Dynamic Optimization

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Contents

1 Define the problem

The provided labor demand model can be solved is 2 different ways:

1. Bellman equation with 2 decision variables
2. Bellman equation with 1 decision variable by finding the other one numerically

Both methods will be shown.

1.1 2 decision variables

Bellman Equation:

$$\begin{aligned} v(e_t,h_t) = \max_{e_{t+1},h_{t+1}} \quad & \{R(e_t,h_t) - \omega(e_t,h_t) - C(e_{t+1},e_t) + \beta v(e_{t+1},h_{t+1})\} \\ \text{s.t.} \quad & e_t, h_t \; given \end{aligned} \tag{1}$$

Where:

- e_t, h_t (number of workers and the hours worked at time t) are the states variables
- e_{t+1}, h_{t+1} are the decision variables
- $R(e_0, h_0)$ revenues function
- $\omega(e_0, h_0)$ cost of hiring function
- $C(e_1, e_0)$ adjustment cost function
- $v(e_0, h_0)$ value function given the states at time zero
- $v(e_0, h_0)$ value function given the states at time one

1.2 1 decision variable

In order to write down the problem with 1 decision variable it is possible to find the optimal "h*" given a certain "e" by maximizing the difference $R(e,h) - \omega(e,h)$ with respect to "h".

In the following all the steps:

$$\Pi(e,h) = R(e,h) - \omega(e,h) \tag{2}$$

First Order Condition:

$$\frac{\partial R(e,h)}{\partial h} = \frac{\partial \omega(e,h)}{\partial h} \tag{3}$$

$$Ah^{\alpha-1}\alpha e^{\alpha} = we + wew_1 + 2wew_2 h - 80wew_2 \tag{4}$$

$$Ah^{\alpha-1}\alpha e^{\alpha} - 2wew_2 h = we + wew_1 - 80wew_2 \tag{5}$$

We should solve the last equation by isolating "h".
However, since it is not easy to find a closed-form solution, we will proceed by numerical approximation.

Regarding the algorithm it is important to highlight that the optimal "h*" is chosen among the grid of integer numbers 1:70, and this brings sub-optimal choices for "e" values in the range 1:10
Below some examples of the algorithm, specifically the difference between the RHS and the LHS of the equation for different values of "h" and given a different "e":

