ZIGAM with Repeated Sampling in a Short Period

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1 Proof the Distribution of Latent n and Detections d is in Exponential Family

Proof. First, set $(n, \vec{d})$ is the latent population and detection vectors at site i. I intend to prove that the distribution of this vector is with in exponential family witch has form:

$$f(y|\theta) = b(y)exp(\eta^T T(y) - a(\eta)) \tag{1}$$

Assume detections given latent population n were binomial distributed with parameter p_j, thus:

$$\begin{aligned} P(\vec{d}|n) &= \prod_{j=1}^{w} \binom{n}{d_j} p_j^{d_j}(1-p_j)^{d_j} \\ &= \prod_{j=1}^{w} \binom{n}{d_j} exp(\sum_{j=1}^{w} d_j log\frac{p_j}{1-p_j} + n\sum_{j=1}^{w} log(1-p_j)) \end{aligned} \tag{2}$$

which showed that binomial distribution given n belongs to exponential family.

Then assume latent population was Poisson distributed with rate λ

$$\begin{aligned} P(n|\theta) &= e^{-\lambda}\frac{\lambda^n}{n!} \\ &= \frac{1}{n!} exp(nlog(\lambda) - \lambda) \end{aligned} \tag{3}$$

Thus the total probability contains the latent and detections is given by:

$$\begin{aligned} P(n, \vec{d}|\theta) &= P(d|n, \theta)P(n|\theta) \\ &= \frac{1}{n!}\prod_{j=1}^{w} \binom{n}{d_j} exp(\sum_{j=1}^{w} d_j log\frac{p_j}{1-p_j} + n\sum_{j=1}^{w} log(1-p_j))exp(nlog(\lambda) - \lambda) \\ &= \frac{1}{n!}\prod_{j=1}^{w} \binom{n}{d_j} exp[\sum_{j=1}^{w} d_j log\frac{p_j}{1-p_j} + n(\sum_{j=1}^{w} log(1-p_j) + log(\lambda)) - \lambda] \\ &= \frac{1}{n!}\prod_{j=1}^{w} \binom{n}{d_j} exp(\eta^T(n, \vec{d}) - \lambda) \end{aligned} \tag{4}$$

in which

$$\eta^T = (\sum_{j=1}^{w} log(1-p_j) + log(\lambda), log\frac{\vec{p}}{1-\vec{p}})$$

$$T(y) = y$$

λ can be calculated using η since it contains all p_j and λ itself. Later on, we note this function as

$$f(n, d|\theta)$$

$\square$

2 EM Algorithm to Deal with Missing Occupancy and Latent N

2.1 Expected logL

We first write down the total probability assuming knowing occupancy status z and latent population size n:

$$\begin{aligned} P(\vec{d}, n, z) &= [\psi Pois(n|\lambda) \prod_{j=1}^{w} Bin(d_j|n, p_j)]^z (I_{\vec{d}=0}(1-\psi))^{1-z} \\ &= (\psi f(n, \vec{d}|\theta))^z (I_{\vec{d}=0}(1-\psi))^{1-z} \end{aligned} \tag{5}$$

In E step of EM algorithm, we need $P(n, z|\vec{d})$, which need to sum all n up, here, we truncated it with some large N.

$$\begin{aligned} P(n, z|\vec{d}) &= \frac{P(n, z, \vec{d})}{\sum_{n=max(d_j)}^{N} \sum_{z=0}^{1} P(n, z, \vec{d})} \\ &= \frac{(\psi f(n, \vec{d}|\theta))^z (I_{\vec{d}=0}(1-\psi))^{1-z}}{\psi \sum_{n=max(d_j)}^{N} f(n, d|\theta) + (1-\psi) I_{\vec{d}=0}} \end{aligned} \tag{6}$$

Take expectation of logL of total likelihood under r^{th} θ given by summing every z and n up:

$$\mathbb{E}(l_p|\vec{d}, \theta^{[r]}) = \frac{\sum_n^N \psi^{[r]} f(n, \vec{d}|\theta^{[r]}) log[\psi f(n, \vec{d}|\theta)] + (1-\psi^{[r]}) I_{\vec{d}=0} log[(1-\psi) I_{\vec{d}=0}]}{\sum_n^N \psi^{[r]} f(n, \vec{d}|\theta^{[r]}) + (1-\psi^{[r]}) I_{\vec{d}=0}} \tag{7}$$

This involves large number of GAMs, but still can be solved via PRILS