Modified Field Theories and Gravity

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Запись не предназначена для распространения.

Обсуждаются детали особых теорий поля.

Цели:

1) Создаю 1 часть!!! на 23.12.2023 2 дня только ее и буду создавать!

Содержание

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1 Предисловие

Обсудим минимальное, что хорошо бы понимать для занятий предметом.

1.1 Основная мотивация

Обсудим основную мотивацию, которая позволит нам познать предмет без проблем.

Если кто-то умничает и нужно тоже поумничать, и если у кого-то есть сомнения в профессионализме, то после прохождения основ суперсимметрий и гравитации с этим не будет проблем

Можно всего за месяц научиться некоторым методам, после которых все будут думать, что ты очень много знаешь в теорфизе. Что приятно, это повышает самооценку и авторитет, и, возможно, это одно из самых основных применений этих тем.

Результаты

Обсудим, что полезного и важного человечество получило, поняв этот предмет.

Идейные головоломки для мотивации

(тут много будет, мб пару разделов сделаю)

Технические головоломки для мотивации

(тут только 1 раздел, потому что технические задачи редко мотивируют)

Часть I

Different Fields and Gravity in a Nutshell

2 Typical Modified Theories in a Nutshell

2.1 Maths tricks of Proyen's type for derivations (!!!)

(тут самые типичные ходы по преобразованиям, соберу их.)

2.2 Supergravity in a Nutshell

2.3 On tests and experimental constraints (!?!?!??!?)

(потом много указаний будет, это же ответ на вопрос, что же может быть верным, а что нет и почему? В ближайший год (2024й) вряд ли буду это прописывать.)

3 On details about relativistic field theory in Minkowski space-time

3.1 1 Scalar field theory and its symmetries

3.1.1 1 The scalar field system

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3.1.4 4 Symmetries in the canonical formalism

3.1.5 5 Quantum operators

3.1.6 6 The Lorentz group for D = 4

3.2 The Dirac field

3.2.1 The homomorphism of SL(2, C) SO(3, 1)

(это же в теории групп и алгебре, типичная тема)

First we note that a general 2×2 hermitian matrix can be parametrized as

$$\mathbf{x} = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix}$$

and that $\det \mathbf{x} = -x^\mu \eta_{\mu\nu} x^\nu$, which is the negative of the Minkowski norm of the 4-vector x^μ. This suggests a close relation between the linear space of hermitian 2×2 matrices and four-dimensional Minkowski space. Indeed, there is an isomorphism between these spaces, which we now elucidate.

$$\sigma_\mu = (-\mathbb{1}, \sigma_i), \quad \bar{\sigma}_\mu = \sigma^\mu = (\mathbb{1}, \sigma_i),$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\sigma_\mu \bar{\sigma}_\nu + \sigma_v \bar{\sigma}_\mu = 2\eta_{\mu v} \mathbb{1},$$
$$\operatorname{tr}(\sigma^\mu \bar{\sigma}_v) = 2\delta^\mu{}_\nu.$$

$$\mathbf{x} = \bar{\sigma}_\mu x^\mu, \quad x^\mu = \frac{1}{2} \operatorname{tr}(\sigma^\mu \mathbf{x})$$

$$\phi(A)^\mu_v = \frac{1}{2} \operatorname{tr}\left(\sigma^\mu \mathrm{A} \bar{\sigma}_v \ \mathrm{A}^\dagger\right)$$

$$\sigma_{\mu\nu} = \frac{1}{4}(\sigma_\mu \bar{\sigma}_\nu - \sigma_\nu \bar{\sigma}_\mu),$$
$$\bar{\sigma}_{\mu\nu} = \frac{1}{4}(\bar{\sigma}_\mu \sigma_\nu - \bar{\sigma}_\nu \sigma_\mu).$$

Note that $\sigma^{\mu\nu\dagger} = -\bar{\sigma}^{\mu\nu}$. The finite Lorentz transformation (1.32) is then represented as

$$L(\lambda) = \mathrm{e}^{\frac{1}{2}\lambda^{\mu v}\sigma_{\mu v}},$$
$$\bar{L}(\lambda) = \mathrm{e}^{\frac{1}{2}\lambda^{\mu v}\bar{\sigma}_{\mu v}}.$$

Exercise 2.5 Show that

$$L(\lambda)^\dagger = \bar{L}(-\lambda) = \bar{L}(\lambda)^{-1}$$

3.2.2 The Dirac equation

$$\partial\!\!\!/ \Psi(x) \equiv \gamma^\mu \partial_\mu \Psi(x) = m\Psi(x).$$

The quantities $\gamma^\mu, \mu = 0, 1, \ldots, D-1$, are a set of square matrices, which act on the indices of the spinor field Ψ. Applying the Dirac operator again, one finds

$$\partial^2 \Psi = m^2 \Psi$$

$$\frac{1}{2}\{\gamma^\mu \gamma^v + \gamma^v \gamma^\mu\} \partial_\mu \partial_v \Psi = m^2 \Psi$$

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu \gamma^\nu + \gamma^v \gamma^\mu = 2\eta^{\mu\nu} \mathbb{1}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

$$\Sigma^{\mu\nu} \equiv \frac{1}{4}[\gamma^\mu, \gamma^\nu]$$

$$[\Sigma^{\mu\nu}, \gamma^\rho] = 2\gamma^{[\mu}\eta^{\nu]\rho} = \gamma^\mu \eta^{\nu\rho} - \gamma^v \eta^{\mu\rho}.$$

$$L(\lambda) = \mathrm{e}^{\frac{1}{2}\lambda^{\mu v}\Sigma_{\mu\nu}}$$
$$L(\lambda)\gamma^\rho L(\lambda)^{-1} = \gamma^\sigma \Lambda(\lambda)^\rho_\sigma$$
$$\Psi'(x) = L(\lambda)^{-1}\Psi(\Lambda(\lambda)x)$$

$$
\begin{aligned}
\Psi(x) &= \Psi_+(x) + \Psi_-(x), \\
\Psi_+(x) &= \int \frac{\mathrm{d}^{(D-1)}\vec{p}}{(2\pi)^{D-1}2E} \mathrm{e}^{\mathrm{i}(\vec{p}\cdot\vec{x}-Et)} \sum_s u(\vec{p},s)c(\vec{p},s), \\
\Psi_-(x) &= \int \frac{\mathrm{d}^{(D-1)}\vec{p}}{(2\pi)^{D-1}2E} \mathrm{e}^{-\mathrm{i}(\vec{p}\cdot\vec{x}-Et)} \sum_s v(\vec{p},s)d(\vec{p},s)^*.
\end{aligned}
$$

The * indicates complex conjugation in the classical theory and an operator adjoint after quantization.

3.2.3 Dirac adjoint and bilinear form

$$
\begin{aligned}
\delta\Psi(x) &= -\frac{1}{2}\lambda^{\mu v}\left(\Sigma_{\mu\nu} + L_{[\mu\nu]}\right)\Psi(x) = -\frac{1}{2}\lambda^{\mu v}\Sigma_{\mu\nu}\Psi(x) + \lambda^{\mu}{}_{\nu}x^{v}\partial_\mu\Psi(x), \\
\delta\Psi^\dagger(x) &= -\frac{1}{2}\lambda^{\mu v}\Psi^\dagger\Sigma_{\mu\nu}{}^\dagger + \lambda^{\mu}{}_{\nu}x^{v}\partial_\mu\Psi(x)^\dagger.
\end{aligned}
$$

$$\Psi^\dagger\beta\Psi$$

$$\Sigma_{\mu\nu}^\dagger\beta + \beta\Sigma_{\mu\nu} = 0$$

$$\Sigma_{\mu\nu}^\dagger = -\Sigma_{\mu\nu}$$

$$
\begin{aligned}
\beta\gamma_\mu\beta^{-1} &= -\gamma_\mu^\dagger, \\
\beta\Sigma_{\mu\nu}\beta^{-1} &= -\Sigma_{\mu\nu}^\dagger.
\end{aligned}
$$

The last relation is a rewriting of (2.27), which shows that the Dirac spinor representation of the Lorentz group is equivalent to the transposed, complex conjugate representation. It is convenient to make the specific choice

$$\beta = \mathrm{i}\gamma^0.$$

The Dirac adjoint is a row vector:

$$\bar{\Psi} := \Psi^\dagger\beta = \Psi^\dagger\mathrm{i}\gamma^0,$$

(!!!!!!!!!!!!! вот это важное определение!!!)

and write our invariant bilinear form as

$$\bar{\Psi}\Psi.$$

$$\left(\bar{\Psi}_1\Psi_2\right)^\dagger = \bar{\Psi}_2\Psi_1$$

$\bar{\Psi}\Psi$ has signature $(2,2)$ in four dimensions.

3.2.4 Dirac action

(мб уменьшу раздел)

$$S[\bar{\Psi}, \Psi] = -\int \mathrm{d}^D x \bar{\Psi} \left[\gamma^\mu \partial_\mu - m\right] \Psi(x)$$

It is stationary reads (including integration by parts in the second term):

$$\delta S[\bar{\Psi}, \Psi] = -\int \mathrm{d}^D x \left\{ \overline{\delta\Psi} \left[\gamma^\mu \partial_\mu - m\right] \Psi - \bar{\Psi} \left[\gamma^\mu \overleftarrow{\partial}_\mu + m\right] \delta\Psi \right\} = 0.$$

$$\bar{\Psi} \left[\gamma^\mu \overleftarrow{\partial}_\mu + m\right] = 0$$

3.2.5 The spinors $u(\vec{p}, s)$ and $v(\vec{p}, s)$ for $D = 4$

Since $\Psi_\pm(x)$ in (2.24) satisfy the Dirac equation (2.16) and the plane waves $\mathrm{e}^{\mathrm{i}p\cdot x}$ for different 4-vectors p^μ are linearly independent, it follows that the spinors satisfy the algebraic equations

$$\begin{aligned} \gamma \cdot p u(\vec{p}, s) &= -\mathrm{i} m u(\vec{p}, s), \\ \gamma \cdot p v(\vec{p}, s) &= +\mathrm{i} m v(\vec{p}, s). \end{aligned}$$

We will solve these equations using the Weyl representation (2.19) in which the equation for $u(\vec{p}, s)$ in (2.35) becomes

$$\begin{pmatrix} 0 & \sigma \cdot p \\ \bar{\sigma} \cdot p & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -\mathrm{i} m \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

$$-\sigma \cdot p \bar{\sigma} \cdot p = -\bar{\sigma} \cdot p \sigma \cdot p = m^2.$$

Using this information it is easy to see that

$$u(p) = \begin{pmatrix} \sqrt{-\sigma \cdot p} \xi \\ \mathrm{i} \sqrt{\bar{\sigma} \cdot p} \xi \end{pmatrix}$$

is a solution of (2.36) for any two-component spinor ξ. Similarly

$$v(p) = \begin{pmatrix} \sqrt{-\sigma \cdot p} \eta \\ -\mathrm{i} \sqrt{\bar{\sigma} \cdot p} \eta \end{pmatrix}$$

It is convenient to choose spin states which are eigenstates of the helicity, the component of angular momentum in the direction of motion of the particle. Therefore we define spinors $\xi(\vec{p}, \pm)$ that satisfy

$$\vec{\sigma} \cdot \vec{p} \xi(\vec{p}, \pm) = \pm |\vec{p}| \xi(\vec{p}, \pm)$$

Note that $\vec{\sigma} \cdot \vec{p} \equiv \sigma^i p^i$ is summed over the spatial components only. We assume these spinors to be normalized:

$$\xi(\vec{p}, \pm)^\dagger \xi(\vec{p}, \pm) = 1, \quad \xi(\vec{p}, \pm)^\dagger \xi(\vec{p}, \mp) = 0$$

Since the angular momentum operator is $\vec{J} = \frac{1}{2}\vec{\sigma}$, the spinor $\xi(\vec{p}, \pm)$ is an eigenstate of $\vec{p} \cdot \vec{J} / |\vec{p}|$ with eigenvalue $\pm 1/2$. We also choose

$$\eta(\vec{p}, \pm) = -\sigma_2 \xi(\vec{p}, \pm)^*$$

$$\vec{\sigma} \cdot \vec{p} \eta(\vec{p}, \pm) = \mp |\vec{p}| \eta(\vec{p}, \pm)$$

.

$$
\begin{aligned}
u(\vec{p}, \pm) &= \begin{pmatrix} \sqrt{E \mp |\vec{p}|}\xi(\vec{p}, \pm) \\ \mathrm{i}\sqrt{E \pm |\vec{p}|}\xi(\vec{p}, \pm) \end{pmatrix} \\
v(\vec{p}, \pm) &= \begin{pmatrix} \sqrt{E \pm |\vec{p}|}\eta(\vec{p}, \pm) \\ -\mathrm{i}\sqrt{E \mp |\vec{p}|}\eta(\vec{p}, \pm) \end{pmatrix}
\end{aligned}
$$

$$
u(\vec{p}, -) = \sqrt{2E}\begin{pmatrix} \xi(\vec{p}, -) \\ 0 \end{pmatrix}, \quad u(\vec{p}, +) = \sqrt{2E}\begin{pmatrix} 0 \\ \mathrm{i}\xi(\vec{p}, +) \end{pmatrix}
$$

Similarly, one finds for massless v spinors

$$
v(\vec{p}, -) = \sqrt{2E}\begin{pmatrix} 0 \\ -\mathrm{i}\eta(\vec{p}, -) \end{pmatrix}, \quad v(\vec{p}, +) = \sqrt{2E}\begin{pmatrix} \eta(\vec{p}, +) \\ 0 \end{pmatrix}
$$

$$
\begin{aligned}
\bar{u}(\vec{p}, s)u(\vec{p}, s') &= -\bar{v}(\vec{p}, s)v(\vec{p}, s') = -2m\delta_{ss'}, \\
\bar{u}(\vec{p}, s)v(\vec{p}, s') &= \bar{v}(\vec{p}, s)u(\vec{p}, s') = 0, \\
\bar{u}(\vec{p}, s)\gamma^\mu u(\vec{p}, s) &= \bar{v}(\vec{p}, s)\gamma^\mu v(\vec{p}, s) = -2\mathrm{i}p^\mu.
\end{aligned}
$$

3.2.6 Weyl spinor fields in even spacetime dimension

Let's return to the case of even dimension $D = 2$ m. We saw in Sec. 2.2 that the Dirac representation of the Lorentz group is reducible if $D = 4$. Since there is a 'block offdiagonal' representation for any even dimension, the same is true for all $D = 2m$ with two irreducible subrepresentations, each of dimension $2^{(m-1)}$. This suggests that a Dirac spinor $\Psi(x)$ is not the simplest Lorentz covariant field. In a sense this is correct. One can define a Weyl field $\psi(x)$ with $2^{(m-1)}$ components, which is defined to transform as

$$
\psi(x) \to \psi'(x) = L(\lambda)^{-1}\psi(\Lambda(\lambda)x),
$$

where $L(\lambda)$ is defined as in (2.11), but now for any even dimension. One can also define a field $\bar{\chi}(x)$ that transforms in the conjugate representation, namely as

$$
\bar{\chi}(x) \to \bar{\chi}'(x) = \bar{L}(\lambda)^{-1}\bar{\chi}(\Lambda(\lambda)x),
$$

with $\bar{L}(\lambda)$ defined as in (2.12). There are Lorentz invariant wave equations for these fields:

$$
\begin{aligned}
\bar{\sigma}^\mu\partial_\mu\psi(x) &= 0, \\
\sigma^\mu\partial_\mu\bar{\chi}(x) &= 0.
\end{aligned}
$$

$$
\psi(x)^\dagger\bar{\sigma}^\mu\psi(x) \to \Lambda^{-1\mu}{}_\nu\psi(\Lambda x)^\dagger\bar{\sigma}^v\psi(\Lambda x)
$$

The Lorentz invariant hermitian action is then

$$
S[\psi, \bar{\psi}] = -\int \mathrm{d}^D x \mathrm{i}\psi^\dagger\bar{\sigma}^\mu\partial_\mu\psi
$$

The candidate wave equation

$$
\bar{\sigma}^\mu\partial_\mu\psi(x) = m\psi(x)
$$

is not Lorentz invariant.

One can describe massive particles using both $\psi(x)$ and $\bar{\chi}(x)$. In fact this is the secret content of a single Dirac field in any even dimension, and this can be exhibited using a Weyl representation of the γ-matrices.

Exercise 2.19 Show this! Write the Dirac field as the column

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \bar{\chi}(x) \end{pmatrix}$$

and show that the Dirac equation (2.16) in the representation (2.19) is equivalent to the pair of equations

$$\bar{\sigma}^\mu \partial_\mu \psi(x) = m\bar{\chi}(x), \quad \sigma^\mu \partial_\mu \bar{\chi}(x) = m\psi(x).$$

The Dirac Lagrangian in (2.32) can be rewritten in terms of $\psi, \bar{\chi}$ and their adjoints as

$$\mathcal{L} = \mathrm{i}\left[-\psi^\dagger \bar{\sigma} \cdot \partial\psi + \bar{\chi}^\dagger \sigma \cdot \partial\bar{\chi} - m\bar{\chi}^\dagger \psi + m\psi^\dagger \bar{\chi}\right].$$

Show that each of the four terms is a Lorentz scalar. Note that the result in (2.13) holds for all even $D = 2m$.

3.2.7 Conserved currents

2.7.1 Conserved U(1) current 36

In this section we discuss the global $U(1)$ symmetry property of the Dirac field in any spacetime dimension. The free Dirac action (2.32) is invariant under the global U(1) phase transformation $\Psi(x) \rightarrow \Psi'(x) \equiv \mathrm{e}^{\mathrm{i}\theta}\Psi(x)$. The conserved Noether current for this symmetry is

$$J^\mu = \mathrm{i}\bar{\Psi}\gamma^\mu\Psi$$

The time component is positive in all Lorentz frames for commuting complex spinors,

$$J^0 = \Psi^\dagger\Psi > 0.$$

Thus, the vector J^μ is generically future-directed time-like.

2.7.2 Energy-momentum tensors for the Dirac field 37

$$T_{\mu\nu} = \bar{\Psi}\gamma_\mu\partial_\nu\Psi + \eta_{\mu\nu}\mathcal{L}$$

Exercise 2.21 It is well known that the Lagrangian density of a field theory can be changed by adding a total divergence $\partial_\mu B^\mu$, since the Euler-Lagrange equations are unaffected. Show that the addition of $\frac{1}{2}\partial_\mu\left(\bar{\Psi}\gamma^\mu\Psi\right)$ brings the action to the form

$$S' = -\int \mathrm{d}^D x \left[\frac{1}{2}\bar{\Psi}\gamma^\mu \overleftrightarrow{\partial}_\mu \Psi - m\bar{\Psi}\Psi\right]$$

Note that the antisymmetric derivative is defined as

$$A \overleftrightarrow{\partial}_\mu B \equiv A\left(\partial_\mu B\right) - \left(\partial_\mu A\right) B$$

The advantage of the form (2.61) of the Dirac theory is that the Lagrangian density $\mathcal{L}'$ is hermitian as an operator in Hilbert space.

Exercise 2.22 Show that the Noether stress tensor obtained using $\mathcal{L}'$ is

$$T'_{\mu\nu} = \frac{1}{2}\bar{\Psi}\gamma_\mu \overleftrightarrow{\partial}_\nu \Psi + \eta_{\mu\nu}\mathcal{L}'$$

Show that $T'_{\mu\nu} - T_{\mu\nu} = \partial^\rho S_{\rho\mu\nu}$ where the tensor $S_{\rho\mu\nu}$ satisfies $S_{\rho\mu\nu} = -S_{\mu\rho v}$, as in the discussion of improved Noether currents in Sec. 1.3.

Exercise 2.23 Show that the addition of $\Delta T_{\mu\nu} = \frac{1}{4}\partial^{\rho}\left(\bar{\Psi}\left\{\Sigma_{\rho\mu}, \gamma_{\nu}\right\}\Psi\right)$ to $T'_{\mu\nu}$ produces the symmetric energy-momentum tensor

$$\Theta_{\mu\nu} = \frac{1}{4}\bar{\Psi}\left(\gamma_{\mu}\overset{\leftrightarrow}{\partial}_{\nu} + \gamma_{\nu}\overset{\leftrightarrow}{\partial}_{\mu}\right)\Psi + \eta_{\mu\nu}\mathcal{L}'$$

Note that symmetry currents are evaluated 'on-shell', i.e. one should assume that Ψ and $\bar{\Psi}$ satisfy the Dirac equation. The last term $\mathcal{L}'$ then vanishes.

3.3 Clifford algebras and spinors

$$\gamma^{\mu}\gamma^{v} + \gamma^{v}\gamma^{\mu} = 2\eta^{\mu v}\mathbb{1}.$$

3.3.1 The Clifford algebra in general dimension

3.1.1 The generating γ-matrices 39

$$\begin{aligned}
\gamma^{1} &= \sigma_{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\ldots, \\
\gamma^{2} &= \sigma_{2}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\ldots, \\
\gamma^{3} &= \sigma_{3}\otimes\sigma_{1}\otimes\mathbb{1}\otimes\ldots, \\
\gamma^{4} &= \sigma_{3}\otimes\sigma_{2}\otimes\mathbb{1}\otimes\ldots, \\
\gamma^{5} &= \sigma_{3}\otimes\sigma_{3}\otimes\sigma_{1}\otimes\ldots, \\
\ldots &= \ldots
\end{aligned}$$

$$\gamma^{\mu\dagger} = \gamma^{0}\gamma^{\mu}\gamma^{0}$$

It is fundamental to the Dirac theory that the physics of a spinor field is the same in all equivalent representations of the Clifford algebra. Thus we are really concerned with classes of representations related by conjugacy, i.e.

$$\gamma'^{\mu} = S\gamma^{\mu}S^{-1}$$

3.1.2 The complete Clifford algebra 40

$$\gamma^{\mu_1\cdots\mu_r} = \gamma^{[\mu_1}\ldots\gamma^{\mu_r]}, \quad \text{e.g.} \quad \gamma^{\mu v} = \frac{1}{2}\left(\gamma^{\mu}\gamma^{v} - \gamma^{v}\gamma^{\mu}\right),$$

where the antisymmetrization indicated with [...] is always with total weight 1. Thus the right-hand side of (3.5) contains the overall factor $1/r$! times a sum of r ! signed permutations of the indices. Non-vanishing tensor components can be written as the products

$$\gamma^{\mu_1\mu_2\cdots\mu_r} = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_r} \quad \text{where } \mu_1 \neq \mu_2 \neq \cdots \neq \mu_r$$

The higher rank γ-matrices can be defined as the alternate commutators or anti-commutators

$$\begin{aligned}
\gamma^{\mu\nu} &= \frac{1}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right], \\
\gamma^{\mu_1\mu_2\mu_3} &= \frac{1}{2}\left\{\gamma^{\mu_1}, \gamma^{\mu_2\mu_3}\right\}, \\
\gamma^{\mu_1\mu_2\mu_3\mu_4} &= \frac{1}{2}\left[\gamma^{\mu_1}, \gamma^{\mu_2\mu_3\mu_4}\right], \\
&\text{etc.}
\end{aligned}$$

(!?!??! чередуются)

$$\gamma^{\mu_1\cdots\mu_D} = \frac{1}{2}\left(\gamma^{\mu_1}\gamma^{\mu_2\cdots\mu_D} - (-)^D\gamma^{\mu_2\cdots\mu_D}\gamma^{\mu_1}\right).$$

Thus, $\operatorname{Tr}\gamma^{\mu_1\cdots\mu_D}$ vanishes for even D.

3.1.3 Levi-Civita symbol 41

$$\varepsilon_{012(D-1)} = 1, \quad \varepsilon^{012(D-1)} = -1.$$

Indices are raised using the Minkowski metric which leads to the difference in sign above (due to the single time-like direction).

Exercise 3.3 Prove the contraction identity for these tensors:

$$\varepsilon_{\mu_1\ldots\mu_n v_1\ldots v_p}\varepsilon^{\mu_1\cdots\mu_n\rho_1\ldots\rho_p} = -p!n!\delta^{\rho_1\cdots\rho_p}_{v_1\cdots v_p}, \quad p = D-n.$$

The antisymmetric p-index Kronecker δ is in turn defined by

$$\delta^{\rho_1\cdots\rho_p}_{v_1\ldots v_q} \equiv \delta^{[\rho_1}_{v_1}\delta^{\rho_2}_{v_2}\cdots\delta^{\rho_p]}_{v_q},$$

which includes a signed sum over p ! permutations of the lower indices, each with a coefficient $1/p$!, such that the 'total weight' is 1 (as in (A.8)).

The Schouten identity:

$$0 = 5\delta^{[v}_{\mu}\varepsilon^{\rho\sigma\tau\lambda]} \equiv \delta_\mu{}^v\varepsilon^{\rho\sigma\tau\lambda} + \delta_\mu{}^\rho\varepsilon^{\sigma\tau\lambda\nu} + \delta_\mu{}^\sigma\varepsilon^{\tau\lambda\nu\rho} + \delta_\mu{}^\tau\varepsilon^{\lambda\nu\rho\sigma} + \delta_\mu{}^\lambda\varepsilon^{\nu\rho\sigma\tau}$$

3.1.4 Practical γ-matrix manipulation 42

Consider first products with index contractions such as

$$\gamma^{\mu\nu}\gamma_v = (D-1)\gamma^\mu$$

You can memorize this rule, but it is easier to recall the simple logic behind it: v runs over all values except μ, so there are $(D-1)$ terms in the sum. Similar logic explains the result

$$\gamma^{\mu v\rho}\gamma_\rho = (D-2)\gamma^{\mu v}$$

or even more generally

$$\gamma^{\mu_1\ldots\mu_r v_1\ldots v_s}\gamma_{v_s\ldots v_1} = \frac{(D-r)!}{(D-r-s)!}\gamma^{\mu_1\ldots\mu_r}$$

$$\gamma^{v_1\ldots v_r} = (-)^{r(r-1)/2}\gamma^{v_r\ldots v_1}.$$

The sign factor $(-)^{r(r-1)/2}$ is negative for $r = 2, 3 \bmod 4$. Even if one does not sum over indices, similar combinatorial tricks can be used. For example, when calculating

$$\gamma^{\mu_1\mu_2}\gamma_{v_1\ldots v_D},$$

one knows that the index values μ_1 and μ_2 appear in the set of $\{v_i\}$. There are D possibilities for μ_2, and since μ_1 should be different, there remain $D-1$ possibilities for μ_1. Hence the result is

$$\gamma^{\mu_1\mu_2}\gamma_{v_1\ldots v_D} = D(D-1)\delta^{\mu_2\mu_1}_{[v_1v_2}\gamma_{v_3\ldots v_D]}.$$

Note that such generalized δ-functions are always normalized with 'weight 1 ', i.e.

$$\delta^{\mu_2\mu_1}_{v_1v_2} = \frac{1}{2}\left(\delta^{\mu_2}_{v_1}\delta^{\mu_1}_{v_2} - \delta^{\mu_1}_{v_1}\delta^{\mu_2}_{v_2}\right)$$

This makes contractions easy; e.g. we obtain from (3.17)

$$\gamma^{\mu_1\mu_2}\gamma_{v_1\ldots v_D}\varepsilon^{v_1\ldots v_D} = D(D-1)\varepsilon^{\mu_2\mu_1\nu_3\ldots v_D}\gamma_{v_3\ldots v_D}.$$

We now consider products of γ-matrices without index contractions. The very simplest case is

$$\gamma^{\mu}\gamma^{v} = \gamma^{\mu v} + \eta^{\mu v}.$$

$$\gamma^{\mu\nu\rho}\gamma_{\sigma\tau} = \gamma^{\mu\nu\rho}_{\sigma\tau} + 6\gamma^{[\mu\nu}_{[\tau}\delta^{\rho]}_{\sigma]} + 6\gamma^{[\mu}\delta^{\nu}_{[\tau}\delta^{\rho]}_{\sigma]}.$$

$$\gamma^{\mu_1\cdots\mu_4}\gamma_{v_1v_2} = \gamma^{\mu_1\cdots\mu_4}{}_{v_1v_2} + 8\gamma^{[\mu_1\cdots\mu_3}{}_{[v_2}\delta^{\mu_4]}{}_{v_1]}\big] + 12\gamma^{[\mu_1\mu_2}\delta^{\mu_3}{}_{[v_2}\delta^{\mu_4]}{}_{v_1]}.$$

Finally, we consider products with both contracted and uncontracted indices. Consider $\gamma^{\mu_1\cdots\mu_4\rho}\gamma_{\rho v_1v_2}$. The result should contain terms similar to (3.22), but each term has an extra numerical factor reflecting the number of values that ρ can take in this sum. For example, in the second term above there is now one contraction between an upper and lower index, and therefore ρ can run over all D values except the four values $\mu_1,\ldots\mu_4$, and the two values v_1, v_2. This counting gives

$$\begin{aligned}\gamma^{\mu_1\cdots\mu_4\rho}\gamma_{\rho v_1v_2} =&(D-6)\gamma^{\mu_1\cdots\mu_4}{}_{v_1v_2} + 8(D-5)\gamma^{[\mu_1\cdots\mu_3}{}_{[v_2}\delta^{\mu_4]}{}_{v_1]}\\ &+ 12(D-4)\gamma^{[\mu_1\mu_2}\delta^{\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{v_1]}.\end{aligned}$$

Exercise 3.5 Show that

$$\gamma_{\nu}\gamma^{\mu}\gamma^{v} = (2-D)\gamma^{\mu},$$
$$\gamma_{\rho}\gamma^{\mu v}\gamma^{\rho} = (D-4)\gamma^{\mu\nu}.$$

Derive the general form $\gamma_{\rho}\gamma^{\mu_1\mu_2\cdots\mu_r}\gamma^{\rho} = (-)^r(D-2r)\gamma^{\mu_1\mu_2\cdots\mu_r}$.

3.1.5 Basis of the algebra for even dimension D = 2m 43

The basis is denoted by the following list $\{\Gamma^A\}$ of matrices chosen from those defined in Sec. 3.1.2:

$$\left\{\Gamma^A = \mathbb{1}, \gamma^{\mu}, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \cdots, \gamma^{\mu_1\cdots\mu_D}\right\}.$$

Index values satisfy the conditions $\mu_1 < \mu_2 < \ldots < \mu_r$. There are C^D_r distinct index choices at each rank r and a total of 2^D matrices. To see that this is a basis, it is convenient to define the reverse order list

$$\{\Gamma_A = \mathbb{1}, \gamma_{\mu}, \gamma_{\mu_2\mu_1}, \gamma_{\mu_3\mu_2\mu_1}, \ldots, \gamma_{\mu_D\cdots\mu_1}\}.$$

By (3.15) the matrices of this list differ from those of (3.25) by sign factors only. Exercise 3.6 Show that $\Gamma^A\Gamma^B = \pm\Gamma^C$, where Γ^C is the basis element whose indices are those of A and B with common indices excluded. Derive the trace orthogonality property

$$\mathrm{Tr}\left(\Gamma^A\Gamma_B\right) = 2^m\delta^A_B.$$

The list (3.25) contains 2^D trace orthogonal matrices in an algebra of total dimension 2^D. Therefore it is a basis of the space of matrices M of dimension $2^m \times 2^m$.

Exercise 3.7 Show that any matrix M can be expanded in the basis $\{\Gamma^A\}$ as

$$M = \sum_A m_A\Gamma^A, \quad m_A = \frac{1}{2^m\breve{a}}\,\mathrm{Tr}\left(M\Gamma_A\right).$$

$$\mathrm{Tr}\,\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma} = 2^m\left[\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right].$$

3.1.6 The highest rank Clifford algebra element 44

$$\gamma_* \equiv (-\mathrm{i})^{m+1}\gamma_0\gamma_1\dots\gamma_{D-1}$$

which satisfies $\gamma_*^2 = \mathbb{1}$ in every even dimension and is hermitian. For spacetime dimension $D = 2m$, the matrix γ_* is frequently called γ_{D+1} in the physics literature, as in four dimensions where it is called γ_5.

This matrix occurs as the unique highest rank element in (3.25). For any order of components μ_i, one can write

$$\gamma_{\mu_1\mu_2\cdots\mu_D} = \mathrm{i}^{m+1}\varepsilon_{\mu_1\mu_2\cdots\mu_D}\gamma_*,$$

Exercise 3.10 Show that γ_* commutes with all even rank elements of the Clifford algebra and anti-commutes with all odd rank elements. Thus, for example,

$$\{\gamma_*, \gamma^\mu\} = 0,$$
$$[\gamma_*, \gamma^{\mu\nu}] = 0.$$

Since $\gamma_*^2 = \mathbb{1}$ and $\mathrm{Tr}\,\gamma_* = 0$, it follows that one can choose a representation in which

$$\gamma_* = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

Some exercises follow, which illustrate the properties of a representation of the full Clifford algebra in which γ_* takes the form in (3.34).

Exercise 3.11 Assume a general block form,

$$\gamma^\mu = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

for the generating elements in a basis where (3.34) holds. Show that (3.32) implies the block off-diagonal form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar\sigma^\mu & 0 \end{pmatrix},$$

in which the matrices σ^μ and $\bar\sigma^\mu$ are $2^{m-1}\times 2^{m-1}$ generalizations of the explicit Weyl matrices of (2.2).

$$P_L = \frac{1}{2}\left(\mathbb{1}+\gamma_*\right), \quad P_R = \frac{1}{2}\left(\mathbb{1}-\gamma_*\right).$$

Thus

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} \equiv P_L\Psi, \quad \begin{pmatrix} 0 \\ \bar\chi \end{pmatrix} \equiv P_R\Psi.$$

3.1.7 Odd spacetime dimension D = 2m + 1 46

$$\gamma^\mu_\pm = \left(\gamma^0, \gamma^1, \dots, \gamma^{(2m-1)}, \gamma^{2m} = \pm\gamma_*\right)$$

$$\gamma_\pm^{\mu_1\dots\mu_r} = \pm\mathrm{i}^{m+1}\frac{1}{(D-r)!}\varepsilon^{\mu_1\dots\mu_D}\gamma_{\pm\mu_D\dots\mu_{r+1}}.$$

Exercise 3.16 Prove the relation (3.41) and the analogous but different relation for even dimension:

$$\gamma^{\mu_1\mu_2\dots\mu_r}\gamma_* = -(-\mathrm{i})^{m+1}\frac{1}{(D-r)!}\varepsilon^{\mu_r\mu_{r-1}\dots\mu_1 v_1\nu_2\dots v_{D-r}}\gamma_{\nu_1\nu_2\dots\nu_{D-r}}.$$

You can use the tricks explained in Sec. 3.1.4. Show that in four dimensions

$$\gamma_{\mu\nu\rho} = \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\gamma^{\sigma}\gamma_{*}.$$

Thus, a basis of the Clifford algebra in $D = 2m + 1$ dimensions contains the matrices in (3.25) only up to rank m. This agrees with the counting argument in Ex. 3.9. For example, the set $\{\mathbb{1}, \gamma^{\mu}, \gamma^{\mu\nu}\}$ of $1+5+10 = 16$ matrices is a basis of the Clifford algebra for $D = 5$. Ex. 3.16 shows that it is a rearrangement of the basis $\{\Gamma^{A}\}$ for $D = 4$.

3.1.8 Symmetries of γ-matrices 47

In the Clifford algebra of the $2^m \times 2^m$ matrices, for both $D = 2m$ and $D = 2m + 1$, one can distinguish between the symmetric and the antisymmetric matrices where the symmetry property is defined in the following way. There exists a unitary matrix, C, called the charge conjugation matrix, such that each matrix $C\Gamma^{A}$ is either symmetric or antisymmetric. Symmetry depends only on the rank r of the matrix Γ^{A}, so we can write:

$$\left(C\Gamma^{(r)}\right)^{T} = -t_r C\Gamma^{(r)}, \quad t_r = \pm 1,$$

where $\Gamma^{(r)}$ is a matrix in the set (3.25) of rank r. (The $-$ sign in (3.44) is convenient for later manipulations.) For rank $r = 0$ and 1, one obtains from (3.44)

$$C^{T} = -t_0 C, \quad \gamma^{\mu T} = t_0 t_1 C\gamma^{\mu}C^{-1}.$$

These relations suffice to determine the symmetries of all $C\gamma^{\mu_1\cdots\mu_r}$ and thus all coefficients t_r : e.g. $t_2 = -t_0$ and $t_3 = -t_1$. Further, $t_{r+4} = t_r$.

Exercise 3.17 A formal proof of the existence of C can be found in [10, 11], but you can check that the following two matrices satisfy (3.45) for even D. They are given in the product representation of (3.2): [1]

$$\begin{aligned} C_{+} &= \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \ldots, \quad t_0 t_1 = 1, \\ C_{-} &= \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \ldots, \quad t_0 t_1 = -1. \end{aligned}$$

Table 3.1 Symmetries of γ-matrices. The entries contain the numbers r mod 4 for which $t_r = \pm 1$. For even dimensions, in bold face are the choices that are most convenient for supersymmetry.

D(mod8)	$t_r = -1$	$t_r = +1$
0	0,3	2, 1
	0, 1	**2, 3**
1	0, 1	2,3
2	0, 1	2,3
	1,2	**0, 3**
3	1,2	0,3
4	**1,2**	**0, 3**
	2,3	0, 1
5	2,3	0, 1
6	**2,3**	**0, 1**
	0, 3	1, 2
7	0,3	1,2

(!!!!!!!!!!!!!!!!!!)

$$B = \mathrm{i}t_0 C\gamma^0$$

$$\gamma^{\mu *} = -t_0 t_1 B\gamma^\mu B^{-1}.$$

$$B^* B = -t_1 \mathbb{1}.$$

In the Weyl representation (2.19), one can choose $B = \gamma^0\gamma^1\gamma^3$, which is real, symmetric, and satisfies $B^2 = \mathbb{1}$. Then $C = \mathrm{i}\gamma^3\gamma^1$.

In another representation, related by (3.4), the C and B matrices are given by

$$C' = S^{-1T} C S^{-1}, \quad B' = S^{-1T} B S^{-1}.$$

3.3.2 Spinors in general dimensions

$$\bar{\lambda} \equiv \lambda^T C.$$

3.2.1 Spinors and spinor bilinears 49

$$\bar{\lambda}\gamma_{\mu_1\ldots\mu_r}\chi = t_r \bar{\chi}\gamma_{\mu_1\ldots\mu_r}\lambda.$$

Clifford matrices:

$$\bar{\lambda}\Gamma^{(r_1)}\Gamma^{(r_2)}\cdots\Gamma^{(r_p)}\chi = t_0^{p-1} t_{r_1} t_{r_2}\cdots t_{r_p}\bar{\chi}\Gamma^{(r_p)}\cdots\Gamma^{(r_2)}\Gamma^{(r_1)}\lambda,$$

where $\Gamma^{(r)}$ stands for any rank r matrix $\gamma_{\mu_1\ldots\mu_r}$. Note that the prefactor t_0^{p-1} is not relevant in four dimensions, where $t_0 = 1$.

Exercise 3.22 One often encounters the special case that the bilinear contains the product of individual γ^μ-matrices. Prove that for the Majorana dimensions $D = 2, 3, 4 \bmod 8$,

$$\bar{\lambda}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_p}\chi = (-)^p \bar{\chi}\gamma^{\mu_p}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\lambda.$$

The previous relations imply also the following rule. For any relation between spinors that includes γ-matrices, there is a corresponding relation between the barred spinors,

$$\chi_{\mu_1\ldots\mu_r} = \gamma_{\mu_1\ldots\mu_r}\lambda \Longrightarrow \bar{\chi}_{\mu_1\ldots\mu_r} = t_0 t_r \bar{\lambda}\gamma_{\mu_1\ldots\mu_r},$$

and similar for longer chains,

$$\chi = \Gamma^{(r_1)}\Gamma^{(r_2)}\cdots\Gamma^{(r_p)}\lambda \Longrightarrow \bar{\chi} = t_0^p t_{r_1} t_{r_2}\cdots t_{r_p}\bar{\lambda}\Gamma^{(r_p)}\cdots\Gamma^{(r_2)}\Gamma^{(r_1)}.$$

$$\chi = P_L\lambda \to \bar{\chi} = \begin{cases} \bar{\lambda}P_L, & \text{for } D = 0, 4, 8, \ldots \\ \bar{\lambda}P_R, & \text{for } D = 2, 6, 10, \ldots \end{cases}$$

Exercise 3.23 Using the 'spin part' of the infinitesimal Lorentz transformation (2.25),

$$\delta\chi = -\frac{1}{4}\lambda^{\mu\nu}\gamma_{\mu\nu}\chi$$

prove that the spinor bilinear $\bar{\lambda}\chi$ is a Lorentz scalar.

3.2.2 Spinor indices

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta.$$

We also introduce a lowering matrix such that (again NW-SE contraction)

$$\lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha}$$

In order for these two equations to be consistent, we must require

$$\mathcal{C}^{\alpha\beta}\mathcal{C}_{\gamma\beta} = \delta_\gamma{}^\alpha, \quad \mathcal{C}_{\beta\alpha}\mathcal{C}^{\beta\gamma} = \delta_\alpha{}^\gamma$$

Hence $\mathcal{C}_{\alpha\beta}$ are the components of C^{-1}, and the unitarity of C implies then $(\mathcal{C}_{\alpha\beta})^* = \mathcal{C}^{\alpha\beta}$.

When we write a covariant spinor bilinear with components explicitly indicated, the γ-matrices are written as $(\gamma_\mu)_\alpha{}^\beta$. For example, for the simplest case,

$$\bar{\chi}\gamma_\mu\lambda = \chi^\alpha (\gamma_\mu)_\alpha{}^\beta \lambda_\beta,$$

These γ-matrices with indices at the 'same level' have a definite symmetry or antisymmetry property, which follows from (3.44):

$$(\gamma_{\mu_1\ldots\mu_r})_{\alpha\beta} = -t_r (\gamma_{\mu_1\ldots\mu_r})_{\beta\alpha}.$$

An interesting property is that

$$\lambda^\alpha\chi_\alpha = -t_0\lambda_\alpha\chi^\alpha$$

3.2.3 Fierz rearrangement 52

In this subsection we study an important consequence of the completeness of the Clifford algebra basis $\{\Gamma^A\}$ in (3.25). As we saw in Ex. 3.7 completeness means that any matrix M has a unique expansion in the basis with coefficients obtained using trace orthogonality. The expansion was derived for even $D = 2m$ in Ex. 3.7, but it is also valid for odd $D = 2m + 1$ provided that the sum is restricted to rank $r \leq m$. We saw at the end of Sec. 3.1.7 that the list of (3.25) is complete for odd D when so restricted. The rearrangement properties we derive using completeness are frequently needed in supergravity. These involve changing the pairing of spinors in products of spinor bilinears, which is called a 'Fierz rearrangement'.

Let's proceed to derive the basic Fierz identity. Using spinor indices, we can regard the quantity $\delta_\alpha{}^\beta\delta_\gamma{}^\delta$ as a matrix in the indices $\gamma\beta$ with the indices $\alpha\delta$ as inert 'spectators'. We apply (3.28) in the detailed form $\delta_\alpha^\beta\delta_\gamma^\delta = \sum_A (m_A)_\alpha^\delta (\Gamma_A)_\gamma{}^\beta$. The coefficients are $(m_A)_\alpha^\delta = 2^{-m}\delta_\alpha{}^\beta\delta_\gamma{}^\delta (\Gamma_A)_\beta{}^\gamma = 2^{-m}(\Gamma_A)_\alpha{}^\delta$. Therefore, we obtain the basic rearrangement lemma

$$\delta_\alpha{}^\beta\delta_\gamma{}^\delta = \frac{1}{2^m}\sum_A (\Gamma_A)_\alpha{}^\delta (\Gamma^A)_\gamma{}^\beta.$$

Note that the 'column indices' on the left- and right-hand sides have been exchanged. Exercise 3.26 Derive the following result:

$$(\gamma^\mu)_\alpha^\beta (\gamma_\mu)_\gamma^\delta = \frac{1}{2^m}\sum_A v_A (\Gamma_A)_\alpha{}^\delta (\Gamma^A)_\gamma{}^\beta$$

and prove that the explicit values of the expansion coefficients are given by $v_A = (-)^{r_A}(D - 2r_A)$, where r_A is the tensor rank of the Clifford basis element Γ_A.

Exercise 3.27 Lower the β and δ index in the result of the previous exercise and consider the completely symmetric part in $(\beta\gamma\delta\ )$. The left-hand side is only non-vanishing for dimensions in which $t_1 = -1$. Consider the right-hand side and use Table 3.1 and the result for v_A to prove that for $D = 3$ and $D = 4$ only rank 1γ-matrices contribute to the right-hand side. For $D = 4$ you have to use the bold face row in the table to arrive at this result. You can also check that there are no other dimensions where this occurs. The previous exercise implies that, for $D = 3$ and $D = 4$,

$$(\gamma_\mu)_{\alpha(\beta}(\gamma^\mu)_{\gamma\delta)} = 0.$$

$$\gamma_\mu\lambda_{[1}\bar{\lambda}_2\gamma^\mu\lambda_{3]} = 0$$

$$\bar{\lambda}_1\lambda_2\bar{\lambda}_3\lambda_4 = -\frac{1}{2^m}\sum_A \bar{\lambda}_1\Gamma^A\lambda_4\bar{\lambda}_3\Gamma_A\lambda_2$$

This can be generalized to include general matrices M, M' of the Clifford algebra.

Exercise 3.28 Show that

$$\begin{aligned}\bar{\lambda}_1 M\lambda_2\bar{\lambda}_3 M'\lambda_4 &= -\frac{1}{2^m}\sum_A \bar{\lambda}_1 M\Gamma_A M'\lambda_4\bar{\lambda}_3\Gamma^A\lambda_2\\ &= -\frac{1}{2^m}\sum_A \bar{\lambda}_1\Gamma_A M'\lambda_4\bar{\lambda}_3\Gamma^A M\lambda_2.\end{aligned}$$

When $\lambda_{1,2,3,4}$ are not all independent, it is frequently the case that some terms in the rearranged sum vanish due to symmetry relations such as (3.51). One can write the Fierz relation (3.65) in the alternative form:

$$M = 2^{-m}\sum_{k=0}^{[D]}\frac{1}{k!}\Gamma_{\mu_1\ldots\mu_k}\operatorname{Tr}\left(\Gamma^{\mu_k\cdots\mu_1}M\right)$$

Exercise 3.29 Prove the following chiral Fierz identities for $D = 4$:

$$\begin{aligned}P_L\chi\bar{\lambda}P_L &= -\frac{1}{2}P_L\left(\bar{\lambda}P_L\chi\right) + \frac{1}{8}P_L\gamma^{\mu\nu}\left(\bar{\lambda}\gamma_{\mu\nu}P_L\chi\right),\\ P_L\chi\bar{\lambda}P_R &= -\frac{1}{2}P_L\gamma^\mu\left(\bar{\lambda}\gamma_\mu P_L\chi\right).\end{aligned}$$

You will need (3.42) to combine terms in (3.71).

Exercise 3.30 Prove that for $D = 5$ the matrix $\chi\bar{\lambda} - \lambda\bar{\chi}$ can be written as

$$\chi\bar{\lambda} - \lambda\bar{\chi} = \gamma_{\mu v}\left(\bar{\lambda}\gamma^{\mu v}\chi\right).$$

(!?????????!?!?!!!!?!?!?!?!??!)

Complex conjugation can be replaced by charge conjugation, an operation that acts as complex conjugation on scalars, and has a simple action on fermion bilinears. For example, it preserves the order of spinor factors.

(!?????????!?!?!!!!?!?!?!?!??!)

3.2.4 Reality 54

First we define the charge conjugate of any spinor as

$$\lambda^C \equiv B^{-1}\lambda^*.$$

The barred charge conjugate spinor is then, using (3.50) and (3.47),

$$\overline{\lambda^C} = (-t_0 t_1)\,\mathrm{i}\lambda^\dagger\gamma^0.$$

Note that this is the Dirac conjugate as defined in (2.30) except for the numerical factor $(-t_0t_1)$. The meaning of this will become clear below when we discuss Majorana spinors. Note that $(-t_0t_1) = +1$ in $2, 3, 4, 10$ or 11 dimensions. [5] The charge conjugate of any $2^m \times 2^m$ matrix M is defined as

$$M^C \equiv B^{-1}M^*B.$$

Charge conjugation does not change the order of matrices: $(MN)^C = M^C N^C$. In practice the matrices M we deal with are products of γ-matrices. Hence, we need only the charge conjugation property of the generating γ-matrices, which is

$$(\gamma_\mu)^C \equiv B^{-1}\gamma_\mu^* B = (-t_0t_1)\,\gamma_\mu$$

$$(\gamma_\mu)^C \equiv B^{-1}\gamma_\mu^* B = (-t_0t_1)\,\gamma_\mu$$

Exercise 3.31 Start from (3.77) (and note that charge conjugation on any number is just complex conjugation). Prove that

$$(\gamma_*)^C = (-)^{D/2+1}\gamma_*.$$

$$(\bar\chi M\lambda)^* \equiv (\bar\chi M\lambda)^C = (-t_0t_1)\,\overline{\chi^C}M^C\lambda^C.$$

3.3.3 Majorana spinors

3.3.1 Definition and properties 56

$$\psi = \psi^C = B^{-1}\psi^*, \qquad \text{i.e.} \qquad \psi^* = B\psi$$

The two possible values $t_0 = \pm 1$ must be considered, and we begin with the case $t_0 = +1$. Consulting Table 3.1, we see that $t_0 = +1$ holds for spacetime dimension $D = 2, 3, 4$, mod 8. In this case we call the spinors that satisfy (3.80) Majorana spinors. It is clear from (3.75) that if $t_0 = 1$ and $t_1 = -1$, the barred (3.50) and Dirac adjoint spinors (2.30) agree for Majorana spinors. In fact, this gives an alternative definition of a Majorana spinor.

Another fact about the Majorana case is that there are representations of the γ-matrices that are explicitly real and may be called really real representations. Here is a really real representation for $D = 4$:

$$\begin{aligned}
\gamma^0 &= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \mathrm{i}\sigma_2 \otimes \mathbb{1}, \\
\gamma^1 &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \sigma_3 \otimes \mathbb{1}, \\
\gamma^2 &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_1, \\
\gamma^3 &= \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_3.
\end{aligned}$$

Note that the γ^i are symmetric, while γ^0 is antisymmetric. This is required by hermiticity in any real representation. We construct really real representations in all allowed dimensions $D = 2, 3, 4 \bmod 8$ in Appendix 3A.5.

In such representations (3.48) implies that $B = \mathbb{1}$ (up to a phase). The relation (3.47) then gives $C = \mathrm{i}\gamma^0$. Further, a Majorana spinor field is really real since (3.80) reduces to $\Psi^* = \Psi$.

Really real representations are sometimes convenient, but we emphasize that the physics of Majorana spinors is the same in, and can be explored in, any representation of the Clifford algebra, replacing complex conjugation with charge conjugation.

$$(\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi)^* = (\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi)^C = \bar{\chi}\left(\gamma_{\mu_1\ldots\mu_r}\right)^C\psi = \bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi.$$

When $t_0 = -1$ (and still $t_1 = -1$) spinors that satisfy (3.80) are called pseudo-Majorana spinors. They are mostly relevant for $D = 8$ or 9. There are no really real representations in these dimensions; instead there are representations of the Clifford algebra in which the generating γ-matrices are imaginary, $(\gamma^\mu)^* = -\gamma^\mu$. In any representation (3.79) and (3.77) hold with $t_0 = t_1 = -1$. This implies that the reality properties of bilinears are different from those of Majorana spinors. Although these differences are significant, the essential property that a complex spinor can be reduced to a real one still holds, and it is common in the literature not to distinguish between Majorana and pseudo-Majorana spinors. However, note the following.

Exercise 3.35 Show that the mass term $m\bar{\chi}\chi = 0$ for a single pseudo-Majorana field. Pseudo-Majorana spinors must be massless (unless paired).

We now consider (pseudo-)Majorana spinors in even dimensions $D = 0, 2, 4 \bmod 8$. We can quickly show using (3.78) that these cases are somewhat different. For $D = 2 \bmod 8$ we have $(\gamma_*\psi)^C = \gamma_*\psi^C$. Thus the two constraints

$$\text{Majorana:} \quad \psi^C = \psi, \qquad \text{Weyl:} \quad P_{L,R}\psi = \psi,$$

are compatible. It is equivalent to observe that the chiral projections of a Majorana spinor ψ satisfy

$$(P_L\psi)^C = P_L\psi, \quad (P_R\psi)^C = P_R\psi$$

Thus the chiral projections of a Majorana spinor are also Majorana spinors. Each chiral projection satisfies both constraints in (3.83) and is called a Majorana-Weyl spinor. Such spinors have 2^{m-1} independent 'real' components in dimension $D = 2m = 2 \bmod 8$ and are the 'most fundamental' spinors available in these dimensions. It is not surprising that supergravity and superstring theories in $D = 10$ dimensions are based on Majorana-Weyl spinors.

For $D = 4 \bmod 4$ dimensions we have $(\gamma_*\psi)^C = -\gamma_*\psi^C$, so that the equations of (3.84) are replaced by

$$(P_L\psi)^C = P_R\psi, \quad (P_R\psi)^C = P_L\psi$$

For $D = 4 \bmod 4$ dimensions we have $(\gamma_*\psi)^C = -\gamma_*\psi^C$, so that the equations of (3.84) are replaced by

$$(P_L\psi)^C = P_R\psi, \quad (P_R\psi)^C = P_L\psi.$$

3.3.2 Symplectic Majorana spinors 58

When $t_1 = 1$ we cannot define Majorana spinors, but we can define 'symplectic Majorana spinors'. These consist of an even number of spinors χ^i, with $i = 1, \ldots, 2k$, which satisfy a 'reality condition' containing a non-singular antisymmetric matrix ε^{ij}. The inverse matrix ε_{ij} satisfies $\varepsilon^{ij}\varepsilon_{kj} = \delta^i_k$. Symplectic Majorana spinors satisfy the condition

$$\chi^i = \varepsilon^{ij}\left(\chi^j\right)^C = \varepsilon^{ij}B^{-1}\left(\chi^j\right)^*.$$

The consistency check discussed after (3.80) now works for $t_1 = 1$ because of the antisymmetric ε^{ij}.

Exercise 3.36 Check that, in five dimensions with symplectic Majorana spinors, $\bar{\psi}^i\chi_i \equiv \bar{\psi}^i\chi^j\varepsilon_{ji}$ is pure imaginary while $\bar{\psi}^i\gamma_\mu\chi_i$ is real.

3.3.3 Dimensions of minimal spinors 58

$$\begin{aligned} t_1 = -1, \quad t_0 = 1: &\quad \text{Majorana,} \\ t_0 = -1: &\quad \text{pseudo-Majorana,} \\ t_1 = 1: &\quad \text{symplectic Majorana.} \end{aligned}$$

Irreducible spinors, number of components and symmetry properties.

dim	spinor	min # components	antisymmetric
2	MW	1	1
3	M	2	$1,2$
4	M	4	$1,2$
5	S	8	$2,3$
6	SW	8	3
7	S	16	$0,3$
8	M	16	$0,1$
9	M	16	$0,1$
10	MW	16	1
11	M	32	$1,2$

3.3.4 Majorana spinors in physical theories

3.4.1 Variation of a Majorana Lagrangian 59

In this section we consider a prototype action for a Majorana spinor field in dimension $D = 2, 3, 4 \bmod 8$. Majorana and Dirac fields transform the same way under Lorentz transformations, but Majorana spinors have half as many degrees of freedom, so we write

$$S[\Psi] = -\frac{1}{2}\int \mathrm{d}^D x \bar{\Psi}\left[\gamma^\mu\partial_\mu - m\right]\Psi(x).$$

There is an immediate and curious subtlety due to the symmetries of the matrices C and $C\gamma^\mu$. Using (3.50), we see that the mass and kinetic terms are proportional to $\Psi^T C\Psi$ and $\Psi^T C\gamma^\mu\partial_\mu\Psi$. Suppose that the field components Ψ are conventional commuting numbers. Since C is antisymmetric, the mass term vanishes. Since $C\gamma^\mu$ is symmetric, the kinetic term is a total derivative and thus vanishes when integrated in the action. For commuting field components, there is no dynamics! To restore the dynamics we must assume that Majorana fields are anti-commuting Grassmann variables, which we always assume unless stated otherwise.

$$\delta S[\Psi] = -\int \mathrm{d}^D x \delta\bar{\Psi}\left[\gamma^\mu\partial_\mu - m\right]\Psi(x)$$

3.4.2 Relation of Majorana and Weyl spinor theories 60

In even dimensions $D = 0, 2, 4 \bmod 8$, both Majorana and Weyl fields exist and both have legitimate claims to be more fundamental than a Dirac fermion. In fact both fields describe equivalent physics. Let's show this for $D=4$. We can rewrite the action (3.88) as

$$\begin{aligned} S[\psi] &= -\frac{1}{2}\int \mathrm{d}^4x \left[\bar{\Psi}\gamma^\mu\partial_\mu - m\right](P_L+P_R)\Psi \\ &= -\int \mathrm{d}^4x \left[\bar{\Psi}\gamma^\mu\partial_\mu P_L\Psi - \frac{1}{2}m\bar{\Psi}P_L\Psi - \frac{1}{2}m\bar{\Psi}P_R\Psi\right]. \end{aligned}$$

We obtained the second line by a Majorana flip and partial integration. In the second form of the action, the Majorana field is replaced by its chiral projections. In our treatment of chiral multiplets in supersymmetry, we will exercise the option to write Majorana fermion actions in this way.

Exercise 3.40 Show that the Euler-Lagrange equations that follow from the variation of the second form of the action in (3.91) are

$$\not{\partial}P_L\Psi = mP_R\Psi, \quad \not{\partial}P_R\Psi = mP_L\Psi.$$

Derive $\Box P_{L,R}\Psi = m^2 P_{L,R}\Psi$ from the equations above.

Let's return to the Weyl representation (2.19) for the final step in the argument to show that the equation of motion for a Majorana field can be reexpressed in terms of a Weyl field and its adjoint. The Majorana condition $\Psi = B^{-1}\Psi^* = \gamma^0\gamma^1\gamma^3\Psi^*$ requires that Ψ take the form

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_2^* \\ -\psi_1^* \end{pmatrix}.$$

With (3.93) and (2.55) in view we define the two-component Weyl fields

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \tilde{\psi} = \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix}.$$

Using the form of γ^μ (2.19) and γ_* (3.34) in the Weyl representation, we see that we can identify

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} = P_L\Psi, \quad \begin{pmatrix} 0 \\ \tilde{\psi} \end{pmatrix} = (P_L\Psi)^C = P_R\Psi.$$

The equations of motion (3.92) can then be rewritten as

$$\bar{\sigma}^\mu\partial_\mu\psi = m\tilde{\psi}, \quad \sigma^\mu\partial_\mu\tilde{\psi} = m\psi.$$

These are equivalent to the pair of Weyl equations in (2.56) with the restriction $\tilde{\psi} = \bar{\chi}$ which comes because we started in this section with a Majorana rather than a Dirac field.

3.4.3 U(1) symmetries of a Majorana field 61

In Sec. 2.7.1 we considered the U(1) symmetry operation $\Psi \to \Psi' = \mathrm{e}^{\mathrm{i}\theta}\Psi$. This symmetry is obviously incompatible with the Majorana condition (3.80). Thus the simplest internal symmetry of a Dirac fermion cannot be defined in a field theory of a (single) Majorana field. However, it is easy to see that $(\mathrm{i}\gamma_*)^C = \mathrm{i}\gamma_*$, so the chiral transformation $\Psi \to \Psi' = \mathrm{e}^{\mathrm{i}\gamma_*\theta}\Psi$ preserves the Majorana condition. Let's ask whether the infinitesimal limit of this transformation is a symmetry of the free massive Majorana action (3.88).

Exercise 3.41 Use $\delta\bar{\Psi} = \mathrm{i}\theta\bar{\Psi}\gamma_*$ and partial integration to derive the variation

$$\delta S[\Psi] = \mathrm{i}\theta m\int \mathrm{d}^4x\bar{\Psi}\gamma_*\Psi,$$

which vanishes only for a massless Majorana field.

Exercise 3.42 Show that the axial current

$$J_*^\mu = \frac{1}{2}\mathrm{i}\bar{\Psi}\gamma^\mu\gamma_*\Psi$$

is the Noether current for the chiral symmetry defined above. Use the equations of motion to show that

$$\partial_\mu J_*^\mu = -\mathrm{i}m\bar{\Psi}\gamma_*\Psi$$

The current is conserved only for massless Majorana fermions. The dynamics of a Majorana field Ψ can be expressed in terms of its chiral projections $P_{L,R}\Psi$. So can the chiral transformation, which becomes $P_{L,R}\Psi \to P_{L,R}\Psi' = \mathrm{e}^{\pm\mathrm{i}\theta}\Psi$.

Throughout this section we used the simple dynamics of a free massive fermion to illustrate the relation between Majorana and Weyl fields and to explore their U(1) symmetries. It is straightforward to extend these ideas to interacting field theories with nonlinear equations of motion.

3.3.5 Appendix 3A Details of the Clifford algebras for D = 2m

3A.1 Traces and the basis of the Clifford algebra 62

The trace properties of the matrices are important for proofs of these properties which are independent of the explicit construction in (3.2). The matrices Γ^A for tensor rank $1 \leq r \leq D-1$ are traceless. One simple way to see this is to use the Lorentz transformations (2.22) and its extension to general rank

$$L(\lambda)\gamma^{\mu_1\mu_2\ldots\mu_r}L(\lambda)^{-1} = \gamma^{v_1v_2\ldots v_r}\Lambda_{v_1}^{\mu_1}\ldots\Lambda_{v_r}^{\mu_r}.$$

Traces then satisfy the Lorentz transformation law as suggested by their free indices:

$$\operatorname{Tr}\gamma^{\mu_1\mu_2\ldots\mu_r} = \operatorname{Tr}\gamma^{v_1v_2\ldots v_r}\Lambda_{v_1}^{\mu_1}\ldots\Lambda_{v_r}^{\mu_r}$$

This means that the traces must be totally antisymmetric Lorentz invariant tensors.

$$\sum_A x_A\Gamma^A = 0.$$

Multiply by Γ_B from the right. Take the trace and use the trace orthogonality to obtain

$$\sum_A x_A \operatorname{Tr}\Gamma^A\Gamma^B = \pm x_B \operatorname{Tr}\mathbb{1} = 0.$$

Hence all $x_A = 0$ and linear independence is proven.

3A.2 Uniqueness of the γ-matrix representation 63

We must now show that there is exactly one irreducible representation up to equivalence. We use the basic properties of representations of finite groups. However, the Clifford algebra is not quite a group because the minus signs that necessarily occur in the set of products $\Gamma^A\Gamma^B = \pm\Gamma^C$ are not allowed by the definition of a group. This problem is solved by doubling the basis in (3.25) to the larger set $\left\{\Gamma^A, -\Gamma^A\right\}$. This set is a group of order 2^{2m+1} since all products are contained within the larger set. For $m=1$, the group obtained is isomorphic to the quaternions, so the groups defined by doubling the Clifford algebras are called generalized quaternionic groups.

(пока мне это не интересно)

3A.3 The Clifford algebra for odd spacetime dimensions 65

We gave in (3.40) two different sets of γ-matrices for odd dimensions. They are inequivalent as representations of the generating elements. Indeed it is easily seen that $S\gamma_+{}^\mu S^{-1} = \gamma_-{}^\mu$ cannot be satisfied. This requires $S\gamma^\mu S^{-1} = \gamma^\mu$ for the first $2m$ components. But then, from the product form in (3.6) and (3.30), we obtain $S\gamma^{2m}S^{-1} = +\gamma^{2m}$, rather than the opposite sign needed.

It follows from Ex. 2.8 that the two sets of second rank elements constructed from the generating elements above, namely

$$\begin{aligned}\Sigma^{\mu\nu}_{\pm} &= \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right], \quad \mu, v = 0, \ldots, 2m-1, \\ &= \frac{1}{4}\left[\gamma^\mu, \pm\gamma_*\right], \quad \mu = 0, \ldots, 2m-1, \quad v = 2m,\end{aligned}$$

are each representations of the Lie algebra $\mathfrak{so}(2m, 1)$. The two representations are equivalent, however, since $\gamma_*\Sigma_+{}^{\mu\nu}\gamma_* = \Sigma_-{}^{\mu\nu}$. This representation is irreducible; indeed it is a copy of the unique 2^{2m}-dimensional fundamental irreducible representation with Dynkin designation $(0, 0, \ldots, 0, 1)$. It is associated with the short simple root of the Dynkin diagram for B_m.

