Transition from Stagnation to Growth:

Unified Growth Theory (UGT)

What we have so far:

$\ln(C_t) + \ln(n_t) + \ln(h_t)$

2 version budget constraints. $C_t = y_t - n_t(p + e_t)$

$C_t = y_t(1 - n_t(\phi + e_t))$

change to change fraction of income to consume

Food constraint: $F + C_t = y_t(1 - n_t(\phi + e_t))$

don't scale up/down.

$\ln[y_t(1-n_t(\phi + e_t)) - F] + \ln(n_t) + \ln(h_t)$

F make y_t in the first order condition.

The Production Side of the Model: Agriculture and Industry

- General version with different productivity levels and different productivity growth rates.
- The two technologies:

$$Y_t^a = (A_t^a X)^{\alpha} (N_t^a)^{1-\alpha},$$
$$Y_t^i = A_t^i N_t^i.$$

- Laws of motion:

$$A_{t+1}^a = (1 + g^a) A_t^a,$$
$$A_{t+1}^i = (1 + g^i) A_t^a.$$

- Labor used in the two sectors has to add up to total population size:

$$N_t^a + N_t^i = N_t.$$

Special Role of Agriculture: The Food Consumption Constraint

- Every person has to eat at least f units of food, which can only be produced using agriculture.
- Given population N_t, the food consumption constraint is:

$$fN_t \leq (A_t^a X)^\alpha (N_t^a)^{1-\alpha}.$$

- If constraint cannot be met, immediate reduction in population size until constraint is met.

$$fN_t^s = (A_t^a X)^\alpha (N_t^s)^{1-\alpha}$$

Household Decisions: Tradeoff Between Child Quantity (i.e., Fertility) and Child Quality (i.e. Education)

- Every person has to eat f units of food.
- Preferences defined over consumption c_t over and above basic food consumption f, fertility n_t, and human capital h_t of children.
- Utility function is:

$$\log(c_t) + \log(n_t) + \log(h_t).$$

- Budget constraint is:

or y_t income

$$c_t + f = (1 - (\phi + e_t)n_t)w_t.$$

education expenditure.

- Here ϕ is basic time cost of children, e_t is chosen time to educate each child, w_t is the wage, and f is food consumption.
- Production function for human capital is:

$$h_t = e_t^{\theta_t}.$$

- NOTE: Food consumption f enters budget constraint, and θ_t may vary over time.

Modeling the Return to Education

- Return to education θ_t (i.e., the elasticity of human capital with respect to education) is key parameter driving fertility choice.
- We assume that return to education is **higher** in industry than in agriculture.
- How can the model capture this parsimoniously? By making the return to education a function of the fraction of people working in industry:

$$\theta_t = \bar{\theta}\frac{N_t^i}{N_t}.$$

- *Remark*: In principle, should adjust labor input for human capital. We don't do this here to keep things simple.

Linking the Different Parts of the Model

- Population tomorrow depends on today's fertility decisions:

$$N_{t+1} = n_t N_t.$$

- Wage (i.e., income per person) is given by income per capita:

$$w_t = y_t = \frac{(A_t^a X)^\alpha (N_t^a)^{1-\alpha} + A_t^i N_t^i}{N_t}.$$

- Allocation of people across agriculture and industry to maximize income per capita subject to food production constraint.

Outline of Steps of Analysis

- Analyze household choice problem.
- Characterize Malthusian steady state.
- Put full model on computer.

The Household Choice Problem

- After plugging the constraints into the utility function, the household's utility maximization problem is:

$$\max_{(e_t, n_t)} \left\{ \overset{\ln}{\cancel{\log}}((1-(\phi+e_t)n_t)y_t - f) + \overset{\ln}{\cancel{\log}}(n_t) + \theta_t \overset{\ln}{\cancel{\log}}(e_t) \right\}.$$

- The first-order conditions for n_t and e_t are:

1^{st} condition WRT

$$n_t \quad \frac{(\phi+e_t)y_t}{(1-(\phi+e_t)n_t)y_t - f} = \frac{1}{n_t},$$

$$e_t \quad \frac{n_t y_t}{(1-(\phi+e_t)n_t)y_t - f} = \frac{\theta_t}{e_t}.$$

- Ratio of the first-order conditions:

$$\frac{(\phi+e_t)\cancel{y_t}}{n_t \cancel{y_t}} = \frac{e_t}{\theta_t \cancel{n_t}}. \Rightarrow (\phi + e_t)\theta_t = e_t$$

$$e_t = \frac{\phi\theta_t}{(1-\theta_t)}$$

Solving for Optimal Education

- Canceling terms:

$$\phi + e_t = \frac{e_t}{\theta_t}.$$

- Solving for e_t:

$$\begin{aligned} \phi\theta_t + \theta_t e_t =& e_t, \\ (1-\theta)e_t =& \phi\theta_t, \\ e_t =& \frac{\phi\theta_t}{1-\theta_t}. \end{aligned}$$

- Result: Education is increasing in return to education θ_t.
- Given that θ_t is linked to industrialization, education will rise once industrialization takes place.

$\Downarrow$

quality $\Leftrightarrow$ quantity trade off.

Solving for Optimal Fertility

- Starting from the first-order condition for n_t:

$$\begin{aligned}
\frac{(\phi + e_t)y_t}{(1-(\phi+e_t)n_t)y_t - f} =& \frac{1}{n_t}, \\
(\phi + e_t)y_t n_t =& (1-(\phi+e_t)n_t)y_t - f, \\
2(\phi + e_t)y_t n_t =& y_t - f, \\
n_t =& \frac{1}{2(\phi+e_t)}\left(1 - \frac{f}{y_t}\right).
\end{aligned}$$

- Fertility is decreasing in education e_t and increasing in income per capita y_t.

Optimal Fertility Choice in Malthusian Phase

- Consider fertility choice in Malthusian phase. When all employment is in agriculture, we have $\theta_t = 0$, hence $e_t = 0$ and:

$$n_t = \frac{1}{2\phi}\left(1 - \frac{f}{y_t}\right).$$

- Fertility is increasing in income per capita:
  - When $y_t = f$, we have:

$$n_t = \frac{1}{2\phi}\left(1 - \frac{f}{f}\right) = 0.$$

  - When $y_t \to \infty$, we have:

$\downarrow$

fertility increase in income is bounded. + $e_t = 0$

$$n_t = \frac{1}{2\phi}.$$

Optimal Fertility Choice

- As long as $\theta_t = 0$, positive relationship between income per capita and fertility gives rise to usual Malthusian dynamics.
- However, fertility is also decreasing in e_t: Once industrialization raises θ_t, there is a new downward force on fertility.
- Hence, model can account for both Malthusian population dynamics and the demographic transition.

The Malthusian Steady State

$$y_t = \frac{(A_t^a X)^\alpha}{(N_T^a)^i}$$

if y_t is constant,

$$\left(\frac{A_t^a}{N_T^a}\right)^\alpha \cdot X^\alpha$$

is constant

- Consider economy in which only agricultural technology is used.
- Economy displays negative Malthusian relationship between population size and income per capita.
- Economy converges to steady state in which income per capita is constant and in which population growth just offsets productivity growth.
- Steady-state condition is that population growth is equal to productivity growth in agriculture:

$$\bar{n} = 1 + g^a.$$

The Malthusian Steady State

- Solving for steady-state income per capita $\bar{y}$:

$$
\begin{aligned}
\bar{n} &= 1 + g^a, \\
\frac{1}{2\phi}\left(1 - \frac{f}{\bar{y}}\right) &= 1 + g^a, \\
1 - 2\phi(1 + g^a) &= \frac{f}{\bar{y}}, \\
\bar{y} &= \frac{f}{1 - 2\phi(1 + g^a)}.
\end{aligned}
$$

increase in

$g^a \uparrow \to \bar{y} \uparrow$

$f \uparrow \to \bar{y} \uparrow$

keep $\bar{n}$ small

Condition for Introduction of Industry

- The industrial sector will be introduced when the marginal product of labor in industry reaches the marginal product in agriculture:

$$A_t^i \geq \frac{\partial Y_t^a}{\partial N_t^a} = (1-\alpha)\left(\frac{A_t^a X}{N_t^a}\right)^{\alpha}.$$

- The marginal product of labor in agriculture is $1-\alpha$ times income per capita in agriculture.
- If the economy starts out in Malthusian steady state, condition for introduction of agriculture can therefore be expressed as:

does depend on g, but must introduce industry eventually. → growth

$$A_t^i \geq (1-\alpha)\bar{y} = \frac{(1-\alpha)f}{1-2\phi(1+g^a)}$$ ← constant.

