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Problem 1. *When viewed as metric subspaces of* $\mathbb{R}$*, are* $\{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ *and* $\{1, 2, 3, \dots\}$ *homeomorphic? Isometric?*

Proof: Consider the map $f : H := \{1, \frac{1}{2}, \frac{1}{3}, \dots\} \to G := \{1, 2, 3, \dots\}$ by $f(x) = \frac{1}{x}$, and obviously it is a bijection. Notice that any subset in H (or G) is open in H (or G), since for arbitrary $x \in H$, Consider the ball $B_{1/2}(x)$, then $B_{1/2}(x) \cap H = \{x\} \subseteq H$. For $O \subseteq G$, which is automatically open, $f^{-1}(O)$ is open, and vice versa. Thus is homeomorphic.

For isometry, notice that for $x, y \in H$, $d_H(x, y) = |x - y| < 1$. However, $2 - 1 = 1$, then they are not isometric. $\square$

Problem 2. *Given a metric space* X*, and any point* x *and* y *in* X*, define*

$$\Lambda_X(x, y) := \{z \in X : d_X(x, y) = d_X(x, z) + d_X(z, y)\}.$$

a). Show that if f *is an isometry from* X *onto a metric space* Y*, then*

$$\Lambda_Y(f(x), f(y)) = f(\Lambda_X(x, y)), \text{ for all } x, y \in X.$$

b). Show that $\mathbb{R}^n_1$ *and* $\mathbb{R}^n_2$ *are not isometric for any* $n \geq 2$

Proof: a). For $z \in \Lambda_x(x, y)$, then $d_X(x, y) = d_X(x, z) + d_X(z, y)$, then by isometry of f, $d_Y(f(x), f(y)) = d_Y(f(x), f(z)) + d_Y(f(z), f(y))$, which means that $f(z) \in \Lambda_Y(f(x), f(y))$, implying $f(\Lambda_x(x, y)) \subseteq \Lambda_Y(f(x), f(y))$. Conversely, for $z' \in \Lambda_Y(f(x), f(y))$, $d_Y(f(x), f(y)) = d_Y(f(x), z') + d_Y(z', f(y))$. By injectivity condition, there exists unique $z \in X$ such that $f(z) = z'$, and $d_X(x, y) = d_X(x, z) + d_X(z, y)$ by isometry, then $z \in \Lambda_X(x, y)$, which implies $z' \in f(\Lambda_X(x, y))$, hence we have $\Lambda_Y(f(x), f(y)) = f(\Lambda_x(x, y))$ for every $x, y \in X$.

b). If f is an isometry from $\mathbb{R}^n_1$ onto $\mathbb{R}^n_2$, then $f - f(\mathbf{0})$ is also a isometry, then we may assume $f(\mathbf{0}) = \mathbf{0}$ without loss of generality, since we can always do translation as above. Consider $C_1 = \{x \in \mathbb{R}^n_1 : \|x\|_1 = 1\}$ and $C_2 = \{x \in \mathbb{R}^n_1 : \|x\|_2 = 1\}$, then f send C_1 onto C_2 by isometry to the origin. Consider vertices of C_1, $v_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$, where only the i-th component is 1, then notice that every two distinct vertices in C_1 has distance 2. Now without loss of generality, assume that f maps $(1, 0, \dots, 0)$ to $(1, 0, \dots, 0) =: x$, then for $y = (y_1, y_2, \dots, y_n) \in C_2$, $d_2(x, y) = \sqrt{(y_1 - 1)^2 + y_2^2 + \cdots + y_n^2} = 2$, combining with the fact that $y_1^2 + \cdots + y_n^2 = 1$, obtain that $y_1 = -1$ and $y_i = 0$ otherwise. However, we have more than one candidate for that point, which breaks the injectivity of f. Thus f cannot be an isometry. $\square$

Problem 3. *Let* A *and* B *be two finite subsets of* $\mathbb{R}^n$ *with* $|A| = |B|$*. Prove that* $\mathbb{R}^n \setminus A \simeq \mathbb{R}^n \setminus B$*.*

