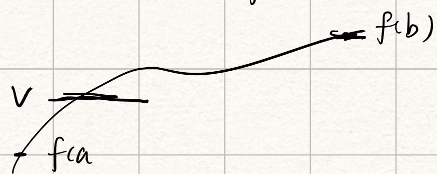
Continuity

Def $f : [a,b] \to \mathbb{R}$ is cts on $[a,b]$ if $\lim_{x\to a} f(x) = f(y)$ $\forall y \in (a,b)$ and $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

IVT Let $f : [a,b] \to \mathbb{R}$ be cts on $[a,b]$. Suppose that $f(a) < f(b)$, then $\forall v \in (f(a), f(b))$, $\exists c \in (a,b)$ s.t. $f(c) = v$.



Prove: $c = \sup A$ $A = \{x \in [a,b] \mid f(y) < 0 \ \forall y \in [a,x]\}$.

Fact Suppose f is cts at x & $f(x) < 0$, Then $\exists \delta > 0$ s.t. $\forall y \in (x-\delta, x+\delta)$, we have $f(y) < 0$.

Ex. Let $f : [0,1] \to [0,1]$ be cts. Then $\exists x \in [0,1]$ s.t. $f(x) = x$.

Proof: First if $f(0) = 0$ or $f(1) = 1$ Done.

Suppose not. Then as $f(x) \in [0,1]$ $\forall x$, we must have $f(0) > 0$, $f(1) < 1$.

Let $g(x) = x - f(x)$

Then as f, x are cts, so is g.

Moreover, $g(0) = -f(0) < 0$, $g(1) = 1 - f(1) > 0$.

Thus, by IVT, $\exists x \in [0,1]$ s.t. $g(x) = 0$, i.e. $x = f(x)$ ☒

EVT Let $f : [a,b] \to \mathbb{R}$ be cts, then f is bdd & moreover, $\exists x_-, x_+ \in [a,b]$ s.t. $f(x_-) \le f(x) \le f(x_+)$ $\forall x \in [a,b]$

In other words, $f(x_-) = \inf \{f(x) \mid x \in [a,b]\}$

$f(x_+) = \sup \{f(x) \mid x \in [a,b]\}$.

If we work on an an open (a,b). f may not be bdd.

e.g. $f(x)=\frac{1}{x}$ on $(0,1)$ is cts on $(0,1)$ but not bdd.

OR: f may be bdd, but not attain its sup.

e.g. $f(x)=x$ on $(0,1)$ $\sup f = 1$, but $f(x)<1$ $\forall x \in (0,1)$.

If we do not have bdd interval, consider $f(x)=x$ on $\mathbb{R}$, for each $n \in \mathbb{N}$, f is bdd on $[-n,n]$ by n, but not bdd on $\mathbb{R}$.

If we drop cts at one point, f may not be bdd.

e.g. $f(x)=\begin{cases} \frac{1}{x} & x \neq 0 \\ 0 & x=0 \end{cases}$ f is cts on $(-1,1)\setminus\{0\}$, not bdd.

f may be bdd but not attain its max

$f(x)=\begin{cases} x & x\in[0,1) \\ 0 & x\in[1,2] \end{cases}$ Then f is cts on $[0,2]\setminus\{1\}$.

but not attain its max.

Differentiability

Def: f is diff'ble at a if $\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$ exists.

In this case, we say that the derivative of f at a is $f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h}$

Ex. If $f(x)=c \in \mathbb{R}$. $f'(x)=0$ $\forall x\in\mathbb{R}$.

If $f(x)=x^n$ $f'(x)=nx^{n-1}$ $\forall x\in\mathbb{R}$.

Lemma Suppose f is diff'ble at x, then f is cts at x. (Converse not true)

e.g. $f(x)=|x|$ cts at 0, not diff'ble at 0.

Sum, Product, Quotient Rules.

Let f, g be diff'ble at a,

i) Then $f+g$ is diff'ble at a & $(f+g)'(a) = f'(a) + g'(a)$

ii) Then $f \cdot g$ is diff'ble at a, & $(f \cdot g)'(a) = f'(a)g(a) + f(a)g'(a)$

iii) If $g(a) \neq 0$, then $\frac{f}{g}$ is diff'ble at a & $(\frac{f}{g})'(a) = \frac{g(a)f'(a) - f(a)g'(a)}{g(a)^2}$

Chain Rule.

compositions can not be treated as easily as product.

Suppose f is diff'ble at a, g diff'ble at $f(a)$, then $g \circ f$ diff'ble at a

& $(g \circ f)'(a) = g'(f(a))\, f'(a)$.

Ex. Let $f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & x \neq 0 \\ 0 & 0 \end{cases}$

By chain rule, f is diff'ble $\forall x \neq 0$. & $f'(x) = 2x\sin(\frac{1}{x}) - \cos(1/x)$ $\forall x \neq 0$.

At $x=0$. consider $h \neq 0$. $\frac{f(h) - f(0)}{h} = \frac{h^2 \sin(\frac{1}{x}) - 0}{h} = h\sin(\frac{1}{x})$

As $|\sin(y)| \leq 1$ $\forall y \in \mathbb{R}$, $\left|\frac{f(h) - f(0)}{h}\right| \leq |h|$

So, $\lim_{h \to 0} \frac{f(h) - f(0)}{h} = 0$, $f'(0) = 0$.

$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & x \neq 0 \\ 0 & x = 0. \end{cases}$

To get a derivative, need f to be defined at a & on both side of a, so that $f(a+h)$ make sense for $h>0$ & $h<0$.

Lemma: Suppose f has a max at a & f is diff'ble at a, then $f'(a) = 0$.

Idea: $h>0$ $\frac{f(a+h) - f(a)}{h} \leq 0$ $\quad h \leq 0$ $\frac{f(a+h) - f(a)}{h} \geq 0$

So, $f'(a) \geq 0$ and $f'(a) \leq 0$ so $f'(a) = 0$

Note: $f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a}$ think $x = a+h$.