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Problem 1. *Let X be a nonempty set, and let $\mathcal{O}$ stand for the collection of all subsets of X whose complements in X are countable, plus the empty set. Show that $\mathcal{O}$ is a topology on X which reduces to the discrete topology when X is countable. Now, assume that X is uncountable, and endow it with the countable complement topology. Prove:*

a. A sequence in X converges iff it is eventually constant.

b. X is not Hausdorff, but a sequence in X may converge to at most one point in X.

c. A nonempty proper subset of X is closed iff it is countable.

d. For any point x in X, the closure of $X \setminus \{x\}$ is X even though no sequence in $X \setminus \{x\}$ can possibly converge to x.

Proof: To show that $\mathcal{O}$ is a topology. Firstly, $X \in \mathcal{O}$ because $X \setminus X = \emptyset$ is countable, and $\emptyset \in X$ by definition. Secondly, let $\mathcal{O}' \subseteq \mathcal{O}$, $X \setminus \bigcup_{O \in \mathcal{O}} O = \bigcap_{O \in \mathcal{O}} (X \setminus O)$, which is at most countable. Finally, let $O_1, \dots O_n \in \mathcal{O}$. $X \setminus \bigcap_{i=1}^n O_i = \bigcup_{i=1}^n (X \setminus O_i)$, which is a finite union of countable sets, thus is countable. Hence $\mathcal{O}$ is indeed a topology. When X itself is countable, want to show that $\mathcal{O} = 2^X$, left inclusion is immediate, for $S \in 2^X$, $X \setminus S$ is at most countable given X is countable, $S \in \mathcal{O}$, thus $\mathcal{O} = 2^X$ is discrete topology.

a. Left implication is obvious since all constant sequences converge. Now suppose that $\{x_m\}_{m=1}^{\infty} \in X^{\infty}$ converges to $x \in X$ (rule out all x_i such that $x_i = x$), and $\{x_m\}$ is not eventually constant, then choose $O' = O \cup \{x_m\}$ for $O \in \mathcal{O}$, notice that $X \setminus O' = (X \setminus O) \cap (X \setminus \{x_m\})$, which is at most countable, also $x \in O'$. However, there is no $M \in \mathbb{N}$ such that O' contains $\{x_m\}_{m \geq M}$, contradicting the fact that it converges, hence must be eventually constant.

b. For arbitrary $x, y \in X$. Suppose there exists O_1 containing x, and O_2 containing y such that $O_1 \cap O_2 = \emptyset$, then $O_1 \subseteq X \setminus O_2$ at most countable, implying that $X \setminus O_1$ is uncountable, contradicting the fact that $O \in \mathcal{O}$, hence X is not Hausdorff. Since by question a, a convergent sequence is eventually constant, so it can only converge at one point.

c. Let $\emptyset \neq S \subsetneq X$ be closed, then by definition it's equivalent to $X \setminus S$ is open and not empty, then by definition of countable complement topology, it's equivalent to $X \setminus (X \setminus S) = S$ is countable.

d. First, notice that $X \setminus \{x\}$ is open, because $X \setminus (X \setminus \{x\}) = \{x\}$ countable. By definition of closure, $\overline{X \setminus \{x\}} := \bigcap \{C \in \mathcal{C}_X : S \subseteq C\}$. Since the only possible $C \in \mathcal{C}_X$ containing $X \setminus \{x\}$ is X itself, then its closure has to be X. $\square$

Problem 2. *Let X be a topological space and S a subset of X. We say that a point x in X is a limit point of S if every open neighborhood of x contains a point of S other than x. Prove:*

a. S is closed iff it contains all of its limit points.

b. cl(S) equals S plus all limit points of S.

c. If X is a T_1-space and x is a limit point of S, then every open neighborhood of x contains infinitely many points of S.

