L3 01.27.2020

§3 Functions

Ex i) the Fxn that assigns to each real # its square.

ii) - - - - - - take each real x to $\frac{x^3-7}{x^2+1}$

iii) - - - - - - Tokyo to London, London to

$\mapsto$ maps to

f: {Tokyo, London} $\mapsto$ {London, }.

i) $x \mapsto x^2$ for $x \in \mathbb{R}$.

ii) $x \mapsto \frac{x^3-7}{x^2+1}$ for $x \in \mathbb{R}$

iii) Tokyo $\mapsto$ London, London $\mapsto$.

write fxns w/ f, g, h.

For each x s.t. $f(x)$ makes sense, we call $f(x)$ the value of f at x.

We call f the fxn.

Given 2 sets A and B, a fxn $f: A \mapsto B$ has domain A and values in B.

$A = \{x \mid f(x) \text{ exists}\} = \mathrm{dom}(f)$.

We can compose fxns f and g provided $\{f(x) \mid x \in \mathrm{dom}(g)\} \subseteq \mathrm{dom}(g)$.

define $f \circ g$ by $(f \circ g)(x) = g(f(x))$ for $x \in \mathrm{dom}(f)$.

L4 01.29

A Function from A to B is a rule that assigns to each $a \in A$ a unique $b \in B$.

We write $b = f(a)$ and call this a value of f

$A = \mathrm{dom}(f)$ is the domain of f

The range of f is $\{f(a) \mid a \in A\}$.

Ex.

i) Polynomials : $p: \mathbb{R} \to \mathbb{R}$, $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$

where $a_j \in \mathbb{R}$, $j = 1, 2, \cdots, n$

if $a_n \neq 0$, we say the degree of P is n

ii) rational functions : $r = \frac{p}{q}$ where p and q are polynomials

$\mathrm{dom}(r) = \{x \in \mathbb{R} \mid q(x) \neq 0\}$.

Sums of functions

Let $f: A \to \mathbb{R}$, $g: B \to \mathbb{R}$ where A, B are sets. s.t. $A \cap B \neq \phi$

Then $(f+g) = A \cap B \to \mathbb{R}$ given by $(f+g)(x) = f(x) + g(x)$ $\forall x \in A \cap B$.

So, $\mathrm{dom}(f+g) = \mathrm{dom}(f) \cap \mathrm{dom}(g)$.

Product

$(fg)(x) = f(x)\, g(x)$ for $x \in \mathrm{dom}(f) \cap \mathrm{dom}(g)$

where f and g are real-valued.

Quotient : $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

$\operatorname{dom}(\frac{f}{g}) = (\operatorname{dom}(f) \cap \operatorname{dom}(g)) \setminus \{x \mid g(x) = 0\}$.

Real-valued functions satisfy anthretic axiam properties

Ex. Commutavity.

Let $f, g : A \to \mathbb{R}$. Then $f+g = g+f$.

Pf: we need to show $\forall x \in A$

$(f+g)(x) = (g+f)(x)$

Let $x \in A$

Then $(f+g)(x) = f(x) + g(x)$

$= g(x) + f(x)$ by A2 in $\mathbb{R}$.

$= (g+f)(x)$. $\blacksquare$

Ex. Additive identity function.

$0 : A \to \mathbb{R}$. defined by $0(x) = 0 \quad \forall x \in A$.

Composition.

$(f \circ g)(x) = f(g(x))$

$\operatorname{range}(g) \subset \operatorname{dom}(f)$

Associative: s.t. $\operatorname{range}(h) \subset \operatorname{dom}(g)$, $\operatorname{range}(g) \subset \operatorname{dom}(f)$

Then Let $x \in \operatorname{dom}(h)$.

Then $(f \circ (g \circ h))(x) = f(g \circ h) = f(g(h(x)))$

$= (f \circ g)(h(x)) = ((f \circ g) \circ h)(x)$.

So $(f \circ (g \circ h))(x) = ((f \circ g) \circ h)(x)$.

Composition is not commutative.

E.x. $f, g : \mathbb{R} \to \mathbb{R}$ s.t. $f \circ g \neq g \circ f$.

Let $g(x) = x^2 \quad \forall x \in \mathbb{R}$.

$f(x) = x^2 + 1 \quad \forall x \in \mathbb{R}$.

Then $(f \circ g)(x) = f(g(x)) = f(x^2) = x^4 + 1. \quad \forall x \in \mathbb{R}$,

$(g \circ f)(x) = g(f(x)) = (x^2+1)^2 = x^4 + 2x^2 + 1, \quad \forall x \in \mathbb{R}$.

So. if $x \neq 0$, $f \circ g \neq g \circ f$

So, $f \circ g \neq g \circ f$.

Def (not examinable)

Let A, B be non empty sets.

A function from A to B is a set of ordered pairs $\{(a,b) \mid a \in A, b \in B\}$ s.t.

if (a,b) and (a,c) are in the function, then $b = c$.

Domain is $\{a \in A \mid \exists b \in B \text{ s.t. } (a,b) \in f\}$.

Given $a \in \mathrm{dom}(f)$ the associated b s.t. $(a,b) \in f$. we write $b = f(a)$.

Def

An ordered pair (a,b) is the set $\{\{a\}, \{a,b\}\}$.

Note if $b = a$, $(a,b) = \{\{a\}\}$.

See appendix to §3 of Spivak for details