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Normalized power prior Bayesian analysis

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ABSTRACT

The elicitation of power priors, based on the availability of historical data, is realized by raising the likelihood function of the historical data to a fractional power δ , which quantifies the degree of discounting of the historical information in making inference with the current data. When δ is not pre-specified and is treated as random, it can be estimated from the data using Bayesian updating paradigm. However, in the original form of the joint power prior Bayesian approach, certain positive constants before the likelihood of the historical data could be multiplied when different settings of sufficient statistics are employed. This would change the power priors with different constants, and hence the likelihood principle is violated.

In this article, we investigate a normalized power prior approach which obeys the likelihood principle and is a modified form of the joint power prior. The optimality properties of the normalized power prior in the sense of minimizing the weighted Kullback–Leibler divergence are investigated. By examining the posteriors of several commonly used distributions, we show that the discrepancy between the historical and the current data can be well quantified by the power parameter under the normalized power prior setting. Efficient algorithms to compute the scale factor is also proposed. In addition, we illustrate the use of the normalized power prior Bayesian analysis with three data examples, and provide an implementation with an R package NPP.

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1. Introduction

In applying statistics to real experiments, it is common that the sample size in the current study is inadequate to provide enough precision for parameter estimation, while plenty of the historical data or data from similar research settings are available. For example, when designing a clinical study, historical data of the standard care might be available from other clinical studies or a patient registry. Due to the nature of sequential information updating, it is natural to use a Bayesian approach with an informative prior on the model parameters to incorporate these historical data. Though the current and historical data are usually assumed to follow distributions from the same family, the population parameters may change somewhat over different time and/or experimental settings. How to adaptively incorporate the historical data considering the data heterogeneity becomes a major concern for the informative prior elicitation.

To address this issue, Ibrahim and Chen (1998), and thereafter Chen et al. (2000), Ibrahim and Chen (2000), and Ibrahim et al. (2003) proposed the concept of *power priors*, based on the availability of historical data. The basic idea is to raise

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the likelihood function based on the historical data to a power parameter δ ($0 \leq \delta \leq 1$) that controls the influence of the historical data. Its relationship with hierarchical models is also shown by [Chen and Ibrahim \(2006\)](#). For a comprehensive review of the power prior, we refer the readers to the seminar article ([Ibrahim et al., 2015](#)). The power parameter δ can be prefixed according to external information. It is also possible to search for a reasonable level of information borrowing from the prior-data conflict via sensitivity analysis according to certain criteria. For example, [Ibrahim et al. \(2012a\)](#) suggested the use of deviance information criterion ([Spiegelhalter et al., 2002](#)) or the logarithm of pseudo-marginal likelihood. The choice of δ would depend on the criterion of interest.

[Ibrahim and Chen \(2000\)](#) and [Chen et al. \(2000\)](#) generalized the power prior with a fixed δ to a random δ by introducing the *joint power priors*. They specified a joint prior distribution directly for both δ and θ , the parameters in consideration, in which an independent proper prior for δ was considered in addition to the original form of the power prior. Hypothetically, when the initial prior for δ is vague, the magnitude of borrowing would be mostly determined by the heterogeneity between the historical and the current data. However, under the joint power priors, the posterior distributions vary with the constants before the historical likelihood functions, which violates the likelihood principle ([Birnbbaum, 1962](#)). It raises a critical question regarding which likelihood function should be used in practice. For example, the likelihood function based on the raw data and the likelihood function based on the sufficient statistics could differ by a multiplicative constant. This would likely yield different posteriors. Therefore, it may not be appropriate ([Neuenschwander et al., 2009](#)). Furthermore, the power parameter has a tendency to be close to zero empirically, which suggests that much of a historical data may not be used in decision making ([Neelon and O'Malley, 2010](#)).

In this article, we investigate a modified power prior which was initially proposed by [Duan et al. \(2006\)](#) for a random δ . It is named as the *normalized power prior* since it includes a scale factor. The normalized power prior obeys the likelihood principle. As a result, the posteriors can quantify the compatibility between the current and historical data automatically, and hence control the influence of historical data on the current study in a more sensible way.

The goals of this work are threefold. First, we review the joint power prior and the normalized power prior that have been proposed in literature. We aim to show that the joint power prior may not be appropriate for a random δ . Second, we carry out a comprehensive study on properties of the normalized power prior both theoretically and numerically, shed light on the posterior behavior in response to the data compatibility. Finally, we design efficient computational algorithms and provide practical implementations along with three data examples.

2. A normalized power prior approach

2.1. The normalized power prior

Suppose that θ is the parameter (vector or scalar) of interest and $L(\theta|D_0)$ is the likelihood function of θ based on the historical data D_0 . In this article, we assume that the historical data D_0 and current data D are independent random samples. Furthermore, denote by $\pi_0(\theta)$ the *initial prior* for θ . Given the power parameter δ , [Ibrahim and Chen \(2000\)](#) defined the *power prior* of θ for the current study as

$$\pi(\theta|D_0, \delta) \propto L(\theta|D_0)^\delta \pi_0(\theta). \quad (2.1)$$

The power parameter δ , a scalar in $[0, 1]$, measures the influence of historical information on the current study.

The power prior $\pi(\theta|D_0, \delta)$ in (2.1) was initially elicited for a fixed δ . As the value of δ is not necessarily pre-determined and typically unknown in practice, the full Bayesian approach extends the case to a random δ by assigning a reasonable initial prior $\pi_0(\delta)$ on it. A natural prior for δ would be a $\text{Beta}(\alpha_\delta, \beta_\delta)$ distribution since $0 \leq \delta \leq 1$. [Ibrahim and Chen \(2000\)](#) constructed the *joint power prior* of (θ, δ) as

$$\pi(\theta, \delta|D_0) \propto L(\theta|D_0)^\delta \pi_0(\theta) \pi_0(\delta), \quad (2.2)$$

with the posterior, given the current data D , as

$$\pi(\theta, \delta|D_0, D) = \frac{L(\theta|D)L(\theta|D_0)^\delta \pi_0(\theta) \pi_0(\delta)}{\int_0^1 \pi_0(\delta) \left\{ \int_{\Theta} L(\theta|D)L(\theta|D_0)^\delta \pi_0(\theta) d\theta \right\} d\delta}, \quad (2.3)$$

where Θ denotes the parameter space of θ . The prior in (2.2) is constructed by directly assigning a prior for (θ, δ) jointly ([Ibrahim et al., 2015](#)). However, if we integrate θ out in (2.2) we have $\pi(\delta|D_0) \propto \pi_0(\delta) \int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta$, which does not equal to $\pi_0(\delta)$. This meant that the initial prior for δ is updated after one observes the historical data alone. Moreover, in the posterior (2.3), any constant before $L(\theta|D_0)$ cannot be canceled out between the numerator and the denominator. This could yield different posteriors if different forms of the likelihood functions are used. For example, the likelihood based on the raw data and the likelihood based on the distribution of sufficient statistics could result in different posteriors. Also, the prior in (2.2) could be improper. Once the historical information is available, a prior elicited from such information would better be proper. Propriety conditions for four commonly used classes of regression models can be found in [Ibrahim and Chen \(2000\)](#) and [Chen et al. \(2000\)](#).

Alternatively, one can first specify a conditional prior distribution on θ given δ , then specify a marginal distribution for δ . The normalizing constant in the first step is therefore a function of δ . Since δ is a parameter, this scale factor

$C(\delta) = \int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta$ should not be ignored. Therefore, a modified power prior formulation, called the *normalized power prior*, was proposed by Duan et al. (2006) which included this scale factor. Consequently, for (θ, δ) , the normalized power prior is

$$\pi(\theta, \delta|D_0) \propto \frac{L(\theta|D_0)^\delta \pi_0(\theta) \pi_0(\delta)}{\int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta}, \quad (2.4)$$

in the region of δ such that the denominator of (2.4) is finite.

When $\int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta < \infty$, the prior in (2.4) is always proper given that $\pi_0(\delta)$ is proper, whereas it is not necessarily the case for that of the joint power prior (2.2). More importantly, multiplying the likelihood function in (2.2) by an arbitrary positive constant, which could be a function of D_0 , may change the joint power prior, whereas the constant is canceled out in the normalized power prior in (2.4).

Using the current data to update the prior distribution $\pi(\theta, \delta|D_0)$ in (2.4), we derive the joint posterior distribution for (θ, δ) as

$$\pi(\theta, \delta|D_0, D) \propto L(\theta|D) \pi(\theta, \delta|D_0) \propto \frac{L(\theta|D) L(\theta|D_0)^\delta \pi_0(\theta) \pi_0(\delta)}{\int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta}.$$

Integrating θ out from the expression above, the marginal posterior distribution of δ can be expressed as

$$\pi(\delta|D_0, D) \propto \pi_0(\delta) \frac{\int_{\Theta} L(\theta|D) L(\theta|D_0)^\delta \pi_0(\theta) d\theta}{\int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta}. \quad (2.5)$$

If we integrate δ out in (2.4), we obtain a new prior for θ , a prior that is updated by the historical information,

$$\pi(\theta|D_0) = \int_0^1 \pi(\theta, \delta|D_0) d\delta \propto \pi_0(\theta) \int_0^1 \frac{L(\theta|D_0)^\delta \pi_0(\delta)}{\int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta} d\delta. \quad (2.6)$$

With historical data appropriately incorporated, (2.6) can be viewed as an informative prior for the Bayesian analysis to the current data. Consequently, the posterior distribution of θ can be written as

$$\pi(\theta|D_0, D) \propto \pi(\theta|D_0) L(\theta|D) \propto \pi_0(\theta) L(\theta|D) \int_0^1 \frac{L(\theta|D_0)^\delta \pi_0(\delta)}{\int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta} d\delta.$$

Below we describe some variations of the normalized power prior. A primary extension deals with the presence of multiple historical studies. Similar to Ibrahim and Chen (2000), the prior defined in (2.4) can be easily generalized. Suppose there are m historical studies, denote by D_{0j} the historical data for the j th study, $j = 1, \dots, m$ and $\mathbf{D}_0 = (D_{01}, \dots, D_{0m})$. The power parameter for each historical study can be different, and we can further assume they follow the same independent initial prior. Let $\delta = (\delta_1, \dots, \delta_m)$, the normalized power prior of the form (2.4) can be generalized to

$$\pi(\theta, \delta|\mathbf{D}_0) \propto \frac{\left\{ \prod_{j=1}^m L(\theta|D_{0j})^{\delta_j} \pi_0(\delta_j) \right\} \pi_0(\theta)}{\int_{\Theta} \left\{ \prod_{j=1}^m L(\theta|D_{0j})^{\delta_j} \right\} \pi_0(\theta) d\theta}.$$

This framework would accommodate the potential heterogeneity among historical datasets from different sources or collected at different time points. Data collected over a long period may be divided into several historical datasets to ensure the homogeneity within each data. Examples of implementing the power prior approach using multiple historical studies can be found in Duan et al. (2006), Gamalo et al. (2014), Gravestock and Held (2019) and Banbata et al. (2019).

An important extension is based on the *partial borrowing power prior* (Ibrahim et al., 2012b; Chen et al., 2014b), in which the historical data can be borrowed only through some common parameters with fixed δ . For instance, when evaluating cardiovascular risk in new therapies, priors for only a subset of the parameters are constructed based on the historical data (Chen et al., 2014a). Below we describe the *partial borrowing normalized power prior*, which is an extension of the partial borrowing power prior. Let $\theta = (\theta_c, \theta_1)$ be the parameter of interest in the current study, and let (θ_c, θ_0) be the parameter in a historical study, where θ_c is a subset of the common parameters. Now

$$\pi(\theta, \delta|D_0) \propto \frac{\left\{ \int_{\Theta_0} L(\theta_c, \theta_0|D_0)^\delta \pi_0(\theta_c, \theta_0) d\theta_0 \right\} \pi_0(\theta_1) \pi_0(\delta)}{\int_{\Theta_c} \left\{ \int_{\Theta_0} L(\theta_c, \theta_0|D_0)^\delta \pi_0(\theta_c, \theta_0) d\theta_0 \right\} d\theta_c} \quad (2.7)$$

defines the partial borrowing normalized power prior, where Θ_0 and Θ_c denote the parameter spaces of θ_0 and θ_c , respectively. In this case, the dimensions of Θ_0 and Θ_c can be different, which is another advantage of using the prior in (2.7).

In addition, for model with latent variables ξ , one can also extend the fixed borrowing to a random δ under the normalized power prior framework. Denote $g(\xi)$ the distribution of ξ and assume θ is the parameter of interest, we have

two strategies to construct a power prior for θ when δ is fixed. One way is to discount directly on the likelihood of D_0 expressed as $\int_{\Xi} L(\theta|D_0, \xi)g(\xi)d\xi$, where Ξ denotes the domain of ξ . The normalized power prior is of the form

$$\pi(\theta, \delta|D_0) \propto \frac{\left\{ \int_{\Xi} L(\theta|D_0, \xi)g(\xi)d\xi \right\}^{\delta} \pi_0(\theta)\pi_0(\delta)}{\int_{\Theta} \left\{ \int_{\Xi} L(\theta|D_0, \xi)g(\xi)d\xi \right\}^{\delta} \pi_0(\theta)d\theta}. \quad (2.8)$$

Another borrowing strategy is to discount the likelihood of D_0 conditional on ξ , while $g(\xi)$ is not discounted such that the power prior with δ fixed has the form $\pi_0(\theta) \int_{\Xi} L(\theta|D_0, \xi)^{\delta} g(\xi)d\xi$. Ibrahim et al. (2015) named such a prior *partial discounting power prior*. We propose its counterpart beyond a fixed δ , the *partial discounting normalized power prior*, which is formulated as

$$\pi(\theta, \delta|D_0) \propto \frac{\left\{ \int_{\Xi} L(\theta|D_0, \xi)^{\delta} g(\xi)d\xi \right\} \pi_0(\theta)\pi_0(\delta)}{\int_{\Theta} \left\{ \int_{\Xi} L(\theta|D_0, \xi)^{\delta} g(\xi)d\xi \right\} \pi_0(\theta)d\theta}. \quad (2.9)$$

Ibrahim et al. (2015) argued that the partial discounting power prior is preferable due to both practical reasons and computational advantages. Both of (2.8) and (2.9) can be extended to models with random effects, in which the distribution $g(\xi)$ may depend on additional unknown variance parameters.

