SUSTech Southern University of Science and Technology

高数期中复习

致诚书院学长学姐带飞课

Chapter 10 - Chapter 14

By 苏宇鹏 3.18 2025

高数带飞课提纲一览（复习专用）

- Ch10 无限数列与无穷级数
  - (数列极限定义 a_n) & (数列极限计算 @ 运算法则 & 三明治 & 重要极限)
  - (数列的形式 @ 迭代型/递归型) & (数列的性质 @ 有界性/单调性)
  - (无穷级数定义 S_n) & (无穷级数计算 @ (好求级数 @ 几何级数 & p级数) & 运算法则)
  - 级数敛散性判别 (正项级数) @ 直接判别 & 必要条件 & 积分判别 & 比较判别
  - 级数敛散性判别 @ (绝对收敛 @ ratio判别 & root判别) & (条件收敛 @ 莱布尼茨判别)
  - (幂级数 @ 收敛定理 & 运算法则 & 微/积分敛散性) & 二次项级数
  - 泰勒级数 @ 泰勒/麦克劳林级数 & 泰勒展开求极限 & 间接法求泰勒级数(五)
- Ch11 参数方程与极坐标
  - 参数方程 @ 定义 & 微积分 & 求弧长 & 求旋转体表面积
  - 极坐标 @ 定义 & 微积分 & 求扇形面积 & 求弧长

Chapter10

无限数列与无穷级数

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数列极限的定义

DEFINITIONS The sequence $\{a_n\}$ **converges** to the number L if for every positive number ϵ there corresponds an integer N such that for all n,

$$n > N \quad \Rightarrow \quad |a_n - L| < \epsilon.$$

If no such number L exists, we say that $\{a_n\}$ **diverges**.

If $\{a_n\}$ converges to L, we write $\lim_{n\to\infty} a_n = L$, or simply $a_n \to L$, and call L the **limit** of the sequence (Figure 10.2).

DEFINITION The sequence $\{a_n\}$ **diverges to infinity** if for every number M there is an integer N such that for all n larger than N, $a_n > M$. If this condition holds we write

$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty.$$

Similarly if for every number m there is an integer N such that for all $n > N$ we have $a_n < m$, then we say $\{a_n\}$ **diverges to negative infinity** and write

$$\lim_{n\to\infty} a_n = -\infty \quad \text{or} \quad a_n \to -\infty.$$

极限与初始项无关！

1. 数列极限的定义未给出求极限的方法；
2. 极限的精确定义不做要求。

(a) $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$

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数列极限的运算法则

THEOREM 1 Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers, and let A and B be real numbers. The following rules hold if $\lim_{n\to\infty} a_n = A$ and $\lim_{n\to\infty} b_n = B$.

1. *Sum Rule:* $\lim_{n\to\infty}(a_n + b_n) = A + B$
2. *Difference Rule:* $\lim_{n\to\infty}(a_n - b_n) = A - B$
3. *Constant Multiple Rule:* $\lim_{n\to\infty}(k \cdot b_n) = k \cdot B$ (any number k)
4. *Product Rule:* $\lim_{n\to\infty}(a_n \cdot b_n) = A \cdot B$
5. *Quotient Rule:* $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \dfrac{A}{B}$ if $B \neq 0$

数列极限 VS 函数极限

函数极限的运算法则

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k\cdot f(x)) = k\cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x)\cdot g(x)) = L\cdot M$
5. *Quotient Rule:* $\lim_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

1. 有 + 无 = 无
2. 无 + 无 = 不定
3. 有 * 无 = 不定
4. 无 * 无 = 不定

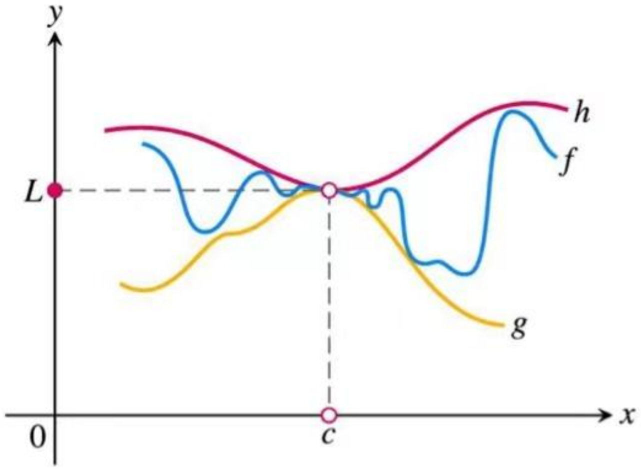
Sandwich Theorem

THEOREM 2—The Sandwich Theorem for Sequences Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences of real numbers. If $a_n \le b_n \le c_n$ holds for all n beyond some index N, and if $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$, then $\lim_{n\to\infty} b_n = L$ also.

(a) $\dfrac{\cos n}{n} \to 0$

(b) $\dfrac{1}{2^n} \to 0$

(c) $(-1)^n \dfrac{1}{n} \to 0$



求 $\lim\limits_{n\to\infty}\left(\dfrac{1}{\sqrt{n^2+1}}+\dfrac{1}{\sqrt{n^2+2}}+\cdots+\dfrac{1}{\sqrt{n^2+n}}\right)$.

数列与函数的关系

THEOREM 3—The Continuous Function Theorem for Sequences Let $\{a_n\}$ be a sequence of real numbers. If $a_n \to L$ and if f is a function that is continuous at L and defined at all a_n, then $f(a_n) \to f(L)$.

可以摆脱外函数的限制，直接求带 n 项的极限。

Show that $\sqrt{(n+1)/n} \to 1$.

THEOREM 4 Suppose that $f(x)$ is a function defined for all $x \geq n_0$ and that $\{a_n\}$ is a sequence of real numbers such that $a_n = f(n)$ for $n \geq n_0$. Then

$$\lim_{x\to\infty} f(x) = L \qquad \Rightarrow \qquad \lim_{n\to\infty} a_n = L.$$

可以使用洛必达法则等求函数极限方法

$$\lim_{n\to\infty} \frac{\ln n}{n} = 0.$$

重要数列极限

THEOREM 5 The following six sequences converge to the limits listed below:

1. $\lim_{n\to\infty} \frac{\ln n}{n} = 0$

2. $\boxed{\lim_{n\to\infty} \sqrt[n]{n} = 1}$

3. $\boxed{\lim_{n\to\infty} x^{1/n} = 1 \qquad (x > 0)}$

4. $\lim_{n\to\infty} x^n = 0 \qquad (|x| < 1)$

5. $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x \qquad (\text{any } x)$

6. $\boxed{\lim_{n\to\infty} \frac{x^n}{n!} = 0} \qquad (\text{any } x)$

In Formulas (3) through (6), x remains fixed as $n \to \infty$.

(b) $\sqrt[n]{n^2}$ (c) $\sqrt[n]{3n}$ (e) $\left(\frac{n-2}{n}\right)^n$ (f) $\frac{100^n}{n!}$

斯特林公式

$$n! \approx \sqrt{2\pi n}\left(\frac{n}{e}\right)^n$$

or

$$\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}\left(\frac{n}{e}\right)^n}=1$$

$$\ln\ln n \ll \ln n \ll n^{\alpha} \ll a^n \ll n! \ll n^n$$

Proof of Stirling's formula requires monotone bounded convergence theorem and

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n=e$$

证明？

沃利斯公式

Find the limit

$$\lim_{n\to\infty}\frac{(2n-1)!!}{(2n)!!}=\lim_{n\to\infty}\frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}$$

Notice that $(2n-1)(2n+1)=4n^2-1<(2n)^2$,

$$[(2n-1)!!]^2(2n+1)=\prod_{k=1}^{n}(2k-1)(2k+1)<\prod_{k=1}^{n}(2k)^2=((2n)!!)^2$$

$$0<\frac{(2n-1)!!}{(2n)!!}<\frac{1}{\sqrt{2n+1}}\to 0$$

$$\lim_{n\to\infty}\left(\frac{(2n)!!}{(2n-1)!!}\right)^2\frac{1}{2n+1}=\frac{\pi}{2}$$

求数列极限（迭代型）

1. 求极限 $\lim\limits_{n\to\infty}\sqrt[n]{1^n+2^n+3^n}$.

2. 设$0\le a_1\le a_2\le\cdots\le a_k$

   则 $\lim\limits_{n\to\infty}\sqrt[n]{a_1^n+a_2^n+\cdots+a_k^n}=a_k$

3. $\lim\limits_{n\to\infty}\sqrt[n]{n^2-n+2}=$

4. $\sum\limits_{n=1}^{\infty}\frac{1}{n(n+1)(n+2)}$ $\sum\limits_{k=1}^{\infty}\frac{1}{k(k+2)(k+6)}$

5. $\lim\limits_{n\to\infty}\left(\sqrt{n+\sqrt{n+2\sqrt{n}}}-\sqrt{n}\right)$

6. $(n!)^{\frac{1}{n^2}}$

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数列的形式

迭代形式 (iterative form) ⟶ 递归形式 (recursive form)

$$\frac{\ln(n^2)}{n} = \frac{2\ln n}{n} \to 2\cdot 0 = 0$$

$$\sqrt[n]{n^2} = n^{2/n} = (n^{1/n})^2 \to (1)^2 = 1$$

$$\sqrt[n]{3n} = 3^{1/n}(n^{1/n}) \to 1\cdot 1 = 1$$

$$\left(-\frac{1}{2}\right)^n \to 0$$

$$\left(\frac{n-2}{n}\right)^n = \left(1 + \frac{-2}{n}\right)^n \to e^{-2}$$

$$\frac{100^n}{n!} \to 0$$

$a_1 = 1$ and $a_n = a_{n-1} + 1$

$a_1 = 1,\ a_2 = 1,$ and $a_{n+1} = a_n + a_{n-1}$

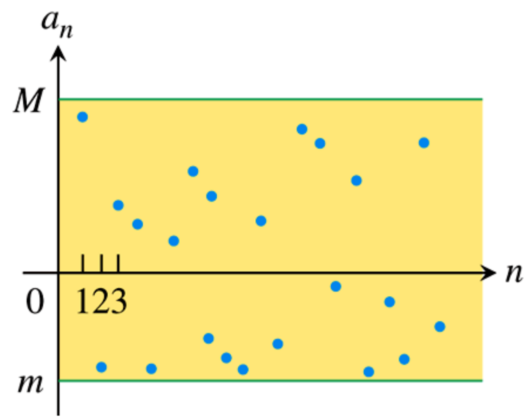
$a_1 = -4,\quad a_{n+1} = \sqrt{8 + 2a_n}$

数列的有界性

DEFINITIONS A sequence $\{a_n\}$ is **bounded from above** if there exists a number M such that $a_n \leq M$ for all n. The number M is an **upper bound** for $\{a_n\}$. If M is an upper bound for $\{a_n\}$ but no number less than M is an upper bound for $\{a_n\}$, then M is the **least upper bound** for $\{a_n\}$.

