T x \geq 0$ for all $x \geq 0$. Thus $A^T a \geq 0$. Let $y = a$, then $A^T y \geq 0$. Put $z = 0$ in (3.20), to deduce that $c > 0$. Then (3.21) says

$$b^T y = b^T a < -c < 0. \tag{3.22}$$

□

3.2 Convex functions

A convex function is a real-valued function such that the region above its graph is a convex set. Convex functions therefore inherit many useful properties from convex sets.

3.2.1 Convex functions of one variable

Definition 3.2.1 *A function $f : \mathbb{R} \to \mathbb{R}$ is called convex if*

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta) f(x_2) \tag{C_1}$$

for all $x_1, x_2 \in \mathbb{R}, 0 \leq \theta \leq 1$.

A function $g : \mathbb{R} \to \mathbb{R}$ is called concave if $-g$ is convex.

If f is convex, then for all points x_1, x_2, the graph of f lies below the line segment connecting $[x_1\ f(x_1))]^T$ and $[x_2\ f(x_2)]^T$.

It is easy to see that $f : \mathbb{R} \to \mathbb{R}$ is a convex function if and only if its epigraph

$$E = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y \geq f(x), x \in \mathbb{R} \right\} \subset \mathbb{R}^2 \tag{3.23}$$

is a convex set.

Remark 3.2.1 It follows by induction that if $f : \mathbb{R} \to \mathbb{R}$ is convex, then

$$f(\sum_i \theta_i x_i) \leq \sum_i \theta_i f(x_i) \tag{3.24}$$

for all $x_i \in \mathbb{R}$, $\theta_i \geq 0$ and $\sum_i \theta_i = 1$.

Theorem 3.2.1 (Equivalent characterizations of convexity) *If $f : \mathbb{R} \to \mathbb{R}$ is continuously differentiable, then f is convex if and only if*

$$f(x_1) + f'(x_1)(x_2 - x_1) \leq f(x_2) \tag{C_2}$$

for all $x_1, x_2 \in \mathbb{R}$.

If f *is twice continuously differentiable, then* f *is convex if and only if*

$$f''(x) \geq 0 \tag{C_3}$$

for all $x \in \mathbb{R}$.

Proof. Step 1: Assume f is continuously differentiable, and let us show (C_1) holds if and only if (C_2) holds. So, suppose (C_1). For $\theta > 0$,

$$f(\theta x_2 + (1-\theta)x_1) \leq \theta f(x_2) + (1-\theta) f(x_1) \tag{3.25}$$

$$\frac{f(x_1 + \theta(x_2 - x_1)) - f(x_1)}{\theta} \leq f(x_2) - f(x_1) \tag{3.26}$$

Let $\theta \longrightarrow 0$, to deduce that $f'(x_1)(x_2 - x_1) \leq f(x_2) - f(x_1)$ which is (C_1).

Now assume (C_2). Then if $w = \theta x_1 + (1-\theta)x_2$, we have

$$f(x_1) \geq f(w) + f'(w)(x_1 - w) \tag{3.27}$$

$$f(x_2) \geq f(w) + f'(w)(x_2 - w). \tag{3.28}$$

So

$$\theta f(x_1) + (1-\theta) f(x_2) \geq f(w) + f'(w) \underbrace{(\theta(x_1 - w) + (1-\theta)(x_2 - w))}_{=0} \tag{3.29}$$

$$= f(\theta x_1 + (1-\theta)x_2). \tag{3.30}$$

So we get (C_1).

Step 2: Suppose now that f is twice continuously differentiable. We will show that (C_2) holds if and only if (C_3) holds. TODO □

The condition (C_3) is especially convenient for checking if a given function is convex or not. But the graphs of convex functions can have corners, and so convex functions need not be twice or even once continuously differentiable. However, they are always continuous.