Finally, we note that in the complex data analysis practice, the extensions described above might be combined. For example, one can consider a partial borrowing normalized power prior with multiple historical data, where the borrowing is carried out only through some selected mutual parameters. Another example is in Chen et al. (2014a), where the partial borrowing power prior is used in the presence of latent variables. Further variations for specific problems will be explored elsewhere.

2.2. Computational considerations in the normalized power prior

For the normalized power prior, the only computational effort in addition to that of the joint power prior is to calculate the scale factor $C(\delta) = \int_{\Theta} L(\theta|D_0)^{\delta} \pi_0(\theta)d\theta$. In some models the integral can be calculated analytically up to a normalizing constant, so $\pi(\theta, \delta|D_0, D)$ can be expressed in closed forms. The posterior sample from $\pi(\theta, \delta|D_0, D)$ can be obtained by first sampling from $\pi(\delta|\theta, D_0, D)$ or $\pi(\delta|D_0, D)$, then from $\pi(\theta_i|\theta_{-i}, \delta, D_0, D)$, where θ_{-i} is θ without the i th element. It is typically achieved by using a Metropolis–Hastings algorithm (Chib and Greenberg, 1995) for δ , followed by Gibbs sampling for each θ_i .

However, $C(\delta)$ needs to be calculated numerically in some models. General Monte Carlo methods to calculate the normalizing constant in the Bayesian computation can be applied. Since the integrand includes a likelihood function powered to $\delta \in [0, 1]$, we consider the following approach, which is best tailored to the specific form of the integral. It is based on a variant of the algorithm in Friel and Pettitt (2008) and Van Rosmalen et al. (2018) using the idea of path sampling (Gelman and Meng, 1998). The key observation is that $\log C(\delta)$ can be expressed as an integral of the expected log-likelihood of historical data, where the integral is calculated with respect to a bounded one-dimensional parameter. This identity can be written as

$$\log C(\delta) = \int_0^{\delta} E_{\pi(\theta|D_0, \delta^*)} \{ \log [L(\theta|D_0)] \} d\delta^*, \quad (2.10)$$

which is an adaptive version of the results from Friel and Pettitt (2008). Proof is shown in Appendix A. For given δ^* , the expectation in (2.10) is evaluated with respect to the density $\pi(\theta|D_0, \delta^*) \propto L(\theta|D_0)^{\delta^*} \pi_0(\theta)$. Therefore the integrand can be calculated numerically if we can sample from $\pi(\theta|D_0, \delta^*)$. This is the prerequisite to implement the power prior with a fixed power parameter; hence no extra condition is required to calculate $\log C(\delta)$ using (2.10). By choosing an appropriate sequence of δ^* we can approximate the integral numerically.

When sampling from the posterior $\pi(\theta, \delta|D_0, D)$ using the normalized power prior, $C(\delta)$ needs to be calculated for every iteration. Van Rosmalen et al. (2018) suggested that the function $\log C(\delta)$ can be well approximated by linear interpolation. Since δ is bounded, it is recommended to calculate a sufficiently large number of the $\log C(\delta)$ for different δ on a fine grid before the posterior sampling, then use a piecewise linear interpolation at each iteration during the posterior sampling. In addition to the power prior with fixed δ , the only computational cost is to determine $\log C(\delta)$ for selected values of $\delta \in [0, 1]$ as *knots*. Details of a sampling algorithm are provided in Appendix B.

Sampling from the density $\pi(\theta|D_0, \delta^*)$ can be computationally intensive in some models. Therefore the knots should be carefully selected given limited computational budget. A rule of thumb based on our empirical evidence is to select more grid points close to 0, to account for the larger deviation from piecewise linearity in $\log C(\delta)$ when $\delta \rightarrow 0$. An example is to use $\{\delta_s = (s/S)^c\}_{s=0}^S$ with $c > 1$. Recently, Carvalho and Ibrahim (2020) noted that $C(\delta)$ is a strictly convex function but not necessarily monotonic. They design primary grid points by prioritizing the region where the derivative $C'(\delta)$ is close to 0, then use a generalized additive model to interpolate values on a larger grid. In practice, one may consider combining the two strategies above by adding some grid points used by Carvalho and Ibrahim (2020) into the original design $\{\delta_s = (s/S)^c\}_{s=0}^S$. In addition, when $C(\delta)$ is not monotone, piecewise linear interpolation with limited number of grid points also needs to be cautious, especially around the region where $C'(\delta)$ change signs.

2.3. Normalized power prior approach for exponential family

In this section we discuss how to make inference on parameter θ (scalar or vector-valued) in an exponential family, incorporating both the current data $D = (x_1, \dots, x_n)$ and the historical data $D_0 = (x_{01}, \dots, x_{0n_0})$. Suppose that the data comes from an exponential family with probability density function or probability mass function of the form (Casella and Berger, 2002)

$$f(x|\theta) = h(x) \exp \left\{ \sum_{i=1}^k w_i(\theta) t_i(x) + \tau(\theta) \right\}, \quad (2.11)$$

where the dimension of θ is no larger than k . Here $h(x) \geq 0$ and $t_1(x), \dots, t_k(x)$ are real-valued functions of the observation x , and $w_1(\theta), \dots, w_k(\theta)$ are real-valued functions of the parameter θ . Define $\underline{w}(\theta) = (w_1(\theta), \dots, w_k(\theta))'$. Furthermore, define

$$\underline{T}(x) = \left(\frac{1}{n} \sum_{j=1}^n t_1(x_j), \dots, \frac{1}{n} \sum_{j=1}^n t_k(x_j) \right)' \quad (2.12)$$

as the *compatibility statistic* to measure how compatible a sample $\underline{x} = (x_1, \dots, x_n)$ is with other samples in providing information about θ . The density function of the current data can be expressed as

$$f(D|\theta) = h(D) \exp \{ n[\underline{T}(D)'\underline{w}(\theta) + \tau(\theta)] \}, \quad (2.13)$$

where $h(D) = \prod_{j=1}^n h(x_j)$ and $\underline{T}(D)$ stands for the compatibility statistic related to the current data D . Accordingly, the compatibility statistic and the density function similar to (2.12) and (2.13) for the historical data D_0 can be defined as well. The joint posterior of (θ, δ) can be written as

$$\pi(\theta, \delta|D_0, D) \propto \frac{\exp \{ [\delta n_0 \underline{T}(D_0)' + n \underline{T}(D)'] \underline{w}(\theta) + (\delta n_0 + n) \tau(\theta) \} \pi_0(\theta) \pi_0(\delta)}{\int_{\Theta} \exp \{ \delta n_0 [\underline{T}(D_0)' \underline{w}(\theta) + \tau(\theta)] \} \pi_0(\theta) d\theta}. \quad (2.14)$$

Integrating θ out from (2.14), the marginal posterior distribution of δ is given by

$$\pi(\delta|D_0, D) \propto \pi_0(\delta) \frac{\int_{\Theta} \exp \{ [\delta n_0 \underline{T}(D_0)' + n \underline{T}(D)'] \underline{w}(\theta) + (\delta n_0 + n) \tau(\theta) \} \pi_0(\theta) d\theta}{\int_{\Theta} \exp \{ \delta n_0 [\underline{T}(D_0)' \underline{w}(\theta) + \tau(\theta)] \} \pi_0(\theta) d\theta}.$$

The behavior of the power parameter δ can be examined from this marginal posterior distribution. Similarly, the marginal posterior distribution of θ can be derived by integrating δ out in $\pi(\theta, \delta|D_0, D)$, but it often does not have a closed form. Instead the posterior distribution of θ given D_0, D and δ is often in a more familiar form. Therefore we may learn the characteristic of the marginal posterior of θ by studying the conditional posterior distribution $\pi(\theta|D_0, D, \delta)$, together with $\pi(\delta|D_0, D)$.

In the following subsections we provide three examples of the commonly used distributions, where the posterior marginal density (up to a normalizing constant) of δ can be expressed in closed forms. It can be extended to many other distributions as well by choosing appropriate initial priors $\pi_0(\theta)$.

2.3.1. Bernoulli population

Suppose we are interested in making inference on the probability of success p from a Bernoulli population with multiple replicates. Assume the total number of successes in the historical and the current data are $y_0 = \sum_{i=1}^{n_0} x_{0i}$ and $y = \sum_{i=1}^n x_i$ respectively, with the corresponding total number of trials n_0 and n . The joint posterior distribution of p and δ can be easily derived as the result below and the proof is omitted.

Result 1. Assume that the initial prior distribution of p follows a Beta(α, β) distribution, the joint posterior distribution of (p, δ) can be expressed as

$$\pi(p, \delta|D_0, D) \propto \pi_0(\delta) \frac{p^{\delta y_0 + y + \alpha - 1} (1-p)^{\delta(n_0 - y_0) + n - y + \beta - 1}}{B(\delta y_0 + \alpha, \delta(n_0 - y_0) + \beta)},$$

where $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ stands for the beta function.

Integrating p out in $\pi(p, \delta|D_0, D)$, the marginal posterior distribution of δ can be expressed as

$$\pi(\delta|D_0, D) \propto \pi_0(\delta) \frac{B(\delta y_0 + y + \alpha, \delta(n_0 - y_0) + n - y + \beta)}{B(\delta y_0 + \alpha, \delta(n_0 - y_0) + \beta)}.$$

The conditional posterior distribution of p given δ follows a Beta($\delta y_0 + y + \alpha, \delta(n_0 - y_0) + n - y + \beta$) distribution. However, the marginal posterior distribution of p does not have a closed form.

2.3.2. Multinomial population

As a generalization of the Bernoulli/binomial to $k \geq 3$ categories, in a multinomial population assume we observe historical data $D_0 = (y_{01}, y_{02}, \dots, y_{0k})$ and the current data $D = (y_1, y_2, \dots, y_k)$, where each element represents the number of success in that category. Let $n_0 = \sum_{i=1}^k y_{0i}$ and $n = \sum_{i=1}^k y_i$. Suppose the parameter of interest is $\theta = (\theta_1, \theta_2, \dots, \theta_k)$ which adds up to 1. We have the following results below.

Result 2. Assume the initial prior of θ follows a Dirichlet distribution with $\pi_0(\theta) \sim \text{Dir}(\alpha_1, \alpha_2, \dots, \alpha_k)$, the joint posterior of (θ, δ) can be expressed as

$$\pi(\theta, \delta | D_0, D) \propto \pi_0(\delta) \prod_{i=1}^k \theta_i^{y_{0i}\delta + y_i + \alpha_i - 1} \frac{\Gamma(n_0\delta + \sum_{i=1}^k \alpha_i)}{\prod_{i=1}^k \Gamma(y_{0i}\delta + \alpha_i)},$$

where $\Gamma(\cdot)$ stands for the gamma function.

The marginal posterior of δ can be derived by integrating θ out as

$$\pi(\delta | D_0, D) \propto \pi_0(\delta) \frac{\Gamma(n_0\delta + \sum_{i=1}^k \alpha_i) \prod_{i=1}^k \Gamma(y_{0i}\delta + y_i + \alpha_i)}{\Gamma(n + n_0\delta + \sum_{i=1}^k \alpha_i) \prod_{i=1}^k \Gamma(y_{0i}\delta + \alpha_i)}.$$

Similar to the Bernoulli case, the marginal posterior distribution of θ does not have a closed form. The conditional posterior distribution of θ given δ follows a Dirichlet distribution with $\text{Dir}(\delta y_{01} + y_1 + \alpha_1, \dots, \delta y_{0k} + y_k + \alpha_k)$.

2.3.3. Normal linear model and normal population

Suppose we are interested in making inference on the regression parameters β from a linear model with current data

$$\mathbf{Y} = \mathbf{X}\beta + \epsilon, \text{ with } \epsilon \sim \text{MVN}(\mathbf{0}, \sigma^2 I_n), \quad (2.15)$$

where the dimension of vector \mathbf{Y} is n and that of β is k . Similarly, we assume the historical data has the form $\mathbf{Y}_0 = \mathbf{X}_0\beta + \epsilon_0$, with $\epsilon_0 \sim \text{MVN}(\mathbf{0}, \sigma^2 I_{n_0})$. Assume that both $\mathbf{X}_0'\mathbf{X}_0$ and $\mathbf{X}'\mathbf{X}$ are positive definite. Define

$$\begin{aligned} \hat{\beta}_0 &= (\mathbf{X}_0'\mathbf{X}_0)^{-1}\mathbf{X}_0'\mathbf{Y}_0, \quad S_0 = (\mathbf{Y}_0 - \mathbf{X}_0\hat{\beta}_0)'(\mathbf{Y}_0 - \mathbf{X}_0\hat{\beta}_0), \\ \hat{\beta} &= (\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y}, \quad \text{and } S = (\mathbf{Y} - \mathbf{X}\hat{\beta})'(\mathbf{Y} - \mathbf{X}\hat{\beta}). \end{aligned}$$

Now, let us consider a conjugate initial prior for (β, σ^2) as the following. $\pi_0(\sigma^2) \propto \sigma^{-2a}$, with $a > 0$, and $\beta|\sigma^2$ either has a $\text{MVN}(\mu_0, \sigma^2 \mathbf{R}^{-1})$ distribution, which includes the Zellner's g -prior (Zellner, 1986) or $\pi_0(\beta|\sigma^2) \propto 1$, which is a noninformative prior. Here we assume \mathbf{R} as a known positive definite matrix. Hence, the initial prior can be written as

$$\pi_0(\beta, \sigma^2) \propto \frac{1}{(\sigma^2)^{a+\frac{kb}{2}}} \exp \left\{ -\frac{b}{2\sigma^2} (\beta - \mu_0)' \mathbf{R} (\beta - \mu_0) \right\}, \text{ with } b = 0 \text{ or } 1. \quad (2.16)$$

We have the following theorem whose proof is given in Appendix A.

