
Problem 1. *Let f be a Borel function on $[0,1]$ with the following property: for each $\epsilon > 0$, there exists $p \in (0,\epsilon)$ such that*

$$f(x) = f(x+p),\ 0 \le x \le 1-p$$

The goal of this problem is to show that such f is constant almost everywhere.

(i). Show that it suffices to prove: For every Borel set $B \subseteq \mathbb{R}$,

$$\mathcal{L}(f^{-1}(B)) \in \{0,1\}$$

where $\mathcal{L}$ is the Lebesgue measure on the unit interval.

(ii). For every Borel set B and $x \in [0,1]$, prove that $f^{-1}(B)$ and $[0,x]$ is independent with respect to $\mathcal{L}$. Conclude that f is constant almost everywhere.

(iii). Give an example showing that f need not be constant.

Proof: (i). For fixed $y \in \mathbb{R}$ and consider the sequence $A_n := (y-1/n, y+1/n) \in \mathcal{B}$ for $n \in \mathbb{N}$, then

$$\mathcal{L}\left(f^{-1}(y)\right) = \mathcal{L}\left(f^{-1}\left(\bigcap_{n\in\mathbb{N}} A_n\right)\right) = \mathcal{L}\left(\bigcap_{n\in\mathbb{N}} f^{-1}(A_n)\right)$$

by continuity of measure and the fact that each $f^{-1}(A_n)$ can only take measure 0 or 1,

$$\mathcal{L}(f^{-1}(y)) = \lim_{n\to\infty} \mathcal{L}(f^{-1}(A_n))$$

equals either 0 or 1. Suppose that there exists $y_1 \neq y_2$ such that $\mathcal{L}(f^{-1}(y_1)) = \mathcal{L}(f^{-1}(y_2)) = 1$, then $\mathcal{L}(f^{-1}(y_1) \cup f^{-1}(y_2)) = 2 > \mathcal{L}[0,1] = 1$, which is absurd. Thus, we have for some unique $y \in \mathbb{R}$ such that $\mathcal{L}(f^{-1}(y)) = 1$, which indicates that f is constant y almost everywhere on $[0,1]$.

(ii). Fix $x \in [0,1]$ and B Borel set, by property of f, we can find some ϵ and corresponding $p \in (0,\epsilon)$ such that $p < 1-x$. Define N by $Np \le 1 \le (N+1)p$ and denote $f^{-1}(B)$ as A, then

$$\mathcal{L}(A) = \mathcal{L}(A \cap [0,Np]) + \mathcal{L}(A \cap [Np,1]) \le N\mathcal{L}(A \cap [0,p]) + p =: N\delta + p$$

Also, define m by $mp \le x \le (m+1)p$, then

$$\mathcal{L}(A \cap [0,x]) = \mathcal{L}(A \cap [0,mp]) + \mathcal{L}(A \cap [mp,x]) =: m\delta + \gamma$$

where $0 < \gamma < p$. Since $m = \frac{x}{p} - \left\{\frac{x}{p}\right\}$, where $\left\{\frac{x}{p}\right\}$ is the decimal part of $\frac{x}{p}$, then

$$\left| m\delta + \gamma - \frac{x\delta}{p} \right| \le \left| m\delta - \frac{x\delta}{p} \right| + p \le \delta + p \le 2p$$

Also, we can do the following estimate:

$$\begin{aligned}\left|\mathcal{L}(A)-\frac{\delta}{p}\right| &\leq \left|\mathcal{L}(A)-\frac{\mathcal{L}(A)}{Np}\right|+\left|\frac{\mathcal{L}(A)}{Np}-\frac{\delta}{p}\right| \\ &\leq \left|\frac{1}{NP}-1\right|+\frac{1}{N} \\ &\leq \frac{1}{1-p}-1+\frac{1}{N} \leq \frac{p}{1-p}+\frac{p}{1-p} \leq 2p+2p=4p\end{aligned}$$

for p small. Hence

$$|\mathcal{L}(A\cap[0,x])-x\mathcal{L}(A)| \leq \left|m\delta+\gamma-\frac{x\delta}{p}\right|+\left|\frac{x\delta}{p}-x\mathcal{L}(A)\right| \leq 2p+4px \leq 6p$$

we can let $p\to 0$ by choosing $\epsilon\to 0$ and we obtain the desired result.

