Special Field Theories and Gravity

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This note is not intended for distribution.

Modified field and gravity theories are discussed in detail. Links below show contents of solved problems and summary of special topics. Readers who have opened this note for the first time are highly recommended to read the preface.

Goals: 1) By 05.08.24 two weeks of practice are needed.

Contents

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Preface and main motivation

Let's discuss some minimum knowledge and motivation that would be good to understand for studying the subject.

Overview of the note

(write links to the table of contents)

Also, there are TASI Lectures by Tong on solitons, instantons, monopoles, vortices and kinks. Currently this note follows mostly Freedman's and Van Proeyen's book, but also I used the following literature.

I used the following literature.

Idea of the note in a nutshell

(I'll write very shortly the main idea of this approach later)

Main motivation

Modified field theories - a cool field with many interesting and elaborated theories

Cool formulas and ideas are the main motivation to study it.

(раскрою эту мысль с примерами, очень много интересного и удивительного в этих разных теориях)

Подавляющее большинство теоретиков занимаются именно разными теориями поля с разной закрученной математикой

Поэтому начать ориентироваться в модифицированных теориях поля - это начать ориентироваться в теоретической физики конца 20 и начала 21го века. Там просто все -

следствия этой записи.

Поэтому можно многое узнать и про современную физику и про современных ученых! Те, кто не занимались такими моделями и представить себе не могут, что современные ученые делаю, какие они, как работают. Вот, изучая этот предмет лучше всего можно научиться современной физике. Конечно, можно критиковать эту физику, однако тем не менее, именно в этом предмете и очень большая часть современной теоретической физики.

Есть огромные гранты на теоретические исследования, и они как раз в особых теориях поля

Есть гранты от 1 500 000 евро. Конечно, их получает 1 из 100 000 занимающихся физикой, наверное, тем не менее, это есть и это очень поражает. Просто позаниматься теорями поля - и можно но большие доходы и выйти. Наверное. Не стоит в это вкладываться, но такое направление есть.

Если у кого-то есть сомнения в профессионализме, то после прохождения основ суперсимметрий и гравитации с высказыванием каких-то казалось бы умных мыслей не будет проблем

Можно всего за месяц научиться некоторым методам, после которых все будут думать, что ты очень много знаешь в теорфизе. Что приятно, это повышает самооценку и авторитет, и, возможно, это одно из самых основных применений этих тем.

Результаты новых теорий поля (????)

Обсудим, что полезного и важного человечество получило, поняв этот предмет.

Идейные головоломки для мотивации

(тут много будет, мб пару разделов сделаю)

Технические головоломки для мотивации

(тут только 1 раздел, потому что технические задачи редко мотивируют)

Which special theories are promising and which are not more than just interesting

String theories are just interesting

(I see no point in studying them in advance)

Matrix models are just interesting

(I see no point in studying them in advance. in 2010-s, 2020-s they are popular. Maybe because people just don't know what to do better, not because of any physical reason.)

Part I
Main Special Fields and Gravity Theories in a Nutshell

1 Deep physical knowledge for modified theories

1.1 Constants of nature (?)

(I'll think a lot about them later!!)

On Newtonian's constant

(там Проен говорил, что она связана с конформной длиной волны, но я не понял, почему. конформная теория поля мб что-то про нее особое говорит, пока не знаю.)

1.2 Main fundamental experiments (???)

(later I'll think about it)

2 Important Special Physical and Mathematical Properties of Proven Theories

2.1 Special mathematical ideas and methods

2.1.1 General method of symmetries

Idea and main formulas

Key examples

(там типичное утверждение, что мы можем взять теорию с большой симметрией и понижать симметрии. как? напишу тут пример. хороший пока вопрос, симметрии не так как нужно понял.)

2.1.2 Lie algebras background for field theories

(for now see my note on algebra, later I'll add important basic ideas here. See also my note on CFT)

2.1.3 Manifolds, forms, metric

(diff geom section)

7.1 Manifolds 135

7.2 Scalars, vectors, tensors, etc. 137

7.3 The algebra and calculus of differential forms 140

7.4 The metric and frame field on a manifold 142 7.4.1 The metric 142 7.4.2 The frame field 143 7.4.3 Induced metrics 145

2.1.4 Properties of spinors

(see my note on QFT also for it!)

Spinors and spinor bilinears

Majorana flip relations

$$\bar{\lambda}\gamma_{\mu_1\ldots\mu_r}\chi = t_r\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\lambda.$$

$$\bar{\lambda}\Gamma^{(r_1)}\Gamma^{(r_2)}\cdots\Gamma^{(r_p)}\chi = t_0^{p-1}t_{r_1}t_{r_2}\cdots t_{r_p}\bar{\chi}\Gamma^{(r_p)}\cdots\Gamma^{(r_2)}\Gamma^{(r_1)}\lambda,$$

$\Gamma^{(r)}$ stands for any rank r matrix $\gamma_{\mu_1\ldots\mu_r}$. The prefactor t_0^{p-1} is not relevant in four dimensions, $t_0 = 1$.

Exercise 3.22 One often encounters the special case that the bilinear contains the product of individual γ^μ-matrices. Prove that for the Majorana dimensions $D = 2, 3, 4 \bmod 8$,

$$\bar{\lambda}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_p}\chi = (-)^p\bar{\chi}\gamma^{\mu_p}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\lambda.$$

The previous relations imply also the following rule. For any relation between spinors that includes γ-matrices, there is a corresponding relation between the barred spinors,

$$\chi_{\mu_1\ldots\mu_r} = \gamma_{\mu_1\ldots\mu_r}\lambda \Longrightarrow \bar{\chi}_{\mu_1\ldots\mu_r} = t_0t_r\bar{\lambda}\gamma_{\mu_1\ldots\mu_r},$$

and similar for longer chains,

$$\chi = \Gamma^{(r_1)}\Gamma^{(r_2)}\cdots\Gamma^{(r_p)}\lambda \Longrightarrow \bar{\chi} = t_0^p t_{r_1}t_{r_2}\cdots t_{r_p}\bar{\lambda}\Gamma^{(r_p)}\cdots\Gamma^{(r_2)}\Gamma^{(r_1)}.$$

$$\chi = P_L\lambda \to \bar{\chi} = \begin{cases} \bar{\lambda}P_L, & \text{for } D = 0, 4, 8, \ldots \\ \bar{\lambda}P_R, & \text{for } D = 2, 6, 10, \ldots \end{cases}$$

Exercise 3.23 Using the 'spin part' of the infinitesimal Lorentz transformation (2.25),

$$\delta\chi = -\frac{1}{4}\lambda^{\mu\nu}\gamma_{\mu\nu}\chi$$

prove that the spinor bilinear $\bar{\lambda}\chi$ is a Lorentz scalar.

Spinor indices

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta.$$

We also introduce a lowering matrix such that (again NW-SE contraction)

$$\lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha}$$

In order for these two equations to be consistent, we must require

$$\mathcal{C}^{\alpha\beta}\mathcal{C}_{\gamma\beta} = \delta_\gamma{}^\alpha, \quad \mathcal{C}_{\beta\alpha}\mathcal{C}^{\beta\gamma} = \delta_\alpha{}^\gamma$$

Hence $\mathcal{C}_{\alpha\beta}$ are the components of C^{-1}, and the unitarity of C implies then $(\mathcal{C}_{\alpha\beta})^* = \mathcal{C}^{\alpha\beta}$.

When we write a covariant spinor bilinear with components explicitly indicated, the γ-matrices are written as $(\gamma_\mu)_\alpha{}^\beta$. For example, for the simplest case,

$$\bar{\chi}\gamma_\mu\lambda = \chi^\alpha (\gamma_\mu)_\alpha{}^\beta \lambda_\beta,$$

These γ-matrices with indices at the 'same level' have a definite symmetry or antisymmetry property, which follows from (3.44):

$$(\gamma_{\mu_1\ldots\mu_r})_{\alpha\beta} = -t_r (\gamma_{\mu_1\ldots\mu_r})_{\beta\alpha}.$$

$$\lambda^\alpha\chi_\alpha = -t_0\lambda_\alpha\chi^\alpha$$

(!!! write examples, tricky point!! why it can be not like usual?????? need an explanation!!!)

Thus, in four dimensions, raising and lowering a contracted index produces a minus sign. The same property can be used when the contracted indices involved are on γ-matrices, e.g. $\gamma_{\mu\alpha}{}^\beta\gamma_{\nu\beta}{}^\gamma = -t_0\gamma_{\mu\alpha}\beta\gamma_\nu{}^{\beta\gamma}$.

(????? I'll ask Proeyen about it!!!!! A very strange statement!!!)

(I'll add it to field theory later)

Fierz rearrangement

In this subsection we study an important consequence of the completeness of the Clifford algebra basis $\left\{\Gamma^A\right\}$ in $\{\Gamma^A\}$. As we saw in Ex. 3.7 completeness means that any matrix M has a unique expansion in the basis with coefficients obtained using trace orthogonality. The expansion was derived for even $D = 2m$ in Ex. 3.7, but it is also valid for odd $D = 2m + 1$ provided that the sum is restricted to rank $r \leq m$. We saw at the end of Sec. 3.1.7 that the list of $\{\Gamma^A\}$ is complete for odd D when so restricted. The rearrangement properties we derive using completeness are frequently needed in supergravity. These involve changing the pairing of spinors in products of spinor bilinears, which is called a 'Fierz rearrangement'.

Let's proceed to derive the basic Fierz identity. Using spinor indices, we can regard the quantity $\delta_\alpha{}^\beta\delta_\gamma{}^\delta$ as a matrix in the indices $\gamma\beta$ with the indices $\alpha\delta$ as inert 'spectators'. We apply $M = \sum\limits_A m_A\Gamma^A$, $m_A = \frac{1}{2^m \text{ă}}\operatorname{Tr}(M\Gamma_A)$ in the detailed form $\delta_\alpha^\beta\delta_\gamma^\delta = \sum\limits_A (m_A)_\alpha^\delta (\Gamma_A)_\gamma{}^\beta$. The coefficients are $(m_A)_\alpha^\delta = 2^{-m}\delta_\alpha{}^\beta\delta_\gamma{}^\delta (\Gamma_A)_\beta{}^\gamma = 2^{-m}(\Gamma_A)_\alpha{}^\delta$. Therefore,

$$\delta_\alpha{}^\beta\delta_\gamma{}^\delta = \frac{1}{2^m}\sum_A (\Gamma_A)_\alpha{}^\delta (\Gamma^A)_\gamma{}^\beta.$$

Note that the 'column indices' on the left- and right-hand sides have been exchanged.

$$(\gamma^\mu)_\alpha^\beta (\gamma_\mu)_\gamma^\delta = \frac{1}{2^m}\sum_A v_A (\Gamma_A)_\alpha{}^\delta \left(\Gamma^A\right)_\gamma{}^\beta,$$

$v_A = (-)^{r_A}(D-2r_A)$, r_A is the tensor rank of the Clifford basis element Γ_A.

Exercise 3.27 Lower the β and δ index in the result of the previous exercise and consider the completely symmetric part in $(\beta\gamma\delta)$. The left-hand side is only non-vanishing for dimensions in which $t_1=-1$. Consider the right-hand side and use Table 3.1 and the result for v_A to prove that for $D=3$ and $D=4$ only rank 1γ-matrices contribute to the right-hand side. For $D=4$ you have to use the bold face row in the table to arrive at this result. You can also check that there are no other dimensions where this occurs. This implies s.c. cyclic identity, for $D=3$ and $D=4$,

$$(\gamma_\mu)_{\alpha(\beta}(\gamma^\mu)_{\gamma\delta)} = 0.$$

$$\gamma_\mu\lambda_{[1}\bar\lambda_2\gamma^\mu\lambda_{3]} = 0$$

$$\bar\lambda_1\lambda_2\bar\lambda_3\lambda_4 = -\frac{1}{2^m}\sum_A \bar\lambda_1\Gamma^A\lambda_4\bar\lambda_3\Gamma_A\lambda_2$$

This can be generalized to include general matrices M, M' of the Clifford algebra.

$$\begin{aligned}\bar\lambda_1 M\lambda_2\bar\lambda_3 M'\lambda_4 &= -\frac{1}{2^m}\sum_A \bar\lambda_1 M\Gamma_A M'\lambda_4\bar\lambda_3\Gamma^A\lambda_2 \\ &= -\frac{1}{2^m}\sum_A \bar\lambda_1\Gamma_A M'\lambda_4\bar\lambda_3\Gamma^A M\lambda_2.\end{aligned}$$

When $\lambda_{1,2,3,4}$ are not all independent, it is frequently the case that some terms in the rearranged sum vanish due to symmetry relations such as $\bar\lambda\gamma_{\mu_1\ldots\mu_r}\chi = t_r\bar\chi\gamma_{\mu_1\ldots\mu_r}\lambda$.

One can write the Fierz relation (3.65) in the alternative form:

$$M = 2^{-m}\sum_{k=0}^{[D]}\frac{1}{k!}\Gamma_{\mu_1\ldots\mu_k}\operatorname{Tr}\left(\Gamma^{\mu_k\cdots\mu_1}M\right)$$

$$\begin{cases}[D]=D, & \text{for even } D,\\ [D]=(D-1)/2, & \text{for odd } D.\end{cases}.$$

For $D=4$:

$$\begin{aligned}P_L\chi\bar\lambda P_L &= -\frac{1}{2}P_L\left(\bar\lambda P_L\chi\right) + \frac{1}{8}P_L\gamma^{\mu\nu}\left(\bar\lambda\gamma_{\mu\nu}P_L\chi\right),\\ P_L\chi\bar\lambda P_R &= -\frac{1}{2}P_L\gamma^\mu\left(\bar\lambda\gamma_\mu P_L\chi\right).\end{aligned}$$

For $D=5$:

$$\chi\bar\lambda - \lambda\bar\chi = \gamma_{\mu\nu}\left(\bar\lambda\gamma^{\mu\nu}\chi\right).$$

(I'll also later add properties from article about spinors, see my note on QFT)

Reality and charge conjugation

(!?????????!?!?!!!!?!?!?!?!??!)

Complex conjugation can be replaced by charge conjugation, an operation that acts as complex conjugation on scalars, and has a simple action on fermion bilinears. For example, it preserves the order of spinor factors.

(!?????????!?!?!!!!?!?!?!?!??!)

The charge conjugate:

$$\lambda^C := B^{-1}\lambda^*.$$

$B = \mathrm{i}t_0 C\gamma^0$,

$$\overline{\lambda^C} = (-t_0 t_1)\,\mathrm{i}\lambda^\dagger\gamma^0.$$

This is the Dirac conjugate as defined in (2.30) except for the numerical factor $(-t_0t_1)$. The meaning of this will become clear below when we discuss Majorana spinors. $(-t_0t_1) = +1$ in $2, 3, 4, 10$ or 11 dimensions. For these dimensions the spinor bilinears of Chs. 2 and 3 are related by $(\bar{\lambda}\chi)_{\mathrm{Ch.2}} = \left(\overline{\lambda^C}\chi\right)_{\mathrm{Ch.3}}$.

The charge conjugate of any $2^m \times 2^m$ matrix M:

$$M^C := B^{-1}M^*B.$$

Charge conjugation does not change the order of matrices: $(MN)^C = M^C N^C$. In practice the matrices M we deal with are products of γ-matrices. Hence, we need only the charge conjugation property of the generating γ-matrices, which is

$$(\gamma_\mu)^C \equiv B^{-1}\gamma_\mu^* B = (-t_0t_1)\,\gamma_\mu$$

$$(\gamma_*)^C = (-)^{D/2+1}\gamma_*.$$

$$(\bar{\chi}M\lambda)^* \equiv (\bar{\chi}M\lambda)^C = (-t_0t_1)\,\overline{\chi^C}M^C\lambda^C.$$

Majorana spinors

1 Definition and properties 56

$$\psi = \psi^C = B^{-1}\psi^*, \qquad \text{i.e.} \qquad \psi^* = B\psi$$

$B = \mathrm{i}t_0 C\gamma^0$

(!! допишу явный вид матриц!!!)

The two possible values $t_0 = \pm 1$ must be considered, and we begin with the case $t_0 = +1$. Consulting Table 3.1, we see that $t_0 = +1$ holds for spacetime dimension $D = 2, 3, 4$, mod 8. In this case we call the spinors that satisfy (3.80) Majorana spinors. It is clear from (3.75) that if $t_0 = 1$ and $t_1 = -1$, the barred (3.50) and Dirac adjoint spinors (2.30) agree for Majorana spinors. In fact, this gives an alternative definition of a Majorana spinor.

Another fact about the Majorana case is that there are representations of the γ-matrices that are explicitly real and may be called really real representations. Here is a really real representation for $D = 4$:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \mathrm{i}\sigma_2 \otimes \mathbb{1},$$

$$\gamma^1 = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \sigma_3 \otimes \mathbb{1},$$

$$\gamma^2 = \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_1,$$

$$\gamma^3 = \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_3.$$

Note that the γ^i are symmetric, while γ^0 is antisymmetric. This is required by hermiticity in any real representation. We construct really real representations in all allowed dimensions $D = 2, 3, 4$ mod 8 in Appendix 3A.5.

In such representations (3.48) implies that $B = \mathbb{1}$ (up to a phase). The relation (3.47) then gives $C = \mathrm{i}\gamma^0$. Further, a Majorana spinor field is really real since (3.80) reduces to $\Psi^* = \Psi$.

Really real representations are sometimes convenient, but we emphasize that the physics of Majorana spinors is the same in, and can be explored in, any representation of the Clifford algebra, replacing complex conjugation with charge conjugation.

$$(\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi)^* = (\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi)^C = \bar{\chi}\left(\gamma_{\mu_1\ldots\mu_r}\right)^C\psi = \bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi.$$

When $t_0 = -1$ (and still $t_1 = -1$) spinors that satisfy (3.80) are called pseudo-Majorana spinors. They are mostly relevant for $D = 8$ or 9. There are no really real representations in these dimensions; instead there are representations of the Clifford algebra in which the generating γ-matrices are imaginary, $(\gamma^\mu)^* = -\gamma^\mu$. In any representation (3.79) and (3.77) hold with $t_0 = t_1 = -1$. This implies that the reality properties of bilinears are different from those of Majorana spinors. Although these differences are significant, the essential property that a complex spinor can be reduced to a real one still holds, and it is common in the literature not to distinguish between Majorana and pseudo-Majorana spinors. However, note the following.

Exercise 3.35 Show that the mass term $m\bar{\chi}\chi = 0$ for a single pseudo-Majorana field. Pseudo-Majorana spinors must be massless (unless paired).

We now consider (pseudo-)Majorana spinors in even dimensions $D = 0, 2, 4$ mod 8. We can quickly show using (3.78) that these cases are somewhat different. For $D = 2$mod 8 we have $(\gamma_*\psi)^C = \gamma_*\psi^C$. Thus the two constraints

$$\text{Majorana:}\quad \psi^C = \psi, \qquad \text{Weyl:}\quad P_{L,R}\psi = \psi,$$

are compatible. It is equivalent to observe that the chiral projections of a Majorana spinor ψ satisfy

$$(P_L\psi)^C = P_L\psi, \quad (P_R\psi)^C = P_R\psi$$

Thus the chiral projections of a Majorana spinor are also Majorana spinors. Each chiral projection satisfies both constraints in (3.83) and is called a Majorana-Weyl spinor. Such spinors have 2^{m-1} independent 'real' components in dimension $D = 2m = 2$ mod 8 and are the 'most fundamental' spinors available in these dimensions. It is not surprising that supergravity and superstring theories in $D = 10$ dimensions are based on Majorana-Weyl spinors.

For $D = 4$ mod 4 dimensions we have $(\gamma_*\psi)^C = -\gamma_*\psi^C$, so that the equations of (3.84) are replaced by

$$(P_L\psi)^C = P_R\psi, \quad (P_R\psi)^C = P_L\psi$$

For $D = 4$ mod 4 dimensions we have $(\gamma_*\psi)^C = -\gamma_*\psi^C$, so that the equations of (3.84) are replaced by

$$(P_L\psi)^C = P_R\psi, \quad (P_R\psi)^C = P_L\psi.$$

2 Symplectic Majorana spinors 58

When $t_1 = 1$ we cannot define Majorana spinors, but we can define 'symplectic Majorana spinors'. These consist of an even number of spinors χ^i, with $i = 1, ..., 2k$, which satisfy a 'reality condition' containing a non-singular antisymmetric matrix ε^{ij}. The inverse matrix ε_{ij} satisfies $\varepsilon^{ij}\varepsilon_{kj} = \delta^i_k$. Symplectic Majorana spinors satisfy the condition

$$\chi^i = \varepsilon^{ij}\left(\chi^j\right)^C = \varepsilon^{ij}B^{-1}\left(\chi^j\right)^*.$$

The consistency check discussed after (3.80) now works for $t_1 = 1$ because of the antisymmetric ε^{ij}.

Exercise 3.36 Check that, in five dimensions with symplectic Majorana spinors, $\bar{\psi}^i\chi_i \equiv \bar{\psi}^i\chi^j\varepsilon_{ji}$ is pure imaginary while $\bar{\psi}^i\gamma_\mu\chi_i$ is real.

3 Dimensions of minimal spinors 58

$$\begin{aligned} t_1 = -1, \quad t_0 = 1: &\quad \text{Majorana,} \\ t_0 = -1: &\quad \text{pseudo-Majorana,} \\ t_1 = 1: &\quad \text{symplectic Majorana.} \end{aligned}$$

Irreducible spinors, number of components and symmetry properties.

dim	spinor	min # components	antisymmetric
2	MW	1	1
3	M	2	$1, 2$
4	M	4	$1, 2$
5	S	8	$2, 3$
6	SW	8	3
7	S	16	$0, 3$
8	M	16	$0, 1$
9	M	16	$0, 1$
10	MW	16	1
11	M	32	$1, 2$

Majorana spinors in physical theories

1 Variation of a Majorana Lagrangian 59

In this section we consider a prototype action for a Majorana spinor field in dimension $D = 2, 3, 4 \bmod 8$. Majorana and Dirac fields transform the same way under Lorentz transformations, but Majorana spinors have half as many degrees of freedom, so we write

$$S[\Psi] = -\frac{1}{2} \int \mathrm{d}^D x \bar{\Psi} \left[\gamma^\mu \partial_\mu - m\right] \Psi(x).$$

There is an immediate and curious subtlety due to the symmetries of the matrices C and $C\gamma^\mu$. Using (3.50), we see that the mass and kinetic terms are proportional to $\Psi^T C \Psi$ and $\Psi^T C\gamma^\mu \partial_\mu \Psi$. Suppose that the field components Ψ are conventional commuting numbers. Since C is antisymmetric, the mass term vanishes. Since $C\gamma^\mu$ is symmetric, the kinetic term is a total derivative and thus vanishes when integrated in the action. For commuting field components, there is no dynamics! To restore the dynamics we must assume that Majorana fields are anticommuting Grassmann variables, which we always assume unless stated otherwise.

$$\delta S[\Psi] = -\int \mathrm{d}^D x \delta \bar{\Psi} \left[\gamma^\mu \partial_\mu - m\right] \Psi(x)$$

2 Relation of Majorana and Weyl spinor theories 60

In even dimensions $D = 0, 2, 4 \bmod 8$, both Majorana and Weyl fields exist and both have legitimate claims to be more fundamental than a Dirac fermion. In fact both fields describe equivalent physics. Let's show this for $D = 4$. We can rewrite the action (3.88) as

$$\begin{aligned} S[\psi] &= -\frac{1}{2} \int \mathrm{d}^4 x \left[\bar{\Psi}\gamma^\mu \partial_\mu - m\right] (P_L + P_R) \Psi \\ &= -\int \mathrm{d}^4 x \left[\bar{\Psi}\gamma^\mu \partial_\mu P_L \Psi - \frac{1}{2} m \bar{\Psi} P_L \Psi - \frac{1}{2} m \bar{\Psi} P_R \Psi\right]. \end{aligned}$$

We obtained the second line by a Majorana flip and partial integration. In the second form of the action, the Majorana field is replaced by its chiral projections. In our treatment of chiral multiplets in supersymmetry, we will exercise the option to write Majorana fermion actions in this way.

Exercise 3.40 Show that the Euler-Lagrange equations that follow from the variation of the second form of the action in (3.91) are

$$\not{\partial} P_L \Psi = m P_R \Psi, \quad \not{\partial} P_R \Psi = m P_L \Psi.$$

Derive $\Box P_{L,R}\Psi = m^2 P_{L,R}\Psi$ from the equations above.

Let's return to the Weyl representation (2.19) for the final step in the argument to show that the equation of motion for a Majorana field can be reexpressed in terms of a Weyl field and its adjoint. The Majorana condition $\Psi = B^{-1}\Psi^* = \gamma^0\gamma^1\gamma^3\Psi^*$ requires that Ψ take the form

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_2^* \\ -\psi_1^* \end{pmatrix}.$$

With (3.93) and (2.55) in view we define the two-component Weyl fields

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \tilde{\psi} = \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix}.$$

Using the form of γ^μ (2.19) and γ_* (3.34) in the Weyl representation, we see that we can identify

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} = P_L\Psi, \quad \begin{pmatrix} 0 \\ \tilde{\psi} \end{pmatrix} = (P_L\Psi)^C = P_R\Psi.$$

The equations of motion (3.92) can then be rewritten as

$$\bar{\sigma}^\mu \partial_\mu \psi = m\tilde{\psi}, \quad \sigma^\mu \partial_\mu \tilde{\psi} = m\psi.$$

These are equivalent to the pair of Weyl equations in (2.56) with the restriction $\tilde{\psi} = \bar{\chi}$ which comes because we started in this section with a Majorana rather than a Dirac field.

3 U(1) symmetries of a Majorana field 61

In Sec. 2.7.1 we considered the U(1) symmetry operation $\Psi \to \Psi' = \mathrm{e}^{\mathrm{i}\theta}\Psi$. This symmetry is obviously incompatible with the Majorana condition (3.80). Thus the simplest internal symmetry of a Dirac fermion cannot be defined in a field theory of a (single) Majorana field. However, it is easy to see that $(\mathrm{i}\gamma_*)^C = \mathrm{i}\gamma_*$, so the chiral transformation $\Psi \to \Psi' = \mathrm{e}^{\mathrm{i}\gamma_*\theta}\Psi$ preserves the Majorana condition. Let's ask whether the infinitesimal limit of this transformation is a symmetry of the free massive Majorana action (3.88).

$\delta\bar{\Psi} = \mathrm{i}\theta\bar{\Psi}\gamma_*$, variation $\delta S[\Psi] = \mathrm{i}\theta m \int \mathrm{d}^4x \bar{\Psi}\gamma_*\Psi$, vanishes only for a massless Majorana field.

$$J^\mu_{*\text{Noether}} = \frac{1}{2}\mathrm{i}\bar{\Psi}\gamma^\mu\gamma_*\Psi$$
$$\partial_\mu J^\mu_* = -\mathrm{i}m\bar{\Psi}\gamma_*\Psi$$

The current is conserved only for massless Majorana fermions. The dynamics of a Majorana field Ψ can be expressed in terms of its chiral projections $P_{L,R}\Psi$. So can the chiral transformation, which becomes $P_{L,R}\Psi \to P_{L,R}\Psi' = \mathrm{e}^{\pm\mathrm{i}\theta}\Psi$.

Throughout this section we used the simple dynamics of a free massive fermion to illustrate the relation between Majorana and Weyl fields and to explore their U(1) symmetries. It is straightforward to extend these ideas to interacting field theories with nonlinear equations of motion.

Majorana conjugate (??)

$$\bar{\lambda} \equiv \lambda^T C$$

For Majorana spinors, it is equivalent to $\bar{\Psi} = \Psi^\dagger \beta = \Psi^\dagger \mathrm{i}\gamma^0$.
(?? I'll add details later)

2.1.5 Clifford algebras for fields

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbb{1}.$$

The Clifford algebra in general dimension

1 The generating γ-matrices

$$\text{Euclidean:} \qquad \begin{aligned} \gamma^1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ..., \\ \gamma^2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ..., \\ \gamma^3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes ..., \\ \gamma^4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes ..., \\ \gamma^5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes ..., \\ ... &= ... \end{aligned}$$

For odd $D = 2m+1$ we need one additional matrix, and we take γ^{2m+1} from the list above, but we keep only the first m factors, i.e. deleting a σ_1 (?????). Thus there is no increase in the dimension of the representation in going from $D = 2m$ to $D = 2m+1$, and we can say in general that the construction above gives a representation of dimension $2^{[D/2]}$, where [D/2] means the integer part of $D/2$.

For Lorentzian γ we pick any single matrix from the Euclidean construction, multiply it by i and label it γ^0 for the time-like direction. This matrix is anti-hermitian and $\left(\gamma^0\right)^2 = -\mathbb{1}$. We then relabel the remaining $D-1$ matrices to obtain the Lorentzian set $\gamma^\mu, 0 \le \mu \le D-1$. The hermiticity properties of the Lorentzian γ are

$$\gamma^{\mu\dagger} = \gamma^0 \gamma^\mu \gamma^0$$

2 The complete Clifford algebra

$$\gamma^{\mu_1 \cdots \mu_r} := \gamma^{[\mu_1} ... \gamma^{\mu_r]},$$

$$\begin{aligned} \gamma^{\mu\nu} &= \frac{1}{2}\left[\gamma^\mu, \gamma^\nu\right], \\ \gamma^{\mu_1\mu_2\mu_3} &= \frac{1}{2}\left\{\gamma^{\mu_1}, \gamma^{\mu_2\mu_3}\right\}, \\ \gamma^{\mu_1\mu_2\mu_3\mu_4} &= \frac{1}{2}\left[\gamma^{\mu_1}, \gamma^{\mu_2\mu_3\mu_4}\right], \\ & .... \end{aligned}$$

$$\gamma^{\mu_1 \cdots \mu_D} = \frac{1}{2}\left(\gamma^{\mu_1}\gamma^{\mu_2\cdots\mu_D} - (-)^D \gamma^{\mu_2\cdots\mu_D}\gamma^{\mu_1}\right).$$

So, $\mathrm{Tr}\, \gamma^{\mu_1 \cdots \mu_D} = 0$ for even D.

3 Levi-Civita symbol in high dimensions

$$\varepsilon_{012(D-1)} = 1, \quad \varepsilon^{012(D-1)} = -1.$$

Indices are raised using the Minkowski metric which leads to the difference in sign above (due to the single time-like direction).

$$\varepsilon_{\mu_1\ldots\mu_n v_1\ldots\nu_p}\varepsilon^{\mu_1\cdots\mu_n\rho_1\cdots\rho_p} = -p!n!\delta^{\rho_1\cdots\rho_p}_{\nu_1\cdots v_p}, \quad p = D-n.$$

The antisymmetric p-index Kronecker δ:

$$\delta^{\rho_1\cdots\rho_p}_{\nu_1\ldots\nu_q} := \delta^{[\rho_1}_{\nu_1}\delta^{\rho_2}_{\nu_2}\cdots\delta^{\rho_p]}_{\nu_q},$$

The Schouten identity:

$$0 = 5\delta^{[v}_{\mu}\varepsilon^{\rho\sigma\tau\lambda]} \equiv \delta_\mu{}^\nu\varepsilon^{\rho\sigma\tau\lambda} + \delta_\mu{}^\rho\varepsilon^{\sigma\tau\lambda\nu} + \delta_\mu{}^\sigma\varepsilon^{\tau\lambda\nu\rho} + \delta_\mu{}^\tau\varepsilon^{\lambda\nu\rho\sigma} + \delta_\mu{}^\lambda\varepsilon^{\nu\rho\sigma\tau}$$

4 Practical γ-matrix manipulation

For both even and odd D:

• Products of γ-matrices with a contraction of index:

$$\gamma^{\mu\nu}\gamma_\nu = (D-1)\gamma^\mu$$

$$\gamma^{\mu\nu\rho}\gamma_\rho = (D-2)\gamma^{\mu\nu}$$

$$\gamma^{\mu_1\ldots\mu_r v_1\ldots\nu_s}\gamma_{\nu_s\ldots\nu_1} = \frac{(D-r)!}{(D-r-s)!}\gamma^{\mu_1\ldots\mu_r}$$

(?? idea???)

$$\gamma^{\nu_1\ldots\nu_r} = (-)^{r(r-1)/2}\gamma^{\nu_r\ldots\nu_1}.$$

• Products of γ-matrices with no sum over indices: some combinatorial tricks can be used. For example, when calculating

$$\gamma^{\mu_1\mu_2}\gamma_{\nu_1\ldots\nu_D},$$

one knows that the index values μ_1 and μ_2 appear in the set of $\{v_i\}$. There are D possibilities for μ_2, and since μ_1 should be different, there remain $D-1$ possibilities for μ_1. Hence the result is

$$\gamma^{\mu_1\mu_2}\gamma_{\nu_1\ldots\nu_D} = D(D-1)\delta^{\mu_2\mu_1}_{[v_1 v_2}\gamma_{v_3\ldots\nu_D]}.$$

Such generalized δ-functions are always normalized with 'weight 1', i.e. $\delta^{\mu_2\mu_1}_{\nu_1\nu_2} = \frac{1}{2}\left(\delta^{\mu_2}_{\nu_1}\delta^{\mu_1}_{\nu_2} - \delta^{\mu_1}_{\nu_1}\delta^{\mu_2}_{\nu_2}\right)$, so (???)

$$\gamma^{\mu_1\mu_2}\gamma_{\nu_1\ldots\nu_D}\varepsilon^{\nu_1\ldots\nu_D} = D(D-1)\varepsilon^{\mu_2\mu_1\nu_3\ldots\nu_D}\gamma_{\nu_3\ldots\nu_D}.$$

• Products of γ-matrices without index contractions. (???!!! cool method?) The very simplest case is

$$\gamma^\mu\gamma^\nu = \gamma^{\mu\nu} + \eta^{\mu\nu}.$$

This follows directly from the definitions: the antisymmetric part of the product is defined in $\gamma^{\mu_1\cdots\mu_r} = \gamma^{[\mu_1}\ldots\gamma^{\mu_r]}$ to be $\gamma^{\mu\nu}$, while the symmetric part of the product is $\eta^{\mu\nu}$, by virtue of $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}$. This already illustrates the general approach: one first writes the totally antisymmetric Clifford matrix that contains all the indices and then adds terms for all possible index pairings. Another example:

$$\gamma^{\mu\nu\rho}\gamma_{\sigma\tau} = \gamma^{\mu\nu\rho}_{\sigma\tau} + 6\gamma^{[\mu\nu}_{[\tau}\delta^{\rho]}_{\sigma]} + 6\gamma^{[\mu}\delta^\nu_{[\tau}\delta^{\rho]}_{\sigma]}.$$

$$\gamma^{\mu_1\cdots\mu_4}\gamma_{\nu_1\nu_2} = \gamma^{\mu_1\cdots\mu_4}{}_{\nu_1\nu_2} + 8\gamma^{[\mu_1\cdots\mu_3}{}_{[v_2}\,\delta^{\mu_4]}{}_{v_1]} + 12\gamma^{[\mu_1\mu_2}\,\delta^{\mu_3}{}_{[v_2}\,\delta^{\mu_4]}{}_{v_1]}.$$

• Products of γ-matrices with both contracted and uncontracted indices. Consider $\gamma^{\mu_1\cdots\mu_4\rho}\gamma_{\rho v_1 v_2}$. The result should contain terms similar to $\gamma^{\mu_1\cdots\mu_4}\gamma_{\nu_1\nu_2}$, but each term has an extra numerical factor reflecting the number of values that ρ can take in this sum. For example, in the second term above there is now one contraction between an upper and lower index, and therefore ρ can run over all D values except the four values $\mu_1, ...\mu_4$, and the two values v_1, v_2. This counting gives

$$\gamma^{\mu_1\cdots\mu_4\rho}\gamma_{\rho v_1 v_2} = (D-6)\gamma^{\mu_1\cdots\mu_4}{}_{\nu_1\nu_2} + 8(D-5)\gamma^{[\mu_1\cdots\mu_3}{}_{[v_2}\,\delta^{\mu_4]}{}_{v_1]} + 12(D-4)\gamma^{[\mu_1\mu_2}\,\delta^{\mu_3}{}_{[\nu_2}\,\delta^{\mu_4]}{}_{v_1]}.$$

Examples:

$$\gamma_\nu\gamma^\mu\gamma^\nu = (2-D)\gamma^\mu,$$
$$\gamma_\rho\gamma^{\mu\nu}\gamma^\rho = (D-4)\gamma^{\mu\nu}.$$

5 Basis of Clifford algebra for even dimension $D = 2m$

$$\left\{\Gamma^A := \mathbb{1}, \gamma^\mu, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \cdots, \gamma^{\mu_1\cdots\mu_D}\right\}.$$
$$\left\{\Gamma_A := \mathbb{1}, \gamma_\mu, \gamma_{\mu_2\mu_1}, \gamma_{\mu_3\mu_2\mu_1}, ..., \gamma_{\mu_D\cdots\mu_1}\right\}. \qquad \gamma^{\nu_1\ldots\nu_r} = (-)^{r(r-1)/2}\gamma^{\nu_r\ldots\nu_1}$$

$\mu_1 < \mu_2 < ... < \mu_r$. This list contains 2^D trace orthogonal matrices in an algebra of total dimension 2^D. Therefore it is a basis of the space of matrices M of dimension $2^m \times 2^m$.

$$\Gamma^A\Gamma^B = \pm\Gamma^C,$$

Γ^C is the basis element whose indices are those of A and B with common indices excluded.

$$\mathrm{Tr}\left(\Gamma^A\Gamma_B\right) = 2^m\delta^A_B.$$

$$\forall M \qquad\qquad M \equiv \sum_A m_A\Gamma^A, \qquad m_A = \frac{1}{2^m ă}\,\mathrm{Tr}\left(M\Gamma_A\right).$$

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 2^m\left[\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right].$$

6 The highest rank Clifford algebra element

$$\gamma_* := (-\mathrm{i})^{m+1}\gamma_0\gamma_1\cdots\gamma_{D-1}$$
$$\gamma_*^2{}_{\ D=2m} = \mathbb{1}$$

For $D = 2m$, $\gamma_* \equiv \gamma_{D+1}$, e.g. $\gamma_{*\ D=4} = \gamma_5$.

