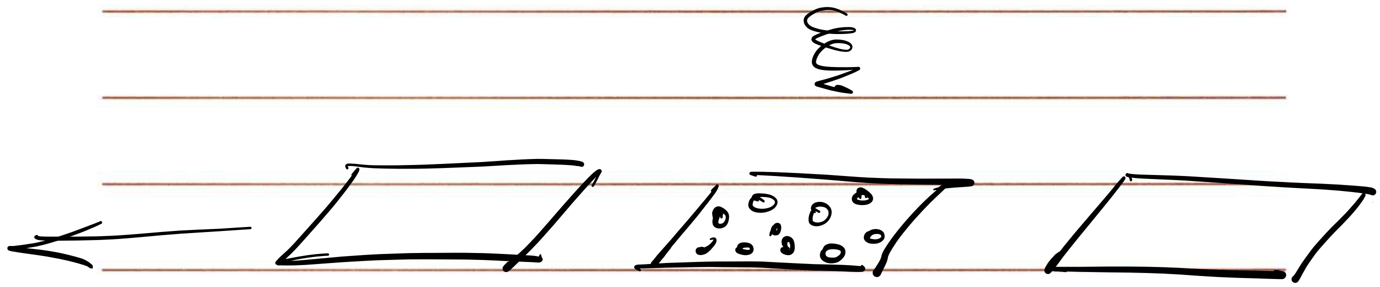
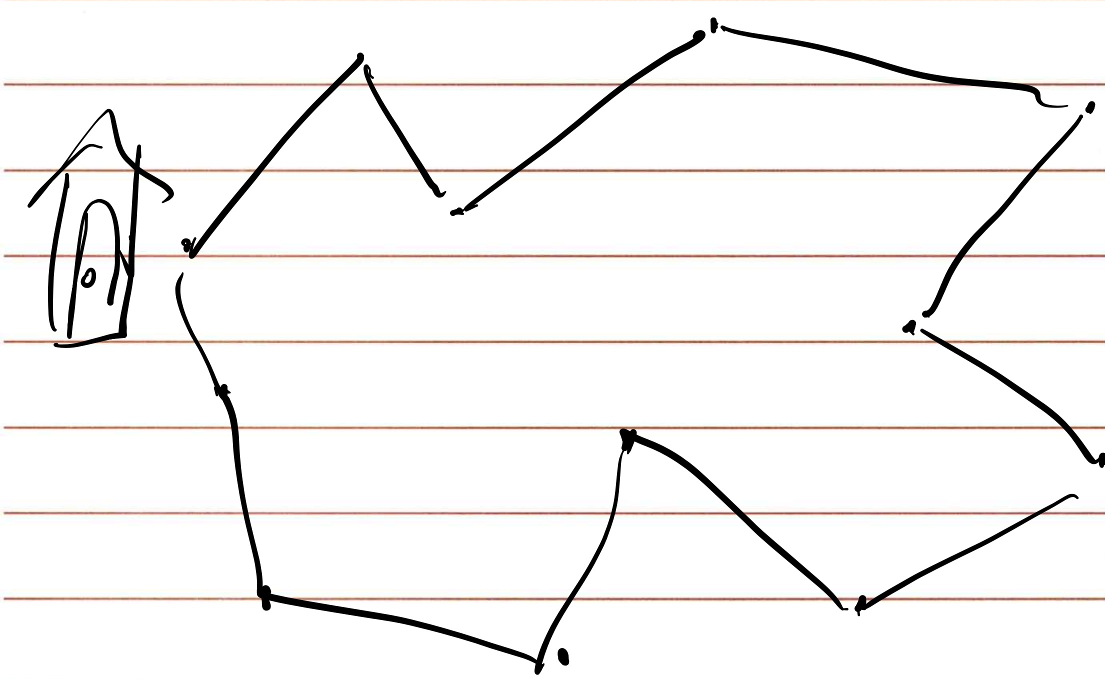
Traveling Salesman Problem (TSP) & Hamiltonian Cycle





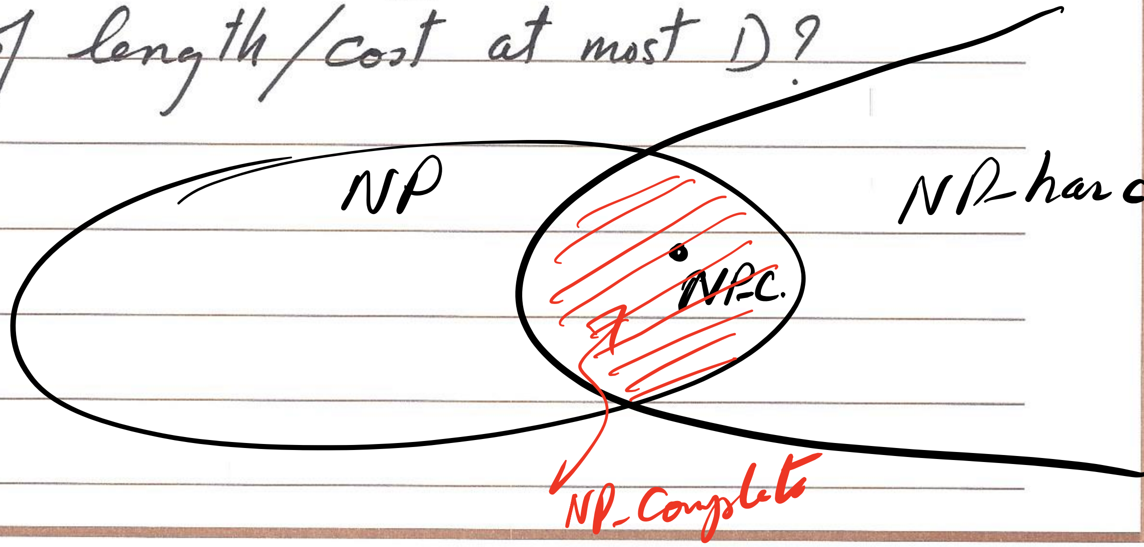
Problem Statement

Given the set of distances, order n cities in a tour $V_{i_1}, V_{i_2}, \ldots V_{i_n}$ with $i_1 = 1$, so it minimizes

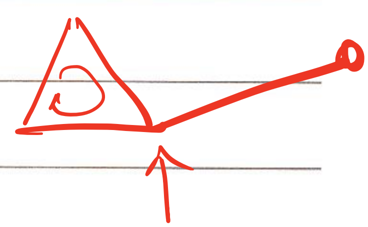
$$\underbrace{\sum d(V_{i_j}, V_{i_{j+1}})} + \underbrace{d(V_{i_n}, V_{i_1})}$$

Decision version of TSP:

Given a set of distances on n cities and a bound D, is there a tour of length/cost at most D?



