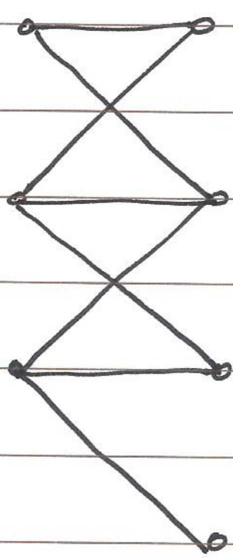
Network Flow

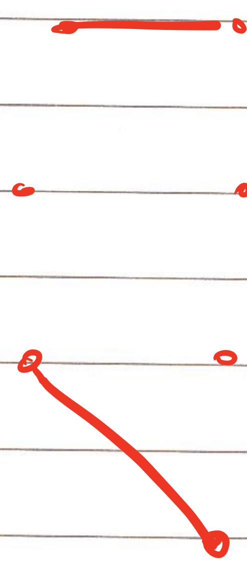
Bipartite Matching Problem

Def. A bipartite graph $G = (V, E)$ is an undirected graph whose node set can be partitioned as $V = X \cup Y$ with property that every edge $e \in E$ has one end in X and the other in Y.

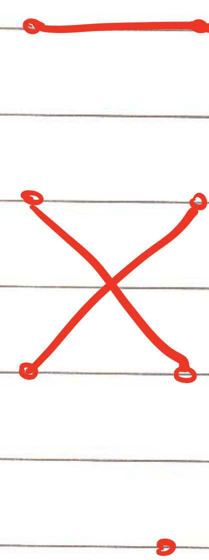
Def. A matching M in G is a subset of the edges $M \subseteq E$ such that each node appears in at most one edge in M.



G



A matching



A max. size matching

Problem Statement:

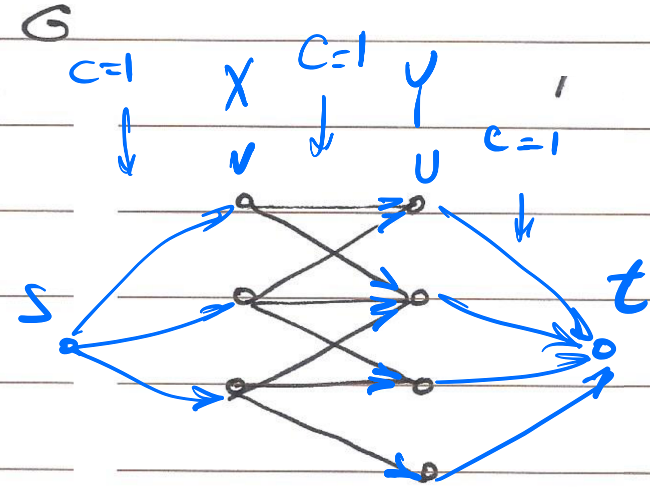
Find a matching M of largest possible size in G.

General Plan:

Design a flow network G' that will have a flow value $v(f) = k$ iff there is a ~~max.~~ ~~size~~ matching of size k in G. Moreover, flow f in G' should identify the matching M in G.

Construction of G'

G



G'

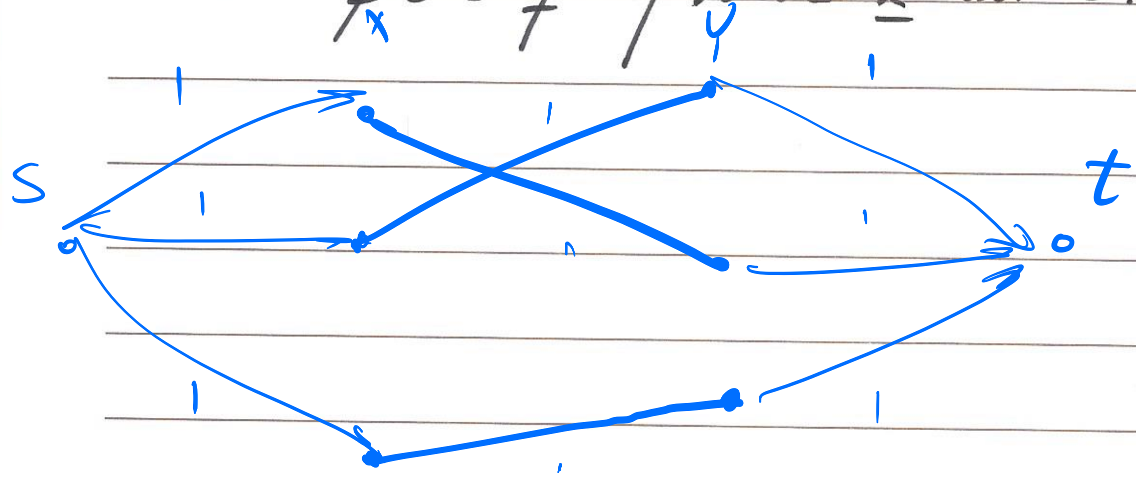
Solution

Run Max Flow on G'. Say max. flow is f.
Edges carrying flow between sets X & Y will correspond to our max. size matching in G.