Theorem 3.2.2 (Convex functions are continuous) *If* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *is convex, then* f *is continuous.*

Proof. Step 1: First we show that f is bounded on each interval $[a, b]$ of finite length. For each $a \leq x \leq b$, we can write $x = \theta a + (1-\theta)b$ where $0 \leq \theta \leq 1$. Therefore,

$$f(x) \leq \theta f(a) + (1-\theta) f(b) \leq \max\{|f(a)|, |f(b)|\}. \tag{3.31}$$

Now if $\frac{a+b}{2} \leq x \leq b$, then $\frac{a+b}{2} = \theta x + (1-\theta)a$ for $\theta = \frac{b-a}{2(x-a)}$. Then $1/2 \leq \theta \leq 1$ and convexity implies $f(\frac{a+b}{2}) \leq \theta f(x) + (1-\theta) f(a)$. Hence

$$f(x) \geq \frac{1}{\theta}\left(f(\frac{a+b}{2}) - (1-\theta) f(a)\right) \geq -2\left(|f(\frac{a+b}{2})| + |f(a)|\right) \tag{3.32}$$

since $\frac{1}{\theta} \leq 2$. Likewise, if $a \leq x \leq \frac{a+b}{2}$, we also have

$$f(x) \geq -2\left(|f(\frac{a+b}{2})| + |f(a)|\right). \tag{3.33}$$

Therefore

$$\sup_{[a,b]} |f(x)| \leq 4\max\{|f(a)|, |f(\frac{a+b}{2})|, |f(b)|\} \leq \infty. \tag{3.34}$$

Step 2: Then we prove the continuity for f on $[0,1]$. Let $0 \leq x < y \leq 1$. Then $y = \theta x + (1-\theta)2$ for $\theta = \frac{2-y}{2-x}$. Thus $f(y) \leq \theta f(x) + (1-\theta)f(2)$, and so

$$\begin{aligned} f(y) - f(x) &\leq (1-\theta)(f(2) - f(x)) && (3.35)\\ &= \underbrace{\frac{1}{2-x}}_{\in[1/2,1]}(y-x)\underbrace{(f(2)-f(x))}_{\leq 2\sup_{[0,2]}|f|} && (3.36)\\ &\leq 2|y-x|\sup_{[0,2]}|f(x)| && (3.37)\end{aligned}$$

Similarly, we have $x = \theta y + (1-\theta)(-2)$ for $\theta = \frac{y+2}{x+2}$. Then

$$f(x) - f(y) \leq 2|y-x| \sup_{[-2,1]} |f(x)|. \tag{3.38}$$

Together we have

$$|f(y) - f(x)| \leq 2|y-x| \sup_{[-2,2]} |f(x)|. \tag{3.39}$$

Since $\sup_{[-2,2]}|f(x)|$ is bounded, this inequality implies f is continuous on $[0,1]$.

Step 3: We will finish by shifting f on any finite interval $[a,b]$ to $\hat{f}$ that defined on $[0,1]$. Let $\hat{f}(x) = f(\frac{x-a}{b-a})$. The continuity for $\hat{f}$ is also applied for f. If f is continuous, then $\hat{f}(x) = f(ax+b)$ is also continuous. For all $\epsilon > 0$, there exists a $r_0 > 0$ such that with given x_0 for all $y \in B_{r_0}(x_0)$, $|f(y) - f(x_0)| < \epsilon$. So if we have $x_0 = ax + b$ and $r = r_0/a$, then within $y \in B_r(x)$,

$$|\hat{f}(y) - \hat{f}(x)| = |f(ay+b) - f(ax+b)| < \epsilon. \tag{3.40}$$

So that $\hat{f}$ is continuous. □

3.2.2 Convex functions of more variables

Definition 3.2.2 *A function $f : \mathbb{R}^n \to \mathbb{R}$ is called convex if*

$$f(\theta x_1 + (1-\theta)x_2) \leq \theta f(x_1) + (1-\theta)f(x_2) \tag{C_1'}$$

for all $x_1, x_2 \in \mathbb{R}, 0 \leq \theta \leq 1$.

