Introduction to Optimization and Nonlinear Equations

Zeyu Lu & Yuqiu Yang

Safe Univariate Methods:

Lattice Search

Golden Search

Bisection

Root finding

Newton's Method

The Secant Method

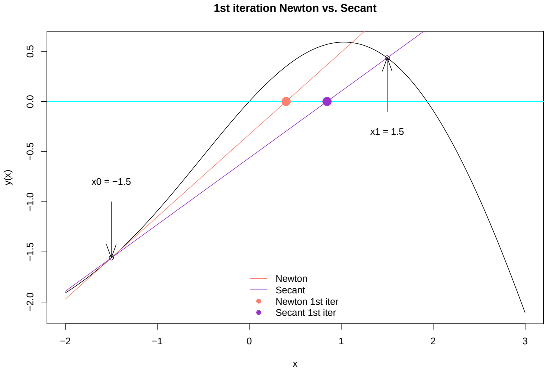
If g' is hard or even impossible to find, we can approximate $g'(x) \approx \dfrac{g(x+h) - g(x)}{h}$.

The iteration formula now becomes
$x_{n+1} = x_n - g(x_n)\dfrac{x_n - x_{n-1}}{g(x_n) - g(x_{n-1})}$

Notice two initial approximations are required instead of one like the Newton's method.

The Secant Method: Geometrical Interpretation

Let $f(x) = sin(x) - (\frac{x}{2})^2$, $x_0 = -1.5$ and $x_1 = 1.5$

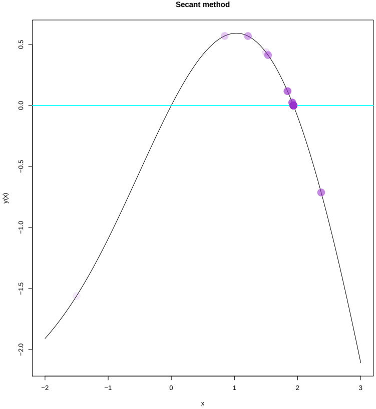


x_{n+1} is taken to be the abscissa of the point of intersection between the secant through $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$ and the x-axis.

The Secant Method: An Example

Let $f(x) = sin(x) - (\frac{x}{2})^2$

$x_0 = -1.5$ and $x_1 = 1.5$



n	xn	fn
0	-1.5000000	-1.5599950
1	1.5000000	0.4349950
2	0.8458689	0.5696740
3	3.6127549	-3.7169217
4	1.2135787	0.5686801
5	1.5319385	0.4125362
6	2.3730538	-0.7127606
7	1.8402932	0.1172352
8	1.9155445	0.0238328
9	1.9347459	-0.0013123
10	1.9337439	0.0000131
11	1.9337538	0.0000000

The Secant Method: Several Definitions

Given $n+1$ distinct pairs $\{(x_0, g(x_0)), (x_1, g(x_1)), \ldots, (x_n, g(x_n))\}$, we will define:

$int(x_0, x_1, \ldots, x_n)$: the smallest interval that contains $x_0, \ldots, x_n$

The divided differences

$$g[x_0, x_1, \ldots, x_j, x] = \frac{g[x_0, x_1, \ldots, x_{j-1}, x] - g[x_0, x_1, \ldots, x_j]}{x - x_j}$$

, and

$$g[x_0, x] = \frac{g(x) - g(x_0)}{x - x_0}$$

The Secant Method: Newton's Interpolation Formula

Given $n+1$ distinct pairs $\{(x_0, g(x_0)), (x_1, g(x_1)), \ldots, (x_n, g(x_n))\}$, we can interpolate these points using a polynomial $q(x)$ of degree n. Specifically,

$$q(x) = g(x_0) + \sum_{j=1}^{n} g[x_0, x_1, \ldots, x_j] \prod_{i=0}^{j-1} (x - x_i)$$

, with the remainder

$$g(x) - q(x) = \frac{g^{n+1}(\xi) \prod_{i=0}^{n} (x - x_i)}{(n+1)!}$$

, where $\xi \in int(x_0, x_1, \ldots, x_n, x)$

The Secant Method: Order of convergence

According to Newton's interpolation formula, we have

$$g(x) = g(x_n) + (x - x_n)g[x_{n-1}, x_n] + \frac{1}{2}(x - x_n)(x - x_{n-1})g''(\xi)$$

where $g[x_{n-1}, x_n] = \dfrac{g(x_n) - g(x_{n-1})}{x_n - x_{n-1}}$, and $\xi \in int(x, x_n, x_{n-1})$

By the Secant Method, we have

$$x_{n+1} = x_n - g(x_n)\frac{x_n - x_{n-1}}{g(x_n) - g(x_{n-1})} \Rightarrow$$

$$0 = g(x_n) + (x_{n+1} - x_n)g[x_{n-1}, x_n]$$

The Secant Method: Order of convergence

Let the root the Secant Method approaches be c, then

$$0 = g(c) - g(x_n) + (x_{n+1} - x_n)g[x_{n-1}, x_n] =$$

$$g[x_{n-1}, x_n](c - x_{n+1}) + \frac{1}{2}(c - x_n)(c - x_{n-1})g''(\xi)$$

By the mean value theorem, we have

$$g[x_{n-1}, x_n] = g'(\eta), \eta \in (x_{n-1}, x_n)$$

Let $\epsilon_n = c - x_n$, we get $0 = g'(\eta)\epsilon_{n+1} + \frac{1}{2}\epsilon_n\epsilon_{n-1}g''(\xi) \Rightarrow$
$\epsilon_{n+1} = \frac{g''(\xi)}{2g'(\eta)}\epsilon_n\epsilon_{n-1}$

