Trinomial Model Construction

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Hedging in a 1-Period Trinomial Model

We consider a 1-period model where the underlying has an initial value $S_0 = S$ and can take 3 possible (future) values at time T that we note and with probability $P(S_T = S_i) = p_i$ and where we assume that $S_1 < S_2 < Se^{rT} < S_3, r$ being the (continuous) interest rate from 0 to T.

1 Basic Model Considerations (10 pts)

(a) Is the Model arbitrage free?
(b) How many martingale measures (i.e. risk neutral measure) are there?
(c) Is the market complete?

Solution:

$$\begin{cases} p_1 + p_2 + p_3 = 1 \\ p_1 S_1 + p_2 S_2 + p_3 S_3 = S \cdot e^{rT} \end{cases}$$

This is a case that 2 equations contain 3 variables (p_1, p_2, p_3). Therefore, there are infinite sets of solutions (p_1, p_2, p_3) . So,
(a) The model is arbitrage free.
(b) There are infinite sets of martingale measures.
(c) The market is not complete.

2 Mean-Variance Hedging (50 pts)

(a) Let's imagine you sold a derivatives with (generic) payoff $g(S_T)$ that you will hedge with a self-financing portfolio with initial endowment p and quantity of underlying Δ. We note $P_T = P_T(p, \Delta)$ the value of the herding portfolio at time T. To hedge our option, we want to minimize

$$\varepsilon(p, \Delta) = E\left[(P_T - g(S_T))^2\right]$$

that is, we want to find the initial portfolio value and delta-hedging strategy that minimizes our quadratic risk.
i. What is the expression P_T of the self-financing portfolio at time T ?
ii. What are the optimal (p, Δ) that minimize $\varepsilon(p, \Delta)$? (give explicit mathematical formulae)

iii. What is the expression of residual error, namely $\varepsilon(p, \Delta)$ for the optimal solution?

Solution:

i. Given that the portfolio is self-financing,

$$P_0 = p = \Delta S + B_0 \Rightarrow B_0 = p - \Delta S$$
$$P_T = \Delta S_T + B_0 e^{rT} = \Delta S_T + (p - \Delta S)e^{rT}$$

ii. The $\varepsilon(p, \Delta)$ expression is:

$$\varepsilon(p, \Delta) = E\left[(p_T - g(S_T))^2\right] = \textstyle\sum_{i=1}^{3} p_i \left(\Delta S_i + (p - \Delta S)e^{rT} - g(S_i)\right)^2$$
$$= \sum_{i=1}^{3} p_i \left(\left(S_i - Se^{rT}\right)\Delta + e^{rT}p - g(S_i)\right)^2$$

The $\frac{\partial \varepsilon(p,\Delta)}{\partial p}$ expression is:

$$\frac{\partial \varepsilon(p,\Delta)}{\partial p} = 2e^{rT} \textstyle\sum_{i=1}^{3} p_i \left(\left(S_i - se^{rT}\right)\Delta + e^{rT}p - g(S_i)\right)$$
$$= 2e^{rT} \textstyle\sum_{i=1}^{3} p_i \left(S_i - Se^{rT}\right) \cdot \Delta + 2e^{2rT} \sum_{i=1}^{3} p_i \cdot p - 2e^{rT} \sum_{i=1}^{3} p_i g(s_i)$$
$$= 2e^{rT} \left[\textstyle\sum_{i=1}^{3} p_i \left(S_i - Se^{rT}\right) \cdot \Delta + e^{rT} \cdot p - \sum_{i=1}^{3} p_i g(S_i)\right]$$

The $\frac{\partial \varepsilon(p,\Delta)}{\partial \Delta}$ expression is:

$$\frac{\partial \varepsilon(p,\Delta)}{\partial \Delta} = 2 \textstyle\sum_{i=1}^{3} p_i \left(S_i - Se^{rT}\right) \left(\left(S_i - Se^{rT}\right)\Delta + e^{rT}p - g(S_i)\right)$$
$$= 2 \textstyle\sum_{i=1}^{3} P_i \left(S_i - Se^{rT}\right)^2 \Delta + 2e^{rT} \sum_{i=1}^{3} P_i \left(S_i - Se^{rT}\right) \cdot p - 2 \sum_{i=1}^{3} P_i \left(S_i - Se^{rT}\right) g(S_i)$$

In order to find the minimum, the necessary condition is to Let $\frac{\partial \varepsilon(p,\Delta)}{\partial p} = \frac{\partial \varepsilon(p,\Delta)}{\partial \Delta} = 0$, and get