This matrix occurs as the unique highest rank element in $\{\Gamma^A\}$. For any order of components μ_i, one can write

$$\gamma_{\mu_1\mu_2\cdots\mu_D} = \mathrm{i}^{m+1}\varepsilon_{\mu_1\mu_2\cdots\mu_D}\gamma_*,$$

γ_* commutes with all even rank elements of the Clifford algebra and anti-commutes with all odd rank elements, for example,

$$\{\gamma_*, \gamma^\mu\} = 0,$$
$$[\gamma_*, \gamma^{\mu\nu}] = 0.$$

(????)

$$P_L := \frac{1}{2}\left(\mathbb{1} + \gamma_*\right), \qquad P_R := \frac{1}{2}\left(\mathbb{1} - \gamma_*\right).$$

Thus

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} \equiv P_L\Psi, \quad \begin{pmatrix} 0 \\ \bar{\chi} \end{pmatrix} \equiv P_R\Psi.$$

(?!?!?!? how exactly higher rank element is used???)

7 Clifford algebra in odd spacetime dimension $D = 2m + 1$

$$\gamma^\mu_\pm = \left(\gamma^0, \gamma^1, ..., \gamma^{(2m-1)}, \gamma^{2m} = \pm\gamma_*\right)$$

Duality relations:

$$\gamma_\pm^{\mu_1\cdots\mu_r} = \pm \mathrm{i}^{m+1}\frac{1}{(D-r)!}\varepsilon^{\mu_1\cdots\mu_D}\gamma_{\pm\mu_D\cdots\mu_{r+1}}.$$

$$(\mathcal{D} = 2m) \qquad \gamma^{\mu_1\mu_2\cdots\mu_r}\gamma_* = -(-\mathrm{i})^{m+1}\frac{1}{(D-r)!}\varepsilon^{\mu_r\mu_{r-1}\cdots\mu_1 v_1\nu_2\cdots\nu_{D-r}}\gamma_{\nu_1\nu_2\cdots\nu_{D-r}}.$$

You can use the tricks explained in Sec. 3.1.4. Show that in four dimensions

$$\gamma_{\mu\nu\rho} = \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\gamma^\sigma\gamma_*.$$

Thus, a basis of the Clifford algebra in $D = 2m + 1$ dimensions contains the matrices in $\{\Gamma^A\}$ only up to rank m. This agrees with the counting argument in Ex. 3.9. For example, the set $\{\mathbb{1}, \gamma^\mu, \gamma^{\mu\nu}\}$ of $1 + 5 + 10 = 16$ matrices is a basis of the Clifford algebra for $D = 5$. Ex. 3.16 shows that it is a rearrangement of the basis $\left\{\Gamma^A\right\}$ for $D = 4$.

8 Symmetries of γ-matrices

In the Clifford algebra of the $2^m \times 2^m$ matrices, for both $D = 2m$ and $D = 2m + 1$, one can distinguish between the symmetric and the antisymmetric matrices where the symmetry property is defined in the following way. There exists a unitary matrix, C, called the charge conjugation matrix, such that each matrix $C\Gamma^A$ is either symmetric or antisymmetric. Symmetry depends only on the rank r of the matrix Γ^A, so we can write:

$$\left(C\Gamma^{(r)}\right)^T = -t_r C\Gamma^{(r)}, \quad t_r = \pm 1,$$

$\Gamma^A \equiv \left\{\mathbb{1}, \gamma^\mu, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \cdots, \gamma^{\mu_1\cdots\mu_D}\right\}$ of rank r. (The $-$ sign here is convenient for later manipulations.) For rank $r = 0$ and 1:

$$C^T = -t_0 C, \qquad \gamma^{\mu T} = t_0 t_1 C\gamma^\mu C^{-1}.$$

(??? what is a rank here???)

These relations suffice to determine the symmetries of all $C\gamma^{\mu_1\cdots\mu_r}$ and thus all coefficients t_r: e.g. $t_2 = -t_0$ and $t_3 = -t_1$. Further, $t_{r+4} = t_r$.

For example, for even D:

$$\begin{aligned} C_+ &= \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes ..., \quad t_0 t_1 = 1, \\ C_- &= \sigma_2 \otimes \sigma_1 \otimes \sigma_2 \otimes \sigma_1 \otimes ..., \quad t_0 t_1 = -1. \end{aligned}$$

Symmetries of γ-matrices:

$D(\text{mod}8)$	$t_r = -1$	$t_r = +1$
0	0,3	2, 1
	0, 1	**2, 3**
1	0, 1	2,3
2	0, 1	2,3
	1,2	**0, 3**
3	1,2	0,3
4	**1,2**	**0, 3**
	2,3	0, 1
5	2,3	0, 1
6	**2,3**	**0, 1**
	0, 3	1, 2
7	0,3	1,2

The entries contain the numbers r mod 4 for which $t_r = \pm 1$. For even dimensions, in bold face are the choices that are most convenient for supersymmetry.

(????? why two options here????)

(???? how to prove this table??????)

$$B := \mathrm{i} t_0 C \gamma^0$$

$$\gamma^{\mu *} = -t_0 t_1 B \gamma^\mu B^{-1}.$$

$$B^* B = -t_1 \mathbb{1}.$$

(!!!! very important paragraph about why exactly are we using such notation!!! seems too complicated, proof that it is important are needed!)

In the Weyl representation $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$, one can choose $B = \gamma^0 \gamma^1 \gamma^3$, which is real, symmetric, and $B^2 = \mathbb{1}$. Then $C = \mathrm{i}\gamma^3\gamma^1$.

In another representation $\gamma'^\mu = S \gamma^\mu S^{-1}$,

$$C' = S^{-1T} C S^{-1}, \quad B' = S^{-1T} B S^{-1}.$$

Appendix 3A Details of the Clifford algebras for $D = 2m$

A.1 Traces and the basis of the Clifford algebra 62

The trace properties of the matrices are important for proofs of these properties which are independent of the explicit construction in (3.2). The matrices Γ^A for tensor rank $1 \le r \le D-1$ are traceless. One simple way to see this is to use the Lorentz transformations (2.22) and its extension to general rank

$$L(\lambda)\gamma^{\mu_1\mu_2\ldots\mu_r} L(\lambda)^{-1} = \gamma^{\nu_1\nu_2\ldots\nu_r} \Lambda_{\nu_1}^{\mu_1} \ldots \Lambda_{\nu_r}^{\mu_r}.$$

Traces then satisfy the Lorentz transformation law as suggested by their free indices:

$$\mathrm{Tr}\, \gamma^{\mu_1\mu_2\ldots\mu_r} = \mathrm{Tr}\, \gamma^{\nu_1\nu_2\ldots\nu_r} \Lambda_{\nu_1}^{\mu_1} \ldots \Lambda_{\nu_r}^{\mu_r}$$

This means that the traces must be totally antisymmetric Lorentz invariant tensors.

$$\sum_A x_A \Gamma^A = 0.$$

Multiply by Γ_B from the right. Take the trace and use the trace orthogonality to obtain

$$\sum_A x_A \operatorname{Tr} \Gamma^A \Gamma^B = \pm x_B \operatorname{Tr} \mathbb{1} = 0.$$

Hence all $x_A = 0$ and linear independence is proven.

A.2 Uniqueness of the γ-matrix representation 63

We must now show that there is exactly one irreducible representation up to equivalence. We use the basic properties of representations of finite groups. However, the Clifford algebra is not quite a group because the minus signs that necessarily occur in the set of products $\Gamma^A \Gamma^B = \pm \Gamma^C$ are not allowed by the definition of a group. This problem is solved by doubling the basis in $\{\Gamma^A\}$ to the larger set $\left\{\Gamma^A, -\Gamma^A\right\}$. This set is a group of order 2^{2m+1} since all products are contained within the larger set. For $m = 1$, the group obtained is isomorphic to the quaternions, so the groups defined by doubling the Clifford algebras are called generalized quaternionic groups.

(пока мне это не интересно)

A.3 The Clifford algebra for odd spacetime dimensions 65

We gave in (3.40) two different sets of γ-matrices for odd dimensions. They are inequivalent as representations of the generating elements. Indeed it is easily seen that $S\gamma_+{}^\mu S^{-1} = \gamma_-{}^\mu$ cannot be satisfied. This requires $S\gamma^\mu S^{-1} = \gamma^\mu$ for the first $2m$ components. But then, from the product form in (3.6) and (3.30), we obtain $S\gamma^{2m}S^{-1} = +\gamma^{2m}$, rather than the opposite sign needed.

It follows from Ex. 2.8 that the two sets of second rank elements constructed from the generating elements above, namely

$$\begin{aligned} \Sigma_\pm^{\mu\nu} &= \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right], \quad \mu, v = 0, ..., 2m-1, \\ &= \frac{1}{4}\left[\gamma^\mu, \pm\gamma_*\right], \quad \mu = 0, ..., 2m-1, \quad v = 2m, \end{aligned}$$

are each representations of the Lie algebra $\mathfrak{so}(2m, 1)$. The two representations are equivalent, however, since $\gamma_* \Sigma_+{}^{\mu\nu} \gamma_* = \Sigma_-{}^{\mu\nu}$. This representation is irreducible; indeed it is a copy of the unique 2^{2m}-dimensional fundamental irreducible representation with Dynkin designation $(0, 0, ..., 0, 1)$. It is associated with the short simple root of the Dynkin diagram for B_m.

A.4 Determination of symmetries of γ-matrices 65

(не думаю, что это актуально, понадобится - запишу)

A.5 Friendly representations 66

(не думаю, что это актуально, понадобится - запишу)

We start in $D = 2$ and write

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i}\sigma_2, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1,$$

which is a really real, hermitian, and friendly representation. The matrix γ_* is also real:

$$\gamma_* = -\gamma_0 \gamma_1 = \sigma_3$$

Adding it to (3.107) as γ^2 gives a real representation in $D=3$. The recursion relation for moving from a $D=2m-2$ representation with $\tilde{\gamma}$ to $D=2m$ is

$$\gamma^\mu = \tilde{\gamma}^\mu \otimes \mathbb{1}, \quad \mu = 0, ..., 2m-3,$$
$$\gamma^{2m-2} = \tilde{\gamma}_* \otimes \sigma_1, \quad \gamma^{2m-1} = \tilde{\gamma}_* \otimes \sigma_3.$$

This gives

$$\gamma_* = -\tilde{\gamma}_* \otimes \sigma_2.$$

2.2 Symmetries of field theories

(I'll collect all main methods about them)

$SO(n)$ internal symmetry

(??? я пока не понял, неужели в таком формате серьезно лучше всего записывать???)

$$R^i{}_k \delta_{ij} R^j{}_\ell = \delta_{k\ell}, \quad \det R = 1.$$

It is quite obvious that the linear map,

$$\phi^i(x) \to \phi'^i(x) = R^i{}_j \phi^j(x),$$

a matrix of SO(n) depends continuously on $\frac{1}{2}n(n-1)$ parameters.

$$R^i{}_j = \delta^i_j - \epsilon r^i{}_j.$$

$$[r, r'] = rr' - r'r.$$

$$r_{[\hat{i}\hat{j}]}{}^i{}_j \equiv \delta^i_{\hat{i}} \delta_{\hat{j}j} - \delta^i_{\hat{j}} \delta_{\hat{i}j} = -r_{[\hat{j}\hat{i}]}{}^i{}_j.$$

$$\left[r_{[\hat{i}\hat{j}]}, r_{[\hat{k}\hat{l}]}\right] = \delta_{\hat{j}\hat{k}} r_{[\hat{i}\hat{l}]} - \delta_{\hat{i}\hat{k}} r_{[\hat{j}\hat{l}]} - \delta_{\hat{j}\hat{l}} r_{[\hat{i}\hat{k}]} + \delta_{\hat{i}\hat{l}} r_{[\hat{j}\hat{k}]}.$$

SO(n) is determined by $\frac{1}{2}n(n-1)$ real parameters $\theta^{\hat{i}\hat{j}}$,

$$\forall R \in \mathrm{SO}(n) \qquad R \equiv \mathrm{e}^{-\frac{1}{2}\theta^{\hat{i}\hat{j}}} r_{[\hat{i}\hat{j}]}$$

General internal symmetry and Proeyen's notation of variation

$$[t_A, t_B] = f_{AB}{}^C t_C$$
$$\Theta = \theta^A t_A.$$
$$U(\Theta) = \mathrm{e}^{-\Theta} = \mathrm{e}^{-\theta^A t_A}.$$

$$\phi^i(x) \to \phi'^i(x) \equiv U(\Theta)^i{}_j \phi^j(x)$$

$$S\left[\phi^i\right] = \int \mathrm{d}^D x \mathcal{L}\left(\phi^i, \partial_\mu \phi^i\right),$$
$$\mathcal{L}\left(\phi^i, \partial_\mu \phi^i\right) = \mathcal{L}\left(\phi'^i, \partial_\mu \phi'^i\right).$$

(???? дальше нотация, которую нужно особым образом понимать, пока я не пользуюсь ей, но чуть что знаю, что такая есть)

$$\delta_1\delta_2\phi \equiv -\Theta_2\delta_1\phi = \Theta_2\Theta_1\phi$$

$$\delta_1\delta_2\phi = \theta_1^A\theta_2^B t_B t_A\phi$$

$$\begin{aligned}\left[\delta_1, \delta_2\right]\phi = -\left[\Theta_1, \Theta_2\right]\phi \equiv \delta_3\phi \\ \Theta_3 \equiv \left[\Theta_1, \Theta_2\right] = f_{AB}{}^C\theta_1^A\theta_2^B t_C.\end{aligned}$$

$$\phi \xrightarrow{\theta_2} \phi' = U\left(\Theta_2\right)\phi \xrightarrow{\theta_1} \phi'' = U\left(\Theta_2\right)U\left(\Theta_1\right)\phi.$$

Noether currents and charges (????!!)

$\frac{\partial}{\partial x^\mu}\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i(x)} - \frac{\delta\mathcal{L}}{\delta\phi^i(x)} = 0,$

$$\begin{aligned}\delta\phi^i(x) := \epsilon^A\Delta_A\phi^i(x), \qquad \epsilon^A = \text{const} \\ \text{internal } \epsilon^A\Delta_A\phi^i \to -\theta^A\left(t_A\right)^i{}_j\phi^j, \\ \text{spacetime } \epsilon^A\Delta_A\phi^i \to \left[a^\mu\partial_\mu - \frac{1}{2}\lambda^{\rho\sigma}\left(x_\rho\partial_\sigma - x_\sigma\partial_\rho\right)\right]\phi^i.\end{aligned}$$

$$\delta\mathcal{L} \equiv \epsilon^A\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\partial_\mu\Delta_A\phi^i + \frac{\delta\mathcal{L}}{\delta\phi^i}\Delta_A\phi^i\right] = \epsilon^A\partial_\mu K_A^\mu$$

Using (1.62) we can rearrange (1.66) to read $\partial_\mu J_A^\mu = 0$, J_A^μ is the Noether current

$$J_A^\mu = -\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\Delta_A\phi^i + K_A^\mu.$$

This is a conserved current, by which we mean that $\partial_\mu J_A^\mu = 0$ for all solutions of the equations of motion of the system.

We temporarily assume that the symmetry parameters are arbitrary functions $\epsilon^A(x)$.

$$\begin{aligned}\delta S &= \int \mathrm{d}^D x\left[\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\partial_\mu\left(\epsilon^A\Delta_A\phi^i\right) + \frac{\delta\mathcal{L}}{\delta\phi^i}\epsilon^A\Delta_A\phi^i\right] \\ &= \int \mathrm{d}^D x\left[\epsilon^A\partial_\mu K_A^\mu + \left(\partial_\mu\epsilon^A\right)\frac{\delta\mathcal{L}}{\delta\partial_\mu\phi^i}\Delta_A\phi^i\right] \\ &= -\int \mathrm{d}^D x\left(\partial_\mu\epsilon^A\right)J_A^\mu.\end{aligned}$$

$$Q_A = \int\limits_{\Sigma(\tau)} \mathrm{d}\Sigma_\mu J_A^\mu(x),$$

$$Q_A = \int \mathrm{d}^{D-1}\vec{x} J_A^0(\vec{x}, t)$$

Main examples of Noether theorem

$$\mathcal{L} = -\frac{1}{2}\partial_\mu \phi^i \partial^\mu \phi^i - V\left(\phi^i\right).$$

For internal symmetry, substitution of the first line of (1.64) in (1.67) gives

$$J_A^\mu = -\partial^\mu \phi t_A \phi,$$

$$\delta V = \partial_i V \delta \phi^i = -\partial_i V \left(t_A\right)^i_j \phi^j = 0.$$

$$T^\mu{}_\nu = \partial^\mu \phi \partial_\nu \phi + \delta^\mu_\nu \mathcal{L}.$$

$$M^\mu_{[\rho\sigma]} = -x_\rho T^\mu_\sigma + x_\sigma T^\mu_\rho.$$

$$\begin{aligned} T_A &= \int \mathrm{d}^{D-1}\vec{x} J^0_A, \\ P_\mu &= \int \mathrm{d}^{D-1}\vec{x} T^0{}_\mu, \\ M_{[\rho\sigma]} &= \int \mathrm{d}^{D-1}\vec{x} M^0_{[\rho\sigma]}. \end{aligned}$$

2.3 On Lorentz and Poicare groups in details

The Lorentz group for $D = 4$

(тут подробно будет написано все!!)

Let $m_{[\mu\nu]}$ denote the matrices of a representation of the Lie algebra (1.34) for $D = 4$. The six independent matrices consist of three spatial rotations $J_i = -\frac{1}{2}\varepsilon_{ijk} m_{[jk]}$ (ε_{ijk} is the alternating symbol with $\varepsilon_{123} = 1$) and three boosts $K_i = m_{[0i]}$. It is a straightforward and important exercise to show, using (1.34), that the linear combinations

$$\begin{aligned} I_k &= \frac{1}{2}\left(J_k - \mathrm{i}K_k\right), \quad k = 1, 2, 3, \\ I'_k &= \frac{1}{2}\left(J_k + \mathrm{i}K_k\right), \end{aligned}$$

satisfy the commutation relations of two independent copies of the Lie algebra $\mathfrak{su}(2)$, viz.

$$\begin{aligned} \left[I_i, I_j\right] &= \varepsilon_{ijk} I_k, \\ \left[I'_i, I'_j\right] &= \varepsilon_{ijk} I'_k, \\ \left[I_i, I'_j\right] &= 0. \end{aligned}$$

(потом выпишу из теорпола)

The homomorphism of $SL(2, C) \to SO(3, 1)$

First we note that a general 2×2 hermitian matrix can be parametrized as

$$\mathbf{x} = \begin{pmatrix} x^0 + x^3 & x^1 - \mathrm{i}x^2 \\ x^1 + \mathrm{i}x^2 & x^0 - x^3 \end{pmatrix}$$

and that $\det \mathbf{x} = -x^\mu \eta_{\mu\nu} x^\nu$, which is the negative of the Minkowski norm of the 4-vector x^μ. This suggests a close relation between the linear space of hermitian 2×2 matrices and four-dimensional Minkowski space. Indeed, there is an isomorphism between these spaces, which we now elucidate.

$$\sigma_\mu = (-\mathbb{1}, \sigma_i)\,, \quad \bar\sigma_\mu = \sigma^\mu = (\mathbb{1}, \sigma_i)\,,$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

$$\sigma_\mu \bar\sigma_\nu + \sigma_\nu \bar\sigma_\mu = 2\eta_{\mu\nu}\mathbb{1},$$
$$\mathrm{tr}\,(\sigma^\mu \bar\sigma_\nu) = 2\delta^\mu{}_\nu.$$

$$\mathbf{x} = \bar\sigma_\mu x^\mu, \quad x^\mu = \frac{1}{2}\,\mathrm{tr}\,(\sigma^\mu \mathbf{x})$$

$$\phi(A)^\mu_\nu = \frac{1}{2}\,\mathrm{tr}\left(\sigma^\mu \mathrm{A} \bar\sigma_\nu\ \mathrm{A}^\dagger\right)$$

$$\sigma_{\mu\nu} = \frac{1}{4}\left(\sigma_\mu \bar\sigma_\nu - \sigma_\nu \bar\sigma_\mu\right),$$
$$\bar\sigma_{\mu\nu} = \frac{1}{4}\left(\bar\sigma_\mu \sigma_\nu - \bar\sigma_\nu \sigma_\mu\right).$$

Note that $\sigma^{\mu\nu\dagger} = -\bar\sigma^{\mu\nu}$. The finite Lorentz transformation (1.32) is then represented as

$$L(\lambda) = \mathrm{e}^{\frac{1}{2}\lambda^{\mu\nu}\sigma_{\mu\nu}},$$
$$\bar L(\lambda) = \mathrm{e}^{\frac{1}{2}\lambda^{\mu\nu}\bar\sigma_{\mu\nu}}.$$

Exercise 2.5 Show that

$$L(\lambda)^\dagger = \bar L(-\lambda) = \bar L(\lambda)^{-1}$$

2.4 Scalar field

The scalar field system

We will use $\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}(-, +, ..., +)$.

$$\Box \phi^i(x) = m^2 \phi^i(x),$$

$\Box = \eta^{\mu\nu}\partial_\mu \partial_\nu$ is the Lorentz invariant d'Alembertian wave operator.

$$\phi^i(x) = \phi^i_+(x) + \phi^i_-(x),$$
$$\phi^i_+(x) = \int \frac{\mathrm{d}^{D-1}\vec p}{(2\pi)^{(D-1)} 2E} \mathrm{e}^{\mathrm{i}(\vec p \cdot \vec x - Et)} a^i(\vec p),$$
$$\phi^i_-(x) = \int \frac{\mathrm{d}^{D-1}\vec p}{(2\pi)^{(D-1)} 2E} \mathrm{e}^{-\mathrm{i}(\vec p \cdot \vec x - Et)} a^{i*}(\vec p).$$

In the classical theory the quantities $a^i(\vec p), a^{i*}(\vec p)$ are simply complex valued functions of the spatial momentum $\vec p$. After quantization one arrives at the true quantum field theory [1] in which $\mathbf{a}^i(\vec p), \mathbf{a}^{i*}(\vec p)$ are annihilation and creation operators [2] for the particles described by the field operator $\phi^i(\vec x)$. The Klein-Gordon equation (1.2) is the variational derivative $\delta S/\delta\phi^i(x)$ of the action

$$S = \int \mathrm{d}^D x \mathcal{L}(x) = -\frac{1}{2}\int \mathrm{d}^D x \left[\eta^{\mu\nu}\partial_\mu \phi^i \partial_\nu \phi^i + m^2 \phi^i \phi^i\right].$$

Spacetime symmetries the Lorentz and Poincare groups

Типичные формулы

(то, что уже известно должно быть)

$$x^\mu = \Lambda^\mu_\nu x'^\nu \quad \text{or} \quad x'^\mu = \Lambda^{-1\mu}{}_\nu x^\nu,$$
$$x^\mu \eta_{\mu\nu} x^\nu = x'^\mu \eta_{\mu\nu} x'^\nu.$$
$$x'^\mu = \Lambda^{-1\mu}_\nu{}_\nu \left(x^\nu - a^\nu\right).$$

$$\Lambda_{\mu\nu} = \left(\Lambda^{-1}\right)_{\nu\mu}, \quad \Lambda^\mu{}_\nu = \left(\Lambda^{-1}\right)_\nu{}^\mu,$$
$$x'_\mu = \left(\Lambda^{-1}\right)_\mu{}^\nu x_\nu = x_\nu \Lambda^\nu{}_\mu.$$

$$\Lambda^\mu{}_\nu = \delta^\mu_\nu + \epsilon m^\mu{}_\nu + \cdots.$$

It is straightforward to see that (1.29) satisfies (1.27) to first order in ϵ as long as the generator (m with two lower indices) is antisymmetric, viz.

$$m_{\mu\nu} \equiv \eta_{\mu\rho} m^\rho{}_\nu = -m_{\nu\mu}.$$

The Lie algebra is the real linear space spanned by the $\frac{1}{2}D(D-1)$ independent generators, with the commutator product $[m, m'] = mm' - m'm$. These matrices must also be multiplied as $m^\mu{}_\rho m'{}_\nu$, but the forms with both indices down, as in (1.30), (or both up) are often convenient.

A useful basis for the Lie algebra is to choose generators that act in each of the $\frac{1}{2}D(D-1)$ coordinate 2 -planes. For the 2 -plane in the directions ρ, σ this generator is given by

$$m_{[\rho\sigma]}{}^\mu{}_\nu \equiv \delta^\mu_\rho \eta_{\nu\sigma} - \delta^\mu_\sigma \eta_{\rho v} = -m_{[\sigma\rho]}{}^\mu{}_\nu.$$

$$\Lambda = \mathrm{e}^{\frac{1}{2}\lambda^{\rho\sigma} m_{[\rho\sigma]}}.$$

When matrix indices are restored, we have, with the representation (1.31), the series [8]

$$\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \lambda^\mu{}_\nu + \frac{1}{2}\lambda^\mu{}_\rho \lambda^\rho{}_\nu + \ldots$$

The commutators of the generators defined in (1.31) are

$$\left[m_{[\mu\nu]}, m_{[\rho\sigma]}\right] = \eta_{\nu\rho} m_{[\mu\sigma]} - \eta_{\mu\rho} m_{[v\sigma]} - \eta_{\nu\sigma} m_{[\mu\rho]} + \eta_{\mu\sigma} m_{[v\rho]}.$$

These equations specify the structure constants of the Lie algebra, which may be written as

$$f_{[\mu\nu][\rho\sigma]^{[\kappa\tau]}} = 8\eta_{[\rho[\nu} \delta^{[\kappa}_{\mu]} \delta^{\tau]}_{\sigma]}.$$

$$[m_A, m_B] = f_{AB}{}^C m_C \to \left[m_{[\mu\nu]}, m_{[\rho\sigma]}\right] = \frac{1}{2} f_{[\mu\nu][\rho\sigma]}{}^{[\kappa\tau]} m_{[\kappa\tau]}.$$

der a symmetry, each group element is mapped to a transformation of the configurapace of the dynamical fields. This map must give a group homomorphism. For the tz matrix Λ, the transformation of the scalar fields is defined as

$$\phi^i(x) \xrightarrow{\Lambda} \phi'^i(x) \equiv \phi^i(\Lambda x).$$

Typical paticular cases of Lorentz and Poincare transformations

(I'll add later)

Symmetries in the canonical formalism

scalar fields ϕ^i, the canonical coordinates at fixed time $t = 0$ are the field variables $\phi(\vec{x}, 0)$ at each point $\vec{x}$ of space, and the canonical momenta are given by $\pi(\vec{x}, 0) = \delta S/\delta\partial_t\phi(\vec{x}, 0)$. For the action (1.71), the canonical momentum is $\pi_i = \partial_0\phi^i = -\partial^0\phi^i$.

We consider explicitly the special cases of internal symmetry, space translations and rotations in which the vector K_A^μ of (1.66) has vanishing time component. In these cases the formula (1.70) for the Noether charge simplifies to

$$
\begin{aligned}
Q_A &= -\int \mathrm{d}^{D-1}\vec{x}\frac{\delta\mathcal{L}}{\delta\partial_0\phi^i}\Delta_A\phi^i \\
&= -\int \mathrm{d}^{D-1}\vec{x}\pi_i\Delta_A\phi^i.
\end{aligned}
$$

We remind readers that the basic (equal-time) Poisson bracket is $\{\phi^i(\vec{x}), \pi_j(\vec{y})\} = \delta^i_j\delta^{D-1}(\vec{x}-\vec{y})$. The Poisson bracket of two observables $A(\phi, \pi)$ and $B(\phi, \pi)$ is

$$
\{A, B\} \equiv \int \mathrm{d}^{D-1}\vec{x}\left(\frac{\delta A}{\delta\phi^i(\vec{x})}\frac{\delta B}{\delta\pi_i(\vec{x})} - \frac{\delta A}{\delta\pi_i(\vec{x})}\frac{\delta B}{\delta\phi^i(\vec{x})}\right).
$$

Poisson brackets $\{A, \{B, C\}\}$ obey the Jacobi identity

$$
\{A, \{B, C\}\} + \{B, \{C, A\}\} + \{C, \{A, B\}\} = 0.
$$

It is now easy to see that the infinitesimal symmetry transformation $\Delta_A\phi^i$ is generated by its Poisson bracket with the Noether charge Q_A. In detail

$$
\Delta_A\phi^i(x) = \left\{Q_A, \phi^i(x)\right\} = -\int \mathrm{d}^{D-1}\vec{y}\left\{\pi_j(\vec{y})\Delta_A\phi^j(\vec{y}), \phi^i(x)\right\}.
$$

Further, Poisson brackets of the conserved charges obey the Lie algebra of the symmetry group,

$$
\{Q_A, Q_B\} = f_{AB}^C Q_C.
$$

$$
\delta_1\delta_2\phi^i = \epsilon_1^A\epsilon_2^B\left\{Q_A, \left\{Q_B, \phi^i\right\}\right\}.
$$

Using the Jacobi identity (1.78) one easily obtains the commutator

$$
[\delta_1, \delta_2]\,\phi^i = f_{AB}{}^C\epsilon_1^A\epsilon_2^B\left\{Q_C, \phi^i\right\}.
$$

Note that the symmetry parameters compose exactly as in (1.22) and (1.60).
$[\Delta_A, \Delta_B] = f_{AB}{}^C\Delta_C$.

When Poisson brackets are available, we define symmetry operators as in (1.79). However, in practice we streamline the notation by omitting explicit Poisson brackets and simply use the notation $\Delta_A\phi^i$ to indicate the transformation rules. In the three cases of interest we replace $\Delta_A\phi$ by $T_A\phi, P_\mu\phi$ and $M_{\mu\nu}\phi$ and write the explicit transformation rules as

$$
\begin{aligned}
T_A\phi^i &= -\left(t_A\right)^i_j\phi^j, \\
P_\mu\phi^i &= \partial_\mu\phi^i, \\
M_{[\mu\nu]}\phi^i &= -J_{[\mu\nu]}\phi^i.
\end{aligned}
$$

Quantum operators

After quantization the symmetry operators become the operator commutators

$$\Delta_A \mathbf{\Phi}^i = -\mathrm{i}\left[\mathbf{Q}_A, \mathbf{\Phi}^i\right]_{\mathrm{qu}},$$
$$\left[\mathbf{Q}_A, \mathbf{Q}_B\right]_{\mathrm{qu}} = \mathrm{i} f_{AB}{}^C \mathbf{Q}_C.$$

$$\mathbf{\Phi}^i(x) \to \mathrm{e}^{-\mathrm{i}\epsilon^A \mathbf{Q}_A} \mathbf{\Phi}^i(x) \mathrm{e}^{\mathrm{i}\epsilon^A \mathbf{Q}_A} = U(\epsilon)\mathbf{\Phi}^i(x).$$

Here $U(\epsilon)$ is a generic notation for a finite group transformation. More specifically, for internal symmetry $U(\epsilon) \to U(\Theta)$ of (1.15), for translations $U(\epsilon) \to U(a)$ of (1.53), and for Lorentz $U(\epsilon) \to U(\Lambda)$ of (1.49). For finite transformations of an internal symmetry group G or the Poincaré group, (1.85) reads

$$\mathrm{e}^{-\mathrm{i}\theta^A \mathbf{T}_A} \Phi^i(x) \mathrm{e}^{\mathrm{i}\theta^A \mathbf{T}_A} = \mathrm{e}^{-\Theta}\Phi^i(x),$$
$$\mathrm{e}^{-\mathrm{i}\left[a^\mu \mathbf{P}_\mu + \frac{1}{2}\lambda^{\rho\sigma}\mathbf{M}_{[\rho\sigma]}\right]} \Phi^i(x) \mathrm{e}^{\mathrm{i}\left[a^\mu \mathbf{P}_\mu + \frac{1}{2}\lambda^{\rho\sigma}\mathbf{M}_{[\rho\sigma]}\right]} = \Phi^i(\Lambda x + a).$$

Exercise 1.14 Verify the corresponding quantum operator relation

$$\left[\delta_1, \delta_2\right] \Phi^i = -\mathrm{i} f_{AB}{}^C \epsilon_1^A \epsilon_2^B \left[\mathbf{Q}_C, \Phi^i\right]_{\mathrm{qu}}.$$

It is also useful to verify the composition of finite group transformations. A transformation with parameters ϵ_2^A followed by another one with parameters ϵ_1^A is found by applying (1.85) twice. One obtains

$$\begin{aligned} \mathrm{e}^{-\mathrm{i}\epsilon_1^A \mathbf{Q}_A} \mathrm{e}^{-\mathrm{i}\epsilon_2^B \mathbf{Q}_B} \mathbf{\Phi}^i(x) \mathrm{e}^{\mathrm{i}\epsilon_2^B \mathbf{Q}_B} \mathrm{e}^{\mathrm{i}\epsilon_1^A \mathbf{Q}_A} &= U\left(\epsilon_2\right) \mathrm{e}^{-\mathrm{i}\epsilon_1^A \mathbf{Q}_A} \mathbf{\Phi}^i(x) \mathrm{e}^{\mathrm{i}\epsilon_1^A \mathbf{Q}_A} \\ &= U\left(\epsilon_2\right) U\left(\epsilon_1\right) \mathbf{\Phi}^i(x). \end{aligned}$$

$$P^0 \equiv H = \frac{1}{2}\int \mathrm{d}^{D-1}\vec{x}\left[\pi^2 + (\vec{\partial}\phi)^2\right].$$

$$\begin{aligned} [H, \Phi]_{\mathrm{qu}} &= -\mathrm{i}\pi = -\mathrm{i}\partial_0\phi \\ &= \int \frac{\mathrm{d}^{D-1}\vec{p}}{(2\pi)^{(D-1)}2E} E\left(-\mathrm{e}^{\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}a(\vec{p}) + \mathrm{e}^{-\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}a^*(\vec{p})\right). \end{aligned}$$

$$H = \frac{1}{2}\int \frac{\mathrm{d}^{D-1}\vec{p}E}{(2\pi)^{(D-1)}2E}\left[a^*(\vec{p})a(\vec{p}) + a(\vec{p})a^*(\vec{p})\right].$$

$$\left[a(\vec{p}), a^*\left(\vec{p}'\right)\right]_{\mathrm{qu}} = (2\pi)^3 2E(\vec{p})\delta^3\left(\vec{p} - \vec{p}'\right).$$

you can then reobtain (1.90).

