Solow Growth Model with Technological Progress

Long-Term Growth vs. Transition

- In simple Solow model economies converge to steady state in long run and growth rate of GDP per capita drops to zero.
- If countries were in steady state, they would not grow. But countries *do* grow!
- Are countries growing because they are still converging to their steady states? Or is there something that generates growth in the long run?
- Can data tell us if a country is in transition or on long-run growth path?

Long-Term Growth vs. Transition

MPK in Steady State

$$MPK = \frac{\partial Y}{\partial K} = \alpha A K_t^{\alpha-1} L_t^{1-\alpha} = \alpha A \left(\frac{K_t}{L_t}\right)^{1-\alpha} = \alpha A k_t^{1-\alpha}.$$

In steady state, $k_t = k^* \quad \forall t$.

This imply: MPK = constant in steady state.

$$MPK_t = r_t + \delta$$

Allow productivity changes. $A \to A_t$.

Solow Model with Technological Progress

- Production function with labor-augmenting technological progress:
$$Y = F(K, AL) = K^{\alpha}(AL)^{1-\alpha}$$
- Let $\tilde{k} = \frac{K}{AL}$. Then corresponding fundamental equation is:

$\dot{\tilde{k}}(t) = sf(\tilde{k}(t)) - (\delta + n + g)\tilde{k}(t)$

$\hookrightarrow$ growth rate of $A(t)$

steady state:

$0 = s\tilde{k}^{*\alpha} - (\delta + n + g)\tilde{k}^{\alpha}$.

$(\delta + n + g)\tilde{k}^{*} = s\tilde{k}^{*\alpha}$.

$\tilde{k}^{*1-\alpha} = \frac{s}{\delta + n + g}$.

$\tilde{k}^{*} = \left(\frac{s}{\delta + n + g}\right)^{\frac{1}{1-\alpha}}$ ← efficiency unit.

↑ adding people. divide k across people.

$k(t) = \tilde{k}(t) \cdot A$

$\ln(k) = \ln(\tilde{k}) + \ln(A)$.

In steady state.

$g_k = \frac{\dot{\tilde{k}}}{\tilde{k}}\Big|_{\tilde{k}=\tilde{k}^*} + g$.

$g_k = g$.

- With technological progress, the economy converges to a *balanced growth path* rather than a steady state.
- Along the balanced growth path:

$$g = \frac{\dot{k}(t)}{k(t)} =$$

$\frac{\dot{k}}{k} - g = s\tilde{k}^{\alpha-1} - (\delta + g + n)$.

$\frac{\dot{\tilde{k}}}{\tilde{k}} = s\tilde{k}^{\alpha-1} - (\delta + n + g)$.

$\frac{\dot{k}}{k} = s\left(\frac{k}{A}\right)^{\alpha-1} - (\delta + n)$

$= \frac{sf(\frac{k}{A})}{\frac{k}{A}} - (\delta + n)$.

Solow Model with Technological Progress

Derivation of the Fundamental Equation

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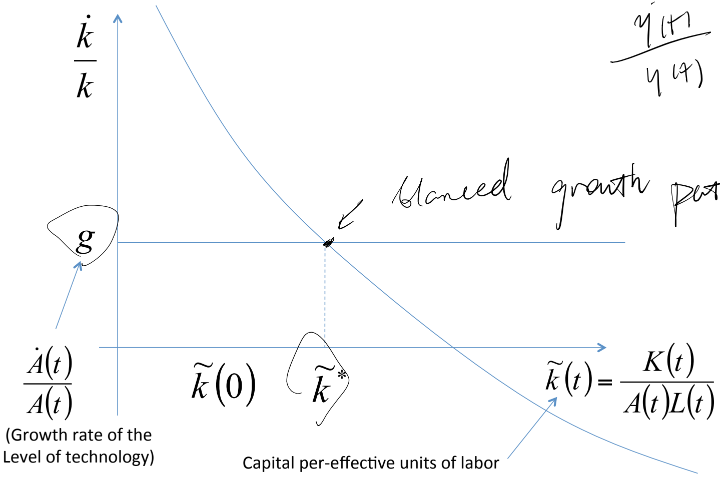
$y(t) = k(t)^{\alpha}$

$\frac{\dot{y}(t)}{y(t)} = \alpha \frac{\dot{k}(t)}{k(t)}$



balanced growth path × steady state.

