Topology Assignment 4
Yunzhe Zheng. (2025/03/18)

Problem 1. *Let $X := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 \geq 0\}$. For any $x := (x_1, 0) \in X$, let $\mathcal{B}(x)$ be the collection of all sets that are obtained by adding x to the usual open balls that are contained in X and are tangent to the horizontal axis at x. (That is, any member of $\mathcal{B}(x)$ looks like $\{x\} \cup B((x_1, \epsilon), \epsilon)$ where $\epsilon > 0$). For any $x := (x_1, x_2) \in X$ with $x_2 > 0$, put $\mathcal{B}(x) := \{B(x, \epsilon) : \epsilon \in (0, |x_2|)\}$. Finally, put $\mathcal{B} := \bigcup\{\mathcal{B}(x) : x \in X\}$.*

a. Show that $\mathcal{B}$ is a basis for a topology on X.

b. Show that the topology generated by $\mathcal{B}$ on X is first-countable and separable.

c. Endow X with the topology generated by $\mathcal{B}$, and show that the subspace topology on $\mathbb{R}_+ \times \{0\}$ is discrete. Conclude that X is not second-countable.

Proof: a. It's clear that $\bigcup \mathcal{B} = X$ by definition. Now consider $B(x, \epsilon_1) \cap B(x', \epsilon_2)$, where $B(x, \epsilon_1) \in \mathcal{B}(x)$ and $B(x', \epsilon_2) \in \mathcal{B}(x')$. If $x_2 > 0$ and $x_2' > 0$, then the intersection is open in Euclidean sense, for $x'' \in B(x, \epsilon_1) \cap B(x', \epsilon_2)$, there exists $B(x'', \epsilon) \subseteq \mathcal{B}(x) \cap \mathcal{B}(x')$, where we let $\epsilon < x_2''$, then $B(x'', \epsilon) \in \mathcal{B}$. Now suppose that $x_2 = 0$ and $x_2' > 0$, and notice that $B(x, \epsilon_1) \cap B(x', \epsilon_2)$ never touches the x-axis, then by the same argument as in the first case, for $x'' \in B(x, \epsilon_1) \cap B(x', \epsilon_2)$, there exists $B(x'', \epsilon) \subseteq \mathcal{B}(x) \cap \mathcal{B}(x')$, and $B(x'', \epsilon) \in \mathcal{B}$. Finally, if $x_2 = 0 = x_2'$, then also discover that $B(x, \epsilon_1) \cap B(x', \epsilon_2)$ never touches x-axis, thus $\mathcal{B}$ is a basis for a topology on X.

b. First show that the topology is first-countable. Suppose that $x = (x_1, x_2)$, where $x_2 = 0$. For any $O \in \mathcal{O}_X(x)$, O must contain $\{x\} \cup B((x_1, \epsilon), \epsilon)$, then there exists $1/m$, $m \in \mathbb{N}$ such that $x \in \{x\} \cup B((x_1, 1/m), 1/m) \subseteq O$, there is a countable local basis for such x. Now for the case where $x_2 > 0$, the countable local basis is $B(x, \min\{x_2, 1/m\})$, because for any $O \in \mathcal{O}_X(x)$, deleting possible points on the x-axis, it is trivial that such local basis is valid. Finally, to prove separability, consider point set $S = \mathbb{Q} \times \mathbb{Q}_{\geq 0}$. For $x = (x_1, x_2)$ where $x_2 > 0$, by discussion in the normal Euclidean space, for any open set O containing x, $O \cap A \neq \emptyset$. When $x_2 = 0$, any open set containing x must contain $\{x\} \cup B(x_1, \epsilon)$ for $\epsilon > 0$, then it must intersect points in S, hence we find a countable dense set in X, Moore Plane is separable.

c. For any subset $O \subseteq \mathbb{R}_+ \times \{0\}$, $O = \left(\bigcup_{x \in \mathbb{R}_+ \times \{0\}} \{x\} \cup B((x, \epsilon), \epsilon)\right) \cap \mathbb{R}_+ \times \{0\}$ for any $\epsilon > 0$, then every subset in $\mathbb{R}_+ \times \{0\}$ is open, thus is discrete. Since $\mathbb{R} \times \{0\}$ is uncountable, it is not second-countable, then the ground space X cannot be second-countable. $\square$

Problem 2. *Let $\mathcal{F}$ stand for the set of all $\{0, 1\}$-valued functions f on $\mathbb{R}$ such that $f^{-1}(0)$ is a finite set. Let $\boldsymbol{0}$ denote the self-map on $\mathbb{R}$ that equals to zero everywhere.*

a. Show that $\boldsymbol{0}$ is not in cl($\mathcal{F}$), if we view $\mathcal{F}$ as residing in $\boldsymbol{B}(\mathbb{R})$.

b. Show that $\boldsymbol{0} \in$ cl($\mathcal{F}$), if we view $\mathcal{F}$ as residing in $\mathbb{R}^{\mathbb{R}}$ with the topology on $\mathbb{R}^{\mathbb{R}}$ being the topology of pointwise convergence.