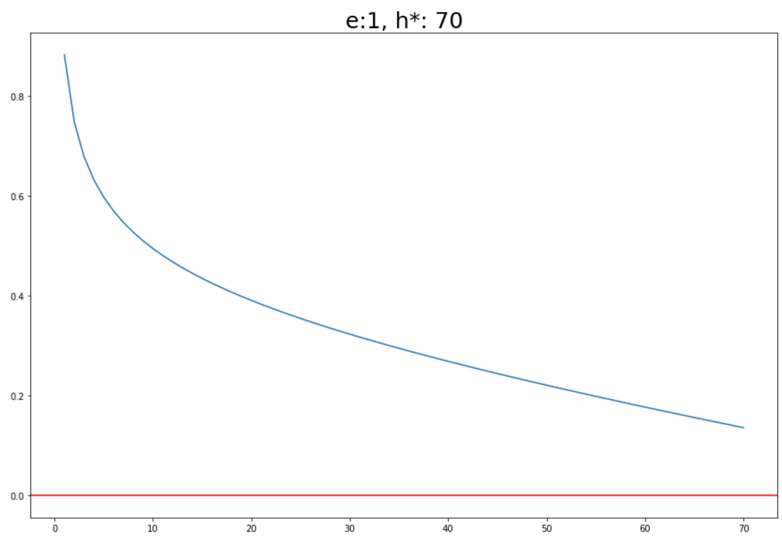

Figure 1: Right hand side - Left hand side, for e=1

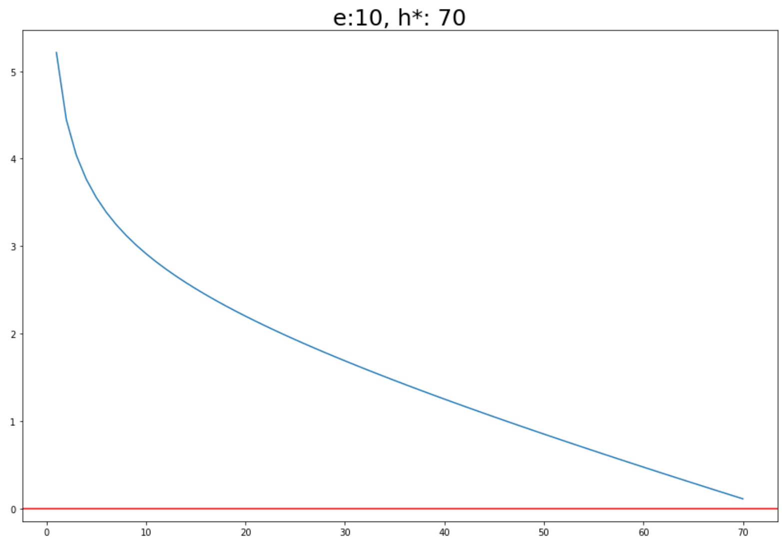

Figure 2: Right hand side - Left hand side, for e=10

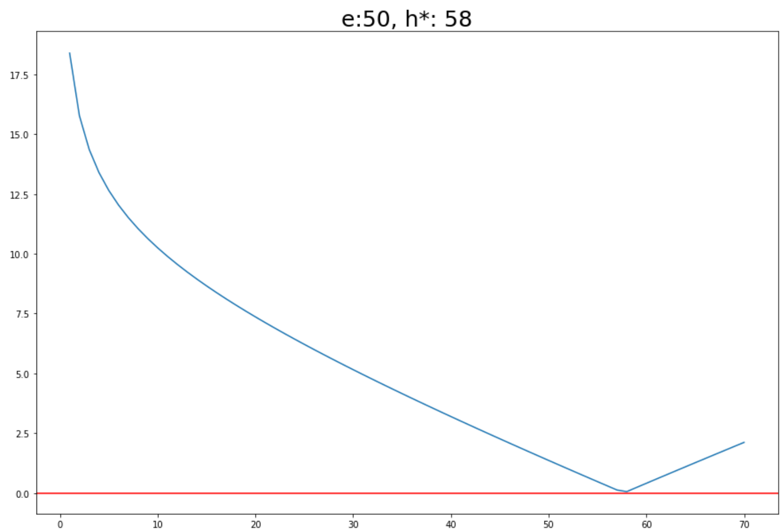

Figure 3: Right hand side - Left hand side, for e=50

The resulting Bellman Equation used in the project is the following:
(it is not a formal notation due to the lack of a closed-form solution for h*)

$$\begin{aligned} v(e_t) = \max_{e_{t+1}} \quad & \{R(e_t, h*) - \omega(e_t, h*) - C(e_{t+1}, e_t) + \beta v(e_{t+1})\} \\ \text{s.t.} \quad & e_t \; given \end{aligned} \tag{6}$$