Solow Model with Technological Progress

$$g = s\frac{f\left(\frac{k^*(t)}{A(t)}\right)}{\frac{k^*(t)}{A(t)}} - (\delta + n)$$

With a Cobb-Douglas production function

$$k^*(t) = \left(\frac{s}{n+\delta+g}\right)^{\frac{1}{1-\alpha}} A(t).$$

and output per-capita equals

$$\begin{aligned} y^*(t) &= k^*(t)^{\alpha} A(t)^{1-\alpha} \\ &= \left(\frac{s}{n+\delta+g}\right)^{\frac{\alpha}{1-\alpha}} A(t). \end{aligned}$$

Solow Model with Technological Progress

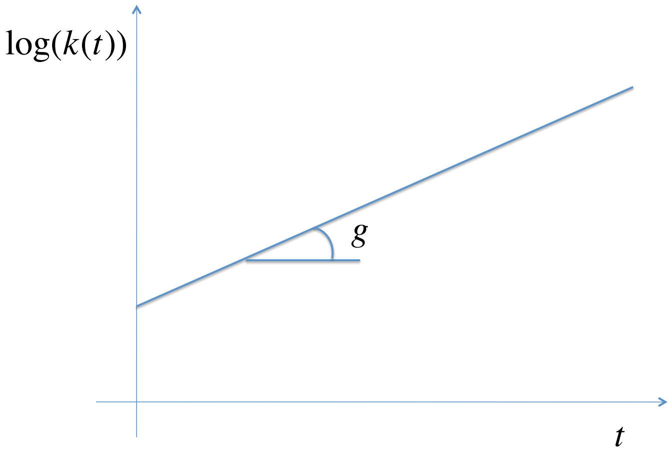
Taking logs and using $A(t) = e^{gt}$, $A(0) = 1$, we obtain

$$\log k^*(t) = \frac{1}{1-\alpha} \ln\left(\frac{s}{\delta+n+g}\right) + gt$$

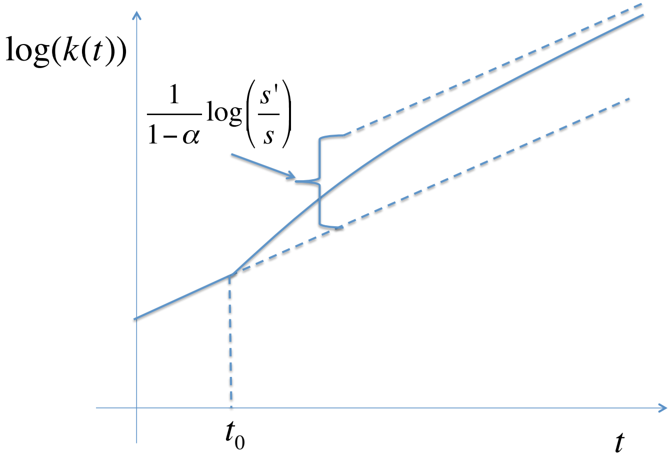
and

$$\log y^*(t) = \underbrace{\frac{\alpha}{1-\alpha} \ln\left(\frac{s}{\delta+n+g}\right)}_{\text{intercept.}} + \underbrace{g}_{\text{slope.}}t$$