2.5 Dirac field

The Dirac equation

$$\partial\!\!\!/\Psi(x) \equiv \gamma^\mu\partial_\mu\Psi(x) = m\Psi(x).$$

$$\partial^2\Psi = m^2\Psi$$
$$\frac{1}{2}\left\{\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu\right\}\partial_\mu\partial_\nu\Psi = m^2\Psi$$
$$\left\{\gamma^\mu, \gamma^\nu\right\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}$$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}.$$

$$\Sigma^{\mu\nu} \equiv \frac{1}{4}\left[\gamma^\mu, \gamma^\nu\right]$$

$$[\Sigma^{\mu\nu}, \gamma^\rho] = 2\gamma^{[\mu}\eta^{\nu]\rho} = \gamma^\mu\eta^{\nu\rho} - \gamma^\nu\eta^{\mu\rho}.$$

$$L(\lambda) = \mathrm{e}^{\frac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}}$$
$$L(\lambda)\gamma^\rho L(\lambda)^{-1} = \gamma^\sigma \Lambda(\lambda)^\rho_\sigma$$
$$\Psi'(x) = L(\lambda)^{-1}\Psi(\Lambda(\lambda)x)$$

$$\Psi(x) = \Psi_+(x) + \Psi_-(x),$$
$$\Psi_+(x) = \int \frac{\mathrm{d}^{(D-1)}\vec{p}}{(2\pi)^{D-1}2E}\mathrm{e}^{\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}\sum_s u(\vec{p},s)c(\vec{p},s),$$
$$\Psi_-(x) = \int \frac{\mathrm{d}^{(D-1)}\vec{p}}{(2\pi)^{D-1}2E}\mathrm{e}^{-\mathrm{i}(\vec{p}\cdot\vec{x}-Et)}\sum_s v(\vec{p},s)d(\vec{p},s)^*.$$

The * indicates complex conjugation in the classical theory and an operator adjoint after quantization.

Dirac adjoint and bilinear form

Under an infinitesimal Lorentz transformation in the $[\mu\nu]$-plane,

$$\delta\Psi(x) = -\frac{1}{2}\lambda^{\mu\nu}\left(\Sigma_{\mu\nu} + L_{[\mu\nu]}\right)\Psi(x) = -\frac{1}{2}\lambda^{\mu\nu}\Sigma_{\mu\nu}\Psi(x) + \lambda^\mu{}_\nu x^\nu\partial_\mu\Psi(x),$$
$$\delta\Psi^\dagger(x) = -\frac{1}{2}\lambda^{\mu\nu}\Psi^\dagger\Sigma_{\mu\nu}{}^\dagger + \lambda^\mu{}_\nu x^\nu\partial_\mu\Psi(x)^\dagger.$$

$$\beta := \mathrm{i}\gamma^0.$$
$$\bar{\Psi} := \Psi^\dagger\beta = \Psi^\dagger\mathrm{i}\gamma^0,$$
$$\text{Lor-inv bil form } = \Psi^\dagger\beta\Psi, = \bar{\Psi}\Psi.$$

$$\left(\bar{\Psi}_1\Psi_2\right)^\dagger = \bar{\Psi}_2\Psi_1$$

$\bar{\Psi}\Psi$ has signature (2,2) in 4D.

Dirac action

$$S[\bar{\Psi},\Psi] := -\int \mathrm{d}^D x\bar{\Psi}\left[\gamma^\mu\partial_\mu - m\right]\Psi(x)$$

$$\delta S[\bar{\Psi},\Psi] = -\int \mathrm{d}^D x\left\{\overline{\delta\Psi}\left[\gamma^\mu\partial_\mu - m\right]\Psi - \bar{\Psi}\left[\gamma^\mu\overleftarrow{\partial}_\mu + m\right]\delta\Psi\right\} = 0.$$

$$\bar{\Psi}\left[\gamma^\mu\overleftarrow{\partial}_\mu + m\right] = 0$$

The spinors $u(\vec{p}, s)$ and $v(\vec{p}, s)$ for $D = 4$

$$\begin{aligned}\gamma \cdot p u(\vec{p}, s) &= -\mathrm{i} m u(\vec{p}, s), \\ \gamma \cdot p v(\vec{p}, s) &= +\mathrm{i} m v(\vec{p}, s).\end{aligned} \qquad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & \sigma \cdot p \\ \bar{\sigma} \cdot p & 0 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = -\mathrm{i} m \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}.$$

$-\sigma \cdot p \bar{\sigma} \cdot p = -\bar{\sigma} \cdot p \sigma \cdot p = m^2,$

$$u(p) = \begin{pmatrix} \sqrt{-\sigma \cdot p} \xi \\ \mathrm{i} \sqrt{\bar{\sigma} \cdot p} \xi \end{pmatrix}$$

$$v(p) = \begin{pmatrix} \sqrt{-\sigma \cdot p} \eta \\ -\mathrm{i} \sqrt{\bar{\sigma} \cdot p} \eta \end{pmatrix}$$

$$\xi(\vec{p}, \pm) : \qquad \vec{\sigma} \cdot \vec{p} \xi(\vec{p}, \pm) = \pm |\vec{p}| \xi(\vec{p}, \pm)$$

$$\xi^\dagger(\vec{p}, \pm) \xi(\vec{p}, \pm) = 1, \quad \xi(\vec{p}, \pm)^\dagger \xi(\vec{p}, \mp) = 0$$

$$\eta(\vec{p}, \pm) := -\sigma_2 \xi(\vec{p}, \pm)^*$$

$$\vec{\sigma} \cdot \vec{p} \eta(\vec{p}, \pm) = \mp |\vec{p}| \eta(\vec{p}, \pm)$$

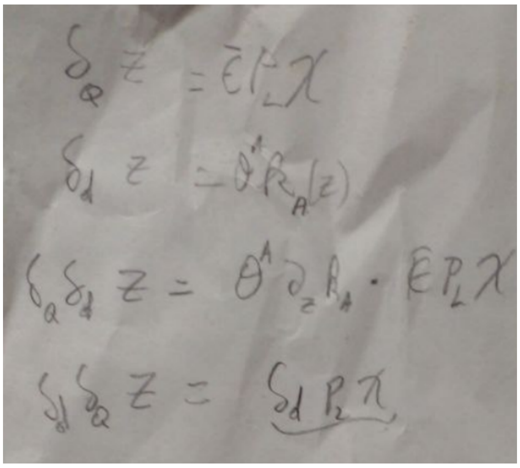
$$\begin{aligned} u(\vec{p}, \pm) &= \begin{pmatrix} \sqrt{E \mp |\vec{p}|}\ \xi(\vec{p}, \pm) \\ \mathrm{i} \sqrt{E \pm |\vec{p}|}\ \xi(\vec{p}, \pm) \end{pmatrix} \\ v(\vec{p}, \pm) &= \begin{pmatrix} \sqrt{E \pm |\vec{p}|}\ \eta(\vec{p}, \pm) \\ -\mathrm{i} \sqrt{E \mp |\vec{p}|}\ \eta(\vec{p}, \pm) \end{pmatrix} \end{aligned}$$

$$u_{m=0}(\vec{p}, -) = \sqrt{2E} \begin{pmatrix} \xi(\vec{p}, -) \\ 0 \end{pmatrix}, \qquad u_{m=0}(\vec{p}, +) = \sqrt{2E} \begin{pmatrix} 0 \\ \mathrm{i} \xi(\vec{p}, +) \end{pmatrix}$$

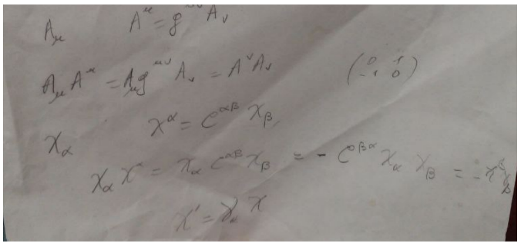
$$v(\vec{p}, -) = \sqrt{2E} \begin{pmatrix} 0 \\ -\mathrm{i} \eta(\vec{p}, -) \end{pmatrix}, \qquad v_{m=0}(\vec{p}, +) = \sqrt{2E} \begin{pmatrix} \eta(\vec{p}, +) \\ 0 \end{pmatrix}$$

$$\begin{aligned} \bar{u}(\vec{p}, s) u(\vec{p}, s') &= -\bar{v}(\vec{p}, s) v(\vec{p}, s') = -2m \delta_{ss'}, \\ \bar{u}(\vec{p}, s) v(\vec{p}, s') &= \bar{v}(\vec{p}, s) u(\vec{p}, s') = 0, \\ \bar{u}(\vec{p}, s) \gamma^\mu u(\vec{p}, s) &= \bar{v}(\vec{p}, s) \gamma^\mu \nu(\vec{p}, s) = -2\mathrm{i} p^\mu. \end{aligned}$$

Spinor metric and transformations (??)



Change position of spinors and get minus sign: (?)



Weyl spinor fields in even spacetime $D \equiv 2$ m

One can define Weyl fields with $2^{(m-1)}$ components (?? why so many???):

$$\psi(x): \qquad \psi(x) \to \psi'(x) = L(\lambda)^{-1}\psi(\Lambda(\lambda)x), \qquad L(\lambda) = \mathrm{e}^{\frac{1}{2}\lambda^{\mu\nu}}\sigma_{\mu\nu}$$

$$\chi(x): \qquad \bar{\chi}(x) \to \bar{\chi}'(x) = \bar{L}(\lambda)^{-1}\bar{\chi}(\Lambda(\lambda)x), \qquad \bar{L}(\lambda) = \mathrm{e}^{\frac{1}{2}\lambda^{\mu\nu}}\bar{\sigma}_{\mu\nu}$$

$$\bar{\sigma}^{\mu}\partial_{\mu}\psi(x) = 0, \qquad \Rightarrow \qquad \Box\psi(x) = 0$$
$$\sigma^{\mu}\partial_{\mu}\bar{\chi}(x) = 0. \qquad \Rightarrow \qquad \Box\bar{\chi}(x) = 0$$

$$\psi_{D=4}(x) = \int \frac{\mathrm{d}^{(D-1)}\vec{p}}{(2\pi)^{\frac{1}{2}(D-1)}\sqrt{2E}} \left[\mathrm{e}^{\mathrm{i}p\cdot x}c(\vec{p},-) + \mathrm{e}^{-\mathrm{i}p\cdot x}d(\vec{p},+)^*\right]\xi(\vec{p},-)$$

$$\psi(x)^{\dagger}\bar{\sigma}^{\mu}\psi(x) \to \Lambda^{-1\mu}{}_{\nu}\psi(\Lambda x)^{\dagger}\bar{\sigma}^{\nu}\psi(\Lambda x)$$

$$S[\psi,\bar{\psi}] = -\int \mathrm{d}^D x \mathrm{i}\psi^{\dagger}\ \bar{\sigma}^{\mu}\ \partial_{\mu}\psi$$

With a single Weyl field (ψ or $\bar{\chi}$), there is no way to introduce a mass. The candidate wave equation $\bar{\sigma}^{\mu}\partial_{\mu}\psi(x) = m\psi(x)$ is not Lorentz invariant.

One can describe massive particles using both $\psi(x)$ and $\bar{\chi}(x)$. In fact this is the secret content of a single Dirac field in any even dimension, and this can be exhibited using a Weyl representation of the γ-matrices. To show this write the Dirac field as $\Psi(x) = \begin{pmatrix} \psi(x) \\ \bar{\chi}(x) \end{pmatrix}$

and show that the Dirac equation $\partial\!\!\!/\Psi(x) \equiv \gamma^\mu\partial_\mu\Psi(x) = m\Psi(x)$ in the representation $\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$ is equivalent to the pair of equations

$$\bar{\sigma}^\mu\partial_\mu\psi(x) = m\bar{\chi}(x),$$
$$\sigma^\mu\partial_\mu\bar{\chi}(x) = m\psi(x).$$

$$\mathcal{L}_{\mathrm{Dir}}(\psi, \bar{\chi}) \equiv \mathrm{i}\left[-\psi^\dagger\bar{\sigma}\cdot\partial\psi + \bar{\chi}^\dagger\sigma\cdot\partial\bar{\chi} - m\bar{\chi}^\dagger\psi + m\psi^\dagger\bar{\chi}\right].$$

Each of the four terms is a Lorentz scalar.

Об уравнении Вейля

Уравнение Вейля - уравнение движения для безмассовой двухкомпонентной частицы со спином 1/2. Оно представляет собой частный случай уравнения Дирака для безмассовой частицы:

$$\frac{\partial\psi_+}{\partial x^0} + \sigma\nabla\psi_+ = 0$$
$$\frac{\partial\psi_-}{\partial x^0} - \sigma\nabla\psi_- = 0$$

(I'll work more with it later!)

Conserved currents for Dirac field

7.1 Conserved U(1) current for Dirac field

$\Psi(x) \to \Psi'(x) \equiv \mathrm{e}^{\mathrm{i}\theta}\Psi(x),$

$$J^\mu = \mathrm{i}\bar{\Psi}\gamma^\mu\Psi$$
$$J^0 = \Psi^\dagger\Psi > 0.$$

7.2 Energy-momentum tensors for the Dirac field

$$T_{\mu\nu\mathrm{can}} = \bar{\Psi}\gamma_\mu\partial_\nu\Psi + \eta_{\mu\nu}\mathcal{L}$$

$L = -\bar{\Psi}\left[\gamma^\mu\partial_\mu - m\right]\Psi(x)$

$$S' = -\int \mathrm{d}^D x \left[\frac{1}{2}\bar{\Psi}\gamma^\mu \overset{\leftrightarrow}{\partial}_\mu \Psi - m\bar{\Psi}\Psi\right]$$

$A \overset{\leftrightarrow}{\partial}_\mu B := A\left(\partial_\mu B\right) - \left(\partial_\mu A\right)B$ by addition of $\frac{1}{2}\partial_\mu\left(\bar{\Psi}\gamma^\mu\Psi\right)$. The advantage of this form is that the Lagrangian density $\mathcal{L}'$ is hermitian as an operator in Hilbert space. For it

$$T'_{\mu\nu} = \frac{1}{2}\bar{\Psi}\gamma_\mu \overset{\leftrightarrow}{\partial}_\nu \Psi + \eta_{\mu\nu}\mathcal{L}'$$
$$T'_{\mu\nu} - T_{\mu\nu} = \partial^\rho S_{\rho\mu\nu},$$

$S_{\rho\mu\nu}: \; S_{\rho\mu\nu} = -S_{\mu\rho\upsilon}.$

Addition of $\Delta T_{\mu\nu} = \frac{1}{4}\partial^\rho\left(\bar{\Psi}\left\{\Sigma_{\rho\mu}, \gamma_\nu\right\}\Psi\right)$ to $T'_{\mu\nu}$ produces the symmetric EMT:

$$\Theta_{\mu\nu} = \frac{1}{4}\bar{\Psi}\left(\gamma_\mu \overset{\leftrightarrow}{\partial}_\nu + \gamma_\nu \overset{\leftrightarrow}{\partial}_\mu\right)\Psi + \eta_{\mu\nu}\mathcal{L}'$$

Symmetry currents are evaluated 'on-shell', i.e. one should assume that Ψ and $\bar{\Psi}$ satisfy the Dirac equation. The last term $\mathcal{L}'$ then vanishes. (??)

On other formalisms for Dirac equation (???)

(тут по Редькову некоторые конструкции когда-то добавлю мб)

Как можно записывать уравнения Дирака?

2.6 Maxwell and Yang-Mills gauge fields

The abelian gauge field $A_\mu(x)$

Gauge invariance and fields with electric charge

For a Dirac spinor field the local gauge transformation:

$$\Psi(x) \to \Psi'(x) \equiv \mathrm{e}^{\mathrm{i}q\theta(x)}\Psi(x).$$

The goal is to formulate field equations that transform covariantly under the gauge transformation. This requires

$$A_\mu(x) \to A'_\mu(x) \equiv A_\mu(x) + \partial_\mu\theta(x).$$

One then defines the covariant derivative $D_\mu\Psi(x) \equiv (\partial_\mu - \mathrm{i}qA_\mu(x))\,\Psi(x)$, which transforms with the same phase factor as $\Psi(x)$, namely $D_\mu\Psi(x) \to \mathrm{e}^{\mathrm{i}q\theta(x)}D_\mu\Psi(x)$. The desired field equation is obtained by replacing $\partial_\mu\Psi \to D_\mu\Psi$:

$$[\gamma^\mu D_\mu - m]\,\Psi \equiv [\gamma^\mu\left(\partial_\mu - \mathrm{i}qA_\mu\right) - m]\,\Psi = 0.$$

The same procedure can be applied to a complex scalar field $\phi(x)$, to which we assign an electric charge q (which may differ from the charge of Ψ). We extend the global U(1) symmetry discussed in Ch. 1 to the local gauge symmetry $\phi(x) \to \phi'(x) = \mathrm{e}^{\mathrm{i}q\theta(x)}\phi(x)$ by defining the covariant derivative $D_\mu\phi = \left(\partial_\mu - \mathrm{i}qA_\mu\right)\phi$ and modifying the Klein-Gordon equation to the form

$$\left[D^\mu D_\mu - m^2\right]\phi = 0.$$

Degrees of freedom
On-shell degrees of freedom = number of helicity states.
Off-shell degrees of freedom = number of field components - gauge transformations.

The free gauge field

(обычная максвелловская эд, потом напишу, в теорполе почти в точности это же)

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x),$$

is invariant under the gauge transformation, a fact that is trivial to verify. In four dimensions $F_{\mu\nu}$ has six components, which split into the electric $E_i = F_{i0}$ and magnetic $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$ fields.

Since A_μ is a bosonic field, we expect it to satisfy a second order wave equation. The only Lorentz covariant and gauge invariant quantity available is $\partial^\mu F_{\mu\nu}$, so the free electromagnetic field satisfies

$$\partial^\mu F_{\mu\nu} = 0.$$

$$\begin{aligned} \nabla^2 A_0 - \partial_0\left(\partial^i A_i\right) &= 0, \\ \Box A_i - \partial_i\partial^0 A_0 - \partial_i\left(\partial^j A_j\right) &= 0. \end{aligned}$$

It is instructive to write the solution of $\Box A_i = 0$ as the Fourier transform

$$A_i(x) = \int \frac{\mathrm{d}^{(D-1)}k}{(2\pi)^{(D-1)}2k^0} \sum_\lambda \left[\mathrm{e}^{\mathrm{i}k\cdot x}\epsilon_i(\vec{k},\lambda)a(\vec{k},\lambda) + \mathrm{e}^{-\mathrm{i}k\cdot x}\epsilon_i^*(\vec{k},\lambda)a^*(\vec{k},\lambda) \right],$$

$\vec{k}, k^0 = |\vec{k}|$, is the on-shell energy-momentum vector. The $\epsilon_i(\vec{k},\lambda)$ are called polarization vectors, which are constrained by the Coulomb gauge condition to satisfy $k^i\epsilon_i(\vec{k},\lambda) = 0$. So there are $(D-2)$ independent polarization vectors, indexed by λ, and there are $2(D-2)$ independent real degrees of freedom contained in the complex quantities $a(\vec{k},\lambda)$. As in the case of the plane wave expansions of the free Klein-Gordon and Dirac fields, $a(\vec{k},\lambda)$ and $a^*(\vec{k},\lambda)$ are interpreted as Fourier amplitudes in the classical theory and as annihilation and creation operators for particle states after quantization. There are $D-2$ particle states.

Also

$$[D_\mu, D_\nu]\,\Psi \equiv (D_\mu D_\nu - D_\nu D_\mu)\,\Psi = -\mathrm{i}qF_{\mu\nu}\Psi.$$

The charged Dirac field also satisfies the second order equation

$$\left[D^\mu D_\mu - \frac{1}{2}\mathrm{i}q\gamma^{\mu\nu}F_{\mu\nu} - m^2 \right]\Psi = 0.$$

The field strength tensor satisfies the equation $\Box F_{\mu\nu} = 0$. This is a gauge invariant derivation of the fact that the free electromagnetic field describes massless particles.

Sources and Greens function

$$\partial^\mu F_{\mu\nu} = -J_\nu.$$

$$\Box F_{\nu\rho} = -\left(\partial_\nu J_\rho - \partial_\rho J_\nu\right).$$

Consider first the analogous problem of the scalar field coupled to a source $J(x)$:

$$\left(\Box - m^2\right)\phi(x) = -J(x).$$

The response is determined by the Green's function $G(x-y)$: $(\Box - m^2)\,G(x-y) = -\delta(x-y)$. The Euclidean Green's function:

$$G(x-y) = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\mathrm{e}^{\mathrm{i}k\cdot(x-y)}}{k^2+m^2}.$$

The integral can be expressed in terms of modified Bessel functions (? how?). In the massless case the result simplifies to the power law (? prove?)

$$G(x-y) = \frac{\Gamma\left(\frac{1}{2}(D-2)\right)}{4\pi^{\frac{1}{2}D}(x-y)^{(D-2)}}.$$

Here $(x-y)^2 = \delta_{\mu\nu}(x-y)^\mu(x-y)^\nu$ is the Euclidean distance between source point y and observation point x. Given $G(x-y)$, the solution:

$$\phi(x) = \int \mathrm{d}^D y G(x-y)J(y).$$

$$\left(\delta^{\mu\rho}\Box - \partial^\mu\partial^\rho\right)G_{\rho v}(x,y) = -\delta^\mu_\nu\delta(x-y).$$

$$\left(\delta^{\mu\rho}\Box - \partial^\mu\partial^\rho\right) G_{\rho v}(x,y) = -\delta^\mu_\nu \delta(x-y) + \frac{\partial}{\partial y^\nu}\Omega^\mu(x,y),$$

$\Omega^\mu(x,y)$ is an arbitrary vector function. If $\Omega^\mu(x,y)$ and $J_\nu(y)$ are suitably damped at large distance, the effect of the second term in (4.22) cancels (after partial integration) in the formula

$$A_\mu(x) = \int \mathrm{d}^D y G_{\mu\nu}(x,y) J^\nu(y),$$

which is the analogue of (4.20). We now derive the precise form of $G_{\mu\nu}(x,y)$. By Euclidean symmetry, we can assume the tensor form

$$G_{\mu\nu}(x,y) = \delta_{\mu\nu} F(\sigma) + (x-y)_\mu (x-y)_\nu \hat{S}(\sigma),$$

$\sigma = \frac{1}{2}(x-y)^2$. It is more useful, but equivalent, to take advantage of gauge invariance and rewrite this ansatz as

$$G_{\mu\nu}(x,y) = \delta_{\mu\nu} F(\sigma) + \partial_\mu \partial_\nu S(\sigma),$$

because the pure gauge term involving $S(\sigma)$ has no effect in (4.23) and cancels in (4.22). We may also assume that the gauge term in (4.22) has the Euclidean invariant form $\partial^\mu \partial_\nu \Omega(\sigma)$. Substituting (4.25) in (4.22) we find the two independent tensors δ^μ_ν and $(x-y)^\mu (x-y)_\nu$ and thus two independent differential equations involving F and Ω, namely

$$\begin{aligned} 2\sigma F''(\sigma) + (D-1)F'(\sigma) &= \Omega'(\sigma), \\ F''(\sigma) &= -\Omega''(\sigma). \end{aligned}$$

Note that $F'(\sigma) = \mathrm{d}F(\sigma)/\mathrm{d}\sigma$, etc. We have dropped the δ-function term in (4.22), because we will first solve these equations for $\sigma \neq 0$. The second equation in (4.26) may be integrated immediately, giving $F'(\sigma) = -\Omega'(\sigma)$; a possible integration constant is chosen to vanish, so that $F'(\sigma)$ vanishes at large distance. The first equation then becomes $2\sigma F''(\sigma) + DF'(\sigma) = 0$, which has the power-law solution $F(\sigma) \sim \sigma^{1-\frac{1}{2}D}$. However, on any function of σ, the Laplacian acts as $\Box F(\sigma) = 2\sigma F''(\sigma) + DF'(\sigma)$. In our case there is a hidden δ-function in $\Box F(\sigma)$ because the power law is singular. The effect of the δ-function in (4.22) is automatically incorporated if we take $F(\sigma) = G(x-y)G$ is the massless scalar Green's function in (4.19). The result of this analysis is the gauge field Green's function

$$G_{\mu\nu}(x,y) = \delta_{\mu\nu} G(x-y) + \partial_\mu \partial_\nu S(\sigma).$$

It may not be obvious why this method works. To see why, apply $\partial/\partial x^\mu$ to both sides of (4.22), obtaining

$$0 = -\partial_\nu \delta(x-y) - \partial_\nu \Box \Omega(\sigma),$$

in which $\partial_\nu = \partial/\partial x^\nu$. This consistency condition is satisfied because the analysis above led us the result $\Omega(\sigma) = -F(\sigma) = -G\left((x-y)^2\right)$.

Exercise 4.4 In $D = 4$ dimensions, consider a point charge at rest, i.e. $J^\mu(x) = \delta^\mu_0 q \delta(\vec{x})$. Obtain, using (4.23), that the resulting value of A^0, and therefore of the electric field, is

$$A^0(x) = \frac{q}{4\pi}\frac{1}{|\vec{x}|}, \quad \vec{E} = \frac{q}{4\pi}\frac{\vec{x}}{|x|^3}.$$

Quantum electrodynamics

It is also advantageous to change notation from that of Sec. 4.1.1 by scaling the vector potential, $A_\mu \to eA_\mu$, e is the conventional coupling constant of the electromagnetic field to charged fields; $e^2/4\pi \approx 1/137$ is called the fine structure constant. In this notation the relevant equations of Sec. 4.1 read:

$$\begin{aligned} F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \\ A_\mu \to A'_\mu &\equiv A_\mu + \frac{1}{e}\partial_\mu \theta, \\ D_\mu \Psi &\equiv (\partial_\mu - \mathrm{i}eqA_\mu)\,\Psi, \\ [D_\mu, D_\nu]\,\Psi &= -\mathrm{i}eqF_{\mu\nu}\Psi. \end{aligned}$$

The electric charges q of the various charged fields are then simple rational numbers, for example $q = 1$ for the electron. [5]

$$S\left[A_\mu, \bar\Psi, \Psi\right] = \int \mathrm{d}^D x \left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \bar\Psi\left(\gamma^\mu D_\mu - m\right)\Psi\right].$$

Euler variation of (4.31) with respect to the gauge potential A_ν is

$$\frac{\delta \mathcal{L}}{\delta A^\nu} = \partial^\mu F_{\mu\nu} + \mathrm{i}eq\bar\Psi\gamma_\nu\Psi = 0$$

The stress tensor and gauge covariant translations

$$\Theta_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho + \frac{1}{4}\bar\Psi\left(\gamma_\mu \overleftrightarrow{D}_\nu + \gamma_\nu \overleftrightarrow{D}_\mu\right)\Psi + \eta_{\mu\nu}\mathcal{L}$$

Why? Because the canonical stress tensor, calculated from the Noether formula (1.67) with $\Delta_A\phi^i \to \partial_\nu A_\rho, \partial_\nu\bar\Psi, \partial_\nu\Psi$ for the three independent fields, is

$$T^\mu{}_\nu = F^{\mu\rho}\partial_\nu A_\rho + \bar\Psi\gamma^\mu\partial_\nu\Psi + \delta^\mu_\nu\mathcal{L}.$$

It is conserved on the index μ, but not on ν, not symmetric and not gauge invariant. The situation can be improved by treating fermion terms as in Sec. 2.7.2 and then adding $\Delta T^\mu{}_\nu = -\partial_\rho\left(F^{\mu\rho}A_\nu\right)$ in accord with the discussion in Sec. 1.3. The final result is the gauge invariant and symmetric.

Об использовании (???)

(не знаю, пока только пару раз в жизни их видел)

Non-abelian gauge symmetry

Global internal symmetry

$$[t_A, t_B] = f_{AB}{}^C t_C.$$

The array of real numbers $f_{AB}{}^C$ are structure constants of the algebra (the same in all representations). They obey the Jacobi identity

$$f_{AD}{}^E f_{BC}{}^D + f_{BD}{}^E f_{CA}{}^D + f_{CD}{}^E f_{AB}{}^D = 0.$$

The indices can be lowered by the Cartan-Killing metric defined in Appendix B (see (B.6)), and then the f_{ABC} are totally antisymmetric. For simple algebras, the generators can be chosen to be trace orthogonal, $\mathrm{Tr}\,(t_A t_B) = -c\delta_{AB}$, with c positive for compact groups, and the Cartan-Killing metric is then proportional to this expression.

A theory with global non-abelian internal symmetry contains scalar and spinor fields, each of which transforms in an irreducible representation R. For example, there may be a Dirac spinor [13] field $\Psi^\alpha(x), \alpha = 1, ..., \dim_R$, that transforms in the complex representation R as

$$\Psi^\alpha(x) \to \left(\mathrm{e}^{-\theta^A t_A}\right)^\alpha{}_\beta \Psi^\beta(x).$$

The conjugate spinor [14] is denoted by $\bar{\Psi}_\alpha$ and transforms as

$$\bar{\Psi}_\alpha \to \bar{\Psi}_\beta \left(\mathrm{e}^{\theta^A t_A}\right)^\beta{}_\alpha.$$

For most of our discussion it is sufficient to restrict attention to the infinitesimal transformations,

$$\begin{aligned}
\delta\Psi &= -\theta^A t_A \Psi,\\
\delta\bar{\Psi} &= \bar{\Psi}\theta^A t_A,\\
\delta\phi^A &= \theta^C f_{BC}{}^A \phi^B.
\end{aligned}$$

For $S[\bar{\Psi}, \Psi] = -\int \mathrm{d}^D x \bar{\Psi} \left[\gamma^\mu \partial_\mu - m\right] \Psi$,

$$J_{A\mu} = -\bar{\Psi} t_A \gamma_\mu \Psi, \quad A = 1, ..., \dim_G .$$

The current transforms as a field in the adjoint representation, i.e.

$$\delta J_{A\mu} = \theta^C f_{CA}{}^B J_{B\mu}.$$

$\delta\left(\phi^A J_{A\mu}\right) = 0.$

Gauging the symmetry

$$\delta A_\mu^A(x) = \frac{1}{g}\partial_\mu \theta^A + \theta^C(x) A_\mu^B(x) f_{BC}{}^A.$$

$$\begin{aligned}
D_\mu \Psi &= \left(\partial_\mu + g t_A A_\mu^A\right)\Psi,\\
D_\mu \bar{\Psi} &= \partial_\mu \bar{\Psi} - g\bar{\Psi} t_A A_\mu^A,\\
D_\mu \phi^A &= \partial_\mu \phi^A + g f_{BC}^A A_\mu^B \phi^C.\\
\frac{\delta S}{\delta \bar{\Psi}_\alpha} &= -\left[\gamma^\mu D_\mu - m\right]\Psi^\alpha = 0.
\end{aligned}$$

Yang-Mills field strength and action

$$\begin{aligned}
\left[D_\mu, D_\nu\right]\Psi &= g F_{\mu\nu}^A t_A \Psi,\\
F_{\mu\nu}^A &:= \partial_\mu A_\nu^A - \partial_\nu A_\mu^A + g f_{BC}{}^A A_\mu^B A_\nu^C.
\end{aligned}$$

$$\delta F_{\mu\nu}^A = \theta^C F_{\mu\nu}^B f_{BC}{}^A.$$

$$D_\mu F^A_{\nu\rho} + D_\nu F^A_{\rho\mu} + D_\rho F^A_{\mu\nu} = 0,$$

$D_\mu F^A_{\nu\rho} := \partial_\mu F^A_{\nu\rho} + g f_{BC}{}^A A^B_\mu F^C_{\nu\rho}.$

$$S\left[A^A_\mu, \bar{\Psi}_\alpha, \Psi^\alpha\right] = \int \mathrm{d}^D x \left[-\frac{1}{4} F^{A\mu\nu} F^A_{\mu\nu} - \bar{\Psi}_\alpha \left(\gamma^\mu D_\mu - m\right) \Psi^\alpha\right].$$

Yang-Mills theory for $G = SU(N)$ (??)

The generators of the fundamental representation of Lie algebra of SU(N) are a set of N^2-1 traceless antihermitian $N \times N$ matrices t_A, which are normalized by

$$\mathrm{Tr}\left(t_A t_B\right) = -\frac{1}{2}\delta_{AB}.$$

$U(x) \equiv \mathrm{e}^{-\Theta(x)}$, with $\Theta(x) = \theta^A(x) t_A$

$$\Psi(x) \to U(x)\Psi(x). \qquad \text{(fund. repr.)}$$

$$U(x) t_A U(x)^{-1} = t_B R(x)^B{}_A,$$

$R(x)^B{}_A$ is a real $(N^2-1) \times (N^2-1)$ matrix.

Given any set of N^2-1 real quantities X^A, that is any element of the vector space $\mathbb{R}^{N^2-1}$, we can form the matrix $\mathbf{X} = t_A X^A$. For any group element U, we have $U\mathbf{X}U^{-1} = t_B R^B{}_A X^A$. Thus the unitary transformation of the matrix $\mathbf{X}$ contains the information that the quantities $X^A = -2\delta^{AB}\,\mathrm{Tr}\left(t_B \mathbf{X}\right)$ transform in the adjoint representation, that is as $X^A \to R^A{}_B X^B$. Thus, given any field in the adjoint representation, such as $\phi^A(x)$, we can form the matrix $\Phi(x) = t_A \phi^A(x)$. Gauge transformations can then be implemented as

$$\Phi(x) \to U(x)\Phi(x)U(x)^{-1}.$$

$$\mathbf{A}_\mu(x) \to \mathbf{A}'_\mu(x) \equiv \frac{1}{g} U(x)\partial_\mu U(x)^{-1} + U(x)\mathbf{A}_\mu(x)U(x)^{-1}$$

$$t_A A^A_\mu(x) \equiv \delta A_\mu(x) = \frac{1}{g}\partial_\mu \Theta(x) + \left[A_\mu(x), \Theta(x)\right],$$

$$D_\mu \Psi \equiv \left(\partial_\mu + g\mathbf{A}_\mu\right)\Psi,$$
$$D_\mu \bar{\Psi} \equiv \partial_\mu \bar{\Psi} - g\bar{\Psi}\mathbf{A}_\mu.$$

For a field in the adjoint representation Φ,

$$D_\mu \Phi := \partial_\mu \Phi + g\left[\mathbf{A}_\mu, \Phi\right],$$

$$D_\mu \Psi \to U(x) D_\mu \Psi, \qquad D_\mu \bar{\Psi} \to D_\mu \bar{\Psi} U(x)^{-1}, \qquad D_\mu \Phi \to U(x) D_\mu \Phi U(x)^{-1}.$$

The non-abelian field strength can also be converted to matrix form:

$$\mathbf{F}_{\mu\nu} = t_A F^A_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu + g\left[\mathbf{A}_\mu, \mathbf{A}_\nu\right].$$

$$S\left[\mathbf{A}_\mu, \bar{\Psi}, \Psi\right] = \int \mathrm{d}^D x \left[\frac{1}{2}\,\mathrm{Tr}\left(\mathbf{F}^{\mu\nu}\mathbf{F}_{\mu\nu}\right) - \bar{\Psi}\left(\gamma^\mu \mathrm{D}_\mu - \mathrm{m}\right)\Psi\right]$$

The N^2-1 matrix generators $(t_A)^\alpha{}_\beta$ of the fundamental representation, normalized by $\mathrm{Tr}\left(t_A t_B\right) = -\frac{1}{2}\delta_{AB}$, together with the matrix $\mathrm{i}\delta^\alpha_\beta$ form a complete set of $N \times N$ anti-hermitian matrices, which are orthogonal in the trace norm. Therefore one can expand any $N \times N$ anti-hermitian matrix $H^{\alpha}{}_{\beta}$ in this set as

$$
\begin{aligned}
H^{\alpha}{}_{\beta} &= \mathrm{i} h_0 \delta^{\alpha}{}_{\beta} + h^A \left(t_A\right)^{\alpha}{}_{\beta}, \\
h_0 = -\frac{\mathrm{i}}{N} \operatorname{Tr} H, &\quad h^A = -2\delta^{AB} \operatorname{Tr}\left(H t_B\right).
\end{aligned}
$$

Internal symmetry for Majorana spinors

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Let χ^{α} denote a set of Majorana spinors to which we assign the group transformation rule

$$
\chi^{\alpha} \rightarrow \chi'^{\alpha} \equiv \left(\mathrm{e}^{-\theta^A\left(t_A P_L + t_A^* P_R\right)}\right)^{\alpha}{}_{\beta} \chi^{\beta}.
$$

The matrices $t_A P_L + t_A^* P_R$ are generators of a representation of an explicitly real representation of the Lie algebra, so the transformed spinors χ'^{α} also satisfy the Majorana condition.

$$
\begin{aligned}
P_L \chi \rightarrow P_L \chi' &\equiv \left(\mathrm{e}^{-\theta^A t_A}\right) P_L \chi, \\
P_R \chi \rightarrow P_R \chi' &\equiv \left(\mathrm{e}^{-\theta^A t_A^*}\right) P_R \chi.
\end{aligned}
$$

$$
\delta(\bar{\chi}\chi) = -\theta^A \bar{\chi}\left(t_A + t_A^T\right)\gamma_* \chi.
$$

The mass term is invariant only for the subset of generators that are antisymmetric, and thus real. This condition defines a subalgebra that contains only parity conserving vector-like gauge transformations. For $\mathfrak{g} = \mathfrak{su}(N)$, the subalgebra is isomorphic to $\mathfrak{so}(N)$. Non-invariance of the Majorana mass term is a special case of the general idea that chiral symmetry requires massless fermions.

$$
\frac{1}{2}\int \mathrm{d}^4 x \bar{\chi}\gamma^{\mu} D_{\mu}\chi = \int \mathrm{d}^4 x \bar{\chi}\gamma^{\mu} P_L D_{\mu}\chi = \int \mathrm{d}^4 x \bar{\chi}\gamma^{\mu} P_R D_{\mu}\chi,
$$

$P_{L,R} D_{\mu}\chi = D_{\mu} P_{L,R}\chi$.