A sequence $\{a_n\}$ is **bounded from below** if there exists a number m such that $a_n \geq m$ for all n. The number m is a **lower bound** for $\{a_n\}$. If m is a lower bound for $\{a_n\}$ but no number greater than m is a lower bound for $\{a_n\}$, then m is the **greatest lower bound** for $\{a_n\}$.

If $\{a_n\}$ is bounded from above and below, the $\{a_n\}$ is **bounded**. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an **unbounded** sequence.



上界；下界；最小上界；最大上界

数列的单调性

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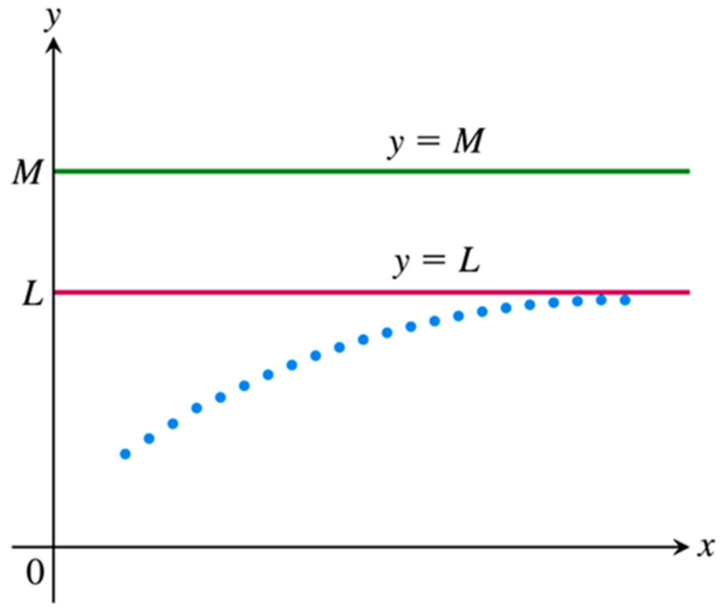
DEFINITIONS A sequence $\{a_n\}$ is **nondecreasing** if $a_n \le a_{n+1}$ for all n. That is, $a_1 \le a_2 \le a_3 \le \ldots$. The sequence is **nonincreasing** if $a_n \ge a_{n+1}$ for all n. The sequence $\{a_n\}$ is **monotonic** if it is either nondecreasing or nonincreasing.

THEOREM 6—The Monotonic Sequence Theorem If a sequence $\{a_n\}$ is both bounded and monotonic, then the sequence converges.

1. 单调有界推出一个数列收敛
2. 数列收敛不能推出单调或是有界
3. 单调增有上界，单调减有下界



例 证明数列 $x_n = \sqrt{3+\sqrt{3+\sqrt{\cdots+\sqrt{3}}}}$ (n重根式)的极限存在.

求数列极限（递归型）

1. Assume that sequence $\{a_n\}$ converges, find its limit.

$$a_1 = -4, \quad a_{n+1} = \sqrt{8 + 2a_n}.$$

2. $x_1 > 0$, $x_{n+1} = 1 + \frac{x_n}{x_n + 1}$. Prove that the sequence $\{x_n\}$ is convergent and find its limit.

3. $a_1 = 3$, $a_{n+1} = 12 - \sqrt{a_n}$. Prove that the sequence $\{a_n\}$ is convergent and find its limit.

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无穷级数的定义

DEFINITIONS Given a sequence of numbers $\{a_n\}$, an expression of the form

$$a_1 + a_2 + a_3 + \cdots + a_n + \cdots$$

is an **infinite series**. The number a_n is the ***n*th term** of the series. The sequence $\{s_n\}$ defined by

$$\begin{aligned} s_1 &= a_1 \\ s_2 &= a_1 + a_2 \\ &\vdots \\ s_n &= a_1 + a_2 + \cdots + a_n = \sum_{k=1}^{n} a_k \\ &\vdots \end{aligned}$$

is the **sequence of partial sums** of the series, the number s_n being the ***n*th partial sum**. If the sequence of partial sums converges to a limit L, we say that the series **converges** and that its **sum** is L. In this case, we also write

$$a_1 + a_2 + \cdots + a_n + \cdots = \sum_{n=1}^{\infty} a_n = L.$$

If the sequence of partial sums of the series does not converge, we say that the series **diverges**.

常数项级数收敛(发散)

$\Downarrow\Uparrow$

$\lim\limits_{n\to\infty} s_n$ 存在(不存在)

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好求级数（直接判别法）

Geometric series are series of the form

$$a + ar + ar^2 + \cdots + ar^{n-1} + \cdots = \sum_{n=1}^{\infty} ar^{n-1}$$

in which a and r are fixed real numbers and $a \neq 0$.

几何级数

If $|r| < 1$, the geometric series $a + ar + ar^2 + \cdots + ar^{n-1} + \cdots$ converges to $a/(1 - r)$:

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

If $|r| \geq 1$, the series diverges.

“裂项”级数

the “telescoping” series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

级数收敛的必要条件（排除法）

THEOREM 7 If $\sum_{n=1}^{\infty} a_n$ converges, then $a_n \to 0$.

Conversely, if $a_n \to 0$, $\sum_{n=1}^{\infty} a_n$ may fail to converge.

调和级数 $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}+\cdots$

1. 如果级数的一般项不趋于零, 则级数发散;
2. 必要条件不充分.

$\sum_{n=1}^{\infty} n^2$ diverges because $n^2 \to \infty$.

$\sum_{n=1}^{\infty} \frac{n+1}{n}$ diverges because $\frac{n+1}{n} \to 1$. $\lim_{n\to\infty} a_n \neq 0$

$\sum_{n=1}^{\infty} (-1)^{n+1}$ diverges because $\lim_{n\to\infty}(-1)^{n+1}$ does not exist.

$\sum_{n=1}^{\infty} \frac{-n}{2n+5}$ diverges because $\lim_{n\to\infty} \frac{-n}{2n+5} = -\frac{1}{2} \neq 0$.

The nth-Term Test for Divergence

$\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n$ fails to exist or is different from zero.

数列级数的运算法则

THEOREM 8 If $\sum a_n = A$ and $\sum b_n = B$ are convergent series, then

1. *Sum Rule:* $\sum(a_n + b_n) = \sum a_n + \sum b_n = A + B$
2. *Difference Rule:* $\sum(a_n - b_n) = \sum a_n - \sum b_n = A - B$
3. *Constant Multiple Rule:* $\sum ka_n = k\sum a_n = kA$ (any number k).

只会用到加减和数乘！

1. Every nonzero constant multiple of a divergent series diverges.
2. If $\sum a_n$ converges and $\sum b_n$ diverges, then $\sum(a_n + b_n)$ and $\sum(a_n - b_n)$ both diverge.

函数极限的运算法则

THEOREM 1—Limit Laws If L, M, c, and k are real numbers and

$$\lim_{x\to c} f(x) = L \quad \text{and} \quad \lim_{x\to c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:* $\lim_{x\to c}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to c}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to c}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to c}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to c}\frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to c}[f(x)]^n = L^n$, n a positive integer
7. *Root Rule:* $\lim_{x\to c}\sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer

(If n is even, we assume that $\lim_{x\to c} f(x) = L > 0$.)

1. 有 + 无 = 无
2. 无 + 无 = 不定
3. 有 * 无 = 不定
4. 无 * 无 = 不定

加减项与重建索引

We can add a finite number of terms to a series or delete a finite number of terms without altering the series' convergence or divergence, although in the case of convergence this will usually change the sum.

$$\sum_{n=1}^{\infty} \frac{1}{5^n} = \frac{1}{5} + \frac{1}{25} + \frac{1}{125} + \sum_{n=4}^{\infty} \frac{1}{5^n}$$

and

$$\sum_{n=4}^{\infty} \frac{1}{5^n} = \left(\sum_{n=1}^{\infty} \frac{1}{5^n} \right) - \frac{1}{5} - \frac{1}{25} - \frac{1}{125}.$$

加减有限的项
不会改变级数敛散性

As long as we preserve the order of its terms, we can reindex any series without altering its convergence.

项的次序不变
不会改变级数敛散性

$$a_1 + a_2 + a_3 + \cdots = \sum_{n=1}^{\infty} a_n = \sum_{n=1+h}^{\infty} a_{n-h} = \sum_{n=1-h}^{\infty} a_{n+h}$$

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  - (数列的形式 @ 迭代型/递归型) & (数列的性质 @ 有界性/单调性)
  - (无穷级数定义 S_n) & (无穷级数计算 @ (好求级数 @ 几何级数 & p级数) & 运算法则)
  - 级数敛散性判别 (正项级数) @ 直接判别 & 必要条件 & 积分判别 & 比较判别
  - 级数敛散性判别 @ (绝对收敛 @ ratio判别 & root判别) & (条件收敛 @ 莱布尼茨判别)
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积分判别法

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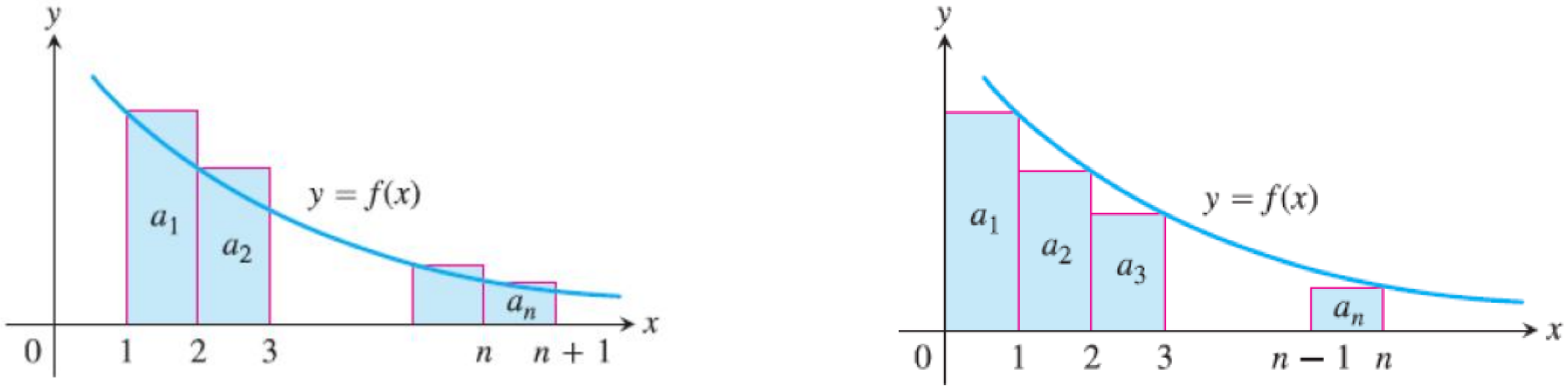
THEOREM 9—The Integral Test Let $\{a_n\}$ be a sequence of positive terms. Suppose that $a_n = f(n)$, where f is a $\boxed{\text{continuous, positive, decreasing}}$ function of x for all $x \geq N$ (N a positive integer). Then the series $\sum_{n=N}^{\infty} a_n$ and the integral $\int_N^{\infty} f(x)\,dx$ both converge or both diverge.