A function $g : \mathbb{R} \to \mathbb{R}$ is called concave if $-g$ is convex.

If f is convex, then for all points x_1, x_2, the graph of f lies below the line segment connecting $[x_1\ f(x)1)]^T$ and $[x_2\ f(x_2)]^T$.

It is easy to see that $f : \mathbb{R}^n \to \mathbb{R}$ is a convex function if and only if its epigraph

$$E = \left\{ \begin{bmatrix} x \\ y \end{bmatrix} : y \geq f(x), x \in \mathbb{R}^n \right\} \subset \mathbb{R}^{n+1} \tag{3.41}$$

is a convex set.

Remark 3.2.2 It follows by induction that if $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then

$$f(\sum_i \theta_i x_i) \leq \sum_i \theta_i f(x_i) \tag{3.42}$$

for all $x_i \in \mathbb{R}^n$, $\theta_i \geq 0$ and $\sum_i \theta_i = 1$.

Theorem 3.2.3 (Equivalent characterizations of multivariable convexity) *If $f : \mathbb{R}^n \to \mathbb{R}$ is continuously differentiable, then f is convex if and only if*

$$f(x_1) + \nabla f(x_1)^T (x_2 - x_1) \leq f(x_2) \tag{C_2'}$$

for all $x_1, x_2 \in \mathbb{R}^n$.

If f is twice continuously differentiable, then f is convex if and only if

$$\nabla^2 f(x) \geq 0 \tag{C_3'}$$

for all $x \in \mathbb{R}^n$.

Proof. Step 1: We claim that $f : \mathbb{R}^n \to \mathbb{R}$ is convex if and only if $\phi(t) = f(x + ty)$ is a convex function of t for all $x, y \in \mathbb{R}^n$.

Assume f is convex. Select $x, y \in \mathbb{R}^n$, $0 \leq \theta \leq 1$, $t_1, t_2 \in \mathbb{R}$, and $z^1 = x + t_1 y, z^2 = x + t_2 y$. Then

$$\begin{aligned} \phi(\theta t_1 + (1-\theta)t_2) &= f(\theta z_1 + (1-\theta)z_2) && (3.43) \\ &\leq \theta f(z_1) + (1-\theta) f(z_2) && (3.44) \\ &= \theta\phi(t_1) + (1-\theta)\phi(t_2). && (3.45) \end{aligned}$$

Thus ϕ is convex.

Conversely, let $0 \leq \theta \leq 1$, $x^1, x^2 \in \mathbb{R}^n$ and $\phi(t) = f(x_2 + t(x_1 - x_2))$. If phi is convex, then

$$\begin{aligned} f(\theta x_1 + (1-\theta)x_2) = \phi(\theta) &= \phi(\theta \cdot 1 + (1-\theta)\cdot 0) && (3.46) \\ &\leq \theta\phi(1) + (1-\theta)\phi(0) && (3.47) \\ &= \theta f(x_1) + (1-\theta) f(x_2) && (3.48) \end{aligned}$$

So if $\phi(t) = f(x + ty)$ is convex for all $x, y \in \mathbb{R}^n$, f is convex.

Step 2: The one dimensional version (C_2) implies that for all $t_1, t_2 \in \mathbb{R}$, $\phi(t_1) + \phi(t_1)(t_2 - t_1) \leq \phi(t_2)$ for convex function $\phi(t) = f(x + ty)$. Also $\phi'(t) = \nabla f(x+ty)^T y$. Let $t_1 = 0, t_2 = 1$ to get $f(x) = \nabla f(x)^T y \leq f(x+y)$ which is equivalent to $f(x_1) + \nabla f(x_1)^T (x_2 - x_1) \leq f(x_2)$. This is (C_2') for the function f.