Theorem 2.1. With the set up above for the normal linear model (2.15) and the initial prior of (β, σ^2) as in (2.16), suppose the initial prior of δ is $\pi_0(\delta)$. Then, the following results can be shown.

(a) The normalized power prior distribution of $(\beta, \sigma^2, \delta)$ is

$$\pi(\beta, \sigma^2, \delta | D_0) \propto \frac{\pi_0(\delta) M_0(\delta)}{(\sigma^2)^{\frac{\delta n_0 + kb}{2} + a}} \exp \left\{ -\frac{1}{2\sigma^2} [\delta \{S_0 + bH_0(\delta)\} + Q(\delta, \beta)] \right\},$$

where

$$\begin{aligned} Q(\delta, \beta) &= (\beta - \beta^*)' (b\mathbf{R} + \delta \mathbf{X}_0'\mathbf{X}_0) (\beta - \beta^*), \\ \beta^* &= (b\mathbf{R} + \delta \mathbf{X}_0'\mathbf{X}_0)^{-1} (b\mathbf{R}\mu_0 + \delta \mathbf{X}_0'\mathbf{X}_0\hat{\beta}_0), \\ H_0(\delta) &= (\mu_0 - \hat{\beta}_0)' \mathbf{X}_0'\mathbf{X}_0 (b\mathbf{R} + \delta \mathbf{X}_0'\mathbf{X}_0)^{-1} \mathbf{R} (\mu_0 - \hat{\beta}_0), \text{ and} \\ M_0(\delta) &= \frac{|\mathbf{R} + \delta \mathbf{X}_0'\mathbf{X}_0|^{\frac{1}{2}}}{\Gamma\left(\frac{\delta n_0 + (b-1)k}{2} + a - 1\right)} \left\{ \delta \frac{S_0 + bH_0(\delta)}{2} \right\}^{\frac{\delta n_0 + (b-1)k}{2} + a - 1}. \end{aligned}$$

(b) The marginal posterior density of δ , given (D_0, D) , can be expressed as

$$\pi(\delta | D_0, D) \propto \frac{\pi_0(\delta) |\mathbf{R} + \delta \mathbf{X}_0'\mathbf{X}_0|^{\frac{1}{2}} \Gamma\left(\frac{n + \delta n_0 + (b-1)k}{2} + a - 1\right)}{|\mathbf{R} + \delta \mathbf{X}_0'\mathbf{X}_0 + \mathbf{X}'\mathbf{X}|^{\frac{1}{2}} \Gamma\left(\frac{\delta n_0 + (b-1)k}{2} + a - 1\right) M(\delta)},$$

where

$$M(\delta) = [\delta \{S_0 + bH_0(\delta)\} + S + H(\delta)]^{\frac{n}{2}} \left[1 + \frac{S + H(\delta)}{\delta \{S_0 + bH_0(\delta)\}} \right]^{\frac{\delta n_0 + (b-1)k}{2} + a - 1},$$

$$\text{and } H(\delta) = (\beta^* - \hat{\beta})' \mathbf{X}' \mathbf{X} (b\mathbf{R} + \delta \mathbf{X}_0' \mathbf{X}_0 + \mathbf{X}' \mathbf{X})^{-1} (b\mathbf{R} + \delta \mathbf{X}_0' \mathbf{X}_0) (\beta^* - \hat{\beta}).$$

(c) The conditional posterior distribution of β , given (δ, D_0, D) , is a multivariate Student t -distribution with location parameters μ , shape matrix Σ , and the degrees of freedom ν as

$$\begin{aligned} \mu &= (b\mathbf{R} + \delta \mathbf{X}_0' \mathbf{X}_0 + \mathbf{X}' \mathbf{X})^{-1} \{ (b\mathbf{R} + \delta \mathbf{X}_0' \mathbf{X}_0) \beta^* + \mathbf{X}' \mathbf{X} \hat{\beta} \}, \\ \Sigma &= \frac{S + H(\delta) + \delta \{S_0 + bH_0(\delta)\}}{\nu} (b\mathbf{R} + \delta \mathbf{X}_0' \mathbf{X}_0 + \mathbf{X}' \mathbf{X})^{-1}, \text{ and} \\ \nu &= (b-1)k + \delta n_0 + n + 2a - 2. \end{aligned}$$

(d) The conditional posterior distribution of σ^2 , given (δ, D_0, D) , follows an inverse-gamma distribution with shape parameter $\frac{(b-1)k + \delta n_0 + n}{2} + a - 1$, and scale parameter $\frac{1}{2} [S + H(\delta) + \delta \{S_0 + bH_0(\delta)\}]$.

Theorem 2.1 provides a general case for the normal linear model with certain conjugate prior structure. We can easily obtain the results for a regular normal population with such conjugate structure. One of the results for a normal population $N(\mu, \sigma^2)$ with $\pi_0(\mu, \sigma^2) \propto \sigma^{-2a}$ and $\pi_0(\delta) \sim \text{Beta}(\alpha_\delta, \beta_\delta)$ can be found in [Duan et al. \(2006\)](#).

3. Optimality properties of the normalized power prior

In investigating the optimality properties of the normalized power priors, we use the idea of minimizing the weighted Kullback–Leibler (KL) divergence ([Kullback and Leibler, 1951](#)) that is similar to, but not the same as in [Ibrahim et al. \(2003\)](#).

Recall the definition of the KL divergence,

$$K(g, f) = \int_{\Theta} \log \left(\frac{g(\theta)}{f(\theta)} \right) g(\theta) d\theta,$$

where g and f are two densities with respect to Lebesgue measure. In [Ibrahim et al. \(2003\)](#), a loss function related to a target density g , denoted by K_g , is defined as the convex sum of the KL divergence between g and two posterior densities. One is the posterior density without using any historical data, denoted by $f_0 \propto L(\theta|D)\pi_0(\theta)$, and the other is the posterior density with the historical and current data equally weighted, denoted by $f_1 \propto L(\theta|D_0)L(\theta|D)\pi_0(\theta)$. The loss is defined as

$$K_g = (1 - \delta)K(g, f_0) + \delta K(g, f_1),$$

where the weight for f_1 is δ . It is showed that, when δ is given, the unique minimizer of K_g is the posterior distribution derived using the power prior, i.e.,

$$\pi(\theta|D_0, D, \delta) \propto L(\theta|D_0)^\delta L(\theta|D)\pi_0(\theta).$$

Furthermore, [Ibrahim et al. \(2003\)](#) claim that the posterior derived from the joint power prior also minimizes $E_{\pi_0(\delta)}(K_g)$ when δ is random.

We look into the problem from a different angle. Since the prior for θ without the historical data is $\pi_0(\theta)$ with $\int_{\Theta} \pi_0(\theta) d\theta = 1$, we further denote the prior for θ when fully utilizing the historical data as $\pi_1(\theta) \propto \pi_0(\theta)L(\theta|D_0)$, with $\int_{\Theta} \pi_1(\theta) d\theta = 1$. Clearly

$$\pi_1(\theta) = Q(D_0)\pi_0(\theta)L(\theta|D_0), \quad (3.1)$$

where $Q^{-1}(D_0) = \int_{\Theta} \pi_0(\theta)L(\theta|D_0)d\theta$ is a normalizing constant.

Suppose we have a prior $\pi_0(\delta)$. For any function $g(\theta|\delta)$, define the expected weighted KL divergence between g and π_0 , and between g and π_1 as

$$L_g = E_{\pi_0(\delta)} \{ (1 - \delta)K(g, \pi_0) + \delta K(g, \pi_1) \}, \quad (3.2)$$

where $0 \leq \delta \leq 1$. We have the following theorem whose proof is given in [Appendix A](#).

Theorem 3.1. Suppose $\pi(\delta|D_0) = \pi_0(\delta)$. The function $g(\theta|\delta, D_0)$ that minimizes the expected weighted KL divergence defined in (3.2) is

$$\pi^*(\theta|\delta, D_0) = \frac{L(\theta|D_0)^\delta \pi_0(\theta)}{\int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta},$$

from which we deduce the normalized power prior $\pi(\theta, \delta|D_0)$ in (2.4).

Note that the last claim in [Theorem 3.1](#) comes from

$$\pi(\theta, \delta|D_0) = \pi(\theta|\delta, D_0)\pi(\delta|D_0) = \pi(\theta|\delta, D_0)\pi_0(\delta).$$

The assumption of $\pi(\delta|D_0) = \pi_0(\delta)$ indicates that the original prior of δ does not depend on D_0 , which is reasonable.

4. Posterior behavior of the normalized power prior

In this section we investigate the posteriors of both θ and δ under different settings of the observed statistics. We show that by using the normalized power prior, the resulting posteriors can respond to the compatibility between D_0 and D in an expected way. However, the posteriors are sensitive to different forms of the likelihoods under same data and model using the joint power priors.

4.1. Results on the marginal posterior mode of the power parameter

Some theoretical results regarding the relationship between the posterior mode of δ and the *compatibility statistic* defined in (2.12) are given as follows. Their proofs are given in [Appendix A](#).

Theorem 4.1. Suppose that historical data D_0 and current data D are two independent random samples from an exponential family given in (2.11). The compatibility statistic for D_0 and D are $\underline{T}(D_0)$ and $\underline{T}(D)$ respectively as defined in (2.12). Then the marginal posterior mode of δ is always 1 under the normalized power prior approach, if

$$\frac{d}{d\delta} \log \pi_0(\delta) + h_1(D_0, D, \delta) + n_0[\underline{T}(D_0) - \underline{T}(D)]'h_2(D_0, D, \delta) \geq 0, \quad (4.1)$$

for all $0 \leq \delta \leq 1$, where

$$h_1(D_0, D, \delta) = \frac{n_0}{n} \int_{\Theta} \log L(\theta|D)[\pi(\theta|D_0, D, \delta) - \pi(\theta|D_0, \delta)]d\theta,$$

and

$$h_2(D_0, D, \delta) = \int_{\Theta} \underline{w}(\theta)[\pi(\theta|D_0, D, \delta) - \pi(\theta|D_0, \delta)]d\theta.$$

The first term in (4.1) is always non-negative if the prior of δ is a nondecreasing function. Hence, if one uses uniform prior on δ , this term is zero. The second term, $h_1(D_0, D, \delta)$, is always non-negative by using the property of KL divergence. It is 0 if and only if $\pi(\theta|D_0, D, \delta) = \pi(\theta|D_0, \delta)$, which means given δ and D_0 , current data D does not contribute to any information for θ . This could be a rare case. The third term in (4.1) depends on how close $\underline{T}(D_0)$ and $\underline{T}(D)$ are to each other. When $\underline{T}(D_0) = \underline{T}(D)$, the third term is zero, and hence the posterior mode of δ is 1. Since $h_1(D_0, D, \delta)$ is non-negative, the posterior mode of δ may also achieve 1 as long as the difference between $\underline{T}(D_0)$ and $\underline{T}(D)$ is negligible from a practical point of view. On the other hand, for the joint power prior approach, we have the following result.

Theorem 4.2. Suppose that current data D comes from a population with density function $f(x|\theta)$, and D_0 is a related historical data. Furthermore, suppose that the initial prior $\pi_0(\delta)$ is a non-increasing function and the conditional posterior distribution of θ given δ is proper for any δ . Then for any D_0 and D , if

$$\max_{0 \leq \delta \leq 1} \frac{\int \pi_0(\theta)f(D|\theta)f(D_0|\theta)^\delta \log f(D_0|\theta)d\theta}{\int \pi_0(\theta)f(D|\theta)f(D_0|\theta)^\delta d\theta} < \infty, \quad (4.2)$$

then there exists at least one positive constant k_0 such that $\pi(\delta|D_0, D)$ has mode at $\delta = 0$ under the joint power prior, where $L(\theta|x) = k_0 f(x|\theta)$.

The assumption in (4.2) is valid in the case that all the integrals are finite positive values when δ is either 0 or 1. Usually this condition satisfies when $\pi_0(\theta)$ is smooth. The proof of this result is also given in the [Appendix A](#). For a normal or a Bernoulli population, our research reveals that $\pi(\delta|D_0, D)$ has mode at $\delta = 0$ in many scenarios regardless of the level of compatibility between D and D_0 . Note that the results in [Theorem 4.2](#) are not limited to exponential family distributions.

A primary objective of considering δ as random is to let the posterior inform the compatibility between the historical and the current data, given a vague initial prior on δ . This allows adaptive borrowing according to the prior-data conflict. [Theorem 4.1](#) indicates that, when the uniform initial prior of δ is used, the posterior of δ could potentially suggest borrowing more information from D_0 as long as D is compatible with D_0 . In practice, this has the potential to reduce the sample size required in D in the design stage, and to provide estimates with high precision in the analysis stage. [Theorem 4.2](#) shows that, on the other hand, if one considers the joint power prior with an arbitrary likelihood form and a smooth initial prior $\pi_0(\theta)$, it is possible that the posterior of δ could not inform the data compatibility. This suggests the opposite, meaning that adaptive borrowing might not be true when using the joint power prior; see [Section 4.2](#) for more details.