Further, suppose that for fixed Borel set B, $\mathcal{L}(f^{-1}(B)\cap[0,x]) = x\mathcal{L}(f^{-1}(B))$, then consider it as function f of x, we can write alternatively

$$\int_0^x \mathbf{1}_{f^{-1}(B)} d\mathcal{L} = x\cdot\mathcal{L}(f^{-1}(B))$$

by taking derivative on x, we have for almost every $x\in[0,1]$,

$$\mathbf{1}_{f^{-1}(B)}(x) = \mathcal{L}(f^{-1}(B))$$

then $\mathcal{L}(f^{-1}(B))$ can only take value 0 or 1, which implies that f is constant.

(iii). Consider the Dirichlet function

$$f(x)=\begin{cases}1 & \text{if } x \text{ is irrational}\\ 0 & \text{if } x \text{ is rational}\end{cases}$$

then for arbitrary $\epsilon>0$, there exists a rational number $p\in(0,\epsilon)$ such that for x rational, $f(x)=f(x+p)=0$ since $x+p$ is also rational; for x irrational, $f(x)=f(x+p)=1$ since $x+p$ remains irrational. $\square$

Problem 2. *Let $F:\mathbb{R}\to[0,1]$ be a function satisfying:*

(i). F is monotone increasing.

(ii). $\lim_{x\to\infty} F(x)=1$, $\lim_{x\to-\infty} F(x)=0$.

(iii). F is right-continuous.

The goal is to prove that there exists a unique probability measure μ on $(\mathbb{R},\mathcal{B}(\mathbb{R}))$ such that

$$\mu((-\infty,x]) = F(x),\ \forall x$$

(i). Show that if μ exists, then it is unique.

(ii). Define $G:(0,1)\to\mathbb{R}$ by

$$G(y)=\inf\{x\in\mathbb{R}: F(x)\geq y\}$$

Show that G is non-decreasing, that

$$F(x) \geq y \iff x \geq G(y)$$

and deduce that G is left-continuous.

(iii). Let $\Omega = (0,1)$, $\mathcal{F} = \mathcal{B}((0,1))$, and let $\mathbb{P}$ be the restriction of the Lebesgue measure. Show that $X := G$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$, and that its image measure $\mu_X := \mathbb{P} \circ X^{-1}$ satisfies

$$\mu_X((-\infty, x]) = \mathbb{P}(X \leq x) = F(x).$$

Proof: (i). Consider the π-system $\pi(\mathbb{R}) := \{(-\infty, x] : x \in \mathbb{R}\}$, then suppose that there exist two measure μ, ν satisfying $\mu((-\infty, x]) = \nu((-\infty, x]) = F(x)$. Since two measures agree on a π-system, we have $\mu = \nu$, proving uniqueness.

(ii). Suppose that $F(x) \geq y$, then $x \in \{x \in \mathbb{R} : F(x) \geq y\} \geq \inf\{x \in \mathbb{R} : F(x) \geq y\} = G(y)$. Conversely, if $x \geq G(y)$, then by monotonicity of F, $F(x) \geq y$. Since G is non-decreasing function, $\lim\limits_{y \to y_0^-} G(y) \leq G(y_0)$. If $\lim\limits_{y \to y_0^-} G(y) = G(y_0)$, then there is nothing to prove; If $\lim\limits_{y \to y_0^-} G(y) < G(y_0)$, then there exists x such that $\lim\limits_{y \to y_0^-} G(y) \leq x < G(y_0)$, which means that $x \geq G(y)$ for all $y < y_0$. Hence $F(x) \geq y$ for all $y < y_0$, then $F(x) \geq y_0$, which implies $x \geq G(y_0)$, leading to a contradiction. To conclude, G is left-continuous.

