Basic Counting

Product rule

Suppose a procedure can be broken up into a sequence of 2 tasks. If there are n_1 ways to do the 1st task, and for each of these ways there are n_2 ways to do the 2nd task, then there are $n_1 \cdot n_2$ ways to do the procedure

More generally, if there are k tasks in our procedure, and n_1 choices for the 1st task,
once that task is chosen we have n_2 choices for the 2nd task,
once that task is chosen we have n_3 choices for the 3rd task,
etc., then the total number of ways to do the procedure is $n_1 \cdot n_2 \cdot n_3 \ldots \cdot n_k = \prod_{i=1}^{k} n_i$

product rule and sets : Let S_i be the set of options for task i

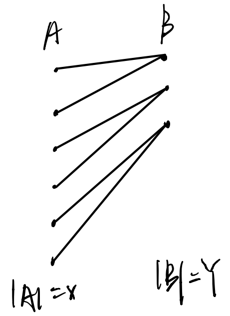
$$|S_1 \times S_2 \times S_3 \times \cdots \times S_k| = |S_1| \times |S_2| \times \cdots \times |S_k| = \prod_{i=1}^{k} |S_i|$$

Examples

1. How many different license plates can be made if each plate contains a sequence of 3 uppercase letters followed by 3 digits (and no sequences are prohibited)?

   26 choices for 1st letter, 26 choices for 2nd Letter, 26 for 3rd letter
   10 choice for each digit
   So total is $26^3 \times 10^3$

2. How many total functions are there from a set with X elements to a set with Y elements?



   Each total function corresponds to picking an element of B for each ele of A
   Y choices for the first choice of A, 2nd, 3rd . . . of A
   Total choices: Y^X

3. How many total one-to-one functions are there from a set with X elements to a set with Y elements?

   If $X > Y$, 0 total one-to-one function from A to B ($|A| = X$, $|B| = Y$)
   If $X \leq Y$, then $A = \{a_1, a_2, \ldots, a_X\}$.
   consider constructing func of $f: A \to B$

   $\frac{Y!}{(Y-X)!}$ choices.

   e.g. $|A| = 3$, $|B| = 5$.
   total $5^3 = 125$
   total, one-to-one $5 \times 4 \times 3 = 60$

Sum rule

If a task can be done in one of n_1 ways or in one of n_2 ways,
where there is no overlap between the n_1 ways and the n_2 ways,
then there are $n_1 + n_2$ ways to do the task.

sum rule and sets : Suppose S is the disjoint union of $S_1, S_2, \ldots, S_j$

partition of S

$S = \bigcup_{i=1}^{k} S_i$ where $S_i \cap S_j = \phi$ if $i \neq j$.

$|S| = |S_1| + |S_2| + \cdots + |S_k|$

Examples

4. Suppose that either a CS faculty associate or CS faculty member is to be chosen as a representative to a university committee. How many different choices are there for this representative given that there are 9 faculty associates and 40 faculty members in the CS department?

   No fauc associate is a faculty number and vice versa
   disjoint. By sum rule, there are 40 + 9 = 49 choices.

5. In a version of BASIC, the name of a variable is a string of one or two alphanumeric characters, where uppercase and lowercase letters are not distinguished. A variable name must begin with a letter and must be different from the five strings of two characters that are reserved for programming use. How many different variable names are there in this version of BASIC?

   V_1: 1 character var V_2: 2 character var disjoint
   num of vars $V_1 + V_2 = 26 + (26^2 - \underset{\text{reserved}}{5}) = 957$.

6. Suppose a computer system requires users to create a password with the following requirements: a password must be 6 to 8 characters long, can only contain lowercase letters and digits, and must contain at least one digit. How many unique passwords can be created?

   P = total # of password that meet the criteria
   P_i : # of i-character password.
   $P = P_6 + P_7 + P_8$
   $= (36^6 - 26^6) + (36^7 - 26^7) + (36^8 - 26^8)$

Basic Counting (continued)

Inclusion-exclusion (subtraction rule)

If a task can be done either n_1 ways or n_2 ways, then the total number of ways to do the task is

$n_1 + n_2$ - (# ways to do the task that are common to the 2 different ways).

inclusion-exclusion and sets

$$|S_1 \cup S_2| = |S_1| + |S_2| - |S_1 S_2|$$

Example : How many bit strings of length 8 either start with a 1 bit (a) or end with the two bits 00 (b)?

$$\text{Total } \# = \#a + \#b - \#a \cap b$$
$$= 2^7 + 2^6 - 2^5 = 128 + 64 - 32 = 160$$

Division rule

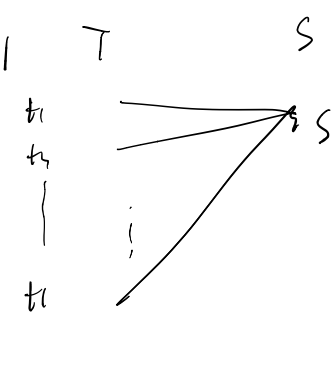
If a task can be done using a procedure that can be carried out in n ways, and for every way w, exactly k of the n ways correspond to w, then there are n/k ways to do the task.

division rule and total functions

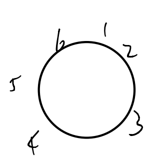
If $f : T \to S$ is total function that is onto and k to 1

k-to-1 means for each $s \in S$

$\exists$ exactly k elts $t_1, t_2 \cdots t_k \in T$ such that $f(t_i) = s$



Example : How many ways are there to seat 6 people around a circular table, where two seatings are considered the same if each person has the same left neighbor and right neighbor?



$$\frac{6!}{5} = 120$$