SUSTech Southern University of Science and Technology

高数期末复习

致诚书院学长学姐带飞课

Chapter 14, Chapter 15， Chapter 16

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Chapter14

偏导数 Cont.

多元函数

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \ldots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

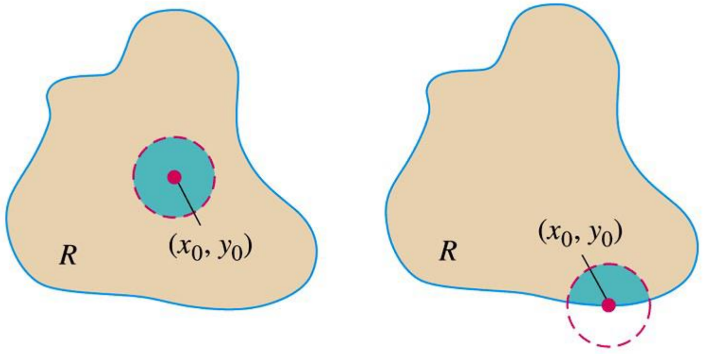
$$w = f(x_1, x_2, \ldots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

二元函数的点

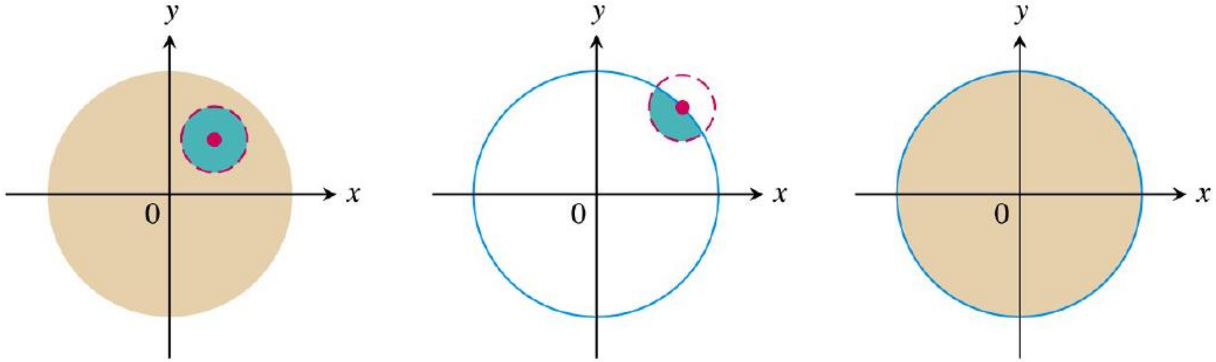
DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).



(a) Interior point (b) Boundary point

内点； 边界点； 开放； 封闭

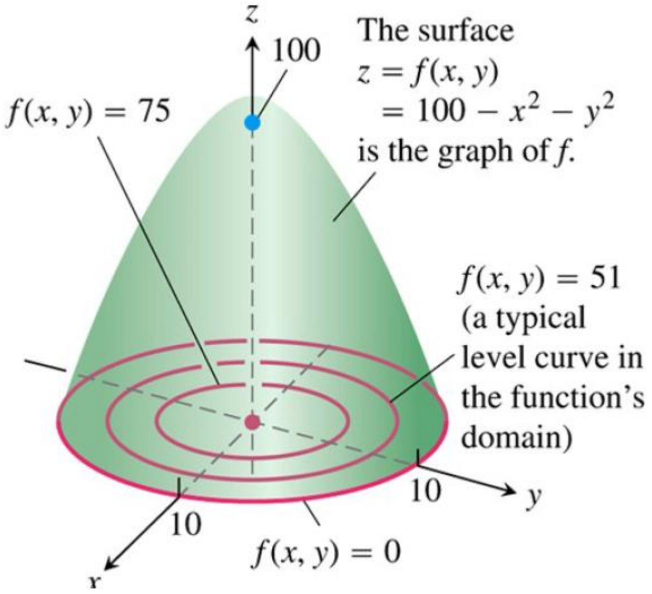


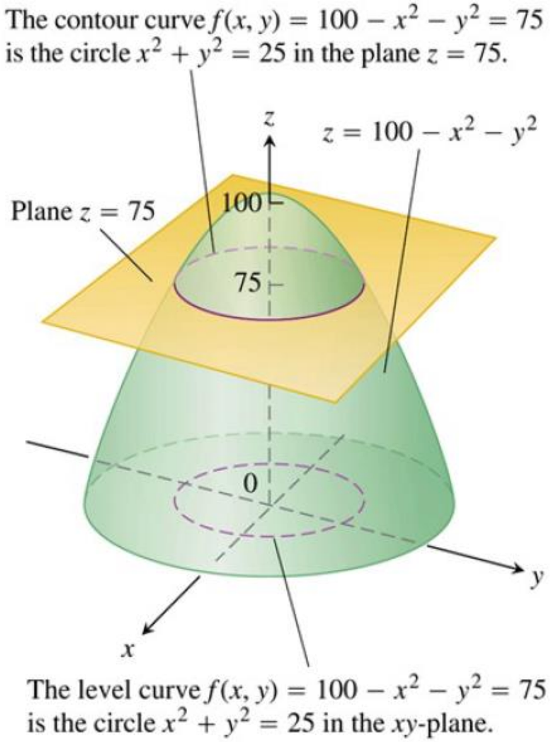
有界 VS. 无界

DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

等高线与曲面

DEFINITIONS The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $\boldsymbol{z = f(x, y)}$.





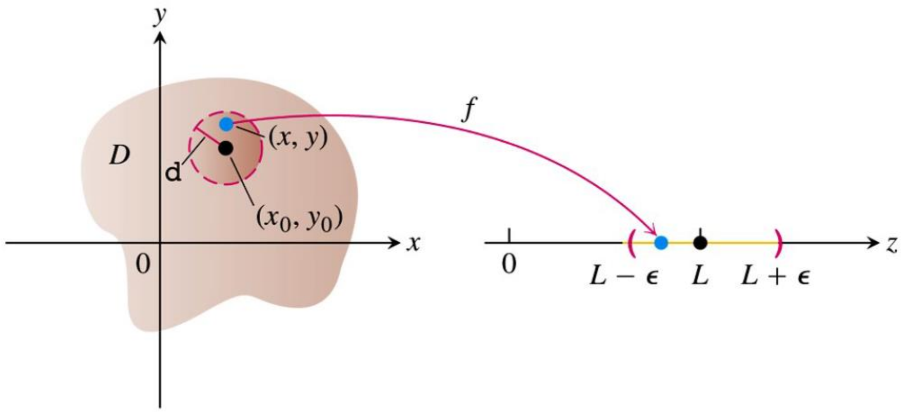
多元函数的极限

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$



多元函数极限运算法则

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THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$
5. *Quotient Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$,

   n a positive integer, and if n is even, we assume that $L > 0$.

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + y^2}$ if it exists.

求极限 $\displaystyle\lim_{\substack{x\to0\\y\to0}} \frac{(y-x)x}{\sqrt{x^2+y^2}}$.

多元函数的连续性

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

$$f(x, y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Two-Path Test for Nonexistence of a Limit

If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

Having the same limit along all straight lines approaching (x_0, y_0) does not imply a limit exists at (x_0, y_0).

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

排除法！

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

一阶偏导数

DEFINITION The **partial derivative of $f(x, y)$ with respect to x** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0, y_0)} = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h},$$

provided the limit exists.

DEFINITION The **partial derivative of $f(x, y)$ with respect to y** at the point (x_0, y_0) is

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0, y_0)} = \left.\frac{d}{dy} f(x_0, y)\right|_{y=y_0} = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h},$$

provided the limit exists.

例 设$z = x^y\ (x > 0, x \neq 1)$，

求证 $$\frac{x}{y}\frac{\partial z}{\partial x} + \frac{1}{\ln x}\frac{\partial z}{\partial y} = 2z.$$

二阶偏导数

The defining equations are

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

Find $\partial^2 w/\partial x \partial y$ if

$$w = xy + \frac{e^y}{y^2 + 1}.$$

Notice the order in which the mixed partial derivatives are taken:

$\frac{\partial^2 f}{\partial x \partial y}$	Differentiate first with respect to y, then with respect to x.
$f_{yx} = (f_y)_x$	Means the same thing.

注意运算顺序

THEOREM 2—The Mixed Derivative Theorem If $f(x, y)$ and its partial derivatives f_x, f_y, f_{xy}, and f_{yx} are defined throughout an open region containing a point (a, b) and are all continuous at (a, b), then

$$f_{xy}(a, b) = f_{yx}(a, b).$$

多元函数可微的定义

DEFINITION A function $z = f(x, y)$ is **differentiable at** (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and Δz satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$. We call f **differentiable** if it is differentiable at every point in its domain, and say that its graph is a **smooth surface**.

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by

$$\Delta y = f'(a)\,\Delta x + \epsilon\,\Delta x \tag{1}$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

多元函数可微的充分条件

THEOREM 3—The Increment Theorem for Functions of Two Variables Suppose that the first partial derivatives of $f(x, y)$ are defined throughout an open region R containing the point (x_0, y_0) and that f_x and f_y are continuous at (x_0, y_0). Then the change

$$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$$

in the value of f that results from moving from (x_0, y_0) to another point $(x_0 + \Delta x, y_0 + \Delta y)$ in R satisfies an equation of the form

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y$$

in which each of $\epsilon_1, \epsilon_2 \to 0$ as both $\Delta x, \Delta y \to 0$.

COROLLARY OF THEOREM 3 If the partial derivatives f_x and f_y of a function $f(x, y)$ are continuous throughout an open region R, then f is differentiable at every point of R.

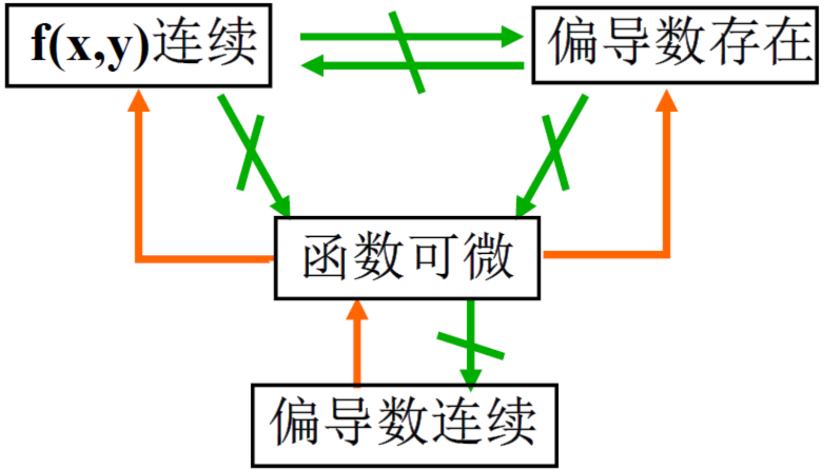
多元函数可微的必要条件

THEOREM 4—Differentiability Implies Continuity If a function $f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

函数 $f(x,y)$ 在点 (x_0,y_0) 处连续，且两个偏导数 $f_x(x_0,y_0), f_y(x_0,y_0)$ 存在是 $f(x,y)$ 在该点可微的(　　).