- As long as $g^i > 0$, this threshold will be reached for sure at some point.

in finite time

The Long-Run Evolution of the Economy after Industry is Introduced

- Once industry is introduced, alternative forces are at work:
  - Population growth might rise because of higher income per capita. This could lead to insufficient food supply and a development failure.
  - But on the other hand, education is going to rise, which puts downward pressure on fertility. If fertility falls, long-run growth might be possible.
- Put model on computer to see what determines whether the takeoff to growth succeeds or fails.

Putting the Model on the Computer

- Similar to Malthus-to-Solow model with food requirement, but have to add new features of education and quantity-quality fertility choice.
- Maximum population that can be sustained given current productivity:

$$fN_t^{\max} = (A_t^a X)^\alpha (N_t^{\max})^{1-\alpha},$$
$$N_t^{\max} = f^{-\frac{1}{\alpha}} A_t^a X.$$

- Surviving population N_t^s is given by:

$$N_t^s = \min\{N_t, N_t^{\max}\}.$$

- Interpretation: If maximum population $N_t^{\max}$ is smaller than the actual population N_t at the beginning of the period, there will be a famine, and $N_t - N_t^{\max}$ people will die of starvation.

Putting the Model on the Computer

- Number of agricultural workers required to produce sufficient amount of food:

$$fN_t^s = (A_t^a X)^\alpha (N_t^{a\,\min})^{1-\alpha},$$

$$N_t^{a\,\min} = \frac{(fN_t^s)^{\frac{1}{1-\alpha}}}{(A_t^a X)^{\frac{\alpha}{1-\alpha}}}.$$

- Labor that would be used in agriculture if marginal product of labor were equalized between agriculture and industry:

$$A_t^i = (1-\alpha)\left(\frac{A_t^a X}{\tilde{N}_t^a}\right)^\alpha,$$

$$\tilde{N}_t^a = \left(\frac{1-\alpha}{A_t^i}\right)^{\frac{1}{\alpha}} A_t^a X.$$

- Labor actually used in agriculture:

$$N_t^a = \min\left\{N_t^s, \max\{N_t^{a\,\min}, \tilde{N}_t^a\}\right\}.$$

Putting the Model on the Computer

- Labor used in industry:

$$N_t^i = N_t^s - N_t^a.$$

- Income per capita:

$$y_t = \frac{(A_t^a X)^\alpha (N_t^a)^{1-\alpha} + A_t^i N_t^i}{N_t^s}.$$

- Return to education and education choice:

$$\theta_t = \bar{\theta}\frac{N_t^i}{N_t^s}, \qquad e_t = \frac{\phi\theta_t}{1-\theta_t}.$$

- Fertility choice:

$$n_t = \frac{1}{2(\phi + e_t)}\left(1 - \frac{f}{y_t}\right).$$

- Population at the beginning of the next period:

$$N_{t+1} = n_t N_t^s.$$

Putting the Model on the Computer

- Initial conditions needed for X, A_0, and N_0.
- Assume that economy starts out in Malthusian steady state:

$$\bar{y} = \frac{f}{1 - 2\phi(1 + g^a)} = \left(\frac{A_0^a X}{N_0} \right)^{\alpha},$$

so that:

$$N_0 = \left(\frac{1 - 2\phi(1 + g^a)}{f} \right)^{\frac{1}{\alpha}} A_0^a X.$$

Some Findings from the Computed Model

- Economy starts out in Malthusian steady state.
- Ultimately, industrial sector is introduced.
- Population growth can rise or fall during transition, depending on whether income effect or education effect dominates.
- Development may ultimately fail if the education effect is too weak.