SUSTech Southern University of Science and Technology

高数期末复习

致诚书院学长学姐带飞课

Chapter 2, Chapter 3

By 苏宇鹏 12.22 2024

SUSTech Southern University of Science and Technology

Chapter2

极限与连续性

高数带飞课提纲一览（复习专用）

- Ch02 极限与连续性
  - 极限 @ 定义 & 性质 @ (加乘 & 三明治 & 保号) & 重要极限 (等价无穷小)
  - 连续性 @ 定义 & 判别 (两类间断点)
  - 复合函数 & 介值定理 & 无穷极限/极限无穷 (渐近线)
- Ch03 导数与微分
  - 导数 @ 定义 & 可导性 & 计算 @ (三角函数 & 链式法则 & 隐函数求导)
  - 微分 @ 线性化 & 定义 & 判别
- Ch04 导数的应用
  - 函数的点 @ 最值 (最值定理) & 极值 (极值定理) & 极值点判别 & 中值定理
  - 函数性质 @ 单调性 (一阶检验) & 凹凸性 (二阶检验) & 拐点 & 画图
- Ch05 积分
  - 积分(定/不定) @ 定义 (黎曼和) & 可积性 & 定理 @ (积分中值定理 & 微积分基本定理)
  - 积分计算 @ 换元法 & 奇偶性 & 曲线面积

极限的定义

If $f(x)$ is arbitrarily close to the number L (as close to L as we like) for all x sufficiently close to c, we say that f approaches the **limit** L and write

$$\lim_{x \to c} f(x) = L,$$

which is read "the limit of $f(x)$ as x approaches c is L."

1. Arbitrarily close

2. For all x

极限的定义

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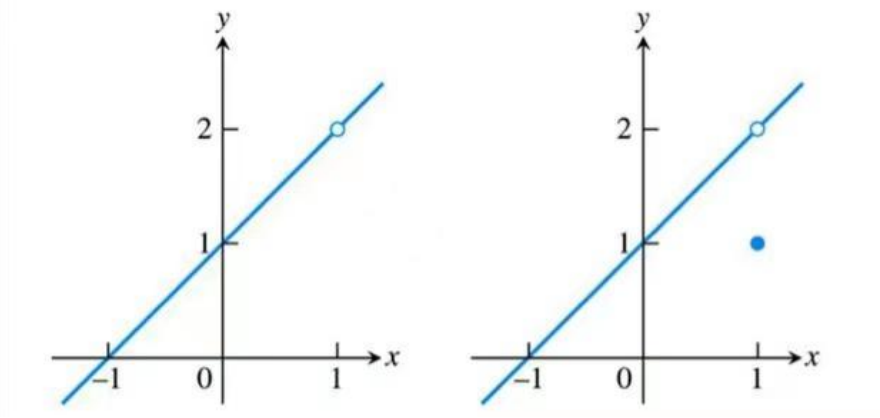
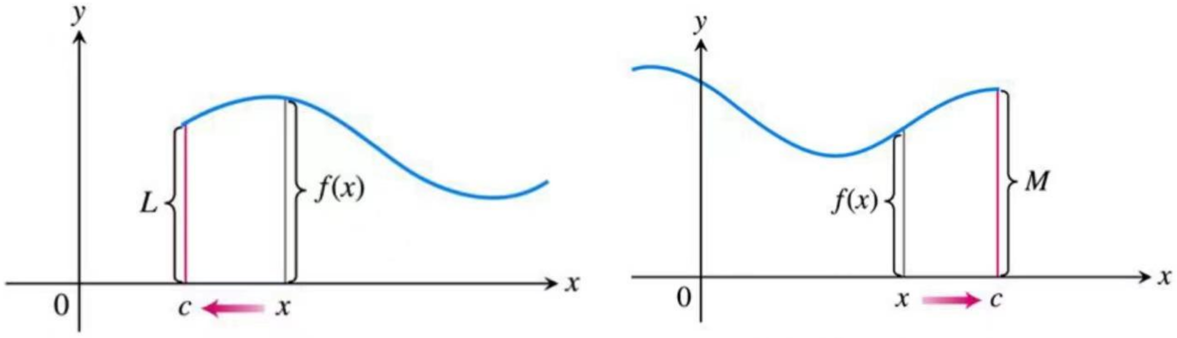
1. Arbitrarily close

都有极限！
极限与该点函数值无关
X = 1 处的极限值为 2



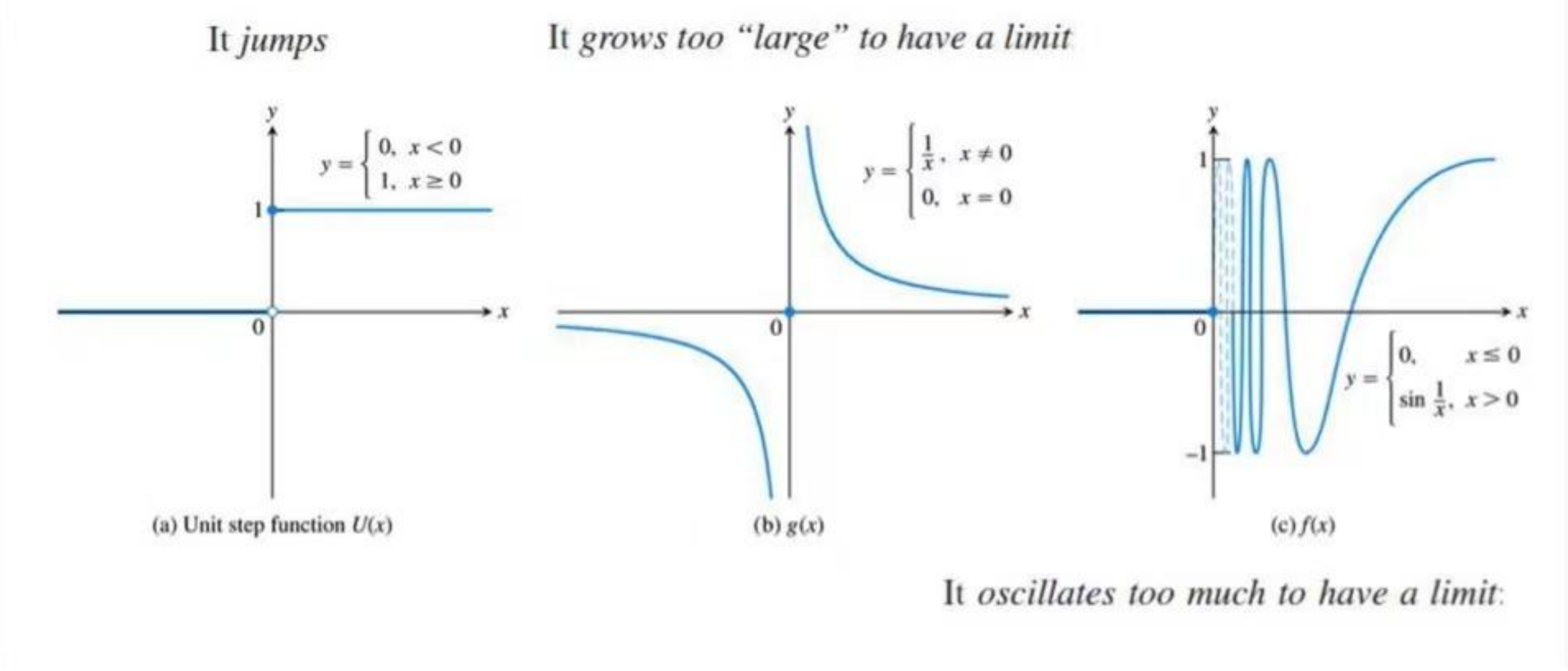
2. For all x

左极限 与 右极限
当左极限等于右极限 函数有极限



(a) $\lim_{x \to c^+} f(x) = L$ (b) $\lim_{x \to c^-} f(x) = M$

无极限的例子


It jumps
It grows too "large" to have a limit
$y = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$
(a) Unit step function $U(x)$
$y = \begin{cases} \frac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$
(b) $g(x)$
$y = \begin{cases} 0, & x \leq 0 \\ \sin\frac{1}{x}, & x > 0 \end{cases}$
(c) $f(x)$
It oscillates too much to have a limit

极限的性质——加乘法则

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\boxed{\lim_{x\to c}(f(x) + g(x)) = L + M}$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\boxed{\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M}$
5. *Quotient Rule:* $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

   (If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

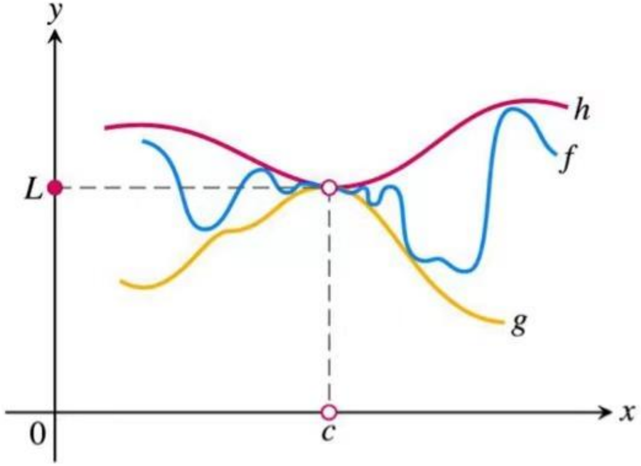
1. 有 + 无 = 无
2. 无 + 无 = 不定
3. 有 * 无 = 不定
4. 无 * 无 = 不定

极限的性质——Sandwich

THEOREM 4—The Sandwich Theorem Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself. Suppose also that

$$\lim_{x\to c} g(x) = \lim_{x\to c} h(x) = L.$$

Then $\lim_{x\to c} f(x) = L$.



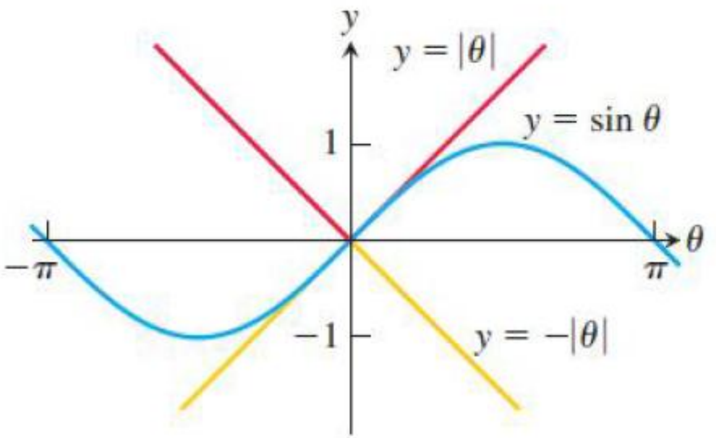
正弦/余弦函数的有界性！

$$(1)\ \lim_{x\to 0} x \sin\frac{1}{x} \qquad\qquad (2)\ \lim_{x\to 0}\left[\frac{1}{x}\right]\sin x$$

极限的性质——保号性

THEOREM 5 If $f(x) \le g(x)$ for all x in some open interval containing c, except possibly at $x = c$ itself, and the limits of f and g both exist as x approaches c, then

$$\lim_{x\to c} f(x) \le \lim_{x\to c} g(x).$$



$$-|\theta| < \sin\theta < |\theta| \qquad \text{for } \theta \ne 0,$$

$\lim_{\theta\to 0} \sin\theta = 0 = \lim_{\theta\to 0}|\theta|$, not $\lim_{\theta\to 0}\sin\theta < \lim_{\theta\to 0}|\theta|$.