2.7 Rarita-Schwinger field

Basic thery of RS field

$$
\Psi_{\mu}(x) \rightarrow \Psi_{\mu}(x) + \partial_{\mu}\epsilon(x).
$$

We want an action, that (a) Lorentz invariant, (b) first order in spacetime derivatives, (c) invariant under the gauge transformation $\Psi_{\mu}(x) \rightarrow \Psi_{\mu}(x) + \partial_{\mu}\epsilon(x)$ and the simultaneous conjugate transformation of $\bar{\Psi}_{\mu}$, and (d) hermitian, so that the Euler-Lagrange equation for $\bar{\Psi}_{\mu}$ is the Dirac conjugate of that for Ψ_{μ}:

$$
S = -\int \mathrm{d}^D x \bar{\Psi}_{\mu}\gamma^{\mu\nu\rho}\partial_{\nu}\Psi_{\rho},
$$

$$
\gamma^{\mu\nu\rho}\partial_{\nu}\Psi_{\rho} = 0
$$

For $D = 3$, $\gamma^{\mu\nu\rho}\partial_{\nu}\Psi_{\rho} = 0$ implies that $\partial_{\nu}\Psi_{\rho} - \partial_{\rho}\Psi_{\nu} = 0$. This means that the field has no gauge invariant degrees of freedom and thus no propagating particle modes. This is the supersymmetric counterpart of the situation in gravity for $D = 3$, where the field equation $R_{\mu\nu} = 0$ implies that the full curvature tensor $R_{\mu\nu\rho\sigma} = 0$. Hence no degrees of freedom.

$\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho = 0$ can be rewritten by $\gamma_\mu\gamma^{\mu\nu\rho} = (D-2)\gamma^{\nu\rho}$, which implies that $\gamma^{\nu\rho}\partial_\nu\Psi_\rho = 0$ in spacetime dimension $D > 2$, and $\gamma^{\mu\nu\rho} = \gamma^\mu\gamma^{\nu\rho} - 2\eta^{\mu[v}\gamma^{\rho]}$:

$$\gamma^\mu \left(\partial_\mu\Psi_\nu - \partial_\nu\Psi_\mu\right) = 0.$$

If we apply ∂_ρ and antisymmetrize in $\rho\nu$, we obtain

$$\not{\partial}\left(\partial_\rho\Psi_v - \partial_\nu\Psi_\rho\right) = 0.$$

The initial value problem for RS field

$$\gamma^i\Psi_i = 0,$$

which will play the same role as the Coulomb gauge condition we used in Sec. 4.1.2. Exercise 5.3 Show by an argument analogous to that in Sec. 4.1.2 that this condition does fix the gauge uniquely.

We use the equivalent form (5.4) of the field equations. The $v = 0$ and $v \to i$ components are

$$\begin{aligned} \gamma^i\partial_i\Psi_0 - \partial_0\gamma^i\Psi_i &= 0, \\ \gamma\cdot\partial\Psi_i - \partial_i\gamma\cdot\Psi &= 0. \end{aligned}$$

Using the gauge condition one can see that $\nabla^2\Psi_0 = 0$, so $\Psi_0 = 0$ according to the discussion on p. 69. The spatial components Ψ_i then satisfy the Dirac equation

$$\gamma\cdot\partial\Psi_i = 0,$$

which is a time evolution equation. However, there is an additional constraint, $\partial^i\Psi_i = 0$, obtained by contracting (5.8) with γ^i. Thus from the gauge condition and the equation of motion, we find $3\times 2^{[D/2]}$ independent constraints on the initial data, namely

$$\begin{aligned} \gamma^i\Psi_i(\vec{x},0) &= 0, \\ \Psi_0(\vec{x},0) &= 0, \\ \partial^i\Psi_i(\vec{x},0) &= 0. \end{aligned}$$

Degrees of freedom of the massless Rarita-Schwinger field:
On-shell degrees of freedom $= \frac{1}{2}(D-3)2^{[D/2]}$.
Off-shell degrees of freedom $= (D-1)2^{[D/2]}$.
Let's derive that this field has states of helicity $\lambda = \pm 3/2$ starting from the plane wave

$$\Psi_i(x) = \mathrm{e}^{\mathrm{i}p\cdot x}v_i(\vec{p})u(\vec{p}),$$

for a positive null energy-momentum vector $p^\mu = (|\vec{p}|, \vec{p})$. Since $\Psi_i(x)$ satisfies the Dirac equation (5.8), the four-component spinor $u(\vec{p})$ must be a superposition of the massless helicity spinors $u(\vec{p},\pm)$ given in (2.44). Thus we use the Weyl representation (2.19) of the γ-matrices. The vector $v_i(\vec{p})$ may be expanded in the complete set

$$v_i(\vec{p}) = ap_i + b\epsilon_i(\vec{p},+) + c\epsilon_i(\vec{p},-),$$

$\epsilon_i(\vec{p},\pm)$ are the transverse polarization vectors of Sec. 4.1.2, i.e. they satisfy $p^i\epsilon_i(\vec{p},\pm) = 0$. The constraint (5.11) requires that $a = 0$. Thus (5.12) is reduced to the form

$$\Psi_i(x) = \mathrm{e}^{\mathrm{i}p\cdot x}\Big[b_+\epsilon_i(\vec{p},+)u(\vec{p},+) + c_+\epsilon_i(\vec{p},-)u(\vec{p},+) + b_-\epsilon_i(\vec{p},+)u(\vec{p},-) + c_-\epsilon_i(\vec{p},-)u(\vec{p},-)\Big].$$

$$\Psi_\mu(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0} \sum_\lambda \left[\mathrm{e}^{\mathrm{i}p\cdot x} \epsilon_\mu(\vec{p},\lambda) u(\vec{p},\lambda) c(\vec{p},\lambda) + \mathrm{e}^{-\mathrm{i}p\cdot x} \epsilon^*_\mu(\vec{p},\lambda) v(\vec{p},\lambda) d^*(\vec{p},\lambda) \right].$$

$$T_{\mu\nu\mathrm{RS}} = \bar{\Psi}_\rho \gamma^{\rho\sigma}{}_\mu \partial_\nu \Psi_\sigma - \eta_{\mu\nu}\mathcal{L}$$

It is neither symmetric nor gauge invariant under $\Psi_\mu(x) \to \Psi_\mu(x) + \partial_\mu \epsilon(x)$ (and its Dirac conjugate). It can be made symmetric, but gauge non-invariance is intrinsic and cannot be restored by adding terms of the form $\partial_\sigma S^{\sigma\mu\nu}$. The reason is that the gravitino must be joined with gravity in the gauge multiplet of SUSY. In a gravitational theory there is no well-defined energy density (???!? add it to notes in gravity).

The total energy-momentum $P^\nu = \int \mathrm{d}^3\vec{x} T^{0v}(\vec{x},t)$ is gauge invariant and

$$P^\nu_{D=4} = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0} p^\nu \sum_\lambda \left[c^*(\vec{p},\lambda) c(\vec{p},\lambda) - d(\vec{p},\lambda) d^*(\vec{p},\lambda) \right].$$

Sources and Greens function for RS field

$$\gamma^{\mu\nu\rho} \partial_\nu \Psi_\rho = J^\mu.$$

$$(\not{\partial} - m)\Psi(x) = J(x).$$

$S(x-y): \quad (\not{\partial}_x - m)\, S(x-y) = -\delta(x-y),$

$$\Psi(x) = -\int \mathrm{d}^D y S(x-y) J(y).$$

$$S(x-y) \overset{\text{Fourier}}{=} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \mathrm{e}^{\mathrm{i}p\cdot(x-y)} S(p).$$

$$(\mathrm{i}\not{p} - m) S(p) = -1,$$

$$S(p) = -\frac{1}{\mathrm{i}\not{p} - m} = \frac{\mathrm{i}\not{p} + m}{p^2 + m^2 - \mathrm{i}\epsilon}.$$

$$S(x-y) = (\partial_x + m)\, G(x-y).$$

$$\Psi_\mu(x) = -\int \mathrm{d}^D y S_{\mu\nu}(x-y) J^\nu(y),$$

$S_{\mu\nu}(x-y)$ is a tensor bispinor. A bispinor has two spinor indices, which are suppressed in our notation, and it can be regarded as a matrix of the Clifford algebra. As in the electromagnetic case, the Rarita-Schwinger operator is not invertible, but we can assume that the Green's function satisfies

$$\gamma^{\mu\sigma\rho} \frac{\partial}{\partial x^\sigma} S_{\rho v}(x-y) = -\delta^\mu_\nu \delta(x-y) + \frac{\partial}{\partial y^\nu} \Omega^\mu(x-y).$$

$$\mathrm{i}\gamma^{\mu\sigma\rho} p_\sigma S_{\rho v}(p) = -\delta^\mu_\nu - \mathrm{i}p_\nu \Omega^\mu(p).$$

We will solve (5.28) by writing an appropriate ansatz for $S_{\rho v}(p)$ and then find the unknown functions in the ansatz. The matrix $\gamma^{\mu\sigma\rho} p_\sigma$ in (5.28) contains an odd rank element of the Clifford algebra and it is odd under the reflection $p_\sigma \to -p_\sigma$. It is reasonable to guess that the ansatz we need should also involve odd rank Clifford elements and be odd under the reflection. We would also expect that terms that contain the momentum vectors p_ρ or p_ν are 'pure gauges' and thus arbitrary additions to the propagator, which would not be determined by the equation (5.28). So we omit such terms and postulate the ansatz

$$\mathrm{i}\ S_{\rho v}(p) = A\left(p^2\right)\eta_{\rho v}\not{p} + B\left(p^2\right)\gamma_\rho\not{p}\gamma_\nu.$$

$$\begin{aligned}\mathrm{i}\gamma^{\mu\sigma\rho}p_\sigma S_{\rho v}(p) =& A\gamma^{\mu\sigma}{}_\nu\not{p}p_\sigma + (D-2)B\gamma^{\mu\sigma}\not{p}\gamma_\nu p_\sigma\\ =& A\left(p^\mu\gamma^\sigma{}_\nu - p^\sigma\gamma^\mu{}_\nu\right)p_\sigma + (D-2)B\left(-p^\mu\gamma^\sigma + p^\sigma\gamma^\mu\right)\gamma_\nu p_\sigma + ...\\ =& [A-(D-2)B]\left(p^\mu\gamma^\sigma{}_\nu - p^\sigma\gamma^\mu{}_\nu\right)p_\sigma + (D-2)Bp^2\delta^\mu_\nu + ...\end{aligned}$$

$A = -1/p^2, B = -1/\left((D-2)p^2\right)$,

$$S_{\mu\nu}(p) = \mathrm{i}\frac{1}{p^2}\left[\eta_{\mu\nu}\ \not{p} + \frac{1}{D-2}\gamma_\mu\ \not{p}\gamma_v + Cp_\mu\gamma_v + E\gamma_\mu p_v + Fp_\mu\ \not{p}p_v\right]$$

$$S_{\mu\nu}(x-y) = \left[\eta_{\mu\nu}\not{\partial} + \frac{1}{D-2}\gamma_\mu\not{\partial}\gamma_\nu + C\partial_\mu\gamma_\nu + E\gamma_\mu\partial_\nu - F\partial_\mu\not{\partial}\partial_\nu\right]G(x-y),$$

$G(x-y) = \frac{\Gamma\left(\frac{1}{2}(D-2)\right)}{4\pi^{\frac{1}{2}D}(x-y)^{(D-2)}}$ is the massless scalar propagator, and all derivatives are with respect to x.

For $E = -1/(D-2)$, and $\forall C, F$,

$$\mathrm{i}\gamma^{\mu\sigma\rho}p_\sigma S_{\rho v}(p) = -\left(\delta^\mu_\nu - \frac{p^\mu p_\nu}{p^2}\right).$$

For $D = 4$, the propagator, with $C = -1$, takes the 'reverse index' form

$$S_{\mu\nu}(p) = -\mathrm{i}\frac{1}{2}\gamma_\nu\not{p}\gamma_\mu,$$

which is the form used in most of the literature on perturbative studies in supergravity.

Massive gravitinos from dimensional reduction

(??? maybe I'll add it it section of special methods)

Dimensional reduction for scalar fields (!!??)

Let's change to a more convenient notation and rename the coordinates of the $(D+1)$ dimensional product spacetime $x^0 = t, x^1, ..., x^{D-1}, y$, y is the coordinate of S^1 with range $0 \le y \le 2\pi L$. We consider a massive complex scalar field $\phi\left(x^\mu, y\right)$ that obeys the Klein-Gordon equation

$$\left[\Box_{D+1} - m^2\right]\phi = \left[\Box_D + \left(\frac{\partial}{\partial y}\right)^2 - m^2\right]\phi = 0.$$

$$\phi\left(x^\mu, y\right) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky/L}\phi_k\left(x^\mu\right).$$

$$\left[\Box_D - \left(\frac{k}{L}\right)^2 - m^2\right]\phi_k = 0.$$

It describes a particle of mass $m_k^2 = (k/L)^2 + m^2$. So the spectrum of the theory, as viewed in Minkowski$_D$, contains an infinite tower of massive scalars!

There is an even simpler way to find the mass spectrum. Just substitute the plane wave $\mathrm{e}^{\mathrm{i}p^\mu x_\mu}\mathrm{e}^{\mathrm{i}ky/L}$ directly in the $(D+1)$-dimensional equation (5.34). The D-component energy-momentum vector p^μ must satisfy $p^\mu p_\mu = (k/L)^2 + m^2$. The mass shift due to the Fourier wave on S^1 is immediately visible.

Dimensional reduction for spinor fields

We will consider the dimensional reduction process for a complex spinor $\Psi(x^\mu, y)$ for even $D = 2m$ (so that the spinors in $D+1$ dimensions have the same number of components). Two new ideas enter the game. The first just involves the Dirac equation in D dimensions. We remark that if $\Psi(x)$ satisfies

$$[\partial_D - m]\,\Psi(x) = 0,$$

$\tilde{\Psi} \equiv \mathrm{e}^{-\mathrm{i}\gamma_* \beta}\Psi$:

$$[\partial_D - m\,(\cos 2\beta + \mathrm{i}\gamma_* \sin 2\beta)]\,\tilde{\Psi} = 0.$$

Physical quantities are unchanged by the field redefinition, so both equations describe particles of mass m. One simple implication is that the sign of m in (5.37) has no physical significance, since it can be changed by field redefinition with $\beta = \pi/2$.

Fermion fields can be either periodic or anti-periodic $\Psi(x^\mu, y) = \pm\Psi(x^\mu, y + 2\pi L)$). Anti-periodic behavior is permitted because a fermion field is not observable. Rather, bilinear quantities such as the energy density $T^{00} = -\bar{\Psi}\gamma^0\partial^0\Psi$ are observables and they are periodic even when Ψ is anti-periodic. Thus

$$\Psi(x^\mu, y) = \sum_k \mathrm{e}^{\mathrm{i}ky/L}\Psi_k(x^\mu),$$

where the mode number k is integer or half-integer for periodic or anti-periodic fields, respectively. In either case when we substitute the series in the $(D+1)$-dimensional Dirac equation $[\partial_{D+1} - m]\,\Psi(x^\mu, y) = 0$,

$$\left[\partial_D - \left(m - \mathrm{i}\gamma_* \frac{k}{L}\right)\right]\Psi_k(x^\mu) = 0.$$

By applying a chiral transformation with phase $\tan 2\beta = k/(mL)$, we see that $\Psi_k(x^\mu)$ describes particles of mass ${m_k}^2 = (k/L)^2 + m^2$. Again we would observe an infinite tower of massive spinor particles with distinct spectra for the periodic and anti-periodic cases.

Dimensional reduction for the vector gauge field

We now apply circular dimensional reduction to Maxwell's equation

$$\partial^\nu F_{\nu\mu} = \Box_{D+1} A_\mu - \partial_\mu(\partial^\nu A_\nu) = 0$$

in $D+1$ dimensions, and we assume a periodic Fourier series representation

$$A_\mu(x,y) = \sum_k \mathrm{e}^{\mathrm{i}ky/L} A_{\mu k}(x), \quad A_D(x,y) = \sum_k \mathrm{e}^{\mathrm{i}ky/L} A_{Dk}(x),$$

with k an integer. The analysis simplifies greatly if we assume the gauge conditions $A_{Dk}(x) = 0$ for $k \neq 0$ and vector component D tangent to S^1. It is easy to see that this gauge can be achieved and uniquely fixes the Fourier modes $\theta_k(x), k \neq 0$, of the gauge function. The gauge invariant Fourier mode $A_{D0}(x)$ remains a physical field in the dimensionally reduced theory. A quick examination of the $\mu \to D$ component of (5.41) shows that it reduces to

$$\begin{aligned} &k = 0 : \Box_{D+1} A_{D0} = \Box_D A_{D0} = 0, \\ &k \neq 0 : \partial^\mu A_{\mu k} = 0, \end{aligned}$$

so the mode $A_{D0}(x)$ simply describes a massless scalar in D dimensions. For $\mu \leq D-1$, the wave equation (5.41) implies that the vector modes $A_{\mu k}(x)$ satisfy

$$\left[\Box_D - \frac{k^2}{L^2}\right] A_{\mu k} - \partial_\mu(\partial^\nu A_{\nu k}) = 0.$$

For mode number $k=0$ this is just the Maxwell equation in D dimensions with its gauge symmetry under $A_{\mu 0} \rightarrow A_{\mu 0}+\partial_\mu \theta_0$ intact, since the Fourier mode $\theta_0(x)$ remained unfixed in the process above. For mode number $k \neq 0,(5.44)$ is the standard equation [4] for a massive vector field with mass $m_k^2=k^2/L^2$, namely the equation of motion of the action

$$S=\int \mathrm{d}^D x\left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu}-\frac{1}{2} m^2 A_\mu A^\mu\right].$$

Dimensional reduction for RS field $\Phi_\mu(x,y)$ $\quad (D=2m)$

Let's write out the $\mu=D$ and $\mu \leq D-1$ components of (5.3) with $\Psi_D=0$ (using $\gamma^D=\gamma_*$) :

$$\begin{aligned}
&\gamma^{\nu\rho} \partial_\nu \Psi_{\rho k}=0, \\
&\left[\gamma^{\mu\nu\rho} \partial_\nu-\mathrm{i} \frac{k}{L} \gamma_* \gamma^{\mu\rho}\right] \Psi_{\rho k}=0.
\end{aligned}$$

Note that the first equation of (5.46) follows by application of ∂^μ to the second one.

Exercise 5.8 Show that the chiral transformation $\Psi_{\rho k}=\mathrm{e}^{(-\mathrm{i}\pi\gamma_*/4)} \Psi_{\rho k}^{\prime}$ leads, after replacing $\Psi^{\prime} \rightarrow \Psi$, to the equation of motion

$$\left(\gamma^{\mu\nu\rho} \partial_\nu-m \gamma^{\mu\rho}\right) \Psi_\rho=0.$$

The last equation is the Euler-Lagrange equation of the action

$$S=-\int \mathrm{d}^D x \bar{\Psi}_\mu\left[\gamma^{\mu\nu\rho} \partial_\nu-m \gamma^{\mu\rho}\right] \Psi_\rho$$

$$\begin{aligned}
&\gamma^\mu \Psi_\mu=0, \\
&\left(\gamma^{ij} \partial_i-m \gamma^j\right) \Psi_j=0, \\
&[\not{\partial}+m] \Psi_\mu=0.
\end{aligned}$$

From Kaluza-Klein reduction for the Rarita-Schwinger field with assumption of periodicity $\Psi_\mu(x, y+2\pi)=\Psi_\mu(x,y)$ in y it can be shown that the spectrum seen in Minkowski$_D$ consists of a massive gravitino for each Fourier mode $k \neq 0$ plus a massless gravitino and massless Dirac particle for the zero mode. (!??!?)

$$S=-\int \mathrm{d}^D x \bar{\Psi}_\mu\Big[\gamma^{\mu\nu\rho} \partial_\nu-\underbrace{m \gamma^{\mu\rho}}_{\text{(mass)}}-\underbrace{m' \eta^{\mu\rho}}_{\text{(Lor inv)}}\Big] \Psi_\rho,$$

2.8 The first and second order formulations of general relativity

3 Typical Special Methods in a Nutshell

3.1 Duality methods (!?)

(see my note about duality, it is a big method and I am still learning it)

Dual tensors

$$\tilde{H}^{\mu\nu} := -\frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}H_{\rho\sigma}$$

In our conventions the dual tensor is imaginary, но некоторые определяют его иначе, тогда чуть другие свойства. Also

$$H^{\pm}_{\mu\nu} := \frac{1}{2}\left(H_{\mu\nu} \pm \tilde{H}_{\mu\nu}\right), \quad H^{\pm}_{\mu\nu} := \left(H^{\mp}_{\mu\nu}\right)^* .$$

The dual of the dual is the identity:

$$-\frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\tilde{H}_{\rho\sigma} = H^{\mu\nu}.$$

(??? тут указание про вывод! You will need (3.9).) The validity of this property is the reason for the "i" in the definition (4.35).

$H^+_{\mu\nu}$ and $H^-_{\mu\nu}$ are, respectively, self-dual and anti-self-dual:

$$-\frac{1}{2}\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}H^{\pm}_{\rho\sigma} = \pm H^{\pm}_{\mu\nu}.$$

Let $G_{\mu\nu}$ be another antisymmetric tensor with $G^{\pm}_{\mu\nu}$ defined the same way. Prove the following relations:

$$G^{+\mu\nu}H^{-}{}_{\mu\nu} = 0, \qquad G^{\pm\rho(\mu}H^{\pm\nu)}{}_{\rho} = -\frac{1}{4}\eta^{\mu\nu}G^{\pm\rho\sigma}H^{\pm}{}_{\rho\sigma}, \qquad G^{+}{}_{\rho[\mu}H^{-}{}_{\nu]}{}^{\rho} = 0,$$

$(\mu\nu)$ means symmetrization. Hint: you could first prove $\tilde{G}^{\rho\mu}\tilde{H}^{\nu}{}_{\rho} = -\frac{1}{2}\eta^{\mu\nu}G^{\rho\sigma}H_{\rho\sigma} - G^{\rho v}H^{\mu}{}_{\rho}$.

Duality for one free electromagnetic field

We know that $\partial_\mu F^{\mu\nu} = 0, \partial_\mu \tilde{F}^{\mu\nu} = 0$. There is a dual symmetry - change of variables:

$$F^{\mu\nu} \to F'^{\mu\nu} = \mathrm{i}\tilde{F}^{\mu\nu}$$

(the "i" is included to make the transformation real).

Since $F'^{\mu\nu}$ also obeys both equations of (4.41) we have defined a symmetry of the free electromagnetic field.

The symmetry (4.42) exchanges the electric and magnetic fields: $E_i \to E'_i = -B_i$ and $B_i \to B'_i = E_i$.

$$\partial_\mu F^{\mu\nu} = 0, \quad \partial_\mu \tilde{F}^{\mu\nu} = 0.$$

We can now consider the change of variables (the i is included to make the transformation real):

$$F^{\mu\nu} \to F'^{\mu\nu} = \mathrm{i}\tilde{F}^{\mu\nu}.$$

Since $F'^{\mu\nu}$ also obeys both equations of (4.41) we have defined a symmetry of the free electromagnetic field.

The symmetry (4.42) exchanges the electric and magnetic fields: $E_i \to E'_i = -B_i$ and $B_i \to B'_i = E_i$.

Exercise 4.9 Show that the self-dual combinations $F^{\pm}_{\mu\nu}$contain only photons of one polarization in their plane wave expansions:

$$F^{\pm}_{\mu\nu} = 2\mathrm{i}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\left[\mathrm{e}^{\mathrm{i}k\cdot x}k_{[\mu}\epsilon_{\nu]}(\vec{k},\pm)a(\vec{k},\pm) - \mathrm{e}^{-\mathrm{i}k\cdot x}k_{[\mu}\epsilon^*_{\nu]}(\vec{k},\mp)a^*(\vec{k},\mp)\right].$$

To perform this exercise, check first that with the polarization vectors given in Sec. 4.1.2, one has

$$-\frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}k_\rho\epsilon_\sigma(\vec{k},\pm) = \pm k^{[\mu}\epsilon^{\nu]}(\vec{k},\pm).$$

Exercise 4.10 Show that the quantity $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total derivative, i.e.

$$F_{\mu\nu}\tilde{F}^{\mu\nu} = -\mathrm{i}\partial_\mu\left(\varepsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}\right).$$

Show, using (1.45), that under a Lorentz transformation

$$\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)(x) \to \det\Lambda^{-1}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)(\Lambda x).$$

Thus $F_{\mu\nu}\tilde{F}^{\mu\nu}$ transforms as a scalar under proper Lorentz transformations but changes sign under space or time reflections. Use the Schouten identity (3.11) to prove that

$$F_{\mu\rho}\tilde{F}_\nu^\rho = \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}\tilde{F}^{\rho\sigma}.$$

Duality for gauge field and complex scalar

$$\mathcal{L} = -\frac{1}{4}(\operatorname{Im} Z)F_{\mu\nu}F^{\mu\nu} - \frac{1}{8}(\operatorname{Re} Z)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}.$$

Actions in which the gauge field kinetic term is multiplied by a function of complex scalar fields are quite common in supersymmetry and supergravity. We now define an extension of the duality transformation (4.42) which gives a non-abelian global $\mathrm{SL}(2,\mathbb{R})$ symmetry of the gauge field equations of this theory. In Sec. 7.12.2 we will discuss a generalized scalar kinetic term that is invariant under $\mathrm{SL}(2,\mathbb{R})$. The field $Z(x)$ carries dynamics, and the equations of motion of the combined vector and scalar theory are also invariant. The gauge Bianchi identity and equation of motion of our theory are

$$\partial_\mu\tilde{F}^{\mu\nu} = 0, \quad \partial_\mu\left[(\operatorname{Im} Z)F^{\mu\nu} + \mathrm{i}(\operatorname{Re} Z)\tilde{F}^{\mu\nu}\right] = 0.$$

It is convenient to define the real tensor

$$G^{\mu\nu} \equiv \varepsilon^{\mu\nu\rho\sigma}\frac{\delta S}{\delta F^{\rho\sigma}} = -\mathrm{i}(\operatorname{Im} Z)\tilde{F}^{\mu\nu} + (\operatorname{Re} Z)F^{\mu\nu},$$

and to consider the self-dual combinations $F^{\mu\nu\pm}$ and $G^{\mu\nu\pm}$. Note that these are related by

$$G^{\mu\nu-} = ZF^{\mu\nu-}, \quad G^{\mu\nu+} = \bar{Z}F^{\mu\nu+}.$$

The information in (4.49) can then be reexpressed as

$$\partial_\mu \operatorname{Im} F^{\mu\nu-} = 0, \quad \partial_\mu \operatorname{Im} G^{\mu\nu-} = 0.$$

We define a matrix of the group $\mathrm{SL}(2,\mathbb{R})$ by

$$\mathcal{S} \equiv \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \quad ad - bc = 1$$

The group $\mathrm{SL}(2,\mathbb{R})$ acts on the tensors $F^-$and G^-as follows:

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S}\begin{pmatrix} F^- \\ G^- \end{pmatrix}.$$

Since $\mathcal{S}$ is real, the conjugate tensors $\mathrm{F}^{+}$and G^{+}also transform in the same way.

Exercise 4.11 Assume that $\operatorname{Im} F^{-}$and $\operatorname{Im} G^{-}$satisfy (4.52), and show that $\operatorname{Im} F^{\prime -}$ and $\operatorname{Im} G^{\prime -}$ also obey the same equations. Show that $G^{\prime -}$ and a transformed scalar $Z^{\prime}$ satisfy $G^{\prime\mu\nu -}=Z^{\prime} F^{\prime\mu\nu -}$, if $Z^{\prime}$ is defined as the following nonlinear transform of Z:

$$Z^{\prime}=\frac{a Z+b}{c Z+d}.$$

Exercise 4.12 Show that the Lagrangian (4.48) can be rewritten as

$$\mathcal{L}(F, Z)=-\frac{1}{2} \operatorname{Im}\left(Z F_{\mu\nu}^{-} F^{\mu\nu -}\right).$$

Consider the $\mathrm{SL}(2, \mathbb{R})$ transformation with parameters $a=d=1$ and $b=0$. Show that

$$\mathcal{L}\left(F^{\prime}, Z^{\prime}\right)=-\frac{1}{2} \operatorname{Im}\left(Z(1+c Z) F_{\mu\nu}^{-} F^{\mu\nu -}\right) \neq \mathcal{L}(F, Z).$$

The symmetric gauge invariant stress tensor of this theory is

$$\Theta^{\mu\nu}=(\operatorname{Im} Z)\left(F^{\mu\rho} F_{\rho}^{\nu}-\frac{1}{4} \eta^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right).$$

As we will see in Ch. 8, when the theory is coupled to gravity, it is this stress tensor that is the source of the gravitational field; see (8.4). It is then important that $\operatorname{Im} Z$ is positive, which restricts the domain of Z to the upper half-plane. It is also important that the stress tensor is invariant under the duality transformations (4.54) and (4.55). This is the reason for the duality symmetry of many black hole solutions of supergravity,

Exercise 4.13 Prove that the energy-momentum tensor (4.58) is invariant under duality. Here are some helpful relations which you will need:

$$\operatorname{Im} Z^{\prime}=\frac{\operatorname{Im} Z}{(c Z+d)(c \bar{Z}+d)}.$$

Further you need again (4.47) and a similar identity (proven by contracting ε-tensors)

$$\tilde{F}_{\mu\rho} \tilde{F}_{\nu}{}^{\rho}=-F_{\mu\rho} F_{\nu}{}^{\rho}+\frac{1}{2} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}.$$

This leads to

$$F_{\mu\rho}^{\prime} F_{\nu}^{\prime\rho}-\frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma}^{\prime} F^{\prime\rho\sigma}=|c Z+d|^{2}\left[F_{\mu\rho} F_{\nu}{}^{\rho}-\frac{1}{4} \eta_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}\right].$$

Exercise 4.14 The free Maxwell theory is the special case of (4.48) with fixed $Z=\mathrm{i}$. Suppose that the gauge field is coupled to a conserved current as in (4.14). Check that the electric charge can be expressed in terms of F or G by

$$q \equiv \int \mathrm{d}^{3} \vec{x} J^{0}=\int \mathrm{d}^{3} \vec{x} \partial_{i} F^{0 i}=-\frac{1}{2} \int \mathrm{d}^{3} \vec{x} \varepsilon^{i j k} \partial_{i} G_{j k}.$$

A magnetic charge can be introduced in Maxwell theory as the divergence of $\vec{B}$ (recall $E^{i}=F^{0 i}$ and $B^{i}=\frac{1}{2} \varepsilon^{i j k} F_{j k}$). This leads to a definition [7]

$$p \equiv-\frac{1}{2} \int \mathrm{d}^{3} \vec{x} \varepsilon^{i j k} \partial_{i} F_{j k}.$$

Show that $\begin{pmatrix} p \\ q \end{pmatrix}$ is a vector that transforms under $\mathrm{SL}(2,\mathbb{R})$ in the same way as the tensors $F^-$and G^-in (4.54).

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$Z' = Z + 1, \quad Z' = -\frac{1}{Z}.$$

This means that one can express any element of $\mathrm{SL}(2,\mathbb{Z})$ as the product of (finitely many) factors of the two generators above and their inverses.

In Secs. 4.1 and 4.2.2 we considered $Z = \mathrm{i}g = \mathrm{i}$. Check that general duality transformations in this case are of the form

$$F'^-_{\mu\nu} = (d + \mathrm{i}c)F^-_{\mu\nu}, \qquad \text{i.e.} \qquad F'_{\mu\nu} = dF_{\mu\nu} - \mathrm{i}c\tilde{F}_{\mu\nu}.$$

Electromagnetic duality for coupled Maxwell fields (!?!)

$$\mathcal{L} := -\frac{1}{4}\left(\mathrm{Re}\, f_{AB}\right) F^A_{\mu\nu} F^{\mu\nu B} + \frac{1}{4}\mathrm{i}\left(\mathrm{Im}\, f_{AB}\right) F^A_{\mu\nu} \tilde{F}^{\mu\nu B},$$

$$\mathcal{L}\left(F^+, F^-\right) = -\frac{1}{2}\,\mathrm{Re}\left(f_{AB} F^{-A}_{\mu\nu} F^{\mu\nu - B}\right) \equiv -\frac{1}{4}\left(f_{AB} F^{-A}_{\mu\nu} F^{\mu\nu - B} + f^*_{AB} F^{+A}_{\mu\nu} F^{\mu\nu + B}\right),$$

$$\begin{aligned}
G_A^{\mu\nu} &:= \varepsilon^{\mu\nu\rho\sigma} \frac{\delta S}{\delta F^{\rho\sigma A}} = -\left(\mathrm{Im}\, f_{AB}\right) F^{\mu\nu B} - \mathrm{i}\left(\mathrm{Re}\, f_{AB}\right) \tilde{F}^{\mu\nu B} = G_A^{\mu\nu +} + G_A^{\mu\nu -}, \\
G_A^{\mu\nu -} &:= -2\mathrm{i} \frac{\delta S\left(F^+, F^-\right)}{\delta F^{-A}_{\mu\nu}} = \mathrm{i} f_{AB} F^{\mu\nu - B}, \\
G_A^{\mu\nu +} &:= 2\mathrm{i} \frac{\delta S\left(F^+, F^-\right)}{\delta F^{+A}_{\mu\nu}} = -\mathrm{i} f^*_{AB} F^{\mu\nu + B}.
\end{aligned}$$

Since the field equation for the action containing (4.67) is

$$0 = \frac{\delta S}{\delta A^A_\nu} = -2\partial_\mu \frac{\delta S}{\delta F^A_{\mu\nu}},$$

$$\begin{aligned}
\partial^\mu\, \mathrm{Im}\, F^{A-}_{\mu\nu} &= 0 \quad \text{Bianchi identities,} \\
\partial_\mu\, \mathrm{Im}\, G_A^{\mu\nu -} &= 0 \quad \text{equations of motion.}
\end{aligned}$$

(The same equations hold for $\mathrm{Im}\, F^{A+}$ and $\mathrm{Im}\, G_A{}^+$.)

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S} \begin{pmatrix} F^- \\ G^- \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} F^- \\ G^- \end{pmatrix},$$

with real $m \times m$ submatrices A, B, C, D. Owing to the reality of these matrices, the same relations hold for the self-dual tensors $F^+$and G^+. In Sec. 4.2.3, these matrices were just numbers:

$$A = d, \quad B = c, \quad C = b, \quad D = a.$$

We require that the transformed field tensors F'^{A} and G'_{A} are also related by the definitions (4.68), with appropriately transformed f_{AB}. We work out this requirement in the following steps:

$$G'^{-} = (C + \mathrm{i}Df)F^{-} = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1}F'^{-},$$

such that we conclude that

$$\mathrm{i}f' = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1}.$$

$$A^{T}C - C^{T}A = 0, \quad B^{T}D - D^{T}B = 0, \quad A^{T}D - C^{T}B = \mathbb{1}.$$

These relations among A, B, C, D are the defining conditions of a matrix of the symplectic group in dimension $2m$ so we reach the conclusion that

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2m, \mathbb{R}).$$

The conditions (4.75) may be summarized as

$$\mathcal{S}^{T}\Omega\mathcal{S} = \Omega \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}.$$

(!!???!??!?!?!?!??!??!??!?!?)

The duality transformations in four dimensions are transformations in the symplectic group Sp(2m, R).

(!?!!?!??!?!!??!?!?!?!??)

$$\mathcal{L} = -\frac{1}{2}\,\mathrm{Re}\left(f_{AB}F^{-A}_{\mu\nu}F^{\mu\nu-B}\right) = -\frac{1}{2}\,\mathrm{Im}\left(F^{-A}_{\mu\nu}G^{\mu\nu-}_{A}\right),$$

we obtain

$$\mathrm{Im}\,F'^{-}G^{-} = \mathrm{Im}\left(F^{-}G^{-}\right) + \mathrm{Im}\left[2F^{-}\left(C^{T}B\right)G^{-} + F^{-}\left(C^{T}A\right)F^{-} + G^{-}\left(D^{T}B\right)G^{-}\right].$$

If $C \neq 0, B = 0$ the Lagrangian is invariant up to a 4-divergence, since $\mathrm{Im}\,F^{-}F^{-} = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ and the matrices A and C are real constants. For $B \neq 0$ neither the Lagrangian nor the action is invariant.