Step 3: The one dimensional version (C_3) for convex ϕ says $\phi''(t) \geq 0$ where $\phi(t) = f(x + ty)$. Then $\phi'(t) = \nabla f(x + ty)^T y$ and $\phi''(t) = y^T \nabla^2 f(x + ty)y$. Let $t = 0$, so that $y^T \nabla^2 f(x)y \geq 0$ for all $y \in \mathbb{R}^n$. This is (C_3') for the function f. □

Theorem 3.2.4 (Multivariable convex functions are continuous) *If $f : \mathbb{R}^n \to \mathbb{R}$ is convex, then f is continuous.*

3.2.3 Subdifferentials

Definition 3.2.3 (Subdifferential) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. For each $x_1 \in \mathbb{R}^n$, we define*

$$\partial f(x_1) = \{r \in \mathbb{R}^n : f(x_1) + r^T(x_2 - x_1) \leq f(x_2), \forall x_2 \in \mathbb{R}^n\}. \tag{3.49}$$

This set is called the subdifferential of f at x_1.

Example 3.2.1 Let $n = 1$ and $f(x) = |x|$. Then

$$\partial f(x) = \begin{cases} \{-1\}, & x < 0 \\ [-1, 1], & x = 0 \\ \{1\}, & x > 0 \end{cases} \tag{3.50}$$

Theorem 3.2.5 (Properties of subdifferential) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be convex. Then for each $x \in \mathbb{R}^n$, $\partial f(x)$ is a closed, convex, and non-empty set.*

Proof. Step 1: Convexity. For any $r_1, r_2 \in \partial f(x)$ and $0 \leq \theta \leq 1$,

$$f(x) + [\theta r_1 + (1-\theta) r_2]^T (x_1 - x) = \theta [f(x) + r_1(x_1 - x)] + (1-\theta)[f(x) + r_2(x_1 - x)] \tag{3.51}$$

$$\leq f(x_1) \tag{3.52}$$

Thus, $[\theta r_1 + (1-\theta) r_2] \in \partial f(x)$.

Step 2: Closeness. Assume now $\{r_k\}_{k=1}^{\infty} \subset \partial f(x)$ and $r_0 = \lim_{k\to\infty} r_k$. Then for each k and each x_1, $f(x) + r_k^T(x_1 - x) \leq f(x_1)$. Let $k \to \infty$ to deduce that $f(x) + r_0^T(x_1 - x) \leq f(x_1)$ for each $x_1 \in \mathbb{R}^n$ and hence $r_0 \in \partial f(x)$. Consequently, $\partial f(x)$ is closed.

Step 3: Non-emptiness. Select any point $x \in \mathbb{R}^n$. Will show that $\partial f(x) \neq \emptyset$. TODO.

□

3.2.4 Dual convex functions

For this section, assume $f : \mathbb{R}^n \to \mathbb{R}$ is convex, with

$$\lim_{|x|\to\infty} \frac{f(x)}{|x|} = +\infty. \tag{3.53}$$

This is called a super-linear growth condition.

Definition 3.2.4 *For $y \in \mathbb{R}^n$, the dual convex function (or Legendre transform) of f is*

$$f^*(y) = \max_{x \in \mathbb{R}^n} \{x^T y - f(x)\}. \tag{3.54}$$

Example 3.2.2 Let $f(x) = \frac{x^2}{2}$ for $x \in \mathbb{R}$. Then

$$f^*(y) = \max_{x \in \mathbb{R}} \{xy - \frac{x^2}{2}\} = \frac{y^2}{2} \tag{3.55}$$

Lemma 3.2.6 (Fenchel-Young inequality) *For all $x, y \in \mathbb{R}^n$ we have*

$$x^T y \leq f(x) + f^*(y). \tag{3.56}$$

Theorem 3.2.7 *The function $f^* : \mathbb{R}^n \to \mathbb{R}$ is convex.*

Proof.