数列级数敛散性 <==> 函数广义积分敛散性

证明？



$$\int_1^{n+1} f(x)\,dx \leq a_1 + a_2 + \cdots + a_n \leq a_1 + \int_1^{n} f(x)\,dx.$$

P级数

EXAMPLE 3 Show that the ***p*-series**

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

(p a real constant) converges if $p > 1$, and diverges if $p \leq 1$.

$$\frac{1}{n^{\alpha}(\ln n)^{\beta}} \begin{cases} \text{converge if } \alpha > 1 \text{ or } \alpha = 1 \ \beta > 1 \\ \text{diverge if } \alpha < 1 \text{ or } \alpha = 1 \ \beta \leq 1 \end{cases}$$

证明？

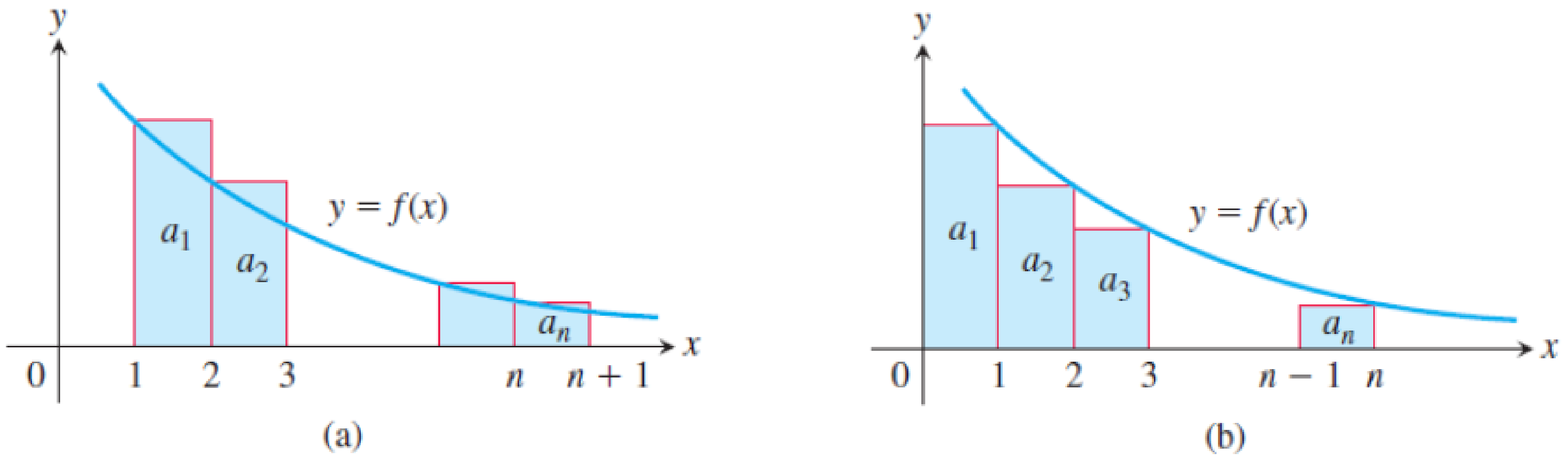
(a) $\sum_{n=1}^{\infty} n e^{-n^2}$ (b) $\sum_{n=1}^{\infty} \frac{1}{2^{\ln n}}$

余数的界限

Bounds for the Remainder in the Integral Test

Suppose $\{a_k\}$ is a sequence of positive terms with $a_k = f(k)$, where f is a continuous positive decreasing function of x for all $x \geq n$, and that Σa_n converges to S. Then the remainder $R_n = S - s_n$ satisfies the inequalities

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_{n}^{\infty} f(x)\,dx. \qquad (1)$$



直接比较判别法

THEOREM 10—The Comparison Test Let $\sum a_n$, $\sum c_n$, and $\sum d_n$ be series with $\boxed{\text{nonnegative terms.}}$ Suppose that for some integer N

$$d_n \le a_n \le c_n \qquad \text{for all} \qquad n > N.$$

(a) If $\sum c_n$ converges, then $\sum a_n$ also converges.

(b) If $\sum d_n$ diverges, then $\sum a_n$ also diverges.

$$\sum_{n=1}^{\infty} \frac{5}{5n-1}$$

the partial sums of $\sum a_n$ are bounded above by

$$M = a_1 + a_2 + \cdots + a_N + \sum_{n=N+1}^{\infty} c_n.$$

$$\frac{1}{2^n + \sqrt{n}}$$

证明？

the partial sums for $\sum d_n$ would be bounded by

$$M^* = d_1 + d_2 + \cdots + d_N + \sum_{n=N+1}^{\infty} a_n$$

极限比较判别法

THEOREM 11—Limit Comparison Test Suppose that $\boxed{a_n > 0 \text{ and } b_n > 0}$ for all $n \geq N$ (N an integer).

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = c > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.

3. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.

Does $\sum_{n=1}^{\infty} \frac{\ln n}{n^{3/2}}$ converge?

几何级数/等比级数

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}, \qquad |r| < 1.$$

P级数（或 ln 形式）

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

敛散性判别 *

THEOREM 2—Direct Comparison Test Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. $\displaystyle\int_a^\infty f(x)\,dx$ converges if $\displaystyle\int_a^\infty g(x)\,dx$ converges.

2. $\displaystyle\int_a^\infty g(x)\,dx$ diverges if $\displaystyle\int_a^\infty f(x)\,dx$ diverges.

判断：

1. $\displaystyle\int_1^\infty \frac{\arctan x}{(1+x^2)^{\frac{3}{2}}}\,dx$

2. $\displaystyle\int_1^\infty x^{p-1}e^{-x}\,dx$ 敛散性

3. $\displaystyle\int_0^1 \frac{1}{\sqrt[4]{1-x^4}}\,dx$

THEOREM 3—Limit Comparison Test If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

用p级数比较非常重要！

$\displaystyle\lim_{x\to\infty}\frac{f(x)}{g(x)} = L$ (positive)

$$\begin{cases} L=\infty & f(x)\text{收敛 则 } g(x)\text{同} \\ 0<L<\infty & \text{同敛散} \\ L=0 & g(x)\text{收敛 则 } f(x)\text{同} \end{cases}$$

判别级数敛散性（比较审敛）

判别 $\sum_{n=1}^{\infty}(1-\cos\frac{x}{n})$的敛散性

(1) $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n^2+n}}$ (3) $\sum_{n=1}^{\infty}\frac{1}{1+a^n}$ $(a>0)$

(2) $\sum_{n=1}^{\infty}\sin\frac{1}{n}$, $\sum_{n=1}^{\infty}\frac{n+1}{n^2-n+3}$,

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级数绝对收敛 (Absolutely)

只含非负项的级数 $\Longrightarrow$ 具有正负项的级数

DEFINITION A series $\sum a_n$ **converges absolutely** (is **absolutely convergent**) if the corresponding series of absolute values, $\sum |a_n|$, converges.

THEOREM 12—The Absolute Convergence Test If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

级数绝对收敛 ==> 级数收敛

证明？

For each n,

$$-|a_n| \leq a_n \leq |a_n|, \quad \text{so} \quad 0 \leq a_n + |a_n| \leq 2|a_n|.$$

Ratio判别法与Root判别法

THEOREM 13—The Ratio Test Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

THEOREM 14—The Root Test Let $\sum a_n$ be any series and suppose that

$$\lim_{n\to\infty} = \sqrt[n]{|a_n|} = \rho.$$

Then **(a)** the series *converges absolutely* if $\rho < 1$, **(b)** the series *diverges* if $\rho > 1$ or ρ is infinite, **(c)** the test is *inconclusive* if $\rho = 1$.

1. 更适用于带有阶乘，n 次方或者连乘号的级数
2. 不必找参考级数

Consider again the series with terms $a_n = \begin{cases} n/2^n, & n \text{ odd} \\ 1/2^n, & n \text{ even}. \end{cases}$

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交替级数

A series in which the terms are alternately positive and negative is an **alternating series**.

examples:

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots + \frac{(-1)^{n+1}}{n} + \cdots \tag{1}$$

$$-2 + 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots + \frac{(-1)^n 4}{2^n} + \cdots \tag{2}$$

the nth term of an alternating series is of the form

$$a_n = (-1)^{n+1} u_n \qquad \text{or} \qquad a_n = (-1)^n u_n$$

where $u_n = |a_n|$ is a positive number.

正负交替的级数为交替级数

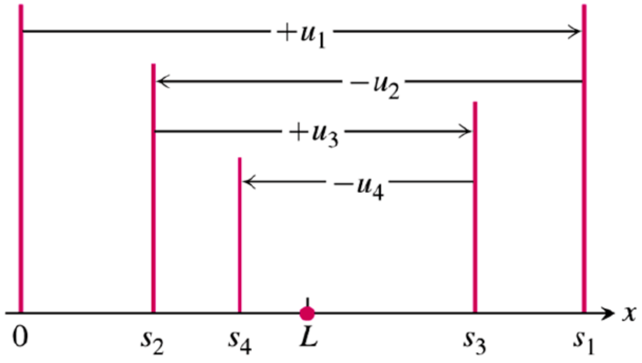
莱布尼茨判别法

THEOREM 14—The Alternating Series Test (Leibniz's Test) The series

$$\sum_{n=1}^{\infty}(-1)^{n+1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$$

converges if all three of the following conditions are satisfied:

1. The u_n's are all positive.
2. The positive u_n's are (eventually) nonincreasing: $u_n \geq u_{n+1}$ for all $n \geq N$, for some integer N.
3. $u_n \to 0$.



THEOREM 15—The Alternating Series Estimation Theorem If the alternating series $\sum_{n=1}^{\infty}(-1)^{n+1}u_n$ satisfies the three conditions of Theorem 14, then for $n \geq N$,

$$s_n = u_1 - u_2 + \cdots + (-1)^{n+1}u_n$$

approximates the sum L of the series with an error whose absolute value is less than u_{n+1}, the absolute value of the first unused term. Furthermore, the sum L lies between any two successive partial sums s_n and s_{n+1}, and the remainder, $L - s_n$, has the same sign as the first unused term.