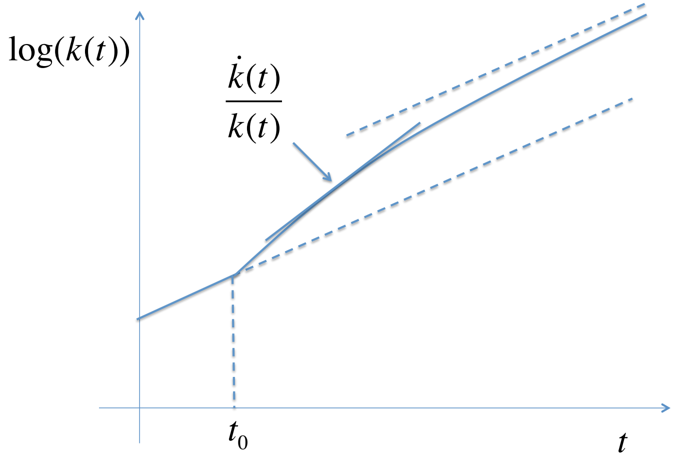
Solow Model with Technological Progress



Increase in the Saving Rate, $s' > s$ saving rate jump



Increase in the Saving Rate, $s' > s$



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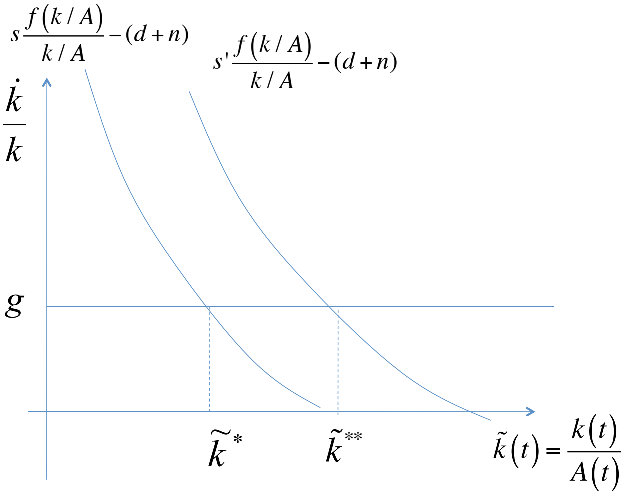

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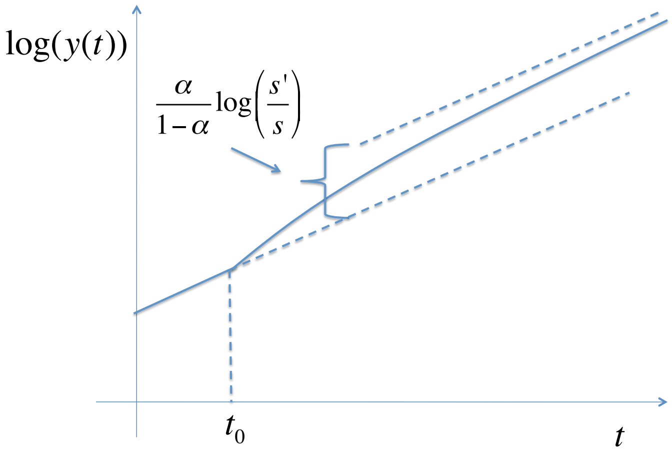


Failure of the Solow Model

Example: US vs. Kenya

$$s^{USA} = 0.20 \qquad n^{USA} = 0.01$$
$$s^{KEN} = 0.11 \qquad n^{KEN} = 0.03$$

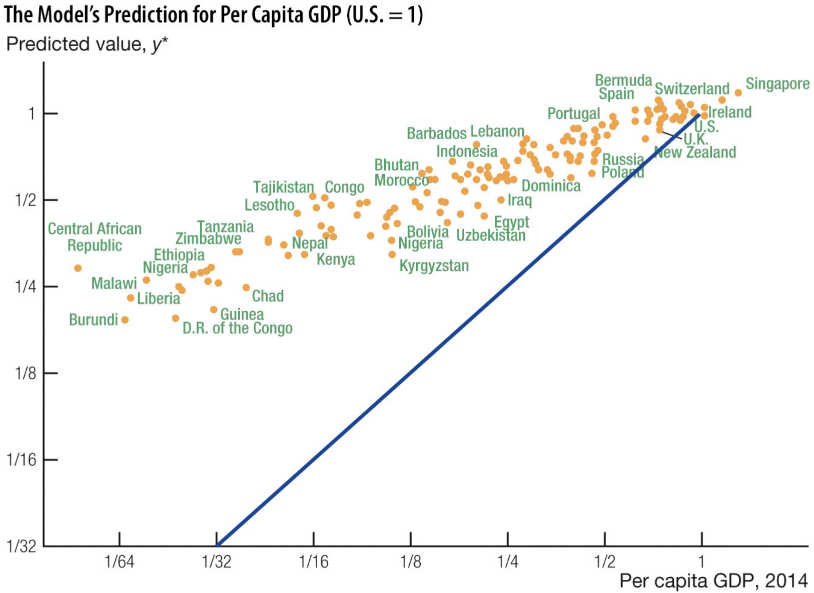
$$\left(\frac{y^{KEN}}{y^{USA}}\right)^{\text{predicted}} = \frac{A^{KEN}}{A^{USA}} \cdot \left(\frac{\frac{s^{KEN}}{s^{USA}}}{\frac{\delta+n^{KEN}+g}{\delta+n^{USA}+g}}\right)^{\frac{\alpha}{1-\alpha}}$$

