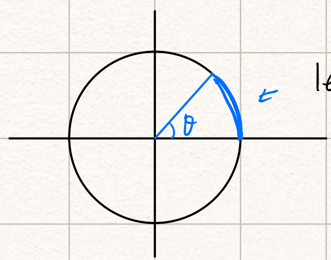
L38 04.27

§15. triangometric functions

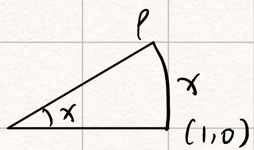
non-examble



length = θ in radians

How is length defined?

Easier to work with area under a graph.



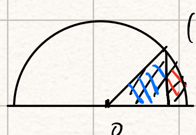
area = $\dfrac{x}{\text{total circumtance}} \cdot$ total area $= \dfrac{x}{2\pi} \cdot \pi = \dfrac{x}{2}$

So $\cos x$, $\sin x$ should determine area of a segment as $\frac{x}{2}$

Def 15.1 $\pi = 2\int_{-1}^{1} \sqrt{1-x^2}\,dx$

$\int_{-1}^{1} f = \frac{\pi}{2}$

Area of secta:

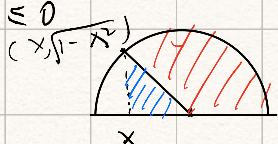


$(x, \sqrt{1-x^2})$ $\quad x \in [0,1]$

For $0 \le x \le 1$ area in /// = area of \\\\\\\\ + area of ////

$= \frac{1}{2} x\sqrt{1-x^2} + \int_x^1 \sqrt{1-t^2}\,dt$

For $-1 \le x \le 0$



Area $= \int_x^1 \sqrt{1-t^2}\,dt - \frac{1}{2}|x|\sqrt{1-x^2}$

$= \int_x^1 \sqrt{1-t^2}\,dt + \frac{1}{2} x\sqrt{1-x^2}$

We define $A(x) = \dfrac{x\sqrt{1-x^2}}{2} + \int_x^1 \sqrt{1-t^2}\,dt \quad \forall x \in [-1,1]$

By FTC, A is diff'ble on $(-1,1)$

$A'(x) = \frac{1}{2}\left(\sqrt{1-x^2} - \dfrac{x^2}{\sqrt{1-x^2}}\right) - \sqrt{1-x^2} = \dfrac{-1}{2\sqrt{1-x^2}} < 0$

So, A is strictly decreasing from $A(-1) = 0 + \int_{-1}^{1} \sqrt{1-t^2}\, dt = \frac{\pi}{2}$ by def

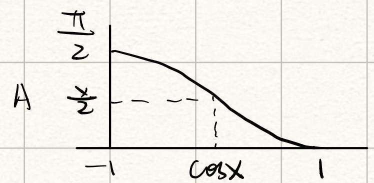
to $A(1) = 0$

We define cos

Def 15.2: For $0 \le x \le \pi$, we define $\cos x$ as the unique number in $[-1,1]$

s.t. $A(\cos x) = \frac{x}{2}$

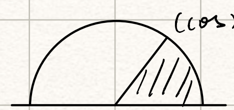
We define $\sin x = \sqrt{1 - (\cos x)^2}$



By IVT, $\cos x$ exists.

$\cos x$ is unique as A is strictly decreasing

Recall:



Area(////) $= A(y) = \frac{y}{2}$ $\qquad y = \cos x$

With area function A, $\cos x$ satisfies $A(\cos x) = \frac{x}{2}$

So if $B = 2A$ $\quad B(\cos x) = x$, i.e. $B = \cos^{-1}$

Thm 15.3 If $0 < x < \pi$, then $\cos'(x) = -\sin x \quad \sin'(x) = \cos x$

Proof: As $A'(y) = -\frac{1}{2\sqrt{1-y^2}}$, so $B'(y) = -\frac{1}{\sqrt{1-y^2}}$

As $\cos = B^{-1}$

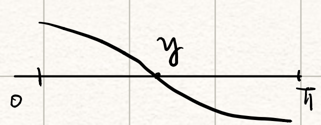
$$\cos'(x) = (B^{-1})'(x) = \frac{1}{B'(B^{-1}(x))} = -\sqrt{1 - (B^{-1}(x))^2}$$

$$= -\sqrt{1 - (\cos x)^2} = -\sin x$$

As $\sin x = \sqrt{1 - \cos^2 x}$ $\quad \sin'(x) = \frac{1}{2\sqrt{1-\cos^2 x}} \cdot (-2\cos x \cos' x)$

$$= \frac{1}{\sin x} \cos x \sin x = \cos x$$

sketch: $\cos x$



What is y? $A(\cos x) = \frac{x}{2}$ $\forall x$

So $A(0) = \frac{y}{2}$

(by def 15.1)

So $y = 2A(0) = 2\int_0^1 \sqrt{1-t^2}\, dt = \int_{-1}^1 \sqrt{1-t^2}\, dt = \frac{\pi}{2}$

So, $\cos(\frac{\pi}{2}) = 0$

Hence sin is increasing on $[0, \frac{\pi}{2}]$, decreasing on $[\frac{\pi}{2}, \pi]$

To extend to $\mathbb{R}$.

set, for $\pi \le x \le 2\pi$, $\sin x = -\sin(2\pi - x)$

$\cos x = \cos(2\pi - x)$

This defines sin, cos on $[0, 2\pi]$

For $x = 2k\pi + x'$, $k \in \mathbb{Z}$, $x' \in [0, 2\pi]$.

define $\sin(x) = \sin(x')$

$\cos(x) = \cos(x')$

Excersice: $\sin' = \cos$, $\cos' = -\sin$ on $\mathbb{R}$.

Def $\tan = \frac{\sin}{\cos}$, arcsin, arccos, sec etc and check their derivatives.