$$f^*(\theta y_1 + (1-\theta) y_2) = \max_x \left\{ x^T (\theta y_1 + (1-\theta) y_2) - f(x) \right\} \tag{3.57}$$

$$= \max_x \left\{ \theta \left[x^T y_1 - f(x) \right] + (1-\theta) \left[x^T y_2 - f(x) \right] \right\} \tag{3.58}$$

$$\leq \theta \max_x \left\{ x^T y_1 - f(x) \right\} + (1-\theta) \max_x \left\{ x^T y_2 - f(x) \right\} \tag{3.59}$$

$$= \theta f^*(y_1) + (1-\theta) f^*(y_2) \tag{3.60}$$

□

Theorem 3.2.8 $\lim_{|y| \to \infty} \frac{f^*(y)}{|y|} = +\infty$.

Proof. According to Lemma 3.2.6, $f(x) + f^*(y) \geq x^T y$ for all x, y. Fix $y \neq 0, \mu > 0$ and let $x = \frac{\mu y}{|y|}$. Then

$$f^*(y) \geq \left(\frac{\mu y}{|y|} \right)^T y - f(\frac{\mu y}{|y|}) \qquad \geq \mu |y| - \max_{x \in B_\mu(0)} f(x) \tag{3.61}$$

So

$$\frac{f^*(y)}{|y|} \geq mu - \frac{1}{|y|} \max_{x \in B_\mu(0)} f(x) \tag{3.62}$$

$$\lim_{|y| \to \infty} \frac{f^*(y)}{|y|} \geq \mu \tag{3.63}$$

for all $\mu > 0$. □

Theorem 3.2.9 $f^{**} = f$.

Proof. Step 1: According to Lemma 3.2.6, $f(x) \geq x^T y - f^*(y)$ for all x, y.

$$f(x) \geq \max_y \left\{ x^T y - f^*(y) \right\} = f^{**}(x) \tag{3.64}$$

Step 2: Theorem 3.2.5 tells us that $\partial f(x)$ is non-empty. Select $r \in \partial f(x)$, then $f(z) \geq f(x) + r^T(z - x)$ for all $z \in \mathbb{R}^n$. Consequently[1],

1: FIRST =,WHY???

$$r^T x - f(x) = \max_z \{r^T z - f(z)\} = f^*(r) \tag{3.65}$$

and so

$$f^{**}(x) = \max_y \left\{x^T y - f^*(y)\right\} \geq x^T r - f^*(r) = f(x) \tag{3.66}$$

□

Theorem 3.2.10 (Subdifferentials and dual functions) *For all points $x, y \in \mathbb{R}^n$, the following are equivalent.*

1. $x^T y = f(x) + f^*(y)$;
2. $y \in \partial f(x)$;
3. $x \in \partial f^*(y)$.

Proof. TODO □

3.3 Applications

TODO

Nonlinear Optimization 4

In this chapter we examine minimization problems with inequality constraints and study when and how Lagrange multipliers can be used to characterize minimizers.

4.1 Inequality constraints

Assume $f, h_1, \dots, h_p : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable.

As usual, we write

$$h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_p \end{bmatrix} \tag{4.1}$$

and

$$\nabla h = \begin{bmatrix} (\nabla h_1)^T \\ (\nabla h_2)^T \\ \vdots \\ (\nabla h_p)^T \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \frac{\partial h_1}{\partial x_2} & \cdots & \frac{\partial h_1}{\partial x_n} \\ \frac{\partial h_2}{\partial x_1} & \frac{\partial h_2}{\partial x_2} & \cdots & \frac{\partial h_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \frac{\partial h_p}{\partial x_2} & \cdots & \frac{\partial h_p}{\partial x_n} \end{bmatrix}. \tag{4.2}$$

We study in this section the constrained optimization problem to find $x_0 \in \mathbb{R}^n$ to

$$\begin{aligned} &\min f(x) \\ &\text{s.t. } h(x) \le 0 \end{aligned} \tag{NLP}$$

The requirement that $h_j(x) \le 0$ for $j = 1, \dots, p$ are inequality constraints. The j-th constraint is active if $h_j(x) = 0$. A point x is feasible for (NLP) if $h(x) \le 0$.