$$= \frac{A^{KEN}}{A^{USA}} \cdot \left(\frac{\frac{0.11}{0.20}}{\frac{0.06+0.03+0.02}{0.06+0.01+0.02}}\right)^{\frac{1/3}{1-1/3}}$$

$$= \frac{A^{Ken}}{A^{USA}} \quad 2/3$$

diminishing marginal of return,

Failure of the Solow Model



Failure of the Solow Model

Mechanically, we can account for the differences in income by saying that the relative labor productivity is higher in the US:

$$\frac{A^{KEN}}{A^{USA}} = \frac{\left(\frac{y^{KEN}}{y^{USA}}\right)^{\text{actual}}}{\left(\frac{y^{KEN}}{y^{USA}}\right)^{\text{predicted}}} = \frac{0.046}{0.67} = 0.07.$$

How can we interpret these productivity differences?

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Human Capital

Basic idea:

- individuals spend time and resources (tuition) to go to school in order to accumulate knowledge
- by accumulating knowledge individuals get to be more productive, and therefore, earn higher wages
- in other words, individuals "save" by investing in education, which raises the productivity of their labor and therefore raises the market value of their labor

Human Capital

Think about education as an investment: you deposit yourself into a school, university,...and at the end of each year of additional education, you compound a return to your efforts:

$$\begin{aligned} h(1) &= (1+\psi)\, h(0) \\ h(2) &= (1+\psi)^2\, h(0) \\ &\dots \\ h(u) &= (1+\psi)^u\, h(0) \end{aligned}$$

where u denote the maximum schooling level attained (as in Becker's 1964 book *Human Capital*)

Human Capital

More generally, think about your life as a complex investment project:

$$\underbrace{I_6, I_7, ..., I_{23}}_{\cdots}, \underbrace{\Delta w_{24}, \Delta w_{25}, ..., \Delta w_{64}}_{\cdots}$$

Human Capital

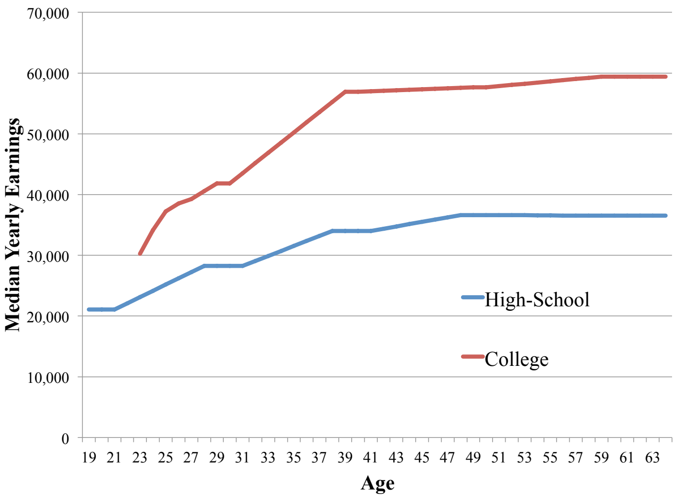
What is I_t?

- Tuition costs, materials,...
- But most importantly, foregone wages (when you are in college you give up the wage you could earn as a high school graduate or college drop-out).

What's Δw?

Wage differential compared to worker with *benchmark* education (typically high school diploma in the U.S.)

Earnings Profile: High-School vs. College Grads



Returns to Schooling?

How can we learn something about ψ (return to schooling) from available data?

$$\begin{aligned}\frac{\text{earning college graduate}}{\text{earning high-school graduate}} &= (1+\psi)^{16-12} \\ &= (1+\psi)^{4}\end{aligned}$$

so

$$1+\psi = \left(\frac{\text{earning college graduate}}{\text{earning high-school graduate}}\right)^{1/4}$$

Returns to Schooling?

- In our simple example,

$$\begin{aligned} E_u &= wh(0)\,e^{\psi u} \\ \ln E_u &= \ln w + \ln h(0) + \psi u \end{aligned}$$

- E_u denotes earnings after u years of schooling.
- This relationship is called a *Mincer Regression* since it was introduced by Mincer in 1974.
- In the US, the *ex tuition* return to schooling is $\psi \approx 10\%$.