The Δ is:

$$\Delta = \frac{-2e^{rT} \sum p_i \left(S_i - Se^{rT}\right) g(S_i) + \sum p_i g(s_i) \cdot 2e^{rT} \sum p_i \left(S_i - Se^{rT}\right)}{\sum p_i \left(S_i - S_{e^{rT}}\right) \cdot 2e^{rT} \sum p_i \left(S_i - Se^{rT}\right) - 2e^{rT} \sum p_i \left(S_i - Se^{rT}\right)^2}$$
$$= \frac{E(g(S_T)) \cdot E(S_T) - E(g(S_T) \cdot S_T)}{E\left(S_T - Se^{rT}\right)^2 - E^2\left(S_T - S_{e^{rT}}\right)} = \frac{Cov(g(S_T), S_T)}{Var\left(S_T - S_{e^{rT}}\right)} = \frac{Cov(g(S_T), S_T)}{Var(S_T)}$$

The p is:

$$p = e^{-rT}\left(\sum_{i=1}^{3} P_i g\left(S_i\right) - \sum_{i=1}^{3} P_i \left(S_i - Se^{rT}\right) \cdot \frac{Cov\left(g\left(S_T\right), S_T\right)}{Var\left(S_T - Se^{rT}\right)}\right)$$
$$= e^{-rT}\left(E\left(g\left(S_T\right)\right) - E\left(S_T - Se^{rT}\right) \cdot \frac{Cov\left(g\left(S_T\right), S_T\right)}{Var\left(S_T\right)}\right)$$

iii. By plugging the above Δ and p in the expression of $\varepsilon(p, \Delta)$, the $\varepsilon(p, \Delta)$ is:

$$\varepsilon(p, \Delta) = \sum_{i=1}^{3} p_i \left(\left(S_i - Se^{rT}\right)\Delta + e^{rT}p - g\left(S_i\right)\right)^2$$
$$= \sum P_i \left(\left(S_i - Se^{rT}\right) \cdot \Delta + E\left(g\left(S_T\right)\right) - E\left(S_T - Se^{rT}\right)\Delta - g\left(S_i\right)\right)^2$$
$$= Var(g(T)) - 2\Delta Cov\left(g\left(S_T\right), S_T\right) + \Delta^2 Var\left(S_T - Se^{rT}\right)$$
$$= Var(g(T)) - \frac{2Cov^2\left(g\left(S_T\right), S_T\right)}{Var\left(S_T\right)} + \frac{Cov^2\left(g\left(S_T\right), S_T\right)}{Var\left(S_T\right)}$$
$$= Var(g(T)) - \frac{Cov^2\left(g\left(S_T\right), S_T\right)}{Var\left(S_T\right)}$$

(b) Find a martingale measure $Q = (q_1, q_2, q_3)$ which does not depend on g (at all) such that

$$p = E^Q\left(g\left(S_T\right)e^{-rT}\right) = \sum_i g\left(S_i\right)e^{-rT}q_i$$

Solution:

$$p = e^{-rT}\left(E\left(g\left(S_T\right)\right) - E\left(S_T - Se^{rT}\right) \cdot \frac{Cov\left(g\left(S_T\right), S_T\right)}{Var\left(S_T\right)}\right)$$
$$= e^{-rT}\left(E\left(g\left(S_T\right)\right) - E\left(S_T - Se^{rT}\right)\frac{Cov\left(g\left(S_T\right), S_T - e^{rT}S\right)}{Var\left(S_T\right)}\right)$$
$$= e^{-rT}\left(E\left(g\left(S_T\right)\right) - E\left(S_T - Se^{rT}\right) \cdot \frac{E\left[g\left(S_T\right) \cdot \left(S_T - e^{rT}S\right)\right] - E\left(g\left(S_T\right)\right) \cdot E\left(S_T - e^{rT}S\right)}{Var\left(S_T\right)}\right.$$
$$= e^{-rT}\left(\sum_{i=1}^{3} p_i g\left(S_i\right) - E\left(S_T - Se^{rT}\right)\frac{\sum_{i=1}^{3} p_i g\left(S_i\right)\left(S_i - e^{rT}S\right) - \sum_{i=1}^{3} p_i g\left(S_i\right) \cdot E\left(S_T - e^{rT}S\right)}{Var\left(S_T\right)}\right)$$
$$= e^{-rT}\left(\sum_{i=1}^{3} p_i g\left(S_i\right) - E\left(S_T - Se^{rT}\right)\frac{\sum_{i=1}^{3} p_i g\left(S_i\right)\left[S_i - e^{rT}S - E\left(S_T - e^{rT}S\right)\right]}{Var\left(S_T\right)}\right.$$
$$= e^{-rT}\sum_{i=1}^{3} p_i g\left(S_i\right)\left[1 - E\left(S_T - Se^{rT}\right) \cdot \frac{S_i - E\left(S_T\right)}{Var\left(S_T\right)}\right]$$

