L2. 01.24.2020

§2 Naturals, Rationals, Reals

Q. what is $\mathbb{R}$?

We define $\mathbb{N}$ as $\mathbb{N} = \{1, 2, 3, \ldots\}$

Note, $\mathbb{N}$ does not satisfies any of A1)-A4), M1)-M4),

Principle of Induction

Let $P(x)$ mean property P holds for number x, if

i) $P(1)$ is true

ii) if $P(k)$ is true, then $P(k+1)$ is true.

then $P(x)$ is true $\forall x \in \mathbb{N}$.

Prop 2.1

$\forall n \in \mathbb{N}$, $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

Proof by induction.

Base case. if $n=1$, $\displaystyle\sum_{k=1}^{n} k = 1 = \frac{1 \cdot 2}{2}$, so $\displaystyle\sum_{k=1}^{n} k = \frac{1 \cdot (1+1)}{2}$

Let $n \in \mathbb{N}$ be s.t. $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

We will show this implies $\displaystyle\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$

Note:
$$\sum_{k=1}^{n+1} k = \sum_{k=1}^{n} k + n+1$$
$$= \frac{n(n+1)}{2} + n+1$$
$$= \frac{n^2+n+2n+2}{2}$$
$$= \frac{(n+1)(n+2)}{2}$$

Thus we are done by induction ■

In set notation

Po I.

Let $A \subseteq \mathbb{N}$. If

i) $1 \in A$

ii) if $k \in A$ implies $k+1 \in A$

then $A = \mathbb{N}$.

So prop 2.2 says Let $A = \{ n \in \mathbb{N} \mid \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \}$.

Then $A = \mathbb{N}$.

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In set notation: Principle of Induction

Let $A \subseteq \mathbb{N}$, if i) $1 \in A$

ii) if $k \in A$ implies $k+1 \in A$, then $A = \mathbb{N}$

So prop 2.1 says let $A = \{ n \in \mathbb{N} \mid \sum_{k+1}^{n} k = \frac{n(n+1)}{2} \}$, then $A = \mathbb{N}$

Well-ordering Principle (empty set)

Let $A \subseteq \mathbb{N}$ s.t. $A \neq \emptyset$. Then A has a least element

i.e. $\exists k \in A$ s.t. if $l \in A$, then $l \geq k$

Theorem 2.2 POI is equivalent to WOP

Proof:

i) We prove PoI $\Rightarrow$ WOP

Let $A \subseteq \mathbb{N}$ s.t. $A \neq \emptyset$. Suppose for a contradiction that A has no least element.

Note in particular $1 \notin A$.

Define $B = \{k \in \mathbb{N} \mid 1, 2, \cdots, k \notin A\}$.

We know $1 \in B$.

We suppose for induction that $k \in B$, so $1, \cdots k \notin A$.

If $k+1 \in A$, then as $1, \cdots k \notin A$, $k+1$ is the least element of A.

But A has no least element, so $k+1 \notin A$. Hence $k+1 \in B$.

So by induction. $B = \mathbb{N}$. Thus $A = \emptyset$

※ to $A \neq \emptyset$.
contradiction

ii) WOP $\Rightarrow$ PoI is on HW1.

Principle of complete induction

Let $A \subseteq \mathbb{N}$ s.t. 1) $1 \in A$

2) if $1, \cdots, k \in A$, then $k+1 \in A$, then $A = \mathbb{N}$.

Thm 2.3. PoI implies PoCI.

Ex: Hint: if $1 \in A$ and $1, \cdots, k \in A \Rightarrow k+1 \in A$.

consider $B = \{k \in \mathbb{N} \mid 1, \cdots, k \in A\}$.

[PoI to B or POCI to A]

Recursive Definition

We can also make definitions inductively

e.g. defining the n^{th} power of a by

i) $a^1 = a$

ii) $\forall n \in \mathbb{N}$ s.t. $n > 1$, $a^n = a \cdot a^{n-1}$

Integers.

Recall $\mathbb{N}$ does not contain additive identity or inverses.

We define integers to be $\mathbb{Z} = \cdots, -2, -1, 0, 1, 2, \cdots$

$= \mathbb{N} \cup \{0\} \cup \{-n \mid n \in \mathbb{N}\}$.

Integers are either odd or even.

Every integer m may either be written $m = 2k$ or $m = 2k+1$, $k \in \mathbb{Z}$.

Proof by induction (Ex).

Rationals

To obtain multiplicative axioms, we use rationals $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z}, n \neq 0\}$.

$\mathbb{Q}$ satisfies A1) - A4), M1) - M4), D. P1) - P3)

Recall: $\nexists p \in \mathbb{Q}$ s.t. $p^2 = 2$. so $\sqrt{2} \notin \mathbb{Q}$. so $\mathbb{Q} \neq \mathbb{R}$.