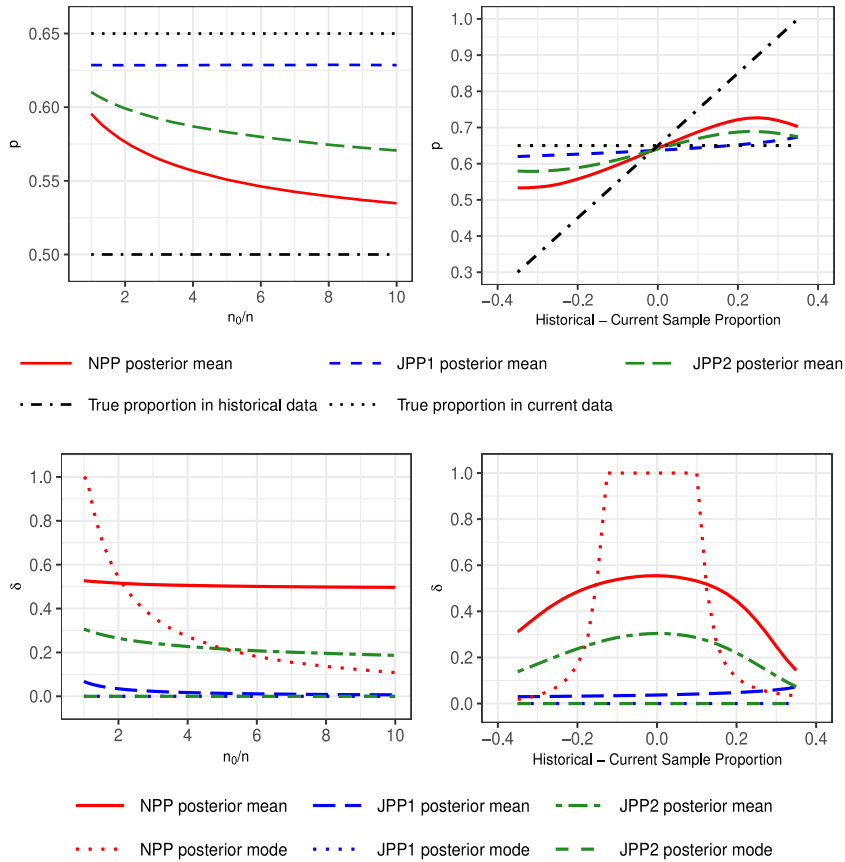


Fig. 1. Posterior behavior of p (top) and δ (bottom) for Bernoulli population when $n = 20$, $\hat{p} = 0.65$. Left: $\hat{p}_0 = 0.5$ fixed and n_0 varies. Right: $n_0 = 40$ fixed and \hat{p}_0 varies.

4.2. Posteriors of model parameters

We investigate the posteriors of all model parameters in Bernoulli and normal populations, to illustrate that different forms of the likelihoods could result in different posteriors, which affects the borrowing strength.

For independent Bernoulli trials, two different forms of the likelihood functions are commonly used. One is based on the product of independent Bernoulli densities such that $L_{J1}(p|D_0) = p^{y_0}(1-p)^{n_0-y_0}$, and another is based on the sufficient statistic, the summation of the binary outcomes, which follows a binomial distribution $L_{J2}(p|D_0) = c_1 p^{y_0}(1-p)^{n_0-y_0}$, where $c_1 = \binom{n_0}{y_0}$. Assuming $\pi_0(p) \sim \text{Beta}(\alpha, \beta)$, the corresponding posteriors are

$$\pi_{J1}(p, \delta|D_0, D) \propto \pi_0(\delta) p^{\delta y_0 + y + \alpha - 1} (1-p)^{\delta(n_0 - y_0) + n - y + \beta - 1}$$

and

$$\pi_{J2}(p, \delta|D_0, D) \propto c_1^\delta \pi_{J1}(p, \delta|D_0, D),$$

respectively. After marginalization we have

$$\pi_{J1}(\delta|D_0, D) \propto \pi_0(\delta) B(\delta y_0 + y + \alpha, \delta(n_0 - y_0) + n - y + \beta),$$

$$\pi_{J2}(\delta|D_0, D) \propto c_1^\delta \pi_{J1}(\delta|D_0, D).$$

We denote these two scenarios as JPP1 and JPP2 in Fig. 1.

For the normal population, we also consider two different forms of the likelihood functions. One uses the product of n_0 independent normal densities

$$L_{J1}(\mu, \sigma^2|D_0) = (2\pi\sigma^2)^{-\frac{n_0}{2}} \exp \left\{ -\frac{\sum_{i=1}^{n_0} (x_{0i} - \mu)^2}{2\sigma^2} \right\},$$

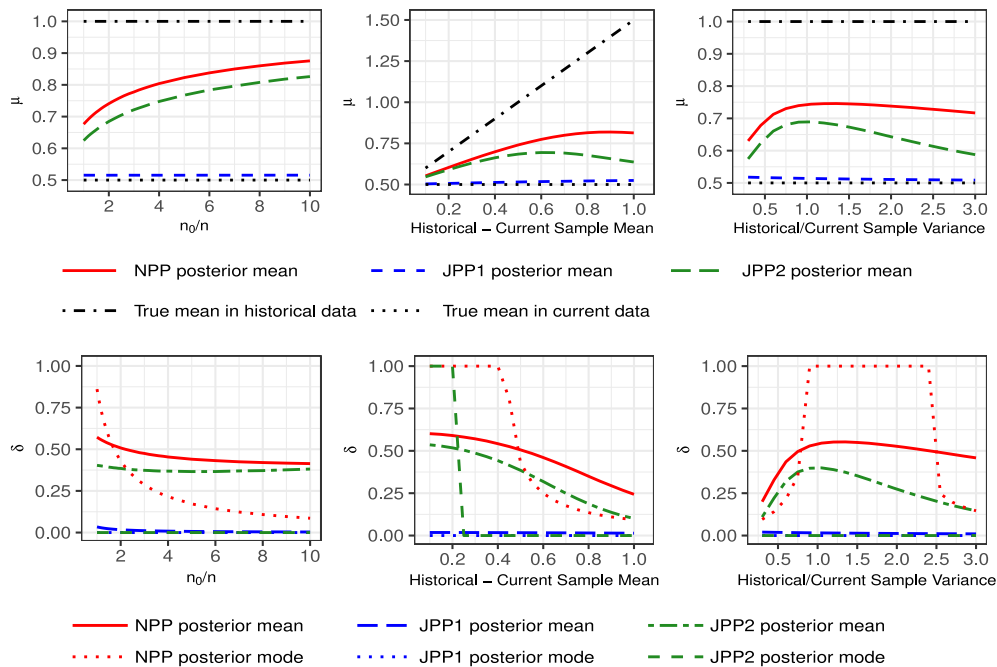


Fig. 2. Posterior behavior of μ (top) and δ (bottom) for normal population when $n = 20$, $\bar{x} = 0.5$, $\hat{\sigma}^2 = 1$. Left: $\bar{x}_0 = 1$ and $\hat{\sigma}_0^2 = 0.8$ fixed, n_0 varies. Middle: $n_0 = 40$ and $\hat{\sigma}_0^2 = 0.8$ fixed, \bar{x}_0 varies. Right: $n_0 = 40$ and $\bar{x}_0 = 1$ fixed, $\hat{\sigma}_0^2$ varies.

where x_{0i} is the value of the i th observation in D_0 . Another less frequently used form is the density of sufficient statistics $f(\bar{x}_0, s_0^2 | \mu, \sigma^2)$, where \bar{x}_0 and s_0^2 are the sample mean and variance of D_0 , respectively. Since $\bar{x}_0 \sim N(\mu, \frac{\sigma^2}{n_0})$ and $\frac{(n_0-1)s_0^2}{\sigma^2} \sim \chi_{n_0-1}^2$, so $s_0^2 \sim \text{Gamma}(\frac{n_0-1}{2}, \frac{2\sigma^2}{n_0-1})$ under the shape-scale parameterization. Then

$$L_{j2}(\mu, \sigma^2 | D_0) = c_2(\sigma^2)^{-\frac{n_0}{2}} \exp \left\{ -\frac{n_0(\bar{x}_0 - \mu)^2 + (n_0 - 1)s_0^2}{2\sigma^2} \right\},$$

where $\log c_2 = (n_0 - 3) \log s_0 + \frac{n_0-1}{2} \log \left(\frac{n_0-1}{2} \right) + \frac{1}{2} \log n_0 - \frac{1}{2} \log(2\pi) - \log \Gamma(\frac{n_0-1}{2})$.

Similar to the Bernoulli case, we can easily derive their joint power priors and the corresponding posteriors denoted as JPP1 and JPP2. As a result, their log posteriors are differed by $-\frac{n_0\delta}{2} \log(2\pi) - \delta \log(c_2)$. In the numerical experiment we use a Beta(1, 1) as the initial prior for δ , and the reference prior $\pi_0(\mu, \sigma^2) \propto 1/\sigma^2$ (Berger and Bernardo, 1992) as the initial prior for (μ, σ^2) .

Fig. 1 shows how the posteriors of p and δ change with n_0/n and $\hat{p}_0 - \hat{p}$ in data simulated from the Bernoulli population, in which a Beta(1, 1) is used as the initial prior for both p and δ . Fig. 2 shows how the posterior of μ and δ changes with n_0/n , $\hat{\mu}_0 - \hat{\mu}$ (for fixed $\hat{\sigma}_0^2$ and $\hat{\sigma}^2$), and $\hat{\sigma}_0^2/\hat{\sigma}^2$ (for fixed $\hat{\mu}_0$ and $\hat{\mu}$) in the normal population.

From both Figs. 1 and 2, we observe, under the normalized power prior, the posterior mean of the parameter of interest (p in the Bernoulli population and μ in the normal population) are sensitive to the change of compatibility between D and D_0 . As the difference between the observed sample average of D_0 and D increases, the posterior mean of both p and μ are getting closer to the parameter estimate based on D_0 at the beginning, then going back to the parameter estimate based on D . For increasing n_0/n , the posterior mean are getting closer to the parameter estimate based on D_0 . Both of the posterior mean and mode of δ respond to the compatibility between D_0 and D as expected. In addition, when the two samples are not perfectly homogeneous, the posterior mode of δ can still attain 1. This is reasonable because the historical population is subjectively believed to have similarity with the current population with a modest amount of heterogeneous. These findings imply that the power parameter δ responds to data in a sensible way in the normalized power prior approach.

When using the joint power prior approach, we observe that the posteriors of the parameters p , μ and δ behave differently with different forms of the likelihoods. Despite a violation of the likelihood principle, the joint power prior might provide moderate adaptive borrowing under certain form of the likelihood. The degree of the adaptive borrowing is less than using the normalized power prior. Under another likelihood form in our illustration, the posteriors suggest almost no borrowing, regardless of how compatible these two samples are.

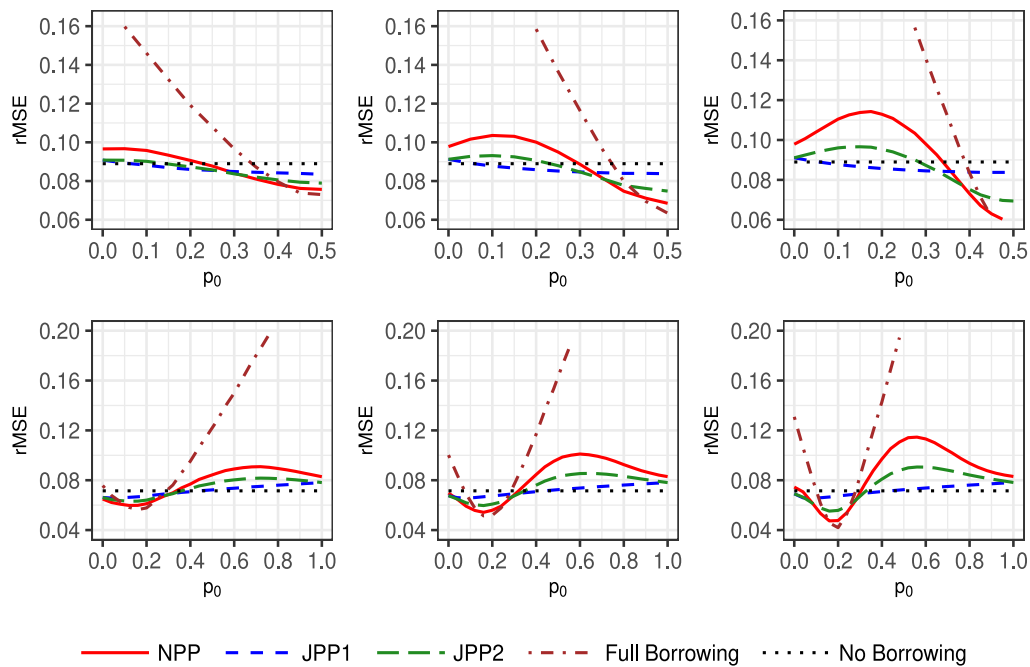


Fig. 3. Square root of the MSE of \hat{p} when $n = 30$. Top: $p = 0.5$; Bottom: $p = 0.2$. Left: $n_0 = 15$; Middle: $n_0 = 30$; Right: $n_0 = 60$.

5. Behavior of the square root of mean square error under the normalized power prior

We now investigate the influence of borrowing historical data in parameter estimation using the square root of the mean square error (rMSE) as the criteria. Several different approaches are compared, including the full borrowing (pooling), no borrowing, normalized power prior, and joint power prior. Two different likelihood forms are used for D_0 in the joint power priors, with the same notation as in Section 4. The rMSE obtained by the Monte Carlo method, defined as $\sqrt{\frac{1}{m} \sum_{i=1}^m (\hat{\theta}^{(i)} - \theta)^2}$, is used for comparison, where m is the number of Monte Carlo samples, θ is the true parameter and $\hat{\theta}^{(i)}$ is the estimate in the i th sample. We choose $m = 5000$ in all experiments.

5.1. Bernoulli population

We first compute the rMSE of estimated p in independent Bernoulli trials, where p is the probability of success in the current population. Suppose the current data comes from a binomial(n, p) distribution and the historical data comes from a binomial(n_0, p_0) distribution, with both p and p_0 unknown. The posterior mean of p is used as the estimate. In the simulation experiment we choose $n = 30$, $p = 0.2$ or 0.5 , and $n_0 = 15, 30$ or 60 . We use the Beta(1, 1) as the initial prior for both p and δ .

Based on the results in Fig. 3, the normalized power prior approach yields the rMSE comparable to the full borrowing when the divergence between the current and the historical population is small or mild. As $|p - p_0|$ increases from 0, both the posterior mean and the mode of δ will decrease on average. The rMSE of the posterior mean of p will increase with $|p - p_0|$ when p_0 is near p . As the $|p_0 - p|$ further increases, the posterior mean and mode of δ will automatically drop toward 0 (Fig. 5), so the rMSE will then decrease and eventually drop to the level comparable to no borrowing. Also, when $|p - p_0|$ is small, the rMSE will decrease as n_0 increase, which implies when the divergence between the current and the historical populations is mild, incorporating more historical data would result in better estimates using the normalized power prior. However, when $|p - p_0|$ is large, the rMSE will increase with n_0 in most scenarios. All plots from Figs. 3 and 5 indicate that the normalized power prior approach provides adaptive borrowing.