Def. A cycle C in G is a Hamiltonian Cycle, if it visits each vertex exactly once.



Problem Statement:

Given an undirected graph G, is there a Hamiltonian cycle in G?

Show that the Hamiltonian Cycle Problem is NP-complete

1- Show that the problem is in NP.

a. polyn. length certificate:
ordered list of nodes on the H C.

b. polyn. time certifier

- All nodes visited only once

- edge exists between every pair of adjacent nodes in the list

- first & last nodes are connected w/ an edge

2- Choose a problem that we already know is NP-Complete

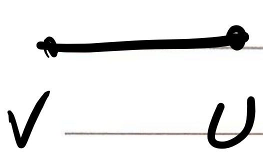
Choose Vertex Cover

3- Show that Vertex Cover $\leq_p$ HC.

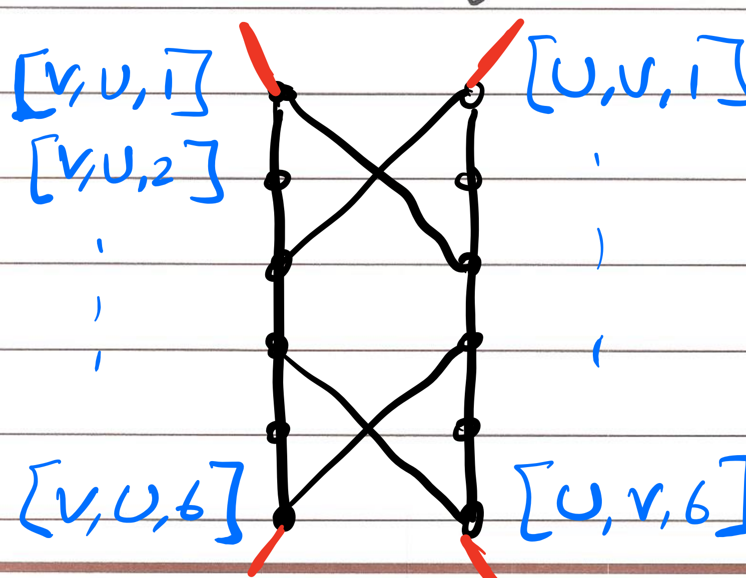
Plan: Given an undirected graph $G=(V,E)$ and an integer k, we construct $G'=(V',E')$ that has a Hamiltonian Cycle iff G has a vertex cover of size at most k.

Construction of G'

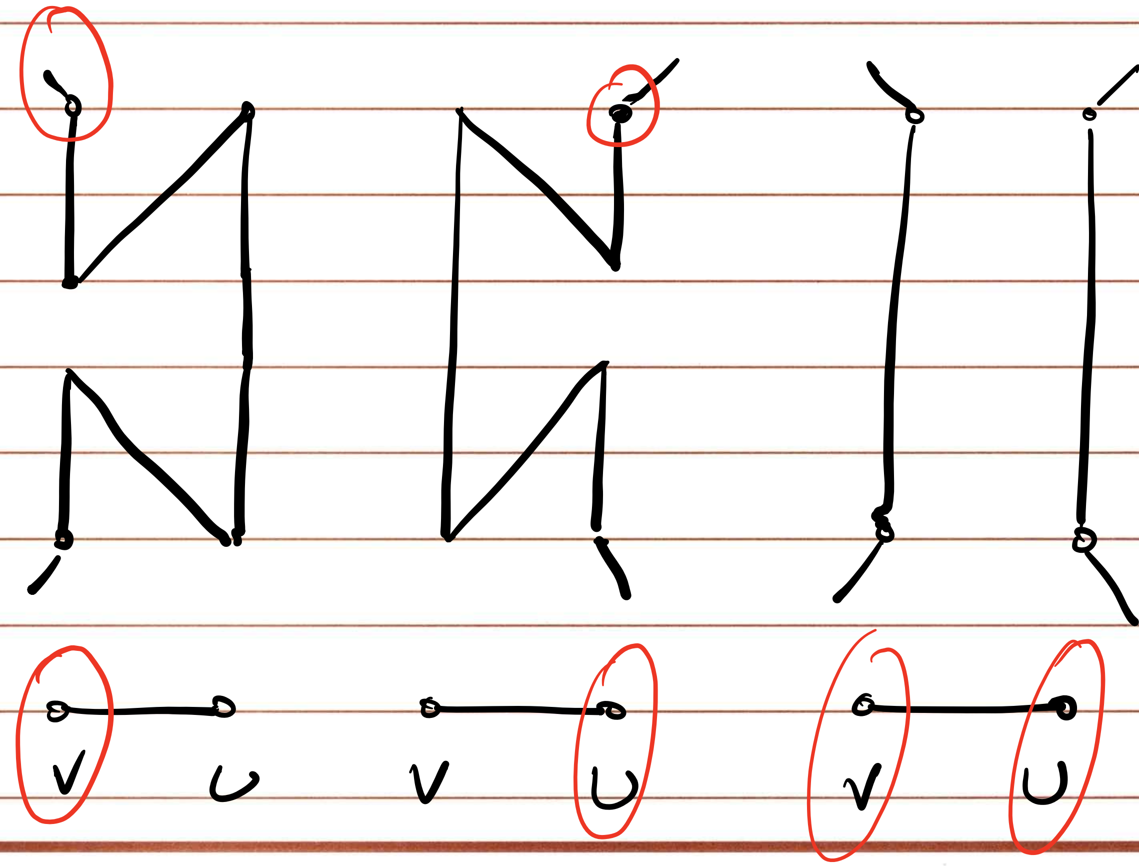
For each edge (V,U) in G, G' will have one gadget W_{VU} with following node labeling:



G



G'

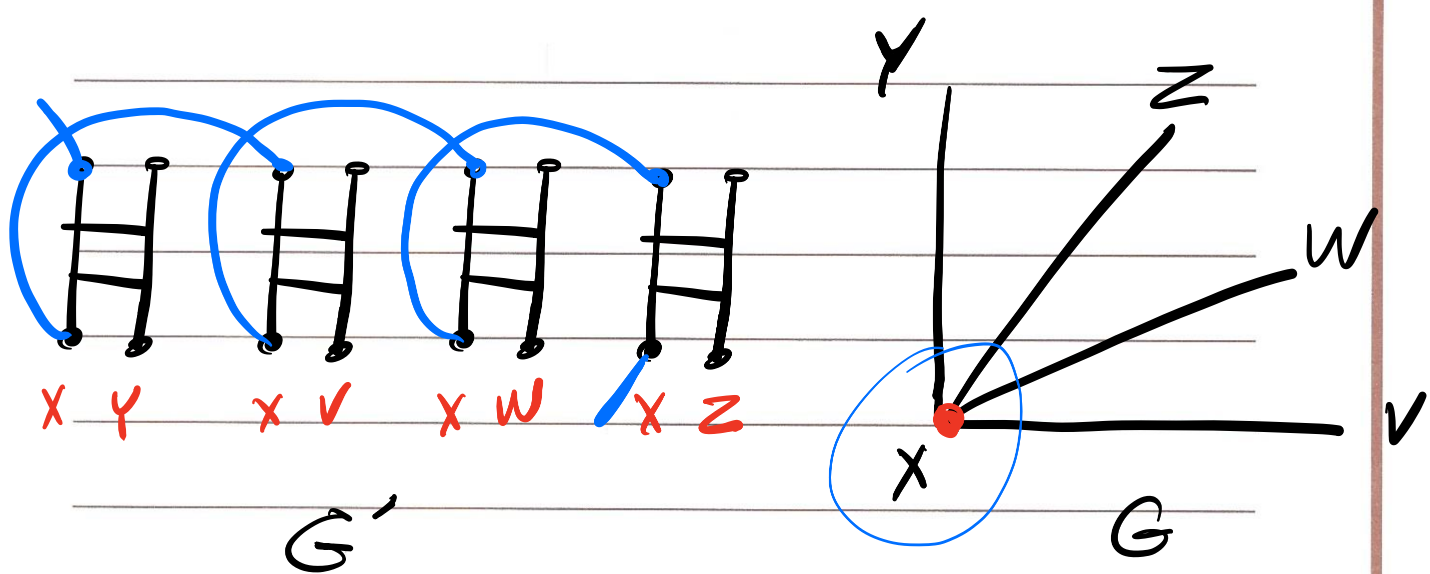

v
u
v
u
v
u

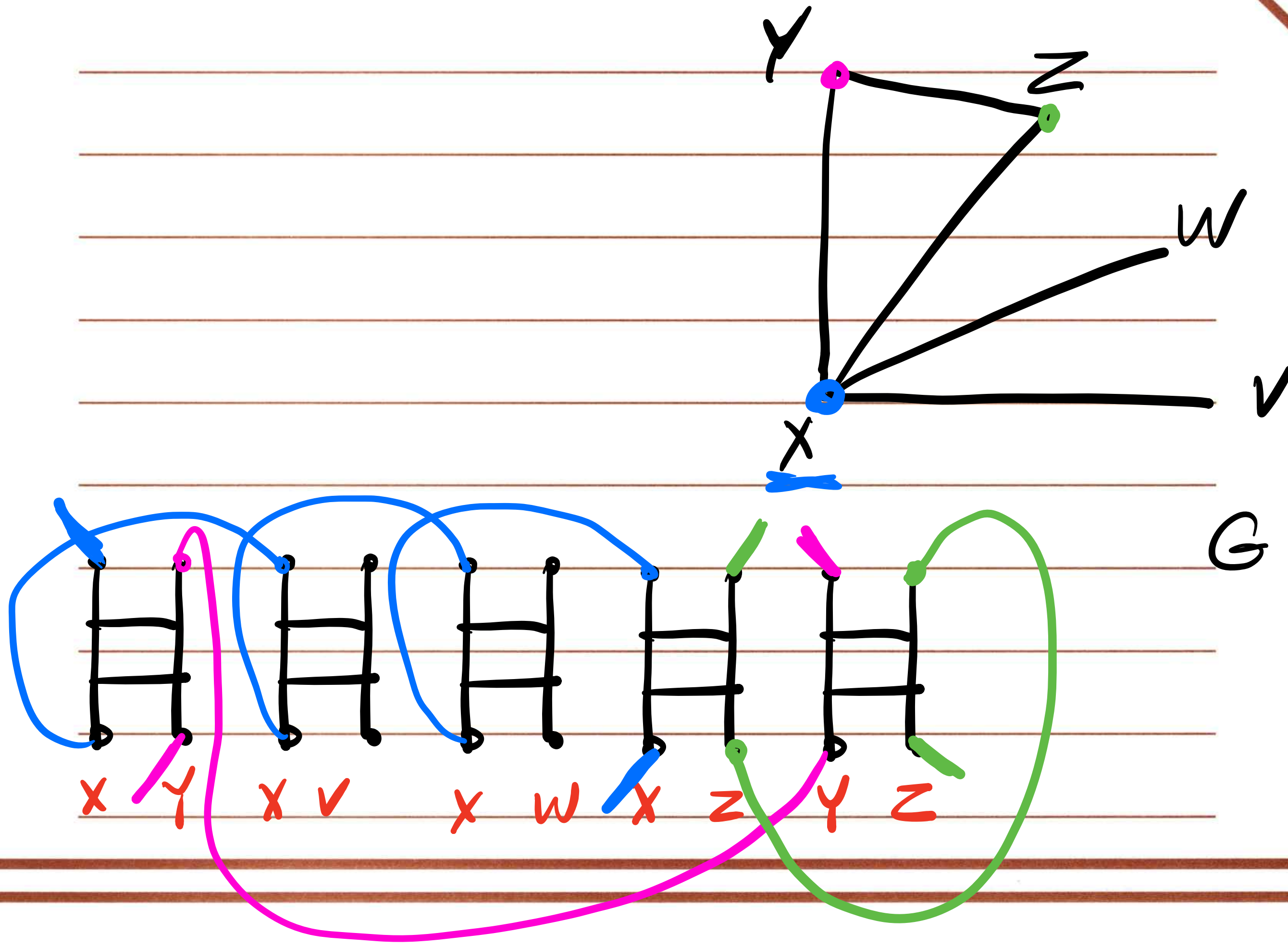
Other vertices in G'