Meanwhile,

$$p = \sum_{i=1}^{3} g\left(S_i\right) e^{-rT} q_i$$

Therefore,

$$q_i = p_i \left[1 - E\left(S_T - Se^{rT}\right) \cdot \frac{S_i - E\left(S_T\right)}{Var\left(S_T\right)}\right]$$

(c) Show that $\sum_i q_i = 1$ and that.

Solution:

$$\sum_{i=1}^{3} q_i = \sum_{i=1}^{3} p_i \left[1 - E\left(S_T - Se^{rT}\right) \cdot \frac{S_i - E\left(S_T\right)}{Var\left(S_T\right)}\right]$$

$$= \sum_{i=1}^{3} P_i - E\left(S_T - Se^{rT}\right) \frac{\sum_{i=1}^{3} P_i S_i - E\left(S_T\right) \cdot \sum_{i=1}^{3} P_i}{Var\left(S_T\right)}$$

$$= 1 - E\left(S_T - Se^{rT}\right) \frac{E\left(S_T\right) - E\left(S_T\right) \times 1}{Var\left(S_T\right)} = 1$$

(d) Compute $E^Q\left(S_T\right)$. What does it mean?

Solution:

$$E^Q\left(S_T\right) = \sum_{i=1}^{3} q_i S_i = \sum_{i=1}^{3} p_i S_i \left[1 - E\left(S_T - Se^{rT}\right) \cdot \frac{S_i - E\left(S_T\right)}{Var\left(S_T\right)}\right]$$

$$= \sum_{i=1}^{3} p_i S_i - E\left(S_T - Se^{rT}\right) \frac{\sum_{i=1}^{3} p_i S_i^2 - E\left(S_T\right) \cdot \sum_{i=1}^{3} p_i S_i}{Var\left(S_T\right)}$$

$$= \sum_{i=1}^{3} p_i S_i - E\left(S_T - Se^{rT}\right) \frac{E\left(S_T^2\right) - E^2\left(S_T\right)}{Var\left(S_T\right)}$$

$$= \sum_{i=1}^{3} p_i S_i - E\left(S_T - Se^{rT}\right) \frac{Var\left(S_T\right)}{Var\left(S_T\right)}$$

$$= E\left(S_T\right) - \left(E\left(S_T\right) - Se^{rT}\right) = Se^{rT}$$

It indicates that $Q = (q_1, q_2, q_3)$ is a matingale measure both for S (the price of the underlying asset) and $g(S)$ (the payoff of derivatives).

(e) Show that $0 < q_i < 1$ for all i.

Solution:

$$\forall i, \quad S_i - E(S_T) < \frac{Var\left(S_T - S_{e^{rT}}\right)}{E\left(S_T - S_{e^{rT}}\right)}$$

Therefore,

$$q_i = p_i \left[1 - E\left(S_T - S_e^{rT}\right) \cdot \frac{S_i - E(S_T)}{Var(S_T)}\right] > 0$$

Plus, in (d) I prove:

$$\sum_{i=1}^{3} q_i = 1$$

Therefore, $\forall i, o < q_i < 1$

3 Comparison with Binomial Tree (60 pts)

We set $S = 100, S_1 = 92, S_2 = 98, S_3 = 105, r = 0$ and $P(S_T = S_i) = p_i$ with $(p_1, p_2, p_3) = (1/3, 1/4, 5/12)$ and assume that we sold a put option with strike $K = 100$.

(a) What are the values of p and Δ in this specific trinomial model?