(iii). Consider $\{G(y) \leq b\}$ for arbitrary $b \in \mathbb{R}$, which can be equivalently written as $\{y : 0 < y \leq F(b)\} \in \mathcal{B}((0,1))$, hence G is a random variable. Further, $\mu_X((-\infty, x]) = \mathbb{P}(X^{-1}(x)) = \mathbb{P}((0, F(x))) = F(x)$. $\square$

Problem 3. *For a fixed $d \geq 1$, let E be the set of edges of the lattice $\mathbb{Z}^d$, i.e.*

$$E := \{\{x, y\} : x, y \in \mathbb{Z}^d \text{ and } \|x - y\|_1 = 1\}$$

For a fixed $p \in [0,1]$, let $\{X_e\}_{e \in E}$ be an independent identically distributed sequence of random variables on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ satisfying

$$\mathbb{P}(X_e = 1) = p, \ \mathbb{P}(X_e = 0) = 1 - p$$

We say that an edge $e \in E$ is open if $X_e = 1$. For fixed two vertices x and y, an open path is a sequence of pairwise distinct vertices $x_0, x_1, \ldots, x_k$ such that $x_0 = x$, $x_k = y$, and for every $0 \leq i < k$, $\|x_i - x_{i+1}\|_1 = 1$ and each edge $\{x_i, x_{i+1}\}$ is open. Two vertices x and y are connected if there exists an open path from x to y. A connected open component is a maximal connected subset of open edges.

For $x \in \mathbb{Z}^d$, let C_x be the connected open component containing x. Denote by $|C_x|$ the number of vertices in C_x. Define the events

$$J_x := \{|C_x| = \infty\}, \ I := \{\omega \in \Omega : \exists \text{ infinite open connected component in } \omega\} = \bigcup_{x \in \mathbb{Z}^d} J_x$$

(i). Prove that $I \in \mathcal{F}$ and $J_x \in \mathcal{F}$ for every $x \in \mathbb{Z}^d$.

(ii). Consider the event

$$I_n := \{\text{the restriction of } (X_e)_{e \in E} \text{ to } E \setminus E_{B(n)} \text{ contains an infinite open connected component}\}$$

where $E_{B(n)}$ is the set of edges in $B(n)$. Prove that $I_n = I$ and conclude

$$I \in \sigma(X_e : e \in E \setminus E_{B(n)}) \ \forall n \geq 1$$

(iii). Prove that $\mathbb{P}(I) \in \{0, 1\}$.

Proof: (i). Let $B(n) = [-n, n]^d \cap \mathbb{Z}^d$, and

$$A_n = \{0 \text{ is connected by an open path to the set } B(n)^c\}$$

then we see that $A_n \in \mathcal{F}$ since it contains finite edges, and since $J_0 = \bigcap_n A_n$, $J_0 \in \mathcal{F}$. J_x is simply a translation, then $J_x \in \mathcal{F}$ for any $x \in \mathbb{Z}^d$. Further, we have $I = \bigcup_{x \in \mathbb{Z}^d} J_x \in \mathcal{F}$.

(ii). $I_n \subseteq I$ since $(X_e)_{e \in E}$ contains an infinite open connected component in E immediately after knowing that it also contains one in the restriction $E \setminus E_{B(n)}$. Conversely, Suppose $(X_e)_{e \in E}$ contains an infinite connected component in E, if the connected component doesn't intersect $E_{B(n)}$, then there is nothing to prove, otherwise since $E_{B(n)}$ contains finite edges at most, then consider the connected components cut by $B(n)$, one of them must be infinite, thus $I \subseteq I_n$.

(iii). Since $I \in \sigma(X_e : e \in E \setminus E_{B(n)})$ for all $n \geq 1$, then $I \in \bigcap_n \sigma(X_e : e \in E \setminus E_{B(n)})$, and by Kolmogorov's 0-1 law, $\mathbb{P}(I) = \{0, 1\}$. □

Problem 4. *Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of independent identically distributed exponential random variables with parameter λ (i.e. with distribution function $F(x) = \mathbb{P}(X_n \leq x) = 1 - e^{-\lambda x}$). Prove that with probability one,*

$$\limsup_{n \to \infty} \frac{X_n}{\log n} = \frac{1}{\lambda}$$

Proof: We consider $\mathbb{P}(X_n > \mu \log n)$ for $\mu > 0$,

$$\mathbb{P}(X_n > \mu \log n) = 1 - \mathbb{P}(X_n \leq \mu \log n) = e^{-\lambda \mu \log n} = \frac{1}{n^{\lambda \mu}}$$