- Selector vertices: There are k selector vertices in G', $s_1, \dots, s_k$

Other edges in G'

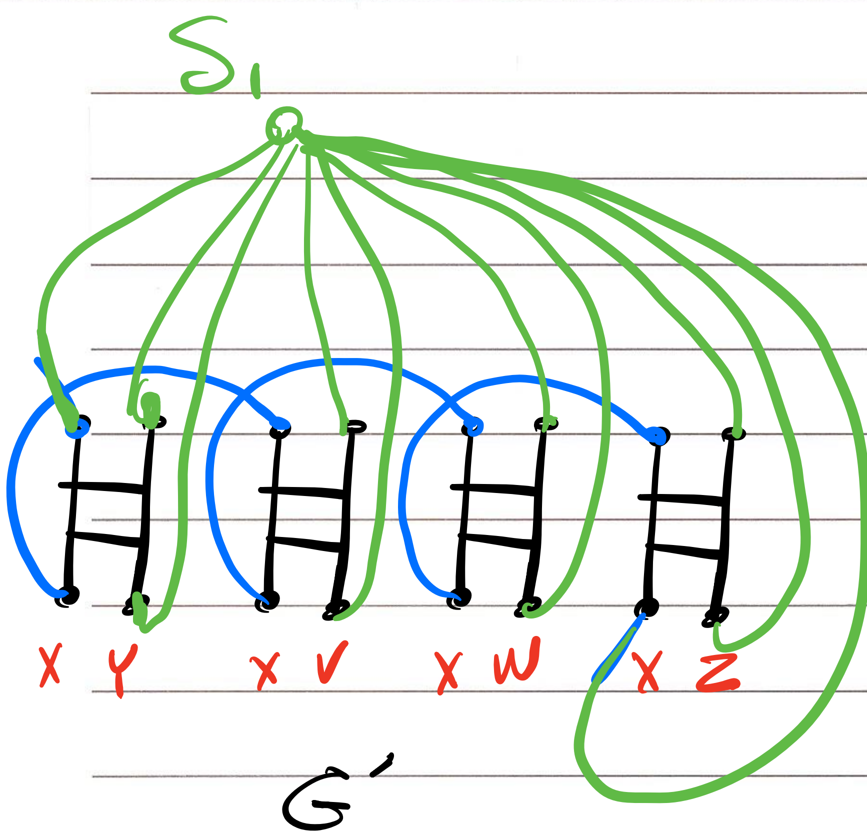
1. For each vertex $v \in V$ we add edges to join pairs of gadgets in order to form a path going through all the gadgets corresponding to edges incident on v in G.

