Directed Graphs

Terminology

- **multiset :** a set-like collection that allows duplicate elements

  e.g $\{1, 2, 2, 2, 3\}$ - all sets are multisets

- **directed graph (digraph) :** *G* = (*V*, *E*) where *V* is a set of **vertices** (or **nodes**) and *E* is a multiset of elements of *V*×*V*

  - elements in E are edges
  Given $(a, b) \in E$, a is start vertex (or start node)
  b is end vertex (or end node)

- **simple graph:** a graph *G* = (*V*, *E*) where *E* is a set (and not a multiset)

  i.e. no duplicate edges
  note: E is then a relation on V

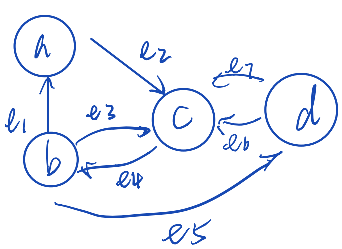
- **path :** Given graph *G* = (*V*, *E*) and *u*, *v* ∈ *V*, the **path from *u* to *v*** is a sequence of edges such that

  - u is start vertex of 1^{st} edge,
  - v is end vertex of last edge,
  - end vertex of i^{th} edge $(a, \underline{b})$ = start vertex of $(i+1)^{th}$ edge $(\underline{b}, c)$

- **length of a path :** the number of edges in the path

- **cycle :** a path of length > 0 that begins and ends at the same vertex

Directed graph example



$V = \{a, b, c, d\}$

$E = \{(b, a), (a, c), (b, c), (c, b), (b, d), \underline{(d, c), (d, c)}\}$.

E is multiset.

$G = (V, E)$ is not a simple graph

Path	start vertex	end vertex	sequence of edges	length
P1	a	a	empty sequence	0
P2	a	b	e2, e4	2
P3	a	d	e2, e4, e5, e6, e4, e5	6
P4	a	a	e2, e4, e1	3

⇐ not a cycle

⇐ cycle.

Transitive closure

transitive closure : the relation *R* on *V* where *aRb* ($(a,b) \in R$) if and only if there exists a path in *G* from *a* to *b*

Is transitive closure reflexive? $\forall a \in V, (a,a) \in R$

Yes, there always exists a path of length 0 from a vertex to itself

$(a,b), (d,a)$ are in transitive closure of example

Is transitive closure transitive? $(a,b) \in R \wedge (b,c) \in R \Rightarrow (a,c) \in R$.

Yes. If $\exists$ a path from a to b and $\exists$ a path from b to c, there is a path from a to c.

Is transitive closure symmetric? $(a,b) \in R \Leftrightarrow (b,a) \in R$.

No, if $\exists$ is a path from a to b, not necessarily a path from b to a.

$$a \to c \to b$$

anti-symmetric ?

No

$$\begin{tikzcd} a \arrow[r] & c \arrow[r] & b \arrow[ll, bend left] \end{tikzcd}$$

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More terminology

- **DAG :** Directed Acyclic Graph = digraph with no cycles
- **in-degree** of a vertex *v* = # edges in E with V as an end vertex.

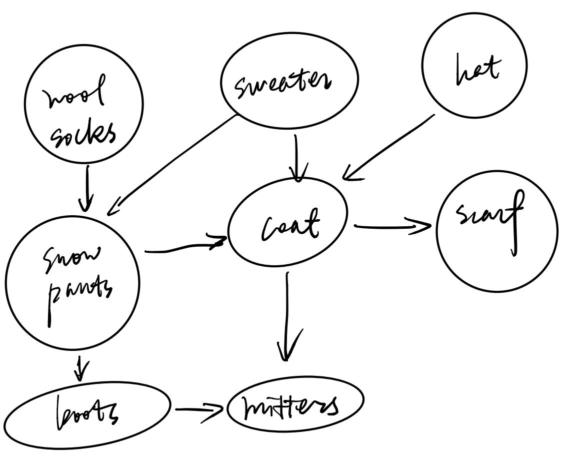
  $= |\{(x,v) \mid (x,v) \in E\}|$
- **out-degree** of a vertex *v* = # edges in E with V as start vertex.

