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Problem 1. *Give a direct proof of the fact that every compact subset of a metric space is bounded. Next use the fact to prove that $GL(n)$ is not compact for any $n \in \mathbb{N}$*

Proof: Let S be compact subset of metric space M, then consider the open cover $\{B(x,\epsilon)\}_{x\in S}$ for some $\epsilon > 0$, then there exists a finite subcover $\{B(x_i,\epsilon)\}_{i=1}^k$. Fix $x \in S$, for any $y \in S$, $d(x,y) < 2k\epsilon$, hence $S \subseteq B(x, 2k\epsilon)$, and is bounded. $GL(n)$ is not bounded, then it is not compact for any $n \in \mathbb{N}$. $\square$

Problem 2. *(The Local-to-Global Method) Let X be a topological space, and suppose P is a property that a subspace of X may or may not satisfy. Assume that i). P is satisfied by an open neighborhood of every point in X; and ii). if P is satisfied by two open sets in X, then it is also satisfied by the union of these sets. Show that if X is compact, then it satisfies the property P.*

Proof: Since $X = \bigcup_{x\in X} O_x$, then there exists a finite subcover $\{O_{x_i}\}_{i=1}^k$ such that $X = \bigcup_{i=1}^k O_{x_i}$, Since by assumption P is satisfied on all these O_{x_i}, then it is satisfied on the finite union of these sets by assumption ii, hence the conclusion follows. $\square$

Problem 3. *(Dini's Theorem) Let X be a compact metric space, and take any $f_1, f_2, \cdots \in C(X)$. Use the local-to-global method to prove that if $f_1 \geq f_2 \geq \dots$ and $f_m \to 0$ pointwise, then $f_m \to 0$ uniformly.*

Proof: For any $x \in X$, since $f_m \to 0$ pointwise, then for all $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that $|f_m(x)| < \epsilon/2$ for all $m > M$. Also, by continuity, for the same ϵ as above, there exists $\delta > 0$ such that $|f_m(x') - f_m(x)| < \epsilon/2$ for $x' \in B(x,\delta)$, then $|f_m(x')| < \epsilon$ for all $x' \in B(x,\delta)$, implying that $\{f_m\}$ converges uniformly on $B(x,\delta)$ for some $\delta > 0$ and every $x \in X$. Also, if uniform convergence property is satisfied on two open sets U_1 and U_2, then choose $M = \max(M_1, M_2)$ where M_1 and M_2 are chosen from each uniform convergence property, then clearly $|f_m(x)| < \epsilon$ for all $x \in U_1 \cup U_2$. Apply the Local-to-Global Method on the property of uniformly convergence, we obtain the uniform convergence on X. $\square$

Problem 4. *Let X be a compact Hausdorff space, and let f be a continuous self-map on X. Then, there is a nonempty compact fixed set of f, that is, $f(S) = S$. Prove this by showing that $S := \bigcap_{i=1}^\infty X_i$ is indeed such a set, where $X_1 = f(X)$ and $X_i = f(X_{i-1})$ for each $i \geq 2$*

Proof: Indeed $S = \bigcap_{i=1}^\infty X_i$ is a fixed set of f, since $f(S) = f(\bigcap_{i=1}^\infty X_i) = \bigcap_{i=1}^\infty f(X_i) = \bigcap_{i=1}^\infty X_i = S$. Now we are left to prove that S is nonempty and compact. Notice that X_i are all closed and compact since they are image of compact sets under a Hausdorff space. First prove that for $\{X_i\}$ has finite intersection property: Suppose there exists a finite collection $\mathcal{B} = \{X_{i_j}\}_{j=1}^k \subseteq \{X_i\}$ such that $\bigcap \mathcal{B} = \emptyset$, then $\bigcap_{j=1}^k \{f^{-1}(X_{i_i})\} = \bigcap_{j=1}^k \{X_{i_j-1}\} = \emptyset$, continue this process for finitely many time, and finally we hit X as the preimage of one element in $\mathcal{B}$, which contains all other X_i's, then the intersection can no longer be $\emptyset$, that leads to a contradiction, so $\{X_i\}$ indeed has finite intersection property. By Proposition 1.1 from Chapter 7, S is nonempty. S is compact since S is the infinite intersection of closed sets in a compact space. □

Problem 5. *A topological space X is said to be countably compact if we can extract a finite cover of X from any given countable open cover of X.*

a. Show that a topological space is compact iff it is Lindelöf and countably compact.

b. $\mathbb{R}$ is Lindelöf but not countably compact.

c. Let X stand for the Sorgenfrey line. Show that X is Lindelöf, but $X \times X$ is not.

d. Show that if X and Y are two topological spaces, one compact and the other Lindelöf, then $X \times Y$ is Lindelöf.