Electromagnetic duality has important applications to black hole solutions of extended supergravity theories. Supergravity is also very relevant to the analysis of black hole solutions of string theory. Many black holes are dyons; they carry both magnetic and electric charges for the gauge fields of the system. The general situation is a generalization of what was discussed at the end of Sec. 4.2.3. The charges form a symplectic vector $\begin{pmatrix} q^{A}_{\mathrm{m}} \\ q_{\mathrm{e}A} \end{pmatrix}$ which must transform as in (4.71). The Dirac-Schwinger-Zwanziger quantization condition restricts these charges to a lattice. Invariance of this lattice restricts the symplectic transformations of (4.71) to a discrete subgroup $\mathrm{Sp}(2m, \mathbb{Z})$, which is analogous to the $\mathrm{SL}(2, \mathbb{Z})$ group discussed previously.

On extensions and applications (????)

(про приложения абзац)

(про другие дуальности. где нужно - не раскрыто еще)

4 Typical Modified Theories in a Nutshell

4.1 On typical generalizations of electrodynamics

(maybe I'll change the structure later)

4.1.1 On Born-Infeld theory

Main Formulas and ideas

4.2 Supergravity in a Nutshell

4.3 On typical gauge theories (!?!?!??!?)

(for now see my note on CFT.)

4.4 On typical CFT (!?!?!??!?)

(for now see my note on CFT.)

4.5 On tests and experimental constraints for field modifications (!?!?!??!?)

(потом много указаний будет, это же ответ на вопрос, что же может быть верным, а что нет и почему? В ближайший год (2024й) вряд ли буду это прописывать.)

4.5.1 Overview of predictions and physical results of modified theories (!!?!?!??)

(??? important)

4.5.2 Main experimental tests

4.6 On N=1 global supersymmetry in D=4

4.6.1 Basic SUSY field theory

SUSY theories contain both bosons and fermions, which are the basis states of a particle representation of the SUSY algebra (6.1)-(6.4). We give a systematic treatment of these representations in Sec. 6.4, but start with an informal discussion here. The states of particles with momentum $\vec{p}$ and energy $E(\vec{p}) = \sqrt{\vec{p}^2 + m_{B,F}^2}$ are denoted by $|\vec{p}, B\rangle$ and $|\vec{p}, F\rangle$, where the labels B and F include particle helicity. SUSY transformations connect these states. Since the spinor Q_α carries angular momentum 1/2, it transforms bosons into fermions and fermions into bosons. Hence $Q_\alpha|\vec{p}, B\rangle = |\vec{p}, F\rangle$ and $Q_\alpha|\vec{p}, F\rangle \propto |\vec{p}, B\rangle$. Since $[P^\mu, Q_\alpha] = 0$, the transformed states have the same momentum and energy, hence the same mass, so $m_B^2 = m_F^2$. We show in Sec. 6.4.1 that a representation of the algebra contains the same number of boson and fermion states.

The simplest representations of the algebra that lead to the most basic SUSY field theories are:

(i) the chiral multiplet, which contains a self-conjugate spin-1/2 fermion described by the Majorana field $\chi(x)$ plus a complex spin- 0 boson described by the scalar field $Z(x)$. Alternatively, $\chi(x)$ may be replaced by the Weyl spinor $P_L\chi$ and/or $Z(x)$ by the combination $Z(x) = (A(x) + \mathrm{i}B(x))/\sqrt{2}A$ and B are a real scalar and pseudo-scalar, respectively. A chiral multiplet can be either massless or massive.

(ii) the gauge multiplet consisting of a massless spin-1 particle, described by a vector gauge field $A_\mu(x)$, plus its spin-1/2 fermionic partner, the gaugino, described by a Majorana spinor $\lambda(x)$ (or the corresponding Weyl field $P_L\lambda$).

Main formulas

$$\begin{aligned}\{Q_\alpha, \bar{Q}^\beta\} &= -\frac{1}{2}\left(\gamma_\mu\right)_\alpha{}^\beta P^\mu, \\ \left[M_{[\mu\nu]}, Q_\alpha\right] &= -\frac{1}{2}\left(\gamma_{\mu\nu}\right)_\alpha{}^\beta Q_\beta, \\ \left[P_\mu, Q_\alpha\right] &= 0.\end{aligned}$$

(?? why are they so???)

Note that these are the classical (anti-)commutator relations; see Secs. 1.4 and 1.5. We will discuss this further in Ch.11.

Exercise 6.1 Use $\bar{\Psi} = \Psi^\dagger \beta = \Psi^\dagger \mathrm{i}\gamma^0$ to reexpress the supercharge anti-commutator in terms of Q and $Q^\dagger$. Then use the correspondence principle, that is multiply by the imaginary i, to obtain the quantum anti-commutator from the classical relation. This procedure gives the operator relation

$$\left\{Q_\alpha, \left(Q^\dagger\right)^\beta\right\}_{\text{qu}} = \frac{1}{2}\left(\gamma_\mu\gamma^0\right)_\alpha{}^\beta P^\mu$$

Trace on the spinor indices to obtain the positivity condition

$$\operatorname{Tr}\left(QQ^\dagger + Q^\dagger Q\right) = 2P^0.$$

Many SUSY theories, but not all, are invariant under a chiral U(1) symmetry called *R*-symmetry. Denote the generator by T_R,

$$\left[T_R, Q_\alpha\right] = -\mathrm{i}\left(\gamma_*\right)_\alpha{}^\beta Q_\beta$$

but this generator T_R is not required. Other internal symmetries, which commute with Q_α and are frequently called outside charges, can also be included.

According to the ColemanMandula (CM) theorem, in the presence of massive particles, bosonic charges are limited to $M_{[\mu\nu]}$ and P_μ plus (optional) scalar internal symmetry charges, and the Lie algebra is the direct sum of the Poincaré algebra and a (finite-dimensional) compact Lie algebra for internal symmetry.

There are two important theorems that severely limit the type of charges and algebras that can be realized in an interacting relativistic quantum field theory in $D = 4$ (strictly speaking in a theory with a non-trivial S-matrix in flat space). According to the ColemanMandula (CM) theorem [29, 30], in the presence of massive particles, bosonic charges are limited to $M_{[\mu\nu]}$ and P_μ plus (optional) scalar internal symmetry charges, and the Lie algebra is the direct sum of the Poincaré algebra and a (finite-dimensional) compact Lie algebra for internal symmetry.

If superalgebras are admitted, the situation is governed by the Haag-ŁopuszańskiSohnius (HLS) theorem [31,30], and the algebra of symmetries admits spinor charges Q^i_α. If there is only one Q_α, then the superalgebra must agree with the $\mathcal{N} = 1$ Poincaré SUSY algebra in (6.1). When $\mathcal{N} > 1$, the possibilities are restricted to the extended SUSY algebras discussed in Appendix 6A. The main thought that we wish to convey is that SUSY theories realize the most general symmetry possible within the framework of the few assumptions made in the hypotheses of the CM and HLS theorems. [2] They also unify bosons and fermions, the two broad classes of particles found in Nature.

Physical motivation for SUSY (????)

(very important question)

6.1.1 Conserved supercurrents 109

$$Q_\alpha = \int \mathrm{d}^3x \mathcal{J}^0_\alpha(\vec{x}, t).$$

If the current is conserved for all solutions of the equations of motion of a theory, then the theory has a fermionic symmetry. By the HLS theorem this symmetry must be supersymmetry!

Therefore we begin the technical discussion of SUSY in quantum field theory by displaying such conserved currents, [3] first for free fields and then for one non-trivial interacting system. Consider a free scalar field $\phi(x)$ satisfying the Klein-Gordon equation $(\Box - m^2)\,\phi = 0$ and a spinor field $\Psi(x)$ satisfying the Dirac equation $(\not{\partial} m)\Psi = 0$.

Exercise 6.2 Show that the current $\mathcal{J}^\mu = (\not{\partial}\phi - m\phi)\gamma^\mu\Psi$ is conserved for all field configurations satisfying the Klein-Gordon and Dirac equations.

As the second example let's look at the free gauge multiplet with vector potential A_μ and field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ satisfying the Maxwell equation $\partial^\mu F_{\mu\nu} = 0$ and a spinor λ satisfying $\partial\lambda = 0$. Let's show that the current $\mathcal{J}^\mu = \gamma^{\nu\rho}F_{\nu\rho}\gamma^\mu\lambda$ is conserved. We have

$$\partial_\mu\mathcal{J}^\mu = \partial_\mu F_{\nu\rho}\gamma^{\nu\rho}\gamma^\mu\lambda + \gamma^{\nu\rho}F_{\nu\rho}\not{\partial}\lambda.$$

The last term vanishes. To treat the first term we manipulate the γ-matrices as discussed in Sec. 3.1.4:

$$\gamma^{\nu\rho}\gamma^\mu = \gamma^{\nu\rho\mu} + 2\gamma^{[v}\eta^{\rho]\mu}$$

6.1.2 SUSY YangMills theory 110

$$S = \int \mathrm{d}^4x \left[-\frac{1}{4}F^{\mu\nu A}F^A_{\mu\nu} - \frac{1}{2}\bar{\lambda}^A\gamma^\mu D_\mu\lambda^A \right]$$

For details of the notation see Secs. 3.4.1 and 4.3. Note that the gaugino action vanishes unless $\lambda^A(x)$ is anti-commuting! The Euler-Lagrange equations (and gauge field Bianchi identity) are

$$\begin{aligned}
&D^\mu F^A_{\mu\nu} = -\frac{1}{2}g f_{BC}{}^A\bar{\lambda}^B\gamma_\nu\lambda^C,\\
&D_\mu F^A_{\nu\rho} + D_\nu F^A_{\rho\mu} + D_\rho F^A_{\mu\nu} = 0,\\
&\gamma^\mu D_\mu\lambda^A = 0.
\end{aligned}$$

The supercurrent is

$$\mathcal{J}^\mu = \gamma^{\nu\rho}F^A_{\nu\rho}\gamma^\mu\lambda^A.$$

The proof that it is conserved begins as in the free (abelian) case:

$$\begin{aligned}
\partial_\mu\mathcal{J}^\mu &= D_\mu F^A_{\nu\rho}\gamma^{\nu\rho}\gamma^\mu\lambda^A + \gamma^{\nu\rho}F^A_{\nu\rho}\gamma^\mu D_\mu\lambda^A\\
&= -2D^\mu F^A_{\mu\nu}\gamma^\nu\lambda^A\\
&= g f_{ABC}\gamma^\nu\lambda^A\bar{\lambda}^B\gamma_\nu\lambda^C.
\end{aligned}$$

The right-hand side vanishes due to (3.68) and the supercurrent (6.10) is conserved!

Exercise 6.3 Study the appropriate Fierz rearrangement and, using the results of Ex. 3.27, show that the supercurrent is conserved in the following cases:

(i) Majorana spinors in $D = 3$,
(ii) Majorana (or Weyl) spinors in $D = 4$, which is the case analyzed above,
(iii) symplectic Weyl spinors in $D = 6$, and
(iv) Majorana-Weyl spinors in $D = 10$.

6.1.3 SUSY transformation rules 111

(!?!?!?!? мб это очень важно, так что раньше это поставлю где-то??)

$$\delta\Phi(x) = \{\bar{\epsilon}^{\alpha} Q_{\alpha}, \Phi(x)\}_{\mathrm{PB}} = -\mathrm{i}\, [\bar{\epsilon}^{\alpha} Q_{\alpha}, \Phi(x)]_{\mathrm{qu}}\,,$$

Φ denotes any field of the system under study. A brief description of Poisson brackets (PB) and commutation relations in the canonical formalism is given in Secs. 1.4 and 1.5. A link in the opposite direction is provided by the Noether formalism, which produces a conserved supercurrent given field variations under which the action is invariant. One reason to emphasize the field variations, *ab* initio, is that this avoids some subtleties in the canonical formalism for gauge theories and for Majorana spinors.

The next exercise illustrates the link between the supercurrent and field variations. It involves the free scalar-spinor $\phi - \Psi$ system of Ex. 6.2. The spinors Ψ, the supersymmetry parameters ϵ and the supersymmetry generator Q are Majorana spinors. They all mutually anti-commute. For the canonical formalism, one can either treat Ψ and $\bar{\Psi}$ as independent variables, or use Dirac brackets to obtain

$$\begin{aligned}
\{\phi(x), \partial_0\phi(y)\}_{\mathrm{PB}} &= -\{\partial_0\phi(x), \phi(y)\}_{\mathrm{PB}} = \delta^3(\vec{x}-\vec{y}),\\
\left\{\Psi_{\alpha}(x), \bar{\Psi}^{\beta}(y)\right\}_{\mathrm{PB}} &= \left\{\bar{\Psi}^{\beta}(x), \Psi_{\alpha}(y)\right\}_{\mathrm{PB}} = \left(\gamma^0\right)_{\alpha}{}^{\beta}\delta^3(\vec{x}-\vec{y}).
\end{aligned}$$

Exercise 6.4 Use $\bar{Q} = (1/\sqrt{2})\int \mathrm{d}^3\vec{x}\bar{\Psi}\gamma^0(\not{\partial}+m)\phi$ or $Q = (1/\sqrt{2})\int \mathrm{d}^3\vec{x}(\not{\partial}-m\phi)\gamma^0\Psi$ to obtain

$$\begin{aligned}
\delta\phi(x) &= \{\bar{\epsilon}Q, \phi(x)\}_{\mathrm{PB}} = \frac{1}{\sqrt{2}}\bar{\epsilon}\Psi(x),\\
\delta\Psi(x) &= \{\bar{\epsilon}Q, \Psi(x)\}_{\mathrm{PB}} = \frac{1}{\sqrt{2}}(\not{\partial} m)\phi\epsilon.
\end{aligned}$$

Note that $\left[\bar{Q}\epsilon, \Psi_{\alpha}(x)\right]_{\mathrm{PB}} = -\left\{\bar{Q}^{\beta}, \Psi_{\alpha}(x)\right\}_{\mathrm{PB}}\epsilon_{\beta}$.

4.6.2 SUSY field theories of the chiral multiplet

Теория

$$\begin{aligned}
\delta Z &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi,\\
\delta P_L\chi &= \frac{1}{\sqrt{2}}P_L(\not{\partial}Z + F)\epsilon,\\
\delta F &= \frac{1}{\sqrt{2}}\bar{\epsilon}\not{\partial}P_L\chi.
\end{aligned}$$

The anti-chiral multiplet transformation rules are

$$\begin{aligned}
\delta\bar{Z} &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_R\chi,\\
\delta P_R\chi &= \frac{1}{\sqrt{2}}P_R(\partial\bar{Z} + \bar{F})\epsilon,\\
\delta\bar{F} &= \frac{1}{\sqrt{2}}\bar{\epsilon}\not{\partial}P_R\chi.
\end{aligned}$$

Note that the form of the transformation rules for the physical components is similar to those of the 'toy model' in Ex. 6.4.

Exercise 6.5 Show that the variations $\delta\bar{Z}, \delta P_R\chi, \delta\bar{F}$ are the complex conjugates of δZ, $\delta P_L\chi, \delta F$.

There are two basic actions, which are separately invariant under the transformation rules above. The first is the free kinetic action

$$S_{\text{kin}} = \int \mathrm{d}^4x \left[-\partial^\mu \bar{Z}\partial_\mu Z - \bar{\chi}\not{P}_L\chi + \bar{F}F\right],$$

6.2.1 U(1)R symmetry 115

The SUSY algebra

$$\begin{aligned}
[\delta_1, \delta_2]\,\Phi(x) &= \left[\bar{\epsilon}_1 Q, \left[\bar{Q}\epsilon_2, \Phi(x)\right]\right] - (\epsilon_1 \leftrightarrow \epsilon_2) \\
&= \bar{\epsilon}_1^\alpha \left[\left\{Q_\alpha, \bar{Q}^\beta\right\}, \Phi(x)\right]\epsilon_{2\beta} \\
&= -\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu\Phi(x).
\end{aligned}$$

6.2.3 More chiral multiplets

4.6.3 SUSY gauge theories

6.3.1 SUSY YangMills vector multiplet 121

6.3.2 Chiral multiplets in SUSY gauge theories 122

4.6.4 Massless representations of $\mathcal{N}$-extended supersymmetry

Superalgebras that containing $\mathcal{N} > 1$ Majorana spinor charges $Q_{i\alpha}, i = 1, ..., \mathcal{N}$ are called $\mathcal{N}$-extended supersymmetry algebras.

$$\begin{aligned}
\left\{Q_{i\alpha}, Q^{\dagger j\beta}\right\}_{\text{qu}} &= \frac{1}{2}\delta_i^j\left(\gamma_\mu\gamma^0\right)_\alpha{}^\beta P^\mu, \quad \alpha = 1, 2, \\
\left[M_{[\mu\nu]}, Q_{i\alpha}\right]_{\text{qu}} &= -\frac{1}{2}\mathrm{i}\left(\gamma_{\mu\nu}\right)_\alpha{}^\beta Q_{i\beta}, \\
\left[P_\mu, Q_{i\alpha}\right]_{\text{qu}} &= 0.
\end{aligned}$$

6.4.1 Particle representations of $\mathcal{N}$-extended supersymmetry

6.4.2 Structure of massless representations

$$\begin{aligned}
\left\{Q_{i\alpha}, Q^{\dagger j\beta}\right\}_{\text{qu}} &= \frac{1}{2}\delta_i^j\left(\mathbb{1}P^0 - \vec{\sigma}\cdot\vec{P}\right)_\alpha^\beta, \\
\left\{Q_{i\alpha}, Q_{j\beta}\right\}_{\text{qu}} &= 0, \quad \left\{Q^{\dagger i\alpha}, Q^{\dagger j\beta}\right\}_{\text{qu}} = 0, \\
\left[\vec{J}, Q_{i\alpha}\right]_{\text{qu}} &= -\frac{1}{2}(\vec{\sigma})_\alpha{}^\beta Q_{i\beta}.
\end{aligned}$$

$\vec{J}$ stands for the space components $J^i = -\frac{1}{2}\varepsilon^{ijk}M_{jk}$.

Since SUSY transformations do not change the 4-momentum, it is sufficient to consider the action of the supercharges on a set of particle states $|\bar{p}, h\rangle$ of fixed energy-momentum $\bar{p}^\mu = (E, 0, 0, E)$. On states of 4-momentum $\bar{p}^\mu$, we find from (6.74) that

$$\begin{aligned}
\left\{Q_{i1}, Q^{\dagger j1}\right\}_{\text{qu}} &= 0, \\
\left\{Q_{i2}, Q^{\dagger j2}\right\}_{\text{qu}} &= E\delta_i^j.
\end{aligned}$$

4.6.5 Appendix 6A Extended supersymmetry and Weyl spinors

4.6.6 Appendix 6B On- and off-shell multiplets and degrees of freedom 130

4.7 Typical supergravities

4.7.1 N = 1 pure supergravity in four dimensions

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9.3 The 1.5 order formalism 193

$$\begin{aligned}
\omega_{\mu ab} &= \omega_{\mu ab}(e) + K_{\mu ab}, \\
K_{\mu\nu\rho} &= -\frac{1}{4}\left(\bar{\psi}_\mu \gamma_\rho \psi_\nu - \bar{\psi}_\nu \gamma_\mu \psi_\rho + \bar{\psi}_\rho \gamma_\nu \psi_\mu\right),
\end{aligned}$$

Summarize the prescription for δS in the 1.5 order formalism:

1. Use the first order form of the action $S[e, \omega, \psi]$ and the transformation rules δe and $\delta\psi$ with connection ω unspecified.

2. Ignore the connection variation and calculate

$$\delta S = \int \mathrm{d}^D x \left[\frac{\delta S}{\delta e}\delta e + \frac{\delta S}{\delta \psi}\delta\psi\right].$$

3. Substitute ω from (9.21) in the result, which must vanish for a consistent supergravity theory.

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9.5 The algebra of local supersymmetry 197 9.6 Anti-de Sitter supergravity 199

4.7.2 D = 11 supergravity

10.1 D 11 from dimensional reduction 201

10.2 The field content of D = 11 supergravity 203

10.3 Construction of the action and transformation rules 203

Ansatz

$$\begin{aligned}
S &= \frac{1}{2\kappa^2}\int \mathrm{d}^{11}x e\left[e^{a\mu}e^{b\nu}R_{\mu\nu ab} - \bar{\psi}_\mu \gamma^{\mu\nu\rho} D_\nu \psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} + \ldots\right] \\
\delta(\epsilon) e_\mu^a &= \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu, \\
\delta(\epsilon)\psi_\mu &= D_\mu\epsilon + \left(a\gamma^{\alpha\beta\gamma\delta}{}_\mu + b\gamma^{\beta\gamma\delta}\delta_\mu^\alpha\right)F_{\alpha\beta\gamma\delta}\epsilon, \\
\delta(\epsilon)A_{\mu\nu\rho} &= c\bar{\epsilon}\gamma_{[\mu\nu}\psi_{\rho]}, \quad \delta(\theta)A_{\mu\nu\rho} = 3\partial_{[\mu}\theta_{\nu\rho]}
\end{aligned}$$

Initially torsion free. Graviton-gravitino system as our 'universal' calculation. Then check $\delta S \propto \psi F \partial$. After integrations by parts, Bianchi identities and γ-matrix algebra: a and b determined in terms of c Check algebra

$$[\delta(\epsilon_1), \delta(\epsilon_2)] A_{\mu\nu\rho} = \frac{8}{9}c^2\xi^\sigma F_{\sigma\mu\nu\rho} = \frac{8}{9}c^2\xi^\sigma\left(\partial_\sigma A_{\mu\nu\rho} + 3\partial_{[\mu}A_{\nu\rho]\sigma}\right)$$

$$b = -8a$$
$$3^2 2^{9/2} a = 1$$

First term $\xi^\sigma \partial_\sigma A_{\mu\nu\rho}$ is the GCT, covariantized by the remainder. $\theta_{\mu\nu} = \xi^\sigma A_{\mu\nu\sigma}$ To have same result as on the frame field: $\to c^2 = 9/8$.

There is then a remaining transformation

$$\delta(\epsilon)S = \int \mathrm{d}^{11}xe\,(\partial_{\nu\bar{\epsilon}})\,\mathcal{I}^\nu, \quad \mathcal{J}^\nu \propto \left(\gamma^{\alpha\beta\gamma\delta\nu\rho}F_{\alpha\beta\gamma\gamma} + 12\gamma^{\alpha\beta}F_{\alpha\beta\beta}{}^{\nu\rho}\right)\psi_\rho$$

- Add Noether term $S_{\text{Noether}} = -\frac{1}{2}\int \mathrm{d}^{11}xe\bar{\psi}_\nu \mathcal{J}^\nu$
- But its variation contains also $\epsilon FF\psi$-terms: $\bar{\epsilon}\gamma^{(r)}\psi_\mu F_{\alpha\beta\gamma\delta}F_{\alpha'\beta'\gamma'\delta'}$ with $r = 1, 3, 5, 7, 9$.
- all cancel except $\gamma^{(9)}$, which can be written as

$$\delta(\epsilon)S = \frac{1}{32 \cdot 144}\int \mathrm{d}^{11}x\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\delta\rho\mu\nu}\underbrace{\bar{\varepsilon}\gamma_{\nu\mu}\psi_\rho}_{\delta A_{\nu\mu\rho}}F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}$$

The Chern-Simons term (from lectures)

Suggest to add

$$\begin{aligned} S_{\text{C-S}} &= -\frac{\sqrt{2}}{(144\kappa)^2}\int \mathrm{d}^{11}x\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\mu\nu\rho}F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\gamma}A_{\mu\nu\rho} \qquad \text{('Ch-S term')} \\ &= -\frac{\sqrt{2}}{6\kappa^2}\int F^{(4)} \wedge F^{(4)} \wedge A^{(3)}, \end{aligned}$$

No frame fields: 'topological' !!

$$\begin{aligned} \delta\int F^{(4)} \wedge F^{(4)} \wedge A^{(3)} &= \int \big[2\ \mathrm{d}\delta A^{(3)} \wedge F^{(4)} \wedge A^{(3)} + F^{(4)} \wedge F^{(4)} \wedge \delta A^{(3)}. \\ &= 3\int F^{(4)} \wedge F^{(4)} \wedge \delta A^{(3)}, \end{aligned}$$

- Gauge invariant: see $\delta(\theta)A^{(3)} = d\theta^{(2)}$.
- No other variations!

Final result

Further: transformation laws that are 'covariant'. Like covariant derivatives: transform without derivative on ϵ.

$$\begin{aligned} \omega_{\mu ab} &= \omega_{\mu ab}(e) + K_{\mu ab}, \\ \hat{\omega}_{\mu ab} &= \omega_{\mu ab}(e) - \frac{1}{4}\left(\bar{\psi}_\mu\gamma_b\psi_a - \bar{\psi}_a\gamma_\mu\psi_b + \bar{\psi}_b\gamma_a\psi_\mu\right), \\ K_{\mu ab} &= -\frac{1}{4}\left(\bar{\psi}_\mu\gamma_b\psi_a - \bar{\psi}_a\gamma_\mu\psi_b + \bar{\psi}_b\gamma_a\psi_\mu\right) + \frac{1}{8}\bar{\psi}_\nu\gamma^{\nu\rho}{}_{\mu ab}\psi_\rho, \\ \hat{F}_{\mu\nu\rho\sigma} &= 4\partial_{[\mu}A_{\nu\rho\sigma]} + \frac{3}{2}\sqrt{2}\bar{\psi}_{[\mu}\gamma_{\nu\rho\rho}\psi_{\sigma]}. \end{aligned}$$

$\delta\hat{F}$ and $\delta\hat{\omega}$ have no $\partial_\mu\epsilon$. In action:

$$\begin{aligned} S = \frac{1}{2\kappa^2}\int \mathrm{d}^{11}xc\bigg[&c^{a\mu}e^{b\nu}R_{\mu\nu ab}(\omega) - \bar{\psi}_\mu\gamma^{\mu\nu\rho}D_\nu\left(\frac{1}{2}(\omega+\hat{\omega})\right)\psi_\rho - \frac{1}{24}F^{\mu\nu\rho\sigma}F_{\mu\nu\rho\sigma} - \\ -\frac{\sqrt{2}}{192}\bar{\psi}_\nu&\left(\gamma^{\alpha\beta\gamma\delta\nu\rho} + 12\gamma^{\alpha\beta}g^{\gamma\nu}g^{\delta\rho}\right)\psi_\rho\left(F_{\alpha\beta\gamma\delta} + \hat{F}_{\alpha\beta\gamma\delta}\right) - \frac{2\sqrt{2}}{(144)^2}\varepsilon^{\alpha'\beta'\gamma'\delta'\alpha\beta\gamma\delta\mu\nu\rho}F_{\alpha'\beta'\gamma'\delta'}F_{\alpha\beta\gamma\delta}A_{\mu\nu\rho}\bigg]. \end{aligned}$$

Such that field equations are covariant. E.g.

$$\gamma^{\mu\nu\rho} D_\nu(\hat{\omega})\psi_\rho - \frac{\sqrt{2}}{288}\gamma^{\mu\nu\rho}\left(\gamma^{\alpha\beta\gamma\delta}{}_\nu - 8\gamma^{\beta\gamma\delta}\delta^\alpha_\nu\right)\psi_\rho \hat{F}_{\alpha\beta\gamma\delta} = 0.$$

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4.7.3 General gauge theory

11.1 Symmetries 212

11.1.1 Global symmetries 213

11.1.2 Local symmetries and gauge fields 217

$$\delta(\epsilon) B_\mu{}^A \equiv \partial_\mu \epsilon^A + \epsilon^C B_\mu{}^B f_{BC}$$

generic gauge symmetry T_A	parameter ϵ^A	gauge field B^A_μ
local translations P_a	ξ^a	e^a_μ
Lorentz transformations $M_{\{ab\}}$	λ^{ab}	ω^{ab}_μ
Supersymmetry Q_α	$\bar{\epsilon}^\alpha$	$\bar{\psi}^\alpha_u$
Internal symmetry T_A	θ^A	A^A_μ

Theories of gravity: Distinguish between coordinate indices $\mu, v, ...$ and local frame indices a,b,c.... - We now use $\mathrm{M}_{[ab]}$ and P_a to denote the generators of local Lorentz transformations and translations. Thus (forgetting internal symmetries): $A = (a, [ab], \alpha)$ There are some subtleties for local translations.

(anti)commutators	structure constants	third parameter
$\left[M_{\{ab\}}, M_{\{cd\}}\right] = 4\eta_{[a[c}M_{\{d]b]\}}$	$f_{\{ab\}\{cd\}}\{cf\} = 8\eta_{[c\|b}\delta^{[e}_{a]}\delta^{f]}_{d]}$	$\lambda^{ab}_3 = 2\lambda_1{}^a{}_c\lambda^{cb}_2$
$\left[P_a, M_{\{bc\}}\right] = 2\eta_{a[b}P_{c]}$	$f_{a,\{bc\}}{}^d = 2\eta_{a[b}\delta^d_{c]}$	$\xi^a_3 = -\lambda^{ab}_2\xi_{1b} + \lambda^{ab}_1\xi_{2b}$
$[P_a, P_b] = 0$		
$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left(\gamma^a\right)_{\alpha\beta} P_a$	$f_{\alpha\beta}{}^a = -\frac{1}{2}\left(\gamma^a\right)_{\alpha\beta}$	$\xi^a_3 = \frac{1}{2}\bar{\epsilon}_2\gamma^a\epsilon_1$
$\left[M_{\{ab\}}, Q\right] = -\frac{1}{2}\gamma_{ab}Q$	$f_{\{ab\},\alpha}{}^\beta = -\frac{1}{2}\left(\gamma_{ab}\right)_\alpha{}^\beta$	
$[P_a, Q] = 0$		

Express ϵ_3 in function of ϵ_1 and λ_2. Write the result as a spinor without spinor indices. Solution

$$\begin{aligned}
\epsilon^\gamma_3 &= \frac{1}{2}\lambda^{ab}_2\epsilon^\alpha_1 f^\gamma_{\alpha,[ab]} \\
&= \frac{1}{2}\lambda^{ab}_2\epsilon^\alpha_1\frac{1}{2}\left(\gamma_{ab}\right)_\alpha{}^\gamma \\
\epsilon_{3\gamma} &= \frac{1}{4}\lambda^{ab}_2\left(\gamma_{ab}\right)_{\alpha\gamma}\epsilon^\alpha_1 \\
\epsilon_{3\gamma} &= -\frac{1}{4}\lambda^{ab}_2\left(\gamma_{ab}\right)_\gamma{}^\alpha\epsilon_{1\alpha} \\
\epsilon_3 &= -\frac{1}{4}\lambda^{ab}_2\gamma_{ab}\epsilon_1
\end{aligned}$$

$$t_0 = 1, t_1 = -1$$

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(see lectures)

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Algebra with central charges: for $N=2$:

$$\begin{aligned}
&\left\{Q_{i\alpha}, Q^{\dagger j\beta}\right\}_{\text{qu}} = \frac{1}{2}\delta_i^j \left(P_L\gamma_\mu\gamma^0\right)_\alpha{}^\beta P^\mu, \\
&\left\{Q_{i\alpha}, Q_j{}^\beta\right\}_{\text{qu}} = -\frac{1}{2}\mathrm{i}\varepsilon_{ij}P_{L\alpha}{}^\beta \mathcal{Z}, \quad \left\{Q^{\dagger i}{}_\alpha, Q^{\dagger j\beta}\right\}_{\text{qu}} = \frac{1}{2}\mathrm{i}\varepsilon^{ij}P_{L\alpha}{}^\beta\overline{\mathcal{Z}}.
\end{aligned}$$

implies for any θ:

$$A_{i\alpha} \equiv Q_{i\alpha} + \mathrm{e}^{\mathrm{i}\theta}\varepsilon_{ij}Q^{\dagger j}{}_\alpha, \quad A^{\dagger i\alpha} \equiv (A_{i\alpha})^\dagger = Q^{\dagger i\alpha} + \mathrm{i}\mathrm{e}^{-\mathrm{i}\theta}\varepsilon^{ij}Q_j{}^\alpha.$$

r.h.s. should be positive for any θ, and on a state $P^0 = M$:

This implies the Bogomol'nyi-Prasad-Sommerfield (BPS) bound:

$$M \geq |\mathcal{Z}|$$

supersymmetric solutions (on which some combination of Q and $Q^\dagger$ vanishes) are BPS solutions! i.e $M = |\mathcal{Z}|$

Central charges: other symmetries in {Q, Q}:

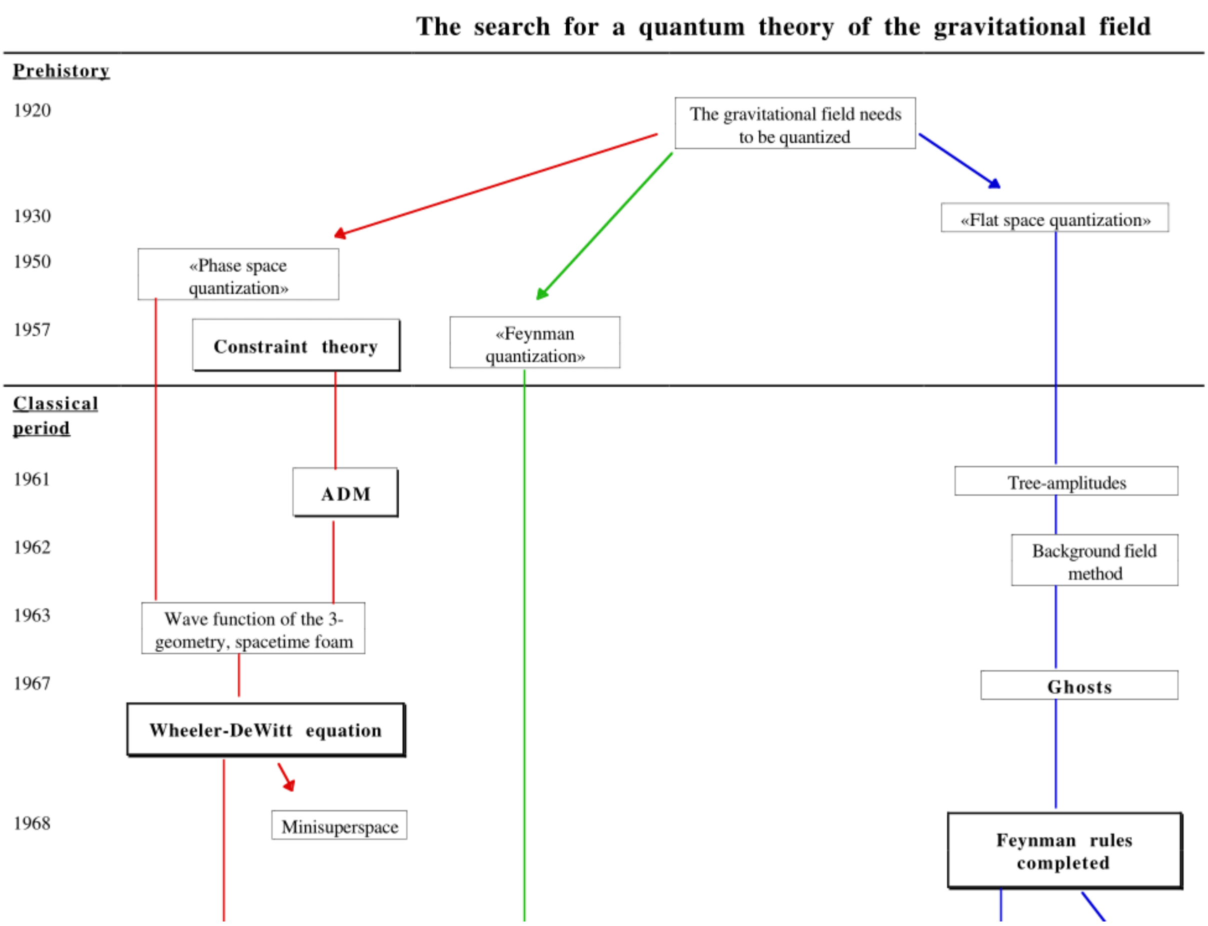
$$\text{e.g. } D=4, \quad \{Q^i_\alpha, Q^j_\beta\} = \gamma^\mu_{\alpha\beta}\delta^i_j P_\mu + \varepsilon^{ij}\left[\mathcal{C}_{\alpha\beta}Z_1 + (\gamma_*)_{\alpha\beta}Z_2\right]$$
$$D=11: \{Q_\alpha, Q_\beta\} = \gamma^\mu_{\alpha\beta}P_\mu + \gamma^{\mu\nu}_{\alpha\beta}Z_{\mu\nu} + \gamma^{\mu_1\cdots\mu_5}_{\alpha\beta}Z_{\mu_1\cdots\mu_5}$$

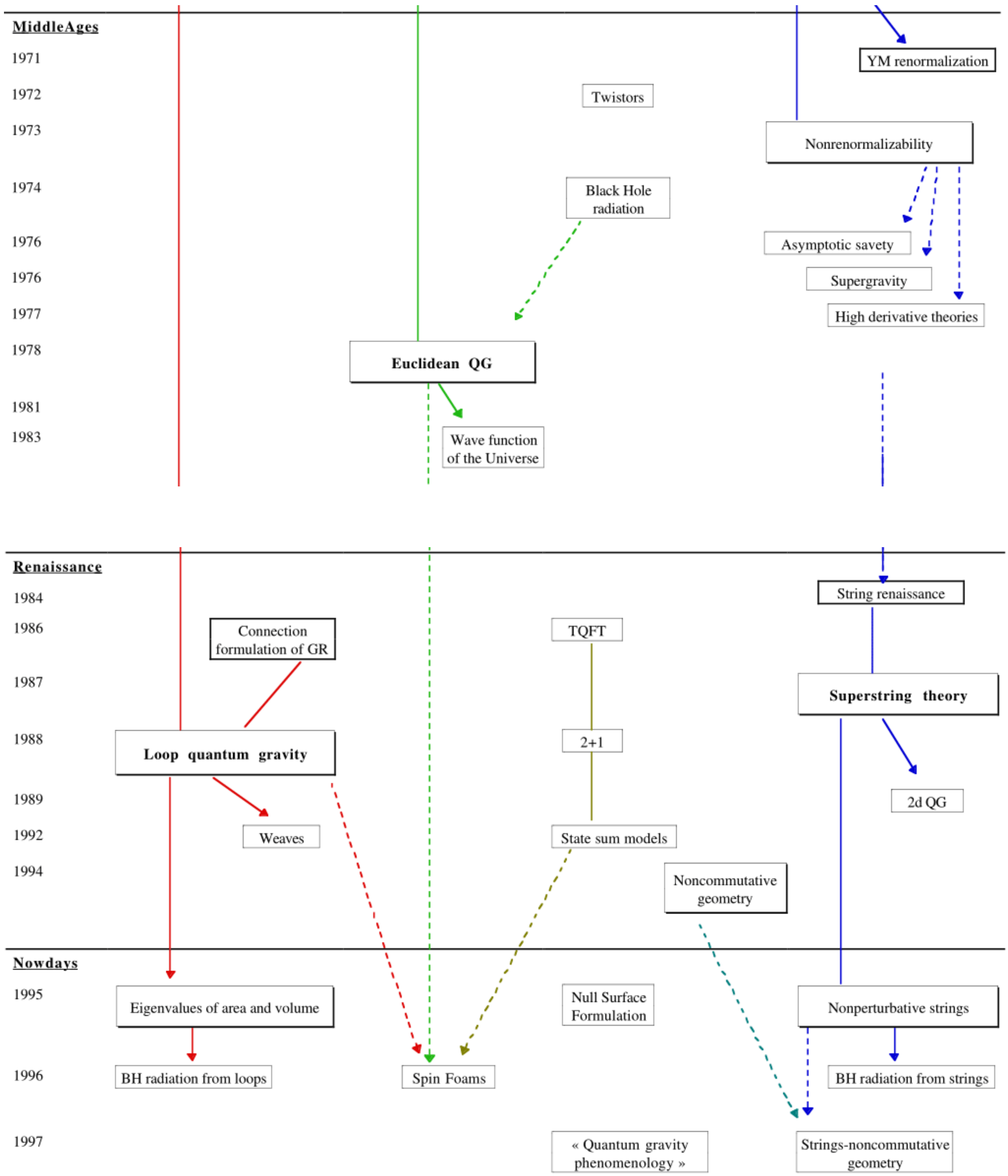
-The algebra gives limits on solutions.
-The left-hand side can be written as $QQ^\dagger$, hence positive !
-Without central charges: solution with preserved supersymmetry, i.e. $Q \mid \text{soln} >= 0 \to$ mass zero.
-With central charges: limit, typically $Z^2 \leq M^2$. BPS bound For preserved supersymmetry: bound saturated BPS states

12.6.3 Central charges in higher dimensions 253

(это последняя лекция, которую нам читали)

4.8 Typical quantum gravities





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Part II
Overview of Other Special Theories and Methods

5 Other Formalisms for Proven Theories

(see Red'kov's articles with many of it. Problem that it is not needed)

5.1 Other special formalisms of electrodynamics

5.1.1 Maxwell Equations in Complex Form of Majorana

Maxwell Equations in Complex Form of Majorana (????)

