Predicates and Predicate Logic

Motivating example: Consider

If $(x + y \geq 300)$ then $(x \geq 150$ or $y \geq 150)$

De Morgen Law.

Is this true or false?

$P: x+y \geq 300$
$Q: x \geq 150$
$R: y \geq 150$

technically not propositions.

$P \Rightarrow (Q \vee R) \equiv (\neg Q \wedge \neg R) \Rightarrow \neg P$ counterpositive.

$\neg P: x+y < 30$
$\neg Q: x < 150$
$\neg R: y < 150$

if $x<150$ and $y<150$, then $x+y<300$

Predicates

universe of discourse

predicate : a mapping from some some underlying domain D to propositions

propositional function.

parametrized proposition

Examples:

Domain is numbers

- $P(x) : x^2 \geq x$
- $Q(x) : x = x + 1$

Domain is strings (words)

- $S(x)$: x ends in the letter 'y'
- $T(x)$: x has at least 9 letters

Domain is integers.

- Even(x) : x is even
- Costars(a, b, m) : a and b both appeared in movie m

domain of a, b is actors
- - - m is movies

$P(2)$ is proposition $2^2 \geq 2$ True
$Q(3)$ is proposition $3=4$ False
$S(\text{hello})$ false

Creating propositions from predicates:

1. subbing in values from domain(s) for parameter(s)/variable(s)
2. using quantification (operators)

Quantification

Universal quantification

universal qualifier $\forall$

$(\forall x) P(x)$ → For all x, $P(x)$ holds. i.e. is true.

- *true* if $P(x)$ is true for all values of x in the domain.
- *false* if $P(x)$ is false for some values of x in the domain
  counter example

If domain is unclear, specify it: $\forall x \in D$
Common numeric domains:

$\mathbb{Z}$ integers $\mathbb{N}$ natural numbers
$\mathbb{R}$ real numbers $\mathbb{Q}$ rational numbers

Examples

Suppose we have the domain Days = {Monday, Tuesday, Wednesday, Thursday, Friday, Saturday, Sunday}

($\forall$ x ∊ Days) S(x) end in "y". $\Leftarrow$ Everyday of the week end in y

$\hookrightarrow \equiv S(\text{Monday}) \wedge \cdots \wedge S(\text{Sunday})$

($\forall$ x) P(x) $(\forall x)\ x^2 \geq x$

$(\forall x \in \mathbb{R})\ x^2 \geq x \Rightarrow$ false
because we have a counter-example
$(0.1)^2 \ngeq 0.1 \Rightarrow$ false

$(\forall x \in \mathbb{Z})\ x^2 \geq x \Rightarrow$ true (but we need to prove).

Existential quantification

existential quantifier $\exists$

($\exists$ x) P(x) There exisit an x for which P(x) is true (holds).

- *true* if we can find an x where P(x) is true
- *false* if P(x) is false for every x in the domain

Examples

recall predicates:
$Q(x) = x = x+1$ $T(x)$: x has at least 9 letters

($\exists$ x ∊ Days) T(x) : At least one day of the week has 9 letters
True. becaus Tuednesday

($\exists$ x) Q(x)
$(\exists x \in \mathbb{R})(x = x+1)$
False. because: let a be an arbitrary real number
Suppose $a = a+1$
$a - a = a + 1 - a$ substract a from both side
$\underbrace{0 = 1}_{\text{contradiction}}$ simplify

Relationship between universal and existential quantification

$\forall$ $\exists$ $D = \{x_0, x_1 \cdots, x_n\}$ finite domain

¬ ($\forall$ x) P(x) ≡ $(\exists x) \neg P(x)$

¬ ($\exists$ x) P(x) ≡ $(\forall x) \neg P(x)$

Demorgen's Law

$(\forall x) P(x) \equiv \neg(P(x_0) \wedge P(x_1) \cdots \wedge P(x_n)) \equiv \neg P(x_0) \vee P(x_1) \cdots P(x_n)$
$\equiv (\exists x) \neg P(x)$

$\neg(\exists x) Q(x) \equiv \neg(P(x_0) \vee P(x_1) \vee \cdots \vee P(x_n)) \equiv \neg P(x_0) \wedge \neg P(x_1) \cdots \wedge \neg P(x_n) = (\forall x) \neg P(x).$

Predicates and Predicate Logic (cont.)

