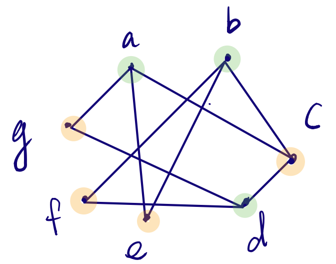
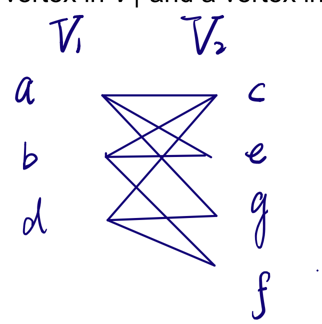
Undirected Graphs

Terminology

- **(undirected) graph** $G=(V,E)$ graph where edges don't have a direction
  - denote edges as $\{u,v\} \in E$ since order doesn't matter.
  - self loop $\{v\}$
- **incident :** If edge *e* has vertex *v* as one if its endpoints, then we say *e* and *v* are ***incident***.
- **adjacent :** If *u* and *v* are connected by an edge, we say *u* and *v* are ***adjacent*** or ***neighbors***.
- **simple :** A graph *G* = (*V*, *E*) is ***simple*** if *E* does not contain duplicates. i.e. E is a set
- **bipartite graph :** A simple graph is ***bipartite*** if its vertex set *V* can be partitioned into two disjoint sets V_1 and V_2 such that every edge in the graph contains a vertex in V_1 and a vertex in V_2.

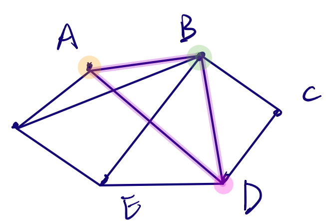
Graph 1





bipartite.

Graph 2



not bipartite
consider trying to partition a, b, d

Theorem : A simple graph is bipartite if and only if it is possible to color the vertices of the graph using 2 colors s.t. no 2 adjacent vertices have the same color.

Connectedness

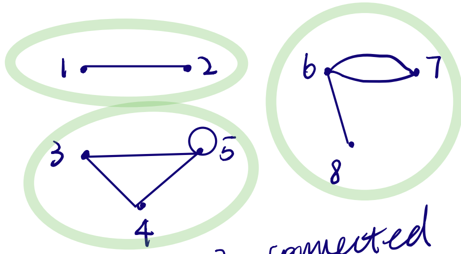
transitive closure = { (*u*, *v*) ∈ *V*×*V* | ∃ a path from *u* to *v* in *G* }

last week: saw this is reflexive and transitive for digraphs
also for undirected graphs.
on undirected graph: this is also symmetric. → equivalence graph.

connected component : Consider the transitive closure of *G*. Let V_1, V_2, V_3, ..., V_r be the equivalence classes in the partition of *V* induced by the transitive closure.

For each equivalence class V_i, the subgraph of G with vertex set V_i & edges from V involving V_i is a connected component

connected : a graph with at most one connected component



3 connected components

Equivalence class of transitive closure.

$V_1 = \{1, 2\}$

$V_2 = \{3, 4, 5\}$

$V_3 = \{6, 7, 8\}$.

Degree

$\{v\}$

degree of vertex *v* = # of edges containing *v*, counting a self-loop twice

deg(v) = k means vertex v has degree k.

Examples: $\deg(3) = 6$ $\deg(2) = 1$, $\deg(5) = 4$

Theorem: In every graph $G = (V, E)$, $\Sigma_{v \in V} \deg(v) = 2 \cdot |E|$ $(*)$

Proof: Consider each edge $e \in E$

Then, $e = \{v\}$ for $v \in V$ or $e = \{u, v\}$, $u, v \in V$, $u \neq v$

$\Sigma_{v \in V} \deg(v)$ if $e = \{v\}$, then e contributes 2 to count for $\deg(v)$

if $e = \{v, u\}$, then e contributes 1 to $\deg(v)$ and 1 to $\deg(u)$

So edge e contributes 2 to LHS of $(*)$

$2 \cdot |E|$ edge e contributes 1 to $|E|$

so edge e contributes 2 to RHS of $(*)$

This is true $\forall e \in E$, so RHS & LHS of $(*)$ calculate the same value. ■

Corollary: The number of vertices of odd degree in a graph is even.

Proof: Divide V into 2 disjoint sets

$V_{odd} = \{v \in V \mid \deg(v) \text{ is odd}\}$

$V_{even} = \{v \in V \mid \deg(v) \text{ is even}\}$

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_{odd}} \deg(v) + \sum_{v \in V_{even}} \deg(v) = \overbrace{2 \cdot |E|}^{\text{from Theorem}}$$

$$\sum_{v \in V} \deg(v) = \underbrace{2 \cdot |E|}_{\text{even}} - \underbrace{\sum_{v \in V_{even}} \deg(v)}_{\text{ev}}.$$

So sum of odd #s = even #

The only way for a sum of odd #s to be even is if the # items being summed is even.

i.e., if the # of vertices with odd degree ($|V_{odd}|$) is even.