For the joint power prior approaches, the prior with the likelihood expressed as the product of independent Bernoulli densities is similar to no borrowing while using the prior based on a binomial likelihood tends to provide some adaptive borrowing, with less information incorporated than using the normalized power prior. This is consistent with what we observed regarding their posteriors in Section 4.

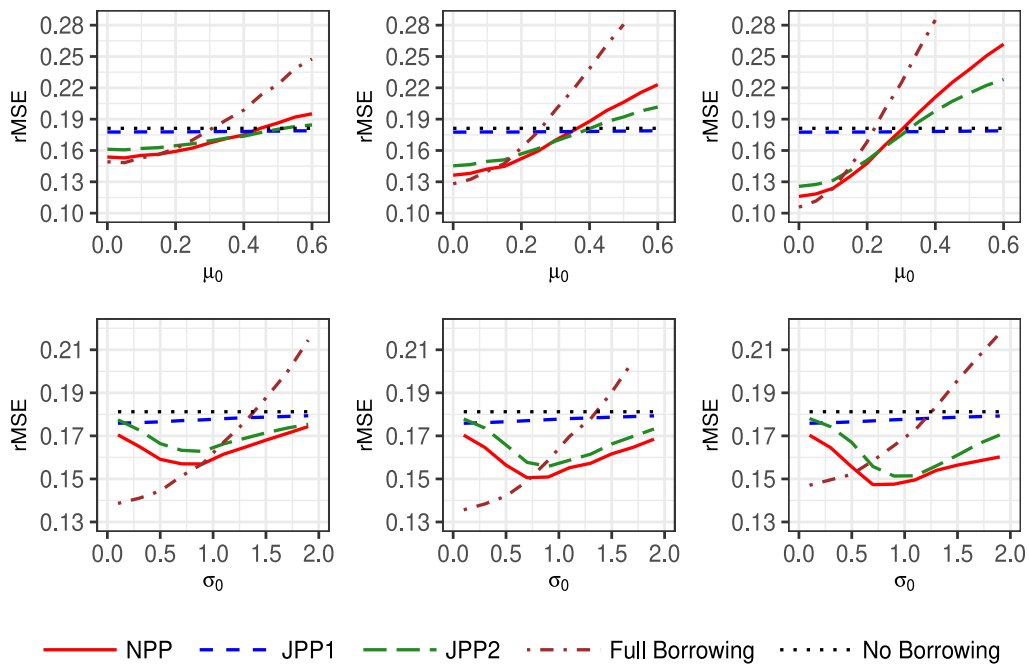


Fig. 4. Square root of the MSE of $\hat{\mu}$ when $n = 30$, $\mu = 0$, $\sigma = 1$. Top: $\sigma_0 = 1$. Bottom: $\mu_0 = 0.2$. Left: $n_0 = 15$; Middle: $n_0 = 30$; Right: $n_0 = 60$.

5.2. Normal population

We also investigate the rMSE of estimated μ in a normal population with unknown variance. Suppose that the current and historical samples are from normal $N(\mu, \sigma^2)$ and $N(\mu_0, \sigma_0^2)$ populations respectively, with both mean and variance unknown. Furthermore, the population mean μ is the parameter of interest, and the posterior mean is used as the estimate of μ .

It can be shown that the marginal posterior distribution of δ only depends on n_0 , n_0/n , σ_0/σ , and $(\mu_0 - \mu)/\sigma$, and so does the rMSE. Therefore we design two simulation settings, with $n = 30$, $\mu = 0$, $\sigma = 1$, and $n_0 = 15, 30$ or 60 under both settings. In the first experiment we fix $\sigma_0 = 1$, the heterogeneity is reflected by varying μ_0 and therefore $(\mu_0 - \mu)/\sigma$. In the second experiment, we fix $\mu_0 = 0.2$ so $(\mu_0 - \mu)/\sigma$ is fixed at 0.2 . We change σ_0 at various levels resulting in changes in σ_0/σ .

Figs. 4 and 5 display the results. The trend of the rMSE in the normalized power prior is generally consistent with the findings in a Bernoulli population. For the joint power prior approaches, the one with the likelihood based on the original data is similar to no borrowing. The one based on the product of densities using sufficient statistics tends to provide some adaptive borrowing, while less information is incorporated than using the normalized power prior. We conclude that the normalized power prior can also provide adaptive borrowing under the normal population.

6. Applications

6.1. Water-quality assessment

In this example, we use measurements of pH to evaluate impairment of four sites in Virginia individually. pH data collected over a two-year or three-year period are treated as the current data, while pH data collected over the previous nine years represents one single historical data. Of interest is the determination of whether the pH values at a site indicate that the site violates a (lower) standard of 6.0 more than 10% of the time. For each site, larger sample size is associated with the historical data and smaller with the current data. We apply the normalized power prior approach, a traditional Bayesian approach for current data only using the reference prior, and the joint power prior approaches. Assume that the measurements of water quality follow a normal distribution, and for ease of comparison, the normal model with a simple mean is considered. Since the data is used as an illustration to implement the normalized power prior, other factors, such as spatial and temporal features, are not considered. The current data and historical data are plotted side by side for each

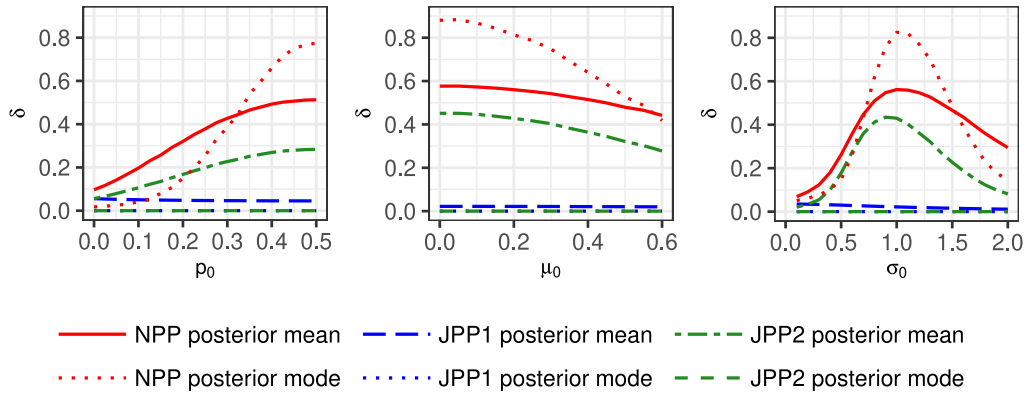


Fig. 5. Average value of the posterior quantities for δ in simulated data with $n = n_0 = 30$. Left: Bernoulli population with $p = 0.5$; Middle: Normal population with $\mu = 0$ and $\sigma = \sigma_0 = 1$; Right: Normal population with $\mu = 0$, $\mu_0 = 0.2$ and $\sigma = 1$.

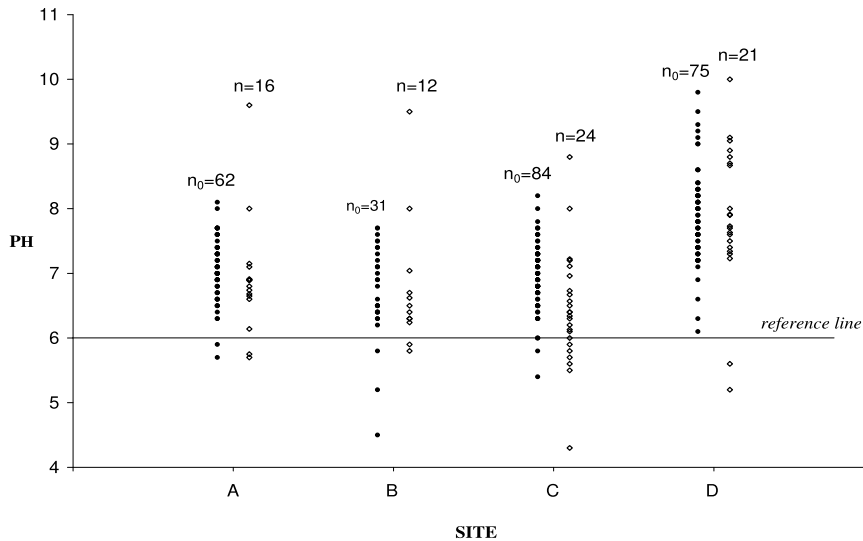


Fig. 6. pH data collected at four stations. For each site, historical data are on the left (circle) and current data on the right (diamond).

site in Fig. 6. A violation is evaluated using a Bayesian test of

$$H_0 : L \geq 6.0 \text{ (no impairment),}$$

$$H_1 : L < 6.0 \text{ (impairment),}$$

where L is the lower 10th percentile of the distribution for pH.

Table 1 summarizes the current and the historical data, and the test results using the reference prior analysis (without incorporating historical data), the normalized power prior, and the joint power prior analyses (with reference prior as the initial prior for (μ, σ^2) , i.e., $a = 1$ in Section 2.3.3). Similar to Sections 4 and 5, results from the joint power priors are calculated using different likelihood functions: (1) joint density of sufficient statistics; (2) product of n_0 independent normal densities; (3) product of n_0 independent normal densities multiplied by an arbitrary large constant $(2\pi)^{n_0/2} \exp(200)$.

The posterior probability of H_0 is calculated based on the posterior of $L = \mu + \Phi^{-1}(0.1)\sigma$, where $\Phi^{-1}(\cdot)$ is the quantile function of a standard normal distribution. If the 0.05 significance level is used, the Bayesian test using the reference prior and the current data would only indicate site C as impaired. Here we use the posterior probability of H_0 as equivalent to the p -value (Berger, 2013). Using historical data does lead to different conclusions for site B. The test using normalized power prior results in significance for both sites B & C. The test using joint power prior with likelihood (1) results in significance for site C, and the posterior probability of H_0 for site B is very close to 0.05. In the case of site B, there are around 10% of historical observations below 6.0. Hence our prior opinion of the site is suggestive of impairment. Less information is therefore required to declare impairment relative to a reference prior and the result is a smaller p -value.

Table 1

Model fitting results in evaluating site impairment with historical data available. In the table n and n_0 are sample sizes, mean (s.d.) refers to sample mean (sample standard deviation), and s.d. of L is the posterior standard deviation of L .

Site	Current data		Historical data		Posterior probability of H_0 (s.d. of L)				
	n	Mean (s.d.)	n_0	Mean (s.d.)	Reference prior	Normalized power prior	Joint power prior		
							(1)	(2)	(3)
A	16	6.91 (0.90)	62	7.05 (0.47)	0.177 (0.34)	0.488 (0.26)	0.385 (0.31)	0.201 (0.32)	0.997 (0.09)
B	12	6.78 (1.03)	31	6.73 (0.71)	0.069 (0.47)	0.047 (0.26)	0.051 (0.30)	0.070 (0.45)	0.033 (0.17)
C	24	6.43 (0.88)	84	6.95 (0.49)	0.001 (0.26)	0.004 (0.24)	0.003 (0.25)	0.002 (0.25)	0.592 (0.08)
D	21	7.87 (1.11)	75	7.88 (0.67)	0.865 (0.36)	0.986 (0.25)	0.959 (0.30)	0.886 (0.35)	1.000 (0.11)

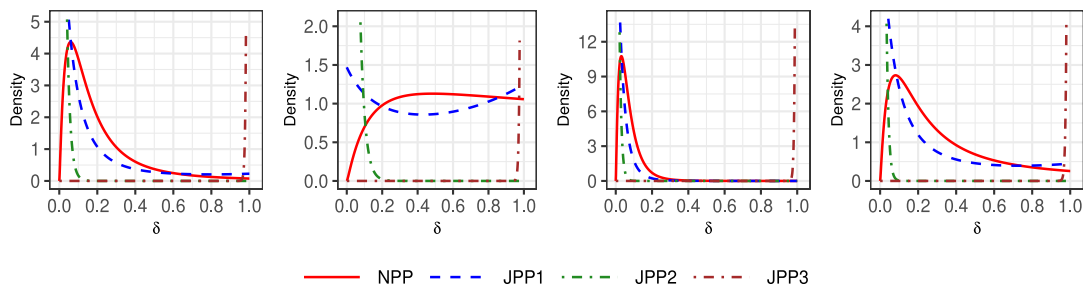


Fig. 7. Marginal posterior density plot for δ using different priors. JPP 1 to 3 refer to the joint power priors with different likelihood forms as described in the example.

However, if one uses the likelihood function in case (2) of the joint power prior method, the test result is similar to no borrowing. Furthermore, if we use an arbitrary constant as in case (3) of the joint power prior, results will be completely different. The standard deviations of L will become very small, and it is similar to a full borrowing; see Fig. 7. We will conclude site B impaired, but site C not, due to the strong influence of the historical data.

Hence, this example shows that the inference results are sensitive to the likelihood form in employing the joint power prior. On the other hand, normalized power prior provides adaptive borrowing in all scenarios. It is more reasonable to conclude that both site B and site C are impaired.

6.2. Noninferiority trial in vaccine development

In a vaccine clinical trial, it is commonly required to demonstrate that the new vaccine does not interfere with other routine recommended vaccines concomitantly. In addition to the phase 3 efficacy and safety trials, a noninferiority trial is commonly designed to demonstrate that the effect (in this example, the response rate) of a routine recommended vaccine (vaccine A) can be preserved when concomitantly used with the experimental vaccine (vaccine B). If the differences in the response rate of vaccine A when concomitantly used with vaccine B and the response rate of using vaccine A alone are within a certain prespecified margin, then we may conclude that they do not interfere each other. The prespecified positive margin d_m , known as the noninferiority margin, reflects the maximum acceptable extent of clinical noninferiority in an experimental treatment.

A simple frequentist approach of conducting such noninferiority test is to calculate the 95% confidence interval of $p_t - p_c$, where p_t and p_c are the response rates for test and control groups respectively. Given a positive noninferiority margin d_m , we conclude that the experimental treatment is not inferior to the control if the lower bound of the 95% confidence interval is greater than $-d_m$. When a Bayesian approach is applied, the 95% confidence interval can be replaced by the 95% credible interval (CI) based on the highest posterior density (Gamalo et al., 2011).