Proof: Denote $A = \{a_1, \ldots, a_n\}$, and $B = \{b_1, \ldots, b_n\}$. It is sufficient to show that there exists a homeomorphism between $\mathbb{R}^n$ and $\mathbb{R}^n$ such that it sends A to B. In order to utilize induction, we should prove that for $A' \subset \mathbb{R}^n$, which is a set of finite points, and $x, y \notin A'$, there exists a homeomorphism f such that $f(x) = y$. Since A' is finite, there exists $z \in \mathbb{R}^n$ such that the union of two line segments $\overline{xz} \cup \overline{yz}$ contains no point in A', then there exits ϵ such that $U := \{p \in \mathbb{R}^n : d(p, \overline{xz} \cup \overline{yz}) < \epsilon\}$ and $U \cap A' = \emptyset$. Since $\overline{U}$ is homeomorphic to a closed ball, then we will prove later that there exists a homeomorphism φ on $\overline{U}$ such that $\varphi(x) = y$ and $\varphi(u) = u$ when $x \in \partial U$. Now, induct on the cardinality of A. When $n = 1$, apply translation. Suppose for $n \leq N$, we have the conclusion, then for $n = N + 1$, first use induction step, there exists homeomorphism $f(a_i) = b_i$ for $1 \leq i \leq N$, also, by previous claim, we have homeomorphism $\varphi(a_{N+1}) = b_{N+1}$ and φ is identity outside of the interior of some neighborhood not intersecting any previous points a_i, $1 \leq i \leq N$. Hence $f \circ \varphi$ is the desired homeomorphism.

We are left to show that such φ does exist. We first consider the map from closed unit ball to closed unit ball. Given $t\alpha \in \overline{B}$, $\|\alpha\| = 1$ and $t \in [0, 1]$, define $\varphi(t\alpha) = t\alpha + (1 - t)p$, then it satisfies our criterion. $\overline{B}$ after homeomorphism generalize this φ to the U mentioned before in the proof, and thanks to problem 6, on the boundary of U, the map is still identity. $\square$

Problem 4. *Let n be a positive integer. Show that $\{x \in \mathbb{R}^n : \|x\|_p = 1\} \simeq \mathbb{S}^{n-1}$ for any $p \in [1, \infty]$*

Proof: Consider the map $f : \{x \in \mathbb{R}^n : \|x\|_p = 1\} \to \mathbb{S}^{n-1}$ as $f(x) = \frac{x}{\|x\|_2} \in \mathbb{S}^{n-1}$, and claim its inverse is $g : \mathbb{S}^{n-1} \to \{x \in \mathbb{R}^n : \|x\|_p = 1\}$ as $g(y) = \frac{y}{\|y\|_p}$. Indeed, $f(g(y)) = \frac{y/\|y\|_P}{\|y/\|y\|_p\|_2} = \frac{y}{\|y\|_2} = y$, and similarly with $g(f(x))$. Also, map f, g are continuous given that the norm is a continuous function, thus the two spaces are homeomorphic. $\square$

Problem 5. *Show that the products of countably many homeomorphic metric spaces is homeomorphic. Conclude that if we metrized $[0, 1] \times [0, \frac{1}{2}] \times \ldots$ by the product metric, we would obtain a metric space that is homeomorphic to the Hilbert cube. Now combine this fact with Theorem 3.1 to conclude: Every separable metric space can be embedded in l^2.*

Proof: Suppose we have $A_i \simeq B_i$ for $i = 1, 2, 3 \ldots$, and denote each homeomorphism by $f_i : A_i \to B_i$. Consider $F : \prod_{i=1}^{\infty} A_i \to \prod_{i=1}^{\infty} B_i$ by $F(a_1, a_{2,\ldots}) = (f_1(a_1), f_2(a_2), \ldots)$, and this map is clearly bijective since each component function is a homeomorphism. Continuity of F is given by choosing any $(x_m)_{m=1}^{\infty} \in \prod_{i=1}^{\infty} A_i$ that converges to x, then $x_{m,i} \to x_i$ for all i, hence $F(x_m) \to (f_1(x_1), f_2(x_2), \ldots) = F(x)$ continuous, the proof for continuity of F^{-1} is identical once we realize each f_i^{-1} is continuous by homeomorphism condition.