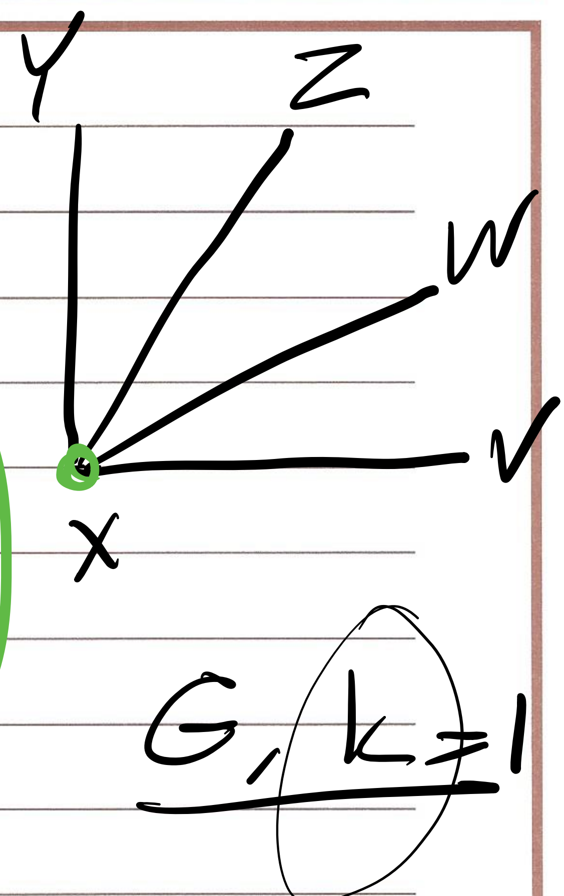


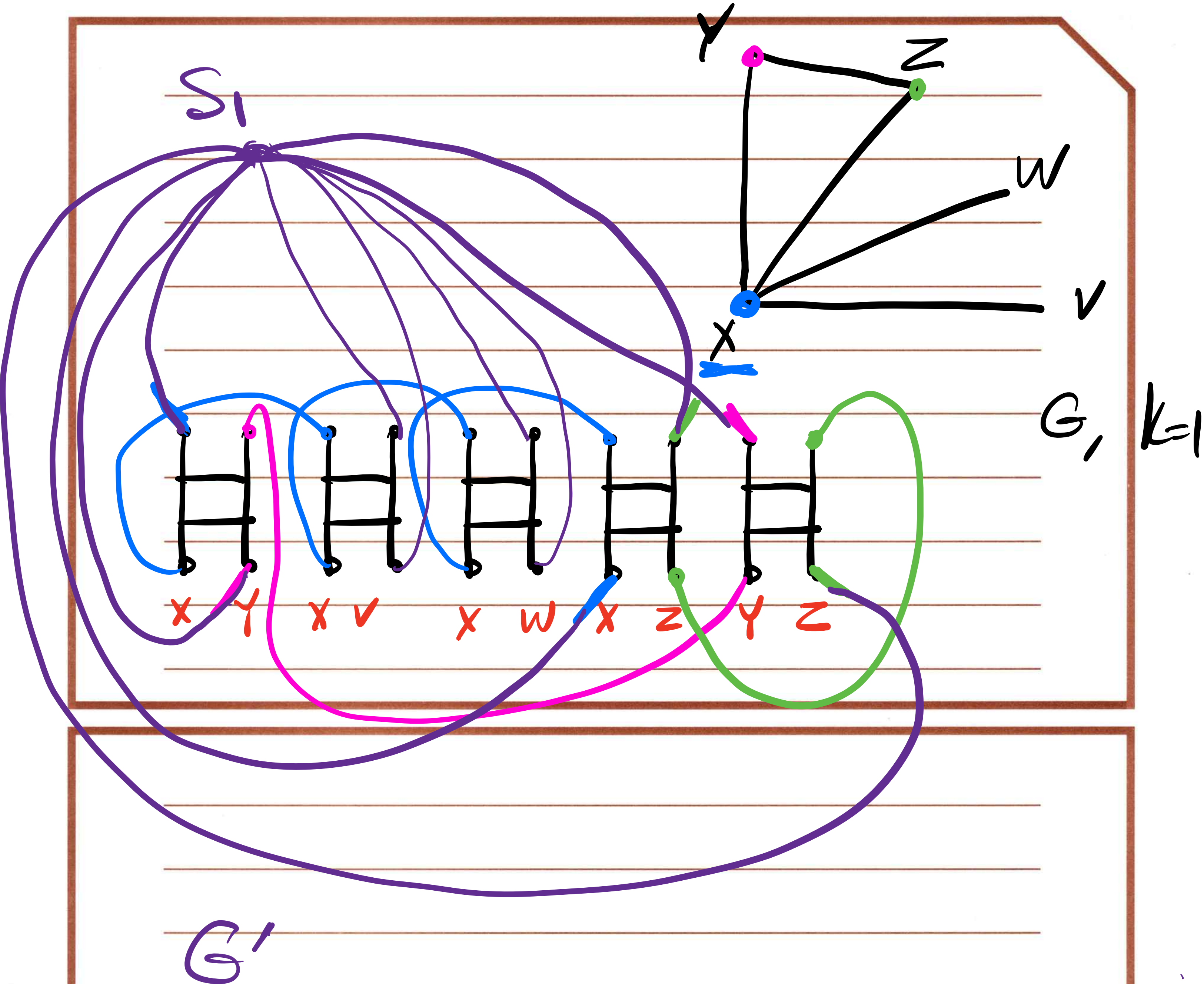


Y
Z
W
V
X
G
X Y X V X W X Z Y Z

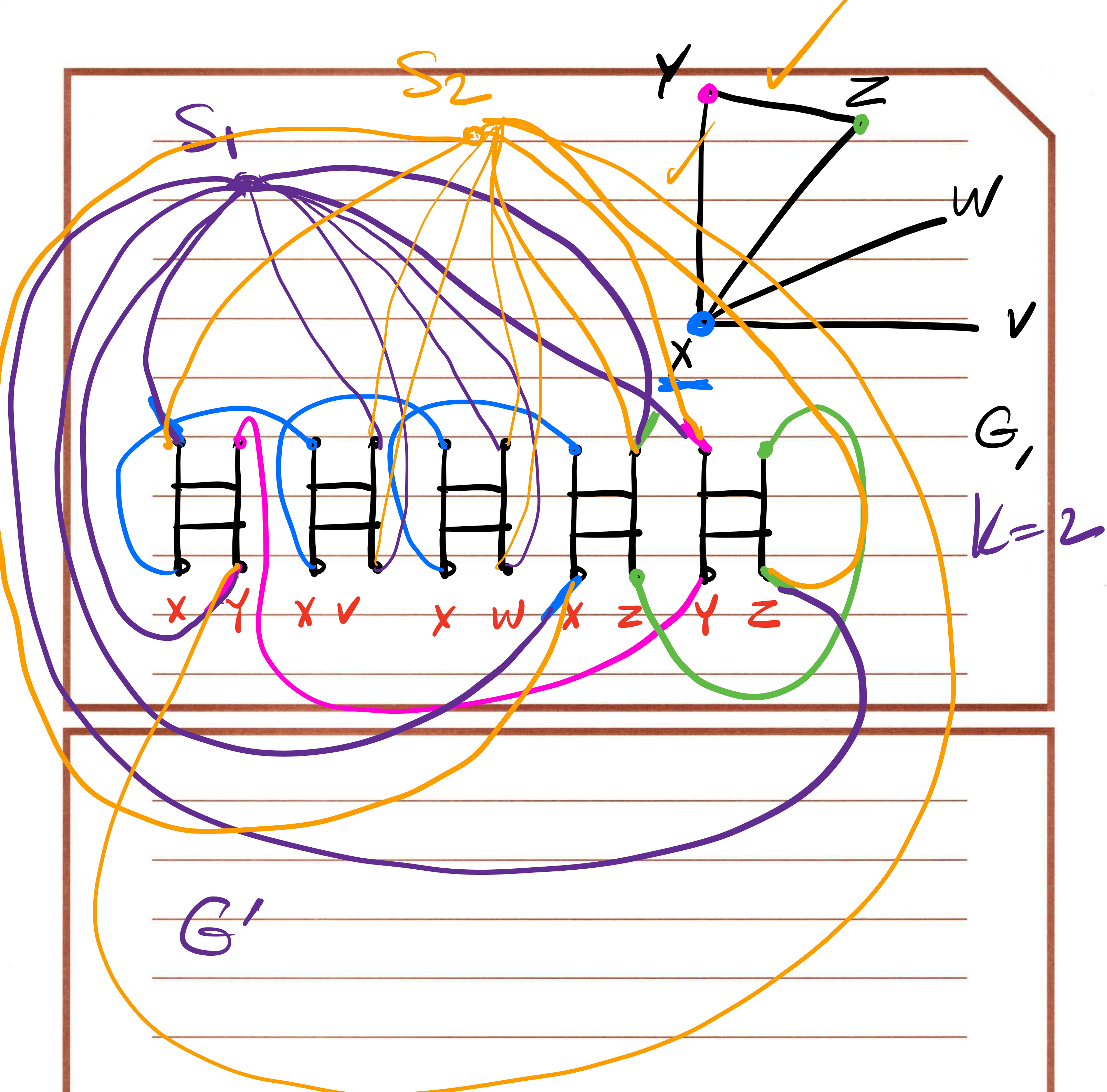
2- Final set of edges in G′ join the first vertex [X, Y, 1] and last vertex [X, Y(deg(x)), 6] of each of these paths to each of the selector vertices.