Solution:

$S = K = 100$
$S_1 = 92, \quad p_1 = \frac{1}{3}, g(S_1) = (K - S_1)^+ = 8$
$S_2 = 98, \quad p_2 = \frac{1}{4}, g(S_2) = (K - S_2)^+ = 2$
$S_3 = 105, \quad p_3 = \frac{5}{12}, g(S_3) = (K - S_3)^+ = 0$

According to discussions and solutions in Task 2,

$$\begin{cases} \frac{\partial \varepsilon(p,\Delta)}{\partial p} = 0 \\ \frac{\partial \varepsilon(p,\Delta)}{\partial \Delta} = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \frac{1}{3}(-8\Delta + p - 8) + \frac{1}{4}(-2\Delta + p - 2) + \frac{5}{12}(5\Delta + p - 0) = 0 \\ -\frac{8}{3}(-8\Delta + p - 8) - \frac{2}{4}(-2\Delta + p - 2) + \frac{25}{12}(5\Delta + p - 0) = 0 \end{cases}$$

$$\Rightarrow \begin{cases} \Delta = -\frac{2722}{4547} \approx -0.5986 \\ p = \frac{11450}{4547} \approx 2.5181 \end{cases}$$

(b) Because $S_2 < Se^{rT} < S_3$, we think of keeping $S_3 = 105$ the same and look for a value $\tilde{S}_1$ such that $\Delta_{bin} = \Delta$ and the binomial model would have two

possible outcomes, namely $\tilde{S}_1$ that needs to be computed and $S_3 = 105$.
i. Explain why there is a solution $\tilde{S}_1$ such that $\Delta_{bin} = \Delta$ and how to compute it.
ii. What is the value of p_{bin} ?

Solution:

i. For the equal-delta binomial model, there is:

$$\begin{cases} g\left(\tilde{S}_1\right) = \Delta_{bin}\tilde{S}_1 + (P_{bin} - \Delta_{bin}S) \cdot e^{rT} \\ g\left(S_3\right) = \Delta_{bin}S_3 + \left(P_{bin} - \Delta_{bin} \cdot S\right) e^{rT} \end{cases}$$

There are 2 equations and 2 variables $\tilde{S}_1$ and o_{bin}, so there must be a solution $(\tilde{S}_1, o_{bin})$, whose values are shown later.

$\Rightarrow \Delta_{bin} = \frac{g(\tilde{S}_1) - g(S_3)}{\tilde{S}_1 - S_3}$

Plus, given that $\begin{cases} \Delta_{bin} = \Delta \\ g\left(\tilde{S}_1\right) = \left(K - \tilde{S}_1\right)^+ = K - \tilde{S}_1 \end{cases}$

where the second equation is because there must be $\tilde{S}_1 < K < S_3$ in the binomial model.

Plugging these conditions in the equations above, we get:

$$\tilde{S}_1 = \frac{33778}{365} \approx 92.5425$$

ii. It is aforementioned in i. that

$$g\left(S_3\right) = \Delta_{bin}S_3 + \left(P_{bin} - \Delta_{bin} \cdot S\right) e^{rT}$$

Therefore,

$$P_{bin} = e^{-0T}\left(0 + \frac{2722}{4547} \cdot 105\right) - \frac{2722}{4547} \cdot 100 = \frac{13610}{4547} \approx 2.9932$$

(c) We assume that in reality, the stock realized at 92 , that is $S_T = 92$, which was a possible value in the trinomial model. The P&L of the hedge position is the difference between the value of the hedging portfolio and the derivatives payoff (we are supposed to pay to our client), that is $P\&L = P_T - (K - S_T)^+$.
i. What if the P&L of the hedged position if we used the binomial model?
ii. What if the P&L of the hedged position if we used the trinomial model?
iii. What do you conclude?

Solution:

i.
$$P\&L_{bin} = \Delta_{bin} S_T + (P_{bin} - \Delta_{bin} S) \cdot e^{rT} - (K - S_T)^+$$
$$= -\frac{2722}{4547} \times 92 + \left(\frac{13610}{4547} + \frac{2722}{4547} \times 100 \right) \cdot e^0 - (100 - 92)^+$$
$$= -\frac{990}{4547} \approx -0.2177$$

ii.
$$P\&L_{tri} = \Delta S_T + (P - \Delta S) \cdot e^{rT} - (K - S_T)^+$$
$$= -\frac{2722}{4547} \times 92 + \left(\frac{11450}{4547} + \frac{2722}{4547} \times 100 \right) \cdot e^0 - (100 - 92)^+$$
$$= -\frac{3150}{4547} \approx -0.6928$$

iii. When $S_T = 92$, the equal-delta binomial model makes fewer loss than trinomial model.

(d) We now consider "out-of-sample" scenarii for the first model:
i. What if the underlying drops even more, namely $S_T = 90$: what is the PL from using each model?
ii. What if the underlying goes up to $S_T = 105$: what is the PL from using each model?