重要极限

THEOREM 7—Limit of the Ratio sin θ/θ as $\theta \to 0$

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1 \qquad (\theta \text{ in radians}) \qquad (1)$$

$$\lim_{\square \to 0} \frac{\sin \square}{\square} = 1$$

把趋近零时比值为1的量称之为**等价无穷小**

等价无穷小

当 $\text{🐶} \to 0$ 时

$\sin \text{🐶} \sim \text{🐶}$　　　　$\tan \text{🐶} \sim \text{🐶}$

$\ln(1+\text{🐶}) \sim \text{🐶}$　　　　$e^{\text{🐶}} - 1 \sim \text{🐶}$

$\arcsin \text{🐶} \sim \text{🐶}$　　　　$\arctan \text{🐶} \sim \text{🐶}$

$\log_a(1+\text{🐶}) \sim \dfrac{\text{🐶}}{\ln a}$　　　　$\text{🐶} - \ln(1+\text{🐶}) \sim \dfrac{1}{2}\text{🐶}^2$

$1-\cos \text{🐶} \sim \dfrac{1}{2}\text{🐶}^2$　　　　$\ln(\text{🐶} + \sqrt{1+\text{🐶}^2}) \sim \text{🐶}$

$\text{🐶} - \sin \text{🐶} \sim \dfrac{1}{6}\text{🐶}^3$　　　　$\tan \text{🐶} - \text{🐶} \sim \dfrac{1}{3}\text{🐶}^3$

$(1+\text{🐶})^\alpha - 1 \sim \alpha\text{🐶}$　　　　$\arcsin \text{🐶} - \text{🐶} \sim \dfrac{1}{6}\text{🐶}^3$

$\text{🐶} - \arctan \text{🐶} \sim \dfrac{1}{3}\text{🐶}^3$　　　　$\tan \text{🐶} - \sin \text{🐶} \sim \dfrac{1}{2}\text{🐶}^3$

在求极限时，
可以直接用等价无穷小替换，
使过程更简洁，更直接。

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连续性的定义

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"Without lifting the pencil."

DEFINITION

Interior point: A function $y = f(x)$ is **continuous at an interior point c** of its domain if

$$\lim_{x \to c} f(x) = f(c).$$

Endpoint: A function $y = f(x)$ is **continuous at a left endpoint a** or is **continuous at a right endpoint b** of its domain if

$$\lim_{x \to a^+} f(x) = f(a) \qquad \text{or} \qquad \lim_{x \to b^-} f(x) = f(b), \quad \text{respectively.}$$

连续性的判别

Continuity Test

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

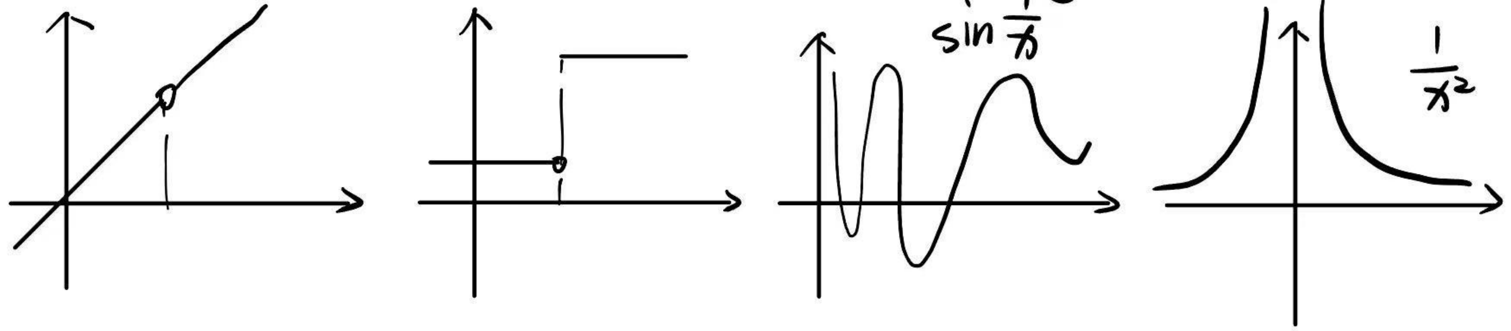
1. $f(c)$ exists (c lies in the domain of f).
2. $\lim_{x\to c} f(x)$ exists (f has a limit as $x \to c$).
3. $\lim_{x\to c} f(x) = f(c)$ (the limit equals the function value).

极限关注的是趋近值，连续性关注的是趋近值与该点的值是否相同！

两类间断点

☆ 第一类间断点 / 第二类间断点

① removable ② jump ③ osilatting ④ infinite



⑤ 判断：普通极限
- 存在 → $f(c)$
  - 存在且相等 → 连续
  - 不存在或存在但不等 → 可去间断点
- 不存在
  - 左右不等 → 跳跃间断点
  - 趋于 $\pm\infty$ → 无穷间断点
  - 振荡 → 振荡间断点

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复合函数的极限与连续性

THEOREM 9—Composite of Continuous Functions If f is continuous at c and g is continuous at $f(c)$, then the composite $g \circ f$ is continuous at c.

THEOREM 10—Limits of Continuous Functions If g is continuous at the point b and $\lim_{x\to c} f(x) = b$, then

$$\lim_{x\to c} g(f(x)) = g(b) = g(\lim_{x\to c} f(x)).$$

外连续 + 内连续 = 复合连续

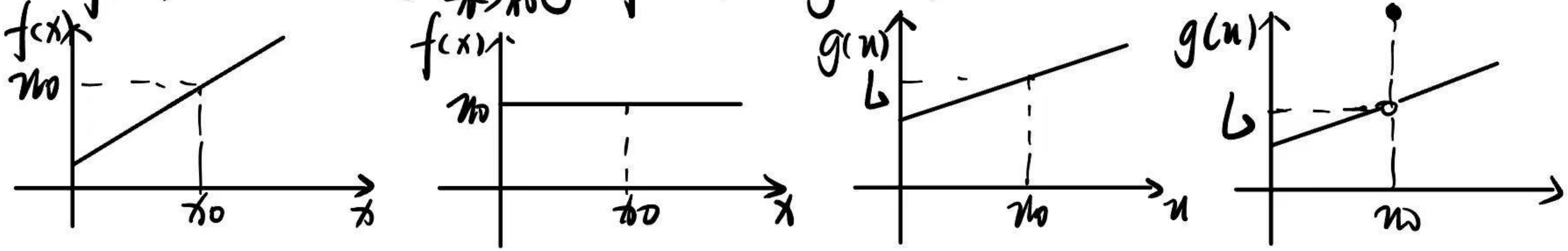
外连续 + 内有极 = 复合有极

复合函数的极限与连续性

Q：若 $\lim\limits_{x\to x_0} f(x) = u_0$ $\quad \lim\limits_{u\to u_0} g(u) = L$ 则 $\lim\limits_{x\to x_0} g(f(x)) = L$?

外有极 + 内有极 = 啥也不是！

若：$\begin{cases} f(x) \neq u_0，则 \lim\limits_{x\to x_0} g(f(x)) = \lim\limits_{u\to u_0} g(u) = L \\ f(x) = u_0，则 \lim\limits_{x\to x_0} g(f(x)) = g(u_0) \end{cases}$



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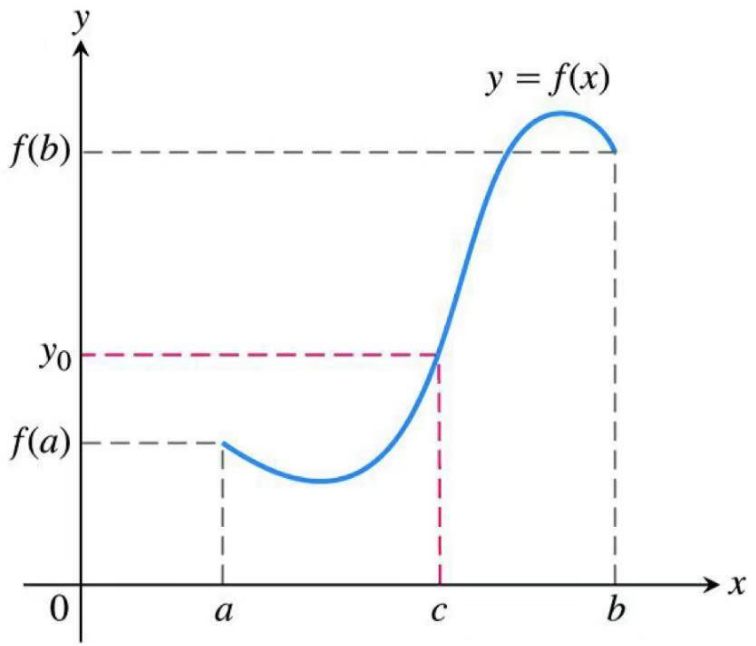
介值定理

THEOREM 11—The Intermediate Value Theorem for Continuous Functions If f is a continuous function on a closed interval $[a, b]$, and if y_0 is any value between $f(a)$ and $f(b)$, then $y_0 = f(c)$ for some c in $[a, b]$.



二分法、牛顿迭代法

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无穷极限/极限无穷

1. We say that $f(x)$ has the **limit L as x approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L$$

2. We say that $f(x)$ has the **limit L as x approaches minus infinity** and write

$$\lim_{x\to-\infty} f(x) = L$$

1. We say that **$f(x)$ approaches infinity as x approaches x_0**, and write

$$\lim_{x\to x_0} f(x) = \infty,$$

2. We say that **$f(x)$ approaches minus infinity as x approaches x_0**, and write

$$\lim_{x\to x_0} f(x) = -\infty,$$

有理函数在无穷处的极限

小结：当$a_0 \neq 0, b_0 \neq 0, m$和$n$为非负整数时有

$$\lim_{x\to\infty}\frac{a_0x^m + a_1x^{m-1} + \cdots + a_m}{b_0x^n + b_1x^{n-1} + \cdots + b_n} = \begin{cases} \dfrac{a_0}{b_0} & n = m \\ 0 & n > m \\ \infty\text{（自己决定符号）} & n < m \end{cases}$$

除最大幂

渐近线

1. 水平渐近线

DEFINITION A line $y = b$ is a **horizontal asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = b \qquad \text{or} \qquad \lim_{x \to -\infty} f(x) = b.$$

2. 垂直渐近线

DEFINITION A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x \to a^-} f(x) = \pm\infty.$$

渐近线

3. 斜渐近线

If the degree of the numerator of a rational function is 1 greater than the degree of the denominator, the graph has an **oblique** or **slant line asymptote**.

① 带余除法求 $ax+b$.

② $\lim\limits_{x\to\infty}\dfrac{f(x)}{x}=a \qquad \lim\limits_{x\to\infty}(f(x)-ax)=b$

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Chapter3

导数与微分

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导数的定义

DEFINITION The **derivative of a function f at a point x_0**, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

provided this limit exists.

A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h \to 0^+} \frac{f(a + h) - f(a)}{h} \quad \text{Right-hand derivative at } a$$

$$\lim_{h \to 0^-} \frac{f(b + h) - f(b)}{h} \quad \text{Left-hand derivative at } b$$

exist at the endpoints (Figure 3.7).

