Problem 1. *Show that the following sets of subsets of $\mathbb{R}$ generates the same σ-algebra:*

$$B := \{(a,b) : a < b\}, B_1 := \{(a,b] : a < b\}, B_2 := \{(-\infty, b] : b \in \mathbb{R}\}$$

Proof: To show that $\mathcal{B} := \sigma(B) = \sigma(B_1) =: \mathcal{B}_1$. For any $(a,b) \in B$, $(a,b) = \bigcup_n (a, b - \frac{1}{n}]$, where n choose sufficiently large such that $a < b - \frac{1}{n}$, thus $\mathcal{B} \subseteq \mathcal{B}_1$. Conversely, For any $(a,b] \in B_1$, $(a,b] = \bigcap_{n\in\mathbb{N}} \left(a, b + \frac{1}{n}\right)$, then $\mathcal{B}_1 \subseteq \mathcal{B}$, hence $\mathcal{B} = \mathcal{B}_1$.

To show $\mathcal{B}_1 = \mathcal{B}_2$. Similarly, for any $(a,b] \in B_1$, $(a,b] = (-\infty, b] \cup (-\infty, a]^c$, thus $\mathcal{B}_1 \subseteq \mathcal{B}_2$. Conversely, for $(-\infty, b] \in B_2$, $(-\infty, b] = \bigcup_n (-n, b]$, where $-n$ is chosen such that $-n < b$, thus $\mathcal{B}_2 \subseteq \mathcal{B}_1$, hence $\mathcal{B}_1 = \mathcal{B}_2$. □

Problem 2. *Show that every open subset G of $\mathbb{R}$ is a disjoint countable union of open intervals.*

Proof: For fixed $x \in G$, consider the largest interval such that it is contained in G, namely $I_x = (a,b)$ where

$$a = \inf\{y : y < x, y \in G\},\ b = \sup\{z : x < z, z \in G\}$$

then $G = \bigcup_{x\in G} I_x$. Notice that for I_x, I_y defined as above for $x \neq y$ such that $I_x \cap I_y \neq \emptyset$, for x, $I_x \cup I_y$ would be a larger interval containing x and contained in G, hence $I_x = I_y$, indicating that $\{I_x : x \in G\}$ are collection of disjoint open intervals automatically. Finally, we are left to show that it is countable union. Indeed, for each I_x (assuming the Axiom of Choice implicitly makes the specification valid), it contains a rational number, then we create a one-to-one correspondence between $\{I_x : x \in G\}$ and a subset of $\mathbb{Q}$, suggesting that is indeed a countable union. □

Problem 3. *Let τ be the set of all open subsets of $\mathbb{R}$, and let the π-system*

$$\pi(\mathbb{R}) := \{(-\infty, x] : x \in \mathbb{R}\}$$

Deduce from Problem 1 and 2 that $\mathcal{B}(\mathbb{R}) := \sigma(\tau) = \sigma(\pi(\mathbb{R}))$.

Proof: Thanks to conclusion in Problem 1, $\sigma(\pi(\mathbb{R})) = \sigma(\{(a,b) : a < b\})$, then we are left to show that $\sigma(\tau) = \sigma(\{(a,b) : a < b\})$. $\sigma(\{(a,b) : a < b\}) \subseteq \sigma(\tau)$ is immediate since $\{(a,b) : a < b\} \subseteq \tau$. Conversely, applying conclusion in Problem 2, we see that for any open subset $G \in \tau$, $G = \bigcup_{i=1}^n I_i$ for disjoint sequence of open intervals $\{I_i\}$, thus $\sigma(\tau) \subseteq \sigma(\{(a,b) : a < b\})$. To conclude, $\mathcal{B}(\mathbb{R}) = \sigma(\pi(\mathbb{R}))$. □

Problem 4. *Let μ be a finite-valued, additive set function on an algebra $\mathcal{A}$. Show that μ is countably additive iff for any sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ with*

$$A_n \supseteq A_{n+1}\ \forall n \in \mathbb{N},\ \bigcap_{n\in\mathbb{N}} A_n = \emptyset \implies \mu(A_n) \to 0 \text{ as } n \to \infty.$$