Proof: a. Suppose that S is closed, and if there exists a limit point s of S such that $x \notin S$, then by definition, for every open neighborhood of x, it contains a point of S, thus S^c

is not open, contradicting the fact that S is closed. Conversely, if S contains all of its limit points, the for $x \in S^c$, x cannot possibly be a limit point of S, meaning that there exists open neighborhood of x not intersecting S, implying S^c is open, thus S is closed.

b. $\text{cl}(S) = \bigcap\{C \in \mathcal{C}_X : S \subseteq C\} = X \setminus \bigcup\{O \in \mathcal{O}_X : O \subseteq S^c\}$. For $x \in \text{cl}(S)^c$, then $x \in \bigcup\{O \in \mathcal{O}_X : O \subseteq S^c\}$, then there exists a neighborhood B_x containing x such that $B_x \subseteq \bigcup\{O \in \mathcal{O}_X : O \subseteq S^c\}$, then x is not a limit point of S and surely not a point in S, thus $H := S \cup \{\text{limit points of } S\} \subseteq \text{cl}(S)$. Conversely, if $x \notin H$, then $x \notin S$ and $x \notin \{\text{limit points of } S\}$, then there exists a open neighborhood of x that it doesn't contain any point of S, so $x \notin \text{cl}(S)$, thus $\text{cl}(S) \subseteq H$.

c. Let O_1 be any open neighborhood of x, then by definition of limit point, there exists a point $x_1 \in S$ such that $x_1 \in O_1$. Consider $O_2 = O_1 \setminus \{x_1\}$, O_2 is open since $X \setminus O_2 = (X \setminus O_1) \cup \{x_1\}$ is closed ($\{x_1\}$ is closed since X is T_1-space), then O_2 must contain $x_2 \in S$ such that $x_2 \in O_2$, $x_1 \neq x_2$. Continuing this process, we conclude that O_1 must contain infinitely many points of S (Notice that this process can indeed be executed infinitely many times given that x is a limit point). $\square$

Problem 3. *Prove: A topological space X is Hausdorff iff for every $x \in X$, the intersection of all closed neighborhoods of x in X equals $\{x\}$.*

Proof: $\{x\} \subseteq \bigcap\{C \in \mathcal{C}_X : x \in C\}$ is obvious. If X is Hausdorff, for $y \in \bigcap\{C \in \mathcal{C}_X : x \in C\}$, if $y \neq x$, there exists open B_x containing x, open B_y containing y such that B_x and B_y are disjoint. Consider $C' = X \setminus B_y$, then C' is closed containing x, however, $y \notin C'$, contradicting the fact that $y \in \bigcap\{C \in \mathcal{C}_X : x \in C\}$. Conversely, for any $x, y \in X$, since $\{x\} = \bigcap\{C \in \mathcal{C}_X : x \in C\}$, then there exists a closed neighborhood C' such that $y \notin C'$, then $y \in X \setminus C'$ open. By definition of neighborhood, C' contains an open neighborhood of x, say B_x, then we obtain $X \setminus C'$ and B_x satisfying criteria for being Hausdorff. $\square$

Problem 4. *For any real number a, let S_a stand for the set $\{x \in \mathbb{R}^2 : x_1 > a\}$. Show that $\mathcal{B} := \{S_a : a \in \mathbb{R}\}$ is a basis for a topology on $\mathbb{R}^2$. Also show that the topology generated by $\mathcal{B}$ is $\mathcal{B} \cup \{\emptyset, \mathbb{R}^2\}$.*

Proof: First is to show that $\bigcup_{a\in\mathbb{R}} S_a = \mathbb{R}^2$, indeed for any $(x, y) \in \mathbb{R}^2$, there exists $a \in \mathbb{R}$ such that $x > a$, so $(x, y) \in \bigcup_{a\in\mathbb{R}} S_a$, the other direction is immediate. Secondly, suppose that $p = (x, y) \in S_{a_1} \cap S_{a_2}$ for $S_{a_1}, S_{a_2} \in \mathcal{B}$, we may assume that $a_1 < a_2$, then $a_1 < a_2 < x$. Let $a_3 = (a_2 + x)/2$, then $S_{a_3} \subseteq S_{a_1} \cap S_{a_2}$ and $p \in S_{a_3}$, thus $\mathcal{B}$ is a basis for a topology on $\mathbb{R}^2$. Now we are left to show that $\{\text{all unions of elements in } \mathcal{B}\} \cup \{\emptyset\} = \mathcal{B} \cup \{\emptyset, \mathbb{R}^2\}$. $\mathcal{B} \cup \{\emptyset, \mathbb{R}^2\} \subseteq \{\text{all unions of elements in } \mathcal{B}\} \cup \{\emptyset\}$ is clear since $\mathbb{R}^2 = \bigcup_{S\in\mathcal{B}} S$. Conversely, let $\mathcal{S}$ be the collection of elements on $\mathcal{B}$, and let $\mathcal{A}_\mathcal{S}$ be the corresponding collection of a for $S_a \in \mathcal{S}$. If $\mathcal{A}_\mathcal{S}$ is finite, then $\bigcup_{S\in\mathcal{S}} S = S_a \in \mathcal{B}$ for minimum $a \in \mathcal{A}_\mathcal{S}$. If $\mathcal{A}_\mathcal{S}$ is infinite, then consider $a = \inf \mathcal{A}_\mathcal{S}$. Suppose that $a = -\infty$, then $\bigcup_{S\in\mathcal{S}} S = \mathbb{R}^2$, or if $a < -\infty$, then $\bigcup_{S\in\mathcal{S}} S = S_a$ ($\bigcup_{S\in\mathcal{S}} S \subseteq S_a$ is obvious. Conversely for $x \in S_a$, there exists $y \in \mathcal{A}$ such that $a \leq y < x$). To conclude, the topology generated is $\mathcal{B} \cup \{\emptyset, \mathbb{R}^2\}$. $\square$