Proof: a. Consider the set $O := \{f \in \mathbf{B}(\mathbb{R}) : \|f - \mathbf{0}\|_\infty < 2\}$, then O is open in $\mathbf{B}(\mathbb{R})$. Also, O never intersects $\mathcal{F}$, since if so, $f^{-1}(1)$ is a finite set, and $f^{-1}(0)$ will be a infinite set for $f \in \mathcal{F}$, which contradicts the definition of $\mathcal{F}$.

b. Write any basis element as $U(x_1, O_1) \cap U(x_2, O_2) \cap \cdots \cap U(x_n)$, where $U(x, O) := \{f \in \mathbb{R}^{\mathbb{R}} : f(x) \in O\}$ for $O \in \mathcal{O}_{\mathbb{R}}$, by the definition of topology of pointwise convergence. Then for any open set $\mathcal{U} \ni \mathbf{0}$, $\mathcal{U} = \bigcup_{i \in I} U(x_{1,i}, O_{1,i}) \cap \cdots \cap U(x_{n_i,i}, O_{n_i,i})$. Notice that since $\mathbf{0} \in \mathcal{U}$, there must exists a $i \in I$ such that $0 \in O_{j,i}$ for $j = 1, \ldots, n_i$, then choose f such that $f \in U(x_{1,i}, O_{1,i}) \cap \cdots \cap U(x_{n_i,i}, O_{n_i,i})$ for that particular i, specifically $f(x_{j,i}) = 0$ for $j = 1, \ldots, n_i$, then let $f(x) = 1$ otherwise, then $f \in \mathcal{F}$, hence $\mathbf{0} \in \mathrm{cl}(\mathcal{F})$. $\square$

Problem 3. *Let I be a nonempty set, X_i a topological space for each $i \in I$, and X the cartesian product of $\{X_i : i \in I\}$. Then, it is easily verified that $\{\prod_{i \in I} O_i : i \in I\}$ is a basis for a topology on X. The topology that is generated by this basis is Hausdorff, and it is called the box topology on X.*

a. Explain why the box topology is finer than the product topology on X. Are the two topologies the same when I is finite?

b. Show that the map $f : \mathbb{R} \to \mathbb{R}^\infty$, defined by $f(x) := (x, x, \ldots)$, is continuous when we view $\mathbb{R}^\infty$ as a topological space relative to the product topology, but not relative to the box topology.

c. Show that the sequence (x_m) in $\mathbb{R}^\infty$, where $x_m := (0, \ldots, 0, m, m, \ldots)$ with the first nonzero term being the m-th one, converges relative to the product topology, but not relative to the box topology.

Proof: a. For product topology, only for finite index do we have O_i rather than the whole space X_i, so the box topology has more open sets than product topology, thus is finer. If I is finite, then the two are the same.

b. First show that f is not continuous in box topology, consider the preimage of $O := (-1, 1) \times (-\frac{1}{2}, \frac{1}{2}) \times \cdots$, then $f^{-1}(O) = \{0\}$, since if $f(x) = x \in O$ for $x \neq 0$, then there exists $1/m$ such that $x \notin (-1/m, 1/m)$, hence f cannot be continuous because $\{x\}$ is not open in $\mathbb{R}$. Secondly, for product topology, for any open set O, we may assume without loss of generality that it is of the form $O_1 \times O_2 \times \cdots \times O_m \times \mathbb{R} \times \cdots$, then $f^{-1}(O) = \bigcap_{i=1}^m O_i$, thus is open in $\mathbb{R}$, conclude that f is not continuous in box topology but continuous in product topology.

c. First prove for the product space. Claim that the sequence converges to $\mathbf{0} = (0, 0, \ldots)$, then for any open set O containing $\mathbf{0}$, write $O = \prod_{i=1}^\infty O_i$, and for all but finitely many i, $O_i = \mathbb{R}$, each O_i contains 0. Suppose that $k \in \mathbb{N}$ is the largest index where $O_i \neq \mathbb{R}$, then for $x_n = (0, 0, \ldots, n, n \ldots)$, $n \geq k$, obviously $x_n \in O$, hence (x_m) converges in product topology. For box topology, suppose that (x_m) converges to x, then consider the open set in box topology $O := (-2x, 2x) \times (-2x, 2x) \times \cdots$, for $m > 2x$, $x_m \notin O$, thus not convergent. $\square$

Problem 4. *Prove: For any positive integer n, prove that $O(n) \simeq SO(n) \times \{-1, 1\}$.*

Proof: Notice that for $A \in O(n)$, $AA^T = I$, which implies that $\det(A) = \pm 1$. Consider the function $f : O(n) \to SO(n) \times \{-1, 1\}$ by $M \mapsto (T(M), \det(M))$, where

$$T(M) = \begin{pmatrix} \det(M) & 0 & \cdots & 0 \\ 0 & 1 & \ldots & 0 \\ 0 & 0 & \ddots & \\ 0 & 0 & \cdots & 1 \end{pmatrix} \circ M$$

then clearly T is bijective, since for any $(O, i) \in SO(n) \times \{-1, 1\}$, there exists $M \in O(n)$ such that $M \in O(n)$, where all entries are the same except possibly for the first row where all entries are negative of O, injectivity is given by matrix multiplication. Also, continuity of f and its inverse is clear from continuity of linear transformation and determinant, hence $O(n) \simeq SO(n) \times \{1, -1\}$. $\square$

Problem 5. *Let X and Y be topological spaces, and take any $f : X \to Y$. Consider the map $F : X \to graph(f)$, defined by $F(x) := (x, f(x))$, and show that F is a homeomorphism iff $f \in \boldsymbol{C}(X, Y)$.*

Proof: Suppose that F is a homeomorphism, then $F(x) = (x, f(x))$ is continuous, then $f(x) \in \mathbf{C}(X, Y)$ as a component map. Conversely, if $f \in \mathbf{C}(X, Y)$, then $F(x) = (x, f(x))$ is continuous. Note that F is bijective, since if $x_1 \neq x_2$, then $(x_1, f(x_1)) \neq (x_2, f(x_2))$ for sure, and surjectivity is given by definition. For any open set $O = O_X \times O_Y \subseteq \text{graph}(f)$ for $O_X \in \mathcal{O}_X, O_Y \in \mathcal{O}_Y$, $F^{-1}(O) = O_X \cap f^{-1}(O_Y)$ is open , then F^{-1} is continuous, thus is a homeomorphism. $\square$