However, a problem with either the frequentist or the Bayesian approach using noninformative priors is, when the sample size is too small, the confidence interval or the credible interval will become too wide. Therefore inferiority could be inappropriately concluded. For this reason, historical evidence, especially historical data for the control group, can be incorporated. Examples of Bayesian noninferiority trials design based on power prior can be found in Lin et al. (2016) and Li et al. (2018).

We illustrate the use of normalized power prior approach to adaptively borrow data from historical controls in the development of RotaTeg, a live pentavalent rotavirus vaccine. A study was designed to investigate the concomitant use of RotaTeg and some routine pediatric vaccines between 2001–2005 (Liu, 2018). Specifically, the test was conducted

Table 2

Summary of historical and current studies.

Study		Study years	N	Responders	Response rate
Historical studies	Study 1	1992–1993	576	417	72.4%
	Study 2	1993–1995	111	90	81.1%
	Study 3	1993–1995	62	49	79.9%
	Study 4	1997–2000	487	376	77.2%
Current study	Control	2001–2005	592	426	72.0%
	Test	2001–2005	558	415	74.4%

Table 3

Summary of study results.

Prior	\hat{p}_c (%)	95% CI for $p_t - p_c$ (%)	$\bar{\delta}$	Mode of δ
Jeffrey's Prior	71.92	(−2.61, 7.58)	–	–
JPP1	71.93	(−2.89, 7.31)	0.001	0
JPP2	72.68	(−3.26, 6.59)	0.166	0
NPP	73.50	(−3.76, 5.54)	0.482	0.181

to evaluate the anti-polyribosylribitol phosphate response (a measure of vaccination against invasive disease caused by *Haemophilus influenzae* type b) to COMVAX (a combination vaccine for *Haemophilus influenzae* type b and hepatitis B), in concomitant use with RotaTeq. Since our goal is to assess whether the experimental vaccine RotaTeq will affect the response rate of the routine recommended COMVAX or not, the endpoint is the response rate of COMVAX. The per-protocol population included 558 subjects from the test group (COMVAX+RotaTeq) and 592 from the control group (COMVAX+placebo).

Since COMVAX was used for a few years, data from historical trials with similar features can be incorporated. Table 2 provides a summary of the available datasets (Liu, 2018). We pool the four historical datasets, and applying (1) non-informative Bayesian analysis with Jeffrey's prior; (2) joint power prior with the likelihood written as the product of Bernoulli densities, denoted as JPP1; (3) joint power prior with likelihood written as the binomial density, denoted as JPP2; (4) normalized power prior. Results are summarized in Table 3.

Since the normalized power prior incorporates the most information from the control group of the historical studies, its 95% CI of $p_t - p_c$ is the shortest. On the other hand, using the joint power prior with the product of Bernoulli densities as the likelihood results in almost no borrowing, while using a binomial density as the likelihood will slightly improves the borrowing. Since the average response rate in historical controls is slightly larger than that of the current control, the estimated response rate of the control group is the largest under the normalized power prior. This will result in a more conservative decision making when concluding noninferiority. Under a commonly used noninferiority margin $d_m = 5\%$, we can conclude noninferiority under all approaches, but in very rare cases, when a smaller margin is chosen, say $d_m = 3\%$, the noninferiority might be questionable when considering more historical information with a normalized power prior.

The posterior distribution of δ is skewed, therefore the posterior mean is not close to the posterior mode of δ . In the normalized power prior approach, the posterior mean of δ is 0.482, indicating that on average, approximately $1236 \times 48.2\%$ subjects are borrowed from the historical data. On the other hand, if one considers the power prior with a fixed δ for ease of interpretation, the posterior mode and posterior mean of δ can serve as the guided values, since they provide some useful information regarding the data compatibility. For example, considering a fixed $\delta = 0.95$ in practice might be anti-conservative, while a fixed $\delta = 0.05$ might be too conservative from the prior-data conflict point of view.

6.3. Diagnostic test evaluation

The U.S. Food and Drug Administration (FDA) has released a guidance¹ for the use of Bayesian methods in medical device clinical trials. This guidance specifies that the power prior could be one of the methodologies to borrow strength from other studies. In this example, the proposed normalized power prior is applied to evaluate the diagnostic test for spontaneous preterm delivery (SPD). The binary diagnostic test may result in one of the four possible outcomes: true positive (Cell 1), false positive (Cell 2), false negative (Cell 3) and true negative (Cell 4); see Table 4. Let $\theta = (\theta_1, \theta_2, \theta_3, \theta_4)$ denote the cell probabilities and let $\mathbf{n} = (n_1, n_2, n_3, n_4)$ denote the corresponding number of subjects in Table 4. The sensitivity η and specificity λ of a test can be expressed in terms of the cell probabilities θ as

$$\eta \equiv \Pr(T^+ | D^+) \equiv \frac{\theta_1}{\theta_1 + \theta_3}, \quad \text{and} \quad \lambda \equiv \Pr(T^- | D^-) \equiv \frac{\theta_4}{\theta_2 + \theta_4},$$

respectively, where D stands for disease status and T stands for test status.

¹ The complete version of the guidance can be freely downloaded at: <https://www.fda.gov/media/71512/download> [Accessed 03 June 2019].

Table 4
Possible outcomes of a binary diagnostic test.

	Disease status	
	Yes	No
Test positive	Cell 1 (TP)	Cell 2 (FP)
Test negative	Cell 3 (FN)	Cell 4 (TN)

Table 5
 2×2 performance tables with the US study and the European study.

US study	Disease status		Total	European study	Disease status		Total
	Yes	No			Yes	No	
Test positive	3	11	14	Test positive	9	20	29
Test negative	3	669	672	Test negative	9	473	482
Total	6	680	686	Total	18	493	511

Table 6
Summary of study results.

Prior	$100\hat{\eta}$	95% CI for η (%)	$100\hat{\lambda}$	95% CI for λ (%)	$\bar{\delta}$	Mode of δ
Fixed $\delta = 0$	50.04	(16.67, 82.80)	98.31	(97.32, 99.22)	–	–
Fixed $\delta = 1$	49.85	(31.40, 68.70)	97.32	(96.38, 98.17)	–	–
JPP	49.98	(18.94, 83.05)	98.24	(97.27, 99.18)	0.044	0
NPP	49.88	(21.60, 78.84)	98.02	(96.93, 99.00)	0.216	0.085

A simple frequentist approach to evaluate such binary test is to compute the 95% confidence intervals of η and λ , denoted by (η_L, η_U) and (λ_L, λ_U) . Then we compare the lower bounds η_L and λ_L to the value of 50% which is the sensitivity and specificity of a random test. We may conclude that the diagnostic test outweighs a random test on the diseased group if η_L is greater than 50%. Similarly, the diagnostic test outweighs a random test on non-diseased group if λ_L is greater than 50%.

In practice, however, the diseased group's data are difficult to collect leading to a relatively small $n_1 + n_3$. As a result, the confidence interval of η tends to be too wide to make any conclusions. For the purpose of this agreement, the sequential Bayesian updating and the power prior can be used to incorporate the historical/external information.

A diagnostic test based on a medical device (PartoSure Test-P160052) was developed to aid in rapidly assessing the risk of spontaneous preterm delivery within 7 days from the time of diagnosis in pre-pregnant women with signs and symptoms.² Table 5 lists the dataset of 686 subjects from the US study and the dataset of 511 subjects from the European study. The test was approved by FDA based on the US study, so the European study is regarded as the external information in this example. The joint power prior (with the full multinomial likelihood), the normalized power prior, no borrowing and full borrowing are applied, with Jeffrey's prior $\text{Dir}(0.5, 0.5, 0.5, 0.5)$ as the initial prior for θ . Table 6 summarizes the results. It is found that the posterior mean under the power prior is always between the posterior mean of no borrowing and full borrowing. Also, the result of using joint power prior is close to the one of no borrowing since only 4.4% of the external information is incorporated on average. Using the normalized power prior will on average increase the involved external information to 21.6%, making its result closer to the full borrowing. In practice, the posterior mean of δ (e.g., 4.4% and 21.6%) could be important to clinicians because it not only reflects the information amount that is borrowed, but also indicates the average sample size (e.g., $511 \times 4.4\%$ and $511 \times 21.6\%$) that is incorporated. The joint power prior suggests very little borrowing while the normalized power prior suggests a moderate level of borrowing. In general, these two datasets are compatible since they have similar sensitivity (50% and 50%) and specificity (96% and 98%). The value obtained by the normalized power prior is more persuasive and reflects the data compatibility.

Computer code in R for all three examples above is available from <https://github.com/hanzifei/CodeNPPJSPI>.

7. Summary and discussion

As a general class of the informative priors for Bayesian inference, the power prior provides a framework to incorporate data from alternative sources, whose influence on statistical inference can be adjusted according to its availability and its discrepancy between the current data. It is semi-automatic, in the sense that it takes the form of raising the likelihood function based on the historical data to a fractional power regardless of the specific form of heterogeneity. As a consequence of using more data, the power prior has advantages in terms of the estimation with small sample sizes.

² The dataset used in this example is freely available at: https://www.accessdata.fda.gov/cdrh_docs/pdf16/P160052C.pdf [Accessed 03 June 2019].

When we do not have enough knowledge to model such heterogeneity and cannot specify a fixed power parameter in advance, a power prior with a random δ is especially attractive in practice.

In this article we provide a framework of using the normalized power prior approach, in which the degree of borrowing is dynamically adjusted through the prior-data conflict. The subjective information about the difference in two populations can be incorporated by adjusting the hyperparameters in the prior for δ , and the discrepancy between the two samples is automatically taken into account through a random δ . Theoretical justification is provided based on the weighted KL divergence. The controlling role of the power parameter in the normalized power prior is adjusted automatically based on the congruence between the historical and the current samples and their sample sizes; this is shown using both the analytical and numerical results. On the other hand, we revisit some undesirable properties of using the joint power prior for a random δ ; this is shown by theoretical justifications and graphical examples. Efficient algorithms for posterior sampling using the normalized power prior are also discussed and implemented.

We acknowledge when δ is considered random and estimated with a Bayesian approach, the normalized power prior is more appropriate. The violation of likelihood principle under the joint power prior was discussed in [Duan et al. \(2006\)](#) and [Neuenschwander et al. \(2009\)](#). However, a comprehensive study on the joint power prior and the normalized power prior is not available in literature. As a result, the joint power priors with random δ were still used afterwards, for example, [Zhao et al. \(2014\)](#), [Gamalo et al. \(2014\)](#), [Lin et al. \(2016\)](#) and [Zhang et al. \(2019\)](#). This might be partially due to the fact that the undesirable behavior of the joint power priors were not fully studied and recognized. Although under certain likelihood forms, the joint power priors would provide limited adaptive borrowing, its mechanism is unclear. We conclude that the joint power prior is not recommended with a random δ .

On the other hand, the power prior with δ fixed is widely used in both clinical trial design and observational studies. It can be viewed as a special case of the normalized power prior with initial prior of δ coming from a degenerate distribution. We conjecture that a similar sensitivity analysis used in a power prior with δ fixed ([Ibrahim et al., 2015](#)) might be carried out to search for the initial prior of δ in the normalized power prior context. Since the normalized power prior generalizes the power prior with δ fixed, most inferential results in power prior with δ fixed could be easily adopted. Further studies will be carried out elsewhere.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Proofs and theorems

Proof of Identity (2.10). Taking derivative of $\log \int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta$ with respect to δ we have:

$$\begin{aligned} \frac{d}{d\delta} \log \int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta &= \frac{1}{\int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta} \frac{d}{d\delta} \int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta \\ &= \frac{1}{\int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta} \int_{\Theta} L(\theta|D_0)^\delta \log[L(\theta|D_0)] \pi_0(\theta) d\theta \\ &= \int_{\Theta} \frac{L(\theta|D_0)^\delta \pi_0(\theta)}{\int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta} \log[L(\theta|D_0)] d\theta \\ &= E_{\pi(\theta|D_0, \delta)} \{\log[L(\theta|D_0)]\}. \end{aligned}$$

So Eq. (2.10) can be obtained by integrating with respect to δ .

Proof of Theorem 2.1. To prove Theorem 2.1, we first state two simple identities of linear algebra and multivariate integral without proof. For positive-definite $k \times k$ matrices \mathbf{A} and \mathbf{B} , and $k \times 1$ vectors \mathbf{x} , \mathbf{y} , and \mathbf{z} ,

$$\begin{aligned} (\mathbf{x} - \mathbf{y})' \mathbf{A} (\mathbf{x} - \mathbf{y}) + (\mathbf{x} - \mathbf{z})' \mathbf{B} (\mathbf{x} - \mathbf{z}) &= (\mathbf{y} - \mathbf{z})' \mathbf{B} (\mathbf{A} + \mathbf{B})^{-1} \mathbf{A} (\mathbf{y} - \mathbf{z}) \\ &+ [\mathbf{x} - (\mathbf{A} + \mathbf{B})^{-1} (\mathbf{A} \mathbf{y} + \mathbf{B} \mathbf{z})]' (\mathbf{A} + \mathbf{B}) [\mathbf{x} - (\mathbf{A} + \mathbf{B})^{-1} (\mathbf{A} \mathbf{y} + \mathbf{B} \mathbf{z})]. \end{aligned} \quad (\text{A.1})$$

On the other hand, for \mathbf{A} being a positive-definite $k \times k$ matrix, \mathbf{x} and \mathbf{x}_0 $k \times 1$ vectors, with positive constants t , a and b where $a > \frac{k}{2} + 1$,

$$\begin{aligned} &\int_0^\infty \int_{\mathcal{R}^k} \frac{1}{t^a} \exp \left\{ -\frac{b + (\mathbf{x} - \mathbf{x}_0)' \mathbf{A} (\mathbf{x} - \mathbf{x}_0)}{2t} \right\} d\mathbf{x} dt \\ &= (2\pi)^{\frac{k}{2}} \Gamma \left(a - \frac{k}{2} - 1 \right) |\mathbf{A}|^{-\frac{1}{2}} \left(\frac{b}{2} \right)^{-(a - \frac{k}{2} - 1)}. \end{aligned} \quad (\text{A.2})$$

For the current data D , the likelihood function of $(\boldsymbol{\beta}, \sigma^2)$ using (2.15) can be written as

$$L(\boldsymbol{\beta}, \sigma^2 | D) \propto \frac{1}{(\sigma^2)^{\frac{n}{2}}} \exp \left\{ -\frac{1}{2\sigma^2} [S + (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})' \mathbf{X}' \mathbf{X} (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})] \right\},$$

where S is defined in Section 2.3.3. Accordingly, adding subscript 0 to data and all other quantities except for the parameters $(\boldsymbol{\beta}, \sigma^2)$ would give similar form to $L(\boldsymbol{\beta}, \sigma^2 | D_0)$.

