Project Report: Numerical Solution of the Heat Equation

Yuqi Jia

301354524

1. Problem Description

1.1 Partial Differential Equations (PDEs)

This project studies the one-dimensional Heat Equation on the domain [0, L], which is of the form:

$$u_t = \alpha u_{xx}$$

Where $u(x,t)$ represents the temperature distribution at position x and time t, and α is the thermal diffusion coefficient.

1.2 Boundary conditions and initial conditions

- **Boundary conditions:** The boundary conditions are Dirichlet conditions with both ends fixed at 0. At both ends of the interval [0,L], the temperature is fixed at 0, that is:

$$u(0, t) = u(L, t) = 0$$
$$t \geq 0$$

- **Initial conditions:** The initial condition is a sinusoidal temperature distribution given by the equation above. At time t=0, the temperature distribution is a sinusoidal function, that is:

$$u(x,0) = sin(\frac{\pi x}{L})$$, for $x \in [0, L]$

1.3 Complexity of the problem and the necessity of numerical calculation

The analytical solutions of the Heat Equation are typically limited to simple boundary conditions and initial conditions. In this project, the initial condition is a sinusoidal function, and the boundary conditions are fixed at 0. In this case, the analytical solution is difficult to obtain, so numerical methods are essential to solve it. In addition, in actual engineering and scientific computing, complex boundary conditions and initial conditions are more common, and analytical solutions are often not directly available, so numerical methods become a necessary means to solve such problems.

2. Test Problem

In order to verify the correctness of the code and the efficiency of the numerical method, we consider a simple test problem:

$$u_t = \alpha u_{xx}$$

Where the boundary conditions and initial conditions are the same as those in section 1, we solve it numerically and compare it with the analytical solution. For this specific problem, the analytical solution is:

$$u(x,t) = sin(\frac{\pi x}{L}) exp(-\alpha (\frac{\pi}{L})^2 t)$$

By comparing the numerical and analytical solutions, we can verify the accuracy of the numerical method.

The error metric used to verify the accuracy of numerical methods is the Maximum Error. Specifically, the code evaluates the accuracy of numerical methods by calculating the maximum absolute difference between the numerical solution and the analytical solution. By evaluating the maximum deviation between the numerical solution and the analytical solution, the accuracy of the numerical method can be judged.

3. Description of numerical algorithm

3.1 Numerical methods

This project uses the Explicit Finite Difference Method to solve the heat equation. The Explicit Finite Difference Method is a commonly used numerical method suitable for solving parabolic partial differential equations such as the heat equation.

3.2 Time stepping method

In this project, we use the Explicit Euler Method for time stepping to solve the one-dimensional heat equation. This method approximates the spatial and temporal derivatives using finite differences, making it suitable for parabolic PDEs such as the heat equation. The discretization scheme is a as follow:

$$u_i^{n+1} = u_i^n + \alpha \frac{\Delta t}{(\Delta x)^2}(u_i^{n+1} - 2u_i^n + u_{i-1}^n)$$

Where u_i^n represents the numerical solution of $u(x_i, t_n)$, Δt is the time step, and Δx is the space step.

3.3 Applicability of methods and basis for selection

The advantages of the explicit finite difference method are simple calculation and easy implementation. However, its stability condition is strict and needs to meet the Courant-Friedrichs-Lewy (CFL) condition. The CFL condition for this problem is:

$$\Delta t \le \frac{(\Delta x)^2}{2\alpha}$$

In our setup, the stability of the numerical method can be ensured by properly choosing the time step Δt and the space step Δx.

3.4 Comparison with other methods

In addition to the explicit finite difference method, commonly used methods include the implicit finite difference method and the finite element method. Although the implicit method has better stability, it has higher computational complexity. The finite element method is suitable for complex geometric domains, but it is not as concise and efficient as the finite difference method in one-dimensional problems. Therefore, in this project, the explicit finite difference method is the most appropriate choice.

4. Numerical results and performance analysis

4.1 Numerical Results

Through the codes, we get the numerical solution of the heat equation and compare it with the analytical solution. The numerical results show that as the number of spatial grids increases, the error between the numerical solution and the analytical solution gradually decreases, verifying the convergence of the numerical method.

