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* § 4.4 Change of Variable

X, pdf, f_X ✓

$Y = g(X)$ pdf of Y?

- Distribution function technique
  - step: find cdf of Y, $F_Y(y)$
  - step 2: take derivative of $F_Y(y)$, then pdf $f_Y(y) = F_Y'(y)$.

E.g. $Y = aX + b$ $(a \neq 0)$, with pdf of X is $f_X(x)$

$\cdot F_Y(y) = P(Y \leq y) = P(aX + b \leq y)$

$X \leq \frac{y-b}{a}$ $a > 0$

$X \geq \frac{y-b}{a}$ $a < 0$.

If $a > 0$ $F_Y(y) = P(X \leq \frac{y-b}{a}) = F_X(\frac{y-b}{a})$

$f_Y(y) = (F_X(\frac{y-b}{a}))' = f_X(\frac{y-b}{a}) \cdot (\frac{y-b}{a})'$

by "chain rule".

$= f_X(\frac{y-b}{a}) \cdot \frac{1}{a}$.

if $a < 0$, $f_Y(y) = f_X(\frac{y-b}{a}) \cdot (-\frac{1}{a})$

Thus, $f_Y(y) = f_X(\frac{y-b}{a}) \cdot \left|\frac{1}{a}\right|$

• (Monotone functions) Change-of-variable

Theorem: Let X be a continuous r.v. with pdf of f_X.

suppose that $g(x)$ is a strictly monotone differentiable of x, then $Y = g(X)$ has a pdf

$$f_Y(y) = \begin{cases} f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|, & \text{if } y = g(x) \text{ for some } x. \\ 0, & \text{otherwise} \end{cases}$$

Alternate Formula. $f_Y(y) = f_X(x) / \left| \frac{dy}{dx} \right|$, where $y = g(x)$.

step 1 $= f_X(x) / |g'(x)|$

step 2 $= f_X(g^{-1}(y)) / |g'(g^{-1}(y))|$ where $y = g(x)$

i.e. solve for x from $y = g(x)$
$x = g^{-1}(y)$

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E.g. $Y = X^2$. X is a r.v. w/ pdf f_X on $[0,1]$.

Find pdf of Y

Sol: (i) $f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right|$ $\quad y = x^2 \Rightarrow x = \underset{g^{-1}(y)}{\sqrt{y}}$ $\quad 0 \le y \le 1$.

$= f_X(\sqrt{y}) \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{y}}$

(ii) alternate formula

$f_Y(y) = f_X(x) / \left| \frac{dy}{dx} \right| = f_X(\sqrt{y}) / |2x|$

$= f_X(\sqrt{y}) / |2\sqrt{y}|$

$= \frac{1}{2} \cdot \frac{1}{\sqrt{y}} f_X(\sqrt{y})$

E.g. Let X be a continuous nonnegative r.v. with pdf f_X.

Find the pdf of $Y = X^n$. (n is a positive integer).

Sol: 1-1 func.

$$f_Y(y) = f_X(g^{-1}(y)) \cdot \left|\frac{d}{dy} g^{-1}(y)\right|$$

$$= f_X(y^{\frac{1}{n}}) \left|\frac{1}{n} \cdot y^{\frac{1}{n}-1}\right|$$

$$= \frac{1}{n} \cdot y^{\frac{1-n}{n}} f_X(y^{\frac{1}{n}})$$

E.g. log of uniform. $\quad f_X(x) = 1 \quad 0 \le x \le 1$

Suppose $X \sim$ uniform $(0,1)$. Find the distribution of

$Y = -\frac{1}{\lambda} \ln X$, where $\lambda > 0$.

Sol: Possible values of Y or range of Y.

$Y \geq 0$. $\quad g(x) = -\frac{1}{\lambda} \ln x \qquad$ is $g(x)$ 1-1?

$g^{-1}(y) \quad y = -\frac{1}{\lambda} \ln x \Rightarrow x = e^{-\lambda y}$, $y \geq 0$

$$f_Y(y) = f_X(e^{-\lambda y}) \cdot \left|\frac{d}{dy} e^{-\lambda y}\right|$$

$$= 1 \cdot |-\lambda \cdot e^{-\lambda y}|$$

$$= \lambda e^{-\lambda y}$$

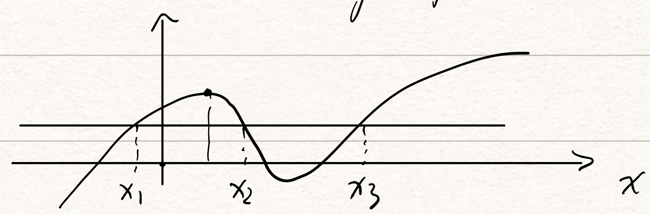
$Y \sim \exp(\lambda)$.

Lognormal Distribution

Def: A r.v. X is said to have a lognormal distribution if the r.v. $Y = \log X$ has a normal distribution $N(\mu, \sigma^2)$. The pdf of X is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma x} \cdot e^{-\frac{(\log - \mu)^2}{2\sigma^2}}, \quad x \geq 0.$$

Many-to-one change of variable.



Suppose $y = g(x)$ has a derivative that is zero at only a finite number of points. Now some values of y may come from more than one value of x. consider $Y = g(X)$ for a r.v. X then

$$f_Y(y) = \sum f_X(x) / \left|\frac{dy}{dx}\right| \qquad f_X(x_1) / \left|\frac{dy}{dx}\Big|_{x=x_1}\right| + f_X(x_2) \left|\frac{dy}{dx}\Big|_{x=x_2}\right| \cdots$$

E.g. Suppose X has pdf f_X. Find the distribution of $Y = X^2$

Sol: use many-to-one formula. $x = \pm\sqrt{y}$

$$f_Y(y) = f_X(\sqrt{y}) \left|\frac{dy}{dx}\Big|_{x=\sqrt{y}}\right| + f_X(-\sqrt{y}) \left|\frac{dy}{dx}\Big|_{x=-\sqrt{y}}\right|$$

$$= f_X(\sqrt{y})/|2\sqrt{y}| + f_X(-\sqrt{y})/|2\sqrt{y}|$$

$$= \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] \qquad y \geq 0.$$

$$f_Y(y) = \begin{cases} \frac{1}{2\sqrt{y}}\left[f_X(\sqrt{y}) + f_X(-\sqrt{y})\right] & y \geq 0 \\ 0 & \text{otherwise.} \end{cases}$$