By the Borel-Contelli lemma,

$$\mathbb{P}(X_n > \mu \log n \text{ i.o.}) = \begin{cases} 1 & \text{if } \lambda\mu \leq 1 \\ 0 & \text{if } \lambda\mu > 1 \end{cases}$$

Pick $\mu = \frac{1}{\lambda}$, we have

$$\mathbb{P}\left(\limsup_n \frac{X_n}{\log n} \geq \frac{1}{\lambda}\right) \geq \mathbb{P}\left(\frac{X_n}{\log n} > \frac{1}{\lambda} \text{ i.o.}\right) = 1$$

Now pick $\mu = \frac{1}{\lambda} + \frac{1}{k}$ for any $k \in \mathbb{N}^+$, then $\mu\lambda > 1$ and

$$\mathbb{P}\left(\frac{X_n}{\log n} > \frac{1}{\lambda} + \frac{1}{k} \text{ i.o.}\right) = 0$$

We also notice that

$$\left\{\limsup_n \frac{X_n}{\log n} > \frac{1}{\lambda}\right\} \subseteq \bigcup_k \left\{\frac{X_n}{\log n} > \frac{1}{\lambda} + \frac{1}{k} \text{ i.o.}\right\} =: \bigcup_k A_k$$

Since for fixed $\omega \in \left\{\limsup_n \frac{X_n}{\log n} > \frac{1}{\lambda}\right\}$, suppose we have $\omega \notin \bigcup_k \left\{\frac{X_n}{\log n} > \frac{1}{\lambda} + \frac{1}{k} \text{ i.o.}\right\}$, then for all $k \in \mathbb{N}$, there exists n such that for all $m > n$, $\frac{X_n}{\log n} \leq \frac{1}{\lambda} + \frac{1}{k}$, let $k \to \infty$, we have $\limsup_n \frac{X_n}{\log n} \leq \frac{1}{\lambda}$. However, $\mathbb{P}(A_k) = 0$ for each k, then $\mathbb{P}\left(\limsup_n \frac{X_n}{\log n} > \frac{1}{\lambda}\right) = 0$, implying that

$$\mathbb{P}\left(\limsup_{n\to\infty} \frac{X_n}{\log n} = \frac{1}{\lambda}\right) = 1 - 0 = 1$$

□

Problem 5. *Let $s > 1$, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ and $\mathbb{P}(X = n) = \frac{n^{-s}}{\zeta(s)}$. For $m \geq 1$, let E_m be the event that X is divisible by m.*

(i). Prove that the events E_p are independent for p prime.

(ii). Prove Euler's formula

$$\frac{1}{\zeta(s)} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)$$

probabilistically.

(iii). A number n is squar-free if there is no natural number $m > 1$ such that m^2 divides n. Prove that the probability that X is square-free is $\frac{1}{\zeta(2s)}$.

Proof: (i). We want to show that

$$\mathbb{P}(E_{p_1} \cap E_{p_2}) = \mathbb{P}(E_{p_1}) \cdot \mathbb{P}(E_{p_2})$$

for any $p_1 \neq p_2$ primes. We also notice that

$$\mathbb{P}(E_{p_1} \cap E_{p_2}) = \sum_{n=1}^{\infty} \mathbb{P}(X = np_1p_2) = \sum_{n=1}^{\infty} \frac{(p_1p_2n)^{-s}}{\zeta(s)}$$

and

$$\begin{aligned}\mathbb{P}(E_{p_1}) \cdot \mathbb{P}(E_{p_2}) = \sum_{n=1}^{\infty} \frac{(p_1n)^{-s}}{\zeta(s)} \sum_{n=1}^{\infty} \frac{(p_2n)^{-s}}{\zeta(s)} &= \frac{(p_1p_2)^{-s}}{\zeta(s)^2} \left(\sum_{n=1}^{\infty} n^{-s}\right)^2 \\ &= \sum_{n=1}^{\infty} \frac{(p_1p_2n)^{-s}}{\zeta(s)}\end{aligned}$$

and the conclusion follows from induction.