(A) 充分条件，但不是必要条件；
(B) 必要条件，但不是充分条件；
(C) 充分必要条件；
(D) 既不是充分条件，也不是必要条件.

多元函数连续、偏导数存在、可微的关系



多元函数可微的充要条件*

1. 写出全增量$\Delta z = f(x_0 + \Delta x, y_0 + \Delta y) - f(x_0, y_0)$;

2. 写出线性增量$A\Delta x + B\Delta y$，其中$A = f'_x(x_0, y_0)$，$B = f'_y(x_0, y_0)$;

3. 作极限 $\lim\limits_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \dfrac{\Delta z - (A\Delta x + B\Delta y)}{\sqrt{(\Delta x)^2 + (\Delta y)^2}} = 0$，则$z = f(x, y)$在$(x_0, y_0)$处可微。

总结：

1. 充分条件：偏导数存在且连续
2. 必要条件：导数连续且偏导数存在
3. 充要条件：满足上述条件

多元函数的链式法则 只有一个独立变量的情况

THEOREM 5—Chain Rule for Functions of Two Independent Variables If $w = f(x, y)$ is differentiable and if $x = x(t)$, $y = y(t)$ are differentiable functions of t, then the composite $w = f(x(t), y(t))$ is a differentiable function of t and

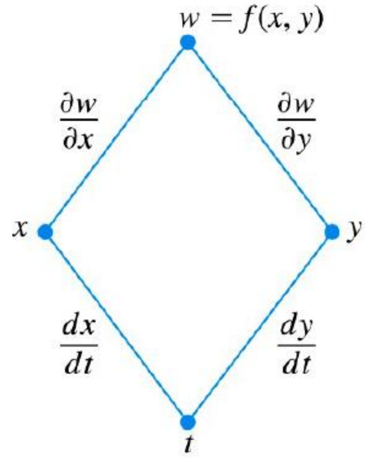
$$\frac{dw}{dt} = f_x(x(t), y(t)) \cdot x'(t) + f_y(x(t), y(t)) \cdot y'(t),$$

or

$$\frac{dw}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

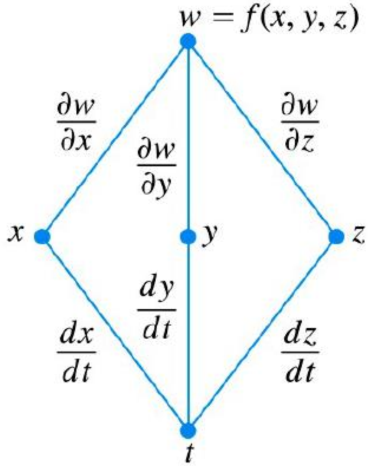
更经典的形式：

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}.$$



THEOREM 6—Chain Rule for Functions of Three Independent Variables If $w = f(x, y, z)$ is differentiable and x, y, and z are differentiable functions of t, then w is a differentiable function of t and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}.$$

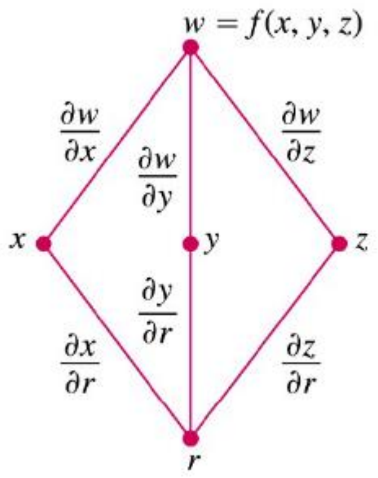


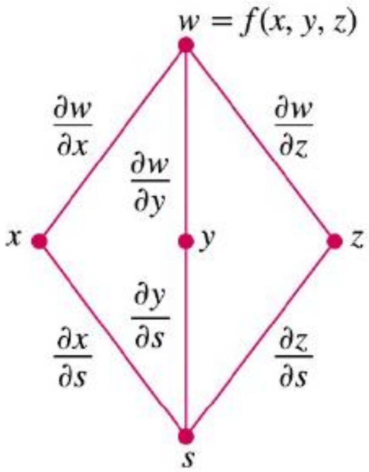
多元函数的链式法则 有多个独立变量的情况

THEOREM 7—Chain Rule for Two Independent Variables and Three Intermediate Variables Suppose that $w = f(x, y, z)$, $x = g(r, s)$, $y = h(r, s)$, and $z = k(r, s)$. If all four functions are differentiable, then w has partial derivatives with respect to r and s, given by the formulas

$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial s}.$$





特殊的:

$z = f(u, x, y)$ $u = \varphi(x, y)$

则有: $\frac{\partial z}{\partial x} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial f}{\partial x}$

$\boxed{\frac{\partial z}{\partial y}} = \frac{\partial f}{\partial u} \cdot \frac{\partial u}{\partial y} + \boxed{\frac{\partial f}{\partial y}}$

隐函数微分公式

1. The function $F(x, y)$ is differentiable and
2. The equation $F(x, y) = 0$ defines y implicitly as a differentiable function of x, say $y = h(x)$.

Computing the derivative from the Chain Rule

$$0 = \frac{dw}{dx} = F_x \frac{dx}{dx} + F_y \frac{dy}{dx}$$

$$= F_x \cdot 1 + F_y \cdot \frac{dy}{dx}.$$

THEOREM 8—A Formula for Implicit Differentiation Suppose that $F(x, y)$ is differentiable and that the equation $F(x, y) = 0$ defines y as a differentiable function of x. Then at any point where $F_y \neq 0$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y}.$$

例 设 $x^2 + y^2 + z^2 - 4z = 0$， $z \neq 2$， 求 $\frac{\partial^2 z}{\partial x^2}$.

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}.$$

偏导数习题

1、设 $f(x,y)=\begin{cases}(x^2+y^2)\sin\dfrac{1}{x^2+y^2}, & x^2+y^2\neq 0\\ 0, & x^2+y^2=0\end{cases}$

则在原点(0,0)处 $f(x,y)$ (　　).

(A) 偏导数不存在；　　(B) 不可微；

(C) 偏导数存在且连续；　　(D) 可微 .

2、设 $z=f(x,v), v=v(x,y)$ 其中 f,v 具有二阶连续偏导数. 则 $\dfrac{\partial^2 z}{\partial y^2}=$ (　　).

(A) $\dfrac{\partial^2 f}{\partial v\partial y}\cdot\dfrac{\partial v}{\partial y}+\dfrac{\partial f}{\partial v}\cdot\dfrac{\partial^2 v}{\partial y^2}$；　(B) $\dfrac{\partial f}{\partial v}\cdot\dfrac{\partial^2 v}{\partial y^2}$；

(C) $\dfrac{\partial^2 f}{\partial v^2}\left(\dfrac{\partial v}{\partial y}\right)^2+\dfrac{\partial f}{\partial v}\cdot\dfrac{\partial^2 v}{\partial y^2}$；　(D) $\dfrac{\partial^2 f}{\partial v^2}\cdot\dfrac{\partial v}{\partial y}+\dfrac{\partial f}{\partial v}\cdot\dfrac{\partial^2 v}{\partial y^2}$.

二、设 $z=(u,x,y), u=xe^y$, 其中 f 具有连续的二阶偏导数, 求 $\dfrac{\partial^2 z}{\partial x\partial y}$.

方向导数的定义

DEFINITION The **derivative of f at $P_0(x_0, y_0)$ in the direction of the unit vector u $= u_1$i $+ u_2$j** is the number

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \lim_{s\to 0}\frac{f(x_0 + su_1, y_0 + su_2) - f(x_0, y_0)}{s}, \qquad (1)$$

provided the limit exists.

The **directional derivative** defined by Equation (1) is also denoted by

$$(D_{\mathbf{u}}f)_{P_0}.$$

"The derivative of f at P_0 in the direction of $\mathbf{u}$"

The partial derivatives $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ are the directional derivatives of f at P_0 in the **i** and **j** directions.

梯度的定义

DEFINITION The **gradient vector (gradient)** of $f(x, y)$ at a point $P_0(x_0, y_0)$ is the vector

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$$

obtained by evaluating the partial derivatives of f at P_0.

For a differentiable function $f(x, y, z)$ and a unit vector $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ in space,

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

“一个标量的梯度是一个矢量”

方向导数与梯度的关系

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = \left(\frac{\partial f}{\partial x}\right)_{P_0}\frac{dx}{ds} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\frac{dy}{ds} = \left(\frac{\partial f}{\partial x}\right)_{P_0}u_1 + \left(\frac{\partial f}{\partial y}\right)_{P_0}u_2$$

$$= \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)_{P_0}\mathbf{i} + \left(\frac{\partial f}{\partial y}\right)_{P_0}\mathbf{j}\right]}_{\text{Gradient of } f \text{ at } P_0} \cdot \underbrace{\left[u_1\mathbf{i} + u_2\mathbf{j}\right]}_{\text{Direction } \mathbf{u}}.$$

THEOREM 9—The Directional Derivative Is a Dot Product If $f(x, y)$ is differentiable in an open region containing $P_0(x_0, y_0)$, then

$$\left(\frac{df}{ds}\right)_{\mathbf{u},P_0} = (\nabla f)_{P_0} \cdot \mathbf{u}, \tag{4}$$

the dot product of the gradient ∇f at P_0 and $\mathbf{u}$.

这里必须为方向向量

方向导数与梯度的关系

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Properties of the Directional Derivative $D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f| \cos \theta$

1. The function f increases most rapidly when $\cos \theta = 1$ or when $\theta = 0$ and $\mathbf{u}$ is the direction of ∇f. That is, at each point P in its domain, f increases most rapidly in the direction of the gradient vector ∇f at P. The derivative in this direction is
$$D_{\mathbf{u}}f = |\nabla f| \cos (0) = |\nabla f|.$$
函数上升最快的方向
2. Similarly, f decreases most rapidly in the direction of $-\nabla f$. The derivative in this direction is $D_{\mathbf{u}}f = |\nabla f| \cos (\pi) = -|\nabla f|$. 函数下降最快的方向
3. Any direction $\mathbf{u}$ orthogonal to a gradient $\nabla f \neq 0$ is a direction of zero change in f because θ then equals $\pi/2$ and
$$D_{\mathbf{u}}f = |\nabla f| \cos (\pi/2) = |\nabla f| \cdot 0 = 0.$$

等高线的切线 (两个独立变量)

At every point (x_0, y_0) in the domain of a differentiable function $f(x, y)$, the gradient of f is normal to the level curve through (x_0, y_0) (Figure 14.30).

