Recurrences

Recurrence relation

Definition: Given a sequence of numbers a_0, a_1, a_2, a_3, ..., a **recurrence relation** is an equation that expresses a_n in terms of one or more previous terms in the sequence for all $n \geq n_0$ where n, $n_0 \in \mathbf{Z}^+$.

= inductive def of a sequence

= recursive def of a sequence

n_0 = initial value. (initial condition)

Example: compound interest

Suppose you deposit $5,000 at 3% interest (compounded annually). How much money will you have after 30 years?

P_n = amount after n years $= P_{n-1} + 0.03 \times P_{n-1} = 1.03\, P_{n-1}$ for $n > 0$

$P_0 = 5000$

$P_1 = 1.03 \cdot P_0 = 1.03 \times 5000$

⋮

formula for P_n only including n, P_0 initial value.

$P_n = 1.03^n \times 5000 = (1.03)^n P_0$

Example: vending machine

Suppose a vending machine accepts only \$1 coins, \$1 bills, and \$5 bills. **Find a recurrence relation for the number of ways to deposit *n* dollars into the vending machine**, where the order in which the coins/bills are deposited matters.

let a_n = # ways to deposit n dollars into vending machine.

$a_0 = 1$

$a_1 = 2$

$a_2 = 4$ [(c, b) (b, c) (b, b) (c, c)]

three $a_3 = 8$. 2 choices for 1st dollar, 2 choices for 2nd dollar, 2 choices for 3rd \$.

$a_4 = 16$

$a_5 = 1\$5$ or $5\$1 = 1 + 2^5 = 33 = a_0 + a_4 + a_4$.

$a_6 = a_1$ deposit \$5 1st

$+ a_5$ deposit \$1 bill first

$+ a_5$ deposit \$1 coin first

⋮

solution $a_0 = 1$, $a_1 = 2$, $a_2 = 4$

$a_3 = 8$, $a_4 = 16$,

$a_n = a_{n-5} + 2 \cdot a_{n-1}$ for $n \geq 5$

$$a_n = \underset{\substack{\uparrow \\ \text{1st is \$5 bill}}}{a_{n-5}} + \underset{\substack{\uparrow \\ \text{1st is}}}{a_{n-1}} + \underset{\substack{\uparrow \\ \text{1st is \$1 coin.}}}{a_{n-1}} = a_{n-5} + 2 \cdot a_{n-1}$$

Solving a recurrence relation

solving the recurrence = coming up with a formula for P_n with only n and constants in it.

Solving linear homogeneous recurrence relations of degree 2

Given a sequence $\{a_n\}$ described by the recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$ where $c_1, c_2 \in \mathbf{R}$ and where $r^2 - c_1r - c_2 = 0$ has 2 distinct roots r_1 and r_2

characteristic equation

The solution to the recurrence relation is $a_n = \alpha\, r_1^n + \beta\, r_2^n$ for $n = 0, 1, 2, \ldots$ where α and β are constants.

Example:

Solve the recurrence relation $a_n = 5a_{n-1} - 6a_{n-2}$ for $n \geq 2$, $a_0 = 1$, $a_1 = 0$

$c_1 = 5$ $c_2 = -6$ characteristic equation $\rightarrow r^2 - 5r + 6 = 0$

$r_1 = 3$ $r_2 = 2$ $(r^2 - c_1r - c_2)$ $(r-3)(r-2) = 0$

Solution looks like: $a_n = \alpha \cdot 3^n + \beta \cdot 2^n$ (what is α, & β)? ← use the initial condition ($a_0 = 1$, $a_1 = 0$)

Using $a_0 = 1$: $1 = \alpha \cdot 3^0 + \beta \cdot 2^0 = \alpha + \beta \rightarrow \alpha = 1 - \beta$

$a_1 = 0$: $0 = \alpha \cdot 3^1 + \beta \cdot 2^1 = 3\alpha + 2\beta$ $\beta = 3$.

$\alpha = -2$.

Solution is $a_n = -2 \cdot 3^n + 3 \cdot 2^n$

Double-check answer:

n	$a_n = 5a_{n-1} - 6a_{n-2}$	$a_n = -2 \cdot 3^n + 3 \cdot 2^n$
0	1	$(-2) \cdot 3^0 + 3 \cdot 2^0 = 1$
1	0	$-2 \cdot 3^1 + 3 \cdot 2^1 = 0$.
2	$5a_1 - 6a_0 = -6$	$-2 \cdot 3^2 + 3 \cdot 2^2 = -18 + 12 = -6$.
3	$5a_2 - 6a_1 = -30$	$-2 \cdot 3^3 + 3 \cdot 2^3 = -30$.

Recurrences (continued)

Example: Towers of Hanoi

The Towers of Hanoi is a game that consists of *n* disks and three poles: *A* (the source), *B* (the destination), and *C* (the spare). The disks are of different sizes and each has a hole in the middle so that it may be placed on a pole. The game starts with all the disks on pole *A* arranged by size with the largest disk on the bottom. The goal of the game is to move all the disks to pole *B*, with the following restrictions:

- only one disk at a time may be moved from one pole to another pole
- a disk can only be placed on top of a larger disk

Disks can be placed on pole *C* during the transfer, as long as the above restrictions are met.

How many moves does it take to solve the Towers of Hanoi with *n* disks?

Step 1) Find a recurrence relation for the number of moves.

