Binomial Theorem

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Let *x* and *y* be variables and let *n* be a non-negative integer. Then

$$\begin{aligned}(x+y)^n &= (x+y)\cdots(x+y)\\ &= \binom{n}{n}x^n + \binom{n}{n-1}x^{n-1}y + \cdots\cdots \binom{n}{0}x^0y^n\\ &= \sum_{k=0}^{n}\binom{n}{k}x^k y^{n-k}\end{aligned}$$

Examples

Example 1: What is the coefficient of x^7y^3 in the expansion of $(x + y)^{10}$?

$\binom{10}{7} = \frac{10!}{3!\,7!} = 120 = \binom{10}{3}$ also coef for x^3y^7

Example 2: Show $\Sigma^n{}_{k=0}C(n, k) = 2^n$ $\quad \sum_{k=0}^{n}\binom{n}{k} = 2^n$

Using the binomial theorem:

$$2^n = (1+1)^n = \sum_{k=0}^{n}\binom{n}{k}1^k 1^{n-k} = \sum_{k=0}^{n}\binom{n}{k}$$

Using a *combinatorial proof*: show 2 different ways of counting something

Suppose S is a set of n element.

\# of subset of S $= |P(S)| = 2^{|S|} = 2^n$

# of subsets with	is
0 elts	$1 = \binom{n}{0}$
1	$n = \binom{n}{1}$
2	$\binom{n}{2}$
$\vdots$	
k	$\binom{n}{k}$
$\vdots$	$\vdots$
n	$\binom{n}{n} = 1$

Total # of subsets of

$$= \binom{n}{0} + \binom{n}{1} + \cdots\cdots \binom{n}{n} = \sum_{k=0}^{n}\binom{n}{k}$$

Count the subset with 2 different technique, 2 value must be equal.

i.e. $\sum_{k=0}^{n}\binom{n}{k} = 2^n$

Pigeonhole Principle

Theorem

Let S and T be sets such that $|T| > |S|$. Then for every total function $f : T \to S$, there are at least two elements in T that map to the same element of S under f.

Or, put another way: Let k ($\to |S|$) be a positive integer. If $k + 1$ (or more) objects are placed into k boxes, then there is at least one box containing 2 or more objects.

Examples

Example 1: Show that at least two people in this lecture got the same score on the first exam.

- 60 points possible on the exam, no one get a score < 10, half points allowed, so there are 101 different possibilities.
- There are 187 people in Lec 1

Since there are more people than exam scores possible, by pigeonhole principle, at least 2 people get the same score.

Example 2: Show that for every positive integer n, there is a multiple of n that has only 0s and 1s in its decimal expansion (i.e., when expressed in base 10).

e.g. 11,111,100 is evenly divisible by 7.

Let $n \in \mathbb{Z}^+$

x	$x \bmod n = k$
1	1
11	⋮
111	
1111	
⋮	
$\underbrace{11\cdots1}_{n+1\ 1\text{'s}}$	

$0 \le k \le n-1$

n possible values.

$\sim n+1$ values

By p.p., $x \bmod n$ must be same for at least 2 different values of x. ← A, B where $A < B$

$A = a \cdot n + k$

$B = b \cdot n + k$

$\to B - A = (b-a) \cdot n$

So $B-A$ is evenly divisible by n, i.e. it is a multiple of n.

$$\begin{array}{rl} 111\cdots1 & B \\ -\ \ 11\cdots1 & A \\ \hline 1100\cdots0 & B-A \end{array}$$

$B-A$ has only 0s and 1s in it.