Proof: Suppose that μ is countably additive, and consider $\{A_n^c\}$, then $A_n^c \subseteq A_{n+1}^c$. Define $B_i = A_{i+1}^c \setminus A_i^c$, then $\{B_i\}$ are disjoint, and

$$\mu\left(\bigcup_i A_i^c\right) = \mu\left(\bigcup_i B_i\right) = \sum_i \mu(B_i) = \lim_{n\to\infty}\left(\sum_{i=2}^n \left(\mu(A_i^c) - \mu(A_{i-1}^c)\right) + \mu(A_1^c)\right)$$
$$= \lim_{n\to\infty} \mu(A_n^c)$$

Now, consider $E_i = A_1 \setminus A_i$, then $E_n \subseteq E_{n+1}$

$$\mu\left(\bigcup_i E_i\right) = \lim_{i\to\infty} \mu(E_i) = \lim_{i\to\infty}(\mu(A_1) - \mu(A_i)) = \mu(A_1) - \lim_{i\to\infty} \mu(A_i)$$

while at the same time,

$$\mu\left(\bigcup_i E_i\right) = \mu\left(A_1 \cap \left(\bigcap_i A_i\right)^c\right) = \mu(A_1) - \mu\left(\bigcap_{i=1}^\infty A_i\right)$$

hence

$$0 = \mu\left(\bigcap_{i=1}^\infty A_i\right) = \lim_{i\to\infty} \mu(A_i)$$

Conversely, for $\{E_i\}$ collection of disjoint sets such that $\bigcup_{i=1}^\infty E_i \in \mathcal{A}$, consider the collection $\{A_n\}$ where $A_n = (\bigcup_{i=1}^\infty E_i)\setminus(\bigcup_{i=1}^n E_i)$, then obviously A_n is decreasing and $\bigcap_{i=1}^\infty A_i = \emptyset$, then

$$\lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^\infty E_i \setminus \bigcup_{i=1}^n E_i\right) = \mu\left(\bigcup_{i=1}^\infty E_i\right) - \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^n E_i\right)$$
$$= \mu\left(\bigcup_{i=1}^\infty E_i\right) - \sum_{i=1}^\infty \mu(E_i) = 0$$

then we obtain the countable additivity. □

Problem 5. *Given $\alpha < \beta$, let $\mathcal{A}$ be the algebra of finite unions of disjoint intervals of the form*

$$A = (a_1, b_1] \cup \cdots \cup (a_n, b_n],\ \ \alpha \le a_k, b_k \le \beta,\ \ 1 \le k \le n$$

and define, for every $A \in \mathcal{A}$, $\mu(A) := \sum_{i=1}^n (b_i - a_i)$.

(i). Check that μ is well-defined.

(ii). Prove, using Problem 4, that μ is countably additive on $\mathcal{A}$.

Proof: (i). If $A := \bigcup_{i=1}^n (a_i, b_i] = \bigcup_{i=1}^m (a_i', b_i'] =: A'$, then we can obtain a refined partition $\alpha \le p_1 < p_2 < \cdots < p_N \le \beta$, where p_i are endpoints of A and A' such that $A = A' =$

$\bigcup_{i=1}^{N}(p_i, p_{i+1}]$

$$\mu(A) = \sum_{i=1}^{N}(p_{i+1} - p_i) = \mu(A')$$

which proves that μ is well-defined.

(ii). First, finite additivity is immediate from definition of μ, suppose we have collection of disjoint $\{E_i\}_{i=1}^{n}$ in $\mathcal{A}$, each of the form $E_i = \bigcup_{i=1}^{n_i}(a_{i,k}, b_{i,k}]$, then

$$\mu\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n}\sum_{k=1}^{n_i}(b_{i,k} - a_{i,k}) = \sum_{i=1}^{n}\mu(E_i)$$

Now suppose we have a decreasing sequence $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{A}$ such that $\mu(A_n) \geq \epsilon$ for some $\epsilon > 0$, we aim to find a decreasing bounded subsets $B_n \in \mathcal{A}$ such that $\overline{B_n} \subseteq A_n$ with $\bigcap_{n\in\mathbb{N}} B_n \neq \emptyset$. For each $A_n = \bigcup_{i=1}^{n}(a_{n,i}, b_{n,i}]$, we can shrink each interval by a small margin to $C_n = \bigcup_{i=1}^{n}(a_{n,i} + \delta_{n,i}, b_{n,i}]$ such that $\mu(A_n \setminus C_n) \leq \epsilon/2^{n+1}$, and define $B_n = \bigcap_{i=1}^{n} C_n$, then B_n is decreasing sequence of bounded subsets, we also have

