Introduction to Optimization and Nonlinear Equations

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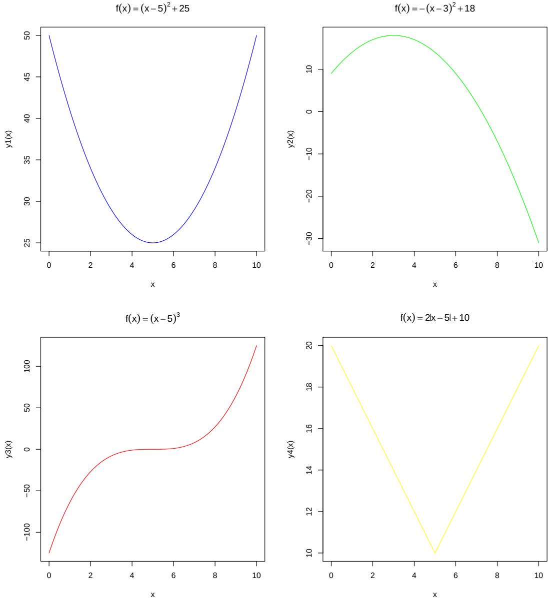
Safe Univariate Methods:

Optimization Problem:Definition

In mathematics, computer science and economics, an optimization problem is the problem of finding the best solution from all feasible solutions.–wiki

Such as finding the maximum/minimum value for a certain function that is defined on a discrete set/continuum

Optimization Problem:examples



Optimization Problem:Assumptions and efficiency

What assumptions needed for each algorithm?

which algorithm is less restrictive?

How to evaluate the efficiency of an algorithm?

by counting how many evaluations needed to reach the maximum/root.

also in computer program, some minor steps also need to be counted in.

Lattice Search

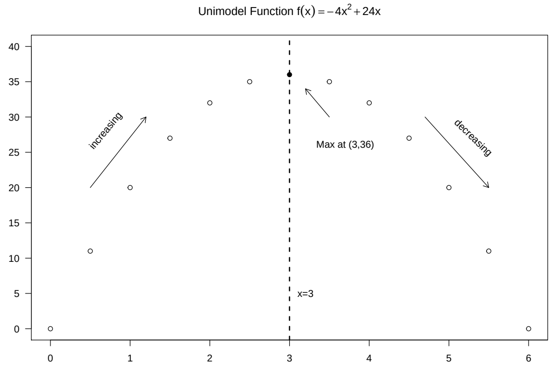
Finding the maximum of a unimodel function f on a discrete set of points $1, 2, ..., m$ a lattice

(i)finding good end strategies for finding the mode on a small set of points

(ii)employing backwards induction to start with the right strategy to match the optimal ending

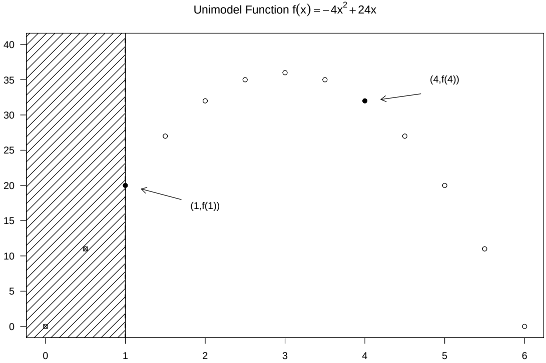
Lattice Search:Unimodel function on discrete points

Graph for $f(x) = -4x^2 + 24x$, this is a unimodel function.



for $x < 3$, function value $f(x)$ is monotonically increasing, and for $x \geq 3$, $f(x)$ is monotonically decreasing.

Lattice Search:How to choose points to compare?



