

Problem 1. Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in \mathcal{L}^\infty$ with the bound

$$\|u\|_{\mathcal{L}^\infty(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n . (Evans 5.10.20)

Proof: Since

$$u = \check{u} = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{ixy} \hat{u} dy$$

we have

$$|u| \leq \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\hat{u}| \cdot |e^{ixy}| dx = \frac{1}{(2\pi)^n} \|\hat{u}\|_{\mathcal{L}^1(\mathbb{R}^n)}$$

Then we are left to prove $\|\hat{u}\|_{\mathcal{L}^1(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}$, and we do this by noticing

$$\begin{aligned} \int_{\mathbb{R}^n} |\hat{u}| dx &= \int_{\mathbb{R}^n} \frac{1}{|1 + |y|^s|} \cdot |1 + |y|^s| \cdot |\hat{u}| dy \\ &\leq \left\| \frac{1}{|1 + |y|^s|} \right\|_{\mathcal{L}^2} \cdot \|(1 + |y|^s)\hat{u}\|_{\mathcal{L}^2} \end{aligned}$$

by applying the Holder inequality. Notice that $\frac{1}{1+|y|^s}$ is \mathcal{L}^2 -integrable when $s > n/2$, and to see that, we calculate

$$\begin{aligned} \int_{\mathbb{R}^n} \frac{1}{|1 + |y|^2|} dy &= c(n) \int_0^\infty \frac{r^{n-1}}{(1 + r^s)^2} dy \\ &\leq c(n) \int_0^\infty \frac{r^{n-1}}{1 + r^{2s}} dy \\ &\leq c(n) \int_0^\infty r^{n-2s-1} dy \leq C(n, p) \end{aligned}$$

where $c(n)$ is the area of a unit sphere. □

Problem 2. Show that if $u, v \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $uv \in H^s(\mathbb{R}^n)$ and

$$\|uv\|_{H^s(\mathbb{R}^n)} \leq C\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}$$

the constant C depending only on s and n . (Evans 5.10.21)

Proof: Since we know that

$$\widehat{uv} = \frac{1}{(2\pi)^n} \hat{u} * \hat{v}$$

we estimate the term

$$\begin{aligned} \left| (1 + |y|^s) \int_{\mathbb{R}^n} \hat{u}(x) \hat{v}(x - y) dx \right|^2 &= \left| \int_{\mathbb{R}^n} \frac{(1 + |y|^s) \cdot (1 + |x|^s) \hat{u}(x) \cdot (1 + |x - y|^s) \hat{v}(x - y)}{(1 + |x|^s)(1 + |x - y|^s)} dx \right|^2 \\ &\leq \int_{\mathbb{R}^n} \left| \frac{1 + |y|^s}{(1 + |x|^s)(1 + |x - y|^s)} \right|^2 dx \cdot \int_{\mathbb{R}^n} |G(x) \cdot H(x - y)|^2 dx \end{aligned}$$

where $G(x) = (1 + |x|^s) \hat{u}(x)$ and $H(x) = (1 + |x|^s) \hat{v}(x)$.

The Cauchy-Schwarz inequality holds since first of all $G^2(x)$ and $H^2(x - y)$ are \mathcal{L}^1 given $G, H \in \mathcal{L}^2$, then the second term of the right hand side is the convolution of two \mathcal{L}^1 function, which is in \mathcal{L}^1 . Same reason for the first term if we consider the function $h(x) = \frac{1}{1 + |x|^s}$.

Now integrate both sides over y , and before doing so, we want the first term of the right hand side to not only integrable, but also bounded by a constant independent of x, y . We do so (for fixed y) by

$$\int_{\mathbb{R}^n} \left| \frac{1 + |y|^s}{(1 + |x|^s)(1 + |x - y|^s)} \right|^2 dx = \int_{|y| > |x|/2} \left| \frac{1 + |y|^s}{(1 + |x|^s)(1 + |x - y|^s)} \right|^2 dx \quad (1)$$

$$+ \int_{|y| \leq |x|/2} \left| \frac{1 + |y|^s}{(1 + |x|^s)(1 + |x - y|^s)} \right|^2 dx \quad (2)$$

For $|x| \leq |y|/2$, $|x - y| \geq |y| - |x| \geq |y|/2$

$$\frac{(1 + |y|^s)^2}{(1 + |x|^s)^2(1 + |x - y|^s)^2} \leq \frac{(1 + |y|^s)^2}{(1 + |x|^s)^2(1 + |y/2|^s)^2} \leq \frac{2^{s+1}(1 + |y|^s)^2}{(1 + |x|^s)^2(1 + |y|^s)^2} = \frac{2^{s+1}}{(1 + |x|^s)^2}$$

For $|x| > |y|/2$,

$$\frac{(1 + |y|^s)^2}{(1 + |x|^s)^2(1 + |x - y|^s)^2} \leq \frac{2^{s+1}(1 + |y|^s)^2}{(1 + |y|^s)^2(1 + |x - y|^s)^2} = \frac{2^{s+1}}{(1 + |x - y|^s)^2}$$

Thus

$$\int_{\mathbb{R}^n} \left| \frac{1 + |y|^s}{(1 + |x|^s)(1 + |x - y|^s)} \right|^2 dx \leq C(n, s)$$

Finally, we calculate

$$\begin{aligned} \int_{\mathbb{R}^n} \left| (1 + |y|^s) \int_{\mathbb{R}^n} \hat{u}(x) \hat{v}(x - y) dx \right|^2 dy &\leq C(n, s) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |G(x)|^2 |H(x - y)|^2 dx dy \\ &= C(n, s) \int_{\mathbb{R}^n} |G(x)|^2 \int_{\mathbb{R}^n} |H(x - y)|^2 dy dx \\ &= C(n, s) \|u\|_{H^s(\mathbb{R}^n)} \|v\|_{\mathbb{R}^n} \end{aligned}$$

□

Remark: Now we prove the following:

$$\widehat{uv} = \frac{1}{(2\pi)^n} \cdot \hat{u} * \hat{v}$$

Proof:

$$\begin{aligned}\widehat{uv}(y) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x)v(x)e^{-ixy}dx = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x)\check{u}(x)e^{-ixy}dx \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{-ixy}u(x) \int_{\mathbb{R}^n} \hat{v}(t)e^{itx}dtdx \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{v}(t) \int_{\mathbb{R}^n} u(x)e^{-ix(y-t)}dxdt \\ &= \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{v}(t)\hat{u}(y-t)dt \\ &= \frac{1}{(2\pi)^n} \cdot \hat{u} * \hat{v}\end{aligned}$$

□