
Problem 1. *Let X be a nonempty set. Suppose that there exists a function $d : X \times X \to \mathbb{R}_+$ which satisfies the separation and symmetry properties of being a metric, and in addition, has the property that $d(x, y) \geq d(x, z) + d(z, y)$ for every $x, y, z \in X$. Show that X must then be a singleton.*

Proof: For $x, y \in X$, by assumption we have $0 = d(x, x) \geq d(x, y) + d(y, x) = 2d(x, y)$, which implies that $d(x, y) = 0 \iff x = y$. Thus it must be a singleton. □

Problem 2. *Let X be a nonempty set and $d : X \times X \to \mathbb{R}_+$ a function which satisfies the separation and symmetry properties of being a metric, and in addition, has the property that $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for every $x, y, z \in X$. Such a function is said to be an ultrametric on X, and when d is an ultrametric on X, we refer to (X, d) as an ultrametric space. Clearly, every ultrametric space is a metric space. Give an example to show that the converse of this is false.*

Proof: Consider the $\mathbb{R}^2$ space with Euclidean metric, then $\sqrt{2} = d((0, 0), (1, 1)) \leq d((0, 0), (1, 0)) + d((1, 1), (1, 0)) = 2$. However, $\sqrt{2} = d((0, 0), (1, 1)) > \max\{d((0, 0), (1, 0)), d((1, 1), (1, 0))\} = 1$. □

Problem 3. *Let X be a nonempty set. For any distinct x and y in X^∞, let $k(x, y)$ be the first term at which the sequence x and y differ. Consider the function $d : X^\infty \times X^\infty \to \mathbb{R}_+$ defined by $d(x, y) := \frac{1}{k(x,y)}$ for every distinct $x, y \in X^\infty$, and by $d(x, x) := 0$ for every $x \in X^\infty$. Show that d is an ultrametric on X^∞.*

Proof: The separation and symmetry properties automatically hold by the definition of $k(x, y)$. For $x, y, z \in X^\infty$, suppose $k(x, y) = k$. First case is that $k' = k(x, z) \leq k$, which means that $k(y, z) = k' \leq k$, then $d(x, y) = \frac{1}{k(x,y)} \leq \frac{1}{k'} = d(x, z) = d(y, z) = \max\{d(x, z), d(y, z)\}$. The second case is that $k(x, z) > k$, then we claim that $k(y, z) \leq k$, because otherwise, at least for $i = k$, $x_i = z_i = y_i$, which contradict the fact that $k = k(x, y)$ is the smallest term where x, y differs. Thus $d(x, y) = \frac{1}{k(x,y)} \leq \frac{1}{k(y,z)} = \max\{d(x, z), d(y, z)\}$. In conclusion, d is an ultrametric on X^∞. □

Problem 4. *Let X be an ultrametric space, and take any $x \in X$ and $\epsilon > 0$. Show that if $y \in B(x, \epsilon)$, then $B(y, \epsilon) = B(x, \epsilon)$. (So, an open ball in an ultrametric space may have several centers) Also show that if $B(x, \epsilon_1)$ and $B(y, \epsilon_2)$ overlaps for some $\epsilon_1, \epsilon_2 > 0$, then either $B(x, \epsilon_1)$ is contained in $B(y, \epsilon_2)$ or vice versa.*

Proof: Suppose $z \in B(y, \epsilon)$, then $d(z, x) \leq \max\{d(z, y), d(x, y)\} < \epsilon$, the last inequality is given by the fact that $y \in B(x, \epsilon)$. Thus $B(y, \epsilon) \subset B(x, \epsilon)$. $B(x, \epsilon) \subset B(y, \epsilon)$ is given by exact same argument, hence $B(x, \epsilon) = B(y, \epsilon)$.

Now, without loss of generality, we assume that $\epsilon_1 > \epsilon_2$. For $z \in B(x,\epsilon_1) \cap B(y,\epsilon_2)$, we have $d(x,z) < \epsilon_1$ and $d(y,z) < \epsilon_2$, then $d(x,y) \le \max\{d(x,z), d(y,z)\} < \epsilon_1$, which means that $y \in B(x,\epsilon_1)$. By our previous result, $B(y,\epsilon_1) = B(x,\epsilon_1)$, then $B(y,\epsilon_2) \subset B(y,\epsilon_1) = B(x,\epsilon_1)$. $\square$

Problem 5. *Let X be an ultrametric space, and take any $x \in X$ and $\epsilon > 0$. Show that $B(x,\epsilon)$ is clopen, and conclude that $\partial B(x,\epsilon) = \emptyset$.*

