Fundamental Algorithms
Assignment 11
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1. Consider Dumb Prim for MST. The high level idea is the same but to find the minimal weight of an edge $\{v, w\}$, $v \in S$, $w \notin S$, one looks at all the weights and finds the minimum in the usual way. (There is no updating in Dumb Prim.) Assume that all pairs $\{v, w\}$ have a weight. Let n be the number of vertices.

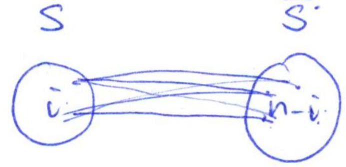
   (a) When $|S| = i$ what is the time to add a vertex to S as a function of n and i.

   (b) What is the total time for Dumb Prim as a sum over i.

   (c) Evaluate the above sum as $\Theta(g(n))$ for some nice function $g(n)$. (Caution: The time is *not* an increasing function of i. For example, when $i = n - 1$ the time is quite quick.)

   (d) Compare the time for Dumb Prim with Prim as discussed in class

(a)



$\Omega(i(n-i))$

(b) $\sum_{i=1}^{n}(i(n-i)) = \sum_{i=1}^{n} in - \sum_{i=1}^{n} i^2 = (n+2n+3n+\cdots n^2) - (1^2+2^2+\cdots n^2)$

$= \frac{n(n+1)n}{2} - \frac{n(n+1)(2n+1)}{6}$

$\sim O(n^3) \quad = \frac{n(n+1)(n-1)}{6}$

(c) $g(n) = n^3$

(d) Prim is faster, which uses $O(n\lg n)$ time.
Dumb Prim takes more time. $O(n^3)$.

2. Consider Prim's Algorithm for MST on the complete graph with vertex set $\{1, \ldots, n\}$. Assume that edge $\{i, j\}$ has weight $(j-i)^2$. Let the root vertex $r = 1$. Show the pattern as Prim's Algorithm is applied. In particular, Let $n = 100$ and consider the situation when the tree created has 73 elements and π and *key* have been updated.

   (a) What are these 73 elements.

   (b) What are $\pi[84]$ and $key[84]$.

(a) These elements are 1.2.3 ... 73 which edge $\{1,2\}=1$, $\{2,3\}=1$ ...
$\{72,73\}=1$

(b) $\pi[84] = 73$.
$k[84] = (84-73)^2 = 11^2 = 121$

3. Find $d = \gcd(89, 55)$ and x, y with $89x + 55y = 1$. [Remark: This is part of a pattern with two consecutive numbers from the Fibonacci sequence $0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, \ldots$]

$\gcd(89, 55) = 1$

Euclid $(89, 55)$ $\quad 89 = 1\times55 + 34$

Euclid $(55, 34)$ $\quad 55 = 1\times34 + 21$

Euclid $(34, 21)$ $\quad 34 = 1\times21 + 13$

Euclid $(21, 13)$ $\quad 21 = 1\times13 + 8$

Euclid $(13, 8)$ $\quad 13 = 1\times8 + 5$

Euclid $(8, 5)$ $\quad 8 = 1\times5 + 3$

Euclid $(5, 3)$ $\quad 5 = 1\times3 + 2$

Euclid $(3, 2)$ $\quad 3 = 1\times2 + 1$

Euclid $(2, 1)$ $\quad 2 = 2\times1 + 0$

Euclid $(1, 0) = 1$

$= 13\times55 - 21(89-55) = \boxed{(-21)\times89 + (34)\times55}$

$= 13\times55 - 21\times34$

$= 13\times(55-34) - 8\times34$

$= 13\times21 - 8\times34$

$= 2[2\times21 \cancel{-34+21} - 3(34-21)] - [2(34-21)-21]$

$= 2[(21-13)-(13-21+13)] - (13-21+13)$

$= 2[8-(13-8)] - (13-8)$

$= (8-(13-8)) - ((13-8) - 8\times(13-8))$

$= (8-1\times5) - (5 - 8\times5)$

$= 3 - 1\times(5 - 1\times3)$

$1 = 3 - 1\times2$

$\therefore$ $x = -21$ $\quad y = 34$

4. Find $\frac{211}{507}$ in Z_{1000}.

$211 \times (507)^{-1}$ in Z_{1000}

$\gcd(1000, 507) = 1$

Euclid $(1000, 507)$ $\quad 1000 = 1 \times 507 + 493$

Euclid $(507, 493)$ $\quad 507 = 1 \times 493 + 14$

Euclid $(493, 14)$ $\quad 493 = 35 \times 14 + 3$

Euclid $(14, 3)$ $\quad 14 = 4 \times 3 + 2$.