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n}$$

$$\sum_{n=0}^{\infty}(-1)^{n}\frac{1}{2^n}$$

级数条件收敛 (Conditionally)

DEFINITION A series that converges but does not converge absolutely **converges conditionally.**

级数绝对收敛 VS. 级数条件收敛？

本质区别：是否满足重排定理。

THEOREM 17—The Rearrangement Theorem for Absolutely Convergent Series If $\sum_{n=1}^{\infty} a_n$ converges absolutely, and $b_1, b_2, \ldots, b_n, \ldots$ is any arrangement of the sequence $\{a_n\}$, then $\sum b_n$ converges absolutely and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n .$$

设正项级数$\sum_{n=1}^{\infty} u_n$收敛，能否推得$\sum_{n=1}^{\infty} u_n{}^2$收敛?反之是否成立?

设级数$\sum_{n=1}^{\infty} u_n$收敛，能否推得$\sum_{n=1}^{\infty} u_n{}^2$收敛?反之是否成立?

敛散性判别

Southern University of Science and Technology

If $0 < a_n < \frac{1}{n}$, which series must be convergent?

$$\sum_{n=1}^{\infty} a_n, \quad \sum_{n=1}^{\infty} (-1)^n a_n, \quad \sum_{n=1}^{\infty} \sqrt{a_n}, \quad \sum_{n=1}^{\infty} \frac{a_n}{n}$$

If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, prove that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ is also absolutely convergent.

$a \in \mathbb{R}$, $\sum_{n=1}^{\infty} \left(\frac{\sin a}{n^2} - \frac{1}{\sqrt{n}} \right)$ is divergent.

Justify the convergence of

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n + (-1)^n}}$$

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幂级数

DEFINITIONS A **power series about $x = 0$** is a series of the form

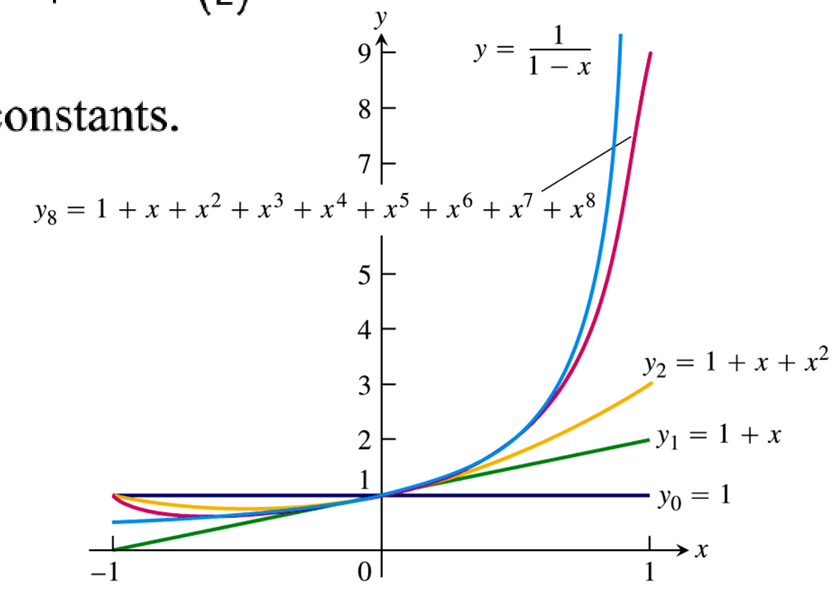
$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n + \cdots. \quad (1)$$

A **power series about $x = a$** is a series of the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots \quad (2)$$

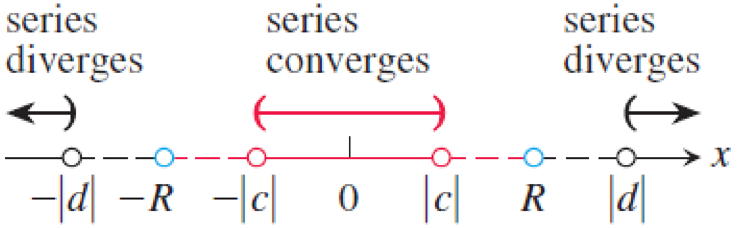
in which the **center** a and the **coefficients** $c_0, c_1, c_2, \ldots, c_n, \ldots$ are constants.

几何级数： $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots + x^n + \cdots.$



幂级数收敛定理

THEOREM 18—The Convergence Theorem for Power Series If the power series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$ converges at $x = c \neq 0$, then it converges absolutely for all x with $|x| < |c|$. If the series diverges at $x = d$, then it diverges for all x with $|x| > |d|$.



COROLLARY TO THEOREM 18 The convergence of the series $\sum c_n(x-a)^n$ is described by one of the following three cases:

1. There is a positive number R such that the series diverges for x with $|x-a| > R$ but converges absolutely for x with $|x-a| < R$. The series may or may not converge at either of the endpoints $x = a - R$ and $x = a + R$.
2. The series converges absolutely for every x $(R = \infty)$.
3. The series converges at $x = a$ and diverges elsewhere $(R = 0)$.

1. 用判断方法划定收敛范围（找到R）
2. |R|内部**绝对收敛**
3. |R|外部**发散**
4. 对于R点（端点值）单独判断

幂级数收敛定理

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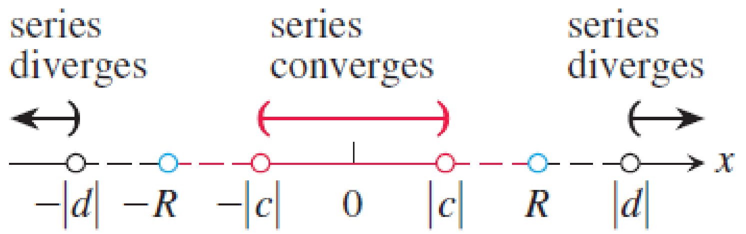
How to Test a Power Series for Convergence

1. *Use the Ratio Test (or Root Test) to find the interval where the series converges absolutely.* Ordinarily, this is an open interval

$$|x-a|<R \qquad \text{or} \qquad a-R<x<a+R.$$

2. *If the interval of absolute convergence is finite, test for convergence or divergence at each endpoint*, as in Examples 3a and b. Use a Comparison Test, the Integral Test, or the Alternating Series Test.

3. *If the interval of absolute convergence is* $a-R<x<a+R$, *the series diverges for* $|x-a|>R$ (it does not even converge conditionally) because the nth term does not approach zero for those values of x.



1. 用判断方法划定收敛范围（找到R）
2. |R|内部**绝对收敛**
3. |R|外部**发散**
4. 对于R点（端点值）单独判断

求幂级数$\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2^n}$的收敛区间.

幂级数的运算法则

设 $\sum_{n=0}^{\infty} a_n x^n$ 和 $\sum_{n=0}^{\infty} \mathrm{b}_n x^n$ 的收敛半径各为 R_1 和 R_2， $R = min\,\{R_1, R_2\}$

(1) 加减法

$$\sum_{n=0}^{\infty} a_n\, x^n \pm \sum_{n=0}^{\infty} b_n\, x^n = \sum_{n=0}^{\infty} c_n\, x^n \quad x \in (-R, R) \quad c_n = a_n \pm b_n$$

(2) 乘法

THEOREM 19—The Series Multiplication Theorem for Power Series If $A(x) = \sum_{n=0}^{\infty} a_n x^n$ and $B(x) = \sum_{n=0}^{\infty} b_n x^n$ converge absolutely for $|x| < R$, and

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \cdots + a_{n-1} b_1 + a_n b_0 = \sum_{k=0}^{n} a_k b_{n-k},$$

then $\sum_{n=0}^{\infty} c_n x^n$ converges absolutely to $A(x)B(x)$ for $|x| < R$:

$$\left(\sum_{n=0}^{\infty} a_n x^n\right) \cdot \left(\sum_{n=0}^{\infty} b_n x^n\right) = \sum_{n=0}^{\infty} c_n x^n.$$

幂级数的衍生函数敛散性

THEOREM 20 If $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely for $|x| < R$, then $\sum_{n=0}^{\infty} a_n (f(x))^n$ converges absolutely for any continuous function f on $|f(x)| < R$.

THEOREM 21—The Term-by-Term Differentiation Theorem If $\sum c_n(x-a)^n$ has radius of convergence $R > 0$, it defines a function

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n \quad \text{on the interval} \quad a - R < x < a + R.$$

This function f has derivatives of all orders inside the interval, and we obtain the derivatives by differentiating the original series term by term:

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1},$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)c_n(x-a)^{n-2},$$

and so on. Each of these derived series converges at every point of the interval $a - R < x < a + R$.

THEOREM 22—The Term-by-Term Integration Theorem Suppose that

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$$

converges for $a - R < x < a + R$ $(R > 0)$. Then

$$\sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$$

converges for $a - R < x < a + R$ and

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$$

for $a - R < x < a + R$.

幂级数逐项求导后，收敛半径不变，但它的收敛域的端点可能改变（越求导越小）。

求幂级数

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例 求$\sum\limits_{n=1}^{\infty}\dfrac{n(n+1)}{2^n}$的和.

求幂级数$\sum\limits_{n=0}^{\infty}\left(\dfrac{4n^2+4n+3}{2n+1}\right)x^{2n}$的和函数.

Find the radius and interval of convergence of the power series

$$\sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)x^n$$

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泰勒级数及麦克劳林级数

DEFINITIONS Let f be a function with derivatives of all orders throughout some interval containing a as an interior point. Then the **Taylor series generated by f at $x = a$** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots.$$

The **Maclaurin series generated by f** is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots,$$

the Taylor series generated by f at $x = 0$.