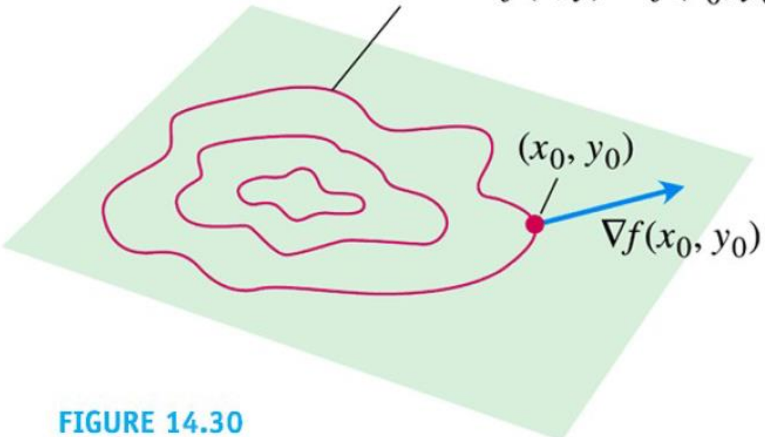
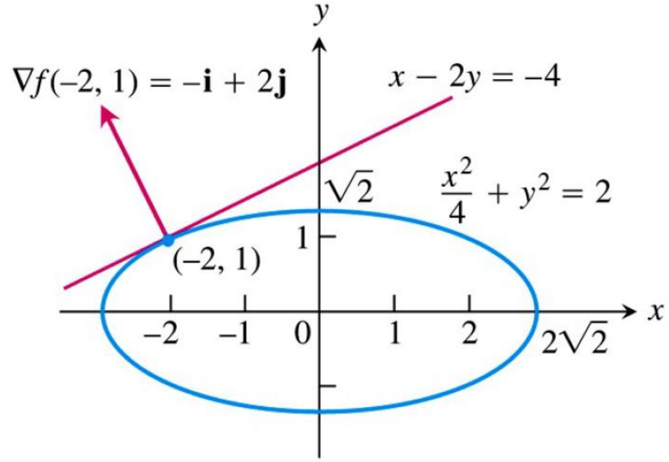


FIGURE 14.30



Tangent Line to a Level Curve

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) = 0 \qquad (6)$$

梯度的运算法则

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Algebra Rules for Gradients

1. *Sum Rule:* $\nabla(f+g) = \nabla f + \nabla g$
2. *Difference Rule:* $\nabla(f-g) = \nabla f - \nabla g$
3. *Constant Multiple Rule:* $\nabla(kf) = k\nabla f$ (any number k)
4. *Product Rule:* $\nabla(fg) = f\nabla g + g\nabla f$
5. *Quotient Rule:* $\nabla\left(\dfrac{f}{g}\right) = \dfrac{g\nabla f - f\nabla g}{g^2}$

Scalar multipliers on left of gradients

曲面的切平面 (三个独立变量)

DEFINITIONS The **tangent plane** at the point $P_0(x_0, y_0, z_0)$ on the level surface $f(x, y, z) = c$ of a differentiable function f is the plane through P_0 normal to $\nabla f|_{P_0}$.

The **normal line** of the surface at P_0 is the line through P_0 parallel to $\nabla f|_{P_0}$.

Tangent Plane to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$f_x(P_0)(x - x_0) + f_y(P_0)(y - y_0) + f_z(P_0)(z - z_0) = 0$$

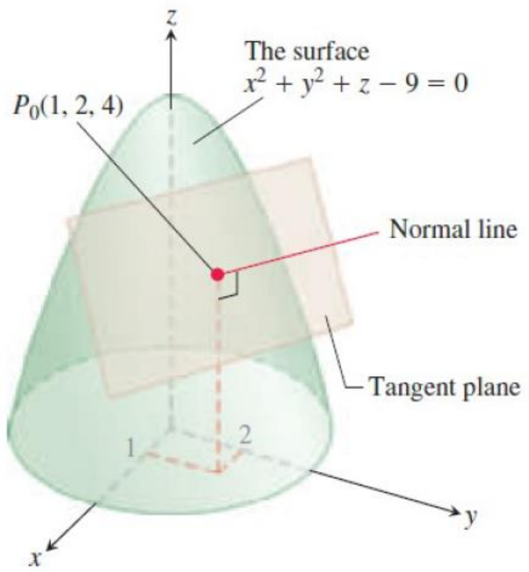
Plane Tangent to a Surface $z = f(x, y)$ at $(x_0, y_0, f(x_0, y_0))$

The plane tangent to the surface $z = f(x, y)$ of a differentiable function f at the point $P_0(x_0, y_0, z_0) = (x_0, y_0, f(x_0, y_0))$ is

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) - (z - z_0) = 0. \qquad (4)$$

Normal Line to $f(x, y, z) = c$ at $P_0(x_0, y_0, z_0)$

$$x = x_0 + f_x(P_0)t, \qquad y = y_0 + f_y(P_0)t, \qquad z = z_0 + f_z(P_0)t \qquad (3)$$



多元函数线性化

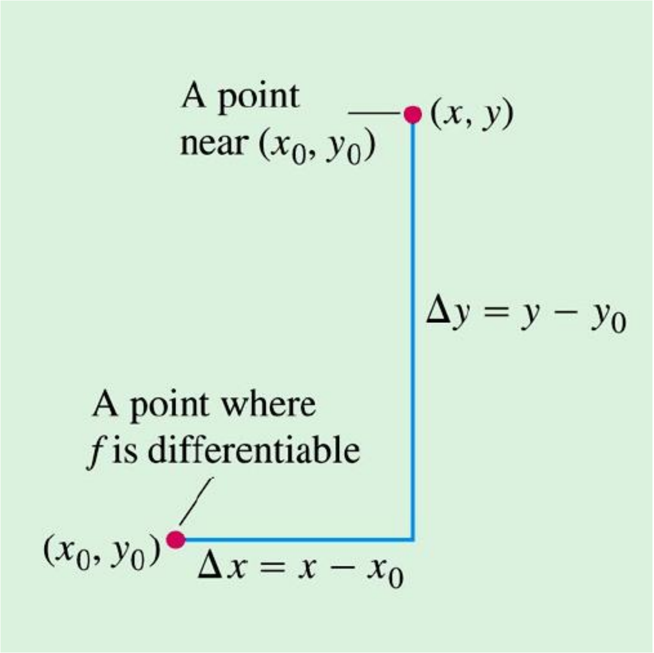
DEFINITIONS The **linearization** of a function $f(x, y)$ at a point (x_0, y_0) where f is differentiable is the function

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0). \qquad (5)$$

The approximation

$$f(x, y) \approx L(x, y)$$

is the **standard linear approximation** of f at (x_0, y_0).



The Error in the Standard Linear Approximation (derived in Section 14.9).
If f has continuous first and second partial derivatives throughout an open set containing a rectangle R centered at (x_0, y_0) and if M is any upper bound for the values of $|f_{xx}|, |f_{yy}|$, and $|f_{xy}|$ on R, then the error $E(x, y)$ incurred in replacing $f(x, y)$ on R by its linearization

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

satisfies the inequality

$$|E(x, y)| \le \frac{1}{2} M(|x - x_0| + |y - y_0|)^2.$$

全微分

DEFINITION If we move from (x_0, y_0) to a point $(x_0 + dx, y_0 + dy)$ nearby, the resulting change

$$df = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

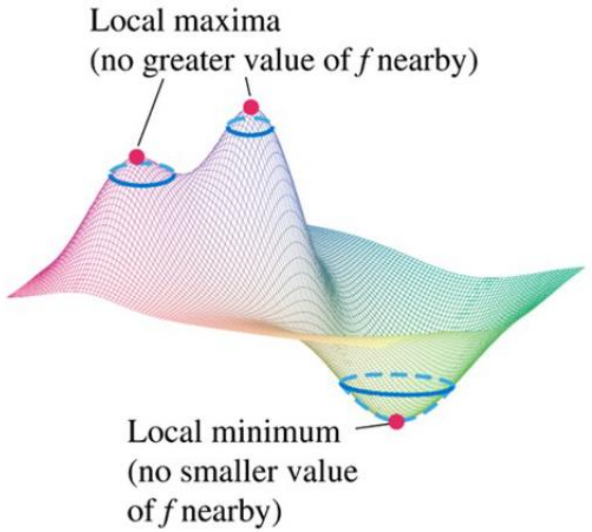
in the linearization of f is called the **total differential of f**.

$$df = f_x(P_0)\,dx + f_y(P_0)\,dy + f_z(P_0)\,dz$$

多元函数极值的定义

DEFINITIONS Let $f(x, y)$ be defined on a region R containing the point (a, b). Then

1. $f(a, b)$ is a **local maximum** value of f if $f(a, b) \geq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).
2. $f(a, b)$ is a **local minimum** value of f if $f(a, b) \leq f(x, y)$ for all domain points (x, y) in an open disk centered at (a, b).



可能产生极值点的位置

1. interior points where $f' = 0$, 驻点
2. interior points where f' is undefined, 不可导点
3. endpoints of the domain of f. 端点

驻点与不可导点统称为可疑点 (Critical Point)

求最值方法：比较可疑点与端点的函数值

多元函数极值的一阶判别

THEOREM 10—First Derivative Test for Local Extreme Values If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain and if the first partial derivatives exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

DEFINITION An interior point of the domain of a function $f(x, y)$ where both f_x and f_y are zero or where one or both of f_x and f_y do not exist is a **critical point** of f.

极值点 $\Rightarrow$ 驻点

DEFINITION A differentiable function $f(x, y)$ has a **saddle point** at a critical point (a, b) if in every open disk centered at (a, b) there are domain points (x, y) where $f(x, y) > f(a, b)$ and domain points (x, y) where $f(x, y) < f(a, b)$. The corresponding point $(a, b, f(a, b))$ on the surface $z = f(x, y)$ is called a saddle point of the surface (Figure 14.42).

若 $f(x_0, y)$ 及 $f(x, y_0)$ 在 (x_0, y_0) 点均取得极值，则 $f(x, y)$ 在点 (x_0, y_0) 是否也取得极值？

多元函数极值的二阶判别

THEOREM 11—Second Derivative Test for Local Extreme Values Suppose that $f(x, y)$ and its first and second partial derivatives are continuous throughout a disk centered at (a, b) and that $f_x(a, b) = f_y(a, b) = 0$. Then

i) f has a **local maximum** at (a, b) if $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

ii) f has a **local minimum** at (a, b) if $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b).

iii) f has a **saddle point** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b).

iv) **the test is inconclusive** at (a, b) if $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b). In this case, we must find some other way to determine the behavior of f at (a, b).

The expression $f_{xx}f_{yy} - f_{xy}^2$ is called the **discriminant** or **Hessian** of f. It is sometimes easier to remember it in determinant form,

$$f_{xx}f_{yy} - f_{xy}^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix}.$$

多元函数极值判别步骤

求函数 $z = f(x,y)$ 极值的一般步骤:

第一步 解方程组 $\begin{cases} f_x(x,y)=0 \\ f_y(x,y)=0 \end{cases}$

求出得驻点.

第二步 对于每一个驻点 (x_0, y_0),

求出二阶偏导数的值.

