Sets

Terms

set = a well-defined collection of elements from some domain ← roster notation.

$\{1, 2, 7\}$ is the same as $\{1, 7, 2\}$, no duplication allowed.

well-defined = for every element of the domain, it can be determined whether the element belongs in the set or not

x ∈ A   means x is in the set A

x ∉ A   – – – is not – – – – –.

subset : A ⊆ B   iff $(\forall x \in A)(x \in B)$

strict subset : A ⊂ B   iff $A \subseteq B \land (\exists x)(x \in B \land x \notin A)$

set equality : A = B   iff $A \subseteq B \land B \subseteq A$

S240(x)
CS(x)

UW = set of all UW-Madison students.

$CS240 = \{x \in UW \mid S240(x)\}$ ← set builder notation.

↑ such that.

$CS240 \subset UW$

empty set : $\{\ \}$ or $\emptyset$

cardinality : |A| = # elements in A ← can be finite or infinite.

$|\emptyset| = 0$
$|CS240| = 499$

Operations on sets

Given domain D

union : A ∪ B = $\{x \in D \mid x \in A \lor x \in B\}$

intersection : A ∩ B = $\{x \in D \mid x \in A \land x \in B\}$

set difference : A – B = $\{x \in D \mid x \in A \land x \notin B\}$

complement : $\overline{A}$ = $\{x \in D \mid x \notin A\} = D - A$.

Cartesian product : A × B = $\{(a, b) \mid a \in A \land b \in B\}$

↑ order matters.

Sets and propositions

Prove the proposition: Let A, B, and C be sets. Then A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)

Proof (using propositional logic): & set builder notation.

Recall for propositions P, Q, R: $P \vee (Q \wedge R) \equiv (P \vee Q) \wedge (P \vee R)$

$A \cup (B \cap C) = \{x \mid x \in A \vee (B \cap C)\}$ set builder notation.

$= \{x \mid x \in A \vee x \in B \cap C\}$ ← def of ∪

$= \{x \mid x \in A \vee (x \in B \wedge x \in C)\}$ ← def of ∩

distributive law.

$= \{x \mid (x \in A \vee x \in B) \wedge (x \in A \vee x \in C)\}$

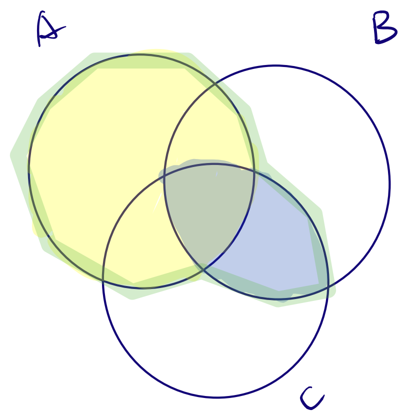
$= \{x \mid (x \in A \cup B) \wedge (x \in A \cup C)\}$ def of ∪

$= \{x \mid x \in (A \cup B) \cap (A \cup C)\}$ def of ∩

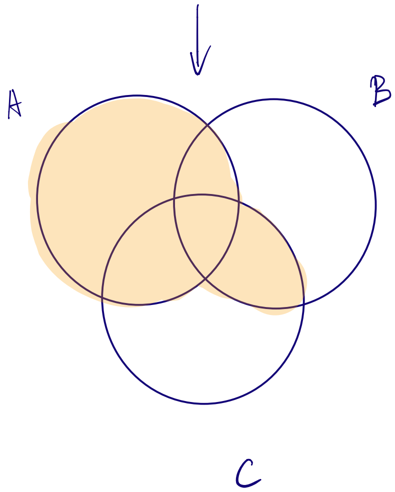
$= (A \cup B) \cap (A \cup C)$

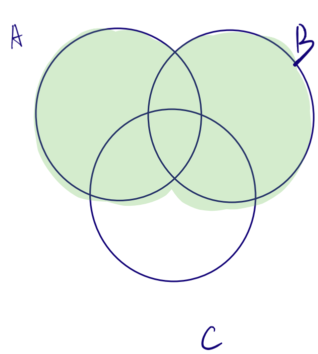
Using Venn diagrams to understand the proposition:

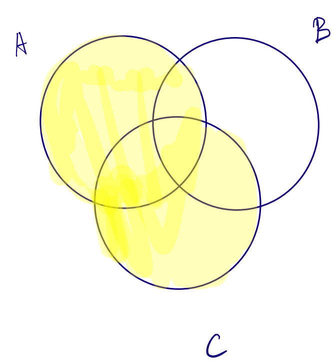
$A \cup (B \cap C)$



$(A \cup B) \cap (A \cup C)$







Sets (continued)

Power Set $P(A)$

$A \subseteq A \Rightarrow A \in P(A)$

$\emptyset \subseteq A \Rightarrow \emptyset \in P(A)$

The **power set** of a set A is the set of all subsets of A.

Example: Let A = { x ∈ **Z** | x is prime ∧ x < 10 } $A = \{2, 3, 5, 7\}$

1. What is |A|? cardinality 4.
2. What is *P*(A)?
3. What is |*P*(A)|?

2. $\{\emptyset, \{2\}, \{3\}, \{5\}, \{7\}, \{2,3\}, \{2,5\}, \{2,7\}, \{3,5\}, \{3,7\}, \{5,7\}, \{2,3,5\}, \{2,3,7\},$
$\{2,5,7\}, \{3,5,7\}, \{2,3,5,7\}\}$

3. $2^4 = 16$

$|P(A)| = 2^{|A|}$

$\in A$

$\in P(A)$	7	5	3	2	
$\emptyset$	0	0	0	0	0
$\{2\}$	0	0	0	1	1
$\{3\}$	0	0	1	0	2
$\{2,3\}$	0	0	1	1	
⋮	⋮	⋮	⋮		⋮
$\{2,3,5,7\}$	1	1	1	1	15

1 : in the subset

0 : not in the subset.

\# subsets = $2 \times 2 \times 2 \times 2 = 16$

Countable Sets

Definition : A set is **countable** iff

- it is finite, or
- it is infinite and there is an enumeration consisting exactly of all elements in that set

enumeration : a listing of elements in the set. i.e. an ordering of elements in the set.
1^{st} elt, 2^{nd} elt, 3^{rd} elt ... . — — —

$\mathbb{N} = \{0, 1, 2, \ldots\ldots, \ldots\}$ $\Rightarrow \mathbb{N}$ is countable

Set S is countable means there is a one-to-one correspondence (or matching):

position in enumeration: 1^{st} 2^{nd} 3^{rd} 4^{th} 5^{th} ...... i^{th} ?

elements of S given as an enumeration: e_1 e_2 e_3 e_4 e_5 ...... ? k

To show an enumeration is a matching, determine:

- Given $i \in \mathbf{Z}^+$, what is the i^{th} element in the enumeration of S?
- Given $k \in S$, at what position does k show up in the enumeration?

Examples of Countable Sets

Example: The set of all positive even numbers is countable.

$S = \{x \in \mathbb{Z} \mid x > 0 \wedge x \text{ is even}\} = \{2, 4, 6, 8, 10 \ldots\}$

To show S is countable, find a matching.

pos	1	2	3	4	5	...	i	$k/2$
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$		$\updownarrow$	$\uparrow$
enu	2	4	6	8	10	...	$2i$	k

a): what is the i^{th} elt of S in this enumeration? $2i$

b). Given $k \in S$, where does k appear in the enumeration? $k/2$.

Example: The integers are countable. $\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, 4 \ldots\}$

Consider the following enumeration: $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3 \ldots\}$

show that this is a matching (1-1 correspondense):

position: 1, 2, 3, 4, 5, 6 ...... i b

elts of: 0 1 −1 2 −2 3 ...... a k.

what is the ith elt of enumeration:

- if i is even, then element is $i/2$
- if i is odd, then element is $-\left(\frac{i-1}{2}\right)$

where does $k \in \mathbb{Z}$ appear in enumeration?

- if $k > 0$, then position is $2 \cdot k$.
- if $k \le 0$, then position is $k = -\left(\frac{i-1}{2}\right) \Rightarrow i = -2k+1$

Uncountable Sets

$\mathbb{R}$ is uncountable

A set is **uncountable** if it is not countable.

Example: The power set of the natural numbers is not countable. $\mathbb{P}(\mathbb{N})$