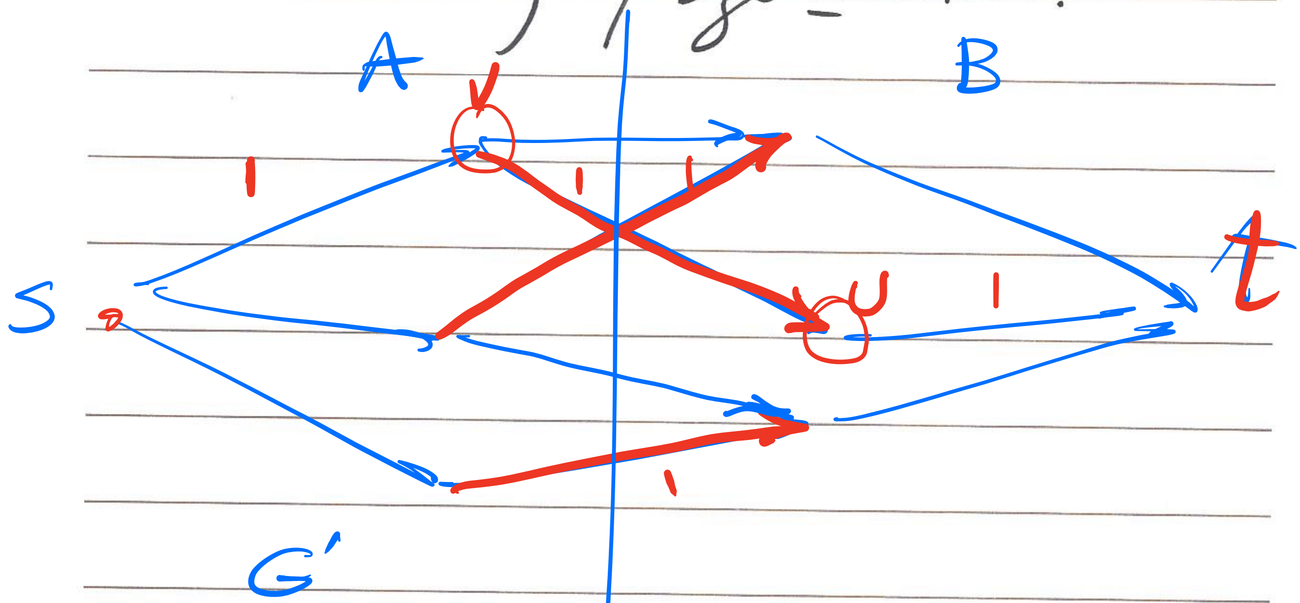
To prove this, we will show that G' will have a ~~max.~~ flow of value $\underline{k}$ iff G has a ~~max~~ ~~siz~~ matching of size $\underline{k}$.

Proof:

A- If we have a matching of size $\underline{k}$ in G, we can find an s-t flow f of value $\underline{k}$ in G'.



B- If we have an s-t flow of value $\underline{k}$ in G', we can find a matching of size $\underline{k}$ in G.



Say we use Ford Fulkerson to find Max Flow

Complexity will be $O(Cm)$

$O(nm)$

Network Flow

Edge-Disjoint Paths

Def. A set of paths is edge-disjoint if their edge sets are disjoint

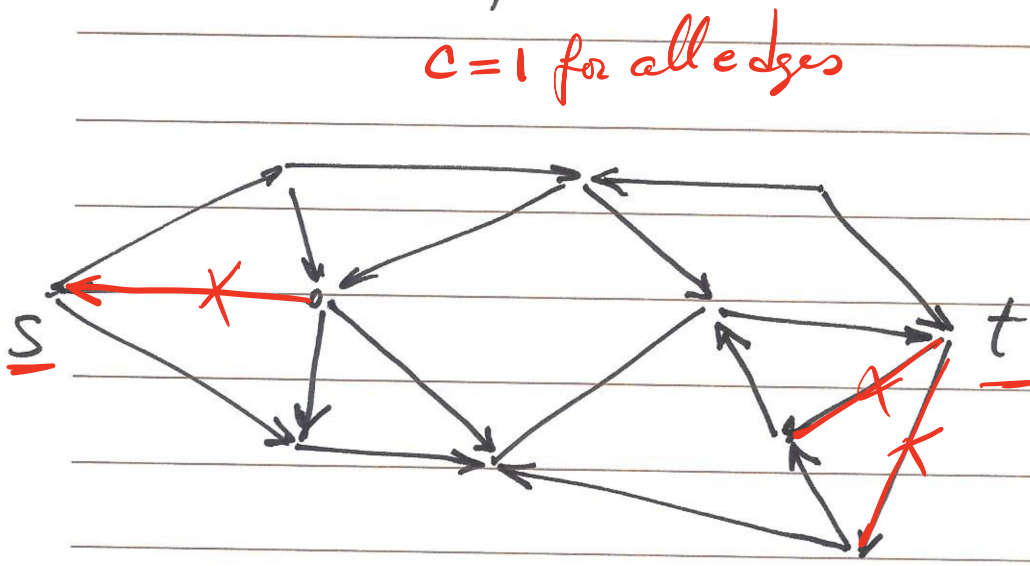
Problem Statement

Given a directed graph G with $s \& t \in V$, find max. number of edge-disjoint s-t paths in G.

General Plan:

Design a flow network G' that will have a flow value $v(f) = k$ iff there are $\underline{k}$ edge-disjoint s-t paths in G.
Moreover, flow f in G' should identify the set of edge-disjoint paths in G.