If $f(4) > f(1)$, then we immediately drop the points that are less than $x = 1$, otherwise it will violate the assumption of unimodel function

Lattice Search:Fibonacci Numbers

The optimal strategy is by applying Fibonacci numbers

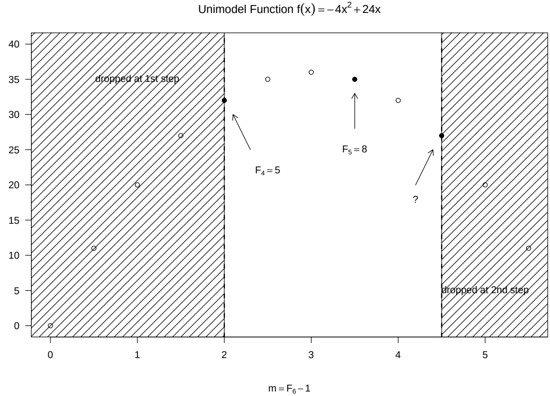
$F_n = \{1, 2, 3, 5, 8, 13, ..., F_n = F_{n-1} + F_{n-2}\}$

suppose we have a set of discrete points$\{1, 2, 3, ..., m = F_n - 1\}$, and we begin the searching by evaluating at the pointsF_{n-2}andF_{n-1}.

if $f(F_{n-2}) < f(F_{n-1})$, then the sub-problem is $\{F_{n-2} + 1, ..., F_n - 1\}$ with $f(F_{n-1})$ has already been evaluated, thus a problem with $F_n - 1$ elements needs $n - 1$ evaluations to solve.

Lattice Search:Fibonacci Numbers

$F_n = \{1, 2, 3, 5, 8, 13\}$



For $m = F_6 - 1 = 12$, 5 evaluations are enough to reach the maximum value.
after each step, we got a sub-problem with $F_{n-1} - 1$ points.

Lattice Search:Details

(i)If the values of the function are the same at F_{n-2} and F_{n-1}, the mode must be between the two points according to our assumption, then it doesn't matter which part is dropped.

(ii)If the number of points m is not one fewer than a Fibonacci number, then add some points at one side. where the value of additional points is $-\infty$.

Golden Section

A more common problem is searching for the maximum on a continuum. so without losing generality, we set the interval (0,1).

Divided the interval and use lattice search by placing m points in the interval, the set is

$\{0, \frac{1}{m-1}, \frac{2}{m-1}, ..., 1\}$.

First two points

$lim \frac{F_{n-2}-1}{F_n-1}$ and $lim \frac{F_{n-1}-1}{F_n-1}$

Golden Section

And since we knew that the lattice search is defined by the first two evaluations, let m goes to infinity, then

set$lim\frac{F_{n-1}}{F_n} = \phi$, so that $lim\frac{F_n}{F_{n-1}} = \frac{1}{\phi} = lim\frac{F_{n-2}+F_{n-1}}{F_{n-1}} = \phi + 1$

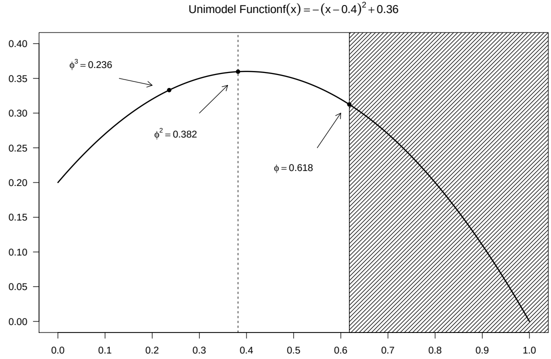
$\phi^2 + \phi + 1 = 0, \phi = \frac{\sqrt{5}-1}{2} \approx .0618$

which also known as the golden ratio. thus the starting points of the search are

$X_1 = \phi^2 \approx 0.382, X_2 = \phi \approx 0.618$

The limit of the lattice search is called the golden section search.

Golden Section



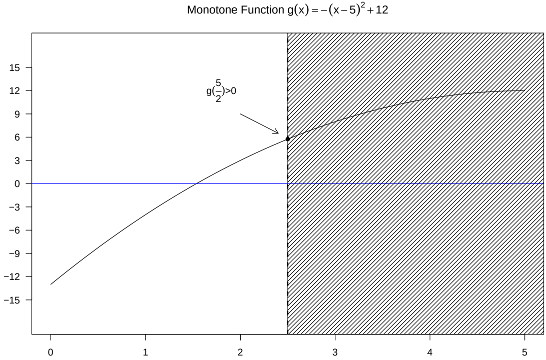
After the first step, the uncertainty of interval is $(0, \phi)$ and the point $\phi^2 = 0.382$ has already been evaluated. Noticed it is also the right point in the second step, which is very similar to lattice search, and we only need to evaluate one more point at $\phi^3 \approx 0.236$.

