Relations

Relation

A **relation** R between A and B is a subset of $A \times B$

Recall $A \times B = \{(a,b) \mid a \in A, b \in B\}$ order pairs

If A, B are finite, then $|A \times B| = |A| \cdot |B|$

A: domain of R. B = codomain of R

If $A = B$, then R is a relation on A

If $A = B$, then R is a relation on A

Notation: $(a,b) \in R$ or aRb

How many relations on A are there if |A| = n? i.e. $|\mathcal{P}(A \times A)|$

$= 2^{|A|^2} = 2^{n^2}$

Examples

The examples below are relations on integers (i.e., A = **Z**) unless otherwise specified.

- R_1 = { (1, 1), (1, 2), (2, 1), (3, 3), (2, 2) }
- R_2 = { (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4) } – if $A = \{1,2,3,4\}$. then $R_2 = \{(a,b) \mid A \mid b\}$.
- R_3 = { (a, b) | a ≤ b }
- R_4 = { (a, b) | a > b }
- R_5 = { (a, b) | a = b or a = -b }
- R_6 = { (a, b) | a | b } → a divides b i.e $\exists k \in \mathbb{Z}$, s.t. $b = a \cdot k$.
- R_7 = { (a, b) | a + b ≤ 3}
- R_8 = { (a, b) | a $\equiv_4$ b } where A = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

Congruence modulo *m*

$A \equiv b \pmod{m}$ or $a \equiv_m b$

$\Leftrightarrow m \mid (A - b)$

$\Leftrightarrow a \bmod m = b \bmod m$

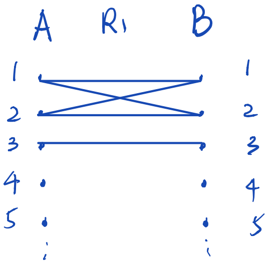
i.e. a and b have the same remainder when divided by m

Example $A = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

$1 \equiv_4 b \quad b = 4n + 1$

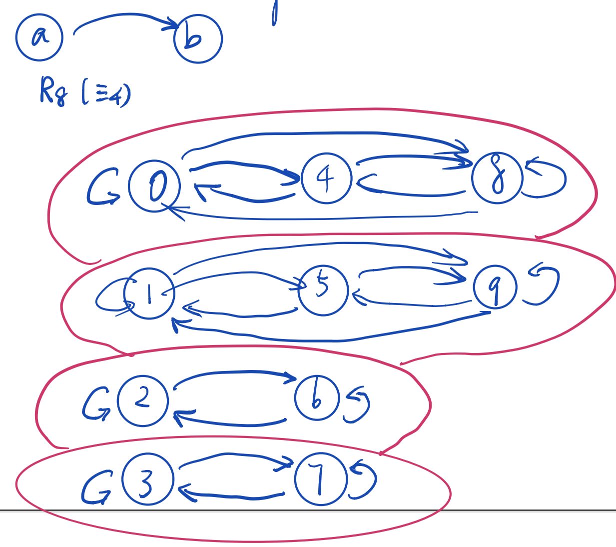
Representing relations

Bipartite graph



line from a on the left to b on the right means $(a,b) \in R$

Directed graph only when $A = B$.



Properties of relations

reflexive : A releation ***R*** on a set ***A*** is ***reflexive*** iff $(\forall a \in A)\, aRa$ i.e $(a,a) \in R$

every node has self loop

antireflexive : A releation ***R*** on a set ***A*** is ***antireflexive*** iff $(\forall a \in A)\, \neg(aRa)$ i.e $(a,a) \notin R$.

direct graph has no self loops at all.

symmetric : A releation ***R*** on a set ***A*** is ***symmetric*** iff $(\forall a, b \in A)$ $aRb \Leftrightarrow bRa$ i.e $(a,b) \in R \Leftrightarrow (b,a) \in R$.

i.e. $a \to b \Leftrightarrow a \leftarrow b$

antisymmetric : A releation ***R*** on a set ***A*** is ***antisymmetric*** iff

$(\forall a, b \in A)(aRb \wedge bRa) \Rightarrow (a = b)$ i.e. $((a,b) \in R \wedge (b,a) \in R) \Rightarrow (a = b)$

no Consider This is antisymmetric.

Recall: $P \Rightarrow Q \equiv \neg P \vee Q$

So $(\forall a, b \in A)\, (a,b) \notin R \vee (b,a) \notin R \vee (a = b)$

Consider both symmetric and antisymmetric

transitive : A releation ***R*** on a set ***A*** is ***transitive*** iff

Examples : Consider the examples above where we further restrict the domain of R_1 to A = {1, 2, 3} and the domain of R_2 to A = {1, 2, 3, 4}. Which of the examples are:

- reflexive R_1, R_2, R_3, R_5, R_6, R_8 } R_7 is neither.
- antireflexive R_4
- symmetric R_1, R_5, R_7, R_8
- antisymmetric R_2, R_3, R_4, (R_6 if $A = \mathbb{Z}^+$)
- transitive

R_6: if $A = \mathbb{Z}$, neither symmetric or antisymmetric $2 | -2$, $-2 | 2$.
$= \mathbb{Z}^+$, antisymmetric

Equivalence and Order Relations

Examples : The examples below are relations on integers (i.e., A = **Z**) unless otherwise specified.