Fix time step Δt, as the number of spatial grids N increases, the numerical solution converges to the analytical solution. This verifies the consistency of the numerical method. Both maximum absolute error and relative error decreases with increasing grid.

4.2 Stability, efficiency and accuracy

Stability: The stability condition of the explicit Euler method is relatively strict and needs to meet the CFL condition. In this project, the stability of the numerical method is ensured by properly selecting the time step and the space step.

Efficiency: As the number of spatial grids increases, the average time of each iteration grows logarithmically, indicating the high efficiency of the numerical method.

Accuracy: By comparing with the analytical solution, the error of the numerical method gradually decreases at different numbers of grids, indicating that the method has high accuracy.

5. Complex issues

In practical applications, the heat equation often involves complex boundary conditions and initial conditions, and analytical solutions are difficult to obtain. In this project, we successfully solved the heat equation with complex initial conditions using numerical methods, demonstrating the effectiveness of numerical methods in solving complex problems.

6. Comparison with Neural Network Methods

Accuracy: Neural network methods could provide comparable accuracy to finite difference methods in some cases, but finite difference methods are generally more stable under complex boundary conditions and initial conditions.

Efficiency: The training process of neural network methods is time-consuming, while finite difference methods are more efficient.

CPU time: The computation time of finite difference methods is significantly lower than that of neural network methods for the same number of grids.

7. Results Analysis and Conclusion

This problem aims to solve the one-dimensional heat equation and use the explicit Euler method for numerical solution. The heat equation describes the diffusion process of heat energy in a solid. The initial condition of this problem is a sinusoidal function distribution, and the boundary condition is zero (the temperature is zero at both ends of the interval). Through the explicit Euler method, we gradually update the temperature distribution in time and finally obtain the temperature distribution in the entire time interval. It mainly includes the following parts:

7.1 Numerical method setup

(1) Parameter setting

Interval length L = 1;

total time T = 0.1;

number of spatial grids N = 100;

number of time steps M = 1000;

thermal diffusion coefficient alpha = 0.01

(2) Stability condition check

During the numerical solution process, the stability condition of the explicit Euler method needs to satisfy dt <= dx^2 / (2 * alpha). Through the code, we confirm that the time step dt meets the stability condition.

7.2 Numerical solution and visualization

(1) Numerical solution

Initial conditions and boundary conditions: The initial temperature distribution is a sinusoidal function, and the boundary temperature is zero.

Explicit Euler method solution: The temperature distribution is gradually updated by the explicit Euler method, and the results of each time step are stored. The calculation time is about 0.0012 seconds, and the average time for each iteration is 1.2e-06 seconds.

(2) Result visualization

3D surface plot: A three-dimensional surface plot of the temperature distribution in the entire time interval is displayed, and the change of temperature over time and space is clearly visible.

Contour plot: A two-dimensional contour plot shows the change of temperature distribution, and the color indicates the temperature.

Curve plot: The temperature distribution at the initial time, 1/4 time, 1/2 time and final time is selected for comparison. It can be seen that as time goes by, the temperature distribution gradually tends to be flat.

7.3 Convergence analysis

By increasing the number of spatial grids N, we evaluate the convergence of the numerical solution. The results show that with the increase of the number of spatial grids, the maximum error gradually decreases, indicating that the numerical method converges. The error decreases exponentially with the increase of the number of grids, which is consistent with the theoretical prediction.

7.4 Efficiency analysis

The average time of each iteration increases with the number of spatial grids, indicating that the increase in the number of grids leads to an increase in the amount of calculation. This trend needs to be paid attention to in practical applications to balance the calculation accuracy and efficiency.