3A.4 Determination of symmetries of γ-matrices 65

(не думаю, что это актуально, понадобится - запишу)

3A.5 Friendly representations 66

(не думаю, что это актуально, понадобится - запишу)

We start in $D = 2$ and write

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i}\sigma_2, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1,$$

which is a really real, hermitian, and friendly representation. The matrix γ_* is also real:

$$\gamma_* = -\gamma_0\gamma_1 = \sigma_3$$

Adding it to (3.107) as γ^2 gives a real representation in $D = 3$. The recursion relation for moving from a $D = 2m-2$ representation with $\tilde{\gamma}$ to $D = 2m$ is

$$\begin{aligned}&\gamma^\mu = \tilde{\gamma}^\mu \otimes \mathbb{1}, \quad \mu = 0, \ldots, 2m-3, \\ &\gamma^{2m-2} = \tilde{\gamma}_* \otimes \sigma_1, \quad \gamma^{2m-1} = \tilde{\gamma}_* \otimes \sigma_3.\end{aligned}$$

This gives

$$\gamma_* = -\tilde{\gamma}_* \otimes \sigma_2.$$

3.4 The Maxwell and Yang-Mills gauge fields

3.4.1 The abelian gauge field Aμ(x)

4.1.1 Gauge invariance and fields with electric charge

For a Dirac spinor field the local gauge transformation:

$$\Psi(x) \to \Psi'(x) \equiv \mathrm{e}^{\mathrm{i}q\theta(x)}\Psi(x).$$

The goal is to formulate field equations that transform covariantly under the gauge transformation. This requires

$$A_\mu(x) \to A'_\mu(x) \equiv A_\mu(x) + \partial_\mu\theta(x).$$

One then defines the covariant derivative $D_\mu\Psi(x) \equiv (\partial_\mu - \mathrm{i}qA_\mu(x))\,\Psi(x)$, which transforms with the same phase factor as $\Psi(x)$, namely $D_\mu\Psi(x) \to \mathrm{e}^{\mathrm{i}q\theta(x)}D_\mu\Psi(x)$. The desired field equation is obtained by replacing $\partial_\mu\Psi \to D_\mu\Psi$:

$$[\gamma^\mu D_\mu - m]\,\Psi \equiv [\gamma^\mu\,(\partial_\mu - \mathrm{i}qA_\mu) - m]\,\Psi = 0.$$

The same procedure can be applied to a complex scalar field $\phi(x)$, to which we assign an electric charge q (which may differ from the charge of Ψ). We extend the global U(1) symmetry discussed in Ch. 1 to the local gauge symmetry $\phi(x) \to \phi'(x) = \mathrm{e}^{\mathrm{i}q\theta(x)}\phi(x)$ by defining the covariant derivative $D_\mu\phi = (\partial_\mu - \mathrm{i}qA_\mu)\,\phi$ and modifying the Klein-Gordon equation to the form

$$\left[D^\mu D_\mu - m^2\right]\phi = 0.$$

Degrees of freedom

On-shell degrees of freedom = number of helicity states.

Off-shell degrees of freedom = number of field components - gauge transformations.

4.1.2 The free gauge field

(обычная максвелловская эд, потом напишу, в теорполе почти в точности это же)

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x),$$

is invariant under the gauge transformation, a fact that is trivial to verify. In four dimensions $F_{\mu\nu}$ has six components, which split into the electric $E_i = F_{i0}$ and magnetic $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$ fields.

Since A_μ is a bosonic field, we expect it to satisfy a second order wave equation. The only Lorentz covariant and gauge invariant quantity available is $\partial^\mu F_{\mu\nu}$, so the free electromagnetic field satisfies

$$\partial^\mu F_{\mu\nu} = 0.$$

$$\nabla^2 A_0 - \partial_0\left(\partial^i A_i\right) = 0,$$
$$\square A_i - \partial_i\partial^0 A_0 - \partial_i\left(\partial^j A_j\right) = 0.$$

It is instructive to write the solution of $\square A_i = 0$ as the Fourier transform

$$A_i(x) = \int \frac{\mathrm{d}^{(D-1)}k}{(2\pi)^{(D-1)}2k^0}\sum_\lambda \left[\mathrm{e}^{\mathrm{i}k\cdot x}\epsilon_i(\vec{k},\lambda)a(\vec{k},\lambda) + \mathrm{e}^{-\mathrm{i}k\cdot x}\epsilon_i^*(\vec{k},\lambda)a^*(\vec{k},\lambda)\right],$$

where $\vec{k}, k^0 = |\vec{k}|$, is the on-shell energy-momentum vector. The $\epsilon_i(\vec{k},\lambda)$ are called polarization vectors, which are constrained by the Coulomb gauge condition to satisfy $k^i\epsilon_i(\vec{k},\lambda) = 0$. So there are $(D-2)$ independent polarization vectors, indexed by λ, and there are $2(D-2)$ independent real degrees of freedom contained in the complex quantities $a(\vec{k},\lambda)$. As in the case of the plane wave expansions of the free Klein-Gordon and Dirac fields, $a(\vec{k},\lambda)$ and $a^*(\vec{k},\lambda)$ are interpreted as Fourier amplitudes in the classical theory and as annihilation and creation operators for particle states after quantization. There are $D-2$ particle states.

Also

$$[D_\mu, D_v]\,\Psi \equiv (D_\mu D_v - D_\nu D_\mu)\,\Psi = -\mathrm{i}qF_{\mu\nu}\Psi.$$

The charged Dirac field also satisfies the second order equation

$$\left[D^{\mu}D_{\mu}-\frac{1}{2}\mathrm{i}q\gamma^{\mu v}F_{\mu\nu}-m^{2}\right]\Psi=0.$$

The field strength tensor satisfies the equation $\Box F_{\mu\nu}=0$. This is a gauge invariant derivation of the fact that the free electromagnetic field describes massless particles.

4.1.3 Sources and Greens function

$$\partial^{\mu}F_{\mu v}=-J_{v}.$$

$$\Box F_{v\rho}=-\left(\partial_{v}J_{\rho}-\partial_{\rho}J_{v}\right).$$

Consider first the analogous problem of the scalar field coupled to a source $J(x)$:

$$\left(\Box-m^{2}\right)\phi(x)=-J(x).$$

The response is determined by the Green's function $G(x-y)$: $(\Box-m^{2})\,G(x-y)=-\delta(x-y)$. The Euclidean Green's function:

$$G(x-y)=\int\frac{\mathrm{d}^{D}k}{(2\pi)^{D}}\frac{\mathrm{e}^{\mathrm{i}k\cdot(x-y)}}{k^{2}+m^{2}}.$$

The integral can be expressed in terms of modified Bessel functions (? how?). In the massless case the result simplifies to the power law (? prove?)

$$G(x-y)=\frac{\Gamma\left(\frac{1}{2}(D-2)\right)}{4\pi^{\frac{1}{2}D}(x-y)^{(D-2)}}.$$

Here $(x-y)^{2}=\delta_{\mu\nu}(x-y)^{\mu}(x-y)^{\nu}$ is the Euclidean distance between source point y and observation point x. Given $G(x-y)$, the solution:

$$\phi(x)=\int\mathrm{d}^{D}yG(x-y)J(y).$$

$$\left(\delta^{\mu\rho}\Box-\partial^{\mu}\partial^{\rho}\right)G_{\rho v}(x,y)=-\delta_{v}^{\mu}\delta(x-y).$$

$$\left(\delta^{\mu\rho}\Box-\partial^{\mu}\partial^{\rho}\right)G_{\rho v}(x,y)=-\delta_{v}^{\mu}\delta(x-y)+\frac{\partial}{\partial y^{v}}\Omega^{\mu}(x,y),$$

where $\Omega^{\mu}(x,y)$ is an arbitrary vector function. If $\Omega^{\mu}(x,y)$ and $J_{v}(y)$ are suitably damped at large distance, the effect of the second term in (4.22) cancels (after partial integration) in the formula

$$A_{\mu}(x)=\int\mathrm{d}^{D}yG_{\mu\nu}(x,y)J^{v}(y),$$

which is the analogue of (4.20). We now derive the precise form of $G_{\mu\nu}(x,y)$. By Euclidean symmetry, we can assume the tensor form

$$G_{\mu\nu}(x,y)=\delta_{\mu\nu}F(\sigma)+(x-y)_{\mu}(x-y)_{\nu}\hat{S}(\sigma),$$

where $\sigma=\frac{1}{2}(x-y)^{2}$. It is more useful, but equivalent, to take advantage of gauge invariance and rewrite this ansatz as

$$G_{\mu v}(x,y)=\delta_{\mu\nu}F(\sigma)+\partial_{\mu}\partial_{\nu}S(\sigma),$$

because the pure gauge term involving $S(\sigma)$ has no effect in (4.23) and cancels in (4.22). We may also assume that the gauge term in (4.22) has the Euclidean invariant form $\partial^\mu\partial_v\Omega(\sigma)$. Substituting (4.25) in (4.22) we find the two independent tensors δ^μ_v and $(x-y)^\mu(x-y)_\nu$ and thus two independent differential equations involving F and Ω, namely

$$\begin{aligned}2\sigma F''(\sigma)+(D-1)F'(\sigma)&=\Omega'(\sigma),\\ F''(\sigma)&=-\Omega''(\sigma).\end{aligned}$$

Note that $F'(\sigma)=\mathrm{d}F(\sigma)/\mathrm{d}\sigma$, etc. We have dropped the δ-function term in (4.22), because we will first solve these equations for $\sigma\neq 0$. The second equation in (4.26) may be integrated immediately, giving $F'(\sigma)=-\Omega'(\sigma)$; a possible integration constant is chosen to vanish, so that $F'(\sigma)$ vanishes at large distance. The first equation then becomes $2\sigma F''(\sigma)+DF'(\sigma)=0$, which has the power-law solution $F(\sigma)\sim\sigma^{1-\frac{1}{2}D}$. However, on any function of σ, the Laplacian acts as $\Box F(\sigma)=2\sigma F''(\sigma)+DF'(\sigma)$. In our case there is a hidden δ-function in $\Box F(\sigma)$ because the power law is singular. The effect of the δ-function in (4.22) is automatically incorporated if we take $F(\sigma)=G(x-y)$ where G is the massless scalar Green's function in (4.19). The result of this analysis is the gauge field Green's function

$$G_{\mu\nu}(x,y)=\delta_{\mu\nu}G(x-y)+\partial_\mu\partial_\nu S(\sigma).$$

It may not be obvious why this method works. To see why, apply $\partial/\partial x^\mu$ to both sides of (4.22), obtaining

$$0=-\partial_v\delta(x-y)-\partial_v\Box\Omega(\sigma),$$

in which $\partial_v=\partial/\partial x^\nu$. This consistency condition is satisfied because the analysis above led us the result $\Omega(\sigma)=-F(\sigma)=-G\left((x-y)^2\right)$.

Exercise 4.4 In $D=4$ dimensions, consider a point charge at rest, i.e. $J^\mu(x)=\delta^\mu_0 q\delta(\vec{x})$. Obtain, using (4.23), that the resulting value of A^0, and therefore of the electric field, is

$$A^0(x)=\frac{q}{4\pi}\frac{1}{|\vec{x}|},\qquad \vec{E}=\frac{q}{4\pi}\frac{\vec{x}}{|x|^3}.$$

4.1.4 Quantum electrodynamics

It is also advantageous to change notation from that of Sec. 4.1.1 by scaling the vector potential, $A_\mu\to eA_\mu$, where e is the conventional coupling constant of the electromagnetic field to charged fields; $e^2/4\pi\approx 1/137$ is called the fine structure constant. In this notation the relevant equations of Sec. 4.1 read:

$$\begin{aligned}F_{\mu\nu}&\equiv\partial_\mu A_\nu-\partial_\nu A_\mu,\\ A_\mu\to A'_\mu&\equiv A_\mu+\frac{1}{e}\partial_\mu\theta,\\ D_\mu\Psi&\equiv\left(\partial_\mu-\mathrm{i}eqA_\mu\right)\Psi,\\ \left[D_\mu,D_\nu\right]\Psi&=-\mathrm{i}eqF_{\mu\nu}\Psi.\end{aligned}$$

The electric charges q of the various charged fields are then simple rational numbers, for example $q=1$ for the electron. [5]

$$S\left[A_\mu,\bar\Psi,\Psi\right]=\int\mathrm{d}^Dx\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\bar\Psi\left(\gamma^\mu D_\mu-m\right)\Psi\right].$$

Euler variation of (4.31) with respect to the gauge potential A_v is

$$\frac{\delta\mathcal{L}}{\delta A^\nu}=\partial^\mu F_{\mu\nu}+\mathrm{i}eq\bar\Psi\gamma_\nu\Psi=0$$

4.1.5 The stress tensor and gauge covariant translations

$$\Theta_{\mu v} = F_{\mu\rho} F_v{}^\rho + \frac{1}{4}\bar{\Psi}\left(\gamma_\mu \overset{\leftrightarrow}{D}_v + \gamma_\nu \overset{\leftrightarrow}{D}_\mu\right)\Psi + \eta_{\mu\nu}\mathcal{L}$$

Why? Because the canonical stress tensor, calculated from the Noether formula (1.67) with $\Delta_A \phi^i \to \partial_\nu A_\rho, \partial_\nu \bar{\Psi}, \partial_\nu \Psi$ for the three independent fields, is

$$T^\mu{}_\nu = F^{\mu\rho}\partial_\nu A_\rho + \bar{\Psi}\gamma^\mu \partial_\nu \Psi + \delta^\mu_\nu \mathcal{L}.$$

It is conserved on the index μ, but not on ν, not symmetric and not gauge invariant. The situation can be improved by treating fermion terms as in Sec. 2.7.2 and then adding $\Delta T^\mu{}_v = -\partial_\rho\left(F^{\mu\rho} A_v\right)$ in accord with the discussion in Sec. 1.3. The final result is the gauge invariant and symmetric.

3.4.2 Electromagnetic duality (!?!??!?!??!?!?!)

4.2.1 Dual tensors

$$\tilde{H}^{\mu\nu} := -\frac{1}{2}\mathrm{i}\varepsilon^{\mu v\rho\sigma} H_{\rho\sigma}$$

In our conventions the dual tensor is imaginary, но некоторые определяют его иначе, тогда чуть другие свойства. Also

$$H^{\pm}_{\mu\nu} := \frac{1}{2}\left(H_{\mu\nu} \pm \tilde{H}_{\mu\nu}\right), \quad H^{\pm}_{\mu\nu} := \left(H^{\mp}_{\mu\nu}\right)^*.$$

The dual of the dual is the identity:

$$-\frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\tilde{H}_{\rho\sigma} = H^{\mu\nu}.$$

(??? тут указание про вывод! You will need (3.9).) The validity of this property is the reason for the "i" in the definition (4.35).

$H^+_{\mu\nu}$ and $H^-_{\mu\nu}$ are, respectively, self-dual and anti-self-dual:

$$-\frac{1}{2}\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma} H^{\pm}_{\rho\sigma} = \pm H^{\pm}_{\mu\nu}.$$

Let $G_{\mu\nu}$ be another antisymmetric tensor with $G^{\pm}_{\mu\nu}$ defined the same way. Prove the following relations:

$$G^{+\mu v} H^-{}_{\mu\nu} = 0, \qquad G^{\pm\rho(\mu} H^{\pm\nu)}{}_\rho = -\frac{1}{4}\eta^{\mu\nu} G^{\pm\rho\sigma} H^{\pm}{}_{\rho\sigma}, \qquad G^+{}_{\rho[\mu} H^-{}_{\nu]}{}^\rho = 0,$$

$(\mu\nu)$ means symmetrization. Hint: you could first prove $\tilde{G}^{\rho\mu}\tilde{H}^v{}_\rho = -\frac{1}{2}\eta^{\mu v} G^{\rho\sigma} H_{\rho\sigma} - G^{\rho v} H^\mu{}_\rho$.

4.2.2 Duality for one free electromagnetic field

We know that $\partial_\mu F^{\mu\nu} = 0, \partial_\mu \tilde{F}^{\mu v} = 0$. There is a dual symmetry - change of variables:

$$F^{\mu\nu} \to F'^{\mu\nu} = \mathrm{i}\tilde{F}^{\mu\nu}$$

(the "i" is included to make the transformation real).

Since $F'^{\mu\nu}$ also obeys both equations of (4.41) we have defined a symmetry of the free electromagnetic field.

The symmetry (4.42) exchanges the electric and magnetic fields: $E_i \to E_i' = -B_i$ and $B_i \to B_i' = E_i$.

$$\partial_\mu F^{\mu v} = 0, \quad \partial_\mu \tilde{F}^{\mu v} = 0.$$

We can now consider the change of variables (the i is included to make the transformation real):

$$F^{\mu v} \to F'^{\mu\nu} = \mathrm{i}\tilde{F}^{\mu v}.$$

Since $F'^{\mu\nu}$ also obeys both equations of (4.41) we have defined a symmetry of the free electromagnetic field.

The symmetry (4.42) exchanges the electric and magnetic fields: $E_i \to E_i' = -B_i$ and $B_i \to B_i' = E_i$.

Exercise 4.9 Show that the self-dual combinations $F^{\pm}_{\mu\nu}$contain only photons of one polarization in their plane wave expansions:

$$F^{\pm}_{\mu\nu} = 2\mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\left[\mathrm{e}^{\mathrm{i}k\cdot x}k_{[\mu}\epsilon_{v]}(\vec{k},\pm)a(\vec{k},\pm) - \mathrm{e}^{-\mathrm{i}k\cdot x}k_{[\mu}\epsilon^*_{\nu]}(\vec{k},\mp)a^*(\vec{k},\mp)\right].$$

To perform this exercise, check first that with the polarization vectors given in Sec. 4.1.2, one has

$$-\frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}k_\rho\epsilon_\sigma(\vec{k},\pm) = \pm k^{[\mu}\epsilon^{\nu]}(\vec{k},\pm).$$

Exercise 4.10 Show that the quantity $F_{\mu\nu}\tilde{F}^{\mu v}$ is a total derivative, i.e.

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -\mathrm{i}\partial_\mu\left(\varepsilon^{\mu\nu\rho\sigma}A_v F_{\rho\sigma}\right).$$

Show, using (1.45), that under a Lorentz transformation

$$\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)(x) \to \det\Lambda^{-1}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)(\Lambda x).$$

Thus $F_{\mu\nu}\tilde{F}^{\mu\nu}$ transforms as a scalar under proper Lorentz transformations but changes sign under space or time reflections. Use the Schouten identity (3.11) to prove that

$$F_{\mu\rho}\tilde{F}^{\rho}_{\nu} = \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}\tilde{F}^{\rho\sigma}.$$

4.2.3 Duality for gauge field and complex scalar

$$\mathcal{L} = -\frac{1}{4}(\operatorname{Im} Z)F_{\mu\nu}F^{\mu\nu} - \frac{1}{8}(\operatorname{Re} Z)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}.$$

Actions in which the gauge field kinetic term is multiplied by a function of complex scalar fields are quite common in supersymmetry and supergravity. We now define an extension of the duality transformation (4.42) which gives a non-abelian global $\mathrm{SL}(2,\mathbb{R})$ symmetry of the gauge field equations of this theory. In Sec. 7.12.2 we will discuss a generalized scalar kinetic term that is invariant under $\mathrm{SL}(2,\mathbb{R})$. The field $Z(x)$ carries dynamics, and the equations of motion of the combined vector and scalar theory are also invariant. The gauge Bianchi identity and equation of motion of our theory are

$$\partial_\mu\tilde{F}^{\mu v} = 0, \quad \partial_\mu\left[(\operatorname{Im} Z)F^{\mu v} + \mathrm{i}(\operatorname{Re} Z)\tilde{F}^{\mu v}\right] = 0.$$

It is convenient to define the real tensor

$$G^{\mu v} \equiv \varepsilon^{\mu\nu\rho\sigma}\frac{\delta S}{\delta F^{\rho\sigma}} = -\mathrm{i}(\operatorname{Im} Z)\tilde{F}^{\mu v} + (\operatorname{Re} Z)F^{\mu\nu},$$

and to consider the self-dual combinations $F^{\mu\nu\pm}$ and $G^{\mu\nu\pm}$. Note that these are related by

$$G^{\mu\nu-} = ZF^{\mu\nu-}, \quad G^{\mu\nu+} = \bar{Z}F^{\mu v+}.$$

The information in (4.49) can then be reexpressed as

$$\partial_\mu \operatorname{Im} F^{\mu v-} = 0, \quad \partial_\mu \operatorname{Im} G^{\mu\nu-} = 0.$$

We define a matrix of the group $\mathrm{SL}(2,\mathbb{R})$ by

$$\mathcal{S} \equiv \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \quad ad - bc = 1$$

The group $\mathrm{SL}(2,\mathbb{R})$ acts on the tensors $F^-$and G^-as follows:

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S} \begin{pmatrix} F^- \\ G^- \end{pmatrix}.$$

Since $\mathcal{S}$ is real, the conjugate tensors $\mathrm{F}^+$and G^+also transform in the same way.

Exercise 4.11 Assume that $\operatorname{Im} F^-$and $\operatorname{Im} G^-$satisfy (4.52), and show that $\operatorname{Im} F'^-$ and $\operatorname{Im} G'^-$ also obey the same equations. Show that G'^- and a transformed scalar Z' satisfy $G'^{\mu\nu-} = Z'F'^{\mu\nu-}$, if Z' is defined as the following nonlinear transform of Z :

$$Z' = \frac{aZ + b}{cZ + d}.$$

Exercise 4.12 Show that the Lagrangian (4.48) can be rewritten as

$$\mathcal{L}(F, Z) = -\frac{1}{2} \operatorname{Im}\left(ZF_{\mu\nu}^- F^{\mu\nu-}\right).$$

Consider the $\mathrm{SL}(2,\mathbb{R})$ transformation with parameters $a = d = 1$ and $b = 0$. Show that

$$\mathcal{L}\left(F', Z'\right) = -\frac{1}{2} \operatorname{Im}\left(Z(1 + cZ)F_{\mu\nu}^- F^{\mu v-}\right) \neq \mathcal{L}(F, Z).$$

The symmetric gauge invariant stress tensor of this theory is

$$\Theta^{\mu v} = (\operatorname{Im} Z)\left(F^{\mu\rho} F_\rho^\nu - \frac{1}{4}\eta^{\mu v} F_{\rho\sigma} F^{\rho\sigma}\right).$$

As we will see in Ch. 8, when the theory is coupled to gravity, it is this stress tensor that is the source of the gravitational field; see (8.4). It is then important that $\operatorname{Im} Z$ is positive, which restricts the domain of Z to the upper half-plane. It is also important that the stress tensor is invariant under the duality transformations (4.54) and (4.55). This is the reason for the duality symmetry of many black hole solutions of supergravity,

Exercise 4.13 Prove that the energy-momentum tensor (4.58) is invariant under duality. Here are some helpful relations which you will need:

$$\operatorname{Im} Z' = \frac{\operatorname{Im} Z}{(cZ + d)(c\bar{Z} + d)}.$$

Further you need again (4.47) and a similar identity (proven by contracting ε-tensors)

$$\tilde{F}_{\mu\rho} \tilde{F}_\nu{}^\rho = -F_{\mu\rho} F_\nu{}^\rho + \frac{1}{2}\eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}.$$

This leads to

$$F'_{\mu\rho}F'^{\rho}_{\nu} - \frac{1}{4}\eta_{\mu\nu}F'_{\rho\sigma}F'^{\rho\sigma} = |cZ+d|^2\left[F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right].$$

Exercise 4.14 The free Maxwell theory is the special case of (4.48) with fixed $Z = \mathrm{i}$. Suppose that the gauge field is coupled to a conserved current as in (4.14). Check that the electric charge can be expressed in terms of F or G by

$$q \equiv \int \mathrm{d}^3\vec{x} J^0 = \int \mathrm{d}^3\vec{x}\partial_i F^{0i} = -\frac{1}{2}\int \mathrm{d}^3\vec{x}\varepsilon^{ijk}\partial_i G_{jk}.$$

A magnetic charge can be introduced in Maxwell theory as the divergence of $\vec{B}$ (recall $E^i = F^{0i}$ and $B^i = \frac{1}{2}\varepsilon^{ijk}F_{jk}$). This leads to a definition [7]

$$p \equiv -\frac{1}{2}\int \mathrm{d}^3\vec{x}\varepsilon^{ijk}\partial_i F_{jk}.$$

Show that $\begin{pmatrix} p \\ q \end{pmatrix}$ is a vector that transforms under $\mathrm{SL}(2,\mathbb{R})$ in the same way as the tensors $F^-$and G^-in (4.54).

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$Z' = Z+1, \quad Z' = -\frac{1}{Z}.$$

This means that one can express any element of $\mathrm{SL}(2,\mathbb{Z})$ as the product of (finitely many) factors of the two generators above and their inverses.

In Secs. 4.1 and 4.2.2 we considered $Z = \mathrm{i}g = \mathrm{i}$. Check that general duality transformations in this case are of the form

$$F'^{-}_{\mu\nu} = (d+\mathrm{i}c)F^{-}_{\mu\nu}, \qquad \text{i.e.} \qquad F'_{\mu\nu} = dF_{\mu\nu} - \mathrm{i}c\tilde{F}_{\mu\nu}.$$

4.2.4 Electromagnetic duality for coupled Maxwell fields (!?!??!?!??!?!?!)

$$\mathcal{L} = -\frac{1}{4}(\mathrm{Re}\, f_{AB})\, F^{A}_{\mu\nu}F^{\mu\nu B} + \frac{1}{4}\mathrm{i}\,(\mathrm{Im}\, f_{AB})\, F^{A}_{\mu\nu}\tilde{F}^{\mu\nu B},$$

$$\begin{aligned}\mathcal{L}\left(F^+,F^-\right) &= -\frac{1}{2}\,\mathrm{Re}\left(f_{AB}F^{-A}_{\mu\nu}F^{\mu\nu-B}\right)\\ &= -\frac{1}{4}\left(f_{AB}F^{-A}_{\mu\nu}F^{\mu\nu-B} + f^*_{AB}F^{+A}_{\mu\nu}F^{\mu\nu+B}\right),\end{aligned}$$

and define the new tensors

$$\begin{aligned}G^{\mu\nu}_{A} &= \varepsilon^{\mu\nu\rho\sigma}\frac{\delta S}{\delta F^{\rho\sigma A}} = -\left(\mathrm{Im}\, f_{AB}\right)F^{\mu\nu B} - \mathrm{i}\left(\mathrm{Re}\, f_{AB}\right)\tilde{F}^{\mu\nu B} = G^{\mu\nu+}_{A} + G^{\mu\nu-}_{A},\\ G^{\mu\nu-}_{A} &= -2\mathrm{i}\frac{\delta S\left(F^+,F^-\right)}{\delta F^{-A}_{\mu\nu}} = \mathrm{i}f_{AB}F^{\mu\nu-B},\\ G^{\mu\nu+}_{A} &= 2\mathrm{i}\frac{\delta S\left(F^+,F^-\right)}{\delta F^{+A}_{\mu\nu}} = -\mathrm{i}f^*_{AB}F^{\mu\nu+B}.\end{aligned}$$

Since the field equation for the action containing (4.67) is

$$0 = \frac{\delta S}{\delta A^{A}_{\nu}} = -2\partial_\mu\frac{\delta S}{\delta F^{A}_{\mu\nu}},$$

the Bianchi identity and the equation of motion can be expressed in the concise form

$$\begin{aligned}\partial^\mu \operatorname{Im} F^{A-}_{\mu\nu} &= 0 \quad \text{Bianchi identities,}\\ \partial_\mu \operatorname{Im} G^{\mu\nu-}_{A} &= 0 \quad \text{equations of motion.}\end{aligned}$$

(The same equations hold for $\operatorname{Im} F^{A+}$ and $\operatorname{Im} G_A{}^{+}$.)

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S} \begin{pmatrix} F^- \\ G^- \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^- \\ G^- \end{pmatrix},$$

with real $m \times m$ submatrices A, B, C, D. Owing to the reality of these matrices, the same relations hold for the self-dual tensors $F^+$and G^+. In Sec. 4.2.3, these matrices were just numbers:

$$A = d, \quad B = c, \quad C = b, \quad D = a.$$

We require that the transformed field tensors F'^A and G'_A are also related by the definitions (4.68), with appropriately transformed f_{AB}. We work out this requirement in the following steps:

$$G'^- = (C + \mathrm{i}Df)F^- = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1}F'^-,$$

such that we conclude that

$$\mathrm{i}f' = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1}.$$

$$A^T C - C^T A = 0, \quad B^T D - D^T B = 0, \quad A^T D - C^T B = \mathbb{1}.$$

These relations among A, B, C, D are the defining conditions of a matrix of the symplectic group in dimension $2m$ so we reach the conclusion that

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2m, \mathbb{R}).$$

The conditions (4.75) may be summarized as

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

(!!???!??!?!?!?!??!??!??!?!?)

The duality transformations in four dimensions are transformations in the symplectic group Sp(2m, R).

(!?!!?!??!?!!??!?!?!?!??)

$$\mathcal{L} = -\frac{1}{2} \operatorname{Re} \left(f_{AB} F^{-A}_{\mu\nu} F^{\mu\nu-B} \right) = -\frac{1}{2} \operatorname{Im} \left(F^{-A}_{\mu\nu} G^{\mu\nu-}_{A} \right),$$

we obtain

$$\operatorname{Im} F'^- G^- = \operatorname{Im} \left(F^- G^- \right) + \operatorname{Im} \left[2F^- \left(C^T B \right) G^- + F^- \left(C^T A \right) F^- + G^- \left(D^T B \right) G^- \right].$$

If $C \neq 0, B = 0$ the Lagrangian is invariant up to a 4-divergence, since $\operatorname{Im} F^- F^- = -\frac{1}{4} \varepsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}$ and the matrices A and C are real constants. For $B \neq 0$ neither the Lagrangian nor the action is invariant.

Electromagnetic duality has important applications to black hole solutions of extended supergravity theories. Supergravity is also very relevant to the analysis of black hole solutions of string theory. Many black holes are dyons; they carry both magnetic and electric charges for the gauge fields of the system. The general situation is a generalization of what was discussed at the end of Sec. 4.2.3. The charges form a symplectic vector $\begin{pmatrix} q^A_{\mathrm{m}} \\ q_{\mathrm{e}A} \end{pmatrix}$ which must transform as in (4.71). The Dirac-Schwinger-Zwanziger quantization condition restricts these charges to a lattice. Invariance of this lattice restricts the symplectic transformations of (4.71) to a discrete subgroup $\mathrm{Sp}(2m, \mathbb{Z})$, which is analogous to the $\mathrm{SL}(2, \mathbb{Z})$ group discussed previously.

On extensions and applications (????)

(про приложения абзац)
(про другие дуальности. где нужно - не раскрыто еще)

3.4.3 Non-abelian gauge symmetry

4.3.1 Global internal symmetry

$$[t_A, t_B] = f_{AB}{}^C t_C.$$

The array of real numbers $f_{AB}{}^C$ are structure constants of the algebra (the same in all representations). They obey the Jacobi identity

$$f_{AD}{}^E f_{BC}{}^D + f_{BD}{}^E f_{CA}{}^D + f_{CD}{}^E f_{AB}{}^D = 0.$$

The indices can be lowered by the Cartan-Killing metric defined in Appendix B (see (B.6)), and then the f_{ABC} are totally antisymmetric. For simple algebras, the generators can be chosen to be trace orthogonal, $\mathrm{Tr}\,(t_A t_B) = -c\delta_{AB}$, with c positive for compact groups, and the Cartan-Killing metric is then proportional to this expression.

A theory with global non-abelian internal symmetry contains scalar and spinor fields, each of which transforms in an irreducible representation R. For example, there may be a Dirac spinor [13] field $\Psi^\alpha(x), \alpha = 1, \ldots, \dim_R$, that transforms in the complex representation R as

$$\Psi^\alpha(x) \to \left(\mathrm{e}^{-\theta^A t_A}\right)^\alpha{}_\beta \Psi^\beta(x).$$

The conjugate spinor [14] is denoted by $\bar{\Psi}_\alpha$ and transforms as

$$\bar{\Psi}_\alpha \to \bar{\Psi}_\beta \left(\mathrm{e}^{\theta^A t_A}\right)^\beta{}_\alpha.$$

For most of our discussion it is sufficient to restrict attention to the infinitesimal transformations,

$$\begin{aligned}
\delta\Psi &= -\theta^A t_A \Psi, \\
\delta\bar{\Psi} &= \bar{\Psi}\theta^A t_A, \\
\delta\phi^A &= \theta^C f_{BC}{}^A \phi^B.
\end{aligned}$$

Actions, such as the kinetic action for massive fermion fields,

$$S[\bar{\Psi}, \Psi] = -\int \mathrm{d}^D x \bar{\Psi}\left[\gamma^\mu \partial_\mu - m\right]\Psi,$$

are required to be invariant under (4.82). Exercise 4.18 Show that (4.85) is invariant under the transformation (4.82) and (4.83). Consider an infinitesimal transformation and derive the conserved current

$$J_{A\mu} = -\bar{\Psi} t_A \gamma_\mu \Psi, \quad A = 1, \ldots, \dim_G.$$

Show that the current transforms as a field in the adjoint representation, i.e.

$$\delta J_{A\mu} = \theta^C f_{CA}{}^B J_{B\mu}.$$

Show that $\delta\left(\phi^A J_{A\mu}\right) = 0$.

4.3.2 Gauging the symmetry

$$\delta A_\mu^A(x) = \frac{1}{g}\partial_\mu\theta^A + \theta^C(x)A_\mu^B(x)f_{BC}{}^A.$$

$$\begin{aligned}
D_\mu\Psi &= \left(\partial_\mu + gt_AA_\mu^A\right)\Psi,\\
D_\mu\bar{\Psi} &= \partial_\mu\bar{\Psi} - g\bar{\Psi}t_AA_\mu^A,\\
D_\mu\phi^A &= \partial_\mu\phi^A + gf_{BC}^AA_\mu^B\phi^C.\\
\frac{\delta S}{\delta\bar{\Psi}_\alpha} &= -\left[\gamma^\mu D_\mu - m\right]\Psi^\alpha = 0.
\end{aligned}$$

4.3.3 YangMills field strength and action

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$$[D_\mu, D_\nu]\Psi = gF_{\mu\nu}^At_A\Psi,$$

where

$$F_{\mu\nu}^A = \partial_\mu A_v^A - \partial_\nu A_\mu^A + gf_{BC}{}^AA_\mu^BA_v^C.$$

The properties of the covariant derivative guarantee that the right-hand side of (4.91) transforms as a field in the same representation as Ψ. Thus $F_{\mu\nu}^A$ should have simple transformation properties. Indeed, one can derive

$$\delta F_{\mu\nu}^A = \theta^CF_{\mu\nu}^Bf_{BC}{}^A.$$

$$D_\mu F_{\nu\rho}^A + D_vF_{\rho\mu}^A + D_\rho F_{\mu\nu}^A = 0,$$

where $D_\mu F_{v\rho}^A = \partial_\mu F_{v\rho}^A + gf_{BC}{}^AA_\mu^BF_{v\rho}^C$.

$$S\left[A_\mu^A, \bar{\Psi}_\alpha, \Psi^\alpha\right] = \int \mathrm{d}^Dx\left[-\frac{1}{4}F^{A\mu v}F_{\mu\nu}^A - \bar{\Psi}_\alpha\left(\gamma^\mu D_\mu - m\right)\Psi^\alpha\right].$$

The action is gauge invariant. The Euler variation with respect to A_v^A gives (4.94) with current source (4.86), and the variation with respect to $\bar{\Psi}_\alpha$ gives (4.90).

4.3.4 YangMills theory for G = SU(N) 90

The most commonly studied gauge group for Yang-Mills theory is $\mathrm{SU}(N)$. The generators of the fundamental representation of its Lie algebra are a set of N^2-1 traceless antihermitian $N\times N$ matrices t_A, which are normalized by the bilinear trace relation

$$\operatorname{Tr}\left(t_At_B\right) = -\frac{1}{2}\delta_{AB}.$$

$$U(x)t_AU(x)^{-1} = t_BR(x)^B{}_A,$$

where $R(x)^B{}_A$ is a real $(N^2-1)\times(N^2-1)$ matrix. Exercise 4.23 Consider the product of two gauge group elements U_1 and U_2, which gives a third via $U_1U_2 = U_3$. For each element U_i, there is an associated matrix $\left(R_i\right)^B{}_A$, defined by $U_it_AU_i^{-1} = t_B\left(R_i\right)^B{}_A$. Prove that $\left(R_3\right)^B{}_A = \left(R_1\right)^B{}_C\left(R_2\right)^C{}_A$, which shows that the matrices $R^B{}_A$ defined by (4.99) are the matrices of an (N^2-1)-dimensional representation of $\mathrm{SU}(N)$. Use (4.99) to show that, to first order in the gauge parameters θ^C, $R^B{}_A = \delta^B_A + \theta^C f_{AC}{}^B + \ldots$. This shows that the matrices $R^B{}_A$ are exactly those of the adjoint representation. [15]

Given any set of N^2-1 real quantities X^A, that is any element of the vector space $\mathbb{R}^{N^2-1}$, we can form the matrix $\mathbf{X} = t_A X^A$. For any group element U, we have $U\mathbf{X}U^{-1} = t_B R^B{}_A X^A$. Thus the unitary transformation of the matrix $\mathbf{X}$ contains the information that the quantities $X^A = -2\delta^{AB}\,\mathrm{Tr}\,(t_B\mathbf{X})$ transform in the adjoint representation, that is as $X^A \to R^A{}_B X^B$. Thus, given any field in the adjoint representation, such as $\phi^A(x)$, we can form the matrix $\Phi(x) = t_A\phi^A(x)$. Gauge transformations can then be implemented as

$$\Phi(x) \to U(x)\Phi(x)U(x)^{-1}.$$

$$D_\mu\Psi \equiv (\partial_\mu + g\mathbf{A}_\mu)\,\Psi,$$
$$D_\mu\bar{\Psi} \equiv \partial_\mu\bar{\Psi} - g\bar{\Psi}\mathbf{A}_\mu.$$

For a field in the adjoint representation, such as Φ, we define

$$D_\mu\Phi = \partial_\mu\Phi + g\,[\mathbf{A}_\mu, \Phi]\,,$$

which involves the matrix commutator.

Exercise 4.25 Demonstrate that these covariant derivatives transform correctly, specifically that

$$D_\mu\Psi \to U(x)D_\mu\Psi, \quad D_\mu\bar{\Psi} \to D_\mu\bar{\Psi}U(x)^{-1}, \quad D_\mu\Phi \to U(x)D_\mu\Phi U(x)^{-1}.$$

The non-abelian field strength can also be converted to matrix form as

$$\mathbf{F}_{\mu v} = t_A F^A_{\mu v} = \partial_\mu\mathbf{A}_v - \partial_v\mathbf{A}_\mu + g\,[\mathbf{A}_\mu, \mathbf{A}_v]\,.$$

The matrix formalism is a convenient way to express quantities of interest in the theory. For example the Yang-Mills action (4.96) can be written as

$$S\left[\mathbf{A}_\mu, \bar{\Psi}, \Psi\right] = \int \mathrm{d}^D x \left[\frac{1}{2}\,\mathrm{Tr}\,(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}) - \bar{\Psi}\,(\gamma^\mu \mathrm{D}_\mu - \mathrm{m})\,\Psi\right]$$

The N^2-1 matrix generators $(t_A)^\alpha{}_\beta$ of the fundamental representation, normalized as in (4.97), together with the matrix $\mathrm{i}\delta^\alpha_\beta$ form a complete set of $N\times N$ anti-hermitian matrices, which are orthogonal in the trace norm. Therefore one can expand any $N\times N$ anti-hermitian matrix $H^\alpha{}_\beta$ in this set as

$$H^\alpha{}_\beta = \mathrm{i}h_0\delta^\alpha{}_\beta + h^A\,(t_A)^\alpha{}_\beta,$$
$$h_0 = -\frac{\mathrm{i}}{N}\,\mathrm{Tr}\,H, \quad h^A = -2\delta^{AB}\,\mathrm{Tr}\,(Ht_B)\,.$$

4.4 Internal symmetry for Majorana spinors

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representation. Let χ^α denote a set of Majorana spinors to which we assign the group transformation rule

$$\chi^\alpha \to \chi'^\alpha \equiv \left(\mathrm{e}^{-\theta^A\left(t_A P_L + t^*_A P_R\right)}\right)^\alpha{}_\beta\chi^\beta.$$

The matrices $t_A P_L + t^*_A P_R$ are generators of a representation of an explicitly real representation of the Lie algebra, so the transformed spinors χ'^α also satisfy the Majorana condition. This is the transformation rule used for Majorana spinors in supersymmetric gauge theories in Ch. 6.

By applying the projectors to (4.110), one can see that the chiral and anti-chiral projections of χ transform as

$$P_L\chi \to P_L\chi' \equiv \left(\mathrm{e}^{-\theta^A t_A}\right) P_L\chi,$$
$$P_R\chi \to P_R\chi' \equiv \left(\mathrm{e}^{-\theta^A t_A^*}\right) P_R\chi.$$

$$\delta(\bar{\chi}\chi) = -\theta^A \bar{\chi}\left(t_A + t_A^T\right)\gamma_*\chi.$$

The mass term is invariant only for the subset of generators that are antisymmetric, and thus real. This condition defines a subalgebra of the original Lie algebra $\mathfrak{g}$ of the theory, specifically the subalgebra that contains only parity conserving vector-like gauge transformations. For the case $\mathfrak{g} = \mathfrak{su}(N)$, the subalgebra is isomorphic to $\mathfrak{so}(N)$. Non-invariance of the Majorana mass term is a special case of the general idea that chiral symmetry requires massless fermions.

Exercise 4.29 Show that

$$\frac{1}{2}\int \mathrm{d}^4x\bar{\chi}\gamma^\mu D_\mu\chi = \int \mathrm{d}^4x\bar{\chi}\gamma^\mu P_L D_\mu\chi = \int \mathrm{d}^4x\bar{\chi}\gamma^\mu P_R D_\mu\chi.$$

Note that $P_{L,R}D_\mu\chi = D_\mu P_{L,R}\chi$.

3.5 5 The free RaritaSchwinger field

3.5.1 Basic thery

$$\Psi_\mu(x) \to \Psi_\mu(x) + \partial_\mu\epsilon(x).$$

$$S = -\int \mathrm{d}^D x\bar{\Psi}_\mu\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho,$$

$$\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho = 0$$

Exercise 5.1 Show directly that for $D = 3$, the field equation (5.3) implies that $\partial_\nu\Psi_\rho - \partial_\rho\Psi_v = 0$. This means that the field has no gauge invariant degrees of freedom and thus no propagating particle modes. This is the supersymmetric counterpart of the situation in gravity for $D = 3$, where the field equation $R_{\mu\nu} = 0$ implies that the full curvature tensor $R_{\mu\nu\rho\sigma} = 0$. Hence no degrees of freedom.

We notice that (5.3) can be rewritten in an equivalent but simpler form. For this purpose, we use the γ-matrix relation $\gamma_\mu\gamma^{\mu\nu\rho} = (D-2)\gamma^{\nu\rho}$, which implies that $\gamma^{\nu\rho}\partial_\nu\Psi_\rho = 0$ in spacetime dimension $D > 2$. We also note that $\gamma^{\mu v\rho} = \gamma^\mu\gamma^{v\rho} - 2\eta^{\mu[v}\gamma^{\rho]}$. Using this information, it is easy to see that (5.3) implies that

$$\gamma^\mu\left(\partial_\mu\Psi_\nu - \partial_\nu\Psi_\mu\right) = 0.$$

3.5.2 The initial value problem

$$\gamma^i\Psi_i = 0,$$

which will play the same role as the Coulomb gauge condition we used in Sec. 4.1.2. Exercise 5.3 Show by an argument analogous to that in Sec. 4.1.2 that this condition does fix the gauge uniquely.

We use the equivalent form (5.4) of the field equations. The $v = 0$ and $v \to i$ components are

$$\gamma^i\partial_i\Psi_0 - \partial_0\gamma^i\Psi_i = 0,$$
$$\gamma\cdot\partial\Psi_i - \partial_i\gamma\cdot\Psi = 0.$$

Using the gauge condition one can see that $\nabla^2\Psi_0 = 0$, so $\Psi_0 = 0$ according to the discussion on p. 69. The spatial components Ψ_i then satisfy the Dirac equation

$$\gamma \cdot \partial\Psi_i = 0,$$

which is a time evolution equation. However, there is an additional constraint, $\partial^i\Psi_i = 0$, obtained by contracting (5.8) with γ^i. Thus from the gauge condition and the equation of motion, we find $3 \times 2^{[D/2]}$ independent constraints on the initial data, namely

$$\begin{aligned} \gamma^i\Psi_i(\vec{x}, 0) &= 0, \\ \Psi_0(\vec{x}, 0) &= 0, \\ \partial^i\Psi_i(\vec{x}, 0) &= 0. \end{aligned}$$

Degrees of freedom of the massless Rarita-Schwinger field

On-shell degrees of freedom $= \frac{1}{2}(D-3)2^{[D/2]}$.

Off-shell degrees of freedom $= (D-1)2^{[D/2]}$.

According to the discussion for $D = 4$ at the beginning of Ch.4, we would expect the Fourier expansion of the field to contain annihilation and creation operators for states of helicity $\lambda = \pm 3/2$. Let's derive this fact starting from the plane wave

$$\Psi_i(x) = \mathrm{e}^{\mathrm{i}p\cdot x} v_i(\vec{p}) u(\vec{p}),$$

for a positive null energy-momentum vector $p^\mu = (|\vec{p}|, \vec{p})$. Since $\Psi_i(x)$ satisfies the Dirac equation (5.8), the four-component spinor $u(\vec{p})$ must be a superposition of the massless helicity spinors $u(\vec{p}, \pm)$ given in (2.44). Thus we use the Weyl representation (2.19) of the γ-matrices. The vector $v_i(\vec{p})$ may be expanded in the complete set

$$v_i(\vec{p}) = a p_i + b\epsilon_i(\vec{p}, +) + c\epsilon_i(\vec{p}, -),$$

where $\epsilon_i(\vec{p}, \pm)$ are the transverse polarization vectors of Sec. 4.1.2, i.e. they satisfy $p^i\epsilon_i(\vec{p}, \pm) = 0$. The constraint (5.11) requires that $a = 0$. Thus (5.12) is reduced to the form

$$\begin{aligned} \Psi_i(x) = \mathrm{e}^{\mathrm{i}p\cdot x}[& b_+\epsilon_i(\vec{p}, +)u(\vec{p}, +) + c_+\epsilon_i(\vec{p}, -)u(\vec{p}, +) \\ & + b_-\epsilon_i(\vec{p}, +)u(\vec{p}, -) + c_-\epsilon_i(\vec{p}, -)u(\vec{p}, -)]\,. \end{aligned}$$

$$\Psi_\mu(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0} \sum_\lambda \left[\mathrm{e}^{\mathrm{i}p\cdot x}\epsilon_\mu(\vec{p}, \lambda)u(\vec{p}, \lambda)c(\vec{p}, \lambda) + \mathrm{e}^{-\mathrm{i}p\cdot x}\epsilon_\mu^*(\vec{p}, \lambda)v(\vec{p}, \lambda)d^*(\vec{p}, \lambda)\right].$$

$$T_{\mu\nu} = \bar{\Psi}_\rho\gamma^{\rho\sigma}{}_\mu\partial_\nu\Psi_\sigma - \eta_{\mu\nu}\mathcal{L}$$

It is neither symmetric nor gauge invariant under (5.1) (and its Dirac conjugate). It can be made symmetric (see [27]), but gauge non-invariance is intrinsic and cannot be restored by adding terms of the form $\partial_\sigma S^{\sigma\mu\nu}$. The reason is that the gravitino must be joined with gravity in the gauge multiplet of SUSY. In a gravitational theory there is no well-defined energy density.

Exercise 5.6 Show that the total energy-momentum $P^v = \int \mathrm{d}^3\vec{x}T^{0v}(\vec{x}, t)$ is gauge invariant and given (for $D = 4$) by

$$P^v = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0} p^\nu \sum_\lambda \left[c^*(\vec{p}, \lambda)c(\vec{p}, \lambda) - d(\vec{p}, \lambda)d^*(\vec{p}, \lambda)\right].$$

3.5.3 Sources and Greens function

Theory

$$\gamma^{\mu v \rho}\partial_v \Psi_\rho = J^\mu.$$

$$(\not{\partial} - m)\Psi(x) = J(x).$$

Given a Green's function $S(x-y)$ that satisfies

$$(\not{x} - m)\, S(x-y) = -\delta(x-y),$$

the solution of (5.19) is given by

$$\Psi(x) = -\int \mathrm{d}^D y S(x-y) J(y).$$

Let's solve this problem using the Fourier transform. The symmetries of Minkowski spacetime allow us to assume the Fourier representation

$$S(x-y) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \mathrm{e}^{\mathrm{i}p\cdot(x-y)} S(p).$$

In momentum space, (5.20) reads

$$(\mathrm{i}pp - m) S(p) = -1,$$

and the solution (with Feynman's causal structure) is

$$S(p) = -\frac{1}{\mathrm{i}\not{p} - m} = \frac{\mathrm{i}\not{p} + m}{p^2 + m^2 - \mathrm{i}\epsilon}.$$

Comparing with (4.18), we see that we can express $S(x-y)$ in terms of the scalar Green's function as

$$S(x-y) = (\partial_x + m)\, G(x-y).$$

$$\Psi_\mu(x) = -\int \mathrm{d}^D y S_{\mu\nu}(x-y) J^v(y),$$

where $S_{\mu v}(x-y)$ is a tensor bispinor. A bispinor has two spinor indices, which are suppressed in our notation, and it can be regarded as a matrix of the Clifford algebra. As in the electromagnetic case, the Rarita-Schwinger operator is not invertible, but we can assume that the Green's function satisfies

$$\gamma^{\mu\sigma\rho}\frac{\partial}{\partial x^\sigma} S_{\rho v}(x-y) = -\delta^\mu_v \delta(x-y) + \frac{\partial}{\partial y^\nu}\Omega^\mu(x-y).$$

$$\mathrm{i}\gamma^{\mu\sigma\rho} p_\sigma S_{\rho v}(p) = -\delta^\mu_v - \mathrm{i}p_v \Omega^\mu(p).$$

We will solve (5.28) by writing an appropriate ansatz for $S_{\rho v}(p)$ and then find the unknown functions in the ansatz. The matrix $\gamma^{\mu\sigma\rho}p_\sigma$ in (5.28) contains an odd rank element of the Clifford algebra and it is odd under the reflection $p_\sigma \to -p_\sigma$. It is reasonable to guess that the ansatz we need should also involve odd rank Clifford elements and be odd under the reflection. We would also expect that terms that contain the momentum vectors p_ρ or p_v are 'pure gauges' and thus arbitrary additions to the propagator, which would not be determined by the equation (5.28). So we omit such terms and postulate the ansatz

$$\mathrm{i}\, S_{\rho v}(p) = A\left(p^{2}\right) \eta_{\rho v}\ \not{p} + B\left(p^{2}\right) \gamma_{\rho}\ \not{p} \gamma_{v}.$$

The next step is to substitute the ansatz in (5.28) and simplify the products of γ-matrices that appear. This process yields

$$\begin{aligned}
\mathrm{i}\gamma^{\mu\sigma\rho} p_{\sigma} S_{\rho v}(p) =& A\gamma^{\mu\sigma}{}_{\nu}\ \not{p} p_{\sigma} + (D-2) B \gamma^{\mu\sigma}\ \not{p}\gamma_{\nu} p_{\sigma} \\
=& A\left(p^{\mu}\gamma^{\sigma}{}_{\nu} - p^{\sigma}\gamma^{\mu}{}_{\nu}\right) p_{\sigma} + (D-2) B\left(-p^{\mu}\gamma^{\sigma} + p^{\sigma}\gamma^{\mu}\right)\gamma_{\nu} p_{\sigma} \\
&+ \ldots \\
=& [A - (D-2)B]\left(p^{\mu}\gamma^{\sigma}{}_{\nu} - p^{\sigma}\gamma^{\mu}{}_{\nu}\right) p_{\sigma} + (D-2) B p^{2} \delta_{v}^{\mu} \\
&+ \ldots
\end{aligned}$$

$$S_{\mu\nu}(x-y) = \left[\eta_{\mu\nu}\ \not{\partial} + \frac{1}{D-2}\gamma_{\mu}\ \not{\partial}\gamma_{\nu} + C\partial_{\mu}\gamma_{\nu} + E\gamma_{\mu}\partial_{\nu} - F\partial_{\mu}\ \not{\partial}\partial_{\nu}\right] G(x-y),$$

where $G(x-y)$ is the massless scalar propagator (4.19), and all derivatives are with respect to x.

Exercise 5.7 Include the omitted p_{v} terms in (5.30) and $\Omega(p)$ in the analysis and verify that the gauge terms in the propagator are arbitrary. Show that, for the choice $E = -1/(D-2)$, and arbitrary C and F, the propagator satisfies

$$\mathrm{i}\gamma^{\mu\sigma\rho} p_{\sigma} S_{\rho v}(p) = -\left(\delta_{v}^{\mu} - \frac{p^{\mu} p_{v}}{p^{2}}\right).$$

Show that, for $D = 4$, the propagator, with $C = -1$, takes the 'reverse index' form $S_{\mu\nu}(p) = -\mathrm{i}\frac{1}{2}\gamma_{\nu}\ \not{p}\gamma_{\mu}$, which is the form used in most of the literature on perturbative studies in supergravity [28].

3.5.4 Massive gravitinos from dimensional reduction

5.3.1 Dimensional reduction for scalar fields 102 (!!??!?!?! научусь этому, выглядит просто!!!)

Let's change to a more convenient notation and rename the coordinates of the $(D+1)$ dimensional product spacetime $x^{0} = t, x^{1}, \ldots, x^{D-1}, y$, where y is the coordinate of S^{1} with range $0 \leq y \leq 2\pi L$. We consider a massive complex scalar field $\phi\left(x^{\mu}, y\right)$ that obeys the Klein-Gordon equation

$$\left[\square_{D+1} - m^{2}\right]\phi = \left[\square_{D} + \left(\frac{\partial}{\partial y}\right)^{2} - m^{2}\right]\phi = 0.$$

Acceptable solutions must be single-valued on S^{1} and thus have a Fourier series expansion

$$\phi\left(x^{\mu}, y\right) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky/L}\phi_{k}\left(x^{\mu}\right).$$

5.3 Massive gravitinos from dimensional reduction 103 It is immediate that the spacetime function associated with the k th Fourier mode, namely $\phi_{k}\left(x^{\mu}\right)$, satisfies

$$\left[\square_{D} - \left(\frac{k}{L}\right)^{2} - m^{2}\right]\phi_{k} = 0.$$

Thus it describes a particle of mass $m_k^2 = (k/L)^2 + m^2$. So the spectrum of the theory, as viewed in Minkowski $_D$, contains an infinite tower of massive scalars!

There is an even simpler way to find the mass spectrum. Just substitute the plane wave $\mathrm{e}^{\mathrm{i}p^\mu x_\mu}\mathrm{e}^{\mathrm{i}ky/L}$ directly in the $(D+1)$-dimensional equation (5.34). The D-component energy-momentum vector p^μ must satisfy $p^\mu p_\mu = (k/L)^2 + m^2$. The mass shift due to the Fourier wave on S^1 is immediately visible.

5.3.2 Dimensional reduction for spinor fields 103

We will consider the dimensional reduction process for a complex spinor $\Psi(x^\mu, y)$ for even $D = 2m$ (so that the spinors in $D+1$ dimensions have the same number of components). Two new ideas enter the game. The first just involves the Dirac equation in D dimensions. We remark that if $\Psi(x)$ satisfies

$$[\partial_D - m]\,\Psi(x) = 0,$$

then the new field $\tilde{\Psi} \equiv \mathrm{e}^{-\mathrm{i}\gamma_*\beta}\Psi$, obtained by applying a chiral phase factor, satisfies

$$[\partial_D - m\,(\cos 2\beta + \mathrm{i}\gamma_* \sin 2\beta)]\,\tilde{\Psi} = 0.$$

Physical quantities are unchanged by the field redefinition, so both equations describe particles of mass m. One simple implication is that the sign of m in (5.37) has no physical significance, since it can be changed by field redefinition with $\beta = \pi/2$.

The second new idea is that a fermion field can be either periodic or anti-periodic $\Psi(x^\mu, y) = \pm\Psi(x^\mu, y +$ Anti-periodic behavior is permitted because a fermion field is not observable. Rather, bilinear quantities such as the energy density $T^{00} = -\bar{\Psi}\gamma^0\partial^0\Psi$ are observables and they are periodic even when Ψ is anti-periodic. Thus we consider the Fourier series

$$\Psi(x^\mu, y) = \sum_k \mathrm{e}^{\mathrm{i}ky/L}\Psi_k(x^\mu),$$

where the mode number k is integer or half-integer for periodic or anti-periodic fields, respectively. In either case when we substitute (5.39) in the $(D+1)$-dimensional Dirac equation $[\partial_{D+1} - m]\,\Psi((x^\mu, y) = 0$ we find that $\Psi_k(x^\mu)$ satisfies [3]

$$\left[\partial_D - \left(m - \mathrm{i}\gamma_* \frac{k}{L}\right)\right]\Psi_k(x^\mu) = 0.$$

5.3.3 Dimensional reduction for the vector gauge field 104

We now apply circular dimensional reduction to Maxwell's equation

$$\partial^v F_{v\mu} = \Box_{D+1}A_\mu - \partial_\mu(\partial^v A_v) = 0$$

in $D+1$ dimensions, and we assume a periodic Fourier series representation

$$A_\mu(x,y) = \sum_k \mathrm{e}^{\mathrm{i}ky/L}A_{\mu k}(x), \quad A_D(x,y) = \sum_k \mathrm{e}^{\mathrm{i}ky/L}A_{Dk}(x),$$

with k an integer. The analysis simplifies greatly if we assume the gauge conditions $A_{Dk}(x) = 0$ for $k \neq 0$ and vector component D tangent to S^1. It is easy to see that this gauge can be achieved and uniquely fixes the Fourier modes $\theta_k(x), k \neq 0$, of the gauge function. The gauge invariant Fourier mode $A_{D0}(x)$ remains a physical field in the dimensionally reduced theory. A quick examination of the $\mu \to D$ component of (5.41) shows that it reduces to

$$\begin{aligned} &k = 0 : \Box_{D+1}A_{D0} = \Box_D A_{D0} = 0, \\ &k \neq 0 : \partial^\mu A_{\mu k} = 0, \end{aligned}$$

so the mode $A_{D0}(x)$ simply describes a massless scalar in D dimensions. For $\mu \leq D-1$, the wave equation (5.41) implies that the vector modes $A_{\mu k}(x)$ satisfy

$$\left[\square_D - \frac{k^2}{L^2}\right] A_{\mu k} - \partial_\mu \left(\partial^v A_{vk}\right) = 0.$$

For mode number $k=0$ this is just the Maxwell equation in D dimensions with its gauge symmetry under $A_{\mu 0} \rightarrow A_{\mu 0} + \partial_\mu \theta_0$ intact, since the Fourier mode $\theta_0(x)$ remained unfixed in the process above. For mode number $k \neq 0$, (5.44) is the standard equation [4] for a massive vector field with mass $m_k^2 = k^2/L^2$, namely the equation of motion of the action

$$S = \int \mathrm{d}^D x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu\right].$$

5.3.4 Finally $\Phi_\mu(x, y)$ 104

Let's write out the $\mu = D$ and $\mu \leq D-1$ components of (5.3) with $\Psi_D = 0$ (using $\gamma^D = \gamma_*$) :

$$\begin{aligned} &\gamma^{v\rho} \partial_\nu \Psi_{\rho k} = 0, \\ &\left[\gamma^{\mu\nu\rho} \partial_\nu - \mathrm{i} \frac{k}{L} \gamma_* \gamma^{\mu\rho}\right] \Psi_{\rho k} = 0. \end{aligned}$$

Note that the first equation of (5.46) follows by application of ∂^μ to the second one. Exercise 5.8 Show that the chiral transformation $\Psi_{\rho k} = \mathrm{e}^{(-\mathrm{i}\pi\gamma_*/4)} \Psi'_{\rho k}$ leads, after replacing $\Psi' \rightarrow \Psi$, to the equation of motion

$$\left(\gamma^{\mu\nu\rho} \partial_\nu - m \gamma^{\mu\rho}\right) \Psi_\rho = 0.$$

The last equation is the Euler-Lagrange equation of the action

$$S = -\int \mathrm{d}^D x \bar{\Psi}_\mu \left[\gamma^{\mu v \rho} \partial_v - m \gamma^{\mu\rho}\right] \Psi_\rho$$

$$\begin{aligned} &\gamma^\mu \Psi_\mu = 0, \\ &\left(\gamma^{ij} \partial_i - m \gamma^j\right) \Psi_j = 0, \\ &[\not{\partial} + m] \Psi_\mu = 0. \end{aligned}$$

Exercise 5.11 Study the Kaluza-Klein reduction for the Rarita-Schwinger field assuming periodicity $\Psi_\mu(x, y+2\pi) = \Psi_\mu(x, y)$ in y. Show that the spectrum seen in Minkowski $_D$ consists of a massive gravitino for each Fourier mode $k \neq 0$ plus a massless gravitino and massless Dirac particle for the zero mode.

The dimensional reduction process has thus taught us the correct action for a massive gravitino. In particular the mass term is $m \bar{\Psi}_\mu \gamma^{\mu v} \Psi_v$. There is a more general action, namely

$$S = -\int \mathrm{d}^D x \bar{\Psi}_\mu \left[\gamma^{\mu\nu\rho} \partial_v - m \gamma^{\mu\rho} - m' \eta^{\mu\rho}\right] \Psi_\rho,$$

3.6 On N=1 global supersymmetry in D=4

3.6.1 Basic SUSY field theory

SUSY theories contain both bosons and fermions, which are the basis states of a particle representation of the SUSY algebra (6.1)-(6.4). We give a systematic treatment of these representations in Sec. 6.4, but start with an informal discussion here. The states of particles with momentum

$\vec{p}$ and energy $E(\vec{p}) = \sqrt{\vec{p}^2 + m_{B,F}^2}$ are denoted by $|\vec{p}, B\rangle$ and $|\vec{p}, F\rangle$, where the labels B and F include particle helicity. SUSY transformations connect these states. Since the spinor Q_α carries angular momentum 1/2, it transforms bosons into fermions and fermions into bosons. Hence $Q_\alpha|\vec{p}, B\rangle = |\vec{p}, F\rangle$ and $Q_\alpha|\vec{p}, F\rangle \propto |\vec{p}, B\rangle$. Since $[P^\mu, Q_\alpha] = 0$, the transformed states have the same momentum and energy, hence the same mass, so $m_B^2 = m_F^2$. We show in Sec. 6.4.1 that a representation of the algebra contains the same number of boson and fermion states.

The simplest representations of the algebra that lead to the most basic SUSY field theories are: (i) the chiral multiplet, which contains a self-conjugate spin-1/2 fermion described by the Majorana field $\chi(x)$ plus a complex spin- 0 boson described by the scalar field $Z(x)$. Alternatively, $\chi(x)$ may be replaced by the Weyl spinor $P_L\chi$ and/or $Z(x)$ by the combination $Z(x) = (A(x) + \mathrm{i}B(x))/\sqrt{2}$ where A and B are a real scalar and pseudo-scalar, respectively. A chiral multiplet can be either massless or massive. (ii) the gauge multiplet consisting of a massless spin-1 particle, described by a vector gauge field $A_\mu(x)$, plus its spin-1/2 fermionic partner, the gaugino, described by a Majorana spinor $\lambda(x)$ (or the corresponding Weyl field $P_L\lambda$).

Main formulas

$$\begin{aligned}
\left\{Q_\alpha, \bar{Q}^\beta\right\} &= -\frac{1}{2}\left(\gamma_\mu\right)_\alpha{}^\beta P^\mu, \\
\left[M_{[\mu\nu]}, Q_\alpha\right] &= -\frac{1}{2}\left(\gamma_{\mu\nu}\right)_\alpha{}^\beta Q_\beta, \\
\left[P_\mu, Q_\alpha\right] &= 0.
\end{aligned}$$

Note that these are the classical (anti-)commutator relations; see Secs. 1.4 and 1.5. We will discuss this further in Ch.11.

Exercise 6.1 Use (2.30) to reexpress the supercharge anti-commutator in terms of Q and $Q^\dagger$. Then use the correspondence principle, that is multiply by the imaginary i, to obtain the quantum anti-commutator from the classical relation. This procedure gives the operator relation

$$\left\{Q_\alpha, \left(Q^\dagger\right)^\beta\right\}_{\mathrm{qu}} = \frac{1}{2}\left(\gamma_\mu\gamma^0\right)_\alpha{}^\beta P^\mu$$

Trace on the spinor indices to obtain the positivity condition

$$\operatorname{Tr}\left(QQ^\dagger + Q^\dagger Q\right) = 2P^0.$$

Many SUSY theories, but not all, are invariant under a chiral U(1) symmetry called R-symmetry. We denote the generator by T_R. This acts on Q_α via

$$\left[T_R, Q_\alpha\right] = -\mathrm{i}\left(\gamma_*\right)_\alpha{}^\beta Q_\beta$$

but this generator T_R is not required. Other internal symmetries, which commute with Q_α and are frequently called outside charges, can also be included.

There are two important theorems that severely limit the type of charges and algebras that can be realized in an interacting relativistic quantum field theory in $D = 4$ (strictly speaking in a theory with a non-trivial S-matrix in flat space). According to the ColemanMandula (CM) theorem [29, 30], in the presence of massive particles, bosonic charges are limited to $M_{[\mu\nu]}$ and P_μ plus (optional) scalar internal symmetry charges, and the Lie algebra is the direct sum of the Poincaré algebra and a (finite-dimensional) compact Lie algebra for internal symmetry.

If superalgebras are admitted, the situation is governed by the Haag-ŁopuszańskiSohnius (HLS) theorem [31,30], and the algebra of symmetries admits spinor charges Q_α^i. If there is only one Q_α, then the superalgebra must agree with the $\mathcal{N}=1$ Poincaré SUSY algebra in (6.1). When $\mathcal{N}>1$, the possibilities are restricted to the extended SUSY algebras discussed in Appendix 6A. [1] The main thought that we wish to convey is that SUSY theories realize the most general symmetry possible within the framework of the few assumptions made in the hypotheses of the CM and HLS theorems. [2] They also unify bosons and fermions, the two broad classes of particles found in Nature.

6.1.1 Conserved supercurrents 109

$$Q_\alpha = \int \mathrm{d}^3 x \mathcal{J}^0_\alpha(\vec{x},t).$$

If the current is conserved for all solutions of the equations of motion of a theory, then the theory has a fermionic symmetry. By the HLS theorem this symmetry must be supersymmetry!

Therefore we begin the technical discussion of SUSY in quantum field theory by displaying such conserved currents, [3] first for free fields and then for one non-trivial interacting system. Consider a free scalar field $\phi(x)$ satisfying the Klein-Gordon equation $(\Box - m^2)\,\phi = 0$ and a spinor field $\Psi(x)$ satisfying the Dirac equation $(\not{\partial} - m)\Psi = 0$.

Exercise 6.2 Show that the current $\mathcal{J}^\mu = (\not{\partial}\phi - m\phi)\gamma^\mu\Psi$ is conserved for all field configurations satisfying the Klein-Gordon and Dirac equations.