A basic question is how to characterize x_0 solving (NLP).

4.1.1 Constraint qualification

Suppose hereafter x_0 solves (NLP). Our plan is to make a first variation calculation, but for this we need to be careful in designing an appropriate curve of variations staying within the feasible region.

We write $J = \{j \in \{1, \dots, p\} : h_j(x_0) = 0\}$. These are the indices of the active constraints for x_0.

We write $o(t)$ to denote any vector function $r(t)$ such that $\lim_{t\to 0_+} \frac{|r(t)|}{t} = 0$.

Definition 4.1.1 *We say the constraint qualification condition (CQ) holds at x_0 holds at x_0 if for each vector $y \in \mathbb{R}^n$ satisfying $y^T \nabla h_j(x_0 \leq 0$ for all $j \in J$, there exists a continuous curve $\{x(t) : 0 \leq t < t_0\}$ for some $t_0 > 0$ such that $h(x(t)) \leq 0$ within $0 \leq t < t_0$; and $x(t) = x_0 + ty + o(t)$ as $t \to 0_+$.*

4.1.2 Karush-Kuhn-Tucker conditions

Theorem 4.1.1 *Let x_0 solve (NLP) and suppose the constraint qualification condition holds at x_0. Then there exist a vector of real numbers $\mu_0 = [\mu_0^1, \ldots, \mu_0^p]^T \geq 0$ such that*

$$\nabla f(x_0) + \mu_0^T \nabla h(x_0) = 0. \tag{4.3}$$

Furthermore, the vector μ_0 satisfies

$$\mu_0^T h(x_0) = 0. \tag{4.4}$$

We interpret μ_0^j as the Lagrange multiplier for the constraint $h_j(x_0) \leq 0$. So for our inequality constrained problem (NLP), we are asserting both that Lagrange multipliers exist and that they are nonnegative.

In addition, if $h_j(x_0) < 0$ for some index j, that constraint is inactive and so the corresponding Lagrange multiplier μ_0^j equals zero. This is a complementary slackness condition.

Proof. Step 1. Assume that the vector y satisfies $y^T \nabla h_j(x_0) \leq 0$ for $j \in J$. Let $\{x(t) : 0 \leq t < t_0\}$ be the corresponding curve, whose existence is assured according to CQ. Write $\phi(t) = f(x(t))$. Then $\phi(0) = f(x_0) \leq f(x(t)) = \phi(t)$ for $0 \leq t < t_0$, since x_0 solves (NLP). Thus phi has a minimum at $t = 0$ and hence $\phi'(0) \geq 0$. Now $\phi'(0) = \nabla f(x_0)^T x'(0) = \nabla f(x_0)^T y$ and therefore $\nabla f(x_0)^T y \geq 0$ for all y satisfying $y^T \nabla h_j(x_0) \leq 0$ for $j \in J$.

Step 2. Recall that Theorem 3.1.7 states: For an $m \times n$ matrix A and $b \in \mathbb{R}^m$, either

1. $Ax = b, x \geq 0$ has a solution $x \in \mathbb{R}^n$; or
2. $A^T y \geq 0, b^T y < 0$ has a solution $y \in \mathbb{R}^m$,

but not both.

We apply this to

$$A = -\underbrace{[\nabla h_{j_1}(x_0)\ \nabla h_{j_2}(x_0)\ \cdots\ \nabla h_{j_k}(x_0)]}_{\text{columns}}, \tag{4.5}$$

$$b = \nabla f(x_0), \tag{4.6}$$

where $J = \{j_1, j_2, \ldots, j_k\}$. In Step 1, we have shown that $A^T y \geq 0$ implies $y^T b \geq 0$ and thus fails.