Construction of G'



Solution:

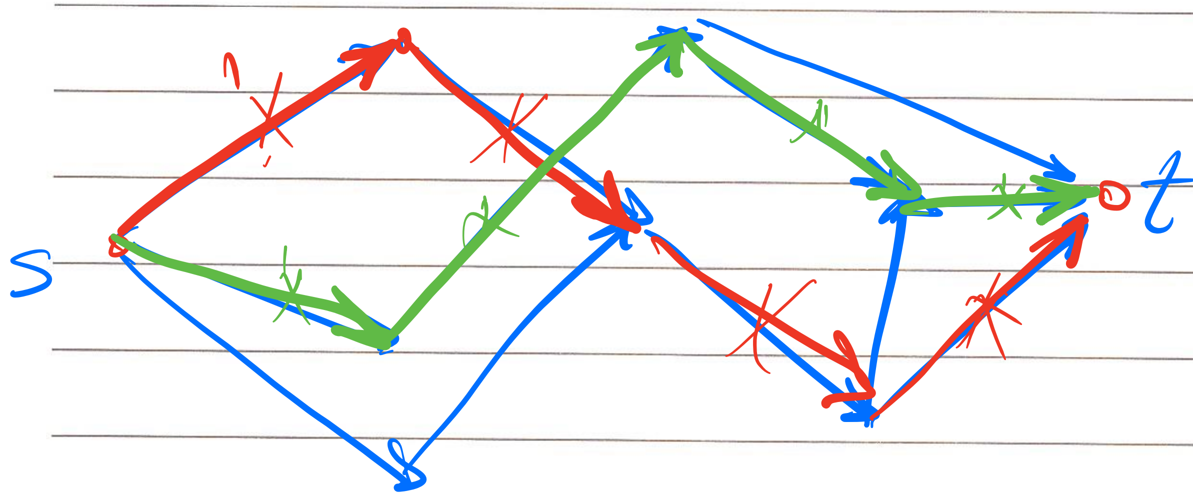
- Run Max flow in G'
- v(f) will equal the max. number of edge-disjoint s-t paths
- f will identify edges on these paths

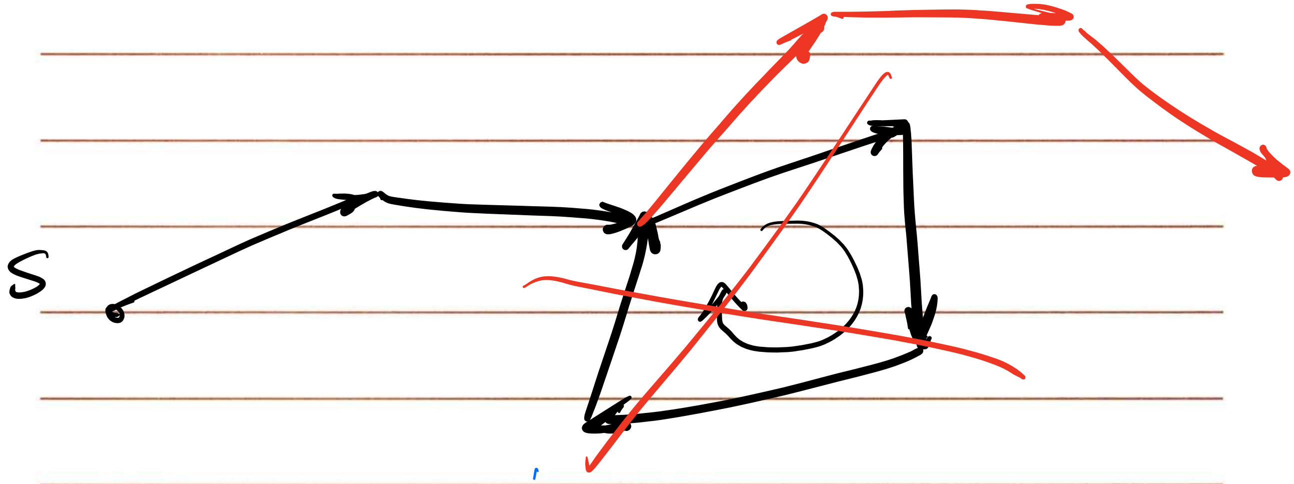
To prove this, we will show that there are k edge disjoint paths in G iff there is a flow of value k in G'.

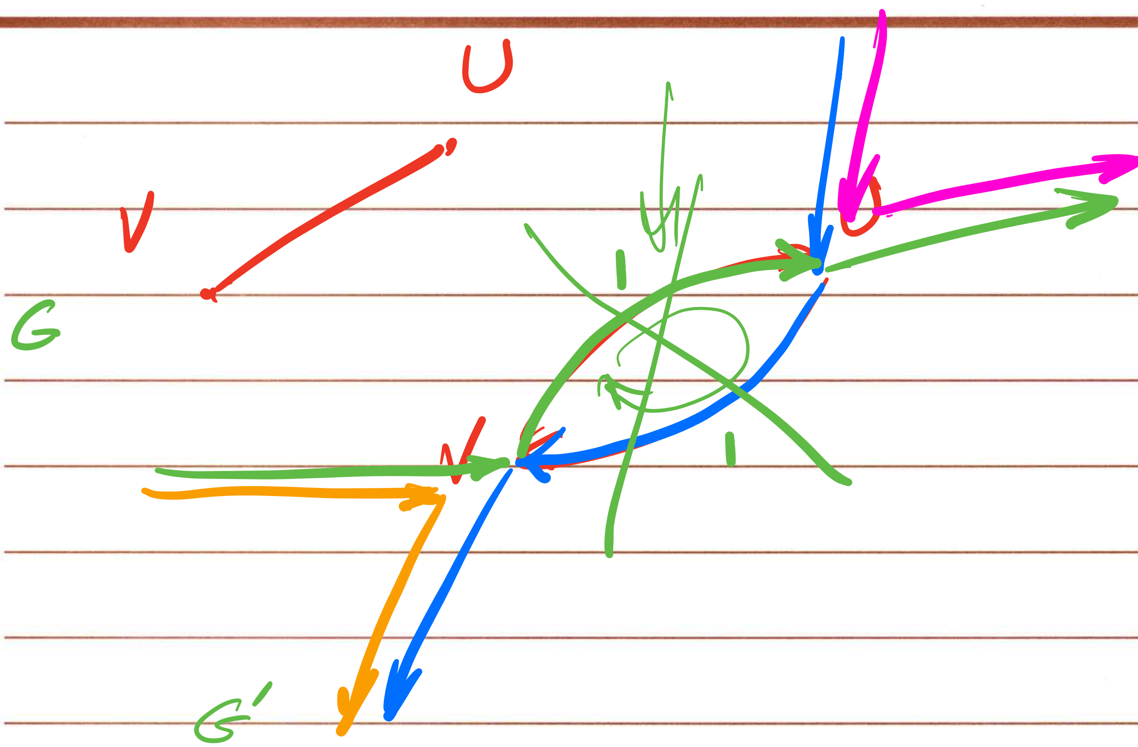
Proof:

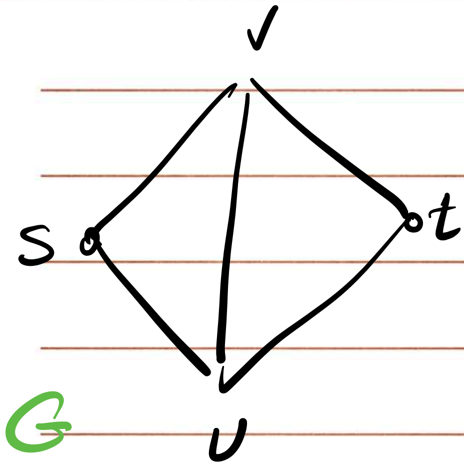
A) If we have k edge disjoint s-t paths in G, we can find a flow of value k in G'.

B) If we have a flow of value k in G', we can find k edge-disjoint s-t paths in G.




S


U
V
G
G'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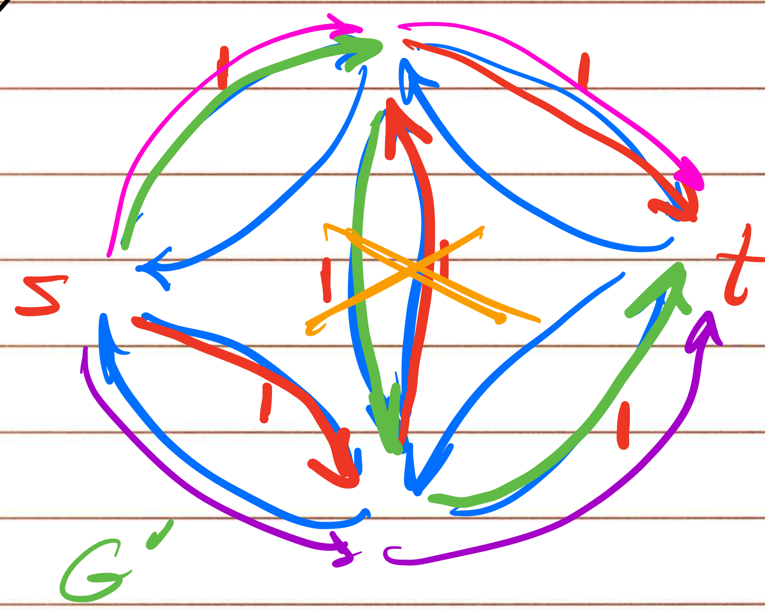

s
t
1
1
1
1
1
1
1
G'

Network Flow

Node-disjoint Paths

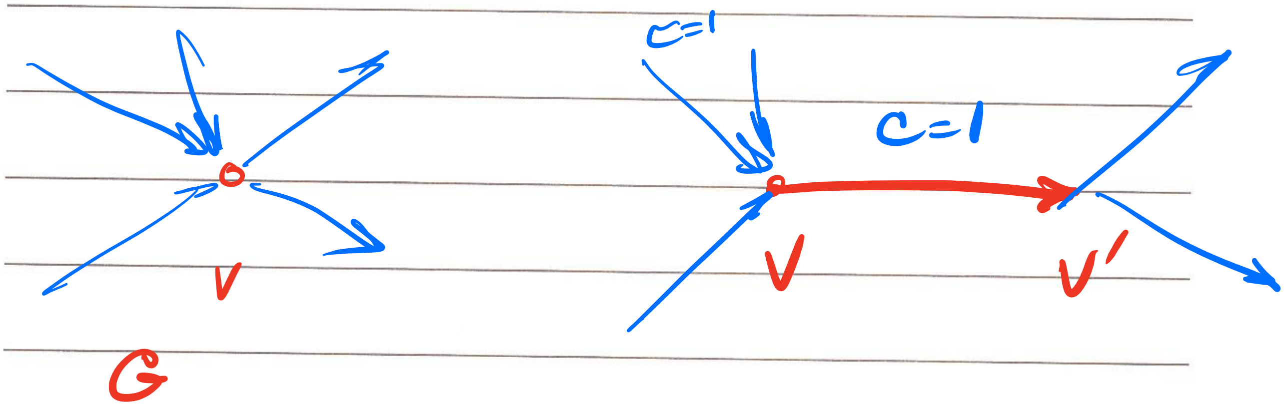
Def. A set of paths is node-disjoint if their node sets (except for starting & ending nodes) are disjoint

Problem Statement

Given a directected graph G with $s \& t \in V$, find the max. number of node-disjoint s-t paths in G.

