MACM 316 – Computing Assignment 5

Due Date: Friday March 13 at 11:00pm.

Submission Instructions: You must upload one .pdf file in Crowdmark that consists of two pages: page 1 is your report which should fit all discussions, data and figures into a single page; and page 2 is a listing of your code. The deadline is **11:00pm** on the due date. The actual due time is set to 11:05pm and if Crowdmark indicates that you submitted late, you will be assigned a grade of 0 on this assignment. Your TA has emailed you a Crowdmark link that you should save since it will allow you to upload your completed assignments.

- Please review the **Guidelines for Assignments** carefully.

- Acknowledge any collaborations or assistance from colleagues/TAs/instructor.

- If you have any questions about Matlab or aspects of this assignment, then you are strongly encouraged to attend tutorials and drop-in workshops.

Computing Assignment – Banded Matrices From Solid Mechanics

Consider a horizontal cantilevered beam of length L that is clamped at one end but free to move vertically along the rest of its length. Divide the the beam into n equal sections of length $h = \frac{L}{n}$ and separated by points $x_i = ih$, with $i = 1, 2, \ldots, n$. A discrete model for the forces and solid deformations along the beam yields a system of linear equations $Az = b$, where the $n \times n$ matrix A has the banded form

$$
\begin{bmatrix}
9 & -4 & 1 & 0 & \cdots & \cdots & 0 \\
-4 & 6 & -4 & 1 & \ddots & & 0 \\
1 & -4 & 6 & -4 & 1 & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & 1 & -4 & 6 & -4 & 1 \\
\vdots & & \ddots & 1 & -4 & 5 & -2 \\
0 & \cdots & \cdots & 0 & 1 & -2 & 1
\end{bmatrix}
$$

Then the n-vector $b = [b_i]$ is the given load force acting along the bar (which includes its own weight) and $z = [z_i]$ is the n-vector specifying the vertical deflection (away from $z = 0$) caused by the load force; both b_i and z_i are measured at location x_i. We take the bar to be uniformly loaded, which means that there is a constant upward force F distributed along the entire beam and so each component of the load vector is a constant, $b_i = F \cdot h^4$. For this problem, assume that the values of the physical constants are $L = 10\,m$ and $F = 0.005\,N/m$.
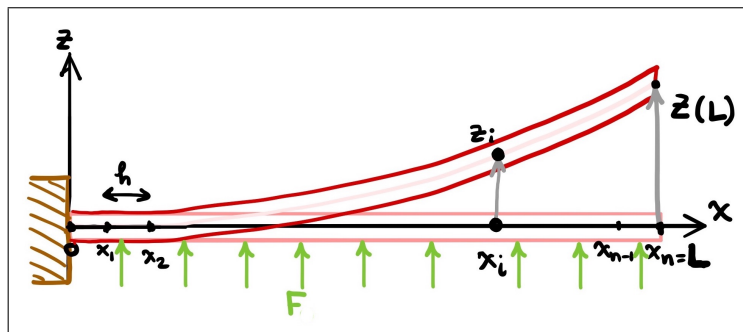
(a) Solve the linear system $Az = b$ using the following three methods:

    I. The GE+PP algorithm for sparse (banded) linear systems, which is the default algorithm used by Matlab's "\" operator when the matrix (call it `Asparse`) is of sparse type. You may find it easiest to set up the matrix using the `spdiags` command.

    II. The GE+PP algorithm for dense linear systems, again using "\". Here, you need a dense version of A which you can obtain either with the `diag` command or more simply by typing

        `Adense = full(Asparse)`

    III. The Gauss-Seidel iterative algorithm, also using the matrix `Asparse`. You may make use of the `gs2` code from lectures, setting the parameters `tol = 1e-8` and `maxiter = 1e5`, and taking initial guess $z_0 = (1, 1, \ldots, 1)^T$.

For each method, compute the solution with $n = 100$ and $n = 1000$ points. How do the three methods compare in terms of cost? (use elapsed time from Matlab's `tic` / `toc` as a proxy for cost). How well do your three answers agree with each other? (use the vector 2-norm to compare the differences). Which result do you think is most accurate?

(b) Next, verify that the matrix A has a UL factorization (yes, that's UL and not LU) of the form $A = UU^T$, where U is the upper triangular matrix

$$\begin{bmatrix} 2 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & -2 & 1 \\ \vdots & & & \ddots & 1 & -2 \\ 0 & \cdots & \cdots & \cdots & 0 & 1 \end{bmatrix}$$

For the same two values of n as in the previous part (100, 1000) solve the linear system using the UU^T factorization via a sequence of two triangular solves, corresponding to forward and backward substitution. You can use the "\" command for both solves, but make sure your matrix U is set up as a sparse matrix. Compare your answer to the corresponding solution from Method I (sparse GE+PP) in terms of both accuracy and cost.

(c) Compute the condition number for both A and U, as well as the spectral radius $\rho(T)$ for the Gauss-Seidel iteration matrix T. Use this information to help explain your results from parts (a) and (b). The condition number for a large sparse matrix can only be estimated, and for this purpose you should use the Matlab function `condest`.

(d) Choose what you consider to be your most accurate solution from parts (a)–(c), and use it to generate a plot of the deflection z versus x. In any introductory course in engineering solid mechanics, students learn about deflection of elastic beams and derive the following formula for the maximum deflection:

$$z_{max} = z(L) = \frac{FL^4}{6EI}$$

where EI is a material property of the beam called *flexural rigidity* or *bending stiffness* (with units $N \cdot m^2$). Use your solution to estimate the bending stiffness.

Extra reading: You can find out more about models for beam deflection at
`https://en.wikipedia.org/wiki/Deflection_(engineering)`