  $= |\{(v,y) \mid (v,y) \in E\}|$
- **topological ordering** of a graph *G* = (*V*, *E*) is a total order $\preccurlyeq$ on *V* such that if $(u, v) \in E$ for $u, v \in V$, $u \neq v$, then $u \preccurlyeq v$.

  transitive & antisymmetric.

  any 2 distinct element can be compared.

  i.e. $\forall u,v \in V, u \neq v$, either $u \leq v$ or $v \leq u$



ready means orders is.

$\{ (ws, sw), (ws, sp),$

$(ws, b), (ws, h), (ws, c),$

$(ws, m)\ (ws, sc), (sw, sp) \dots \}$.

all the edges points to the right.

topological ordering

wool sweater ≤ sweater ≤ snow pants ≤ boots ≤ hat ≤ coat ≤ mittens ≤ scarf.

not necessarily unique!

Theorem! every DAG has a topological ordering.

Proving a Graph Property

Recall

- **in-degree** of vertex *v* : # of edges in E with *v* as the end vertex (x, v)
- **out-degree** of vertex *v* : # of edges in E with *v* as the start vertex (v, y)

Lemma : Every finite non-empty DAG has a vertex with in-degree 0.

Proof by contradiction

Suppose $G = (V, E)$ is a finite non-empty DAG. Assume G has **no** vertices that have in-degree 0. Let $n = |V|$.

The following invariant holds at the beginning of stage i for $1 \le i \le n$:

> There exist distinct vertices $v_1, v_2, \ldots v_i$ such that there exists a path from v_i to v_1.

$$v_i \to v_{i-1} \to \cdots \to v_3 \to v_2 \to v_1$$

Let $i = 1$: picks any vertexes to be V_1

$\exists$ empty path (of length 0) from V_1 to V_1 so invariants holds

Suppose the invariant holds at the beginning of stage i where $i < n$, so there exists a path from v_i to v_1.

In degree $V_i \geq 1$, $\exists u \in V$ s.t. $(u, V_i) \in E$.

$$V_i \to V_{i-1} \to \cdots \; V_{k+1} \to \underset{=u}{V_k} \to V_{k-1} \cdots \to V_1$$

If $u = V_k$ for some $k \in \{1, 2, \ldots, i\}$, then

$(V_i, V_{i-1})\ (V_{i-1}, V_{i-2}) \cdots (V_{k+1}, V_k), (V_k, V_i)$. is a cycle.

which contradict G being DAG.

- so assume $u \notin V_k$ for any $k \in \{1, 2, 3, \ldots, i\}$.
  then define $V_{i+1} = u$ & now we have the path $(\underset{u}{V_{i+1}}, V_i)\ (V_i, V_{i-1}) \cdots (V_2, V_1)$
  & invariant holds.

Consider the beginning of stage n. By the invariant, there exists a path from v_n to v_1.

$$V_n \to V_{n-1} \to V_{n-2} \cdots \to V_2 \to V_1$$

Since $|V| = n$, this path includes all the vertices of V.

By assumption, in-degree of $V_n \geq 1$, so $\exists V_k \in V$ s.t. $(V_k, V_n) \in E$.

$$V_n \to V_{n-1} \to \cdots \; V_{k+1} \to V_k \to V_{k-1} \cdots \to V_1$$

This creates cycle! $(V_n, V_{n-1})\ (V_{n-1}, V_{n-2}) \cdots (V_{k+1}, V_k), (V_k, V_n)$

which contradict with G being acyclic.

All situations lead to a contradiction, so initial assumption is false.

$\therefore \exists$ at least one vertex in V with in-degree of 0 ■