Efficient Belief Propagation List Decoding of Polar Codes

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***Abstract*—As known to all, polar codes have been chosen as the control channel codes for the enhance mobile broadband (eMBB) scenario in the 3GPP RAN1. Due to its excellent performance, polar codes also have caused widespread concern in the academia and industry. In general, polar codes can be decoded by two methods: successive cancellation (SC) and belief propagation (BP) algorithms. However, compared with the series of SC algorithms, the error-correction performance of BP decoding is not satisfactory, even though it has excellent parallel throughput. Hence, this paper proposes an efficient BP list (EBPL) decoding of polar codes which can enhance the performance to the same scale of successive cancellation list (SCL) without sacrificing decoding throughput. With reasonable complexity, the proposed EBPL decoding can achieve comparable BER and FER compared to SCL in simulation results.**

***Index Terms*—Polar codes, belief propagation list (BPL), cyclic redundancy check (CRC)**

I. Introduction

Polar codes can be proved to fully achieve Shannon limit in the binary discrete memoryless channel (B-DMC). In [1], SC decoding was proposed by Arıkan. And then SCL and CA-SCL decoding which is combined with CRC can outperform the low-density parity-check (LDPC) codes used in the WiMAX standard [2]. Compared with the development of performance, the latency of original SCL will grow linearly with the code length. After that, simplified successive cancellation (SSC) and simplified successive cancellation list (SSCL) were put forward to deal with the delay brought by serial decoding.

In addition, BP decoding as a parallel decoding was proposed in [3]. Correspondingly, its hardware design has been addressed in [4], [5]. Due to the parallelism of BP algorithm, it can easily complete high rate decoding. However, BP decoding has two neural disadvantages. Firstly, a huge number of iterations requires BP decoders to undertake large computation complexity which is caused by slow convergence. Early stopping criteria which can effectively reduce the iterations are proposed [6], [7]. Secondly, although many efforts have been spent on enhancing the performance of BP decoding, it still can't compare with that of SCL and CA-SCL decoding. In [8], it utilized a short LDPC as a outer codes to protect intermediate bit channels, which can significantly improve the error-correction performance. And then [9] can gain a bit forward according to the alteration of information bits and frozen bits by the evolution of log-likelihood ratios (LLRs). In [10], it changes the permutations of factor graph to construct different paths to decode polar codes which path is decoded by BP algorithm directly. However, for long code length ($N \geq 2048$), large path size ($L \geq 200$) and moderate E_b/N_0 ($2 \sim 3$ dB) rarely, [10] has a similar performance as the same size of SCL decoding.

In this paper, we propose an efficient belief propagation list (EBPL) decoding to reach at the similar level of SCL in the common cases. Firstly, transforming the specific bits of R initialization layer, the proposed renew min-sum (RMS) decoding is aimed to enhance the speed of convergence. Then the EBPL decoding which is combined with RMS and BPL decoding can reach the above objectives actually.

The reminder of this paper is organized as follows. Section II reviews the related knowledge of polar codes and BPL decoding. Section III proposes two novel BP decoding algorithms: RMS decoding and EBPL decoding. The simulations of BER and FER are described in Section IV.

II. Preliminaries

A. Polar Codes

By channel polarization phenomenon which means recursively combining and spitting channels, the performance of N binary channels is extremely distributed [1]. Selecting K most reliable coordinate channels to transmit information by Gaussian approximation [11], we usually use $(N, K, \mathcal{A}, u_{\mathcal{A}^c})$ to represent polar codes, where $\mathcal{A}$ is the set of corresponding indices of information bits and $u_{\mathcal{A}^c}$ denotes the remaining frozen bits. Then, u_1^N is multiplied by generator matrix $\mathbf{G}$ to generate N-bit transmitted codeword x_1^N as follows Eq. (1):

$$x_1^N = u_1^N \mathbf{G} = u_1^N \mathbf{F}^{\otimes \log_2^N}, \mathbf{F} = \left[\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right], \tag{1}$$

where Fig. 1 is the construction of $N = 8$ polar codes.