Proof: Suppose that $y \in B(x,\epsilon)$, then by Problem 4, $B(y,\epsilon) = B(x,\epsilon) \subset B(x,\epsilon)$, thus being open. Choose an arbitrary $z \in X \setminus B(x,\epsilon)$, suppose there doesn't exist a $\epsilon' > 0$ such that $B(z,\epsilon') \subset X \setminus B(x,\epsilon)$, then it's equivalent to say that for every $\epsilon' > 0$, $B(z,\epsilon')$ intersect with $B(x,\epsilon)$, then from Problem 4 we've already known that either $B(x,\epsilon)$ contains $B(z,\epsilon')$ or vice versa. Surely $B(z,\epsilon')$ can't be contained in $B(x,\epsilon)$ since $z \notin B(x,\epsilon)$ for a start. Now suppose that $B(x,\epsilon)$ is contained in $B(z,\epsilon')$ for all $\epsilon' > 0$, then for $\epsilon' < \epsilon$, $d(x,z) \ge \epsilon$ given that $z \notin B(x,\epsilon)$, on the other side, $d(x,z) < \epsilon'$, which is absurd. Thus, $X \setminus B(x,\epsilon)$ is open, suggesting that $B(x,\epsilon)$ is closed. To conclude, $B(x,\epsilon)$ is clopen, and by the definition of boundary, $\partial B(x,\epsilon) = \overline{B}(x,\epsilon) \setminus B(x,\epsilon) = \emptyset$. $\square$

Problem 6. *We say that an ordered pair (X,μ) is an oriented semimetric space if X is a nonempty set and $\mu : X \times X \to [0,\infty)$ is a function that satisfies the triangular inequality and $\mu(x,x) = 0$ for all $x \in X$. We say that a subset S of X is open (relative to μ) if for every $x \in S$ there is an $\epsilon > 0$ such that $y \in S$ for every $y \in X$ with $\mu(x,y) < \epsilon$. We say that S is closed (relative to μ) if $X \setminus S$ is open.*

a). Show that $(\mathbb{R},\mu)$ is an oriented semimetric space where $\mu : \mathbb{R}\times\mathbb{R} \to [0,\infty)$ is defined by $\mu(x,y) := \max\{y-x, 0\}$.

b). In the following part of this problem, (X,μ) is an arbitrarily oriented semimetric space. Define $d : X \times X \to [0,\infty)$ by $d(x,y) := \mu(x,y) + \mu(y,x)$. Is d a semimetric on X?

c). For any $\epsilon > 0$ and $x \in X$, show that $B(x,\epsilon) := \{y \in X : \mu(x,y) < \epsilon\}$ is open.

d). Prove or disprove: For any $\epsilon > 0$ and $x \in X$, $B[x,\epsilon] := \{y \in X : \mu(x,y) \le \epsilon\}$ is closed.

Proof: a). Consider $x,y,z \in \mathbb{R}$, if $x \ge y$, then $\mu(x,y) = 0 \le \mu(x,z) + \mu(z,y)$, else if $x < y$, then $\mu(x,y) + \mu(z,y) = \max\{z-x, 0\} + \max\{y-z, 0\} \ge z - x + y - z = y - x > 0$, concluding the proof of triangular inequality. $\mu(x,x) = 0$ for all $x \in X$ is obvious.

b). First, it's obvious that $d(x,x) = 0$ for all $x \in X$. Symmetry is given by $d(x,y) = \mu(x,y) + \mu(y,x) = \mu(y,x) + \mu(x,y) = d(y,x)$. Finally, to prove the triangular inequality. For any $x,y,z \in X$, $d(x,y) = \mu(x,y) + \mu(y,x) \le \mu(x,z) + \mu(z,y) + \mu(y,z) + \mu(z,x) = d(x,z) + d(y,z)$.

c). For any $z \in B(x,\epsilon)$, choose $\delta = \epsilon - \mu(x,z)$, then we claim that $B(z,\delta) \subset B(x,\epsilon)$. Indeed, choose any $a \in B(z,\delta)$, $\mu(x,a) \le \mu(x,z) + \mu(z,a) < \mu(x,z) + \delta = \epsilon$.

d). Consider the following function on $\mathbb{R}$: $\mu(x,y) = 0$ if $x \le y$, $\mu(x,y) = 1$ if $x > y$, then $\mu(x,x) = 0$ for all $x \in \mathbb{R}$, and $\mu(x,y) \le \mu(x,z) + \mu(z,y)$ for every $x,y,z \in \mathbb{R}$, then it's an oriented semimetric space. Now, consider $B[0,1/2] = \{y \in \mathbb{R} : \mu(0,y) \le 1/2\} = [0,+\infty)$, $\mathbb{R} \setminus [0,\infty) = (-\infty,0)$, and consider $B(-1,\delta) = \{z : \mu(-1,z) < \delta\}$, then for arbitrary $\delta > 0$, $z \ge -1$ contains in the ball, which obviously isn't contained in $(-\infty,0)$. $\square$