(??? still need to understand it better!)

При изучении квантовомеханических свойств фотона удобно представление уравнений Максвелла для пустоты в форме Майорана, которое аналогично уравнению Дирака для безмассовой частицы. Уравнения Максвелла в форме Майорана имеют вид:

$$
\begin{aligned}
i\frac{\partial \xi}{\partial t} &= \vec{s}\vec{p}\xi, \qquad \vec{p}\vec{\xi} = 0,\\
i\frac{\partial \eta}{\partial t} &= \vec{s}\vec{p}\eta, \qquad \vec{p}\vec{\eta} = 0, ,
\end{aligned}
$$

$\vec{\xi} = \vec{E} + i\vec{H}, \quad \vec{\eta} = \vec{E} - i\vec{H}$ $\vec{E}, \vec{H}$ - векторы электрического и магнитного поля в уравнениях Максвелла для пустоты (в релятивистской системе единиц):

$$
\begin{aligned}
\frac{\partial \vec{H}}{\partial t} &= -\operatorname{rot}\vec{E}, \operatorname{div}\vec{H} = 0,\\
\frac{\partial \vec{E}}{\partial t} &= \operatorname{rot}\vec{H}, \operatorname{div}\vec{E} = 0,
\end{aligned}
$$

$\vec{p} = \left(\frac{1}{i}\frac{\partial}{\partial x_1}, \frac{1}{i}\frac{\partial}{\partial x_2}, \frac{1}{i}\frac{\partial}{\partial x_3}\right)$ - оператор импульса, $\vec{s}$ - вектор с матричными компонентами:

$$
s_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \qquad s_2 = \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix} \qquad s_3 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

Main examples and applications of уравнения Максвелла в форме Майорана (????)

Maxwell equation from Poincare group (????)

(??? add source and ideas here???)

$$
\mathcal{L}(\boldsymbol{\alpha}) = -\frac{1}{2}\left(\bar{\psi}(\boldsymbol{\alpha})\Gamma_\mu \frac{\partial \psi(\boldsymbol{\alpha})}{\partial x_\mu} - \frac{\partial \bar{\psi}(\boldsymbol{\alpha})}{\partial x_\mu}\Gamma_\mu \psi(\boldsymbol{\alpha})\right) - \\
-\frac{1}{2}\left(\bar{\psi}(\boldsymbol{\alpha})\Upsilon_\nu \frac{\partial \psi(\boldsymbol{\alpha})}{\partial \mathfrak{g}_\nu} - \frac{\partial \bar{\psi}(\boldsymbol{\alpha})}{\partial \mathfrak{g}_\nu}\Upsilon_\nu \psi(\boldsymbol{\alpha})\right),
$$

where $\psi(\boldsymbol{\alpha}) = \psi(x)\psi(\mathfrak{g})(\mu = 0, 1, 2, 3, \nu = 1, ..., 6)$, and

$$\Gamma_0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} 0 & -\alpha_1 \\ \alpha_1 & 0 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & -\alpha_2 \\ \alpha_2 & 0 \end{pmatrix}, \quad \Gamma_3 = \begin{pmatrix} 0 & -\alpha_3 \\ \alpha_3 & 0 \end{pmatrix}$$

$$\Upsilon_1 = \begin{pmatrix} 0 & \Lambda_1^* \\ \Lambda_1 & 0 \end{pmatrix}, \quad \Upsilon_2 = \begin{pmatrix} 0 & \Lambda_2^* \\ \Lambda_2 & 0 \end{pmatrix}, \quad \Upsilon_3 = \begin{pmatrix} 0 & \Lambda_3^* \\ \Lambda_3 & 0 \end{pmatrix}$$

$$\Upsilon_4 = \begin{pmatrix} 0 & i\Lambda_1^* \\ i\Lambda_1 & 0 \end{pmatrix}, \quad \Upsilon_5 = \begin{pmatrix} 0 & i\Lambda_2^* \\ i\Lambda_2 & 0 \end{pmatrix}, \quad \Upsilon_6 = \begin{pmatrix} 0 & i\Lambda_3^* \\ i\Lambda_3 & 0 \end{pmatrix}$$

where

$$\alpha_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & -i & 0 \end{pmatrix}, \quad \alpha_2 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad \alpha_3 = \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\Lambda_1 = \frac{c_{11}}{\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad \Lambda_2 = \frac{c_{11}}{\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad \Lambda_3 = c_{11} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

5.1.2 Maxwell Equations in quaternions

Maxwell Equations in quaternions

Main examples and applications of Maxwell Equations in quaternions (????)

6 Special Modified Theories in a Nutshell

6.1 Extended N = 2 supergravity

6.1.1 Construction of the matter-coupled N = 2 supergravity

20.1 Global supersymmetry

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$$\begin{aligned} \lambda^i &= P_L \lambda^i, \quad \bar{\lambda}^i = \bar{\lambda}^i P_R, \\ \lambda_i &= \left(\lambda^i\right)^C, \quad \lambda^i = -\left(\lambda_i\right)^C, \\ \lambda_i &= \lambda^j \varepsilon_{ji}, \quad \lambda^i = \varepsilon^{ij} \lambda_j, \end{aligned}$$

where one replaces λ with ϵ for the supersymmetry parameters.

$$\begin{aligned} \delta A_\mu^I &= \frac{1}{2} \bar{\epsilon}^i \gamma_\mu \lambda_i^I, \\ \delta \lambda^{iI} &= -\frac{1}{4} \gamma^{\mu\nu} F_{\mu\nu}{}^I \epsilon^i - Y^{ijI} \epsilon_j, \\ \delta Y^{ijI} &= -\frac{1}{2} \bar{\epsilon}^{(i} \not{D} \lambda^{j)I} \end{aligned}$$

realize the algebra

$$\left[\delta\left(\epsilon_1\right), \delta\left(\epsilon_2\right)\right] = \delta_{\text{cgct}}\left(\xi^\mu\right), \quad \xi^\mu = \frac{1}{2} \bar{\epsilon}_2^i \gamma^\mu \epsilon_{1i}.$$

$$S = \int \mathrm{d}^6 x \delta_{IJ} \left[-\frac{1}{4} F_{\mu\nu}^I F^{\mu\nu J} - \frac{1}{2} \bar{\lambda}^{iI} \not{D} \lambda_i^J + 2 \vec{Y}^I \cdot \vec{Y}^J \right],$$

20.1.2 Gauge multiplets for D = 5 413

$$
\begin{aligned}
\delta A_\mu^I &= \frac{1}{2}\bar{\epsilon}^i\gamma_\mu\lambda_i{}^I,\\
\delta\sigma^I &= \frac{1}{2}\mathrm{i}\bar{\epsilon}^i\lambda_i^I,\\
\delta\lambda^{iI} &= -\frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}{}^I\epsilon^i - \frac{1}{2}\mathrm{i}\not{D}\sigma^I\epsilon^i - Y^{ijI}\epsilon_j,\\
\delta\vec{Y}^I &= -\frac{1}{4}\vec{\tau}_{ij}\bar{\epsilon}^i\left[\not{D}\lambda^{jI} + \mathrm{i}f_{JK}{}^I\sigma^J\lambda^{jK}\right].
\end{aligned}
$$

The gauge part in the δ_{cgct} in (20.5) also contains A_μ, which is the origin of a new term in five dimensions, coming from the last component of the $D = 6$ gauge field, which is here σ^I:

$$
[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_{\text{cgct}}(\xi^\mu) + \delta_G\left(\theta^I\right), \quad \theta^I = -\frac{1}{2}\mathrm{i}\sigma^I\bar{\epsilon}_2^i\epsilon_{1i} = -\frac{1}{2}\mathrm{i}\sigma^I\bar{\epsilon}_2^i\epsilon_1^j\varepsilon_{ji}
$$

The action that has superconformal symmetry depends on a symmetric tensor C_{IJK}. The gauge multiplets contain real scalars σ^I in the adjoint of the gauge group, whose kinetic terms define (rigid) 'very special real geometry'.

$$
\begin{aligned}
\mathcal{L}_{5v} =& C_{IJK}\Bigg[\left(-\frac{1}{4}F_{\mu\nu}^I F^{\mu\nu J} - \frac{1}{2}\bar{\lambda}^{iI}\not{D}\lambda_i{}^J - \frac{1}{2}D_\mu\sigma^I D^\mu\sigma^J + 2\vec{Y}^I\cdot\vec{Y}^J\right)\sigma^K\\
& - \frac{1}{24}\varepsilon^{\mu\nu\rho\sigma\tau}A_\mu^I\left[F_{\nu\rho}^J F_{\sigma\tau}^K + f_{LM}{}^J A_\nu^L A_\rho^M\left(-\frac{1}{2}F_{\sigma\tau}^K + \frac{1}{10}f_{NP}{}^K A_\sigma^N A_\tau^P\right)\right]\\
& -\frac{1}{8}\mathrm{i}\bar{\lambda}^{iI}\gamma^{\mu\nu}F_{\mu\nu}^J\lambda_i{}^K - \frac{1}{2}\mathrm{i}\bar{\lambda}^{-iI}\lambda^{jJ}\vec{\tau}_{ij}\cdot\vec{Y}^K + \frac{1}{4}\mathrm{i}\sigma^I\sigma^J\bar{\lambda}^{iL}\lambda_i{}^M f_{LM}{}^K\Bigg].
\end{aligned}
$$

20.1.3 Gauge multiplets for D = 4

$$
\Omega_i{}^I = P_L\Omega_i{}^I, \quad \Omega^{iI} = P_R\Omega^{iI}, \quad \epsilon^i = P_L\epsilon^i, \quad \epsilon_i = P_R\epsilon_i.
$$

The relation with the spinors of $D = 5$ is given in Sec. 20B.2. The supersymmetry transformations are

$$
\begin{aligned}
\delta X^I &= \frac{1}{2}\bar{\epsilon}^i\Omega_i^I,\\
\delta\Omega_i^I &= \not{D}X^I\epsilon_i + \frac{1}{4}\gamma^{\mu\nu}F_{\mu\nu}{}^I\varepsilon_{ij}\epsilon^j + Y_{ij}{}^I\epsilon^j + X^J\bar{X}^K f_{JK}{}^I\varepsilon_{ij}\epsilon^j,\\
\delta A_\mu^I &= \frac{1}{2}\varepsilon^{ij}\bar{\epsilon}_i\gamma_\mu\Omega_j{}^I + \text{ h.c. },\\
\delta\vec{Y}^I &= \frac{1}{4}\vec{\tau}^{ij}\bar{\epsilon}_i\not{D}\Omega_j^I - \frac{1}{2}f_{JK}{}^I\vec{\tau}_i^j\bar{\epsilon}_j X^J\Omega^{iK} + \text{ h.c. },
\end{aligned}
$$

where we included the case of non-abelian gauge multiplets [211], in which all the fields transform in the adjoint of the gauge group. The supersymmetry transformations realize the algebra

$$
\begin{aligned}
&[\delta(\epsilon_1), \delta(\epsilon_2)] = \delta_{\text{cgct}}(\xi^\mu) + \delta_G\left(\theta^I\right),\\
&\qquad \xi^\mu = \frac{1}{2}\bar{\epsilon}_2^i\gamma^\mu\epsilon_{1i} + \text{ h.c. }, \quad \theta^I = X^I\varepsilon^{ij}\bar{\epsilon}_{2i}\epsilon_{1j} + \bar{X}^I\varepsilon_{ij}\epsilon_2^i\epsilon_1^j.
\end{aligned}
$$

Similar to the case of $D = 5$, the additional terms contained in $\delta_G\left(\theta^I\right)$ can lead to a central charge for solutions when $X^I \neq 0$.

The kinetic terms are determined by a holomorphic prepotential function $F(X)$. The latter determines the Kähler potential of the manifold of the scalars. The Kähler manifolds that are of this type are called rigid special Kähler manifolds.

The construction of an action is very similar to the procedure in $\mathcal{N}=1$ supergravity. An arbitrary holomorphic function $F(X)$ determines another $\mathcal{N}=2$ chiral multiplet. It is not a constrained one: it has $16+16$ components. The last component of the latter defines an invariant action, very similar to the procedure of the F-terms and D terms in $\mathcal{N}=1$. Hence, we find an action that depends on this holomorphic function, which is called the prepotential. This Lagrangian is

$$\begin{aligned}\mathcal{L}_{4v} =& \mathrm{i}F_{IJ}D_\mu X^I D^\mu \bar{X}^J + \frac{1}{4}\mathrm{i}F_{IJ}F_{\mu\nu}{}^{-I}F^{-\mu\nu J} + \frac{1}{2}\mathrm{i}F_{IJ}\bar{\Omega}_i^I \not{D}\Omega^{iJ} - \mathrm{i}F_{IJ}\vec{Y}^I\cdot\vec{Y}^J \\ &+ \frac{1}{4}\mathrm{i}F_{IJK}\vec{Y}^I\cdot\vec{\tau}^{ij}\bar{\Omega}_i^J\Omega_j^K \\ &- \frac{1}{16}\mathrm{i}F_{IJK}\varepsilon^{ij}\bar{\Omega}_i^I\gamma^{\mu\nu}F_{\mu\nu}{}^{-J}\Omega_j^K + \frac{1}{48}\mathrm{i}F_{IJKL}\bar{\Omega}_i^I\Omega_\ell^J\bar{\Omega}_j^K\Omega_k^L\varepsilon^{ij}\varepsilon^{k\ell} \\ &\quad + \frac{1}{2}\mathrm{i}F_I f_{JK}{}^I\bar{\Omega}^{iJ}\Omega^{jK}\varepsilon_{ij} - \frac{1}{2}\mathrm{i}F_{IJ}f_{KL}{}^I\bar{X}^K\bar{\Omega}_i^J\Omega_j^L\varepsilon^{ij} \\ &\quad - \mathrm{i}F_I f_{JK}{}^I f_{LM}{}^J\bar{X}^K\bar{X}^L X^M + \text{ h.c.}\end{aligned}$$

Here $F_I, F_{IJ}, \ldots$ denote the derivatives of $F(X)$. The $F_{\mu\nu}{}^{-I}$ are the anti-self-dual field strengths (see Sec. 4.2.1). Let us first consider the action without gauging.

The reader will recognize that the kinetic terms of the scalars X^I are of the hermitian form. In fact, the scalars form a Kähler manifold, with Kähler potential

$$K = \mathrm{i}X^I\bar{F}_I(\bar{X}) - \mathrm{i}\bar{X}^I F_I(X).$$

The kinetic terms are determined by a holomorphic prepotential function $F(X)$. The latter determines the Kähler potential of the manifold of the scalars. The Kähler manifolds that are of this type are called rigid special Kähler manifolds.

20.1.4 Hypermultiplets

$$f^{i\mathcal{A}}{}_Y f^X{}_{i\mathcal{A}} = \delta_Y^X, \quad f^{i\mathcal{A}}{}_X f^X{}_{j\mathcal{B}} = \delta^i{}_j\delta_{\mathcal{B}}^{\mathcal{A}}.$$

The frame field satisfies a reality condition

$$\left(f^{i\mathcal{A}}{}_X\right)^* = f^{j\mathcal{B}}{}_X\varepsilon_{ji}\rho_{\mathcal{B}\overline{\mathcal{A}}}, \quad \left(f^X{}_{i\mathcal{A}}\right)^* = \varepsilon^{ij}\rho^{\overline{\mathcal{A}}\mathcal{B}}f^X{}_{j\mathcal{B}},$$

in terms of a non-degenerate tensor $\rho_{\mathcal{A}\overline{\mathcal{B}}}$ that satisfies

$$\rho_{\mathcal{A}\bar{\mathcal{B}}}\rho^{\overline{\mathcal{B}}\mathcal{C}} = -\delta_{\mathcal{A}}^{\mathcal{C}}, \quad \rho^{\overline{\mathcal{A}}\mathcal{B}} = \left(\rho_{\mathcal{A}\overline{\mathcal{B}}}\right)^*.$$

The above conditions have as consequence that

$$2f^{i\mathcal{A}}{}_X f^Y{}_{j\mathcal{A}} = \delta_X^Y\delta_j^i + \vec{\tau}_j{}^i\cdot\vec{J}_X{}^Y, \quad \vec{J}_X{}^Y = \left(\vec{J}_X{}^Y\right)^* = -f^{i\mathcal{A}}{}_X f^Y{}_{j\mathcal{A}}\vec{\tau}_i{}^j,$$

$$\vec{A}\cdot\vec{J}_X{}^Z\vec{B}\cdot\vec{J}_Z{}^Y = -\delta_X{}^Y\vec{A}\cdot\vec{B} + (\vec{A}\times\vec{B})\cdot\vec{J}_X{}^Y.$$

The frame field is covariantly constant using a connection related to the indices $\mathcal{A}$, and a torsionless connection related to the indices X:

$$\begin{aligned}\nabla_Y f^X{}_{i\mathcal{A}} &\equiv \partial_Y f^X{}_{i\mathcal{A}} - \omega_{Y\mathcal{A}}{}^{\mathcal{A}}(q) f^X{}_{i\mathcal{B}} + \Gamma^X_{YZ}(q) f^Z{}_{i\mathcal{A}} = 0,\\ \nabla_Y f^{i\mathcal{A}}{}_X &\equiv \partial_Y f^{i\mathcal{A}}{}_X + f^{i\mathcal{B}}{}_X \omega_{Y\mathcal{B}}\mathcal{A}^{\mathcal{A}}(q) - \Gamma^Z_{YX}(q) f^{i\mathcal{A}}{}_Z = 0,\end{aligned}$$

where the second equation follows from the first using (20.21). We can solve these equations for $\omega_{X\mathcal{A}}\mathcal{B}$,

$$\omega_{X\mathcal{A}}\mathcal{B} = \frac{1}{2} f^{i\mathcal{B}}{}_Y \left(\partial_X f^Y{}_{i\mathcal{A}} + \Gamma^Y_{XZ} f^Z{}_{i\mathcal{A}}\right),$$

such that the independent connection is $\Gamma_{XY}{}^Z$. The latter is the unique connection on the manifold of the scalars q^X that 'preserves' the complex structures in (20.23):

$$\nabla_Z \vec{J}_X{}^Y \equiv \partial_Z \vec{J}_X{}^Y - \Gamma_{ZX}{}^U \vec{J}_U{}^Y + \Gamma_{ZU}{}^Y \vec{J}^U_X = 0.$$

$$\left(\omega_{X\mathcal{A}}{}^{\mathcal{B}}\right)^* \equiv \bar{\omega}_X \overline{\mathcal{A}}_{\overline{\mathcal{B}}} = -\rho^{\overline{\mathcal{A}}\mathcal{C}} \omega_{X\mathcal{C}}{}^{\mathcal{D}} \rho_{\mathcal{D}\overline{\mathcal{B}}}$$

The matrix $\rho_{\mathcal{A}\mathcal{B}}$ used in these reality conditions, which should also be covariantly constant, enters also in the definition of the symplectic Majorana spinors of hypermultiplets in five and six dimensions. [5] The spinors of the $D = 6$ hypermultiplets are moreover righthanded chiral. For $D = 4$ on the other hand, charge conjugation raises or lowers the indices on the fermions (and changes the chirality):

$$\begin{aligned} D = 6: &\quad \left(\zeta^{\mathcal{A}}\right)^C = \zeta^{\mathcal{B}} \rho_{\mathcal{B}\overline{\mathcal{A}}}, \quad \zeta^{\mathcal{A}} = P_R \zeta^{\mathcal{A}},\\ D = 5: &\quad \left(\zeta^{\mathcal{A}}\right)^C = \zeta^{\mathcal{B}} \rho_{\mathcal{B}\overline{\mathcal{A}}},\\ D = 4: &\quad \left(\zeta^{\mathcal{A}}\right)^C = \zeta_{\overline{\mathcal{A}}} = P_R \zeta_{\overline{\mathcal{A}}}, \quad (\zeta_{\overline{\mathcal{A}}})^C = \zeta^{\mathcal{A}} = P_L \zeta^{\mathcal{A}}.\end{aligned}$$

The supersymmetry transformation laws are simplest for $D = 6$:

$$\begin{aligned} D = 6: \delta q^X &= \bar{\epsilon}^i \zeta^{\mathcal{A}} f^X{}_{i\mathcal{A}}\\ \delta\zeta^{\mathcal{A}} &= \frac{1}{2} f^{i\mathcal{A}}{}_X \not{D} q^X \epsilon_i - \zeta^{\mathcal{B}} \omega_{X\mathcal{B}} \mathcal{A}_{\delta q^X}.\end{aligned}$$

$$\begin{aligned} D = 5: \quad \delta q^X &= -\mathrm{i}\bar{\epsilon}^i \zeta^{\mathcal{A}} f^X{}_{i\mathcal{A}},\\ \delta\zeta^{\mathcal{A}} &= \tfrac{1}{2}\mathrm{i} f^{i\mathcal{A}}{}_X \not{D} q^X \epsilon_i - \zeta^{\mathcal{B}} \omega_{X\mathcal{B}}{}^{\mathcal{A}} \delta q^X + \tfrac{1}{2}\sigma^I k_I{}^X f^{i\mathcal{A}}{}_X \epsilon_i.\end{aligned}$$

For $D = 4$, the transformation laws are [6][95, 219]

$$\begin{aligned} D = 4: \quad \delta q^X &= -\mathrm{i}\bar{\epsilon}^i \zeta^{\mathcal{A}} f^X{}_{i\mathcal{A}} + \mathrm{i}\varepsilon^{ij} \rho^{\overline{\mathcal{A}}\mathcal{B}} \bar{\epsilon}_i \zeta_{\overline{\mathcal{A}}} f^X{}_{j\mathcal{B}},\\ \delta\zeta^{\mathcal{A}} &= \tfrac{1}{2}\mathrm{i} f^{i\mathcal{A}}{}_X \not{D}^X \epsilon_i \epsilon_i - \zeta^{\mathcal{B}} \omega_{X\mathcal{B}} \mathcal{A} \delta q^X + \mathrm{i}\bar{X}^I k_I{}^X f^{i\mathcal{A}}{}_X \varepsilon_{ij} \epsilon^j\end{aligned}$$

$$\mathcal{R}_{XYB'} \equiv 2\partial_{[X} \omega_{Y]\mathcal{B}}{}^{\mathcal{A}} + 2\omega_{[X|\mathcal{C}|} \mathcal{A}^{\omega_{Y]\mathcal{B}}\mathcal{C}}$$

It satisfies reality equations similar to (20.28). The integrability condition of ditions of $C_{\mathcal{A}\mathcal{B}}$ and $d^{\overline{\mathcal{A}}} \bar{B}_{\mathcal{B}}$ implies relations on the curvature:

$$\begin{aligned} 0 &= [\nabla_X, \nabla_Y] C_{\mathcal{A}\mathcal{B}} = 2\mathcal{R}_{XY[\mathcal{A}}{}^{\mathcal{C}} C_{\mathcal{B}]\mathcal{C}} = -2\mathcal{R}_{XY[\mathcal{A}\mathcal{B}]},\\ 0 &= [\nabla_X, \nabla_Y] d^{\overline{\mathcal{A}}}{}_{\mathcal{B}} = -\left(\mathcal{R}_{XY\mathcal{A}}\mathcal{C}\right)^* d^{\overline{\mathcal{C}}}{}_{\mathcal{B}} - \mathcal{R}_{XY\mathcal{B}}{}^{\mathcal{C}} d^{\overline{\mathcal{A}}}\mathcal{C}\end{aligned}$$

$$\begin{aligned} R_{XY}{}^W{}_Z &= f^W{}_{i\mathcal{A}} f^{i\mathcal{B}}{}_Z \mathcal{R}_{XY\mathcal{B}}{}^{\mathcal{A}} = \frac{1}{2} f^{\mathcal{A}i}{}_X f_i \mathcal{B}_Y f^{k\mathcal{C}}{}_Z f^W{}_{k\mathcal{D}} W_{\mathcal{A}\mathcal{B}\mathcal{C}}{}^{\mathcal{D}},\\ W_{\mathcal{A}\mathcal{B}\mathcal{C}\mathcal{D}} &\equiv f^{Xi}{}_{\mathcal{A}} f^Y{}_{i\mathcal{B}} \mathcal{R}_{XY\mathcal{C}\mathcal{D}} = \frac{1}{2} f^{Xi}{}_{\mathcal{A}} f^Y{}_{i\mathcal{B}} f^{Zk}{}_{\mathcal{C}} f^W{}_{k\mathcal{D}} R_{XYZW}.\end{aligned}$$

The kinetic terms of the hypermultiplets are governed by a metric that respects three complex structures. The corresponding scalar manifold is a hyper-Kähler manifold, which has vanishing Ricci tensor. Gauging is determined by a triplet of moment maps, defining triholomorphic Killing vectors.

The tensor $W_{\mathcal{ABCD}} \equiv W_{\mathcal{ABC}}{}^{\mathcal{E}} C_{\mathcal{ED}}$ is symmetric in its four indices. This will be consistent with its appearance in the four-fermion term of the action, similar to the appearance of the curvature term in the kinetic action of the $\mathcal{N} = 1$ chiral multiplet (14.29). Moreover, it implies that the manifold is Ricci flat:

$$R_{YZ} = R_{XYZ} = 0$$

The Lagrangians for the different dimensions are

$$\begin{aligned}
\mathcal{L}_6 =& \mathcal{L}_5 = -\frac{1}{2} g_{XY} \partial_\mu q^X \partial^\mu q^Y + \bar{\zeta}_{\mathcal{A}} \not{\zeta}^{\mathcal{A}} - \frac{1}{4} W_{\mathcal{ABCD}} \bar{\zeta}^{\mathcal{A}} \zeta^{\mathcal{B}} \bar{\zeta}^{\mathcal{CD}}_{\zeta} \\
\mathcal{L}_4 =& -\frac{1}{2} g_{XY} \partial_\mu q^X \partial^\mu q^Y - d^{\overline{\mathcal{A}}} \mathcal{B}^{(} \left(\bar{\zeta}_{\overline{\mathcal{A}}} \nabla \zeta^{\mathcal{B}} + \bar{\zeta}^{\mathcal{B}} \nabla \zeta_{\overline{\mathcal{A}}} \right) \\
&+ \frac{1}{2} W_{\mathcal{AB}}{}^{\mathcal{EF}} d^{\overline{\mathcal{C}}}{}_{\mathcal{E}} d^{\overline{\mathcal{D}}}{}_{\mathcal{F}} \bar{\zeta}_{\overline{\mathcal{C}}} \zeta_{\overline{\mathcal{D}}} \bar{\zeta}^{\mathcal{A}} \zeta^{\mathcal{B}}
\end{aligned}$$

where $\nabla_\mu \zeta^{\mathcal{A}} = \partial_\mu \zeta^{\mathcal{A}} + \partial_\mu q^X \omega_X \mathcal{B}^{\mathcal{A}} \zeta^{\mathcal{B}}$.

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20.2 N = 2 superconformal calculus

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$$\begin{array}{lll}
 & [M_{ab}, Q^i_\alpha] = -\frac{1}{2} (\gamma_{ab} Q^i)_\alpha \,, & [M_{ab}, S^i_\alpha] = -\frac{1}{2} (\gamma_{ab} S^i)_\alpha \,, \\
 & [D, Q_\alpha{}^i] = \frac{1}{2} Q_\alpha{}^i, & [D, S_\alpha{}^i] = -\frac{1}{2} S^i_\alpha, \\
 & \left[U_i{}^j, Q_\alpha{}^k\right] = \delta_i{}^k Q^j_\alpha - \frac{1}{2} \delta_i{}^j Q_\alpha{}^k, & \left[U_i{}^j, S_\alpha{}^k\right] = \delta_i{}^k S^j_\alpha - \frac{1}{2} \delta_i{}^j S_\alpha{}^k, \\
D = 4: & [U_i{}^j, Q_{\alpha k}] = -\delta_k{}^j Q_{\alpha i} + \frac{1}{2} \delta_i{}^j Q_{\alpha k}, & [U_i{}^j, S_{\alpha k}] = \delta_k{}^j S_{\alpha i} - \frac{1}{2} \delta_i{}^j S_{\alpha k}, \\
D = 4: & [T, S_\alpha{}^i] = \mathrm{i} \frac{1}{2} Q_\alpha{}^i, S_\alpha{}^i, & \\
D = 5: & [K_a, Q^i_\alpha] = (\gamma_a S^i)_\alpha \,, S_\alpha{}^i] = (\gamma_a Q^i)_\alpha \,, & \\
D = 6: & [K_a, Q^i_a, Q^i_\alpha] = -(\gamma_a S^i)_\alpha \,, & [P_a, S_\alpha{}^i] = -\mathrm{i} \left(\gamma_a Q^i)^i\right)_\alpha \,, \\
D, & [P_a, S_\alpha{}^i] = -(\gamma_a Q^i)_\alpha \,. &
\end{array}$$

The anti-commutation relations between the fermionic generators are

$$\begin{aligned}
&\left\{Q_{i\alpha}, Q^{j\beta}\right\} = -\frac{1}{2} \delta_i^j (\gamma^a)_\alpha{}^\beta P_a, \quad \left\{S_{i\alpha}, S^{j\beta}\right\} = -\frac{1}{2} \delta_i^j (\gamma^a)_\alpha{}^\beta K_a, \\
&D = 4: \left\{Q_\alpha{}^i, Q_j{}^\beta\right\} = -\frac{1}{2} \delta_j{}^i (\gamma^a)_\alpha{}^\beta P_a, \quad \left\{S_\alpha{}^i, S_j{}^\beta\right\} = -\frac{1}{2} \delta_j{}^i (\gamma^a)_\alpha{}^\beta K_a, \\
&D = 4: \left\{Q_\alpha{}^i, Q^{j\beta}\right\} = 0, \quad \left\{S_\alpha{}^i, S^{j\beta}\right\} = 0, \quad \left\{Q_\alpha{}^i, S^{j\beta}\right\} = 0, \\
&D = 4: \left\{Q_{i\alpha}, S^{j\beta}\right\} = -\frac{1}{2} \delta_i{}^j \delta_\alpha{}^\beta D - \frac{1}{4} \delta_i{}^j \left(\gamma^{ab}\right)_\alpha{}^\beta M_{ab} + \mathrm{i} \frac{1}{2} \delta_i \delta_\alpha{}^\beta T - \delta_\alpha{}^\beta U_i{}^j, \\
&D = 4: \left\{Q_\alpha{}^i, S_j{}^\beta\right\} = -\frac{1}{2} \delta_j{}^j \delta_\alpha{}^\beta D - \frac{1}{4} \delta_j{}^i \left(\gamma^{ab}\right)_\alpha{}^\beta M_{ab} - \mathrm{i} \frac{1}{2} \delta_j \delta_\alpha{}^\beta T + \delta_\alpha{}^\beta U_j{}^j, \\
&D = 5: \left\{Q_{i\alpha}, S^{j\beta}\right\} = -\mathrm{i} \frac{1}{2} \left(\delta_i{}^j \delta_\alpha{}^\beta D + \frac{1}{2} \delta_i{}^j \left(\gamma^{ab}\right)_\alpha{}^\beta M_{ab} + 3 \delta_\alpha{}^\beta U_i{}^j \right), \\
&D = 6: \left\{Q_{i\alpha}, S^{j\beta}\right\} = \frac{1}{2} \left(\delta_i{}^j \delta_\alpha{}^\beta D + \frac{1}{2} \delta_i{}^j \left(\gamma^{ab}\right)_\alpha{}^\beta M_{ab} + 4 \delta_\alpha{}^\beta U_i{}^j \right).
\end{aligned}$$

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$$\delta = \epsilon^A T_A = \frac{1}{2}\lambda^{ab} M_{[ab]} + \lambda_{\mathrm{D}} D + \lambda_{\mathrm{K}}^a K_a + \lambda_i{}^j U_j{}^i + \lambda_T T \\ + \bar{\epsilon}^i Q_i + \bar{\epsilon}_i Q^i + \bar{\epsilon}^i S_i + \bar{\epsilon}_i S^i.$$

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20.2.4 Superconformal actions 432

20.2.5 Partial gauge fixing 434

20.2.6 Elimination of auxiliary fields 436

20.2.7 Complete action 439

20.2.8 D = 5 and D = 6, N = 2 supergravities 440

6.1.2 The physical N = 2matter-coupled supergravity

6.2 Classical solutions and the AdS/CFT correspondence

6.2.1 Classical solutions of gravity and supergravity

6.2.2 The AdS/CFT correspondence

6.3 Other famous hypothetical fundamental theories in a nutshell

6.3.1 On universe-antiuniverse entangled by Naman Kumar

(очень интересная теория, потом укажу, что за основные формулы и идеи. Пока нет времени изучать её.)