Translate sentences to logical expressions

Example 1: If (x + y >= 300) then (x >= 150 or y >= 150)

$A(x,y): x+y \geq 300$

$B(z): z \geq 150$

$\underbrace{(\forall x \in \mathbb{R})(\forall y \in \mathbb{R})}_{\text{same qualifier and domain}}\ [A(x,y) \Rightarrow (B(x) \vee B(y))]$

$\forall x, y \in \mathbb{R}$.

For the next two examples, the universe of discourse (domain) is all students. UW
We'll define the following predicates:

- S240(x) : x is a student in CS 240
- Calc(x) : x has studied calculus
- CS(x) : x is a CS major

Example 2: Every student in this class has studied calculus.

For every student in this class, that student has studied calculus

For every student at UW-Madison, if they are in this class then they has studied calculus

$(\forall x \in UW)(CS240(x) \Rightarrow Calc(x))$

- why not $(\forall x \in UW)(CS240(x) \wedge Calc(x))$
  → not every one is in this class, has studied calculus.
- why not $(\forall x \in CS240(x))\ Calc(x)$
  used with sets → predicate → T/F value

Example 3: Some students in this class are not CS majors.

There exists a student in this class who is not a CS major.

There is a student at UW who is in CS 240 and is not in CS major.

$(\exists x \in UW)(CS240(x) \wedge \neg CS(x))$

why not: $(\exists x \in UW)(CS240(x) \Rightarrow \neg CS(x))$

recall: $P \Rightarrow Q \equiv \neg P \vee Q$

$\equiv (\exists x \in UW)(\neg CS240(x) \vee \neg CS(x))$

There is a student at UW who is not in CS240 or not in CS major (or both)

Multiple quantifiers and domains

Example: The *Bacon number* of an actor is the number of degrees of separation they have from Kevin Bacon. Kevin Bacon has a Bacon number of 0. Actors who have co-starred with Kevin Bacon have a Bacon number of 1. Actors who have co-starred with actors who have co-starred with Kevin Bacon have a Bacon number of 2, and so on.

Use propositional logic to define what it means to have a Bacon number of 2. Use the following domains and predicates:

- Actors = domain of all actors
- Movies = domain of all movies
- Bacon2(x) : x has a Bacon number of 2
- Costars(a, b, m) : a and b both appeared (co-starred) in movie m $\quad a \xrightarrow[m]{} b$

Kevin Bacon $\xrightarrow[\text{(2010)}]{\text{Sup}}$ Michael Rooker $\xrightarrow[\text{(1987)}]{\text{Light of Day}}$ Joan Jett $\xrightarrow[\text{(2001)}]{\text{By Hook or by Crook}}$ Silas Howard.

Bacon2(x): $KB \xrightarrow[m_1]{} y \xrightarrow[m_2]{} x$. and not $KB \xrightarrow[m_3]{} x$

$$\text{Bacon2}(x) \equiv (\exists y)\big((\exists m)\,\text{Costars}(KB, y, m)\big) \wedge \big((\exists m)\,\text{Costars}(y, x, m)\big)$$
$$\wedge \neg (\exists m)\,\text{Costars}(KB, x, m)$$

or $(\forall m)\,\neg\,\text{Costars}(KB, x, m)$ This means $x \neq y$ and $x \neq KB$.

$(\exists m_1, m_2)$ or $(\exists m_1, m_2 \in \text{movies})$

$$\equiv (\exists y)(\exists m_1)(\exists m_2)\big(\text{Costars}(KB, y, m_1) \wedge \text{Costars}(x, y, m_2)\big)$$
$$\wedge \neg (\exists m)\,\text{Costars}(KB, x, m).$$

Translate to predicate logic: Every integer has an additive inverse.

$$(\forall x \in \mathbb{Z})(\exists y \in \mathbb{Z})(x + y = 0)$$

Consider: $(\exists y \in \mathbb{Z})(\forall x \in \mathbb{Z})(x + y = 0)$

i.e. There exists an integer y such that $x + y = 0$ for every possible integer x (i.e. no matter what x is).

Take away: $\forall x \exists y \neq \exists y \forall x$

can't change order of mixed quantifiers.