Plan: As usual...

Construction of G':



Circulation &

Circulation with Lower Bounds

Circulation Network

- We are given a directed graph $G=(V,E)$ with capacities on the edges.
- Associated with each node $v \in V$ is a demand d_v
  - if $d_v > 0$, node v has demand of d_v for flow (sink)
  - if $d_v < 0$, node v has a supply of $|d_v|$ for flow (source)
  - if $d_v = 0$ v is neither a sink nor a source

Def. A circulation with demand $\{d_v\}$ is a function f that assigns non-negative real numbers to each edge and satisfies:

1) Capacity condition
   for each edge $e \in E$ $0 \leq f(e) \leq C_e$

2) Demand condition
   for each node $v \in V$, $f^{in}(v) - f^{out}(v) = d_v$

FACT: If there is a feasible circulation w/ demands $\{d_v\}$ then $\sum_v d_v = 0$

Proof: $f^{in}(v) - f^{out}(v) = d_v$

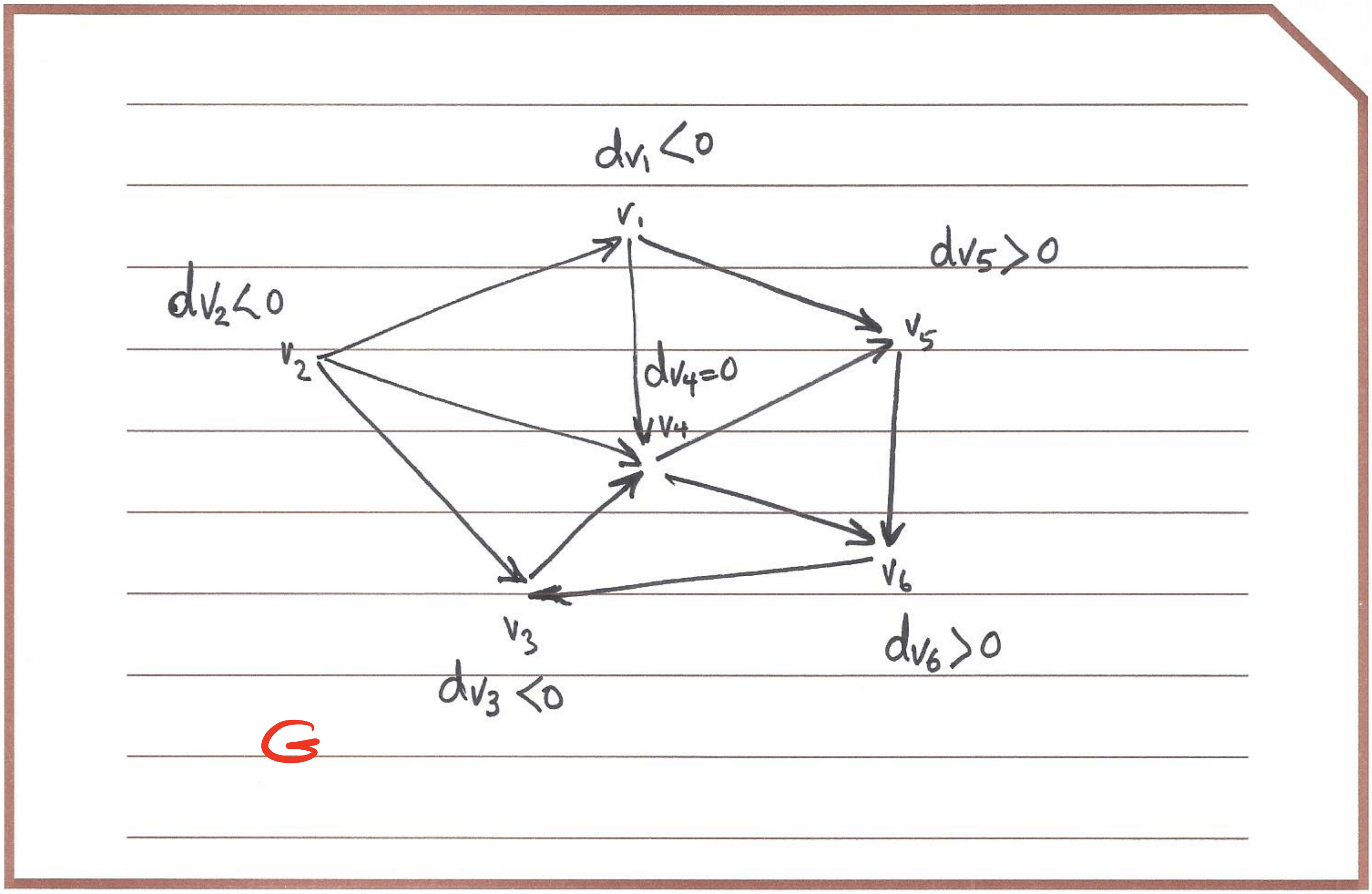
$$\sum_v d_v = \sum \cancel{f^{in}(v)} - \cancel{f^{out}(v)} = 0$$