左导数 = 右导数 则 函数在该点可导

函数的可导性

THEOREM 1—Differentiability Implies Continuity If f has a derivative at $x = c$, then f is continuous at $x = c$.

可导必连续

连续不一定可导

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三角函数的导数

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

一定要记牢！

链式法则

THEOREM 2—The Chain Rule If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

隐函数求导

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.

2. Collect the terms with dy/dx on one side of the equation and solve for dy/dx.

① 计算函数某点 (x_0, y_0) 的切线与法线

1、求该点 $k = \frac{dy}{dx}\Big|_{x=x_0}$

2、切线 $y = kx + b$，法线 $y = -\frac{1}{k}x + b$

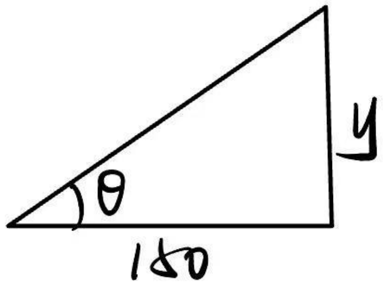
② 求高阶导数（以二阶为例）

1、求一阶导数

2、求二阶导数形式，将式中一阶导数代换.

相关变化

The problem of finding a rate of change from other known rates of change is called a *related rates problem*.



$$\left.\frac{d\theta}{dt}\right|_{\theta=\frac{\pi}{4}} = 1 \qquad \frac{dy}{dt} = \frac{d\,150\tan\theta}{dt} = 150\sec^2\theta\cdot\frac{d\theta}{dt} = 150$$

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  - 连续性 @ 定义 & 判别 (两类间断点)
  - 复合函数 & 介值定理 & 无穷极限/极限无穷 (渐近线)
- Ch03 导数与微分
  - 导数 @ 定义 & 可导性 & 计算 @ (三角函数 & 链式法则 & 隐函数求导)
  - 微分 @ 线性化 & 定义 & 判别
- Ch04 导数的应用
  - 函数的点 @ 最值 (最值定理) & 极值 (极值定理) & 极值点判别 & 中值定理
  - 函数性质 @ 单调性 (一阶检验) & 凹凸性 (二阶检验) & 拐点 & 画图
- Ch05 积分
  - 积分(定/不定) @ 定义 (黎曼和) & 可积性 & 定理 @ (积分中值定理 & 微积分基本定理)
  - 积分计算 @ 换元法 & 奇偶性 & 曲线面积

线性化

DEFINITIONS If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

重要线性近似 $(1+x)^k \approx 1 + kx$

微分的定义

DEFINITION Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is

$$dy = f'(x)\,dx.$$

可微 $\rightarrow$ 可导

在一维下，可微 $\leftrightarrow$ 可导

微分的判定

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a)\,\Delta x + \epsilon\,\Delta x \tag{1}$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

一维可微的判定 $\varepsilon = \lim\limits_{\Delta x \to 0} \dfrac{\Delta y - f'(a)\Delta x}{\Delta x} = 0$ （唯一切线）

二维 $f(x+\Delta x, y+\Delta y) - f(x,y) = \Delta z = f_x \Delta x + f_y \Delta y + o(\sqrt{\Delta x^2 + \Delta y^2})$

二维可微的判定 $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \dfrac{\Delta z - f_x \Delta x - f_y \Delta y}{\sqrt{\Delta x^2 + \Delta y^2}} = 0$ （唯一切平面）

Chapter 4

导数的应用

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  - 积分计算 @ 换元法 & 奇偶性 & 曲线面积

函数的最值 (Absolute)

DEFINITIONS Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$

and an **absolute minimum** value on D at c if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

不同的定义域会出现不同的最值！

最值定理

THEOREM 1—The Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$.

1. 函数有连续性

2. 函数在闭区间

函数的极值 (Local)

DEFINITIONS A function f has a **local maximum** value at a point c within its domain D if $f(x) \leq f(c)$ for all $x \in D$ lying in some open interval containing c.

A function f has a **local minimum** value at a point c within its domain D if $f(x) \geq f(c)$ for all $x \in D$ lying in some open interval containing c.

Larger than neighbor

“极值定理”

THEOREM 2—The First Derivative Theorem for Local Extreme Values If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$ **驻点**

极值点是驻点，但驻点不一定是极值点

可能产生极值点的位置

1. interior points where $f' = 0$, 驻点
2. interior points where f' is undefined, 不可导点
3. endpoints of the domain of f. 端点

驻点与不可导点统称为可疑点 (Critical Point)

求最值方法：比较可疑点与端点的函数值

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The Mean Value Theorem

THEOREM 3—Rolle's Theorem Suppose that $y = f(x)$ is continuous over the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) at which $f'(c) = 0$.

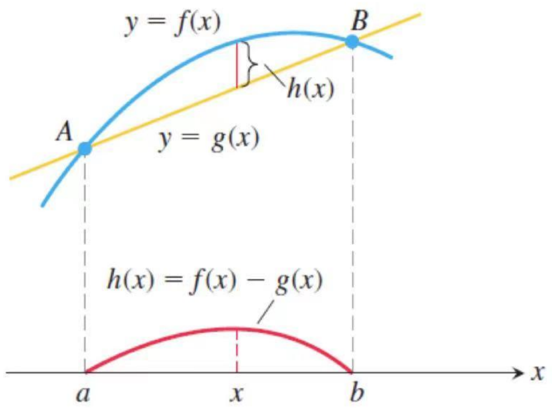
THEOREM 4—The Mean Value Theorem Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

成立条件均为闭区间连续，开区间可导

证明

请用罗尔定理证明拉格朗日中值定理。



$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a)$$

$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \end{aligned}$$

The function h satisfies the hypotheses of Rolle's Theorem on $[a, b]$.

中值定理的推论

COROLLARY 1 If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

COROLLARY 2 If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant function on (a, b).

要会推导！这可能会是简单的证明题

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函数的单调性

COROLLARY 3 Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.

If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

单调性是区间的性质

一阶导数检验

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

求单调区间：找可疑点划分区间，在判断导数符号

函数的凹凸性

DEFINITION The graph of a differentiable function $y = f(x)$ is

(a) **concave up** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

二阶导数判断凹凸性

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

拐点

DEFINITION A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

二阶导数改变符号的点，二阶导数等于 0 或者不存在

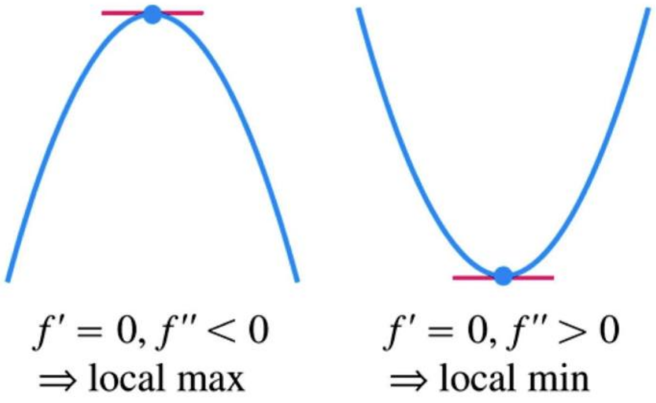
Comparison: first derivative VS second derivative

二阶导数检验

THEOREM 5—Second Derivative Test for Local Extrema Suppose f'' is continuous on an open interval that contains $x = c$.

1. If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
2. If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
3. If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

判断一阶导函数左右符号



函数画图

① 定义域确定，

② 求一阶导数与二阶导数

③ 根据区间列表格

④ 画图，确定细节（特殊点，$\boxed{\text{渐近线}}$，对称性，周期性等）

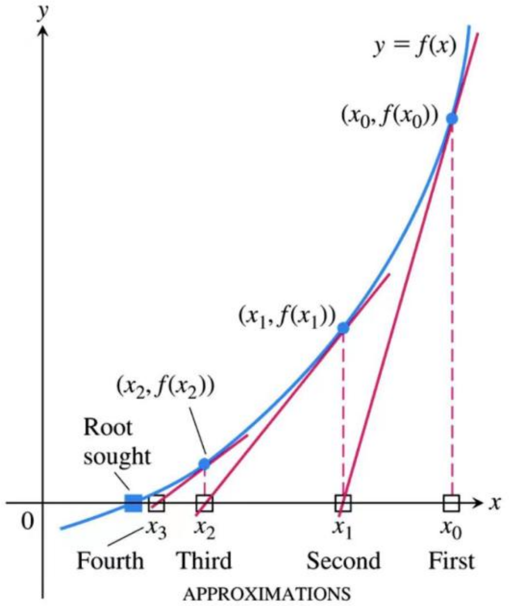
19期中 $f(x)=\frac{x^3}{x^2+1}$

牛顿迭代法

Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{if } f'(x_n) \neq 0.$$



牛顿迭代法不能保证收敛

SUSTech Southern University of Science and Technology

Chapter 5

积分

黎曼和

The first n squares: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

The first n cubes: $\displaystyle\sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2$

Finally we sum all these products to get

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k\,.$$

The sum S_P is called a **Riemann sum for f on the interval $[a, b]$**.

定积分

$$\lim_{\|p\|\longrightarrow 0}\sum_{k=1}^{n} f(c_k)\Delta x_k = \int_a^b f(x)dx$$

1. 积分值仅与被积函数与积分区间有关，与积分变量无关
2. 区间的分法与Ci的取法是任意的

一般的，我们区间均分，Ci取相对应的位置

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{k=1}^{n} f\left(a + k\frac{(b-a)}{n}\right)\left(\frac{b-a}{n}\right) \qquad (1)$$

函数的可积性

THEOREM 1—Integrability of Continuous Functions If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)\,dx$ exists and f is integrable over $[a, b]$.

闭区间上连续，至多有有限个第一类间断点

积分中值定理

THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

一定有一个点的高度为平均高度

例：$f(x)$ 可导 且 $\lim\limits_{x\to\infty} f(x) = 1$ 求 $\lim\limits_{x\to\infty}\int_x^{x+2} t\sin\frac{3}{t} f(t)\,dt$

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微积分基本定理

THEOREM 4—The Fundamental Theorem of Calculus, Part 1 If f is continuous on $[a, b]$, then $F(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{2}$$

若 $f(t)$ 连续 $a(x)$ $b(x)$ 可导 $F(x) = \int_{a(x)}^{b(x)} f(t)\,dt$

$$F'(x) = f[b(x)]\,b'(x) - f[a(x)]\,a'(x)$$

THEOREM 4 (Continued)—The Fundamental Theorem of Calculus, Part 2
If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)\,dx = F(b) - F(a).$$ **(牛顿—莱布尼茨公式)**

证明

积分中值定理中的c可在开区间 (a, b) 取到

THEOREM 3—The Mean Value Theorem for Definite Integrals If f is continuous on $[a, b]$, then at some point c in $[a, b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

不定积分

DEFINITION The collection of all antiderivatives of f is called the **indefinite integral** of f with respect to x, and is denoted by

$$\int f(x)\,dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

换元法

THEOREM 6—The Substitution Rule If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

THEOREM 7—Substitution in Definite Integrals If g' is continuous on the interval $[a, b]$ and f is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du.$$

奇偶性

THEOREM 8 Let f be continuous on the symmetric interval $[-a, a]$.

