
Problem 1. *We may deduce from eqns (6.7) and (6.9) that*

$$t_i = -pn_i + \mu n_j \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)$$

Show that it is identical to

$$\boldsymbol{t} = -p\boldsymbol{n} + \mu[2(\boldsymbol{n} \cdot \nabla)\boldsymbol{u} + \boldsymbol{n} \times (\nabla \times \boldsymbol{u})]$$

Proof: We utilize the Levi-Civita symbol, where

$$\begin{aligned}
[\mathbf{n} \times (\nabla \times \mathbf{u})]_i &= \varepsilon_{ijk} n_j \left(\varepsilon_{kmn} \frac{\partial u_n}{x_m} \right) = \varepsilon_{kij}\varepsilon_{kmn} n_j \frac{\partial u_n}{\partial x_m} \\
&= (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm}) n_j \frac{\partial u_n}{\partial x_m} \\
&= n_j \frac{\partial u_j}{\partial x_i} - n_j \frac{\partial u_i}{\partial x_j}
\end{aligned}$$

Also, since we also have $[(\mathbf{n} \cdot \nabla)\mathbf{u}]_i = n_j \frac{\partial u_i}{\partial x_j}$, combining with the calculation above, we obtain that

$$\begin{aligned}
t_i &= -pn_i + \mu \left(2n_j \frac{\partial u_i}{\partial x_j} + n_j \frac{\partial u_j}{\partial x_i} - n_j \frac{\partial u_i}{\partial x_j} \right) \\
&= -pn_i + \mu n_j \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)
\end{aligned}$$

□

Problem 2. *Show that the net force exerted on a finite blob of fluid by the surrounding fluid is*

$$\int_S \boldsymbol{t} dS = \int_V (-\nabla p + \mu \nabla^2 \boldsymbol{u}) dV$$

where S is the surface of the blob and V is the region occupied by the blob. Deduce that if the blob is small the net force on it, excluded gravity, is $-\nabla p + \mu \nabla^2 \boldsymbol{u}$ per unit volume.

Proof: Calculate

$$\left[\int_S \mathbf{t} dS \right]_i = \int_S T_{ij} n_j dS = \int_S \mathbf{T}_i \cdot \mathbf{n} dS$$

where $\mathbf{T}_i$ is the i-th row of the stress tensor, then by Divergence Theorem,

$$\begin{aligned}
\left[\int_S \mathbf{t} dS\right]_i &= \int_V \nabla \cdot \mathbf{T}_i dV = \int_V \frac{\partial T_{ij}}{\partial x_j} dV \\
&= \int_V -\frac{\partial p \delta_{ij}}{\partial x_j} + \mu \frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) dV \\
&= \int_V -\frac{\partial p}{\partial x_i} + \mu \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} + \mu \frac{\partial^2 u_i}{\partial x_j^2} dV
\end{aligned}$$

Since $\nabla \cdot \mathbf{u} = 0$, $\mu \frac{\partial}{\partial x_i}\frac{\partial u_j}{\partial x_j} = 0$, we then obtain

$$\int_S \mathbf{t} dS = \int_V (-\nabla p + \mu \nabla^2 \mathbf{u}) dV$$

as desired. And when the volume is sufficiently small, we may approximate the integration by

$$(-\nabla p + \mu \nabla^2 \mathbf{u}) \cdot V$$

then the net force per unit volume is

$$\frac{1}{V}(-\nabla p + \mu \nabla^2 \mathbf{u}) \cdot V = -\nabla p + \mu \nabla^2 \mathbf{u}$$

□

Problem 3. *Verify in the case of a simple shear flow,*

$$\boldsymbol{u} = [u(y), 0, 0]$$

equation $\boldsymbol{t} = -p\boldsymbol{n} + \mu[2(\boldsymbol{n} \cdot \nabla)\boldsymbol{u} + \boldsymbol{n} \times (\nabla \times \boldsymbol{u})]$ *reduces, when* $\boldsymbol{n} = (0, 1, 0)$*, to*

$$\boldsymbol{t} = \left[\mu \frac{du}{dy}, -p, 0\right]$$

Proof: Calculate $\mathbf{t}$ component-wise,

$$\begin{aligned}
t_1 &= \mu\left(\frac{\partial u_1}{\partial y} + \frac{\partial u_2}{\partial x}\right) = \mu \frac{du}{dy} \\
t_2 &= -p + 2\mu \frac{\partial u_2}{\partial y} = -p \\
t_3 &= \mu\left(\frac{\partial u_3}{\partial y} + \frac{\partial u_2}{\partial z}\right) = 0
\end{aligned}$$

□

Problem 4. *Give an order of magnitude estimate of the Reynolds number for:*
(i). flow past the wing of a jumbo jet at 150 ms^{-1} (roughly half speed of sound).
(ii). experiment in 1.1 with $L = 2$ cm and $U = 5$ cm s^{-1}
(iii). a thick layer of golden syrup draining of the spoon.
(iv). a spermatozoa with a tail length of 10^{-3} cm swimming at 10^{-2} cm s^{-1} in water.