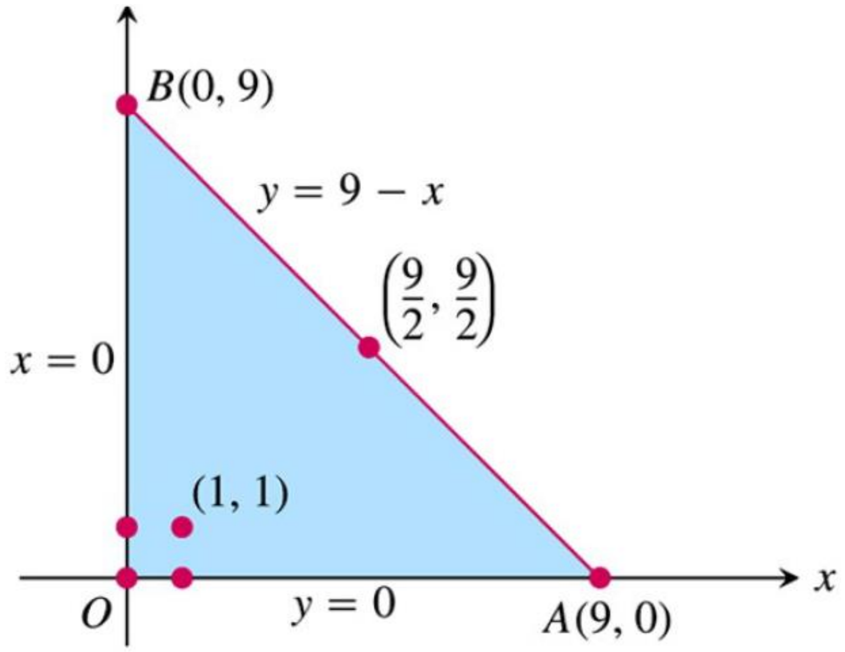
第三步 定出 $\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = AC - B^2$ 的符号,

再判定是否是极值.

EXAMPLE 5 Find the absolute maximum and minimum values of

$$f(x, y) = 2 + 2x + 2y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, $y = 9 - x$.



多元函数各类值判别

Summary of Max-Min Tests

The extreme values of $f(x, y)$ can occur only at

i. **boundary points** of the domain of f

ii. **critical points** (interior points where $f_x = f_y = 0$ or points where f_x or f_y fail to exist).

If the first- and second-order partial derivatives of f are continuous throughout a disk centered at a point (a, b) and $f_x(a, b) = f_y(a, b) = 0$, the nature of $f(a, b)$ can be tested with the **Second Derivative Test**:

i. $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local maximum**

ii. $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$ at (a, b) $\Rightarrow$ **local minimum**

iii. $f_{xx}f_{yy} - f_{xy}^2 < 0$ at (a, b) $\Rightarrow$ **saddle point**

iv. $f_{xx}f_{yy} - f_{xy}^2 = 0$ at (a, b) $\Rightarrow$ **test is inconclusive.**

拉格朗日乘数法原理

THEOREM 12—The Orthogonal Gradient Theorem Suppose that $f(x, y, z)$ is differentiable in a region whose interior contains a smooth curve

函数表达式

$$C: \quad \mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}.$$

限制条件

If P_0 is a point on C where f has a local maximum or minimum relative to its values on C, then ∇f is orthogonal to C at P_0.

COROLLARY At the points on a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$ where a differentiable function $f(x, y)$ takes on its local maxima and minima relative to its values on the curve, $\nabla f \cdot \mathbf{r}' = 0$.

拉格朗日乘数法

SUSTech Southern University of Science and Technology

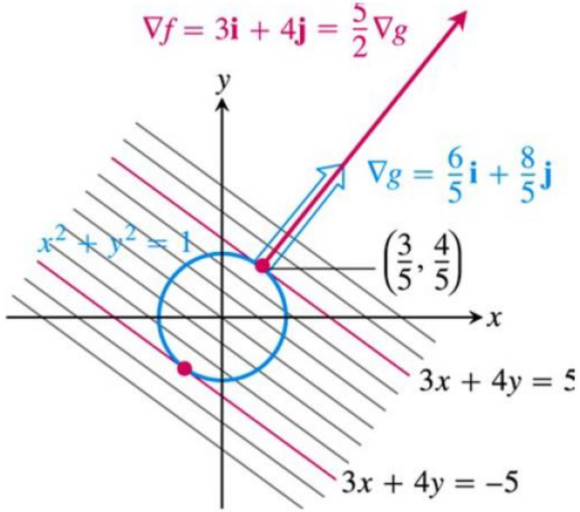
The Method of Lagrange Multipliers

Suppose that $f(x, y, z)$ and $g(x, y, z)$ are differentiable and $\nabla g \neq \mathbf{0}$ when $g(x, y, z) = 0$. To find the local maximum and minimum values of f subject to the constraint $g(x, y, z) = 0$ (if these exist), find the values of x, y, z, and λ that simultaneously satisfy the equations

$$\nabla f = \lambda \nabla g \qquad \text{and} \qquad g(x, y, z) = 0. \tag{1}$$

For functions of two independent variables, the condition is similar, but without the variable z.

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \qquad g_1(x, y, z) = 0, \qquad g_2(x, y, z) = 0$$



EXAMPLE 4 Find the maximum and minimum values of the function $f(x, y) = 3x + 4y$ on the circle $x^2 + y^2 = 1$.

二元函数泰勒展开

Taylor's Formula for $f(x, y)$ at the Point (a, b)

Suppose $f(x, y)$ and its partial derivatives through order $n + 1$ are continuous throughout an open rectangular region R centered at a point (a, b). Then, throughout R,

$$
\begin{aligned}
f(a+h, b+k) = {} & f(a, b) + (hf_x + kf_y)\big|_{(a,b)} + \frac{1}{2!}(h^2 f_{xx} + 2hkf_{xy} + k^2 f_{yy})\big|_{(a,b)} \\
& + \frac{1}{3!}(h^3 f_{xxx} + 3h^2 k f_{xxy} + 3hk^2 f_{xyy} + k^3 f_{yyy})\big|_{(a,b)} + \cdots + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f\bigg|_{(a,b)} \\
& + \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f\bigg|_{(a+ch,\, b+ck)}.
\end{aligned}
\tag{7}
$$

Taylor's Formula for $f(x, y)$ at the Origin

$$
\begin{aligned}
f(x, y) = {} & f(0, 0) + xf_x + yf_y + \frac{1}{2!}(x^2 f_{xx} + 2xyf_{xy} + y^2 f_{yy}) \\
& + \frac{1}{3!}(x^3 f_{xxx} + 3x^2 y f_{xxy} + 3xy^2 f_{xyy} + y^3 f_{yyy}) + \cdots + \frac{1}{n!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^n f \\
& + \frac{1}{(n+1)!}\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)^{n+1} f\bigg|_{(cx,\, cy)}
\end{aligned}
\tag{8}
$$

Chapter15

多重积分

二重积分计算 (直角坐标下)

THEOREM 1—Fubini's Theorem (First Form) If $f(x, y)$ is continuous throughout the rectangular region R: $a \le x \le b$, $c \le y \le d$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_a^b f(x, y)\, dx\, dy = \int_a^b \int_c^d f(x, y)\, dy\, dx.$$

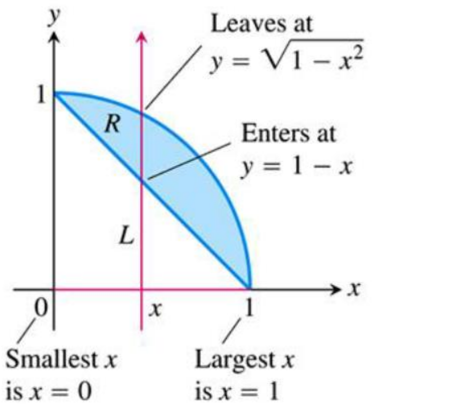
THEOREM 2—Fubini's Theorem (Stronger Form) Let $f(x, y)$ be continuous on a region R.

1. If R is defined by $a \le x \le b$, $g_1(x) \le y \le g_2(x)$, with g_1 and g_2 continuous on $[a, b]$, then

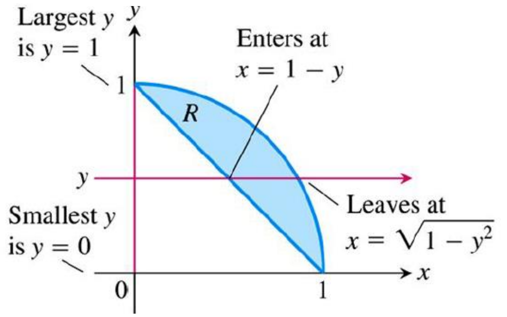
$$\iint_R f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx.$$

2. If R is defined by $c \le y \le d$, $h_1(y) \le x \le h_2(y)$, with h_1 and h_2 continuous on $[c, d]$, then

$$\iint_R f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy.$$



Using Vertical Cross-sections



Using Horizontal Cross-sections

二重积分计算 (极坐标下)

Fubini's Theorem says that the limit approached by these sums can be evaluated by repeated single integrations with respect to r and θ as

$$\iint_R f(r,\theta)\,dA = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=g_1(\theta)}^{r=g_2(\theta)} f(r,\theta)\,r\,dr\,d\theta.$$

适用极坐标计算的情形：

积分区域：圆域，圆环或扇形区域

被积函数：$f(x, y)$ 中含有(x^2+y^2)的因式

二重积分性质

If $f(x, y)$ and $g(x, y)$ are continuous on the bounded region R, then the following properties hold.

1. *Constant Multiple:* $\iint_R cf(x, y)\, dA = c \iint_R f(x, y)\, dA \quad$ (any number c)

2. *Sum and Difference:*

$$\iint_R (f(x, y) \pm g(x, y))\, dA = \iint_R f(x, y)\, dA \pm \iint_R g(x, y)\, dA$$

3. *Domination:*

(a) $\iint_R f(x, y)\, dA \geq 0 \quad$ if $\quad f(x, y) \geq 0$ on R

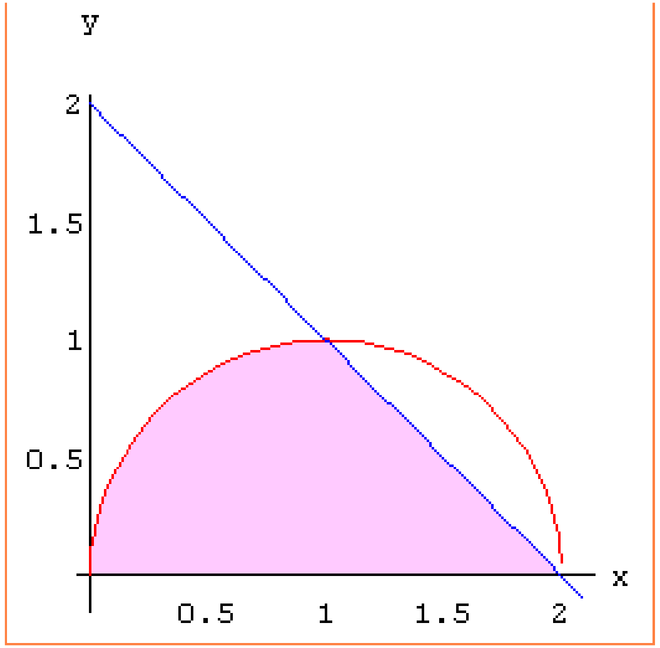
(b) $\iint_R f(x, y)\, dA \geq \iint_R g(x, y)\, dA \quad$ if $\quad f(x, y) \geq g(x, y)$ on R

4. *Additivity:* $\iint_R f(x, y)\, dA = \iint_{R_1} f(x, y)\, dA + \iint_{R_2} f(x, y)\, dA$

if R is the union of two nonoverlapping regions R_1 and R_2

例 改变积分

$\int_0^1 dx \int_0^{\sqrt{2x-x^2}} f(x, y)dy + \int_1^2 dx \int_0^{2-x} f(x, y)dy$的次序.