敛散性？

麦克劳林级数就是在 x = 0 处的泰勒级数

泰勒多项式与泰勒定理

DEFINITION Let f be a function with derivatives of order k for $k = 1, 2, \ldots, N$ in some interval containing a as an interior point. Then for any integer n from 0 through N, the **Taylor polynomial of order n** generated by f at $x = a$ is the polynomial

$$P_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(k)}(a)}{k!}(x-a)^k + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

缺少后面项的泰勒级数

THEOREM 23—Taylor's Theorem If f and its first n derivatives $f', f'', \ldots, f^{(n)}$ are continuous on the closed interval between a and b, and $f^{(n)}$ is differentiable on the open interval between a and b, then there exists a number c between a and b such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \boxed{\frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.}$$

将后面项用一个残差项替代

泰勒公式与残差项

Taylor's Formula

If f has derivatives of all orders in an open interval I containing a, then for each positive integer n and for each x in I,

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

泰勒公式 = 泰勒多项式 + 残差项

$$+ \frac{f^{(n)}(a)}{n!}(x-a)^n + R_n(x), \qquad (1)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1} \qquad \text{for some } c \text{ between } a \text{ and } x. \qquad (2)$$

If $R_n(x) \to 0$ as $n \to \infty$ for all $x \in I$, we say that the Taylor series generated by f at $x = a$ **converges** to f on I, and we write

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

残差项趋于零，泰勒级数收敛

估算残差项

THEOREM 24—The Remainder Estimation Theorem If there is a positive constant M such that $\boxed{|f^{(n+1)}(t)| \leq M}$ for all t between x and a, inclusive, then the remainder term $R_n(x)$ in Taylor's Theorem satisfies the inequality

$$|R_n(x)| \leq M \frac{|x-a|^{n+1}}{(n+1)!}.$$

If this inequality $\boxed{\text{holds for every } n}$ and the other conditions of Taylor's Theorem are satisfied by f, then the series converges to $f(x)$.

Show that the Taylor series for $\sin x$ at $x = 0$ converges for all x.

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二次项级数

The Binomial Series

For $-1 < x < 1$,

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \binom{m}{k} x^k,$$

where we define

$$\binom{m}{1} = m, \qquad \binom{m}{2} = \frac{m(m-1)}{2!},$$

and

$$\binom{m}{k} = \frac{m(m-1)(m-2)\cdots(m-k+1)}{k!} \qquad \text{for } k \geq 3.$$

记住这个形式，很多时候可以直接展开来解决很多问题！

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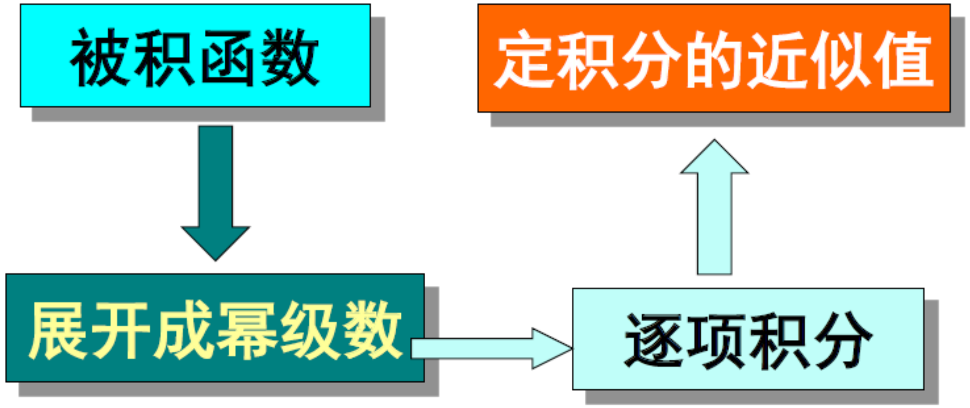
泰勒级数应用——近似计算

两类问题：

1. 给定项数，求近似值并估计精度；2. 给出精度，确定项数.

关键：通过**估计余项**，确定精度或项数.



$$e^{-x^2},\ \frac{\sin x}{x},\ \frac{1}{\ln x},$$

泰勒级数应用——求极限

$$\frac{1}{1-x} = 1 + x + x^2 + \cdots + x^n + \cdots = \sum_{n=0}^{\infty} x^n, \qquad |x| < 1$$

$$\frac{1}{1+x} = 1 - x + x^2 - \cdots + (-x)^n + \cdots = \sum_{n=0}^{\infty} (-1)^n x^n, \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \qquad |x| < \infty$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}, \qquad |x| < \infty$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}, \qquad |x| < \infty$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots + (-1)^{n-1} \frac{x^n}{n} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n}, \qquad -1 < x \le 1$$

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \cdots + (-1)^n \frac{x^{2n+1}}{2n+1} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}, \qquad |x| \le 1$$

$$\lim_{x\to 0} \frac{\sin x - \tan x}{x^3}.$$

Find $\displaystyle\lim_{x\to 0}\left(\frac{1}{\sin x} - \frac{1}{x}\right)$.

$$\lim_{x\to 1} \frac{\ln x}{x-1}.$$

Taylor展开、洛必达法则为求极限最powerful的工具

欧拉公式——数学最美的公式

DEFINITION

For any real number θ, $e^{i\theta} = \cos\theta + i\sin\theta$. (4)

$$e^{i\pi} = -1.$$

集两个自然常数，无理数与有理数为一体的公式

间接法求泰勒级数

变量代换, 四则运算, 恒等变形, 逐项求导, 逐项积分

$$\cos x = (\sin x)'$$

$$\arctan x = \int_0^x \frac{dx}{1+x^2}$$

$$\ln(1+x) = \int_0^x \frac{dx}{1+x}$$

将$f(x) = \sin x \cos 2x$展开成 x的幂级数.

将$f(x) = \dfrac{x-1}{4-x}$在$x=1$处展开成泰勒级数

Find the Taylor series for $f(x) = \ln\left(x + \sqrt{x^2+1}\right)$ at $x = 0$.

请写出函数$f(x) = \ln\left(x + \sqrt{x^2+1}\right)$在$x = 0$处的Taylor级数.

(2019 Midterm exam)

SUSTech Southern University of Science and Technology

Chapter11

参数方程与极坐标

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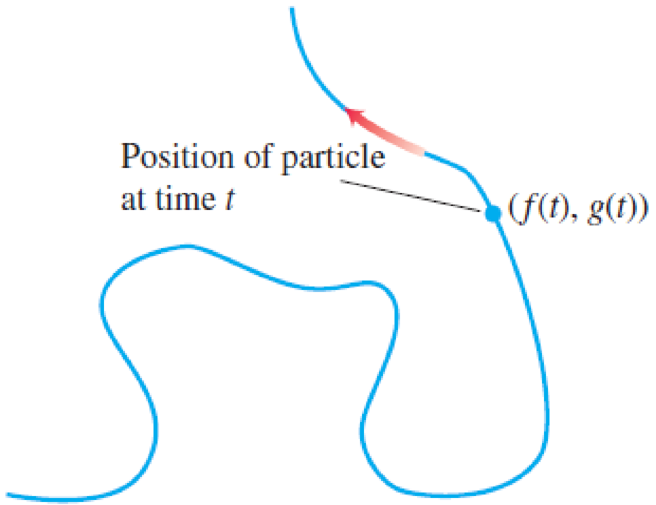
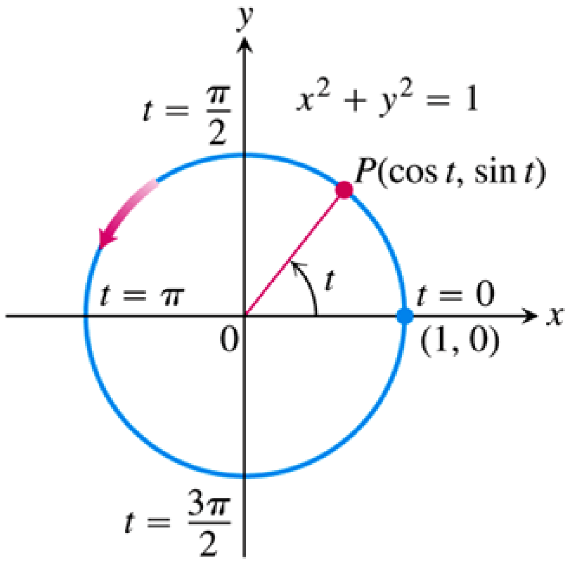
参数方程

DEFINITION If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval I of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.





$$x = a \cos t,$$

$$y = a \sin t,$$

$$0 \leq t \leq 2\pi.$$

参数方程微分

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \qquad (1)$$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t), y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$ and $y' = dy/dx$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \qquad (2)$$

注意两公式的“联动”

Find d^2y/dx^2 as a function of t if $x = t - t^2, y = t - t^3$.

弧长公式 Arc Length

DEFINITION If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

Length of a Curve $y = f(x)$

$$x = t \qquad \text{and} \qquad y = f(t), \qquad a \le t \le b,$$

The Arc Length Differential

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

求弧长 *

DEFINITION If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \qquad (3)$$

Formula for the Length of $x = g(y), c \le y \le d$

If g' is continuous on $[c, d]$, the length of the curve $x = g(y)$ from $A = (g(c), c)$ to $B = (g(d), d)$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \qquad (4)$$

旋转体表面积公式

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Area of Surface of Revolution for Parametrized Curves

If a smooth curve $x = f(t), y = g(t), a \leq t \leq b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis ($y \geq 0$):

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{5}$$

2. Revolution about the y-axis ($x \geq 0$):

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \tag{6}$$

求旋转体的表面积 *

DEFINITION If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\, dx. \qquad (3)$$

Surface Area for Revolution About the y-Axis

If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the graph of $x = g(y)$ about the y-axis is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\, dy. \qquad (4)$$

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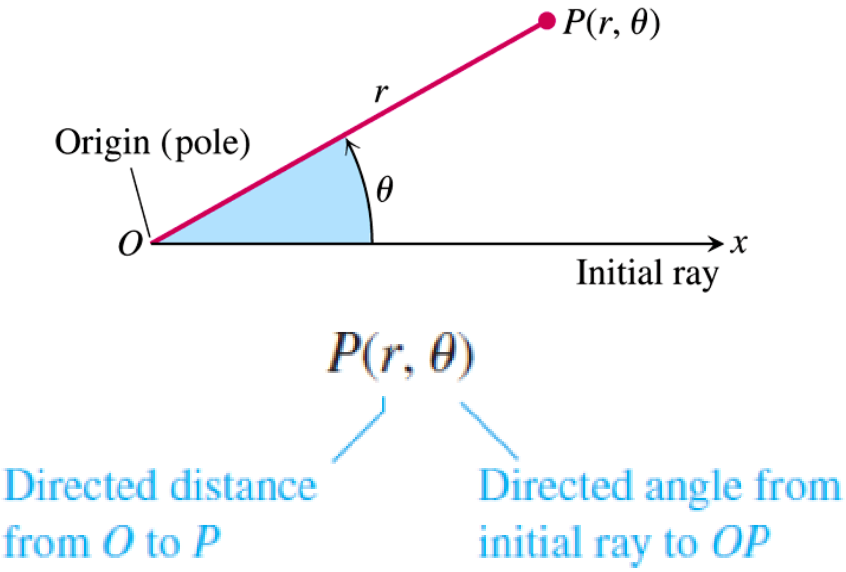
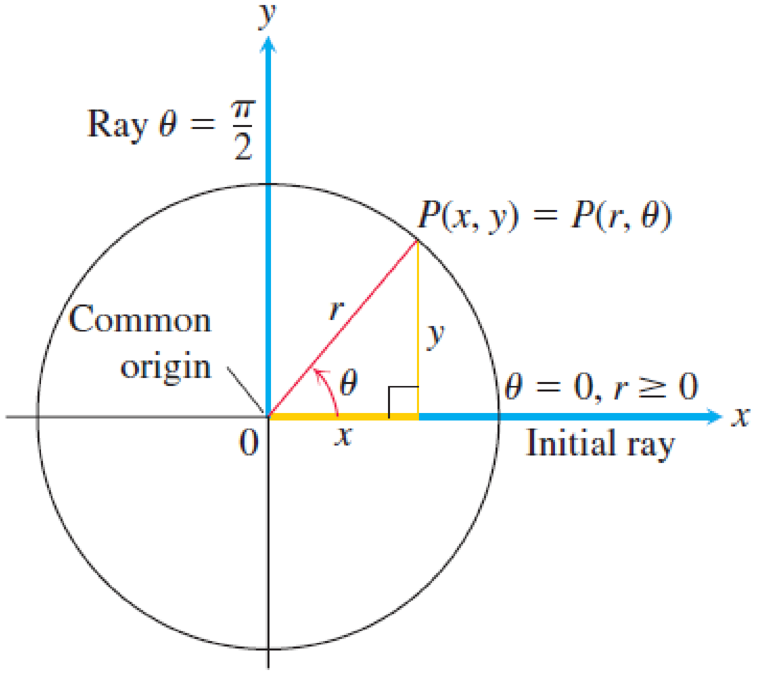
极坐标





Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$

极坐标微分

Slope of the Curve $r = f(\theta)$

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta},$$

provided $dx/d\theta \neq 0$ at (r, θ).

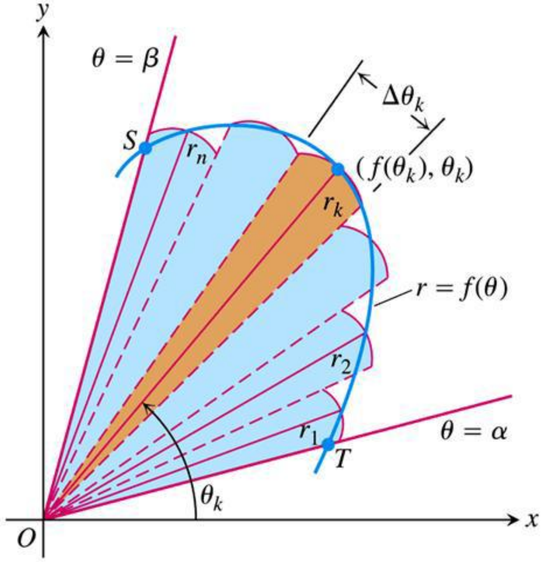
扇形区域面积公式

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta)$, $\alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

This is the integral of the **area differential** (Figure 11.32)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2} \left(f(\theta) \right)^2 d\theta.$$



Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r_2^2 \, d\theta - \int_{\alpha}^{\beta} \frac{1}{2} r_1^2 \, d\theta = \int_{\alpha}^{\beta} \frac{1}{2} \left(r_2^2 - r_1^2 \right) d\theta \qquad (1)$$

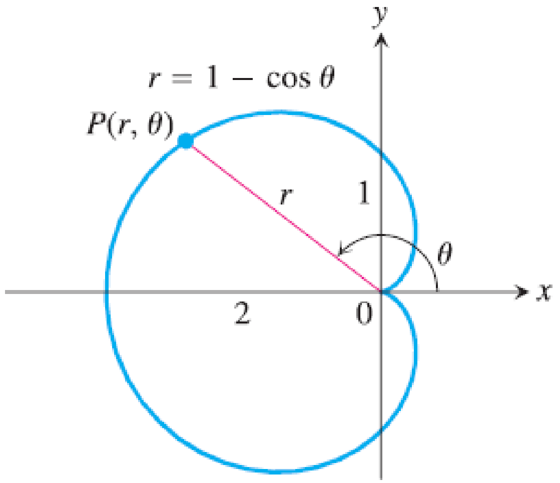
弧长公式

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3}$$

Find the length of the cardioid $r = 1 - \cos\theta$.



高数带飞课提纲一览（复习专用）

- Ch10 无限数列与无穷级数
  - (数列极限定义 a_n) & (数列极限计算 @ 运算法则 & 三明治 & 重要极限)
  - (数列的形式 @ 迭代型/递归型) & (数列的性质 @ 有界性/单调性)
  - (无穷级数定义 S_n) & (无穷级数计算 @ (好求级数 @ 几何级数 & p级数) & 运算法则)
  - 级数敛散性判别 (正项级数) @ 直接判别 & 必要条件 & 积分判别 & 比较判别
  - 级数敛散性判别 @ (绝对收敛 @ ratio判别 & root判别) & (条件收敛 @ 莱布尼茨判别)
  - (幂级数 @ 收敛定理 & 运算法则 & 微/积分敛散性) & 二次项级数
  - 泰勒级数 @ 泰勒/麦克劳林级数 & 泰勒展开求极限 & 间接法求泰勒级数(五)
- Ch11 参数方程与极坐标
  - 参数方程 @ 定义 & 微积分 & 求弧长 & 求旋转体表面积
  - 极坐标 @ 定义 & 微积分 & 求扇形面积 & 求弧长

高数带飞课提纲一览（复习专用）

- Ch12 向量与空间几何
  - 向量 @ (向量运算 @ 加和 & 数乘 & 点乘 & 叉乘) & 二次曲面
  - 空间几何 @ 二次曲面
- Ch13 向量函数与空间运动
  - 向量函数 @ 定义 & 极限 & 连续性 & 微积分 & 切向量 & 法向量 & 曲率圆方程
- Ch14 偏导数
  - 多元函数 @ (定义 @ 点/线/面) & 极限 & 运算法则 & 连续性
  - …

Chapter12

向量与空间几何

高数带飞课提纲一览（复习专用）

- Ch12 向量与空间几何
  - 向量 @ (向量运算 @ 加和 & 数乘 & 点乘 & 叉乘) & 二次曲面
  - 空间几何 @ 二次曲面
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  - 向量函数 @ 定义 & 极限 & 连续性 & 微积分 & 切向量 & 法向量 & 曲率圆方程
- Ch14 偏导数
  - 多元函数 @ (定义 @ 点/线/面) & 极限 & 运算法则 & 连续性
  - …

向量

DEFINITIONS The vector represented by the directed line segment $\overrightarrow{AB}$ has **initial point** A and **terminal point** B and its **length** is denoted by $|\overrightarrow{AB}|$. Two vectors are **equal** if they have the same length and direction.

DEFINITION

If $\mathbf{v}$ is a **two-dimensional** vector in the plane equal to the vector with initial point at the origin and terminal point (v_1, v_2), then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2 \rangle .$$

If $\mathbf{v}$ is a **three-dimensional** vector equal to the vector with initial point at the origin and terminal point (v_1, v_2, v_3), then the **component form** of $\mathbf{v}$ is

$$\mathbf{v} = \langle v_1, v_2, v_3 \rangle .$$

The **magnitude** or **length** of the vector $\mathbf{v} = \overrightarrow{PQ}$ is the nonnegative number

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2} = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

(See Figure 12.10.)

向量运算

DEFINITIONS Let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ be vectors with k a scalar.

$\boxed{\textbf{Addition:}}$ $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2, u_3 + v_3 \rangle$

$\boxed{\textbf{Scalar multiplication}}$: $k\mathbf{u} = \langle ku_1, ku_2, ku_3 \rangle$

DEFINITION The $\boxed{\textbf{dot product}}$ $\mathbf{u} \cdot \mathbf{v}$ ("$\mathbf{u}$ dot $\mathbf{v}$") of vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3 .$$

向量投影： $\operatorname{proj}_{\mathbf{v}} \mathbf{u} = \left(\dfrac{\mathbf{u} \cdot \mathbf{v}}{|\mathbf{v}|^2} \right) \mathbf{v}.$

DEFINITION The $\boxed{\textbf{cross product}}$ $\mathbf{u} \times \mathbf{v}$ ("$\mathbf{u}$ cross $\mathbf{v}$") is the vector

$$\mathbf{u} \times \mathbf{v} = (|\mathbf{u}||\mathbf{v}| \sin \theta)\, \mathbf{n}.$$

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

点到直线距离： $d = \left| \overrightarrow{PS} \cdot \dfrac{\mathbf{n}}{|\mathbf{n}|} \right|$

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- Ch13 向量函数与空间运动
  - 向量函数 @ 定义 & 极限 & 连续性 & 微积分 & 切向量 & 法向量 & 曲率圆方程
- Ch14 偏导数
  - 多元函数 @ (定义 @ 点/线/面) & 极限 & 运算法则 & 连续性
  - …

二次曲面

[二次曲面 - 维基百科，自由的百科全书 (wikipedia.org)](wikipedia.org)

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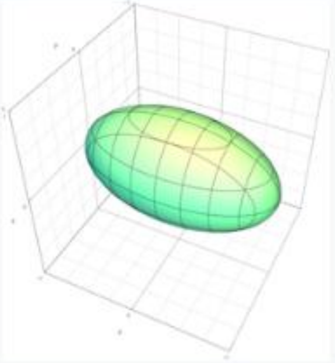
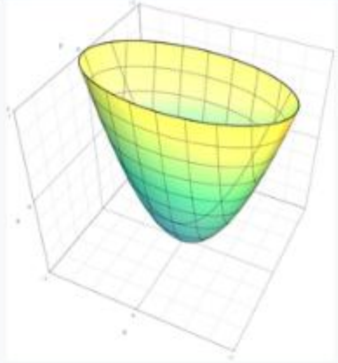
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

ellipsoid



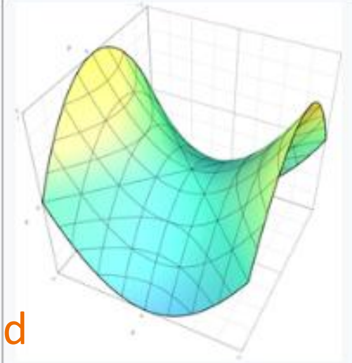
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-z=0$$

elliptical paraboloid



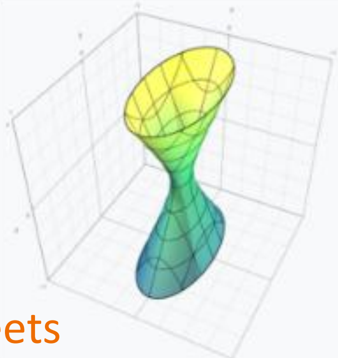
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}-z=0$$