Solution:

i. Similar to (c) above, I can get:

$$P\&L_{bin} = \Delta_{bin} S_T + (P_{bin} - \Delta_{bin} S) \cdot e^{rT} - (K - S_T)^+$$
$$= -\frac{2722}{4547} \times 90 + \left(\frac{13610}{4547} + \frac{2722}{4547} \times 100 \right) \cdot e^0 - (100 - 90)^+$$
$$= -\frac{4640}{4547} \approx -1.0204$$

$$P\&L_{tri} = \Delta S_T + (P - \Delta S) \cdot e^{rT} - (K - S_T)^+$$
$$= -\frac{2722}{4547} \times 90 + \left(\frac{11450}{4547} + \frac{2722}{4547} \times 100 \right) \cdot e^0 - (100 - 90)^+$$
$$= -\frac{6800}{4547} \approx -1.4955$$

ii. Similarly:

$$P\&L_{bin} = \Delta_{bin} S_T + (P_{bin} - \Delta_{bin} S) \cdot e^{rT} - (K - S_T)^+$$

$$= -\frac{2722}{4547} \times 105 + \left(\frac{13610}{4547} + \frac{2722}{4547} \times 100 \right) \cdot e^0 - (100 - 105)^+$$
$$= 0$$

This is because I use the $P\&L_{bin} = 0$ when $S_T = 105$ when constructing the binomial model.

$$P\&L_{tri} = \Delta S_T + (P - \Delta S) \cdot e^{rT} - (K - S_T)^+$$
$$= -\frac{2722}{4547} \times 105 + \left(\frac{11450}{4547} + \frac{2722}{4547} \times 100 \right) \cdot e^0 - (100 - 105)^+$$
$$= -\frac{2160}{4547} \approx -0.4750$$

(e) What if instead we choose $\tilde{S}_1$ so that $p_{bin} = p$.
i. How to compute the solution for such $\tilde{S}_1$ in general and what is its numerical value in this specific example?
ii. What is the corresponding Δ_{bin} ?
iii. Compare the P&L (in a table) of this model for each previous senarii against the trinomial model.

Solution:

i. For the equal-p binomial model, there is:

$$\begin{cases} g\left(\tilde{S}_1\right) = \Delta_{bin}\tilde{S}_1 + (P_{bin} - \Delta_{bin} S) \cdot e^{rT} \\ g\left(S_3\right) = \Delta_{bin} S_3 + \left(P_{bin} - \Delta_{bin} \cdot S\right) e^{rT} \end{cases}$$

$\Rightarrow \Delta_{bin} = \frac{g\left(\tilde{S}_1\right) - g(S_3)}{\tilde{S}_1 - S_3} = \frac{g\left(\tilde{S}_1\right) - p_{bin} \cdot e^{rT}}{\tilde{S}_1 - S e^{rT}}$

Plus, given that $\begin{cases} P_{bin} = P \\ g\left(\tilde{S}_1\right) = \left(K - \tilde{S}_1\right)^+ = K - \tilde{S}_1 \end{cases}$

where the second equation is because there must be $\tilde{S}_1 < K < S_3$ in the binomial model.

Plugging these conditions in the equations above, we get:

$$\frac{100 - \tilde{S}_1 - \frac{11450}{4547} e^{0T}}{\tilde{S}_1 - 100 \cdot e^{0T}} = \frac{100 - \tilde{S}_1 - 0}{\tilde{S}_1 - 105}$$

Solving this function, we get:

$$\tilde{S}_1 = \frac{214250}{2257} \approx 94.9269$$

ii. According to the expression of Δ_{bin} in i.,

$$\Delta_{bin} = \frac{g\left(\tilde{S}_1\right) - g\left(S_3\right)}{\widetilde{S_1} - S_3} = \frac{100 - \frac{214250}{2257} - 0}{\frac{214250}{2257} - 105} = -\frac{2290}{4547} \approx 0.5036$$

iii. Similarly to (c) and (d), I can get:

When $S_T = 92$,

$$P\&L_{bin} = \Delta_{bin} S_T + (P_{bin} - \Delta_{bin} S) \cdot e^{rT} - (K - S_T)^+$$
$$= -\frac{2290}{4547} \times 92 + \left(\frac{11450}{4547} + \frac{2290}{4547} \times 100\right) \cdot e^0 - (100 - 92)^+$$
$$= -\frac{6606}{4547} \approx -1.4528$$