B. BP Decoding of Polar Codes

BP decoding is a process to refine the estimations of the codeword x_1^N and the message u_1^N by iterative operation in a $\mathbf{m}$ $=\log_2 N$ stage factor graph which represents an (N, K) polar code. Every node in the factor graph undertakes the function to propagate left-to-right and right-to-left likelihood

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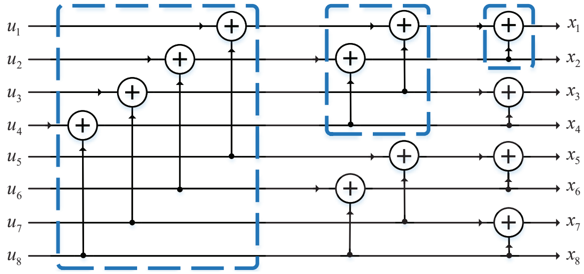

Fig. 1. Construction of polar codes with $N = 8$.

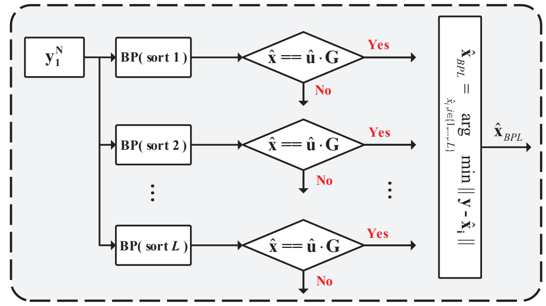

Fig. 2. Structure of Belief Propagation List decoder.

messages which are expressed by $R^t_{i,j}$ and $L^t_{i,j}$ [4]. The formulas are written as follows Eq. (2):

$$\begin{aligned} L^t_{i,j} &= \mathbf{g}(L^{t-1}_{i,j+1}, L^{t-1}_{i+N/2^j,j+1} + R^t_{i+N/2^j,j}), \\ L^t_{i+N/2^j,j} &= \mathbf{g}(L^{t-1}_{i,j+1}, R^t_{i,j}) + L^{t-1}_{i+N/2^j,j+1}, \\ R^t_{i,j+1} &= \mathbf{g}(R^t_{i,j}, L^{t-1}_{i+N/2^j,j+1} + R^t_{i+N/2^j,j}), \\ R^t_{i+N/2^j,j+1} &= \mathbf{g}(R^t_{i,j}, L^{t-1}_{i,j+1}) + R^t_{i+N/2^j,j}, \end{aligned} \tag{2}$$

where $\mathbf{g}(a,b) = sign(a)sign(b)min(|a|,|b|)$, t denotes the number of iteration. In [12], it proposes a scaled min-sum algorithm (SMS) which introduces a parameter α, $\mathbf{g}(a,b) = \alpha * sign(a)sign(b)min(|a|,|b|)$. In this paper, our algorithm is based on SMS decoding.

Moreover, the most right column of factor graph $R(i,1)$ is mapped to 0 and ∞ based on the indices of information bits and frozen bits, and the most left column is the received signals LLRs as Eq. (3).

$$R_{i,1} = \begin{cases} 0, \text{ if } i \in \mathcal{A}, \\ \infty, \text{ if } i \in \mathcal{A}^c, \end{cases} \quad L_{i,m+1} = In\frac{P(y_i|x_i=0)}{P(y_i|x_i=1)}. \tag{3}$$

C. BPL Decoding of Polar Codes

For any kind of polar codes with definite construction, its factor graph like Fig. 1 exists $\mathbf{m}!$ different permutations. According to transforming the order of stages, BP decoding can obtain different paths, which path contains different loops correspondingly. BPL decoder in [10] consists of L parallel independent BP decoder with different permuted factor graph respectively, which is slightly similar with ML decoder. In the end of BPL decoder, there is an Euclidean distance computation as Eq. (4) to pick the codewords most closed to channel output y_1^N as the output of BPL decoder.

$$\hat{\boldsymbol{x}}_{BPL} = \underset{\hat{\boldsymbol{x}}_i, i\in 1\cdots L}{\arg} \min ||\boldsymbol{y} - \hat{\boldsymbol{x}}_i||. \tag{4}$$

The gain of BPL decoding is mainly from selecting the better permuted factor graph for the specific set of inputs.