利用二重积分计算面积

DEFINITION The **area** of a closed, bounded plane region R is

$$A = \iint_R dA.$$

Area in Polar Coordinates

The area of a closed and bounded region R in the polar coordinate plane is

$$A = \iint_R r\,dr\,d\theta.$$

计算函数在一个区间上的均值

$$\textbf{Average value of } f \textbf{ over } R = \frac{1}{\text{area of } R}\iint_R f\,dA. \tag{3}$$

三重积分计算 (直角坐标系下)

$$\iiint_D F(x, y, z)\, dV$$

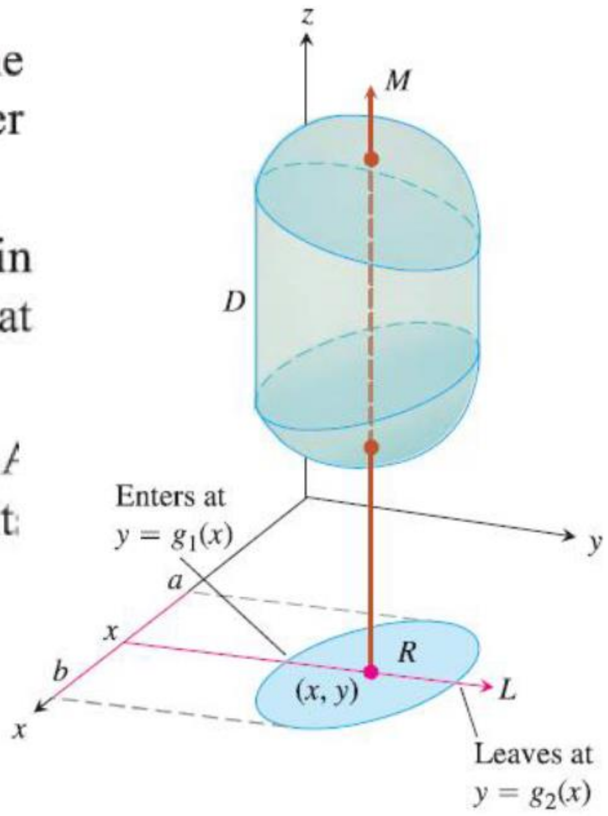
1. *Sketch*. Sketch the region D along with its "shadow" R (vertical projection) in the xy-plane. Label the upper and lower bounding surfaces of D and the upper and lower bounding curves of R.

2. *Find the z-limits of integration*. Draw a line M passing through a typical point (x, y) in R parallel to the z-axis. As z increases, M enters D at $z = f_1(x, y)$ and leaves at $z = f_2(x, y)$. These are the z-limits of integration.

3. *Find the y-limits of integration*. Draw a line L through (x, y) parallel to the y-axis. A increases, L enters R at $y = g_1(x)$ and leaves at $y = g_2(x)$. These are the y-limit integration.

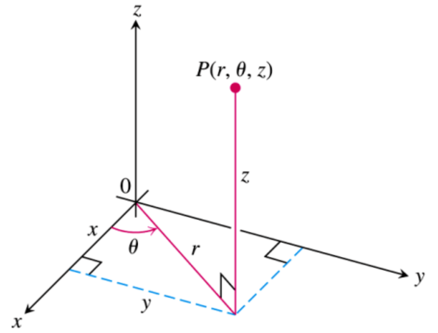
4. *Find the x-limits of integration*. Choose x-limits that include all lines through R parallel to the y-axis ($x = a$ and $x = b$ in the preceding figure). These are the x-limits of integration. The integral is

$$\int_{x=a}^{x=b} \int_{y=g_1(x)}^{y=g_2(x)} \int_{z=f_1(x, y)}^{z=f_2(x, y)} F(x, y, z)\, dz\, dy\, dx.$$



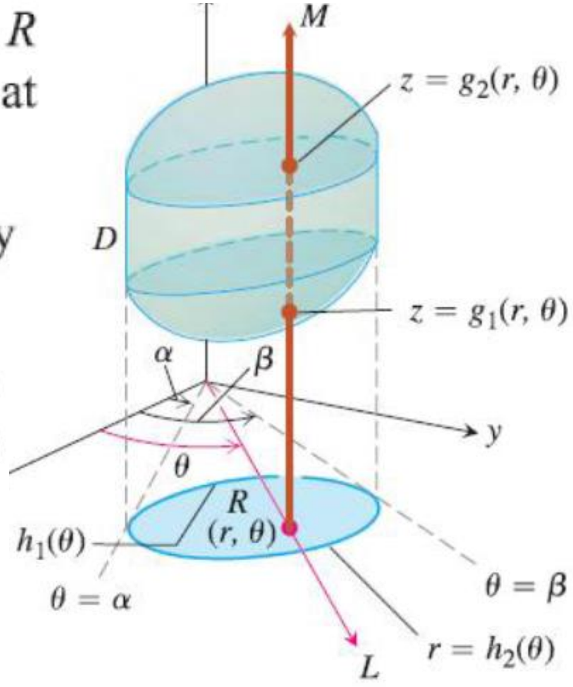
三重积分计算 (柱坐标系下)

$$\lim_{n\to\infty} S_n = \iiint_D f\,dV = \iiint_D f\,dz\,r\,dr\,d\theta.$$

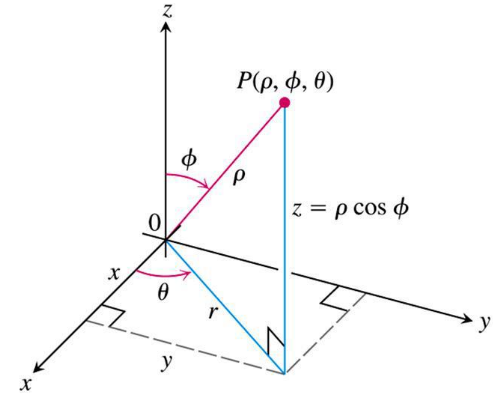


1. *Sketch*. Sketch the region D along with its projection R on the xy-plane. Label the surfaces and curves that bound D and R.
2. *Find the z-limits of integration*. Draw a line M through a typical point (r, θ) of R parallel to the z-axis. As z increases, M enters D at $z = g_1(r, \theta)$ and leaves at $z = g_2(r, \theta)$. These are the z-limits of integration.
3. *Find the r-limits of integration*. Draw a ray L through (r, θ) from the origin. The ray enters R at $r = h_1(\theta)$ and leaves at $r = h_2(\theta)$. These are the r-limits of integration.
4. *Find the θ-limits of integration*. As L sweeps across R, the angle θ it makes with the positive x-axis runs from $\theta = \alpha$ to $\theta = \beta$. These are the θ-limits of integration. The integral is

$$\iiint_D f(r, \theta, z)\,dV = \int_{\theta=\alpha}^{\theta=\beta}\int_{r=h_1(\theta)}^{r=h_2(\theta)}\int_{z=g_1(r,\theta)}^{z=g_2(r,\theta)} f(r, \theta, z)\,dz\,r\,dr\,d\theta.$$



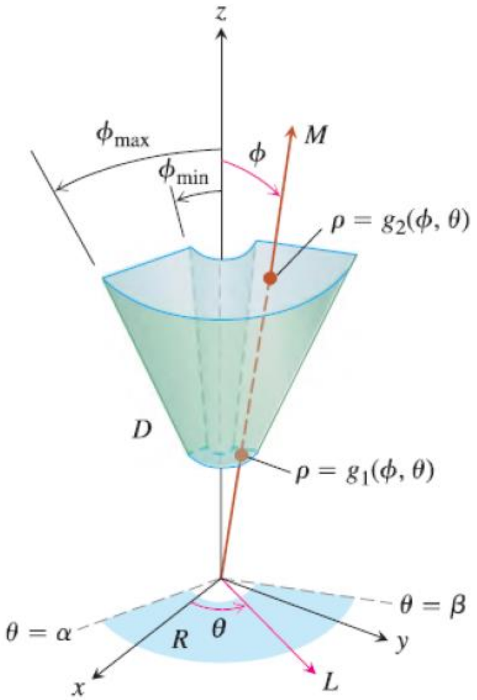
三重积分计算 (球坐标系下)



$$\lim_{n\to\infty} S_n = \iiint_D f(\rho, \phi, \theta)\, dV = \iiint_D f(\rho, \phi, \theta)\, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$

1. *Sketch*. Sketch the region D along with its projection R on the xy-plane. Label the surfaces that bound D.
2. *Find the ρ-limits of integration*. Draw a ray M from the origin through D making an angle ϕ with the positive z-axis. Also draw the projection of M on the xy-plane (call the projection L). The ray L makes an angle θ with the positive x-axis. As ρ increases, M enters D at $\rho = g_1(\phi, \theta)$ and leaves at $\rho = g_2(\phi, \theta)$. These are the ρ-limits of integration.
3. *Find the ϕ-limits of integration*. For any given θ, the angle ϕ that M makes with the z-axis runs from $\phi = \phi_{\min}$ to $\phi = \phi_{\max}$. These are the ϕ -limits of integration.
4. *Find the θ-limits of integration*. The ray L sweeps over R as θ runs from α to β. These are the θ-limits of integration. The integral is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_{\theta=\alpha}^{\theta=\beta} \int_{\phi=\phi_{\min}}^{\phi=\phi_{\max}} \int_{\rho=g_1(\phi, \theta)}^{\rho=g_2(\phi, \theta)} f(\rho, \phi, \theta)\, \rho^2 \sin \phi \, d\rho \, d\phi \, d\theta.$$



三重积分习题

例　　计算 $I=\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$

Ω: $z=xy$ 与 $x+y=1, z=0$ 所围成的区域

例　　计算 $I=\iiint\limits_{\Omega} f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z$

Ω: 抛物柱面 $y=\sqrt{x}$ 与平面 $y=0, z=0, x+z=\dfrac{\pi}{2}$ 所围成的区域。

利用三重积分计算体积

DEFINITION The **volume** of a closed, bounded region D in space is

$$V = \iiint_D dV.$$

Corresponding formulas for dV in triple integrals:

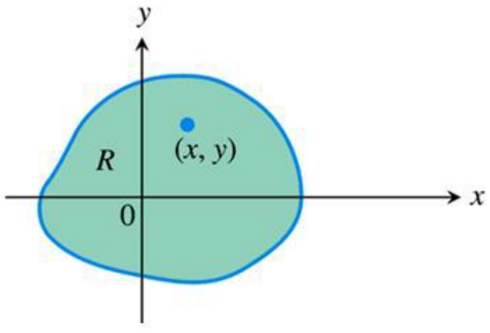
$$\begin{aligned} dV &= dx\,dy\,dz \\ &= dz\,r\,dr\,d\theta \\ &= \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta \end{aligned}$$

计算函数在一个区间上的均值

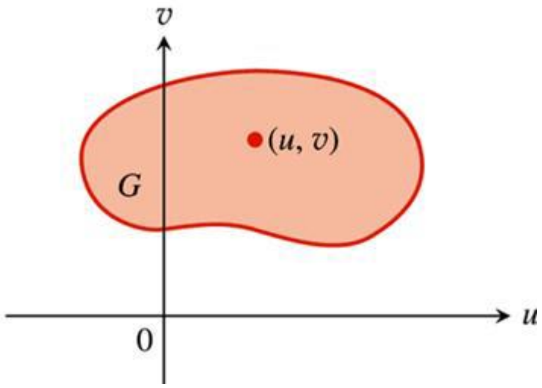
$$\textbf{Average value of } F \text{ over } D = \frac{1}{\text{volume of } D} \iiint_D F\,dV.$$