The Secant Method: Order of convergence

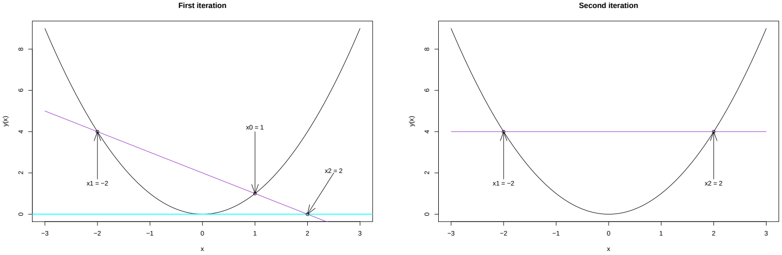
Now suppose the Secant Method converges, then when $n \to \infty$, $\xi \approx c$ and $\eta \approx c$. Let $C = \dfrac{g''(c)}{2g'(c)}$, then $|\epsilon_{n+1}| = C|\epsilon_n||\epsilon_{n-1}|$

To find the order of convergence, we find p such that
$|\epsilon_{n+1}| \approx M|\epsilon_n|^p \Rightarrow$
$M|\epsilon_n|^p = MM|\epsilon_{n-1}|^p|\epsilon_{n-1}| \Rightarrow |\epsilon_n| = M|\epsilon_{n-1}|^{(1+p)/p}$
This implies $p = (1+p)/p \Rightarrow p = 1 + \phi \approx 1.618$

Since the exponent 1.618 lies between 1 (linear convergence) and 2 (quadratic convergence), the convergence rate of the Secant Method is called *superlinear*.

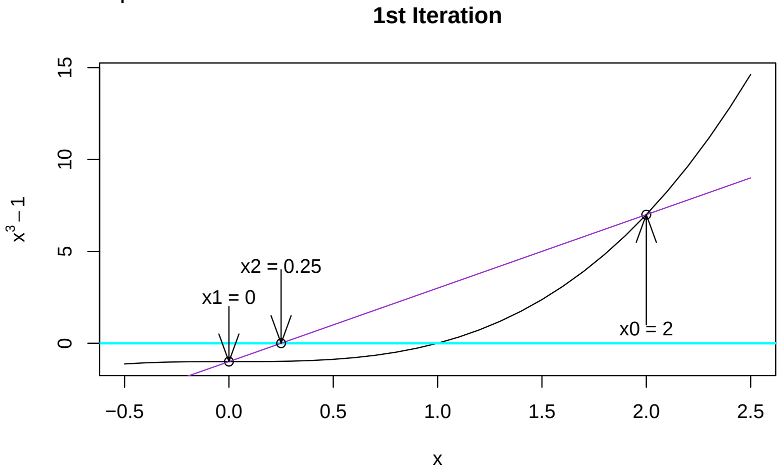
The Secant Method: Pros and Cons

1. Pros:
   - Superlinear convergence
   - No need to evaluate derivatives
2. Cons:
   - Convergence is not guaranteed
   - Not well behaved when g is relatively flat



Regula Falsi: A Motivative Example

Borrowing the idea of the Bisection Method, what if we start with two points that straddle the root?



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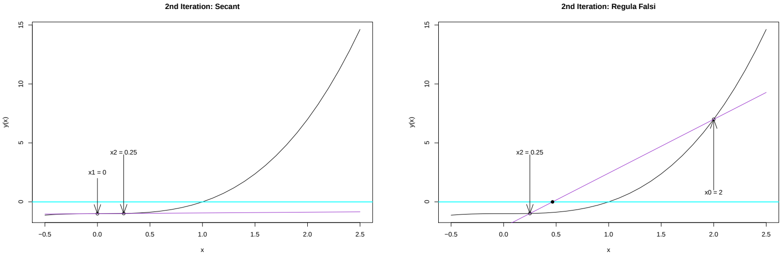
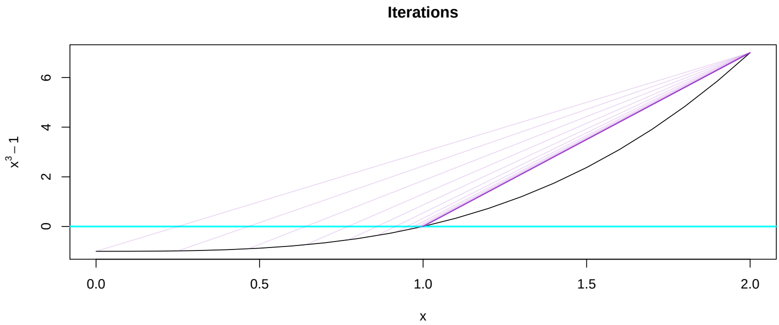
Stopping and Condition

Regula Falsi: A Motivative Example



Regula Falsi

A variant of the Secant Method where instead of choosing the secant through $(x_n, g(x_n))$ and $(x_{n-1}, g(x_{n-1}))$, one finds the secant through $(x_n, g(x_n))$ and $(x_{n'}, g(x_{x'}))$ where $n' < n$ is the largest index for which $g(x_n)g(x_{n'}) < 0$.



