L23 12/3

Sum of 2 variables (5.4)

If X and Y are continuous r.v.s with joint pdf $f(x,y)$, then $X+Y$ has the pdf. $f_{X+Y}(z) = \int_{-\infty}^{\infty} f(x, z-x)\,dx$.

(not required
proof: use distribution function technique.

step 1. find cdf of $X+Y$.

$$F_{X+Y}(z) = P(X+Y \le z) = \iint_{x+y\le z} f(x,y)\,dx\,dy.$$
$$= \int_{-\infty}^{\infty} \int_{y=-\infty}^{y=-x+z} f(x,y)\,dy\,dx.$$
$$= \int_{-\infty}^{\infty} \int_{-\infty}^{z} f(x, t-x)\,dt\,dx$$
$$= \int_{-\infty}^{z} \int_{-\infty}^{\infty} f(x, t-x)\,dx\,dt$$

step 2: $f_{X+Y}(z) = F_{X+Y}'(z) = \int_{-\infty}^{\infty} f(x, z-x)\,dx$

If X, Y are independent, then the pdf of $X+Y$ is

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x)$$

$$f_{X+Y}(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\,dx.$$

convolution of f_X and f_Y, denoted by $f_X * f_Y$.

E.g. if X and Y are independent Uniform $(0,1)$ random variables.
Find the pdf of $X+Y$.

Sol: $f_{X+Y} = f_X * f_Y$. $\quad f_X(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{o/w} \end{cases} \doteq f_Y(y)$.

$$= \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z-x)\,dx$$
$$= \int_{\max(0, z-1)}^{\min(1, z)} 1\,dx.$$

$0<x<1$
$0<z-x<1 \Rightarrow z-1<x<z$

$0<z<2$ If $z<1$ $\quad = \int_0^z 1\,dx = z.$

if $z>1$ $= \int_{z-1}^{1} 1\,dx = 2-z$

$$f_{X+Y}(z) = \begin{cases} z & 0<z\le 1 \\ 2-z & 1<z<2 \\ 0 & o/w \end{cases}$$

3.3 Limit Theorem

Markov's inequality.

Suppose X is a random variable that takes only non-negative values.

Then, for any $a>0$ $\quad P(X\geq a) \leq \frac{E(X)}{a}$

Proof: $E(X) = \int_0^\infty x f(x)\,dx \geq \int_a^\infty x f(x)\,dx \geq \int_a^\infty a f(x)\,dx = aP(X\geq a)$

$E(X) \geq aP(X\geq a) \quad P(X\geq a) \leq \frac{E(X)}{a}.$

Chebyshev's inequality

Suppose X is a random variable with finite mean μ and variance σ^2,

then for any $k>0$, $\quad P(|X-\mu|>k) \leq \frac{\sigma^2}{k^2}$

Weak Law of Large Numbers

Let $X_1, X_2 \ldots$ be a sequence of iid r.v.s, each with a finite mean $E(X_i)=\mu$.

Then, for any $\varepsilon>0$

$$P\left(\left|\frac{X_1+\cdots X_n}{n} - \mu\right| \geq \varepsilon\right) \to 0 \text{; as } n\to\infty.$$

* limits of probabilities $\to 0$ $\quad \frac{X_1+\cdots X_n}{n} \to \mu$, in probability as $n\to\infty$.

Strong law of large number

Let $x_1, x_2 \cdots$ be a sequence of iid r.v.s, each with a finite mean $E(x_i)=\mu$.

Then $P(\lim_{n\to\infty} \frac{x_1+\cdots x_n}{n} = \mu) = 1$

* prob involving limits. $\frac{x_1+\cdots x_n}{n} \to \mu$ with probabilities 1.

Central limit Theorem

Let $x_1, x_2 \cdots$ be a sequence of iid r.v.s. each with mean μ, variance σ^2.

then the distribution of $\frac{x_1+x_2+\cdots x_n - n\mu}{\sqrt{n}\,\sigma}$

tends to the standard normal as $n\to\infty$.

This is, for all z,

$$P\left(\frac{x_1+x_2+\cdots+x_n - n\mu}{\sqrt{n}\,\sigma} \le z\right) \to \underset{\substack{\uparrow \\ \text{cdf of } N(0,1).}}{\Phi(z)}, \text{ as } n\to\infty$$

Remark. $$\frac{\overset{\downarrow \text{ avg}}{\frac{x_1+\cdots+x_n}{n}} - \mu}{\sigma/\sqrt{n}} \to N(0,1)$$

E.g. Suppose $x_1, x_2 \cdots$ iid $\exp(\frac{1}{2})$, then

$$\frac{x_1+x_2+\cdots x_n - 2n}{\sqrt{n}\cdot 2} \to N(0,1) \text{ as } n\to\infty.$$

Find the probability approximately the avg of $x_1, \cdots x_n$ is less than 2.

$x_i \sim \exp(\frac{1}{2})$

$$\frac{\frac{x_1+\cdots x_{100}}{100} - 2}{2/\sqrt{100}} \qquad \frac{\overline{X_{100}} - 2}{1/5} \to N(0,1)$$

$$P(\overline{X_{100}} < 2) = P\left(\frac{\overline{100} - 2}{1/5} < \frac{2-2}{1/5}\right)$$

$$= P\left(\frac{\overline{X_{100}} - 2}{1/5} < 0\right) \approx \Phi(0) = 0.5.$$