二重积分中换元

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(g(u, v), h(u, v))\,|J(u, v)|\, du\, dv.$$



Cartesian xy-plane



DEFINITION The **Jacobian determinant** or **Jacobian** of the coordinate transformation $x = g(u, v), y = h(u, v)$ is

$$J(u, v) = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}. = \frac{\partial(x, y)}{\partial(u, v)} \qquad (2)$$

THEOREM 3—Substitution for Double Integrals Suppose that $f(x, y)$ is continuous over the region R. Let G be the preimage of R under the transformation $x = g(u, v), y = h(u, v)$, assumed to be one-to-one on the interior of G. If the functions g and h have continuous first partial derivatives within the interior of G, then

$$\iint_R f(x, y)\, dx\, dy = \iint_G f(g(u, v), h(u, v)) \left|\frac{\partial(x, y)}{\partial(u, v)}\right| du\, dv. \qquad (2)$$

三重积分中换元

$$\iiint_D F(x, y, z)\, dx\, dy\, dz = \iiint_G H(u, v, w)\,|J(u, v, w)|\, du\, dv\, dw.$$

柱坐标变换时： $x = r\cos\theta,\quad y = r\sin\theta,\quad z = z$

三重积分雅可比行列式：

$$J(u, v, w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{\partial(x, y, z)}{\partial(u, v, w)}.$$

$$J(r, \theta, z) = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial z} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

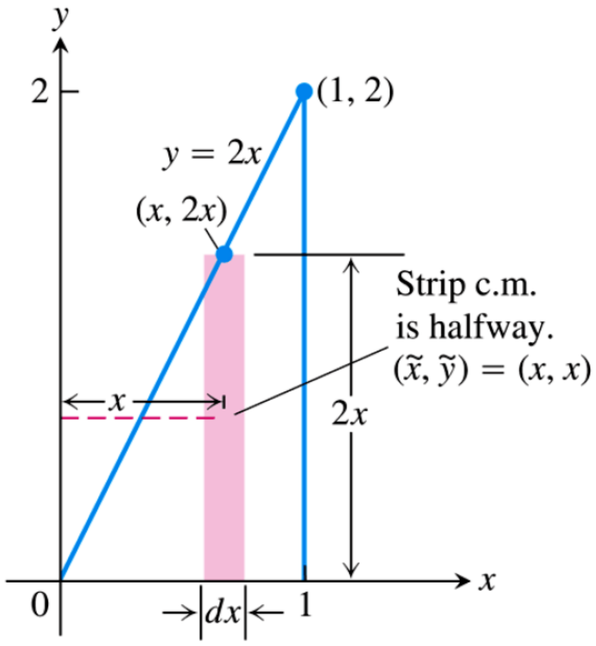
球坐标变换时： $x = \rho\sin\phi\cos\theta,\quad y = \rho\sin\phi\sin\theta,\quad z = \rho\cos\phi$

$$J(\rho, \phi, \theta) = \begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix} = \rho^2\sin\phi.$$

求质心 *

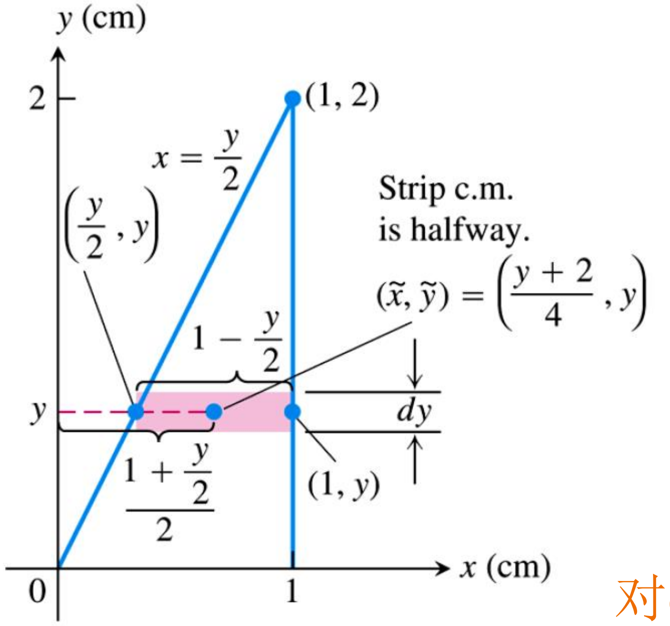
Method 1: Vertical Strips

对dx积分



Units in centimeters

Method 2: Horizontal Strips



对dy积分

推导：

$$\bar{x} = \frac{1}{M}\int_a^b \delta x\,[f(x) - g(x)]\,dx \qquad (6)$$

$$\bar{y} = \frac{1}{M}\int_a^b \frac{\delta}{2}\,[f^2(x) - g^2(x)]\,dx \qquad (7)$$

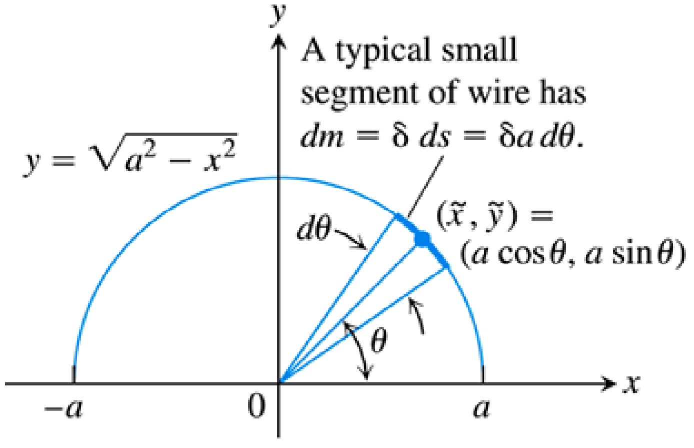
COM = Centroids ?

When the density function is constant, engineers may call the center of mass the **centroid**

对于三角形：三条中位线的交点

EXAMPLE 4 Find the center of mass (centroid) of a thin wire of constant density δ shaped like a semicircle of radius a.

利用极坐标系去求重心(质心)



(a)

length: $ds = a\, d\theta$

mass: $dm = \delta\, ds = \delta a\, d\theta$

distance of c.m. to x-axis: $\tilde{y} = a \sin\theta.$

求质量与一次矩

TABLE 15.1 Mass and first moment formulas

Three-Dimensional Solid

Mass: $M = \iiint_D \delta dV$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x\delta dV, \qquad M_{xz} = \iiint_D y\delta dV, \qquad M_{xy} = \iiint_D z\delta dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

当密度是一个常值函数时

COM 就是这个物体的 Centroid

Two-Dimensional Plate

Mass: $M = \iint_R \delta dA$ $\quad \delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint_R x\delta dA, \; M_x = \iint_R y\delta dA$

Center of mass:

$$\bar{x} = \frac{M_y}{M}, \qquad \bar{y} = \frac{M_x}{M}$$

Chapter16

向量场积分

一型曲线积分的计算（标量）

DEFINITION If f is defined on a curve C given parametrically by $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$, $a \le t \le b$, then the **line integral of f over C** is

$$\int_C f(x, y, z)\, ds = \lim_{n \to \infty} \sum_{k=1}^{n} f(x_k, y_k, z_k)\, \Delta s_k, \qquad (1)$$

provided this limit exists.

How to Evaluate a Line Integral

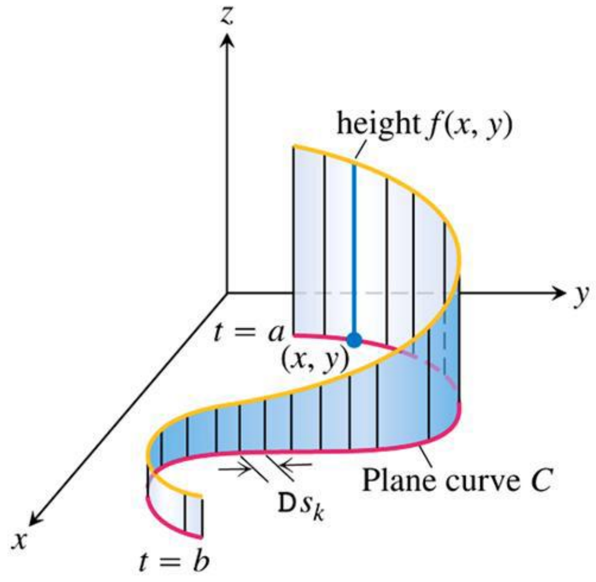
To integrate a continuous function $f(x, y, z)$ over a curve C:

1. Find a smooth parametrization of C,

$$\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}, \qquad a \le t \le b.$$

2. Evaluate the integral as

$$\int_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))\, |\mathbf{v}(t)|\, dt.$$

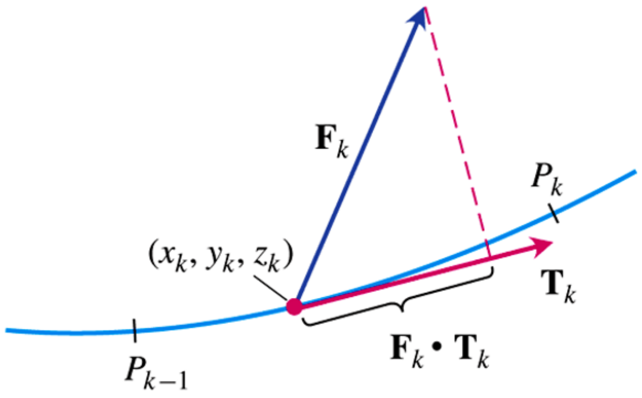


一型曲线积分的几何意义：Wall

二型曲线积分的计算 (矢量场)

DEFINITION Let $\mathbf{F}$ be a vector field with continuous components defined along a smooth curve C parametrized by $\mathbf{r}(t)$, $a \le t \le b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F}\cdot\mathbf{T}\,ds = \int_C \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\right)ds = \int_C \mathbf{F}\cdot d\mathbf{r}.$$



Evaluating the Line Integral of $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ along C: $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$

1. Express the vector field $\mathbf{F}$ in terms of the parametrized curve C as $\mathbf{F}(\mathbf{r}(t))$ by substituting the components $x = g(t)$, $y = h(t)$, $z = k(t)$ of $\mathbf{r}$ into the scalar components $M(x, y, z)$, $N(x, y, z)$, $P(x, y, z)$ of $\mathbf{F}$.
2. Find the derivative (velocity) vector $d\mathbf{r}/dt$.
3. Evaluate the line integral with respect to the parameter t, $a \le t \le b$, to obtain

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t))\cdot\frac{d\mathbf{r}}{dt}\,dt.$$

线积分的性质与应用（矢量场）

在 x y z 轴上做线积分：

$$\int_C M(x, y, z)\, dx = \int_a^b M(g(t), h(t), k(t))\, g'(t)\, dt \qquad (1)$$

$$\int_C N(x, y, z)\, dy = \int_a^b N(g(t), h(t), k(t))\, h'(t)\, dt \qquad (2)$$

$$\int_C P(x, y, z)\, dz = \int_a^b P(g(t), h(t), k(t))\, k'(t)\, dt \qquad (3)$$

将组合形式的先积分可写为：

$$\int_C M(x, y, z)\, dx + \int_C N(x, y, z)\, dy + \int_C P(x, y, z)\, dz = \int_C M\, dx + N\, dy + P\, dz.$$

做功

DEFINITION Let C be a smooth curve parametrized by $\mathbf{r}(t)$, $a \le t \le b$, and $\mathbf{F}$ be a continuous force field over a region containing C. Then the **work** done in moving an object from the point $A = \mathbf{r}(a)$ to the point $B = \mathbf{r}(b)$ along C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt. \qquad (4)$$

流量积分与环量

DEFINITIONS If $\mathbf{r}(t)$ parametrizes a smooth curve C in the domain of a continuous velocity field $\mathbf{F}$, the **flow** along the curve from $A = \mathbf{r}(a)$ to $B = \mathbf{r}(b)$ is

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds. \tag{5}$$

The integral in this case is called a **flow integral.** If the curve starts and ends at the same point, so that $A = B$, the flow is called the **circulation** around the curve.