Bisection

Fibonacci search is less restrictive, since the derivative of the function f doesn't need to exist, but suppose the derivative of the function f is available, which would convert the problem from finding maximum of a unimodel function to finding the root of a monotone function g on the same interval

suppose $g(x)$ is defined on interval (a, b), and let$g(a) < 0 < g(b)$. with a single evaluation at $g(\frac{a+b}{2})$, the uncertainty of interval will be halved.

Bisection



Thus, after the first step, we reset the right endpoint as $\frac{a+b}{2}$, and repeat this procedure to get the root.

Comparison:Golden Section and Bisection

Golden Section:

less restrictive, requiring only a strictly unimodel function.

reduce the interval of uncertainty to $(0, \phi)$ in each iteration.

Bisection:

moew restrictive, requiring the derivative exist and be available.

halve the interval of uncertainty, that is $\frac{1}{2}$.

Root finding

Newton's Method:Iteration Formula

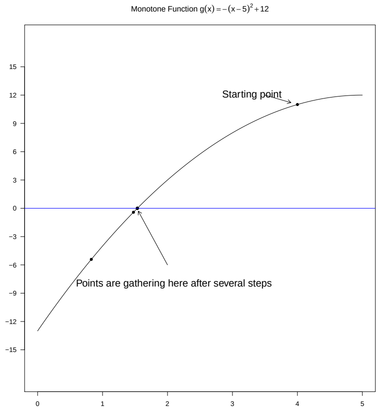
The more common problem is finding a root for a single nonlinear equation $g(x) = 0$.
for function g, set its derivative as g', we have
$g_t(x) = g(x_{old}) + g'(x_{old})(x - x_{old})$

$g_t(x) = 0$ is at

$x_{new} = x_{old} - \frac{g(x_{old})}{g'(x_{old})}$
by using n, the iteration formula is:

$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}$

Newton's Method:



n	xn	gn
1	4.0000000	11.0000000
2	-1.5000000	-30.2500000
3	0.8269231	-5.4145710
4	1.4756735	-0.4208771
5	1.5353838	-0.0035653
6	1.5358983	-0.0000003
7	1.5358984	0.0000000
8	1.5358984	0.0000000
9	1.5358984	0.0000000
10	1.5358984	0.0000000

Newton's Method:Quadratic Convergence

If we denote the root by c and the error at iteration n by $e_n = x_n - c$

the relative error is $d_n = \frac{e_n}{c} = \frac{(x_n - c)}{c}$

By using Taylor expansion:

$$g(c) = 0 = g(x_n) + (c - x_n)g'(x_n) + (c - x_n)^2 \frac{g''(t)}{2}$$

where t lies between x_n and c.

Newton's Method:Quadratic Convergence

noticed $e_{n+1} = x_{n+1} - c = x_n - \frac{g(x_n)}{g'(x_n)} - c$

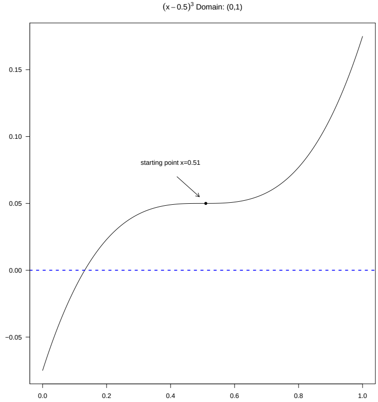
substitute into the equation.

$$x_n - c - \frac{g(x_n)}{g'(x_n)} = (x_n - c)^2 [\frac{g''(t)}{2g'(x_n)}]$$

$$e_{n+1} = e_n^2 [\frac{g''(t)}{2g'(x_n)}]$$

This expression reveals the quadratic convergence of Newton's Method.

