HW 1 Yuqi Jin

1. 3.1-4

Yes

$2^{n+1} = O(2^n)$

$0 \le 2^{n+1} \le c \cdot 2^n$ for $n \ge n_0$

$2^{n+1} \le 2 \cdot 2^n$ for $n \ge 1, n_0 = 1$

for $c = 2$

$2^{n+1} = O(2^n)$

No

$2^{2n} \neq O(2^n)$

Assume $2^{2n} = O(2^n)$

$0 \le 2^{2n} \le c \cdot 2^n$

$0 \le 2^n \cancel{2^n} \le c \cdot \cancel{2^n}$

$c \ge 2^n$

2^n is unbounded, no such c can exist.

3.2-7

Prove by induction

Fibonacci

$F_0 = 0$

$F_1 = 1$

$F_i = F_{i-1} + F_{i-2}$ for $i \ge 2$

Let's assume equality holds for $i = k-1$, $i = k-2$

$\phi^2 - \phi - 1 = 0 \quad \phi^2 = \phi + 1$

$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$

$F_k = F_{k-1} + F_{k-2}$

$= \frac{(\phi^{k-1} - \hat{\phi}^{k-1}) + (\phi^{k-2} - \hat{\phi}^{k-2})}{\sqrt{5}}$

$= \frac{(\phi^{k-1} + \phi^{k-2}) - (\hat{\phi}^{k-1} + \hat{\phi}^{k-2})}{\sqrt{5}}$

$$= \frac{\phi^{k-2}(\phi+1) - [\hat{\phi}^{k-2}(\hat{\phi}+1)]}{\sqrt{5}}$$

$$= \frac{\phi^{k-2} \cdot \phi^2 - \hat{\phi}^{k-2}\,\hat{\phi}^2}{\sqrt{5}}$$

$$= \frac{\phi^k - \hat{\phi}^k}{\sqrt{5}}$$

so the inequality holds for k,

4.3-9

$T(n) = 3T(\sqrt{n}) + \log n$

$m = \log n$.

$m = \lg n$

$2^m = n$.

$\sqrt{n} = 2^{m/2}$.

$T(2^m) = 3T(2^{m/2}) + m$

Let $T(2^m) = S(m)$.

$S(m) = 3S(m/2) + m$

Based on the formula, $T(n) = aT(n/b) + f(n)$, the solution will be $O(n^{\log_b a})$

So, $S(m) = 3S(m/2) + m$

$a = 3$, $b = 2$, $\Rightarrow O(m^{\log_2 3}) = O(m^{\lg 3})$

Guess $S(m) \le C m^{\lg 3}$ $\exists C > 0$, $\forall m \ge m_0$

So, $S(m) = 3 \cdot C(m/2)^{\lg 3} + m$

$$= 3 \cdot C \cdot \frac{m^{\lg 3}}{2^{\lg 3}} + m$$

$= \cancel{3} c \cdot \frac{m^{\lg 3}}{\cancel{3}} + m$ (3)

$= c \cdot m^{\lg 3} + m$

It shows that the assumption is wrong

So, assume $S(m) \leq c \cdot m^{\lg 3} - bm$

$$S(m) = 3\left(c(m/2)^{\lg 3} - b(m/2)\right) + m$$

$$= 3 \cdot c \frac{m^{\lg 3}}{\cancel{2^{\lg 3}}} - \frac{3bm}{2} + m$$

$$= cm^{\lg 3} - \frac{3}{2}bm + m$$

$$= cm^{\lg 3} - bm - \frac{b}{2}m + m$$

$$= cm^{\lg 3} - bm - (b/2 - 1)m \quad , \quad b \geq 2$$

$$\leq cm^{\lg 3} - bm.$$

$$T(n) = O(m^{\lg 3})$$

$$= O(\lg n^{\lg 3})$$

$$= O(\lg^{\lg 3} n)$$

3-2

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^{ε}	Yes	Yes	No	No	No
b.	n^k	c^n	Yes	Yes	No	No	No
c	$\sqrt{n}$	$n^{\sin n}$	No	No	No	No	No
d	2^n	$2^{n/2}$	No	No	Yes	Yes	No
e	$n^{\lg c}$	$c^{\lg n}$	Yes	No	Yes	No	Yes
f	$\lg(n!)$	$\lg(n^n)$	Yes	No	Yes	No	Yes

