

Introduction to Optimization and Nonlinear Equations

Zeyu Lu & Yuqiu Yang

1 Safe Univariate Methods:

2 Root finding

3 Stopping and Condition

Safe Univariate Methods:

Lattice Search

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The Secant Method

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If g' is hard or even impossible to find, we can approximate

$$g'(x) \approx \frac{g(x+h) - g(x)}{h}.$$

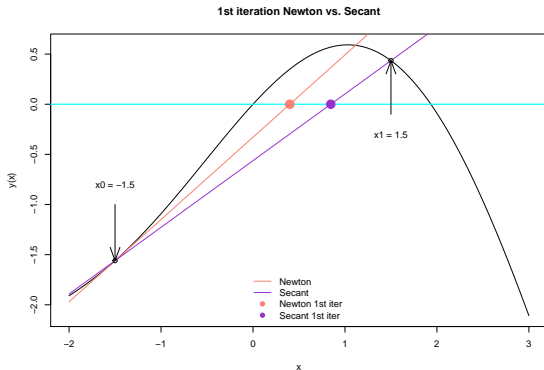
The iteration formula now becomes

$$x_{n+1} = x_n - g(x_n) \frac{x_n - x_{n-1}}{g(x_n) - g(x_{n-1})}$$

Notice two initial approximations are required instead of one like the Newton's method.

The Secant Method: Geometrical Interpretation

Let $f(x) = \sin(x) - (\frac{x}{2})^2$, $x_0 = -1.5$ and $x_1 = 1.5$



x_{n+1} is taken to be the abscissa of the point of intersection between the secant through $(x_{n-1}, f(x_{n-1}))$ and $(x_n, f(x_n))$ and the x-axis.

The Secant Method: An Example

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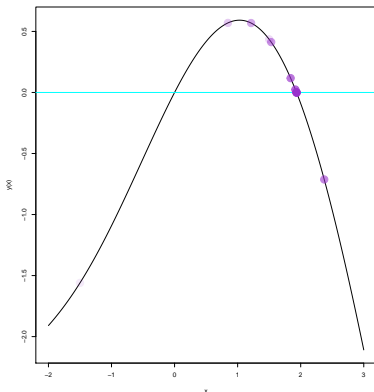
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$$\text{Let } f(x) = \sin(x) - \left(\frac{x}{2}\right)^2$$

$$x_0 = -1.5 \text{ and } x_1 = 1.5$$

Secant method



n	x_n	f_n
0	-1.5000000	-1.5599950
1	1.5000000	0.4349950
2	0.8458689	0.5696740
3	3.6127549	-3.7169217
4	1.2135787	0.5686801
5	1.5319385	0.4125362
6	2.3730538	-0.7127606
7	1.8402932	0.1172352
8	1.9155445	0.0238328
9	1.9347459	-0.0013123
10	1.9337439	0.0000131
11	1.9337538	0.0000000

The Secant Method: Several Definitions

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Given $n + 1$ distinct pairs

$\{(x_0, g(x_0)), (x_1, g(x_1)), \dots, (x_n, g(x_n))\}$, we will define:

$int(x_0, x_1, \dots, x_n)$: the smallest interval that contains x_0, \dots, x_n

The divided differences

$$g[x_0, x_1, \dots, x_j, x] = \frac{g[x_0, x_1, \dots, x_{j-1}, x] - g[x_0, x_1, \dots, x_j]}{x - x_j}$$

, and

$$g[x_0, x] = \frac{g(x) - g(x_0)}{x - x_0}$$

The Secant Method: Newton's Interpolation Formula

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Given $n + 1$ distinct pairs

$\{(x_0, g(x_0)), (x_1, g(x_1)), \dots, (x_n, g(x_n))\}$, we can interpolate these points using a polynomial $q(x)$ of degree n .

Specifically,

$$q(x) = g(x_0) + \sum_{j=1}^n g[x_0, x_1, \dots, x_j] \prod_{i=0}^{j-1} (x - x_i)$$

, with the remainder

$$g(x) - q(x) = \frac{g^{n+1}(\xi) \prod_{i=0}^n (x - x_i)}{(n+1)!}$$

, where $\xi \in \text{int}(x_0, x_1, \dots, x_n, x)$

The Secant Method: Order of convergence

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According to Newton's interpolation formula, we have

$$g(x) = g(x_n) + (x - x_n)g[x_{n-1}, x_n] + \frac{1}{2}(x - x_n)(x - x_{n-1})g''(\xi)$$

$$\text{where } g[x_{n-1}, x_n] = \frac{g(x_n) - g(x_{n-1})}{x_n - x_{n-1}}, \text{ and } \xi \in \text{int}(x, x_n, x_{n-1})$$