Let $m(n)$ = # moves to solve with n disks

1 disk ⇒ 1 move

n disks ⇒ move $n-1$ disks from A to C ($m(n-1)$ moves)

move 1 disk from A to B (1 move)

move $n-1$ disks from C to B ($m(n-1)$ move)

Recurrence relation: $m(1)=1$

$m(n) = m(n-1) + 1 + m(n-1) = 1 + 2\cdot m(n-1)$ for $n>1$.

Step 2) Find a pattern and propose a solution.

$m(1)=1$

$m(n) = 1+2m(n-1)$ for $n>1$

n	$m(n)$
1	$1 = 2^1-1$
2	$1+2m(1)=3 = 2^2-1$
3	$1+2m(2)=7 = 2^3-1$
4	$1+2m(3)=15 = 2^4-1$
5.	$1+2m(4)=31 = 2^5-1$

Proposed Solution

$m(n) = 2^n - 1$ $(\forall n \geq 1)$

Step 3) Prove the solution works using induction.

$P(n) = m(n) = 2^n-1$ show $P(n)$ holds $\forall n \in \mathbb{N}^+$

Base case: show $P(1)$ holds

$m(1)=1$ by def of recurrence relation

$2^1-1=1$ so $m(1)=2^1-1$ & $P(1)$ holds.

Inductive step: show $P(k) \Rightarrow P(k+1)$

Assume $P(k)$ holds, i.e. $\underline{m(k) = 2^k-1}$ (IH)

Now consider $m(k+1)$

$m(k+1) = 1 + 2\cdot \underline{m(k)}$ using recurrance relation

$= 1 + 2(2^k-1)$ using (IH)

$= 1 + 2^{k+1} - 2$

$= 2^{k+1} - 1$ so $P(k+1)$ holds.

$\therefore$ by induction $m(n) = 2^n - 1$ $\forall n \in \mathbb{N}^+$ $\square$

Recurrences and program analysis

Program analysis goal: analyze the ***complexity*** of algorithms

= how # of operations performed by algorithm grows a function of the input size of sequences → # of ops performs in input on size of 1, 2, 3..
recursive programs → recurrence relations.

Given the following program specification:

b: input size.

Input: $a \in \mathbf{Z}$, $b \in \mathbf{Z}^+$
Output: $a^b = a \cdot a \cdot \ldots \cdot a$ (i.e., a multiplied b times)

count multiplication

Consider power 3 (3, 4)

Consider the following 3 implematations:

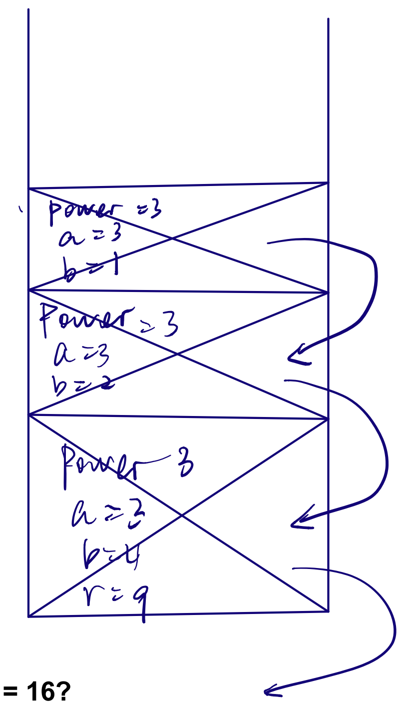
current multiplications
b = input size

m_1 procedure *power*1(a, b)
(1) if b = 1 then return a
(2) else return a·power1(a, b - 1)

m_2 procedure *power*2(a, b)
(1) if b = 1 then return a
(2) else if b is odd then return a·power2(a, $\lfloor b/2 \rfloor$)·power2(a, $\lfloor b/2 \rfloor$)
(3) else return power2(a, $\lfloor b/2 \rfloor$)·power2(a, $\lfloor b/2 \rfloor$)

floor $\lfloor x \rfloor$
= largest integer k s.t. $k \le x$.

m_3 procedure *power*3(a, b)
(1) if b = 1 then return a
(2) else r = power3(a, $\lfloor b/2 \rfloor$)
(3) if b is odd then return a·r·r
(4) else return r·r



How many multiplications are done by each implementation when a = 5 and b = 16?

Let $m_1(n)$ = # numbers done in power 1 with input size n = b Return 81

power (5, 16) : $m_1(16) = 15$
power (5, 16) : $m_2(16) = 15$
power (5, 16) : $m_3(16) =$

$m_1(16) = 1 + m_1(15) = 15$
↳ $1 + m_1(14) = 14$
↳ $1 + m_1(13) = 13$
... ↳ $1 + m_1(1) = 1$
↳ 0 multiplications

$m_2(16) = 1 - 2 \cdot m_2(8) \quad = 15$

$m_2(8) = 1 + 2 \cdot m_2(4) \quad = 7$

$m_2(4) = 1 + 2 \cdot m_2(2) \quad = 3$

$m_2(2) = 1 + 2 \cdot m_2(1) \quad = 1$

$m_2(1) = 0$ multiplication

$m_3(16) = m_3(8) + 1 \quad = 4.$

$m_3(8) = m_3(4) + 1 \quad = 3$

$m_3(4) = m_3(2) + 1 \quad = 2$

$m_3(2) = m_3(1) + 1 \quad = 1$

$m_3(1) = 0$ multiplication.