Problem 5. *The following is a proof of the fact that there are infinitely many primes. An arithmetic progression is a set of the form*

$$B_{a,b} := a + b\mathbb{Z}$$

that is, $B_{a,b} = \{a + kb : k \in \mathbb{Z}\}$, for any $a, b \in \mathbb{Z}$ with $b \neq 0$. Let $\mathcal{B}$ denote the set of all arithmetic progressions.

a. Show that $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.

b. Show that every element of $\mathcal{B}$ is closed with respect to this topology.

c. Put $\mathcal{S} := \{p\mathbb{Z} : p \in \mathcal{P}\}$, and find $\mathbb{Z} \setminus \bigcup \mathcal{S}$.

d. Using part (b) and (c), show that there cannot be finitely many prime numbers.

Proof: a. First we want to show that $\bigcup_{B\in\mathcal{B}} B = \mathbb{Z}$. Indeed, $\bigcup_{B\in\mathcal{B}} B \subseteq \mathbb{Z}$ is obvious, and conversely, notice that $\mathbb{Z} = B_{0,1}$. Secondly, for $z \in B_{a,b} \cap B_{a',b'} = \{a+kb\} \cap \{a'+kb'\}$, where $a \neq a'$ and $b \neq b'$, then $z \in \{z + k \cdot b \cdot b'\} \subseteq B_{a,b} \cap B_{a',b'}$, where $\{z + k \cdot b \cdot b' : k \in \mathbb{Z}\} \in \mathcal{B}$. Hence $\mathcal{B}$ is a basis for a topology on $\mathbb{Z}$.

b. Fix $B_{a,b} = \{a + kb : k \in \mathbb{Z}\}$ for some $a, b \in \mathbb{Z}$. Without loss of generality, we can assume that $0 \leq a < b$, then for $z \in \mathbb{Z} \setminus B_{a,b}$, then we can write $z = a' + kb$, where $0 \leq a' \neq a < b$, thus $z \in \{a' + kb : k \in \mathbb{Z}\} \in \mathcal{B}$, and $\{a' + kb : k \in \mathbb{Z}\} \cap B_{a,b} = \emptyset$. Hence $\mathbb{Z} \setminus B_{a,b}$ is open, $B_{a,b}$ is closed.

c. Suppose there exists $z \in \mathbb{Z} \setminus \bigcup \mathcal{S}$, $z \neq 1, -1$. If z is prime, then it must be in one of $\mathcal{S}$, else if z is not prime, then prime factorize z into product of one prime and an integer, it still belongs to one of $\mathcal{S}$, thus $\mathbb{Z} \setminus \bigcup \mathcal{S} \subseteq \{1, -1\}$. Conversely, both 1 and -1 can not be expressed as $p\mathbb{Z}$ for some p primes. Hence $\mathbb{Z} \setminus \bigcup \mathcal{S} = \{1, -1\}$.

