The uniform continuity.

Rmk: continuity is about the local behavior of f at a.

In particular, is dependent on a, on $\varepsilon > 0$ as well.

A stronger notion: uniform continuity (⇒ continuity itself)

Def: $f: A \to \mathbb{R}$, we say f is uniformly continuous on A if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|f(x) - f(y)| < \varepsilon \quad \forall x, y \in A \text{ s.t. } |x-y| < \delta$$

key different: $\delta > 0$ only depend on $\varepsilon > 0$, but not depend on locations of points x, y any more.

Rmk: there are many situations that continuous functions are not uniformly continuous.

Ex 1. $f(x) = x^2$, $f: \mathbb{R} \to \mathbb{R}$.

Still not uniformly continuous on $\mathbb{R}$

proof: Assume otherwise that f is uniformly continuous.

Pick $\varepsilon = 1$, by def, $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon = 1$ $\forall x, y \in \mathbb{R}$ where $|x-y| < \delta$.

$\Leftrightarrow |x^2 - y^2| < 1$ for $x, y \in \mathbb{R}$ with $|x-y| \le \delta$

Pick $x = n \in \mathbb{N}$, $y = n + \delta$. of course $|x-y| \le \delta$.

In this case, $(x = n, y = n+\delta)$ $\quad |x^2 - y^2| = y^2 - x^2 = (n+\delta)^2 - n^2 = 2n\delta + \delta^2 < 1$

This is absurd if we pick $n > \frac{1}{2\delta}$ ∎

Mean value theorem: $f(y) - f(x) = (y-x) f'(c)$ for some $c \in (x, y)$

In this case $f(x+\delta) - f(x) = \delta \cdot f'(c) = \delta \cdot 2c$

fixed $\quad \infty$ as $\delta \to \infty$

2. $g: (0,1) \to \mathbb{R}$ $\quad g(x) = \frac{1}{x}$

Theorem: Given $a<b$, $a,b \in \mathbb{R}$

Let $f: [a,b] \to \mathbb{R}$ be a continuous function.

Then, f is uniformly continuous.

Proof: Fix an arbitrary $\varepsilon > 0$, we need to show $\exists \delta > 0$ s.t. $|f(x) - f(y)| < \varepsilon$ for all $x,y \in [a,b]$, $|x-y| < \delta$.

Let $C = \{x \in [a,b]$: this property holds for f on $[a,x]$ only$\}$.

Clearly, we are done if we have $b \in C$.

Note, $C \neq \phi$ as $a \in C$ and C is bounded. In fact $C \in [a,b]$.

We have $\sup C$ exist. denoted by $c = \sup C$

We'll deduce a contradiction if $c<b$.

Note f is continuous at c

$\frac{\varepsilon}{2} > 0$, there is a constant $\delta_c > 0$ s.t. $|f(x) - f(c)| < \frac{\varepsilon}{2}$ for $|x-c| \leq \delta_c$.

$|f(x) - f(c)| < \frac{\varepsilon}{2}$ for $c - \delta_c < x < c + \delta_c$.

We claim that $(c + \delta_c) \in C$, which is a contradiction

Earlier, we know the property holds for f on $[a,c]$ with $\delta > 0$ choose new $\bar{\delta} = \min(\delta, \delta_c)$

Claim: $|f(x) - f(y)| < \varepsilon$ $\forall x,y \in [a, c+\delta_c]$, $|x-y| < \bar{\delta}$

only need to consider $x < c < y$, $|x-y| < \bar{\delta}$

Here, $|f(x) - f(y)| \leq |f(x) - f(c)| + |f(c) - f(y)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ ∎