6.3.2 Compactification from 5D: action and manifold with boundary

(one of the topics from KUL exam)

6.3.3 Superfields

(one of the topics from KUL exam)

6.3.4 Group manifold approach

(one of the topics from KUL exam)

6.3.5 Gravity actions of higher order in the curvature

(one of the topics from KUL exam)

6.3.6 Anti-de Sitter supergravity

(one of the topics from KUL exam)

6.3.7 The type IIA supergravity

(one of the topics from KUL exam)

6.3.8 Non-relativistic gravity

(one of the topics from KUL exam)

6.3.9 Coset manifold

(one of the topics from KUL exam)

6.3.10 Complex and Kahler manifolds

(one of the topics from KUL exam)

6.3.11 General actions with N = 1 supersymmetry

(one of the topics from KUL exam)

6.3.12 Supergravity from conformal methods

(one of the topics from KUL exam)

6.3.13 Kaluza-Klein reduction of D = 11 on tori

(one of the topics from KUL exam)

6.3.14 The matter-coupled N = 1 supergravity

(one of the topics from KUL exam)

6.3.15 Classical solutions of gravity and supergravity

(one of the topics from KUL exam)

6.3.16 Supersymmetric Black Holes

(one of the topics from KUL exam)

Part III
Filtering Out Bad Theories

7 Examples and Methods of Finding Out Shitty Theory

(a big discussion here about how to find out what theory is shitty and worth nothing and what is an important one?)

Part IV
Theoretical Field Theories

8 Clifford algebras and spinors

The Dirac equation is a relativistic wave equation that is first order in space and time derivatives. The key to this remarkable property is the set of γ-matrices, which satisfy the anti-commutation relations (2.18):

$$\gamma^\mu\gamma^\nu + \gamma^\upsilon\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}$$

These matrices are the generating elements of a Clifford algebra which plays an important role in supersymmetric and supergravity theories. In the first part of this chapter we discuss the structure of this Clifford algebra for general spacetime dimension D. For a generic value of D, the γ-matrices are intrinsically complex. This is why we assumed that the Dirac field is complex in the previous chapter. In certain spacetime dimensions the representation of the Clifford algebra is real, which means that the γ-matrices are conjugate to real matrices. In this case the basic spinor field may be taken to be real and is called a Majorana spinor field. Since the Majorana field has a smaller number of independent components, it is fair to say that, when it exists, it is more fundamental than the Dirac field. For this reason Majorana spinors are selected in supersymmetry and supergravity. We study the special properties of Majorana spinors in the second part of this chapter.

In the body of the chapter we take a practical approach, intended as a guide to the applications needed later in the book. Further supporting arguments are collected in Appendix 3 A at the end of the chapter.

8.1 The Clifford algebra in general dimension

8.1.1 The generating γ-matrices

The main purpose of this section is to discuss the Clifford algebra associated with the Lorentz group in D dimensions. To be concrete, we start with a general and explicit construction of the generating γ-matrices. It is simplest first to construct Euclidean γ-matrices, which satisfy (3.1) with Minkowski metric $\eta_{\mu\nu}$ replaced by $\delta_{\mu\nu}$:

$$\begin{aligned}
\gamma^1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ..., \\
\gamma^2 &= \sigma_2 \otimes \mathbb{1} \otimes \mathbb{1} \otimes ..., \\
\gamma^3 &= \sigma_3 \otimes \sigma_1 \otimes \mathbb{1} \otimes ... \\
\gamma^4 &= \sigma_3 \otimes \sigma_2 \otimes \mathbb{1} \otimes ... \\
\gamma^5 &= \sigma_3 \otimes \sigma_3 \otimes \sigma_1 \otimes ... \\
... &= ...
\end{aligned}$$

These matrices are all hermitian with squares equal to $\mathbb{1}$, and they mutually anti-commute. Suppose that $D = 2m$ is even. Then we need m factors in the construction (3.2) to obtain $\gamma^\mu, 1 \le \mu \le D = 2m$. Thus we obtain a representation of dimension $2^{D/2}$.

For odd $D = 2m + 1$ we need one additional matrix, and we take γ^{2m+1} from the list above, but we keep only the first m factors, i.e. deleting a σ_1. Thus there is no increase in the dimension of the representation in going from $D = 2m$ to $D = 2m + 1$, and we can say in general that the construction (3.2) gives a representation of dimension $2^{[D/2]}$, where $[D/2]$ means the integer part of $D/2$.

Euclidean γ-matrices do have physical applications, but we need Lorentzian γ for the subject matter of this book. To obtain these, all we need to do is pick any single matrix from the Euclidean construction, multiply it by i and label it γ^0 for the time-like direction. This matrix is anti-hermitian and satisfies $(\gamma^0)^2 = -\mathbb{1}$. We then relabel the remaining $D-1$ matrices to obtain the Lorentzian set $\gamma^\mu, 0 \leq \mu \leq D-1$. The hermiticity properties of the Lorentzian γ are summarized by

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0.$$

It is fundamental to the Dirac theory that the physics of a spinor field is the same in all equivalent representations of the Clifford algebra. Thus we are really concerned with classes of representations related by conjugacy, i.e.

$$\gamma'^\mu = S\gamma^\mu S^{-1}.$$

Since we consider only hermitian representations, in which (3.3) holds, the matrix S must be unitary. Given any two equivalent representations, the transformation matrix S is unique up to a phase factor. Up to this conjugation, there is a unique irreducible representation (irrep) of the Clifford algebra by $2^m \times 2^m$ matrices for even dimension $D = 2m$. Any other representation is reducible and equivalent to a direct sum of copies of the irrep above. One can always choose a hermitian irrep, defined as one which satisfies (3.3). In odd dimensions there are two mathematically inequivalent irreps, which differ only in the sign of the 'final' $\pm\gamma^{2m+1}$. In this book we will always use a hermitian irrep of the γ-matrices. Physical consequences are independent of the particular representation chosen.

8.1.2 The complete Clifford algebra

The full Clifford algebra consists of the identity $\mathbb{1}$, the D generating elements γ^μ, plus all independent matrices formed from products of the generators. Since symmetric products reduce to a product containing fewer γ-matrices by (3.1), the new elements must be antisymmetric products. We thus define

$$\gamma^{\mu_1\cdots\mu_r} = \gamma^{[\mu_1}\ldots\gamma^{\mu_r]}, \qquad \text{e.g.} \qquad \gamma^{\mu\nu} = \frac{1}{2}\left(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu\right)$$

where the antisymmetrization indicated with [...] is always with total weight 1. Thus the right-hand side of (3.5) contains the overall factor $1/r\,!$ times a sum of $r\,!$ signed permutations of the indices. Non-vanishing tensor components can be written as the products

$$\gamma^{\mu_1\mu_2\cdots\mu_r} = \gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_r} \qquad \text{where } \mu_1 \neq \mu_2 \neq \cdots \neq \mu_r$$

All these matrices are traceless (a proof can be found in Appendix 3A), except for the lowest rank $r = 0$, which is the unit matrix, and the highest rank matrix with $r = D$, which is traceless only for even D as we will see below.

There are C_r^D (binomial coefficients) independent index choices at rank r. For even space-time dimension the matrices are linearly independent, so that the Clifford algebra is an algebra of dimension 2^D.

Exercise 3.1 Show that the higher rank γ-matrices can be defined as the alternate commutators or anti-commutators

$$\gamma^{\mu\nu} = \frac{1}{2}\left[\gamma^{\mu}, \gamma^{\nu}\right],$$
$$\gamma^{\mu_1\mu_2\mu_3} = \frac{1}{2}\left\{\gamma^{\mu_1}, \gamma^{\mu_2\mu_3}\right\},$$
$$\gamma^{\mu_1\mu_2\mu_3\mu_4} = \frac{1}{2}\left[\gamma^{\mu_1}, \gamma^{\mu_2\mu_3\mu_4}\right],$$

etc.

Exercise 3.2 Show that $\gamma^{\mu_1\cdots\mu_D} = \frac{1}{2}\left(\gamma^{\mu_1}\gamma^{\mu_2\cdots\mu_D} - (-)^D\gamma^{\mu_2\cdots\mu_D}\gamma^{\mu_1}\right)$. Thus, $\operatorname{Tr}\gamma^{\mu_1\cdots\mu_D}$ vanishes for even D.

8.1.3 Levi-Civita symbol

We need a short technical digression to introduce the Levi-Civita symbol and derive some of its properties. In every dimension D this is defined as the totally antisymmetric rank D tensor $\varepsilon_{\mu_1\mu_2\ldots\mu_D}$ or $\varepsilon^{\mu_1\mu_2\cdots\mu_D}$ with

$$\varepsilon_{012(D-1)} = 1, \quad \varepsilon^{012(D-1)} = -1.$$

Indices are raised using the Minkowski metric which leads to the difference in sign above (due to the single time-like direction).

Exercise 3.3 Prove the contraction identity for these tensors:

$$\varepsilon_{\mu_1\ldots\mu_n\nu_1\ldots v_p}\varepsilon^{\mu_1\cdots\mu_n\rho_1\cdots\rho_p} = -p!n!\delta^{\rho_1\cdots\rho_p}_{v_1\cdots v_p}, \quad p = D-n.$$

The antisymmetric p-index Kronecker δ is in turn defined by

$$\delta^{\rho_1\cdots\rho_p}_{v_1\ldots\nu_q} \equiv \delta^{[\rho_1}_{v_1}\delta^{\rho_2}_{v_2}\cdots\delta^{\rho_p]}_{v_q}$$

which includes a signed sum over $p\,!$ permutations of the lower indices, each with a coeffcient $1/p\,!$, such that the 'total weight' is 1 (as in (A.8)).

In four dimensions the totally antisymmetric Levi-Civita tensor symbol is written as $\varepsilon^{\mu\nu\rho\sigma}$. Because an antisymmetric tensor of rank 5 necessarily vanishes when $D = 4$, this satisfies the Schouten identity

$$0 = 5\delta^{[v}_{\mu}\varepsilon^{\rho\sigma\tau\lambda]} \equiv \delta^{\nu}_{\mu}\varepsilon^{\rho\sigma\tau\lambda} + \delta^{\rho}_{\mu}\varepsilon^{\sigma\tau\lambda\nu} + \delta^{\sigma}_{\mu}\varepsilon^{\tau\lambda\nu\rho} + \delta^{\tau}_{\mu}\varepsilon^{\lambda\nu\rho\sigma} + \delta^{\lambda}_{\mu}\varepsilon^{\nu\rho\sigma\tau}$$

8.1.4 Practical γ-matrix manipulation

Supersymmetry and supergravity theories emerge from the concept of fermion spin. It should be no surprise that intricate features of the Clifford algebra are needed to establish and explore the physical properties of these field theories. In this section we explain some useful tricks to multiply γ-matrices. The results are valid for both even and odd D.

Consider first products with index contractions such as

$$\gamma^{\mu\nu}\gamma_{\nu} = (D-1)\gamma^{\mu}.$$

You can memorize this rule, but it is easier to recall the simple logic behind it: v runs over all values except μ, so there are $(D-1)$ terms in the sum. Similar logic explains the result

$$\gamma^{\mu\nu\rho}\gamma_{\rho} = (D-2)\gamma^{\mu\nu}$$

or even more generally

$$\gamma^{\mu_1\ldots\mu_r v_1\ldots v_s}\gamma_{v_s\ldots v_1} = \frac{(D-r)!}{(D-r-s)!}\gamma^{\mu_1\ldots\mu_r}$$

Indeed, first we can write $\gamma_{v_s\ldots\nu_1}$ as the product $\gamma_{v_s}\ldots\gamma_{v_1}$ since the antisymmetry is guaranteed by the first factor. Then the index v_s has $(D-(r+s-1))$ values, while v_{s-1} has $(D-(r+s-2))$ values, and this pattern continues to $(D-r)$ values for the last one. Note that the second γ on the left-hand side has its indices in opposite order, so that no signs appear when contracting the indices. It is useful to remember the general order reversal symmetry, which is

$$\gamma^{v_1\ldots v_r} = (-)^{r(r-1)/2}\gamma^{v_r\ldots v_1}.$$

The sign factor $(-)^{r(r-1)/2}$ is negative for $r = 2, 3 \bmod 4$.

Even if one does not sum over indices, similar combinatorial tricks can be used. For example, when calculating

$$\gamma^{\mu_1\mu_2}\gamma_{\nu_1\ldots v_D}$$

one knows that the index values μ_1 and μ_2 appear in the set of $\{v_i\}$. There are D possibilities for μ_2, and since μ_1 should be different, there remain $D-1$ possibilities for μ_1. Hence the result is

$$\gamma^{\mu_1\mu_2}\gamma_{v_1\ldots v_D} = D(D-1)\delta^{\mu_2\mu_1}_{[v_1 v_2}\gamma_{v_3\ldots v_D]}.$$

Note that such generalized δ-functions are always normalized with 'weight 1 ', i.e.

$$\delta^{\mu_2\mu_1}_{v_1 v_2} = \frac{1}{2}\left(\delta^{\mu_2}_{v_1}\delta^{\mu_1}_{v_2} - \delta^{\mu_1}_{v_1}\delta^{\mu_2}_{v_2}\right)$$

This makes contractions easy; e.g. we obtain from (3.17)

$$\gamma^{\mu_1\mu_2}\gamma_{\nu_1\ldots\nu_D}\varepsilon^{\nu_1\ldots\nu_D} = D(D-1)\varepsilon^{\mu_2\mu_1\nu_3\ldots\nu_D}\gamma_{\nu_3\ldots\nu_D}.$$

We now consider products of γ-matrices without index contractions. The very simplest case is

$$\gamma^\mu\gamma^\nu = \gamma^{\mu\nu} + \eta^{\mu\nu}$$

This follows directly from the definitions: the antisymmetric part of the product is defined in (3.5) to be $\gamma^{\mu\nu}$, while the symmetric part of the product is $\eta^{\mu\nu}$, by virtue of (3.1). This already illustrates the general approach: one first writes the totally antisymmetric Clifford matrix that contains all the indices and then adds terms for all possible index pairings.

Here is another example:

$$\gamma^{\mu\nu\rho}\gamma_{\sigma\tau} = \gamma^{\mu\nu\rho}{}_{\sigma\tau} + 6\gamma^{[\mu\nu}{}_{[\tau}\delta^{\rho]}{}_{\sigma]} + 6\gamma^{[\mu}\delta^{\nu}_{[\tau}\delta^{\rho]}{}_{\sigma]}.$$

This follows the same pattern. We write the indices $\sigma\tau$ in down position to make it easier to indicate the antisymmetry. The second term contains one contraction. One can choose three indices from the first factor and two indices from the second one, which gives the factor 6. For the third term there are also six ways to make two contractions. The δ-functions contract indices that were adjacent, or separated by already contracted indices, so that no minus signs appear.

Exercise 3.4 As a similar exercise, derive

$$\gamma^{\mu_1\ldots\mu_4}\gamma_{v_1 v_2} = \gamma^{\mu_1\ldots\mu_4}{}_{v_1 v_2} + 8\gamma^{[\mu_1\ldots\mu_3}{}_{[v_2}\delta^{\mu_4]}{}_{v_1]} + 12\gamma^{[\mu_1\mu_2}\delta^{\mu_3}{}_{[v_2}\delta^{\mu_4]}{}_{v_1]}]$$

Finally, we consider products with both contracted and uncontracted indices. Consider $\gamma^{\mu_1\cdots\mu_4\rho}\gamma_{\rho v_1 v_2}$. The result should contain terms similar to (3.22), but each term has an extra numerical factor reflecting the number of values that ρ can take in this sum. For example, in the second term above there is now one contraction between an upper and lower index, and therefore ρ can run over all D values except the four values $\mu_1, ...\mu_4$, and the two values v_1, v_2. This counting gives

$$\begin{aligned}\gamma^{\mu_1\cdots\mu_4\rho}\gamma_{\rho v_1 v_2} =&(D-6)\gamma^{\mu_1\cdots\mu_4}{}_{v_1 v_2} + 8(D-5)\gamma^{[\mu_1\cdots\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{v_1]}\\ &+ 12(D-4)\gamma^{[\mu_1\mu_2}\delta^{\mu_3}{}_{[\nu_2}\delta^{\mu_4]}{}_{\nu_1]}.\end{aligned}$$

Exercise 3.5 Show that

$$\begin{aligned}\gamma_\nu\gamma^\mu\gamma^\nu &= (2-D)\gamma^\mu,\\ \gamma_\rho\gamma^{\mu\nu}\gamma^\rho &= (D-4)\gamma^{\mu\nu}\end{aligned}$$

Derive the general form $\gamma_\rho\gamma^{\mu_1\mu_2\cdots\mu_r}\gamma^\rho = (-)^r(D-2r)\gamma^{\mu_1\mu_2\cdots\mu_r}$.

8.1.5 Basis of the algebra for even dimension $D = 2m$

To continue our study we restrict to even-dimensional spacetime and construct an orthogonal basis of the Clifford algebra. It will be easy to extend the results to odd D later.

The basis is denoted by the following list $\left\{\Gamma^A\right\}$ of matrices chosen from those defined in Sec. 3.1.2:

$$\left\{\Gamma^A = \mathbb{1}, \gamma^\mu, \gamma^{\mu_1\mu_2}, \gamma^{\mu_1\mu_2\mu_3}, \cdots, \gamma^{\mu_1\cdots\mu_D}\right\}$$

Index values satisfy the conditions $\mu_1 < \mu_2 < ... < \mu_r$. There are C_r^D distinct index choices at each rank r and a total of 2^D matrices. To see that this is a basis, it is convenient to define the reverse order list

$$\{\Gamma_A = \mathbb{1}, \gamma_\mu, \gamma_{\mu_2\mu_1}, \gamma_{\mu_3\mu_2\mu_1}, ..., \gamma_{\mu_D\cdots\mu_1}\}$$

By (3.15) the matrices of this list differ from those of (3.25) by sign factors only.

Exercise 3.6 Show that $\Gamma^A\Gamma^B = \pm\Gamma^C$, where Γ^C is the basis element whose indices are those of A and B with common indices excluded. Derive the trace orthogonality property

$$\mathrm{Tr}\left(\Gamma^A\Gamma_B\right) = 2^m\delta^A_B$$

The list (3.25) contains 2^D trace orthogonal matrices in an algebra of total dimension 2^D. Therefore it is a basis of the space of matrices M of dimension $2^m \times 2^m$.

Exercise 3.7 Show that any matrix M can be expanded in the basis $\left\{\Gamma^A\right\}$ as

$$M = \sum_A m_A\Gamma^A, \quad m_A = \frac{1}{2^m ă}\,\mathrm{Tr}\left(M\Gamma_A\right)$$

Readers may already have noted that the signature of spacetime has played little role in the discussion above. The basic conclusion that there is a unique representation of the Clifford algebra of dimension 2^m is true for pseudo-Euclidean metrics of any signature (p, q). Another general fact is that the second rank Clifford elements $\gamma^{\mu\nu}$ are the generators of a representation of the Lie algebra $\mathfrak{so}(p, q)$, with $p + q = D = 2m$; see (1.34) with the metric signature (p, q). Only the hermiticity properties depend on the signature in an obvious fashion.

Exercise 3.8 Show that

$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 2^m\left[\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho}\right]$$

Exercise 3.9 Count the number of elements in the basis (3.25) for odd dimensions $D = 2m+1$, and see that it contains twice the number of independent $2^m \times 2^m$ matrices. Check that we already have enough matrices if we consider the matrices up to $\gamma_{\mu_1 \dots \mu_{(D-1)/2}}$. Therefore the results of this section hold only for even dimensions and will have to be modified for odd dimensions; see Sec. 3.1.7.

8.1.6 6 The highest rank Clifford algebra element

For several reasons it is useful to study the highest rank tensor element of the Clifford algebra. It provides the link between even and odd dimensions and it is closely related to the chirality of fermions, an important physical property. We define

$$\gamma_* \equiv (-\mathrm{i})^{m+1}\gamma_0\gamma_1 \dots \gamma_{D-1},$$

which satisfies $\gamma_*^2 = \mathbb{1}$ in every even dimension and is hermitian. For spacetime dimension $D = 2m$, the matrix γ_* is frequently called γ_{D+1} in the physics literature, as in four dimensions where it is called γ_5.

This matrix occurs as the unique highest rank element in (3.25). For any order of components μ_i, one can write

$$\gamma_{\mu_1\mu_2\cdots\mu_D} = \mathrm{i}^{m+1}\varepsilon_{\mu_1\mu_2\cdots\mu_D}\gamma_*,$$

where the Levi-Civita tensor introduced in Sec. 3.1.3 is used.

Exercise 3.10 Show that γ_* commutes with all even rank elements of the Clifford algebra and anti-commutes with all odd rank elements. Thus, for example,

$$\{\gamma_*, \gamma^\mu\} = 0,$$
$$[\gamma_*, \gamma^{\mu\nu}] = 0.$$

Since $\gamma_*^2 = \mathbb{1}$ and $\mathrm{Tr}\,\gamma_* = 0$, it follows that one can choose a representation in which

$$\gamma_* = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

Some exercises follow, which illustrate the properties of a representation of the full Clifford algebra in which γ_* takes the form in (3.34).

Exercise 3.11 Assume a general block form,

$$\gamma^\mu = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for the generating elements in a basis where (3.34) holds. Show that (3.32) implies the block off-diagonal form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

in which the matrices σ^μ and $\bar{\sigma}^\mu$ are $2^{m-1}\times 2^{m-1}$ generalizations of the explicit Weyl matrices of (2.2).

Exercise 3.12 Show that the matrices σ^μ and $\bar{\sigma}^\mu$ satisfy (2.4) and that $\mathrm{Tr}\,(\sigma^\mu\bar{\sigma}_v) = 2^{(m-1)}\delta^\mu_\nu$.

Exercise 3.13 Show similarly that (3.33) implies that the second rank matrices take the block diagonal form

$$\Sigma^{\mu\nu} = \frac{1}{2}\gamma^{\mu\nu} = \frac{1}{4}\begin{pmatrix} \sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu & 0 \\ 0 & \bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu \end{pmatrix}$$

This exercise shows explicitly that the Dirac representation of $\mathfrak{so}(D-1,1)$, which is generated by $\Sigma^{\mu\nu}$, is reducible (for even D). The matrices of the upper and lower blocks in (3.37) are generators of two subrepresentations, which are inequivalent and irreducible.

(Indeed they are related to the two fundamental spinor representations of D_m denoted by Dynkin integers $(0,0,...,1,0)$ and $(0,0,...,0,1)$.)

Exercise 3.14 Show that all requirements are satisfied by generalized Weyl matrices in which the spatial matrices are $\sigma^i = \bar\sigma^i$, where the σ^i are hermitian generators of the Clifford algebra in odd dimension $2m-1$ Euclidean space, and the time matrices are $\sigma^0 = -\bar\sigma^0 = \mathbb{1}$. Thus the form of the Weyl matrices in $D = 2m$ dimensions is the same as in $D = 4$.

It is frequently useful to note that the Weyl fields ψ and χ can be obtained from a Dirac Ψ field by applying the chiral projectors

$$P_L = \frac{1}{2}\left(\mathbb{1} + \gamma_*\right), \quad P_R = \frac{1}{2}\left(\mathbb{1} - \gamma_*\right).$$

Thus

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} \equiv P_L\Psi, \quad \begin{pmatrix} 0 \\ \bar\chi \end{pmatrix} \equiv P_R\Psi$$

The specific Weyl representation (3.36) will rarely be used in the rest of this book. However, we will use the projectors P_L and P_R to define the chiral parts of Dirac (and Majorana) spinors in a general representation of the γ-matrices.

Exercise 3.15 Show that the matrices (3.38) project to orthogonal subspaces, i.e. $P_L P_L = P_L, P_R P_R = P_R$ and $P_L P_R = 0$. No specific choice of the Clifford algebra representation is needed.

8.1.7 7 Odd spacetime dimension $D = 2m + 1$

The basic idea that we need is that the Clifford algebra for dimension $D = 2m + 1$ can be obtained by reorganizing the matrices in the Clifford algebra for dimension $D = 2$ m. In particular we can define two sets of $2m + 1$ generating elements by adjoining the highest rank γ_* as follows:

$$\gamma^\mu_\pm = \left(\gamma^0, \gamma^1, ..., \gamma^{(2m-1)}, \gamma^{2m} = \pm\gamma_*\right)$$

This gives us two sets of matrices, which each satisfy (2.18) for dimension $D = 2m+1$. The two sets $\{\gamma^\mu_\pm\}$ are not equivalent, but they lead to equivalent representations of the Lorentz group; see Appendix 3A.3.

The main difference with the case of even dimensions is that the matrices in the list (3.25) are not all independent and are thus an over-complete set. Indeed, the highest element of that list, which is the product of all γ-matrices, is, due to (3.40), a phase factor times the unit matrix. More generally, the rank r and rank $D - r$ sectors are related by the duality relations

$$\gamma_\pm^{\mu_1\cdots\mu_r} = \pm\mathrm{i}^{m+1}\frac{1}{(D-r)!}\varepsilon^{\mu_1\cdots\mu_D}\gamma_{\pm\mu_D\cdots\mu_{r+1}}$$

Note that the order of the indices in the γ-matrix on the right-hand side is reversed. Otherwise there would be different sign factors.

Exercise 3.16 Prove the relation (3.41) and the analogous but different relation for even dimension:

$$\gamma^{\mu_1\mu_2\ldots\mu_r}\gamma_* = -(-\mathrm{i})^{m+1}\frac{1}{(D-r)!}\varepsilon^{\mu_r\mu_{r-1}\ldots\mu_1 v_1 v_2\ldots v_{D-r}}\gamma_{\nu_1 v_2\ldots\nu_{D-r}}$$

You can use the tricks explained in Sec. 3.1.4. Show that in four dimensions

$$\gamma_{\mu\nu\rho} = \mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\gamma^{\sigma}\gamma_*.$$

Thus, a basis of the Clifford algebra in $D=2m+1$ dimensions contains the matrices in (3.25) only up to rank m. This agrees with the counting argument in Ex. 3.9. For example, the set $\{\mathbb{1},\gamma^{\mu},\gamma^{\mu\nu}\}$ of $1+5+10=16$ matrices is a basis of the Clifford algebra for $D=5$. Ex. 3.16 shows that it is a rearrangement of the basis $\left\{\Gamma^{A}\right\}$ for $D=4$.

8.1.8 8 Symmetries of γ-matrices

In the Clifford algebra of the $2^m \times 2^m$ matrices, for both $D=2m$ and $D=2m+1$, one can distinguish between the symmetric and the antisymmetric matrices where the symmetry property is defined in the following way. There exists a unitary matrix, C, called the charge conjugation matrix, such that each matrix $C\Gamma^{A}$ is either symmetric or antisymmetric. Symmetry depends only on the rank r of the matrix Γ^{A}, so we can write:

$$\left(C\Gamma^{(r)}\right)^{T} = -t_r C\Gamma^{(r)}, \quad t_r = \pm 1$$

where $\Gamma^{(r)}$ is a matrix in the set (3.25) of rank r. (The $-$ sign in (3.44) is convenient for later manipulations.) For rank $r=0$ and 1, one obtains from (3.44)

$$C^{T} = -t_0 C, \quad \gamma^{\mu T} = t_0 t_1 C\gamma^{\mu}C^{-1}$$

These relations suffice to determine the symmetries of all $C\gamma^{\mu_1\cdots\mu_r}$ and thus all coefficients t_r: e.g. $t_2=-t_0$ and $t_3=-t_1$. Further, $t_{r+4}=t_r$.

Exercise 3.17 A formal proof of the existence of C can be found in [10, 11], but you can check that the following two matrices satisfy (3.45) for even D. They are given in the product representation of (3.2): [1]

$$\begin{aligned} C_+ &= \sigma_1\otimes\sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\ldots, \quad t_0t_1=1 \\ C_- &= \sigma_2\otimes\sigma_1\otimes\sigma_2\otimes\sigma_1\otimes\ldots, \quad t_0t_1=-1 \end{aligned}$$

The values of t_0 and t_1 (thus all t_r) depend on the spacetime dimension D modulo 8 and on the rank r modulo 4, and are given in Table (3.1). The entries in the table are determined by counting the number of independent symmetric and antisymmetric matrices in every dimension; see Appendix 3A.4. An exercise for the simple case $D=5$ follows below. For even dimension both $C_+$and $C_-$are possible choices. One can go from one to

1 We consider here only the Minkowski signature of spacetime. A full treatment is given in [12], for which you should set $\epsilon=t_0$ and $\eta=-t_0t_1$.

$D(\mathrm{mod} 8)$	$t_r = -1$	$t_r = +1$
0	0, 3 **0**, **1**	2, 1 2, 3
1	0, 1	2, 3
2	0, 1 **1**, **2**	2, 3 **0**, **3**
3	1,2	0, 3
4	1, 2 2, 3	**0**, **3** 0, 1
5	2, 3	0, 1
6	2, 3 **0**, 3	0, 1 **1**, **2**
7	0, 3	1,2

the other by replacing the charge conjugation matrix C by $C\gamma_*$ (up to a normalizing phase factor). For applications in supersymmetry we need the choice indicated in bold face. For odd dimension, C is unique (again up to a phase factor).

Exercise 3.18 Check that in five dimensions, where the Clifford algebra basis contains only matrices of rank 0,1 and 2, the numbers in the table are fixed by counting the number of matrices of each rank. The count must conform to the requirement that there are 10 symmetric and 6 antisymmetric matrices in a basis of 4×4 matrices.

Since we use hermitian representations, which satisfy (3.3), the symmetry property of a γ-matrix determines also its complex conjugation property. To see this, we define the unitary matrix

$$B = \mathrm{i} t_0 C \gamma^0$$

Exercise 3.19 Derive

$$\gamma^{\mu *} = -t_0 t_1 B \gamma^\mu B^{-1}$$

Exercise 3.20 Prove that $B^* B = -t_1 \mathbb{1}$.

Exercise 3.21 Show that, in the Weyl representation (2.19), one can choose $B = \gamma^0\gamma^1\gamma^3$, which is real, symmetric, and satisfies $B^2 = \mathbb{1}$. Then $C = \mathrm{i}\gamma^3\gamma^1$.

The properties (3.45) and (3.48) hold for the representation (3.2) using the matrices (3.46) and (3.47). In another representation, related by (3.4), the C and B matrices are given by

$$C' = S^{-1T} C S^{-1}, \quad B' = S^{-1T} B S^{-1}.$$

Since symmetries of spinor bilinears are important for supersymmetry, we use the Majorana conjugate to define $\bar\lambda$.

8.2 Spinors in general dimensions

In Ch. 2 we used complex spinors. We defined the Dirac adjoint (2.30), which involves the complex conjugate spinor, and used it to obtain a Lorentz invariant bilinear form. In this section we start rather differently. We define the 'Majorana conjugate' of any spinor λ using its transpose and the charge conjugation matrix,

$$\bar\lambda \equiv \lambda^T C$$

The bilinear form $\bar\lambda\chi$ is Lorentz invariant as readers will show in Ex. 3.23 below. It is appropriate to use (3.50) in supersymmetry and supergravity in which the symmetry properties of γ-matrices and of spinor bilinears are very important and these properties are determined by C. For Majorana spinors, to be defined in Sec. 3.3, the definitions (3.50) and (2.30) are equivalent.

Unless otherwise stated, we assume in this book that spinor components are anticommuting Grassmann numbers. This reflects the important physical relation between spin and statistics.

8.2.1 Spinors and spinor bilinears

Using the definition (3.50) and the property (3.44), we obtain

$$\bar{\lambda}\gamma_{\mu_1\ldots\mu_r}\chi = t_r\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\lambda$$

The minus sign obtained by changing the order of Grassmann valued spinor components has been incorporated. The symmetry property (3.51) is valid for Dirac spinors, but its main application for us will be to Majorana spinors. For this reason we use the term 'Majorana flip relations' to refer to (3.51).

We now give some further relations that are useful for spinor manipulations. In fact, the same sign factors can be used for a longer chain of Clifford matrices:

$$\bar{\lambda}\Gamma^{(r_1)}\Gamma^{(r_2)}\cdots\Gamma^{(r_p)}\chi = t_0^{p-1}t_{r_1}t_{r_2}\cdots t_{r_p}\bar{\chi}\Gamma^{(r_p)}\cdots\Gamma^{(r_2)}\Gamma^{(r_1)}\lambda$$

where $\Gamma^{(r)}$ stands for any rank r matrix $\gamma_{\mu_1\ldots\mu_r}$. Note that the prefactor t_0^{p-1} is not relevant in four dimensions, where $t_0 = 1$.

Exercise 3.22 One often encounters the special case that the bilinear contains the product of individual γ^μ-matrices. Prove that for the Majorana dimensions $D = 2, 3, 4 \bmod 8$,

$$\bar{\lambda}\gamma^{\mu_1}\gamma^{\mu_2}\cdots\gamma^{\mu_p}\chi = (-)^p\bar{\chi}\gamma^{\mu_p}\cdots\gamma^{\mu_2}\gamma^{\mu_1}\lambda.$$

The previous relations imply also the following rule. For any relation between spinors that includes γ-matrices, there is a corresponding relation between the barred spinors,

$$\chi_{\mu_1\ldots\mu_r} = \gamma_{\mu_1\ldots\mu_r}\lambda \Longrightarrow \bar{\chi}_{\mu_1\ldots\mu_r} = t_0t_r\bar{\lambda}\gamma_{\mu_1\ldots\mu_r}$$

and similar for longer chains,

$$\chi = \Gamma^{(r_1)}\Gamma^{(r_2)}\cdots\Gamma^{(r_p)}\lambda \Longrightarrow \bar{\chi} = t_0^p t_{r_1}t_{r_2}\cdots t_{r_p}\bar{\lambda}\Gamma^{(r_p)}\cdots\Gamma^{(r_2)}\Gamma^{(r_1)}.$$

In even dimensions we define left-handed and right-handed parts of spinors using the projection matrices (3.38). The definition (3.50) implies that the chirality of the conjugate spinor depends on t_0t_D, and we obtain [2]

$$\chi = P_L\lambda \to \bar{\chi} = \begin{cases} \bar{\lambda}P_L, & \text{for } D = 0, 4, 8, \ldots \\ \bar{\lambda}P_R, & \text{for } D = 2, 6, 10, \ldots \end{cases}$$

Exercise 3.23 Using the 'spin part' of the infinitesimal Lorentz transformation (2.25),

$$\delta\chi = -\frac{1}{4}\lambda^{\mu\nu}\gamma_{\mu\nu}\chi$$

prove that the spinor bilinear $\bar{\lambda}\chi$ is a Lorentz scalar.

8.2.2 Spinor indices

For most of this book we do not need spinor indices because they appear contracted within Lorentz covariant expressions. However, in some cases indices are necessary, for example, to write (anti-)commutation relations of supersymmetry generators. The components of the basic spinor λ are indicated as λ_α. The components of the barred spinor defined in (3.50) are indicated with upper indices: λ^α. Sometimes we write $\bar{\lambda}^\alpha$ to stress that these are the components of the barred spinor, but in fact the bar can be omitted. We introduce the raising matrix $\mathcal{C}^{\alpha\beta}$ such that

$$\lambda^\alpha = \mathcal{C}^{\alpha\beta}\lambda_\beta$$

Comparing with (3.50) we see that $\mathcal{C}^{\alpha\beta}$ are the components of the matrix C^T. Note that the summation index β in (3.58) appears in a northwest-southeast (NW-SE) line in the

2 The definition (2.30) would always lead to $\bar{\chi} = \bar{\lambda} P_R$. equation when adjacent indices are contracted. Therefore, this convention is frequently called the NW-SE spinor convention. This is relevant when the raising matrix is antisymmetric ($t_0 = 1$ in the terminology of Table 3.1). Most applications in the book are for dimensions in which this is the case, e.g. $D = 2, 3, 4, 5, 10, 11$.