d. Suppose that there are finitely many prime numbers, then $\bigcup \mathcal{S}$ is finite union of closed sets, thus is closed, then $\{1, -1\}$ is open. However, according to the topology induced by $\mathcal{B}$, any open set is the union of arithmetic progressions, thus is infinite. So $\{1, -1\}$ cannot be open, leading to a contradiction. $\square$

Problem 6. *A topological space X is said to be a Lindelöf space if for every $\mathcal{O} \subseteq \mathcal{O}_X$ with $X = \bigcup \mathcal{O}$, there is a countable $\mathcal{U} \subseteq \mathcal{O}$ with $X = \bigcup \mathcal{U}$.*

a. Show that a closed subspace of a Lindelöf space is Lindelöf.

b. Show that a continuous image of a Lindelöf space is Lindelöf.

c. Prove: Every second-countable topological space is Lindelöf.

d. Show that the Sorgenfrey line X is Lindelöf. Thus, even a first-countable and separable Lindelöf space need not be second-countable.

e. Show that a metric space is second-countable iff it is Lindelöf. In particular, $\mathbb{R}^n$ is Lindelöf for any $n \geq 1$.

Proof: a. Let $X' \subseteq X$ be a closed subspace of Lindelöf space X, and let $\mathcal{O}' \subseteq \mathcal{O}_X$ satisfies $X' \subseteq \bigcup \mathcal{O}'$. Since $O := X \setminus X'$ is open, $\mathcal{O}' \cup \{O\}$ covers X. By Lindelöf property, there exists a countable $\mathcal{U} = \mathcal{U}' \cup \{O\} \subseteq \mathcal{O}$ with $X = \bigcup \mathcal{U}$, where $\mathcal{U}' \subseteq \mathcal{O}'$, then $\mathcal{U}'$ is countable. Consider $\mathcal{U}'' = \{X' \cap U : U \in \mathcal{U}'\}$, $\mathcal{U}''$ is countable and $X' = \bigcup \mathcal{U}''$.

b. Let X be a Lindelöf space and $f : X \to X' = \mathrm{Im}(X)$. Fix $\mathcal{O} \subseteq \mathcal{O}_{X'}$ with $X' = \bigcup \mathcal{O}$, by continuity of f, $\mathcal{U} = \{f^{-1}(O) : O \in \mathcal{O}\}$ is a collection of open sets in X such that

$X = \bigcup \mathcal{U}$. By Lindelöf property of X, there exists a countable $\mathcal{U}' \subseteq \mathcal{U}$ with $X = \bigcup \mathcal{U}'$, then $f(\mathcal{U}') = \{f(U) : U \in \mathcal{U}'\} = \{f(f^{-1}(O)) : \text{for some countable } O \in \mathcal{O}\} = \{O : \text{for some countable } O \in \mathcal{O}\}$, the last equality holds because X' is the image of X, also $X' = f(X) = f(\bigcup \mathcal{U}') = \bigcup f(\mathcal{U}')$, so X' is Lindelöf.

c. Let $\mathcal{O}$ be an open cover of X, and $\mathcal{B}$ be a countable basis. Define $\mathcal{B}' = \{B \in \mathcal{B} : B \subseteq O \text{ for some } O \in \mathcal{O}\}$, also define $O_B \in \mathcal{O}$ such that $B \subseteq O_B$. Claim that $\mathcal{U} = \{O_B : B \in \mathcal{B}'\}$ satisfies countability and $X = \bigcup \mathcal{U}$. Indeed, $\bigcup \mathcal{U} \subseteq X$ is obvious. Now suppose that $x \in X$ but $x \notin O_B$ for all $B \in \mathcal{B}'$, then $x \notin B$ for all $B \in \mathcal{B}'$. Notice that $x \in O \in \mathcal{O}$ (since $\mathcal{O}$ is a covering), and $x \in B$ for some $B \notin \mathcal{B}'$ (because $\mathcal{B}$ is a basis), then $x \in B \subseteq O \cup B \in \mathcal{O}$, then $B \in \mathcal{B}'$, leading to a contradiction, thus $X = \bigcup \mathcal{U}$ and is Lindelöf.