$$v \xrightarrow[f(e)]{e} u$$

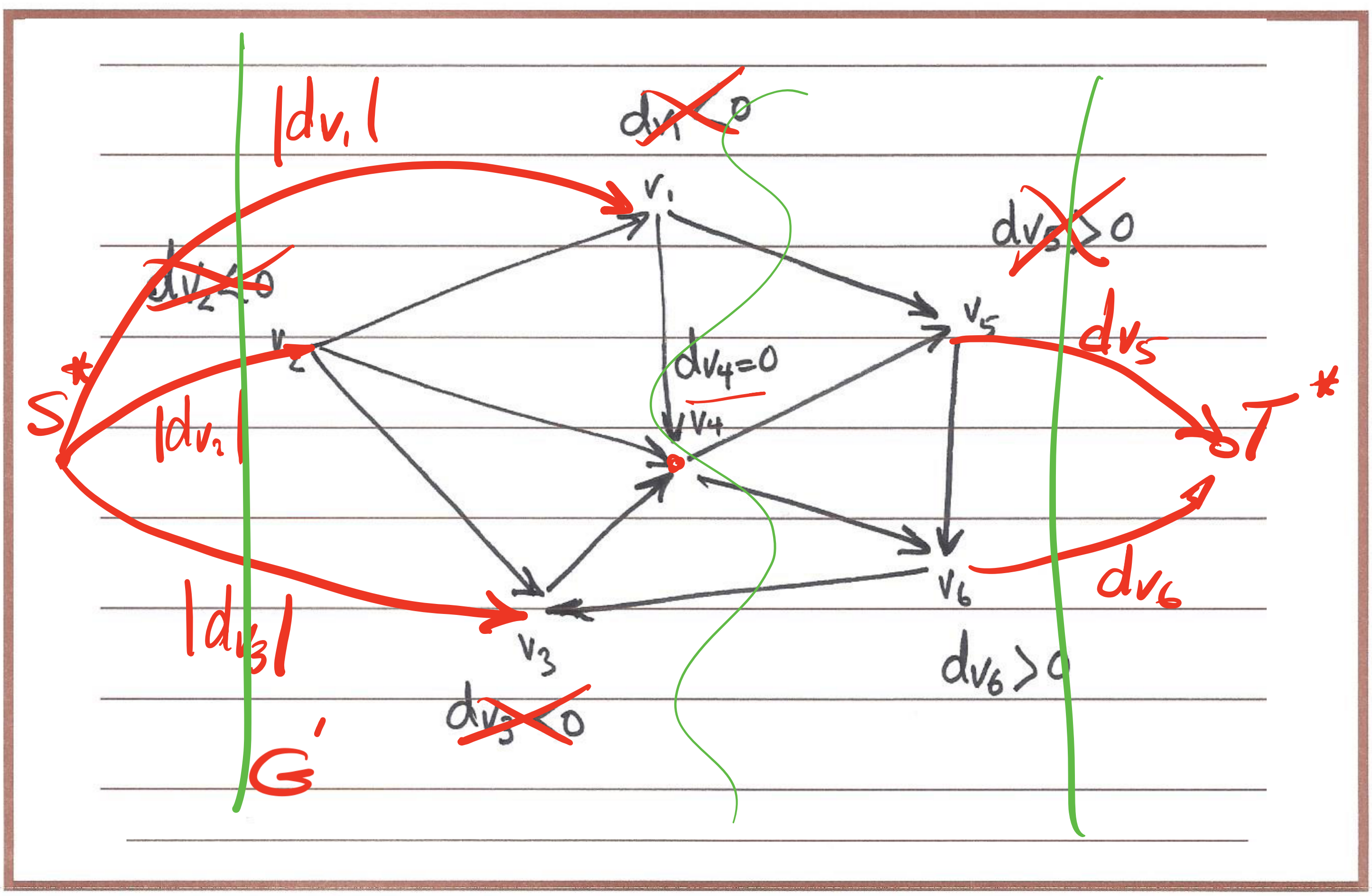
$$\sum_v d_v = 0$$

Total demand value

$$\sum_{v:d_v>0} d_v = \sum_{v:d_v<0} -d_v = \underline{D}$$



$d_{v_1} < 0$
v_1
$d_{v_5} > 0$
$d_{v_2} < 0$
v_2
v_5
$d_{v_4} = 0$
v_4
v_6
v_3
$d_{v_6} > 0$
$d_{v_3} < 0$
G



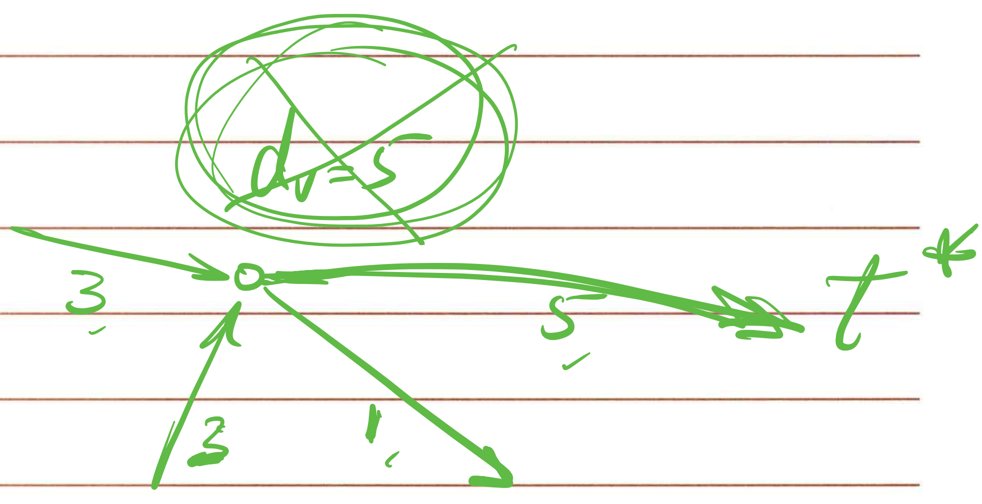
$|d_{v_1}|$
$\cancel{d_{v_1} < 0}$
v_1
$\cancel{d_{v_5} > 0}$
$\cancel{d_{v_2} < 0}$
v_2
v_5
d_{v_5}
S^*
$d_{v_4} = 0$
v_4
T^*
$|d_{v_2}|$
v_6
d_{v_6}
$|d_{v_3}|$
v_3
$d_{v_6} > 0$
$\cancel{d_{v_3} < 0}$
G'