Consequently, 2 holds: there exists $\sigma_j \geq 0$ for $j \in J$ such that

$$-\sum_{j \in J} \sigma_j \nabla h_j(x_0) = \nabla f(x_0). \tag{4.7}$$

Define $\mu_0 \in \mathbb{R}^p$ by

$$\mu_0^j = \begin{cases} \sigma_j, & j \in J \\ 0, & j \notin J \end{cases} \tag{4.8}$$

then $\mu_0 \geq 0$, $\mu_0^T h(x_0) = 0$ and $\nabla f(x_0) + \mu_0^T h(x_0) = 0$. □

4.1.3 When does the constraint qualification condition hold?

The above proof is elegant, but it may be far from clear for particular problems if the constraint qualification condition is valid. We discuss next two important cases.

Linear inequality and equality constraints

Theorem 4.1.2 *If the functions* $\{h_j\}_{j=1}^p$ *are linear (or affine) functions of* x*, then (CQ) holds for each point* x_0*.*

Regular equality constraints

Definition 4.1.2 *We say that* x_0 *is regular for (NLP) if* $\{\nabla h_j(x_0)\}_{j \in J}$ *are linearly independent in* $\mathbb{R}^n$ *where* $J = \{j \in \{1, \ldots, p\} : h_j(x_0) = 0\}$*.*

Theorem 4.1.3 *If* x_0 *is regular for (NLP) then (CQ) holds at* x_0*.*

4.2 More on Lagrange multipliers

Given $f, g_1, \ldots, g_m, h_1, \ldots, h_p : \mathbb{R}^n \to \mathbb{R}$, we write

$$g = \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_m \end{bmatrix} \tag{4.9}$$

$$h = \begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_p \end{bmatrix}. \tag{4.10}$$

Our minimization problem for this section is to find $x_0 \in \mathbb{R}^n$ to

$$\begin{aligned} \min\ & f(x) \\ \text{s.t. } & g(x) = 0 \\ & h(x) \leq 0 \end{aligned} \tag{NLP*}$$

So there are m equality constraints and p inequality constraints.

4.2.1 F. John's formulation

Theorem 4.2.1 *Suppose that x_0 solves the constrained optimization problem (NLP*). Then there exists real number γ_0 and real-valued vectors λ_0, μ_0 not all equal to zero such that*

$$\gamma_0 \nabla f(x_0) + \lambda_0^T \nabla g(x_0) + \mu_0^T \nabla h(x_0) = 0 \tag{4.11}$$

and

$$\gamma_0 \geq 0, \mu_0 \geq 0, \mu_0^T h(x_0) = 0. \tag{4.12}$$

Remark 4.2.1 For $\gamma_0 > 0$, by dividing λ_0, μ_0 by γ_0, we convert F. John's condition to KKT conditions. If $\gamma_0 = 0$, we call it an abnormal multiplier.

4.3 Quadratic programming

The quadratic programming is to find $x_0 \in \mathbb{R}^n$ to

$$\begin{aligned} \min \ & \frac{1}{2}x^T C x + c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned} \tag{QP}$$

where C is a symmetric $n \times n$ matrix.

The problem (QP) is of the form (NLP*) for $f(x) = \frac{1}{2}x^T C x + c^T x$, $g = Ax - b$, $h = -x$.

Convex Optimization 5

We now make additional convexity assumptions, which will let us greatly strengthen the theory from the previous chapter.

5.1 Variational inequalities

Dynamic Optimization

Bibliography

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[1] Theo de Jong. "Lagrange Multipliers and the Fundamental Theorem of Algebra". In: *The American Mathematical Monthly* 116.9 (2009), pp. 828–830. DOI: 10.4169/000298909X474882 (cited on page 8).

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