By the Secant Method, we have

$$x_{n+1} = x_n - g(x_n) \frac{x_n - x_{n-1}}{g(x_n) - g(x_{n-1})} \Rightarrow$$

$$0 = g(x_n) + (x_{n+1} - x_n)g[x_{n-1}, x_n]$$

The Secant Method: Order of convergence

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Let the root the Secant Method approaches be c , then

$$0 = g(c) - g(x_n) + (x_{n+1} - x_n)g[x_{n-1}, x_n] = \\ g[x_{n-1}, x_n](c - x_{n+1}) + \frac{1}{2}(c - x_n)(c - x_{n-1})g''(\xi)$$

By the mean value theorem, we have

$$g[x_{n-1}, x_n] = g'(\eta), \eta \in (x_{n-1}, x_n)$$

$$\text{Let } \epsilon_n = c - x_n, \text{ we get } 0 = g'(\eta)\epsilon_{n+1} + \frac{1}{2}\epsilon_n\epsilon_{n-1}g''(\xi) \Rightarrow \\ \epsilon_{n+1} = \frac{g''(\xi)}{2g'(\eta)}\epsilon_n\epsilon_{n-1}$$

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Now suppose the Secant Method converges, then when $n \rightarrow \infty$, $\xi \approx c$ and $\eta \approx c$. Let $C = \frac{g''(c)}{2g'(c)}$, then $|\epsilon_{n+1}| = C|\epsilon_n||\epsilon_{n-1}|$

To find the order of convergence, we find p such that

$$|\epsilon_{n+1}| \approx M|\epsilon_n|^p \Rightarrow \\ M|\epsilon_n|^p = MM|\epsilon_{n-1}|^p|\epsilon_{n-1}| \Rightarrow |\epsilon_n| = M|\epsilon_{n-1}|^{(1+p)/p}$$

This implies $p = (1 + p)/p \Rightarrow p = 1 + \phi \approx 1.618$

Since the exponent 1.618 lies between 1 (linear convergence) and 2 (quadratic convergence), the convergence rate of the Secant Method is called *superlinear*.

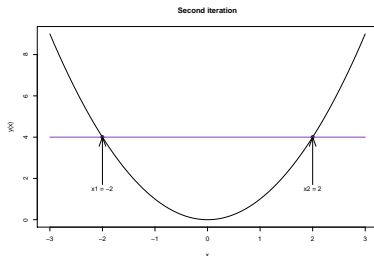
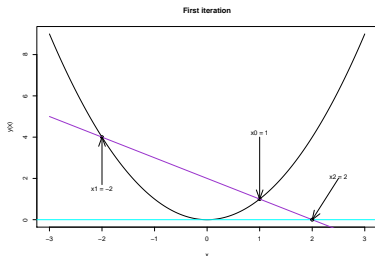
The Secant Method: Pros and Cons

1 Pros:

- Superlinear convergence
- No need to evaluate derivatives

2 Cons:

- Convergence is not guaranteed
- Not well behaved when g is relatively flat



Regula Falsi: A Motivative Example

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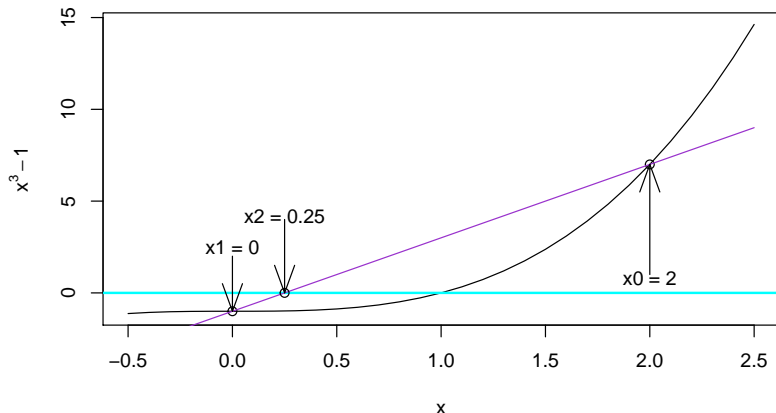
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Borrowing the idea of the Bisection Method, what if we start with two points that straddle the root?