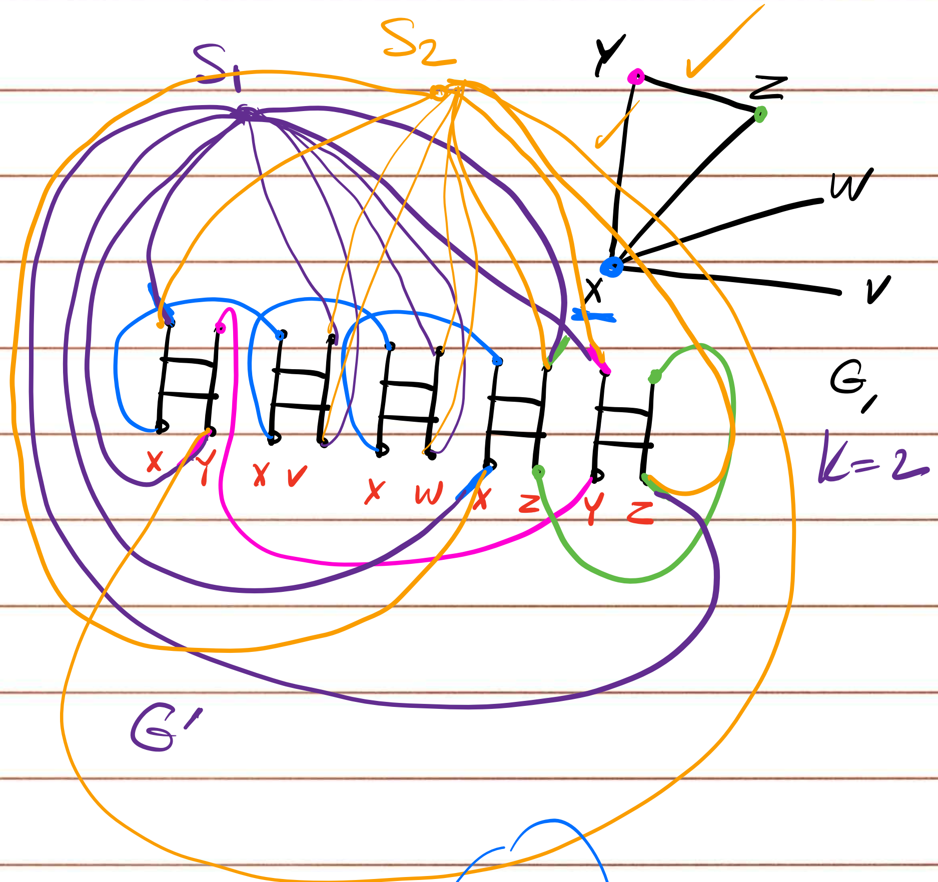


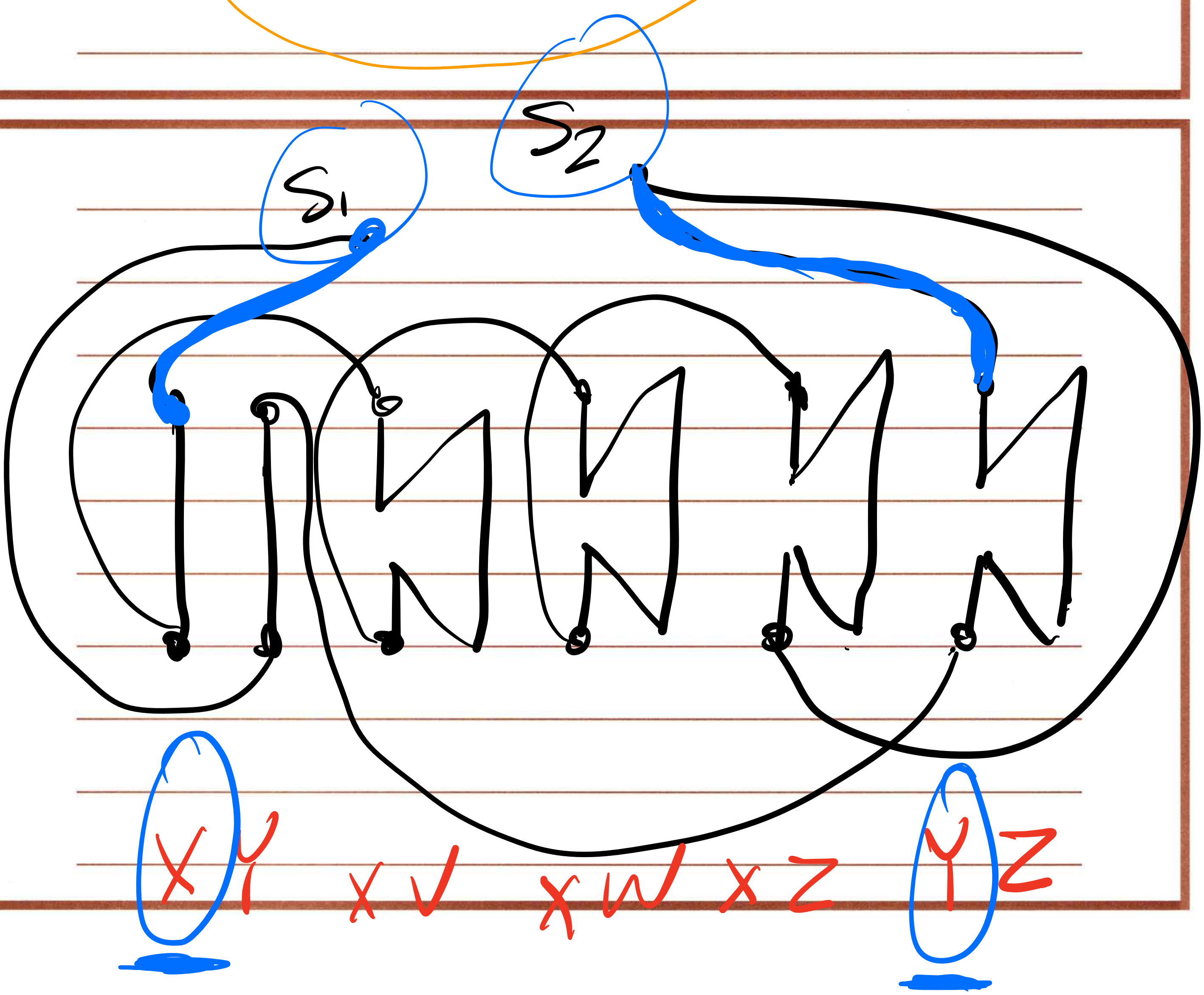




S1
Y
Z
W
V
X
G, k=1
X Y X V X W X Z Y Z
G'


S1
S2
Y
Z
W
V
X
G1
k=2
X Y X V X W X Z Y Z
G'

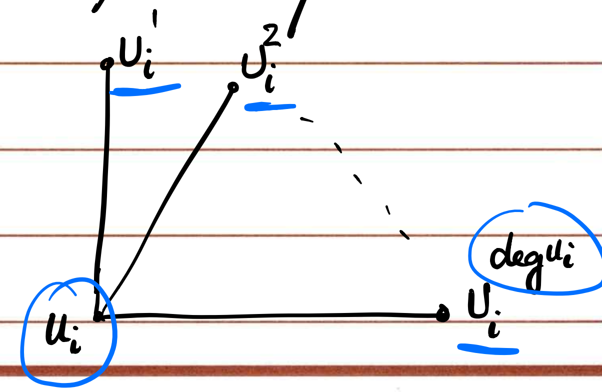

S1
S2
Y
Z
W
V
X
G1
K=2
X Y X V X W X Z Y Z
G'


S1
S2
XY XV XW XZ YZ

Proof: A) Suppose that $G = (V, E)$ has a vertex cover of size k. Let the vertex cover set be

$$S = \{u_1, u_2, \ldots u_k\}$$

We will identify neighbors of u_i as shown here:



Form a Ham. Cycle in G' by following the nodes in G in this order:

start at s_1 and go to

$[u_1, u_1^1, 1] \; \ldots \; [u_1, u_1^1, 6]$

$[u_1, u_1^2, 1] \; \ldots \; [u_1, u_1^2, 6]$

$\vdots$

$[u_1, u_1^{\deg u_1}, 1] \; \ldots \; [u_1, u_1^{\deg u_1}, 6]$

Then go to S_2 and follow the nodes

$[U_2, U_2^1, 1]$ - - - - - $[U_2, U_2^1, 6]$