hyperbolic paraboloid



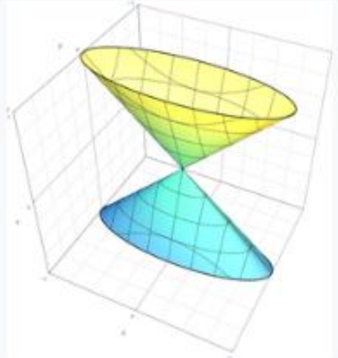
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

hyperboloid of one sheets



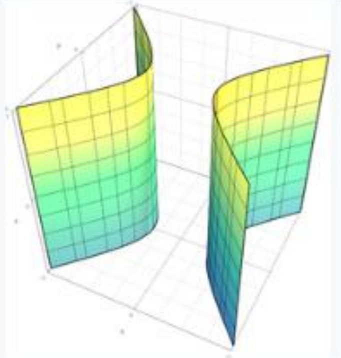
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

elliptical cone



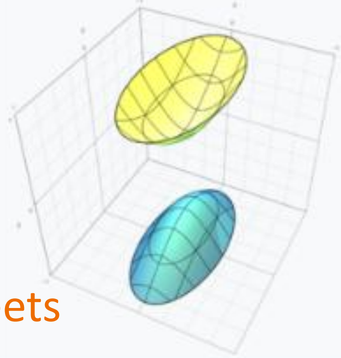
$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

hyperbolic cylinder



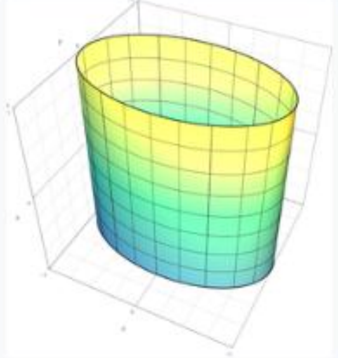
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1$$

hyperboloid of two sheets



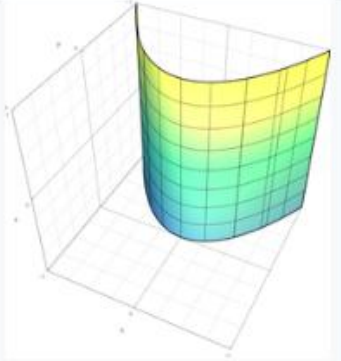
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

elliptical cylinder



$$x^2+2ay=0$$

parabolic cylinder



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Chapter13

向量函数与空间运动

高数带飞课提纲一览（复习专用）

- Ch12 向量与空间几何
  - 向量 @ (向量运算 @ 加和 & 数乘 & 点乘 & 叉乘) & 二次曲面
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- Ch14 偏导数
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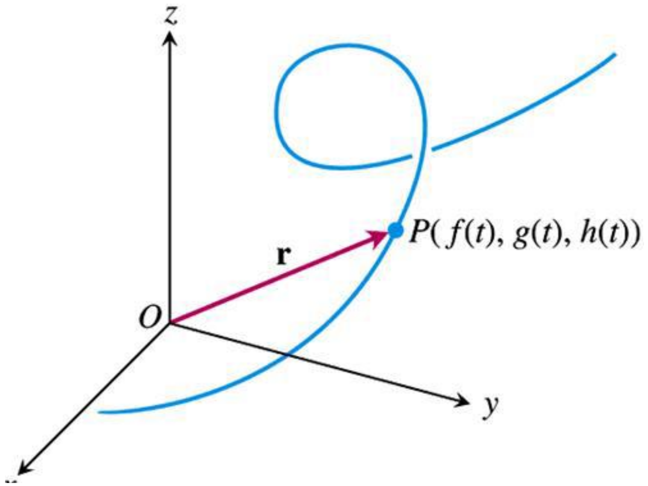
  - …

向量函数

The vector $\mathbf{r}(t) = \overrightarrow{OP} = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$

from the origin to the particle's **position** $P(f(t), g(t), h(t))$ at time t is the particle's **position vector**



极限

DEFINITION Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function with domain D, and $\mathbf{L}$ a vector. We say that $\mathbf{r}$ has **limit** $\mathbf{L}$ as t approaches t_0 and write

$$\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{L}$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all $t \in D$

$$|\mathbf{r}(t) - \mathbf{L}| < \epsilon \quad \text{whenever} \quad 0 < |t - t_0| < \delta.$$

连续性

DEFINITION A vector function $\mathbf{r}(t)$ is **continuous at a point** $t = t_0$ in its domain if $\lim_{t \to t_0} \mathbf{r}(t) = \mathbf{r}(t_0)$. The function is **continuous** if it is continuous at every point in its domain.

向量函数的导数

DEFINITION The vector function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ has a **derivative (is differentiable) at *t*** if f, g, and h have derivatives at t. The derivative is the vector function

$$\mathbf{r}'(t) = \frac{d\mathbf{r}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} = \frac{df}{dt}\mathbf{i} + \frac{dg}{dt}\mathbf{j} + \frac{dh}{dt}\mathbf{k}.$$

Differentiation Rules for Vector Functions

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector functions of t, $\mathbf{C}$ a constant vector, c any scalar, and f any differentiable scalar function.

1. *Constant Function Rule:* $\frac{d}{dt}\mathbf{C} = \mathbf{0}$
2. *Scalar Multiple Rules:* $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

   $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$
3. *Sum Rule:* $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$
4. *Difference Rule:* $\frac{d}{dt}[\mathbf{u}(t) - \mathbf{v}(t)] = \mathbf{u}'(t) - \mathbf{v}'(t)$
5. *Dot Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$
6. *Cross Product Rule:* $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$
7. *Chain Rule:* $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$

向量函数的积分

DEFINITION The **indefinite integral** of $\mathbf{r}$ with respect to t is the set of all antiderivatives of $\mathbf{r}$, denoted by $\int \mathbf{r}(t)\,dt$. If $\mathbf{R}$ is any antiderivative of $\mathbf{r}$, then

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C}.$$

DEFINITION If the components of $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ are integrable over $[a, b]$, then so is $\mathbf{r}$, and the **definite integral** of $\mathbf{r}$ from a to b is

$$\int_a^b \mathbf{r}(t)\,dt = \left(\int_a^b f(t)\,dt\right)\mathbf{i} + \left(\int_a^b g(t)\,dt\right)\mathbf{j} + \left(\int_a^b h(t)\,dt\right)\mathbf{k}.$$

Ideal Projectile Motion Equation

抛体运动方程：

$$\mathbf{r} = (v_0 \cos\alpha)t\mathbf{i} + \left((v_0 \sin\alpha)t - \frac{1}{2}gt^2\right)\mathbf{j}. \tag{5}$$

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  - 多元函数 @ (定义 @ 点/线/面) & 极限 & 运算法则 & 连续性
  - …

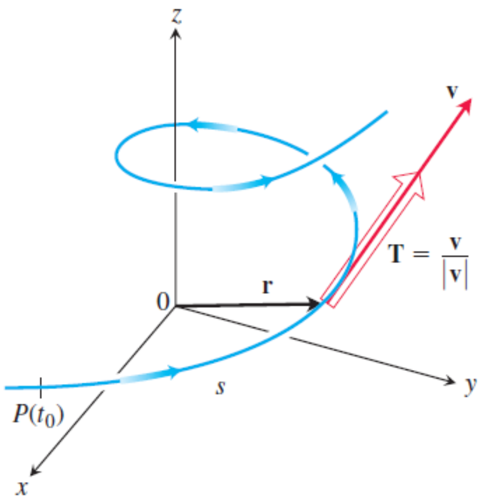
弧长与单位切向量

DEFINITION The **length** of a smooth curve $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, $a \le t \le b$, that is traced exactly once as t increases from $t = a$ to $t = b$, is

弧长： $$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\, dt. = \int_a^b |\mathbf{v}|\, dt \qquad (1)$$

Arc Length Parameter with Base Point $P(t_0)$

$$s(t) = \int_{t_0}^{t} \sqrt{[x'(\tau)]^2 + [y'(\tau)]^2 + [z'(\tau)]^2}\, d\tau = \int_{t_0}^{t} |\mathbf{v}(\tau)|\, d\tau \qquad (3)$$



We already know the velocity vector $\mathbf{v} = d\mathbf{r}/dt$ is tangent to the curve $\mathbf{r}(t)$ and that the vector

单位切向量： $$\mathbf{T} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

is therefore a unit vector tangent to the (smooth) curve, called the **unit tangent vector**

曲率与单位法向量

DEFINITION If $\mathbf{T}$ is the unit vector of a smooth curve, the **curvature** function of the curve is

曲率：

$$\kappa = \left|\frac{d\mathbf{T}}{ds}\right|. \qquad \kappa = \frac{1}{|\mathbf{v}|}\left|\frac{d\mathbf{T}}{dt}\right|,$$

DEFINITION At a point where $\kappa \neq 0$, the **principal unit normal** vector for a smooth curve in the plane is

单位法向量：

$$\mathbf{N} = \frac{1}{\kappa}\frac{d\mathbf{T}}{ds}. \qquad \mathbf{N} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|},$$

曲率圆方程

求曲率圆方程： $y = f(x)$

$$\vec{r}(t) = t\hat{i} + f(t)\hat{j} \qquad \vec{v}(t) = \hat{i} + f'(t)\hat{j}$$

$$\vec{T}(t) = \frac{\vec{v}}{|\vec{v}|} = \frac{\hat{i} + f'(t)\hat{j}}{\sqrt{1+f'(t)^2}} \qquad \vec{T}'(t) = \frac{\sqrt{1+f'(t)^2}\, f''(t)\hat{j} - \frac{f'(t)f''(t)}{\sqrt{1+f'(t)^2}}(\hat{i} + f'(t)\hat{j})}{1+f'(t)^2}$$

$$= \frac{-f'(t)f''(t)\hat{i} + f''(t)\hat{j}}{(1+f'(t)^2)^{\frac{3}{2}}}$$

$$k(t) = \frac{1}{|V(t)|}|T'(t)| = \frac{f''(t)}{(1+f'(t)^2)^{\frac{3}{2}}} \qquad R = \frac{1}{k(t)} = \frac{(1+f'(t))^{\frac{3}{2}}}{f''(t)}$$

$$x_D = x_0 - R \cdot \frac{f'(t)}{\sqrt{1+f'(t)^2}} \qquad y_D = y_0 + R \cdot \frac{1}{\sqrt{1+f'(t)^2}}$$

EXAMPLE 4 Find and graph the osculating circle of the parabola $y = x^2$ at the origin.