Solution: (i). $Re = \frac{UR}{v} = \frac{15000\times10^3}{0.15} = 10^7$.
(ii). If we approximate the viscosity of salty water as pure water, then $Re = \frac{5\times2}{0.01} = 10^3$.
(iii). $Re = \frac{1\times1}{1200} = 8\times10^{-3}$.
(iv). $Re = \frac{10^{-3}\times10^{-2}}{0.01} = 10^{-3}$.

Problem 5. *The problem of $2-D$ steady viscous flow past a circular cylinder of radius a involves finding a velocity field $\boldsymbol{u} = [u(x,y), v(x,y), 0]$ which satisfies*

$$(\boldsymbol{u}\cdot\nabla)\boldsymbol{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{u},\ \nabla\cdot\boldsymbol{u} = 0$$

together with the boundary conditions

$$\boldsymbol{u} = 0 \ \text{on}\ x^2+y^2 = a^2;\ \boldsymbol{u}\to(U,0,0)\ \text{as}\ x^2+y^2\to\infty$$

Rewrite the problem in dimensionless form by using the dimensionless variables

$$\boldsymbol{x}' = \boldsymbol{x}/a,\ \boldsymbol{u}' = \boldsymbol{u}/U,\ p' = p/\rho U^2$$

Without solving the problem, show that the streamline pattern can depend on v, a, U only in the combination $R = Ua/v$, so that flows at equal Reynolds numbers are geometrically similar.

Proof: We calculate each term in the original equation

$$(\mathbf{u}'\cdot\nabla')\mathbf{u}' = \left(\frac{\mathbf{u}}{U}\cdot a\nabla\right)\frac{\mathbf{u}}{U} = \frac{a}{U^2}(\mathbf{u}\cdot\nabla)\mathbf{u}$$

Second term

$$\nabla' p' = a\nabla\left(\frac{p}{\rho U^2}\right) = \frac{a}{\rho U^2}\nabla p$$

thus

$$\frac{1}{\rho}\nabla p = \frac{U^2}{a}\nabla' p'$$

Third term

$$\nu\nabla'^2\mathbf{u}' = \nu\frac{a^2}{U}\nabla^2\mathbf{u}$$

Thus, by calculating the boundary condition, we transform the problem into

$$\begin{cases} (\mathbf{u}' \cdot \nabla')\mathbf{u}' = \nabla' p' + \frac{\nu}{Ua}\nabla'^2 \mathbf{u}' \\ \nabla' \cdot \mathbf{u}' = 0 \\ \mathbf{u}' = 0 \text{ on } x'^2 + y'^2 = 1 \\ \mathbf{u}' \to (1,0,0) \text{ as } x'^2 + y'^2 \to \infty \end{cases}$$

and notice that the solution to this question depends on $\mathbf{x}'$ and R, then at each fixed $\mathbf{x}'$, the direction will only depend on R, so for same R (Reynolds number), the flow is geometrically similar. □

Problem 6. *Viscous fluid flow between two stationary rigid boundaries $y = \pm h$ under a constant pressure gradient $P = -dp/dx$. Show that*

$$u = \frac{P}{2\mu}(h^2 - y^2), \;\; v = w = 0$$

Proof: We seek solution to Navier-Stokes equation of the form $\mathbf{u} = (u(y), 0, 0)$, then consider N-S equation at x-component

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \mu\nabla^2 u$$

Notice that since $v = w = 0$, and by $\nabla \cdot \mathbf{u} = 0$, we have $\frac{\partial u}{\partial x} = 0$. By some simple cancellation, we obtain

$$\mu\frac{d^2 u}{dy} = -\frac{P}{\rho}$$

the solution to this ODE is $u = -\frac{P}{2\mu\rho}y^2 + C_1 y + C_2$, plugging in two boundary values ($u = 0$ at $y = \pm h$), $C_1 = 0, C_2 = \frac{P}{2\mu\rho}h^2$, thus the solution is

$$\mathbf{u} = \left(\frac{P}{2\mu}(h^2 - y^2), 0, 0\right)$$

as desired. □