2. place functions in order from asymptotically smallest to largest

$n^2 + 3n\log(n) + 5 < O(n^2)$

$1 = O(1)$

$n^2 + n^{-2} < O(n^2)$

$n^2 + 3n + 5 < O(n^2)$

$n^{n^2} + n! < O(n^{n^2})$

left term $n^{n^2} < O(n^{n^2})$

right term $n! < O(n!)$

add term together. is dominated by n^{n^2}

$\log(n!) < O(n\ln(n))$

$\log(n!) = N\log(N) - N\log(e) + \frac{\log(2\pi N)}{2} + \Omega(1/N)$

$n^{\frac{1}{n}}$ is $O(1)$ when n approaches to infinite

$\sum_{k=1}^{n} \frac{1}{k} = \ln n + \gamma_{Euler} + \Theta(1/N) < O(\ln(n))$

$n^{n^2-1} < O(n^{n^2})$

$\ln n = O(\ln n)$

$\prod_{k=1}^{n} (1 - \frac{1}{k^2}) = O(1)$

when k approaches infinite

$\ln(\ln n) = O(\ln\ln(n))$ it is slow growing rate than $O(\ln(n))$ due to its nested logarithmic function

$(1 - 1/n)^n$

$\lim_{n\to\infty} (1 - \frac{1}{n})^n = e$. is $O(1)$

$3^{\ln(n)} = e^{\ln(3^{\ln(n)})}$

$= e^{\ln(n)\cdot\ln(3)}$

$= n^{\ln(3)}$

$3^{\ln(n)} < O(n^{\ln(3)})$

$2^n = O(2^n)$

$(1+n)^n < O(n^n)$

$n^{1+\cos n} \in O(n^2)$

$\because n^{1+\cos n} = n \cdot n^{\cos n}$ the max for $\cos n = 1$, so $n \cdot n^1 = n^2$

$\sum_{k=1}^{\log n} \frac{n^2}{2^k} : S_N(1-x) = 1 - x^{N+1}$

$S_N = \frac{1 - x^{N+1}}{1-x} \approx \frac{1 - (\frac{1}{2})^{N+1}}{2} < O(n^2)$

Smallest to largest, for each with same asymptotic complexity by using semicolon to order slight diff 5

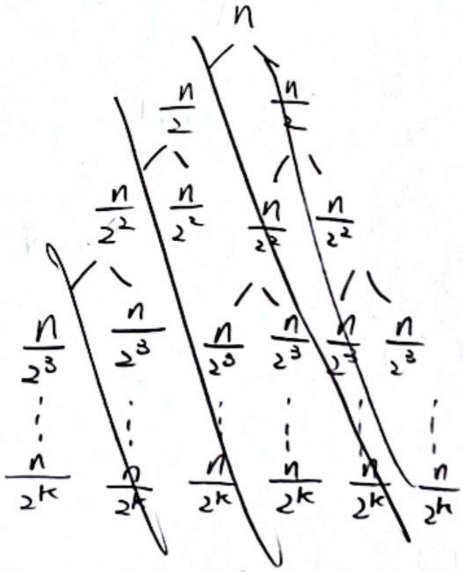
$1 ; \prod_{k=1}^{n}(1-\frac{1}{k^2}) ; (1-\frac{1}{n})^n ; n^{\frac{1}{n}} \ll \ln(\ln(n)) < \ln(n) ; \sum_{k=1}^{n}\frac{1}{k} < \log(n!)$

$< n^2+n^{-2} ; \sum_{k=1}^{\log n}\frac{n^2}{2^k} ; n^{1+\cos n} ; n^2+3n+5 ; n^2+3n\log(n)+5 < 3^{\ln(n)} < 2^n$

$< (1+n)^n < n^{n^2-1} ; n^{n^2}+n!$

1.