III. Proposed RMS and EBPL Decoding

According to the above algorithms, we can find that the reason for the lack of performance of BP decoding is mainly from the loops in the factor graph which are caused by iterative computation. Compared with factor graph, decoding tree [1] in the series of SC decoding indeed can't cause loops. Hence, in this section, we propose a novel algorithm called renew min-sum (RMS) decoding which transforms the $R(i,1)$ in Eq. (3) in every iteration to accelerate the speed of convergence. Then, combining RMS with the above BPL decoder, we propose the final algorithm called efficient belief propagation list (EBPL) decoding as follows Fig. 3. In addition, due to the feature of early stopping criteria [6] which can't enhance the performance, RMS and EBPL are all joint with CRC to assist the final judgement in the extremity of decoder.

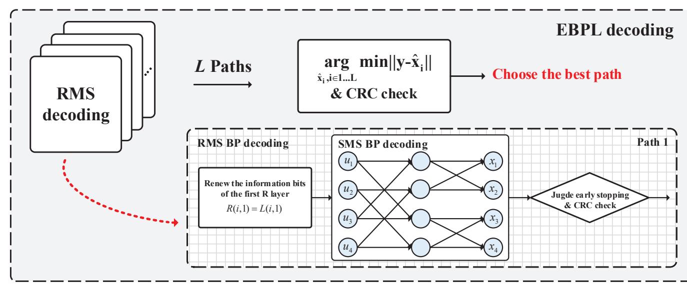

Fig. 3. Framework of the proposed EBPL decoder.

A. Structure of Proposed RMS Decoding

Considering the structure of BP factor graph, we can find that $R(i,1)$ which is the initialization column of R layer as follows Eq. (3) is the major reason for the slow convergence actually. In the process of propagating R information, there are rarely four cases to update R layer message.

$$R_{i,1} = L_{i,1}, \ i \in reverse(\mathcal{A}). \tag{5}$$

From the Table. I, the messages of R layer update only when $R^t_{i,j}$ is the frozen bit and $R^t_{i+N/2^j,j}$ is the information bit directly, that means belief propagation of R layer is an inefficient process and has invalid and redundant operation to slow down the convergence of algorithm. Hence, we propose the RMS decoding to renew the value of information bits in $R(i,1)$ to accelerate the speed as follow Eq. (5). In each iteration, RMS decoding will perform two operations:

- Alter a new $R(i,1)$ which i is the reverse index of $\mathcal{A}$;
- Update the $R_{i,1}$s which have been altered in the previous iterations to the current $L(i,1)$ value correspondingly.

To reach better performance, several modification is utilized in the RMS decoding. Firstly, the order of transforming $R_{i,1}$ follows the reverse order of $\mathcal{A}$ that means the most unreliable

TABLE I
BELIEF PROPAGATION PROCESS OF R LAYER INFORMATION.

$[\boldsymbol{R}^t_{i,j}, \boldsymbol{R}^t_{i+N/2^j,j}]$	[0, 0]	[0, ∞]	[∞, ∞]	[∞, 0]
$\boldsymbol{R}^t_{i,j+1}$	0	0	∞	$L^{t-1}_{i+N/2^j,j+1}$
$\boldsymbol{R}^t_{i+N/2^{j+1},j+1}$	0	∞	∞	$L^{t-1}_{i,j+1}$

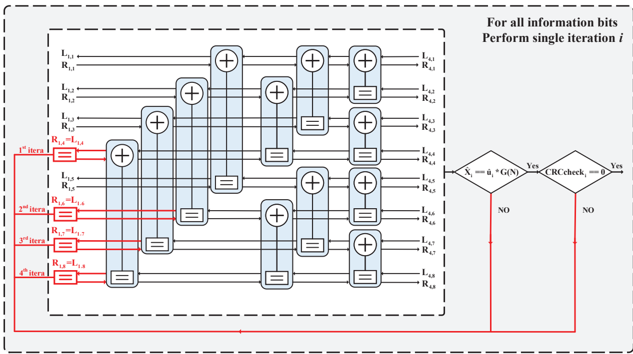

Fig. 4. Hardware of the proposed RMS decoder for $(8, 4)$ polar codes.

information channels will renew $R_{i,1} = L_{i,1}$ more times than other reliable bits. The results of simulation also verify this point. Secondly, the final judgement is designed by combining early stopping criteria with CRC. In [6], $\hat{\boldsymbol{u}}\boldsymbol{G} == \hat{\boldsymbol{x}}$ reduces the complexity by decreasing the number of iterations without enhancing the performance. In the series of SC decoding, polar codes joint with CRC can adapt well to multiple paths of decoding due to the feature of CRC which can assist to choose the most likely codewords in the list, but the series of BP decoding can't show the advantage because of no multiple paths. Actually, keeping the code rate unchanged, adding extra frozen bits to transmit CRC indeed loss the performance of RMS which exists only one path. However, in the case of multiple paths like the below EBPL decoding, the gain from this judgement will far exceed the loss of transmitting the extra CRC. Fig. 4 is an RMS decoder for (8,4) polar codes with $\mathcal{A} = [8, 7, 6, 4]$.