For $[0, \frac{1}{2^n}]$ for $n \geq 0$, consider $f(x) = 2^n x$, then it's obvious that $[0, \frac{1}{2^n}]$ and $[0, 1]$ is homeomorphic, then by previous result $[0, 1] \times [0, \frac{1}{2}] \times \cdots \simeq [0, 1]^{\infty}$. Finally, it's sufficient to prove that $[0, 1] \times [0, \frac{1}{2}] \times \ldots$ is embedded in l^2, and since it is also homeomorphic to $H = \prod_{i=1}^{\infty} [0, \frac{1}{2^i}]$, consider the identity map $I : H \to l^2$ (which obviously is well-defined).

Surely it is injective, and for continuity, fix $\epsilon > 0$, take $\delta < \epsilon^2$, whenever $\rho(x, y) < \delta$, $d_2(x, y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^2\right)^{1/2} \leq \left(\sum_{i=1}^{\infty} 2^{-i}|x_i - y_i|\right)^{1/2} = (\rho(x, y))^{1/2} < \epsilon$, then I is continuous. On the other hand, consider $(x_m)_{m=1}^{\infty} \in l^2$ such that $x_m \to x$, then there exists a $M > 0$ such that whenever $m > M$, we have $\left(\sum_{i=1}^{\infty} |x_{m,i} - x_i|^2\right)^{1/2} < \epsilon$, which implies that $|I^{-1}(x_{m,i}) - I^{-1}(x_i)| = |x_{m,i} - x_i| < \epsilon$ for all i, then $(I^{-1}(x_m))_{m=1}^{\infty}$ converges component-wisely to $I^{-1}(x)$, $I^{-1}(x_m) \to I^{-1}(x)$, I^{-1} is also continuous. To conclude, we find $[0, 1] \times [0, \frac{1}{2}] \times \cdots \simeq H \simeq l^2$, hence by Theorem 3.1 which states that every separable metric space is embedded in $[0, 1]^{\infty}$, they are also embedded in l^2. $\square$

Problem 6. *Let X and Y be two metric spaces, and $f : X \to Y$ a homeomorphism. Prove that $f(cl(S)) = cl(f(S))$ and $f(\partial S) = \partial(f(S))$ for any subset S of X.*

Proof: For $y \in f(\overline{S})$, then $y = f(x)$ for $x \in \mathrm{int}(S)$ or $x \in \partial S$. In the former case, $y \in \overline{f(S)}$, otherwise, for every $\epsilon > 0$, $B(x, \epsilon) \cap S \neq \emptyset$. By setting $\epsilon_n = \frac{1}{n}$, we get a sequence $x_m \to x$, then by continuity of f, $f(x_m) \to f(x)$. We claim that $f(x) \in \overline{f(S)}$, because otherwise there exists a ball centered at $f(x)$ that contained completely in $Y \setminus \overline{f(S)}$, contradicting $f(x_m) \to f(x)$, hence $f(\overline{S}) \subseteq \overline{f(S)}$. Conversely, suppose $y \in \overline{f(S)}$, the $y \in \mathrm{int}(f(S))$ or $y \in \partial(f(S))$, the former case implies $y \in f(\overline{S})$. When $y \in \partial(f(S))$, for all $\epsilon > 0$, there exists $B(y, \epsilon)$ such that $B(y, \epsilon) \cap f(S) \neq \emptyset$, then we obtain $y_m \to y$, and by homeomorphism condition, $f^{-1}(y_m) \to f^{-1}(y)$. $f^{-1}(y)$ must be in $\overline{S}$, for otherwise, there exists a ball containing $f^{-1}(y)$ that is contained in $X \setminus \overline{S}$, contradicting convergence condition. Thus $f(\overline{S}) = \overline{f(S)}$.

For the latter part, since f is bijective, $f(\partial(S)) = f(\overline{S} \setminus \mathrm{int}(S)) = f(\overline{S}) \setminus f(\mathrm{int}(S)) = \overline{f(S)} \setminus f(\mathrm{int}(S))$. We are left to show that $\mathrm{int}(f(S)) = f(\mathrm{int(S)})$, to see that, notice $f(\mathrm{int}(S)) = f(X \setminus \overline{S^c})$, where S^c is the complement of S in X, then $f(\mathrm{int}(S)) = f(X) \setminus f(\overline{S^c}) = f(X) \setminus \overline{f(S^c)} = \mathrm{int}(f(S))$. Hence $f(\partial(S)) = f(\overline{S} \setminus \mathrm{int}(S)) = f(\overline{S}) \setminus f(\mathrm{int}(S)) = \partial(f(S))$. $\square$