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Problem 1. *Find the number of Sylow* 2*-subgroups in* S_5*.*

Solution: Order of Sylow 2-group in S_5 has order $2^3 = 8$. Notice that D_4 also has order 8, and we consider four indices out of five as vertices of a rectangle, then D_4 can be represented as permutations. There are 5 ways to pick four vertices, and within each choice, there are $4! = 24$ ways to name each vertex, 6 distinct namings up to rotation, and finally 3 distinct namings up to both flipping and rotation. Thus the number is 15, given the fact that the only options are $3, 5, 15$ by Sylow Theorem.

Problem 2. *Show that finite group* G *of cardinal* $p^\alpha m$ *with* $(m, p) = 1$ *has a subgroup of order* p^i *for any* $i \leq \alpha$*.*

Proof: Prove by induction, for the base case where $\alpha = 1$, the conclusion is straight from Sylow Theorem. Now suppose for $\alpha \leq n$, the conclusion is true, then for $n + 1$, by Sylow Theorem, there exists a Sylow p-group S, of order p^{n+1}. Since p-group has a non-trivial center, pick $x \in Z(S)$ such that x has order p^k for some $0 < k \leq n + 1$. We raise x by power p^{k-1}, then $x^{p^{k-1}}$ has order p. Now consider $S/\langle x^{p^{k-1}} \rangle$, which has order p^n, and by induction it has a subgroup of order p^i for any $i \leq n = \alpha$, thus by considering the canonical homomorphism $S \to S/\langle x^{p^{k-1}} \rangle$, it's preimage is what we desired. □

Problem 3. *Let* G *be a group of order* 255*. Show that* G *is not simple.*

Proof: Notice that $255 = 3 \times 5 \times 17$. Consider the Sylow 17-subgroup of G. By Sylow Theorem, the number of Sylow 17-group is congruent to 1 mod 17, and it divides $3 \times 5 = 15$, only 1 satisfies both conditions, thus there is only one Sylow 17-subgroup in G, which is normal in G, hence G is not simple. □