Def: f is cts at a if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. if $|x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon$.

Q13: Suppose f is cts on $[a,b]$, g cts on $[b,c]$. $f(b) = g(b)$. Let $h(x) = \begin{cases} f(x) & x \in [a,b] \\ g(x) & x \in (b,c] \end{cases}$

Show h is cts. on $[a,c]$.

Def: f is cts on $[a,b]$ if $f(x)$ is continuous on (a,b) and $\lim_{x \to a^+} f(x) = f(a)$, $\lim_{x \to b^-} f(x) = f(b)$

We need h cts at all $x \in (a,c)$, $\lim_{x \to a^+} h(x) = h(a)$, $\lim_{x \to c} h(x) = h(c)$

Let $x \in (a,b)$ Then $\exists \delta > 0$ s.t. on $(x-\delta, x+\delta)$, $h = f$. $\lim_{y \to x} h(y) = \lim_{y \to x} f(y) = f(x) = h(x)$ as f cts.

So. h is cts on (a,b).

Similary, h is cts at (b,c) as g cts.

$\exists \delta > 0$ s.t. on $[a, a+\delta]$, $h = f$, $\lim_{y \to a^+} h(x) = \lim_{y \to a^+} f(x) = f(a) = h(a)$ as required.

Similarly, $\lim_{x \to c^-} h(x) = \lim_{x \to c^-} g(x) = g(c) = h(c)$

Need to show $\lim_{x \to b} h(x) = h(b)$

Note. $\lim_{x \to b^-} h(x) = \lim_{x \to b^-} f(x) = f(b)$ by cts of f

$= h(b)$

$\lim_{x \to b^+} h(x) = \lim_{x \to b^+} g(x) = g(b)$ by cty of g

$= f(b) = h(b)$.

As $\lim_{x \to b^-} h(x) = \lim_{x \to b^+} h(x) = h(b)$, $\lim_{x \to b} h(x) = h(b)$

Then $h(x)$ cts on $[a,c]$.

Def:

$\lim_{x\to a} f(x) = l$ if $\forall \varepsilon>0$, $\exists \delta>0$, if $0<|x-a|<\delta$, then $|f(x)-l|<\varepsilon$.

$\lim_{x\to b^-} f(x) = k$ if $\forall \varepsilon>0$, $\exists \delta>0$, if $0<b-x<\delta$, then $|f(x)-k|<\varepsilon$.

$\lim_{x\to\infty} f(x) = -\infty$ if $\forall M>0$, $\exists N>0$ s.t. if $x>N$. then $f(x)<-M$.

(x big enough $\Rightarrow$ f(x) small enough).

Q8: Prove if $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{x\to\infty} \frac{1}{f(x)} = 0$.

Let $\varepsilon>0$, as $\lim_{x\to\infty} f(x) = \infty$, $\exists M>0$ s.t. if $x>M$, then $f(x)>\frac{1}{\varepsilon}$.

So, if $x>M$, $\left|\frac{1}{f(x)}\right| = \frac{1}{f(x)} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon$. ■

Q12 $\lim_{x\to b^-} f(x) < \lim_{x\to b^-} g(x)$ Prove $\exists\delta>0$ s.t. if $x<b$ and $|x-b|<\delta$, then $g(x)<f(x)$.

Let $h=g-f$. Need $\delta>0$ s.t. if $0<b-x<\delta$, then $h(x)>0$.

By sum rule, $\lim_{x\to b^-} h(x) > 0$.

Let $L = \lim_{x\to b^-} h(x)$, so $L>0$.

Let $\varepsilon\in(0,L)$.

Then $\exists\delta>0$ s.t. if $0<b-x<\delta$, then $|h(x)-L|<\varepsilon$.

so, $h(x)>0$

Prop: $\lim_{x\to a} x^2 = a^2$.

Let $\varepsilon>0$ Need $\delta>0$ s.t. if $0<|x-a|<\delta$, then $0<|f(x)-a^2|<\varepsilon$.

Note $|x^2-a^2| = |x-a||x+a| < \delta(|x|+|a|) \le \delta\cdot(2|a|+\delta)$

$< \delta\cdot(2|a|+1)$ if $\delta<1$.

Let $\delta = \min\left(1, \frac{\varepsilon}{1+2|a|}\right)$

Let $q(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q}. \end{cases}$ $q(x)$ cts at

discts at other points.

$\lim_{x \to 0} q(x) = 0$ Let $\varepsilon > 0$, Let $\delta = \varepsilon$.

Then if $|x| < \delta$, $|q(x)| \leq |x| < \delta = \varepsilon$ $\blacksquare$