Proof: a. If X is compact, then for any open cover $\mathcal{O}$ of X, we can extract finitely (implies countably) many open sets that covers X, thus is Lindelöf, and countably compact is trivial. Conversely, Suppose X is both Lindelöf and countably compact, then for any open cover $\mathcal{O} \subseteq \mathcal{O}_X$, there exists countable $\mathcal{U} \subseteq \mathcal{O}$ that still covers X, finally use countably compact property, we may extract finite subcover that covers X, hence compact.

b. $\mathbb{R}$ is not countably compact by consider open cover $\{(n/2, n/2+1)\}_{n\in\mathbb{Z}}$, if there exists a finite cover $\mathcal{B}$, then $\operatorname{diam}(\mathcal{B}) < \infty$, which cannot cover $\mathbb{R}$. Now, write $\mathbb{R} = \bigcup_{n\in\mathbb{Z}}[n, n+1]$, and consider any open cover $\mathcal{O} \subseteq \mathcal{O}_X$, define $\mathcal{O}_n := \{O \in \mathcal{O} : O \cap [n, n+1] \neq \emptyset\}$, then clearly $\mathcal{O}_n$ is an open cover of $[n, n+1]$. By compactness of $[n, n+1]$ in $\mathbb{R}$, we can extract an finite subcover for $[n, n+1]$. Continue this process for all $[n, n+1]$, $n \in \mathbb{Z}$, we obtain countable collection of finite open cover, which up to taking union, is a countable subcover $\mathbb{R}$, thus is Lindelöf.

c. The fact that Sorgenfrey line is Lindelöf was proved in Homework 3 Problem 6.d. For $X \times X$, consider the closed subset $S = \{(x, -x) : x \in X\}$ of $X \times X$. Claim that S is a discrete subspace, indeed, for any $(x, -x)$ where $x \in X$, the open neighborhood $[x, x+1) \cup [-x, x+1)$ contains only $(x, -x)$, thus singleton is open in the subset, thus is discrete. However, S is uncountable, the open cover $\{\{x\} : x \in X\}$ has no countable subcover, then $X \times X$ is not Lindelöf.

d. For any open cover $\mathcal{O}_1 \times \mathcal{O}_2 \in \mathcal{O}_X \times \mathcal{O}_Y$ of $X \times Y$, the projection on X is an open cover of a compact set X, then there exists a finite subcover $\{O_{x_i}\}_{i=1}^k$. Now $\{O_{x_i}\} \times \mathcal{O}_2$ is a open cover of $X \times Y$. For each $O_{x_i} \times \mathcal{O}_2$, there exists countable open cover $\{O_{y_i}\}_{i=1}^\infty \subseteq \mathcal{O}_2$, then $\{O_{x_i}\}_{i=1}^k \times \{O_{y_i}\}_{i=1}^\infty$ is a countable open cover of $X \times Y$, hence is Lindelöf. □

Problem 6. *Show that the closed unit ball of $C[0,1]$, that is $\{f \in C[0,1] : \|f\|_\infty \leq 1\}$ is closed and bounded, but not compact subset of $C[0,1]$. Thus, $C[0,1]$ does not have the Heine-Borel property.*

Proof: Since $(C[0,1], \|\cdot\|_\infty)$ is a metric space, $\{f \in C[0,1] : \|f\|_\infty \leq 1\}$ is indeed closed and bounded. Consider $\{x^n : x \in [0,1]\}$, then it converges pointwise to a non-continuous function, so there isn't any subsequence converging uniformly to element in $C[0,1]$, thus is not compact. $\square$

Problem 7. *Let X be a metric space.*

a. Show that if X is compact, then it is separable.

b. We say that X is σ-compact if X can be written as the union of countably many compact subsets of it. Show that if X is σ-compact, then it is separable.

Proof: a. Consider the set $\{B(x, 1/m) : x \in X\}_{m=1}^{\infty}$, then for each m, $\{B(x, 1/m) : x \in X\}$ is a open cover of X, then there exists a finite open subcover $\{B(x_i, 1/m)\}_{i=1}^{k_m}$ that covers X, do this for any $m \in \mathbb{N}$, we obtain a countable open cover of X, namely $\bigcup_{m\in\mathbb{N}}\{B(x_i, 1/m)\}_{i=1}^{k_m}$ and the corresponding $S = \{x_i\}$, then for any $x \in X$, since for each $m \in \mathbb{N}$, x is covered by a open ball of with radius $1/m$, thus we can find a sequence $\{x_{j_i} : x_{j_i} \in S\}$ that converges to x, hence S is countable dense set and X is separable.

b. For each compact subset X_i of X, there exists a countable dense subset S_i, then $\bigcup_{i=1}^{\infty} S_i$ is a countable union of countable sets, which is countable, and since $\bigcup_{i=1}^{\infty} X_i = X$, $\bigcup_{i=1}^{\infty} S_i$ is countably dense in X, thus X is separable. $\square$