- R_1 = { (1, 1), (1, 2), (2, 1), (3, 3), (2, 2) } where A = {1, 2, 3}
- R_2 = { (1, 1), (1, 2), (1, 3), (1, 4), (2, 2), (2, 4), (3, 3), (4, 4) } where A = {1, 2, 3, 4}
- R_3 = { (a, b) | a ≤ b }
- R_4 = { (a, b) | a > b }
- R_5 = { (a, b) | a = b or a = -b }
- R_6 = { (a, b) | a | b } where A = $\mathbf{Z}^+$
- R_7 = { (a, b) | a + b ≤ 3}
- R_8 = { (a, b) | a $\equiv_4$ b } where A = {0, 1, 2, 3, 4, 5, 6, 7, 8, 9}

Properties of relations

- reflexive : (∀ a ∈ A) aRa $R_1, R_2, R_3, R_5, R_6, R_8$
- antireflexive : (∀ a ∈ A) ¬(aRa) R_4
- symmetric : (∀ a, b ∈ A) aRb ⟺ bRa R_1, R_5, R_7, R_8.
- antisymmetric : (∀ a, b ∈ A) (aRb ∧ bRa) ⟹ (a = b) R_2, R_3, R_4, R_6.
- transitive : (∀ a, b, c ∈ A) (aRb ∧ bRc) ⟹ aRc $R_1, R_2, R_3, R_4, R_5, R_6, R_8$

  i.e $(a,b) \in R \wedge (b,c) \in R \rightarrow (a,c) \in R$.

  $a \rightarrow b \rightarrow c \Rightarrow a \rightarrow c$

Equivalence relations

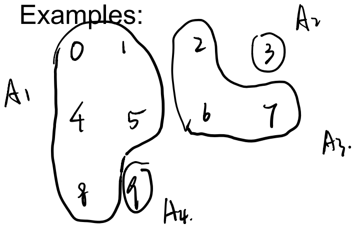
equivalence relation (on *A*) = a relation which is refexive, symmetric & transive.

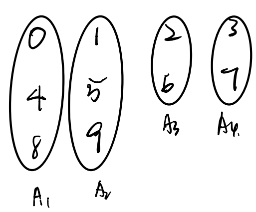
Examples: R_1, R_5, R_8, $=$, $\Leftrightarrow$, $\equiv$, $\equiv_n$

partition of a set *A* = collection of subsets of $A_1, A_2 \cdots$ s.t. $\bigcup_i A_i = A$. and $A_i \cap A_j = \phi$ if $i \neq j$. (i.e. A_i are pairwise disjoint).

each A_i is a cluster or a partition class

Examples:





equivalence class (given an equivalent relation R) $[a] = \{b \in A \mid aRb\}$.

Examples: Consider $\equiv_4 (R_8)$ $A = \{0,1,2,3,4,5,6,7,8,9\}$.

$[0] = \{0, 4, 8\} = [4] = [8]$

$[1] = \{1, 5, 9\} = [5] = [9]$

$[2] = \{2, 6\} = [6]$

$[3] = \{3, 7\} = [7]$.

Relationship between partitions and equivalence classes

1. Given a partition A_1, A_2, A_3, ... of A. Let R be the relation: aRb (i.e $(a,b) \in R$) iff a & b belong to the same partition class (same A_i).
   Then R is a equivalence relation.
2. Given an equivalence relation R on A. Then $P = \{[a] \mid a \in A\}$ is a partition.
3. Given an equivalence relation R on A. Then
   i. $(\forall a \in A)\ a \in [a]$
   ii. $(\forall a, b \in A)\ aRb \Rightarrow ([a] = [b])$
   iii. $(\forall a, b \in A)\ \neg(aRb) \Rightarrow ([a] \cap [b]) = \emptyset$.

What are the equivalence classes for R_1?

1 2

3

$[1] = \{1, 2\} = [2]$.

$[3] = \{3\}$.

What are the equivalence classes for R_5?

$[1] = \{1, -1\}$. $[0] = \{0\}$.

$[2] = \{2, -2\}$

$[3] = \{3, -3\}$.

Order Relations

order relation = a relation that is antisymmetric & transitive.

R_2, R_3, R_4, R_6, $<$, $>$, $\leq$, $\geq$, $\subseteq$, $\subset$, $|$ (divides).

partial order = an order relation that is reflexive.

$R_2, R_3, R_6, \leq, \geq, \subseteq$ canonical example $\leq$.

partially ordered set (poset) (S, R) = a set S along with partial order R and S.

$(\mathbb{Z}, \leq)$ $(\mathbb{Z}^+, |)$ divides.

strict order = an order relation that is antireflexive.

$R_4, <, >, \subset$ canonical example: $<$.

total order = an order relation R with the property

$(\forall a, b \in A)\ a \neq b \Rightarrow (aRb \vee bRa)$.

i.e. all elts in A can be compared with each other via R.

& put "in order"

R_3, R_4 are total orders

R_6 (divides) is not a total order:

$7 \nmid 2$, $2 \nmid 7$, $(7,2) \notin R_6$, $(2,7) \notin R_6$.

R_2? No, because $(2,3) \notin R_2$ and $(3,2) \notin R_2$.

"happens before" relation in computer systems.

$A \to B$ means memory written by stmt A is visible to stmt B

i.e. A completes its write to memory before B starts its read from memory.