(a) If f is even, then $\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

(b) If f is odd, then $\int_{-a}^{a} f(x)\,dx = 0$.

计算：$\int_{-1}^{1} \frac{2x^2 + x\cos x}{1+\sqrt{1-x^2}}\,dx$

曲线围成的面积

DEFINITION If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the **area of the region between the curves $y = f(x)$ and $y = g(x)$ from a to b** is the integral of $(f - g)$ from a to b:

$$A = \int_a^b [f(x) - g(x)]\, dx.$$

$$A = \int_c^d [f(y) - g(y)]\, dy$$

高数带飞课提纲一览（复习专用）

- Ch06 定积分的应用
  - 求旋转体的体积 (截面法) @ Disk/Washer/Shell
  - 求弧长 & 旋转体的表面积
  - 物理应用 @ Work & Moment & Mass & Center of Mass(COM) & Centroids
- Ch07 超越函数
  - 反函数与其导数 @ 对数 (三角函数积分) & 指数 & 指对互换 & 可分离微分方程(01)
  - 洛必达法则 & 柯西中值定理 & 反三角函数 @ (定义 & 导数 & 积分)
- Ch08 积分方法
  - 基本积分 & 分部积分 (表格式) & 三角函数积分3 (三角换元) & 有理函数积分 & 数值积分
  - 反常 (广义) 积分 @ Type1(无穷积分) & Type2 (瑕积分) & P级数及其衍生级数 & 敛散性
- Ch09 一阶微分方程
  - 一阶微分方程及其特解 (数值解) & 一阶线性微分方程 (标准解)
  - 自治微分方程及其画图

Chapter 6

定积分的应用

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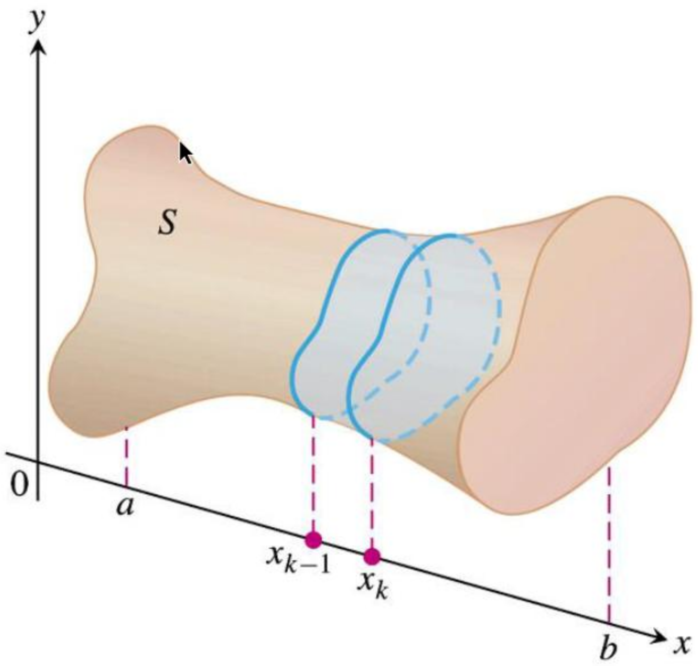
截面法求体积

DEFINITION The **volume** of a solid of integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\, dx.$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\,\Delta x_k = \int_a^b A(x)\, dx.$$



求旋转体的体积 *

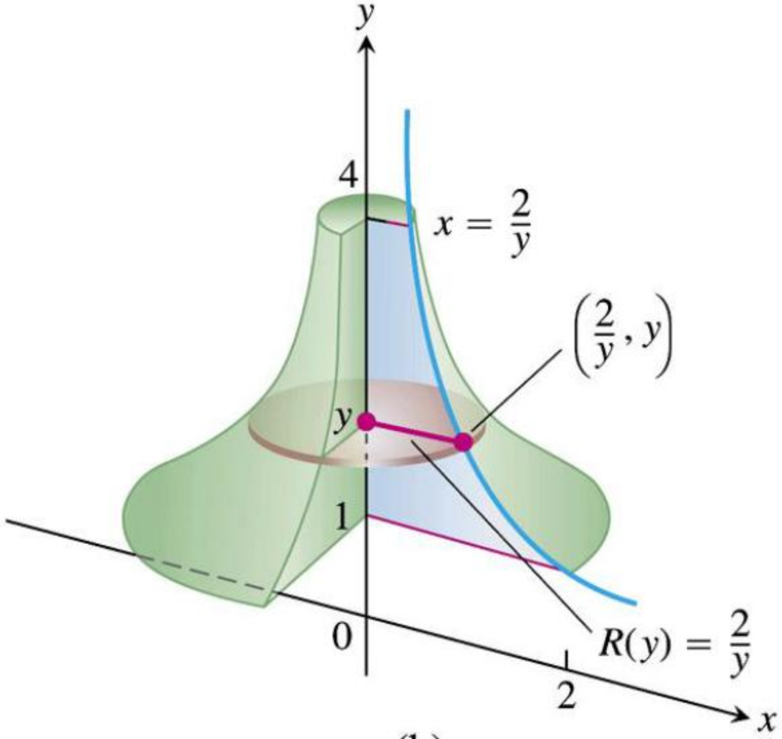
1. The Disk Method (圆盘法)

Volume by Disks for Rotation About the x-axis

绕X轴旋转 $$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.$$

Volume by Disks for Rotation About the y-axis

绕Y轴旋转 $$V = \int_c^d A(y)\,dy = \int_c^d \pi[R(y)]^2\,dy.$$



(b)

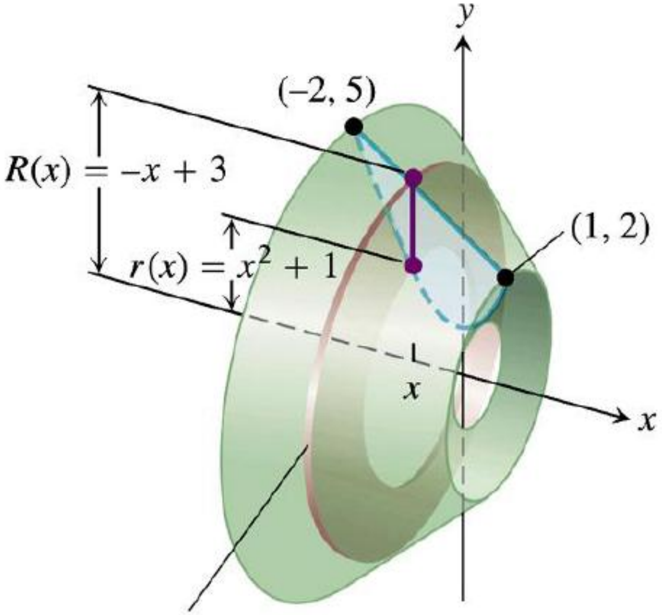
求旋转体的体积 *

2. The Washer Method (垫圈法)

Volume by Washers for Rotation About the x-axis

绕X轴旋转 $$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.$$

绕Y轴旋转 $$V = \int_c^d \pi([R(y)]^2 - [r(y)]^2)\,dy$$



Washer cross section
Outer radius: $R(x) = -x + 3$
Inner radius: $r(x) = x^2 + 1$

求旋转体的体积 *

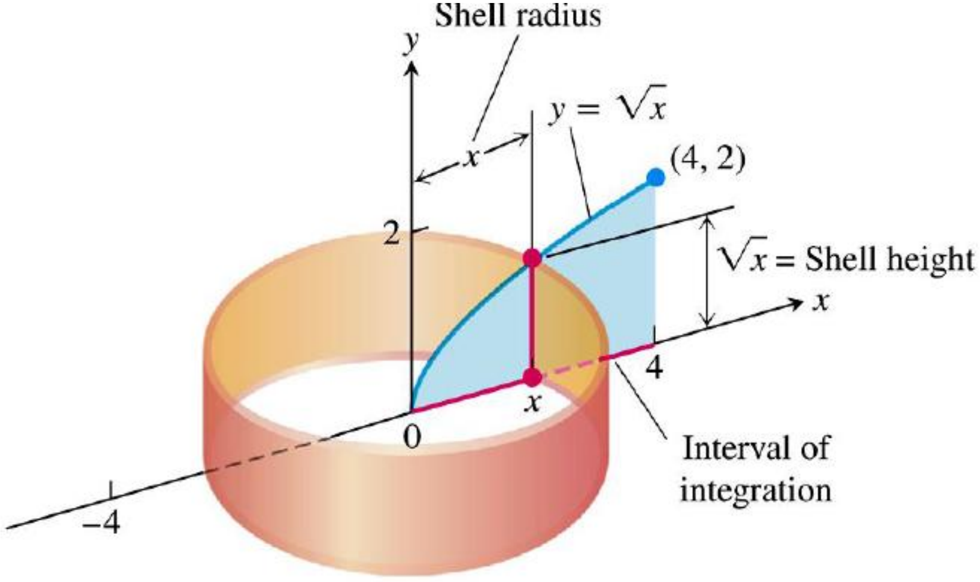
3. The Shell Method (壳层法)

Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x = L$ is

绕Y轴旋转 $$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx.$$

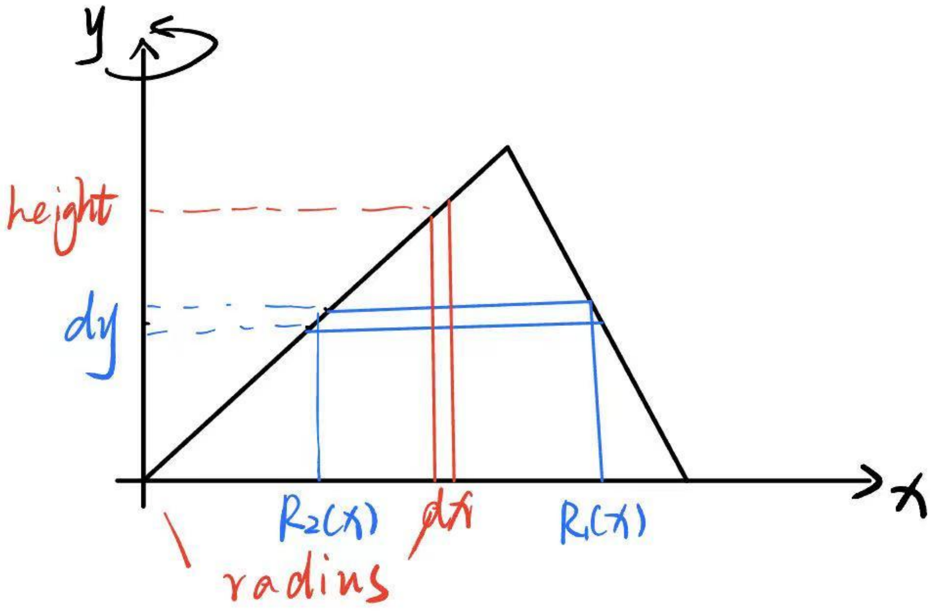
绕X轴旋转 $$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dy$$



(b)

求旋转体的体积 * – 总结

1. 对于圆盘法与垫圈法（蓝色）：
    - 绕哪一个轴旋转就对哪个轴积分；
    - 画垂直轴的截面。
2. 对于壳层法（红色）：
    - 绕哪一个轴旋转就对另一个轴积分；
    - 画平行轴的截面。



