ENGG 5781: Matrix Analysis and Computations **2020-21 First Term**

Final Examination

Time Limit: 2 hours and 5 minutes December 21, 2020

Note:

- This is an online open-book examination.
- For theorems and properties that appear in the course notes, you can use them directly and without proof. Exceptions are when you are asked to prove those theorems and properties.
- Theorems and properties outside the scope of the course notes **may not** be used. Exceptions are when those results can be straightforwardly deduced, and/or your answer also provide the proofs of those results.
- You should use the same conventions, notations, symbols, etc., used in the lecture notes, rather than those in other courses, textbooks, online resources, etc.
- Answer **any nine** of the twelve problems. There will be **no bonus** if you answer more than nine problems.
- On the first page of your answer, write down the problem numbers you choose. If you do not do so and answer more than nine questions, we will choose the **worst nine**.
- The problems are equally weighted, each with 10 points.

Problem 1 Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{B} \in \mathbb{R}^{k \times n}$. Suppose $\mathbf{A}$ has full column rank. Show that

$$\mathbf{C} = \begin{bmatrix} \mathbf{A} \\ \mathbf{B} \end{bmatrix}$$

has full column rank.

Problem 2 Let $\mathbf{A} \in \mathbb{R}^{m \times n}$, $\mathbf{B} \in \mathbb{R}^{m \times k}$. Suppose $\mathbf{A}$ and $\mathbf{B}$ both have full column rank, and $n + k > m$. Show that there exists $\mathbf{x} \in \mathbb{R}^n, \mathbf{y} \in \mathbb{R}^k$, such that

$$\mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{y}.$$

Also, we have $\mathbf{x} \neq \mathbf{0}, \mathbf{y} \neq \mathbf{0}$.

Problem 3 Let $\mathbf{A}_1 \in \mathbb{R}^{m \times n}, \mathbf{A}_2 \in \mathbb{R}^{k \times n}, \mathbf{y}_1 \in \mathbb{R}^m, \mathbf{y}_2 \in \mathbb{R}^k$. Suppose both $\mathbf{A}_1$ and $\mathbf{A}_2$ have full column rank. Show that the solution to

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{y}_1 - \mathbf{A}_1 \mathbf{x}\|_2^2 + \|\mathbf{y}_2 - \mathbf{A}_2 \mathbf{x}\|_2^2$$

is

$$\mathbf{x}^\star = (\mathbf{A}_1^T \mathbf{A}_1 + \mathbf{A}_2^T \mathbf{A}_2)^{-1} (\mathbf{A}_1^T \mathbf{y}_1 + \mathbf{A}_2^T \mathbf{y}_2).$$

Problem 4 Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Suppose $\mathbf{A}^2 = \mathbf{A}$. Show that an eigenvalue of $\mathbf{A}$ is either 0 or 1.

Problem 5 Show that if $\mathbf{A} \in \mathbb{R}^{n \times n}$ is orthogonal, then $|\det(\mathbf{A})| = 1$.

Problem 6 Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\mathbf{A}$, and let k be a positive integer. Show that the eigenvalues of $\mathbf{A}^k$, or more precisely, the roots of the characteristic polynomial of $\mathbf{A}^k$, are $\lambda_1^k, \ldots, \lambda_n^k$. If your answer assumes the existence of eigenvalue decomposition, 5 points will be the maximum number of points you can get.

Problem 7 Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$. Suppose $\mathbf{A}$ is positive definite, and $\mathbf{B}$ is positive semidefinite. Show that $\mathbf{A} + \mathbf{B}$ is positive definite.

Problem 8 Let $\mathbf{A} \in \mathbb{S}^n$. Show that $a_{11} + a_{22} \leq \lambda_1(\mathbf{A}) + \lambda_2(\mathbf{A})$.

Problem 9 Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a square or tall matrix. Show that

$$\min_{\|\mathbf{x}\|_2 = 1} \|\mathbf{A}\mathbf{x}\|_2^2 = (\sigma_{\min}(\mathbf{A}))^2.$$

Problem 10 Let $\mathbf{A} \in \mathbb{R}^{n \times n}$. Show that $|\lambda_i(\mathbf{A})| \leq \sigma_{\max}(\mathbf{A})$ for all i. You may make a simplifying assumption that the eigenvalues and eigenvectors of $\mathbf{A}$ are real-valued.

Problem 11 Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a full row-rank matrix. Show that the solution to

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{x}\|_2^2 \quad \text{s.t. } \mathbf{y} = \mathbf{A}\mathbf{x}$$

is uniquely given by $\mathbf{x} = \mathbf{A}^\dagger \mathbf{y}$.

Problem 12 Describe how you would use QR decomposition to solve the linear system $\mathbf{y} = \mathbf{A}\mathbf{x}$ computationally. Here $\mathbf{A} \in \mathbb{R}^{n \times n}$ is nonsingular.

The End