(ii). We see that

$$\begin{aligned}\frac{1}{\zeta(s)} = \mathbb{P}(X=1) = \mathbb{P}\left(\bigcap_{p \text{ prime}} E_p^c\right) &= \prod_{p \text{ prime}} (1 - \mathbb{P}(E_p)) \\ &= \prod_{p \text{ prime}} \left(\sum_{n=1}^{\infty} \frac{n^{-s}}{\zeta(s)} - \sum_{n=1}^{\infty} \frac{(pn)^{-s}}{\zeta(s)}\right) \\ &= \prod_{p \text{ prime}} (1 - p^{-s}) \left(\sum_{n=1}^{\infty} \frac{n^{-s}}{\zeta(s)}\right) \\ &= \prod_{p \text{ prime}} (1 - p^{-s})\end{aligned}$$

(iii). Since for any natural number m, we can write $m = \prod_{i=1}^{n_m} p_i$ for p_i primes, then n is square-free is equivalent to p^2 doesn't divide n for all p prime. By using the same lines of reasoning as in sub-question (i), E_{p^2} are independent for p prime.

$$\mathbb{P}(\{\text{square-free}\}) = \mathbb{P}\left(\bigcap_{p \text{ prime}} E_{p^2}^c\right) = \prod_{p \text{ prime}} (1 - p^{-2s}) = \frac{1}{\zeta(2s)}$$

□

Problem 6. *Let $X_1, X_2, \dots$ be independent random variables uniformly distributed on $[0,1]$. Let A_n be the event that a record occurs at time n, that is,*

$$X_n > X_m \ \forall m < n$$

Find $\mathbb{P}(A_n)$, and show that $A_1, A_2, \dots$ are independent. Deduce that, with probability one, infinitely many records occurs.

Proof: For fixed n, since $X_1, \dots, X_n$ are independent and identically distributed, the probability of X_n being the largest is $1/n = \mathbb{P}(A_n)$, since each X_i has equal possibility of being the largest. Consider increasing index $1 \le i_1 < i_2 < \cdots < i_n$, then it's sufficient to show that

$$\mathbb{P}(A_{i_1} \cap (A_{i_2} \cap \cdots \cap A_{i_n})) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2} \cap \cdots \cap A_{i_n})$$

Since

$$\mathbb{P}(A_{i_1} \cap (A_{i_2} \cap \cdots \cap A_{i_n})) = \mathbb{P}(A_{i_1,k} \cap (A_{i_2} \cap \cdots \cap A_{i_n}))$$

and

$$\mathbb{P}(A_{i_1}) = \mathbb{P}(\{X_k > X_n, \forall 1 \le n \le i_1, n \neq k\} := A_{i_1,k}) \text{ for all } 1 \le k \le i_1$$

by the fact that X_i's are i.i.d. We also notice that $A_{i_1,k}$'s are disjoint, and

$$\mathbb{P}\left(\bigcup_{k=1}^{i_1} A_{i_1,k}\right) = 1$$

thus,

$$\begin{aligned}
\mathbb{P}(A_{i_2} \cap \cdots \cap A_{i_n}) &= \mathbb{P}\left(\left(\bigcup_{k=1}^{i_1} A_{i_1,k}\right) \cap (A_{i_2} \cap \cdots \cap A_{i_n})\right) \\
&= \mathbb{P}\left(\bigcup_{k=1}^{i_1} (A_{i_1,k} \cap A_2 \cap \cdots \cap A_{i_n})\right) \\
&= \sum_{k=1}^{i_1} \mathbb{P}(A_{i_1,k} \cap A_2 \cap \cdots \cap A_{i_n}) \\
&= i_1 \mathbb{P}(A_{i_1,k} \cap A_2 \cap \cdots \cap A_{i_n})
\end{aligned}$$

hence,

$$\mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_n}) = \frac{1}{i_1}\mathbb{P}(A_{i_2} \cap \cdots \cap A_{i_n}) = \mathbb{P}(A_{i_1})\mathbb{P}(A_{i_2} \cap \cdots \cap A_{i_n})$$

and by induction, we prove independence. Finally, since

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$$

and the fact that A_i's are independent, apply Borel-Contelli lemma, infinitely many records occurs at probability 1. □