$P\&L_{tri}$ is calculated in (c):

$$P\&L_{tri} = \Delta S_T + (P - \Delta S) \cdot e^{rT} - (K - S_T)^+$$
$$= -\frac{3150}{4547} \approx -0.6928$$

When $S_T = 90$,

$$P\&L_{bin} = \Delta_{bin} S_T + (P_{bin} - \Delta_{bin} S) \cdot e^{rT} - (K - S_T)^+$$
$$= -\frac{2290}{4547} \times 90 + \left(\frac{11450}{4547} + \frac{2290}{4547} \times 100\right) \cdot e^0 - (100 - 90)^+$$
$$= -\frac{1112}{4547} \approx -2.4456$$

$P\&L_{tri}$ is calculated in (d):

$$P\&L_{tri} = \Delta S_T + (P - \Delta S) \cdot e^{rT} - (K - S_T)^+$$
$$= -\frac{6800}{4547} \approx -1.4955$$

When $S_T = 105$,

$$P\&L_{bin} = \Delta_{bin} S_T + (P_{bin} - \Delta_{bin} S) \cdot e^{rT} - (K - S_T)^+$$
$$= -\frac{2290}{4547} \times 105 + \left(\frac{11450}{4547} + \frac{2290}{4547} \times 100\right) \cdot e^0 - (100 - 105)^+$$
$$= 0$$

This is because I use the $P\&L_{bin} = 0$ when $S_T = 105$ when constructing the binomial model.

$P\&L_{tri}$ is calculated in (d):

$$P\&L_{tri} = \Delta S_T + (P - \Delta S) \cdot e^{rT} - (K - S_T)^+$$
$$= -\frac{2160}{4547} \approx -0.4750$$

Table 1: P&L of 3 Models

P&L		S_T		
		90	92	105
model	equal-delta binomial	-1.0204	-0.2177	0
	equal-p binomial	-2.4456	-1.4528	0
	trinomial	-1.4955	-0.6928	-0.4750

(f) Summarize a table the P&L of each model for all the previous scenarii (and feel free to add some, best is to code it and show a graph).

Solution:

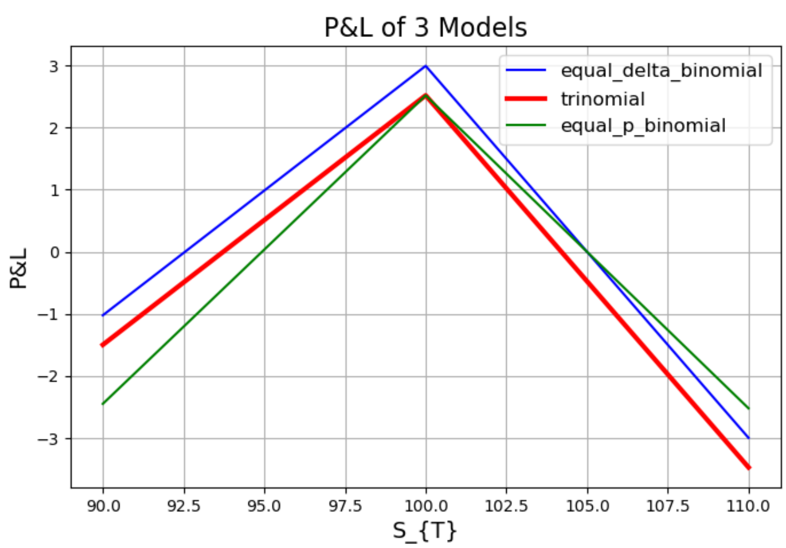

Figure 1: PNL of 3 Models

Conclusions:

1. The equal-delta binomial model, the equal-p binomial model, and the trinomial model reach zero P&L respectively at around (92.5, 105), (95.0, 105),

and (93.7, 104.2). These values of binomial models are exactly what I set when constructing the trees.

2. The equal-delta binomial model always have greater P&L than the trinomial model. And they have the same slope and trending. This is because in their expressions only the y-intercept containing the p_{bin} is different. The equal-delta binomial model has a higher p_{bin}.

3. When $S_T < S = K$, the equal-p binomial model has lower P&L than the trinomial model. However, when $S_T > S = K$, the equal-p binomial model has greater P&L than the trinomial model. When $S_T = S = K$, they reach the same P&L.

4. When $S_T < S_3 = 105$, the equal-p binomial model has lower P&L than the equal-delta binomial model. However, when $S_T > S_3 = 105$, the equal-p binomial model has greater P&L. When $S_T = S_3 = 105$, they reach the same P&L.