1st Iteration



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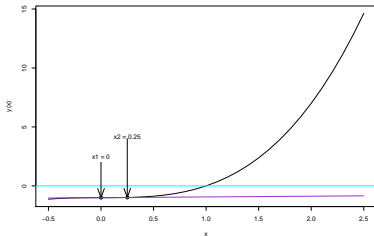
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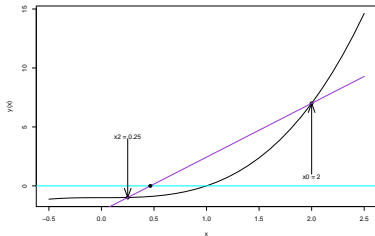
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2nd Iteration: Secant



2nd Iteration: Regula Falsi



Regula Falsi

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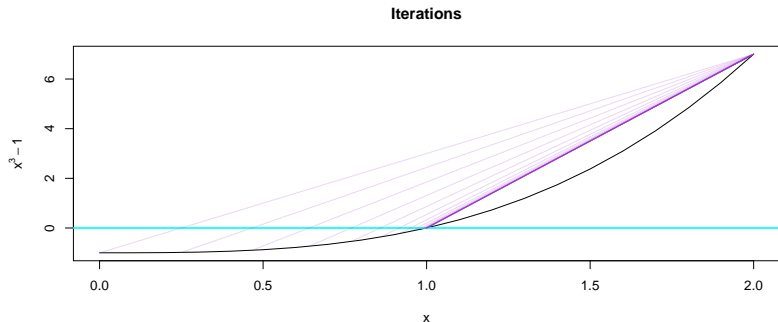
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A variant of the Secant Method where instead of choosing the secant through $(x_n, g(x_n))$ and $(x_{n-1}, g(x_{n-1}))$, one finds the secant through $(x_n, g(x_n))$ and $(x_{n'}, g(x_{n'}))$ where $n' < n$ is the largest index for which $g(x_n)g(x_{n'}) < 0$.



Regula Falsi: Order of Convergence

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Like the Bisection Method, the Regula Falsi is “safe”. However, from the previous example, we see that this method is in general a first-order method.

Especially, if $g(x)$ is convex on $[x_0, x_1]$, then

$$|\epsilon_{n+1}| \approx C|\epsilon_n||\epsilon_0| = C'|\epsilon_n|$$

where $C = \frac{g''(c)}{2g'(c)}$

The Regula Falsi Method tends to retain one end-point for several iterations. As a result, it can be a good “start” method or a part of a “hybrid” method, but it should not be used near a root.

Illinois Algorithm: Building on Regula Falsi

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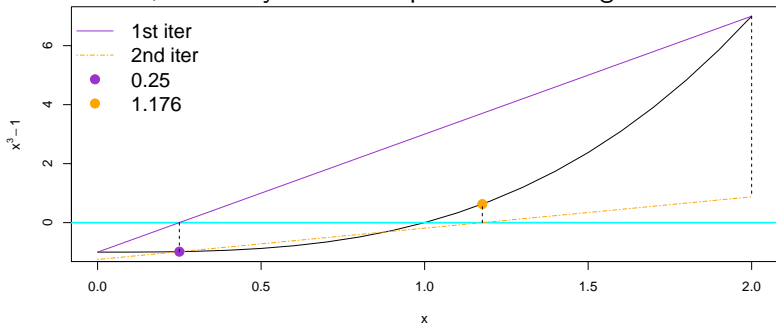
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In the previous example, if we artificially create a shallower secant, then maybe the end-point will no longer be retained.



By dividing the function value at 2 by 8 and calculating the new secant, we find a new root on right of the root. In the next iteration, the new root 1.176 instead of 2 will be used.

Illinois Algorithm

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- 1 During the Regula Falsi procedure, once we find one end-point has been retained more than once, we half the function value at that point, find the secant line and the new root.
- 2 If the point still retains, we repeat Step 1.
- 3 Once the point changes, we proceed with the Regula Falsi

Illinois Algorithm: An Example

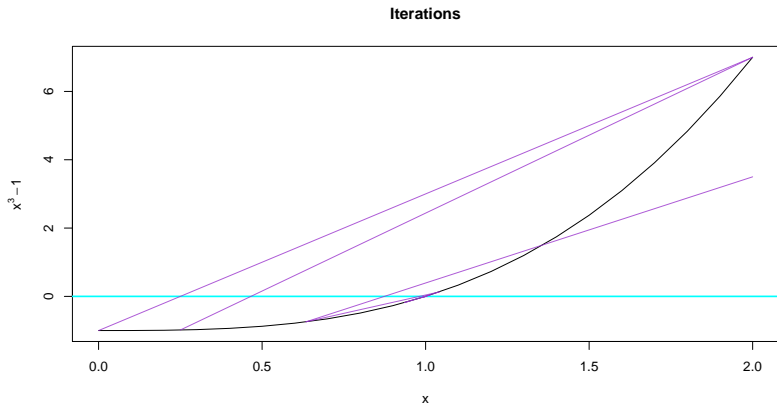
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Compared with the Regula Falsi Method, the Illinois Algorithm gets in a small neighborhood of the root in just 4 or 5 iterations.

Illinois Algorithm: Order of Convergence

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