更常用的计算方法：

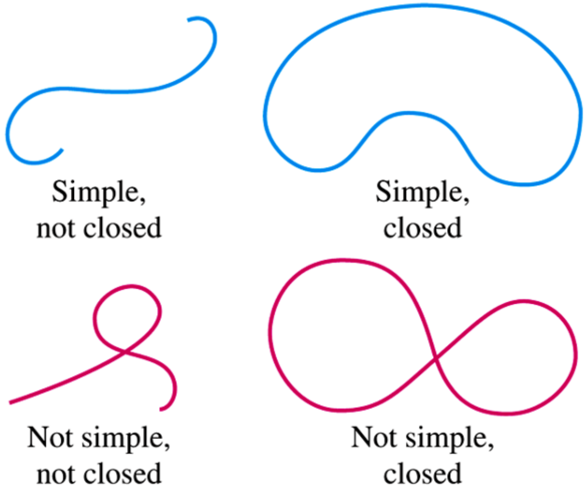
$$\text{Flow} = \int_C M\,dx + N\,dy + P\,dz$$

矢量场沿路径方向线积分的物理意义

通量

DEFINITION If C is a smooth simple closed curve in the domain of a continuous vector field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the plane, and if $\mathbf{n}$ is the outward-pointing unit normal vector on C, the **flux** of $\mathbf{F}$ across C is

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \mathbf{n}\, ds.$$



Calculating Flux Across a Smooth Closed Plane Curve

更常用的计算方法： $$(\text{Flux of } \mathbf{F} = M\mathbf{i} + N\mathbf{j} \text{ across } C) = \oint_C M\,dy - N\,dx$$

The integral can be evaluated from any smooth parametrization $x = g(t), y = h(t)$, $a \le t \le b$, that traces C counterclockwise exactly once.

矢量场垂直路径方向线积分的物理意义

保守场与势函数

DEFINITIONS Let $\mathbf{F}$ be a vector field defined on an open region D in space, and suppose that for any two points A and B in D the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ along a path C from A to B in D is the same over all paths from A to B. Then the integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is **path independent in D** and the field $\mathbf{F}$ is **conservative on D**.

DEFINITION If $\mathbf{F}$ is a vector field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a **potential function for F**.

梯度场是保守的，保守场必然有势函数。

梯度相关定理

THEOREM 1—Fundamental Theorem of Line Integrals Let C be a smooth curve joining the point A to the point B in the plane or in space and parametrized by $\mathbf{r}(t)$. Let f be a differentiable function with a continuous gradient vector $\mathbf{F} = \nabla f$ on a domain D containing C. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A).$$

THEOREM 2—Conservative Fields are Gradient Fields Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components are continuous throughout an open connected region D in space. Then $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ is a gradient field ∇f for a differentiable function f.

THEOREM 3—Loop Property of Conservative Fields The following statements are equivalent.

1. $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ around every loop (that is, closed curve C) in D.
2. The field $\mathbf{F}$ is conservative on D.

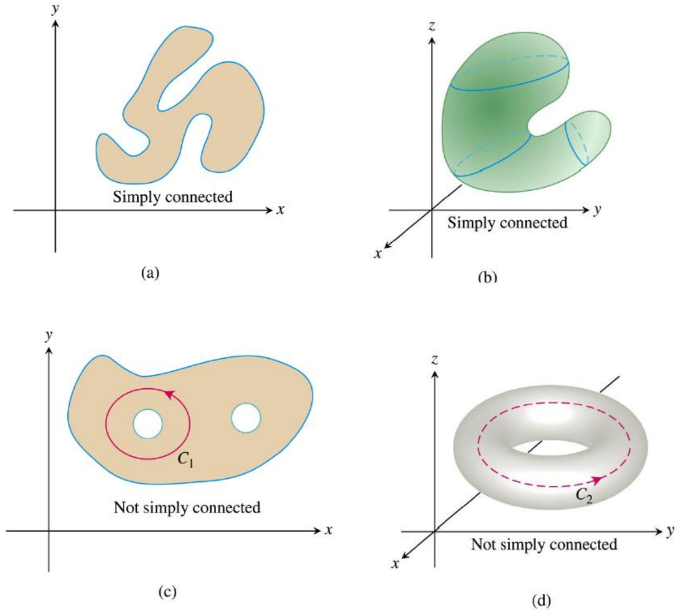
保守场的判别

Component Test for Conservative Fields

Let $\mathbf{F} = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$ be a field on a connected and simply connected domain whose component functions have continuous first partial derivatives. Then, $\mathbf{F}$ is conservative if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}. \tag{2}$$

the Component Test does not apply when the domain of the field is not simply connected.



判断出保守并找出势函数：

$$\frac{\partial f}{\partial x} = M, \qquad \frac{\partial f}{\partial y} = N, \qquad \frac{\partial f}{\partial z} = P,$$

全微分

DEFINITIONS Any expression $M(x, y, z)\, dx + N(x, y, z)\, dy + P(x, y, z)\, dz$ is a **differential form**. A differential form is **exact** on a domain D in space if

$$M\, dx + N\, dy + P\, dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = df$$

for some scalar function f throughout D.

Component Test for Exactness of $M\, dx + N\, dy + P\, dz$

The differential form $M\, dx + N\, dy + P\, dz$ is exact on a connected and simply connected domain if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z}, \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x}, \qquad \text{and} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

This is equivalent to saying that the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is conservative.

计算曲线积分

$$\int_L (1 - 2xy - y^2)dx - (x + y)^2 dy,$$

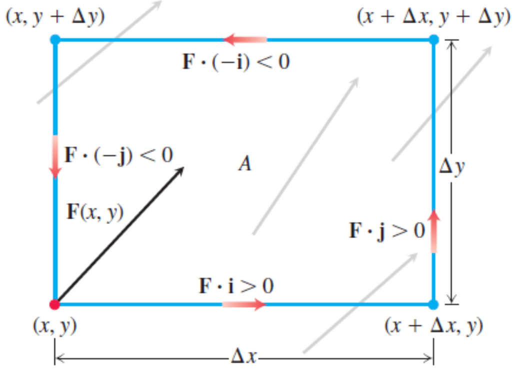
$x^2 + y^2 = 2y$ 上从(0,0)至（1,1)的一段有向弧

环量密度(旋度)与通量密度(散度)

DEFINITION The **circulation density** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar expression

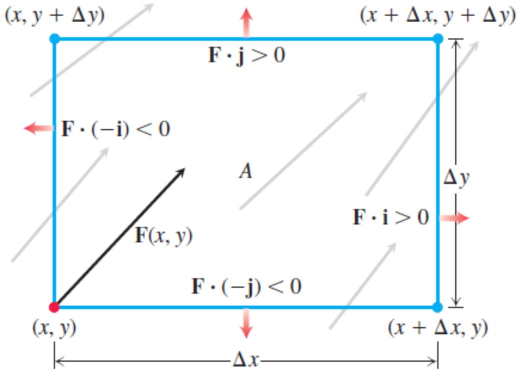
$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}. \tag{2}$$

This expression is also called **the k-component of the curl,** denoted by $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$.



DEFINITION The **divergence (flux density)** of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\text{div } \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}. \tag{1}$$



格林公式 (Green Theorem)

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THEOREM 4—Green's Theorem (Flux-Divergence or Normal Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\mathbf{F}$ across C equals the double integral of div $\mathbf{F}$ over the region R enclosed by C.

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{n}\,ds}_{\text{Outward flux}} = \oint_C M\,dy - N\,dx = \underbrace{\iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}_{\text{Divergence integral}} \tag{3}$$

THEOREM 5—Green's Theorem (Circulation-Curl or Tangential Form) Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of $(\text{curl } \mathbf{F}) \cdot \mathbf{k}$ over R.

$$\underbrace{\oint_C \mathbf{F}\cdot\mathbf{T}\,ds}_{\text{Counterclockwise circulation}} = \oint_C M\,dx + N\,dy = \underbrace{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}_{\text{Curl integral}} \tag{4}$$

曲面面积的计算

DEFINITION The **area** of the smooth surface

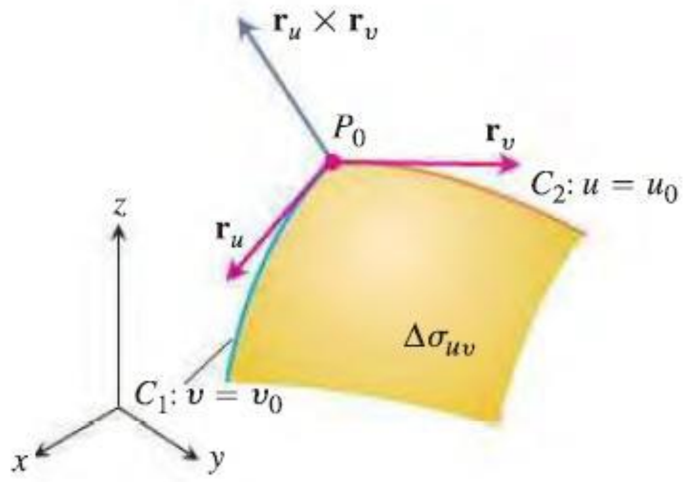
$$\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}, \qquad a \le u \le b, \quad c \le v \le d$$

is

$$A = \iint_R |\mathbf{r}_u \times \mathbf{r}_v|\, dA = \int_c^d \int_a^b |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \tag{4}$$

Surface Area Differential for a Parametrized Surface

$$\underbrace{d\sigma = |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv}_{\text{Surface area differential}} \qquad \underbrace{\iint_S d\sigma}_{\text{Differential formula for surface area}} \tag{5}$$



Formula for the Surface Area of a Graph $z = f(x, y)$

For a graph $z = f(x, y)$ over a region R in the xy-plane, the surface area formula is

$$A = \iint_R \sqrt{f_x^2 + f_y^2 + 1}\, dx\, dy. \tag{8}$$

一型曲面积分的计算（标量）

1. For a smooth surface S defined **parametrically** as $\mathbf{r}(u, v) = f(u, v)\mathbf{i} + g(u, v)\mathbf{j} + h(u, v)\mathbf{k}$, $(u, v) \in R$, and a continuous function $G(x, y, z)$ defined on S, the surface integral of G over S is given by the double integral over R,