Euclid $(3, 2)$ $\quad 3 = 2 \times 1 + 1$

Euclid $(2, 1)$ $\quad 2 = 2 \times 1 + 0$.

$= 181 \times 1000 - 357 \times 507$.

$= 181 \times (1000 - 507) - 176 \times 507$

$= 181 \times 493 - 176 \times 507$

$= 5 \times 493 - 176 \times (507 - 493)$

$= 5 \times 493 - 176 \times 14$

$= 5 \times (493 - 35 \times 14) - 14$

$= 5 \times 3 - 14$

$1 = 3 - 2$.

$\therefore \; 507^{-1} = -357 = 643$

Thus, $211 \times 643 = 135673$.

5. Solve the system
$x \equiv 34 \bmod 101$
$x \equiv 59 \bmod 103$.

$x = 101y + 34 = 101 \times 39 + 34 = 3973 \bmod 10403$.

$101y + 34 = 59 \bmod 103$

$101y = 25 \bmod 103$

$\because \gcd(103, 101) = 1$

$1 = 101 \times 51 + 103 \times (-50)$

$\therefore \; 101^{-1} = 51$

$y = 101^{-1} \cdot 101y = 51 \times 25 = 1275 \bmod 103 = 39 \bmod 103$

6. Using the Island-Hopping Method to find 2^{1000} modulo 1001 using a Calculator but NOT using multiple precision arithmetic. (You should never have an intermediate value more than a million.)

$A^B \bmod M$. $\quad 2^{1000} \bmod 1001$

$B = 1000 = \underset{S_1}{2^9} + \underset{S_2}{2^8} + \underset{S_3}{2^7} + \underset{S_4}{2^6} + \underset{S_5}{2^5} + \underset{S_6}{2^3}$

$A^B = A^{2S_1} \cdot A^{2S_2} \cdots 2^{2S_6} = X_9 X_8 X_7 . X_6 . X_5 X_3$

$X_0 = 2$

$X_1 = 4 \bmod 1001 = 4$

$X_2 = 16 \bmod 1001 = 16$

$X_3 = 256 \bmod 1001 = 256$ ✓

$X_4 = (256)^2 \bmod 1001 = 471$

$X_5 = (471)^2 \bmod 1001 = 620$ ✓

$X_6 = (620)^2 \bmod 1001 = 16$ ✓

$X_7 = (16)^2 \bmod 1001 = 256$ ✓

$X_8 = (256)^2 \bmod 1001 = 471$ ✓

$X_9 = (471)^2 \bmod 1001 = 620$ ✓

$= 620 \times 471 \times 256 \times 16 \times 620 \times 256$

$= (620)^2 \times (256)^2 \times 471 \times 16$

$= 189847201382400$.

Thus. $A^B \bmod M = 189847201382400 \bmod 1001 = 562$.

7. (extra from last week!) Suppose that during Kruskal's Algorithm (for MST) and some point we have $SIZE[v] = 37$. What is the interpretation of that in the case when $\pi[v] = v$? What is the interpretation of that in the case when $\pi[v] = u \neq v$? Let w be a vertex. How many different values can $\pi[w]$ have during the course of Kruskal's algorithm? How many different values (as a function of V, the number of vertices) can $SIZE[w]$ have during the course of Kruskal's algorithm? (That is, the maximal number possible.)

① when $\pi[v] = v$. it is a root.

② $\pi[v] = u \neq v$: size[v] < size[u]
and the root. is updated/reset to u.

③ $\pi[w] = w$. when it is a root itself
$\pi[w] = u$. when size[v] < size[u]

④ V different values.