As the second example let's look at the free gauge multiplet with vector potential A_μ and field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ satisfying the Maxwell equation $\partial^\mu F_{\mu\nu} = 0$ and a spinor λ satisfying $\partial\lambda = 0$. Let's show that the current $\mathcal{J}^\mu = \gamma^{\nu\rho}F_{\nu\rho}\gamma^\mu\lambda$ is conserved. We have

$$\partial_\mu \mathcal{J}^\mu = \partial_\mu F_{\nu\rho}\gamma^{\nu\rho}\gamma^\mu\lambda + \gamma^{v\rho}F_{\nu\rho}\not{\partial}\lambda.$$

The last term vanishes. To treat the first term we manipulate the γ-matrices as discussed in Sec. 3.1.4:

$$\gamma^{v\rho}\gamma^\mu = \gamma^{v\rho\mu} + 2\gamma^{[v}\eta^{\rho]\mu}$$

6.1.2 SUSY YangMills theory 110

$$S = \int \mathrm{d}^4 x \left[-\frac{1}{4}F^{\mu\nu A}F^A_{\mu\nu} - \frac{1}{2}\bar{\lambda}^A\gamma^\mu D_\mu\lambda^A \right]$$

For details of the notation see Secs. 3.4.1 and 4.3. Note that the gaugino action vanishes unless $\lambda^A(x)$ is anti-commuting! The Euler-Lagrange equations (and gauge field Bianchi identity) are

$$\begin{aligned}
&D^\mu F^A_{\mu\nu} = -\frac{1}{2}g f_{BC}{}^A\bar{\lambda}^B\gamma_\nu\lambda^C,\\
&D_\mu F^A_{\nu\rho} + D_v F^A_{\rho\mu} + D_\rho F^A_{\mu\nu} = 0,\\
&\gamma^\mu D_\mu\lambda^A = 0.
\end{aligned}$$

The supercurrent is

$$\mathcal{J}^\mu = \gamma^{v\rho}F^A_{v\rho}\gamma^\mu\lambda^A.$$

The proof that it is conserved begins as in the free (abelian) case:

$$\begin{aligned}
\partial_\mu\mathcal{J}^\mu &= D_\mu F^A_{v\rho}\gamma^{v\rho}\gamma^\mu\lambda^A + \gamma^{v\rho}F^A_{v\rho}\gamma^\mu D_\mu\lambda^A\\
&= -2D^\mu F^A_{\mu\nu}\gamma^v\lambda^A\\
&= g f_{ABC}\gamma^v\lambda^A\bar{\lambda}^B\gamma_\nu\lambda^C.
\end{aligned}$$

The right-hand side vanishes due to (3.68) and the supercurrent (6.10) is conserved!

Exercise 6.3 Study the appropriate Fierz rearrangement and, using the results of Ex. 3.27, show that the supercurrent is conserved in the following cases:

(i) Majorana spinors in $D = 3$,

(ii) Majorana (or Weyl) spinors in $D = 4$, which is the case analyzed above,

(iii) symplectic Weyl spinors in $D = 6$, and

(iv) Majorana-Weyl spinors in $D = 10$.

6.1.3 SUSY transformation rules 111

(!?!?!?!? мб это очень важно, так что раньше это поставлю где-то??)

$$\delta\Phi(x) = \{\bar{\epsilon}^{\alpha} Q_{\alpha}, \Phi(x)\}_{\mathrm{PB}} = -\mathrm{i}\, [\bar{\epsilon}^{\alpha} Q_{\alpha}, \Phi(x)]_{\mathrm{qu}}\,,$$

where Φ denotes any field of the system under study. A brief description of Poisson brackets (PB) and commutation relations in the canonical formalism is given in Secs. 1.4 and 1.5. A link in the opposite direction is provided by the Noether formalism, which produces a conserved supercurrent given field variations under which the action is invariant. One reason to emphasize the field variations, *ab* initio, is that this avoids some subtleties in the canonical formalism for gauge theories and for Majorana spinors.

The next exercise illustrates the link between the supercurrent and field variations. It involves the free scalar-spinor $\phi - \Psi$ system of Ex. 6.2. The spinors Ψ, the supersymmetry parameters ϵ and the supersymmetry generator Q are Majorana spinors. They all mutually anti-commute. For the canonical formalism, one can either treat Ψ and $\bar{\Psi}$ as independent variables, or use Dirac brackets to obtain

$$\begin{aligned}
\{\phi(x), \partial_0\phi(y)\}_{\mathrm{PB}} &= -\{\partial_0\phi(x), \phi(y)\}_{\mathrm{PB}} = \delta^3(\vec{x}-\vec{y}),\\
\left\{\Psi_{\alpha}(x), \bar{\Psi}^{\beta}(y)\right\}_{\mathrm{PB}} &= \left\{\bar{\Psi}^{\beta}(x), \Psi_{\alpha}(y)\right\}_{\mathrm{PB}} = \left(\gamma^0\right)_{\alpha}{}^{\beta}\delta^3(\vec{x}-\vec{y}).
\end{aligned}$$

Exercise 6.4 Use $\bar{Q} = (1/\sqrt{2})\int \mathrm{d}^3\vec{x}\bar{\Psi}\gamma^0(\not{\partial}+m)\phi$ or $Q = (1/\sqrt{2})\int \mathrm{d}^3\vec{x}\ (\not{\phi} - m\phi)\gamma^0\Psi$ to obtain

$$\begin{aligned}
\delta\phi(x) &= \{\bar{\epsilon}Q, \phi(x)\}_{\mathrm{PB}} = \frac{1}{\sqrt{2}}\bar{\epsilon}\Psi(x),\\
\delta\Psi(x) &= \{\bar{\epsilon}Q, \Psi(x)\}_{\mathrm{PB}} = \frac{1}{\sqrt{2}}(\not{A} m)\phi\epsilon.
\end{aligned}$$

Note that $\left[\bar{Q}\epsilon, \Psi_{\alpha}(x)\right]_{\mathrm{PB}} = -\left\{\bar{Q}^{\beta}, \Psi_{\alpha}(x)\right\}_{\mathrm{PB}}\epsilon_{\beta}$.

3.6.2 SUSY field theories of the chiral multiplet

Теория

$$\begin{aligned}
\delta Z &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi,\\
\delta P_L\chi &= \frac{1}{\sqrt{2}}P_L(\not{Z}+F)\epsilon,\\
\delta F &= \frac{1}{\sqrt{2}}\bar{\epsilon}\ \not{P}_L\chi.
\end{aligned}$$

The anti-chiral multiplet transformation rules are

$$\begin{aligned}\delta\bar{Z} &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_R\chi,\\ \delta P_R\chi &= \frac{1}{\sqrt{2}}P_R(\partial\bar{Z}+\bar{F})\epsilon,\\ \delta\bar{F} &= \frac{1}{\sqrt{2}}\bar{\epsilon}\ \not{\partial}P_R\chi.\end{aligned}$$

Note that the form of the transformation rules for the physical components is similar to those of the 'toy model' in Ex. 6.4.

Exercise 6.5 Show that the variations $\delta\bar{Z}, \delta P_R\chi, \delta\bar{F}$ are the complex conjugates of δZ, $\delta P_L\chi, \delta F$.

There are two basic actions, which are separately invariant under the transformation rules above. The first is the free kinetic action

$$S_{\text{kin}} = \int \mathrm{d}^4x \left[-\partial^\mu\bar{Z}\partial_\mu Z - \bar{\chi}\ \not{\partial}P_L\chi + \bar{F}F\right],$$

6.2.1 U(1)R symmetry 115

The SUSY algebra

$$\begin{aligned}\left[\delta_1,\delta_2\right]\Phi(x) &= \left[\bar{\epsilon}_1 Q, \left[\bar{Q}\epsilon_2, \Phi(x)\right]\right] - (\epsilon_1 \leftrightarrow \epsilon_2)\\ &= \bar{\epsilon}_1^\alpha\left[\left\{Q_\alpha, \bar{Q}^\beta\right\}, \Phi(x)\right]\epsilon_{2\beta}\\ &= -\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu\Phi(x).\end{aligned}$$

6.2.3 More chiral multiplets

3.6.3 SUSY gauge theories

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$$\omega_{\mu ab} = \omega_{\mu ab}(e) + K_{\mu ab},$$
$$K_{\mu\nu\rho} = -\frac{1}{4}\left(\bar{\psi}_\mu\gamma_\rho\psi_\nu - \bar{\psi}_\nu\gamma_\mu\psi_\rho + \bar{\psi}_\rho\gamma_\nu\psi_\mu\right),$$

Summarize the prescription for δS in the 1.5 order formalism:

1. Use the first order form of the action $S[e,\omega,\psi]$ and the transformation rules δe and $\delta\psi$ with connection ω unspecified.

2. Ignore the connection variation and calculate

$$\delta S = \int \mathrm{d}^D x \left[\frac{\delta S}{\delta e}\delta e + \frac{\delta S}{\delta \psi}\delta\psi\right].$$

3. Substitute ω from (9.21) in the result, which must vanish for a consistent supergravity theory.

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5.6.2 The AdS/CFT correspondence

Часть II

Theoretical Field Theories

6 Clifford algebras and spinors

The Dirac equation is a relativistic wave equation that is first order in space and time derivatives. The key to this remarkable property is the set of γ-matrices, which satisfy the anti-commutation relations (2.18):

$$\gamma^{\mu}\gamma^{\nu}+\gamma^{v}\gamma^{\mu}=2\eta^{\mu\nu}\mathbb{1}$$

These matrices are the generating elements of a Clifford algebra which plays an important role in supersymmetric and supergravity theories. In the first part of this chapter we discuss the structure of this Clifford algebra for general spacetime dimension D. For a generic value of D, the γ-matrices are intrinsically complex. This is why we assumed that the Dirac field is complex in the previous chapter. In certain spacetime dimensions the representation of the Clifford algebra is real, which means that the γ-matrices are conjugate to real matrices. In this case the basic spinor field may be taken to be real and is called a Majorana spinor field. Since the Majorana field has a smaller number of independent components, it is fair to say that, when it exists, it is more fundamental than the Dirac field. For this reason Majorana spinors are selected in supersymmetry and supergravity. We study the special properties of Majorana spinors in the second part of this chapter.

In the body of the chapter we take a practical approach, intended as a guide to the applications needed later in the book. Further supporting arguments are collected in Appendix 3 A at the end of the chapter.

6.1 3.1 The Clifford algebra in general dimension

6.1.1 3.1.1 The generating γ-matrices

The main purpose of this section is to discuss the Clifford algebra associated with the Lorentz group in D dimensions. To be concrete, we start with a general and explicit construction of the generating γ-matrices. It is simplest first to construct Euclidean γ-matrices, which satisfy (3.1) with Minkowski metric $\eta_{\mu\nu}$ replaced by $\delta_{\mu\nu}$:

$$\begin{aligned}
\gamma^{1}&=\sigma_{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\ldots,\\
\gamma^{2}&=\sigma_{2}\otimes\mathbb{1}\otimes\mathbb{1}\otimes\ldots,\\
\gamma^{3}&=\sigma_{3}\otimes\sigma_{1}\otimes\mathbb{1}\otimes\ldots\\
\gamma^{4}&=\sigma_{3}\otimes\sigma_{2}\otimes\mathbb{1}\otimes\ldots\\
\gamma^{5}&=\sigma_{3}\otimes\sigma_{3}\otimes\sigma_{1}\otimes\ldots\\
\ldots&=\ldots
\end{aligned}$$

These matrices are all hermitian with squares equal to $\mathbb{1}$, and they mutually anti-commute. Suppose that $D=2m$ is even. Then we need m factors in the construction (3.2) to obtain $\gamma^{\mu}, 1\leq\mu\leq D=2m$. Thus we obtain a representation of dimension $2^{D/2}$.

For odd $D=2m+1$ we need one additional matrix, and we take γ^{2m+1} from the list above, but we keep only the first m factors, i.e. deleting a σ_1. Thus there is no increase in the dimension of the representation in going from $D=2m$ to $D=2m+1$, and we can say in general that the construction (3.2) gives a representation of dimension $2^{[D/2]}$, where $[D/2]$ means the integer part of $D/2$.

Euclidean γ-matrices do have physical applications, but we need Lorentzian γ for the subject matter of this book. To obtain these, all we need to do is pick any single matrix from the Euclidean construction, multiply it by i and label it γ^0 for the time-like direction. This matrix is anti-hermitian and satisfies $\left(\gamma^0\right)^2 = -\mathbb{1}$. We then relabel the remaining $D-1$ matrices to obtain the Lorentzian set $\gamma^\mu, 0 \leq \mu \leq D-1$. The hermiticity properties of the Lorentzian γ are summarized by

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0.$$

It is fundamental to the Dirac theory that the physics of a spinor field is the same in all equivalent representations of the Clifford algebra. Thus we are really concerned with classes of representations related by conjugacy, i.e.

$$\gamma'^\mu = S\gamma^\mu S^{-1}.$$

Since we consider only hermitian representations, in which (3.3) holds, the matrix S must be unitary. Given any two equivalent representations, the transformation matrix S is unique up to a phase factor. Up to this conjugation, there is a unique irreducible representation (irrep) of the Clifford algebra by $2^m \times 2^m$ matrices for even dimension $D = 2m$. Any other representation is reducible and equivalent to a direct sum of copies of the irrep above. One can always choose a hermitian irrep, defined as one which satisfies (3.3). In odd dimensions there are two mathematically inequivalent irreps, which differ only in the sign of the 'final' $\pm\gamma^{2m+1}$. In this book we will always use a hermitian irrep of the γ-matrices. Physical consequences are independent of the particular representation chosen.

6.1.2 3.1.2 The complete Clifford algebra

The full Clifford algebra consists of the identity $\mathbb{1}$, the D generating elements γ^μ, plus all independent matrices formed from products of the generators. Since symmetric products reduce to a product containing fewer γ-matrices by (3.1), the new elements must be antisymmetric products. We thus define

$$\gamma^{\mu_1\ldots\mu_r} = \gamma^{[\mu_1}\ldots\gamma^{\mu_r]}, \quad \text{e.g.} \quad \gamma^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$$

where the antisymmetrization indicated with $[\ldots]$ is always with total weight 1 . Thus the right-hand side of (3.5) contains the overall factor $1/r$! times a sum of r ! signed permutations of the indices. Non-vanishing tensor components can be written as the products

$$\gamma^{\mu_1\mu_2\ldots\mu_r} = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_r} \quad \text{where } \mu_1 \neq \mu_2 \neq \cdots \neq \mu_r$$

All these matrices are traceless (a proof can be found in Appendix 3A), except for the lowest rank $r=0$, which is the unit matrix, and the highest rank matrix with $r=D$, which is traceless only for even D as we will see below.

There are C_r^D (binomial coefficients) independent index choices at rank r. For even spacetime dimension the matrices are linearly independent, so that the Clifford algebra is an algebra of dimension 2^D.

Exercise 3.1 Show that the higher rank γ-matrices can be defined as the alternate commutators or anti-commutators

$$
\begin{aligned}
\gamma^{\mu\nu} &= \frac{1}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right], \\
\gamma^{\mu_1\mu_2\mu_3} &= \frac{1}{2}\left\{\gamma^{\mu_1}, \gamma^{\mu_2\mu_3}\right\}, \\
\gamma^{\mu_1\mu_2\mu_3\mu_4} &= \frac{1}{2}\left[\gamma^{\mu_1}, \gamma^{\mu_2\mu_3\mu_4}\right],
\end{aligned}
$$

etc.

Exercise 3.2 Show that $\gamma^{\mu_1\cdots\mu_D} = \frac{1}{2}\left(\gamma^{\mu_1}\gamma^{\mu_2\cdots\mu_D} - (-)^D\gamma^{\mu_2\cdots\mu_D}\gamma^{\mu_1}\right)$. Thus, $\mathrm{Tr}\,\gamma^{\mu_1\cdots\mu_D}$ vanishes for even D.

6.1.3 3.1.3 Levi-Civita symbol

We need a short technical digression to introduce the Levi-Civita symbol and derive some of its properties. In every dimension D this is defined as the totally antisymmetric rank D tensor $\varepsilon_{\mu_1\mu_2\ldots\mu_D}$ or $\varepsilon^{\mu_1\mu_2\ldots\mu_D}$ with

$$
\varepsilon_{012(D-1)} = 1, \quad \varepsilon^{012(D-1)} = -1.
$$

Indices are raised using the Minkowski metric which leads to the difference in sign above (due to the single time-like direction).

Exercise 3.3 Prove the contraction identity for these tensors:

$$
\varepsilon_{\mu_1\ldots\mu_n\nu_1\ldots v_p}\varepsilon^{\mu_1\ldots\mu_n\rho_1\ldots\rho_p} = -p!n!\delta^{\rho_1\cdots\rho_p}_{v_1\cdots v_p}, \quad p = D-n.
$$

The antisymmetric p-index Kronecker δ is in turn defined by

$$
\delta^{\rho_1\cdots\rho_p}_{v_1\ldots\nu_q} \equiv \delta^{[\rho_1}_{v_1}\delta^{\rho_2}_{v_2}\cdots\delta^{\rho_p]}_{v_q}
$$

which includes a signed sum over p ! permutations of the lower indices, each with a coeffcient $1/p$!, such that the 'total weight' is 1 (as in (A.8)).

In four dimensions the totally antisymmetric Levi-Civita tensor symbol is written as $\varepsilon^{\mu\nu\rho\sigma}$. Because an antisymmetric tensor of rank 5 necessarily vanishes when $D=4$, this satisfies the Schouten identity

$$
0 = 5\delta^{[v}_{\mu}\varepsilon^{\rho\sigma\tau\lambda]} \equiv \delta^{\nu}_{\mu}\varepsilon^{\rho\sigma\tau\lambda} + \delta^{\rho}_{\mu}\varepsilon^{\sigma\tau\lambda\nu} + \delta^{\sigma}_{\mu}\varepsilon^{\tau\lambda\nu\rho} + \delta^{\tau}_{\mu}\varepsilon^{\lambda\nu\rho\sigma} + \delta^{\lambda}_{\mu}\varepsilon^{\nu\rho\sigma\tau}
$$

6.1.4 3.1.4 Practical γ-matrix manipulation

Supersymmetry and supergravity theories emerge from the concept of fermion spin. It should be no surprise that intricate features of the Clifford algebra are needed to establish and explore the physical properties of these field theories. In this section we explain some useful tricks to multiply γ-matrices. The results are valid for both even and odd D.

Consider first products with index contractions such as

$$
\gamma^{\mu\nu}\gamma_{\nu} = (D-1)\gamma^{\mu}.
$$

You can memorize this rule, but it is easier to recall the simple logic behind it: v runs over all values except μ, so there are $(D-1)$ terms in the sum. Similar logic explains the result

$$
\gamma^{\mu\nu\rho}\gamma_{\rho} = (D-2)\gamma^{\mu\nu}
$$

or even more generally

$$\gamma^{\mu_1\ldots\mu_r v_1\ldots v_s}\gamma_{v_s\ldots v_1}=\frac{(D-r)!}{(D-r-s)!}\gamma^{\mu_1\ldots\mu_r}$$

Indeed, first we can write $\gamma_{v_s\ldots\nu_1}$ as the product $\gamma_{v_s}\ldots\gamma_{v_1}$ since the antisymmetry is guaranteed by the first factor. Then the index v_s has $(D-(r+s-1))$ values, while v_{s-1} has $(D-(r+s-2))$ values, and this pattern continues to $(D-r)$ values for the last one. Note that the second γ on the left-hand side has its indices in opposite order, so that no signs appear when contracting the indices. It is useful to remember the general order reversal symmetry, which is

$$\gamma^{v_1\ldots v_r}=(-)^{r(r-1)/2}\gamma^{v_r\ldots v_1}.$$

The sign factor $(-)^{r(r-1)/2}$ is negative for $r=2,3 \bmod 4$.

Even if one does not sum over indices, similar combinatorial tricks can be used. For example, when calculating

$$\gamma^{\mu_1\mu_2}\gamma_{\nu_1\ldots v_D}$$

one knows that the index values μ_1 and μ_2 appear in the set of $\{v_i\}$. There are D possibilities for μ_2, and since μ_1 should be different, there remain $D-1$ possibilities for μ_1. Hence the result is

$$\gamma^{\mu_1\mu_2}\gamma_{v_1\ldots v_D}=D(D-1)\delta^{\mu_2\mu_1}_{[v_1v_2}\gamma_{v_3\ldots v_D]}.$$

Note that such generalized δ-functions are always normalized with 'weight 1 ', i.e.

$$\delta^{\mu_2\mu_1}_{v_1v_2}=\frac{1}{2}\left(\delta^{\mu_2}_{v_1}\delta^{\mu_1}_{v_2}-\delta^{\mu_1}_{v_1}\delta^{\mu_2}_{v_2}\right)$$

This makes contractions easy; e.g. we obtain from (3.17)

$$\gamma^{\mu_1\mu_2}\gamma_{\nu_1\ldots\nu_D}\varepsilon^{\nu_1\ldots\nu_D}=D(D-1)\varepsilon^{\mu_2\mu_1\nu_3\ldots\nu_D}\gamma_{\nu_3\ldots\nu_D}.$$

We now consider products of γ-matrices without index contractions. The very simplest case is

$$\gamma^\mu\gamma^\nu=\gamma^{\mu\nu}+\eta^{\mu\nu}$$

This follows directly from the definitions: the antisymmetric part of the product is defined in (3.5) to be $\gamma^{\mu\nu}$, while the symmetric part of the product is $\eta^{\mu\nu}$, by virtue of (3.1). This already illustrates the general approach: one first writes the totally antisymmetric Clifford matrix that contains all the indices and then adds terms for all possible index pairings.

Here is another example:

$$\gamma^{\mu\nu\rho}\gamma_{\sigma\tau}=\gamma^{\mu\nu\rho}{}_{\sigma\tau}+6\gamma^{[\mu\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]}+6\gamma^{[\mu}\delta^{\nu}_{[\tau}\delta^{\rho]}{}_{\sigma]}.$$

This follows the same pattern. We write the indices $\sigma\tau$ in down position to make it easier to indicate the antisymmetry. The second term contains one contraction. One can choose three indices from the first factor and two indices from the second one, which gives the factor 6 . For the third term there are also six ways to make two contractions. The δ-functions contract indices that were adjacent, or separated by already contracted indices, so that no minus signs appear.

Exercise 3.4 As a similar exercise, derive

$$\gamma^{\mu_1\ldots\mu_4}\gamma_{v_1v_2}=\gamma^{\mu_1\ldots\mu_4}{}_{v_1v_2}+8\gamma^{[\mu_1\ldots\mu_3}{}_{[v_2}\delta^{\mu_4]}{}_{v_1]}+12\gamma^{[\mu_1\mu_2}\delta^{\mu_3}{}_{[v_2}\delta^{\mu_4]}{}_{v_1]}]$$

Finally, we consider products with both contracted and uncontracted indices. Consider $\gamma^{\mu_1\cdots\mu_4\rho}\gamma_{\rho v_1 v_2}$. The result should contain terms similar to (3.22), but each term has an extra numerical factor reflecting the number of values that ρ can take in this sum. For example, in the second term above there is now one contraction between an upper and lower index, and therefore ρ can run over all D values except the four values $\mu_1,\ldots\mu_4$, and the two values v_1, v_2. This counting gives

$$\begin{aligned}\gamma^{\mu_1\cdots\mu_4\rho}\gamma_{\rho v_1 v_2} =&(D-6)\gamma^{\mu_1\cdots\mu_4}{}_{v_1 v_2}+8(D-5)\gamma^{[\mu_1\cdots\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{v_1]}\\ &+12(D-4)\gamma^{[\mu_1\mu_2}\delta^{\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{\nu_1]}.\end{aligned}$$

Exercise 3.5 Show that

$$\begin{aligned}\gamma_\nu\gamma^\mu\gamma^\nu &= (2-D)\gamma^\mu,\\ \gamma_\rho\gamma^{\mu\nu}\gamma^\rho &= (D-4)\gamma^{\mu\nu}\end{aligned}$$

Derive the general form $\gamma_\rho\gamma^{\mu_1\mu_2\ldots\mu_r}\gamma^\rho=(-)^r(D-2r)\gamma^{\mu_1\mu_2\ldots\mu_r}$.

6.1.5 3.1.5 Basis of the algebra for even dimension $D=2m$

To continue our study we restrict to even-dimensional spacetime and construct an orthogonal basis of the Clifford algebra. It will be easy to extend the results to odd D later.

The basis is denoted by the following list $\{\Gamma^A\}$ of matrices chosen from those defined in Sec. 3.1.2:

$$\left\{\Gamma^A=\mathbb{1},\gamma^\mu,\gamma^{\mu_1\mu_2},\gamma^{\mu_1\mu_2\mu_3},\cdots,\gamma^{\mu_1\cdots\mu_D}\right\}$$

Index values satisfy the conditions $\mu_1<\mu_2<\ldots<\mu_r$. There are C_r^D distinct index choices at each rank r and a total of 2^D matrices. To see that this is a basis, it is convenient to define the reverse order list

$$\{\Gamma_A=\mathbb{1},\gamma_\mu,\gamma_{\mu_2\mu_1},\gamma_{\mu_3\mu_2\mu_1},\ldots,\gamma_{\mu_D\cdots\mu_1}\}$$

By (3.15) the matrices of this list differ from those of (3.25) by sign factors only.

Exercise 3.6 Show that $\Gamma^A\Gamma^B=\pm\Gamma^C$, where Γ^C is the basis element whose indices are those of A and B with common indices excluded. Derive the trace orthogonality property

$$\operatorname{Tr}\left(\Gamma^A\Gamma_B\right)=2^m\delta_B^A$$

The list (3.25) contains 2^D trace orthogonal matrices in an algebra of total dimension 2^D. Therefore it is a basis of the space of matrices M of dimension $2^m\times 2^m$.

Exercise 3.7 Show that any matrix M can be expanded in the basis $\{\Gamma^A\}$ as

$$M=\sum_A m_A\Gamma^A,\quad m_A=\frac{1}{2^m\breve{a}}\operatorname{Tr}\left(M\Gamma_A\right)$$

Readers may already have noted that the signature of spacetime has played little role in the discussion above. The basic conclusion that there is a unique representation of the Clifford algebra of dimension 2^m is true for pseudo-Euclidean metrics of any signature (p,q). Another general fact is that the second rank Clifford elements $\gamma^{\mu\nu}$ are the generators of a representation of the Lie algebra $\mathfrak{so}(p,q)$, with $p+q=D=2m$; see (1.34) with the metric signature (p,q). Only the hermiticity properties depend on the signature in an obvious fashion.

Exercise 3.8 Show that

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 2^m\left[\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right]$$

Exercise 3.9 Count the number of elements in the basis (3.25) for odd dimensions $D = 2m+1$, and see that it contains twice the number of independent $2^m \times 2^m$ matrices. Check that we already have enough matrices if we consider the matrices up to $\gamma_{\mu_1\cdots\mu_{(D-1)/2}}$. Therefore the results of this section hold only for even dimensions and will have to be modified for odd dimensions; see Sec. 3.1.7.

6.1.6 3.1.6 The highest rank Clifford algebra element

For several reasons it is useful to study the highest rank tensor element of the Clifford algebra. It provides the link between even and odd dimensions and it is closely related to the chirality of fermions, an important physical property. We define

$$\gamma_* \equiv (-\mathrm{i})^{m+1}\gamma_0\gamma_1\ldots\gamma_{D-1},$$

which satisfies $\gamma_*^2 = \mathbb{1}$ in every even dimension and is hermitian. For spacetime dimension $D = 2m$, the matrix γ_* is frequently called γ_{D+1} in the physics literature, as in four dimensions where it is called γ_5.

This matrix occurs as the unique highest rank element in (3.25). For any order of components μ_i, one can write

$$\gamma_{\mu_1\mu_2\cdots\mu_D} = \mathrm{i}^{m+1}\varepsilon_{\mu_1\mu_2\cdots\mu_D}\gamma_*,$$

where the Levi-Civita tensor introduced in Sec. 3.1.3 is used.

Exercise 3.10 Show that γ_* commutes with all even rank elements of the Clifford algebra and anti-commutes with all odd rank elements. Thus, for example,

$$\{\gamma_*, \gamma^\mu\} = 0,$$
$$[\gamma_*, \gamma^{\mu\nu}] = 0.$$

Since $\gamma_*^2 = \mathbb{1}$ and $\mathrm{Tr}\,\gamma_* = 0$, it follows that one can choose a representation in which

$$\gamma_* = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

Some exercises follow, which illustrate the properties of a representation of the full Clifford algebra in which γ_* takes the form in (3.34).

Exercise 3.11 Assume a general block form,

$$\gamma^\mu = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the generating elements in a basis where (3.34) holds. Show that (3.32) implies the block off-diagonal form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

in which the matrices σ^μ and $\bar{\sigma}^\mu$ are $2^{m-1}\times 2^{m-1}$ generalizations of the explicit Weyl matrices of (2.2).

Exercise 3.12 Show that the matrices σ^μ and $\bar{\sigma}^\mu$ satisfy (2.4) and that $\mathrm{Tr}\,(\sigma^\mu\bar{\sigma}_v) = 2^{(m-1)}\delta^\mu_\nu$.

Exercise 3.13 Show similarly that (3.33) implies that the second rank matrices take the block diagonal form

$$\Sigma^{\mu\nu} = \frac{1}{2}\gamma^{\mu\nu} = \frac{1}{4}\begin{pmatrix} \sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu & 0 \\ 0 & \bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu \end{pmatrix}$$

This exercise shows explicitly that the Dirac representation of $\mathfrak{so}(D-1,1)$, which is generated by $\Sigma^{\mu\nu}$, is reducible (for even D). The matrices of the upper and lower blocks in (3.37) are generators of two subrepresentations, which are inequivalent and irreducible.

(Indeed they are related to the two fundamental spinor representations of D_m denoted by Dynkin integers $(0,0,\ldots,1,0)$ and $(0,0,\ldots,0,1)$.)

Exercise 3.14 Show that all requirements are satisfied by generalized Weyl matrices in which the spatial matrices are $\sigma^i = \bar{\sigma}^i$, where the σ^i are hermitian generators of the Clifford algebra in odd dimension $2m-1$ Euclidean space, and the time matrices are $\sigma^0 = -\bar{\sigma}^0 = \mathbb{1}$. Thus the form of the Weyl matrices in $D = 2m$ dimensions is the same as in $D = 4$.

It is frequently useful to note that the Weyl fields ψ and χ can be obtained from a Dirac Ψ field by applying the chiral projectors

$$P_L = \frac{1}{2}\left(\mathbb{1} + \gamma_*\right), \quad P_R = \frac{1}{2}\left(\mathbb{1} - \gamma_*\right).$$

Thus

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} \equiv P_L\Psi, \quad \begin{pmatrix} 0 \\ \bar{\chi} \end{pmatrix} \equiv P_R\Psi$$

The specific Weyl representation (3.36) will rarely be used in the rest of this book. However, we will use the projectors P_L and P_R to define the chiral parts of Dirac (and Majorana) spinors in a general representation of the γ-matrices.

Exercise 3.15 Show that the matrices (3.38) project to orthogonal subspaces, i.e. $P_L P_L = P_L, P_R P_R = P_R$ and $P_L P_R = 0$. No specific choice of the Clifford algebra representation is needed.

6.1.7 3.1.7 Odd spacetime dimension $D = 2m + 1$

The basic idea that we need is that the Clifford algebra for dimension $D = 2m + 1$ can be obtained by reorganizing the matrices in the Clifford algebra for dimension $D = 2$ m. In particular we can define two sets of $2m + 1$ generating elements by adjoining the highest rank γ_* as follows:

$$\gamma_\pm^\mu = \left(\gamma^0, \gamma^1, \ldots, \gamma^{(2m-1)}, \gamma^{2m} = \pm\gamma_*\right)$$

This gives us two sets of matrices, which each satisfy (2.18) for dimension $D = 2m+1$. The two sets $\{\gamma_\pm^\mu\}$ are not equivalent, but they lead to equivalent representations of the Lorentz group; see Appendix 3A.3.

The main difference with the case of even dimensions is that the matrices in the list (3.25) are not all independent and are thus an over-complete set. Indeed, the highest element of that list, which is the product of all γ-matrices, is, due to (3.40), a phase factor times the unit matrix. More generally, the rank r and rank $D - r$ sectors are related by the duality relations

$$\gamma_\pm^{\mu_1\ldots\mu_r} = \pm \mathrm{i}^{m+1}\frac{1}{(D-r)!}\varepsilon^{\mu_1\ldots\mu_D}\gamma_{\pm\mu_D\ldots\mu_{r+1}}$$

Note that the order of the indices in the γ-matrix on the right-hand side is reversed. Otherwise there would be different sign factors.

Exercise 3.16 Prove the relation (3.41) and the analogous but different relation for even dimension:

$$\gamma^{\mu_1\mu_2\ldots\mu_r}\gamma_* = -(-\mathrm{i})^{m+1}\frac{1}{(D-r)!}\varepsilon^{\mu_r\mu_{r-1}\ldots\mu_1 v_1 v_2\ldots v_{D-r}}\gamma_{\nu_1 v_2\ldots\nu_{D-r}}$$

You can use the tricks explained in Sec. 3.1.4. Show that in four dimensions

$$\gamma_{\mu\nu\rho} = \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_*.$$

Thus, a basis of the Clifford algebra in $D=2m+1$ dimensions contains the matrices in (3.25) only up to rank m. This agrees with the counting argument in Ex. 3.9. For example, the set $\{\mathbb{1},\gamma^\mu,\gamma^{\mu\nu}\}$ of $1+5+10=16$ matrices is a basis of the Clifford algebra for $D=5$. Ex. 3.16 shows that it is a rearrangement of the basis $\left\{\Gamma^A\right\}$ for $D=4$.

6.1.8 3.1.8 Symmetries of γ-matrices

In the Clifford algebra of the $2^m \times 2^m$ matrices, for both $D=2m$ and $D=2m+1$, one can distinguish between the symmetric and the antisymmetric matrices where the symmetry property is defined in the following way. There exists a unitary matrix, C, called the charge conjugation matrix, such that each matrix $C\Gamma^A$ is either symmetric or antisymmetric. Symmetry depends only on the rank r of the matrix Γ^A, so we can write:

$$\left(C\Gamma^{(r)}\right)^T = -t_r C\Gamma^{(r)}, \quad t_r = \pm 1$$

where $\Gamma^{(r)}$ is a matrix in the set (3.25) of rank r. (The $-$ sign in (3.44) is convenient for later manipulations.) For rank $r=0$ and 1 , one obtains from (3.44)

$$C^T = -t_0 C, \quad \gamma^{\mu T} = t_0 t_1 C\gamma^\mu C^{-1}$$

These relations suffice to determine the symmetries of all $C\gamma^{\mu_1\cdots\mu_r}$ and thus all coefficients t_r : e.g. $t_2=-t_0$ and $t_3=-t_1$. Further, $t_{r+4}=t_r$.

Exercise 3.17 A formal proof of the existence of C can be found in [10, 11], but you can check that the following two matrices satisfy (3.45) for even D. They are given in the product representation of (3.2): [1]

$$\begin{aligned} C_+ &= \sigma_1\otimes\sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\ldots, \quad t_0t_1=1 \\ C_- &= \sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_1\otimes\ldots, \quad t_0t_1=-1 \end{aligned}$$

The values of t_0 and t_1 (thus all t_r) depend on the spacetime dimension D modulo 8 and on the rank r modulo 4, and are given in Table (3.1). The entries in the table are determined by counting the number of independent symmetric and antisymmetric matrices in every dimension; see Appendix 3A.4. An exercise for the simple case $D=5$ follows below. For even dimension both $C_+$and $C_-$are possible choices. One can go from one to

1 We consider here only the Minkowski signature of spacetime. A full treatment is given in [12], for which you should set $\epsilon=t_0$ and $\eta=-t_0t_1$.

$D(\text{mod}8)$	$t_r=-1$	$t_r=+1$
0	0, 3	2, 1
	0, **1**	2, 3
1	0, 1	2, 3
2	0, 1	2, 3
	1, **2**	**0**, **3**
3	1,2	0, 3
4	1, 2	**0**, **3**
	2, 3	0, 1
5	2, 3	0, 1
6	2, 3	0, 1
	0, 3	**1**, **2**
7	0, 3	1,2

the other by replacing the charge conjugation matrix C by $C\gamma_*$ (up to a normalizing phase factor). For applications in supersymmetry we need the choice indicated in bold face. For odd dimension, C is unique (again up to a phase factor).

Exercise 3.18 Check that in five dimensions, where the Clifford algebra basis contains only matrices of rank 0,1 and 2, the numbers in the table are fixed by counting the number of matrices of each rank. The count must conform to the requirement that there are 10 symmetric and 6 antisymmetric matrices in a basis of 4×4 matrices.

Since we use hermitian representations, which satisfy (3.3), the symmetry property of a γ-matrix determines also its complex conjugation property. To see this, we define the unitary matrix

$$B = \mathrm{i}t_0 C\gamma^0$$

Exercise 3.19 Derive

$$\gamma^{\mu *} = -t_0 t_1 B\gamma^\mu B^{-1}$$

Exercise 3.20 Prove that $B^*B = -t_1\mathbb{1}$.

Exercise 3.21 Show that, in the Weyl representation (2.19), one can choose $B=\gamma^0\gamma^1\gamma^3$, which is real, symmetric, and satisfies $B^2=\mathbb{1}$. Then $C=\mathrm{i}\gamma^3\gamma^1$.

The properties (3.45) and (3.48) hold for the representation (3.2) using the matrices (3.46) and (3.47). In another representation, related by (3.4), the C and B matrices are given by

$$C' = S^{-1T}CS^{-1}, \quad B' = S^{-1T}BS^{-1}.$$

Since symmetries of spinor bilinears are important for supersymmetry, we use the Majorana conjugate to define $\bar\lambda$.

6.2 3.2 Spinors in general dimensions

In Ch. 2 we used complex spinors. We defined the Dirac adjoint (2.30), which involves the complex conjugate spinor, and used it to obtain a Lorentz invariant bilinear form. In this section we start rather differently. We define the 'Majorana conjugate' of any spinor λ using its transpose and the charge conjugation matrix,

$$\bar\lambda \equiv \lambda^T C$$

The bilinear form $\bar\lambda\chi$ is Lorentz invariant as readers will show in Ex. 3.23 below. It is appropriate to use (3.50) in supersymmetry and supergravity in which the symmetry properties of γ-matrices and of spinor bilinears are very important and these properties are determined by C. For Majorana spinors, to be defined in Sec. 3.3, the definitions (3.50) and (2.30) are equivalent.

Unless otherwise stated, we assume in this book that spinor components are anticommuting Grassmann numbers. This reflects the important physical relation between spin and statistics.

6.2.1 3.2.1 Spinors and spinor bilinears

Using the definition (3.50) and the property (3.44), we obtain

$$\bar{\lambda}\gamma_{\mu_1\dots\mu_r}\chi = t_r\bar{\chi}\gamma_{\mu_1\dots\mu_r}\lambda$$

The minus sign obtained by changing the order of Grassmann valued spinor components has been incorporated. The symmetry property (3.51) is valid for Dirac spinors, but its main application for us will be to Majorana spinors. For this reason we use the term 'Majorana flip relations' to refer to (3.51).

We now give some further relations that are useful for spinor manipulations. In fact, the same sign factors can be used for a longer chain of Clifford matrices:

$$\bar{\lambda}\Gamma^{(r_1)}\Gamma^{(r_2)}\cdots\Gamma^{(r_p)}\chi = t_0^{p-1}t_{r_1}t_{r_2}\cdots t_{r_p}\bar{\chi}\Gamma^{(r_p)}\cdots\Gamma^{(r_2)}\Gamma^{(r_1)}\lambda$$

where $\Gamma^{(r)}$ stands for any rank r matrix $\gamma_{\mu_1\dots\mu_r}$. Note that the prefactor t_0^{p-1} is not relevant in four dimensions, where $t_0 = 1$.

Exercise 3.22 One often encounters the special case that the bilinear contains the product of individual γ^μ-matrices. Prove that for the Majorana dimensions $D = 2, 3, 4 \bmod 8$,

$$\bar{\lambda}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_p}\chi = (-)^p\bar{\chi}\gamma^{\mu_p}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\lambda.$$

The previous relations imply also the following rule. For any relation between spinors that includes γ-matrices, there is a corresponding relation between the barred spinors,

$$\chi_{\mu_1\dots\mu_r} = \gamma_{\mu_1\dots\mu_r}\lambda \Longrightarrow \bar{\chi}_{\mu_1\dots\mu_r} = t_0t_r\bar{\lambda}\gamma_{\mu_1\dots\mu_r}$$

and similar for longer chains,

$$\chi = \Gamma^{(r_1)}\Gamma^{(r_2)}\cdots\Gamma^{(r_p)}\lambda \Longrightarrow \bar{\chi} = t_0^p t_{r_1}t_{r_2}\cdots t_{r_p}\bar{\lambda}\Gamma^{(r_p)}\cdots\Gamma^{(r_2)}\Gamma^{(r_1)}.$$

In even dimensions we define left-handed and right-handed parts of spinors using the projection matrices (3.38). The definition (3.50) implies that the chirality of the conjugate spinor depends on t_0t_D, and we obtain [2]

$$\chi = P_L\lambda \to \bar{\chi} = \begin{cases} \bar{\lambda}P_L, & \text{for } D = 0, 4, 8, \dots \\ \bar{\lambda}P_R, & \text{for } D = 2, 6, 10, \dots \end{cases}$$

Exercise 3.23 Using the 'spin part' of the infinitesimal Lorentz transformation (2.25),

$$\delta\chi = -\frac{1}{4}\lambda^{\mu\nu}\gamma_{\mu\nu}\chi$$

prove that the spinor bilinear $\bar{\lambda}\chi$ is a Lorentz scalar.

6.2.2 3.2.2 Spinor indices

For most of this book we do not need spinor indices because they appear contracted within Lorentz covariant expressions. However, in some cases indices are necessary, for example, to write (anti-)commutation relations of supersymmetry generators. The components of the basic spinor λ are indicated as λ_α. The components of the barred spinor defined in (3.50) are indicated with upper indices: λ^α. Sometimes we write $\bar{\lambda}^\alpha$ to stress that these are the components of the barred spinor, but in fact the bar can be omitted. We introduce the raising matrix $\mathcal{C}^{\alpha\beta}$ such that

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta$$

Comparing with (3.50) we see that $\mathcal{C}^{\alpha\beta}$ are the components of the matrix C^T. Note that the summation index β in (3.58) appears in a northwest-southeast (NW-SE) line in the

2 The definition (2.30) would always lead to $\bar{\chi} = \bar{\lambda}P_R$. equation when adjacent indices are contracted. Therefore, this convention is frequently called the NW-SE spinor convention. This is relevant when the raising matrix is antisymmetric ($t_0 = 1$ in the terminology of Table 3.1). Most applications in the book are for dimensions in which this is the case, e.g. $D = 2, 3, 4, 5, 10, 11$.

We also introduce a lowering matrix such that (again NW-SE contraction)

$$\lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha}$$

In order for these two equations to be consistent, we must require

$$\mathcal{C}^{\alpha\beta}\mathcal{C}_{\gamma\beta} = \delta_\gamma{}^\alpha, \quad \mathcal{C}_{\beta\alpha}\mathcal{C}^{\beta\gamma} = \delta_\alpha{}^\gamma$$

Hence $\mathcal{C}_{\alpha\beta}$ are the components of C^{-1}, and the unitarity of C implies then $(\mathcal{C}_{\alpha\beta})^* = \mathcal{C}^{\alpha\beta}$.

When we write a covariant spinor bilinear with components explicitly indicated, the γ-matrices are written as $(\gamma_\mu)_\alpha{}^\beta$. For example, for the simplest case,

$$\bar{\chi}\gamma_\mu\lambda = \chi^\alpha (\gamma_\mu)_\alpha^\beta \lambda_\beta$$

where again all contractions are NW-SE.

One can now raise or lower indices consistently. For example, one can define

$$(\gamma_\mu)_{\alpha\beta} = (\gamma_\mu)_\alpha^\gamma \mathcal{C}_{\gamma\beta}$$

These γ-matrices with indices at the 'same level' have a definite symmetry or antisymmetry property, which follows from (3.44):

$$(\gamma_{\mu_1\ldots\mu_r})_{\alpha\beta} = -t_r (\gamma_{\mu_1\ldots\mu_r})_{\beta\alpha}.$$

An interesting property is that

$$\lambda^\alpha\chi_\alpha = -t_0\lambda_\alpha\chi^\alpha$$

Thus, in four dimensions, raising and lowering a contracted index produces a minus sign. The same property can be used when the contracted indices involved are on γ-matrices, e.g. $\gamma_{\mu\alpha}^{\beta}\gamma_{\nu\beta}^{\gamma} = -t_0\gamma_{\mu\alpha\beta}\gamma_v^{\beta\gamma}$.

Exercise 3.24 Using this property and (3.63) prove the relation (3.52). Do not forget the sign due to interchange of two (anti-commuting) spinors.

Exercise 3.25 Show that, using the index raising and lowering conventions, $\mathcal{C}_\alpha{}^\beta = \delta_\alpha^\beta$, and for $D = 4$ that $\mathcal{C}^\alpha{}_\beta = -\delta_\beta^\alpha$.

6.2.3 3.2.3 Fierz rearrangement

In this subsection we study an important consequence of the completeness of the Clifford algebra basis $\left\{\Gamma^{A}\right\}$ in (3.25). As we saw in Ex. 3.7 completeness means that any matrix M has a unique expansion in the basis with coefficients obtained using trace orthogonality. The expansion was derived for even $D=2m$ in Ex. 3.7, but it is also valid for odd $D=2m+1$ provided that the sum is restricted to rank $r \leq m$. We saw at the end of Sec. 3.1.7 that the list of (3.25) is complete for odd D when so restricted. The rearrangement properties we derive using completeness are frequently needed in supergravity. These involve changing the pairing of spinors in products of spinor bilinears, which is called a 'Fierz rearrangement'.

Let's proceed to derive the basic Fierz identity. Using spinor indices, we can regard the quantity $\delta_{\alpha}{}^{\beta}\delta_{\gamma}{}^{\delta}$ as a matrix in the indices $\gamma\beta$ with the indices $\alpha\delta$ as inert 'spectators'. We apply (3.28) in the detailed form $\delta_{\alpha}^{\beta}\delta_{\gamma}^{\delta} = \sum_{A}\left(m_{A}\right)_{\alpha}^{\delta}\left(\Gamma_{A}\right)_{\gamma}^{\beta}$. The coefficients are $\left(m_{A}\right)_{\alpha}^{\delta} = 2^{-m}\delta_{\alpha}^{\beta}\delta_{\gamma}{}^{\delta}\left(\Gamma_{A}\right)_{\beta}^{\gamma} = 2^{-m}\left(\Gamma_{A}\right)_{\alpha}{}^{\delta}$. Therefore, we obtain the basic rearrangement lemma

$$\delta_{\alpha}^{\beta}\delta_{\gamma}^{\delta} = \frac{1}{2^{m}}\sum_{A}\left(\Gamma_{A}\right)_{\alpha}^{\delta}\left(\Gamma^{A}\right)_{\gamma}^{\beta}$$

Note that the 'column indices' on the left- and right-hand sides have been exchanged.

Exercise 3.26 Derive the following result:

$$\left(\gamma^{\mu}\right)_{\alpha}^{\beta}\left(\gamma_{\mu}\right)_{\gamma}^{\delta} = \frac{1}{2^{m}}\sum_{A} v_{A}\left(\Gamma_{A}\right)_{\alpha}^{\delta}\left(\Gamma^{A}\right)_{\gamma}^{\beta}$$

and prove that the explicit values of the expansion coefficients are given by $v_{A} = (-)^{r_{A}}\left(D-2r_{A}\right)$, where r_{A} is the tensor rank of the Clifford basis element Γ_{A}.

Exercise 3.27 Lower the β and δ index in the result of the previous exercise and consider the completely symmetric part in $(\beta\gamma\delta)$. The left-hand side is only non-vanishing for dimensions in which $t_{1}=-1$. Consider the right-hand side and use Table 3.1 and the result for v_{A} to prove that for $D=3$ and $D=4$ only rank 1γ-matrices contribute to the right-hand side. For $D=4$ you have to use the bold face row in the table to arrive at this result. You can also check that there are no other dimensions where this occurs.

The previous exercise implies that, for $D=3$ and $D=4$,

$$\left(\gamma_{\mu}\right)_{\alpha(\beta}\left(\gamma^{\mu}\right)_{\gamma\delta)} = 0$$

This is called the cyclic identity and is important in the context of string and brane actions. It can be extended to some other dimensions under further restrictions. [3] Multiplying the equations with three spinors $\lambda_{1}^{\beta}, \lambda_{2}^{\gamma}$ and λ_{3}^{γ}, equation (3.67) can be written as

$$\gamma_{\mu}\lambda_{[1}\bar{\lambda}_{2}\gamma^{\mu}\lambda_{3]} = 0$$

3 For $D=2$ and $D=10$ this equation holds when contracted with chiral spinors. Owing to (3.56) only odd rank γ-matrices then occur in the sum over A. This is sufficient to extend the result (3.67) to these cases. With the same restrictions of chirality there is for $D=6$ an analogous identity for the completely antisymmetric part in $[\beta\gamma\delta]$. where the symmetry of the indices in (3.67) is transformed to an antisymmetry between the three spinors due to the anti-commutating nature of spinors. This result is important to construct supersymmetric Yang-Mills theories; see Sec. 6.3.

The following application of Fierz rearrangement is valid for any set of four anticommuting spinor fields. The basic Fierz identity (3.65) immediately gives

$$\bar{\lambda}_1\lambda_2\bar{\lambda}_3\lambda_4 = -\frac{1}{2^m}\sum_A \bar{\lambda}_1\Gamma^A\lambda_4\bar{\lambda}_3\Gamma_A\lambda_2$$

This can be generalized to include general matrices M, M' of the Clifford algebra.

Exercise 3.28 Show that

$$\begin{aligned}
\bar{\lambda}_1 M\lambda_2\bar{\lambda}_3 M'\lambda_4 &= -\frac{1}{2^m}\sum_A \bar{\lambda}_1 M\Gamma_A M'\lambda_4\bar{\lambda}_3\Gamma^A\lambda_2 \\
&= -\frac{1}{2^m}\sum_A \bar{\lambda}_1\Gamma_A M'\lambda_4\bar{\lambda}_3\Gamma^A M\lambda_2.
\end{aligned}$$

When $\lambda_{1,2,3,4}$ are not all independent, it is frequently the case that some terms in the rearranged sum vanish due to symmetry relations such as (3.51).

One can write the Fierz relation (3.65) in the alternative form:

$$M = 2^{-m}\sum_{k=0}^{[D]}\frac{1}{k!}\Gamma_{\mu_1\ldots\mu_k}\operatorname{Tr}\left(\Gamma^{\mu_k\cdots\mu_1}M\right)$$

where

$$\begin{cases}[D]=D, & \text{for even } D\\ [D]=(D-1)/2, & \text{for odd } D\end{cases}$$

The factor $1/k$! compensates for the fact that in the sum over $\mu_1\ldots\mu_k$ each matrix of the basis appears k ! times.

Exercise 3.29 Prove the following chiral Fierz identities for $D=4$:

$$\begin{aligned}
P_L\chi\bar{\lambda}P_L &= -\frac{1}{2}P_L\left(\bar{\lambda}P_L\chi\right)+\frac{1}{8}P_L\gamma^{\mu\nu}\left(\bar{\lambda}\gamma_{\mu\nu}P_L\chi\right)\\
P_L\chi\bar{\lambda}P_R &= -\frac{1}{2}P_L\gamma^{\mu}\left(\bar{\lambda}\gamma_{\mu}P_L\chi\right).
\end{aligned}$$

You will need (3.42) to combine terms in (3.71).

Exercise 3.30 Prove that for $D=5$ the matrix $\chi\bar{\lambda}-\lambda\bar{\chi}$ can be written as

$$\chi\bar{\lambda}-\lambda\bar{\chi}=\gamma_{\mu\nu}\left(\bar{\lambda}\gamma^{\mu v}\chi\right)$$

Readers who understand the Majorana flip properties (3.51) and Fierz rearrangement are well equipped for supersymmetry and supergravity!

Complex conjugation can be replaced by charge conjugation, an operation that acts as complex conjugation on scalars, and has a simple action on fermion bilinears. For example, it preserves the order of spinor factors.

6.2.4 3.2.4 Reality

In this chapter, we have not yet discussed the complex conjugation of the spinor fields we are working with. Complex conjugation is necessary for such purposes as the verification that a term in the Lagrangian involving spinor bilinears is hermitian. But the complex conjugation of a bilinear [4] is an awkward operation since the hermiticity of both the C matrix in (3.50) and γ-matrices is involved. Therefore we present a related operation called charge conjugation which is much simpler in practice. For any scalar, defined here as a quantity whose spinor indices are fully contracted, charge conjugation and complex conjugation are the same. Since the Lagrangian is a scalar, charge conjugation can be used to manipulate the terms it contains.

First we define the charge conjugate of any spinor as

$$\lambda^C \equiv B^{-1}\lambda^*.$$

The barred charge conjugate spinor is then, using (3.50) and (3.47),

$$\overline{\lambda^C} = (-t_0t_1)\,\mathrm{i}\lambda^\dagger\gamma^0$$

Note that this is the Dirac conjugate as defined in (2.30) except for the numerical factor $(-t_0t_1)$. The meaning of this will become clear below when we discuss Majorana spinors. Note that $(-t_0t_1) = +1$ in $2, 3, 4, 10$ or 11 dimensions. [5]

The charge conjugate of any $2^m \times 2^m$ matrix M is defined as

$$M^C \equiv B^{-1}M^*B$$

Charge conjugation does not change the order of matrices: $(MN)^C = M^C N^C$. In practice the matrices M we deal with are products of γ-matrices. Hence, we need only the charge conjugation property of the generating γ-matrices, which is

$$(\gamma_\mu)^C \equiv B^{-1}\gamma_\mu^* B = (-t_0t_1)\,\gamma_\mu$$

Exercise 3.31 Start from (3.77) (and note that charge conjugation on any number is just complex conjugation). Prove that

$$(\gamma_*)^C = (-)^{D/2+1}\gamma_*.$$

4 We use the convention that the order of fermion fields is reversed in the process of complex conjugation. See (A.16) with $\beta = 1$.

[5] For these dimensions the spinor bilinears of Chs. 2 and 3 are related by $(\bar\lambda\chi)_{\text{Ch. 2}} = \left(\bar\lambda^C\chi\right)_{\text{Ch. 3}}$.

With these ingredients, we can derive the following rule for complex conjugation of a spinor bilinear involving an arbitrary matrix M :

$$(\bar\chi M\lambda)^* \equiv (\bar\chi M\lambda)^C = (-t_0t_1)\,\overline{\chi^C}M^C\lambda^C$$

A 'hidden' interchange of the order of the fermion fields is needed in the derivation, but there is no change of order in the final result for the charge conjugate of any bilinear. One may think of this relation as the appropriate conjugation property when the conjugate of a spinor is defined as in (3.50).

Exercise 3.32 It is important that any spinor λ and its conjugate λ^C transform in the same way under a Lorentz transformation. Prove this using (3.57) and the rules above. If the matrix M is a Clifford element of rank r, i.e. $M = \gamma_{\mu_1\ldots\mu_r}$, then both sides of (3.79) transform as tensors of rank r.

Exercise 3.33 Show that for any spinor $\left(\lambda^C\right)^C = -t_1\lambda$, and for any matrix $\left(M^C\right)^C = M$.

Exercise 3.34 Suppose that $\psi(x)$ is a fermion field which satisfies the free massive Dirac equation $\partial\!\!\!/\psi = m\psi$ for $D = 4$. Show that the charge conjugate field ψ^C satisfies the same equation. This exercise gives some physical motivation for the definition of Majorana spinors in the next section.

6.2.5 3.3 Majorana spinors

The concept of supersymmetry is closely tied to the relativistic treatment of particle spin. Indeed the transformation parameters are spinors ϵ_α. It is reasonable to suppose that the simplest supersymmetric field theories in each spacetime dimension D are based on the simplest spinors that are compatible with invariance under the Lorentz group $\mathrm{SO}(D-1,1)$. In even dimension $D=2m$ we already know that Weyl fields, rather than Dirac fields, transform irreducibly under Lorentz transformations. Weyl fields were first discussed in Sec. 2.6. They have 2^{m-1} complex components while a Dirac field has 2^m complex components. Weyl fields can be obtained by applying the chiral projector P_L or P_R to a Dirac field.

In this section we introduce Majorana fields, which are Dirac fields that satisfy an additional 'reality condition'. This condition reduces the number of independent components by a factor of 2. Thus, like Weyl fields, a Majorana spinor field has half the degrees of freedom and can be viewed as more fundamental than a complex Dirac field. Physically the properties of particles described by a Majorana field are similar to Dirac particles, except that particles and anti-particles are identical. The spin states of massive and massless Majorana spinors transform in representations of $\mathrm{SO}(D-1)$ and $\mathrm{SO}(D-2)$, respectively.

6.2.6 3.3.1 Definition and properties

The results of Sec. 3.2.4 suggest that it might be possible to impose the reality constraint

$$\psi=\psi^C=B^{-1}\psi^*, \quad \text{i.e.} \quad \psi^*=B\psi$$

on a spinor field. Ex. 3.32 shows that both sides transform in the same way under Lorentz transformations in any dimension D, so the constraint is compatible with Lorentz symmetry. In fact (3.80) is the defining condition for Majorana spinors. However, there is a subtle and important consistency condition that we now derive, which restricts the spacetime dimension in which Majorana spinors can exist. It is easy to see that the reality condition (3.80) is not automatically consistent. Take the complex conjugate of the second form of the condition and use it again to obtain $\psi=B^*B\psi$. Thus the reality condition is mathematically consistent only if $B^*B=\mathbb{1}$. Using Ex. 3.20, we see that this requires ${}^6t_1=-1$.

The two possible values $t_0=\pm1$ must be considered, and we begin with the case $t_0=+1$. Consulting Table 3.1, we see that $t_0=+1$ holds for spacetime dimension $D=2,3,4$, mod 8. In this case we call the spinors that satisfy (3.80) Majorana spinors. It is clear from (3.75) that if $t_0=1$ and $t_1=-1$, the barred (3.50) and Dirac adjoint spinors (2.30) agree for Majorana spinors. In fact, this gives an alternative definition of a Majorana spinor.

Another fact about the Majorana case is that there are representations of the γ-matrices that are explicitly real and may be called really real representations. Here is a really real representation for $D=4$:

$$\begin{aligned}
\gamma^0 &= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \mathrm{i}\sigma_2\otimes\mathbb{1}, \\
\gamma^1 &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \sigma_3\otimes\mathbb{1}, \\
\gamma^2 &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \sigma_1\otimes\sigma_1, \\
\gamma^3 &= \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} = \sigma_1\otimes\sigma_3.
\end{aligned}$$

Note that the γ^i are symmetric, while γ^0 is antisymmetric. This is required by hermiticity in any real representation. We construct really real representations in all allowed dimensions $D=2,3,4$ mod 8 in Appendix 3A.5.

In such representations (3.48) implies that $B=\mathbb{1}$ (up to a phase). The relation (3.47) then gives $C=\mathrm{i}\gamma^0$. Further, a Majorana spinor field is really real since (3.80) reduces to $\Psi^*=\Psi$.

Really real representations are sometimes convenient, but we emphasize that the physics of Majorana spinors is the same in, and can be explored in, any representation of the Clifford algebra, replacing complex conjugation with charge conjugation. For convenience we

6 This manipulation is the same as working out Ex. 3.33, and this thus leads to the same result. often write 'complex conjugation' when in fact we use 'charge conjugation'. For example, the complex conjugate of $\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi$, where χ and ψ are Majorana, is computed as follows. We follow Sec. 3.2.4 and write

$$(\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi)^* = (\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi)^C = \bar{\chi}\left(\gamma_{\mu_1\ldots\mu_r}\right)^C\psi = \bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi$$

We used the reality conditions $\psi^C = \psi$ and $\bar{\chi}^C = \bar{\chi}^C$ as well as (3.77) to deduce this result. [7] Hence, bilinears such as $\bar{\chi}\psi$ and $\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi$ are real.

When $t_0 = -1$ (and still $t_1 = -1$) spinors that satisfy (3.80) are called pseudo-Majorana spinors. They are mostly relevant for $D = 8$ or 9 . There are no really real representations in these dimensions; instead there are representations of the Clifford algebra in which the generating γ-matrices are imaginary, $(\gamma^\mu)^* = -\gamma^\mu$. In any representation (3.79) and (3.77) hold with $t_0 = t_1 = -1$. This implies that the reality properties of bilinears are different from those of Majorana spinors. Although these differences are significant, the essential property that a complex spinor can be reduced to a real one still holds, and it is common in the literature not to distinguish between Majorana and pseudo-Majorana spinors. However, note the following.

Exercise 3.35 Show that the mass term $m\bar{\chi}\chi = 0$ for a single pseudo-Majorana field. Pseudo-Majorana spinors must be massless (unless paired).

We now consider (pseudo-)Majorana spinors in even dimensions $D = 0, 2, 4 \bmod 8$. We can quickly show using (3.78) that these cases are somewhat different. For $D = 2 \bmod 8$ we have $(\gamma_*\psi)^C = \gamma_*\psi^C$. Thus the two constraints

$$\text{Majorana:} \quad \psi^C = \psi, \quad \text{Weyl:} \quad P_{L,R}\psi = \psi,$$

are compatible. It is equivalent to observe that the chiral projections of a Majorana spinor ψ satisfy

$$(P_L\psi)^C = P_L\psi, \quad (P_R\psi)^C = P_R\psi$$

Thus the chiral projections of a Majorana spinor are also Majorana spinors. Each chiral projection satisfies both constraints in (3.83) and is called a Majorana-Weyl spinor. Such spinors have 2^{m-1} independent 'real' components in dimension $D = 2m = 2 \bmod 8$ and are the 'most fundamental' spinors available in these dimensions. It is not surprising that supergravity and superstring theories in $D = 10$ dimensions are based on Majorana-Weyl spinors.

For $D = 4 \bmod 4$ dimensions we have $(\gamma_*\psi)^C = -\gamma_*\psi^C$, so that the equations of (3.84) are replaced by

$$(P_L\psi)^C = P_R\psi, \quad (P_R\psi)^C = P_L\psi$$

7 Notice that Majorana spinors, which are real in the sense of C-conjugation, are not real for the original complex conjugation, not even in the really real representation. In fact, $\bar{\chi}$ is purely imaginary in the really real representation. However, under complex conjugation we should interchange the order of the spinors, which leads to another - sign, compensating the - sign of complex conjugation of $\bar{\chi}$. Neither sign appears explicitly when one uses charge conjugation, independent of the γ-matrix representation. This illustrates how the use of C simplifies the reality considerations.

These equations state that the 'left' and 'right' components of a Majorana spinor are related by charge conjugation. A direct consequence is that, for any expression involving the lefthanded projection $P_L\psi$ of a Majorana spinor ψ, the corresponding expression for $P_R\psi$ follows by complex conjugation. Of course there are Weyl spinors that are chiral projections $P_{L,R}\psi$ of a Dirac spinor ψ, but these cannot satisfy the Majorana condition since for Majorana fermions $(P_{L,R}\psi)^C = P_{R,L}\psi$.

6.2.7 3.3.2 Symplectic Majorana spinors

When $t_1 = 1$ we cannot define Majorana spinors, but we can define 'symplectic Majorana spinors'. These consist of an even number of spinors χ^i, with $i = 1, \ldots, 2k$, which satisfy a 'reality condition' containing a non-singular antisymmetric matrix ε^{ij}. The inverse matrix ε_{ij} satisfies $\varepsilon^{ij}\varepsilon_{kj} = \delta^i_k$. Symplectic Majorana spinors satisfy the condition

$$\chi^i = \varepsilon^{ij}\left(\chi^j\right)^C = \varepsilon^{ij}B^{-1}\left(\chi^j\right)^*$$

The consistency check discussed after (3.80) now works for $t_1 = 1$ because of the antisymmetric ε^{ij}.

Exercise 3.36 Check that, in five dimensions with symplectic Majorana spinors, $\bar{\psi}^i\chi_i \equiv \bar{\psi}^i\chi^j\varepsilon_{ji}$ is pure imaginary while $\bar{\psi}^i\gamma_\mu\chi_i$ is real.

For dimensions $D = 6 \bmod 8$, one can use (3.78) to show that the symplectic Majorana constraint is compatible with chirality. We can therefore define the symplectic MajoranaWeyl spinors $P_L\chi^i$ or $P_R\chi^i$.

6.2.8 3.3.3 Dimensions of minimal spinors

The various types of spinors we have discussed are linked to the signs of t_0 and t_1 as follows:

$$\begin{array}{lll} t_1 = -1, & t_0 = 1: & \text{Majorana,} \\ & t_0 = -1: & \text{pseudo-Majorana ,} \\ t_1 = 1: & & \text{symplectic Majorana .} \end{array}$$

As explained above we no longer distinguish between Majorana and pseudo-Majorana spinors. In any even dimension one can define Weyl spinors, while in dimensions $D = 2 \bmod 4$, one can combine the (symplectic) Majorana condition and Weyl conditions. These facts are summarized in Table 3.2. [8] For each spacetime dimension it is indicated whether Majorana (M), Majorana-Weyl (MW), symplectic (S) or symplectic Weyl (SW) spinors can be defined as the 'minimal spinor'. The number of components of this minimal spinor is given. The table is for Minkowski signature and has a periodicity of 8 in dimension. When D is changed to $D+8$, the number of spinor components is multiplied by 16 . The

[8] For $D = 4 \bmod 4$ we can also define Weyl spinors, but we omit this in the table.

