
Problem 1. *Describe all possible complex 3-dimensional complex representations of $\mathbb{Z}$ modulo equivalence.*

Solution: Since $\mathbb{Z}$ is generated by 1, we only need to consider ρ_1 given representation $\rho : \mathbb{Z} \to GL_3(\mathbb{C})$. By Jordan Decomposition theorem, it's sufficient to categorize 3×3 Jordan normal form as follows:

1. Diagonal matrix with nonzero distinct diagonal entries. $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$, where λ_i are nonzero and distinct.
2. Diagonal matrix with two identical diagonal entries. $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$, where λ_i are nonzero and distinct.
3. Diagonal matrix with identical diagonal entries. $\begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_1 \end{pmatrix}$, where λ_1 nonzero.
4. One 2×2 Jordan block and one 1×1 block. $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}$, where λ_i are nonzero.
5. One 3×3 Jordan block. $\begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}$, where λ_1 nonzero.

Problem 2. *Let $G = \mathbb{Z}_n$ be a cyclic group of order n coprime to p. Show that any linear representation of G over a field F of characteristic p is a direct sum of irreducible representations.*

Proof: For any representation $\rho : \mathbb{Z}_n \to GL(\mathbb{F}_p)$, only need to consider ρ_1 because $\mathbb{Z}_n$ is generated by 1. Up to choosing a basis, we can write ρ_1 as matrix T, where $T^n = I$, the minimal polynomial of A divides $x^n - 1$, where $x^n - 1 = 0$ has no multiple roots given $\gcd(n, p) = 1$, then $T^n - 1 = f(T)g(T)$, f is irreducible. Take $V = \ker(f(T))$, then it is an invariant subspace, thus the representation can be written as $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}$, where A, B, C are block matrices. Then it's sufficient to show that $B = 0$. Since $T^n = I$, then $\begin{pmatrix} A & B \\ 0 & C \end{pmatrix}^n = \begin{pmatrix} A^n & \sum_{k=0}^{n-1} \binom{p}{n} A^k B C^{n-k-1} \\ 0 & C^n \end{pmatrix}$, then notice that in order to make summation 0, we have to have $B = 0$. Finally by doing the same on matrix C and A, eventually we can obtain that any representation can be decomposed into direct sum of irreducible representation.

Problem 3. *What is going to be the tensor product of two copies of the representation V_2 of the group S_3.*

Solution: Representation V_2 is $\rho : S_3 \to GL(\mathbb{R}^2)$, and we pick standard basis $\{e_1, e_2\}$ of $\mathbb{R}^2$. Let $x = (1\ 2\ 3)$ and $y = (1\ 2)$, then the matrix representations are:

$$M_x = \begin{pmatrix} \cos(2\pi/3) & -\sin(2\pi/3) \\ \sin(2\pi/3) & \cos(2\pi/3) \end{pmatrix}, M_y = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The basis of $\mathbb{R}^2 \otimes \mathbb{R}^2$ is $\{e_1 \otimes e_1, e_1 \otimes e_2, e_2 \otimes e_1, e_2 \otimes e_2\}$, and by the definition of tensor product of two representation ρ^1, ρ^2, $\rho'_s(v_1 \cdot v_2) = \rho^1_s(v_1) \cdot \rho^1_s(v_2)$. So we only need to compute for basis.

$$\begin{aligned}
\rho'_x(e_1 \otimes e_1) &= M_x e_1 \otimes M_x e_1 = \frac{1}{4} e_1 \otimes e_1 - \frac{\sqrt{3}}{4} e_1 \otimes e_2 - \frac{\sqrt{3}}{4} e_2 \otimes e_1 + \frac{3}{4} e_2 \otimes e_2 \\
\rho'_x(e_1 \otimes e_2) &= M_x e_1 \otimes M_x e_2 = \frac{\sqrt{3}}{4} e_1 \otimes e_1 + \frac{1}{4} e_1 \otimes e_2 - \frac{3}{4} e_2 \otimes e_1 - \frac{\sqrt{3}}{4} e_2 \otimes e_2 \\
\rho'_x(e_2 \otimes e_1) &= M_x e_2 \otimes M_x e_1 = \frac{\sqrt{3}}{4} e_1 \otimes e_1 - \frac{3}{4} e_1 \otimes e_2 + \frac{1}{4} e_2 \otimes e_1 - \frac{\sqrt{3}}{4} e_2 \otimes e_2 \\
\rho'_x(e_2 \otimes e_2) &= M_x e_2 \otimes M_x e_2 = \frac{3}{4} e_1 \otimes e_1 + \frac{\sqrt{3}}{4} e_1 \otimes e_2 + \frac{\sqrt{3}}{4} e_2 \otimes e_1 + \frac{1}{4} e_2 \otimes e_2
\end{aligned}$$

then the result is

$$\begin{pmatrix} \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{\sqrt{3}}{4} & \frac{1}{4} & -\frac{3}{4} & \frac{\sqrt{3}}{4} \\ -\frac{\sqrt{3}}{4} & -\frac{3}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} \\ \frac{3}{4} & -\frac{\sqrt{3}}{4} & -\frac{\sqrt{3}}{4} & \frac{1}{4} \end{pmatrix}$$

the other one is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Problem 4. *Show that $PSL(2, \mathbb{Z}_5)$ is a simple group (it is the quotient of $SL(2, \mathbb{Z}_5)$ modulo the central subgroup consisting of diagonal matrices).*

Proof: $PSL(2, \mathbb{Z}_5) \simeq A_5$, which is simple.

Problem 5. *Describe character table of elements of A_4.*

Solution: A_4 has four conjugacy classes, namely $C_1 = [e]$, $C_2 = [(1\ 2)(3\ 4)]$, $C_3 = \{(1\ 2\ 3), (1\ 3\ 4), (1\ 4\ 2), (2\ 4\ 3)\}$ and $C_4 = \{(1\ 3\ 2), (1\ 4\ 3), (1\ 2\ 4), (2\ 3\ 4)\}$. There are four isomorphism classes of irreducible representations, with dimensions satisfying $|A_4| = 12 = d_1^2 + d_2^2 + d_3^2 + d_4^2$, the only solution to that is $d_1 = d_2 = d_3 = 1$, $d_4 = 3$. The characteristic table lookes like:

	C_1	C_2	C_3	C_4
χ_1	1	1	1	1
χ_2	1	a	b	c
χ_3	1	d	e	f
χ_4	3	g	h	i

By orthogonal theorem, $\langle \chi_1, \chi_2 \rangle = \frac{1}{12}(1 + 3a + 4b + 4c) = 0$. Since $\chi_2(x) = a$ is the trace of a 1×1 matrix, and $x^2 = 1$, then $a = \pm 1$. Similarly with b, c, the possible values are $1, \omega, \omega^2$ where ω is the third root of unity. However, b or c equals 1 is impossible from the equation, and hence $a = 1$. Same procedure for χ_3. Finally for χ_4, by checking $\langle \chi_1, \chi_4 \rangle = 0$, $\langle \chi_2, \chi_4 \rangle = 0$ and $\langle \chi_3, \chi_4 \rangle = 0$, we can obtain that $g = -1$, $h = i = 0$. The complete character map is as follows:

	C_1	C_2	C_3	C_4
χ_1	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0