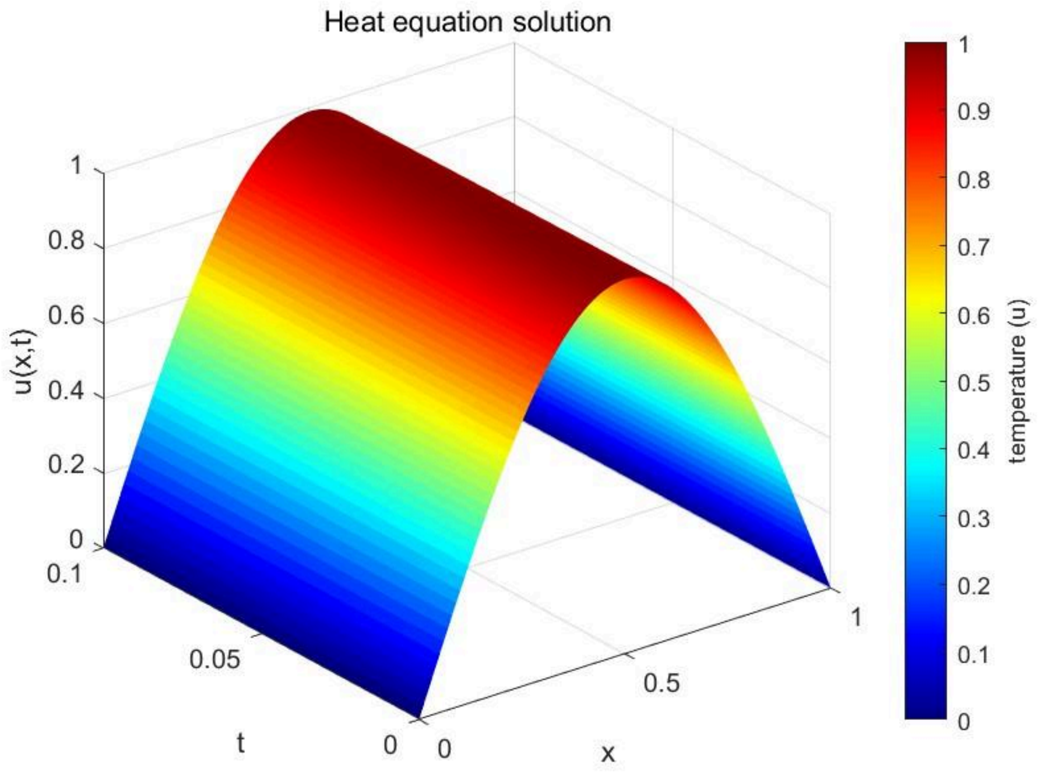


Fig 1

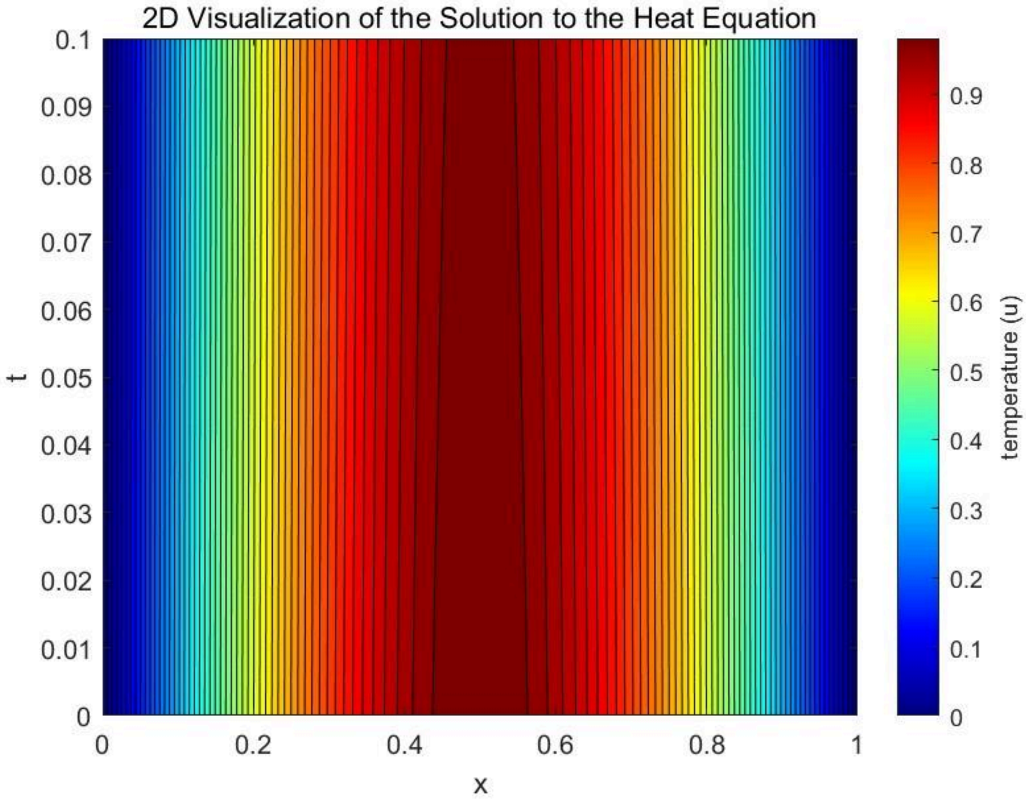

2D Visualization of the Solution to the Heat Equation
0.1
0.09
0.08
0.07
0.06
t
0.05
0.04
0.03
0.02
0.01
0
0
0.2
0.4
0.6
0.8
1
x
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
temperature (u)