Table 3.2			Irreducible spinors, number of components and symmetry properties.
dim	spinor	min # components	antisymmetric
2	MW	1	1
3	M	2	1,2
4	M	4	1,2
5	S	8	2,3
6	SW	8	3
7	S	16	0,3
8	M	16	0,1
9	M	16	0,1
10	MW	16	1
11	M	32	1,2

final column indicates the ranks of the antisymmetric spinor bilinears, e.g. a 0 indicates that $\bar{\epsilon}_2\epsilon_1 = -\bar{\epsilon}_1\epsilon_2$, and a 2 indicates that $\bar{\epsilon}_2\gamma_{\mu\nu}\epsilon_1 = -\bar{\epsilon}_1\gamma_{\mu\nu}\epsilon_2$. This entry is modulo 4, i.e. if rank 0 is antisymmetric, then so are rank 4 and 8 bilinears. Minimal spinors in dimension $D = 2 \bmod 4$ must have the same chirality to define a symmetry for their bilinears. The property (3.56) then implies that non-vanishing bilinears contain an odd number of γ matrices. For $D = 4 \bmod 4$, there are two possibilities for reality conditions and we have chosen the one that includes rank 1 in the column 'antisymmetric'. This property is needed for the supersymmetry algebra.

6.2.9 3.4 Majorana spinors in physical theories

6.2.10 3.4.1 Variation of a Majorana Lagrangian

In this section we consider a prototype action for a Majorana spinor field in dimension $D = 2, 3, 4 \bmod 8$. Majorana and Dirac fields transform the same way under Lorentz transformations, but Majorana spinors have half as many degrees of freedom, so we write

$$S[\Psi] = -\frac{1}{2}\int \mathrm{d}^D x \bar{\Psi}\left[\gamma^\mu\partial_\mu - m\right]\Psi(x)$$

There is an immediate and curious subtlety due to the symmetries of the matrices C and $C\gamma^\mu$. Using (3.50), we see that the mass and kinetic terms are proportional to $\Psi^T C\Psi$ and $\Psi^T C\gamma^\mu\partial_\mu\Psi$. Suppose that the field components Ψ are conventional commuting numbers. Since C is antisymmetric, the mass term vanishes. Since $C\gamma^\mu$ is symmetric, the kinetic term is a total derivative and thus vanishes when integrated in the action. For commuting field components, there is no dynamics! To restore the dynamics we must assume that Majorana fields are anti-commuting Grassmann variables, which we always assume unless stated otherwise.

Let's derive the Euler-Lagrange equation for Ψ. Field variations must satisfy the Majorana condition (3.80), so that $\delta\Psi$ and $\delta\bar{\Psi}$ are related following Sec. 3.2.1. Initially $\delta S[\Psi]$ contains two terms. However, after a Majorana flip and partial integration, one can see that the two terms are equal, so that $\delta S[\Psi]$ can be written as the single expression

$$\delta S[\Psi] = -\int \mathrm{d}^D x \delta\bar{\Psi}\left[\gamma^\mu\partial_\mu - m\right]\Psi(x)$$

Thus a Majorana field satisfies the conventional Dirac equation.

This fact is no surprise, but it is an example of a more general and simplifying rule for the variation of Majorana spinor actions. If integration by parts is valid, it is sufficient to vary $\bar{\Psi}$ and multiply by 2 to account for the variation of Ψ.

Exercise 3.37 Derive (3.89) in full detail.

Exercise 3.38 A Majorana field is simply a Dirac field subject to the reality condition (3.80). Let's impose that constraint on the plane wave expansion (2.24) for $D = 4$ using the relation $v = u^C = Bu^*$, which holds for the u and v spinors defined in (2.37) and (2.38). In this way one derives $d(\vec{p}, s) = c(\vec{p}, s)$ which proves that a Majorana particle is its own anti-particle. Readers should derive this fact!

Exercise 3.39 Show that

$$v(\vec{p}, s) = u(\vec{p}, s)^C$$

holds for the u and v spinors defined for the Weyl representation in Sec. 2.5. This was the motivation for the choice (2.41).

6.2.11 3.4.2 Relation of Majorana and Weyl spinor theories

In even dimensions $D = 0, 2, 4 \bmod 8$, both Majorana and Weyl fields exist and both have legitimate claims to be more fundamental than a Dirac fermion. In fact both fields describe equivalent physics. Let's show this for $D = 4$. We can rewrite the action (3.88) as

$$S[\psi] = -\frac{1}{2}\int \mathrm{d}^4x \left[\bar{\Psi}\gamma^\mu\partial_\mu - m\right](P_L + P_R)\Psi$$
$$= -\int \mathrm{d}^4x \left[\bar{\Psi}\gamma^\mu\partial_\mu P_L\Psi - \frac{1}{2}m\bar{\Psi}P_L\Psi - \frac{1}{2}m\bar{\Psi}P_R\Psi\right]$$

We obtained the second line by a Majorana flip and partial integration. In the second form of the action, the Majorana field is replaced by its chiral projections. In our treatment of chiral multiplets in supersymmetry, we will exercise the option to write Majorana fermion actions in this way.

Exercise 3.40 Show that the Euler-Lagrange equations that follow from the variation of the second form of the action in (3.91) are

$$\partial\!\!\!/ P_L\Psi = mP_R\Psi, \quad \partial\!\!\!/ P_R\Psi = mP_L\Psi.$$

Derive $\Box P_{L,R}\Psi = m^2 P_{L,R}\Psi$ from the equations above.

6.2.12 Majorana and Weyl fields in $D = 4$

Any field theory of a Majorana spinor field Ψ can be rewritten in terms of a Weyl field $P_L\Psi$ and its complex conjugate. Conversely, any theory involving the chiral field $\chi = P_L\chi$ and its conjugate $\chi^C = P_R\chi^C$ can be rephrased as a Majorana equation if one defines the Majorana field $\Psi = P_L\chi + P_R\chi^C$. Supersymmetry theories in $D = 4$ are formulated in both descriptions in the physics literature.

Let's return to the Weyl representation (2.19) for the final step in the argument to show that the equation of motion for a Majorana field can be reexpressed in terms of a Weyl field and its adjoint. The Majorana condition $\Psi = B^{-1}\Psi^* = \gamma^0\gamma^1\gamma^3\Psi^*$ requires that Ψ take the form

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_2^* \\ -\psi_1^* \end{pmatrix}$$

With (3.93) and (2.55) in view we define the two-component Weyl fields

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \tilde{\psi} = \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix}$$

Using the form of γ^μ (2.19) and γ_* (3.34) in the Weyl representation, we see that we can identify

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} = P_L\Psi, \quad \begin{pmatrix} 0 \\ \tilde{\psi} \end{pmatrix} = (P_L\Psi)^C = P_R\Psi$$

The equations of motion (3.92) can then be rewritten as

$$\bar{\sigma}^\mu\partial_\mu\psi = m\tilde{\psi}, \quad \sigma^\mu\partial_\mu\tilde{\psi} = m\psi.$$

These are equivalent to the pair of Weyl equations in (2.56) with the restriction $\tilde{\psi} = \bar{\chi}$ which comes because we started in this section with a Majorana rather than a Dirac field.

3.4.3 U(1) symmetries of a Majorana field

In Sec. 2.7.1 we considered the U(1) symmetry operation $\Psi \to \Psi' = e^{i\theta}\Psi$. This symmetry is obviously incompatible with the Majorana condition (3.80). Thus the simplest internal symmetry of a Dirac fermion cannot be defined in a field theory of a (single) Majorana field. However, it is easy to see that $(i\gamma_*)^C = i\gamma_*$, so the chiral transformation $\Psi \to \Psi' = e^{i\gamma_*\theta}\Psi$ preserves the Majorana condition. Let's ask whether the infinitesimal limit of this transformation is a symmetry of the free massive Majorana action (3.88).

Exercise 3.41 Use $\delta\bar{\Psi} = i\theta\bar{\Psi}\gamma_*$ and partial integration to derive the variation

$$\delta S[\Psi] = i\theta m \int d^4x \bar{\Psi}\gamma_*\Psi$$

which vanishes only for a massless Majorana field.

Thus we have learned the following.

- The conventional vector U(1) symmetry is incompatible with the Majorana condition.
- The axial transformation above is compatible and is a symmetry of the action for a massless Majorana field only.

Exercise 3.42 Show that the axial current

$$J_*^\mu = \frac{1}{2} i\bar{\Psi}\gamma^\mu\gamma_*\Psi$$

is the Noether current for the chiral symmetry defined above. Use the equations of motion to show that

$$\partial_\mu J_*^\mu = -im\bar{\Psi}\gamma_*\Psi$$

The current is conserved only for massless Majorana fermions.

The dynamics of a Majorana field Ψ can be expressed in terms of its chiral projections $P_{L,R}\Psi$. So can the chiral transformation, which becomes $P_{L,R}\Psi \to P_{L,R}\Psi' = e^{\pm i\theta}\Psi$.

Throughout this section we used the simple dynamics of a free massive fermion to illustrate the relation between Majorana and Weyl fields and to explore their U(1) symmetries. It is straightforward to extend these ideas to interacting field theories with nonlinear equations of motion.

6.2.13 Appendix 3A Details of the Clifford algebras for $D = 2m$

6.2.14 3A.1 Traces and the basis of the Clifford algebra

Let us start with the following facts discussed in Sec. 3.1. The Clifford algebra in even dimension $D = 2m$ has a basis of 2^m linearly independent, trace orthogonal matrices, given in (3.25). Any representation by matrices of dimension 2^m is irreducible.

The trace properties of the matrices are important for proofs of these properties which are independent of the explicit construction in (3.2). The matrices Γ^A for tensor rank $1 \le r \le D-1$ are traceless. One simple way to see this is to use the Lorentz transformations (2.22) and its extension to general rank

$$L(\lambda)\gamma^{\mu_1\mu_2\dots\mu_r}L(\lambda)^{-1} = \gamma^{v_1v_2\dots v_r}\Lambda_{\nu_1}^{\mu_1}\dots\Lambda_{v_r}^{\mu_r}$$

Traces then satisfy the Lorentz transformation law as suggested by their free indices:

$$\operatorname{Tr}\gamma^{\mu_1\mu_2\dots\mu_r} = \operatorname{Tr}\gamma^{\nu_1\nu_2\dots v_r}\Lambda_{\nu_1}^{\mu_1}\dots\Lambda_{v_r}{}^{\mu_r}$$

This means that the traces must be totally antisymmetric Lorentz invariant tensors. However the only invariant tensors available are the Minkowski metric $\eta^{\mu v}$ and the Levi-Civita tensor $\varepsilon^{\mu_1\mu_2\cdots\mu_D}$ introduced in Sec. 3.1.3. No totally antisymmetric tensor can be formed from products of $\eta^{\mu\nu}$. This proves that $\operatorname{Tr}\Gamma^A = 0$ for all elements of rank $1 \leq r \leq D-1$.

The argument does not apply to the highest rank element. However, one can see from the pattern of alternation in (3.7) that this is given by a commutator for even $D = 2$ m and by an anti-commutator for odd $D = 2m+1$. Thus the trace of the highest rank element vanishes for $D = 2m$ but need not (and does not) vanish for $D = 2m+1$. This is actually a fundamental distinction between the Clifford algebras for even and odd dimensions. It might have been expected since the second rank elements (see Ex. 2.8) give a representation of the Lorentz algebras $\mathfrak{so}(D-1,1)$ which are real forms of different Lie algebras in the Cartan classification, namely D_m for even $D = 2m$ and B_m for $D = 2m+1$.

There is another way to prove the traceless property, which does not require information concerning invariant tensors. For rank 1, we simply take the trace of the formula derived in Ex. 2.9. Contraction with $\eta_{\nu\rho}$ immediately gives $\operatorname{Tr}\gamma^\mu = 0$. As an exercise, the reader can extend this argument to higher rank.

The trace property leads also to the proof of independence of the elements of the basis (3.25) for even spacetime dimensions. One uses the 'reverse order' basis of (3.26) and the trace orthogonality property (3.27). We suppose that there is a set of coefficients x_A such that

$$\sum_A x_A \Gamma^A = 0$$

Multiply by Γ_B from the right. Take the trace and use the trace orthogonality to obtain

$$\sum_A x_A \operatorname{Tr}\Gamma^A\Gamma^B = \pm x_B \operatorname{Tr}\mathbb{1} = 0$$

Hence all $x_A = 0$ and linear independence is proven.

Furthermore, since we have a linearly independent, indeed trace orthogonal, basis of the algebra, the Γ^A are a complete set in the space of $2^m \times 2^m$ matrices.

It now follows that, in any representation of the Clifford algebra for $D = 2m$ spacetime dimensions, the dimension of the $N \times N$ matrices satisfies $N \geq 2^m$. The reason is that no linearly independent set of matrices of any smaller dimension exists. It also follows that any representation of dimension 2^m is irreducible. It can have no non-trivial invariant subspace, since a set of linearly independent matrices of smaller dimension would be realized by projection to this subspace.

6.2.15 3A.2 Uniqueness of the γ-matrix representation

We must now show that there is exactly one irreducible representation up to equivalence. We use the basic properties of representations of finite groups. However, the Clifford algebra is not quite a group because the minus signs that necessarily occur in the set of products $\Gamma^A\Gamma^B = \pm\Gamma^C$ are not allowed by the definition of a group. This problem is solved by doubling the basis in (3.25) to the larger set $\left\{\Gamma^A, -\Gamma^A\right\}$. This set is a group of order 2^{2m+1} since all products are contained within the larger set. For $m = 1$, the group obtained is isomorphic to the quaternions, so the groups defined by doubling the Clifford algebras are called generalized quaternionic groups.

Every representation of the Clifford algebra by a set of matrices $D\left(\Gamma^A\right)$ extends to a representation of the group if we define $D\left(-\Gamma^A\right) = -D\left(\Gamma^A\right)$. It is not true that every group representation gives a representation of the algebra. For example, in a onedimensional group representation, the matrices $D(\gamma^\mu)$ of the Clifford generators cannot satisfy $\{D(\gamma^\mu), D(\gamma^\nu)\} = 2\eta^{\mu\nu}$.

The three basic facts that we need are discussed in many mathematical texts such as [13, 14]. Consider the set of all finite-dimensional irreducible representations and choose one representative within each class of equivalent representations. The set so formed, which may be called the set of all inequivalent irreducible representations, has the following properties:

1. The sum of the squares of the dimensions of these representations is equal to the order of the group.

2. The number of inequivalent irreducible representations is equal to the number of conjugacy classes in the group.

3. The number of inequivalent one-dimensional representations is equal to the index of the commutator subgroup G_c. The index of a subgroup is the ratio of the order of the group divided by the order of the subgroup.

The conjugacy classes of the group are sets of products $\pm\Gamma^B\Gamma^A\left(\Gamma^B\right)^{-1}$ (with no sum on B).

Exercise 3.43 Show that for rank $r \geq 1$ there is a conjugacy class containing the pair $\left(\Gamma^A, -\Gamma^A\right)$ for each distinct Γ^A, and that $\mathbb{1}$ and $-\mathbb{1}$ belong to different conjugacy classes.

Thus there are a total of $2^{2m}+1$ conjugacy classes.

The commutator subgroup is generated by all products of the form $\pm\Gamma^B\Gamma^A\left(\Gamma^B\right)^{-1}\left(\Gamma^A\right)^{-1}$. But in our case this subgroup contains only $\pm\mathbb{1}$, so the order of the subgroup is 2 and its index is 2^{2m}.

These facts establish that the group has exactly one irreducible representation of dimension 2^m plus 2^{2m} inequivalent one-dimensional representations. We must now show that the 2^m-dimensional representation of the group is also a representation of the algebra. We use the fact that any finite-dimensional algebra has a (reducible) representation called the regular representation for which the algebra itself is the carrier space. The dimension is thus the dimension of the algebra, 2^{2m} in our case. The regular representation $\Gamma^A \to T\left(\Gamma^A\right)$ is defined by $T\left(\Gamma^A\right)\Gamma^B \equiv \Gamma^A\Gamma^B$. This algebra representation, in which $T\left(-\Gamma^A\right) = -T\left(\Gamma^A\right)$ is necessarily satisfied, is also a group representation. Its decomposition into irreducible components thus cannot contain any one-dimensional group representations in which $D\left(-\Gamma^A\right) = +D\left(\Gamma^A\right)$. Thus the only possibility is that the regular representation decomposes into 2^m copies of the 2^m-dimensional irreducible representation. This proves the essential fact that there is exactly one irreducible representation of the Clifford algebra for even spacetime dimension. For dimension $D = 2m$, the dimension of the Clifford representation is 2^m.

Another fact from finite group theory is helpful at this point. Any representation of a finite group is equivalent to a representation by unitary matrices. We can and therefore will choose a representation in which the spatial γ-matrices $\gamma^i, i = 1, \ldots, D-1$, which satisfy $\left(\gamma^i\right)^2 = \mathbb{1}$, are hermitian, and γ^0, which satisfies $\left(\gamma^0\right)^2 = -\mathbb{1}$, is anti-hermitian.

6.2.16 3A.3 The Clifford algebra for odd spacetime dimensions

We gave in (3.40) two different sets of γ-matrices for odd dimensions. They are inequivalent as representations of the generating elements. Indeed it is easily seen that $S{\gamma_+}^\mu S^{-1} = {\gamma_-}^\mu$ cannot be satisfied. This requires $S\gamma^\mu S^{-1} = \gamma^\mu$ for the first $2m$ components. But then, from the product form in (3.6) and (3.30), we obtain $S\gamma^{2m}S^{-1} = +\gamma^{2m}$, rather than the opposite sign needed.

It follows from Ex. 2.8 that the two sets of second rank elements constructed from the generating elements above, namely

$$
\begin{aligned}
\Sigma_{\pm}\mu\nu &= \frac{1}{4}\left[\gamma^{\mu}, \gamma^{\nu}\right], \quad \mu, v = 0, \ldots, 2m-1, \\
&= \frac{1}{4}\left[\gamma^{\mu}, \pm\gamma_{*}\right], \quad \mu = 0, \ldots, 2m-1, \quad v = 2m
\end{aligned}
$$

are each representations of the Lie algebra $\mathfrak{so}(2m, 1)$. The two representations are equivalent, however, since $\gamma_{*}\Sigma_{+}{}^{\mu v}\gamma_{*} = \Sigma_{-}{}^{\mu\nu}$. This representation is irreducible; indeed it is a copy of the unique 2^{2m}-dimensional fundamental irreducible representation with Dynkin designation $(0, 0, \ldots, 0, 1)$. It is associated with the short simple root of the Dynkin diagram for B_m.

We refer readers to [9][10, 11, 12, 15, 16] for alternative discussions of γ-matrices and Majorana spinors.

6.2.17 3A.4 Determination of symmetries of γ-matrices

We will determine the possible symmetries of γ-matrices for each spacetime dimension $D = 2m$ by showing that each matrix $C\Gamma_A$ formed from the basis (3.25) has a definite symmetry that depends only on the tensor rank r. Then we will count the number of symmetric and antisymmetric matrices in the list $\{C\Gamma_A\}$, which must be equal to $2^{m-1}\left(2^m \pm 1\right)$ for $D = 2$ m. For given values of t_0 and t_1, the number of antisymmetric matrices in the list $\{C\Gamma_A\}$ is given, using (3.44), by

$$
\begin{aligned}
N_{-} &= \sum_{r=0}^{2m} \frac{1}{2}\left[1 + t_r\right] C_r^{2m} \\
&= 2^{2m-1} + \frac{1}{2} t_0 \sum_{s=0}^{m} (-)^s C_{2s}^{2m} + \frac{1}{2} t_1 \sum_{s=0}^{m-1} (-)^s C_{2s+1}^{2m} \\
&= 2^{2m-1} + t_0 2^{m-1} \cos\frac{m\pi}{2} + t_1 2^{m-1} \sin\frac{m\pi}{2} \\
&= 2^{m-1}\left(2^m - 1\right).
\end{aligned}
$$

We thus find

$$
t_0 \cos\frac{m\pi}{2} + t_1 2^{m-1} \sin\frac{m\pi}{2} = -1
$$

9 In [15], the discussion of Majorana spinors is in Sec. 4, pp. 843-851. which leads to the solutions that are in Table 3.1 for even dimensions.

To understand the situation in odd $D = 2m + 1$ we note that the highest rank Clifford element γ_* in (3.30) has the symmetry determined by t_{2m}. Since we attach $\pm\gamma_* = \gamma^{2m}$ as the last generating element in (3.40) we must require it to have the same symmetry as the other generating γ^{μ}, and thus t_{2m} should be equal to t_1. This determines which of the two possibilities for even dimensions in Table 3.1 is valid in the next odd dimension.

6.2.18 3A.5 Friendly representations

6.2.19 General construction

In this section we present an explicit recursive construction of the generating γ^{μ} for any even dimension $D = 2m$. In this representation each generating matrix will be either pure real or pure imaginary. A representation of this type will be called a friendly representation. [10] Using this representation it is also possible to prove the existence of Majorana (and pseudo-Majorana) spinors in a quite simple way [17, 12] (see Appendix B in [17]).

We already know that the γ-matrices in dimension $D = 2m$ are $2^m \times 2^m$ matrices. In the recursive construction the generating matrices γ^μ for dimension $D = 2m$ will be written as direct products of the $\tilde{\gamma}^\mu$ and $\tilde{\gamma}_*$ for dimension $D = 2m - 2$ with the Pauli matrices σ_i.

We start in $D = 2$ and write

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i}\sigma_2, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$$

which is a really real, hermitian, and friendly representation. The matrix γ_* is also real:

$$\gamma_* = -\gamma_0\gamma_1 = \sigma_3.$$

Adding it to (3.107) as γ^2 gives a real representation in $D = 3$.

The recursion relation for moving from a $D = 2m - 2$ representation with $\tilde{\gamma}$ to $D = 2m$ is

$$\begin{aligned} \gamma^\mu &= \tilde{\gamma}^\mu \otimes \mathbb{1}, \quad \mu = 0, \dots, 2m-3, \\ \gamma^{2m-2} &= \tilde{\gamma}_* \otimes \sigma_1, \quad \gamma^{2m-1} = \tilde{\gamma}_* \otimes \sigma_3. \end{aligned}$$

This gives

$$\gamma_* = -\tilde{\gamma}_* \otimes \sigma_2.$$

This matrix γ_* can be used as γ^{2m} to define a representation in $D = 2m + 1$ dimensions.

This construction gives a real representation in four dimensions, which is explicitly given in (3.81). This one has an imaginary γ_* and hence this construction will not give real representations for higher dimensions. The matrix B is obtained as the product of all the imaginary γ-matrices.

We thus obtained representations for all dimensions, and really real for $D = 2, 3, 4$. The latter can be extended to any $D = 10, 11, 12$ or any other dimension that differs from it modulo 8 . To see this, consider the following 16×16 matrices:

$$\begin{aligned} E_1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\ E_2 &= \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\ E_3 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \otimes \mathbb{1}, \\ E_4 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \mathbb{1}, \\ E_5 &= \sigma_2 \otimes \sigma_1 \otimes \mathbb{1} \otimes \sigma_2, \\ E_6 &= \sigma_2 \otimes \sigma_3 \otimes \mathbb{1} \otimes \sigma_2, \\ E_7 &= \sigma_2 \otimes \mathbb{1} \otimes \sigma_2 \otimes \sigma_1, \\ E_8 &= \sigma_2 \otimes \mathbb{1} \otimes \sigma_2 \otimes \sigma_3, \\ E_* &= E_1 \dots E_8 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2. \end{aligned}$$

This is a real representation for Euclidean γ-matrices in $D = 8$ (or $D = 9$ if one includes E_*). Using this and a representation $\tilde{\gamma}^\mu$ in any D, one can construct a representation γ^μ in $D + 8$ dimensions by

$$\begin{aligned} \gamma^\mu &= \tilde{\gamma}^\mu \otimes E_*, \quad \mu = 0, \dots, D-1, \\ \gamma^{D-1+i} &= \mathbb{1} \otimes E_i, \quad i = 1, \dots, 8. \end{aligned}$$

When the $\tilde{\gamma}^\mu$ are real, the γ-matrices in $D + 8$ are also real. Hence this gives explicitly real representations in $D = 2, 3, 4$ mod 8. For even dimensions, one obtains

10 All of our friends use friendly representations.

$$\gamma_* = \tilde{\gamma}_* \otimes E_*.$$

Hence this is real if $\tilde{\gamma}_*$ is real. For the real representations we saw that it is real for $D = 2$ and not in $D = 4$. This shows explicitly that we can define real projections P_L and P_R on real spinors if and only if $D = 2 \bmod 8$. These are called Majorana-Weyl representations.

Exercise 3.44 We denote a Clifford algebra in s space-like and t time-like directions as $\mathcal{C}(s,t)$ (the ones discussed above are thus of the form $\mathcal{C}(D-1,1)$, apart from the E_i that correspond to $\mathcal{C}(8,0)$). See that the above construction proves that the reality properties of $\mathcal{C}(s+8,t)$ are the same as $\mathcal{C}(s,t)$. Further, show that the analogous construction starting with (3.107) shows that also $\mathcal{C}(s+1,t+1)$ has the same properties as $\mathcal{C}(s,t)$.

6.3 The Maxwell and Yang-Mills gauge fields

In this chapter we discuss the classical abelian and non-abelian gauge fields. Although our treatment is self-contained, it is best taken as a review for readers who have previously studied the role of the vector potential as the gauge field in Maxwell's electromagnetism and also have some acquaintance with Yang-Mills theory.

We will again take a general dimensional viewpoint, but let's begin the discussion in four dimensions with some remarks about the particle representations of the Poincare group and the fields usually used to describe elementary particles. A particle is classified by its mass m and spin s, and a massive particle of spin s has $2s+1$ helicity states. Massless particles of spin $s = 0$ or $s = 1/2$ have one or two helicity states, respectively, in agreement with the counting for massive particles. However, massless particles of spin $s \geq 1/2$ have two helicity states, for all values of s.

Helicity is defined as the eigenvalue of the component of angular momentum in the direction of motion. For a massless particle of spin s, the two helicity states have eigenvalues $\pm s$. For a massive particle of spin s the helicity eigenvalues, called λ, range in integer steps from $\lambda = s$ to $\lambda = -s$.

Let us compare the count of the helicity states with the number of independent functions that must be specified as initial data for the Cauchy initial value problem of the associated field. The first number can be considered to be the number of on-shell degrees of freedom, or number of quantum degrees of freedom, while the second is the number of classical degrees of freedom.

Let's do the counting for massless particles that are identified with their anti-particles. The associated fields are real for bosons and satisfy the Majorana condition for fermions. The counting is similar for complex fields. We assume that the equations of motion are second order in time for bosons and first order for fermions. A unique solution of the Cauchy problem for the scalar $\phi(x)$ requires the initial data $\phi(\vec{x},0)$ and $\dot{\phi}(\vec{x},0)$, the time derivative. For $\Psi_\alpha(x)$, we must specify the initial values $\Psi_\alpha(\vec{x},0)$ of all four components, and the first order Dirac equation then determines the future evolution of $\Psi_\alpha(\vec{x},t)$ and thus the time derivatives $\dot{\Psi}_\alpha(\vec{x},0)$. The number of helicity states (number of on-shell degrees of freedom) is 1 for $\phi(x)$ and 2 for $\Psi_\alpha(x)$. The number of classical degrees of freedom is twice the number of helicity states.

We continue this counting, in a naive fashion, for vector $A_\mu(x)$, vector-spinor $\psi_{\mu\alpha}(x)$, and symmetric tensor $h_{\mu\nu}(x)$ fields, the latter describing gravitons in Minkowski space. Following the earlier pattern we would expect to need 8, 16, and 20 functions, respectively, as initial data. These numbers greatly exceed the two helicity states for spin-1, spin-3/2 and spin-2 particles. Something new is required to resolve this mismatch.

The lessons from quantum electrodynamics, Yang-Mills theory, general relativity and supergravity teach us that the only way to proceed is to use very special field equations with gauge invariance. Gauge invariance accomplishes the following goals:

(a) Relativistic covariance is maintained.

(b) The field equations do not determine certain 'longitudinal' field components (such as $\partial^\mu A_\mu$ for vector fields).

(c) A subset of the field equations are constraints on the initial data rather than time evolution equations. The independent initial data are contained in four real functions, thus again two for each helicity state.

(d) The field describes a pure spin-s particle with no lower spin admixtures. Otherwise there would be some negative metric ghosts.

(e) Most important, for $s = 1, 3/2, 2$, gauge invariant interactions can be introduced. [1] Classical dynamics is consistent at the nonlinear level and the theories can be quantized (although power-counting renormalizability is expected to fail except for spin 1).

The dynamics of the gauge fields $A_\mu, \psi_{\mu\alpha}$ and $h_{\mu\nu}$ is analyzed in this and subsequent chapters. In every case the purpose is to separate the Euler-Lagrange equations into time evolution equations and constraints and determine the initial data required for a unique solution of the former. In the process we will find that certain field components are harmonic functions in Minkowski space; they satisfy the Laplace equation $\nabla^2\phi(\vec{x}) = 0$, which is time independent. Any combination of gauge field components that satisfies this equation is simply eliminated because the Laplace equation has no normalizable solutions in flat space $\mathbb{R}^{D-1}$. The relevance of the normalizability criterion can be seen by transforming the Laplace equation to momentum space where it becomes $\vec{k}^2\hat{\phi}(\vec{k}) = 0$. The only smooth solution vanishes identically. A smooth solution is one that contains no δ-function-type terms.

6.3.1 4.1 The abelian gauge field $A_\mu(x)$

We now review the elementary features of gauge invariance for spin 1 . One purpose is to set the stage for spin 3/2 in the next chapter.

6.3.2 4.1.1 Gauge invariance and fields with electric charge

In Chs. 1 and 2 we discussed the global U(1) symmetry of complex scalar and spinor fields. The abelian gauge symmetry of quantum electrodynamics is an extension in which the phase parameter $\theta(x)$ becomes an arbitrary function in Minkowski spacetime. We generalize the previous discussion slightly and assign an electric charge q, an arbitrary real

1 There are gauge invariant free fields for massless particles of any spin (see [18], for example). It appears to be impossible to introduce consistent interactions for any finite subset of these, but remarkably one can make progress for certain infinite sets of fields and for background spacetimes different from Minkowski space [19, 20, 21]. number at this stage, to each complex field in the system. For a Dirac spinor field of charge q, the gauge transformation, a local change of the phase of the complex field, is [2]

$$\Psi(x) \to \Psi'(x) \equiv \mathrm{e}^{\mathrm{i}q\theta(x)}\Psi(x).$$

The goal is to formulate field equations that transform covariantly under the gauge transformation. This requires the introduction of a new field, the vector gauge field or vector potential $A_\mu(x)$, which is defined to transform as

$$A_\mu(x) \to A'_\mu(x) \equiv A_\mu(x) + \partial_\mu\theta(x)$$

One then defines the covariant derivative

$$D_\mu\Psi(x) \equiv (\partial_\mu - \mathrm{i}qA_\mu(x))\,\Psi(x)$$

which transforms with the same phase factor as $\Psi(x)$, namely $D_\mu\Psi(x) \to \mathrm{e}^{\mathrm{i}q\theta(x)}D_\mu\Psi(x)$. The desired field equation is obtained by replacing $\partial_\mu\Psi \to D_\mu\Psi$ in the free Dirac equation (2.16), viz.

$$[\gamma^\mu D_\mu - m]\,\Psi \equiv [\gamma^\mu(\partial_\mu - \mathrm{i}qA_\mu) - m]\,\Psi = 0.$$

This equation is gauge covariant; if $\Psi(x)$ satisfies (4.4) with gauge potential $A_\mu(x)$, then $\Psi'(x)$ satisfies the same equation with gauge potential $A'_\mu(x)$.

The same procedure can be applied to a complex scalar field $\phi(x)$, to which we assign an electric charge q (which may differ from the charge of Ψ). We extend the global U(1) symmetry discussed in Ch. 1 to the local gauge symmetry $\phi(x) \to \phi'(x) = \mathrm{e}^{\mathrm{i}q\theta(x)}\phi(x)$ by defining the covariant derivative $D_\mu\phi = (\partial_\mu - \mathrm{i}qA_\mu)\,\phi$ and modifying the Klein-Gordon equation to the form

$$\left[D^\mu D_\mu - m^2\right]\phi = 0.$$

The procedure of promoting the global U(1) symmetry of the Dirac or Klein-Gordon equation to a local or gauge symmetry through the introduction of the vector potential in the covariant derivative is called the principle of minimal coupling. Another part of standard vocabulary is to say that fields with electric charge, such as ϕ or Ψ, are charged 'matter fields', which are minimally coupled to the gauge field A_μ.

On-shell degrees of freedom = number of helicity states.

Off-shell degrees of freedom = number of field components - gauge transformations.

2 In the notation of Ch. 1, the 'matrix' generator is $t = -\mathrm{i}q$. The U(1) transformation in Sec. 2.7.1 corresponds to the choice $q = 1$.

6.3.3 4.1.2 The free gauge field

It is quite remarkable that the promotion of global to gauge symmetry requires a new field $A_\mu(x)$. In some cases one may wish to consider (4.4) or (4.5) in a fixed external background gauge potential, but it is far more interesting to think of $A_\mu(x)$ as a field that is itself determined dynamically by its coupling to charged matter in a gauge invariant fashion. The resulting theory is quantum electrodynamics, the quantum and Lorentz covariant version of Maxwell's theory of electromagnetism. The predictions of this theory, both classical and quantum, are well confirmed by experiment. There can be no doubt that Nature knows about gauge principles.

Although we expect that readers are familiar with classical electromagnetism, we review the construction because there are similar patterns in Yang-Mills theory, gravity, and supergravity. The first step is the observation that the antisymmetric derivative of the gauge potential, called the field strength

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

is invariant under the gauge transformation, a fact that is trivial to verify. In four dimensions $F_{\mu\nu}$ has six components, which split into the electric $E_i = F_{i0}$ and magnetic $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$ fields.

Since A_μ is a bosonic field, we expect it to satisfy a second order wave equation. The only Lorentz covariant and gauge invariant quantity available is $\partial^\mu F_{\mu\nu}$, so the free electromagnetic field satisfies

$$\partial^\mu F_{\mu\nu} = 0$$

Since $\partial^\nu\partial^\mu F_{\mu\nu}$ vanishes identically, [3] (4.7) comprises $D-1$ independent components in D-dimensional Minkowski spacetime. This is not enough to determine the D components of A_μ, which is not surprising because of the gauge symmetry. One can change $A_\mu \to A_\mu + \partial_\mu\theta$ without affecting $F_{\mu\nu}$. So far, we did not yet use the field equations. Therefore, we will call this number, i.e. $(D-1)$ for the gauge vectors, the number of off-shell degrees of freedom.

One deals with this situation by 'fixing the gauge'. This means that one imposes one condition on the D components of A_μ, which eliminates the freedom to change gauge. Different gauge conditions illuminate different physical features of the theory. We will look first at the condition $\partial^i A_i(\vec{x},t)=0$, which is called the Coulomb gauge condition. Although non-covariant it is a useful gauge to explore the structure of the initial value problem and determine the true degrees of freedom of the system. Note that the time-space split implicit in the initial value problem is also non-covariant.

First let's show that this condition does eliminate the gauge freedom. We check whether there are gauge functions $\theta(x)$ with the property that $\partial^i A_i' = \partial^i\left(A_i+\partial_i\theta\right)=0$ when $\partial^i A_i=0$. This requires that $\nabla^2\theta=0$. As explained above, the only smooth solution is $\theta(x)\equiv 0$, so the gauge freedom has been completely fixed.

3 This is the 'Noether identity', a relation between the field equations that is a consequence of the gauge symmetry.

Let's write out the time $(\mu\to 0)$ and space $(\mu\to i)$ components of the Maxwell equation (4.7). Using (4.6) and lowering all indices with the Minkowski metric, one finds

$$\begin{aligned}\nabla^2 A_0-\partial_0\left(\partial^i A_i\right)&=0,\\ \square A_i-\partial_i\partial^0 A_0-\partial_i\left(\partial^j A_j\right)&=0.\end{aligned}$$

In the Coulomb gauge, the first equation simplifies to $\nabla^2 A_0=0$, and we see that ${}^4A_0$ vanishes. The second equation in (4.8) then becomes $\square A_i=0$, so the spatial components A_i satisfy the massless scalar wave equation.

We can now count the classical degrees of freedom, which are the initial data $A_i(\vec{x},0)$ and $\dot{A}_i(\vec{x},0)$ required for a unique solution of $\square A_i=0$. There is a total of $2(D-2)$ independent degrees of freedom, because the initial data must be constrained to obey the Coulomb gauge condition.

This number thus agrees for $D=4$ with the rule that the classical degrees of freedom are twice the number of on-shell degrees of freedom counted as helicity states. In general, we find for the gauge vectors $(D-1)$ off-shell degrees of freedom and $(D-2)$ on-shell degrees of freedom. These numbers are the dimension of the vector representation of $\mathrm{SO}(D-1)$ off-shell and $\mathrm{SO}(D-2)$ on-shell.

It is instructive to write the solution of $\square A_i=0$ as the Fourier transform

$$A_i(x)=\int\frac{\mathrm{d}^{(D-1)}k}{(2\pi)^{(D-1)}2k^0}\sum_\lambda\left[\mathrm{e}^{\mathrm{i}k\cdot x}\epsilon_i(\vec{k},\lambda)a(\vec{k},\lambda)+\mathrm{e}^{-\mathrm{i}k\cdot x}\epsilon_i^*(\vec{k},\lambda)a^*(\vec{k},\lambda)\right]$$

where $\vec{k}, k^0=|\vec{k}|$, is the on-shell energy-momentum vector. The $\epsilon_i(\vec{k},\lambda)$ are called polarization vectors, which are constrained by the Coulomb gauge condition to satisfy $k^i\epsilon_i(\vec{k},\lambda)=0$. So there are $(D-2)$ independent polarization vectors, indexed by λ, and there are $2(D-2)$ independent real degrees of freedom contained in the complex quantities $a(\vec{k},\lambda)$. As in the case of the plane wave expansions of the free Klein-Gordon and Dirac fields, $a(\vec{k},\lambda)$ and $a^*(\vec{k},\lambda)$ are interpreted as Fourier amplitudes in the classical theory and as annihilation and creation operators for particle states after quantization. There are $D-2$ particle states.

To understand these particle states better, we discuss the case $D=4$ and assume that the spatial momentum is in the 3 -direction, i.e. $\vec{k}=(0,0,k)$ with $k>0$. The two polarization vectors may be taken to be $\epsilon_i((0,0,k),\pm)=(1/\sqrt{2})(1,\pm\mathrm{i},0)$. We formally add the 0 -component $\epsilon_0=0$ to form 4-vectors $\epsilon_\mu((0,0,k),\pm)$, which are eigenvectors of the rotation generator $J_3=-m_{[12]}$, about the 3 -axis (see text above (1.93)), with angular momentum $\lambda=\pm 1$. For general spatial momentum $\vec{k}=k(\sin\beta\cos\alpha,\sin\beta\sin\alpha,\cos\beta)$, we define polarization vectors $\epsilon_\mu(\vec{k},\pm)$

by applying the spatial rotation with Euler angles α, β, which rotates the 3 -axis to the direction of $\vec{k}$. The associated particle states are photons with helicity \pm 1 .

The same ideas determine the properties of particle states of the gauge field for $D \geq 5$. For spatial momentum in the direction $D-1$, i.e. $\vec{k}=(0,0, \ldots, k)$, there are $D-2$ independent polarization vectors. We need not specify these in detail; the important point to

4 When a source current is added to the Maxwell equation (4.7), A_0 no longer vanishes, but it is determined by the source. Thus it is not a degree of freedom of the gauge field system. note is that these vectors are a basis of the vector representation of the Lie group $\mathrm{SO}(D-2)$, which is the group that 'fixes' the vector $\vec{k}$. The associated particle states also transform in this representation. On the other hand it is clear from (4.9) that the Coulomb gauge vector potential transforms in the vector representation of $\mathrm{SO}(D-1)$.

It should be noted that the equations of the free electromagnetic field can be formulated as conditions involving only the field strength components $F_{\mu \nu}$, with the gauge potential A_μ appearing as a derived quantity. In this form of the theory one has the pair of equations

$$
\begin{gathered}
\partial^\mu F_{\mu \nu}=0 \\
\partial_\mu F_{\nu \rho}+\partial_\nu F_{\rho \mu}+\partial_\rho F_{\mu \nu}=0 .
\end{gathered}
$$

The second equation is called the Bianchi identity. It is easy to see that it is automatically satisfied if $F_{\mu \nu}$ is expressed in terms of A_μ as in (4.6). In a topologically trivial spacetime such as Minkowski space, this is the general solution. This is a consequence of the Poincaré lemma in the theory of differential forms discussed in Ch. 7. Although the manifestly gauge invariant formalism (4.10) and (4.11) is available for the free gauge field, the vector potential is required *ab* initio to describe the minimal coupling to charged matter fields.

This chapter has progressed too far without exercises for readers, so we must now try to remedy this deficiency.

Exercise 4.1 Derive from (4.3) that

$$
\left[D_\mu, D_\nu\right] \Psi \equiv\left(D_\mu D_\nu-D_\nu D_\mu\right) \Psi=-\mathrm{i} q F_{\mu \nu} \Psi
$$

Derive from (4.4) that the charged Dirac field also satisfies the second order equation

$$
\left[D^\mu D_\mu-\frac{1}{2} \mathrm{i} q \gamma^{\mu \nu} F_{\mu \nu}-m^2\right] \Psi=0
$$

Exercise 4.2 Using only (4.10) and (4.11), show that the field strength tensor satisfies the equation $\square F_{\mu \nu}=0$. This is a gauge invariant derivation of the fact that the free electromagnetic field describes massless particles.

6.3.4 4.1.3 Sources and Green's function

Let us now discuss sources for the electromagnetic field. Conventionally one takes an electric source that appears only in (4.10), which is modified to read

$$
\partial^\mu F_{\mu \nu}=-J_\nu .
$$

The Bianchi identity (4.11) is unchanged, so that $F_{\mu \nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$. Lorentz covariance requires that the source is a vector, which is called the electric current vector. Since $\partial^\nu \partial^\mu F_{\mu \nu}$ vanishes identically, the current must be conserved. The theory is inconsistent unless the current satisfies $\partial^v J_v=0$. This condition simply reflects the conventional idea that electric charge cannot be created or destroyed. It is also possible to include sources that carry magnetic charge and appear on the right-hand side of (4.11). However, this requires more sophisticated considerations, which we postpone to Sec. 4.2.3, so we will confine our attention to electric sources.

Exercise 4.3 Repeat Ex. 4.2 when there is an electric source. Show that

$$\square F_{v\rho}=-\left(\partial_{\nu} J_{\rho}-\partial_{\rho} J_{v}\right)$$

Consider first the analogous problem of the scalar field coupled to a source $J(x)$:

$$\left(\square-m^{2}\right) \phi(x)=-J(x).$$

The response to the source is determined by the Green's function $G(x-y)$, which satisfies the equation

$$\left(\square-m^{2}\right) G(x-y)=-\delta(x-y).$$

The translation symmetry of Minkowski spacetime implies that the Green's function depends only on the coordinate difference $x^{\mu}-y^{\mu}$ between observation point x^{μ} and source point y^{μ}. Lorentz symmetry implies that it depends only on the invariant quantities $(x-y)^{2}=\eta^{\mu\nu}(x-y)^{\mu}(x-y)^{v}$ and $\operatorname{sgn}\left(x^{0}-y^{0}\right)$. In Euclidean space $\mathcal{R}^{D}$, there is a unique solution of the equation analogous to (4.17), which is damped in the limit of large separation of observation and source points. In Lorentzian signature Minkowski space, there are several choices, which differ in their causal structure, that is in the dependence on $\operatorname{sgn}\left(x^{0}-y^{0}\right)$. Many texts on quantum field theory, such as [22, 23, 9], discuss these choices.

The Euclidean Green's function is simplest and sufficient for the purposes of this book. The solution of (4.17) can be written as the Fourier transform

$$G(x-y)=\int \frac{\mathrm{d}^{D} k}{(2 \pi)^{D}} \frac{\mathrm{e}^{\mathrm{i} k \cdot(x-y)}}{k^{2}+m^{2}}.$$

The integral can be expressed in terms of modified Bessel functions. In the massless case the result simplifies to the power law

$$G(x-y)=\frac{\Gamma\left(\frac{1}{2}(D-2)\right)}{4 \pi^{\frac{1}{2} D}(x-y)^{(D-2)}}.$$

Here $(x-y)^{2}=\delta_{\mu\nu}(x-y)^{\mu}(x-y)^{v}$ is the Euclidean distance between source point y and observation point x. Given $G(x-y)$, the solution of (4.16) can be expressed as the integral

$$\phi(x)=\int \mathrm{d}^{D} y G(x-y) J(y)$$

One may note that the Green's function is formally the inverse of the wave operator, i.e. $G=-\left(\square-m^{2}\right)^{-1}$. In Euclidean space $\square=\nabla^{2}$ which is the D-dimensional Laplacian.

Let's continue in Euclidean space and find the Green's function for the gauge field. We might expect to solve (4.14) using a Green's function $G_{\mu\nu}(x-y)$, which is a tensor. However, we run into the immediate difficulty that there is no solution to the equation

$$\left(\delta^{\mu\rho} \square-\partial^{\mu} \partial^{\rho}\right) G_{\rho v}(x, y)=-\delta_{v}^{\mu} \delta(x-y).$$

The Maxwell wave operator $\delta^{\mu\nu} \square-\partial^{\mu} \partial^{\nu}$ is not invertible since any pure gradient $\partial_{\nu} f(x)$ is a zero mode. This problem is easily resolved. Since the source J_{v} is conserved, we can replace (4.21) by the weaker condition

$$\left(\delta^{\mu\rho} \square-\partial^{\mu} \partial^{\rho}\right) G_{\rho v}(x, y)=-\delta_{v}^{\mu} \delta(x-y)+\frac{\partial}{\partial y^{\nu}} \Omega^{\mu}(x, y)$$

where $\Omega^\mu(x,y)$ is an arbitrary vector function. If $\Omega^\mu(x,y)$ and $J_v(y)$ are suitably damped at large distance, the effect of the second term in (4.22) cancels (after partial integration) in the formula

$$A_\mu(x) = \int \mathrm{d}^D y G_{\mu\nu}(x,y) J^\nu(y)$$

which is the analogue of (4.20).

We now derive the precise form of $G_{\mu\nu}(x,y)$. By Euclidean symmetry, we can assume the tensor form

$$G_{\mu\nu}(x,y) = \delta_{\mu\nu} F(\sigma) + (x-y)_\mu (x-y)_\nu \hat{S}(\sigma)$$

where $\sigma = \frac{1}{2}(x-y)^2$. It is more useful, but equivalent, to take advantage of gauge invariance and rewrite this ansatz as

$$G_{\mu\nu}(x,y) = \delta_{\mu\nu} F(\sigma) + \partial_\mu \partial_\nu S(\sigma)$$

because the pure gauge term involving $S(\sigma)$ has no effect in (4.23) and cancels in (4.22). We may also assume that the gauge term in (4.22) has the Euclidean invariant form $\partial^\mu \partial_\nu \Omega(\sigma)$. Substituting (4.25) in (4.22) we find the two independent tensors δ^μ_ν and $(x-y)^\mu (x-y)_\nu$ and thus two independent differential equations involving F and Ω, namely

$$\begin{aligned} 2\sigma F''(\sigma) + (D-1)F'(\sigma) &= \Omega'(\sigma) \\ F''(\sigma) &= -\Omega''(\sigma) \end{aligned}$$

Note that $F'(\sigma) = \mathrm{d}F(\sigma)/\mathrm{d}\sigma$, etc. We have dropped the δ-function term in (4.22), because we will first solve these equations for $\sigma \neq 0$. The second equation in (4.26) may be integrated immediately, giving $F'(\sigma) = -\Omega'(\sigma)$; a possible integration constant is chosen to vanish, so that $F'(\sigma)$ vanishes at large distance. The first equation then becomes $2\sigma F''(\sigma) + DF'(\sigma) = 0$, which has the power-law solution $F(\sigma) \sim \sigma^{1-\frac{1}{2}D}$. However, on any function of σ, the Laplacian acts as $\Box F(\sigma) = 2\sigma F''(\sigma) + DF'(\sigma)$. In our case there is a hidden δ-function in $\Box F(\sigma)$ because the power law is singular. The effect of the δ-function in (4.22) is automatically incorporated if we take $F(\sigma) = G(x-y)$ where G is the massless scalar Green's function in (4.19). The result of this analysis is the gauge field Green's function

$$G_{\mu\nu}(x,y) = \delta_{\mu\nu} G(x-y) + \partial_\mu \partial_\nu S(\sigma)$$

The gauge function $S(\sigma)$ is arbitrary and may be taken to vanish. Then (4.27) becomes the usual Feynman gauge propagator.

The gauge field propagators found above are usually derived after gauge fixing in the path integral formalism in quantum field theory texts. The derivation here is purely classical, as appropriate since the response of the gauge field to a conserved current source is a purely classical phenomenon.

It may not be obvious why this method works. To see why, apply $\partial/\partial x^\mu$ to both sides of (4.22), obtaining

$$0 = -\partial_\nu \delta(x-y) - \partial_\nu \Box \Omega(\sigma)$$

in which $\partial_\nu = \partial/\partial x^\nu$. This consistency condition is satisfied because the analysis above led us the result $\Omega(\sigma) = -F(\sigma) = -G\left((x-y)^2\right)$.

Exercise 4.4 In $D = 4$ dimensions, consider a point charge at rest, i.e. $J^\mu(x) = \delta^\mu_0 q \delta(\vec{x})$. Obtain, using (4.23), that the resulting value of A^0, and therefore of the electric field, is

$$A^0(x) = \frac{q}{4\pi}\frac{1}{|\vec{x}|}, \quad \vec{E} = \frac{q}{4\pi}\frac{\vec{x}}{|x|^3}$$

6.3.5 4.1.4 Quantum electrodynamics

The current vector J_v in (4.14) may describe a piece of laboratory apparatus, such as a magnetic solenoid. However, we are more interested in the case where the source is the field of a charged elementary particle, such as the Dirac spinor Ψ. This is the theory of quantum electrodynamics, which contains equations that determine both the electromagnetic field A_μ, with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and Ψ. In dealing with coupled fields it is generally best to package the dynamics in a Lorentz invariant action. The equations of motion then emerge as the condition for a critical point of the action functional and are guaranteed to be mutually consistent.

It is also advantageous to change notation from that of Sec. 4.1.1 by scaling the vector potential, $A_\mu \to eA_\mu$, where e is the conventional coupling constant of the electromagnetic field to charged fields; $e^2/4\pi \approx 1/137$ is called the fine structure constant. In this notation the relevant equations of Sec. 4.1 read:

$$\begin{aligned}
F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \\
A_\mu \to A'_\mu &\equiv A_\mu + \frac{1}{e}\partial_\mu \theta \\
D_\mu \Psi &\equiv (\partial_\mu - \mathrm{i}eqA_\mu)\Psi \\
[D_\mu, D_\nu]\Psi &= -\mathrm{i}eqF_{\mu\nu}\Psi.
\end{aligned}$$

The electric charges q of the various charged fields are then simple rational numbers, for example $q = 1$ for the electron. [5]

The action functional for the electromagnetic field interacting with a field of charge q, which we take to be a massive Dirac field, is the sum of two terms, each gauge invariant,

5 It is an interesting question why the electric charges of elementary particles in Nature are quantized; that they appear to be integer multiples of a lowest fundamental charge. Two reasons have been found. The first is that quantum theory requires quantization of electric charge if a magnetic monopole exists. Second, electric charge can emerge as an unbroken U(1) generator of a larger non-abelian gauge theory with spontaneous gauge symmetry breaking. These reasons are not independent since monopoles solutions exist when gauge symmetry is broken with residual U(1) symmetry. See Sec. 17A.1 and [24] for discussion of these ideas.

$$S\left[A_\mu, \bar{\Psi}, \Psi\right] = \int \mathrm{d}^D x \left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \bar{\Psi}\left(\gamma^\mu D_\mu - m\right)\Psi\right]$$

The Euler variation of (4.31) with respect to the gauge potential A_v is

$$\frac{\delta \mathcal{L}}{\delta A^\nu} = \partial^\mu F_{\mu\nu} + \mathrm{i}eq\bar{\Psi}\gamma_\nu\Psi = 0$$

This is equivalent to (4.14) with the electric current a multiple of the Noether current of the global U(1) phase symmetry discussed in Sec. 2.7.1. It is a typical feature of the various fundamental gauge symmetries in physics that the Noether current of a system with global symmetry becomes the source for the gauge field introduced when the symmetry is gauged. The Euler variation with respect to $\bar{\Psi}$ gives the gauge covariant Dirac equation (4.4), with $D_\mu\Psi$ given in (4.30).

6.3.6 4.1.5 The stress tensor and gauge covariant translations

The situation of the stress tensor of this system is quite curious. The canonical stress tensor, calculated from the Noether formula (1.67) with $\Delta_A \phi^i \to \partial_\nu A_\rho, \partial_\nu \bar{\Psi}, \partial_\nu \Psi$ for the three independent fields, is

$$T^\mu{}_\nu = F^{\mu\rho}\partial_\nu A_\rho + \bar{\Psi}\gamma^\mu \partial_\nu \Psi + \delta^\mu_v \mathcal{L}$$

It is conserved on the index μ, but not on v, not symmetric and not gauge invariant. The situation can be improved by treating fermion terms as in Sec. 2.7.2 and then adding $\Delta T^\mu{}_\nu = -\partial_\rho \left(F^{\mu\rho} A_v\right)$ in accord with the discussion in Sec. 1.3. The final result is the gauge invariant symmetric stress tensor

$$\Theta_{\mu\nu} = F_{\mu\rho} F_\nu{}^\rho + \frac{1}{4}\bar{\Psi}\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\Psi + \eta_{\mu\nu}\mathcal{L}$$

Exercise 4.5 Consider the gauge covariant translation, defined by $\delta A_\mu = a^\nu F_{v\mu}$ and $\delta\Psi = a^v D_v \Psi$. Show that they differ from a conventional translation by a gauge transformation with gauge dependent parameter $\theta = -e a^v A_v$. Gauge covariant translations are a symmetry of the action (4.31). What is the Noether current for this symmetry? How is it related to the stress tensor (4.34)?

6.3.7 4.2 Electromagnetic duality

The subject of electromagnetic duality has several interesting applications in supergravity theories. For example, the symmetry group of black hole solutions of matter-coupled supergravity theories generally contains duality transformations. We recommend that all readers study Secs. 4.2.1 and 4.2.2. However, because the applications of duality are somewhat advanced, the rest of the section can be omitted in the first reading of the book.

6.3.8 4.2.1 Dual tensors

We begin by discussing the duality property of second rank antisymmetric tensors $H_{\mu\nu}$ in four-dimensional Minkowski spacetime. We use the Levi-Civita tensor introduced in Sec. 3.1.3 to define the dual tensor

$$\tilde{H}^{\mu\nu} \equiv -\frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma} H_{\rho\sigma}$$

In our conventions the dual tensor is imaginary. The indices of $\tilde{H}$ can be raised and lowered with the Minkowski [6] metric $\eta_{\mu\nu}$. It is also useful to define the linear combinations

$$H^\pm_{\mu\nu} = \frac{1}{2}\left(H_{\mu\nu} \pm \tilde{H}_{\mu\nu}\right), \quad H^\pm_{\mu\nu} = \left(H^\mp_{\mu\nu}\right)^*$$

Exercise 4.6 Prove that the dual of the dual is the identity, specifically that

$$-\frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\tilde{H}_{\rho\sigma} = H^{\mu\nu}$$

You will need (3.9). The validity of this property is the reason for the i in the definition (4.35).

Show that $H^+_{\mu\nu}$and $H^-_{\mu\nu}$are, respectively, self-dual and anti-self-dual, i.e.

$$-\frac{1}{2}\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma} H^\pm_{\rho\sigma} = \pm H^\pm_{\mu\nu}$$

Let $G_{\mu\nu}$ be another antisymmetric tensor with $G^{\pm}_{\mu\nu}$defined as in (4.36). Prove the following relations (where $(\mu\nu)$ means symmetrization between the indices):

$$G^{+\mu\nu}H^{-}_{\mu\nu} = 0, \quad G^{\pm\rho(\mu}H^{\pm\nu)}_{\rho} = -\frac{1}{4}\eta^{\mu\nu}G^{\pm\rho\sigma}H^{\pm}_{\rho\sigma}, \quad G^{+}_{\rho[\mu}H^{-}_{\nu]}\rho$$

Hint: you could first prove

$$\tilde{G}^{\rho\mu}\tilde{H}^{\nu}_{\rho} = -\frac{1}{2}\eta^{\mu\nu}G^{\rho\sigma}H_{\rho\sigma} - G^{\rho v}H^{\mu}_{\rho}.$$

Exercise 4.7 The duality operation can also be applied to matrices of the Clifford algebra. Define the quantity $L_{\mu\nu} = \gamma_{\mu\nu}P_L$. Show that this is anti-self-dual. Hint: check first that $\gamma_{\mu\nu}\gamma_* = \frac{1}{2}\mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}$.

6.3.9 4.2.2 Duality for one free electromagnetic field

Duality operates as an interesting symmetry of field theories containing one or more abelian gauge fields which may interact with other fields, principally scalars. In this section we discuss the simplest case, namely a single free gauge field. First note that, after contraction with the ε-tensor, the Bianchi identity (4.11) can be expressed as $\partial_\mu \tilde{F}^{\mu\nu} = 0$.

6 The definition (4.35) is valid in Minkowski space, but must be modified in curved spacetimes as we will discuss in Ch. 7.

So we can temporarily ignore the vector potential and regard $F_{\mu\nu}$ as the basic field variable which must satisfy both the Maxwell and Bianchi equations:

$$\partial_\mu F^{\mu\nu} = 0, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0.$$

We can now consider the change of variables (the i is included to make the transformation real):

$$F^{\mu\nu} \to F'^{\mu\nu} = \mathrm{i}\tilde{F}^{\mu\nu}$$

Since $F'^{\mu\nu}$ also obeys both equations of (4.41) we have defined a symmetry of the free electromagnetic field.

Exercise 4.8 Show that the symmetry (4.42) exchanges the electric and magnetic fields: $E_i \to E'_i = -B_i$ and $B_i \to B'_i = E_i$.

It is not possible to extend the symmetry to the vector potentials $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and $F'_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$ because A_μ and A'_μ are not related by any local transformation.

Here are some basic exercises involving the duality transform of the field strength tensor $F_{\mu\nu}$.

Exercise 4.9 Show that the self-dual combinations $F^{\pm}_{\mu\nu}$contain only photons of one polarization in their plane wave expansions:

$$F^{\pm}_{\mu\nu} = 2\mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\left[\mathrm{e}^{\mathrm{i}k\cdot x}k_{[\mu}\epsilon_{\nu]}(\vec{k},\pm)a(\vec{k},\pm) - \mathrm{e}^{-\mathrm{i}k\cdot x}k_{[\mu}\epsilon^*_{\nu]}(\vec{k},\mp)a^*(\vec{k},\mp)\right]$$

To perform this exercise, check first that with the polarization vectors given in Sec. 4.1.2, one has

$$-\frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}k_\rho\epsilon_\sigma(\vec{k},\pm) = \pm k^{[\mu}\epsilon^{\nu]}(\vec{k},\pm).$$

Exercise 4.10 Show that the quantity $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total derivative, i.e.

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -\mathrm{i}\partial_\mu\left(\varepsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}\right)$$

Show, using (1.45), that under a Lorentz transformation

$$\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)(x) \to \det\Lambda^{-1}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)(\Lambda x)$$

Thus $F_{\mu\nu}\tilde{F}^{\mu\nu}$ transforms as a scalar under proper Lorentz transformations but changes sign under space or time reflections. Use the Schouten identity (3.11) to prove that

$$F_{\mu\rho}\tilde{F}^{\rho}_{\nu} = \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}\tilde{F}^{\rho\sigma}$$

6.3.10 4.2.3 Duality for gauge field and complex scalar

The simplest case of electromagnetic duality in an interacting field theory occurs with one abelian gauge field $A_\mu(x)$ and a complex scalar field $Z(x)$. The electromagnetic part of the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(\operatorname{Im} Z)F_{\mu\nu}F^{\mu\nu} - \frac{1}{8}(\operatorname{Re} Z)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}.$$

Actions in which the gauge field kinetic term is multiplied by a function of complex scalar fields are quite common in supersymmetry and supergravity. We now define an extension of the duality transformation (4.42) which gives a non-abelian global SL(2, $\mathbb{R}$) symmetry of the gauge field equations of this theory. In Sec. 7.12.2 we will discuss a generalized scalar kinetic term that is invariant under $\mathrm{SL}(2,\mathbb{R})$. The field $Z(x)$ carries dynamics, and the equations of motion of the combined vector and scalar theory are also invariant.

The gauge Bianchi identity and equation of motion of our theory are

$$\partial_\mu\tilde{F}^{\mu\nu} = 0, \quad \partial_\mu\left[(\operatorname{Im} Z)F^{\mu\nu} + \mathrm{i}(\operatorname{Re} Z)\tilde{F}^{\mu\nu}\right] = 0.$$

It is convenient to define the real tensor

$$G^{\mu\nu} \equiv \varepsilon^{\mu\nu\rho\sigma}\frac{\delta S}{\delta F^{\rho\sigma}} = -\mathrm{i}(\operatorname{Im} Z)\tilde{F}^{\mu\nu} + (\operatorname{Re} Z)F^{\mu\nu}$$

and to consider the self-dual combinations $F^{\mu\nu\pm}$ and $G^{\mu\nu\pm}$. Note that these are related by

$$G^{\mu\nu-} = ZF^{\mu\nu-}, \quad G^{\mu\nu+} = \bar{Z}F^{\mu\nu+}.$$

The information in (4.49) can then be reexpressed as

$$\partial_\mu \operatorname{Im} F^{\mu\nu-} = 0, \quad \partial_\mu \operatorname{Im} G^{\mu\nu-} = 0$$

We define a matrix of the group $\mathrm{SL}(2,\mathbb{R})$ by

$$\mathcal{S} \equiv \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \quad ad - bc = 1$$

The group $\mathrm{SL}(2,\mathbb{R})$ acts on the tensors $F^-$and G^-as follows:

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S}\begin{pmatrix} F^- \\ G^- \end{pmatrix}$$

Since $\mathcal{S}$ is real, the conjugate tensors $F^+$and G^+also transform in the same way.

Exercise 4.11 Assume that $\operatorname{Im} F^-$and $\operatorname{Im} G^-$satisfy (4.52), and show that $\operatorname{Im} F'^-$ and $\operatorname{Im} G'^-$ also obey the same equations. Show that G'^- and a transformed scalar Z' satisfy $G'^{\mu\nu-} = Z'F'^{\mu\nu-}$, if Z' is defined as the following nonlinear transform of Z :

$$Z' = \frac{aZ+b}{cZ+d}$$

The two equations (4.54) and (4.55) specify the $\mathrm{SL}(2,\mathbb{R})$ duality transformation on the field strength and complex scalar of our system. The exercise shows that the Bianchi identity and generalized Maxwell equations are duality invariant. In general the duality transform is not a symmetry of the Lagrangian or the action integral. The following exercise illustrates this.

Exercise 4.12 Show that the Lagrangian (4.48) can be rewritten as

$$\mathcal{L}(F,Z) = -\frac{1}{2}\operatorname{Im}\left(ZF^{-}_{\mu\nu}F^{\mu\nu-}\right).$$

Consider the $\mathrm{SL}(2,\mathbb{R})$ transformation with parameters $a=d=1$ and $b=0$. Show that

$$\mathcal{L}\left(F',Z'\right) = -\frac{1}{2}\operatorname{Im}\left(Z(1+cZ)F^{-}_{\mu\nu}F^{\mu\nu-}\right) \neq \mathcal{L}(F,Z)$$

The symmetric gauge invariant stress tensor of this theory is

$$\Theta^{\mu\nu} = (\operatorname{Im} Z)\left(F^{\mu\rho}F^{\nu}_{\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$$

As we will see in Ch. 8, when the theory is coupled to gravity, it is this stress tensor that is the source of the gravitational field; see (8.4). It is then important that $\operatorname{Im} Z$ is positive, which restricts the domain of Z to the upper half-plane. It is also important that the stress tensor is invariant under the duality transformations (4.54) and (4.55). This is the reason for the duality symmetry of many black hole solutions of supergravity,

Exercise 4.13 Prove that the energy-momentum tensor (4.58) is invariant under duality. Here are some helpful relations which you will need:

$$\operatorname{Im} Z' = \frac{\operatorname{Im} Z}{(cZ+d)(c\bar{Z}+d)}$$

Further you need again (4.47) and a similar identity (proven by contracting ε-tensors)

$$\tilde{F}_{\mu\rho}\tilde{F}^{\rho}_{\nu} = -F_{\mu\rho}F_{\nu}{}^{\rho} + \frac{1}{2}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}.$$

This leads to

$$F'_{\mu\rho}F'^{\rho}_{\nu} - \frac{1}{4}\eta_{\mu\nu}F'_{\rho\sigma}F'^{\rho\sigma} = |cZ+d|^2\left[F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right].$$

When the $\mathrm{SL}(2,\mathbb{R})$ duality transformation appears in supergravity, there is also a scalar kinetic term in the Lagrangian which is invariant under the symmetry, specifically under the transformation (4.55). The prototype Lagrangian with this symmetry is the nonlinear σ-model whose target space is the Poincaré plane. This model and its $\mathrm{SL}(2,\mathbb{R})$ symmetry group will be discussed in Sec. 7.12; see (7.151) and (7.152). The Poincaré plane is the upper half-plane $\operatorname{Im} Z > 0$. The relation (4.59) shows that duality transformations map the upper half-plane into itself. The positive sign is preserved by $\mathrm{SL}(2,\mathbb{R})$ transformations and the energy density obtained from the stress tensor Θ^{00} above will be positive!