Problem 8. *(An Alternative Version of the Stone-Weierstrass Theorem) Let X be a compact Hausdorff space, and $\mathcal{F}$ is a subset of $C(X)$ such that for every distinct points x and y in X, there is an $f \in \mathcal{F}$ with $f(x) \neq f(y)$. (Such an f is said to separate the points of X.) Suppose that (i) $af + g \in \mathcal{F}$ for all $f, h \in \mathcal{F}$ and $a \in \mathbb{R}$; (ii) $fg \in \mathcal{F}$ for all $f, g \in \mathcal{F}$; and (iii) all constant functions functions on X belong to $\mathcal{F}$. Prove that $\mathcal{F}$ is dense in $C(X)$.*

Proof: Given $x, y \in X$ that are distinct, we may choose $g \in \mathcal{F}$ such that $g(x) \neq g(y)$. Consider $f(z) := a\frac{g(z)-g(y)}{g(x)-g(y)} + b\frac{g(z)-g(x)}{g(y)-g(x)}$, then $f \in \mathcal{F}$ is continuous and satisfies two-points interpolation property. Now we claim that $|f| \in Cl(\mathcal{F})$ for any $f \in \mathcal{F}$. By Weierstrass approximation theorem, there exists polynomial $P(f(z))$ such that $\|P(f(z)) - |f(z)|\|_\infty < \epsilon$ for any $\epsilon > 0$, and since $f \in \mathcal{F}$ satisfies closure in addition, scalar multiplication and function multiplication, $P(f(z)) \in \mathcal{F}$, then $|f| \in Cl(\mathcal{F})$. Notice that $\max(f, g) = \frac{1}{2}(f+g+|f-g|) \in Cl(\mathcal{F})$ and $\min(f, g) = \frac{1}{2}(f + g - |f - g|) \in Cl(\mathcal{F})$, then $Cl(\mathcal{F})$ is a sublattice. Finally by the Stone-Weierstrass Theorem introduced in class, $Cl(\mathcal{F})$ is dense in $C(X)$, which means $Cl(\mathcal{F}) = C(X)$, and $\mathcal{F}$ is dense in $C(X)$. $\square$

Problem 9. *(Féjer's Approximation Theorem) For any positive integer n, a trigonometric polynomial on $\mathbb{R}$ of degree n is self-map on $\mathbb{R}$ of the form*

$$t \mapsto a_0 + \sum_{k=1}^{n}(a_k \cos(kt) + b_k \sin(kt))$$

where $a_0, b_0, \ldots, a_n, b_n$ are real numbers with either $a_n \neq 0$ or $b_n \neq 0$. Let $\mathcal{P}$ be the set of all trigonometric polynomials on $\mathbb{R}$ of any degree, along with all constant self-maps on $\mathbb{R}$. Next, consider the following subspace of $C(\mathbb{R})$:

$$C_{per}[0, 1] := \{f \in C(\mathbb{R}) : f(x) = f(x + 2k\pi) \textit{ for all } x \in \mathbb{R} \textit{ and } k \in \mathbb{N}\}.$$

Prove: $\mathcal{P}$ is dense in $C_{per}[0, 1]$.

Proof: Consider the map $g : \mathbb{R} \to \mathbb{S}^1$ as $t \mapsto (\cos t, \sin t)$, and we define the map $f_a : \mathbb{S}^1 \to \mathbb{R}$ as $x \mapsto \langle a, x\rangle$ for $a \in \mathbb{R}^2$, then the original map can be rewritten as $a_0 + \sum\limits_{k=1}^{n} f_{a_k}(g(kt))$. We only consider the part from $\mathbb{S}^1 \to \mathbb{R}$, then the map can be simplified as $a_0 + \sum\limits_{k=1}^{n} \langle a_k, x_k\rangle$. Clearly $\mathcal{P}$ is closed under addition and scalar multiplication, and for function multiplication, notice that if we have $\varphi = a_0 + \sum\limits_{i=1}^{k}\langle a_i, x_i\rangle$ and $\psi = a_0' + \sum\limits_{i=1}^{k'}\langle a_i', x_i'\rangle$, then we shall have $\varphi \cdot \psi = a_0 a_0' + a_0 \sum\limits_{i=1}^{k'}\langle a_i', x_t'\rangle + a_0' \sum\limits_{i=1}^{k}\langle a_t, x_t\rangle + \sum\langle\langle\cdot,\cdot\rangle, x_t\rangle \in \mathcal{P}$, given the fact that $\langle a, b\rangle\langle c, d\rangle = \langle\langle a, b\rangle c, d\rangle$. We are left to show that the map separates points in $\mathbb{S}^1$, indeed, for $x \neq y \in \mathbb{S}^1$, we can pick $b_1, b_2 \in \mathbb{R}$ such that $\langle b_1, x\rangle \neq \langle b_2, y\rangle$ where left and right parts are both in $\mathcal{P}$. Finally by the previous exercise, since $\mathcal{P}$ is contained in $C_{per}[0,1]$, $\mathcal{P}$ is dense in $C_{per}[0,1]$. $\square$