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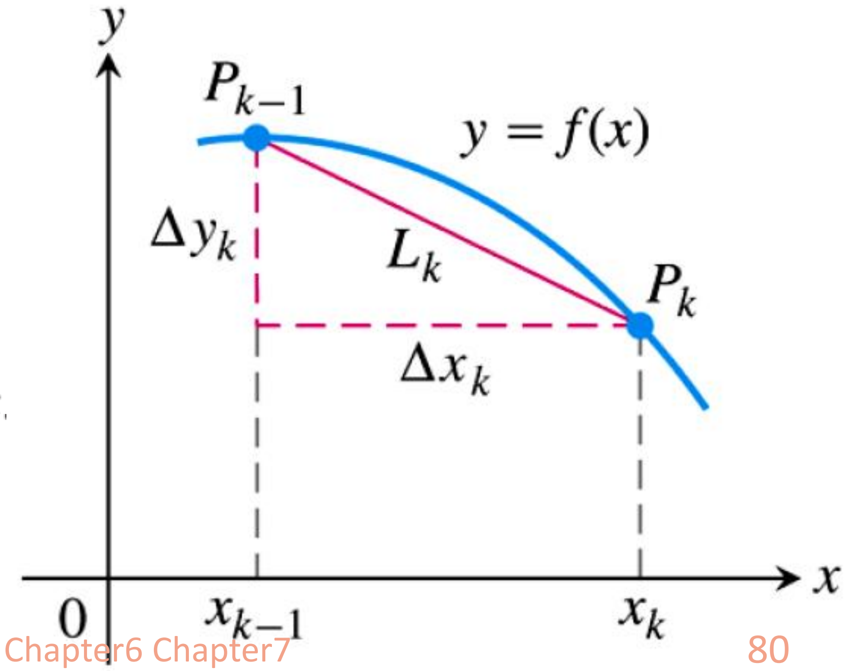
求弧长 *

DEFINITION If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad (3)$$

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

$$\lim_{n\to\infty} \sum_{k=1}^{n} L_k = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\, \Delta x_k = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx.$$



求弧长 *

DEFINITION If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad (3)$$

Formula for the Length of $x = g(y)$, $c \leq y \leq d$

If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from $A = (g(c), c)$ to $B = (g(d), d)$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \qquad (4)$$

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求旋转体的表面积 *

DEFINITION If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

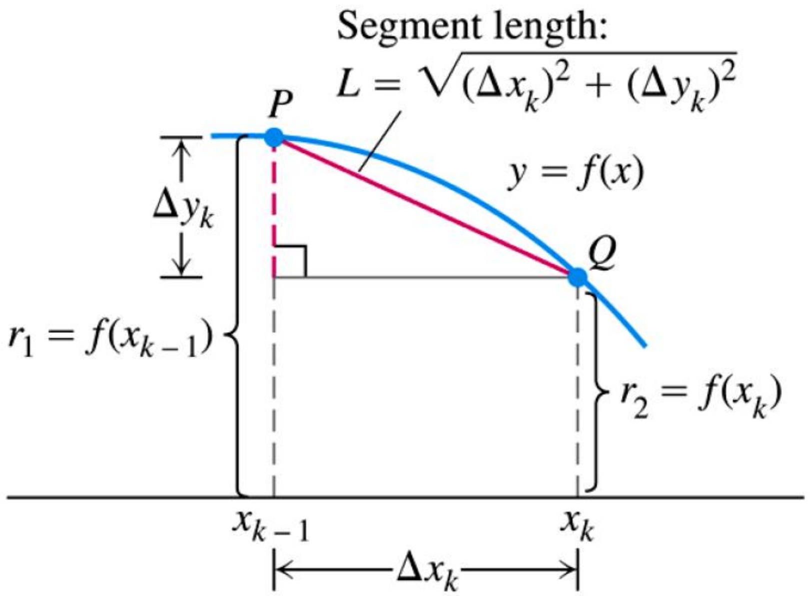
$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx. \qquad (3)$$

$$\text{Frustum surface area} = 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$$

$$= \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.$$

$$\text{Add all areas} = \sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (f'(c_k)\,\Delta x_k)^2}$$

$$= \sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{1 + (f'(c_k))^2}\,\Delta x_k.$$



求旋转体的表面积 *

DEFINITION If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx. \qquad (3)$$

Surface Area for Revolution About the y-Axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

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物理应用

做功

DEFINITION The **work** done by a variable force $F(x)$ in moving an object along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx. \qquad (2)$$

矩

质心

Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the *xy*-Plane

$$\begin{aligned} &\text{Moment about the } x\text{-axis:} && M_x = \int \tilde{y}\,dm \\ &\text{Moment about the } y\text{-axis:} && M_y = \int \tilde{x}\,dm \\ &\text{Mass:} && M = \int dm \\ &\text{Center of mass:} && \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M} \end{aligned} \qquad (5)$$

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求质心 *

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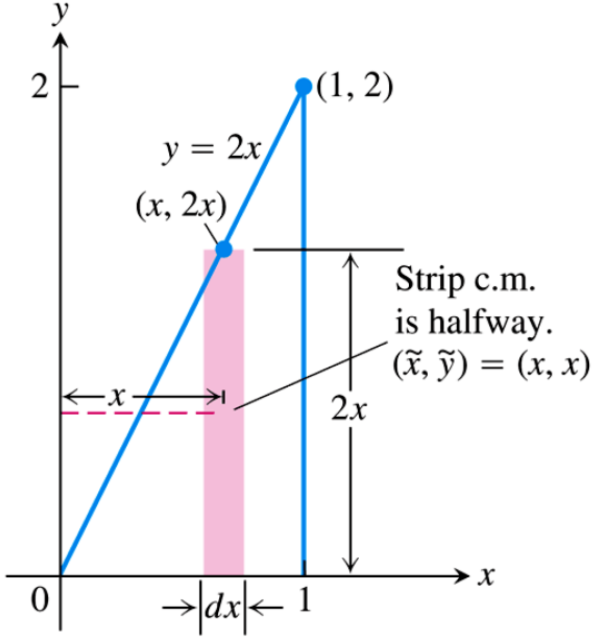
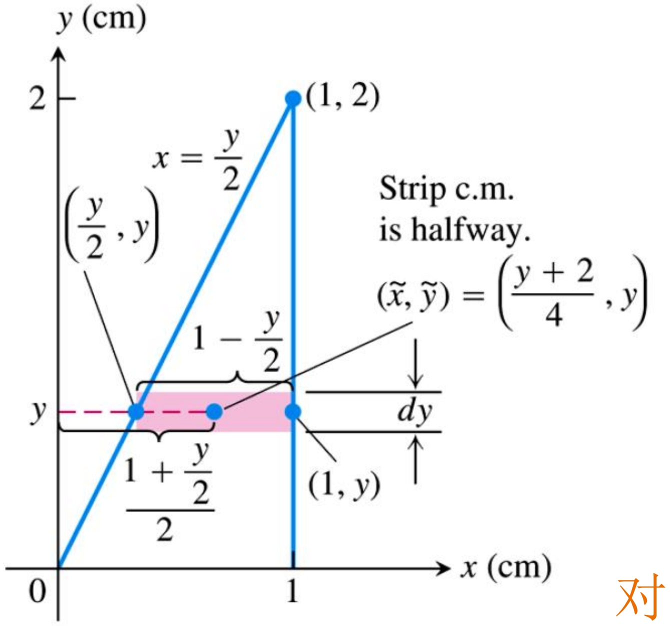
Method 1: Vertical Strips



Units in centimeters

对dx积分

Method 2: Horizontal Strips



对dy积分

推导：

$$\bar{x} = \frac{1}{M}\int_a^b \delta x\,[f(x) - g(x)]\,dx \qquad (6)$$

$$\bar{y} = \frac{1}{M}\int_a^b \frac{\delta}{2}\,[f^2(x) - g^2(x)]\,dx \qquad (7)$$

重心

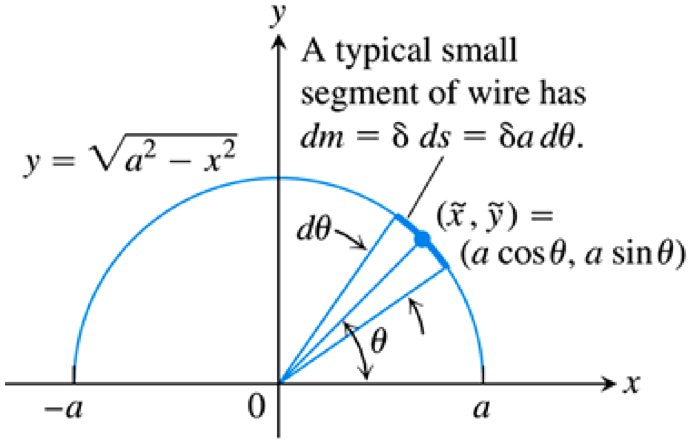
COM = Centroids ?

When the density function is constant, engineers may call the center of mass the **centroid**

对于三角形：三条中位线的交点

EXAMPLE 4 Find the center of mass (centroid) of a thin wire of constant density δ shaped like a semicircle of radius a.

利用极坐标系去求重心(质心)



(a)

$$\text{length:} \quad ds = a\, d\theta$$
$$\text{mass:} \quad dm = \delta\, ds = \delta a\, d\theta$$
$$\text{distance of c.m. to } x\text{-axis:} \quad \tilde{y} = a \sin\theta.$$

Pappus定理

THEOREM 1　Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi\rho A. \tag{9}$$

THEOREM 2　Pappus's Theorem for Surface Areas

If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L. \tag{11}$$

1. 必须是匀质薄盘或细线，密度为常数
2. 旋转轴不能穿过薄盘或细线（旋转后不重叠）

SUSTech Southern University of Science and Technology

Chapter 7

超越函数

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反函数

DEFINITION Suppose that f is a one-to-one function on a domain D with range R. The **inverse function** f^{-1} is defined by

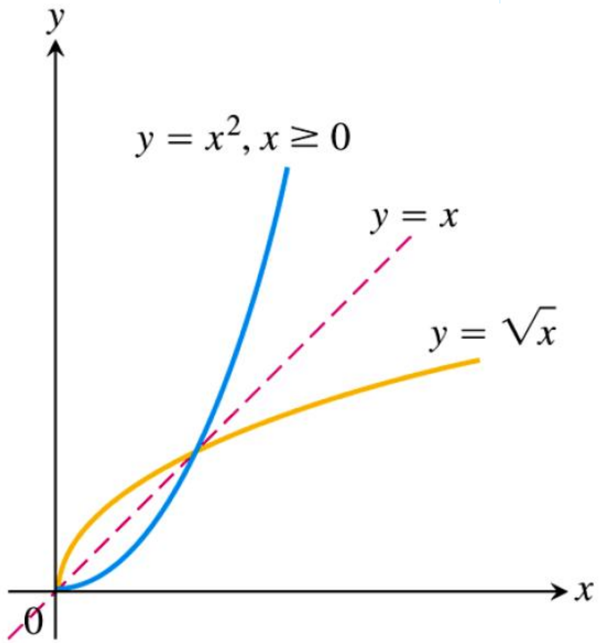
$$f^{-1}(b) = a \text{ if } f(a) = b.$$

The domain of f^{-1} is R and the range of f^{-1} is D.