d. Let $\mathcal{O}$ be a covering of $\mathcal{S}$-line. For every rational number r, there exists $O_r \in \mathcal{O}$ such that $r \in O_r$, then claim that $\mathcal{U} = \{O_r : r \in O_r \text{ for all rational } r\}$ is a countable covering of $\mathcal{S}$-line. $\bigcup \mathcal{U} \subseteq \mathbb{R}$ is obvious. Conversely, suppose there exists $x \in \mathbb{R}$ such that $x \notin \bigcup \mathcal{U}$, then there exists a open ball $B(x, \epsilon)$ (in Euclidean metric sense) such that $B(x, \epsilon) \cap O_r$ for $O_r \in \mathcal{U}$, for otherwise, consider a decreasing rational sequence $\{x_n\}$ approaching x (in Euclidean sense), $\bigcup O_{x_n} = \bigcup [a_{x_n}, b_{x_n}) \ni x$. However, by density of rational numbers in $\mathbb{R}$, there exists $r \in \mathbb{Q}$ such that $x \in B(x, \epsilon)$, contradicting $B(x, \epsilon) \cap O_r$ for $O_r \in \mathcal{U}$. Hence $\bigcup \mathcal{U} = \mathbb{R}$.

e. It's sufficient to show that Lindelöf metric X space is second countable. define $\mathcal{U}_k = \{B(x, 1/k) : x \in X\}$, since X is Lindelöf, there exists $\mathcal{U}_k' \subseteq \mathcal{U}_k$ is countable. Enumerate $\mathcal{U}_k$ for all $k \in \mathbb{N}_+$, each has a countable $\mathcal{U}_k' \subseteq \mathcal{U}_k$. Take the union of all centers of $\bigcup \mathcal{U}_k'$, denote it as D, it's sufficient to show that D is a dense subset of X. Indeed, take any $\epsilon > 0$, pick n sufficiently large such that $1/n < \epsilon$, then by construction of $\mathcal{U}_n'$, $\bigcup_{x \in D} B(x, \epsilon) = X$, so D is a dense subset in X. Finally, by proposition 2.5, a metrizable space containing a countable dense set is separable, and thus second-countable. $\square$

Problem 7. *Let n be a positive integer, and for any nonnegative integer k, put $\Lambda(k) := \{\alpha \in \mathbb{Z}_+^n : \alpha_1 + \cdots + \alpha_n \leq k\}$. For any $k \in \mathbb{Z}_+$, recall that a real polynomial in n-variables of degree at most k is a real map p on $\mathbb{R}^n$ with*

$$p(x) := \sum_{\alpha \in \Lambda(k)} a_\alpha \prod_{i=1}^{n} x_i^{\alpha_i}$$

where a_α is a real number for each $\alpha \in \Lambda(k)$. We denote the set of all such maps by $\mathbb{R}_k[x_1, \ldots, x_n]$, and define $\mathbb{R}[x_1, \ldots, x_n] := \bigcup_{k \geq 0} \mathbb{R}_k[x_1, \ldots, x_n]$. Finally, for any p in this set, we put $O(p) := \{x \in \mathbb{R}^n : p(x) \neq 0\}$.

a. Show that $\{O(p) : p \in \mathbb{R}[x_1, \ldots, x_n]\}$ is a basis for a topology. The topology generated by this basis is called the Zariski topology on $\mathbb{R}^n$. When $\mathbb{R}^n$ is endowed with this topology, we denote it by $\mathbb{A}^n$.

b. For any subset P of $\mathbb{R}[x_1, \ldots, x_n]$, we define

$$Z(P) := \{x \in \mathbb{R}^n : p(x) = 0 \text{ for all } p \in P\}$$

We say that a subset S of $\mathbb{R}^n$ is algebraic if $S = Z(P)$ for some $P \subseteq \mathbb{R}[x_1, \ldots, x_n]$. Prove that a subset of $\mathbb{A}^n$ is closed iff it is algebraic.

c. Show that the Zariski topology on $\mathbb{R}$ coincides with the cofinite topology.

d. Show that $\{(a, \sin a) : a \in \mathbb{R}\}$ is not closed in $\mathbb{R}^2$ relative to the Zariski topology.

e. For any $P \subseteq \mathbb{R}[x_1, \ldots, x_n]$, the ideal generated by P is the set $\langle P \rangle$ that consists of polynomials of the form $\sum_{i=1}^{m} q_i p_i$ where m varies over $\mathbb{N}$, and $q_i \in \mathbb{R}[x_1, \ldots, x_n]$ and $p_i \in P$ for each $i = 1, \ldots, m$. Show that $Z(P) = Z(\langle P \rangle)$ for any such P. Conclude that a subset of $\mathbb{R}^n$ is closed relative to the Zariski topology iff it equals $Z(\langle P \rangle)$ for some $P \subseteq \mathbb{R}[x_1, \ldots, x_n]$.