$$\mu(A_n \setminus B_n) \leq \mu\left(\bigcup_{i=1}^{n}(A_n \setminus C_i)\right) \leq \mu\left(\bigcup_{i=1}^{n}(A_i \setminus C_i)\right) \leq \sum_{i=1}^{\infty}\epsilon/2^i + 1 = \epsilon/2$$

then

$$\mu(B_n) = \mu(A_n) - \epsilon/2 \geq \epsilon/2$$

Also, for each B_n, $\overline{B_n} \subseteq A_n$, since each $\overline{C_n} \subseteq A_n$. $\{\overline{B_n}\}$ is the collection of decreasing compact sets, then the intersection of them is nonempty, thus $\bigcap_{i=1}^{\infty} A_n$ is nonempty. Finally, we can apply Problem 4 to obtain that μ is countable additive. □

Problem 6. *Using Problem 5, prove that there exists, for all $\alpha, \beta \in \mathbb{R}$, a unique Borel measure μ on $(\alpha, \beta]$ such that, for all $a, b \in (\alpha, \beta]$ with $a < b$,*

$$\mu((a, b]) = b - a$$

Prove also there exists a unique such measure on the whole of $\mathbb{R}$

Proof: From Problem 5 we've prove that μ is countable additive on algebra $\mathcal{A}$ with $\mu(\mathcal{A}) = \beta - \alpha < \infty$, then by Caratheodory's extension theorem, there exists a unique Borel measure on $(\alpha, \beta]$.

For the case of $\mathbb{R}$, we consider the algebra $\mathcal{A}$ generated by

$$\varepsilon = \begin{cases} (a, b] & \text{for } -\infty \leq a < b < +\infty \\ (a, +\infty) & \end{cases}$$

and the measure on $\mathcal{A}$ is defined similarly, $\mu((a,b]) = b-a$, $\mu((a,+\infty)) = \infty$, and $\mu((-\infty,a]) = \infty$, then we are left to prove that μ is countable additive. Suppose we have $\{A_i\}$ collection of disjoint elements in $\mathcal{A}$ and that $A := \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$, then if A is unbounded, then the countable additivity is immediate from definition, otherwise, $A = \bigcup_i^n (a_i, b_i]$, then by the countable additivity for μ restricted on bounded A with finite measure, we conclude that μ is indeed countable additive. Then by Caratheodory extension theorem, there exists a measure.

Further, since $\varepsilon \cup \{\emptyset\}$ is a π-system, the measure is indeed unique. $\square$

Problem 7. *Given $A_1, \ldots, A_m$ subsets of a nonempty set E, let $\mathcal{A} := \{A_1, \ldots, A_m\}$ and let $\varepsilon := \sigma(\mathcal{A})$. For $\epsilon \in \{0,1\}^m$, define $A^\epsilon := A_1^{\epsilon_1} \cap \cdots \cap A_m^{\epsilon_m}$ where $A_k^0 = A_k^c$, $A_k^1 = A_k$.*

(i). Prove that

$$A \in \varepsilon \iff A = \bigcup_{\epsilon \in I} A^\epsilon \text{ for some } I \subseteq \{0,1\}^m$$

(ii). Deduce that, for all $\epsilon \in \{0,1\}^m$ and $A \in \varepsilon$, either $A \cap A^\epsilon = \emptyset$ or $A^\epsilon \subseteq A$. The subsets A^ϵ are called atoms of ε.

(iii). Deduce that, if $f : (E, \varepsilon) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is measurable, then there exist $a_\epsilon \in \mathbb{R}$, $\epsilon \in \{0,1\}^m$, such that

$$f = \sum_{\epsilon \in \{0,1\}^m} a_\epsilon \mathbf{1}_{A^\epsilon}$$

Proof: (i). The direction from right to left is trivial from the definition of σ-algebra. Conversely, we only need to show that the collection $\mathcal{C} = \{\bigcup_{\epsilon \in I} A^\epsilon : I \subseteq \{0,1\}^m\}$ is a σ-algebra. Indeed, $\emptyset = \bigcup_{\epsilon \in \emptyset} A^\epsilon$ and $E = \bigcup_{\epsilon \in \{0,1\}^m} A^\epsilon$. For $F = \bigcup_{\epsilon \in I} A^\epsilon$, $F^c = \bigcup_{\epsilon \in \{0,1\}^m \setminus I} A^\epsilon \in \mathcal{C}$, since for each $\epsilon_1 \neq \epsilon_2$, $A^{\epsilon_2} \cap A^{\epsilon_2} = \emptyset$. Now for countable union, notice that since the cardinality of $\{0,1\}^m$ is finite, we are essentially dealing with finite union, and finite union is obviously satisfied by the definition. Finally, since ε is the smallest σ-algebra generated by $\mathcal{A}$, left to right implication is proved.