求反函数方法:

1. 用含y的项表示x

2. 交换x与y



EXAMPLE 4 Find the inverse of the function $y = x^2, x \geq 0$, expressed as a function of x.

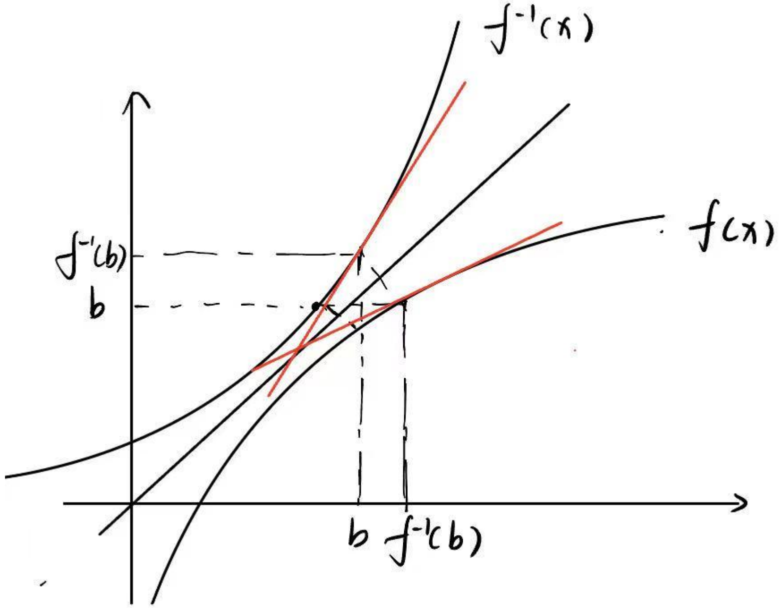
反函数的导数

THEOREM 1—The Derivative Rule for Inverses If f has an interval I as domain and $f'(x)$ exists and is never zero on I, then f^{-1} is differentiable at every point in its domain (the range of f). The value of $(f^{-1})'$ at a point b in the domain of f^{-1} is the reciprocal of the value of f' at the point $a = f^{-1}(b)$:

$$(f^{-1})'(b) = \frac{1}{f'(f^{-1}(b))}$$

or

$$\left.\frac{df^{-1}}{dx}\right|_{x=b} = \frac{1}{\left.\frac{df}{dx}\right|_{x=f^{-1}(b)}}$$



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对数函数

DEFINITION The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \qquad x > 0. \tag{1}$$

求导：

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0. \qquad\qquad \frac{d}{dx}\ln|x| = \frac{1}{x}, \qquad x \neq 0$$

积分：

If u is a differentiable function that is never zero,

$$\int \frac{1}{u}\,du = \ln|u| + C. \tag{5}$$

三角函数积分

$$\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + C$$

Integrals of the tangent, cotangent, secant, and cosecant functions

$$\int \tan u\,du = \ln|\sec u| + C \qquad \int \sec u\,du = \ln|\sec u + \tan u| + C$$

$$\int \cot u\,du = \ln|\sin u| + C \qquad \int \csc u\,du = -\ln|\csc u + \cot u| + C$$

高数上公式总结：https://zhuanlan.zhihu.com/p/106252437

对数微分： 设 $y = x^{\sin x}\ (x>0)$，求 y'

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指数函数

DEFINITION For every real number x, we define the **natural exponential function** to be $e^x = \exp x$.

If u is any differentiable function of x, then

求导:

$$\frac{d}{dx} e^u = e^u \frac{du}{dx}. \tag{2}$$

The general antiderivative of the exponential function

积分:

$$\int e^u \, du = e^u + C$$

指数函数

DEFINITION For any numbers $a > 0$ and x, the **exponential function with base a** is

$$a^x = e^{x \ln a}.$$

Differentiate $f(x) = x^x, x > 0.$

求导： $\dfrac{d}{dx} a^u = a^u \ln a \dfrac{du}{dx}.$ 积分： $\displaystyle\int a^u \, du = \frac{a^u}{\ln a} + C.$

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- Ch06 定积分的应用
  - 求旋转体的体积 (截面法) @ Disk/Washer/Shell
  - 求弧长 & 旋转体的表面积
  - 物理应用 @ Work & Moment & Mass & Center of Mass(COM) & Centroids
- Ch07 超越函数
  - 反函数与其导数 @ 对数 (三角函数积分) & 指数 & 指对互换 & 可分离微分方程(01)
  - 洛必达法则 & 柯西中值定理 & 反三角函数 @ (定义 & 导数 & 积分)
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  - 反常 (广义) 积分 @ Type1(无穷积分) & Type2 (瑕积分) & P级数及其衍生级数 & 敛散性
- Ch09 一阶微分方程
  - 一阶微分方程及其特解 (数值解) & 一阶线性微分方程 (标准解)
  - 自治微分方程及其画图

指对互换

Inverse Equations for a^x and $\log_a x$

$$a^{\log_a x} = x \qquad (x > 0)$$

$$\log_a (a^x) = x \qquad (\text{all } x)$$

$$\log_a x = \frac{\ln x}{\ln a}.$$

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u}\frac{du}{dx}$$

指数级变化

In modeling many real-world situations, a quantity y increases or decreases at a rate proportional to its size at a given time t. Such quantities are said to undergo **exponential change**.

The solution of the initial value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

is

$$y = y_0 e^{kt}. \tag{2}$$

可分离微分方程

suppose we have a differential equation of the form

$$\frac{dy}{dx} = f(x, y),$$

When we rewrite this equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)},$$

its differential form allows us to collect all y terms with dy and all x terms with dx:

$$h(y)\,dy = g(x)\,dx.$$

Now we simply integrate both sides of this equation:

$$\int h(y)\,dy = \int g(x)\,dx.$$

可分离：
可以将含x的项与含y的项分开

微分方程解法：https://zhuanlan.zhihu.com/p/453830266

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洛必达法则 *

THEOREM 5— L'Hôpital's Rule Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$

assuming that the limit on the right side of this equation exists.

使用条件：1. 0 / 0; 2. ∞ / ∞; 3. ∞ * 0; 4. ∞ - ∞

不定型/不定幂

不定型：

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**.

不定幂：

If $\lim_{x \to a} \ln f(x) = L$, then

$$\lim_{x \to a} f(x) = \lim_{x \to a} e^{\ln f(x)} = e^L.$$

Here a may be either finite or infinite.

Find $\lim_{x \to \infty} x^{1/x}$.

柯西中值定理 *

THEOREM 6—Cauchy's Mean Value Theorem Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof1: 用拉格朗日中值定理证明柯西中值定理

Proof2: 用柯西中值定理证明洛必达法则

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反三角函数

DEFINITION

$y = \sin^{-1} x$ is the number in $[-\pi/2, \pi/2]$ for which $\sin y = x$.

$y = \cos^{-1} x$ is the number in $[0, \pi]$ for which $\cos y = x$.

$y = \tan^{-1} x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$.

$y = \cot^{-1} x$ is the number in $(0, \pi)$ for which $\cot y = x$.

$y = \sec^{-1} x$ is the number in $[0, \pi/2) \cup (\pi/2, \pi]$ for which $\sec y = x$.

$y = \csc^{-1} x$ is the number in $[-\pi/2, 0) \cup (0, \pi/2]$ for which $\csc y = x$.

注意定义域范围！

反三角函数的导数 *

1. $$\frac{d(\sin^{-1}u)}{dx}=\frac{1}{\sqrt{1-u^2}}\frac{du}{dx},\quad |u|<1$$
2. $$\frac{d(\cos^{-1}u)}{dx}=-\frac{1}{\sqrt{1-u^2}}\frac{du}{dx},\quad |u|<1$$
3. $$\frac{d(\tan^{-1}u)}{dx}=\frac{1}{1+u^2}\frac{du}{dx}$$
4. $$\frac{d(\cot^{-1}u)}{dx}=-\frac{1}{1+u^2}\frac{du}{dx}$$
5. $$\frac{d(\sec^{-1}u)}{dx}=\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx},\quad |u|>1$$
6. $$\frac{d(\csc^{-1}u)}{dx}=-\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx},\quad |u|>1$$

$$\cos^{-1}x=\pi/2-\sin^{-1}x$$
$$\cot^{-1}x=\pi/2-\tan^{-1}x$$
$$\csc^{-1}x=\pi/2-\sec^{-1}x$$

一共只用记三组公式

每组对应的为相反数

反三角函数的积分 *

TABLE 7.4 Integrals evaluated with inverse trigonometric functions

The following formulas hold for any constant $a \neq 0$.

1. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$ (Valid for $u^2 < a^2$)
2. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$ (Valid for all u)
3. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + C$ (Valid for $|u| > a > 0$)

$$\int \left(\frac{3}{1+x^2} - \frac{2}{\sqrt{1-x^2}}\right)dx. \qquad \int \frac{x}{\sqrt{2-x^4}}dx. \qquad \int \frac{1+2x^2}{x^2(1+x^2)}dx.$$

Little-Oh & Big-Oh

DEFINITION A function f is **of smaller order than** g as $x \to \infty$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0$. We indicate this by writing $\boldsymbol{f} = \boldsymbol{o}(\boldsymbol{g})$ ("f is little-oh of g").

DEFINITION Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive integer M for which

$$\frac{f(x)}{g(x)} \leq M,$$

for x sufficiently large. We indicate this by writing $\boldsymbol{f} = \boldsymbol{O}(\boldsymbol{g})$ ("f is big-oh of g").

在计算机、数学、工程中很重要！

Southern University of Science and Technology

Chapter 8

积分方法

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基本积分公式

TABLE 8.1 Basic integration formulas

1. $\int k\,dx = kx + C \qquad (\text{any number } k)$

2. $\int x^n\,dx = \dfrac{x^{n+1}}{n+1} + C \qquad (n \neq -1)$

3. $\int \dfrac{dx}{x} = \ln|x| + C$

4. $\int e^x\,dx = e^x + C$

5. $\int a^x\,dx = \dfrac{a^x}{\ln a} + C \qquad (a > 0,\ a \neq 1)$

6. $\int \sin x\,dx = -\cos x + C$

7. $\int \cos x\,dx = \sin x + C$

8. $\int \sec^2 x\,dx = \tan x + C$

9. $\int \csc^2 x\,dx = -\cot x + C$

18. $\int \dfrac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\dfrac{x}{a}\right) + C$

19. $\int \dfrac{dx}{a^2 + x^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + C$

分部积分 *

不定积分：

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \qquad (1)$$

$$\int u\,dv = uv - \int v\,du \qquad (2)$$

定积分：

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,dx \qquad (3)$$

例：

Find $\int \ln x\,dx.$ 求 $\int \frac{\arctan e^x}{e^{2x}}dx$ 求 $\int \cos\sqrt{x}dx.$

求 (1) $\int \frac{1+\cos x}{x+\sin x}dx,$

(2) $\int \frac{x+\sin x}{1+\cos x}dx.$

表格式积分 *

$$\int f(x)g(x)\,dx,$$

若f可以逐步求导至0或f，则可以反复运用分部积分去求积分。

例： Evaluate $\displaystyle\int x^2e^x\,dx.$

$f(x)$ and its derivatives		$g(x)$ and its integrals
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

$$\int x^2e^x\,dx = x^2e^x - 2xe^x + 2e^x + C.$$

Evaluate $\displaystyle\int e^x\cos x\,dx.$

Obtain a formula that expresses the integral

$$\int \cos^n x\,dx$$

三角函数积分 * – sin与cos幂之积

$$\int \sin^m x \cos^n x \, dx,$$

Case 1 If ***m*** **is odd**, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \qquad (1)$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x \, dx$ equal to $-d(\cos x)$.