Proof: a. For any $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$, let $p(x) = \prod_{i=1}^{n}(x_i - y_i) + 1 \in \mathbb{R}[x_1, \ldots, x_n]$, $p(y) \neq 0$, then $y \in O(p)$. Also, when $z \in O(p) \cap O(q)$ for $p, q \in \mathbb{R}[x_1, \ldots, x_n]$. Notice that $O(p) \cap O(q) = O(pq)$, where $pq \in \mathbb{R}[x_1, \ldots, x_n]$, indeed $p(z) \neq 0$ and $q(z) \neq 0$ iff $p(z)q(z) \neq 0$. Hence $\{O(p) : p \in \mathbb{R}[x_1, \ldots, x_n]\}$ is a basis.

b. If $S \subseteq \mathbb{A}^n$ is algebraic, then $S = Z(P)$ for some $P \subseteq \mathbb{R}[x_1, \ldots, x_n]$. Consider S^c, for $y \in S^c$, by definition we have $p(y) \neq 0$ for some $P \in P$, then $y \in O(p)$ such that $O(p) \subseteq S^c$. Thus S is closed. Conversely, suppose that $S \subseteq \mathbb{A}^n$ is closed, then S^c is open. For any $x \in S^c$, pick all $p \in \mathbb{R}[x_1, \ldots, x_n]$ such that $x \in O(p)$ and $O(p) \subseteq S^c$ (such choice is always valid because S^c is open), denote such set of p as P_x. Let $P = \bigcup_x P_x$, and we know that S^c is the union of all $O(p)$ where $p \in P$, then we have $S = \{x \in \mathbb{R}^n : p(x) = 0, \forall p \in P\}$, thus algebraic.

c. Let O be an open set in $\mathbb{A}$, then $O = \bigcup_{p \in \mathbb{R}[x_1, \ldots, x_n]} O(p)$, then $\mathbb{A} \setminus O = \bigcap_{p \in \mathbb{R}[x_1, \ldots, x_n]} \mathbb{A} \setminus O(p)$. Since $X \setminus O(p) = \{x \in \mathbb{R} : p(x) = 0\}$, and thus is finite, $\mathbb{A} \setminus O$ is finite. So it coincides with the cofinite topology.

d. Suppose $S = \{(a, \sin a) : a \in \mathbb{R}\}$ is closed, then S is algebraic, then $S = \{(x, y) \in \mathbb{R}^2 : p(x, y) = 0 \text{ for all } p \in P\}$ for some $P \subseteq \mathbb{R}[x_1, x_2]$. If there exists such polynomial p, rewrite it as $p(x, y) = \sum\limits_{k=0}^{n} p_k(x) y^k$, for some $p_k(x) \in \mathbb{R}[x]$. Plugging in $(a, \sin a)$ we get $0 = \sum\limits_{k=0}^{n} p_k(a)(\sin a)^k$ for all a, then $p_k(a) \equiv 0$ for all k, then $p(x, y) \equiv 0$, which implies that $Z(P)$ is the whole space, which is absurd for $S = Z(P)$.

e. Suppose that $x \in Z(P)$, then $p(x) = 0$ for all $p \in P$, then obviously $\sum_{i=1}^{m} q_i(x) p_i(x) = 0$ for $q_i \in \mathbb{R}[x_1, \ldots, x_n]$ and $p_i \in P$. Conversely, if $x \in Z(\langle P \rangle)$, then for any polynomials of the form $\sum_{i=1}^{m} q_i p_i$, $q_i \in \mathbb{R}[x_1, \ldots, x_n]$ and $p_i \in P$, when we choose $m = 1$, and q_i all be constant polynomials, we obtain that $p(x) = 0$ for all $p \in P$. Hence $Z(P) = Z(\langle P \rangle)$. Finally, a subset S of $\mathbb{A}^n$ is closed iff $S = Z(P) = Z(\langle P \rangle)$, conclude. $\square$