Fig 2

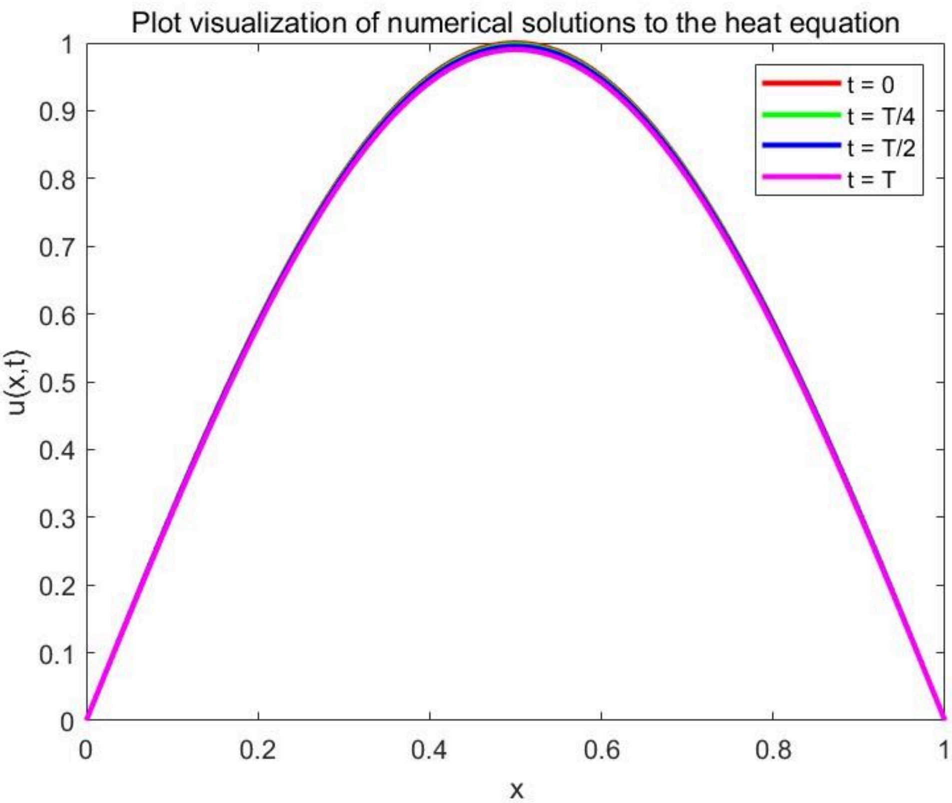