Problem 7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let (X_n) be sequence of i.i.d random variables such that*

$$\mathbb{P}(X = 1) = \mathbb{P}(X = -1) = \frac{1}{2}$$

Let $S_0 := 0$ and, for all $n \in \mathbb{N}$,

$$S_n := \sum_{k=1}^{n} X_k$$

Let all $x \in \mathbb{Z}$, define

$$A_x := \{S_n = x \text{ for infinitely many } n \in \mathbb{N}\}$$

$$B_- := \{\liminf_{n\to\infty} S_n = -\infty\}, B_+ = \{\limsup_{n\to\infty} S_n = \infty\}$$

(i). Using Kolmogorov's 0-1 law, prove that $\mathbb{P}(B_-) \in \{0, 1\}$ and $\mathbb{P}(B_+) \in \{0, 1\}$.
(ii). Prove $\mathbb{P}(B_+) = \mathbb{P}(B_-)$.
(iii). Using the Borel-Cantelli lemma, prove that, for all $k \in \mathbb{N}$,

$$\limsup_{n\to\infty}(S_{n+k} - S_n) = k \text{ a.s.}$$

(iv). Deduce from (iii). that $\mathbb{P}(B_-^c \cap B_+^c) = 0$, and therefore that $\mathbb{P}(B_-) = \mathbb{P}(B_+) = 1$. Conclude that, for all $x \in \mathbb{Z}$, $\mathbb{P}(A_x) = 1$.

Proof: (i).

$$B_- = \{\liminf_{n\to\infty} S_n = -\infty\} = \left\{\liminf_{n\to\infty} \sum_{i=k}^{n} X_i = -\infty\right\}$$
$$B_+ = \{\limsup_{n\to\infty} S_n = \infty\} = \left\{\limsup_{n\to\infty} \sum_{i=k}^{n} X_i = \infty\right\}$$

then both B_- and B_+ are in the tail σ-algebra, then by Kolmogorov's 0-1 law, $\mathbb{P}(B_-) \in \{0,1\}$ and $\mathbb{P}(B_+) \in \{0,1\}$.

(ii). Denote $Y = -X$, then $\mathbb{P}(Y=1) = \mathbb{P}(Y=-1) = \frac{1}{2}$, and

$$B_- = \left\{\liminf_{n\to\infty} \sum_{i=1}^{n} X_i = -\infty\right\} = \left\{\limsup_{n\to\infty} \sum_{i=1}^{n} Y_i = \infty\right\}$$

then

$$\mathbb{P}(B_-) = \mathbb{P}(B_+)$$

(iii). Since $\limsup_{n\to\infty}(S_{n+k} - S_n) \geq S_{n+k} - S_n$,

$$\{\limsup_{n\to\infty}(S_{n+k} - S_n) = k\} \supseteq \{S_{n+k} - S_n = k \text{ i.o.}\} =: \{E_n \text{ i.o.}\}$$

but the problem is that E_n's are not independent, however, $\{E_{n+(k+1)m}\}_m$ are independent given that X_n's are i.i.d, with $\mathbb{P}(E_{n+(k+1)m}) = \frac{1}{2^k}$ and $\sum_{m=1}^{\infty} \mathbb{P}(E_{n+(k+1)m}) = \infty$, then by Borel-Cantelli lemma,

$$\mathbb{P}(\{\limsup_{n\to\infty}(S_{n+k} - S_n) = k\}) \geq \mathbb{P}(E_n \text{ i.o.}) = 1$$

(iv). Since $B_-^c = \bigcup_{m\in\mathbb{Z}}\{\liminf_n S_n = m\} =: \bigcup_{m\in\mathbb{Z}} E_m$ and $B_+^c = \bigcup_{l\in\mathbb{Z}}\{\limsup_n S_n = l\} =: \bigcup_{l\in\mathbb{Z}} F_l$,

$$\mathbb{P}(B_-^c \cap B_+^c) = \mathbb{P}\left(\left(\bigcup_{m\in\mathbb{Z}} E_m\right) \cap \left(\bigcup_{l\in\mathbb{Z}} F_l\right)\right) = \mathbb{P}\left(\bigcup_{m,l\in\mathbb{Z}} (E_m \cap F_l)\right)$$