Exercise 4.14 The free Maxwell theory is the special case of (4.48) with fixed $Z=\mathrm{i}$. Suppose that the gauge field is coupled to a conserved current as in (4.14). Check that the electric charge can be expressed in terms of F or G by

$$q \equiv \int \mathrm{d}^3\vec{x}J^0 = \int \mathrm{d}^3\vec{x}\partial_i F^{0i} = -\frac{1}{2}\int \mathrm{d}^3\vec{x}\varepsilon^{ijk}\partial_i G_{jk}$$

A magnetic charge can be introduced in Maxwell theory as the divergence of $\vec{B}$ (recall $E^i = F^{0i}$ and $B^i = \frac{1}{2}\varepsilon^{ijk}F_{jk}$). This leads to a definition [7]

$$p \equiv -\frac{1}{2}\int \mathrm{d}^3\vec{x}\varepsilon^{ijk}\partial_i F_{jk}$$

Show that $\begin{pmatrix} p \\ q \end{pmatrix}$ is a vector that transforms under $\mathrm{SL}(2,\mathbb{R})$ in the same way as the tensors $F^-$and G^-in (4.54).

In many applications of electromagnetic duality, magnetic and electric charges appear as sources for the Bianchi 'identity' and generalized Maxwell equations of (4.49). As exemplified in Ex. 4.14 this leads to an $\mathrm{SL}(2,\mathbb{R})$ vector of charges. Particles that carry both electric and magnetic charge are called dyons. In quantum mechanics, dyon charges must obey the Schwinger-Zwanziger quantization condition. If a theory contains two dyons with charges (p_1,q_1) and (p_2,q_2), these charges must satisfy $p_1q_2 - p_2q_1 = 2\pi n$, where n is an integer. [8] This condition is invariant under $\mathrm{SL}(2,\mathbb{R})$ transformations of the charges. However, one can show [25] that there is a lowest non-zero value of the electric charge and that all allowed charges are restricted to an infinite discrete set of points called the charge lattice. The allowed SL(2) transformations must take one lattice point to another, and this restricts the group parameters in (4.53) to be integers. This restriction defines the subgroup $\mathrm{SL}(2,\mathbb{Z})$, often called the modular group. [9] One can show that this subgroup is generated by the following choices of $\mathcal{S}$:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$Z' = Z + 1, \quad Z' = -\frac{1}{Z}.$$

This means that one can express any element of $\mathrm{SL}(2,\mathbb{Z})$ as the product of (finitely many) factors of the two generators above and their inverses.

Exercise 4.15 In (4.48), the kinetic terms of the electromagnetic fields are determined by a variable Z that was treated as a scalar field. Z can also be replaced by a coupling constant, and typically one takes Z to be the imaginary number [10]i/g^2, where g is a coupling constant. Observe that the first transformation of (4.64) does not preserve the restriction that Z is imaginary. However, the second one does. Prove that this transformation is of the type (4.42), interchanging the electric and magnetic fields. It transforms g to its inverse, and thus relates the strong and weak coupling descriptions of the theory. In

7 In order to obtain a symplectic vector (p,q) and not $(-p,q)$, we changed the sign of the magnetic charge with respect to some classical works. This implies that we have $\vec{\nabla}\cdot\vec{B} = -j^0_{\mathrm{m}}$, where j^0_{m} is the magnetic charge density.

8 For the case $(p_1,q_1) = (p,0)$ and $(p_2,q_2) = (0,q)$, this reduces to condition $pq = 2\pi n$ found by Dirac in 1933.

9 The modular group generated by the matrices (4.64) is in fact PSL(2, $\mathbb{Z}$). In PSL(2, $\mathbb{Z}$), the elements M and $-M$ of $\mathrm{SL}(2,\mathbb{Z})$ are identified. Both these elements give in fact the same transformation $Z'(Z)$.

10 One often adds an extra term that is a real so-called θ-parameter, but we will omit this here.

Secs. 4.1 and 4.2.2 we considered $Z = \mathrm{i}g = \mathrm{i}$. Check that general duality transformations in this case are of the form

$$F'^-_{\mu\nu} = (d + \mathrm{i}c)F^-_{\mu\nu}, \qquad \text{i.e.} \qquad F'_{\mu\nu} = dF_{\mu\nu} - \mathrm{i}c\tilde{F}_{\mu\nu}.$$

6.3.11 4.2.4 Electromagnetic duality for coupled Maxwell fields

In this section we explore how the duality symmetry is extended to systems containing a set of abelian gauge fields $A_\mu^A(x)$, indexed by $A = 1, 2, \ldots, m$ together with scalar fields ϕ^i. Scalars enter the theory through complex functions $f_{AB}(\phi) = f_{BA}(\phi)$. We consider the action

$$S = \int \mathrm{d}^4 x \mathcal{L}, \quad \mathcal{L} = -\frac{1}{4}\left(\operatorname{Re} f_{AB}\right) F_{\mu\nu}^A F^{\mu\nu B} + \frac{1}{4}\mathrm{i}\left(\operatorname{Im} f_{AB}\right) F_{\mu\nu}^A \tilde{F}^{\mu\nu B}$$

which is real since $\tilde{F}^{\mu\nu}$ is pure imaginary, as defined in (4.35). The first term is a generalized kinetic Lagrangian for the gauge fields, so we usually require that $\operatorname{Re} f_{AB}$ is a positive definite matrix. This ensures that gauge field kinetic energies are positive. Although $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total derivative, the second term does contribute to the equations of motion when $\operatorname{Im} f_{AB}$ is a function of the scalars ϕ^i. Our discussion will not involve the scalars directly. However, as in Sec. 4.2.3, additional terms to specify the scalar dynamics will appear when theories of this type are encountered in extended $D = 4$ supergravity. The treatment that follows is modeled on Sec. 4.2 .3 (where f_{AB} was taken to be $-\mathrm{i}Z$).

Using the self-dual tensors of (4.36), we then rewrite the Lagrangian (4.66) as

$$\begin{aligned}\mathcal{L}\left(F^+, F^-\right) &= -\frac{1}{2}\operatorname{Re}\left(f_{AB} F_{\mu\nu}^{-A} F^{\mu\nu - B}\right)\\ &= -\frac{1}{4}\left(f_{AB} F_{\mu\nu}^{-A} F^{\mu\nu - B} + f_{AB}^* F_{\mu\nu}^{+A} F^{\mu\nu + B}\right),\end{aligned}$$

and define the new tensors

$$\begin{aligned}G_A^{\mu\nu} &= \varepsilon^{\mu\nu\rho\sigma}\frac{\delta S}{\delta F^{\rho\sigma A}} = -\left(\operatorname{Im} f_{AB}\right) F^{\mu\nu B} - \mathrm{i}\left(\operatorname{Re} f_{AB}\right)\tilde{F}^{\mu\nu B} = G_A^{\mu\nu +} + G_A^{\mu\nu -},\\ G_A^{\mu\nu -} &= -2\mathrm{i}\frac{\delta S\left(F^+, F^-\right)}{\delta F_{\mu\nu}^{-A}} = \mathrm{i} f_{AB} F^{\mu\nu - B},\\ G_A^{\mu\nu +} &= 2\mathrm{i}\frac{\delta S\left(F^+, F^-\right)}{\delta F_{\mu\nu}^{+A}} = -\mathrm{i} f_{AB}^* F^{\mu\nu + B}.\end{aligned}$$

Since the field equation for the action containing (4.67) is

$$0 = \frac{\delta S}{\delta A_v^A} = -2\partial_\mu \frac{\delta S}{\delta F_{\mu\nu}^A}$$

the Bianchi identity and the equation of motion can be expressed in the concise form

$$\begin{aligned}\partial^\mu \operatorname{Im} F_{\mu\nu}^{A-} &= 0 && \text{Bianchi identities}\\ \partial_\mu \operatorname{Im} G_A^{\mu\nu -} &= 0 && \text{equations of motion.}\end{aligned}$$

(The same equations hold for $\operatorname{Im} F^{A+}$ and $\operatorname{Im} G_A{}^+$.)

Duality transformations are linear transformations of the $2m$ tensors $F^{A\mu\nu}$ and $G_A^{\mu\nu}$ (accompanied by transformations of the f_{AB}) which mix Bianchi identities and equations of motion, but preserve the structure that led to (4.70). Since the equations (4.70) are real, we can mix them by a real $2m \times 2m$ matrix. We extend these transformations to the (anti-)self-dual tensors, and consider

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S}\begin{pmatrix} F^- \\ G^- \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F^- \\ G^- \end{pmatrix}$$

with real $m \times m$ submatrices A, B, C, D. Owing to the reality of these matrices, the same relations hold for the self-dual tensors $F^+$and G^+. In Sec. 4.2.3, these matrices were just numbers:

$$A = d, \quad B = c, \quad C = b, \quad D = a.$$

We require that the transformed field tensors F'^A and G'_A are also related by the definitions (4.68), with appropriately transformed f_{AB}. We work out this requirement in the following steps:

$$G'^- = (C + \mathrm{i}Df)F^- = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1}F^-,$$

such that we conclude that

$$\mathrm{i}f' = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1}$$

The last equation gives the symmetry transformation relating f'_{AB} to f_{AB}. If $G'^-_{\mu\nu}$ is to be the variational derivative of a transformed action, as (4.68) requires, then the matrix f' must be symmetric. For a generic [11] symmetric f, this requires that the matrices A, B, C, D satisfy

$$A^T C - C^T A = 0, \quad B^T D - D^T B = 0, \quad A^T D - C^T B = \mathbb{1}.$$

These relations among A, B, C, D are the defining conditions of a matrix of the symplectic group in dimension $2m$ so we reach the conclusion that

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2m, \mathbb{R})$$

The conditions (4.75) may be summarized as

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

11 If the initial f_{AB} is non-generic, then the matrix $\mathbb{1}$ in the last equation can be replaced by any matrix which commutes with f_{AB}. For generic f_{AB}, this must be a constant multiple of the unit matrix. The constant, which should be positive to preserve the sign of the kinetic energy of the vectors, can be absorbed by rescaling the matrices A, B, C, D.

The duality transformations in four dimensions are transformations in the symplectic group $\mathrm{Sp}(2m, \mathbb{R})$.

The matrix Ω is often called the symplectic metric, and the transformations (4.71) are then called symplectic transformations. This is the main result originally derived in [26]. Duality transformations in four spacetime dimensions are transformations of the group $\mathrm{Sp}(2m, \mathbb{R})$, which is a non-compact group.

Exercise 4.16 The dimension of the group $\mathrm{Sp}(2m, \mathbb{R})$ is the number of elements of the matrix $\mathcal{S}$, namely $4m^2$ minus the number of independent conditions contained in (4.77). Show that the dimension is $m(2m + 1)$.

Duality transformations have two types of applications: they can describe symmetries of one theory and they can describe transformations from one theory to another. In the first case, the symmetries concerned form a subgroup of the 'maximal' duality group $\mathrm{Sp}(2m, \mathbb{R})$ discussed above. The subgroup consists of transformations (4.74) of $f_{AB}\,(\phi^i)$ induced by the symmetry transformations of the elementary scalars ϕ^i. These scalar transformations must be symmetries of the scalar kinetic term and other parts of the Lagrangian. The model of Sec. 4.2.3 is one example. The transformation of Z defined in (4.55) is the standard $\mathrm{SL}(2, \mathbb{R})$ symmetry of the Poincaré plane. This could be part of the full symmetry group of all the scalar fields of the theory. In extended supergravities it turns out that all the symmetry transformations that act on the scalars appear also as transformations of the vector kinetic matrix. Hence, the symmetry group is then a subgroup of the 'maximal' group $\mathrm{Sp}(2m, \mathbb{R})$ discussed above.

However, another application is of the type that we encountered in Ex. 4.15. In that case constants that specify the theory under consideration change under the duality transformations. The constants that transform are sometimes called 'spurionic quantities'. The transformations thus relate two different theories. Solutions of one theory are mapped into solutions of the other one. This is the basic idea of dualities in M-theory.

Symplectic transformations always transform solutions of (4.70) into other solutions. However, they are not always invariances of the action. Indeed, writing

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Re}\left(f_{AB}F^{-A}_{\mu\nu}F^{\mu\nu-B}\right) = -\frac{1}{2}\,\mathrm{Im}\left(F^{-A}_{\mu\nu}G^{\mu\nu-}_{A}\right)$$

we obtain

$$\mathrm{Im}\,F'^{-}G'^{-} = \mathrm{Im}\left(F^{-}G^{-}\right) + \mathrm{Im}\left[2F^{-}\left(C^{T}B\right)G^{-} + F^{-}\left(C^{T}A\right)F^{-} + G^{-}\left(D^{T}B\right)G^{-}\right].$$

If $C \neq 0, B = 0$ the Lagrangian is invariant up to a 4-divergence, since $\mathrm{Im}\,F^{-}F^{-} = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ and the matrices A and C are real constants. For $B \neq 0$ neither the Lagrangian nor the action is invariant.

Electromagnetic duality has important applications to black hole solutions of extended supergravity theories. Supergravity is also very relevant to the analysis of black hole solutions of string theory. Many black holes are dyons; they carry both magnetic and electric charges for the gauge fields of the system. The general situation is a generalization of what was discussed at the end of Sec. 4.2.3. The charges form a symplectic vector $\begin{pmatrix} q^{A}_{\mathrm{m}} \\ q_{\mathrm{e}A} \end{pmatrix}$ which must transform as in (4.71). The Dirac-Schwinger-Zwanziger quantization condition restricts these charges to a lattice. Invariance of this lattice restricts the symplectic transformations of (4.71) to a discrete subgroup $\mathrm{Sp}(2m, \mathbb{Z})$, which is analogous to the $\mathrm{SL}(2, \mathbb{Z})$ group discussed previously.

Finally, we comment that symplectic transformations with $B \neq 0$ should be considered as non-perturbative for the following reasons. A system with no magnetic charges as in classical electromagnetism is transformed to a system with magnetic charges. The elements of f_{AB} may be regarded as coupling constants (see Ex. 4.15), and a system with weak coupling is transformed to one with strong coupling. A duality transformation which mixes electric and magnetic fields cannot be realized by transformation of the vector potential A_μ. One would need a 'magnetic' partner of A_μ to reexpress the $F'_{\mu\nu}$ and $G'_{\mu\nu}$ in terms of potentials.

The important properties of the matrix f_{AB} are that it is symmetric and that $\mathrm{Re}\,f_{AB}$ define a positive definite quadratic form in order to have positive gauge field energy. These properties are preserved under symplectic transformations defined by (4.74).

6.3.12 4.3 Non-abelian gauge symmetry

Yang-Mills theory is based on a non-abelian generalization of the U(1) gauge symmetry. It is the fundamental idea underlying the standard model of elementary particle interactions. We follow the pattern of Sec. 4.1.1, starting with the global symmetry and then gauging it. The focus of our discussion is the derivation of the basic formulas of the classical gauge theory. Readers may need more information on the underlying geometric ideas and the structure and stunning applications of the quantized theory. They are referred to a modern text in quantum field theory. [12]

6.3.13 4.3.1 Global internal symmetry

Suppose that G is a compact simple Lie group of dimension$^{\dim_G}$. Closely associated with the group is its Lie algebra, denoted by $\mathfrak{g}$, which is a real algebra of dimension$_{\dim_G}$. The theory of Lie algebras and Lie groups is an important subject of mathematics with many applications to physics. With some oversimplification we review only the most essential features required by Yang-Mills theory for compact simple groups.

Each compact simple Lie algebra has an infinite number of inequivalent finitedimensional irreducible representations R of dimension $\dim_R$. In each representation, there is a basis of matrix generators $t_A, A=1,\ldots,\dim_G$, which are anti-hermitian in the case of a compact gauge group. The commutator of the generators determines the local geometrical structure of the group:

12 See, for example, Ch. 15 of [9]. This text also reviews aspects of group theory needed in physical applications.

$$[t_A, t_B] = f_{AB}{}^C t_C$$

The array of real numbers $f_{AB}{}^C$ are structure constants of the algebra (the same in all representations). They obey the Jacobi identity

$$f_{AD}^E f_{BC}{}^D + f_{BD}{}^E f_{CA}^D + f_{CD}^E f_{AB}^D = 0.$$

The indices can be lowered by the Cartan-Killing metric defined in Appendix B (see (B.6)), and then the f_{ABC} are totally antisymmetric. For simple algebras, the generators can be chosen to be trace orthogonal, $\operatorname{Tr}(t_A t_B) = -c\delta_{AB}$, with c positive for compact groups, and the Cartan-Killing metric is then proportional to this expression.

One important representation is the adjoint representation of dimension$_{\dim} = \dim_G$, in which the representation matrices are closely related to the structure constants by $(t_A)^D{}_E = f_{AE}{}^D$. Note that the labels DE denote row and column indices of the matrix t_A. The adjoint representation is a real representation; the representation matrices are real and antisymmetric for compact algebras. For complex representations we will use the notation $(t_A)^\alpha{}_\beta$. Anti-hermiticity then requires $(t_A^*)^\alpha{}_\beta = -(t_A)^\beta$. The row and column indices will often be suppressed when no ambiguity arises.

Exercise 4.17 Use (4.81) to show that the matrices $(t_A)^D{}_E = f_{AE}{}^D$ satisfy (4.80) and therefore give a representation.

The general element of $\mathfrak{g}$ is represented by a superposition of generators $\theta^A t_A$ where the θ^A are $\dim_G$ real parameters. The relation between G and $\mathfrak{g}$ is given by exponentiation, namely $\mathrm{e}^{-\theta^A t_A}$ is an element of G in the representation R.

A theory with global non-abelian internal symmetry contains scalar and spinor fields, each of which transforms in an irreducible representation R. For example, there may be a Dirac spinor [13] field $\Psi^\alpha(x), \alpha=1,\ldots,\dim_R$, that transforms in the complex representation R as

$$\Psi^\alpha(x) \to \left(\mathrm{e}^{-\theta^A t_A}\right)^\alpha{}_\beta \Psi^\beta(x).$$

The conjugate spinor [14] is denoted by $\bar{\Psi}_\alpha$ and transforms as

$$\bar{\Psi}_\alpha \to \bar{\Psi}_\beta \left(\mathrm{e}^{\theta^A t_A}\right)^\beta{}_\alpha$$

For most of our discussion it is sufficient to restrict attention to the infinitesimal transformations,

$$\begin{aligned}
\delta\Psi &= -\theta^A t_A \Psi \\
\delta\bar{\Psi} &= \bar{\Psi}\theta^A t_A, \\
\delta\phi^A &= \theta^C f_{BC}{}^A \phi^B.
\end{aligned}$$

13 Note that we use here indices $\alpha, \ldots$ for the representation of the gauge group. They should not be confused with spinor indices, which we usually omit.

14 The Dirac conjugate (2.30) is used here rather than the Majorana conjugate (3.50).

The first two relations are just the terms of (4.82) and (4.83) that are first order in θ^A. The last relation is the infinitesimal transformation of a field in the adjoint representation, taken here as the set of $\dim_G$ real scalars ϕ^A. Of course, scalars could be assigned to any representation R.

Actions, such as the kinetic action for massive fermion fields,

$$S[\bar{\Psi}, \Psi] = -\int \mathrm{d}^D x \bar{\Psi} \left[\gamma^\mu \partial_\mu - m\right] \Psi$$

are required to be invariant under (4.82).

Exercise 4.18 Show that (4.85) is invariant under the transformation (4.82) and (4.83). Consider an infinitesimal transformation and derive the conserved current

$$J_{A\mu} = -\bar{\Psi} t_A \gamma_\mu \Psi, \quad A = 1, \ldots, \dim_G$$

Show that the current transforms as a field in the adjoint representation, i.e.

$$\delta J_{A\mu} = \theta^C f_{CA}^B J_{B\mu}.$$

Show that $\delta\left(\phi^A J_{A\mu}\right) = 0$.

6.3.14 4.3.2 Gauging the symmetry

In gauged non-abelian internal symmetry, the group parameter $\theta^A(x)$ is promoted to an arbitrary function of x^μ. The first step in the systematic formulation of gauge invariant field equations is to introduce the gauge potentials, namely a set of vectors $A_\mu^A(x)$ whose infinitesimal transformation rule is

$$\delta A_\mu^A(x) = \frac{1}{g} \partial_\mu \theta^A + \theta^C(x) A_\mu^B(x) f_{BC}^A$$

The first term is the gradient term similar to that for the abelian gauge field in (4.2), and the second is exactly the transformation of a field in the adjoint representation, as one can see from the third equation in (4.84). The constant g is the Yang-Mills coupling, which replaces the electromagnetic coupling e of Sec. 4.1.4.

Following the pattern of Sec. 4.1.1, we next define the covariant derivative of a field in the representation R with matrix generators t_A. For the fields Ψ^α, $\bar{\Psi}_\alpha$, and ϕ^A of (4.84) we write

$$\begin{aligned} D_\mu \Psi &= \left(\partial_\mu + g t_A A_\mu^A\right) \Psi, \\ D_\mu \bar{\Psi} &= \partial_\mu \bar{\Psi} - g \bar{\Psi} t_A A_\mu^A, \\ D_\mu \phi^A &= \partial_\mu \phi^A + g f_{BC}{}^A A_\mu^B \phi^C. \end{aligned}$$

Note that the gauge transformation (4.88) can be written as $\delta A_\mu^A(x) = (1/g) D_\mu \theta^A$ using the covariant derivative for the adjoint representation.

Exercise 4.19 Show that the covariant derivatives of the three fields in (4.89) transform in the same way as the fields themselves, and with no derivatives of the gauge parameters. For example $\delta D_\mu \Psi = -\theta^A t_A D_\mu \Psi$.

Given this result it is easy to see that any globally symmetric action for scalar and spinor matter fields becomes gauge invariant if one replaces $\partial_\mu \rightarrow D_\mu$ for all fields. If this is done in (4.85), one obtains the equation of motion

$$\frac{\delta S}{\delta \bar{\Psi}_\alpha} = -\left[\gamma^\mu D_\mu - m\right] \Psi^\alpha = 0$$

6.3.15 4.3.3 Yang-Mills field strength and action

The next step in the development is to define the quantities that determine the dynamics of the gauge field itself. The simplest way to proceed is to compute the commutator of two covariant derivatives acting on a field in the representation R. We would get the same information, no matter which representation, so we will study just the case $[D_\mu, D_v]\,\Psi \equiv (D_\mu D_v - D_v D_\mu)\,\Psi$. A careful computation gives

$$[D_\mu, D_\nu]\,\Psi = gF^A_{\mu\nu} t_A \Psi$$

where

$$F^A_{\mu\nu} = \partial_\mu A^A_\nu - \partial_\nu A^A_\mu + g f_{BC}{}^A A^B_\mu A^C_\nu.$$

The properties of the covariant derivative guarantee that the right-hand side of (4.91) transforms as a field in the same representation as Ψ. Thus $F^A_{\mu\nu}$ should have simple transformation properties. Indeed, one can derive

$$\delta F^A_{\mu\nu} = \theta^C F^B_{\mu\nu} f_{BC}{}^A.$$

We see that $F^A_{\mu\nu}$ is an antisymmetric tensor in spacetime, which transforms as a field in the adjoint representation of $\mathfrak{g}$; $F^A_{\mu\nu}$ is the non-abelian generalization of the electromagnetic field strength (4.6). The principal differences between abelian and non-abelian gauge symmetry are that the non-abelian field strength is not gauge invariant, but transforms in the adjoint representation, and that it is nonlinear in the gauge potential A^A_μ.

Exercise 4.20 Derive (4.93).

Despite these significant differences, it is quite straightforward to formulate the YangMills equations by following the ideas of the electromagnetic case. Since both the current and field strength transform in the adjoint representation, and the covariant derivative does not change the transformation properties, the equation

$$D^\mu F^A_{\mu\nu} = -J^A_v$$

is both gauge and Lorentz covariant. It is the basic dynamical equation of classical YangMills theory and the analogue of (4.14) for electromagnetism. One important difference, however, is that in the absence of matter sources, when the right-hand side of (4.94) vanishes, that equation is still a (much studied!) nonlinear equation for A^A_μ.

There is also a non-abelian analogue of the Bianchi identity (4.11), which takes the form

$$D_\mu F^A_{\nu\rho} + D_\nu F^A_{\rho\mu} + D_\rho F^A_{\mu\nu} = 0,$$

where $D_\mu F^A_{v\rho} = \partial_\mu F^A_{\nu\rho} + g f_{BC}{}^A A^B_\mu F^C_{\nu\rho}$.

Exercise 4.21 Show that (4.95) is satisfied identically if $F^A_{v\rho}$ is written in the form (4.92).

Exercise 4.22 Show that $D^\nu D^\mu F^A_{\mu\nu}$ vanishes identically (despite the nonlinearity). This is again a Noether identity: a relation between field equations that follows from the gauge symmetry.

As in the electromagnetic case, this means that the equation of motion (4.94) is consistent only if the current is covariantly conserved, i.e. only if $D^v J^A_v = 0$. It also means that (4.94) contains $(D-1)\dim_G$ independent equations, which is enough to determine the $D\dim_G$ components of A^A_μ up to a gauge transformation. It is usually convenient to "fix the gauge' by specifying $\dim_G$ conditions on the components of A^A_μ.

Note that, in the limit $g \to 0$, equations (4.92), (4.94), and (4.95) reduce to linear equations, which are $\dim_G$ copies of the corresponding equations for the free electromagnetic field. The count of degrees of freedom of Sec. 4.1.2 can be repeated in the Coulomb gauge $\partial^i A^A_i(\vec{x}, t) = 0$.

For each component $A=1, \ldots, \operatorname{dim}_G, 2(D-2)$ functions must be specified as initial data, and each $A_i^A(x)$ has a Fourier transform identical to (4.9). In this free limit, the gauge field thus describes a particle with $D-2$ polarization states transforming in the adjoint representation of $\mathfrak{g}$.

The equations of motion of the Yang-Mills field A^A coupled to the Dirac field Ψ^α can be obtained from an action functional that is a natural generalization of (4.31):

$$S\left[A_\mu^A, \bar{\Psi}_\alpha, \Psi^\alpha\right]=\int \mathrm{d}^D x\left[-\frac{1}{4} F^{A \mu \nu} F_{\mu \nu}^A-\bar{\Psi}_\alpha\left(\gamma^\mu D_\mu-m\right) \Psi^\alpha\right]$$

The action is gauge invariant. The Euler variation with respect to A_v^A gives (4.94) with current source (4.86), and the variation with respect to $\bar{\Psi}_\alpha$ gives (4.90).

6.3.16 4.3.4 Yang-Mills theory for $G=\mathrm{SU}(N)$

The most commonly studied gauge group for Yang-Mills theory is $\mathrm{SU}(N)$. The generators of the fundamental representation of its Lie algebra are a set of N^2-1 traceless antihermitian $N \times N$ matrices t_A, which are normalized by the bilinear trace relation

$$\operatorname{Tr}\left(t_A t_B\right)=-\frac{1}{2} \delta_{A B}$$

In this section we discuss the special notation that has been developed for this case and is frequently used in the literature. In this notation gauge transformations are explicitly realized at the level of the group $\mathrm{SU}(N)$ rather than just at the level of its Lie algebra $\mathfrak{su}(N)$ as in the previous sections.

We will use the notation $U(x)=\mathrm{e}^{-\Theta(x)}$, with $\Theta(x)=\theta^A(x) t_A$, to denote an element of the gauge group in the fundamental representation. This may be viewed as a map $x^\mu \rightarrow U\left(x^\mu\right)$ from Minkowski spacetime into the group $\mathrm{SU}(N)$. In this notation the gauge transformation of a spinor field Ψ in the fundamental representation can be written (see (4.82))

$$\Psi(x) \rightarrow U(x) \Psi(x).$$

Row and column indices of the fundamental representation are consistently omitted in this notation. Usually we will omit the spacetime argument x^μ also, unless useful for special emphasis.

Given any matrix generator t_A, the unitary transformation $U(x) t_A U(x)^{-1}$ gives another traceless anti-hermitian matrix, which must then be a linear combination of the t_B. Therefore we can write

$$U(x) t_A U(x)^{-1}=t_B R(x)_A^B,$$

where $R(x)^B{}_A$ is a real $\left(N^2-1\right) \times\left(N^2-1\right)$ matrix.

Exercise 4.23 Consider the product of two gauge group elements U_1 and U_2, which gives a third via $U_1 U_2=U_3$. For each element U_i, there is an associated matrix $\left(R_i\right)^B{}_A$, defined by $U_i t_A U_i^{-1}=t_B\left(R_i\right)^B{}_A$. Prove that $\left(R_3\right)^B{}_A=\left(R_1\right)^B{}_C\left(R_2\right)^C{}_A$, which shows that the matrices $R^B{}_A$ defined by (4.99) are the matrices of an $\left(N^2-1\right)$-dimensional representation of $\mathrm{SU}(N)$. Use (4.99) to show that, to first order in the gauge parameters $\theta^C, R_A^B=\delta_A^B+\theta^C f_{A C}{}^B+\ldots$ This shows that the matrices $R^B{}_A$ are exactly those of the adjoint representation. [15]

Given any set of N^2-1 real quantities X^A, that is any element of the vector space $\mathbb{R}^{N^2-1}$, we can form the matrix $\mathbf{X}=t_A X^A$. For any group element U, we have $U \mathbf{X} U^{-1}=t_B R^B{}_A X^A$. Thus the unitary transformation of the matrix $\mathbf{X}$ contains the information that the quantities $X^A=-2 \delta^{A B} \operatorname{Tr}\left(t_B \mathbf{X}\right)$ transform in the adjoint representation, that is as $X^A \rightarrow R^A{}_B X^B$.

Thus, given any field in the adjoint representation, such as $\phi^A(x)$, we can form the matrix $\Phi(x) = t_A \phi^A(x)$. Gauge transformations can then be implemented as

$$\Phi(x) \to U(x)\Phi(x)U(x)^{-1}$$

One can also form the matrix $\mathbf{A}_\mu(x) = t_A A_\mu^A(x)$ for the gauge potential. Quite remarkably, the gauge transformation of the potential can be expressed in matrix form if we define the transformation by

$$\mathbf{A}_\mu(x) \to \mathbf{A}'_\mu(x) \equiv \frac{1}{g} U(x)\partial_\mu U(x)^{-1} + U(x)\mathbf{A}_\mu(x)U(x)^{-1}$$

15 The equation (4.99) is true for the generators t_A of any representation of any Lie algebra $\mathfrak{g}$ and the associated group element $U = \mathrm{e}^{-\theta^C t_C}$. It follows that the matrices $R^B{}_A$ are those of the adjoint representation of G. A matrix description of Yang-Mills theory for a general gauge group can then be constructed by following the procedure discussed below for the fundamental representation of $\mathrm{SU}(N)$.

For infinitesimal transformations this becomes

$$\delta A_\mu(x) = \frac{1}{g}\partial_\mu \Theta(x) + [A_\mu(x), \Theta(x)]$$

which agrees with (4.88).

Exercise 4.24 Suppose that $\mathbf{A}_\mu \to \mathbf{A}'_\mu$ by the gauge transformation $U_2(x)$ followed by $\mathbf{A}'_\mu \to \mathbf{A}''_\mu$ by the gauge transformation $U_1(x)$. Show that the combined transformation $\mathbf{A}_\mu \to \mathbf{A}''_\mu$ is correctly described by the definition (4.101) for the product matrix $U_2(x)U_1(x)$. This result is compatible with (1.23) and with the implementation of gauge transformations by unitary operators in the quantum theory.

It is easy to define covariant derivatives in which the gauge potential appears in matrix form. For fields Ψ in the fundamental and $\bar{\Psi}$ in the anti-fundamental representation (and transforming as $\bar{\Psi} \to \bar{\Psi}U^{-1}$), the previous definitions in (4.89) can simply be rewritten as

$$\begin{aligned} D_\mu \Psi &\equiv (\partial_\mu + g\mathbf{A}_\mu)\Psi \\ D_\mu \bar{\Psi} &\equiv \partial_\mu \bar{\Psi} - g\bar{\Psi}\mathbf{A}_\mu \end{aligned}$$

For a field in the adjoint representation, such as Φ, we define

$$D_\mu \Phi = \partial_\mu \Phi + g[\mathbf{A}_\mu, \Phi],$$

which involves the matrix commutator.

Exercise 4.25 Demonstrate that these covariant derivatives transform correctly, specifically that

$$D_\mu \Psi \to U(x) D_\mu \Psi, \quad D_\mu \bar{\Psi} \to D_\mu \bar{\Psi} U(x)^{-1}, \quad D_\mu \Phi \to U(x) D_\mu \Phi U(x)^{-1}$$

The non-abelian field strength can also be converted to matrix form as

$$\mathbf{F}_{\mu\nu} = t_A F_{\mu\nu}^A = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + g[\mathbf{A}_\mu, \mathbf{A}_\nu]$$

6.3.17 Exercise 4.26 Prove this.

The matrix formalism is a convenient way to express quantities of interest in the theory. For example the Yang-Mills action (4.96) can be written as

$$S[\mathbf{A}_\mu, \bar{\Psi}, \Psi] = \int \mathrm{d}^D x \left[\frac{1}{2}\operatorname{Tr}(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}) - \bar{\Psi}(\gamma^\mu \mathrm{D}_\mu - \mathrm{m})\Psi\right]$$

The N^2-1 matrix generators $(t_A)^\alpha{}_\beta$ of the fundamental representation, normalized as in (4.97), together with the matrix $\mathrm{i}\delta^\alpha_\beta$ form a complete set of $N\times N$ anti-hermitian matrices, which are orthogonal in the trace norm. Therefore one can expand any $N\times N$ anti-hermitian matrix $H^\alpha{}_\beta$ in this set as

$$H^\alpha_\beta = \mathrm{i}h_0\delta^\alpha_\beta + h^A\left(t_A\right)^\alpha_\beta\,,$$
$$h_0 = -\tfrac{\mathrm{i}}{N}\operatorname{Tr}H,\quad h^A = -2\delta^{AB}\operatorname{Tr}\left(Ht_B\right).$$

Note that there is a sum over the N^2-1 values of the repeated indices A,B in (4.108) and in the exercise below.

Exercise 4.27 Use the completeness property (perhaps with Sec. 3.2.3 as a guide) to derive the rearrangement relation

$$\delta^\alpha_\beta\delta^\gamma_\delta = \frac{1}{N}\delta^\alpha_\delta\delta^\gamma_\beta - 2\left(t_A\right)^\alpha_\delta\delta^{AB}\left(t_B\right)^\gamma_\beta$$

6.3.18 4.4 Internal symmetry for Majorana spinors

Majorana spinors play a central role in supersymmetric field theories. In many applications they transform in a representation of a non-abelian internal symmetry group. For example, the spinor fields of super-Yang-Mills theory are denoted as λ^A and transform in the adjoint representation of the gauge group. In the notation of Sec. 4.3.4, we have $\lambda^A\to\lambda'^A = R^A_B\lambda^B$. Since the matrix R^A_B is real, this transformation rule is consistent with the fact that Majorana spinors obey a reality constraint. Indeed, as shown in Sec. 3A.5, there are really real representations of the Clifford algebra in which the spinors are explicitly real. One can consider the more general situation of a set of Majorana spinors Ψ^α transforming as $\Psi^\alpha\to\Psi'^\alpha = \left(\mathrm{e}^{-\theta^A t_A}\right)^\alpha{}_\beta\Psi^\beta$. The transformed Ψ'^α must also satisfy the Majorana condition, and this requires that the matrices $\mathrm{e}^{-\theta^A t_A}$ are those of a really real representation of the group G. (Obviously there is a similar requirement on the symmetry transformation of a set of real scalars, such as the ϕ^A of Sec. 4.3.1.)

In $D=4$ dimensions, the requirement that Majorana spinors transform in a real representation of the gauge group can be bypassed because internal symmetries can include chiral transformations, which involve the highest rank element $\gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$ of the Clifford algebra discussed in Sec. 3.1.6. This matrix is imaginary in a Majorana representation, or in general under the C-operation; see (3.78). We use the chiral projectors P_L and P_R as in (3.38). Suppose that the matrices t_A are generators of a complex representation of the Lie algebra. Then the complex conjugate matrices t^*_A are generators of the conjugate representation. Let χ^α denote a set of Majorana spinors to which we assign the group transformation rule

$$\chi^\alpha\to\chi'^\alpha \equiv \left(\mathrm{e}^{-\theta^A\left(t_A P_L + t^*_A P_R\right)}\right)^\alpha{}_\beta\chi^\beta$$

The matrices $t_AP_L{+}t^*_AP_R$ are generators of a representation of an explicitly real representation of the Lie algebra, so the transformed spinors χ'^α also satisfy the Majorana condition. This is the transformation rule used for Majorana spinors in supersymmetric gauge theories in Ch. 6 .

By applying the projectors to (4.110), one can see that the chiral and anti-chiral projections of χ transform as

$$P_L\chi\to P_L\chi' \equiv \left(\mathrm{e}^{-\theta^A t_A}\right)P_L\chi,$$
$$P_R\chi\to P_R\chi' \equiv \left(\mathrm{e}^{-\theta^A t^*_A}\right)P_R\chi.$$

Exercise 4.28 What is the covariant derivative $D_\mu\chi$? We now use $\bar\chi$ for a Majorana conjugate (3.50), where the transpose includes a transpose in the representation space. When representation indices are needed, $\bar{\chi}$ carries a lower index. Show that then the kinetic Lagrangian density $\bar{\chi}\gamma^{\mu}D_{\mu}\chi$ is invariant under the infinitesimal limit of the transformation (4.110) for anti-hermitian t_A, and that the variation of the mass term is

$$\delta(\bar{\chi}\chi) = -\theta^A \bar{\chi}\left(t_A + t_A^T\right)\gamma_*\chi$$

The mass term is invariant only for the subset of generators that are antisymmetric, and thus real. This condition defines a subalgebra of the original Lie algebra $\mathfrak{g}$ of the theory, specifically the subalgebra that contains only parity conserving vector-like gauge transformations. For the case $\mathfrak{g} = \mathfrak{su}(N)$, the subalgebra is isomorphic to $\mathfrak{so}(N)$. Non-invariance of the Majorana mass term is a special case of the general idea that chiral symmetry requires massless fermions.

Exercise 4.29 Show that

$$\frac{1}{2}\int \mathrm{d}^4x\bar{\chi}\gamma^{\mu}D_{\mu}\chi = \int \mathrm{d}^4x\bar{\chi}\gamma^{\mu}P_L D_{\mu}\chi = \int \mathrm{d}^4x\bar{\chi}\gamma^{\mu}P_R D_{\mu}\chi$$

Note that $P_{L,R}D_{\mu}\chi = D_{\mu}P_{L,R}\chi$.

6.4 The free Rarita-Schwinger field

In this chapter we begin to assemble the ingredients of supergravity by studying the free spin-3/2 field. Supergravity is the gauge theory of global supersymmetry, which we will usually abbreviate as SUSY. The key feature is that the symmetry parameter of global SUSY transformations is a constant spinor ϵ_{α}. In supergravity it becomes a general function in spacetime, $\epsilon_{\alpha}(x)$. The associated gauge field is a vector-spinor $\Psi_{\mu\alpha}(x)$. This field and the corresponding particle have acquired the name 'gravitino'.

Supergravity theories necessarily contain the gauge multiplet, the set of fields required to gauge the symmetry in a consistent interacting theory, and may contain matter multiplets, sets of fields on which global SUSY is realized. The gauge multiplet contains the gravitational field, one or more vector-spinors, and sometimes other fields. This structure is derived from representations of the SUSY algebras in Sec. 6.4.2. In this chapter we are concerned with the free limit, in which the various fields do not interact, and we can consider them separately. In particular we consider $\Psi_{\mu}(x)$ (omitting the spinor index α) as a free field, subject to the gauge transformation

$$\Psi_{\mu}(x) \rightarrow \Psi_{\mu}(x) + \partial_{\mu}\epsilon(x)$$

Furthermore we will assume that Ψ_{μ} and ϵ are complex spinors with $2^{[D/2]}$ spinor components for spacetime dimension D. This is fine for the free theory in any dimension D, but interacting supergravity theories are more restrictive as to the spinor type permitted in a given spacetime dimension (and such theories exist only for $D \leq 11$). We will need to use the required Majorana and/or Weyl spinors when we study these theories in later chapters (and the number $2^{[D/2]}$ must be adjusted to agree with the number of components of each type of spinor).

It is consistent with the pattern set in the previous chapter that the gauge field $\Psi_{\mu}(x)$ is a field with one more vector index than the gauge parameter $\epsilon(x)$. Furthermore, as in the case of electromagnetism, the antisymmetric derivative $\partial_{\mu}\Psi_{\nu} - \partial_{\nu}\Psi_{\mu}$ is gauge invariant. An important difference arises because we now seek a gauge invariant first order wave equation for the fermion field. It is advantageous to start with the action, which must be (a) Lorentz invariant, (b) first order in spacetime derivatives, (c) invariant under the gauge transformation (5.1) and the simultaneous conjugate transformation of $\bar{\Psi}_{\mu}$, and (d) hermitian, so that the Euler-Lagrange equation for $\bar{\Psi}_{\mu}$ is the Dirac conjugate of that for Ψ_{μ}. It is easy to see that the expression

$$S=-\int \mathrm{d}^D x \bar{\Psi}_\mu \gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho$$

which contains the third rank Clifford algebra element $\gamma^{\mu\nu\rho}$, has all these properties. Note that the action is gauge invariant but the Lagrangian density is not. Instead its variation is the total derivative $\delta\mathcal{L}=-\partial_\mu\left(\bar{\epsilon}\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho\right)$. The reason is that the fermionic gauge symmetry is the remnant of supersymmetry, and the anti-commutator of two SUSY transformations is a spacetime symmetry.

It should be noted that a physically equivalent theory can be obtained by rewriting (5.2) in terms of the new field variable $\Psi_\mu^{\prime} \equiv \Psi_\mu+a\gamma_\mu\gamma\cdot\Psi$ where a is an arbitrary parameter. [1] The gauge transformation is modified accordingly. The presentation in (5.1) and (5.2) is universally used in the modern literature, because the gauge transformation is simplest and closely resembles that of electromagnetism. Historically, Rarita and Schwinger invented a wave equation for a massive spin-3/2 particle in 1941. The massless limit of the action is a transformed version of (5.2), and Rarita and Schwinger simply noted that it possesses a fermionic gauge symmetry. [2]

The equation of motion obtained from (5.2) reads

$$\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho=0.$$

One can immediately see that it shares some of the properties of the analogous electromagnetic equation (4.7), which is $\partial^\mu F_{\mu\nu}=0$. Gauge invariance is manifest, and the lefthand side vanishes identically when the derivative ∂_μ is applied. Thus (5.3) comprises $(D-1)2^{[D/2]}$ independent equations, which is enough to determine the $2^{[D/2]}D$ components of Ψ_ρ up to the freedom of a gauge transformation. The difference between the number of components of the gauge field and those of the gauge parameter, in this case $(D-1)2^{[D/2]}$, is called the number of off-shell degrees of freedom.

Exercise 5.1 Show directly that for $D=3$, the field equation (5.3) implies that $\partial_\nu\Psi_\rho-\partial_\rho\Psi_v=0$. This means that the field has no gauge invariant degrees of freedom and thus no propagating particle modes. This is the supersymmetric counterpart of the situation in gravity for $D=3$, where the field equation $R_{\mu\nu}=0$ implies that the full curvature tensor $R_{\mu\nu\rho\sigma}=0$. Hence no degrees of freedom.

We notice that (5.3) can be rewritten in an equivalent but simpler form. For this purpose, we use the γ-matrix relation $\gamma_\mu\gamma^{\mu\nu\rho}=(D-2)\gamma^{\nu\rho}$, which implies that $\gamma^{\nu\rho}\partial_\nu\Psi_\rho=0$ in spacetime dimension $D>2$. We also note that $\gamma^{\mu\nu\rho}=\gamma^\mu\gamma^{\nu\rho}-2\eta^{\mu[\nu}\gamma^{\rho]}$. Using this information, it is easy to see that (5.3) implies that

$$\gamma^\mu\left(\partial_\mu\Psi_\nu-\partial_\nu\Psi_\mu\right)=0$$

1 The case $a=-1/D$ requires special treatment since $\gamma\cdot\Psi^{\prime}=0$.

2 One of the present authors met Prof. Schwinger at a cocktail party in the early 1980s. Supergravity came up in the conversation, and Schwinger remarked lightheartedly 'I should have discovered supergravity.'

This is an alternative form of the equation of motion, equivalent to (5.3), but which cannot be obtained directly from an action. To see that (5.4) is equivalent, note that one can apply γ^ν and obtain $\gamma^{\nu\rho}\partial_\nu\Psi_\rho=0$. The previous steps can then be reversed to obtain (5.3) from (5.4). One can also show that the left-hand side of (5.4) vanishes identically if $\gamma^\nu\ \not{\partial}$ is applied. Finally, let's apply ∂_ρ to (5.4) and antisymmetrize in ρv to obtain

$$\not{\partial}\left(\partial_\rho\Psi_\nu-\partial_\nu\Psi_\rho\right)=0.$$

This is a gauge invariant derivation of the fact that the wave equations, either (5.3) or (5.4), describe massless particles.

Exercise 5.2 Do all the manipulations in the preceding paragraph. Do them backwards and forwards.

6.4.1 5.1 The initial value problem

Let's now study the initial value problem for (5.3) and thus count the number of on-shell degrees of freedom. We must untangle constraints on the initial data from time evolution equations. For this purpose we need to fix the gauge, so we impose the non-covariant condition

$$\gamma^i \Psi_i = 0$$

which will play the same role as the Coulomb gauge condition we used in Sec. 4.1.2.

Exercise 5.3 Show by an argument analogous to that in Sec. 4.1.2 that this condition does fix the gauge uniquely.

We use the equivalent form (5.4) of the field equations. The $v = 0$ and $v \to i$ components are

$$\begin{aligned} \gamma^i \partial_i \Psi_0 - \partial_0 \gamma^i \Psi_i &= 0, \\ \gamma \cdot \partial \Psi_i - \partial_i \gamma \cdot \Psi &= 0. \end{aligned}$$

Using the gauge condition one can see that $\nabla^2 \Psi_0 = 0$, so $\Psi_0 = 0$ according to the discussion on p. 69. The spatial components Ψ_i then satisfy the Dirac equation

$$\gamma \cdot \partial \Psi_i = 0$$

which is a time evolution equation. However, there is an additional constraint, $\partial^i \Psi_i = 0$, obtained by contracting (5.8) with γ^i. Thus from the gauge condition and the equation of motion, we find $3 \times 2^{[D/2]}$ independent constraints on the initial data, namely

$$\begin{aligned} \gamma^i \Psi_i(\vec{x}, 0) &= 0, \\ \Psi_0(\vec{x}, 0) &= 0, \\ \partial^i \Psi_i(\vec{x}, 0) &= 0. \end{aligned}$$

On-shell degrees of freedom $= \frac{1}{2}(D-3)2^{[D/2]}$.

Off-shell degrees of freedom $= (D-1)2^{[D/2]}$.

These constraints imply that there are only $2^{[D/2]}(D-3)$ initial components of Ψ_i to be specified. The time derivatives are already determined by the Dirac equation (5.8). Hence there are $2^{[D/2]}(D-3)$ classical degrees of freedom for the Rarita-Schwinger gauge field in D-dimensional Minkowski space. The on-shell degrees of freedom are half this number. In dimension $D = 4$, with Majorana conditions, we find the two states expected for a massless particle for any spin $s > 0$. We will show below that these states carry helicity $\pm 3/2$. In general dimension, it should be a representation of $\mathrm{SO}(D-2)$ as discussed in Sec. 4.1.2. Indeed, the vector-spinor representation is an irreducible representation after subtraction of the γ-trace. It then contains $\frac{1}{2}(D-3)2^{[D/2]}$ components.

Exercise 5.4 Analyze the degrees of freedom using the original equation of motion (5.3).

According to the discussion for $D = 4$ at the beginning of Ch. 4 , we would expect the Fourier expansion of the field to contain annihilation and creation operators for states of helicity $\lambda = \pm 3/2$. Let's derive this fact starting from the plane wave

$$\Psi_i(x) = \mathrm{e}^{\mathrm{i}p \cdot x} v_i(\vec{p}) u(\vec{p})$$

for a positive null energy-momentum vector $p^\mu = (|\vec{p}|, \vec{p})$. Since $\Psi_i(x)$ satisfies the Dirac equation (5.8), the four-component spinor $u(\vec{p})$ must be a superposition of the massless helicity spinors $u(\vec{p},\pm)$ given in (2.44). Thus we use the Weyl representation (2.19) of the γ-matrices. The vector $v_i(\vec{p})$ may be expanded in the complete set

$$v_i(\vec{p}) = ap_i + b\epsilon_i(\vec{p},+) + c\epsilon_i(\vec{p},-)$$

where $\epsilon_i(\vec{p},\pm)$ are the transverse polarization vectors of Sec. 4.1.2, i.e. they satisfy $p^i\epsilon_i(\vec{p},\pm) = 0$. The constraint (5.11) requires that $a = 0$. Thus (5.12) is reduced to the form

$$\begin{aligned}\Psi_i(x) = \mathrm{e}^{\mathrm{i}p\cdot x}[&b_+\epsilon_i(\vec{p},+)u(\vec{p},+) + c_+\epsilon_i(\vec{p},-)u(\vec{p},+)\\ &+b_-\epsilon_i(\vec{p},+)u(\vec{p},-) + c_-\epsilon_i(\vec{p},-)u(\vec{p},-)]\end{aligned}$$

We must still enforce the constraint $\gamma^i\Psi_i = 0$. Some detailed algebra is needed, which we leave to the reader; the result is that $c_+ = b_- = 0$, while $b_+$and $c_-$are arbitrary. Thus there are two independent physical wave functions $\epsilon_i(\vec{p},\pm)u(\vec{p},\pm)$ for each p^μ.

Exercise 5.5 Do the algebra that was just left for the reader. Show that the resulting vector-spinor wave functions $\epsilon_i(\vec{p},\pm)u(\vec{p},\pm)$ carry helicity $\pm3/2$. Show that the spinor wave functions for the conjugate plane wave are $\epsilon_i^*(\vec{p},\pm)v(\vec{p},\pm)$, where $v(\vec{p},\pm) = B^{-1}u(\vec{p},\pm)^*$ are the massless v spinors of (2.45).

The net result of this analysis is that the Rarita-Schwinger field that satisfies the equation of motion and constraints above has the Fourier expansion (we add the trivial 0-components $\epsilon_0 = 0$ to polarization vectors)

$$\Psi_\mu(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}\sum_\lambda \left[\mathrm{e}^{\mathrm{i}p\cdot x}\epsilon_\mu(\vec{p},\lambda)u(\vec{p},\lambda)c(\vec{p},\lambda) + \mathrm{e}^{-\mathrm{i}p\cdot x}\epsilon_\mu^*(\vec{p},\lambda)v(\vec{p},\lambda)d^*(\vec{p},\lambda)\right].$$

The sum extends over the two physical wave functions of helicity $\pm3/2$. In the quantum theory the Fourier amplitude $c(\vec{p},\lambda)$ becomes the annihilation operator for helicity $\pm3/2$ particles, and $d^*(\vec{p},\lambda)$ becomes the creation operator for anti-particles. The situation is similar to that for the Dirac field in (2.24). A Majorana gravitino has the same expansion, with $d^*(\vec{p},\lambda) = c^*(\vec{p},\lambda)$, since there is no distinction between particles and anti-particles.

In dimension $D > 4$ the allowed gravitino modes are obtained by starting with products of the $D-2$ transverse polarization vectors $\epsilon_i(\vec{p},j)$ and the $\frac{1}{2}2^{[D/2]}$ massless Dirac spinors $u(\vec{p},s)$. The gauge fixing constraint $\gamma^i\Psi_i = 0$ must then be enforced on linear combinations of these products as was done in (5.14). This leads to $\frac{1}{2}2^{[D/2]}(D-3)$ independent wave functions, which describe the on-shell states of the gravitino.

The canonical stress tensor obtained from (5.2) is

$$T_{\mu\nu} = \bar{\Psi}_\rho\gamma_\mu^{\rho\sigma}\partial_\nu\Psi_\sigma - \eta_{\mu\nu}\mathcal{L}.$$

It is neither symmetric nor gauge invariant under (5.1) (and its Dirac conjugate). It can be made symmetric (see [27]), but gauge non-invariance is intrinsic and cannot be restored by adding terms of the form $\partial_\sigma S^{\sigma\mu\nu}$. The reason is that the gravitino must be joined with gravity in the gauge multiplet of SUSY. In a gravitational theory there is no well-defined energy density.

Exercise 5.6 Show that the total energy-momentum $P^v = \int \mathrm{d}^3\vec{x}T^{0v}(\vec{x},t)$ is gauge invariant and given (for $D = 4$) by

$$P^v = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0}p^\nu\sum_\lambda [c^*(\vec{p},\lambda)c(\vec{p},\lambda) - d(\vec{p},\lambda)d^*(\vec{p},\lambda)]$$

6.4.2 5.2 Sources and Green's function

Let's follow the pattern of Sec. 4.1.3 and couple the Rarita-Schwinger field to a vectorspinor source via

$$\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho = J^\mu$$

The contraction of ∂_μ with the left-hand side vanishes identically, which indicates that (5.18) is a consistent equation only if the source current is conserved, i.e. $\partial_\mu J^\mu = 0$. This is the exact analogue of what happens in electromagnetism and Yang-Mills theory. In those theories, the gauge field was later coupled to matter systems, and the source was the Noether current of the global symmetry. Supergravity theories are more complicated.

The same phenomenon occurs, but only as an approximation valid to lowest order in the gravitational coupling. The current J^μ is the Noether supercurrent of the matter multiplets in the theory.

Let's now apply the method of Sec. 4.1.3 to find the Green's function that determines the response of the field to the source. We first solve the simpler problem for the Dirac field,

$$(\not{\partial} - m)\Psi(x) = J(x).$$

Given a Green's function $S(x-y)$ that satisfies

$$(\not{\partial}_x - m)\, S(x-y) = -\delta(x-y),$$

the solution of (5.19) is given by

$$\Psi(x) = -\int \mathrm{d}^D y S(x-y)J(y)$$

Let's solve this problem using the Fourier transform. The symmetries of Minkowski spacetime allow us to assume the Fourier representation

$$S(x-y) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \mathrm{e}^{\mathrm{i}p\cdot(x-y)} S(p)$$

In momentum space, (5.20) reads

$$(\mathrm{i}\not{p} - m)S(p) = -1$$

and the solution (with Feynman's causal structure) is

$$S(p) = -\frac{1}{\mathrm{i}\not{p} - m} = \frac{\mathrm{i}\not{p} + m}{p^2 + m^2 - \mathrm{i}\epsilon}.$$

Comparing with (4.18), we see that we can express $S(x-y)$ in terms of the scalar Green's function as

$$S(x-y) = (\not{\partial}x + m)G(x-y).$$

This result satisfies (5.20) by inspection and could have been guessed at the start. However, the Fourier transform method is useful as a warmup for the more complicated case of the Rarita-Schwinger field.

We expect the Green's function solution of (5.18) to take the form

$$\Psi_\mu(x) = -\int \mathrm{d}^D y S_{\mu\nu}(x-y)J^\nu(y)$$

where $S_{\mu\nu}(x-y)$ is a tensor bispinor. A bispinor has two spinor indices, which are suppressed in our notation, and it can be regarded as a matrix of the Clifford algebra. As in the electromagnetic case, the Rarita-Schwinger operator is not invertible, but we can assume that the Green's function satisfies

$$\gamma^{\mu\sigma\rho}\frac{\partial}{\partial x^\sigma}S_{\rho\nu}(x-y) = -\delta_v^\mu\delta(x-y) + \frac{\partial}{\partial y^\nu}\Omega^\mu(x-y).$$

The last term on the right is a 'pure gauge' in the source point index. In momentum space (5.27) reads

$$\mathrm{i}\gamma^{\mu\sigma\rho}p_\sigma S_{\rho v}(p) = -\delta_v^\mu - \mathrm{i}p_v\Omega^\mu(p)$$

We will solve (5.28) by writing an appropriate ansatz for $S_{\rho\nu}(p)$ and then find the unknown functions in the ansatz. The matrix $\gamma^{\mu\sigma\rho}p_\sigma$ in (5.28) contains an odd rank element of the Clifford algebra and it is odd under the reflection $p_\sigma \to -p_\sigma$. It is reasonable to guess that the ansatz we need should also involve odd rank Clifford elements and be odd under the reflection. We would also expect that terms that contain the momentum vectors p_ρ or p_v are 'pure gauges' and thus arbitrary additions to the propagator, which would not be determined by the equation (5.28). So we omit such terms and postulate the ansatz

$$\mathrm{i}\ S_{\rho\nu}(p) = A\left(p^2\right)\eta_{\rho\nu}\ \not{p} + B\left(p^2\right)\gamma_\rho\ \not{p}\gamma_\nu.$$

The next step is to substitute the ansatz in (5.28) and simplify the products of γ-matrices that appear. This process yields

$$\begin{aligned}
\mathrm{i}\gamma^{\mu\sigma\rho}p_\sigma S_{\rho\nu}(p) =& A\gamma^{\mu\sigma}{}_\nu\ \not{p}p_\sigma + (D-2)B\gamma^{\mu\sigma}\ \not{p}\gamma_\nu p_\sigma\\
=& A\left(p^\mu\gamma_\nu^\sigma - p^\sigma\gamma_\nu^\mu\right)p_\sigma + (D-2)B\left(-p^\mu\gamma^\sigma + p^\sigma\gamma^\mu\right)\gamma_\nu p_\sigma\\
&+\ldots\\
=&[A-(D-2)B]\left(p^\mu\gamma^\sigma{}_\nu - p^\sigma\gamma^\mu{}_\nu\right)p_\sigma + (D-2)Bp^2\delta_v^\mu\\
&+\ldots
\end{aligned}$$

We have omitted terms that are proportional to the vector p_ν, because such terms will be 'matched' in (5.28) by $\Omega^\mu(p)$ rather than by δ_v^μ. It is now easy to see that the δ_v^μ term in (5.28) determines the values $A = -1/p^2$ and $B = -1/\left((D-2)p^2\right)$. Thus we have found the gravitino propagator

$$S_{\mu\nu}(p) = \mathrm{i}\frac{1}{p^2}\left[\eta_{\mu\nu}\ \not{p} + \frac{1}{D-2}\gamma_\mu\ \not{p}\gamma_\nu + Cp_\mu\gamma_\nu + E\gamma_\mu p_v + Fp_\mu\ \not{p}p_v\right]$$

in which we have added possible gauge terms that are not determined by this procedure. In position space the propagator is

$$S_{\mu\nu}(x-y) = \left[\eta_{\mu\nu}\ \not{\partial} + \frac{1}{D-2}\gamma_\mu\ \not{\gamma}_\nu + C\partial_\mu\gamma_\nu + E\gamma_\mu\partial_\nu - F\partial_\mu\ \not{\partial}\partial_\nu\right]G(x-y),$$

where $G(x-y)$ is the massless scalar propagator (4.19), and all derivatives are with respect to x.

Exercise 5.7 Include the omitted p_v terms in (5.30) and $\Omega(p)$ in the analysis and verify that the gauge terms in the propagator are arbitrary. Show that, for the choice $E = -1/(D-2)$, and arbitrary C and F, the propagator satisfies

$$\mathrm{i}\gamma^{\mu\sigma\rho}p_\sigma S_{\rho v}(p) = -\left(\delta_v^\mu - \frac{p^\mu p_v}{p^2}\right)$$

Show that, for $D = 4$, the propagator, with $C = -1$, takes the 'reverse index' form $S_{\mu\nu}(p) = -\mathrm{i}\frac{1}{2}\gamma_\nu \not{p}\gamma_\mu$, which is the form used in most of the literature on perturbative studies in supergravity [28].

6.4.3 5.3 Massive gravitinos from dimensional reduction

Our aim in this section is quite narrow, but the approach will be broad. The narrow goal is to extend the Rarita-Schwinger equation to describe massive gravitinos, but we wish to do it by introducing the important technique of dimensional reduction, which is also called Kaluza-Klein theory. The main idea is that a fundamental theory, perhaps supergravity or string theory, that is formulated in D' spacetime dimensions can lead to an observable spacetime of dimension $D < D'$. In the most common variant of this scenario, there is a stable solution of the equations of the fundamental theory that describes a manifold of the structure $M_{D'} = M_D \times X_d$ with $d = D' - D$. The factor M_D is the spacetime in which we might live, thus non-compact with small curvature, while X_d is a tiny compact manifold of spatial extent L. The compact space X_d can be thought of as hidden dimensions of spacetime that are not accessible to direct observation because of basic properties of wave physics that are coded in quantum mechanics as the uncertainty principle. This principle asserts that it would take wave excitations of energy $E \approx 1/L$ to explore structures of spatial scale L. If L is sufficiently small, this energy scale cannot be achieved by available apparatus. Nevertheless, the dimensional reduction might be confirmed since the presence of X_d has important indirect effects on physics in M_D.

In this section we study an elementary version of dimensional reduction, which still has interesting physics to teach. Instead of obtaining the structure $M_D \times X_d$ from a fundamental theory including gravity, we will simply explore the physics of the various free fields we have studied, assuming that the $(D+1)$-dimensional spacetime is Minkowski $_D \otimes S^1$. The main feature is that Fourier modes of fields on S^1 are observed as infinite 'towers' of massive particles by an observer in Minkowski i_D. The reduction of the free massless gravitino equation in $D+1$ dimensions will then tell us the correct description of massive gravitinos. Massive gravitinos appear in the physical spectrum of $D = 4$ supergravity when SUSY is spontaneously broken.

6.4.4 5.3.1 Dimensional reduction for scalar fields

Let's change to a more convenient notation and rename the coordinates of the $(D+1)$ dimensional product spacetime $x^0 = t, x^1, \ldots, x^{D-1}, y$, where y is the coordinate of S^1 with range $0 \leq y \leq 2\pi L$. We consider a massive complex scalar field $\phi\left(x^\mu, y\right)$ that obeys the Klein-Gordon equation

$$\left[\square_{D+1} - m^2\right]\phi = \left[\square_D + \left(\frac{\partial}{\partial y}\right)^2 - m^2\right]\phi = 0$$

Acceptable solutions must be single-valued on S^1 and thus have a Fourier series expansion

$$\phi\left(x^\mu, y\right) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky/L}\phi_k\left(x^\mu\right)$$

It is immediate that the spacetime function associated with the k th Fourier mode, namely $\phi_k\left(x^\mu\right)$, satisfies

$$\left[\square_D - \left(\frac{k}{L}\right)^2 - m^2\right]\phi_k = 0$$

Thus it describes a particle of mass $m_k^2 = (k/L)^2 + m^2$. So the spectrum of the theory, as viewed in Minkowski $_D$, contains an infinite tower of massive scalars!

There is an even simpler way to find the mass spectrum. Just substitute the plane wave $\mathrm{e}^{\mathrm{i}p^\mu x_\mu}\mathrm{e}^{\mathrm{i}ky/L}$ directly in the $(D+1)$-dimensional equation (5.34). The D-component energy-momentum vector p^μ must satisfy $p^\mu p_\mu = (k/L)^2 + m^2$. The mass shift due to the Fourier wave on S^1 is immediately visible.

6.4.5 5.3.2 Dimensional reduction for spinor fields

We will consider the dimensional reduction process for a complex spinor $\Psi(x^\mu, y)$ for even $D = 2m$ (so that the spinors in $D+1$ dimensions have the same number of components). Two new ideas enter the game. The first just involves the Dirac equation in D dimensions. We remark that if $\Psi(x)$ satisfies

$$[\not{\partial}_D - m]\,\Psi(x) = 0,$$

then the new field $\tilde{\Psi} \equiv \mathrm{e}^{-\mathrm{i}\gamma_*\beta}\Psi$, obtained by applying a chiral phase factor, satisfies

$$[\not{\partial}_D - m(\cos 2\beta + \mathrm{i}\gamma_* \sin 2\beta)]\,\tilde{\Psi} = 0.$$

Physical quantities are unchanged by the field redefinition, so both equations describe particles of mass m. One simple implication is that the sign of m in (5.37) has no physical significance, since it can be changed by field redefinition with $\beta = \pi/2$.

The second new idea is that a fermion field can be either periodic or anti-periodic $\Psi(x^\mu, y) = \pm\Psi(x^\mu, y +$ Anti-periodic behavior is permitted because a fermion field is not observable. Rather, bilinear quantities such as the energy density $T^{00} = -\bar{\Psi}\gamma^0\partial^0\Psi$ are observables and they are periodic even when Ψ is anti-periodic. Thus we consider the Fourier series

$$\Psi(x^\mu, y) = \sum_k \mathrm{e}^{\mathrm{i}ky/L}\Psi_k(x^\mu)$$

where the mode number k is integer or half-integer for periodic or anti-periodic fields, respectively. In either case when we substitute (5.39) in the $(D+1)$-dimensional Dirac equation $[\not{\partial}_{+1} - m]\,\Psi((x^\mu, y) = 0$, we find that $\Psi_k(x^\mu)$ satisfies [3]

$$\left[\not{\partial} - \left(m - \mathrm{i}\gamma_* \frac{k}{L}\right)\right]\Psi_k(x^\mu) = 0$$

By applying a chiral transformation with phase $\tan 2\beta = k/(mL)$, we see that $\Psi_k(x^\mu)$ describes particles of mass $m_k{}^2 = (k/L)^2 + m^2$. Again we would observe an infinite tower of massive spinor particles with distinct spectra for the periodic and anti-periodic cases.

3 Recall from Ch. 3 that for odd spacetime dimension $D = 2m+1, \gamma^D = \pm\gamma_*$, where γ_* is the highest rank Clifford element in $D = 2m$ dimensions.

