Problem 1. *Let $\Omega = \mathbb{R}$ and let $\mathcal{F}$ be the collection of all $A \subseteq \Omega$ such that either A or A^c is countable. Define*

$$\mathbb{P}(A) = \begin{cases} 0 & \text{if } A \text{ is countable} \\ 1 & \text{if } A^c \text{ is countable} \end{cases}$$

Show that $(\Omega, \mathcal{F}, \mathbb{P})$ forms a probability space.

Proof: (Step 1: Show that $\mathcal{F}$ is indeed a σ-algebra). Firstly, $\emptyset \in \mathcal{F}$ is trivial, and $\Omega \in \mathcal{F}$ since its complement $\emptyset$ is countable. Secondly, for any $F \in \mathcal{F}$, if F is countable, then $F^c \in \mathcal{F}$ since $(F^c)^c = F$ is countable; otherwise if F^c is countable, then $F^c \in \mathcal{A}$ trivially. Finally, for sequence of disjoint $\{A_i\} \in \mathcal{F}^{\mathbb{N}}$, if all A_i's are countable, there is nothing to prove, otherwise if for some $j \in \mathbb{N}$, A_j^c is countable, then consider $\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c \subseteq A_j^c$ which is countable, hence the countable union is always in $\mathcal{F}$.

(Step 2: Show that $\mathbb{P}$ is indeed a probability measure). First we have $\mathbb{P}(\emptyset) = 0$ as $\emptyset$ is countable. Secondly, for a sequence of disjoint $\{A_i\} \in \mathcal{F}^{\mathbb{N}}$, we first observe that there can only be one of them, say A_j, such that A^c is countable, for otherwise we have A_j and A_k, $j \neq k$, such that A_j^c, A_k^c are countable and $A_j \cap A_k = \emptyset$, then $(A_j \cap A_k)^c = A_j^c \cup A_k^c = \Omega = \mathbb{R}$, which is absurd since LHS is countable and RHS is uncountable. Therefore, we have for the latter case, $\left(\bigcup_{i=1}^{\infty} A_i\right)^c = \bigcap_{i=1}^{\infty} A_i^c \subseteq A_j^c$, which is countable, and it follows that

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = 1 = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$$

The former case is obvious from the fact that countable union of countable sets is still countable. Finally, immediately from the definition we have $\mathbb{P}(\Omega) = 1$, concluding the proof. $\square$

Problem 2. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X be a measurable function. Define a set function μ on $(\mathbb{R}, \mathcal{B})$ by*

$$\mu(B) = \mathbb{P}(X^{-1}(B)), \ B \in \mathcal{B}$$

Show that μ is a probability measure. (Note: this measure is also called "image measure" or sometimes "pull-back measure".)

Proof: Firstly, $\mu(\emptyset) = \mathbb{P}(X^{-1}(\emptyset)) = \mathbb{P}(\emptyset) = 0$ and $\mu(\mathbb{R}) = \mathbb{P}(X^{-1}(\mathbb{R})) = \mathbb{P}(\Omega) = 1$. Finally, for a sequence of disjoint $\{A_i\}_{i=1}^{\infty} \in \mathcal{B}^{\mathbb{N}}$, we know that $X^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right) = \bigcup_{i=1}^{\infty} X^{-1}(A_i)$, then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(X^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} X^{-1}(A_i)\right) = \sum_{i=1}^{\infty} \mathbb{P}(X^{-1}(A_i)) = \sum_{i=1}^{\infty} \mu(A_i)$$

which shows that μ is indeed a probability measure. $\square$

Problem 3. *Prove that, if $(A_n)_{n\geq 1}$ are independent events, then*

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \prod_{n=1}^{\infty} \mathbb{P}(A_n), \text{ and } \mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = 1 - \prod_{n=1}^{\infty}(1 - \mathbb{P}(A_n))$$

Proof: By the definition of independence, we have

$$\mathbb{P}\left(\bigcap_{n=1}^{N} A_n\right) = \prod_{n=1}^{N} \mathbb{P}(A_n)$$

Notice that $\left\{\bigcap\limits_{n=1}^{N} A_n\right\}_N$ is a decreasing sequence converging to $\bigcap\limits_{n=1}^{\infty} A_n$, then by continuity of measure (Monotone Convergence), we have

$$\mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{N\to\infty} \mathbb{P}\left(\bigcap_{n=1}^{N} A_n\right) = \lim_{N\to\infty} \prod_{n=1}^{N} \mathbb{P}(A_n) = \prod_{n=1}^{\infty} \mathbb{P}(A_n)$$

Second equality holds immediately after what we just proved. $\square$

Problem 4. *Prove that*

$$\limsup_{n\to\infty} \liminf_{k\to\infty} (A_n \cap A_k^c) = 0$$

Proof: From easy set theory we should see that $\limsup\limits_{n\to\infty} \liminf\limits_{k\to\infty} (A_n \cap A_k^c) = \limsup\limits_{n\to\infty} A_n \cap \liminf\limits_{k\to\infty} A_k^c = \emptyset$, as $(\limsup\limits_{n\to\infty} A_n)^c = \liminf\limits_{n\to\infty} A_n^c$. $\square$