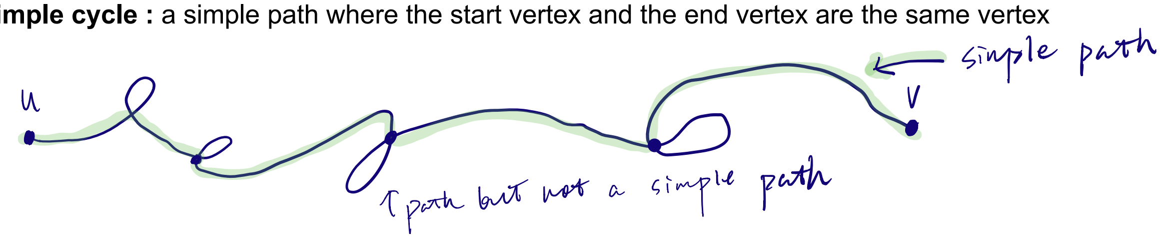
Trees

Simple paths and cycles

simple path : a path where every edge appears at most once and the only vertex that can be visited twice is the start vertext, but only if it is the end vertex

simple cycle : a simple path where the start vertex and the end vertex are the same vertex



Lemma : Let $G = (V, E)$ be a graph. If there is a path from vertex u to vertex v, then there is also a simple path from vertex u to vertex v.

Proof outline (see section 12.3 of Graphs reading for complete details) : Prove by strong induction on path length = #edges in path

P(*n*) : If ∃ a path from u to v of length n, then ∃ a simple path from u to v.

Base case : Show P(0) holds

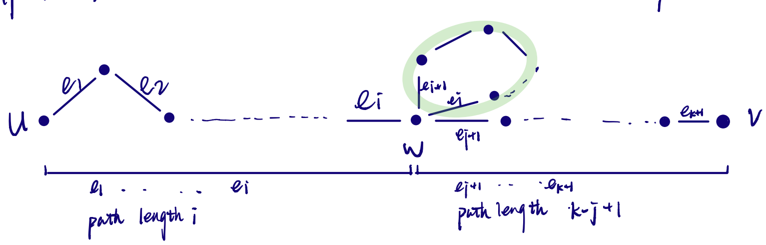
Path length 0 → just a single vertex (no edges)
The empty path from a vertex to itself is a simple path

Inductive step : Assume $\underbrace{P(0), P(1), ..., P(k)}_{IH}$ hold. Show P(k+1) holds.

Consider a path of length k+1 :

path length = k+1.

If this is a simple path, then well done
If not, then ∃ some vertex w that shows up twice.



e_1 . . . . e_{k+1} path length $i+k-j+1$

since $i<j$, $i+1 \leq j$, so $-(i+1) \geq -j$, path length $= i+k-j+1 \leq i+k-(i+1)+1 = k$

so ∃ a path of length ≤ k from u to v so IH applies

and ∃ a simple path from u to v.

Trees

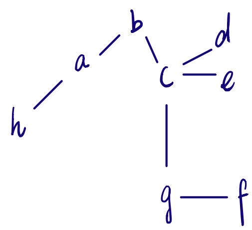
duplicate edges  is a cycle

tree : a connected graph without simple cycles

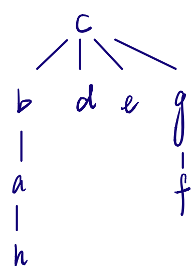
tree can't have duplicate edges
so edges in a tree are a set
tree is a simple graph

root : can be an arbitrary node

leaf : a vertex with degree 1



using c as root

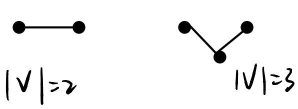


Tree properties

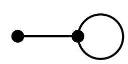
Let $T = (V, E)$ be a tree. Then

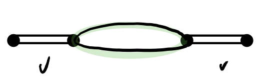
1. If $|V| \geq 2$, then T has at least 2 leaves.
2. There is exactly one simple path beteween any 2 vertices.
3. If V is non-empty, then $|V| = |E| + 1$

Intuition behind properties (formal proofs in on-line reading)

1) 

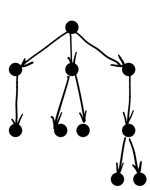
only way not have 2 vertices of degree 1
is to have a cycle.



2) 

if there are 2 distinct simple paths,
then we can find a cycle.

3) To understand, pick arbitrary vertex as root, "re-hang" tree,
& direct edges away from root.



Every vertex has exactly 1 edges going
into it except for the root.
So $|E| = |V| - 1$, $|V| = |E| + 1$