Case 2 If ***m*** **is even and** ***n*** **is odd** in $\int \sin^m x \cos^n x \, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x \, dx$ equal to $d(\sin x)$.

Case 3 If **both** ***m*** **and** ***n*** **are even** in $\int \sin^m x \cos^n x \, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \qquad (2)$$

to reduce the integrand to one in lower powers of $\cos 2x$.

三角函数积分 * – 根号 | tan与sec

二倍角公式： $Cos2\alpha = 2Cos^2\alpha - 1 = 1 - 2Sin^2\alpha$

例： Evaluate $\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$

正切与正割变换： $\tan^2 x = \sec^2 x - 1 \quad \sec^2 x = \tan^2 x + 1$

例： $\int \tan^4 x\, dx. \qquad \int \sec^3 x\, dx.$

三角函数积分 * – sin与cos之积

$$\int \sin mx \sin nx\, dx, \qquad \int \sin mx \cos nx\, dx, \qquad \text{and} \qquad \int \cos mx \cos nx\, dx$$

积化和差：

$$\sin mx \sin nx = \frac{1}{2}\left[\cos(m-n)x - \cos(m+n)x\right],$$

$$\sin mx \cos nx = \frac{1}{2}\left[\sin(m-n)x + \sin(m+n)x\right],$$

$$\cos mx \cos nx = \frac{1}{2}\left[\cos(m-n)x + \cos(m+n)x\right].$$

例：

Evaluate

$$\int \sin 3x \cos 5x\, dx.$$

三角换元 * $\sqrt{a^2+x^2}$, $\sqrt{a^2-x^2}$, and $\sqrt{x^2-a^2}$

$$x = a\tan\theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a\sin\theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a\sec\theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if } \dfrac{x}{a} \ge 1, \\ \dfrac{\pi}{2} < \theta \le \pi & \text{if } \dfrac{x}{a} \le -1. \end{cases}$$

1. 写出对x的三角换元，计算其微分，用dθ代替dx
2. 将换元表达式代入，化简
3. 对积分上下限做出限制，进行积分
4. 画出参考三角形，将结果替换回x

(1) $\sqrt{a^2-x^2}$ 可令$x=a\sin t$;

(2) $\sqrt{a^2+x^2}$ 可令$x=a\tan t$;

(3) $\sqrt{x^2-a^2}$ 可令$x=a\sec t$.

有理函数积分– 部分分式 *

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Method of Partial Fractions when $f(x)/g(x)$ is Proper

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{(x-r)} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}.$$

   Do this for each distinct linear factor of $g(x)$.

2. Let $x^2 + px + q$ be an irreducible quadratic factor of $g(x)$ so that $x^2 + px + q$ has no real roots. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:

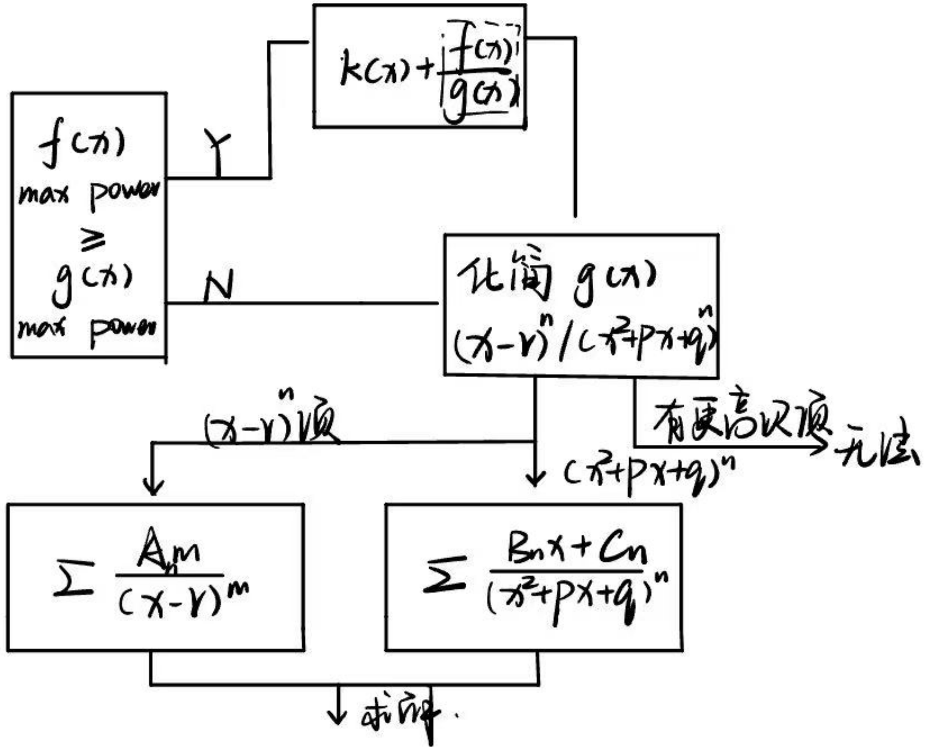
$$\frac{B_1x + C_1}{(x^2+px+q)} + \frac{B_2x + C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx + C_n}{(x^2+px+q)^n}.$$

   Do this for each distinct quadratic factor of $g(x)$.

3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.

4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

部分分式: $h(x) = \frac{f(x)}{g(x)}$ 有理函数.



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Heaviside Method

1. *Write the quotient with g(x) factored:*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_n)}.$$

2. *Cover the factors $(x - r_i)$ of $g(x)$ one at a time*, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1 = \frac{f(r_1)}{(r_1 - r_2)\cdots(r_1 - r_n)}$$

$$A_2 = \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3)\cdots(r_2 - r_n)}$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n - r_1)(r_n - r_2)\cdots(r_n - r_{n-1})}.$$

3. *Write the partial-fraction expansion of $f(x)/g(x)$* as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

$$\frac{1}{x(x-1)^2} = \frac{A}{x} + \frac{B}{(x-1)^2} + \frac{C}{x-1},$$

求积分 $\displaystyle\int \frac{1}{1 + e^{\frac{x}{2}} + e^{\frac{x}{3}} + e^{\frac{x}{6}}}dx.$

求 $\displaystyle\int \frac{1}{\sqrt{x}(1 + \sqrt[3]{x})}dx.$

有理函数积分– 部分分式 cont.

化简后，只会出现三类：

1. 多项式 ✓ 2. $\frac{A}{(x-a)^n}$ ✓ 3. $\frac{Mx+N}{(x^2+px+q)^n}$?

讨论 $\frac{Mx+N}{(x^2+px+q)^n}$：

$$x^2+px+q = (x+\frac{p}{2})^2+q-\frac{p^2}{4}$$

令 $t = x+\frac{p}{2}$ 且 $x^2+px+q = t^2+a^2$ $Mx+N = Mt+b$

$\therefore a=\sqrt{q-\frac{p^2}{4}}$ $b = N-\frac{p}{2}M$

$$\therefore \int\frac{Mx+N}{(x^2+px+q)^n}dx = \int\frac{Mt}{(t^2+a^2)^n}dt + \int\frac{b}{(t^2+a^2)^n}dt$$

$$\begin{cases} \int\frac{Mt}{(t^2+a^2)^n}dt = -\frac{1}{2(n-1)(t^2+a^2)^{n-1}}+C \\ b\int\frac{1}{(t^2+a^2)^n}dt = bI_n \quad (n=1,2\cdots) \end{cases}$$

$n=1$ $I_1 = \int\frac{1}{t^2+a^2}dt = \frac{1}{a}\tan^{-1}(\frac{t}{a})+C$

$n=2$ $I_2 = \int\frac{1}{(t^2+a^2)^2}dt$ 令 $t = a\tan\theta$

$$I_2 = \int\frac{a\sec^2\theta\, d\theta}{a^4\sec^4\theta} = \int\frac{1}{a^3}\cos^2\theta\, d\theta = \int\frac{1}{a^3}\frac{1+\cos 2\theta}{2}d\theta$$

求积分 $\int\frac{1}{x}\sqrt{\frac{1+x}{x}}dx$

数值积分 * – 梯形法则

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

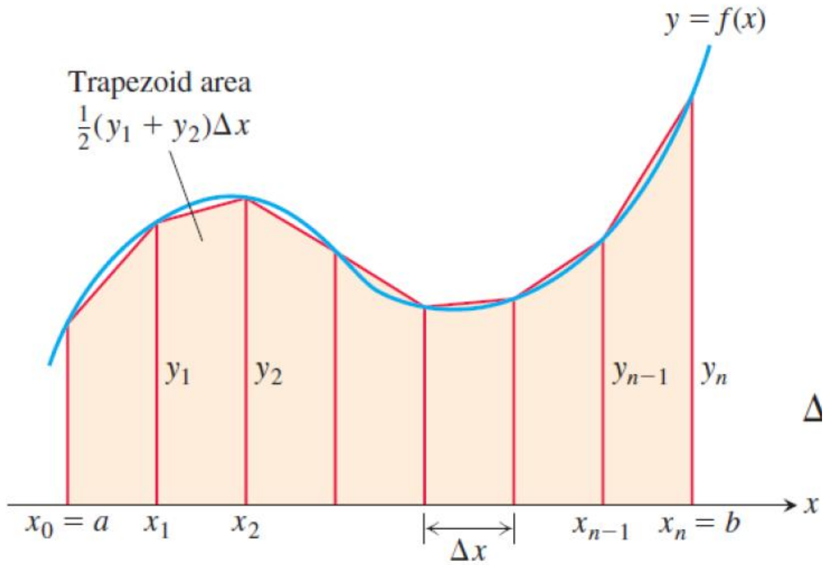
$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

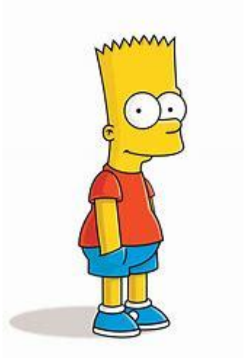
$$x_0 = a,\ x_1 = a + \Delta x,\ x_2 = a + 2\Delta x,\ \ldots,\ x_{n-1} = a + (n-1)\Delta x,\ x_n = b,$$

where $\Delta x = (b - a)/n$.

$$\frac{1}{2}(y_1 + y_2)$$



数值积分 * – Simpson法则



Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

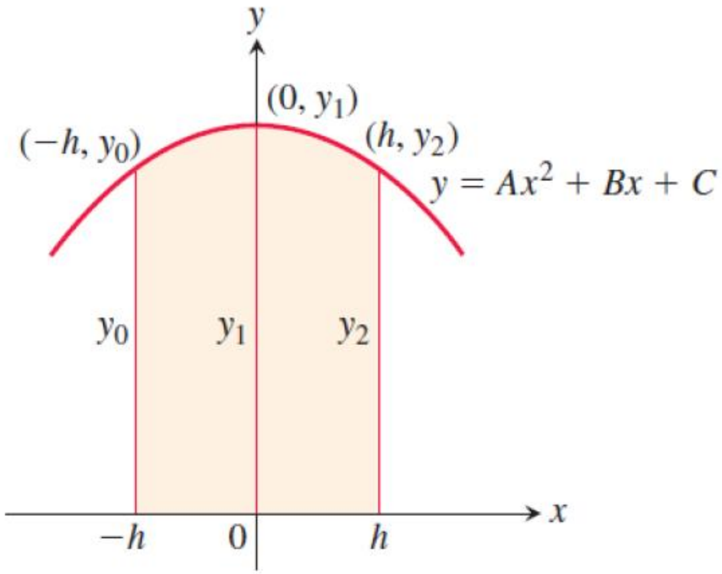
$$S = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a,\ x_1 = a + \Delta x,\ x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x,\ x_n = b.$$

