Honors Algebra II — Assignment 1
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Problem 1. *Prove or disprove $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/4\mathbb{Z}$.*

Disprove: For every element in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$, say $z = (\overline{a}, \overline{b})$, then $z^2 = \overline{e} = (\overline{0}, \overline{0})$, so there is no element of order 4 in $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. However, $\overline{1} \in \mathbb{Z}/4\mathbb{Z}$ has order 4. Thus they cannot be isomorphic to each other.

Problem 2. *Assume that the element a, b, c of a group G satisfies $abc = e$, where e is the neutral element.*
a). does that imply that $bca = e$?
b). does that imply that $bac = e$?

Proof: a). Since $abc = e$, we have $a = c^{-1}b^{-1}$, thus $bca = bcc^{-1}b^{-1} = e$.
b). Consider the group S_3, and let $a = (1, 2)$, $b = (2, 3)$, and $c = (3, 2, 1)$, then $abc = e$. However, $bac = (2, 3)(1, 2)(3, 2, 1) \neq e$, for it permutes 3 to 1 rather than 3. $\square$

Problem 3. *Determine the number of elements of order 2 in the symmetric group S_4.*

Solution: They are (1 2), (1 3), (1 4), (2 3), (2 4), (3 4), (1 2)(3 4), (1 3)(2 4), (1 4)(2 3), and the total number is 9.

Problem 4. *Classify the group of order 6 by analyzing the following three cases:*
a). G contains an element of order 6.
b). G contains an element of order 3 but not of order 6.
c). All elements of G have order 1 or 2.

Proof: Z_6 is a group of order 6 that contains an element of order 6. S_3 is a group of order 6 that contains order 3 but not of order 6. Suppose we have a group of order 6 where all elements of G have order 1 or 2, say the group $G = \{a_1, a_2, a_3, a_4, a_5, a_6\}$, then only one of them can have order 1, which is essentially e, and we let it be a_1, then for $i = 2, 3, 4, 5, 6$, a_i has order 2, which implies that $a_i = a_i^{-1}$. We assume that $a_2 a_3 = a_4$, then $a_4 a_5 = a_6$, otherwise if $a_4 a_5 = a_2$, then $a_2^{-1} a_4 = a_5^{-1}$, which means $a_2 a_4 = a_5$, however, $a_2 a_4 = a_3$, then we would have $a_3 = a_5$, which is absurd. $a_4 a_5 \neq a_3$, $a_4 a_5 \neq a_4$, and $a_4 a_5 \neq a_5$ follows the same reasoning. Consider $a_2 a_5 = a_3 a_4 a_5 = a_3 a_6$, then this product can only be a_4, which means that $a_2 a_5 = a_2 a_3$, then $a_5 = a_3$, which is absurd. Thus the c) case cannot be possible.

For group G of type a, it's easy to see that $G \simeq Z_6$. Take $x \in G$ that has order 6, we must have $x, x^2, x^3, x^4, x^5, e \in G$, and that is all of group G because it has order 6. The isomorphism $f : Z_6 \to G$ is defined as $i \mapsto x^i$.

For group G of type b, the maximum order of its element is 3. Take $x \in G$ of order 3, we have e, x, x^2 in G. Since the order of the element must divide 6, we must have $y \in G$ of order 2, and the number of such y is 3, otherwise the group is not of order 6. Now we write group G explicitly, $\{e, x, x^2, y_1, y_2, y_3\}$, and without loss of generality, we can fix the operation law inside G as $xy_1 = y_2$, $xy_2 = y_3$, $xy_3 = y_1$. Finally the map $f : G \to S_3$ is defined as $x \mapsto (1\ 2\ 3)$, $y_1 \mapsto (1\ 2)$, $y_2 \mapsto (1\ 3)$, and $y_3 \mapsto (2\ 3)$, and it is easy to verify that f is indeed a isomorphism. To conclude, group of order 6 can be classified as either Z_6 or S_3 up to isomorphism. $\square$

Problem 5. *Give an example of a finite group G that satisfies the following both properties:*

a). G is of order 8.

b). G is not abelian.

Solution: Consider the order 8 group of $\{e, r, r^2, r^3, f, fr, fr^2, fr^3\}$ where $f^2 = e$, $r^4 = e$, and $fr = r^3 f$. Verify this is indeed a group, by calculation, it is closed under internal law, and for each element, there is one inverse ($r^{-1} = r^3$, $r^{-2} = r^2$, $f^{-1} = f^1$, $(fr)^{-1} = r^3 f = fr$, $(fr^2)^{-1} = fr^2$, $(fr^3)^{-1} = (fr^3)$). Also, it is non-abelian since $r^3 f = fr \neq fr^3$.