ECON 475
Economics of Growth

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Consider the Romer model with the following production technologies (they are identical to those discussed in class):

$$Y = L_Y^{1-\alpha} \int_0^A x_j^\alpha dj \tag{1}$$
$$x_j = k \tag{2}$$

Equation (1) characterizes production of the homogeneous final good, which can be either consumed or invested, and (2) describes the transformation of one unit of the homogeneous capital good into a differentiated variety indexed by j.

Free entry into the final good sector combined with the fact that the production technology exhibits constant returns to scale implies that final good producers are *price takers*.

Let r denote the rental rate of capital and $p_j(x_j)$ the rental price of the capital variety j.

1. Write down the profit function of a representative final good producer.

profit $= L_Y^{1-\alpha} \int_0^A x_j^a dj - wL_Y - \int_0^A p_j x_j dj$.

$\pi = Y - wL_Y - \int_0^A p_j x_j dj$

final good producers are price takers.

$Y = L_Y^{1-\alpha} \int_0^A x_j^\alpha dj$ no skip any x_j (MP $\to \infty$ f $x_j \to 0$).

2. Write down the profit function of the capital good producer j.

$\pi_j = P_j(x_j) \cdot x_j - r_j \cdot \cancel{\text{x}} k_j$ $x_j = k$ $\int_0^A x_j dj \le K$.

$= P_j(x_j) x_j - r x_j$

producer j is monopolistically competitive.

only j produce x_j. assume x_j does not impact the whole market.

In output market, producer j is a price-setter.

3. Take the two first order conditions (F.O.C.s) associated with the final good producer's problem.

$$\frac{\partial \cancel{Profit}\,\Pi}{\partial L_Y} = \underbrace{(1-\alpha) L_Y^{-\alpha} \int_0^A x_j^{\alpha} dj}_{MP \text{ of } L_Y} - W = 0$$

$$\frac{\partial \cancel{Profit}\,\Pi}{\partial x_j} = L_Y^{1-\alpha} \cdot \alpha \cdot x_j^{\alpha-1} - p_j(x_j) = 0$$

$$p_j(x_j) = \alpha \underbrace{L_Y^{1-\alpha} x_j^{\alpha-1}}_{MP \text{ of } x_j}$$

$$(1-\alpha) L_Y^{-\alpha} \int_0^A x_j^{\alpha} dj = w$$

$$\frac{1}{L_Y}(1-\alpha) \underbrace{L_Y^{1-\alpha} \int_0^A x_j^{\alpha} dj}_{Y} = w$$

$$\frac{Y}{L_Y}(1-\alpha) = w.$$

$$(1-\alpha) = \frac{wL_Y}{Y}.$$

income share of output is $(1-\alpha)$.

4. Using the F.O.C. with respect to x_j, show that the price elasticity of demand is constant (i.e. show that $\frac{\partial p_j(x_j)}{\partial x_j}\frac{x_j}{p_j}$ is constant).

To max profit, final good producer will have $\frac{\partial Profit}{\partial x_j} = L_Y^{1-\alpha} \cdot \alpha \cdot x_j^{\alpha-1} - p_j = 0.$

$$p_j(x_j) = L_Y^{1-\alpha} \cdot \alpha \cdot x_j^{\alpha-1}$$

$$\frac{\partial p_j(x_j)}{\partial x_j} \cdot \frac{x_j}{p_j(x_j)} = \frac{\alpha(\alpha-1) L_Y^{1-\alpha} \cdot x_j^{\alpha-2} \cdot x_j}{L_Y^{1-\alpha} \cdot \alpha \cdot x_j^{\alpha-1}} = \alpha - 1.$$

$$\frac{\partial p_j(x_j)}{\partial x_j} \frac{x_j}{p_j(x_j)} \cdot p_j(x_j) + p_j(x_j) = r.$$

$$p_j(x_j)\left(\frac{\partial p_j(x_j)}{\partial x_j} \frac{x_j}{p_j(x_j)} + 1\right) = r.$$

$$\frac{\partial p_j(x_j)}{\partial x_j} = \frac{\partial p_j(x_j)}{\partial x_j} x_j + p_j(x_j) - r = 0.$$

5. Again, using the F.O.C. with respect to x_j show that $\frac{\partial p_j(x_j)}{\partial x_j}\frac{x_j}{p_j} = \frac{\partial \ln(p_j(x_j))}{\partial \ln(x_j)}$.

$$\frac{\partial \ln(p_j(x_j))}{\partial \ln(x_j)} = \frac{\partial\left[(1-\alpha)\log(L_Y) + \log(\alpha) + (\alpha-1)\log(x_j)\right]}{\partial \ln(x_j)}$$

$$= (\alpha - 1)$$

$$= \frac{\partial p_j(x_j)}{\partial x_j} \cdot \frac{x_j}{p_j(x_j)}$$

6. Use the constant price elasticity to solve for $p_j(x_j)$ as a function of r and model parameters. Show that the rental rates for all varieties $j \in [0, A]$ are equal.

$$\frac{\partial \pi_j}{\partial x_j} = 0.$$

$$\frac{\partial p(x_j) \cdot x_j}{\partial x_j} = r_j$$

$$p(x_j) + x_j \cdot \frac{\partial p(x_j)}{\partial x_j} = r_j.$$

$$p(x_j) = \frac{1}{1 + \frac{x_j}{p(j)} \cdot \frac{\partial p(x_j)}{\partial x_j}} \cdot r_j = \frac{1}{1+\alpha-1} \cdot r_j = \frac{r}{\alpha}.$$

$$p(x_j) - r = \frac{r}{\alpha} - r = \frac{(1-\alpha) r}{\alpha}$$

7. Use your previous result to characterize $x = x_j$ for $j \in [0, A]$ as a function of K and A.

$x = x_j.$ k_j is constant for all j,

$$\int_0^A x_j \, dx = K.$$

$$\int_0^A x \, dx = K.$$

$$Ax = K.$$

$$x = x_j = \frac{K}{A}.$$

8. Use your answer above to characterize the rental rate r as a function of the capital-labor ratio in production $\frac{K}{L_Y}$, A, and parameters of the model. Rewrite the production function for final goods as a function of these aggregate variables.