Algorithm 1 RMS decoding for polar codes

Require:
Channel output: $y_1^N = [y_1, y_2..., y_N]$
Position set: $\mathcal{A}_c$ & $\mathcal{A}$

```
Initialize L^0_{i,m+1} and R^0_{i,1} from Eq. (3)
A_rev = reverse (A)
for i = A_rev do
    Update L^t_{i,j} and R^t_{i,j} based on Eq. (2) for twice
    Transform R^t_{i,1} = L^t_{i,1}
    Update the previous altered R^t_{i,1}s to current L^t_{i,1}s
    Calculate û and x̂ from hard judgment
    if ûG == x̂ then
        if CRCcheck (û)==0 then
            Output û, then break
        else
            Continue & Begin next iteration
        end if
    else
        Continue & Begin next iteration
    end if
end for
```

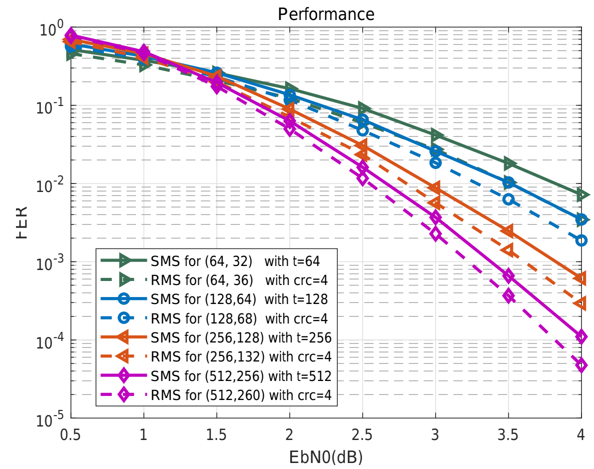

Fig. 5. FER of RMS decoding compared with SMS.

Under the premise of maintaining code rate, RMS decoding adding four extra bits to transmit CRC is still better than SMS as Fig. 5, merely paying a little complexity for replacement.

B. Structure of Proposed EBPL Decoding

Even though BPL decoding proposed in [10] can't be comparable to SCL in the general cases, it indeed provides us a novel idea to construct multiple decoding path reasonably by changing the order of factor graph. Combing this method of constructing paths with the above RMS decoding, we can propose an efficient belief propagation list (EBPL) decoding to rival the performance of SCL in general cases.

Moreover, considering that the codeword with shortest Euclidean distance is not necessarily the most likely one, the final judgment is designed with CRC check which is aimed to choose the shortest Euclidean distance of passing CRC codewords as follows. Hence, there are twice sort operation in the hardware as follows Fig. 6.

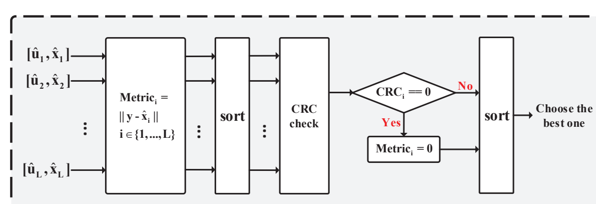

Fig. 6. Hardware design of the final judgement of EBPL decoder.

IV. SIMULATION RESULTS

In this section, we will present the simulation results under the additive white Gaussian noise (AWGN) channels. All polar codes signals are modulated by binary phase-shift keying (BPSK) on the AWGN channels. Considering that polar codes are used as error-correcting codes for short codes in the control

Algorithm 2 EBPL decoding for polar codes

Require:

Channel output: $y_1^N = [y_1, y_2 \cdots y_N]$

Position set: $\mathcal{A}_c$ & $\mathcal{A}$

Factor graph order: $\boldsymbol{order_1 \cdots order_L}$

for $i = L$ **do**

  $[\hat{\boldsymbol{u}}_{\boldsymbol{i,1}}^{\boldsymbol{N}}, \hat{\boldsymbol{x}}_{\boldsymbol{i,1}}^{\boldsymbol{N}}]$ = RMS decoding (y_1^N, $\mathcal{A}$, $\mathcal{A}_c$, $\boldsymbol{order_i}$)

