Introduction to Optimization and Nonlinear Equations

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Safe Univariate Methods:

Lattice Search

Golden Search

Bisection

Root finding

Newton's Method

The Secant Method

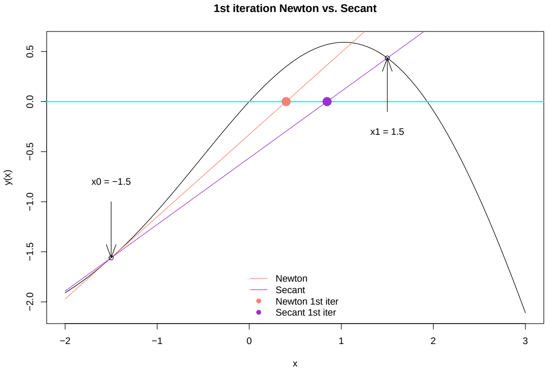
If g' is hard or even impossible to find, we can approximate $g'(x) \approx \dfrac{g(x+h) - g(x)}{h}$.

The iteration formula now becomes
$$x_{n+1} = x_n - g(x_n)\frac{x_n - x_{n-1}}{g(x_n) - g(x_{n-1})}$$

Notice two initial approximations are required instead of one like the Newton's method.

The Secant Method: Geometrical Interpretation

Let $f(x) = sin(x) - (\frac{x}{2})^2$, $x_0 = -1.5$ and $x_1 = 1.5$

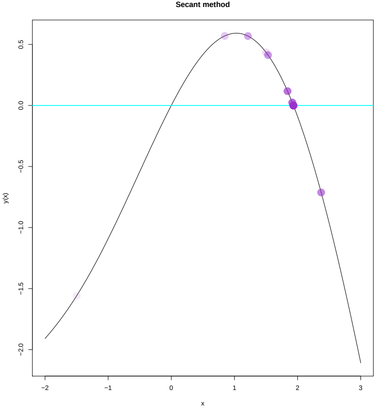


x_{n+1} is taken to be the abscissa of the point of intersection between the secant through $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$ and the x-axis.

The Secant Method: An Example

Let $f(x) = sin(x) - (\frac{x}{2})^2$

$x_0 = -1.5$ and $x_1 = 1.5$



n	xn	fn
0	-1.5000000	-1.5599950
1	1.5000000	0.4349950
2	0.8458689	0.5696740
3	3.6127549	-3.7169217
4	1.2135787	0.5686801
5	1.5319385	0.4125362
6	2.3730538	-0.7127606
7	1.8402932	0.1172352
8	1.9155445	0.0238328
9	1.9347459	-0.0013123
10	1.9337439	0.0000131
11	1.9337538	0.0000000

The Secant Method: Several Definitions

Given $n+1$ distinct pairs $\{(x_0, g(x_0)), (x_1, g(x_1)), \ldots, (x_n, g(x_n))\}$, we will define:

$int(x_0, x_1, \ldots, x_n)$: the smallest interval that contains $x_0, \ldots, x_n$

The divided differences

$$g[x_0, x_1, \ldots, x_j, x] = \frac{g[x_0, x_1, \ldots, x_{j-1}, x] - g[x_0, x_1, \ldots, x_j]}{x - x_j}$$

, and

$$g[x_0, x] = \frac{g(x) - g(x_0)}{x - x_0}$$

The Secant Method: Newton's Interpolation Formula

Given $n+1$ distinct pairs $\{(x_0, g(x_0)), (x_1, g(x_1)), \dots, (x_n, g(x_n))\}$, we can interpolate these points using a polynomial $q(x)$ of degree n. Specifically,

$$q(x) = g(x_0) + \sum_{j=1}^{n} g[x_0, x_1, \dots, x_j] \prod_{i=0}^{j-1} (x - x_i)$$

, with the remainder

$$g(x) - q(x) = \frac{g^{n+1}(\xi) \prod_{i=0}^{n} (x - x_i)}{(n+1)!}$$

, where $\xi \in int(x_0, x_1, \dots, x_n, x)$

The Secant Method: Order of convergence

According to Newton's interpolation formula, we have

$$g(x) = g(x_n) + (x - x_n)g[x_{n-1}, x_n] + \frac{1}{2}(x - x_n)(x - x_{n-1})g''(\xi)$$

where $g[x_{n-1}, x_n] = \dfrac{g(x_n) - g(x_{n-1})}{x_n - x_{n-1}}$, and $\xi \in int(x, x_n, x_{n-1})$

By the Secant Method, we have

$$x_{n+1} = x_n - g(x_n)\frac{x_n - x_{n-1}}{g(x_n) - g(x_{n-1})} \Rightarrow$$

$$0 = g(x_n) + (x_{n+1} - x_n)g[x_{n-1}, x_n]$$

The Secant Method: Order of convergence

Let the root the Secant Method approaches be c, then

$$0 = g(c) - g(x_n) + (x_{n+1} - x_n)g[x_{n-1}, x_n] =$$

$$g[x_{n-1}, x_n](c - x_{n+1}) + \frac{1}{2}(c - x_n)(c - x_{n-1})g''(\xi)$$

By the mean value theorem, we have

$$g[x_{n-1}, x_n] = g'(\eta), \eta \in (x_{n-1}, x_n)$$

Let $\epsilon_n = c - x_n$, we get $0 = g'(\eta)\epsilon_{n+1} + \frac{1}{2}\epsilon_n\epsilon_{n-1}g''(\xi) \Rightarrow$ $\epsilon_{n+1} = \frac{g''(\xi)}{2g'(\eta)}\epsilon_n\epsilon_{n-1}$

The Secant Method: Order of convergence

Now suppose the Secant Method converges, then when $n \to \infty$, $\xi \approx c$ and $\eta \approx c$. Let $C = \dfrac{g''(c)}{2g'(c)}$, then $|\epsilon_{n+1}| = C|\epsilon_n||\epsilon_{n-1}|$

To find the order of convergence, we find p such that
$|\epsilon_{n+1}| \approx M|\epsilon_n|^p \Rightarrow$
$M|\epsilon_n|^p = MM|\epsilon_{n-1}|^p|\epsilon_{n-1}| \Rightarrow |\epsilon_n| = M|\epsilon_{n-1}|^{(1+p)/p}$
This implies $p = (1+p)/p \Rightarrow p = 1 + \phi \approx 1.618$

Since the exponent 1.618 lies between 1 (linear convergence) and 2 (quadratic convergence), the convergence rate of the Secant Method is called *superlinear*.

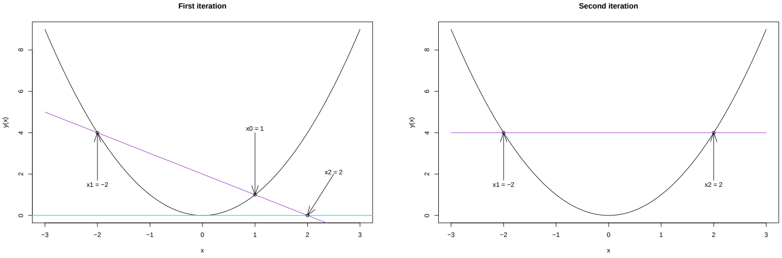
The Secant Method: Pros and Cons

1. Pros:

   - Superlinear convergence
   - No need to evaluate derivatives

2. Cons:

   - Convergence is not guaranteed



Regula Falsi

Illinois Algorithm

Stopping and Condition