Say we run Max flow on G' and $v(f)$ comes out to be $\underline{k}$.

$k < D$? $\longrightarrow$ No feasible circ. in G

$k > D$? $\longrightarrow$ Not possible

$k = D$? $\longrightarrow$ feasible circ. in G



Proof:

A) If there is a feasible circulation f w/ demand values $\{d_v\}$ in G, we can find a Max Flow in G' of value D.

B) If there is a Max Flow in G' of value D, we can find a feasible circulation in G.

Circulation with Demands & Lowerbounds

Conditions:

1) Capacity condition
   for each edge $e \in E$, $l_e \leq f(e) \leq c_e$

2) Demand condition
   for every node $v \in V$, $f^{in}(v) - f^{out}(v) = d_v$

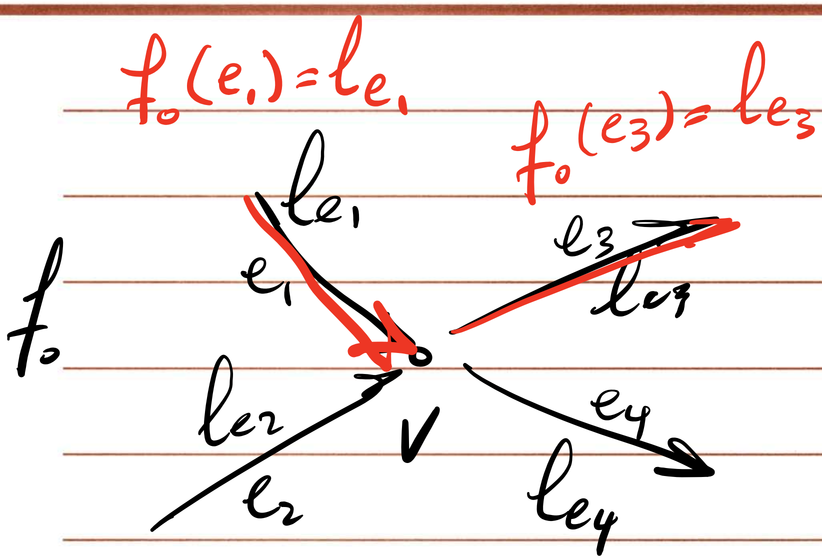
Solution:

Find feasible circulation (if it exists) in two passes.

Pass #1. find f_0 to satisfy all l_e's

Pass #2. Use remaining capacity of the network to find a feasible circulation f_1 (if it exists)

Combine the two flows: $f = f_0 + f_1$



L_v = imbalance at node $\underline{v}$ due to f_0

$$f_0^{in}(v) - f_0^{out}(v) =$$

$$\sum_{e \text{ into } v} l_e - \sum_{e \text{ out of } v} l_e = L_v$$

flow imbalance at node $\underline{v}$.

1- Push flow f_0 thru where
$f_0(e) = l_e$

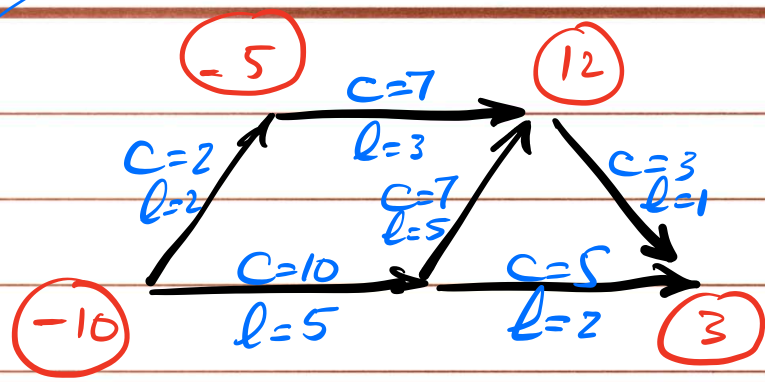
2- Construct G' where
$C_e' = C_e - l_e$

& $d_v' = d_v - L_v$

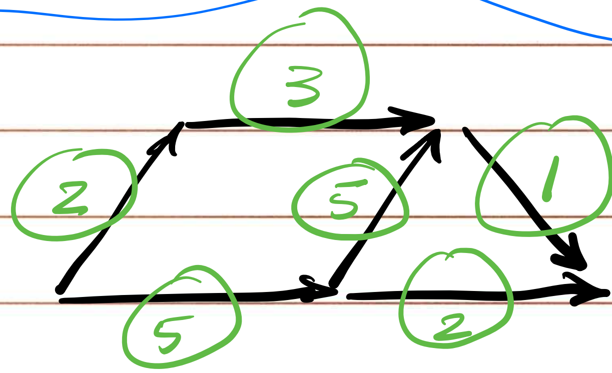
3- Find feasible circ. in G'
call this f_1