6.4.6 5.3.3 Dimensional reduction for the vector gauge field

We now apply circular dimensional reduction to Maxwell's equation

$$\partial^v F_{\nu\mu} = \Box_{D+1}A_\mu - \partial_\mu(\partial^v A_\nu) = 0$$

in $D+1$ dimensions, and we assume a periodic Fourier series representation

$$A_\mu(x, y) = \sum_k \mathrm{e}^{\mathrm{i}ky/L}A_{\mu k}(x), \quad A_D(x, y) = \sum_k \mathrm{e}^{\mathrm{i}ky/L}A_{Dk}(x)$$

with k an integer. The analysis simplifies greatly if we assume the gauge conditions $A_{Dk}(x) = 0$ for $k \neq 0$ and vector component D tangent to S^1. It is easy to see that this gauge can be achieved and uniquely fixes the Fourier modes $\theta_k(x), k \neq 0$, of the gauge function. The gauge invariant Fourier mode $A_{D0}(x)$ remains a physical field in the dimensionally reduced theory. A quick examination of the $\mu \to D$ component of (5.41) shows that it reduces to

$$\begin{aligned} k = 0: \quad & \Box_{D+1} A_{D0} = \Box_D A_{D0} = 0, \\ k \neq 0: \quad & \partial^\mu A_{\mu k} = 0, \end{aligned}$$

so the mode $A_{D0}(x)$ simply describes a massless scalar in D dimensions. For $\mu \leq D-1$, the wave equation (5.41) implies that the vector modes $A_{\mu k}(x)$ satisfy

$$\left[\Box_D - \frac{k^2}{L^2}\right] A_{\mu k} - \partial_\mu \left(\partial^v A_{\nu k}\right) = 0$$

For mode number $k = 0$ this is just the Maxwell equation in D dimensions with its gauge symmetry under $A_{\mu 0} \to A_{\mu 0} + \partial_\mu \theta_0$ intact, since the Fourier mode $\theta_0(x)$ remained unfixed in the process above. For mode number $k \neq 0$, (5.44) is the standard equation [4] for a massive vector field with mass $m_k^2 = k^2/L^2$, namely the equation of motion of the action

$$S = \int \mathrm{d}^D x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu\right]$$

A counting argument similar to that for the massless case in Ch. 4 shows that we have the D-component field $A_{\mu k}$ subject to the single constraint (5.43) and thus giving $D-1$ quantum degrees of freedom for each Fourier mode $k \neq 0$. The $D-1$ particle states for each fixed energy-momentum p^μ transform in the vector representation of $\mathrm{SO}(D-1)$ as appropriate for a massive particle. Note that there are three states for $D = 4$, which agrees with $2s+1$ for spin $s = 1$. The count of states is the same in the massless $k = 0$ sector also, where we have the gauge vector $A_{\mu 0}$ plus the scalar A_{D0} with $(D-2)+1$ on-shell degrees of freedom.

6.4.7 5.3.4 Finally $\Psi_\mu(x, y)$

Let's apply dimensional reduction to the massless Rarita-Schwinger field in $D+1$ dimensions with $D = 2m$. We will assume that the field $\Psi_\mu(x, y)$ is anti-periodic in y so that its Fourier series involves modes $\exp(\mathrm{i}ky/L)\Psi_{\mu k}(x)$ with half-integral k. This assumption simplifies the analysis, since only $k \neq 0$ occurs, and all modes will be massive.

We would like to start with (5.3) in dimension $D+1$ and derive the wave equation of a massive gravitino in Minkowski $_D$. A gauge choice makes this task much easier. All Fourier modes have $k \neq 0$, so we can impose the gauge condition $\Psi_{Dk}(x) = 0$ on all modes and completely eliminate the field component $\Psi_D(x, y)$.

Let's write out the $\mu = D$ and $\mu \leq D-1$ components of (5.3) with $\Psi_D = 0$ (using $\gamma^D = \gamma_*$)

$$\begin{aligned} & \gamma^{\nu\rho} \partial_\nu \Psi_{\rho k} = 0 \\ & \left[\gamma^{\mu\nu\rho} \partial_\nu - \mathrm{i} \frac{k}{L} \gamma_* \gamma^{\mu\rho}\right] \Psi_{\rho k} = 0. \end{aligned}$$

Note that the first equation of (5.46) follows by application of ∂^μ to the second one.

Exercise 5.8 Show that the chiral transformation $\Psi_{\rho k} = \mathrm{e}^{(-\mathrm{i}\pi\gamma_*/4)} \Psi'_{\rho k}$ leads, after replacing $\Psi' \to \Psi$, to the equation of motion

4 Note that the result (5.43) can be obtained by applying ∂^μ to (5.44) and is thus consistent with that equation.

$$\left(\gamma^{\mu\nu\rho}\partial_v - m\gamma^{\mu\rho}\right)\Psi_\rho = 0$$

The last equation is the Euler-Lagrange equation of the action

$$S = -\int \mathrm{d}^D x \bar{\Psi}_\mu \left[\gamma^{\mu\nu\rho}\partial_v - m\gamma^{\mu\rho}\right]\Psi_\rho$$

Exercise 5.9 The equation of motion (5.47) also contains constraints on the initial data. Obtain $\gamma^{\mu\nu}\partial_\mu\Psi_\nu = 0$, which is not a constraint, by contracting the equation with ∂_μ. Then find the constraint $\gamma^\mu\Psi_\mu = 0$ by contracting with γ_μ. Show that the $\mu = 0$ component of the equation of motion gives the constraint $\left(\gamma^{ij}\partial_i - m\gamma^j\right)\Psi_j = 0$.

Exercise 5.10 By analysis similar to that which led from (5.3) to (5.4) in the massless case, derive $(\partial\!\!\!/ + m)\Psi_\mu = 0$, which closely resembles the Dirac equation. The constraints of Ex. 5.9 must still be applied to the initial data, but the new equation clearly shows that the field has definite mass m.

It is useful to recapitulate the equations that we obtained during the analysis (or directly from (5.47)) that determine the counting of the number of initial data and thus the number of degrees of freedom:

$$\begin{aligned}
&\gamma^\mu\Psi_\mu = 0\\
&\left(\gamma^{ij}\partial_i - m\gamma^j\right)\Psi_j = 0,\\
&[\partial\!\!\!/ + m]\Psi_\mu = 0
\end{aligned}$$

As in the massless case, the time derivatives are determined by the Dirac equation (last equation of (5.49)). The initial data are thus the values at $t = 0$ of the Ψ_μ restricted by the first two equations of (5.49). Hence, the complex field $\Psi_\mu(x)$ with $D \times 2^{[D/2]}$ degrees of freedom contains $(D-2) \times 2^{[D/2]}$ independent classical degrees of freedom and thus $\frac{1}{2}(D-2) \times 2^{[D/2]}$ on-shell physical states. For $D = 4$ these are the four helicity states required for a massive particle of spin $s = 3/2$. In the situation of dimensional reduction, there is a massive gravitino with $m = |k|/L$ for every Fourier mode k, each with $\frac{1}{2}(D-2)2^{[D/2]}$ states. Note that this is the same as the number of states of a massless gravitino in $D+1$ dimensions.

Exercise 5.11 Study the Kaluza-Klein reduction for the Rarita-Schwinger field assuming periodicity $\Psi_\mu(x, y+2\pi) = \Psi_\mu(x, y)$ in y. Show that the spectrum seen in Minkowski i_D consists of a massive gravitino for each Fourier mode $k \neq 0$ plus a massless gravitino and massless Dirac particle for the zero mode.

The dimensional reduction process has thus taught us the correct action for a massive gravitino. In particular the mass term is $m\bar{\Psi}_\mu\gamma^{\mu v}\Psi_v$. There is a more general action, namely

$$S = -\int \mathrm{d}^D x \bar{\Psi}_\mu \left[\gamma^{\mu\nu\rho}\partial_\nu - m\gamma^{\mu\rho} - m'\eta^{\mu\rho}\right]\Psi_\rho$$

which contains an additional Lorentz invariant term with a coefficient m' with the dimension of mass. It is curious that this does not give the correct description of a massive gravitino, because it contains too many degrees of freedom. In the following exercise we ask readers to verify this.

Exercise 5.12 Derive the equation of motion for the action (5.50). Analyze this equation as in Ex. 5.9, and show that the previous constraint $\gamma \cdot \Psi = 0$ does not hold if $m' \neq 0$. The field components $\gamma \cdot \Psi$ then describe additional degrees of freedom (which propagate as negative Hilbert space metric 'ghosts'). See [28] for an analysis in terms of projection operators.

6.5 $\mathcal{N}=1$ global supersymmetry in $D=4$

In global SUSY the scope of symmetries included in quantum field theory is extended from Poincaré and internal symmetry transformations, with charges $M_{[\mu\nu]}, P_\mu$, and T_A, to include spinor supercharges Q^i_α, where α is a spacetime spinor index, and $i=1,\ldots,\mathcal{N}$ is an index labeling the distinct supercharges. We will assume that the Q^i_α are four-component Majorana spinors, although an equivalent formulation using two-component Weyl spinors is also commonly used. In this chapter we will mostly study the simplest case $\mathcal{N}=1$ where there is a single spinor charge Q_α. This case is called $\mathcal{N}=1$ SUSY or simple SUSY. Some features of theories with $\mathcal{N}>1$ spinor charges, called extended SUSY theories, are discussed in Sec. 6.4 and Appendix 6A.

In $\mathcal{N}=1$ global SUSY the Poincaré generators and Q_α join in a new algebraic structure, that of a superalgebra. A superalgebra contains two classes of elements, even and odd. From the physics viewpoint, they can be called bosonic (B) and fermionic (F). The structure relations include both commutators and anti-commutators in the pattern $[B,B]=B,[B,F]=F,\{F,F\}=B$. The bosonic charges span a Lie algebra. In SUSY the subalgebra of the bosonic charges $M_{[\mu\nu]}$ and P_μ is the Lie algebra of the Poincaré group discussed in Ch. 1, while the new structure relations involving Q_α are

$$\begin{aligned}\left\{Q_\alpha,\bar{Q}^\beta\right\}&=-\frac{1}{2}\left(\gamma_\mu\right)_\alpha{}^\beta P^\mu,\\ \left[M_{[\mu\nu]},Q_\alpha\right]&=-\frac{1}{2}\left(\gamma_{\mu\nu}\right)_\alpha{}^\beta Q_\beta,\\ \left[P_\mu,Q_\alpha\right]&=0.\end{aligned}$$

Note that these are the classical (anti-)commutator relations; see Secs. 1.4 and 1.5. We will discuss this further in Ch. 11.

Exercise 6.1 Use (2.30) to reexpress the supercharge anti-commutator in terms of Q and $Q^\dagger$. Then use the correspondence principle, that is multiply by the imaginary i, to obtain the quantum anti-commutator from the classical relation. This procedure gives the operator relation

$$\left\{Q_\alpha,\left(Q^\dagger\right)^\beta\right\}_{\mathrm{qu}}=\frac{1}{2}\left(\gamma_\mu\gamma^0\right)_\alpha{}^\beta P^\mu$$

Trace on the spinor indices to obtain the positivity condition

$$\operatorname{Tr}\left(QQ^\dagger+Q^\dagger Q\right)=2P^0$$

The energy $E=P^0$ of any state in the Hilbert space of a global supersymmetric field theory must be positive.

Many SUSY theories, but not all, are invariant under a chiral U(1) symmetry called R-symmetry. We denote the generator by T_R. This acts on Q_α via

$$\left[T_R,Q_\alpha\right]=-\mathrm{i}\left(\gamma_*\right)_\alpha^\beta Q_\beta$$

but this generator T_R is not required. Other internal symmetries, which commute with Q_α and are frequently called outside charges, can also be included.

There are two important theorems that severely limit the type of charges and algebras that can be realized in an interacting relativistic quantum field theory in $D=4$ (strictly speaking in a theory with a non-trivial S-matrix in flat space). According to the ColemanMandula (CM) theorem [29, 30], in the presence of massive particles, bosonic charges are limited to $M_{[\mu\nu]}$ and P_μ plus (optional) scalar internal symmetry charges, and the Lie algebra is the direct sum of the Poincaré algebra and a (finite-dimensional) compact Lie algebra for internal symmetry.

If superalgebras are admitted, the situation is governed by the Haag-ŁopuszańskiSohnius (HLS) theorem [31, 30], and the algebra of symmetries admits spinor charges Q_α^i. If there is only one Q_α, then the superalgebra must agree with the $\mathcal{N}=1$ Poincaré SUSY algebra in (6.1). When $\mathcal{N}>1$, the possibilities are restricted to the extended SUSY algebras discussed in Appendix 6A. [1] The main thought that we wish to convey is that SUSY theories realize the most general symmetry possible within the framework of the few assumptions made in the hypotheses of the CM and HLS theorems. [2] They also unify bosons and fermions, the two broad classes of particles found in Nature.

The parameters of global SUSY transformations are constant anti-commuting Majorana spinors ϵ_α. In supergravity SUSY is gauged, necessarily with the Poincaré generators, since they are joined in the superalgebra (6.1). This means that gravity is included, so the spinor parameters become arbitrary functions $\epsilon_\alpha(x)$ on a curved spacetime manifold. It is logically possible to skip ahead to Ch.9 where $\mathcal{N}=1, D=4$ supergravity is presented. But much important background will be missed, and we encourage only readers quite familiar with global SUSY to do this. We endeavor to give a succinct, pedagogical treatment of classical aspects of SUSY field theories. This material is certainly elegant, and part of the reason that SUSY is so appealing. However, there is much more in the deep results that have been discovered in perturbative and non-perturbative quantum supersymmetry that we cannot include.

The purpose of this chapter is to move as quickly as possible to an understanding of the structure of the major interacting SUSY field theories. At the classical level an interacting field theory is simply one in which the equations of motion are nonlinear. In Sec. 6.4, we give a short survey of the massless particle representations of extended Poincaré SUSY algebras.

1 They also found the extension with central charges, which will be discussed in Sec. 12.6.2.

2 In theories that contain only massless fields and are scale invariant at the quantum level, there are the additional possibilities of conformal and superconformal symmetries. The superconformal algebras contain the Poincaré SUSY algebras as subalgebras. They will be discussed later.

6.5.1 6.1 Basic SUSY field theory

SUSY theories contain both bosons and fermions, which are the basis states of a particle representation of the SUSY algebra (6.1)-(6.4). We give a systematic treatment of these representations in Sec. 6.4, but start with an informal discussion here. The states of particles with momentum $\vec{p}$ and energy $E(\vec{p})=\sqrt{\vec{p}^2+m_{B,F}^2}$ are denoted by $|\vec{p},B\rangle$ and $|\vec{p},F\rangle$, where the labels B and F include particle helicity. SUSY transformations connect these states. Since the spinor Q_α carries angular momentum $1/2$, it transforms bosons into fermions and fermions into bosons. Hence $Q_\alpha|\vec{p},B\rangle=|\vec{p},F\rangle$ and $Q_\alpha|\vec{p},F\rangle\propto|\vec{p},B\rangle$. Since $[P^\mu,Q_\alpha]=0$, the transformed states have the same momentum and energy, hence the same mass, so $m_B^2=m_F^2$. We show in Sec. 6.4.1 that a representation of the algebra contains the same number of boson and fermion states.

The simplest representations of the algebra that lead to the most basic SUSY field theories are:

(i) the chiral multiplet, which contains a self-conjugate spin- $1/2$ fermion described by the Majorana field $\chi(x)$ plus a complex spin- 0 boson described by the scalar field $Z(x)$. Alternatively, $\chi(x)$ may be replaced by the Weyl spinor $P_L\chi$ and/or $Z(x)$ by the combination $Z(x)=(A(x)+\mathrm{i}B(x))/\sqrt{2}$ where A and B are a real scalar and pseudo-scalar, respectively. A chiral multiplet can be either massless or massive.

(ii) the gauge multiplet consisting of a massless spin-1 particle, described by a vector gauge field $A_\mu(x)$, plus its spin- $1/2$ fermionic partner, the gaugino, described by a Majorana spinor $\lambda(x)$ (or the corresponding Weyl field $P_L\lambda$).

6.5.2 6.1.1 Conserved supercurrents

It follows from our discussion of the Noether formalism for symmetries that the spinor charge should be the integral of a conserved vector-spinor current, the supercurrent $\mathcal{J}^{\mu}_{\alpha}$, hence

$$Q_{\alpha}=\int \mathrm{d}^{3}x\mathcal{J}^{0}_{\alpha}(\vec{x},t)$$

If the current is conserved for all solutions of the equations of motion of a theory, then the theory has a fermionic symmetry. By the HLS theorem this symmetry must be supersymmetry!

Therefore we begin the technical discussion of SUSY in quantum field theory by displaying such conserved currents, [3] first for free fields and then for one non-trivial interacting system. Consider a free scalar field $\phi(x)$ satisfying the Klein-Gordon equation $(\Box-m^{2})\,\phi=0$ and a spinor field $\Psi(x)$ satisfying the Dirac equation $(\not{\partial}-m)\Psi=0$.

Exercise 6.2 Show that the current $\mathcal{J}^{\mu}=(\not{\phi}-m\phi)\gamma^{\mu}\Psi$ is conserved for all field configurations satisfying the Klein-Gordon and Dirac equations.

3 The spinor index α on the current and on most spinorial quantities will normally be suppressed.

It is no surprise to find unusual conserved currents in a free theory. In fact the current $\mathcal{J}^{\mu}{}_{\nu}=(\not{\partial}\phi-m\phi)\gamma^{\mu}\partial_{\nu}\Psi$, which gives a charge that violates the HLS theorem, is conserved. Such currents cannot be conserved in an interacting theory. For similar reasons conservation of $\mathcal{J}^{\mu}$ at the free level holds whether ϕ and Ψ are real or complex. To extend to interactions we will have to take $\phi\rightarrow Z$, a complex scalar, and Ψ a Majorana spinor. Note also that the current in Ex. 6.2 is conserved for any spacetime dimension D; this is another property that fails with interactions.

As the second example let's look at the free gauge multiplet with vector potential A_{μ} and field strength $F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$ satisfying the Maxwell equation $\partial^{\mu}F_{\mu\nu}=0$ and a spinor λ satisfying $\not{\partial}\lambda=0$. Let's show that the current $\mathcal{J}^{\mu}=\gamma^{\nu\rho}F_{\nu\rho}\gamma^{\mu}\lambda$ is conserved. We have

$$\partial_{\mu}\mathcal{J}^{\mu}=\partial_{\mu}F_{\nu\rho}\gamma^{\nu\rho}\gamma^{\mu}\lambda+\gamma^{\nu\rho}F_{\nu\rho}\not{\partial}\lambda.$$

The last term vanishes. To treat the first term we manipulate the γ-matrices as discussed in Sec. 3.1.4:

$$\gamma^{\nu\rho}\gamma^{\mu}=\gamma^{\nu\rho\mu}+2\gamma^{[v}\eta^{\rho]\mu}.$$

When inserted in the first term of (6.6) we see that the first term vanishes by the gauge field Bianchi identity (4.11), and the second one by the Maxwell equation.

6.5.3 6.1.2 SUSY Yang-Mills theory

With little more work we can now exhibit an important interacting theory, $\mathcal{N}=1$ SUSY Yang-Mills theory and its conserved supercurrent. The theory contains the gauge boson $A^{A}_{\mu}(x)$ and its SUSY partner, the Majorana spinor gaugino $\lambda^{A}(x)$ in the adjoint representation of a simple, compact, non-abelian gauge group G. The action is [4]

$$S=\int \mathrm{d}^{4}x\left[-\frac{1}{4}F^{\mu\nu A}F^{A}_{\mu\nu}-\frac{1}{2}\bar{\lambda}^{A}\gamma^{\mu}D_{\mu}\lambda^{A}\right]$$

For details of the notation see Secs. 3.4.1 and 4.3. Note that the gaugino action vanishes unless $\lambda^{A}(x)$ is anti-commuting! The Euler-Lagrange equations (and gauge field Bianchi identity) are

$$
\begin{aligned}
&D^{\mu}F^{A}_{\mu\nu} = -\frac{1}{2}gf_{BC}{}^{A}\bar{\lambda}^{B}\gamma_{\nu}\lambda^{C} \\
&D_{\mu}F^{A}_{\nu\rho} + D_{\nu}F^{A}_{\rho\mu} + D_{\rho}F^{A}_{\mu\nu} = 0 \\
&\gamma^{\mu}D_{\mu}\lambda^{A} = 0
\end{aligned}
$$

The supercurrent is

$$\mathcal{J}^{\mu} = \gamma^{\nu\rho}F^{A}_{\upsilon\rho}\gamma^{\mu}\lambda^{A}$$

4 We assume in this chapter that the Lie algebra has an invariant metric δ_{AB}, so that two 'upper' indices can be contracted.

The proof that it is conserved begins as in the free (abelian) case:

$$
\begin{aligned}
\partial_{\mu}\mathcal{J}^{\mu} &= D_{\mu}F^{A}_{\nu\rho}\gamma^{\nu\rho}\gamma^{\mu}\lambda^{A} + \gamma^{\upsilon\rho}F^{A}_{\nu\rho}\gamma^{\mu}D_{\mu}\lambda^{A} \\
&= -2D^{\mu}F^{A}_{\mu\nu}\gamma^{\nu}\lambda^{A} \\
&= gf_{ABC}\gamma^{\nu}\lambda^{A}\bar{\lambda}^{B}\gamma_{\nu}\lambda^{C}.
\end{aligned}
$$

The right-hand side vanishes due to (3.68) and the supercurrent (6.10) is conserved!

Thus we have established the existence of our first interacting SUSY field theory. Notice how the basic relations of non-abelian gauge symmetry such as the Bianchi identity, the relativistic description of spin by the Dirac-Clifford algebra, and the anti-commutativity of fermion fields are all blended in the proof. Readers whose intellectual curiosity is not excited by this are advised to put this book aside permanently and watch television instead of reading it.

The two main approaches to SUSY field theories are the approach of this chapter, in which we deal with the separate field components describing each physical particle in the theory, and the superspace approach, in which the separate fields are grouped in superfields. The latter approach is not used in this book, but is briefly discussed in Appendix 14A (see references there). A Fierz relation is always required to establish supersymmetry in the 'component' approach to any interacting SUSY theory. This is one reason why the existence and field content of SUSY field theories depend so markedly on the spacetime dimension. The Fierz relation also restricts the type of fermion required in the theory.

Here is an exercise in which readers are asked to show that super-Yang-Mills (SYM) theories with gauge field A_{μ} plus a specific type of spinor ψ^{A} and the supercurrent $\mathcal{J}^{\mu} = \gamma^{\nu\rho}F^{A}_{\nu\rho}\gamma^{\mu}\psi^{A}$ do exist in certain spacetime dimensions [32].

Exercise 6.3 Study the appropriate Fierz rearrangement and, using the results of Ex. 3.27, show that the supercurrent is conserved in the following cases:

(i) Majorana spinors in $D = 3$,

(ii) Majorana (or Weyl) spinors in $D = 4$, which is the case analyzed above,

(iii) symplectic Weyl spinors in $D = 6$, and

(iv) Majorana-Weyl spinors in $D = 10$.

Notice that, in every case, the number of on-shell degrees of freedom of the gauge field, namely $D-2$, matches those of the fermion, which are $2 \times 2^{[(D-2)/2]}/k$ (real), where $k = 1$ for a complex Dirac fermion, $k = 2$ for a Majorana, a Weyl or a symplectic Weyl fermion, and $k = 4$ for a Majorana-Weyl fermion. Equality of the total number of boson and fermion states is a necessary condition for SUSY. This basic fact will be proved for massive and massless physical states in Sec. 6.4.1 and in general (on- or off-shell) in Appendix 6B.

6.5.4 6.1.3 SUSY transformation rules

Although global SUSY can be formulated using conserved supercurrents as the primary vehicle, as was done above, it is usually more convenient to emphasize the idea of SUSY field variations involving spinor parameters ϵ_α under which actions must be invariant. The field variations are also called SUSY transformation rules. Given the conserved current one can form the supercharge using (6.5) and use the canonical formalism to compute the field variations, i.e.

$$\delta\Phi(x) = \{\bar{\epsilon}^\alpha Q_\alpha, \Phi(x)\}_{\mathrm{PB}} = -\mathrm{i}\left[\bar{\epsilon}^\alpha Q_\alpha, \Phi(x)\right]_{\mathrm{qu}}$$

where Φ denotes any field of the system under study. A brief description of Poisson brackets (PB) and commutation relations in the canonical formalism is given in Secs. 1.4 and 1.5. A link in the opposite direction is provided by the Noether formalism, which produces a conserved supercurrent given field variations under which the action is invariant. One reason to emphasize the field variations, *ab* initio, is that this avoids some subtleties in the canonical formalism for gauge theories and for Majorana spinors.

The next exercise illustrates the link between the supercurrent and field variations. It involves the free scalar-spinor $\phi - \Psi$ system of Ex. 6.2. The spinors Ψ, the supersymmetry parameters ϵ and the supersymmetry generator Q are Majorana spinors. They all mutually anti-commute. For the canonical formalism, one can either treat Ψ and $\bar{\Psi}$ as independent variables, or use Dirac brackets to obtain

$$\{\phi(x), \partial_0\phi(y)\}_{\mathrm{PB}} = -\{\partial_0\phi(x), \phi(y)\}_{\mathrm{PB}} = \delta^3(\vec{x} - \vec{y})$$
$$\left\{\Psi_\alpha(x), \bar{\Psi}^\beta(y)\right\}_{\mathrm{PB}} = \left\{\bar{\Psi}^\beta(x), \Psi_\alpha(y)\right\}_{\mathrm{PB}} = \left(\gamma^0\right)_\alpha^{\ \beta}\delta^3(\vec{x} - \vec{y})$$

Exercise 6.4 Use $\bar{Q} = (1/\sqrt{2})\int \mathrm{d}^3\vec{x}\bar{\Psi}\gamma^0(\not{\partial} + m)\phi$ or $Q = (1/\sqrt{2})\int \mathrm{d}^3\vec{x}\ (\not{\partial}\phi - m\phi)\gamma^0\Psi$ to obtain

$$\delta\phi(x) = \{\bar{\epsilon}Q, \phi(x)\}_{\mathrm{PB}} = \frac{1}{\sqrt{2}}\bar{\epsilon}\Psi(x)$$
$$\delta\Psi(x) = \{\bar{\epsilon}Q, \Psi(x)\}_{\mathrm{PB}} = \frac{1}{\sqrt{2}}(\not{\partial} + m)\phi\epsilon.$$

Note that $\left[\bar{Q}\epsilon, \Psi_\alpha(x)\right]_{\mathrm{PB}} = -\left\{\bar{Q}^\beta, \Psi_\alpha(x)\right\}_{\mathrm{PB}}\epsilon_\beta$.

6.5.5 6.2 SUSY field theories of the chiral multiplet

The physical fields of the chiral multiplet are a complex scalar $Z(x)$ and the Majorana spinor $\chi(x)$. It simplifies the structure in several ways to bring in a complex scalar auxiliary field $F(x)$. The field equations of F are algebraic, so F can be eliminated from the system at a later stage. The set of fields $Z, P_L\chi, F$ constitute a chiral multiplet, and their conjugates [5] $\bar{Z}, P_R\chi, \bar{F}$ are an anti-chiral multiplet. The treatment is streamlined because we use the chiral projections $P_L\chi$ and $P_R\chi$, but can still regard χ as a Majorana spinor; see Sec. 3.4.2.

Our program is to present the SUSY transformation rules of these multiplets, discuss invariant actions, and then study the SUSY algebra (6.1)-(6.4). The spinor parameter ϵ

5 Here we use $\bar{Z}$ to denote the complex conjugate rather than *, which was used earlier. Both notations will be used later in the book. is a Majorana spinor, whose spinor components anti-commute with each other and with components of χ and $\bar{\chi}$.

The transformation rules of the chiral multiplet are

$$\delta Z = \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi,$$
$$\delta P_L\chi = \frac{1}{\sqrt{2}}P_L(\not{\partial}Z + F)\epsilon,$$
$$\delta F = \frac{1}{\sqrt{2}}\bar{\epsilon}\ \not{\partial}P_L\chi.$$

The anti-chiral multiplet transformation rules are

$$\delta\bar{Z} = \frac{1}{\sqrt{2}}\bar{\epsilon}P_R\chi,$$
$$\delta P_R\chi = \frac{1}{\sqrt{2}}P_R(\partial\!\!\!/\bar{Z} + \bar{F})\epsilon,$$
$$\delta\bar{F} = \frac{1}{\sqrt{2}}\bar{\epsilon}\ \partial\!\!\!/ P_R\chi.$$

Note that the form of the transformation rules for the physical components is similar to those of the 'toy model' in Ex. 6.4.

Exercise 6.5 Show that the variations $\delta\bar{Z}, \delta P_R\chi, \delta\bar{F}$ are the complex conjugates of δZ, $\delta P_L\chi, \delta F$.

There are two basic actions, which are separately invariant under the transformation rules above. The first is the free kinetic action

$$S_{\text{kin}} = \int \mathrm{d}^4x\left[-\partial^\mu\bar{Z}\partial_\mu Z - \bar{\chi}\ \partial\!\!\!/ P_L\chi + \bar{F}F\right]$$

in which we have presented the spinor term in chiral form. The interaction is determined by an arbitrary holomorphic function, the superpotential $W(Z)$. Given this we define the action

$$S_F = \int \mathrm{d}^4x\left[FW'(Z) - \frac{1}{2}\bar{\chi}P_L W''(Z)\chi\right]$$

(The reason for the apparent extra derivative will be explained shortly.) Note that S_F involves only the fields of the chiral multiplet and no anti-chiral components. Thus the action S_F is not hermitian, so we must also consider the conjugate action $S_{\bar{F}} = (S_F)^\dagger$. The complete action of the chiral multiplet is the sum

$$S = S_{\text{kin}} + S_F + S_{\bar{F}}.$$

Exercise 6.6 Consider the superpotential $W = \frac{1}{2}mZ^2 + \frac{1}{3}gZ^3$, which gives the SUSY theory first considered by Wess and Zumino in 1973 [3]. Obtain the equations of motion for all fields, then eliminate F and $\bar{F}$ and show that the correct equations of motion for the physical fields are obtained if you first eliminate F and $\bar{F}$ by solving their algebraic equations of motion and substituting the result in the action. Substitute $Z = (A + \mathrm{i}B)/\sqrt{2}$ and show that the action (after elimination of auxiliary fields) takes the form

$$\begin{aligned} S_{WZ} = \int \mathrm{d}^4x\Bigg[&-\frac{1}{2}\left(\partial_\mu A\partial^\mu A + \partial_\mu B\partial^\mu B\right) - \frac{1}{2}m^2\left(A^2 + B^2\right) - \frac{1}{2}\bar{\chi}(\partial\!\!\!/ + m)\chi\Big) \\ &-\frac{g}{\sqrt{2}}\bar{\chi}\left(A + \mathrm{i}\gamma_* B\right)\chi - \frac{mg}{\sqrt{2}}\left(A^3 + AB^2\right) - \frac{g^2}{4}\left(A^2 + B^2\right)^2\Bigg]. \end{aligned}$$

From the viewpoint of a particle theorist this is a parity conserving, renormalizable theory with equal-mass fields and Yukawa plus quartic interactions.

Let's outline the proof that the actions S_{kin} and S_F are invariant under the SUSY transformation (6.15) and (6.16). It is rather intricate, so trusting readers may wish to move ahead. For S_{kin} the work is simplified by an observation that is correct in any representation of the Clifford algebra, but clearest in the Weyl representation (2.19). The projections $P_L\epsilon$ and $\bar{\epsilon}P_L$ involve the same half of the components of the Majorana ϵ, while $P_R\epsilon$ and $\bar{\epsilon}P_R$ involve the conjugate components. We write the total variation $\delta S = \delta_{P_L\epsilon}S + \delta_{P_R\epsilon}S$, temporarily separating the two chiral projections of ϵ in the transformation rules. Since S_{kin} is hermitian, it is sufficient to compute $\delta_{P_L\epsilon}S_{\text{kin}}$; then $\delta_{P_R\epsilon}S$ is its adjoint. In the calculation we temporarily allow $\epsilon(x)$ to be an arbitrary function in Minkowski spacetime since that provides a simple way [33] to obtain the Noether current for SUSY. We also need $\delta\bar{\chi}P_R = -(1/\sqrt{2})\bar{\epsilon}(\not\partial\bar{Z} - \bar{F})P_R$. Either the Dirac conjugate (2.30) or the Majorana conjugate relations (3.56) and (3.54) can be used. We suggest practice with the latter. (Note that $t_0 = -t_1 = 1$ in four dimensions.)

Now that we have prepared the way, let's calculate

$$\delta_{P_L\epsilon}S_{\text{kin}} = -\frac{1}{\sqrt{2}}\int \mathrm{d}^4x\big[\partial^\mu\bar{Z}\partial_\mu\left(\bar{\epsilon}P_L\chi\right) - \bar{\epsilon}(\not\partial\bar{Z})\not\partial P_L\chi$$
$$+\bar{\chi}\not\partial\left(P_LF\epsilon\right) - \left(\bar{\epsilon}\not\partial P_R\chi\right)F\big].$$

We have included all $P_L\epsilon$ and $\bar{\epsilon}P_L$ terms and dropped others. The $\bar{Z}\chi$ and $F\chi$ terms are independent and must vanish separately if we are to have a symmetry (when ϵ is constant). After a Majorana flip in the last term, we find that the $F\chi$ terms combine to (even for $\epsilon(x)$)

$$-\frac{1}{\sqrt{2}}\int \mathrm{d}^4x\partial_\mu\left(\bar{\chi}\gamma^\mu P_LF\epsilon\right)$$

which vanishes. The $\bar{Z}\chi$ terms can then be processed by substituting

$$\partial_\mu\left(\bar{\epsilon}P_L\chi\right) = \left(\partial_\mu\bar{\epsilon}\right)P_L\chi + \bar{\epsilon}P_L\partial_\mu\chi$$
$$\bar{\epsilon}P_L\gamma^\mu\gamma^\nu\left(\partial_\mu\bar{Z}\right)\partial_\nu\chi = \bar{\epsilon}P_L\left[\left(\partial^\mu\bar{Z}\right)\partial_\mu\chi + \gamma^{\mu\nu}\partial_\nu\left(\partial_\mu\bar{Z}\chi\right)\right]$$

in (6.21). Two of the four terms cancel. After partial integration and use of $\eta^{\mu\nu} - \gamma^{\mu\nu} = \gamma^v\gamma^\mu$, we find the net result

$$\delta_{P_L\epsilon}S_{\text{kin}} = -\frac{1}{\sqrt{2}}\int \mathrm{d}^4x\partial_\mu\bar{\epsilon}P_L(\not\partial\bar{Z})\gamma^\mu\chi$$

This shows that δS_{kin} vanishes for constant ϵ, which is enough to prove SUSY. The remaining term is a contribution to the supercurrent of the complete theory in (6.19), and we will include it below.

Since SUSY for the free action S_{kin} is not worth celebrating, we move on to discuss the interaction term S_F. The variation δS_F under the transformations (6.15) has the structure

$$\delta S_F = \int \mathrm{d}^4x[\delta FW'(Z) + \delta ZFW''(Z)$$
$$\left. -\delta\bar{\chi}P_L\chi W''(Z) - \frac{1}{2}\delta Z\bar{\chi}P_L\chi W'''(Z)\right],$$

where we have taken the derivatives of $W(Z)$ required to include all sources of the δZ variation. After use of (6.15) we combine terms. Two $FP_L\chi$ terms cancel and we are left with the net result

$$\delta S_F = \frac{1}{\sqrt{2}}\int \mathrm{d}^4x\left[\bar{\epsilon}\not\partial\left(W'P_L\chi\right) - \frac{1}{2}W'''\bar{\epsilon}P_L\chi\bar{\chi}P_L\chi\right]$$

The last term vanishes, since $P_L\chi$ has two independent components and any cubic expression vanishes by anti-commutativity! Thus δS_F vanishes for constant ϵ and is supersymmetric. It is clear that $\delta S_{\bar{F}}$ is just the conjugate of (6.26). At last SUSY is established at the interacting level!

The remaining $\partial_\mu\bar{\epsilon}$ terms in (6.24) and (6.26) plus their conjugates combine to give the Noether supercurrent of the interacting theory. This can be written as

$$\mathcal{J}^\mu = \frac{1}{\sqrt{2}}\left[P_L(\not\partial\bar{Z} - F) + P_R(\not\partial Z - \bar{F})\right]\gamma^\mu\chi$$

in which one must use the auxiliary field equations of motion $F = -\bar{W}'(\bar{Z})$ and $\bar{F} = -W'(Z)$.

Exercise 6.7 Show that the current is conserved for all solutions of the equations of motion of the theory (6.19).

Exercise 6.8 Given the component fields $\bar{Z}, P_R\chi, \bar{F}$ of an anti-chiral multiplet, show that $\bar{F}, P_L \not{\partial}\chi, \Box\bar{Z}$ transform in the same way as the $Z, P_L\chi, F$ components of a chiral multiplet; see (6.15) and (6.16).

6.5.6 6.2.1 $\mathrm{U}(1)_R$ symmetry

The R-symmetry is a phase transformation of the fields of the chiral and anti-chiral multiplets. The fields $Z, P_L\chi, F$ carry R-charges $r, r_\chi = r-1, r_F = r-2$ respectively. $\mathrm{U}(1)_R$ is a chiral symmetry, so the charges of component fields of the conjugate anti-chiral multiplet are the opposite of those above. We will discuss below how r is determined.

Infinitesimal transformations with parameter ρ are written as

$$\begin{aligned} \delta_R Z &= \mathrm{i}\rho r Z \\ \delta_R P_L\chi &= \mathrm{i}\rho(r-1)P_L\chi \\ \delta_R F &= \mathrm{i}\rho(r-2)F \end{aligned}$$

There are similar variations, with opposite charges, for $\bar{Z}, P_R\chi, \bar{F}$. The relations $r_\chi = r-1$ and $r_F = r-2$ are required by the commutator (6.4),

$$[\delta_R(\rho), \delta(\epsilon)] = \rho\bar{\epsilon}^\alpha [T_R, Q_\alpha] = -\mathrm{i}\rho\bar{\epsilon}^\alpha (\gamma_*)_\alpha^{\ \beta} Q_\beta$$

where $\delta(\epsilon)$ are the supersymmetry transformations (6.15).

Exercise 6.9 Calculate the variation $\delta_R\mathcal{J}^\mu$ of the supercurrent (6.27) using $\delta_R Z = \mathrm{i}\rho r Z, \delta_R P_L\chi = \mathrm{i}\rho r_\chi P_L\chi, \delta_R F = \mathrm{i}\rho r_F F$. Show that the result agrees with (6.4) if and only if $r_\chi = r-1$ and $r_F = r-2$.

The kinetic action S_{kin} is invariant under (6.28), so it is the interaction S_F that controls the situation. We now study the conditions for invariance of S_F. Clearly FW' and $\bar{\chi}W''\chi$ must be separately invariant. The condition for the vanishing of $\delta(FW')$ is

$$\delta_R(FW') = \mathrm{i}\rho F[(r-2)W' + rZW''] = 0.$$

This must hold for all field configurations, which means that the superpotential must satisfy

$$r(W' + ZW'') = 2W' \Rightarrow ZW' = \frac{2}{r}W$$

(since a constant can be absorbed in the definition of W without changing the physics). Similarly the condition for the vanishing of $\delta_R(\bar{\chi}W''\chi)$ is

$$\delta_R(\bar{\chi}W''\chi) = \mathrm{i}\rho\bar{\chi}[2(r-1)W'' + rZW''']\chi = 0.$$

However, the quantity in square brackets is the derivative of (6.31) and thus vanishes if $W(Z)$ satisfies (6.31).

The conclusion is that $W(Z)$ must be a homogeneous function of order $2/r$. This means that a theory with monomial superpotential $W(Z) = Z^k$ is $\mathrm{U}(1)_R$ invariant provided we assign $r = 2/k$ as the R-charge of the elementary scalar field Z. In the Wess-Zumino model, $W(Z) = mZ^2/2 + gZ^3/3$. If $g = 0$, then we have $\mathrm{U}(1)_R$ symmetry with $r = 1$. If $m = 0$ we have $\mathrm{U}(1)_R$ symmetry with $r = 2/3$. For general values of m and g, the symmetry is absent.

R-symmetry plays an important role in phenomenological applications of global supersymmetry. For example (see Sec. 28.1 of [30] or Sec. 5.2 of [34]), the related discrete R-parity is used to rule out undesired terms in the minimally supersymmetric standard model. $U(1)_R$ plays an important role in models with supersymmetry breaking.

6.5.7 6.2.2 The SUSY algebra

In this section we will study the realization of the SUSY algebra on the components of a chiral multiplet. It is convenient and interesting that the $\{Q, \bar{Q}\}$ anti-commutator in (6.1) is realized in classical manipulations as the commutator of two successive variations of the fields with distinct (anti-commuting) parameters ϵ_1, ϵ_2.

The variation of a generic field $\Phi(x)$ is given in (6.12). The commutator of successive variations δ_1, δ_2 of $\Phi(x)$, with parameters ϵ_1, ϵ_2, respectively, is (recall that $\bar{\epsilon}Q = \bar{Q}\epsilon$ for Majorana spinors)

$$\begin{aligned}
[\delta_1, \delta_2]\, \Phi(x) &= \left[\bar{\epsilon}_1 Q, \left[\bar{Q}\epsilon_2, \Phi(x)\right]\right] - (\epsilon_1 \leftrightarrow \epsilon_2) \\
&= \bar{\epsilon}_1^{\alpha} \left[\left\{Q_\alpha, \bar{Q}^\beta\right\}, \Phi(x)\right] \epsilon_{2\beta} \\
&= -\frac{1}{2}\bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu \Phi(x)
\end{aligned}$$

The standard Jacobi identity has been used to reach the second line and the first relation in (6.1) to obtain the last line. The key result is that the commutator of two SUSY variations is an infinitesimal spacetime translation with parameter $-\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2$.

Let's carry out the computation of $[\delta_1, \delta_2]\, Z(x)$ on the scalar field of a chiral multiplet. Using (6.15) we write

$$\begin{aligned}
[\delta_1, \delta_2]\, Z &= \frac{1}{\sqrt{2}}\delta_1 \left(\bar{\epsilon}_2 P_L \chi\right) - [1 \leftrightarrow 2] \\
&= \frac{1}{2}\bar{\epsilon}_2 P_L(\not{\partial} Z + F)\epsilon_1 - [1 \leftrightarrow 2] \\
&= -\frac{1}{2}\bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu Z
\end{aligned}$$

The symmetry properties of Majorana spinor bilinears (see (3.51)) have been used to reach the final result, which clearly shows the promised infinitesimal translation.

The analogous computation of $[\delta_1, \delta_2]\, P_L\chi(x)$ is more complex because a Fierz rearrangement is required. We outline it here:

$$\begin{aligned}
[\delta_1, \delta_2]\, P_L\chi &= \frac{1}{\sqrt{2}} P_L \left(\not{\partial}\delta_1 Z + \delta_1 F\right) \epsilon_2 - [1 \leftrightarrow 2] \\
&= \frac{1}{2} P_L \gamma^\mu \epsilon_2 \bar{\epsilon}_1 P_L \partial_\mu \chi + \frac{1}{2} P_L \epsilon_2 \bar{\epsilon}_1 \; \not{\partial} P_L \chi - [1 \leftrightarrow 2] \\
&= -\frac{1}{8}\left(\bar{\epsilon}_1 \Gamma_A \epsilon_2\right) P_L \left(\gamma^\mu \Gamma^A + \Gamma^A \gamma^\mu\right) P_L \partial_\mu \chi - [1 \leftrightarrow 2]
\end{aligned}$$

Each term in the second line was reordered as in Ex. 3.28 (with $\bar{\lambda}_1$ of (3.70) removed). We now find a great deal of simplification. Because of the antisymmetrization in $\epsilon_1 \leftrightarrow \epsilon_2$ the only non-vanishing bilinears are $\Gamma_A \to \gamma_\nu$ or $\gamma_{\nu\rho}$. However, only γ^ν survives the chiral projection in the last factor. Thus we find the expected result

$$[\delta_1, \delta_2]\, P_L\chi = -\frac{1}{2}\bar{\epsilon}_1 \gamma^\mu \epsilon_2 P_L \partial_\mu \chi$$

as the just reward for our labor.

Exercise 6.10 It is quite simple to demonstrate that

$$[\delta_1, \delta_2] F = -\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu F.$$

Zealous readers should do it.

The auxiliary field F can be eliminated from the action by substituting the value $F = -\bar{W}'(\bar{Z})$, which is the solution of its equation of motion from (6.19), without affecting the classical (or quantum) dynamics. Here is an exercise to show that SUSY is also maintained after elimination.

Exercise 6.11 Consider the theory after elimination of F and $\bar{F}$. Show that the action

$$S = \int \mathrm{d}^4x \left[-\partial^\mu \bar{Z}\partial_\mu Z - \bar{\chi}\ \not{\partial} P_L\chi - \bar{W}'W' - \frac{1}{2}\bar{\chi}\left(P_L W'' + P_R\bar{W}''\right)\chi\right]$$

is invariant under the transformation rules (6.15) and their conjugates (6.16). Show that $[\delta_1, \delta_2] Z$ is exactly the same as in (6.34), but $[\delta_1, \delta_2] P_L\chi$ is modified as follows:

$$[\delta_1, \delta_2] P_L\chi = \bar{\epsilon}_1\gamma^\mu\epsilon_2 P_L\left[-\frac{1}{2}\partial_\mu\chi + \frac{1}{4}\gamma_\mu\left(\not{\partial} + \bar{W}''\right)\chi\right]$$

We find the spacetime translation plus an extra term that vanishes for any solution of the equations of motion.

Since the commutator of symmetries must give a symmetry of the action [6] and translations are a known symmetry, the remaining transformation, namely

$$\begin{aligned}\delta Z &= 0\\ \delta\chi &= v^\mu\gamma_\mu\left(\not{\partial} + P_L W'' + P_R\bar{W}''\right)\chi,\end{aligned}$$

is itself a symmetry for any constant vector v^μ. However, its Noether charge vanishes when the fermion equation of motion is satisfied, so it has no physical effect. Such a symmetry is sometimes called a 'zilch symmetry'.

Although nothing physically essential is changed by eliminating auxiliary fields, we can nevertheless see that they play a useful role:

(i) It is only with F and $\bar{F}$ included that the form of the SUSY transformation rules (6.15) and (6.16) is independent of the superpotential $W(Z)$.

(ii) The SUSY algebra is also universal on all components of the chiral multiplet when F is included. The phrase used in the literature is that the SUSY algebra is 'closed off-shell' when auxiliary fields are included and 'closed only on-shell' when they are eliminated.

(iii) Auxiliary fields are very useful in determining the terms in a SUSY Lagrangian describing couplings between different multiplets. An example is the general SUSY gauge theory described in the next section.

(iv) In local supersymmetry auxiliary fields simplify the couplings of Faddeev-Popov ghost fields.

6 The argument is easy: a symmetry is a transformation such that $S_{,i}\delta(\epsilon)\phi^i = 0$, where $S_{,i}$ is the functional derivative with respect to the field ϕ^i. Applying a second transformation gives $S_{,ij}\delta\left(\epsilon_1\right)\phi^i\delta\left(\epsilon_2\right)\phi^j + S_{,i}\delta\left(\epsilon_2\right)\delta\left(\epsilon_1\right)\phi^i = 0$. Taking the commutator, the first term vanishes by symmetry, and the second term says that the commutator defines a symmetry.

It is also the case that auxiliary fields are known only for a few extended SUSY theories in four-dimensional spacetime and unavailable for many theories in dimension $D > 4$. Indeed many of the most interesting SUSY theories have no known auxiliary fields.

Although we hope that some readers enjoy the detailed manipulations needed to study SUSY theories, we suspect that many are fed up with Fierz rearrangement and would like a more systematic approach. To a large extent the superspace formalism does exactly that and has many advantages. Unfortunately, complete superspace methods are also unavailable when auxiliary fields are not known.

6.5.8 6.2.3 More chiral multiplets

We conclude this section with a discussion to establish a more general SUSY theory containing several chiral multiplets and their conjugates. We present this theory in the same 'blended' notation used above in which fermions always appear as the chiral projections $P_L\chi$ and $P_R\chi$ of Majorana spinors and the symmetry properties of Majorana bilinears can be used in all manipulations. To make the notation compatible with gauge symmetry in the next section, we denote chiral multiplets [7] by $Z^\alpha, P_L\chi^\alpha, F^\alpha$ and their anti-chiral adjoints by $\bar{Z}_\alpha, P_R\chi_\alpha, \bar{F}_\alpha$. Note that we use lower indices α for the fields of the anti-chiral multiplets.

The interactions of the general theory are determined by an arbitrary holomorphic superpotential $W\left(Z^\alpha\right)$. We denote derivatives of W by $W_\alpha = \partial W/\partial Z^\alpha, W_{\alpha\beta} = \partial^2 W/\partial Z^\alpha \partial Z^\beta$, etc. The kinetic action S_{kin} is the obvious generalization of (6.17) to include a sum over the index α, while the chiral interaction term becomes

$$S_F = \int \mathrm{d}^4 x \left[F^\alpha W_\alpha - \frac{1}{2}\bar{\chi}^\alpha P_L W_{\alpha\beta}\chi^\beta \right]$$

and one must add the conjugate action $S_{\bar{F}}$. The transformation rules of each multiplet are unmodified, but the index α is required, e.g. $\delta Z^\alpha = (1/\sqrt{2})\bar{\epsilon} P_L \chi^\alpha$. This general form of S_F explains why W' and W'' appear in (6.18).

Exercise 6.12 Show that the new actions $S_{\text{kin}}, S_F, S_{\bar{F}}$ are each invariant. The only essential new feature is that a Fierz rearrangement argument is required to show that the cubic term $W_{\alpha\beta\gamma}\bar{\epsilon} P_L \chi^\alpha \bar{\chi}^\beta P_L \chi^\gamma$, which is the analogue of the last term in (6.26), vanishes.

After elimination of the auxiliary field using $F^\alpha = \partial \bar{W}/\partial \bar{Z}_\alpha \equiv \bar{W}^\alpha$ one finds the physically equivalent action

$$\begin{aligned} S = \int \mathrm{d}^4 x [& -\partial^\mu \bar{Z}_\alpha \partial_\mu Z^\alpha - \bar{\chi}_\alpha \not{\partial} P_L \chi^\alpha - \bar{W}^\alpha W_\alpha \\ & \left. -\frac{1}{2}\bar{\chi}^\alpha P_L W_{\alpha\beta}\chi^\beta - \frac{1}{2}\bar{\chi}_\alpha P_R \bar{W}^{\alpha\beta}\chi_\beta \right]. \end{aligned}$$

7 We do not use spinor indices any more, such that the use of $\alpha, \ldots$ to indicate the multiplets should not create confusion.

The $\mathrm{U}(1)_R$ symmetry discussed in Sec. 6.2.1 may be extended to the general chiral multiplet theory provided that the superpotential $W\left(Z^\alpha\right)$ satisfies an appropriate condition. To investigate this we assign charges $r_\alpha, r_\alpha - 1, r_\alpha - 2$ to the fields $Z^\alpha, P_L\chi^\alpha, F^\alpha$, so that infinitesimal transformations with parameter ρ are written as

$$\begin{aligned} \delta_R Z^\alpha &= \mathrm{i}\rho r_\alpha Z^\alpha, \\ \delta_R P_L \chi^\alpha &= \mathrm{i}\rho \left(r_\alpha - 1\right) P_L \chi^\alpha, \\ \delta_R F^\alpha &= \mathrm{i}\rho \left(r_\alpha - 2\right) F^\alpha. \end{aligned}$$

(No sum on α.)

Exercise 6.13 Show that the general S_F of (6.41) is $\mathrm{U}(1)_R$ invariant for any set of charges r_α such that the superpotential satisfies the homogeneity condition

$$\sum_\alpha r_\alpha Z^\alpha W_\alpha = 2W$$

To prove this, you must generalize the argument of Sec. 6.2.1. The condition (6.44) is equivalent to the statement that W has definite R-charge $r_W = 2$.

For each specific theory with superpotential $W\left(Z^{\alpha}\right)$ there are several possibilities. There may not be any choice of the r_{α} for which (6.44) holds. This is the case in the Wess-Zumino model with $m \neq 0$ and $g \neq 0$ discussed at the end of Sec. 6.2.1. In some theories there is a unique set of R-charges, and in others many choices.

6.5.9 6.3 SUSY gauge theories

The basic SUSY gauge theory is the $\mathcal{N}=1$SYM theory containing the gauge multiplet A_{μ}^{A}, λ^{A}, where A is the index of the adjoint representation of a compact, non-abelian gauge group G. This theory was described in Sec. 6.1.2. The discussion there focused on the conserved supercurrent and will now be extended to include field variations, auxiliary fields, and the SUSY algebra.

We assume that the group has an invariant metric that can be chosen as δ_{AB}. This is the case for 'reductive groups', i.e. products of compact simple groups and abelian factors, i.e. $G=G_{1} \otimes G_{2} \otimes \ldots$, where each factor is a simple group or U(1). The normalization of the generators is fixed in each factor, which can lead to different gauge coupling constants $g_{1}, g_{2}, \ldots$ for each of these factors. We have taken here the normalizations of the generators where these coupling constants do not appear explicitly. One can replace everywhere t_{A} with $g_{i} t_{A}$ and $f_{AB}{}^{C}$ with $g_{i} f_{AB}{}^{C}$, where g_{i} can be chosen independently in each simple factor, to re-install these coupling constants. Usually one also redefines then the parameters θ^{A} to $\left(1 / g_{i}\right) \theta^{A}$. This leads to the formulas with coupling constant g in Sec. 4.3. Further note that for these algebras, the structure constants can be written as f_{ABC}, which are completely antisymmetric.

6.5.10 6.3.1 SUSY Yang-Mills vector multiplet

Our first objective is to obtain the SUSY variations δA_{μ}^{A} and $\delta \lambda^{A}$ under which the action (6.8) is invariant. This will give 'on-shell' supersymmetry; then we will add the auxiliary field. We will organize the presentation to make use of previous work in Secs. 6.1.1 and 6.1.2, which established that the supercurrent (6.10) is conserved.

The variation of (6.8) is

$$\delta S=\int \mathrm{d}^{4} x\left[\delta A_{\nu}^{A} D^{\mu} F_{\mu \nu}^{A}-\delta \bar{\lambda}^{A} \gamma^{\mu} D_{\mu} \lambda^{A}+\frac{1}{2} f_{ABC} \delta A_{\mu}^{A} \bar{\lambda}^{B} \gamma^{\mu} \lambda^{C}\right]$$

We first note that the forms

$$\delta A_{\mu}^{A}=-\frac{1}{2} \bar{\epsilon} \gamma_{\mu} \lambda^{A}, \quad \delta \lambda^{A}=\frac{1}{4} \gamma^{\rho \sigma} F_{\rho \sigma}^{A} \epsilon$$

are determined, up to constant factors, by Lorentz and parity symmetry and by the dimensions (in units of l^{-1}) of the quantities involved. Denoting the dimension of any quantity x by $[x]$, we have $[\epsilon]=-1/2,\left[A_{\mu}\right]=1,[\lambda]=3/2$. Note that if we use the assumed form for δA_{μ}, then the last term in (6.45) vanishes by the Fierz rearrangement identity (3.68). We substitute both assumed variations, assuming that $\epsilon(x)$ is a general function, and integrate by parts in the second term of (6.45) to obtain

$$\begin{aligned}
\delta S & =-\frac{1}{2} \int \mathrm{d}^{4} x\left[\bar{\epsilon} \gamma^{\nu} \lambda^{A} D^{\mu} F_{\mu \nu}^{A}+\frac{1}{2} \bar{\epsilon} \gamma^{\rho \sigma} \gamma^{\mu} \lambda^{A} D_{\mu} F_{\rho \sigma}^{A}+\frac{1}{2} \partial_{\mu} \bar{\epsilon} \gamma^{\rho \sigma} \gamma^{\mu} F_{\rho \sigma}^{A} \lambda^{A}\right] \\
& =-\frac{1}{2} \int \mathrm{d}^{4} x\left[\bar{\epsilon} \gamma^{\nu} \lambda^{A} D^{\mu} F_{\mu \nu}^{A}-\bar{\epsilon} \gamma^{\nu} \lambda^{A} D^{\mu} F_{\mu \nu}^{A}+\frac{1}{2} \partial_{\mu} \bar{\epsilon} \gamma^{\rho \sigma} \gamma^{\mu} F_{\rho \sigma}^{A} \lambda^{A}\right]
\end{aligned}$$

where (6.7) and the gauge field Bianchi identity were used to reach the final line. Thus δS vanishes for constant ϵ, establishing supersymmetry, while the supercurrent $\mathcal{J}^{\mu}$ of (6.10) appears in the last term! ![8]

The auxiliary field required for the gauge multiplet is a real pseudo-scalar field D^A in the adjoint representation of G. This fact follows from the superspace formulation. The auxiliary field enters the action and transformation rules in the quite simple fashion

$$
\begin{aligned}
S &= \int \mathrm{d}^4 x \left[-\frac{1}{4} F^{\mu\nu A} F^A_{\mu\nu} - \frac{1}{2}\bar{\lambda}^A \gamma^\mu D_\mu \lambda^A + \frac{1}{2} D^A D^A\right], \\
\delta A^A_\mu &= -\frac{1}{2}\bar{\epsilon}\gamma_\mu \lambda^A, \\
\delta \lambda^A &= \left[\frac{1}{4}\gamma^{\mu\nu} F^A_{\mu\nu} + \frac{1}{2}\mathrm{i}\gamma_* D^A\right]\epsilon, \\
\delta D^A &= \frac{1}{2}\mathrm{i}\bar{\epsilon}\gamma_* \gamma^\mu D_\mu \lambda^A, \quad D_\mu \lambda^A \equiv \partial_\mu \lambda^A + \lambda^C A^B_\mu f_{BC}{}^A.
\end{aligned}
$$

Exercise 6.14 Show that $\delta S = 0$. Only terms involving D^A need to be examined.

8 This term indicates that the true supercurrent should have been written in (6.10) with a factor $\frac{1}{4}$ included. $\frac{1}{4}\mathcal{J}^\mu$ generates correctly normalized SUSY variations.

The fields of the gauge multiplet transform also under an internal gauge symmetry:

$$
\begin{aligned}
\delta(\theta) A^A_\mu &= \partial_\mu \theta^A + \theta^C A^B_\mu f_{BC}{}^A, \\
\delta(\theta)\lambda^A &= \theta^C \lambda^B f_{BC}{}^A \\
\delta(\theta) D^A &= \theta^C D^B f_{BC}{}^A.
\end{aligned}
$$

Let us first remark that the commutator of these internal gauge transformations and supersymmetry vanishes.

Exercise 6.15 Use the transformation rules above to derive the SUSY commutator algebra of the gauge multiplet

$$
\begin{aligned}
\left[\delta_1, \delta_2\right] A^A_\mu &= -\frac{1}{2}\bar{\epsilon}_1 \gamma^\nu \epsilon_2 F^A_{\nu\mu}, \\
\left[\delta_1, \delta_2\right] \lambda^A &= -\frac{1}{2}\bar{\epsilon}_1 \gamma^\nu \epsilon_2 D_\nu \lambda^A \\
\left[\delta_1, \delta_2\right] D^A &= -\frac{1}{2}\bar{\epsilon}_1 \gamma^\nu \epsilon_2 D_\nu D^A
\end{aligned}
$$

It is no surprise that the commutator of two gauge covariant variations from (6.49) is gauge covariant, but at first glance the result seems to disagree with the spacetime translation required by (6.1) and (6.33). Note that in all three cases in Ex. 6.15 the difference between the covariant result in (6.51) and a translation is a gauge transformation by the field dependent gauge parameter $\theta^A = \frac{1}{2}\bar{\epsilon}_1 \gamma^v \epsilon_2 A^A_v$. The covariant forms that occur in (6.51) are called gauge covariant translations. The conclusion is that on the fields of a gauge theory the SUSY algebra closes on gauge covariant translations. See [35] and Sec. 4.1.5 for further information on this issue.

6.5.11 6.3.2 Chiral multiplets in SUSY gauge theories

We now present and briefly discuss the class of SUSY theories in which the gauge multiplet A^A_μ, λ^A, D^A is coupled to a chiral matter multiplet $Z^\alpha, P_L\chi^\alpha, F^\alpha$ in an arbitrary finitedimensional representation $\mathbf{R}$ of G with matrix generators $(t_A)^\alpha{}_\beta$. The representation may be either reducible or irreducible. A reducible representation may be decomposed into a direct sum of irreducible components $\mathbf{R}_i$. The matrix generators in each $\mathbf{R}_i$ are denoted by t_{Ai}. Formally the decomposition is expressed by

$$\mathbf{R} = \bigoplus_i \mathbf{R}_i$$
$$t_A = \bigoplus_i t_{Ai}$$

For most purposes and in most formulas below, the decomposition into irreducible representations need not be indicated explicitly, and we use it only where a more detailed notation is required.

The theory necessarily contains the conjugate anti-chiral multiplet $\bar{Z}_\alpha, P_R\chi_\alpha, \bar{F}_\alpha$, and we use lower indices to indicate that these fields transform in the conjugate representation $\overline{\mathbf{R}}$. Under an infinitesimal gauge transformation with parameters $\theta^A(x)$ the fermions transform as

$$\begin{aligned}
\delta P_L\chi^\alpha &= -\theta^A \left(t_A\right)^\alpha{}_\beta P_L\chi^\beta, \\
\delta P_R\chi_\alpha &= -\theta^A \left(t_A\right)_\alpha^{*\beta} P_R\chi_\beta,
\end{aligned}$$

with similar rules for the other fields. Representation indices are suppressed in most formulas. Thus we can write covariant derivatives of the various fields as

$$\begin{aligned}
D_\mu\lambda^A &= \partial_\mu\lambda^A + f_{BC}^A A_\mu^B \lambda^C, \\
D_\mu Z &= \partial_\mu Z + t_A A_\mu^A Z, \\
D_\mu P_L\chi &= \partial_\mu P_L\chi + t_A A_\mu^A P_L\chi, \\
D_\mu P_R\chi &= \partial_\mu P_R\chi + t_A^* A_\mu^A P_R\chi.
\end{aligned}$$

The system need not contain a superpotential, but superpotentials $W\left(Z^\alpha\right)$, which must be both holomorphic and gauge invariant, are optional. It is useful to express the condition of gauge invariance of $W\left(Z^\alpha\right)$ as

$$\delta_{\text{gauge}} W = W_\alpha \delta_{\text{gauge}} Z^\alpha = -W_\alpha \theta^A \left(t_A\right)^\alpha{}_\beta Z^\beta = 0.$$

The action of the general theory is the sum of several terms

$$S = S_{\text{gauge}} + S_{\text{matter}} + S_{\text{coupling}} + S_W + S_{\bar{W}}.$$

The form of some terms agrees with expressions given earlier in this chapter. Since convenience is a virtue and repetition is no sin, we shall write everything here:

$$\begin{aligned}
S_{\text{gauge}} &= \int \mathrm{d}^4x \left[-\frac{1}{4}F^{\mu\nu A}F_{\mu\nu}^A - \frac{1}{2}\bar{\lambda}^A\gamma^\mu D_\mu\lambda^A + \frac{1}{2}D^A D^A\right], \\
S_{\text{matter}} &= \int \mathrm{d}^4x \left[-D^\mu\bar{Z}D_\mu Z - \bar{\chi}\gamma^\mu P_L D_\mu\chi + \bar{F}F\right], \\
S_{\text{coupling}} &= \int \mathrm{d}^4x \left[-\sqrt{2}\left(\bar{\lambda}^A\bar{Z}t_A P_L\chi - \bar{\chi}P_R t_A Z\lambda^A\right) + \mathrm{i}D^A\bar{Z}t_A Z\right], \\
S_F &= \int \mathrm{d}^4x \left[F^\alpha W_\alpha + \frac{1}{2}\bar{\chi}^\alpha P_L W_{\alpha\beta}\chi^\beta\right], \\
S_{\bar{F}} &= \int \mathrm{d}^4x \left[\bar{F}_\alpha\bar{W}^\alpha + \frac{1}{2}\bar{\chi}_\alpha P_R\bar{W}^{\alpha\beta}\chi_\beta\right].
\end{aligned}$$

The full action is invariant under the SUSY transformation rules given in (6.49) for the gauge multiplet and the following modified gauge covariant transformation rules for the chiral and anti-chiral multiplets:

$$\delta Z=\frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi$$
$$\delta P_L\chi=\frac{1}{\sqrt{2}}P_L\left(\gamma^\mu D_\mu Z+F\right)\epsilon$$
$$\delta F=\frac{1}{\sqrt{2}}\bar{\epsilon}P_R\gamma^\mu D_\mu\chi-\bar{\epsilon}P_R\lambda^A t_A Z$$

and

$$\delta\bar{Z}=\frac{1}{\sqrt{2}}\bar{\epsilon}P_R\chi,$$
$$\delta P_R\chi=\frac{1}{\sqrt{2}}P_R\left(\gamma^\mu D_\mu\bar{Z}+\bar{F}\right)\epsilon,$$
$$\delta\bar{F}=\frac{1}{\sqrt{2}}\bar{\epsilon}P_L\gamma^\mu D_\mu\chi-\bar{\epsilon}P_L\lambda^A\left(t_A\right)^*\bar{Z}.$$

Some modifications in the action and transformation rules above, notably the introduction of gauge covariant derivatives, are clearly required in a gauge theory, but other additions such as the form of the action S_{coupling} are surely not obvious. The best explanation is that they are dictated by the superspace formalism. However, all features can be explained from the component viewpoint. For example, in the SUSY variation δS_{matter} many terms cancel by the same manipulations required to show that δS_{kin} of (6.17) vanishes by simply replacing ∂_μ by D_μ. But there are extra terms due to the variation δA_μ^A,

$$\delta S_{\text{matter}}=\int\mathrm{d}^4x\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A\left(\bar{\chi}t_A\gamma^\mu P_L\chi-\bar{Z}t_A D^\mu Z+D^\mu\bar{Z}t_A Z\right)$$

which involves the gauge current of the matter fields, and there is a correction to (6.23) due to the gauge Ricci identity (4.91) applied to $\bar{Z}$. The correction is proportional to $F_{\mu\nu}^A\bar{\epsilon}P_L\gamma^{\mu\nu}\bar{Z}t_A\chi$. These terms are canceled by the variations of Z and χ in $\delta S_{\text{coupling. A}}$ complete demonstration that the total action (6.56) is invariant under the variations (6.49) and (6.62) requires quite delicate calculations, which we recommend only for sufficiently diligent readers. The reader will also be invited to explain the extra terms in (6.62) from an algebraic viewpoint below in Ex. 14.2

Exercise 6.16 Show that the action (6.56) is supersymmetric. How does the variation $\delta F^\alpha W_\alpha(Z)$ induced by the last term in δF cancel?