$[U_2, U_2^2, 1]$ - - - - - $[U_2, U_2^2, 6]$

$\vdots$

$[U_2, U_2^{\deg U_2}, 1]$ - - - $[U_2, U_2^{\deg U_2}, 6]$

Then go to S_3 - - - -

$\vdots$

$\vdots$

$[U_k, U_k^1, 1]$ - - - $[U_k, U_k^1, 6]$

$[U_k, U_k^2, 1]$ - - - $[U_k, U_k^2, 6]$

$\vdots$

$[U_k, U_k^{\deg U_k}, 1]$ - - - $[U_k, U_k^{\deg U_k}, 6]$

Then return back to S_1.

B) Suppose G' has a Hamiltonian cycle C, then the set

$$S = \{ v_j \in V : (s_j, [v_j, v_j', 1]) \in C \text{ for some } 1 \leq j \leq k \}$$

will be a vertex cover set in G.

We Prove that TSP is NP-Complete

1- Show that $\underline{TSP \in NP}$

a. Certificate:

a tour of cost at most $\underline{D}$

b. Certifier

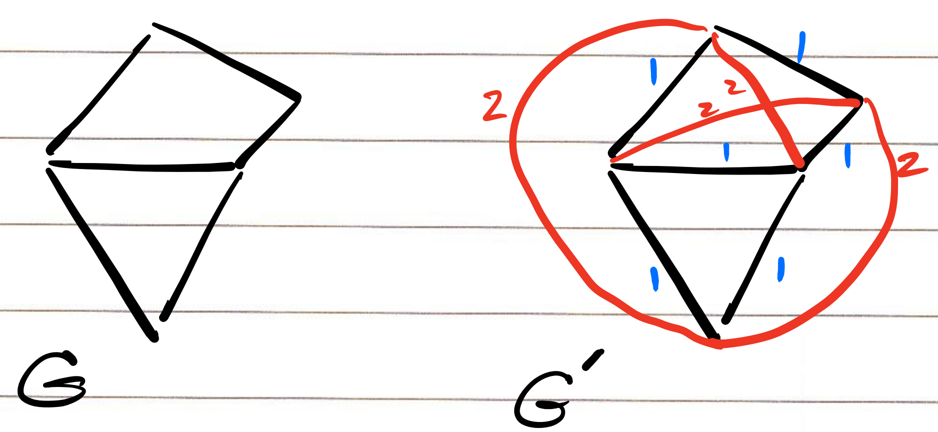
- all that we did for HC

+ check cost of tour $\leq D$

2- Choose an NP-Complete problem:

Hamiltonian Cycle.