Chapter14

偏导数

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  - …

多元函数

DEFINITIONS Suppose D is a set of n-tuples of real numbers $(x_1, x_2, \dots, x_n)$. A **real-valued function** f on D is a rule that assigns a unique (single) real number

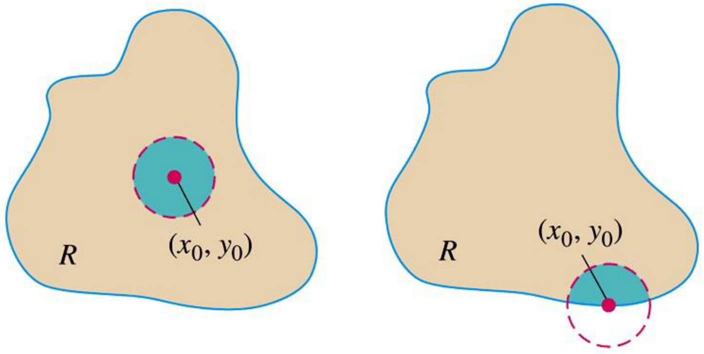
$$w = f(x_1, x_2, \dots, x_n)$$

to each element in D. The set D is the function's **domain**. The set of w-values taken on by f is the function's **range**. The symbol w is the **dependent variable** of f, and f is said to be a function of the n **independent variables** x_1 to x_n. We also call the x_j's the function's **input variables** and call w the function's **output variable**.

二元函数的点

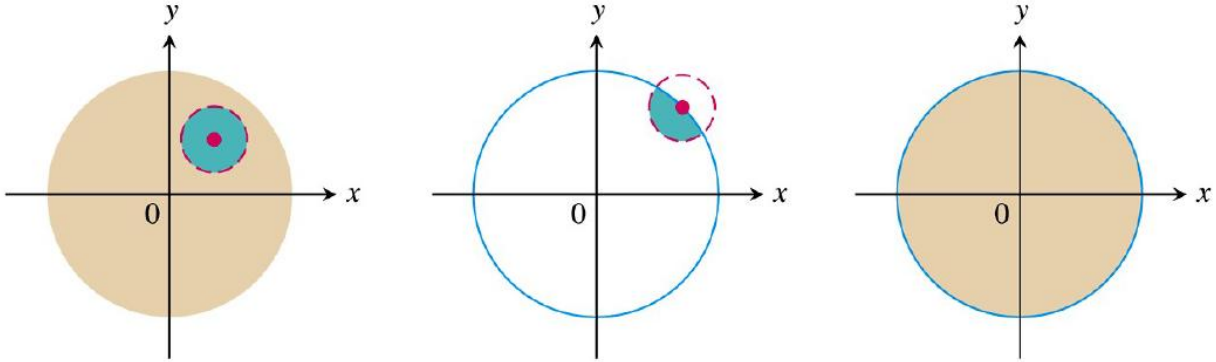
DEFINITIONS A point (x_0, y_0) in a region (set) R in the xy-plane is an **interior point** of R if it is the center of a disk of positive radius that lies entirely in R (Figure 14.2). A point (x_0, y_0) is a **boundary point** of R if every disk centered at (x_0, y_0) contains points that lie outside of R as well as points that lie in R. (The boundary point itself need not belong to R.)

The interior points of a region, as a set, make up the **interior** of the region. The region's boundary points make up its **boundary**. A region is **open** if it consists entirely of interior points. A region is **closed** if it contains all its boundary points (Figure 14.3).



(a) Interior point (b) Boundary point

内点； 边界点； 开放； 封闭

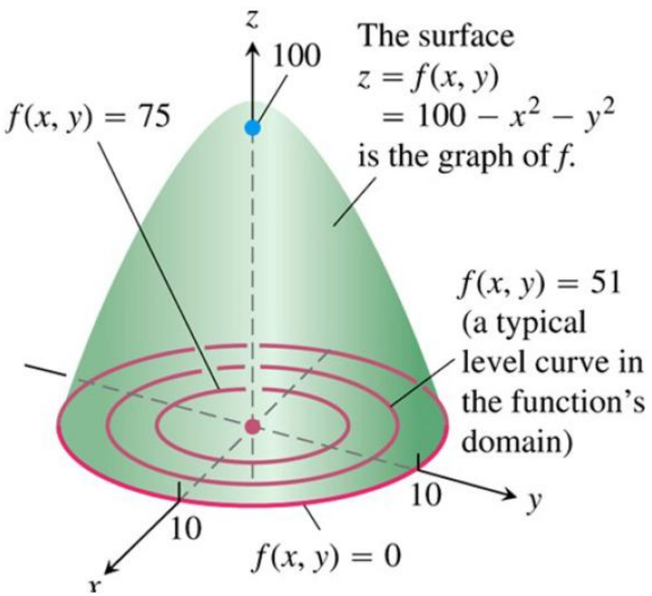


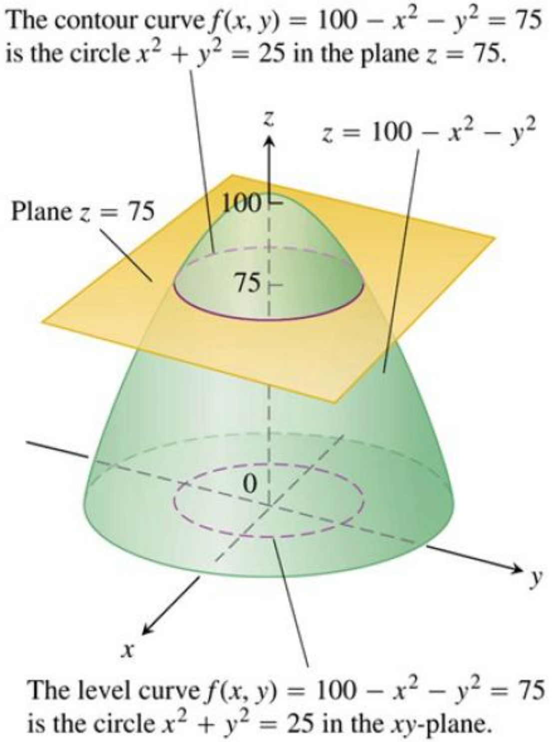
有界 VS. 无界

DEFINITIONS A region in the plane is **bounded** if it lies inside a disk of fixed radius. A region is **unbounded** if it is not bounded.

等高线与曲面

DEFINITIONS The set of points in the plane where a function $f(x, y)$ has a constant value $f(x, y) = c$ is called a **level curve** of f. The set of all points $(x, y, f(x, y))$ in space, for (x, y) in the domain of f, is called the **graph** of f. The graph of f is also called the **surface** $\boldsymbol{z = f(x, y)}$.





高数带飞课提纲一览（复习专用）

- Ch12 向量与空间几何
  - 向量 @ (向量运算 @ 加和 & 数乘 & 点乘 & 叉乘) & 二次曲面
  - 空间几何 @ 二次曲面
- Ch13 向量函数与空间运动
  - 向量函数 @ 定义 & 极限 & 连续性 & 微积分 & 切向量 & 法向量 & 曲率圆方程
- Ch14 偏导数
  - 多元函数 @ (定义 @ 点/线/面) & 极限 & 运算法则 & 连续性
  - …

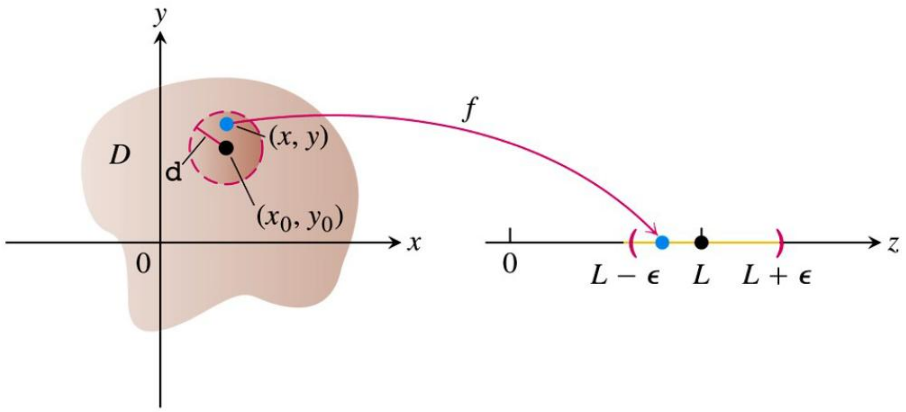
多元函数的极限

DEFINITION We say that a function $f(x, y)$ approaches the **limit** L as (x, y) approaches (x_0, y_0), and write

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if, for every number $\epsilon > 0$, there exists a corresponding number $\delta > 0$ such that for all (x, y) in the domain of f,

$$|f(x, y) - L| < \epsilon \qquad \text{whenever} \qquad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta.$$



多元函数极限运算法则

THEOREM 1—Properties of Limits of Functions of Two Variables The following rules hold if L, M, and k are real numbers and

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \quad \text{and} \quad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = M.$$

1. *Sum Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) + g(x,y)) = L + M$
2. *Difference Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y) - g(x,y)) = L - M$
3. *Constant Multiple Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} kf(x,y) = kL$ (any number k)
4. *Product Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} (f(x,y)\cdot g(x,y)) = L\cdot M$
5. *Quotient Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y)}{g(x,y)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} [f(x,y)]^n = L^n$, n a positive integer
7. *Root Rule:* $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \sqrt[n]{f(x,y)} = \sqrt[n]{L} = L^{1/n}$, n a positive integer, and if n is even, we assume that $L > 0$.

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - xy}{\sqrt{x} - \sqrt{y}}$.

Find $\displaystyle\lim_{(x,y)\to(0,0)} \frac{4xy^2}{x^2 + y^2}$ if it exists.

求极限 $\displaystyle\lim_{\substack{x\to 0\\ y\to 0}} \frac{(y-x)x}{\sqrt{x^2+y^2}}$.

多元函数的连续性

DEFINITION A function $f(x, y)$ is **continuous at the point (x_0, y_0)** if

1. f is defined at (x_0, y_0),
2. $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists,
3. $\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$.

A function is **continuous** if it is continuous at every point of its domain.

$$f(x,y) = \begin{cases} \dfrac{2xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$$

Two-Path Test for Nonexistence of a Limit
If a function $f(x, y)$ has different limits along two different paths in the domain of f as (x, y) approaches (x_0, y_0), then $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ does not exist.

Having the same limit along all straight lines approaching (x_0, y_0) does not imply a limit exists at (x_0, y_0).

$$f(x, y) = \frac{2x^2y}{x^4 + y^2}$$

排除法！

Continuity of Composites

If f is continuous at (x_0, y_0) and g is a single-variable function continuous at $f(x_0, y_0)$, then the composite function $h = g \circ f$ defined by $h(x, y) = g(f(x, y))$ is continuous at (x_0, y_0).

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Q&A

祝大家期中顺利！