The $\mathrm{U}(1)_R$ symmetry extends to SUSY gauge theories as a global symmetry which commutes with gauge transformations. Therefore the R-charges r_α that appear in the transformation (6.43) of chiral fields must be the same for all fields in a given irreducible component $\mathbf{R}_i$ of the full group representation $\mathbf{R}$. Invariance of the Yukawa terms in (6.59) determines the gaugino transformation

$$\delta_R\lambda^A=\mathrm{i}\rho\gamma_*\lambda^A$$

If the superpotential satisfies (6.44), then the full gauge theory is also $\mathrm{U}(1)_R$ invariant at the classical level. However, conservation of the Noether current R^μ is typically violated by the quantum level axial anomaly, and this has important consequences. We refer the reader to Sec. 29.3 of [30] and Sec. 2.C. of [36].

Exercise 6.17 Show that the Noether current R^μ and the gauge currents of the general theory are given by

$$
\begin{aligned}
R^{\mu} &= -\frac{1}{2}\mathrm{i}\bar{\lambda}^{A}\gamma^{\mu}\gamma_{*}\lambda^{A} + \mathrm{i}\sum_{\alpha}\left(r_{\alpha}\left(\bar{Z}_{\alpha}D_{\mu}Z^{\alpha} - D_{\mu}\bar{Z}_{\alpha}Z^{\alpha}\right) - \mathrm{i}\left(r_{\alpha}-1\right)\bar{\chi}_{\alpha}\gamma^{\mu}P_{L}\chi\right)\\
J^{\mu}_{A} &= -f_{ABC}\bar{\lambda}^{B}\gamma^{\mu}\lambda^{C} - \left(\bar{Z}t_{A}D_{\mu}Z - D_{\mu}\bar{Z}t_{A}Z\right) + \mathrm{i}\bar{\chi}\gamma^{\mu}P_{L}t_{A}\chi
\end{aligned}
$$

6.5.12 6.4 Massless representations of $\mathcal{N}$-extended supersymmetry

Up till now we have considered the supersymmetry generated by one Majorana spinor Q_{α}, subject to the structure relations of the superalgebra (6.1). The field theories that realize this algebra are said to possess simple supersymmetry. We now consider superalgebras containing $\mathcal{N} > 1$ Majorana spinor charges $Q_{i\alpha}, i = 1,\ldots,\mathcal{N}$. These algebras are called $\mathcal{N}$-extended supersymmetry algebras.

In the minimal extension, the different supersymmetry generators anti-commute, and each of them separately satisfies (6.1). We will rewrite (6.1) using the chiral components of the Majorana spinors (as justified in Box 3.3). We define $Q_{i\alpha}$ as the left-handed chiral components. They are simplest in the Weyl representation, in which the spinors have the form of (3.93). Then $Q_{i\alpha} = (Q_{i1}, Q_{i2}, 0, 0)$. We write the index i up for their hermitian conjugates, i.e. in Weyl representation: $Q^{\dagger i\alpha} = \left((Q_{i1})^{*}, (Q_{i2})^{*}, 0, 0\right)$. Therefore, we can effectively only use the index range $\alpha = 1, 2$. We will use the quantum expression, which according to Sec. 1.5 implies a multiplication with i; see (1.84). This gives the algebra

$$
\begin{aligned}
\left\{Q_{i\alpha}, Q^{\dagger j\beta}\right\}_{\mathrm{qu}} &= \frac{1}{2}\delta_{i}^{j}\left(\gamma_{\mu}\gamma^{0}\right)_{\alpha}{}^{\beta}P^{\mu}, \quad \alpha = 1,2\\
\left[M_{[\mu\nu]}, Q_{i\alpha}\right]_{\mathrm{qu}} &= -\frac{1}{2}\mathrm{i}\left(\gamma_{\mu\nu}\right)_{\alpha}{}^{\beta}Q_{i\beta}\\
\left[P_{\mu}, Q_{i\alpha}\right]_{\mathrm{qu}} &= 0
\end{aligned}
$$

We refer to Appendix 6A for a more detailed, and representation independent, definition of $Q_{i\alpha}$ and $Q^{\dagger j\beta}$ and the explanation of the anti-commutator in (6.87).

In this chapter, we will restrict attention to the extended supersymmetry algebra in (6.67). More general algebras, e.g. including the concept of 'central charges', will be deferred to Ch. 12.

6.5.13 6.4.1 Particle representations of $\mathcal{N}$-extended supersymmetry

We now discuss the particle representations of the superalgebras (6.67), that is, representations whose carrier space is the Hilbert space of a relativistic quantum field theory. Therefore we use a basis in which particles of energy-momentum $p^{\mu} = \left(p^{0} = E = \sqrt{m^{2} + \vec{p}^{2}}, \vec{p}\right)$ and spin s are described by states $|p^{\mu}, s, h\rangle$. For massive particles the helicity h takes $2s+1$ equally spaced values in the range $-s \le h \le s$. For a massless particle, there are two helicity values $h = \pm s$ if $s > 0$, and the unique value $h = 0$ if $s = 0$.

The carrier space of a particle representation of supersymmetry consists of the states $|p^{\mu}, s, h\rangle$ of a set of bosons, $s = 0, 1, 2, \ldots$, and fermions, $s = 1/2, 3/2, 5/2, \ldots$ The basic observation needed to study these representations is that $[Q, P] = 0$. Supersymmetry preserves the energy-momentum p^{μ} and thus the mass m of any particle state. Thus all that is needed to find the representations of the SUSY algebra is to consider finite sets of Bose and Fermi particles with fixed 4-momentum and various spins s and determine those sets on which the basic anti-commutator of supercharges can be realized irreducibly.

Let's first prove a very general result, namely that any irreducible representation of the SUSY algebra, whether it involves massive or massless particles, contains equal numbers of boson and fermion states. Taking a sum on spinor indices in (6.67) gives

$$Q_{i\alpha}Q^{\dagger j\alpha} + Q^{\dagger j\alpha}Q_{i\alpha} = \delta_i^j P^0$$

Consider the operator $\mathrm{e}^{-2\pi \mathrm{i} J_3}$ which implements rotations by angle 2π. Clearly its effect on boson and fermion states and on the supercharges is

$$\begin{aligned} \mathrm{e}^{-2\pi\mathrm{i}J_3}\,|p^\mu, s, h\rangle &= (-)^{2s}\,|p^\mu, s, h\rangle \\ \left\{Q_{i\alpha}, \mathrm{e}^{-2\pi\mathrm{i}J_3}\right\} &= 0 \end{aligned}$$

Multiply (6.68) on the right by $\mathrm{e}^{-2\pi\mathrm{i}J_3}$ and form the Hilbert space expectation value in a particle state $|p^\mu, s, h\rangle$. Finally sum over the spins and helicities of the particles in the carrier space of a representation. One finds

$$\begin{aligned} &\sum_{s,h} \left\langle p^\mu, s, h \,\middle|\left(Q_{i\alpha}Q^{\dagger j\alpha} + Q^{\dagger j\alpha}Q_{i\alpha}\right)\mathrm{e}^{-2\pi\mathrm{i}J_3}\middle|\, p^\mu, s, h\right\rangle \\ &\quad = \delta_i^j \sum_{s,h} \left\langle p^\mu, s, h \,\middle| P^0 \mathrm{e}^{-2\pi\mathrm{i}J_3}\middle|\, p^\mu, s, h\right\rangle . \end{aligned}$$

The sum over spins and helicities with fixed p^μ is equivalent to a matrix trace in a finitedimensional subspace of the Hilbert space and can be manipulated as a conventional matrix trace. Hence we can rewrite (6.70) as

$$\begin{aligned} \mathrm{Tr}\left(Q_{i\alpha}Q^{\dagger j\alpha}\mathrm{e}^{-2\pi\mathrm{i}J_3} + Q^{\dagger j\alpha}Q_{i\alpha}\mathrm{e}^{-2\pi\mathrm{i}J_3}\right) &= \delta_i^j E\,\mathrm{Tr}\,\mathrm{e}^{-2\pi\mathrm{i}J_3} \\ \mathrm{Tr}\left(Q_{i\alpha}Q^{\dagger j\alpha}\mathrm{e}^{-2\pi\mathrm{i}J_3} - Q^{\dagger j\alpha}\mathrm{e}^{-2\pi\mathrm{i}J_3}Q_{i\alpha}\right) &= \delta_i^j E\,\mathrm{Tr}\,\mathrm{e}^{-2\pi\mathrm{i}J_3} \end{aligned}$$

The left-hand side vanishes by cyclicity of the trace! The trace on the right-hand side can be rewritten as a sum over spins $s = 0, 1/2, 1, \ldots$ weighted by the number of particles n_s of spin s in the representation and the number of helicity states for each s. We thus obtain separate sum rules for massive and massless representations:

$$\begin{aligned} m > 0, &\quad \textstyle\sum_{s\geq 0}(-)^{2s} n_s(2s+1) = 0, \\ m = 0, &\quad 2\textstyle\sum_{s>0}(-)^{2s} n_s + n_0 = 0. \end{aligned}$$

This is the desired result since $(-)^{2s}$ is equal to $+1$ for bosons and -1 for fermions.

There is a small subtlety in the interpretation of (6.73). Lorentz transformations do not change the helicity of a massless particle, so an irreducible representation of the Poincaré group involves the momentum states for a single value of the helicity h. However, the CPT reflection symmetry requires that both helicity states $h = \pm s$ are present in the quantum field theory of a massless particle with spin $s > 0$. This doubling is incorporated in (6.73).

6.5.14 6.4.2 Structure of massless representations

In this section we derive the particle content of unitary irreducible representations involving massless particles. Similar techniques apply to both massless and massive representations, but we focus on the massless representations because they are simpler. The review of Sohnius [37] contains a more complete treatment.

We now use the Weyl representation (2.19) of the γ-matrices explicitly and (6.67) then gives

$$\begin{aligned} \left\{Q_{i\alpha}, Q^{\dagger j\beta}\right\}_{\mathrm{qu}} &= \frac{1}{2}\delta_i^j \left(\mathbb{1}P^0 - \vec{\sigma}\cdot\vec{P}\right)_\alpha^{\ \beta} \\ \left\{Q_{i\alpha}, Q_{j\beta}\right\}_{\mathrm{qu}} &= 0, \quad \left\{Q^{\dagger i\alpha}, Q^{\dagger j\beta}\right\}_{\mathrm{qu}} = 0 \\ \left[\vec{J}, Q_{i\alpha}\right]_{\mathrm{qu}} &= -\frac{1}{2}(\vec{\sigma})_\alpha^{\ \beta} Q_{i\beta} \end{aligned}$$

$\vec{J}$ stands for the space components $J^i = -\frac{1}{2}\varepsilon^{ijk}M_{jk}$.

Since SUSY transformations do not change the 4-momentum, it is sufficient to consider the action of the supercharges on a set of particle states $|\bar{p}, h\rangle$ of fixed energy-momentum $\bar{p}^\mu = (E, 0, 0, E)$. On states of 4-momentum $\bar{p}^\mu$, we find from (6.74) that

$$\begin{aligned}\left\{Q_{i1}, Q^{\dagger j1}\right\}_{\text{qu}} &= 0,\\ \left\{Q_{i2}, Q^{\dagger j2}\right\}_{\text{qu}} &= E\delta_i^j.\end{aligned}$$

We want a unitary representation, one in which the Hilbert space norm of all states is positive. The positivity properties of the anti-commutator then require that Q_{i1} and its adjoint must be represented trivially. They have vanishing action on all states of 4-momentum $\bar{p}^\mu$. The remaining non-trivial anti-commutator in our basis involves the $\mathcal{N}$ supercharge components Q_{i2}. Physicists can immediately recognize that this anti-commutator describes the creation and annihilation operators for $\mathcal{N}$ independent fermions. Equivalently the Q_{i2} anticommutator defines a Clifford algebra with $\mathcal{N}$ complex generators. This is equivalent to the real $2\mathcal{N}$-dimensional Clifford algebra we discussed in Sec. 3.1. The unique irreducible representation of this algebra has dimension $2^{\mathcal{N}}$, so the massless SUSY representation must also have $2^{\mathcal{N}}$ particle states.

The standard Fock space techniques of physics tells us that the unique unitary representation has the following structure. We choose the $Q^{\dagger i2}$ as the creation operators and the Q_{i2} as the annihilators. Note that (6.76) implies that $\left[J^3, Q^{\dagger i2}\right] = -\frac{1}{2}Q^{\dagger i2}$. Thus $Q^{\dagger i2}$ lowers the helicity of a state by 1/2. To specify the representation we define its 'Fock vacuum' $|\bar{p}, h_0\rangle$, with h_0 any positive or negative integer or half-integer, as the state that satisfies

$$Q_{i2}\,|\bar{p}, h_0\rangle = 0, \quad J^3\,|\bar{p}, h_0\rangle = h_0\,|\bar{p}, h_0\rangle\,, \quad \forall i = 1, \ldots, \mathcal{N}$$

The basis [9] of the representation consists of the vacuum state together with all states obtained by applying products of creators. Such products are automatically antisymmetric due to (6.75). In more detail the basis is

9 The particle spin $s = |h|$ is a redundant label, so it is omitted on all states of the Fock basis.

$$\begin{aligned}&|\bar{p}, h_0\rangle\\ &\left|\bar{p}, h_0 - \frac{1}{2}, i\right\rangle = Q^{\dagger i2}\,|\bar{p}, h_0\rangle\\ &|\bar{p}, h_0 - 1, [ij]\rangle = Q^{\dagger i2}Q^{\dagger j2}\,|\bar{p}, h_0\rangle\end{aligned}$$

etc.

States of helicity $h_0 - \frac{1}{2}m$ have multiplicity $\left(\begin{array}{c}\mathcal{N}\\ m\end{array}\right) = \mathcal{N}!/[m!(\mathcal{N}-m)!]$, and the sequence stops at the multiplicity 1 state of lowest helicity $h_0 - \frac{1}{2}\mathcal{N}$. The total number of states is $2^{\mathcal{N}}$ as is required by the representation theory of Clifford algebras. One can check that half the states are bosons and half are fermions.

Thus a massless irreducible representation of supersymmetry contains a 'tower' of helicity states of maximum helicity h_0 and minimum helicity $h_0 - \frac{1}{2}\mathcal{N}$. A local field theory always contains particles in CPT conjugate pairs with helicities $h = \pm s$. Therefore a supersymmetric field theory typically describes a reducible representation of the algebra in which the CPT conjugate states are added to the states of the basis (6.79). These states are obtained by starting from the CPT conjugate Clifford vacuum $|\bar{p}, -h_0\rangle$ and applying products of the operator Q_{i2}, which raises helicity. When $\mathcal{N} = 4\,|h_0|$, the initial sequence (6.79) is already self-conjugate so nothing needs to be added.

Because helicities are always paired, it is simplest to describe the field theory representations in terms of the number of particle states of a given spin. In Table 6.1 [38] we list the spin content [10] of all representations whose maximum spin satisfies $s_{\max} \leq 2$. It is this set of field theories that can incorporate nonlinear interactions.

Exercise 6.18 Show that the spin content of representations with $\mathcal{N} = 4s_{\max}$ and $\mathcal{N} = 4s_{\max} - 1$ is the same (see footnote 10).

The $\mathcal{N} = 1$ multiplets with maximum spin $s_{\max} = 1/2$ and $s_{\max} = 1$ are the chiral and gauge multiplets whose interactions are discussed earlier in this chapter. In principle the next $\mathcal{N} = 1$ multiplet has spins $(1, 3/2)$. There is a corresponding free field theory, but no interacting field theory is known for this multiplet without supergravity. The reason, discussed in Ch. 5, is that field theories for spin-3/2 fields involve a local supersymmetry. The supersymmetry algebra would contain local translations, and hence general relativity. Therefore, we find the spin- 3/2 particle only in the multiplet $(3/2, 2)$. This is the supergravity multiplet that we will consider in Ch. 9 .

The table is limited to $\mathcal{N} \leq 8$ because of the great difficulty of consistent higher spin interactions. For $\mathcal{N} \geq 9$ massless representations necessarily contain some particles of higher spin $s \geq 5/2$. Despite much effort, no interacting field theories in Minkowski spacetime exist. The content of the table may be summarized by the statements:

1. For $\mathcal{N} \leq 4$ there are interacting theories with global SUSY and $s_{\max} \leq 1$.
2. For $\mathcal{N} \leq 8$, there are theories with local SUSY, which involve one spin-2 graviton, $\mathcal{N}$ spin-3/2 gravitinos, and, for $\mathcal{N} \geq 2$, lower spin particles. These are supergravity theories.
3. For $\mathcal{N} \geq 9$ there is the higher spin desert.

10 There is a subtle hermiticity requirement for $\mathcal{N} = 2$, which requires that the multiplet $(-1/2, 0, 0, 1/2)$ must be doubled although it is self-conjugate.

		$s=2$	$s=3/2$	$s=1$	$s=1/2$	$s=0$
	$s_{\max}=2$	1	1			
$\mathcal{N}=1$	$s_{\max}=3/2$		1	1		
	$s_{\max}=1$			1	1	
	$s_{\max}=1/2$				1	$1+1$
	$s_{\max}=2$	1	2	1		
$\mathcal{N}=2$	$s_{\max}=3/2$		1	2	1	
	$s_{\max}=1$			1	2	$1+1$
	$s_{\max}=1/2$				2	$2+2$
	$s_{\max}=2$	1	3	3	1	
$\mathcal{N}=3$	$s_{\max}=3/2$		1	3	3	$1+1$
	$s_{\max}=1$			1	$3+1$	$3+3$
	$s_{\max}=2$	1	4	6	4	$1+1$
$\mathcal{N}=4$	$s_{\max}=3/2$		1	4	$6+1$	$4+4$
	$s_{\max}=1$			1	4	6
$\mathcal{N}=5$	$s_{\max}=2$	1	5	10	$10+1$	$5+5$
	$s_{\max}=3/2$		1	$5+1$	$10+5$	$10+10$
$\mathcal{N}=6$	$s_{\max}=2$	1	6	$15+1$	$20+6$	$15+15$
	$s_{\max}=3/2$		1	6	15	20
$\mathcal{N}=7$	$s_{\max}=2$	1	$7+1$	$21+7$	$35+21$	$35+35$
$\mathcal{N}=8$	$s_{\max}=2$	1	8	28	56	70

Exercise 6.19 The reader should check Table 6.1.

6.5.15 Appendix 6A Extended supersymmetry and Weyl spinors

It is often useful to discuss extended supersymmetry using Weyl spinors and their conjugates rather than Majorana spinors. The equivalence is discussed in Box 3.3, where we showed that one can represent Majorana spinors in terms of Weyl spinors and their conjugates, i.e. $Q = P_L Q + P_R Q$. The chirality of the (Majorana) conjugate spinors can be obtained from (3.56), which implies that

$$\overline{(P_R Q)} \equiv (P_R Q)^T C = \bar{Q} P_R$$

Hence applying a chiral projector to (6.1) teaches us that the two supercharges should have opposite chirality in order to have a non-vanishing anti-commutator:

$$\left\{ (P_L Q)_\alpha , \overline{(P_R Q)}^\beta \right\} = -\frac{1}{2} \left(P_L \gamma^\mu\right)_\alpha^{\ \beta} P_\mu .$$

It is convenient to use the up or down position of the index $i = 1, \ldots, \mathcal{N}$ to indicate at once the chiral projections of the supersymmetry generators for extended supersymmetry:

$$Q_i = P_L Q_i, \quad Q^i = P_R Q^i$$

The Majorana spinors are thus $Q^i + Q_i$, and Q_i is the charge conjugate of Q^i. From (6.80) we obtain

$$\bar{Q}_i = \overline{(P_L Q_i)} = \bar{Q}_i P_L, \quad \bar{Q}^i = \overline{(P_R Q^i)} = \bar{Q}^i P_R$$

The minimal extended algebra is then

$$\begin{aligned}
\left\{Q_{i\alpha}, \bar{Q}^{j\beta}\right\} &= -\frac{1}{2}\delta_i^j \left(P_L \gamma_\mu\right)_\alpha^{\ \beta} P^\mu, & \left\{Q^i_\alpha, \bar{Q}_j^\beta\right\} &= -\frac{1}{2}\delta_j^i \left(P_R \gamma_\mu\right)_\alpha^{\ \beta} P^\mu, \\
\left\{Q_{i\alpha}, \bar{Q}_j^\beta\right\} &= 0, & \left\{Q^i_\alpha, \bar{Q}^{j\beta}\right\} &= 0, \\
\left[M_{[\mu\nu]}, Q_{i\alpha}\right] &= -\frac{1}{2}\left(\gamma_{\mu\nu}\right)_\alpha^{\ \beta} Q_{i\beta}, & \left[M_{[\mu\nu]}, Q^i_\alpha\right] &= -\frac{1}{2}\left(\gamma_{\mu\nu}\right)_\alpha^{\ \beta} Q^i_\beta, \\
\left[P_\mu, Q_{i\alpha}\right] &= 0, & \left[P_\mu, Q^i_\alpha\right] &= 0.
\end{aligned}$$

Exercise 6.20 Lower the spinor indices in the first anti-commutator of (6.84) and check that the equation is consistent with the one obtained by taking the charge conjugate.

In Sec. 6.4 we used the complex conjugate spinors. First, remark that Q is Majorana, and we can thus also use $\bar{Q} = \mathrm{i} Q^\dagger \gamma^0$ in (6.80) to write

$$\overline{(P_R Q)} = \mathrm{i} Q^\dagger \gamma^0 P_R = \mathrm{i} Q^\dagger P_L \gamma^0 = \mathrm{i} \left(P_L Q\right)^\dagger \gamma^0,$$

which illustrates again that $(P_R Q)$ is not Majorana. Using the notation (6.82), this gives

$$\bar{Q}^i = \mathrm{i} \left(Q_i\right)^\dagger \gamma^0 = \mathrm{i} Q^{\dagger i} \gamma^0, \quad Q^{\dagger i\alpha} \equiv \left(Q_{i\alpha}\right)^\dagger .$$

In the last equation we define $Q^{\dagger i}$, with upper i index, as the hermitian conjugate of Q_i, which implies (omitting again spinor indices) $Q^{\dagger i} = Q^{\dagger i} P_L$. This leads to

$$\left\{Q_{i\alpha}, Q^{\dagger j\beta}\right\} = -\frac{1}{2} \mathrm{i} \delta_i^j \left(P_L \gamma_\mu \gamma^0\right)_\alpha^{\ \beta} P^\mu, \quad \alpha = 1, 2$$

In the Weyl representation, $Q_{i\alpha} = (Q_{i1}, Q_{i2}, 0, 0)$, and $\bar{Q}^i$ are their right-handed conjugates, i.e. $\bar{Q}^{i\alpha} = \left(0, 0, (\mathrm{i} Q_{i1})^*, \mathrm{i}\left(Q_{i2}\right)^*\right)$. The extra factor i to go to the quantum bracket thus leads to (6.67).

6.5.16 Appendix 6B On- and off-shell multiplets and degrees of freedom

It has been shown in Sec. 6.4.1 that on-shell multiplets have equal numbers of bosonic and fermionic degrees of freedom. But the reader may have noticed in the explicit examples in

There are equal numbers of bosonic and fermionic degrees of freedom in any realization of a supersymmetry algebra of the form $\{Q, Q\} = P$.

this chapter that also off-shell the multiplets have equal numbers of bosonic and fermionic degrees of freedom. More strictly stated, the theorem of Box 6.1 holds. The theorem is proven in [37]. Off-shell equality of bosonic and fermionic degrees of freedom holds for some extended supersymmetry and higher dimensional theories, but it is not always true. It is valid when the algebra (6.1) holds. In practice this holds when the theory has auxiliary fields which 'close the algebra off-shell'.

Consider first the example of the chiral multiplet. We have discussed the chiral multiplet first with the fields $\{Z, \chi, F\}$, and satisfying this algebra. Z and F are complex fields, and thus there are four real off-shell (since we did not use field equations) bosonic degrees of freedom. These are balanced by the four components of the Majorana spinor χ. We say that the chiral multiplet is a $4 + 4$ off-shell multiplet.

On the other hand, we have seen in Sec. 6.2.2 that the algebra is also valid when the equations of motion are used. Then F is no longer an independent field, we count two bosonic degrees of freedom for the complex Z, and also the fermions have two on-shell degrees of freedom. So the chiral multiplet is also a $2+2$ on-shell multiplet. These two ways of counting are called on-shell counting and off-shell counting.

To interpret this theorem we remind the reader of the terminology of on-shell and offshell degrees of freedom that we introduced in Box 4.1. To illustrate that the relevant definition of off-shell degrees of freedom indeed should contain the subtraction of gauge transformations, we consider the gauge multiplet. The off-shell counting is easily established: the gauge field A_μ and the gaugino λ both describe two on-shell degrees of freedom. To apply the theorem in the off-shell case, we have to remember that at the end of Sec. 6.3.1 we saw that the anti-commutator of two supersymmetries involves also a gauge transformation. Therefore, we can only apply the theorem on gauge invariant states (or identify states that differ by a gauge transformation), i.e. we have to subtract the gauge transformations

field	off-shell		on-shell	
		$D=4$		$D=4$
ϕ	1	1	1	1
λ	$2^{[D/2]}$	4	$\frac{1}{2}2^{[D/2]}$	2
A_μ	$D-1$	3	$D-2$	2
ψ_μ	$(D-1)2^{[D/2]}$	12	$\frac{1}{2}(D-3)2^{[D/2]}$	2
$g_{\mu\nu}$	$\frac{1}{2}D(D-1)$	6	$\frac{1}{2}D(D-3)$	2

in the counting. Thus, the gauge vector A_μ counts off-shell for three degrees of freedom, which together with the one real degree of freedom of the auxiliary field D balance the four off-shell ones of the gaugino.

Since this counting can be used to understand the structure of many realizations of supersymmetry, we end with Table 6.2 that summarizes the results for the degrees of freedom for the scalar field ϕ, a Majorana fermion λ, the gauge field A_μ, the Majorana RaritaSchwinger field ψ_μ and the graviton field $g_{\mu\nu}$.

Exercise 6.21 Check that the entries of the table correspond with the results on degrees of freedom obtained in previous chapters. The results for the graviton will be obtained in Sec. 8.2.

7 PART II DIFFERENTIAL GEOMETRY AND GRAVITY

7.1 Differential geometry

In this chapter we collect the ideas of differential geometry that are required to formulate general relativity and supergravity. There are several books, written for physicists, which explore this subject at greater length and greater depth [39, 40, 41, 42, 43, 44].

In general relativity spacetime is viewed as a differentiable manifold of dimension $D \geq 4$ with a metric of Lorentzian signature $(-,+,+,\ldots,+)$ indicating one time dimension and $D-1$ space dimensions. We assume that readers of this book are not intimidated by the idea of $D-4$ hidden dimensions that are not directly observed. We will also consider manifolds of purely Euclidean signature $(+,+,\ldots,+)$, which may appear in the hidden extra dimensions and as the target space of nonlinear σ-models.

We will give a reasonably rigorous definition of a manifold and then introduce the various quantities that 'live on it' in a less formal manner, emphasizing the way that the quantities transform under changes of coordinates. Invariance under coordinate transformations is one of the key principles that underlie general relativity. The most important structures we need are the metric, connection, and curvature. But other quantities such as vector and tensor fields and differential forms are also very useful. We will discuss them first since they require only the manifold structure.

It would be good if readers have already encountered some of the more elementary ideas before, perhaps in a course on general relativity. Our primary purpose is to collect the necessary ideas and explain them, hopefully clearly albeit non-rigorously, and thus to prepare readers for later chapters where the ideas are applied. Readers who do the suggested exercises will achieve the most thorough preparation.

7.1.1 7.1 Manifolds

A D-dimensional manifold is a topological space M together with a family of open sets M_i that cover it, i.e. $M = \cup_i M_i$. The M_i are called coordinate patches. On each patch there is a $1:1$ map ϕ_i, called a chart, from $M_i \to \mathbb{R}^D$. In more concrete language a point $p \in M_i \subset M$ is mapped to $\phi_i(p) = \left(x^1, x^2, \ldots, x^D\right)$. We say that the set $\left(x^1, x^2, \ldots, x^D\right)$ are the local coordinates of the point p in the patch M_i. If $p \in M_i \cap M_j$, then the map $\phi_j(p) = \left(x'^1, x'^2, \ldots, x'^D\right)$ specifies a second set of coordinates for the point p. The compound map $\phi_j \circ \phi_i^{-1}$ from $\mathbb{R}^D \to \mathbb{R}^D$ is then specified by the set of functions $x'^\mu\left(x^\nu\right)$. These functions, and their inverses $x^\nu\left(x'^\mu\right)$, are required to be smooth, usually C^∞. See Fig. 7.1 for an illustration of the ideas just discussed.

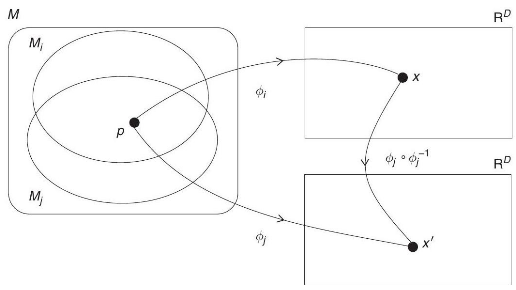

Fig. 7.1 Two charts in $\mathbb{R}^D$ for subsets M_i and M_j of the space M, and the compound map.

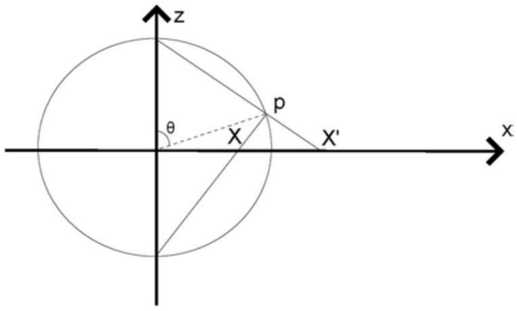

Fig. 7.2 Stereographic projection of the $x-z$ plane of the 2-sphere. The coordinates of the point p are $(x, y, z)=(\sin \theta, 0, \cos \theta)$.

We now describe the unit 2-sphere S^2 as an interesting and useful example of a manifold. Initially it may be defined as the surface $x^2+y^2+z^2=1$ embedded in $\mathbb{R}^3$. It is common to use the usual spherical polar coordinates θ, ϕ with $z=\cos \theta, x=\sin \theta \cos \phi, y=\sin \theta \sin \phi$. This is fine for some purposes, but it does not define a good coordinate chart at the poles $\theta=0, \pi$, since these points have no unique values of ϕ.

There are many ways to introduce coordinate charts to define a manifold structure. One useful way is to use the stereographic projection illustrated in Fig. 7.2. There are two patches whose union covers the sphere, namely M_1, consisting of the sphere with south pole deleted, and M_2, which is the sphere with north pole deleted. From the plane geometry of the triangles in Fig. 7.2, one defines the maps ϕ_1 and ϕ_2 to the central plane in the figure. These maps take the point with polar coordinates θ, ϕ to points X, Y and X', Y' respectively. The maps are given by

$$
\begin{aligned}
& \phi_1: X+\mathrm{i} Y=\mathrm{e}^{\mathrm{i} \phi} \tan (\theta / 2), \\
& \phi_2: X'+\mathrm{i} Y'=\mathrm{e}^{\mathrm{i} \phi} \cot (\theta / 2)
\end{aligned}
$$

On the overlap, we see that

$$
\phi_2 \circ \phi_1^{-1}(X, Y)=X'+\mathrm{i} Y'=1 /(X-\mathrm{i} Y)
$$

Exercise 7.1 Derive (7.1) and (7.2).

7.1.2 7.2 Scalars, vectors, tensors, etc.

The simplest objects to define on a manifold M are scalar functions f that map $M \rightarrow \mathbb{R}$. We say that the point p maps to $f(p)=z \in \mathbb{R}$. On each coordinate patch M_i we can define the compound map $f \circ \phi_i^{-1}$ from $\mathbb{R}^D \rightarrow \mathbb{R}$ as $f_i(x) \equiv f \circ \phi_i^{-1}(x)=z$, where x stands for $\{x^\mu\}$. On the overlap $M_i \cap M_j$ of two patches with local coordinates x^μ and x'^ν of the point p, the two descriptions of f must agree. Thus $f_i(x)=f_j\left(x'\right)$.

We now define the properties of scalar functions in the less formal way we will use for most of the objects that live on M. We no longer refer to a covering by coordinate patches. Instead we conceive of the manifold as a set whose points may be described by many different coordinate systems, say $\left(x^0, x^1, \ldots, x^{D-1}\right)$ and $\left(x'^0, x'^1, \ldots, x'^{D-1}\right)$. Any two sets of coordinates are related by a set of C^{∞} functions, e.g. $x'^\mu\left(x^\nu\right)$ with non-singular Jacobian $\partial x'^\mu / \partial x^\nu$. We refer to such a change of coordinates as a general coordinate transformation. A scalar function, also called a scalar field, is described by $f(x)$ in one set of coordinates and $f'\left(x'\right)$ in the second set. The two functions must be pointwise equal, i.e.

$$
f'\left(x'\right)=f(x) .
$$

Locally, at least, the informal definition agrees with the more formal one above.

In the same fashion, a contravariant vector field is described by D functions $V^{\mu}(x)$ in one coordinate system and D functions $V'^{\mu}(x')$ in the second. They are related by

$$V'^{\mu}(x') = \frac{\partial x'^{\mu}}{\partial x^{\nu}} V^{\nu}(x)$$

with a summation convention on the repeated index v. We go on to define covariant vector fields $\omega_{\mu}(x)$ and (mixed) tensors $T_{\nu}^{\mu}(x)$ by their behavior under coordinate transformations, namely

$$\begin{aligned}
\omega_{\mu}'(x') &= \frac{\partial x^{\nu}}{\partial x'^{\mu}} \omega_{\nu}(x), \\
T_{\nu}'^{\mu}(x') &= \frac{\partial x'^{\mu}}{\partial x^{\sigma}} \frac{\partial x^{\rho}}{\partial x'^{\nu}} T_{\rho}^{\sigma}(x).
\end{aligned}$$

We leave it to the reader to devise the analogous definitions of higher rank tensors such as $T^{\mu\nu}(x), S_{\mu\nu\rho}$, etc. A tensor field with p contravariant and q covariant indices is called a tensor of type (p, q) and rank $p+q$.

At this point in the development, contravariant and covariant quantities are unrelated objects, which transform differently. However a contravariant and covariant index can be contracted (i.e. summed) to define tensorial quantities of lower rank.

Exercise 7.2 Given $V^{\mu}(x), \omega_{\mu}(x), T_{\nu}^{\mu}(x)$, show that $V^{\mu}(x)\omega_{\mu}(x)$ transforms as a scalar field and that $T_{\nu}^{\mu}(x)V^{\nu}(x)$ transforms as a contravariant vector.

One can proceed with concrete local definitions of this type to obtain a physically satisfactory formulation of general relativity. However, there is much richness to be gained, and considerable practical advantage, if we develop the ideas further and incorporate some of the concepts of a more mathematical treatment of differential geometry.

Given a contravariant vector field $V^{\mu}(x)$, one can consider the system of differential equations

$$\frac{\mathrm{d}x^{\mu}}{\mathrm{d}\lambda} = V^{\mu}(x)$$

A solution $x^{\mu}(\lambda)$ is a map from $\mathbb{R} \rightarrow M$, which is a curve on M, called an integral curve of the vector field. There is an integral curve through every point of any open subset of M in which the vector field does not vanish. If the manifold is $\mathbb{R}^{D}$, then we know that the vector $\mathrm{d}x^{\mu}/\mathrm{d}\lambda$ is tangent to the curve $x^{\mu}(\lambda)$, and we make the same interpretation for a general manifold.

Let $x^{\mu}(\lambda)$ be the integral curve through the point p of M with coordinates $x^{\mu}(\lambda_0)$. Then $\mathrm{d}x^{\mu}/\left.\mathrm{d}\lambda\right|_{\lambda_0} = V^{\mu}(x(\lambda_0))$ is the tangent vector to the curve $x^{\mu}(\lambda)$ at p. We can now consider $D-1$ other vector fields $\tilde{V}^{\mu}(x)$ whose values $\tilde{V}^{\mu}(x(\lambda_0))$, together with the first $V^{\mu}(x(\lambda_0))$, fill out a basis of $\mathbb{R}^{D}$. Each $\tilde{V}^{\mu}(x(\lambda_0))$ is the tangent vector of an integral curve $\tilde{x}(\lambda)$ through p. Thus the vector fields evaluated at p determine the D-dimensional vector space $T_p(M)$, the tangent space to the manifold at point p. A vector field $V^{\mu}(x)$ may then be thought of as a smooth assignment of a tangent vector in each $T_p(M)$ as p varies over M. We shall use the notation $T(M)$ to denote the space of contravariant vector fields on M.

One important structure that one can form using the components $V^{\mu}(x)$ of a contravariant vector field is the differential operator $V = V^{\mu}(x)\partial/\partial x^{\mu}$. It follows from the transformation property (7.4) and the chain rule that V is constructed in the same way in all coordinate systems, e.g. $V = V'^{\mu}(x')\,\partial/\partial x'^{\mu}$. In this sense it is invariant under coordinate transformations. The differential operator V acts naturally on a scalar field $f(x)$, yielding another scalar field

$$\mathcal{L}_V f(x) \equiv V^{\mu}(x) \frac{\partial f}{\partial x^{\mu}}$$

On the manifold $\mathbb{R}^D$, this operation is just the directional derivative $V \cdot \nabla f$, and it has the same interpretation on a general manifold M. At each point p with coordinates x^m, $\mathcal{L}_V f(x)$ is the derivative of $f(x)$ in the direction of the tangent of the integral curve of $V^\mu(x)$ through p.

Locally, there is a $1:1$ correspondence between contravariant vector fields $V^\mu(x)$ and differential operators. In mathematical treatments a vector field is viewed as a smooth assignment of a differential operator at each point p. The set of elementary operators $\{\partial/\partial x^\mu, \mu = 1, \ldots, D\}$ are a basis in this view of the tangent space $T_p(M)$. This is consistent with our discussion since $\partial f/\partial x^\mu$ for a given value of μ is the derivative in the direction of the tangent to the curve on which the single coordinate x^μ changes, but the other coordinates x^ν for $v \neq \mu$ are constant. The basis $\{\partial/\partial x^\mu, \mu = 1, \ldots, D\}$ is called a coordinate basis because these operators differentiate along such coordinate curves at each p.

The derivative $\mathcal{L}_V f(x)$ defined in (7.7) may be extended to vector and tensor fields of any type (p, q), always yielding another tensor of the same type. For the vectors and tensors in (7.4) and (7.5), the precise definition is

$$\begin{aligned}
\mathcal{L}_V U^\mu &= V^\rho \partial_\rho U^\mu - (\partial_\rho V^\mu) U^\rho, \\
\mathcal{L}_V \omega_\mu &= V^\rho \partial_\rho \omega_\mu + (\partial_\mu V^\rho) \omega_\rho, \\
\mathcal{L}_V T^\mu_\nu &= V^\rho \partial_\rho T^\mu_\nu - (\partial_\rho V^\mu) T^\rho_\nu + (\partial_\nu V^\rho) T^\mu_\rho
\end{aligned}$$

The derivative defined in this way is called the Lie derivative. Its definition requires a vector field, but not a connection; yet it preserves the tensor transformation property.

Exercise 7.3 Show explicitly that $\mathcal{L}_V U^\mu, \mathcal{L}_V \omega_\mu$, and $\mathcal{L}_V T^\mu_\nu$ defined in (7.8) do transform under coordinate transformations as required by (7.4) and (7.5).

The Lie derivative of a contravariant vector field has special significance because it occurs in the commutator of the corresponding differential operators $U = U^\mu(x)\partial/\partial x^\mu$ and $V = V^\mu(x)\partial/\partial x^\mu$. An elementary calculation gives

$$[U, V] = W = W^\mu(x)\frac{\partial}{\partial x^\mu}$$

with $W^\mu = \mathcal{L}_U V^\mu = -\mathcal{L}_V U^\mu$. The new vector field W^μ is called the Lie bracket of U^μ and V^μ. This discussion also shows that the contravariant tensor fields on M naturally form a Lie algebra.

Let us consider the transformation properties of (7.3)-(7.5) for infinitesimal coordinate transformations, namely those for which $x'^\mu = x^\mu - \xi^\mu(x)$. To first order in $\xi^\mu(x)$, the previous transformation rules can be expressed in terms of Lie derivatives as

$$\begin{aligned}
\delta\phi(x) &\equiv \phi'(x) - \phi(x) = \mathcal{L}_\xi \phi, \\
\delta U^\mu(x) &\equiv U'^\mu(x) - U^\mu(x) = \mathcal{L}_\xi U^\mu, \\
\delta\omega_\mu(x) &\equiv \omega'_\mu(x) - \omega_\mu(x) = \mathcal{L}_\xi \omega_\mu, \\
\delta T^\mu_v(x) &\equiv T'^\mu_v(x) - T^\mu_v(x) = \mathcal{L}_\xi T^\mu_v.
\end{aligned}$$

Thus one of the useful roles of Lie derivatives is in the description of infinitesimal coordinate transformations.

Exercise 7.4 Show that the transformations (7.10) follow from (7.3)-(7.5).

Next we focus attention on covariant vector fields, such as $\omega_\mu(x)$. We already noted in Ex. 7.2 that the contraction $\omega_\mu(x)V^\mu(x)$ with any contravariant vector field gives a scalar field. Thus at any point p with coordinates $x^\nu, \omega_\mu(x)$ can be regarded as an element of the dual space $T^*_p(M)$, a linear functional that maps $T_p(M) \to \mathbb{R}$. The space $T^*_p(M)$ is usually called the cotangent space at p.

In parallel to the way in which we associated contravariant vector fields $V^\mu(x)$ with differential operators $V = V^\mu(x)\partial/\partial x^\mu$, we use the coordinate differentials $\mathrm{d}x^\mu$ to write $\Omega = \omega_\mu(x)\mathrm{d}x^\mu$.

Note that both $\omega_\mu(x)$ and $\mathrm{d}x^\mu$ transform under coordinate transformations, but $\Omega = \omega'_\mu(x')\,\mathrm{d}x'^\mu$ is constructed in the same way in any coordinate system. Ω is called a differential 1-form on M. Note that the gradient $\partial_\mu \phi(x)$ of any scalar transforms as a covariant vector and that the associated differential 1-form $\mathrm{d}\phi = \partial_\mu \phi \mathrm{d}x^\mu$ is just the differential of calculus. We can think of the set of coordinate differentials $\{\mathrm{d}x^\mu, \mu = 1, \ldots, D\}$ as a basis of the space of 1 -forms.

The notion of the cotangent space $T_p^*(M)$ of linear functionals on $T_p(M)$ is naturally extended to the level of 1-forms and differential operators. We define the pairing of basis elements as $\langle \mathrm{d}x^\mu \mid \partial/\partial x^\nu \rangle \equiv \delta^\mu_\nu$. This is extended using linearity to any general 1-form Ω and differential operator V, so that we then have $\langle \Omega \mid V \rangle = \omega_\mu(x) V^\mu(x)$. This agrees with the initial definition as the contraction of component indices.

7.1.3 7.3 The algebra and calculus of differential forms

Among the various fields defined on M, the scalars ϕ, covariant vectors ω_μ, and totally antisymmetric tensors such as $\omega_{\mu\nu} = -\omega_{\nu\mu}$ have a particularly useful structure when considered together. Note that antisymmetry is preserved under coordinate transformations so it is a tensorial property. Using the coordinate differentials $\mathrm{d}x^\mu$, we can construct differential p-forms for $p = 1, 2, \ldots, D$ as

$$
\begin{aligned}
\omega^{(1)} &= \omega_\mu(x)\mathrm{d}x^\mu \\
\omega^{(2)} &= \frac{1}{2}\omega_{\mu\nu}(x)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&\vdots \\
\omega^{(p)} &= \frac{1}{p!}\omega_{\mu_1\mu_2\cdots\mu_p}\,\mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}
\end{aligned}
$$

The wedge product is defined as antisymmetric; that is, $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$, $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho = -\mathrm{d}x^\rho \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$, etc. At each point we have an element of the p-fold antisymmetric tensor product of the cotangent space, so the differential form $\omega^{(p)}$ is a smooth assignment of an element of this tensor product as the point varies over M. The space of p-forms is denoted by $\Lambda^p(M)$. By convention the scalars are considered to be 0 -forms.

There is an exterior algebra and calculus of p-forms, which we will not develop in detail. See [40, 41, 42, 44, 45] for more complete discussions. Rather we will state some key properties without proof and write the specific examples needed later to discuss frames, connections, and curvature. In the exterior algebra, a p-form $\omega^{(p)}$ and a q-form $\omega^{(q)}$ can be multiplied to give a $(p+q)$-form if $p + q \leq D$. The product vanishes if $p + q > D$. The product satisfies $\omega^{(p)} \wedge \omega^{(q)} = (-)^{pq}\omega^{(q)} \wedge \omega^{(p)}$ and it is associative. Some examples are

$$
\begin{aligned}
\omega^{(1)} \wedge \tilde{\omega}^{(1)} &= \omega_\mu \mathrm{d}x^\mu \wedge \tilde{\omega}_\nu \mathrm{d}x^\nu \\
&= \frac{1}{2}\left(\omega_\mu \tilde{\omega}_\nu - \omega_\nu \tilde{\omega}_\mu\right)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
\omega^{(1)} \wedge \omega^{(2)} &= \omega_\mu \mathrm{d}x^\mu \wedge \frac{1}{2}\omega_{\nu\rho}(x)\mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \\
&= \frac{1}{6}\left(\omega_\mu \omega_{\nu\rho} + \omega_\nu \omega_{\rho\mu} + \omega_\rho \omega_{\mu\nu}\right)\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho
\end{aligned}
$$

The explicit antisymmetrization in the second line of each example is not necessary, since it is implicit in the wedge products of the $\mathrm{d}x^\mu$. But it is convenient to indicate that the covariant tensor field associated with each form is antisymmetric.

The exterior calculus is based on the exterior derivative, which maps p-forms into $(p+1)$-forms as follows:

$$d\omega^{(p)} = \frac{1}{p!}\partial_\mu \omega_{\mu_1\mu_2\cdots\mu_p}\, dx^\mu \wedge dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$$

Exercise 7.5 Show that the operation d is nilpotent, i.e. $d\left(d\omega^{(p)}\right) = 0$ on any p-form, and that it satisfies the distributive property

$$d\left(\omega^{(p)} \wedge \omega^{(q)}\right) = d\omega^{(p)} \wedge \omega^{(q)} + (-)^p \omega^{(p)} \wedge d\omega^{(q)}$$

On forms of degree $0, 1, 2$

$$\begin{aligned} d\phi &= \partial_\mu \phi dx^\mu \\ d\omega^{(1)} &= \frac{1}{2}\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) dx^\mu \wedge dx^\nu \\ d\omega^{(2)} &= \frac{1}{6}\left(\partial_\mu \omega_{\nu\rho} + \partial_\nu \omega_{\rho\mu} + \partial_\rho \omega_{\mu\nu}\right) dx^\mu \wedge dx^\nu \wedge dx^\rho \end{aligned}$$

A p-form that satisfies $d\omega^{(p)} = 0$ is called closed. A p-form $\omega^{(p)}$ that can be expressed as $\omega^{(p)} = d\omega^{(p-1)}$ is called exact. The Poincaré lemma implies that locally any closed p-form can be expressed as $d\omega^{(p-1)}$, but $\omega^{(p-1)}$ may not be well defined globally on M.

We saw that the exterior derivative is a map from p-forms into $(p+1)$-forms. There is also an interior derivative, which maps p-forms into $(p-1)$-forms. The latter depends on a vector V and is denoted as i_V. It is defined as follows:

$$\begin{aligned} \left(i_V \omega^{(p)}\right) &= \frac{1}{(p-1)!} V^\mu \omega_{\mu\mu_1\ldots\mu_{p-1}}\, dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_{p-1}} \\ i_V\left(dx^{\mu_1}\right. &\left.\wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}\right) \\ &= V^{\mu_1}\, dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p} - V^{\mu_2}\, dx^{\mu_1} \wedge dx^{\mu_3} \cdots \wedge dx^{\mu_p} + \ldots \end{aligned}$$

Exercise 7.6 Prove that the interior derivative is also nilpotent, i.e. $i_V i_V = 0$.

Like the internal and external derivatives, the Lie derivative, introduced in Sec. 7.2 as a derivative on tensor fields, has a simple action on p-forms. It maps p-forms to p-forms via the formula

$$\mathcal{L}_V = d i_V + i_V\, d$$

It is instructive to work out the example $\mathcal{L}_V \omega^{(1)} = \left(d i_V + i_V\, d\right)\omega^{(1)}$:

$$\begin{aligned} \left(d i_V + i_V\, d\right)\omega^{(1)} &= d\left(V^\mu \omega_\mu\right) + i_V \frac{1}{2}\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) dx^\mu \wedge dx^\nu \\ &= \left(\partial_\nu V^\mu \omega_\mu + V^\mu \partial_\nu \omega_\mu\right) dx^\nu + V^\mu\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) dx^\nu \\ &= \left(V^\mu \partial_\mu \omega_\nu + \partial_\nu V^\mu \omega_\mu\right) dx^\nu = \left(\mathcal{L}_V \omega\right)_\nu dx^\nu \end{aligned}$$

The Lie derivative of the covariant vector field ω_μ, which contains the components of the 1 -form $\omega^{(1)}$, was defined in (7.8) and appears in the final result. [1]

Exercise 7.7 Use the formula (7.17) to calculate the Lie derivative of a 0-form (where the first term vanishes by definition) and a 2-form. The final result should again contain the components of the Lie derivative as defined in Sec. 7.2.

Differential forms have a natural application to the theories of electromagnetism, to Yang-Mills theory, and to the antisymmetric tensor gauge theories that appear in higher dimensional supergravity. However, we need to bring in some other ideas in the next section before discussing these physical applications.

7.1.4 7.4 The metric and frame field on a manifold

We now introduce the additional structure of a metric on a manifold M. In general relativity the metric is of primary importance in describing the geometry of spacetime and the dynamics of gravity. In theories such as supergravity where there are fermions coupled to gravity, one must use an auxiliary quantity, the frame field (more commonly called the vierbein or vielbein), which we discuss in detail. The metric tensor is quadratically related to the frame field.

7.1.5 7.4.1 The metric

A metric or inner product on a real vector space V is a non-degenerate bilinear map from $V \otimes V \rightarrow \mathbb{R}$. The inner product of two vectors $u, v \in V$ is a real number denoted by (u, v). The inner product must satisfy the following properties:

(i) bilinearity, $(u, c_1 v_1 + c_2 v_2) = c_1 (u, v_1) + c_2 (u, v_2)$ and $(c_1 v_1 + c_2 v_2, u) = c_1 (v_1, u) + c_2 (v_2, u)$

(ii) non-degeneracy, if $(u, v) = 0$ for all $v \in V$, then $u = 0$;

(iii) symmetry, $(u, v) = (v, u)$.

1 The distributive formula $\mathcal{L}_V (\omega_\mu \mathrm{d}x^\mu) = (\mathcal{L}_V \omega_\mu) \mathrm{d}x^\mu + \omega_\mu \mathcal{L}_V \, \mathrm{d}x^\mu$ can be used if it interpreted carefully. Both terms are non-vanishing and can be calculated using (7.17) and (7.16). The latter equation requires that each component of $\mathcal{L}_V \omega_\mu = i_V \, \mathrm{d}\omega_\mu$ is calculated as the Lie derivative of a 0 -form.

The metric on a manifold is a smooth assignment of an inner product map on each $T_p(M) \otimes T_p(M) \rightarrow \mathbb{R}$. In local coordinates the metric is specified by a covariant second rank symmetric tensor field $g_{\mu\nu}(x)$, and the inner product of two contravariant vectors $U^\mu(x)$ and $V^\mu(x)$ is $g_{\mu\nu}(x) U^\mu(x) V^\nu(x)$, which is a scalar field. In particular the metric gives a formula for the length s of a curve $x^\mu(\lambda)$ with tangent vector $\mathrm{d}x^\mu / \mathrm{d}\lambda$:

$$s_{12} = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sqrt{g_{\mu\nu}(x(\lambda)) \left(\mathrm{d}x^\mu / \mathrm{d}\lambda\right) \left(\mathrm{d}x^\nu / \mathrm{d}\lambda\right)}$$

Thus it is most convenient to summarize the properties of a given metric by the line element

$$\mathrm{d}s^2 = g_{\mu\nu}(x) \mathrm{d}x^\mu \mathrm{d}x^\nu .$$

Non-degeneracy means that $\det g_{\mu\nu} \neq 0$, so the inverse metric $g^{\mu\nu}(x)$ exists as a rank 2 symmetric contravariant tensor, which satisfies

$$g^{\mu\rho} g_{\rho v} = g_{v\rho} g^{\rho\mu} = \delta_v^\mu .$$

The metric tensor and its inverse may be used to lower and raise indices, e.g. $V_\mu(x) = g_{\mu\nu} V^\nu(x)$ and $\omega^\mu(x) = g^{\mu\nu}(x) \omega_\nu(x)$, thus providing a natural isomorphism between the spaces of contravariant and covariant vectors and tensors.

In a gravity theory in spacetime, the metric has signature $- + + \cdots +$. Concretely this means that the metric tensor $g_{\mu\nu}$ may be diagonalized by an orthogonal transformation, i.e. $\left(O^{-1}\right)_\mu^a = O_\mu^a$ and

$$g_{\mu\nu} = O_\mu^a D_{ab} O_v^b ,$$

with positive eigenvalues λ^a in $D_{ab} = \operatorname{diag}\left(-\lambda^0, \lambda^1, \ldots, \lambda^{D-1}\right)$.

Exercise 7.8 Show that $\lambda^a(x) > 0$ holds throughout M if the metric is non-degenerate. In another coordinate system the transformed metric $g'_{\rho\sigma} = \left(\mathrm{d}x^\mu / \mathrm{d}x'^\rho\right) \left(\mathrm{d}x^\nu / \mathrm{d}x'^\sigma\right) g_{\mu\nu}$ may be diagonalized giving another set of eigenvalues λ'^a, in general different from the λ^a. Show that the $\lambda'^a > 0$. Thus the signature of a metric is a global invariant.

7.1.6 7.4.2 The frame field

The construction above, which involved only matrix linear algebra, allows us to define an important auxiliary quantity in a theory of gravity, namely

$$e_\mu^a(x) \equiv \sqrt{\lambda^a(x)} O_\mu^a(x)$$

In four dimensions this quantity is commonly called the tetrad or vierbein. In general dimension the term vielbein is frequently used, but we prefer the term frame field for reasons that will become clear as we discuss its properties.

Note that

$$g_{\mu\nu}(x) = e_\mu^a(x) \eta_{ab} e_\nu^b(x),$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, \ldots, 1)$ is the metric of flat D-dimensional Minkowski spacetime. It is (7.24) that states the general relation between metric and frame field. For a given metric tensor $g_{\mu\nu}(x)$, the frame field $e_\mu^a(x)$ of (7.23), obtained by diagonalization, is not the only solution. Given any x-dependent matrix $\Lambda_b^a(x)$ which leaves η_{ab} invariant, in other words, given a local Lorentz transformation, we can construct another solution of (7.24), namely

$$e_\mu^{\prime a}(x) = \Lambda^{-1a}{}_b(x) e_\mu^b(x).$$

All choices of frame fields related by local Lorentz transformations are viewed as equivalent. So we require that the frame field and geometrical quantities derived from it must be used in a way that is covariant with respect to the transformation (7.25).

Local Lorentz transformations in curved spacetime differ from the global Lorentz transformations of Minkowski space discussed in Ch. 1 . Only frame indices $a, b, \ldots$ of a quantity transform, coordinate indices $\mu, v, \ldots$ are inert, and the spacetime coordinate does not change. Instead, (7.24) requires that the frame field e_μ^a transforms as a covariant vector under diffeomorphisms (coordinate transformations), viz.

$$e_\mu^{\prime a}\left(x^{\prime}\right) = \frac{\partial x^\rho}{\partial x^{\prime \mu}} e_\rho^a(x)$$

while the frame index is inert. [2]

Since e_μ^a is a non-singular $D \times D$ matrix, with $\det e_\mu^a = \sqrt{-\det g} \neq 0$, there is an inverse frame field $e_a^\mu(x)$, which satisfies $e_\mu^a e_b^\mu = \delta_b^a$ and $e_a^\mu e_v^a = \delta_v^\mu$.

Exercise 7.9 Show that

$$e_a^\mu = g^{\mu\nu} \eta_{ab} e_v^b, \quad e_a^\mu g_{\mu\nu} e_b^v = \eta_{ab}$$

The last relation shows that the (inverse) frame field can be used to relate a general metric of signature $-++\cdots+$ to the Minkowski metric. Show that, under local Lorentz and coordinate transformations,

$$e_a^{\prime \mu}(x) = \Lambda_a^{-1b} e_b^\mu(x), \quad e_a^{\prime \mu}\left(x^{\prime}\right) = \frac{\partial x^{\prime \mu}}{\partial x^\rho} e_a^\rho(x)$$

Frame indices are raised and lowered using the Minkowski metric.

The second relation of (7.27) indicates that the e_a^μ form an orthonormal set of vectors in the tangent space of M at each point. Since $\det e_a^\mu \neq 0$, we have a basis of each tangent space. Any contravariant vector field has a unique expansion in the new basis, i.e. $V^\mu(x) = V^a(x) e_a^\mu(x)$ with $V^a(x) = V^\mu(x) e_\mu^a(x)$. The $V^a(x)$ are the frame components of the original vector field $V^\mu(x)$. They transform as a set of D scalar fields under coordinate transformations, and as a vector under Lorentz transformations, i.e. $V'^a(x) = \Lambda^{-1a_b}(x)V^b(x)$. The same may be done for covariant vectors, i.e. $\omega_\mu(x) = \omega_a(x)e_\mu^a(x)$ with $\omega_a(x) = \omega_\mu(x)e_a^\mu(x)$. These constructions may be extended to tensor fields of any rank in a straightforward way.

Thus we may use e_μ^a and e_a^μ to transform vector and tensor fields back and forth between a coordinate basis with indices $\mu, v, \ldots$ and a local Lorentz basis with indices $a, b, \ldots$ in

2 The relation between local and global Lorentz transformations is discussed further in Sec. 11.3.1. which the metric is η_{ab}. Invariants such as the inner product may be calculated in either basis.

Exercise 7.10 Show that

$$U^\mu(x)V_\mu(x) = g_{\mu\nu}(x)U^\mu(x)V^v(x) = \eta_{ab}U^a(x)V^b(x) = U^a(x)V_a(x).$$

At the level of differential operators the change of basis in the tangent space is expressed as

$$E_a \equiv e_a^\mu(x)\frac{\partial}{\partial x^\mu}$$

This makes it clear that the local Lorentz basis is a non-coordinate basis. If there were local coordinates y^a such that $E_a = \partial/\partial y^a$, these differential operators would commute. However, the commutator

$$[E_a, E_b] = -\Omega_{ab}^c E_c,$$

where $\Omega_{ab}{}^c = -e_\mu^c \mathcal{L}_{e_a} e_b^\mu = e_\mu^c \mathcal{L}_{e_b} e_a^\mu$ are the frame components of the Lie bracket, which do not vanish in a general manifold, and are called 'anholonomy coefficients'.

Exercise 7.11 Show that $\Omega_{ab}{}^c = e_a^\mu e_b^v \left(\partial_\mu e_v^c - \partial_\nu e_\mu^c\right)$.

We can also use the frame field e_μ^a to define a new basis in the spaces $\Lambda^p(M)$ of differential forms. The local Lorentz basis of 1 -forms is

$$e^a \equiv e_\mu^a(x)\mathrm{d}x^\mu$$

This is the dual basis to (7.30), since the pairing is given by $\langle e^a \mid E_b\rangle = \delta_b^a$. For 2-forms the basis consists of the wedge products $e^a \wedge e^b$, and so on.

In a field theory containing only bosonic fields, which are always vectors or tensors, the use of local frames is unnecessary, although it is an option that is convenient for some purposes. Local frames are a necessity to treat the coupling of fermion fields to gravity, because spinors are defined by their special transformation properties under Lorentz transformations.

7.1.7 7.4.3 Induced metrics

In many applications of differential geometry one encounters a manifold of dimension D which can be viewed as a surface embedded in flat Minkowski or Euclidean space of dimension $D+1$. We discuss the Euclidean case for $D=2$. Suppose that our surface is described by the equation

$$f(x, y, z) = 0.$$

On the surface the differential vanishes, viz.

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z = 0$$

The intrinsic geometry of the surface is determined by the Euclidean metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

To find it one can, in principle, solve (7.33) to eliminate one variable and then use (7.34) to find a relation among the coordinate differentials. When this information is inserted in (7.35), one has the induced metric. Voila!

This is often easier said than done, so we confine our discussion to the solvable and instructive example of the unit 2-sphere for which the embedding equation (7.33) is

$$x^2 + y^2 + z^2 = 1.$$

Let's proceed using spherical coordinates:

$$z = r\cos\theta, \quad x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi$$

The embedding equation becomes simply $r^2 = 1$, so we can eliminate the coordinate r and write the differentials:

$$\begin{aligned} \mathrm{d}z &= -\sin\theta \mathrm{d}\theta \\ \mathrm{d}x &= \cos\theta\cos\varphi \mathrm{d}\theta - \sin\theta\sin\varphi \mathrm{d}\varphi \\ \mathrm{d}y &= \cos\theta\sin\varphi \mathrm{d}\theta + \sin\theta\cos\varphi \mathrm{d}\varphi \end{aligned}$$

Upon substitution in (7.35) one finds the induced metric

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$$

This is a commonly used and quite useful metric on S_2, but it is evidently singular at the north and south poles where the metric tensor is not invertible. One can do somewhat better using one of the two sets of coordinates defined by the stereographic projection in Sec. 7.1, and this is the subject of the following exercise.

Exercise 7.12 Reexpress the metric (7.39) in the coordinates $X = \cos\varphi\tan(\theta/2), Y = \sin\varphi\tan(\theta/2)$. Show that the new metric is

$$\mathrm{d}s^2 = \frac{4\left(\mathrm{d}X^2 + \mathrm{d}Y^2\right)}{\left(1 + X^2 + Y^2\right)^2}$$

7.1.8 7.5 Volume forms and integration

The equations of motion in any field theory are most conveniently packaged in the action integral. In a gravitational theory this requires integration over the curved spacetime manifold. We thus need a procedure for integration that is invariant under coordinate transformations. The volume form is the key to this procedure.

On a D-dimensional manifold, one may choose any top degree D-form $\omega^{(D)}$ as a volume form and define the integral

$$\begin{aligned} I &= \int \omega^{(D)} \\ &= \frac{1}{D!}\int \omega_{\mu_1\cdots\mu_D}(x)\mathrm{d}x^{\mu_1}\wedge\cdots\wedge\mathrm{d}x^{\mu_D} \\ &= \int \omega_{01\cdots D-1}\, \mathrm{d}x^0\, \mathrm{d}x^1 \ldots \mathrm{d}x^{D-1}. \end{aligned}$$

The antisymmetric tensor $\omega_{\mu_1\cdots\mu_D}(x)$ has only one independent component, and we have used this fact in the last line above to write the integral so that it may be performed by the rules of multi-variable calculus. For the same reason any two D-forms $\tilde{\omega}(D)$ and $\omega^{(D)}$ must be related by $\tilde{\omega}(D) = f\omega^{(D)}$, where $f(x)$ is a scalar field. Thus the definition (7.41) includes $\int f\omega^{(D)}$.

Exercise 7.13 Show that in a new coordinate system with coordinates $x'^{\mu}\left(x^{\nu}\right)$ the integral I in (7.41) takes the form

$$I=\frac{1}{D!} \int \omega'_{\mu_1 \cdots \mu_D}\left(x'\right) \mathrm{d} x'^{\mu_1} \wedge \cdots \wedge \mathrm{d} x'^{\mu_D}$$

and is thus coordinate invariant.

Although there are many possible volume forms, there are two types that usually appear in the context of physics. The first, which is the more specialized, occurs when the physical theory contains form fields. As an example, on a 3-manifold the wedge product $\omega^{(1)} \wedge \omega^{(2)}$ can be chosen as a volume form. Using (7.12) we see that

$$\begin{aligned} I &=\int \omega^{(1)} \wedge \omega^{(2)} \\ &=\frac{1}{6} \int\left(\omega_\mu \omega_{\nu \rho}+\omega_\nu \omega_{\rho \mu}+\omega_\rho \omega_{\mu \nu}\right) \mathrm{d} x^\mu \wedge \mathrm{d} x^\nu \wedge \mathrm{d} x^\rho \\ &=\int\left(\omega_0 \omega_{12}+\omega_1 \omega_{20}+\omega_2 \omega_{01}\right) \mathrm{d} x^0\ \mathrm{d} x^1\ \mathrm{d} x^2 \end{aligned}$$

The integral is coordinate invariant, and it does not involve the metric on M. The action integral of the simplest Chern-Simons field theory, in which $\omega^{(2)}=\mathrm{d} \omega^{(1)}$, takes this form.