(a) $T(n) = \begin{cases} 1 & n=1 \\ 2T(n/2)+n & n>1 \end{cases}$



$T(\frac{n}{2}) = 2T(\frac{n}{2^2})+\frac{n}{2}$

$T(n) = 2[2T(\frac{n}{2^2})+\frac{n}{2}]+n$

$= 2^2 T(\frac{n}{2^2})+n+n$

$T(\frac{n}{2^2}) = 2T(\frac{n}{2^3})+\frac{n}{2^2}$

$T(n) = 2^2[2T(\frac{n}{2^3})+\frac{n}{2^2}]+n+n$

$= 2^3 T(\frac{n}{2^3})+n+2n$

$\vdots$

$T(n) = 2^k T(\frac{n}{2^k})+kn$ Assume $T(\frac{n}{2^k}) = T(1)$

$\frac{n}{2^k} = 1$ $k=\log_2 n$ $n=2^k$ $\Rightarrow T(n) = 2^k T(1)+kn$

$$\cancel{T(n) = 2^k T(1) + kn}$$

$$\cancel{= n \cdot 1 + n\log_2 n}$$

$$\cancel{T(n) = c_1 n + c_2 n\log_2(n)}$$

$$\cancel{c_1 = 1 \quad c_2 = 1}$$

1.

(a) $T(n) = 2T(n/2) + n$

$$T(n) = c_1 n + c_2 n \log_2(n)$$

$$c_1 n + c_2 n \log_2(n) = 2\left(c_1 \frac{n}{2} + c_2 \left(\frac{n}{2}\right)\log_2\left(\frac{n}{2}\right)\right) + n$$

$$= c_1 n + c_2 n\left(\log_2(n) - 1\right) + n$$

$$T(1) = c_1 + c_2 \cdot 1 \cdot \log_2(1) = c_1$$

$$c_1 n + \cancel{c_2 n \log_2(n)} = c_1 n + \cancel{c_2 n \log_2(n)} - c_2 n + n$$

$$c_1 \cancel{n} = c_1 \cancel{n} - c_2 \cancel{n} + \cancel{n}1$$

$$c_2 = 1$$

$$c_1 = T(1)$$

(b) $T(n) = aT(n/b) + n^k$

$$T(n) = c_1 n^r + c_2 n^k \quad , b^r = a, \ r \neq k$$

First drop the n^k. $T(n) = aT(n/b)$.

$$c_1 n^r + c_2 n^k = a\left(c_1\left(\frac{n}{b}\right)^r + c_2\left(\frac{n}{b}\right)^k\right)$$

$$c_1 n^r + c_2 n^k = \cancel{a}\, c_1 \frac{n^r}{\cancel{b^r}} + a c_2 \frac{n^k}{b^k}$$

$$c_1 n^r + c_2 n^k = c_1 n^r + a c_2 \frac{n^k}{b^k}$$

$$\cancel{c_2}\, \cancel{n^k} = a \cancel{c_2} \frac{\cancel{n^k}}{b^k}$$

$$b^k = a$$

$$k = \log_b a$$

because $c_1 n^r = c_1 n^r$, c_1 is real number

drop $aT(n/b)$ $r \neq k$

$$n^k = c_1 n^r + c_2 n^k$$

c_1 would be 0, $n^k = c_2 n^k$, $c_2 = 1$ to satisfy the equation

So dominate one is n^k

(C)

when n goes to infinity.

they have the same power. $r = \frac{\log(a)}{\log(b)} = k$.

Compare $T(n) = c_1 n^k + c_2 \log(n)\, n^k$, left and right part

take the ratio $\frac{c_1 n^k}{c_2\, n^k \log(n)}$ when $n \to \infty$, $\log(n)$ will be growing faster than constant c_1/c_2, it will become the dominant.

So $T(n) = \Theta(n^k \log(n))$ will be leading solution

part (C) work with classmate Adwait