$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(f(u, v), g(u, v), h(u, v))\, |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv. \qquad (2)$$

2. For a surface S given **explicitly** as the graph of $z = f(x, y)$, where f is a continuously differentiable function over a region R in the xy-plane, the surface integral of the continuous function G over S is given by the double integral over R,

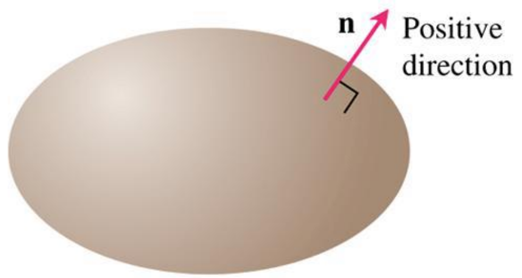
$$\iint_S G(x, y, z)\, d\sigma = \iint_R G(x, y, f(x, y)) \sqrt{f_x^{\,2} + f_y^{\,2} + 1}\, dx\, dy. \qquad (4)$$

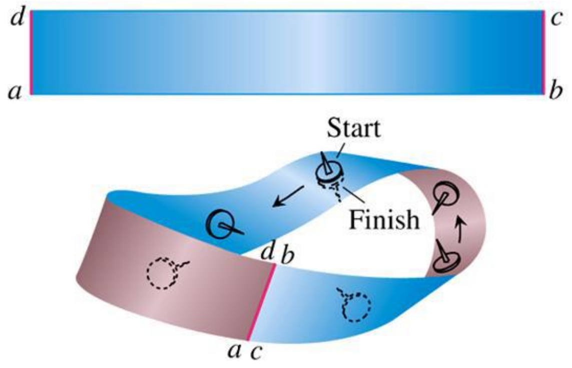
二型曲面积分的计算 (矢量)

DEFINITION Let $\mathbf{F}$ be a vector field in three-dimensional space with continuous components defined over a smooth surface S having a chosen field of normal unit vectors $\mathbf{n}$ orienting S. Then the **surface integral of F over** S is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma. \qquad \mathbf{F} \cdot d\boldsymbol{\sigma} \tag{5}$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R \mathbf{F} \cdot \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} |\mathbf{r}_u \times \mathbf{r}_v|\, du\, dv = \iint_R \mathbf{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, du\, dv.$$



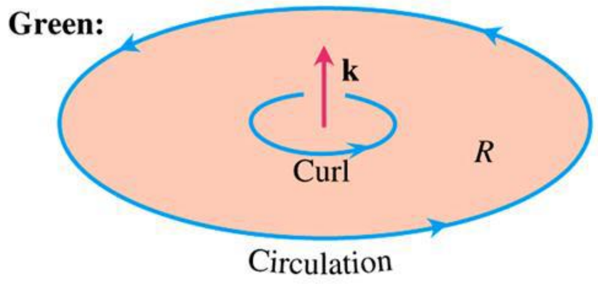


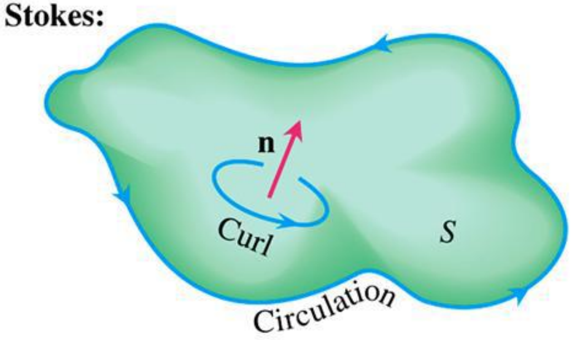
矢量场曲面积分的物理意义

斯托克斯公式 (Stokes Theorem)

THEOREM 6—Stokes' Theorem Let S be a piecewise smooth oriented surface having a piecewise smooth boundary curve C. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ be a vector field whose components have continuous first partial derivatives on an open region containing S. Then the circulation of $\mathbf{F}$ around C in the direction counterclockwise with respect to the surface's unit normal vector $\mathbf{n}$ equals the integral of $\nabla \times \mathbf{F} \cdot \mathbf{n}$ over S.

$$\underbrace{\oint_C \mathbf{F} \cdot d\mathbf{r}}_{\text{Counterclockwise circulation}} = \underbrace{\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma}_{\text{Curl integral}}$$





三维场旋度的定义

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$$

梯度的旋度为零

$$\text{curl grad } f = \mathbf{0} \qquad \text{or} \qquad \nabla \times \nabla f = \mathbf{0}$$

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = (f_{zy} - f_{yz})\mathbf{i} - (f_{zx} - f_{xz})\mathbf{j} + (f_{yx} - f_{xy})\mathbf{k}.$$

$$\begin{array}{ccc} \mathbf{F} \text{ conservative on } D & \Longleftrightarrow & \mathbf{F} = \nabla f \text{ on } D \\ \Updownarrow & & \Downarrow \\ \displaystyle\oint_C \mathbf{F} \cdot d\mathbf{r} = 0 & \Longleftarrow & \nabla \times \mathbf{F} = \mathbf{0} \text{ throughout } D \end{array}$$

over any closed path in D

Domain's simple connectivity and Stokes' Theorem

THEOREM 7—Curl $\mathbf{F} = \mathbf{0}$ Related to the Closed-Loop Property If $\nabla \times \mathbf{F} = \mathbf{0}$ at every point of a simply connected open region D in space, then on any piecewise-smooth closed path C in D,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

高斯公式

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THEOREM 8—Divergence Theorem Let $\mathbf{F}$ be a vector field whose components have continuous first partial derivatives, and let S be a piecewise smooth oriented closed surface. The flux of $\mathbf{F}$ across S in the direction of the surface's outward unit normal field $\mathbf{n}$ equals the integral of $\nabla \cdot \mathbf{F}$ over the region D enclosed by the surface:

$$\underset{S}{\iint} \mathbf{F} \cdot \mathbf{n}\, d\sigma = \underset{D}{\iiint} \nabla \cdot \mathbf{F}\, dV. \tag{2}$$

Outward flux　　Divergence integral

三维场散度的定义　$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}.$$

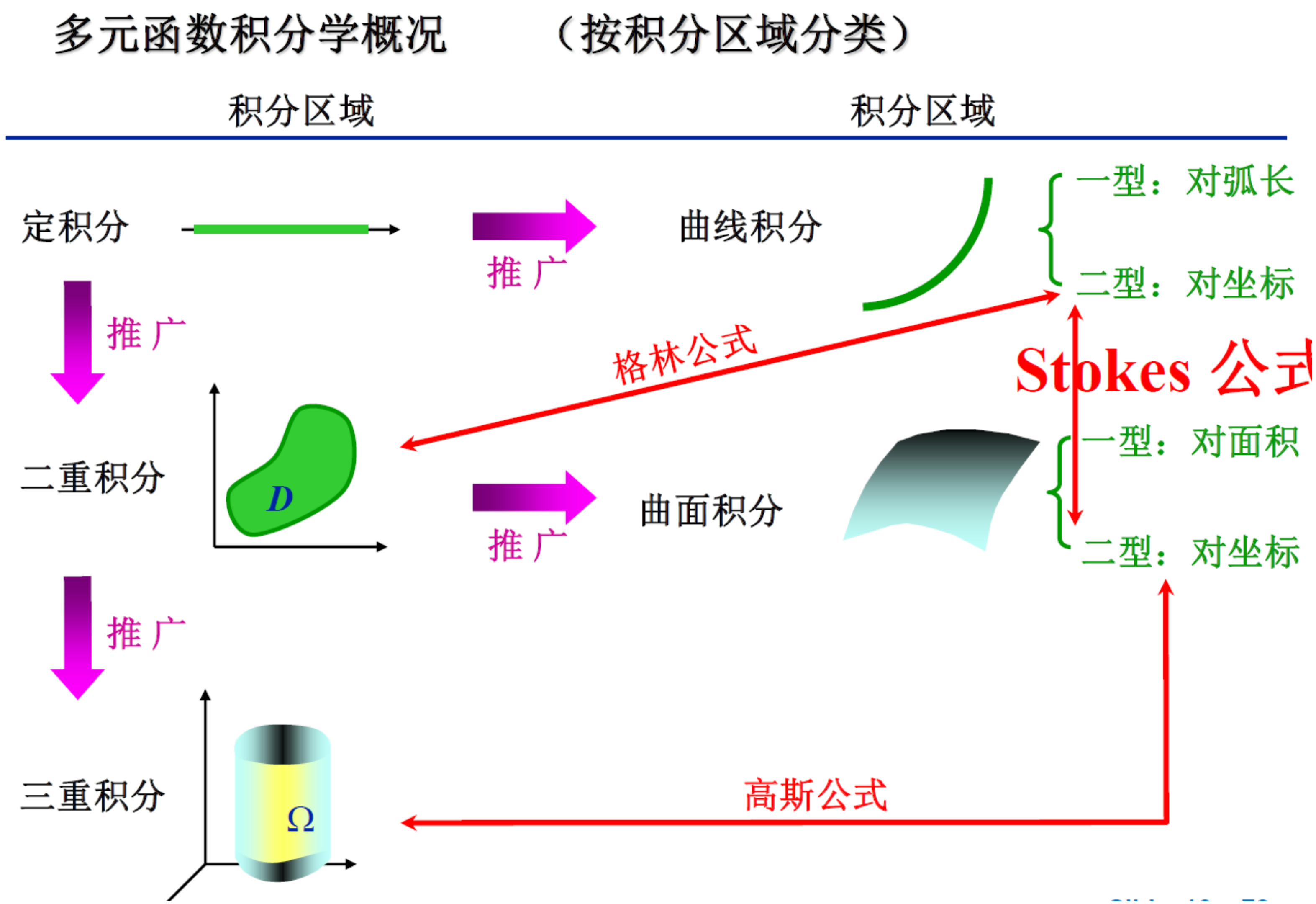
旋度的散度为零

THEOREM 9 If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field with continuous second partial derivatives, then

$$\operatorname{div}(\operatorname{curl}\mathbf{F}) = \nabla\cdot(\nabla\times\mathbf{F}) = 0.$$

$$\begin{aligned}
\operatorname{div}(\operatorname{curl}\mathbf{F}) &= \nabla\cdot(\nabla\times\mathbf{F}) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial P}{\partial y}-\frac{\partial N}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial M}{\partial z}-\frac{\partial P}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right) \\
&= \frac{\partial^2 P}{\partial x\partial y}-\frac{\partial^2 N}{\partial x\partial z}+\frac{\partial^2 M}{\partial y\partial z}-\frac{\partial^2 P}{\partial y\partial x}+\frac{\partial^2 N}{\partial z\partial x}-\frac{\partial^2 M}{\partial z\partial y} \\
&= 0,
\end{aligned}$$

总结



多元函数积分学概况 （按积分区域分类）
积分区域
积分区域
定积分
推 广
曲线积分
一型：对弧长
二型：对坐标
推 广
格林公式
Stokes 公式
二重积分
D
推 广
曲面积分
一型：对面积
二型：对坐标
推 广
三重积分
Ω
高斯公式

Q&A

祝大家期末顺利！