The second type of volume form is far more common in physics and we call it the canonical volume form. There are several ways to introduce it, and we will use the frame field $e_\mu^a(x)$ and the basis of frame 1-forms e^a for this purpose. As a preliminary we define the Levi-Civita alternating symbol in local frame components:

$$\varepsilon_{a_1 a_2 \cdots a_D}=\begin{cases}+1, a_1 a_2 \cdots a_D \text{ an even permutation of } 01 \ldots(D-1) \\ -1, a_1 a_2 \cdots a_D \text{ an odd permutation of } 01 \ldots(D-1) \\ \ \ 0, \text{ otherwise. }\end{cases}$$

Under (proper) Lorentz transformations, i.e. det $\Lambda^a{}_b=1$, this is an invariant tensor that takes the same form in any Lorentz frame. As usual Lorentz indices are raised with η^{ab}. Note that $\varepsilon^{01 \cdots(D-1)}=-1$.

Note that the Levi-Civita symbol provides a useful formula for the determinant of any $D \times D$ matrix A_b^a, namely

$$\det A \varepsilon_{b_1 b_2 \cdots b_D}=\varepsilon_{a_1 a_2 \cdots a_D} A_{b_1}^{a_1} A_{b_2}^{a_2} \cdots A_{b_D}^{a_D},$$

and that there are systematic identities for the contraction of p of the $D=p+q$ indices, as we saw in (3.9).

The Levi-Civita form in the coordinate basis is defined by contracting with frame fields and inserting factors of $e=\det e_\mu^a$ or e^{-1} :

$$\begin{aligned} \varepsilon_{\mu_1 \mu_2 \cdots \mu_D} & \equiv e^{-1} \varepsilon_{a_1 a_2 \cdots a_D} e_{\mu_1}^{a_1} e_{\mu_2}^{a_2} \cdots e_{\mu_D}^{a_D} \\ \varepsilon^{\mu_1 \mu_2 \cdots \mu_D} & \equiv e \varepsilon^{a_1 a_2 \cdots a_D} e_{a_1}^{\mu_1} e_{a_2}^{\mu_2} \cdots e_{a_D}^{\mu_D}\end{aligned}$$

Note that these definitions ensure that $\varepsilon^{\mu_1 \cdots \mu_D}$ and $\varepsilon_{\mu_1 \cdots \mu_D}$ take the constant values given on the right-hand side of (7.44). This can be seen using (7.45). The quantities defined in (7.46) are called tensor densities. It is important to recognize that $\varepsilon^{\mu_1 \mu_2 \cdots \mu_D}$ cannot be obtained by raising the indices of $\varepsilon_{\mu_1 \mu_2 \cdots \mu_D}$ in the usual way using the inverse of the metric. Therefore expressions like $\varepsilon^{\mu_1 \cdots \mu_p} \mu_{p+1} \cdots \mu_D$ are not well defined. There is no such problem for $\varepsilon^{a_1 \cdots a_p} b_{p+1} \cdots b_D$.

Exercise 7.14 Prove, using (7.45), that both $\varepsilon^{\mu_1 \mu_2 \cdots \mu_D}$ and $\varepsilon_{\mu_1 \mu_2 \cdots \mu_D}$ take values \pm 1 for any choice of frame field e_μ^a. This guarantees that they are invariant under infinitesimal changes of the frame field. Show also directly that $\delta \varepsilon^{\mu_1\mu_2\cdots\mu_D} = 0$ for any δe_a^μ using the general matrix formula

$$\delta \det M = (\det M) \operatorname{Tr}\left(M^{-1}\delta M\right),$$

and the Schouten identity; see (3.11).

With these preliminaries, the canonical volume form is defined as

$$\begin{aligned}
\mathrm{d}V &\equiv e^0 \wedge e^1 \wedge \cdots \wedge e^{D-1} \\
&= \frac{1}{D!}\varepsilon_{a_1\cdots a_D} e^{a_1} \wedge \cdots \wedge e^{a_D} \\
&= \frac{1}{D!} e \varepsilon_{\mu_1\cdots\mu_D}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_D} \\
&= e\, \mathrm{d}x^0 \ldots \mathrm{d}x^{D-1} \\
&= \mathrm{d}^D x \sqrt{-\det g}
\end{aligned}$$

Note that the determinant of the frame field e_μ^a appears in a natural fashion. In the last line we give the abbreviated notation we will use in most applications. For example, given the Lagrangian of a system of fields, such as the kinetic Lagrangian $L = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ of a scalar field, the action integral is written as

$$S = \int \mathrm{d}V L = \int \mathrm{d}^D x \sqrt{-\det g} L$$

7.1.9 7.6 Hodge duality of forms

Since p - and q-forms have the same number of components when $p+q = D$, it is possible to define a $1:1$ map between them. This map is the Hodge duality map from $\Lambda^p(M) \to \Lambda^q(M)$, and it is quite useful in the physics of supergravity. The map is denoted by $\Omega^{(q)} = *\omega^{(p)}$

Since the map is linear we can define it on a basis of p-forms and then extend to a general form. It is convenient to use the local frame basis initially and define

$$*e^{a_1} \wedge \cdots \wedge e^{a_p} = \frac{1}{q!} e^{b_1} \wedge \cdots \wedge e^{b_q} \varepsilon_{b_1\cdots b_q}{}^{a_1\cdots a_p}.$$

A general p-form can be expressed in this basis, and we can proceed to define its dual via

$$\begin{aligned}
\Omega^{(q)} = {}^*\omega^{(p)} &= * \left(\frac{1}{p!}\omega_{a_1\cdots a_p} e^{a_1} \wedge \cdots \wedge e^{a_p}\right) \\
&= \frac{1}{p!}\omega_{a_1\cdots a_p} {}^*e^{a_1} \wedge \cdots \wedge e^{a_p}
\end{aligned}$$

Exercise 7.15 Show that the frame components of $\Omega^{(q)}$ are given by

$$\Omega_{b_1\cdots b_q} = \left({}^*\omega\right)_{b_1\cdots b_q} = \frac{1}{p!}\varepsilon_{b_1\cdots b_q}{}^{a_1\cdots a_p}\omega_{a_1\cdots a_p}$$

These formulas are far less complicated than they look since there is only one independent term in each sum. For example, for $D = 4$ the dual of a 3-form is a 1-form. For basis elements we have $*e^1 \wedge e^2 \wedge e^3 = e^0$ and ${}^*e^0 \wedge e^1 \wedge e^2 = e^3$. For components, $({}^*\omega)_0 = \omega_{123}$ and $({}^*\omega)_3 = \omega_{012}$.

The duality has an important involutive property, which can be inferred from the following sequence of operations on basis elements:

$$
\begin{aligned}
*{}^*e^{a_1} \wedge \cdots \wedge e^{a_p}) &= \frac{1}{q!} e^{b_1} \wedge \cdots \wedge e^{b_q} \varepsilon_{b_1 \cdots b_q}{}^{a_1 \cdots a_p} \\
&= \frac{1}{p!q!} e^{c_1} \wedge \cdots \wedge e^{c_p} \varepsilon_{c_1 \cdots c_p}{}^{b_1 \cdots b_q} \varepsilon_{b_1 \cdots b_q}{}^{a_1 \cdots a_p} \\
&= -(-)^{pq} e^{c_1} \wedge \cdots \wedge e^{c_p} \delta^{a_1 \cdots a_p}_{c_1 \cdots c_p} \\
&= -(-)^{pq} e^{a_1} \wedge \cdots \wedge e^{a_p}
\end{aligned}
$$

This leads to the general relation ${}^*\left({}^*\omega^{(p)}\right) = -(-)^{pq}\omega^{(p)}$. This is the correct relation for a Lorentzian signature manifold. For Euclidean signature the involution property is $*\left({}^*\omega^{(p)}\right) = (-)^{pq}\omega^{(p)}$.

For even dimension $D = 2m$, it is possible to impose the constraint of self-duality (or anti-self-duality) on forms of degree m, i.e. $\Omega^{(m)} = \pm{}^*\Omega^{(m)}$. In a given dimension this condition is consistent only if duality is a strict involution, i.e. $-(-)^{m^2} = -(-)^m = +1$ for Lorentzian signature and $(-)^m = +1$ for Euclidean signature. Thus it is possible to have self-dual Yang-Mills instantons in four Euclidean dimensions. A self-dual $F^{(5)}$ is possible in $D = 10$ Lorentzian signature, and it indeed appears in Type IIB supergravity.

The duality relations defined above in a frame basis are easily transformed to a coordinate basis using the relations $e^a = e^a_\mu(x)\mathrm{d}x^\mu$ and $\mathrm{d}x^\mu = e^\mu_a(x)e^a$. For coordinate basis elements the duality map is

$$
*(\,\mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}) = \frac{1}{q!} e g^{\mu_1\rho_1} \cdots g^{\mu_p\rho_p}\, \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_p} \varepsilon_{\nu_1 \cdots \nu_q \rho_1 \cdots \rho_p}.
$$

For antisymmetric tensor components, we have

$$
({}^*\omega)_{\mu_1 \cdots \mu_q} = \frac{1}{p!} e \varepsilon_{\mu_1 \cdots \mu_q \rho_1 \cdots \rho_p} g^{\nu_1\rho_1} \ldots g^{v_p\rho_p} \omega_{\nu_1 \cdots v_p}
$$

Following the discussion in Sec. 7.5, we may take as a volume form on M the wedge product ${}^*\omega^{(p)} \wedge \omega^{(p)}$ of any p-form and its Hodge dual. The integral of this volume form is simply the standard invariant norm of the tensor components of $\omega^{(p)}$, i.e.

$$
\int {}^*\omega^{(p)} \wedge \omega^{(p)} = \frac{1}{p!} \int \mathrm{d}^D x \sqrt{-g}\, \omega^{\mu_1 \cdots \mu_p} \omega_{\mu_1 \cdots \mu_p}
$$

Exercise 7.16 Prove (7.56). Use the definitions above and those in Sec. 7.5 and the fact that

$$
e^{a_1} \wedge \cdots \wedge e^{a_q} \wedge e^{b_1} \wedge \cdots \wedge e^{b_p} = -\varepsilon^{a_1 \cdots a_q b_1 \cdots b_p}\, \mathrm{d}V
$$

where $\mathrm{d}V$ is the canonical volume element of (7.48).

Exercise 7.17 Show that the volume form $\mathrm{d}V$ can also be written as $*1$.

Exercise 7.18 Compare these definitions with Sec. 4.2.1, to obtain

$$
\tilde{F}_{\mu\nu} = -\mathrm{i}\,({}^*F)_{\mu\nu}\,.
$$

Show that the factor i ensures that the tilde operation squares to the identity. Self-duality is then possible for complex 2-forms.

Exercise 7.19 For applications to gauge field theories it is useful to record the relation between the components of the field strength 2-form and its dual:

$$
{}^*F_{\mu\nu} \equiv \frac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \qquad {}^*F^{\mu\nu} = \frac{1}{2\sqrt{-g}}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}
$$

Verify the second relation. Since both $F_{\mu\nu}$ and ${}^*F_{\mu\nu}$ are tensors, their indices are raised by $g^{\mu\nu}$.

7.1.10 7.7 Stokes' theorem and electromagnetic charges

Suppose that M is a manifold of dimension D, and that Σ_p with boundary $\Sigma_{p-1} = \partial\Sigma_p$ is a submanifold of dimension $p \leq D$. Suppose further that ω^{p-1} is a $(p-1)$-form that satisfies certain smoothness properties which we omit here. [3] Stokes' theorem asserts that

$$\int_{\Sigma_p} \mathrm{d}\omega^{p-1} = \int_{\Sigma_{p-1}} \omega^{p-1}$$

The integrals can be evaluated using any choice of coordinates.

In Ex. 4.14 electric and magnetic charges in three-dimensional flat spacetime were expressed as volume integrals. We will use Stokes' theorem to convert these to surface integrals. However, we generalize the discussion and consider a spacetime metric $g_{\mu\nu}(x)$ on M and a conserved current J^v to which the gauge field is coupled. The coordinate invariant action that describes this coupling is

$$S = \int \mathrm{d}^4 x \sqrt{-g} \left[-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} F_{\mu\nu} F_{\rho\sigma} + A_\nu J^\nu \right]$$

The gauge field equation of motion (4.49) and the tensor $G^{\mu v}$ of (4.50) generalize to

$$\begin{aligned}
\frac{\delta S}{\delta A_\nu} &= \partial_\mu \left(\sqrt{-g} F^{\mu\nu} \right) + \sqrt{-g} J^\nu = 0 \\
G_{\mu\nu} \equiv \varepsilon_{\mu\nu\rho\sigma} \frac{\delta S}{\delta F_{\rho\sigma}} &= -\frac{1}{2} \sqrt{-g} \varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma} = -{}^* F_{\mu\nu}.
\end{aligned}$$

The volume integrals (4.62) and (4.63) for the electric and magnetic charge contained in a region Σ_3 with boundary Σ_2 now become

$$\begin{aligned}
q &= \int_{\Sigma_3} \mathrm{d}^3 x \sqrt{-g} J^0 = \int_{\Sigma_3} \mathrm{d}^3 x \partial_i \sqrt{-g} F^{0i} = -\frac{1}{2} \int_{\Sigma_3} \mathrm{d}^3 x \varepsilon^{ijk} \partial_i G_{jk}, \\
p &= -\frac{1}{2} \int_{\Sigma_3} \mathrm{d}^3 x \varepsilon^{ijk} \partial_i F_{jk}.
\end{aligned}$$

The integrands in the final expressions are each the exterior derivatives of 2-forms on Σ_3, so we can apply Stokes' theorem and rewrite them as

$$\begin{pmatrix} p \\ q \end{pmatrix} = -\frac{1}{2} \int_{\Sigma_3} \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k \partial_i \begin{pmatrix} F_{jk} \\ G_{jk} \end{pmatrix} = -\frac{1}{2} \int_{\Sigma^2} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix}$$

The detailed form of $G_{\mu\nu} = -{}^* F_{\mu\nu}$ in terms of the components of $F_{\mu\nu}$ is given in (7.59).

Exercise 7.20 The components of $F_{\mu\nu}$ which describe point charges are quite basic quantities. Derive them for flat spacetime using polar coordinates with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2 \left(\mathrm{d}\theta^2 + \sin^2 \theta \mathrm{d}\phi^2 \right)$$

3 See [43] for details and proof of the theorem. Stokes' theorem dates from 1850 and 1854.

Show that a field configuration whose only non-vanishing component is

$$F_{\theta\phi} = -\frac{p}{4\pi} \sin\theta$$

is a solution of Maxwell's equation (7.62) and has magnetic charge p. Show that a field configuration whose only non-vanishing component is

$$E_r = F_{rt} = \frac{q}{4\pi r^2}$$

is a solution of Maxwell's equation (7.62) which describes an electric point charge q (use $\varepsilon_{\mathrm{tr}\,\theta\phi}=1$).

7.1.11 7.8p-form gauge fields

Using the ideas of Sec. 7.6, we can rewrite the simplest kinetic actions of scalars and gauge vectors as integrals of differential forms:

$$\begin{aligned} S_0 &= -\tfrac{1}{2}\int {}^*F^{(1)}\wedge F^{(1)}, \quad F^{(1)}\equiv \mathrm{d}\phi, \\ S_1 &= -\tfrac{1}{2}\int {}^*F^{(2)}\wedge F^{(2)}, \quad F^{(2)}\equiv \mathrm{d}A^{(1)}. \end{aligned}$$

In each case there is a Bianchi identity, $\mathrm{d}F^{(1)}=0$ and $\mathrm{d}F^{(2)}=0$, which implies that the field strengths can be written as differentials of a lower form. For the form $A^{(1)}$, which describes the photon, there is a gauge transformation that can be written as $\delta A^{(1)}=\mathrm{d}\Lambda^{(0)}$. We can interpret the actions of (7.69) as the definition of field theories for 0 -form and 1 -form 'potentials'.

This suggests a generalization. We can describe a p-form 'potential' in terms of a $(p+1)$-form 'field strength' and write the action

$$S_p=-\frac{1}{2}\int *F^{(p+1)}\wedge F^{(p+1)}, \quad F^{(p+1)}\equiv \mathrm{d}A^{(p)}$$

Again there is a gauge transformation $\delta A^{(p)}=\mathrm{d}\Lambda^{(p-1)}$, and these transformations of the p-form gauge potential leave $F^{(p+1)}$ and the action invariant.

Exercise 7.21 Show that the action (7.70) can be expressed in form components as

$$\begin{aligned} S_p &= -\frac{1}{2(p+1)!}\int \mathrm{d}^D x\sqrt{-g}F^{\mu_1\cdots\mu_{p+1}}F_{\mu_1\cdots\mu_{p+1}}, \\ F_{\mu_1\cdots\mu_{p+1}} &= (p+1)\partial_{[\mu_1}A_{\mu_2\ldots\mu_{p+1}]} \end{aligned}$$

We now determine the number of degrees of freedom of a p-form gauge field. The number of independent components in $\Lambda^{(p-1)}$ is $\begin{pmatrix} D \\ p-1 \end{pmatrix}$. However, not all the components of

Часть III
Modified Gravity Theories

8 Актуальные теории гравитации

пока без структуры, я просто знаю, что они есть, пусть вот и будут.

8.1 Purely affine gravity

(крутая статья от Nikodem J. Pop lawski, потом разберу её, очень много отсылок на разные теории гравитации)

8.1.1 Gravitation, electromagnetism and cosmological constant

(тут нужно будет перечитать статью, потому что там гоитческий лагранжиан, который матпикс не берет, так что переписвать всё нужно будет!!)

Обзор (???)

The Ferraris-Kijowski purely affine Lagrangian for the electromagnetic field, that has the form of the Maxwell Lagrangian with the metric tensor replaced by the symmetrized Ricci tensor, is dynamically equivalent to the metric Einstein-Maxwell Lagrangian, except the zero-field limit, for which the metric tensor is not well-defined. This feature indicates that, for the Ferraris-Kijowski model to be physical, there must exist a background field that depends on the Ricci tensor. The simplest possibility, supported by recent astronomical observations, is the cosmological constant, generated in the purely affine formulation of gravity by the Eddington Lagrangian. In this paper we combine the electromagnetic field and the cosmological constant in the purely affine formulation. We show that the sum of the two affine (Eddington and Ferraris-Kijowski) Lagrangians is dynamically inequivalent to the sum of the analogous (ΛCDM and Einstein-Maxwell) Lagrangians in the metricaffine/metric formulation. We also show that such a construction is valid, like the affine EinsteinBorn-Infeld formulation, only for weak electromagnetic fields, on the order of the magnetic field in outer space of the Solar System. Therefore the purely affine formulation that combines gravity, electromagnetism and cosmological constant cannot be a simple sum of affine terms corresponding separately to these fields. A quite complicated form of the affine equivalent of the metric EinsteinMaxwell-Λ Lagrangian suggests that Nature can be described by a simpler affine Lagrangian, leading to modifications of the Einstein-Maxwell-ΛCDM theory for electromagnetic fields that contribute to the spacetime curvature on the same order as the cosmological constant.

Суть (??)

(вообще хз)

8.1.2 FIELD EQUATIONS IN PURELY AFFINE GRAVITY

В двух словах

The condition for a Lagrangian density to be covariant is that it must be a product of a scalar and the square root of the determinant of a covariant tensor of rank two, or a linear combination of such products [2, 3]. A general purely affine Lagrangian density $\mathfrak{L}$ depends on the affine connection $\Gamma^{\rho}_{\mu\nu}$ (not restricted to be symmetric in the lower indices), the curvature tensor $R^\rho_{\mu\sigma\nu} = \Gamma^\rho_{\mu\nu,\sigma} - \Gamma^\rho_{\mu\sigma,\nu} + \Gamma^\kappa_{\mu\nu}\Gamma^\rho_{\kappa\sigma} - \Gamma^\kappa_{\mu\sigma}\Gamma^\rho_{\kappa\nu}$, and their covariant derivatives (with respect to $\Gamma^\rho_{\mu\nu}$). In order to be generally covariant, the Lagrangian density $\mathfrak{L}$ may depend on $\Gamma^\rho_{\mu\nu}$ only through the covariant derivatives of tensors. If we assume that is of the first differential order with respect to the connection, as usually are Lagrangians in classical mechanics with respect to the configuration, and that derivatives of the connection appear in ? only through the curvature $R^\rho_{\mu\sigma\nu}$, then $= \mathbb{L}(S, R)$.

We assume that the dependence of $\mathfrak{L}$ on the curvature is restricted to the symmetric part $P_{\mu\nu} = R_{(\mu\nu)}$ of the Ricci tensor $R_{\mu\nu} = R^\rho_{\mu\rho\nu}$, as in general relativity: $\boldsymbol{\mathcal{L}} = \boldsymbol{Z}(S, P)$.[2] We also assume that $\boldsymbol{x}$ depends on, in addition to the torsion tensor and the symmetrized Ricci tensor, a matter field ϕ and its covariant derivatives $\nabla\phi : \boldsymbol{L} = \mathfrak{L}(S, P, \phi, \nabla\phi)$. We denote this Lagrangian density as $\mathfrak{L}(\Gamma, P, \phi, \partial\phi)$, bearing in mind that its dependence on the connection Γ and ordinary derivatives $\partial\phi$ is not arbitrary, but such a Lagrangian is a covariant function of the torsion S and $\nabla\phi$. The variation of the corresponding action $I = \frac{1}{c}\int d^4x\mathfrak{L}$ is given by

$$\delta I = \frac{1}{c}\int d^4x \left(\frac{\partial \mathbf{1}}{\partial\Gamma^\rho_{\mu\nu}}\delta\Gamma^\rho_{\mu\nu} + \frac{\partial \mathbf{2}}{\partial P_{\mu\nu}}\delta P_{\mu\nu} + \frac{\partial}{\partial\phi}\delta\phi + \frac{\partial \mathbf{L}}{\partial\phi_{,\mu}}\delta\left(\phi_{,\mu}\right)\right).$$

The fundamental tensor density $g^{\mu\nu}$ associated with a purely affine Lagrangian is obtained using [4, 14, 15, 17, 31, 32]

$$\mathrm{g}^{\mu\nu} \equiv -\mathfrak{L}\kappa\frac{\partial \mathbf{2}}{\partial P_{\mu\nu}},$$

where $\kappa = \frac{8\pi G}{c^4}$ (for purely affine Lagrangians that depend on the symmetric part of the Ricci tensor, this definition is equivalent to that in Refs. $[2,3,42] : \mathrm{g}^{\mu\nu} = -2\kappa\frac{\partial \boldsymbol{L}}{\partial R_{\mu\nu}}\Big)$. This density introduces the metric structure in purely affine gravity by defining the symmetric contravariant metric tensor [4]:

$$g^{\mu\nu} \equiv \frac{\mathrm{g}^{\mu\nu}}{\sqrt{-\det\mathrm{g}^{\rho\sigma}}}.$$

To make this definition meaningful, we must assume $\det\mathrm{g}^{\mu\nu} \neq 0$. The physical signature requirement for $g_{\mu\nu}$ implies that we must take into account only those configurations with $\det\mathrm{g}^{\mu\nu} < 0$, which guarantees that $g_{\mu\nu}$ has the Lorentzian signature $(+,-,-,-)$ or $(-,+,+,+)$[16]. The symmetric covariant metric tensor $g_{\mu\nu}$ is related to the contravariant metric tensor by $g^{\mu\nu}g_{\rho\nu} = \delta^\mu_\rho$.[3] The tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ are used for raising and lowering indices.

We also define the hypermomentum density conjugate to the affine connection [47, 48, 49] :[4]

$$\Pi^{\mu\nu}_\rho \equiv -2\kappa\frac{\partial}{\partial\Gamma^\rho_{\mu\nu}}$$

which has the same dimension as the connection. Consequently, the variation of the action (1) can be written as

$$\delta I = -\frac{1}{2\kappa c}\int d^4x\left(\Pi^\mu_\rho\delta\Gamma^\rho_{\mu\nu} + \mathrm{g}^{\mu\nu}\delta R_{\mu\nu}\right) + \frac{1}{c}\int d^4x\left(\frac{\partial \mathbf{I}}{\partial\phi}\delta\phi + \frac{\partial \mathbf{I}}{\partial\phi_{,\mu}}\delta\left(\phi_{,\mu}\right)\right).$$

If we assume that the field ϕ vanishes at the boundary of integration, then the field equation for ϕ is $\frac{\delta}{\delta\phi} = 0$, where

$$\delta I = \frac{1}{c}\int d^4x\left[-\frac{1}{2\kappa}\left(\Pi^\mu_\rho\delta\Gamma^\rho_{\mu\nu} + \mathrm{g}^{\mu\nu}\delta P_{\mu\nu}\right) + \frac{\delta \mathbf{f}}{\delta\phi}\delta\phi\right]$$

For a general affine connection, the variation of the Ricci tensor is given by the Palatini formula [3, 43, 47]: $\delta R_{\mu\nu} = \delta\Gamma^\rho_{\mu\nu;\rho} - \delta\Gamma^\rho_{\mu\rho;\nu} - 2S^\sigma_{\rho\nu}\delta\Gamma^\rho_{\mu\sigma}$, where the semicolon denotes the covariant differentiation with respect to $\Gamma^\rho_{\mu\nu}$. Using the identity $\int d^4x\,(\mathrm{V}^\mu)_{;\mu} = 2\int d^4x S_\mu \mathrm{V}^\mu$, where V^μ

is an arbitrary contravariant vector density that vanishes at the boundary of the integration and $S_\mu = S^\nu_{\mu\nu}$ is the torsion vector [3, 43], and applying the principle of least action $\delta S = 0$ for arbitrary variations $\delta\Gamma^\rho_{\mu\nu}$, we obtain

$$\mathrm{g}^{\mu\nu}; \rho - \mathrm{g}^{\mu\sigma}{}_{;\sigma}\delta^\nu_\rho - 2\ \mathrm{g}^{\mu\nu}S_\rho + 2\ \mathrm{g}^{\mu\sigma}S_\sigma\delta^\nu_\rho + 2\ \mathrm{g}^{\mu\sigma}S^\nu_{\rho\sigma} = \Pi^{\mu\nu}_\rho.$$

This equation is equivalent to

$$\mathrm{g}^{\mu\nu}{}_{,\rho} + {}^*\Gamma^\mu_{\sigma\rho}\mathrm{g}^{\sigma\nu} + {}^*\Gamma^\nu_{\rho\sigma}\mathrm{g}^{\mu\sigma} - {}^*\Gamma^\sigma_{\sigma\rho}\mathrm{g}^{\mu\nu} = \Pi^{\mu\nu}_\rho - \frac{1}{3}\Pi^\mu_\sigma\sigma^\nu_\rho,$$

where ${}^*\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \frac{2}{3}\delta^\rho_\mu S_\nu$[3, 24]. Contracting the indices μ and ρ in Eq. (7) yields [5]

$$\Pi^{\sigma\nu}_\sigma = 0,$$

which is a constraint on how a purely affine Lagrangian depends on the connection. This unphysical constraint is related to the fact that the gravitational part of this Lagrangian (proportional to $\mathrm{g}^{\mu\nu}P_{\mu\nu}$, see the next section) is invariant under projective transformations of the connection while the matter part, that can depend explicitly on the connection, is generally not invariant [48, 49, 50]. We cannot assume that any form of matter will comply with this condition. Therefore the field equations (7) seem to be inconsistent. To overcome this constraint we can restrict the torsion tensor to be traceless: $S_\mu = 0$[50]. Consequently, ${}^*\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}$. This condition enters the Lagrangian density as a Lagrange multiplier term $-\frac{1}{2\kappa}\mathrm{B}^\mu S_\mu$, where the Lagrange multiplier B $^\mu$ is a vector density. Consequently, there is an extra term $\mathrm{B}^{[\mu}\delta^{\nu]}_\rho$ on the right-hand side of Eq. (7) and Eq. (9) becomes $\frac{3}{2}$ $\mathrm{B}^\nu = \Pi^\sigma{}_\sigma{}^\nu$. Setting this equation to be satisfied identically removes the constraint (9) and brings Eq. (8) into

$$\mathrm{g}^{\mu\nu}{}_{,\rho} + \Gamma^\mu_{\sigma\rho}\mathrm{g}^{\sigma\nu} + \Gamma^\nu_{\rho\sigma}\mathrm{g}^{\mu\sigma} - \Gamma^\sigma_{\sigma\rho}\mathrm{g}^{\mu\nu} = \Pi^{\mu\nu}_\rho - \frac{1}{3}\Pi^{\mu\sigma}_\sigma\delta^\nu_\rho - \frac{1}{3}\Pi^\sigma_{\sigma\rho}\delta^\mu_\rho$$

Equation (10) is an algebraic equation for $\Gamma^\rho_{\mu\nu}$ (with $S_\mu = 0$) as a function of the metric tensor, its first derivatives and the density $\Pi^\mu_\rho 2$. We seek its solution in the form:

$$\Gamma^\rho_{\mu\nu} = \left\{{}^{\rho}_{\mu\nu}\right\}_g + V^\rho_{\mu\nu},$$

which is a constraint on how a purely affine Lagrangian depends on the connection. This unphysical constraint is related to the fact that the gravitational part of this Lagrangian (proportional to $\mathrm{g}^{\mu\nu}P_{\mu\nu}$, see the next section) is invariant under projective transformations of the connection while the matter part, that can depend explicitly on the connection, is generally not invariant [48, 49, 50]. We cannot assume that any form of matter will comply with this condition. Therefore the field equations (7) seem to be inconsistent. To overcome this constraint we can restrict the torsion tensor to be traceless: $S_\mu = 0$[50]. Consequently, ${}^*\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}$. This condition enters the Lagrangian density as a Lagrange multiplier term $-\frac{1}{2\kappa}\mathrm{B}^\mu S_\mu$, where the Lagrange multiplier B $^\mu$ is a vector density. Consequently, there is an extra term $\mathrm{B}^{[\mu}\delta^{\nu]}_\rho$ on the right-hand side of Eq. (7) and Eq. (9) becomes $\frac{3}{2}$ $\mathrm{B}^\nu = \Pi^\sigma{}_\sigma{}^\nu$. Setting this equation to be satisfied identically removes the constraint (9) and brings Eq. (8) into

$$\mathrm{g}^{\mu\nu}{}_{,\rho} + \Gamma^\mu_{\sigma\rho}\mathrm{g}^{\sigma\nu} + \Gamma^\nu_{\rho\sigma}\mathrm{g}^{\mu\sigma} - \Gamma^\sigma_{\sigma\rho}\mathrm{g}^{\mu\nu} = \Pi^{\mu\nu}_\rho - \frac{1}{3}\Pi^{\mu\sigma}_\sigma\delta^\nu_\rho - \frac{1}{3}\Pi^\sigma_{\sigma\rho}\delta^\mu_\rho$$

Equation (10) is an algebraic equation for $\Gamma^\rho_{\mu\nu}$ (with $S_\mu = 0$) as a function of the metric tensor, its first derivatives and the density $\Pi^\mu_\rho 2$. We seek its solution in the form:

$$\Gamma^\rho_{\mu\nu} = \left\{{}^{\rho}_{\mu\nu}\right\}_g + V^\rho_{\mu\nu},$$

where $\left\{ {\rho \atop \mu\nu} \right\}_g = \frac{1}{2} g^{\rho\sigma} \left(g_{\nu\sigma,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma} \right)$ is the Christoffel connection of the metric tensor $g_{\mu\nu}$. Consequently, the Ricci tensor of the affine connection $\Gamma^\rho_{\mu\nu}$ is given by [43]

$$R_{\mu\nu}(\Gamma) = R_{\mu\nu}(g) + V^\rho_{\mu\nu:\rho} - V^\rho_{\mu\rho:\nu} + V^\sigma_{\mu\nu} V^\rho_{\sigma\rho} - V^\sigma_{\mu\rho} V^\rho_{\sigma\nu},$$

where $R_{\mu\nu}(g)$ is the Riemannian Ricci tensor of the metric tensor $g_{\mu\nu}$ and the colon denotes the covariant differentiation with respect to $\left\{ {\rho \atop \mu\nu} \right\}_g$. Substituting Eq. (11) to Eq. (10) gives

$$V^\mu_{\sigma\rho} \mathrm{g}^{\sigma\nu} + V^\nu_{\rho\sigma} \mathrm{g}^{\mu\sigma} - V^\sigma_{\sigma\rho} \mathrm{g}^{\mu\nu} = \Pi^{\mu\nu}_{\ \ \rho} - \frac{1}{3} \Pi^\mu_\sigma \sigma^\nu_\rho - \frac{1}{3} \Pi^\sigma_{\sigma\sigma} \delta^\mu_\rho,$$

which is a linear relation between $V^\rho_{\mu\nu}$ and $\Pi^{\mu\nu}_{\ \ \rho}$ and can be solved [51, 52]. If a purely affine Lagrangian does not depend explicitly on the connection (such Lagrangians are studied later in this paper) then $\Pi^{\mu\nu}_{\ \ \rho} = 0$. In this case, we do not need to introduce the condition $S_\mu = 0$ (or any constraint on four degrees of freedom of the connection) and Eq. (8) becomes

$$\mathrm{g}^{\mu\nu}{}_{,\rho} + {}^*\Gamma^\mu_{\sigma\rho} \mathrm{g}^{\sigma\nu} + {}^*\Gamma^\nu_{\rho\sigma} \mathrm{g}^{\mu\sigma} - {}^*\Gamma^\sigma_{\sigma\rho} \mathrm{g}^{\mu\nu} = 0.$$

The tensor $P_{\mu\nu}$ is invariant under a projective transformation $\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu} + \delta^\rho_\mu W_\nu$. We can use this transformation, with $W_\mu = \frac{2}{3} S_\mu$, to bring the torsion vector S_μ to zero and make ${}^*\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}$. From Eq. (14) it follows that the affine connection is the Christoffel connection of the metric tensor:

$$\Gamma^\rho_{\mu\nu} = \left\{ {\rho \atop \mu\nu} \right\}_g,$$

which is the special case of Eq. (11) with $V^\rho_{\mu\nu} = 0$. The theory based on a general Lagrangian density $\boldsymbol{x}(S, P, \phi, \nabla\phi)$ without any constraints on the affine connection does not determine the connection uniquely because the tensor $P_{\mu\nu}$ is invariant under projective transformations of the connection, $\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu} + \delta^\rho_\mu V_\nu$, where V_ν is a vector function of the coordinates. Therefore at least four degrees of freedom must be constrained to make such a theory consistent from a physical point of view [48, 49]. The condition $S_\mu = 0$ is not the only way to impose such a constraint; other possibilities include vanishing of the Weyl vector $W_\nu = \frac{1}{2} \left(\Gamma^\rho_{\rho\nu} - \left\{ {\rho \atop \rho\nu} \right\}_g \right) = 0$[48, 49] or adding the dependence on the segmental curvature tensor, which is not projectively invariant, to the Lagrangian [48, 49, 51, 52].[6]

If we assume that the affine connection in the purely affine variational principle is symmetric, as in the original formulation of purely affine gravity [4, 14, 15], then instead of Eq. (7) we find [16]:

$$\mathrm{g}^{\mu\nu}; \rho - \mathrm{g}^{\sigma(\mu}; \sigma^{\nu)}_\rho = \Pi^{\mu\nu}_\rho.$$

The hypermomentum density $\Pi^{\mu\nu}_{\ \ \rho}$ is, because of the definition (4), symmetric in the upper indices. Contracting the indices μ and ρ in Eq. (18) does not lead to any algebraic constraint on $\Pi^\mu_\rho 2$. Therefore, for purely affine Lagrangians that depend only on the connection and the symmetric part of the Ricci tensor, the relation $S_\mu = 0$ appears either as a remedy for the unphysical constraint on the hypermomentum density, or results simply from using a symmetric affine connection as a dynamical variable. In this paper we study purely affine Lagrangians that depend on the connection only via the symmetrized Ricci tensor: $\boldsymbol{x} = \boldsymbol{x}(P, \phi, \partial\phi)$, so S_μ decouples from the Einstein equations and can be brought to zero by a projective transformation, leading to Eq. (15).

8.1.3 EQUIVALENCE OF AFFINE, METRIC-AFFINE AND METRIC PICTURES

В двух словах

If we apply to a purely affine Lagrangian $x(\Gamma, P, \phi, \partial\phi)$ the Legendre transformation with respect to $P_{\mu\nu}$[4, 14, 15], defining the Hamiltonian density $\underset{3}{3}$ (or rather the Routhian density [54], since we can also apply the Legendre transformation with respect to $\Gamma^{\rho}_{\mu\nu}$): we find for the differential $d \leqq$:

$$d\sharp\boldsymbol{\xi} = \frac{\partial \mathbf{x}}{\partial \Gamma^{\rho}_{\mu\nu}} d\Gamma^{\rho}_{\mu\nu} + \frac{1}{2\kappa} P_{\mu\nu} d\ \mathrm{g}^{\mu\nu} + \frac{\partial \mathbf{z}}{\partial \phi} d\phi + \frac{\partial \mathbf{x}}{\partial \phi_{,\mu}} d\phi_{,\mu^*}$$

Accordingly, the Hamiltonian density is a covariant function of $\Gamma^{\rho}_{\mu\nu}, \mathrm{g}^{\mu\nu}, \phi$ and $\partial\phi$: $\exists \underset{3}{\mathfrak{j}}(S,\ \mathrm{g}, \phi, \nabla\phi)$, and the action variation (6) takes the form:

$$\begin{aligned}
\delta I &= \frac{1}{c}\delta \int d^4x \left(\mathfrak{j}(\Gamma, g, \phi, \partial\phi) - \frac{1}{2\kappa}\mathrm{g}^{\mu\nu} P_{\mu\nu}\right) \\
&= \frac{1}{c}\int d^4x \left(\frac{\partial \mathfrak{j}}{\partial \Gamma^{\rho}_{\mu\nu}}\delta\Gamma^{\rho}_{\mu\nu} + \frac{\partial \mathfrak{j}}{\partial \mathrm{g}^{\mu\nu}}\delta \mathrm{g}^{\mu\nu} - \frac{1}{2\kappa}\mathrm{g}^{\mu\nu}\delta P_{\mu\nu} - \frac{1}{2\kappa} P_{\mu\nu}\delta \mathrm{g}^{\mu\nu} + \frac{\delta \mathfrak{z}}{\delta\phi}\delta\phi\right)
\end{aligned}$$

The variation with respect to $\mathrm{g}^{\mu\nu}$ yields the first Hamilton equation [4, 14, 15] :

$$P_{\mu\nu} = 2\kappa \frac{\partial \jmath}{\partial \mathrm{g}\mu\nu}$$

8.1.4 EDDINGTON LAGRANGIAN

В двух словах

The simplest purely affine Lagrangian density $\boldsymbol{x} = \boldsymbol{x}\left(P_{\mu\nu}\right)$ was introduced by Eddington [2, 17] :

$$\mathbf{e}_{\mathrm{Edd}} = \frac{1}{\kappa\Lambda}\sqrt{-\det P_{\mu\nu}}$$

To make this Lagrangian density meaningful, we assume $\det P_{\mu\nu} < 0$, that is, the symmetrized Ricci tensor $P_{\mu\nu}$ has the Lorentzian signature. The Eddington Lagrangian does not depend explicitly on the affine connection, which is analogous in classical mechanics to free Lagrangians that depend only on generalized velocities: $L = L\left(\dot{q}^i\right)$. Accordingly, the Lagrangian density (27) describes a free gravitational field. Substituting Eq. (27) into Eq. (2) yields [3]

$$\mathrm{g}^{\mu\nu} = -\frac{1}{\Lambda}\sqrt{-\det P_{\rho\sigma}} P^{\mu\nu},$$

where the symmetric tensor $P^{\mu\nu}$ is reciprocal to the symmetrized Ricci tensor $P_{\mu\nu} : P^{\mu\nu} P_{\rho\nu} = \delta^{\mu}_{\rho}$. Equation (28) is equivalent to

$$P_{\mu\nu} = -\Lambda g_{\mu\nu}.$$

Since the Lagrangian density (27) does not depend explicitly on the connection, the field equations are given by Eq. (15). As a result, Eq. (29) becomes

$$R_{\mu\nu}(g) = -\Lambda g_{\mu\nu},$$

8.1.5 FERRARIS-KIJOWSKI LAGRANGIAN

В двух словах

The purely affine Lagrangian density of Ferraris and Kijowski [17]:

$$_{\mathrm{FK}} = -\frac{1}{4}\sqrt{-\det P_{\mu\nu}}F_{\alpha\beta}F_{\rho\sigma}P^{\alpha\rho}P^{\beta\sigma},$$

where $\det P_{\mu\nu} < 0$, has the form of the metric-affine (or metric, since the connection does not appear explicitly) Maxwell Lagrangian density for the electromagnetic field $F_{\mu\nu}$:[8]

$$\ggg_{\mathrm{Max}}= -\frac{1}{4}\sqrt{-g}F_{\alpha\beta}F_{\rho\sigma}g^{\alpha\rho}g^{\beta\sigma},$$

in which the covariant metric tensor is replaced by the symmetrized Ricci tensor $P_{\mu\nu}$ and the contravariant metric tensor by the tensor $P^{\mu\nu}$ reciprocal to $P_{\mu\nu}$. Substituting Eq. (32) to Eq. (2) gives (in purely affine picture)

$$\mathrm{g}^{\mu\nu} = \kappa\sqrt{-\det P_{\rho\sigma}}P^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma}\left(\frac{1}{4}P^{\mu\nu}P^{\alpha\rho} - P^{\mu\alpha}P^{\nu\rho}\right).$$

From Eqs. (22) and (33) it follows that (in metric-affine/metric picture)

$$P_{\mu\nu} - \frac{1}{2}Pg_{\mu\nu} = \kappa\left(\frac{1}{4}F_{\alpha\beta}F_{\rho\sigma}g^{\alpha\rho}g^{\beta\sigma}g_{\mu\nu} - F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta}\right)$$

which yields $P = 0$. Consequently, Eq. (19) reads $_{\mathrm{Max}} = \underset{\mathrm{Max}}{\mathbf{M}}$, where that is dynamically equivalent to the Maxwell Lagrangian density (33). Similarly, $\bar{\beta}_{\mathrm{FK}} = \boldsymbol{z}_{\mathrm{FK}}$, where metric-affine density that is dynamically equivalent to the Ferraris-Kijowski Lagrangian density (33).

8.1.6 AFFINE EINSTEIN-BORN-INFELD FORMULATION

В двух словах

Since the metric structure in the purely affine Ferraris-Kijowski model of electromagnetism is not well-defined in the zero-field limit, we need to combine the electromagnetic field and the cosmological constant. In the Lagrangian density (32) we used the determinant of the symmetrized Ricci tensor $P_{\mu\nu}$, multiplied by the simplest scalar containing the electromagnetic field tensor and $P_{\mu\nu}$. As an alternative way to add the electromagnetic field into purely affine gravity, we can include the tensor $F_{\mu\nu}$ inside this determinant, [12] constructing a purely affine version of the EinsteinBorn-Infeld theory [38, 39, 40]. For $F_{\mu\nu} = 0$, this construction reduces to the Eddington Lagrangian so the metric structure in the zero-field limit is well-defined. Therefore it describes both the electromagnetic field and cosmological constant. Let us consider the following Lagrangian density:

$$? = \frac{1}{\kappa\Lambda}\sqrt{-\det\left(P_{\mu\nu} + B_{\mu\nu}\right)}$$

where

$$B_{\mu\nu} = i\sqrt{\kappa\Lambda}F_{\mu\nu}$$

and $\Lambda > 0$. Let us also assume

$$|B_{\mu\nu}| \ll |P_{\mu\nu}|$$

where the bars denote the order of the largest (in magnitude) component of the corresponding tensor. [13] Consequently, we can expand the Lagrangian density (47) in small terms $B_{\mu\nu}$. If $s_{\mu\nu}$ is a symmetric tensor and $a_{\mu\nu}$ is an antisymmetric tensor, the determinant of their sum is given by [43, 58]

$$\det(s_{\mu\nu}+a_{\mu\nu}) = \det s_{\mu\nu}\left(1+\frac{1}{2}a_{\alpha\beta}a_{\rho\sigma}s^{\alpha\rho}s^{\beta\sigma}+\frac{\det a_{\mu\nu}}{\det s_{\mu\nu}}\right),$$

where the tensor $s^{\mu\nu}$ is reciprocal to $s_{\mu\nu}$. If we associate $s_{\mu\nu}$ with $P_{\mu\nu}$ and $a_{\mu\nu}$ with $B_{\mu\nu}$, and neglect the last term in Eq. (50), we obtain

$$\det(P_{\mu\nu}+B_{\mu\nu}) = \det P_{\mu\nu}\left(1+\frac{1}{2}B_{\alpha\beta}B_{\rho\sigma}P^{\alpha\rho}P^{\beta\sigma}\right)$$

In the same approximation, the Lagrangian density (47) becomes

$$\boldsymbol{\mathfrak{L}} = \frac{1}{\kappa\Lambda}\sqrt{-\det P_{\mu\nu}}\left(1+\frac{1}{4}B_{\alpha\beta}B_{\rho\sigma}P^{\alpha\rho}P^{\beta\sigma}\right)$$

which is equal to the sum of the Lagrangian densities (27) and (32). Equations (2) and (3) define the contravariant metric tensor, [14] for which we find [15]

$$\sqrt{-g}g^{\mu\nu} = -\frac{1}{\Lambda}\Big[P^{\mu\nu}\left(1+\frac{1}{4}B_{\alpha\beta}B_{\rho\sigma}P^{\alpha\rho}P^{\beta\sigma}\right)$$
$$-P^{\alpha\beta}B_{\alpha\rho}B_{\beta\sigma}P^{\mu\rho}P^{\nu\sigma}\Big]\sqrt{-\det P_{\mu\nu}}.$$

In the terms containing $B_{\mu\nu}$ and in the determinant [16] we can use the relation $P^{\mu\nu} = -\Lambda^{-1}g^{\mu\nu}$ (equivalent to Eq. (30)) valid for $B_{\mu\nu}=0$. As a result, we obtain

$$g^{\mu\nu} = -\Lambda P^{\mu\nu} + \Lambda^{-2}\left(\frac{1}{4}g^{\mu\nu}B_{\rho\sigma}B^{\rho\sigma} - B^{\mu\rho}B^{\nu}_{\rho}\right).$$

Introducing the energy-momentum tensor for the electromagnetic field:

$$T^{\mu\nu} = \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} - F^{\mu\rho}F^{\nu}_{\rho}$$

turns Eq. (54) into

$$P^{\mu\nu} = -\Lambda^{-1}g^{\mu\nu} - \kappa\Lambda^{-2}T^{\mu\nu},$$

which is equivalent, in the approximation (49), to the Einstein equations of general relativity with the cosmological constant in the presence of the electromagnetic field:

$$P_{\mu\nu} = -\Lambda g_{\mu\nu} + \kappa T_{\mu\nu}.$$

As in the case for the gravitational field only, $P_{\mu\nu} = R_{\mu\nu}(g)$. The tensor (55) is traceless, from which it follows that

$$R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu} = \Lambda g_{\mu\nu} + \kappa T_{\mu\nu}.$$

Since the tensor $R_{\mu\nu}(g)$ satisfies the contracted Bianchi identities:

$$\left(R_{\mu\nu}(g) - \frac{1}{2}R(g)g_{\mu\nu}\right)^{;\nu} = 0$$

8.2 Многомерные теории гравитации

Такие есть, к ним у меня есть заготовки, ибо Лашкевича проходил, так что мб потом пару слов впишу.

8.2.1 Заготовки для многомерных теорий гравитаций

(выписка из лекций лашкевича, он в общем виде что-то пробовал решать)

8.2.2 Мотивация заниматься многомерными теориями гравитации

Лашкевич вот занимается
я вот хз, зачем, вот правда.

а вдруг мир не 4-х мерный

фиговая мотивация, другого я придумать не могу.

8.2.3 особенности многомерных теорий гравитаций

выделим, чтобы не тупить.

8.2.4 Двумерная гравитация

подумаем про это

8.3 $f(R)$ гравитация

Она есть, она может объяснить темную материю, я ей заниматься пока что не буду.

8.3.1 Уравнения движения

(!!! потренируюсь сам когда-то, хорошая тренировка!! все ленюсь)

Вывод по Шмидту (!!!??)

1.3. Уравнения движения $f(R)$-гравитации. Хотя изучение модифицированных теорий гравитации в текущем семестре не входит в наши планы, получим полевые уравнения движения в рамках модифицированной $f(R)$-гравитации, которую мы упомянули в конце первого параграфа. Вариационная производная функционала действия $S_{\mathrm{f}}[g]$ имеет вид

$$\delta_g S_{\mathrm{f}} = -\frac{1}{2} m_{\mathrm{pl}}^2 \int_M d_4 x [\delta \sqrt{-\det g} f(R) + \sqrt{-\det g} F(R) \delta R]$$

где под функцией $F(R)$ понимается производная

$$F(R) = \frac{df}{dR}.$$

Подставляя в последнее выражение полученные ранее формулы для вариаций скалярной кривизны и корня из детерминанта метрики, находим

$$\begin{aligned}
\delta_g S_{\mathrm{f}} &= -\frac{1}{2} m_{\mathrm{pl}}^2 \int_M \mathrm{vol}_g \left[1/2 g^{\mu\nu} \delta g_{\mu\nu} f(R) + F(R) \delta g^{\beta\nu} R_{\beta\nu} + F(R) g^{\beta\nu} \delta R_{\beta\nu} \right] = \\
&= -\frac{1}{2} m_{\mathrm{pl}}^2 \int_M \mathrm{vol}_g \left[1/2 g^{\mu\nu} \delta g_{\mu\nu} f(R) - F(R) R^{\mu\nu} \delta g_{\mu\nu} + g^{\alpha\beta} F(R) \left[\nabla_\sigma \delta \Gamma^\sigma_{\alpha\beta} - \nabla_\beta \delta \Gamma^\sigma_{\alpha\sigma} \right] \right]
\end{aligned}$$

Преобразуем немного полученное выражение, а именно, проинтегрируем по частям последнее слагаемое

$$-\frac{1}{2}m_{\text{pl}}^2\int_M \text{vol}_g\, g^{\alpha\beta}F(R)\left[\nabla_\sigma\delta\Gamma^\sigma_{\alpha\beta}-\nabla_\beta\delta\Gamma^\sigma_{\alpha\sigma}\right]=$$
$$=-\frac{1}{2}m_{\text{pl}}^2\int_M \text{vol}_g\,\nabla_\sigma\left[g^{\alpha\beta}F(R)\delta\Gamma^\sigma_{\alpha\beta}\right]+\frac{1}{2}m_{\text{pl}}^2\int_M \text{vol}_g\,\nabla_\beta\left[g^{\alpha\beta}F(R)\delta\Gamma^\sigma_{\alpha\sigma}\right]+$$
$$+\frac{1}{2}m_{\text{pl}}^2\int_M \text{vol}_g\, g^{\alpha\beta}\delta\Gamma^\sigma_{\alpha\beta}\nabla_\sigma F(R)-\frac{1}{2}m_{\text{pl}}^2\int_M \text{vol}_g\, g^{\alpha\beta}\delta\Gamma^\sigma_{\alpha\sigma}\nabla_\beta F(R)=$$
$$=-\frac{1}{2}m_{\text{pl}}^2\int_{\partial M} d\Sigma_\sigma\left[\sqrt{-\det g}g^{\alpha\beta}F(R)\delta\Gamma^\sigma_{\alpha\beta}\right]+\frac{1}{2}m_{\text{pl}}^2\int_{\partial M} d\Sigma_\beta\left[\sqrt{-\det g}g^{\alpha\beta}F(R)\delta\Gamma^\sigma_{\alpha\sigma}\right]+$$
$$+\frac{1}{2}m_{\text{pl}}^2\int_M \text{vol}_g\, g^{\alpha\beta}\delta\Gamma^\sigma_{\alpha\beta}\nabla_\sigma F(R)-\frac{1}{2}m_{\text{pl}}^2\int_M \text{vol}_g\, g^{\alpha\beta}\delta\Gamma^\sigma_{\alpha\sigma}\nabla_\beta F(R)$$

В этом выражении мы воспользовались формулой Остроградского-Гаусса. Поверхностные интегралы

$$\int_{\partial M} d\Sigma_\sigma\left[\sqrt{-\det g}g^{\alpha\beta}F(R)\delta\Gamma^\sigma_{\alpha\beta}\right]=0$$

и

$$\int_{\partial M} d\Sigma_\beta\left[\sqrt{-\det g}g^{\alpha\beta}F(R)\delta\Gamma^\sigma_{\alpha\sigma}\right]=0$$

в силу соответствующих граничных условий: вариации полей $\delta g_{\mu\nu}$ и их производных $\partial_\alpha\delta g_{\mu\nu}$ на поверхности интегрирования обращаются к нуль. С учетом этого вариация действия

$$\delta_g S_{\text{f}}=\frac{1}{2}m_{\text{pl}}^2\int_M \text{vol}_g\left[F(R)R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}f(R)\right]\delta g_{\mu\nu}+$$
$$+\frac{1}{2}m_{\text{pl}}^2\int_M \text{vol}_g\, g^{\alpha\beta}\delta\Gamma^\sigma_{\alpha\beta}\nabla_\sigma F(R)-\frac{1}{2}m_{\text{pl}}^2\int_M \text{vol}_g\, g^{\alpha\beta}\delta\Gamma^\sigma_{\alpha\sigma}\nabla_\beta F(R)=$$
$$=\frac{1}{2}m_{\text{pl}}^2\int_M \text{vol}_g\left[F(R)R^{\mu\nu}-\frac{1}{2}g^{\mu\nu}f(R)\right]\delta g_{\mu\nu}+\frac{1}{2}m_{\text{pl}}^2\int_M \text{vol}_g\,\nabla^\mu F(R)\left[\nabla^\nu\delta g_{\mu\nu}-\nabla_\mu\left(g^{\alpha\beta}\delta g_{\alpha\beta}\right)\right]$$

Повторное интегрирование по частям соответствующего слагаемого в последнем выражении дает

$$\int_M \text{vol}_g\,\nabla^\mu F(R)\left[\nabla^\nu\delta g_{\mu\nu}-\nabla_\mu\left(g^{\alpha\beta}\delta g_{\alpha\beta}\right)\right]=\int_M \text{vol}_g\left[g^{\mu\nu}\Box_g F(R)-\nabla^\mu\nabla^\nu F(R)\right]\delta g_{\mu\nu}$$

Здесь вновь была использована теорема Остроградского-Гаусса. Как и раньше, она позволила занулить соответствующие поверхностные интегралы. Дифференциальный оператор $\Box_g : C^\infty(M)\to C^\infty(M)$ представляет собой т.н. оператор Бельтрами-Лапласа, связанный с метрикой g :

$$\Box_g=g^{\mu\nu}\nabla_\mu\nabla_\nu=\frac{1}{\sqrt{-\det g}}\partial_\mu\left(\sqrt{-\det g}g^{\mu\nu}\partial_\nu\right)$$

Подставляя полученное выше уравнение в выражение для вариации действия, после применения принципа экстремального действия находим динамические уравнения модифицированной $f(R)$-гравитации [1] :

$$F(R)R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}f(R)=\nabla_\mu\nabla_\nu F(R)-g_{\mu\nu}\Box_g F(R)$$

Отсюда видно, что теории гравитации с действием S_{f} не содержат высших производных в уравнениях движения при любом допустимом выборе функции $f(R)$.

Другие вопросы

(там потом создам структуру с обзором приложений, тем, как догадаться, моделями, но пока не до этого.)

8.4 Теории с кручением

ОТО - теория без кручения, а если бы она была - что бы было?
вот тут подумаем, если захочу.

8.5 теория Эйнштейна-Картана

хз, но слышал о ней, мб в космологии нужно.

8.6 метрическая аффинная гравитация

хз.

8.7 Гравитация на бране

(потом мб поизучаю. см. Classical tests of general relativity in brane world models Christian G Bohmer 1, Giuseppe De Risi2, Tiberiu Harko3 and Francisco S N Lobo. по ней многое напишу мб, пока не до этого.)

8.7.1 Gravitational field equations on the brane

(пока лишь пара выгрузок, изучать не скоро буду.)

Описание

(?? тут вообще хз.)

We start by considering a 5D spacetime (the bulk), with a single 4D brane, on which matter is confined. The 4D brane world ${}^{(4)}M, g_{\mu v}\big)$ is located at a hypersurface $\left(B\left(X^A\right)=0\right)$ in the 5D bulk spacetime $\left({}^{(5)}M, g_{AB}\right)$, where the coordinates are described by $X^A, A=0,1,\ldots,4$. The induced 4D coordinates on the brane are $x^\mu, \mu=0,1,2,3$. The action of the system is given by [2]

$$S=S_{\text{bulk }}+S_{\text{brane }},$$

where

$$S_{\text{bulk }}=\int_{(5)M} \sqrt{-{}^{(5)}g}\left[\frac{1}{2k_5^2}{}^{(5)}R+{}^{(5)}L_m+\Lambda_5\right] \mathrm{d}^5X,$$

and

$$S_{\text{brane }}=\int_{(4)M} \sqrt{-{}^{(5)}g}\left[\frac{1}{k_5^2}K^{\pm}+L_{\text{brane }}\left(g_{\alpha\beta},\psi\right)+\lambda_b\right] \mathrm{d}^4x,$$

where $k_5^2=8\pi G_5$ is the 5D gravitational constant, ${}^{(5)}R$ and ${}^{(5)}L_m$ are the 5D scalar curvature and the matter Lagrangian in the bulk $L_{\text{brane }}\left(g_{\alpha\beta},\psi\right)$ is the 4D Lagrangian, which is given by a generic functional of the brane metric $g_{\alpha\beta}$ and of the matter fields $\psi, K^{\pm}$is the trace of the extrinsic curvature on either side of the brane and Λ_5 and λ_b (the constant brane tension) are the negative vacuum energy densities in the bulk and on the brane, respectively. The energy-momentum tensor of bulk matter fields is defined as

$${}^{(5)}\tilde{T}_{IJ}\equiv-2\frac{\delta^{(5)}L_m}{\delta^{(5)}g^{IJ}}+{}^{(5)}g_{IJ}{}^{(5)}L_m,$$

while $T_{\mu\nu}$ is the energy-momentum tensor localized on the brane and is given by

8.7.2 The DMPR brane world vacuum solution

(выгрузки из статьи, потом мб займусь.)

Описание

The first brane solution we consider is a solution of the vacuum field equations, obtained by Dadhich, Maartens, Papadopoulos and Rezania (DMPR) in [6], which represent the simplest generalization of the Schwarzschild solution of GR. We call this type of brane black hole the DMPR black hole. The Solar System tests for the DMPR solutions were extensively analyzed in [29], but we use the general and novel formalism developed above as a consistency check. For this solution the metric tensor components are given by

$$e^{v} = e^{-\lambda} = 1 - \frac{2m}{r} + \frac{Q}{r^2},$$

where Q is the so-called tidal charge parameter. In the limit $Q \to 0$ we recover the usual general relativistic case. In terms of the general equations discussed in section 2, this class of brane world spherical solution is characterized by an equation of state relating dark energy and pressure: $P = -2U$. The metric is asymptotically flat, with $\lim_{r\to\infty} \exp(\nu) = \lim_{r\to\infty} \exp(\lambda) = 1$. There are two horizons, given by

$$r_{\mathrm{h}}^{\pm} = m \pm \sqrt{m^2 - Q}.$$

Both horizons lie inside the Schwarzschild horizon $r_{\mathrm{s}} = 2m, 0 \leqslant r_{\mathrm{h}}^{-} \leqslant r_{\mathrm{h}}^{+} \leqslant r_{\mathrm{s}}$. In the brane world models there is also the possibility of a negative $Q < 0$, which leads to only one horizon $r_{\mathrm{h}+}$ lying outside the Schwarzschild horizon,

$$r_{\mathrm{h}+} = m + \sqrt{m^2 + Q} > r_{\mathrm{s}}$$

Часть IV

Modified Quantum Fields and Gravities

9 title

9.0.1 title

Часть V

Problems

10 General Problems

10.0.1 Questions about understanding nature of fields and gravity

10.0.2 Questions about understanding typical modifications

11 Technical questions

(!! напишу там мотивацию в каждую задачу, почему важно ее решать. часто они что-то дают и где-то используются)

11.0.1 Problems in ...

Часть VI

Другие теории гравитации

(что-то, что не вписывается в предыдущие деления - тут.)

12 Другие теории гравитации

Обсудим теории гравитации, использование которых я пока что нигде не встречал.

12.1 Теория взаимодействия полей с гравитацией

часто об этом заходит речь.

подробнее - в теории поля об этом, тем не менее, укажем некоторые главные моменты тут

(мб подниму куда-то выше главу эту)

12.1.1 Минимальная связь с гравитацией

конструкция минимальной связи

12.1.2 Неминимальная связь с гравитацией

я хз, есть ли такая

проверка на минимальность

как определить, она минимальная или нет?

а вот там теорема римановой геометрии.

потом напишу

12.2 Динамика спин-тензорных полей в гравитационном поле

(таким вопросом тоже можно задаться, потом и задамся. пока хз, не до этого мне абсолютно. мб потом как теоретик поисследую это подробно. типа как описывать в общем случае поля в гравитационных полях?????)

12.3 О дискретной теории гравитации на решетке

(вообще хз абсолютно, просто Вергелес упомянул и всё)

Часть VII
Other Topics

(таких теорий много, так что если будет нечего делать, это раздел для этого создан. минимально актуально, конечно, тем не менее. мб когда-то займусь, все-таки полно самых разных тут теорий)

13 О неверных теориях гравитации

В жизни этот раздел может понадобится при общении с людьми, далекими от научного сообщества.

Для кругозора если не лень, можно почитать теории гравитации, которые признаны неверными, может быть, это даст понимание, как не нужно делать.

13.1 Обличение неверных наукообразных теории

13.1.1 теория Логунова

мб потом посмотрю, суслов говорил про нее

13.2 Обличение неверных псевдонаучных теорий

Существует много явно бреда про гравитацию, обличению которого посвящен этот раздел.

(мб когда-то почитаю разный бред, пропишу ответы на него тут, пока максимально не актуально)

14 Идеи вымерших теорий гравитации

Каждая неверная теория содержала в себе какую-то идею, поэтому интересно обсудить, что в этих идеях хорошего и плохого, чтобы знать, не повторяться, иметь кругозор.

(потом)

Часть VIII
Appendix

A Введение и обзор предмета

Приведу общие важные соображения о том, как я вижу предмет.
(все это уже проявлено выше, тут лишь четкие указания для полноты и для тех, у кого вопросы в том, что мне итак очевидно.)

A.1 Общая мотивация

Обсудим все, что нас мотивирует для изучения предмета.

A.2 Мышление профессионала в модификациях теории поля

(потом раскрою)

A.2.1 Суть предмета

A.2.2 Отношение к предмету (!?)

A.2.3 Насколько вообще полезно заниматься такими теориями? (!?!?)

(большое обсуждение, потому что там абстрактные теории, которые странным образом связаны с реальностью. так что большие вопросы. напишу потом.)

A.2.4 Способы заработать, зная предмет

A.2.5 Использование предмета в обычной жизни (!)

Укажем, как предмет используется в обычной жизни.

A.2.6 Актуальнейшие приложения

A.2.7 Построение с нуля

(потом раскрою, еще я не профессионал, а вопрос этот самый профессиональный)

A.2.8 Способы догадаться до всех главных идей

незаменимая часть нормального понимания предмета.
(потом раскрою)

A.2.9 Мышление для эффективного изучения

Осудим, какое мышление наиболее эффективное для усвоение предмета.

Способы изучения предмета

(потом раскрою)

Необходимые темы для

(потом раскрою)

Дополнительные темы для

(потом раскрою)

A.2.10 Особенности эффективного преподавания (???)

Обсудим, каким образом наиболее эффективно преподавать.

A.3 Литература

A.3.1 Основная

Основная обучающая

[1] Freedman, Daniel Z. and Van Proeyen, Antoine Supergravity
Большая, концентрированная, хорошая для теоретиков книга, где много теоретических методов, компактно все написано. Приложений теории и проверки их не обсуждаются.

Задачники

Литература крепкого минимума

A.3.2 Дополнительная

Дополнительная по теории

В помощь

Статьи о теоретических методах

О приложениях

A.4 Обзор

(потом раскрою)

A.4.1 предмет в двух словах

Обсудим, что из себя представляет предмет наиболее кратко, выделяя самую суть.

появление Предмет в нашей картине мира

один подход

второй подход

один большой раздел

такой-то набор следствий

A.4.2 Итоговые формулы и закономерности

A.4.3 обзор теоретических подходов

такие-то есть, такие полезные, такие - нет.

A.4.4 Обзор дальнейших развитий

A.4.5 Связи с другими науками

Обсудим связи с разделами
(потом раскрою)

A.4.6 Описание записи

Общее описание записи

Общие особенности записи

Особенности глав и разделов

Первая часть про предмет в двух словах

Вторая часть

Часть про приложения какие вообще приложения я разбирал?

Обозначения и константы

A.4.7 Об истории предмета

Обсудим вкратце историю развития ...

A.5 Головоломки

Обсудим в порядке интересности задачки и вопросы

A.5.1 Типичные головоломки

A.5.2 Бытовые головоломки

A.5.3 Принципиальные головоломки

A.5.4 Головоломки о деталях

A.5.5 Головоломки для освоения типичных понятий

Допустим, видим человека, который в принципе хотел бы понять суть, опишем, какие в какой последовательности будем задавать, чтобы в типичном случае ему было бы интереснее въезжать. В частности это могут быть младшие студенты или редко школьники.

Головоломки для освоения школьного уровня

Головоломки для освоения уровня теормина

Головоломки для освоения других особенностей

B Дополнения

B.0.1 title

C Литература

[1] Freedman, Daniel Z. и Van Proeyen, Antoine: *Supergravity.* Cambridge Univ. Press, Cambridge, UK, май 2012, ISBN 978-1-139-36806-3, 978-0-521-19401-3.

[2] Poplawski, Nikodem J.: *Gravitation, Electromagnetism and Cosmological Constant in Purely Affine Gravity.* Foundations of Physics, 39(3):307–330, Feb 2009, ISSN 1572-9516. http://dx.doi.org/10.1007/s10701-009-9284-y.