1.28

proof Thm 1: Induction over size of A.

$1 \times 1 \rightarrow$ basis for $A \in \mathbb{C}^{m \times n}$, $\forall m$

$m > n$: $p \times 1$ $\qquad A \in \mathbb{C}^{m \times n}$

$m < n$: $1 \times p$

$$x = \underset{\hat{U}}{\frac{x}{\|x\|}} \cdot \underset{\hat{\Sigma}}{\|x\|} \cdot \underset{V^*}{1}$$

$$\Sigma = \begin{pmatrix} \sigma_1 \\ \vdots \\ 0 \end{pmatrix}$$

$$U = \begin{pmatrix} \frac{x}{\|x\|} & \cdots \end{pmatrix}$$

augment to be a basis in $\mathbb{C}^{m \times m}$

Hypothesis: any matrix with $\leq m-1$ rows and $\leq n-1$ columns have SVD.

Inductive step:

$S_n :=$ unit sphere in $\mathbb{C}^n = \{x \in \mathbb{C}^n : \|x\|_2 = 1\}$

$$S_n \xrightarrow{A} \mathbb{C}^m \xrightarrow{\|\cdot\|_2} \mathbb{R}^+$$

$\forall x \in S_n$, $Ax \in \mathbb{C}^m$

$$\max_{x \in S_n} \|Ax\|_2 = \|Av_1\|_2, \quad \exists v_1 \in S_n$$

$$\| \quad = \|A\|_2 = \sigma_1$$

$$\sup \frac{\|Ay\|_2}{\|y\|_2}$$

Define: $u_1 := Av_1/\sigma_1 \in \mathbb{C}^m$

$\|u_1\|_2 = 1$, $u_1 \in S_m$

complete to obtain othonormal basis of $\mathbb{C}^n$

$V_1 := (v_1, v_2, \cdots v_n) =$ unitary

similar for $\mathbb{C}^m$

$U_1 := (u_1, u_2, \cdots u_m) =$ unitary

$$M := U_1^* A V_1 = \begin{pmatrix} u_1^* \\ u_2^* \\ \vdots \\ u_m^* \end{pmatrix} (Av_1, Av_2, \cdots Av_n)$$

$$Av_1 \downarrow \sigma_1 u_1$$

$$u_i^* u_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$$

$$= \begin{pmatrix} \sigma_1 & w^* \\ 0 & \\ \vdots & B \\ 0 & \end{pmatrix}$$

$w^* = 0$, $B = U_2 \Sigma_2 V_2^*$ by Hypothesis

$$\|M\|_2 = \sup_{x=0} \frac{\|U_1^* A V_1 x\|_2}{\|x\|_2} = \sup_{x \neq 0} \frac{\|AV_1 x\|_2}{\|V_1 x\|_2}$$

$$= \sup_{y \neq 0} \frac{\|Ay\|_2}{\|y\|_2} = \|A\|_2 = \sigma_1$$

Also: $\left\|M\begin{pmatrix}\sigma_1\\ w\end{pmatrix}\right\|_2 = \left\|\begin{pmatrix}\sigma_1 & w^*\\ 0 & \\ \vdots & B\\ 0 & \end{pmatrix}\begin{pmatrix}\sigma_1\\ w\end{pmatrix}\right\|_2 = \left\|\begin{matrix}\sigma_1^2+\|w\|_2^2\\ Bw\end{matrix}\right\|_2$

(this is a vector)

$$\geq \sigma_1^2 + \|w\|_2^2$$

$$\frac{\left\|M\begin{pmatrix}\sigma_1\\ w\end{pmatrix}\right\|_2}{\left\|\begin{pmatrix}\sigma_1\\ w\end{pmatrix}\right\|_2} \geq \left(\sigma_1^2 + \|w\|_2^2\right)^{\frac{1}{2}}$$

$$M = u_1^* A V_1 = \begin{pmatrix}1 & \\ & U_2\end{pmatrix}\begin{pmatrix}\sigma_1 & \\ & \Sigma_2\end{pmatrix}\begin{pmatrix}1 & \\ & V_2^*\end{pmatrix}$$

$$\sigma_2 = \|B\|_2 = \sup_{\substack{x\in\mathbb{C}^n\\ x\perp v_1}} \frac{\|Ax\|_2}{\|x\|_2}$$

so that v_1 ($\rightsquigarrow \sigma_1$) won't be considered.

$$x = \sum_{j=2}^{n} y_{j-1} v.$$