2 Plots

2.1 Value function

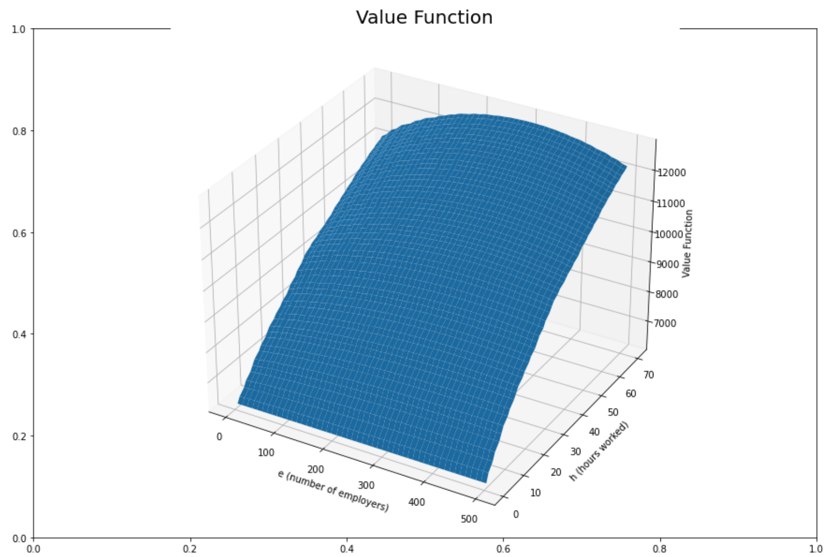

Figure 4: Value function by solving for 2 decision variables

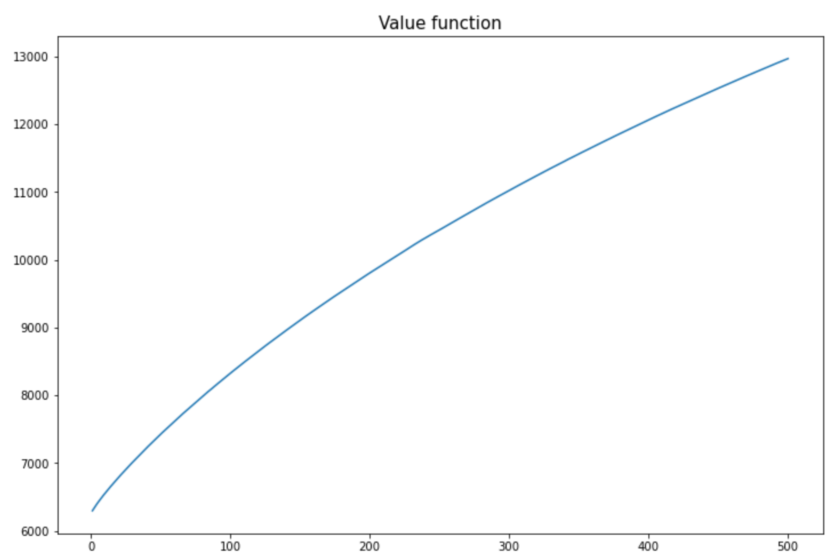

Figure 5: Value function by solving for 1 decision variable

2.2 State-Action

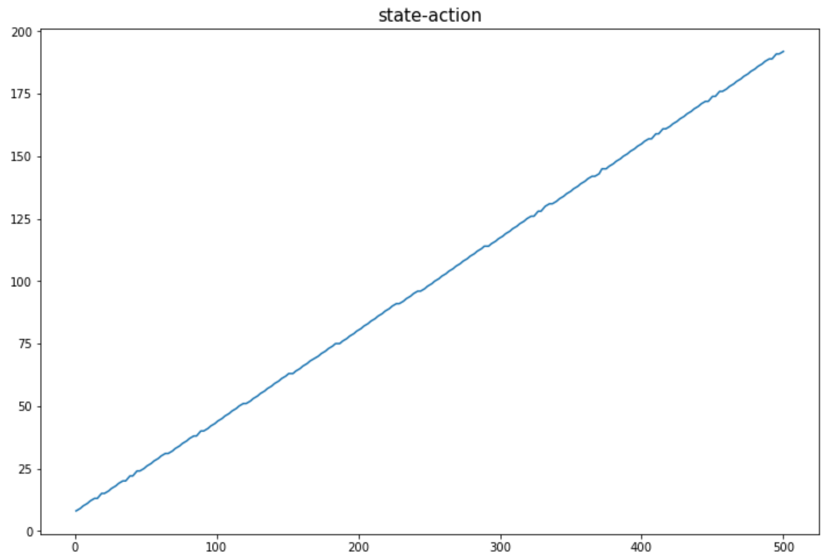

Figure 6: $e_t and e_{t+1}$

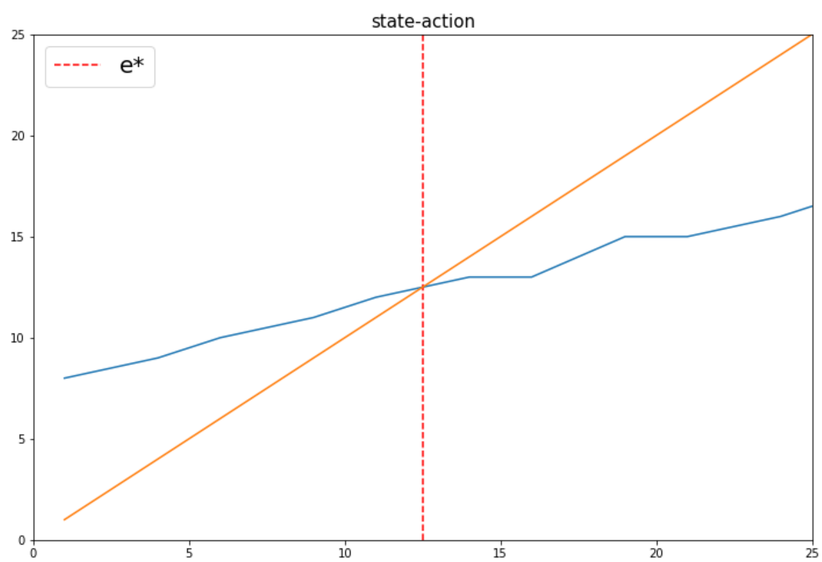

Figure 7: e_t, e_{t+1} and 45 degree line

2.3 Comments

- The value function appears to be concave and monotonically increasing with respect to the initial state ”e”
- It can been seen that, according to this model with parameters and these grids, the ”e” in the long run will converge towards e* which is around 13. Of course, in order to have more precise results, which can change also a lot, it would be required to simulate the model with much denser grids