Newton's Method:steep and flat?



n	xn	ratio
1	0.510	166.670
2	-166.160	-55.553
3	-110.607	-37.036
4	-73.571	-24.690
5	-48.881	-16.460
6	-32.421	-10.973
7	-21.447	-7.316
8	-14.131	-4.877
9	-9.254	-3.251
10	-6.003	-2.167

noticed for a flat point, $g'(x)$ could be very small so that the next point may leap far away from the true root.

Newton's Method:Pros and Cons

Pros:

Newton's method achieves the fastest rate of convergence

Cons:

the derivative function must be available, and finding it can be tedious or impossible.

The Secant Method

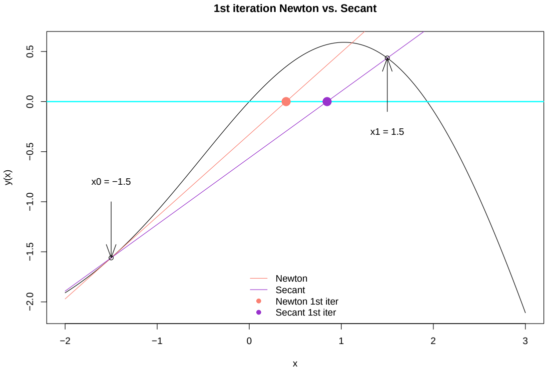
If g' is hard or even impossible to find, we can approximate $g'(x) \approx \frac{g(x+h) - g(x)}{h}$.

The iteration formula now becomes $x_{n+1} = x_n - g(x_n)\frac{x_n - x_{n-1}}{g(x_n) - g(x_{n-1})}$

Notice two initial approximations are required instead of one like the Newton's method.

The Secant Method: Geometrical Interpretation

Let $f(x) = sin(x) - (\frac{x}{2})^2$, $x_0 = -1.5$ and $x_1 = 1.5$

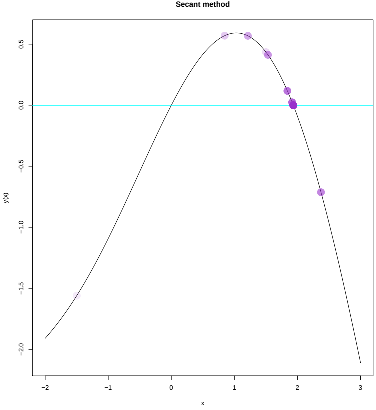


x_{n+1} is taken to be the abscissa of the point of intersection between the secant through $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$ and the x-axis.

The Secant Method: An Example

Let $f(x) = sin(x) - (\frac{x}{2})^2$

$x_0 = -1.5$ and $x_1 = 1.5$



n	xn	fn
0	-1.5000000	-1.5599950
1	1.5000000	0.4349950
2	0.8458689	0.5696740
3	3.6127549	-3.7169217
4	1.2135787	0.5686801
5	1.5319385	0.4125362
6	2.3730538	-0.7127606
7	1.8402932	0.1172352
8	1.9155445	0.0238328
9	1.9347459	-0.0013123
10	1.9337439	0.0000131
11	1.9337538	0.0000000

The Secant Method: Several Definitions

Given $n+1$ distinct pairs $\{(x_0, g(x_0)), (x_1, g(x_1)), \ldots, (x_n, g(x_n))\}$, we will define:

$int(x_0, x_1, \ldots, x_n)$: the smallest interval that contains $x_0, \ldots, x_n$

The divided differences

$$g[x_0, x_1, \ldots, x_j, x] = \frac{g[x_0, x_1, \ldots, x_{j-1}, x] - g[x_0, x_1, \ldots, x_j]}{x - x_j}$$