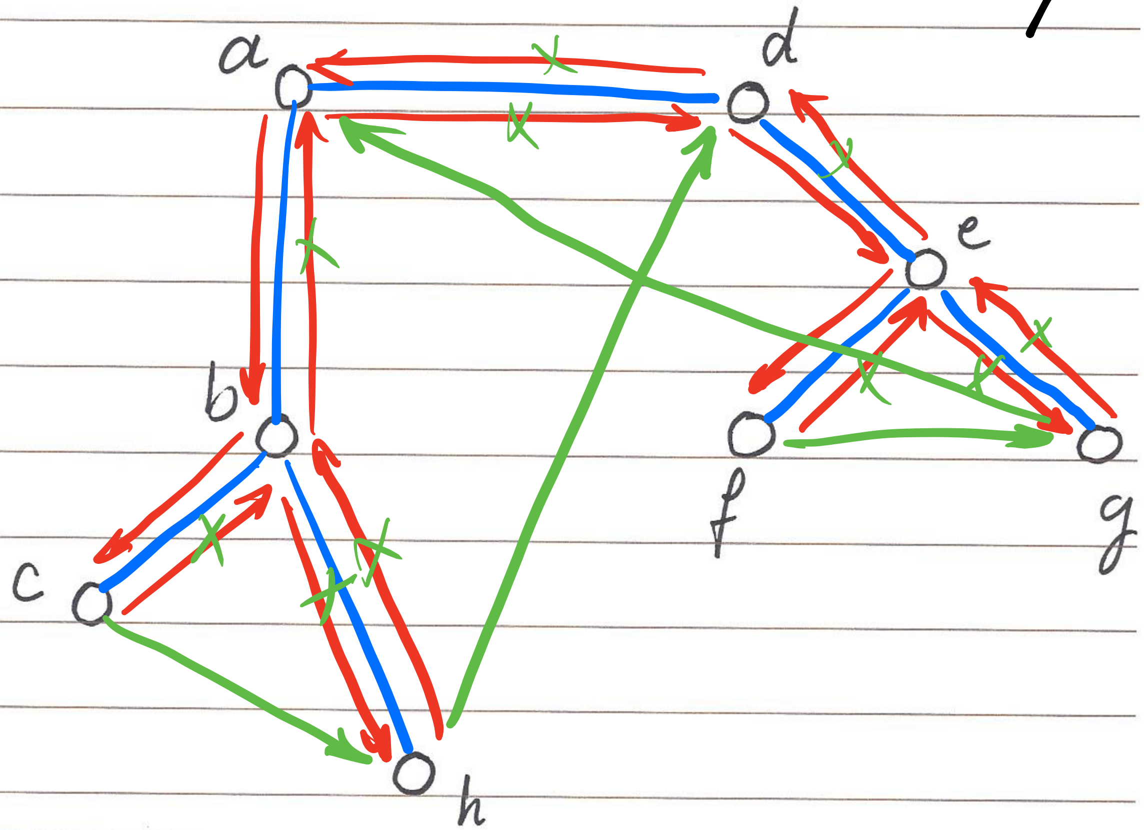
3- Prove that Ham. Cycle $\leq_p$ TSP



a) If we have a HC in G, we can find a tour of cost at most n in G'

b) If we have a tour of cost at most n in G', we can find a HC in G.

assuming triangle inequalities



Cost of the MST < Cost of opt. tour

Cost of our initial tour = 2. Cost of MST

Cost of our approx. Sol. ≤ 2. "

Cost of our approx. sol. < 2. Cost of opt. tour

This is a 2-approximation

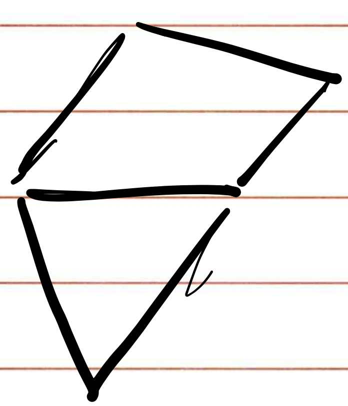
General TSP

Theorem: if $P \neq NP$, then for any constant $\rho \geq 1$, there is no polynomial time approximation algorithm with approximation ratio ρ for the general TSP

Plan: We will assume that such an approximation algorithm exists. We will then use it to solve the HC problem.

Given an instance of the HC problem on graph G, we will construct G' as follows:

- G' has the same set nodes as in G
- G' is a fully connected graph.
- Edges in G' that are also in G have a cost of 1.
- Other edges in G' have a cost of $\rho|V|+1$



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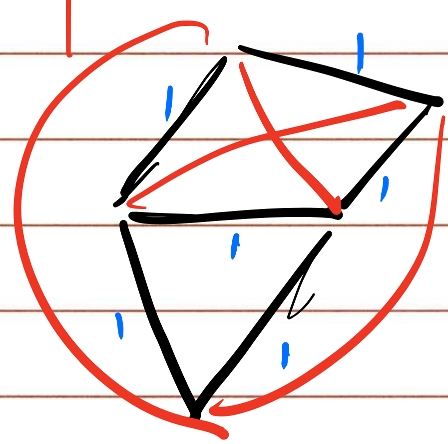

G'

$\rho|V| < \rho|V| + 1$

If G has a Ham. Cycle we can find a tour of cost |V| in G'

If we have a tour of cost $\leq \rho|V|$ in G' we can find a HC in G.