(ii). For $\epsilon \in \{0,1\}^m$ and $A \in \varepsilon$, we know from previous result that we may write

$$A = \bigcup_{\epsilon \in I} A^\epsilon \text{ for some } I \subseteq \{0,1\}^m$$

If $\epsilon' \in I$, then $A^{\epsilon'} \subseteq \bigcup_{\epsilon \in I} A^\epsilon = A$. Otherwise if $\epsilon' \notin I$, we know that, as we also have mentioned in previous proof, from definition if $\epsilon_1 \neq \epsilon_2 \in \{0,1\}^m$ then $A^{\epsilon_1} \cap A^{\epsilon_2} = \emptyset$, that's because there exists some A_i such that $A^{\epsilon_1} \subseteq A_i$ and $A^{\epsilon_2} \subseteq A_i^c$. Thus $A \cap A^{\epsilon'} = \emptyset$.

(iii). Since f is measurable, then consider $H_r := \{x \in E : f(x) > r\} \in \varepsilon$, then according to conclusion part (ii), we must have either $H_r \cap A^\epsilon = \emptyset$ or $H_r \supseteq A^\epsilon$. In other word, for any $x, y \in A^\epsilon$, $f(x), f(y)$ must be either both greater than r or both less or equal to r, and since the choice of r can be arbitrary, we have $f(x) = f(y)$, indicating that f is constant on each atom of ε. Finally, since A^ϵ is a partition of E, we write

$$f = \sum_{\epsilon \in \{0,1\}^m} a_\epsilon \mathbf{1}_{A^\epsilon}$$

where a_ϵ is the constant value on atom A^ϵ. $\square$

Problem 8. *(i). Suppose that $A_1, A_2, \dots$ are independent and $\mathbb{P}(A_n) < 1$ for all $n \geq 1$. Prove that*

$$\mathbb{P}(A_n \ i.o.) = 1 \iff \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = 1$$

(ii). Let $X_1, X_2, \dots$ be independent random variables. Prove that

$$\mathbb{P}\left(\sup_{n\geq 1} X_n < \infty\right) = 1 \iff \sum_{n=1}^{\infty} \mathbb{P}(X_n > a) < \infty$$

for some positive real number a.

Proof: (i). If $\mathbb{P}(A_n \text{ i.o.}) = 1$, then $\mathbb{P}\left(\bigcup_{m=1}^{\infty} \bigcap_{n\geq m} A_n^c\right) = 0 = \sum_{m=1}^{\infty} \mathbb{P}\left(\bigcap_{n\geq m} A_n^c\right)$, thus in particular we have $\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n^c\right) = 0$, hence $\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = 1$. Conversely, we have

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 - \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n^c\right) = 1 - \prod_{n=1}^{\infty}(1 - \mathbb{P}(A_n)) = 1$$

we can do approximation as follows,

$$\prod_{n=1}^{\infty}(1 - \mathbb{P}(A_n)) \leq \prod_{n=1}^{\infty} e^{-\mathbb{P}(A_n)} = e^{-\sum_{n=1}^{\infty} \mathbb{P}(A_n)} = 0$$

then $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$. By Second Borel-Cantelli lemma, $\mathbb{P}(A_n \text{ i.o}) = 1$.

(ii). Suppose that $\sum_{n=1}^{\infty} \mathbb{P}(X_n > a) = \infty$ for all a, then by Borel-Cantelli lemma ($\{X_n > a\}_n$ are independent since $\{X_n\}$ are independent), $\mathbb{P}(X_n > a \text{ i.o.}) = 1$. By the conclusion in Recitation 1, $\mathbb{P}\left(\bigcap_{m=1}^{\infty} \{X_n > m \text{ i.o.}\}\right) = 1$, then we have $\mathbb{P}(\sup_{n\geq 1} X_n = \infty) = 1$, which leads to a contradiction.

Conversely, Since $\sum_{n=1}^{\infty} \mathbb{P}(X_n > a) < \infty$ for some real a, by First Borel-Cantelli lemma, we have $\mathbb{P}(X_n > a \text{ i.o.}) = 0$, then almost surely there exists a N such that for $n \geq N$, $X_n \leq a$, then its immediate that $\mathbb{P}\left(\sup_{n\geq 1} X_n < \infty\right) = 1$. $\square$