(a) To obtain the normalized power prior, we need to find the normalization factor

$$\begin{aligned} C(\delta) &\propto \int_0^\infty \int_{\mathcal{R}^k} \pi_0(\boldsymbol{\beta}, \sigma^2) L(\boldsymbol{\beta}, \sigma^2 | D_0)^\delta d\boldsymbol{\beta} d\sigma^2 \\ &\propto \int_0^\infty \int_{\mathcal{R}^k} \frac{1}{(\sigma^2)^{\frac{\delta n_0 + bk}{2} + a}} \exp \left\{ -\frac{1}{2\sigma^2} [\delta \{S_0 + bH_0(\delta)\} + Q(\delta, \boldsymbol{\beta})] \right\} d\boldsymbol{\beta} d\sigma^2 \\ &\propto 1/M_0(\delta), \end{aligned}$$

where $H_0(\delta)$, $M_0(\delta)$ and $Q(\delta, \boldsymbol{\beta})$ are defined in Theorem 2.1(a). Note that, using (A.1), the second line follows from completing the squares

$$(\boldsymbol{\beta} - \boldsymbol{\mu}_0)' b\mathbf{R} (\boldsymbol{\beta} - \boldsymbol{\mu}_0) + (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0)' \delta \mathbf{X}'_0 \mathbf{X}_0 (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}_0) = Q(\delta, \boldsymbol{\beta}) + \delta bH_0(\delta),$$

while to finish the third line we use the identity in (A.2). Multiplying $\pi_0(\delta) \pi_0(\boldsymbol{\beta}, \sigma^2) L(\boldsymbol{\beta}, \sigma^2 | D_0)^\delta$ by $C(\delta)^{-1}$ above yields the result (a).

(b) Since

$$Q(\delta, \boldsymbol{\beta}) + (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}})' \mathbf{X}' \mathbf{X} (\boldsymbol{\beta} - \hat{\boldsymbol{\beta}}) = H(\delta) + Q^*(\delta, \boldsymbol{\beta}),$$

where $H(\delta)$ is defined in Theorem 2.1(b), and

$$Q^*(\delta, \boldsymbol{\beta}) = (\boldsymbol{\beta} - \boldsymbol{\mu}^*)' (b\mathbf{R} + \delta \mathbf{X}'_0 \mathbf{X}_0 + \mathbf{X}' \mathbf{X}) (\boldsymbol{\beta} - \boldsymbol{\mu}^*),$$

where $\boldsymbol{\mu}^* = (b\mathbf{R} + \delta \mathbf{X}'_0 \mathbf{X}_0 + \mathbf{X}' \mathbf{X})^{-1} [(b\mathbf{R} + \delta \mathbf{X}'_0 \mathbf{X}_0) \hat{\boldsymbol{\beta}}^* + \mathbf{X}' \mathbf{X} \hat{\boldsymbol{\beta}}]$, using the normalized power prior in (a), the posterior $\pi(\boldsymbol{\beta}, \sigma^2, \delta | D_0, D)$ is of the form

$$\pi(\boldsymbol{\beta}, \sigma^2, \delta | D_0, D) \propto \frac{\pi_0(\delta) M_0(\delta)}{(\sigma^2)^{\frac{n + \delta n_0 + bk}{2} + a}} \exp \left\{ -\frac{\delta [S_0 + bH_0(\delta)] + S + H(\delta) + Q^*(\delta, \boldsymbol{\beta})}{2\sigma^2} \right\}.$$

Marginalizing $(\boldsymbol{\beta}, \sigma^2)$ out, we obtain

$$\begin{aligned} \pi(\delta | D_0, D) &\propto \pi_0(\delta) M_0(\delta) \Gamma(v^*) |b\mathbf{R} + \delta \mathbf{X}'_0 \mathbf{X}_0 + \mathbf{X}' \mathbf{X}|^{-\frac{1}{2}} \\ &\times \left\{ \frac{\delta [S_0 + bH_0(\delta)] + S + H(\delta)}{2} \right\}^{-v^*}, \end{aligned}$$

where $v^* = \frac{n + \delta n_0 + (b-1)k}{2} + a - 1$. Plugging in $M_0(\delta)$ we get (b).

(c) Integrating σ^2 out from the joint posterior, we have

$$\pi(\boldsymbol{\beta}, \delta | D_0, D) \propto \pi_0(\delta) M_0(\delta) \Gamma \left(v^* + \frac{k}{2} \right) \left\{ \frac{\delta [S_0 + bH_0(\delta)] + S + H(\delta) + Q^*(\delta, \boldsymbol{\beta})}{2} \right\}^{-(v^* + \frac{k}{2})},$$

where v^* and $Q^*(\delta, \beta)$ are defined above in the proof of part (b). The conditional distribution of β given (δ, D_0, D) satisfies

$$\begin{aligned}\pi(\beta|\delta, D_0, D) &\propto \left\{ \delta [S_0 + bH_0(\delta)] + S + H(\delta) + Q^*(\delta, \beta) \right\}^{-(v^* + \frac{k}{2})} \\ &\propto \left\{ 1 + \frac{1}{v} \left[\frac{(\beta - \mu^*)' v (bR + \delta X_0' X_0 + X' X) (\beta - \mu^*)}{\delta \{S_0 + bH_0(\delta)\} + S + H(\delta)} \right] \right\}^{-\frac{v+k}{2}},\end{aligned}$$

where $v = (b-1)k + \delta n_0 + n + 2a - 2$. This is the kernel of a multivariate Student t -distribution with parameters specified in Theorem 2.1(c).

(d) Using Gaussian integral we can marginalize β out from the joint posterior, then

$$\pi(\sigma^2, \delta|D_0, D) \propto \frac{\pi_0(\delta)M_0(\delta)}{(\sigma^2)^{v^*+1}} \exp \left\{ -\frac{\delta [S_0 + bH_0(\delta)] + S + H(\delta)}{2\sigma^2} \right\} |bR + \delta X_0' X_0 + X' X|^{-\frac{1}{2}},$$

where v^* is defined in the proof of part (b). Conditional on (δ, D_0, D) , $\pi(\sigma^2|\delta, D_0, D)$ is an inverse-gamma kernel with parameters specified in Theorem 2.1(d).

Proof of Theorem 3.1. The quantity L_g in (3.2) can be written as

$$\begin{aligned}L_g &= E_{\pi_0(\delta)} \{ (1-\delta)K(g, \pi_0) + \delta K(g, \pi_1) \} \\ &= E_{\pi_0(\delta)} \left[\int_{\Theta} g(\theta|\delta) \log \left\{ \frac{g(\theta|\delta)^{1-\delta}}{\pi_0(\theta)^{1-\delta}} \cdot \frac{g(\theta|\delta)^\delta}{\pi_1(\theta)^\delta} \right\} d\theta \right] \\ &= E_{\pi_0(\delta)} \left[\int_{\Theta} g(\theta|\delta) \log \left\{ \frac{g(\theta|\delta)}{Q(D_0)^\delta \pi_0(\theta) L(\theta|D_0)^\delta} \right\} d\theta \right] \\ &= E_{\pi_0(\delta)} \{ K[g(\theta|\delta), \pi^*(\theta|\delta, D_0)] \} - E_{\pi_0(\delta)} \left[\log \left\{ \frac{Q^\delta(D_0)}{Q_1(D_0, \delta)} \right\} \right],\end{aligned}\tag{A.3}$$

where

$$\pi^*(\theta|\delta, D_0) = \frac{L(\theta|D_0)^\delta \pi_0(\theta)}{\int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta},\tag{A.4}$$

$Q(D_0)$ is defined in (3.1), and $Q_1(D_0, \delta)^{-1}$ is the denominator in (A.4). The second term of (A.3) in the last line is not related to g , and the inside KL divergence in the first term is clearly minimized when $g(\theta|\delta) = \pi^*(\theta|\delta, D_0)$.

Proof of Theorem 4.1. Applying the property of the KL divergence between two distributions,

$$K(f_1, f_2) = \int f_1(x) \log \frac{f_1(x)}{f_2(x)} dx \geq 0,$$

with equality held if and only if $f_1(x) = f_2(x)$, we conclude that

$$\begin{aligned}\frac{n}{n_0} h_1(D_0, D, \delta) &= \int_{\Theta} \log L(\theta|D) \{ \pi(\theta|D_0, D, \delta) - \pi(\theta|D_0, \delta) \} d\theta \\ &= \int_{\Theta} \log \left\{ \frac{\pi(\theta|D_0, D, \delta)}{\pi(\theta|D_0, \delta)} M(D_0, D|\delta) \right\} \{ \pi(\theta|D_0, D, \delta) - \pi(\theta|D_0, \delta) \} d\theta \\ &= \int_{\Theta} \log \frac{\pi(\theta|D_0, D, \delta)}{\pi(\theta|D_0, \delta)} \pi(\theta|D_0, D, \delta) d\theta + \int_{\Theta} \log \frac{\pi(\theta|D_0, \delta)}{\pi(\theta|D_0, D, \delta)} \pi(\theta|D_0, \delta) d\theta \geq 0,\end{aligned}\tag{A.5}$$

with equality held if and only if $\pi(\theta|D_0, D, \delta) = \pi(\theta|D_0, \delta)$. In (A.5), $M(D_0, D|\delta)$ is a marginal density that does not depend on θ and hence its related term is 0 since both $\pi(\theta|D_0, D, \delta)$ and $\pi(\theta|D_0, \delta)$ are proper.

In order to show that the marginal posterior mode of δ is 1, it is sufficient to show that the derivative of $\pi(\delta|D_0, D)$ in (2.5) is non-negative. Using certain algebra similar to the proof of identity (2.10), we obtain

$$\begin{aligned}\frac{d}{d\delta} \pi(\delta|D_0, D) &= \frac{d}{d\delta} \{ \log \pi_0(\delta) \} \pi(\delta|D_0, D) + \\ &\quad \pi(\delta|D_0, D) \int_{\Theta} \log L(\theta|D_0) \{ \pi(\theta|D_0, D, \delta) - \pi(\theta|D_0, \delta) \} d\theta.\end{aligned}\tag{A.6}$$

Since we are dealing with the exponential family with the form (2.11) and (2.13), considering the likelihood ratio we have

$$\begin{aligned}\log L(\boldsymbol{\theta}|D_0) &= \log h(D_0) + n_0\{\mathbf{T}(D_0)' \underline{w}(\boldsymbol{\theta}) + \tau(\boldsymbol{\theta})\} \\ &= \log h(D_0) - \frac{n_0}{n} \log h(D) + \frac{n_0}{n} \log L(\boldsymbol{\theta}|D) + n_0\{\mathbf{T}(D_0) - \mathbf{T}(D)\}' \underline{w}(\boldsymbol{\theta}).\end{aligned}\quad (\text{A.7})$$

Combining (A.5) and (A.7) into (A.6), we prove Theorem 4.1 by showing the condition (4.1).

Proof of Theorem 4.2. Suppose that k is an arbitrary positive constant. We take the likelihood function of the form $L(\boldsymbol{\theta}|x) = k f(x|\boldsymbol{\theta})$, then $L(\boldsymbol{\theta}|D) = k^n f(D|\boldsymbol{\theta})$ and $L(\boldsymbol{\theta}|D_0) = k^{n_0} f(D_0|\boldsymbol{\theta})$. For the original joint power prior, the marginal posterior distribution of δ can be rewritten as

$$\begin{aligned}\pi(\delta|D_0, D) &\propto \pi_0(\delta) \int_{\boldsymbol{\theta}} L(\boldsymbol{\theta}|D) L(\boldsymbol{\theta}|D_0)^\delta \pi_0(\boldsymbol{\theta}) d\boldsymbol{\theta} \\ &\propto \pi_0(\delta) \int_{\boldsymbol{\theta}} f(D|\boldsymbol{\theta}) [k^{n_0} f(D_0|\boldsymbol{\theta})]^\delta \pi_0(\boldsymbol{\theta}) d\boldsymbol{\theta}.\end{aligned}\quad (\text{A.8})$$

To prove that the marginal posterior mode of δ is 0, it is sufficient to show that the derivative of $\pi(\delta|D_0, D)$ with respect to δ is non-positive for any $\delta \in [0, 1]$.

The derivative contains two parts. The first part is the derivative on $\pi_0(\delta)$. If $\pi_0(\delta)$ is non-increasing as described in the theorem, this part is non-positive. The second part is the derivative in the integral part in (A.8). An equivalent condition to guarantee this part non-positive is

$$\begin{aligned}&\int_{\boldsymbol{\theta}} f(D|\boldsymbol{\theta}) \frac{d[k^{n_0} f(D_0|\boldsymbol{\theta})]^\delta}{d\delta} \pi_0(\boldsymbol{\theta}) d\boldsymbol{\theta} \leq 0 \\ &\iff k^{n_0\delta} \int_{\boldsymbol{\theta}} \pi_0(\boldsymbol{\theta}) f(D|\boldsymbol{\theta}) f(D_0|\boldsymbol{\theta})^\delta \{n_0 \log k + \log f(D_0|\boldsymbol{\theta})\} d\boldsymbol{\theta} \leq 0 \\ &\iff \frac{\int_{\boldsymbol{\theta}} \pi_0(\boldsymbol{\theta}) f(D|\boldsymbol{\theta}) f(D_0|\boldsymbol{\theta})^\delta \log f(D_0|\boldsymbol{\theta}) d\boldsymbol{\theta}}{\int_{\boldsymbol{\theta}} \pi_0(\boldsymbol{\theta}) f(D|\boldsymbol{\theta}) f(D_0|\boldsymbol{\theta})^\delta d\boldsymbol{\theta}} \leq n_0 \log \frac{1}{k},\end{aligned}\quad (\text{A.9})$$

assuming that the derivative and integral are interchangeable.