Problem 7. *Let X and Y be two metric spaces and $f : X \to Y$ a function. If there is a real number $K \geq 0$ such that $d_Y(f(x), f(y)) \leq K d_X(x, y)$ for every $x, y \in X$, we say that f is K - Lipschitz. If f is K - Lipschitz, it is simply referred to as a Lipschitz map. In this case, $\inf\{K \geq 0 : f \text{ is K-Lipschitz}\}$, which is denoted by $Lip(f)$, is called the Lipschitz number of f.*

a). Show that the identity function on any metric space X onto itself is 1 - Lipschitz.

b). A differentiable real function f on a nonempty open interval is Lipschitz continuous, provided that $\sup_{x\in O} |f'(x)| < \infty$. (Recall Mean Value Theorem.)

c). Is the function $t \mapsto \sqrt{t}$ Lipschitz?

d). Let X be a normed linear space. Take any positive integer n and real numbers $\lambda_1, \ldots, \lambda_n$, and define the map $f : X^n \to X$ by $f(x) := \lambda_1 x_1 + \cdots + \lambda_n x_n$. Where X^n is metrized by the product metric ρ, show that f is Lipschitz. (f is $\max\{|\lambda_1|, \ldots, |\lambda_n|\}$ - Lipschitz).

e). Let S be a nonempty subset of a metric space X. Prove that dist $(\cdot, S) := \inf\limits_{z\in S} d(\cdot, z)$ is 1 - Lipschitz.

f). let κ be a bounded, Riemann integrable function on $[0, 1]\times[0, 1]$, and consider map $\Phi :$ $\mathbf{C}[0, 1] \to \mathbf{B}[0, 1]$ defined by $\Phi(f)(x) := \int_0^1 \kappa(x, y)f(y)dy$. Prove that Φ is $\|\kappa\|_\infty$- Lipschitz.

g). Let T be a nonempty set, and X a nonempty subset of $B(T)$ which is closed under addition by positive constant functions. Assume that Φ is an increasing self-map on X. If there exists a $K > 0$ such that $\Phi(f + \alpha) \leq \Phi(f) + K\alpha$ for every $f \in X$ and $\alpha \geq 0$, Φ must be K-Lipschitz.

Proof: a). Let I be the identity function on X, then $d(I(x), I(y)) = d(x, y) \leq 1 \cdot d(x, y)$, hence is a 1 - Lipschitz function.

b). Let x, y be two points in the nonempty open interval O, then by Mean Value Theorem, there exists $z \in (x, y)$, such that $d(f(x), f(y)) = f'(z) \cdot d(x, y) \leq \sup_{x\in O} |f'(x)| \cdot d(x, y) = K \cdot d(x, y)$ with $K \leq \infty$, then f is Lipschitz continuous.

c). It is not Lipschitz. For arbitrary $K > 0$, consider two points $x = 0$, $y = t$ such that $\frac{1}{\sqrt{t}} > K$, then $\frac{|\sqrt{t}-0|}{|t-0|} = \frac{1}{\sqrt{t}} > K$.

d).

$$\begin{aligned} d_X(f(x), f(y)) &= d_X(\lambda_1 x_1 + \cdots + \lambda_n x_n, \lambda_1 y_1 + \cdots + \lambda_n x_n) \\ &= \|\sum_{i=1}^n \lambda_i(x_i - y_i)\| \\ &\leq \sum_{i=1}^n |\lambda_i| \cdot \|x_i - y_i\| \\ &\leq \max\{|\lambda_1|, \ldots, |\lambda_n|\} \sum_{i=1}^n \|x_i - y_i\| \\ &= \max\{|\lambda_1|, \ldots, |\lambda_n|\} \cdot \rho(x, y) \end{aligned}$$

thus is Lipschitz.

e). $\text{dist}(x, S) = \inf\limits_{z\in S} d(x, z) \leq d(x, z')$ for any $z \in S$, then $\text{dist}(x, S) \leq d(x, y) + d(y, z')$, then $\text{dist}(x, S) - \text{dist}(y, S) \leq d(x, y)$. Also, we may have $\text{dist}(y, S) - \text{dist}(x, S) \leq d(x, y)$, thus we have $|\mathrm{dist}(x,S)-\mathrm{dist}(y,S)| \le d(x,y)$.

f). Let $f,g \in \mathbf{C}[0,1]$. $d(\Phi(f),\Phi(g)) = \sup_{x\in[0,1]} \left|\int_0^1 \kappa(x,y)f(y)dy - \int_0^1 \kappa(x,y)g(y)dy\right| = \left|\int_0^1 \kappa(x,y)\left(f(y)-g(y)\right)dy\right| \le \|\kappa\|_\infty \int_0^1 |f(y)-g(y)|\,dy \le \|\kappa\|_\infty \sup_{y\in[0,1]}|f(y)-g(y)| = \|\kappa\|_\infty d(f,g)$.