  $\boldsymbol{Metric_i} = ||y_1^N - \hat{\boldsymbol{x}}_{\boldsymbol{i,1}}^{\boldsymbol{N}}||$

end for

$[\boldsymbol{Metric}]$ = sort($\boldsymbol{Metric_1} \cdots \boldsymbol{Metric_L}$, 'ascend')

for $i = L$ **do**

  if $\boldsymbol{CRC_i == 0}$ **then**

    $\boldsymbol{Metric_i = 0}$

  end if

end for

$[\boldsymbol{Metric}]$ = sort($\boldsymbol{Metric_1} \cdots \boldsymbol{Metric_L}$, 'ascend')

Then, output the corresponding $\hat{\boldsymbol{u}}$ of $\boldsymbol{Metric_1}$

channels of eMBB, polar codes with N ranging from 64 to 256 and $R = 1/2$ are the major concerns in the simulation.

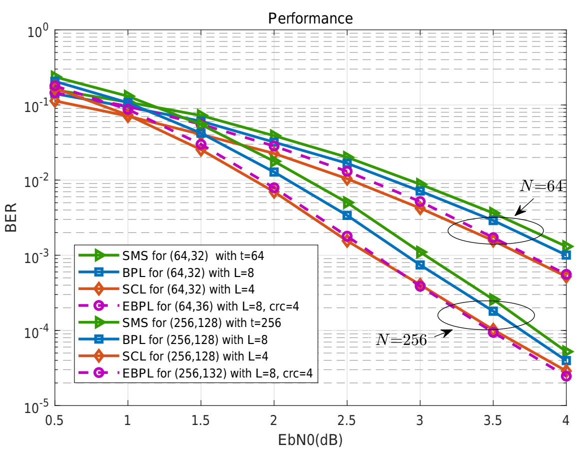

Fig. 7. BER of EBPL decoding compared with other algorithms.

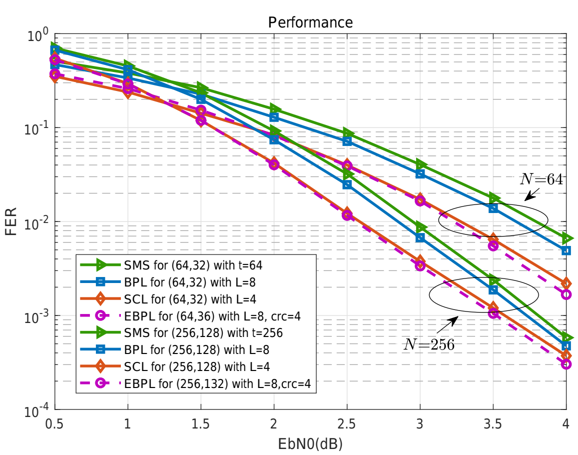

Fig. 8. FER of EBPL decoding compared with other algorithms.

In order to further improve the accuracy of comparison, we don't use the **G**-matrix-based stopping criteria [6] in the scaled min-sum and BP list decoding. In terms of the number of paths in the BPL and EBPL decoding, twice as many as that in SCL decoding is considered to be reasonable because that the series of BP list decoding have no operation to replace L paths in every bit decoding. Hence, eight paths in EBPL decoding are utilized to match the performance of SCL with $L = 4$ in the Fig. 7 and Fig. 8. Moreover, due to **m**! permutations of factor graph which is difficult to select the best $2L$ paths, a simple method by right shifting the stages of factor graph every time is applied to construct $2L$ paths.

As Fig. 7 shows, combining with the CRC-4 in EBPL decoding could rarely influence the BER of short codes like polar codes (64, 32) in the low E_b/N_0 and efficiently enhance the BER in other cases. Actually, the BER of EBPL decoding is close to SCL decoding in the (64, 32) codes, then for the (256, 128) codes, the BER of EBPL decoding has exceeded the SCL decoding with the same scale L. In the comparison of Fig.s 7 and 8, the increase in FER is slightly better than that in BER in EBPL decoding. Moreover, the performance of EBPL decoding is not sensitive to code construction.

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