$$p_j = \alpha L_Y^{1-\alpha} x_j^{\alpha-1} = \frac{r}{\alpha} = \alpha L_Y^{1-\alpha} \left(\frac{K}{A}\right)^{\alpha-1}.$$

$$\underline{r} = \alpha^2 \left(\frac{K}{L_Y}\right)^{\alpha-1} A^{1-\alpha}$$

falling in capital

$$Y = L_Y^{1-\alpha} A x^{\alpha} = L_Y^{1-\alpha} A \left(\frac{K}{A}\right)^{\alpha} = (L_Y A)^{1-\alpha} K^{\alpha}.$$

9. Write a capital producer's profit as a function of aggregate variables (K, L_Y, A) and the parameters of the model.

$\pi = \pi_j = P_j \cdot x_j - r \cdot x_j$

$= P_j \cdot x_j - P_j(x_j) \cdot \alpha \cdot x_j$. $\quad P_j = \alpha L_Y^{1-\alpha} x_j^{\alpha - 1}$

$= (1-\alpha) P_j x_j = (1-\alpha)\alpha L_Y^{1-\alpha} \left(\frac{K}{A}\right)^{\alpha}$

if $\alpha = 0 \Leftarrow Y$ is base on labor, not capital, no profit.

if perfect substitute $\Rightarrow \alpha = 1 \Rightarrow \pi = 0$.

$= (1-\alpha)\alpha \frac{Y}{A}$.

Next, we need to specify the accumulation equations. Since you'll be solving a two-period version of the model we covered in class, we are switching to discrete time.

Instead of the *dot* notation on the slides, we are going to use Δ to denote differences in variables over adjacent time periods, that is, $\Delta a_t \equiv a_{t+1} - a_t$. Since you're solving a two-period problem, we denote period-2 variables with the superscript $'$ (*prime*) rather than $t+1$. For instance, the period-1 capital stock is denoted by K and the period-2 stock is denoted by K'.

$$\Delta A \equiv A' - A = \theta A^{\phi} L_A^{\lambda} \tag{3}$$

$$\Delta K \equiv K' - K = s_K Y - \delta K \tag{4}$$

where $\delta \in (0, 1]$ is the capital depreciation rate, $s_K \in (0, 1)$ is the investment share, and $\phi, \lambda \in (0, 1)$. Labor market clearing implies that $L_A + L_Y = L$.

Going forward, you should assume $\delta = 1$ (i.e. full capital depreciation). This is not a particularly reasonable assumption but it simplifies the solution considerably.

In the two-period problem the world comes to an end after the second period. I'm not much of a "doomsdayer" but this approach will turn out to be instructive for an introduction to dynamic programming later in the course when we solve the neoclassical growth model. In this environment, you anticipate "the end of the world" by allocating all workers to production in the second period. Why? Since there is no third period, new ideas generated in the second period have no economic value and you optimally set $s_R' = 0$.

10. Using $s'_R = 0$, write tomorrow's total output Y' as a function of today's aggregate variables (K, A, L), the population growth rate n, and s_R (today's employment share of research).

    Put differently, use $K' = s_K Y + (1-\delta)K$ (where $\delta = 1$) and $A' = A + \Delta A$ to characterize Y'.

Since $s'_R = 0$. $L'_Y = L'$

$Y' = K'^{\alpha} (A' L')^{1-\alpha}$.

$K' = s_K Y + (1-\delta) K = s_K Y \quad (\delta = 1)$.

$A' = A + \Delta A = A + \theta A^{\phi} (s_R L)^{\lambda}$.

$Y = K^{\alpha} (A \cdot (1-s_R) L)^{1-\alpha}$.

$L' = L(1+n)$.

$$Y' = [s_K K^{\alpha} (A(1-s_R)L)^{1-\alpha}]^{\alpha} [A + \theta A^{\phi} (s_R L)^{\lambda}]^{1-\alpha} \cdot [L(1+n)]^{1-\alpha}$$

11. Take the F.O.C. with respect to s_R and simplify the expression as much as you can. The condition does not have a closed form solution, but try to collect all the s_R terms on one side of the equation and all the terms that don't involve s_R on the other side.

$$\frac{\partial Y'}{\partial s_R} = [L(1+n)]^{1-\alpha} [s_K K^{\alpha} (A(1-s_R)L)^{1-\alpha}]^{\alpha} (1-\alpha) [A + \theta_A^{\phi} (s_R L)^{\lambda}]^{-\alpha} \cdot \lambda \theta A^{\phi}$$

$\frac{\partial Y'}{\partial s_R} = ?$

$\frac{\partial Y}{\partial s_R} = ?$

$\frac{\partial Y}{\partial s_R} + \frac{\partial Y'}{\partial s_R} = 0$

Choice variable s_R

K, L, A

K', L', A'

12. If you can, show that the non-constant side of the equation is monotonically increasing (or decreasing) in s_R. Show also that $s_R = 1$ and $s_R = 0$ do not satisfy the F.O.C.

13. Is s_R increasing or decreasing in today's stock of knowledge A? What's the economic intuition?