, and

$$g[x_0, x] = \frac{g(x) - g(x_0)}{x - x_0}$$

The Secant Method: Newton's Interpolation Formula

Given $n+1$ distinct pairs $\{(x_0, g(x_0)), (x_1, g(x_1)), \ldots, (x_n, g(x_n))\}$, we can interpolate these points using a polynomial $q(x)$ of degree n. Specifically,

$$q(x) = g(x_0) + \sum_{j=1}^{n} g[x_0, x_1, \ldots, x_j] \prod_{i=0}^{j-1} (x - x_i)$$

, with the remainder

$$g(x) - q(x) = \frac{g^{n+1}(\xi) \prod_{i=0}^{n} (x - x_i)}{(n+1)!}$$

, where $\xi \in int(x_0, x_1, \ldots, x_n, x)$

The Secant Method: Order of convergence

According to Newton's interpolation formula, we have

$$g(x) = g(x_n) + (x - x_n)g[x_{n-1}, x_n] + \frac{1}{2}(x - x_n)(x - x_{n-1})g''(\xi)$$

where $g[x_{n-1}, x_n] = \dfrac{g(x_n) - g(x_{n-1})}{x_n - x_{n-1}}$, and $\xi \in int(x, x_n, x_{n-1})$

By the Secant Method, we have

$$x_{n+1} = x_n - g(x_n)\frac{x_n - x_{n-1}}{g(x_n) - g(x_{n-1})} \Rightarrow$$

$$0 = g(x_n) + (x_{n+1} - x_n)g[x_{n-1}, x_n]$$

The Secant Method: Order of convergence

Let the root the Secant Method approaches be c, then

$$0 = g(c) - g(x_n) + (x_{n+1} - x_n)g[x_{n-1}, x_n] =$$

$$g[x_{n-1}, x_n](c - x_{n+1}) + \frac{1}{2}(c - x_n)(c - x_{n-1})g''(\xi)$$

By the mean value theorem, we have

$$g[x_{n-1}, x_n] = g'(\eta), \eta \in (x_{n-1}, x_n)$$

Let $\epsilon_n = c - x_n$, we get $0 = g'(\eta)\epsilon_{n+1} + \frac{1}{2}\epsilon_n\epsilon_{n-1}g''(\xi) \Rightarrow$ $\epsilon_{n+1} = \frac{g''(\xi)}{2g'(\eta)}\epsilon_n\epsilon_{n-1}$

The Secant Method: Order of convergence

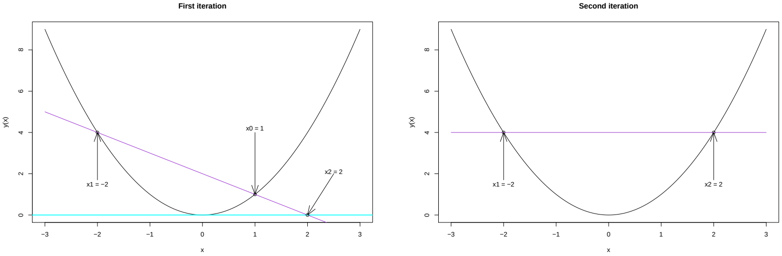
Now suppose the Secant Method converges, then when $n \to \infty$, $\xi \approx c$ and $\eta \approx c$. Let $C = \dfrac{g''(c)}{2g'(c)}$, then $|\epsilon_{n+1}| = C|\epsilon_n||\epsilon_{n-1}|$

To find the order of convergence, we find p such that
$|\epsilon_{n+1}| \approx M|\epsilon_n|^p \Rightarrow$
$M|\epsilon_n|^p = MM|\epsilon_{n-1}|^p|\epsilon_{n-1}| \Rightarrow |\epsilon_n| = M|\epsilon_{n-1}|^{(1+p)/p}$
This implies $p = (1+p)/p \Rightarrow p = 1 + \phi \approx 1.618$

Since the exponent 1.618 lies between 1 (linear convergence) and 2 (quadratic convergence), the convergence rate of the Secant Method is called *superlinear*.

The Secant Method: Pros and Cons

1. Pros:
   - Superlinear convergence
   - No need to evaluate derivatives
2. Cons:
   - Convergence is not guaranteed



Regula Falsi

Illinois Algorithm

Stopping and Condition