Regula Falsi: Order of Convergence

Like the Bisection Method, the Regula Falsi is "safe". However, from the previous example, we see that this method is in general a first-order method.

Especially, if $g(x)$ is convex on $[x_0, x_1]$, then

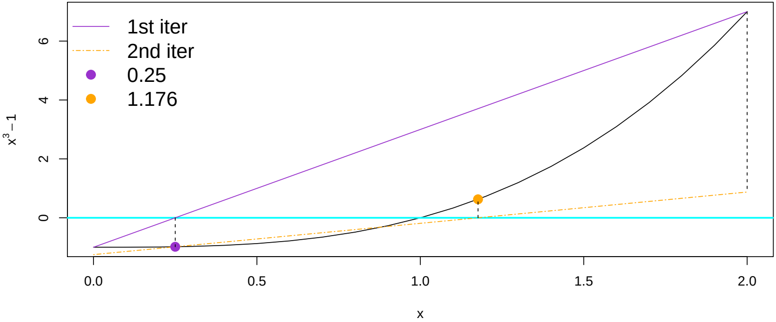
$$|\epsilon_{n+1}| \approx C|\epsilon_n||\epsilon_0| = C'|\epsilon_n|$$

where $C = \dfrac{g''(c)}{2g'(c)}$

The Regula Falsi Method tends to retain one end-point for several iterations. As a result, it can be a good "start" method or a part of a "hybrid" method, but it should not be used near a root.

Illinois Algorithm: Building on Regula Falsi

In the previous example, if we artificially create a shallower secant, then maybe the end-point will no longer be retained.

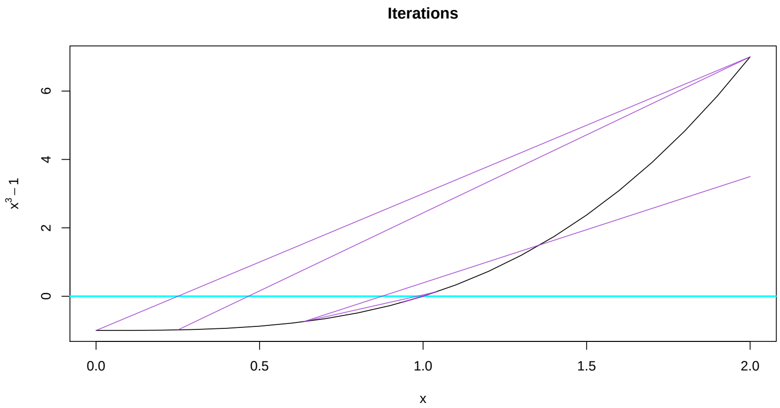


By dividing the function value at 2 by 8 and calculating the new secant, we find a new root on right of the root. In the next iternation, the new root 1.176 instead of 2 will be used.

Illinois Algorithm

1. During the Regula Falsi procedure, once we find one end-point has been retained more than once, we half the function value at that point, find the secant line and the new root.
2. If the point still retains, we repeat Step 1.
3. Once the point changes, we proceed with the Regula Falsi

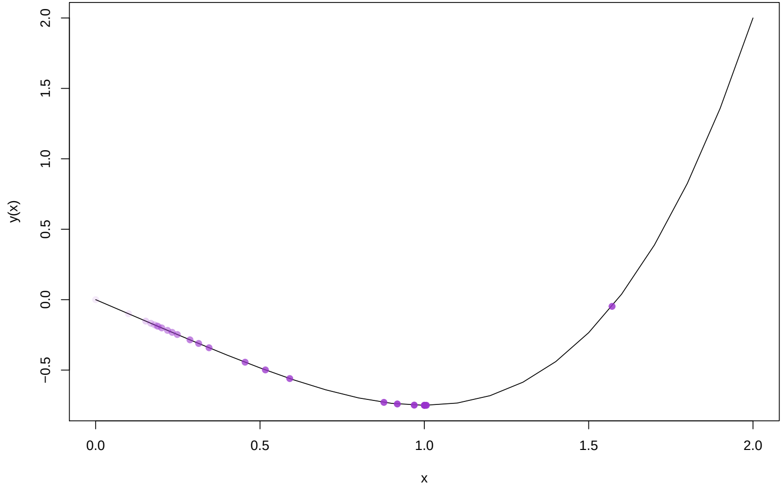
Illinois Algorithm: An Example



Compared with the Regula Falsi Method, the Illinois Algorithm gets in a small neighborhood of the root in just 4 or 5 iterations.

Illinois Algorithm: Order of Convergence

Optimization



Stopping and Condition