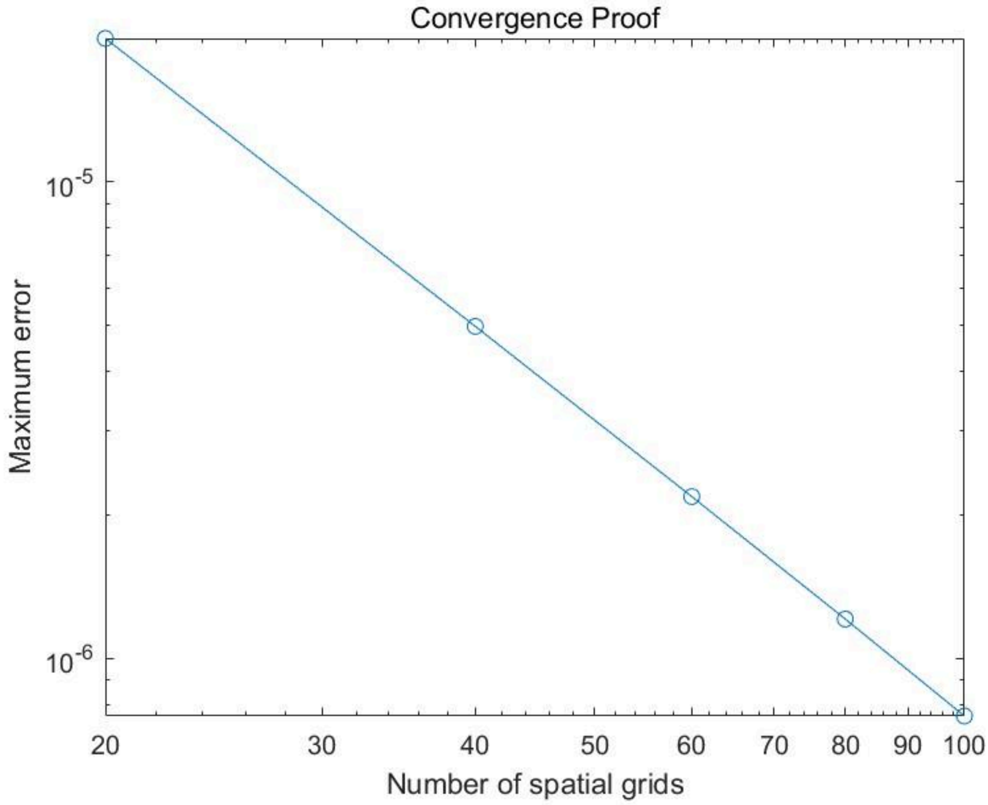
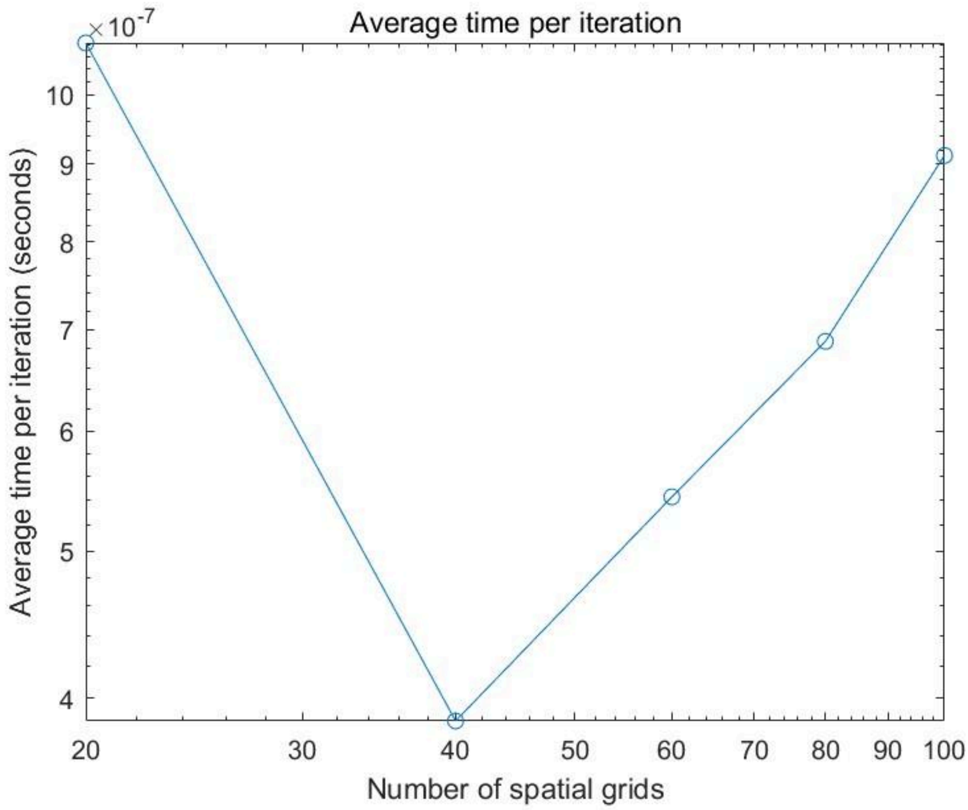

