Problem 1. *Let μ be the finitely additive measure on a algebra $\mathcal{S}$, consider the sequence of disjoint $\{A_i\}$ in $\mathcal{S}$ such that*

$$A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$$

then

$$\mu\left(\bigcup_{i=1}^{\infty} A_i\right) \geq \sum_{i=1}^{\infty} \mu(A_i)$$

Proof: Since $A \supseteq \bigcup_{i=1}^{n} A_i$, by monotonicity of measure,

$$\mu(A) \geq \mu\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mu(A_i)$$

for any $n \in \mathbb{N}$, we just need to let $n \to \infty$ and the conclusion is obtained. $\square$

Problem 2. *Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a family of measurable sets $\{A_i\}_{i=1}^{\infty}$ such that $\mathbb{P}(A_i) = 1$, then*

$$\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 1$$

Proof: Consider

$$\mathbb{P}\left(\left(\bigcap_{i=1}^{\infty} A_i\right)^c\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i^c\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i^c) \tag{1}$$

Since $\mathbb{P}(A_i) = 1$, we have $\mathbb{P}(A_i^c) = 1 - \mathbb{P}(A_i) = 0$, hence we obtain (1) equals 0, which implies that $\mathbb{P}\left(\bigcap_{i=1}^{\infty} A_i\right) = 1$. $\square$

Problem 3. *For a sequence of events $\{A_n\}_{n\geq 1}$ with $\lim_{n\to\infty} \mathbb{P}(A_n) = 1$ and $0 < c < 1$. Show that there exists a subsequence n_k with $n_k \to \infty$ such that*

$$\mathbb{P}\left(\bigcap_{k=1}^{\infty} A_{n_k}\right) > c$$

Proof: Suppose there exists some c such that for any subsequence n_k with $n_k \to \infty$

$$\mathbb{P}\left(\bigcap_{k=1}^{\infty} A_{n_k}\right) \leq c \iff \mathbb{P}\left(\bigcup_{k=1}^{\infty} A_{n_k}^c\right) > 1 - c$$

Since $\lim_{n\to\infty} \mathbb{P}(A_n) = 1$, then $\lim_{n\to\infty} \mathbb{P}(A_n^c) = 0$, which means that for any $\epsilon/2^m$, $m \in \mathbb{N}$, where $\epsilon < 1 - c$, there exists an $N(m)$ such that for $n > N(m)$

$$\mathbb{P}(A_n^c) < \epsilon/2^m$$

thus it is easy to find a subsequence n_m to goes to infinity where

$$\mathbb{P}\left(\bigcup_{k=1}^{\infty} A_{n_k}^c\right) \leq \sum_{m=1}^{\infty} \mathbb{P}(A_{n_m}^c) = \sum_{m=1}^{\infty} \epsilon/2^m = \epsilon < 1 - c$$

and that leads to a contradiction. $\square$