g). For $f,g \in X$ and arbitrary $x \in T$, without loss of generality, assume that $f(x) > g(x)$, write $f(x) = g(x)+\alpha$ for $\alpha > 0$. By assumption, $\Phi(f)(x) = \Phi(g+\alpha)(x) \le \Phi(g)(x) + K\alpha$, then $|\Phi(f)(x)-\Phi(g)(x)| \le K|f(x)-g(x)|$. Taking supreme of x on both sides, we conclude that Φ is k - Lipschitz. □

Problem 8. *Take any metric space X, and let $Lip(X)$ be the set of all bounded and Lipschitz continuous real-valued maps on X.*

a). Show that $Lip(\lambda f+g) \le |\lambda| Lip(f) + Lip(g)$ for every $f,g \in Lip(X)$. Conclude that $Lip(X)$ is a linear subspace of $B(X)$. (Note that $Lip(\mathbb{N}) = l_\infty = B(\mathbb{N})$.)

b). Show that the real map $\|\cdot\|_L$ defined on $Lip(X)$ by $\|f\|_L := \|f\|_\infty + Lip(f)$, is a norm on $Lip(X)$.

c). For any $f,g \in Lip(X)$, show that $Lip(fg) \le \|f\|_\infty Lip(g) + \|g\|_\infty Lip(f)$, and deduce that $\|fg\|_L \le \|f\|_L\|g\|_L$.

Proof: a). For $f,g \in \mathrm{Lip}(X)$, have $d((\lambda f+g)(x),(\lambda f+g)(y)) = |\lambda f(x)+g(x)-\lambda f(y)-g(y)| \le |\lambda|\cdot|f(x)-f(y)| + |g(x)-g(y)|$ for $x,y \in X$, thus $d((\lambda f+g)(x),(\lambda f+g)(y)) \le (|\lambda|\mathrm{Lip}(f)+\mathrm{Lip}(g))\cdot d(y-x)$, then $\mathrm{Lip}(\lambda f+g) \le |\lambda|\mathrm{Lip}(f)+\mathrm{Lip}(g)$.

b). Firstly, $\|f\|_L = 0 \iff \|f\|_\infty = 0$ and $\mathrm{Lip}(f) = 0$, which means that $|f(y)-f(x)| \le 0\cdot d(y,x)$ for all $x,y \in X$, then $f \equiv 0$. Secondly, $\|\lambda f\|_L = |\lambda|\cdot\|f\|_\infty + \mathrm{Lip}(\lambda f)$. Since $|\lambda f(x)-\lambda f(y)| = |\lambda|\cdot|f(x)-f(y)| \le |\lambda|\cdot K\cdot d(x,y)$, then $\mathrm{Lip}(\lambda f) = |\lambda|\mathrm{Lip}(f)$, thus $\|\lambda f\|_L = \lambda(\|f\|_\infty + \mathrm{Lip}(g))$. Finally, for $f,g \in \mathrm{Lip}(X)$, $\|f+g\|_L = \|f+g\|_\infty + \mathrm{Lip}(f+g) \le \|f\|_\infty + \|g\|_\infty + \mathrm{Lip}(f)+\mathrm{Lip}(g) = \|f\|_L + \|g\|_L$.

c). Consider

$$\begin{aligned}
|f(x)g(x)-f(y)g(y)| &= |f(x)g(x)-f(x)g(y)+f(x)g(y)-f(y)g(y)| \\
&\le |f(x)|\cdot|g(x)-g(y)| + |g(y)|\cdot|f(x)-f(y)| \\
&\le (\mathrm{Lip}(g)|f(x)| + \mathrm{Lip}(f)|g(y)|)\cdot d(x,y) \\
&\le (\|f\|_\infty\mathrm{Lip}(f) + \|g\|_\infty\mathrm{Lip}(g))\cdot d(x,y)
\end{aligned}$$

hence $\mathrm{Lip}(fg) \le \|f\|_\infty\mathrm{Lip}(g) + \|g\|_\infty\mathrm{Lip}(f)$. Also,

$$\begin{aligned}
\|fg\|_L &= \|fg\|_\infty + \mathrm{Lip}(fg) \\
&\le \|f\|_\infty\|g\|_\infty + \mathrm{Lip}(fg) + \mathrm{Lip}(f)\mathrm{Lip}(g) \\
&= \|f\|_\infty\|g\|_\infty + \|f\|_\infty\mathrm{Lip}(g) + \mathrm{Lip}(f)\|g\|_\infty + \mathrm{Lip}(f)\mathrm{Lip}(g) \\
&= (\|f\|_\infty + \mathrm{Lip}(f))(\|g\|_\infty + \mathrm{Lip}(g)) \\
&= \|f\|_L\|g\|_L
\end{aligned}$$

□