If we take

$$k_0 = \exp \left\{ -\frac{1}{n_0} \max_{0 \leq \delta \leq 1} \frac{\int_{\boldsymbol{\theta}} \pi_0(\boldsymbol{\theta}) f(D|\boldsymbol{\theta}) f(D_0|\boldsymbol{\theta})^\delta \log f(D_0|\boldsymbol{\theta}) d\boldsymbol{\theta}}{\int_{\boldsymbol{\theta}} \pi_0(\boldsymbol{\theta}) f(D|\boldsymbol{\theta}) f(D_0|\boldsymbol{\theta})^\delta d\boldsymbol{\theta}} \right\} > 0,$$

then the sufficient condition in (A.9) for the marginal posterior mode of δ being 0 is met for any δ .

Appendix B. MCMC sampling scheme

B.1. Algorithm for posterior sampling

Here we describe an algorithm in detail that is applicable in models when $\pi(\delta|\boldsymbol{\theta}, D_0, D)$ is free of any numerical integration, and the full conditional for each θ_i is readily available.

Let $\boldsymbol{\theta} = (\theta_1, \dots, \theta_k)$ denote the parameters of interest in the model, and $\boldsymbol{\theta}_{-i}$ is $\boldsymbol{\theta}$ with the i th element removed. The initial prior $\pi_0(\boldsymbol{\theta})$ can be chosen so that the full conditional posterior of each θ_i , the $\pi(\theta_i|\boldsymbol{\theta}_{-i}, \delta, D_0, D)$, can be sampled directly using the Gibbs sampler (Gelman et al., 2013). However, neither the full conditional posterior $\pi(\delta|\boldsymbol{\theta}, D_0, D)$ nor the marginal posterior $\pi(\delta|D_0, D)$ is readily available. Given that $\pi(\delta|D_0, D)$ is known up to a normalizing constant, the Metropolis–Hastings algorithm (Chib and Greenberg, 1995) is implemented. Here we illustrate the use of a random-walk Metropolis–Hastings algorithm with Gaussian proposals for $\vartheta = \text{logit}(\delta)$, which converges well empirically. Let $q(\cdot | \delta^{\text{old}})$ denote the proposal distribution for δ in the current iteration, given its value in the previous iteration is δ^{old} . The algorithm proceeds as follows:

Step 0: Choose the initial values for the parameters $\boldsymbol{\theta}^{(0)}$ and $\delta^{(0)}$, set the tuning constant as c , and iteration index $l = 0$.

Step 1: The Metropolis–Hastings step. Simulate $\vartheta^* \sim N(\vartheta^{(l)}, c)$ and $U \sim \text{unif}(0, 1)$. Compute $\delta^* = \text{logit}^{-1}(\vartheta^*)$ and the acceptance probability $\alpha = \min\{1, t\}$. After applying a change of variable, we have

$$t = \frac{\pi(\delta^* | D_0, D) q(\delta^{(l)} | \delta^*)}{\pi(\delta^{(l)} | D_0, D) q(\delta^* | \delta^{(l)})} = \frac{\pi(\delta^* | D_0, D) \delta^* (1 - \delta^*)}{\pi(\delta^{(l)} | D_0, D) \delta^{(l)} (1 - \delta^{(l)})}.$$

Then set $\delta^{(l+1)} = \delta^*$, if $U < \alpha$. Otherwise, set $\delta^{(l+1)} = \delta^{(l)}$.

Step 2: The Gibbs sampling step. For $i = 1, \dots, k$, independently sample $\theta_i^{(l+1)}$ from its full conditional posterior $\pi(\theta_i|\boldsymbol{\theta}_{-i}^{(l)}, \delta^{(l+1)}, D_0, D)$.

Step 3: Increase l by 1, and repeat steps 1 and 2 until the states have reached the equilibrium distribution of the Markov chain.

Since $\delta \in [0, 1]$, an independent proposal from a beta distribution might also provide good convergence. In such cases, the proposal distribution $q(\cdot)$ will be the *same* beta distribution evaluated at $\delta^{(l)}$ and δ^* in the nominator and denominator respectively.

B.2. Algorithm to compute the scale factor

Here we describe an algorithm in detail when the scale factor in the denominator, $C(\delta) = \int_{\Theta} L(\theta|D_0)^\delta \pi_0(\theta) d\theta$ needs to be calculated numerically. From identity (2.10), $\log C(\delta) = \int_0^\delta E_{\pi(\theta|D_0, \delta^*)} \{\log[L(\theta|D_0)]\} d\delta^*$, so we only need to calculate the one-dimensional integral.

MCMC samples from $\pi(\theta|D_0, \delta)$ with fixed δ can be easily drawn, since the target density is expressed explicitly up to a normalizing constant. A fast implementation with RStan (Carpenter et al., 2017) and parallel programming is applicable, by including the fixed δ in the target statement. We develop the following algorithm to calculate the scale factor $\log C(\delta)$ up to a true constant. It is an adaptive version of the path sampling based on the results in Van Rosmalen et al. (2018).

- Step 0: Choose a set of $n - 1$ different numbers as knots between 0 and 1, and another knot at 1, with n sufficiently large. Sort them in ascending order $(\delta_1, \dots, \delta_{n-1}, 1)$. Let $\delta_0 = 0$, $\Delta_1 = \delta_1$, $\Delta_i = \delta_i - \delta_{i-1}$ ($1 < i \leq n$), and $\Delta_n = 1 - \delta_{n-1}$. Choose M , the number of MCMC samples in a run when sampling from $\pi(\theta|D_0, \delta)$. Initialize $l = 1$.
- Step 1: Generate M samples from $\pi(\theta|D_0, \delta_l)$ using an appropriate MCMC algorithm. Denote the sample as $(\theta_l^{(1)}, \theta_l^{(2)}, \dots, \theta_l^{(M)})$.
- Step 2: Calculate $h(\delta_l) = \sum_{j=1}^M \log L(\theta_l^{(j)}|D_0)/M$.
- Step 3: Calculate $\log C(\delta_l) \approx \sum_{k=1}^l \Delta_k \{h(\delta_{k-1}) + h(\delta_k)\}$ (the Trapezoidal rule).
- Step 4: Increase l by 1. If $l \leq n$ then repeat Steps 1 to 3.

The output is a vector of n values, $(\log C(\delta_1), \dots, \log C(\delta_{n-1}), \log C(1))$, for selected knots.

Finally, for δ that is not on the knots, it is efficient to linearly interpolate $\log C(\delta)$ based on its nearest two values on the knots (Van Rosmalen et al., 2018). The interpolation can be done quite fast at every iteration when sampling from the posterior $\pi(\theta, \delta|D_0, D)$ using a normalized power prior, so the algorithm similar to the one described in Appendix B.1 can be applied. Compared to the joint power prior, the extra computational cost is to calculate $\log C(\delta)$ on the selected knots, with the capability of parallel computation. Both of the algorithms in Appendices B.1 and B.2 are implemented in R package NPP. An example using Stan with reliability data is also available at <https://github.com/hanzifei/CodeNPPJSPI>.

References

- Ibrahim, J.G., Chen, M.-H., 1998. Prior distributions and Bayesian computation for proportional hazards models. *Sankhya: Indian J. Statist. Ser. B* 48–64.
- Chen, M.-H., Ibrahim, J.G., Shao, Q.-M., 2000. Power prior distributions for generalized linear models. *J. Statist. Plann. Inference* 84, 121–137.
- Ibrahim, J.G., Chen, M.-H., 2000. Power prior distributions for regression models. *Statist. Sci.* 15, 46–60.
- Ibrahim, J.G., Chen, M.-H., Sinha, D., 2003. On optimality properties of the power prior. *J. Amer. Statist. Assoc.* 98, 204–213.
- Chen, M.-H., Ibrahim, J.G., 2006. The relationship between the power prior and hierarchical models. *Bayesian Anal.* 1, 551–574.
- Ibrahim, J.G., Chen, M.-H., Gwon, Y., Chen, F., 2015. The power prior: Theory and applications. *Stat. Med.* 34, 3724–3749.
- Ibrahim, J.G., Chen, M.-H., Chu, H., 2012a. Bayesian methods in clinical trials: a bayesian analysis of ECOG trials E1684 and E1690. *BMC Med. Res. Methodol.* 12, 183.
- Spiegelhalter, D.J., Best, N.G., Carlin, B.P., Van Der Linde, A., 2002. Bayesian Measures of model complexity and fit. *J. R. Stat. Soc. Ser. B Stat. Methodol.* 64, 583–639.
- Birnbaum, A., 1962. On the foundations of statistical inference. *J. Amer. Statist. Assoc.* 57, 269–306.
- Neuenschwander, B., Branson, M., Spiegelhalter, D.J., 2009. A note on the power prior. *Stat. Med.* 28, 3562–3566.
- Neelon, B., O'Malley, A., 2010. Bayesian analysis using power priors with application to pediatric quality of care. *J. Biometr. Biostatist.* 1, 103.
- Duan, Y., Ye, K., Smith, E.P., 2006. Evaluating water quality using power priors to incorporate historical information. *Environmetrics* 17, 95–106.
- Gamalo, M.A., Tiwari, R.C., LaVange, L.M., 2014. Bayesian approach to the design and analysis of non-inferiority trials for anti-infective products. *Pharm. Statist.* 13, 25–40.
- Gravestock, I., Held, L., 2019. Power priors based on multiple historical studies for binary outcomes. *Biom. J.* 1201–1218.
- Banbeta, A., van Rosmalen, J., Dejardin, D., Lesaffre, E., 2019. Modified power prior with multiple historical trials for binary endpoints. *Stat. Med.* 38, 1147–1169.
- Ibrahim, J.G., Chen, M.-H., Xia, H.A., Liu, T., 2012b. Bayesian meta-experimental design: Evaluating cardiovascular risk in new antidiabetic therapies to treat type 2 diabetes. *Biometrics* 68, 578–586.
- Chen, M.-H., Ibrahim, J.G., Zeng, D., Hu, K., Jia, C., 2014b. Bayesian design of superiority clinical trials for recurrent events data with applications to bleeding and transfusion events in myelodysplastic syndrome. *Biometrics* 70, 1003–1013.
- Chen, M.-H., Ibrahim, J.G., Xia, H.A., Liu, T., Hennessey, V., 2014a. Bayesian sequential meta-analysis design in evaluating cardiovascular risk in a new antidiabetic drug development program. *Stat. Med.* 33, 1600–1618.
- Chib, S., Greenberg, E., 1995. Understanding the metropolis-hastings algorithm. *Amer. Statist.* 49, 327–335.
- Friel, N., Pettitt, A.N., 2008. Marginal likelihood estimation via power posteriors. *J. R. Stat. Soc. Ser. B Stat. Methodol.* 70, 589–607.
- Van Rosmalen, J., Dejardin, D., van Norden, Y., Löwenberg, B., Lesaffre, E., 2018. Including historical data in the analysis of clinical trials: Is it worth the effort?. *Stat. Methods Med. Res.* 27, 3167–3182.

- Gelman, A., Meng, X.-L., 1998. Simulating normalizing constants: From importance sampling to bridge sampling to path sampling. *Statist. Sci.* 13, 163–185.
- Carvalho, L.M., Ibrahim, J.G., 2020. On the normalized power prior, [arXiv:2004.14912](https://arxiv.org/abs/2004.14912), v1, 1–31.
- Casella, G., Berger, R.L., 2002. *Statistical Inference*, second ed. Duxbury Pacific Grove, CA.
- Zellner, A., 1986. *On Assessing Prior Distributions and Bayesian Regression Analysis with G-Prior Distributions*. Elsevier, New York.
- Kullback, S., Leibler, R.A., 1951. On information and sufficiency. *Ann. Math. Stat.* 22, 79–86.
- Berger, J.O., Bernardo, J.M., 2013. *Statistical Decision Theory and Bayesian Analysis*. Springer Science & Business Media.
- Gamalo, M.A., Wu, R., Tiwari, R.C., 2011. Bayesian Approach to noninferiority trials for proportions. *J. Biopharm. Statist.* 21, 902–919.
- Lin, J., Gamalo-Siebers, M., Tiwari, R., 2016. Non-inferiority and networks: Inferring efficacy from a web of data. *Pharmaceutical Statistics* 15, 54–67.
- Li, W., Chen, M.-H., Wang, X., Dey, D.K., 2018. Bayesian Design of non-inferiority clinical trials via the Bayes factor. *Stat. Biosci.* 10, 439–459.
- Liu, G.F., 2018. A dynamic power prior for borrowing historical data in noninferiority trials with binary endpoint. *Pharmaceutical Statistics* 17, 61–73.
- Zhao, Y., Zalkikar, J., Tiwari, R.C., LaVange, L.M., 2014. A Bayesian approach for benefit-risk assessment. *Statistics in Biopharmaceutical Research* 6, 326–337.
- Zhang, J., Ko, C.-W., Nie, L., Chen, Y., Tiwari, R.C., 2019. Bayesian Hierarchical methods for meta-analysis combining randomized-controlled and single-arm studies. *Stat. Methods Med. Res.* 28, 1293–1310.
- Gelman, A., Carlin, J.B., Stern, H.S., Dunson, D.B., Vehtari, A., Rubin, D.B., 2013. *Bayesian Data Analysis*, third ed. Chapman and Hall/CRC.
- Carpenter, B., Gelman, A., Hoffman, M.D., Lee, D., Goodrich, B., Betancourt, M., Brubaker, M., Guo, J., Li, P., Riddell, A., 2017. Stan: A probabilistic programming language. *J. Stat. Softw.* 76, 1–32.