4- If there is no feasible circ in G'
$\Rightarrow$ no feasible " in G

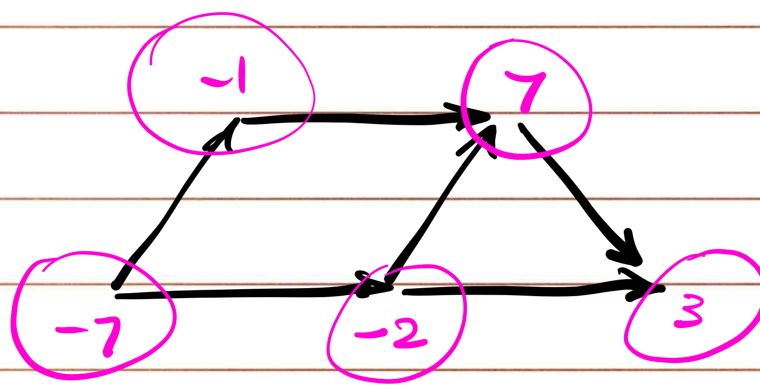
otherwise, feasible circ in G =
$f_0 + f_1$


-5
12
-10
3
c=2 l=2
c=7 l=3
c=7 l=5
c=3 l=1
c=10 l=5
c=5 l=2

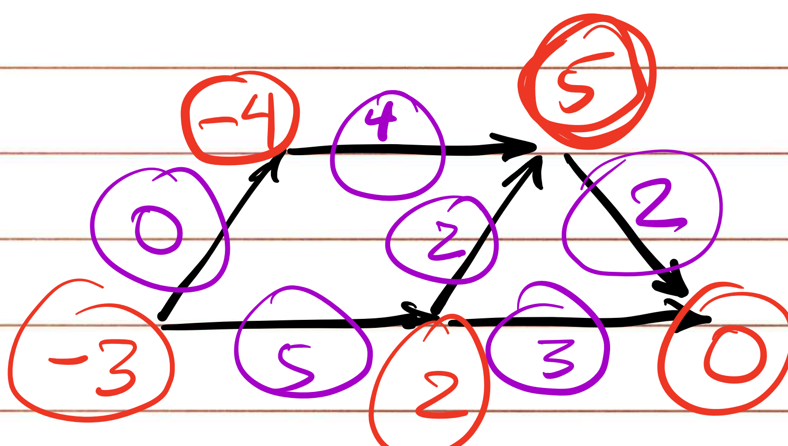
f_0


3
2
5
1
5
2

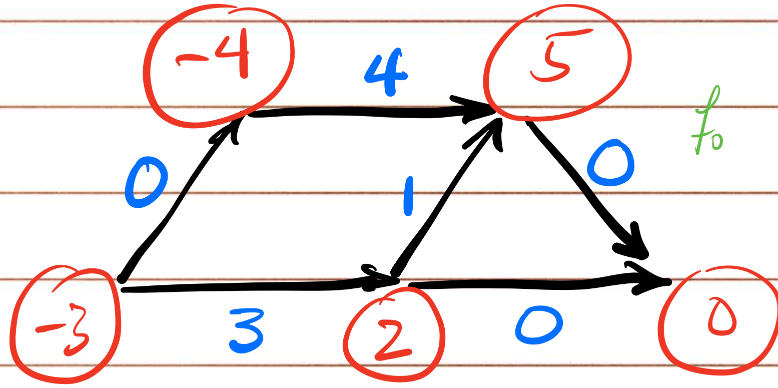
L_v


-1
7
-7
-2
3

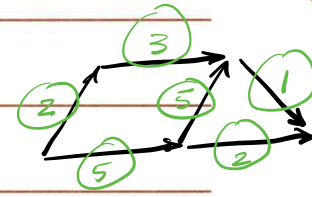
G'


-4
4
5
0
2
2
-3
5
2
3
0

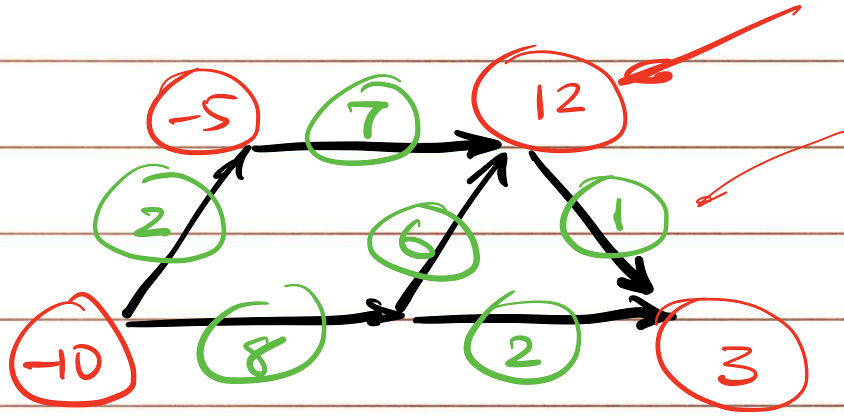
f_1



f_0



$f_0 + f_1$



Survey Design Problem

Survey Design

Input:
- Information on who purchased which products
- Maximum and minimum number of questions to send to customer i
- Maximum and minimum number of questions to ask about product j