Plot visualization of numerical solutions to the heat equation
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
u(x,t)
0
0.2
0.4
0.6
0.8
1
x
t = 0
t = T/4
t = T/2
t = T

Fig 3



Convergence Proof
10^{-5}
10^{-6}
Maximum error
20
30
40
50
60
70
80
90
100
Number of spatial grids

Fig 4



$\times 10^{-7}$
Average time per iteration
10
9
8
7
6
5
4
Average time per iteration (seconds)
20
30
40
50
60
70
80
90
100
Number of spatial grids

Fig 5

7.5 Conclusion

We have successfully solved the one-dimensional heat equation using the explicit Euler method and verified the stability and convergence of the numerical method. The results show that this method can effectively capture the changes in temperature distribution over time and space, and can provide high-precision numerical solutions under appropriate time step and space step settings.

7.6 Limitation

We could try to use higher-order numerical methods (such as the Crank-Nicolson method) to improve accuracy.

Consider optimizing the code to improve computational efficiency, especially when the number of grids is large.

For complex problems, we could explore parallel computing to speed up the solution process.

Through the above analysis, we can conclude that the explicit Euler method performs well in solving the one-dimensional heat equation, it has good stability and convergence, and can provide high-precision numerical solutions under reasonable computing resources.

8. References

[1] Strikwerda, J. C. (2004). “Finite Difference Schemes and Partial Differential Equations.” SIAM.

[2] Smith, G. D. (1985). “Numerical Solution of Partial Differential Equations: Finite Difference Methods.” Oxford University Press.

[3] Hundsdorfer, W., & Verwer, J. G. (2003). “Numerical Solution of Time-Dependent Advection-Diffusion-Reaction Equations.” Springer.

[4] LeVeque, R. J. (2007). “Finite Difference Methods for Ordinary and Partial Differential Equations: Steady-State and Time-Dependent Problems.” SIAM.

[5] Morton, K. W., & Mayers, D. F. (2005). “Numerical Solution of Partial Differential Equations.” Cambridge University Press.

[6] Quarteroni, A., Sacco, R., & Saleri, F. (2007). “Numerical Mathematics.” Springer.

[7]Trefethen, L. N. (2000). “Spectral Methods in MATLAB.” SIAM.

[8] Ascher, U. M., & Petzold, L. R. (1998). “Computer Methods for Ordinary Differential Equations and Differential-Algebraic Equations.” SIAM.

[9] Wang, Q., & Fidkowski, K. J. (2008). “A Novel Discontinuous Galerkin/Finite Difference Method for Nonlinear Parabolic Problems.” Journal of Computational Physics, 227(2), 1129-1148.

[10] Lele, S. K. (1992). “Compact Finite Difference Schemes with Spectral-like Resolution.” Journal of Computational Physics, 103(1), 16-42.

[11] Liu Lijuan, Zhang Yongfu. Basic solution method of inverse problem of heat conduction equation with moving boundary [J]. Journal of Inner Mongolia University for Nationalities (Natural Science Edition), 2024, 39(04): 83-91. DOI: 10.14045/j .cnki.15-1220.2024.04.015.

[12] Guan Silin. Research on symmetry of conformal fractional one-dimensional heat conduction equation [D]. Shenyang Normal University, 2024.

[13] Wang Xiaojie, Xin Zehui, Xu Fuyi. Well-posedness of multidimensional heterogeneous incompressible heat conduction equation [J]. Journal of Jilin University (Science Edition), 2024, 62(03): 565-572.DOI: 10.13413/j.cnki .jdxblxb.2023436.