Consider each $E_m \cap F_l = \{\liminf_n S_n = m\} \cap \{\limsup_n S_n = l\}$, and the fact that

$$\begin{aligned}\limsup_{n\to\infty}(S_{n+k} - S_n) \leq \limsup_{n\to\infty} S_{n+k} + \limsup_{n\to\infty}(-S_n) &= \limsup_{n\to\infty} S_{n+k} - \liminf_{n\to\infty} S_n \\ &= l - k\end{aligned}$$

According to (iii), take $K = l - k + 1$, then $\mathbb{P}(\limsup_{n\to\infty}(S_{n+k} - S_n) \leq l - k) = 0$, then by the fact that $\mathbb{P}(B_+^c) = \mathbb{P}(B_-^c)$, $\mathbb{P}(B_-^c) = 0$, $\mathbb{P}(B_-) = \mathbb{P}(B_+) = 1$. Since $A_x \supseteq B_- \cap B_+$, $\mathbb{P}(A_x) = 1$. $\square$

Problem 8. *Let $(X_n, \geq 2)$ be a sequence of independent random variables such that*

$$\mathbb{P}(X_n = n) = \mathbb{P}(X_n = -n) = \frac{1}{2n\log n},\ \mathbb{P}(X_n = 0) = 1 - \frac{1}{n\log n}$$

Set $S_n = X_2 + \cdots + X_n$. Prove that

$$\frac{S_n}{n} \to 0 \textit{ in probability, but not almost surely.}$$

Proof: We first calculate the mean and variance of S_n, and by virtue of independence,

$$\begin{aligned}
\mathbb{E}(S_n) &= \sum_{k=2}^{n} \mathbb{E}(X_k) = \sum_{k=2}^{n} \left(k \cdot \frac{1}{2k\log k} - k \cdot \frac{1}{2k\log k} + 0\right) = 0 \\
\mathrm{Var}(S_n) &= \sum_{k=2}^{n} \mathrm{Var}(X_k) = \sum_{k=2}^{n} \mathbb{E}(X_n^2) = \sum_{k=2}^{n} \left(k^2 \cdot \frac{1}{2k\log k} + k^2 \cdot \frac{1}{2k\log k}\right) = \sum_{k=2}^{n} \frac{k}{\log k} \\
\mathrm{Var}(S_n) &= \mathbb{E}(S_n^2) - \mathbb{E}(S_n)^2 = \mathbb{E}(S_n^2)
\end{aligned}$$

Apply Chebychev inequality,

$$\begin{aligned}
\mathbb{P}\left(\left|\frac{S_n}{n} - 0\right| > \epsilon\right) &\leq \frac{1}{\epsilon^2}\int \left|\frac{S_n}{n}\right|^2 d\mathbb{P} = \frac{1}{\epsilon^2 n^2}\mathrm{Var}(S_n) \\
&\leq \frac{1}{\epsilon^2 n^2}\left(\frac{2}{\log 2} + \sum_{k=3}^{n} \frac{k}{\log k}\right) \\
&\leq \frac{1}{\epsilon^2 n^2}\left(\frac{2}{\log 2} + \frac{(n-2)n}{\log n}\right) \to 0
\end{aligned}$$

as $n \to \infty$, thus proving convergence in probability.

For convergence a.e, notice that since X_n can only take value $\pm n, 0$,

$$\left\{\frac{S_n}{n} \to 0\right\} \subseteq \{X_n = 0 \text{ eventually}\} =: A$$

However, since

$$\sum_{n=2}^{\infty} \mathbb{P}(X_n = n \text{ or } X_n = -n) = \sum_{n=2}^{\infty} \frac{1}{n\log n} = \infty$$

then by Borel-Cantelli lemma, $\mathbb{P}(\{X_n = n \text{ or } X_n = -n \text{ i.o.}\}) = \mathbb{P}(A^c) = 1$, which means that

$$\mathbb{P}\left(\frac{S_n}{n} \not\to 0\right) = 1$$

□