The number n is even, and $\Delta x = (b - a)/n$.

$$\frac{h}{3}\left(y_0 + 4y_1 + y_2\right).$$



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SUSTech Southern University of Science and Technology

- Ch06 定积分的应用
  - 求旋转体的体积 (截面法) @ Disk/Washer/Shell
  - 求弧长 & 旋转体的表面积
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  - 反常 (广义) 积分 @ Type1(无穷积分) & Type2 (瑕积分) & P级数及其衍生级数 & 敛散性
- Ch09 一阶微分方程
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反常(广义)积分 * Type1

DEFINITION Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\, dx = \lim_{b\to\infty} \int_a^b f(x)\, dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a\to-\infty} \int_a^b f(x)\, dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_c^{\infty} f(x)\, dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

当极限存在时，称为广义积分收敛；

当极限不存在时，称广义积分发散。

广义积分 牛顿莱布尼茨公式

当 $\int_a^{+\infty} f(x)dx$ 收敛时， 设$F(x)$是$f(x)$的原函数，记

$$\lim_{x\to+\infty} F(x) = F(+\infty),$$

则 $$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx$$

$$= \lim_{b\to+\infty} F(x)\Big|_a^b = F(+\infty) - F(a) = F(x)\Big|_a^{+\infty}.$$

此形式与牛顿-莱布尼兹公式的形式相同.

P级数 *

$$\int_1^\infty \frac{dx}{x^p} = \lim_{b\to\infty}\int_1^b \frac{dx}{x^p}$$

$$= \lim_{b\to\infty}\left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}}-1\right)\right] = \begin{cases} \dfrac{1}{p-1}, & p>1 \\ \infty, & p<1 \end{cases}$$

$$\int_1^\infty \frac{dx}{x^p} = \int_1^\infty \frac{dx}{x} = \lim_{b\to\infty}\int_1^b \frac{dx}{x} = \lim_{b\to\infty} \ln x\Big]_1^b = \infty.$$

反常(广义)积分 * Type2

DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then

$$\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

当极限存在时，称为广义积分收敛；

当极限不存在时，称广义积分发散。

反常(广义)积分 * Type2

设函数 $f(x)$ 在区间 $[a,b]$ 上 $\boxed{\text{除点} c\ (a<c<b)}$ 外连续，而在 $\boxed{\text{点} c \text{ 的邻域内无界}}$. 如果两个广义积分 $\int_a^c f(x)dx$ 和 $\int_c^b f(x)dx$ 都收敛，则定义

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$= \lim_{\varepsilon\to+0}\int_a^{\boxed{c-\varepsilon}} f(x)dx + \lim_{\varepsilon'\to+0}\int_{\boxed{c+\varepsilon'}}^b f(x)dx$$

否则，就称广义积分 $\int_a^b f(x)dx$ 发散.

定义中 c 为**瑕点**，以上积分称为**瑕积分.**

C为瑕点，f(x) 为瑕积分

瑕点就是无限间断点

积分 $\int_0^1 \frac{\ln x}{x-1}dx$ 的瑕点是哪几点？

例　求下列广义积分：

(1) $\int_{-\infty}^{+\infty}\frac{dx}{x^2+4x+9}$; (2) $\int_1^2 \frac{dx}{x\sqrt{3x^2-2x-1}}$.

敛散性判别 *

THEOREM 2—Direct Comparison Test Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges.
2. $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.

判断：
1. $\int_1^\infty \frac{\arctan x}{(1+x^2)^{\frac{3}{2}}}dx$
2. $\int_1^\infty x^{p-1}e^{-x}dx$ 敛散性
3. $\int_0^1 \frac{1}{\sqrt[4]{1-x^4}}dx$

THEOREM 3—Limit Comparison Test If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

用p级数比较非常重要！

$\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$ (positive)

$$\begin{cases} L=\infty & f(x)\text{收敛 则 } g(x)\text{同} \\ 0<L<\infty & \text{同敛散} \\ L=0 & g(x)\text{收敛 则 } f(x)\text{同} \end{cases}$$

Chapter 9

一阶微分方程

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一阶微分方程及特解

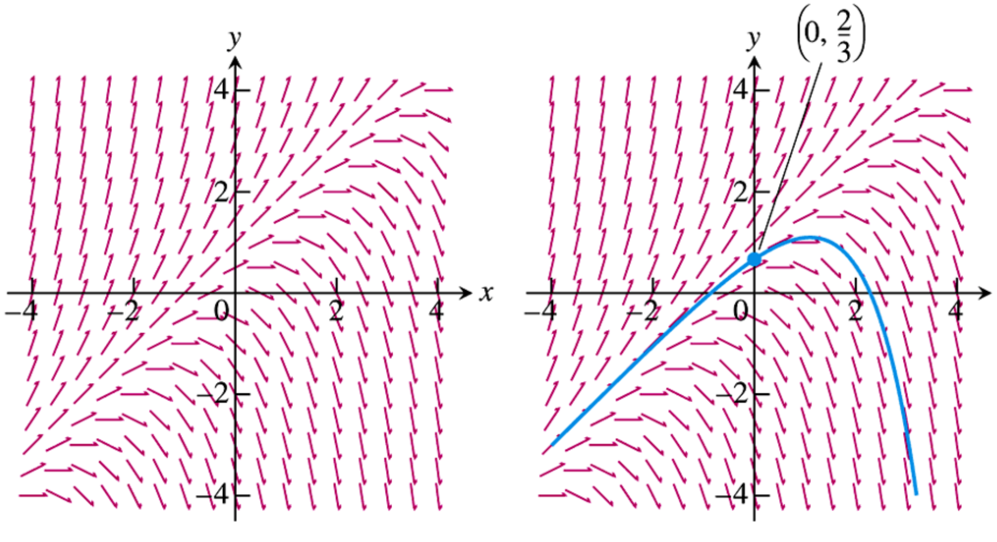
A **first-order differential equation** is an equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane.

As was the case in finding antiderivatives, we often need a *particular* rather than the general solution to a first-order differential equation $y' = f(x, y)$. The **particular solution** satisfying the initial condition $y(x_0) = y_0$ is the solution $y = y(x)$ whose value is y_0 when $x = x_0$. Thus the graph of the particular solution passes through the point (x_0, y_0) in the xy-plane. A **first-order initial value problem** is a differential equation $y' = f(x, y)$ whose solution must satisfy an initial condition $y(x_0) = y_0$.

Solution Curve:



一阶线性微分方程

A first-order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{1}$$

where P and Q are continuous functions of x. Equation (1) is the linear equation's **standard form**.

$$\frac{dy}{dx} = y + x^2, \qquad \frac{dx}{dt} = x\sin t + t^2,$$

$$yy' - 2xy = 3, \qquad y' - \cos y = 1,$$

一阶微分方程解法：

1. 可分离微分方程
2. 对于线性微分方程：标准解法
3. 数值解法

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一阶线性微分方程标准解 *

To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)\,dx}$ and integrate both sides.

求解三步走:

1. 找 $P(x)$, $Q(x)$ 2. factor: $v(x) = e^{\int P(x)dx}$ 3. $y = \frac{1}{v(x)}\int v(x)Q(x)dx$

Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

应用：混合问题

$$\begin{pmatrix}\text{Rate of change}\\ \text{of amount}\\ \text{in container}\end{pmatrix} = \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{departs.}\end{pmatrix}$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t, then the departure rate of the chemical at time t is

$$\begin{aligned}\text{Departure rate} &= \frac{y(t)}{V(t)} \cdot (\text{outflow rate}) \\ &= \begin{pmatrix}\text{concentration in}\\ \text{container at time } t\end{pmatrix} \cdot (\text{outflow rate})\end{aligned}$$

$$\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)} \cdot (\text{outflow rate}).$$

EXAMPLE 3 In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins (Figure 9.14)?

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自治微分方程

A differential equation for which dy/dx is a function of y only is called an **autonomous** differential equation. Let's investigate what happens when the derivative in an autonomous equation equals zero.

DEFINITION If $dy/dx = g(y)$ is an autonomous differential equation, then the values of y for which $dy/dx = 0$ are called **equilibrium values** or **rest points**.

To construct a graphical solution to an autonomous differential equation, we first make a **phase line** for the equation, a plot on the y-axis that shows the equation's equilibrium values along with the intervals where dy/dx and d^2y/dx^2 are positive and negative. Then we know where the solutions are increasing and decreasing, and the concavity of the solution curves.

自治微分方程 – 画图 *

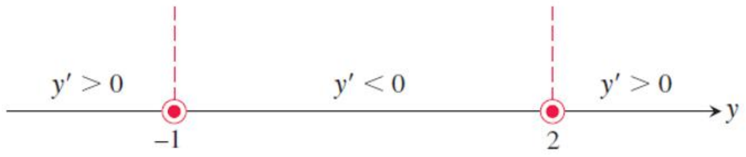
EXAMPLE 1 Draw a phase line for the equation

$$\frac{dy}{dx} = (y + 1)(y - 2)$$

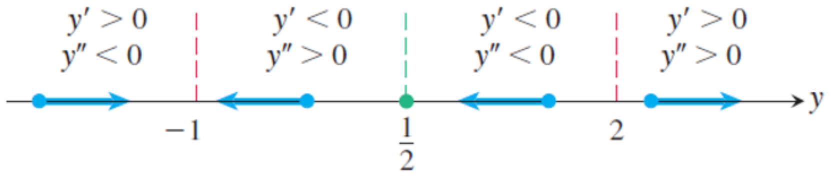
and use it to sketch solutions to the equation.

1. *Draw a number line for y and mark the equilibrium values* $y = -1$ *and* $y = 2$, *where* $dy/dx = 0$.

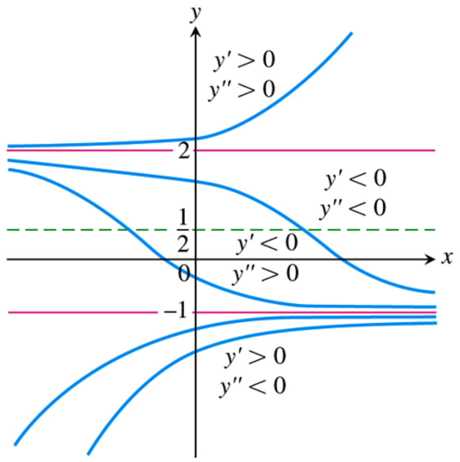
2. *Identify and label the intervals where* $y' > 0$ *and* $y' < 0$.



3. *Calculate* y'' *and mark the intervals where* $y'' > 0$ *and* $y'' < 0$.



4.



Q&A

祝大家期末顺利！