
Problem 1. *Use the Fourier transform to prove that if $u \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $u \in \mathscr{L}^\infty$ with the bound*

$$\|u\|_{\mathscr{L}^\infty(\mathbb{R}^n)} \le C\|u\|_{H^s(\mathbb{R}^n)}$$

for a constant C depending only on s and n. (Evans 5.10.20)

Proof: Since

$$u = \check{\hat{u}} = \int_{\mathbb{R}^n} e^{ixy}\hat{u}dy$$

we have

$$|u| \le \int_{\mathbb{R}^n} |\hat{u}| \cdot |e^{ixy}|dx = \|\hat{u}\|_{\mathscr{L}^1(\mathbb{R}^n)}$$

Then we are left to prove $\|\hat{u}\|_{\mathscr{L}^1(\mathbb{R}^n)} \le C\|u\|_{H^s(\mathbb{R}^n)}$, and we do this by noticing

$$\begin{aligned}\int_{\mathbb{R}^n} |\hat{u}|dx &= \int_{\mathbb{R}^n} \frac{1}{|1+|y|^s|} \cdot |1+|y|^s| \cdot |\hat{u}|dy \\ &\le \left\|\frac{1}{|1+|y|^s|}\right\|_{\mathscr{L}^2} \cdot \|(1+|y|^s)\hat{u}\|_{\mathscr{L}^2}\end{aligned}$$

by applying the Holder inequality. Notice that $\frac{1}{1+|y|^s}$ is $\mathscr{L}^2$-integrable when $s > n/2$, and to see that, we calculate

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{1}{|1+|y||^2}dy &= c(n)\int_0^\infty \frac{r^{n-1}}{(1+r^s)^2}dy \\ &\le c(n)\int_0^\infty \frac{r^{n-1}}{1+r^{2s}}dy \\ &\le c(n)\int_0^\infty r^{n-2s-1}dy \le C(n,p)\end{aligned}$$

where $c(n)$ is the area of a unit sphere. $\square$

Problem 2. *Show that if $u, v \in H^s(\mathbb{R}^n)$ for $s > n/2$, then $uv \in H^s(\mathbb{R}^n)$ and*

$$\|uv\|_{H^s(\mathbb{R}^n)} \le C\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{H^s(\mathbb{R}^n)}$$

the constant C depending only on s and n. (Evans 5.10.21)

Proof: Since we know that

$$\widehat{uv} = \hat{u} * \hat{v}$$

we estimate the term

$$\begin{aligned}\left|(1+|y|^s)\int_{\mathbb{R}^n} \hat{u}(x)\hat{v}(x-y)dx\right|^2 &= \left|\int_{\mathbb{R}^n} \frac{(1+|y|^s)\cdot(1+|x|^s)\hat{u}(x)\cdot(1+|x-y|^s)\hat{v}(x-y)}{(1+|x|^s)(1+|x-y|^s)}dx\right|^2 \\ &\le \int_{\mathbb{R}^n}\left|\frac{1+|y|^s}{(1+|x|^s)(1+|x-y|^s)}\right|^2 dx \cdot \int_{\mathbb{R}^n}|G(x)\cdot H(x-y)|^2\,dx\end{aligned}$$

where $G(x) = (1+|x|^s)\hat{u}(x)$ and $H(x) = (1+|x|^s)\hat{v}(x)$.

The Cauchy-Schwarz inequality holds since first of all $G^2(x)$ and $H^2(x-y)$ are $\mathscr{L}^1$ given $G, H \in \mathscr{L}^2$, then the second term of the right hand side is the convolution of two $\mathscr{L}^1$ function, which is in $\mathscr{L}^1$. Same reason for the first term if we consider the function $h(x) = \frac{1}{1+|x|^s}$.

Now integrate both sides over y, and before doing so, we want the first term of the right hand side to not only integrable, but also bounded by a constant independent of x, y. We do so (for fixed y) by

$$\int_{\mathbb{R}^n} \left|\frac{1+|y|^s}{(1+|x|^s)(1+|x-y|^s)}\right|^2 dx = \int_{|y|>|x|/2} \left|\frac{1+|y|^s}{(1+|x|^s)(1+|x-y|^s)}\right|^2 dx \tag{1}$$

$$+ \int_{|y|\le|x|/2} \left|\frac{1+|y|^s}{(1+|x|^s)(1+|x-y|^s)}\right|^2 dx \tag{2}$$

For $|x| \le |y|/2$, $|x-y| \ge |y| - |x| \ge |y|/2$

$$\frac{(1+|y|^s)^2}{(1+|x|^s)^2(1+|x-y|^s)^2} \le \frac{(1+|y|^s)^2}{(1+|x|^s)^2(1+|y/2|^s)^2} \le \frac{2^{s+1}(1+|y|^s)^2}{(1+|x|^s)^2(1+|y|^s)^2} = \frac{2^{s+1}}{(1+|x|^s)^2}$$

For $|x| > |y|/2$,

$$\frac{(1+|y|^s)^2}{(1+|x|^s)^2(1+|x-y|^s)^2} \le \frac{2^{s+1}(1+|y|^s)^2}{(1+|y|^s)^2(1+|x-y|^s)^2} = \frac{2^{s+1}}{(1+|x-y|^s)^2}$$

Thus

$$\int_{\mathbb{R}^n} \left|\frac{1+|y|^s}{(1+|x|^s)(1+|x-y|^s)}\right|^2 dx \le C(n,s)$$

Finally, we calculate

$$\begin{aligned}\int_{\mathbb{R}^n} \left|(1+|y|^s)\int_{\mathbb{R}^n} \hat{u}(x)\hat{v}(x-y)dx\right|^2 dy &\le C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |G(x)|^2|H(x-y)|^2 dxdy \\ &= C(n,s)\int_{\mathbb{R}^n} |G(x)|^2 \int_{\mathbb{R}^n} |H(x-y)|^2 dydx \\ &= C(n,s)\|u\|_{H^s(\mathbb{R}^n)}\|v\|_{\mathbb{R}^n}\end{aligned}$$

$\square$