We also introduce a lowering matrix such that (again NW-SE contraction)

$$\lambda_\alpha = \lambda^\beta \mathcal{C}_{\beta\alpha}$$

In order for these two equations to be consistent, we must require

$$\mathcal{C}^{\alpha\beta}\mathcal{C}_{\gamma\beta} = \delta_\gamma{}^\alpha, \quad \mathcal{C}_{\beta\alpha}\mathcal{C}^{\beta\gamma} = \delta_\alpha{}^\gamma$$

Hence $\mathcal{C}_{\alpha\beta}$ are the components of C^{-1}, and the unitarity of C implies then $(\mathcal{C}_{\alpha\beta})^* = \mathcal{C}^{\alpha\beta}$.

When we write a covariant spinor bilinear with components explicitly indicated, the γ-matrices are written as $(\gamma_\mu)_\alpha{}^\beta$. For example, for the simplest case,

$$\bar{\chi}\gamma_\mu\lambda = \chi^\alpha \left(\gamma_\mu\right)_\alpha^\beta \lambda_\beta$$

where again all contractions are NW-SE.

One can now raise or lower indices consistently. For example, one can define

$$\left(\gamma_\mu\right)_{\alpha\beta} = \left(\gamma_\mu\right)_\alpha^\gamma \mathcal{C}_{\gamma\beta}$$

These γ-matrices with indices at the 'same level' have a definite symmetry or antisymmetry property, which follows from (3.44):

$$\left(\gamma_{\mu_1\ldots\mu_r}\right)_{\alpha\beta} = -t_r \left(\gamma_{\mu_1\ldots\mu_r}\right)_{\beta\alpha}.$$

An interesting property is that

$$\lambda^\alpha\chi_\alpha = -t_0\lambda_\alpha\chi^\alpha$$

Thus, in four dimensions, raising and lowering a contracted index produces a minus sign. The same property can be used when the contracted indices involved are on γ-matrices, e.g. $\gamma_{\mu\alpha}^\beta\gamma_{\nu\beta}^\gamma = -t_0\gamma_{\mu\alpha\beta}\gamma_v^{\beta\gamma}$.

Exercise 3.24 Using this property and (3.63) prove the relation (3.52). Do not forget the sign due to interchange of two (anti-commuting) spinors.

Exercise 3.25 Show that, using the index raising and lowering conventions, $\mathcal{C}_\alpha{}^\beta = \delta_\alpha^\beta$, and for $D = 4$ that $\mathcal{C}^\alpha{}_\beta = -\delta_\beta^\alpha$.

8.2.3 Fierz rearrangement

In this subsection we study an important consequence of the completeness of the Clifford algebra basis $\{\Gamma^A\}$ in (3.25). As we saw in Ex. 3.7 completeness means that any matrix M has a unique expansion in the basis with coefficients obtained using trace orthogonality. The expansion was derived for even $D = 2m$ in Ex. 3.7, but it is also valid for odd $D = 2m + 1$ provided that the sum is restricted to rank $r \leq m$. We saw at the end of Sec. 3.1.7 that the list of (3.25) is complete for odd D when so restricted. The rearrangement properties we derive using completeness are frequently needed in supergravity. These involve changing the pairing of spinors in products of spinor bilinears, which is called a 'Fierz rearrangement'.

Let's proceed to derive the basic Fierz identity. Using spinor indices, we can regard the quantity $\delta_\alpha{}^\beta \delta_\gamma{}^\delta$ as a matrix in the indices $\gamma\beta$ with the indices $\alpha\delta$ as inert 'spectators'. We apply (3.28) in the detailed form $\delta_\alpha^\beta \delta_\gamma^\delta = \sum_A (m_A)_\alpha^\delta (\Gamma_A)_\gamma^\beta$. The coefficients are $(m_A)_\alpha^\delta = 2^{-m}\delta_\alpha^\beta \delta_\gamma{}^\delta (\Gamma_A)_\beta^\gamma = 2^{-m} (\Gamma_A)_\alpha{}^\delta$. Therefore, we obtain the basic rearrangement lemma

$$\delta_\alpha^\beta \delta_\gamma^\delta = \frac{1}{2^m} \sum_A (\Gamma_A)_\alpha^\delta \left(\Gamma^A\right)_\gamma^\beta$$

Note that the 'column indices' on the left- and right-hand sides have been exchanged.

Exercise 3.26 Derive the following result:

$$(\gamma^\mu)_\alpha^\beta (\gamma_\mu)_\gamma^\delta = \frac{1}{2^m} \sum_A v_A (\Gamma_A)_\alpha^\delta \left(\Gamma^A\right)_\gamma^\beta$$

and prove that the explicit values of the expansion coefficients are given by $v_A = (-)^{r_A} (D - 2r_A)$, where r_A is the tensor rank of the Clifford basis element Γ_A.

Exercise 3.27 Lower the β and δ index in the result of the previous exercise and consider the completely symmetric part in $(\beta\gamma\delta)$. The left-hand side is only non-vanishing for dimensions in which $t_1 = -1$. Consider the right-hand side and use Table 3.1 and the result for v_A to prove that for $D = 3$ and $D = 4$ only rank 1γ-matrices contribute to the right-hand side. For $D = 4$ you have to use the bold face row in the table to arrive at this result. You can also check that there are no other dimensions where this occurs.

The previous exercise implies that, for $D = 3$ and $D = 4$,

$$(\gamma_\mu)_{\alpha(\beta} (\gamma^\mu)_{\gamma\delta)} = 0$$

This is called the cyclic identity and is important in the context of string and brane actions. It can be extended to some other dimensions under further restrictions. [3] Multiplying the equations with three spinors $\lambda_1^\beta, \lambda_2^\gamma$ and λ_3^γ, equation (3.67) can be written as

$$\gamma_\mu \lambda_{[1} \bar{\lambda}_2 \gamma^\mu \lambda_{3]} = 0$$

3 For $D = 2$ and $D = 10$ this equation holds when contracted with chiral spinors. Owing to (3.56) only odd rank γ-matrices then occur in the sum over A. This is sufficient to extend the result (3.67) to these cases. With the same restrictions of chirality there is for $D = 6$ an analogous identity for the completely antisymmetric part in $[\beta\gamma\delta]$. where the symmetry of the indices in (3.67) is transformed to an antisymmetry between the three spinors due to the anti-commutating nature of spinors. This result is important to construct supersymmetric Yang-Mills theories; see Sec. 6.3.

The following application of Fierz rearrangement is valid for any set of four anticommuting spinor fields. The basic Fierz identity (3.65) immediately gives

$$\bar{\lambda}_1\lambda_2\bar{\lambda}_3\lambda_4 = -\frac{1}{2^m}\sum_A \bar{\lambda}_1\Gamma^A\lambda_4\bar{\lambda}_3\Gamma_A\lambda_2$$

This can be generalized to include general matrices M, M' of the Clifford algebra.

Exercise 3.28 Show that

$$\begin{aligned}\bar{\lambda}_1 M\lambda_2\bar{\lambda}_3 M'\lambda_4 &= -\frac{1}{2^m}\sum_A \bar{\lambda}_1 M\Gamma_A M'\lambda_4\bar{\lambda}_3\Gamma^A\lambda_2\\ &= -\frac{1}{2^m}\sum_A \bar{\lambda}_1\Gamma_A M'\lambda_4\bar{\lambda}_3\Gamma^A M\lambda_2.\end{aligned}$$

When $\lambda_{1,2,3,4}$ are not all independent, it is frequently the case that some terms in the rearranged sum vanish due to symmetry relations such as (3.51).

One can write the Fierz relation (3.65) in the alternative form:

$$M = 2^{-m}\sum_{k=0}^{[D]}\frac{1}{k!}\Gamma_{\mu_1\ldots\mu_k}\operatorname{Tr}\left(\Gamma^{\mu_k\cdots\mu_1}M\right)$$

where

$$\begin{cases}[D] = D, & \text{for even } D\\ [D] = (D-1)/2, & \text{for odd } D\end{cases}$$

The factor $1/k$! compensates for the fact that in the sum over $\mu_1 ... \mu_k$ each matrix of the basis appears k ! times.

Exercise 3.29 Prove the following chiral Fierz identities for $D = 4$:

$$\begin{aligned}P_L\chi\bar{\lambda}P_L &= -\frac{1}{2}P_L\left(\bar{\lambda}P_L\chi\right) + \frac{1}{8}P_L\gamma^{\mu\nu}\left(\bar{\lambda}\gamma_{\mu\nu}P_L\chi\right)\\ P_L\chi\bar{\lambda}P_R &= -\frac{1}{2}P_L\gamma^{\mu}\left(\bar{\lambda}\gamma_{\mu}P_L\chi\right).\end{aligned}$$

You will need (3.42) to combine terms in (3.71).

Exercise 3.30 Prove that for $D = 5$ the matrix $\chi\bar{\lambda} - \lambda\bar{\chi}$ can be written as

$$\chi\bar{\lambda} - \lambda\bar{\chi} = \gamma_{\mu\nu}\left(\bar{\lambda}\gamma^{\mu v}\chi\right)$$

Readers who understand the Majorana flip properties (3.51) and Fierz rearrangement are well equipped for supersymmetry and supergravity!

Complex conjugation can be replaced by charge conjugation, an operation that acts as complex conjugation on scalars, and has a simple action on fermion bilinears. For example, it preserves the order of spinor factors.

8.2.4 Reality

In this chapter, we have not yet discussed the complex conjugation of the spinor fields we are working with. Complex conjugation is necessary for such purposes as the verification that a term in the Lagrangian involving spinor bilinears is hermitian. But the complex conjugation of a bilinear [4] is an awkward operation since the hermiticity of both the C matrix in (3.50) and γ-matrices is involved. Therefore we present a related operation called charge conjugation which is much simpler in practice. For any scalar, defined here as a quantity whose spinor indices are fully contracted, charge conjugation and complex conjugation are the same. Since the Lagrangian is a scalar, charge conjugation can be used to manipulate the terms it contains.

First we define the charge conjugate of any spinor as

$$\lambda^C \equiv B^{-1}\lambda^*.$$

The barred charge conjugate spinor is then, using (3.50) and (3.47),

$$\overline{\lambda^C} = (-t_0 t_1)\, \mathrm{i}\lambda^\dagger \gamma^0$$

Note that this is the Dirac conjugate as defined in (2.30) except for the numerical factor $(-t_0 t_1)$. The meaning of this will become clear below when we discuss Majorana spinors. Note that $(-t_0 t_1) = +1$ in $2, 3, 4, 10$ or 11 dimensions. [5]

The charge conjugate of any $2^m \times 2^m$ matrix M is defined as

$$M^C \equiv B^{-1} M^* B$$

Charge conjugation does not change the order of matrices: $(MN)^C = M^C N^C$. In practice the matrices M we deal with are products of γ-matrices. Hence, we need only the charge conjugation property of the generating γ-matrices, which is

$$(\gamma_\mu)^C \equiv B^{-1}\gamma_\mu^* B = (-t_0 t_1)\, \gamma_\mu$$

Exercise 3.31 Start from (3.77) (and note that charge conjugation on any number is just complex conjugation). Prove that

$$(\gamma_*)^C = (-)^{D/2+1}\gamma_*.$$

4 We use the convention that the order of fermion fields is reversed in the process of complex conjugation. See (A.16) with $\beta = 1$.

[5] For these dimensions the spinor bilinears of Chs. 2 and 3 are related by $(\bar\lambda\chi)_{\text{Ch. 2}} = \left(\bar\lambda^C\chi\right)_{\text{Ch. 3}}$.

With these ingredients, we can derive the following rule for complex conjugation of a spinor bilinear involving an arbitrary matrix M:

$$(\bar\chi M\lambda)^* \equiv (\bar\chi M\lambda)^C = (-t_0 t_1)\, \overline{\chi^C} M^C \lambda^C$$

A 'hidden' interchange of the order of the fermion fields is needed in the derivation, but there is no change of order in the final result for the charge conjugate of any bilinear. One may think of this relation as the appropriate conjugation property when the conjugate of a spinor is defined as in (3.50).

Exercise 3.32 It is important that any spinor λ and its conjugate λ^C transform in the same way under a Lorentz transformation. Prove this using (3.57) and the rules above. If the matrix M is a Clifford element of rank r, i.e. $M = \gamma_{\mu_1\ldots\mu_r}$, then both sides of (3.79) transform as tensors of rank r.

Exercise 3.33 Show that for any spinor $\left(\lambda^C\right)^C = -t_1\lambda$, and for any matrix $\left(M^C\right)^C = M$.

Exercise 3.34 Suppose that $\psi(x)$ is a fermion field which satisfies the free massive Dirac equation $\not\partial\psi = m\psi$ for $D = 4$. Show that the charge conjugate field ψ^C satisfies the same equation. This exercise gives some physical motivation for the definition of Majorana spinors in the next section.

8.2.5 Majorana spinors

The concept of supersymmetry is closely tied to the relativistic treatment of particle spin. Indeed the transformation parameters are spinors ϵ_α. It is reasonable to suppose that the simplest supersymmetric field theories in each spacetime dimension D are based on the simplest spinors that are compatible with invariance under the Lorentz group $\mathrm{SO}(D-1,1)$. In even dimension $D = 2m$ we already know that Weyl fields, rather than Dirac fields, transform irreducibly under Lorentz transformations. Weyl fields were first discussed in Sec. 2.6. They have 2^{m-1} complex components while a Dirac field has 2^m complex components. Weyl fields can be obtained by applying the chiral projector P_L or P_R to a Dirac field.

In this section we introduce Majorana fields, which are Dirac fields that satisfy an additional 'reality condition'. This condition reduces the number of independent components by a factor of 2. Thus, like Weyl fields, a Majorana spinor field has half the degrees of freedom and can be viewed as more fundamental than a complex Dirac field. Physically the properties of particles described by a Majorana field are similar to Dirac particles, except that particles and anti-particles are identical. The spin states of massive and massless Majorana spinors transform in representations of $\mathrm{SO}(D-1)$ and $\mathrm{SO}(D-2)$, respectively.

8.2.6 Definition and properties

The results of Sec. 3.2.4 suggest that it might be possible to impose the reality constraint

$$\psi = \psi^C = B^{-1}\psi^*, \qquad \text{i.e.} \qquad \psi^* = B\psi$$

on a spinor field. Ex. 3.32 shows that both sides transform in the same way under Lorentz transformations in any dimension D, so the constraint is compatible with Lorentz symmetry. In fact (3.80) is the defining condition for Majorana spinors. However, there is a subtle and important consistency condition that we now derive, which restricts the spacetime dimension in which Majorana spinors can exist. It is easy to see that the reality condition (3.80) is not automatically consistent. Take the complex conjugate of the second form of the condition and use it again to obtain $\psi = B^*B\psi$. Thus the reality condition is mathematically consistent only if $B^*B = \mathbb{1}$. Using Ex. 3.20, we see that this requires ${}^6t_1 = -1$.

The two possible values $t_0 = \pm 1$ must be considered, and we begin with the case $t_0 = +1$. Consulting Table 3.1, we see that $t_0 = +1$ holds for spacetime dimension $D = 2, 3, 4$, mod 8. In this case we call the spinors that satisfy (3.80) Majorana spinors. It is clear from (3.75) that if $t_0 = 1$ and $t_1 = -1$, the barred (3.50) and Dirac adjoint spinors (2.30) agree for Majorana spinors. In fact, this gives an alternative definition of a Majorana spinor.

Another fact about the Majorana case is that there are representations of the γ-matrices that are explicitly real and may be called really real representations. Here is a really real representation for $D = 4$:

$$\begin{aligned}
\gamma^0 &= \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix} = \mathrm{i}\sigma_2 \otimes \mathbb{1}, \\
\gamma^1 &= \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} = \sigma_3 \otimes \mathbb{1}, \\
\gamma^2 &= \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_1, \\
\gamma^3 &= \begin{pmatrix} 0 & \sigma_3 \\ \sigma_3 & 0 \end{pmatrix} = \sigma_1 \otimes \sigma_3.
\end{aligned}$$

Note that the γ^i are symmetric, while γ^0 is antisymmetric. This is required by hermiticity in any real representation. We construct really real representations in all allowed dimensions $D = 2, 3, 4$ mod 8 in Appendix 3A.5.

In such representations (3.48) implies that $B = \mathbb{1}$ (up to a phase). The relation (3.47) then gives $C = \mathrm{i}\gamma^0$. Further, a Majorana spinor field is really real since (3.80) reduces to $\Psi^* = \Psi$.

Really real representations are sometimes convenient, but we emphasize that the physics of Majorana spinors is the same in, and can be explored in, any representation of the Clifford algebra, replacing complex conjugation with charge conjugation. For convenience we

6 This manipulation is the same as working out Ex. 3.33, and this thus leads to the same result. often write 'complex conjugation' when in fact we use 'charge conjugation'. For example, the complex conjugate of $\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi$, where χ and ψ are Majorana, is computed as follows. We follow Sec. 3.2.4 and write

$$(\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi)^* = (\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi)^C = \bar{\chi}\,(\gamma_{\mu_1\ldots\mu_r})^C\,\psi = \bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi$$

We used the reality conditions $\psi^C = \psi$ and $\bar{\chi}^C = \bar{\chi}^C$ as well as (3.77) to deduce this result. [7] Hence, bilinears such as $\bar{\chi}\psi$ and $\bar{\chi}\gamma_{\mu_1\ldots\mu_r}\psi$ are real.

When $t_0 = -1$ (and still $t_1 = -1$) spinors that satisfy (3.80) are called pseudo-Majorana spinors. They are mostly relevant for $D = 8$ or 9. There are no really real representations in these dimensions; instead there are representations of the Clifford algebra in which the generating γ-matrices are imaginary, $(\gamma^\mu)^* = -\gamma^\mu$. In any representation (3.79) and (3.77) hold with $t_0 = t_1 = -1$. This implies that the reality properties of bilinears are different from those of Majorana spinors. Although these differences are significant, the essential property that a complex spinor can be reduced to a real one still holds, and it is common in the literature not to distinguish between Majorana and pseudo-Majorana spinors. However, note the following.

Exercise 3.35 Show that the mass term $m\bar{\chi}\chi = 0$ for a single pseudo-Majorana field. Pseudo-Majorana spinors must be massless (unless paired).

We now consider (pseudo-)Majorana spinors in even dimensions $D = 0, 2, 4$ mod 8. We can quickly show using (3.78) that these cases are somewhat different. For $D = 2$mod 8 we have $(\gamma_*\psi)^C = \gamma_*\psi^C$. Thus the two constraints

$$\text{Majorana:} \quad \psi^C = \psi, \qquad \text{Weyl:} \quad P_{L,R}\psi = \psi,$$

are compatible. It is equivalent to observe that the chiral projections of a Majorana spinor ψ satisfy

$$(P_L\psi)^C = P_L\psi, \quad (P_R\psi)^C = P_R\psi$$

Thus the chiral projections of a Majorana spinor are also Majorana spinors. Each chiral projection satisfies both constraints in (3.83) and is called a Majorana-Weyl spinor. Such spinors have 2^{m-1} independent 'real' components in dimension $D = 2m = 2$ mod 8 and are the 'most fundamental' spinors available in these dimensions. It is not surprising that supergravity and superstring theories in $D = 10$ dimensions are based on Majorana-Weyl spinors.

For $D = 4$ mod 4 dimensions we have $(\gamma_*\psi)^C = -\gamma_*\psi^C$, so that the equations of (3.84) are replaced by

$$(P_L\psi)^C = P_R\psi, \quad (P_R\psi)^C = P_L\psi$$

7 Notice that Majorana spinors, which are real in the sense of C-conjugation, are not real for the original complex conjugation, not even in the really real representation. In fact, $\bar{\chi}$ is purely imaginary in the really real representation. However, under complex conjugation we should interchange the order of the spinors, which leads to another - sign, compensating the - sign of complex conjugation of $\bar{\chi}$. Neither sign appears explicitly when one uses charge conjugation, independent of the γ-matrix representation. This illustrates how the use of C simplifies the reality considerations.

These equations state that the 'left' and 'right' components of a Majorana spinor are related by charge conjugation. A direct consequence is that, for any expression involving the lefthanded projection $P_L\psi$ of a Majorana spinor ψ, the corresponding expression for $P_R\psi$ follows by complex conjugation. Of course there are Weyl spinors that are chiral projections $P_{L,R}\psi$ of a Dirac spinor ψ, but these cannot satisfy the Majorana condition since for Majorana fermions $(P_{L,R}\psi)^C = P_{R,L}\psi$.

8.2.7 Symplectic Majorana spinors

When $t_1 = 1$ we cannot define Majorana spinors, but we can define 'symplectic Majorana spinors'. These consist of an even number of spinors χ^i, with $i = 1, ..., 2k$, which satisfy a 'reality condition' containing a non-singular antisymmetric matrix ε^{ij}. The inverse matrix ε_{ij} satisfies $\varepsilon^{ij}\varepsilon_{kj} = \delta^i_k$. Symplectic Majorana spinors satisfy the condition

$$\chi^i = \varepsilon^{ij}\left(\chi^j\right)^C = \varepsilon^{ij}B^{-1}\left(\chi^j\right)^*$$

The consistency check discussed after (3.80) now works for $t_1 = 1$ because of the antisymmetric ε^{ij}.

Exercise 3.36 Check that, in five dimensions with symplectic Majorana spinors, $\bar{\psi}^i\chi_i \equiv \bar{\psi}^i\chi^j\varepsilon_{ji}$ is pure imaginary while $\bar{\psi}^i\gamma_\mu\chi_i$ is real.

For dimensions $D = 6 \bmod 8$, one can use (3.78) to show that the symplectic Majorana constraint is compatible with chirality. We can therefore define the symplectic MajoranaWeyl spinors $P_L\chi^i$ or $P_R\chi^i$.

8.2.8 Dimensions of minimal spinors

The various types of spinors we have discussed are linked to the signs of t_0 and t_1 as follows:

$$\begin{array}{lll} t_1 = -1, & t_0 = 1: & \text{Majorana,} \\ & t_0 = -1: & \text{pseudo-Majorana ,} \\ t_1 = 1: & & \text{symplectic Majorana .} \end{array}$$

As explained above we no longer distinguish between Majorana and pseudo-Majorana spinors. In any even dimension one can define Weyl spinors, while in dimensions $D = 2 \bmod 4$, one can combine the (symplectic) Majorana condition and Weyl conditions. These facts are summarized in Table 3.2. [8] For each spacetime dimension it is indicated whether Majorana (M), Majorana-Weyl (MW), symplectic (S) or symplectic Weyl (SW) spinors can be defined as the 'minimal spinor'. The number of components of this minimal spinor is given. The table is for Minkowski signature and has a periodicity of 8 in dimension. When D is changed to $D+8$, the number of spinor components is multiplied by 16. The

[8] For $D = 4 \bmod 4$ we can also define Weyl spinors, but we omit this in the table.

Table 3.2			Irreducible spinors, number of components and symmetry properties.
dim	spinor	min # components	antisymmetric
2	MW	1	1
3	M	2	1,2
4	M	4	1,2
5	S	8	2,3
6	SW	8	3
7	S	16	0,3
8	M	16	0,1
9	M	16	0,1
10	MW	16	1
11	M	32	1,2

final column indicates the ranks of the antisymmetric spinor bilinears, e.g. a 0 indicates that $\bar{\epsilon}_2\epsilon_1 = -\bar{\epsilon}_1\epsilon_2$, and a 2 indicates that $\bar{\epsilon}_2\gamma_{\mu\nu}\epsilon_1 = -\bar{\epsilon}_1\gamma_{\mu\nu}\epsilon_2$. This entry is modulo 4, i.e. if rank 0 is antisymmetric, then so are rank 4 and 8 bilinears. Minimal spinors in dimension $D = 2 \bmod 4$ must have the same chirality to define a symmetry for their bilinears. The property (3.56) then implies that non-vanishing bilinears contain an odd number of γ matrices. For $D = 4 \bmod 4$, there are two possibilities for reality conditions and we have chosen the one that includes rank 1 in the column 'antisymmetric'. This property is needed for the supersymmetry algebra.

8.2.9 Majorana spinors in physical theories

8.2.10 Variation of a Majorana Lagrangian

In this section we consider a prototype action for a Majorana spinor field in dimension $D = 2, 3, 4 \bmod 8$. Majorana and Dirac fields transform the same way under Lorentz transformations, but Majorana spinors have half as many degrees of freedom, so we write

$$S[\Psi] = -\frac{1}{2}\int \mathrm{d}^D x \bar{\Psi}\left[\gamma^\mu\partial_\mu - m\right]\Psi(x)$$

There is an immediate and curious subtlety due to the symmetries of the matrices C and $C\gamma^\mu$. Using (3.50), we see that the mass and kinetic terms are proportional to $\Psi^T C\Psi$ and $\Psi^T C\gamma^\mu\partial_\mu\Psi$. Suppose that the field components Ψ are conventional commuting numbers. Since C is antisymmetric, the mass term vanishes. Since $C\gamma^\mu$ is symmetric, the kinetic term is a total derivative and thus vanishes when integrated in the action. For commuting field components, there is no dynamics! To restore the dynamics we must assume that Majorana fields are anti-commuting Grassmann variables, which we always assume unless stated otherwise.

Let's derive the Euler-Lagrange equation for Ψ. Field variations must satisfy the Majorana condition (3.80), so that $\delta\Psi$ and $\delta\bar{\Psi}$ are related following Sec. 3.2.1. Initially $\delta S[\Psi]$ contains two terms. However, after a Majorana flip and partial integration, one can see that the two terms are equal, so that $\delta S[\Psi]$ can be written as the single expression

$$\delta S[\Psi] = -\int \mathrm{d}^D x \delta\bar{\Psi}\left[\gamma^\mu\partial_\mu - m\right]\Psi(x)$$

Thus a Majorana field satisfies the conventional Dirac equation.

This fact is no surprise, but it is an example of a more general and simplifying rule for the variation of Majorana spinor actions. If integration by parts is valid, it is sufficient to vary $\bar{\Psi}$ and multiply by 2 to account for the variation of Ψ.

Exercise 3.37 Derive (3.89) in full detail.

Exercise 3.38 A Majorana field is simply a Dirac field subject to the reality condition (3.80). Let's impose that constraint on the plane wave expansion (2.24) for $D = 4$ using the relation $v = u^C = Bu^*$, which holds for the u and v spinors defined in (2.37) and (2.38). In this way one derives $d(\vec{p}, s) = c(\vec{p}, s)$ which proves that a Majorana particle is its own anti-particle. Readers should derive this fact!

Exercise 3.39 Show that

$$v(\vec{p}, s) = u(\vec{p}, s)^C$$

holds for the u and v spinors defined for the Weyl representation in Sec. 2.5. This was the motivation for the choice (2.41).

8.2.11 Relation of Majorana and Weyl spinor theories

In even dimensions $D = 0, 2, 4 \bmod 8$, both Majorana and Weyl fields exist and both have legitimate claims to be more fundamental than a Dirac fermion. In fact both fields describe equivalent physics. Let's show this for $D = 4$. We can rewrite the action (3.88) as

$$\begin{aligned} S[\psi] &= -\frac{1}{2}\int \mathrm{d}^4x \left[\bar{\Psi}\gamma^\mu\partial_\mu - m\right](P_L + P_R)\Psi \\ &= -\int \mathrm{d}^4x \left[\bar{\Psi}\gamma^\mu\partial_\mu P_L\Psi - \frac{1}{2}m\bar{\Psi}P_L\Psi - \frac{1}{2}m\bar{\Psi}P_R\Psi\right] \end{aligned}$$

We obtained the second line by a Majorana flip and partial integration. In the second form of the action, the Majorana field is replaced by its chiral projections. In our treatment of chiral multiplets in supersymmetry, we will exercise the option to write Majorana fermion actions in this way.

Exercise 3.40 Show that the Euler-Lagrange equations that follow from the variation of the second form of the action in (3.91) are

$$\not{\partial} P_L\Psi = mP_R\Psi, \quad \not{\partial} P_R\Psi = mP_L\Psi.$$

Derive $\Box P_{L,R}\Psi = m^2 P_{L,R}\Psi$ from the equations above.

8.2.12 Majorana and Weyl fields in $D = 4$

Any field theory of a Majorana spinor field Ψ can be rewritten in terms of a Weyl field $P_L\Psi$ and its complex conjugate. Conversely, any theory involving the chiral field $\chi = P_L\chi$ and its conjugate $\chi^C = P_R\chi^C$ can be rephrased as a Majorana equation if one defines the Majorana field $\Psi = P_L\chi + P_R\chi^C$. Supersymmetry theories in $D = 4$ are formulated in both descriptions in the physics literature.

Let's return to the Weyl representation (2.19) for the final step in the argument to show that the equation of motion for a Majorana field can be reexpressed in terms of a Weyl field and its adjoint. The Majorana condition $\Psi = B^{-1}\Psi^* = \gamma^0\gamma^1\gamma^3\Psi^*$ requires that Ψ take the form

$$\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_2^* \\ -\psi_1^* \end{pmatrix}$$

With (3.93) and (2.55) in view we define the two-component Weyl fields

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}, \quad \tilde{\psi} = \begin{pmatrix} \psi_2^* \\ -\psi_1^* \end{pmatrix}$$

Using the form of γ^μ (2.19) and γ_* (3.34) in the Weyl representation, we see that we can identify

$$\begin{pmatrix} \psi \\ 0 \end{pmatrix} = P_L\Psi, \quad \begin{pmatrix} 0 \\ \tilde{\psi} \end{pmatrix} = (P_L\Psi)^C = P_R\Psi$$

The equations of motion (3.92) can then be rewritten as

$$\bar{\sigma}^\mu\partial_\mu\psi = m\tilde{\psi}, \quad \sigma^\mu\partial_\mu\tilde{\psi} = m\psi.$$

These are equivalent to the pair of Weyl equations in (2.56) with the restriction $\tilde{\psi} = \bar{\chi}$ which comes because we started in this section with a Majorana rather than a Dirac field.

3.4.3 U(1) symmetries of a Majorana field

In Sec. 2.7.1 we considered the U(1) symmetry operation $\Psi \to \Psi' = \mathrm{e}^{\mathrm{i}\theta}\Psi$. This symmetry is obviously incompatible with the Majorana condition (3.80). Thus the simplest internal symmetry of a Dirac fermion cannot be defined in a field theory of a (single) Majorana field. However, it is easy to see that $(\mathrm{i}\gamma_*)^C = \mathrm{i}\gamma_*$, so the chiral transformation $\Psi \to \Psi' = \mathrm{e}^{\mathrm{i}\gamma_*\theta}\Psi$ preserves the Majorana condition. Let's ask whether the infinitesimal limit of this transformation is a symmetry of the free massive Majorana action (3.88).

Exercise 3.41 Use $\delta\bar{\Psi} = \mathrm{i}\theta\bar{\Psi}\gamma_*$ and partial integration to derive the variation

$$\delta S[\Psi] = \mathrm{i}\theta m \int \mathrm{d}^4x \bar{\Psi}\gamma_*\Psi$$

which vanishes only for a massless Majorana field.

Thus we have learned the following.

- The conventional vector U(1) symmetry is incompatible with the Majorana condition.
- The axial transformation above is compatible and is a symmetry of the action for a massless Majorana field only.

Exercise 3.42 Show that the axial current

$$J_*^\mu = \frac{1}{2}\mathrm{i}\bar{\Psi}\gamma^\mu\gamma_*\Psi$$

is the Noether current for the chiral symmetry defined above. Use the equations of motion to show that

$$\partial_\mu J_*^\mu = -\mathrm{i}m\bar{\Psi}\gamma_*\Psi$$

The current is conserved only for massless Majorana fermions.

The dynamics of a Majorana field Ψ can be expressed in terms of its chiral projections $P_{L,R}\Psi$. So can the chiral transformation, which becomes $P_{L,R}\Psi \to P_{L,R}\Psi' = \mathrm{e}^{\pm\mathrm{i}\theta}\Psi$.

Throughout this section we used the simple dynamics of a free massive fermion to illustrate the relation between Majorana and Weyl fields and to explore their U(1) symmetries. It is straightforward to extend these ideas to interacting field theories with nonlinear equations of motion.

8.2.13 Appendix 3A Details of the Clifford algebras for $D = 2m$

8.2.14 A.1 Traces and the basis of the Clifford algebra

Let us start with the following facts discussed in Sec. 3.1. The Clifford algebra in even dimension $D = 2m$ has a basis of 2^m linearly independent, trace orthogonal matrices, given in (3.25). Any representation by matrices of dimension 2^m is irreducible.

The trace properties of the matrices are important for proofs of these properties which are independent of the explicit construction in (3.2). The matrices Γ^A for tensor rank $1 \leq r \leq D-1$ are traceless. One simple way to see this is to use the Lorentz transformations (2.22) and its extension to general rank

$$L(\lambda)\gamma^{\mu_1\mu_2\dots\mu_r}L(\lambda)^{-1} = \gamma^{v_1v_2\dots v_r}\Lambda_{\nu_1}^{\mu_1}\dots\Lambda_{v_r}^{\mu_r}$$

Traces then satisfy the Lorentz transformation law as suggested by their free indices:

$$\operatorname{Tr}\gamma^{\mu_1\mu_2\cdots\mu_r} = \operatorname{Tr}\gamma^{\nu_1\nu_2\cdots v_r}\Lambda^{\mu_1}_{\nu_1}\ldots\Lambda_{v_r}{}^{\mu_r}$$

This means that the traces must be totally antisymmetric Lorentz invariant tensors. However the only invariant tensors available are the Minkowski metric $\eta^{\mu v}$ and the Levi-Civita tensor $\varepsilon^{\mu_1\mu_2\cdots\mu_D}$ introduced in Sec. 3.1.3. No totally antisymmetric tensor can be formed from products of $\eta^{\mu\nu}$. This proves that $\operatorname{Tr}\Gamma^A = 0$ for all elements of rank $1 \leq r \leq D-1$.

The argument does not apply to the highest rank element. However, one can see from the pattern of alternation in (3.7) that this is given by a commutator for even $D = 2$ m and by an anti-commutator for odd $D = 2m+1$. Thus the trace of the highest rank element vanishes for $D = 2m$ but need not (and does not) vanish for $D = 2m+1$. This is actually a fundamental distinction between the Clifford algebras for even and odd dimensions. It might have been expected since the second rank elements (see Ex. 2.8) give a representation of the Lorentz algebras $\mathfrak{so}(D-1,1)$ which are real forms of different Lie algebras in the Cartan classification, namely D_m for even $D = 2m$ and B_m for $D = 2m+1$.

There is another way to prove the traceless property, which does not require information concerning invariant tensors. For rank 1, we simply take the trace of the formula derived in Ex. 2.9. Contraction with $\eta_{\nu\rho}$ immediately gives $\operatorname{Tr}\gamma^\mu = 0$. As an exercise, the reader can extend this argument to higher rank.

The trace property leads also to the proof of independence of the elements of the basis (3.25) for even spacetime dimensions. One uses the 'reverse order' basis of (3.26) and the trace orthogonality property (3.27). We suppose that there is a set of coefficients x_A such that

$$\sum_A x_A\Gamma^A = 0$$

Multiply by Γ_B from the right. Take the trace and use the trace orthogonality to obtain

$$\sum_A x_A \operatorname{Tr}\Gamma^A\Gamma^B = \pm x_B \operatorname{Tr}\mathbb{1} = 0$$

Hence all $x_A = 0$ and linear independence is proven.

Furthermore, since we have a linearly independent, indeed trace orthogonal, basis of the algebra, the Γ^A are a complete set in the space of $2^m \times 2^m$ matrices.

It now follows that, in any representation of the Clifford algebra for $D = 2m$ spacetime dimensions, the dimension of the $N \times N$ matrices satisfies $N \geq 2^m$. The reason is that no linearly independent set of matrices of any smaller dimension exists. It also follows that any representation of dimension 2^m is irreducible. It can have no non-trivial invariant subspace, since a set of linearly independent matrices of smaller dimension would be realized by projection to this subspace.

8.2.15 A.2 Uniqueness of the γ-matrix representation

We must now show that there is exactly one irreducible representation up to equivalence. We use the basic properties of representations of finite groups. However, the Clifford algebra is not quite a group because the minus signs that necessarily occur in the set of products $\Gamma^A\Gamma^B = \pm\Gamma^C$ are not allowed by the definition of a group. This problem is solved by doubling the basis in (3.25) to the larger set $\left\{\Gamma^A, -\Gamma^A\right\}$. This set is a group of order 2^{2m+1} since all products are contained within the larger set. For $m = 1$, the group obtained is isomorphic to the quaternions, so the groups defined by doubling the Clifford algebras are called generalized quaternionic groups.

Every representation of the Clifford algebra by a set of matrices $D\left(\Gamma^A\right)$ extends to a representation of the group if we define $D\left(-\Gamma^A\right) = -D\left(\Gamma^A\right)$. It is not true that every group representation gives a representation of the algebra. For example, in a onedimensional group representation, the matrices $D\left(\gamma^\mu\right)$ of the Clifford generators cannot satisfy $\left\{D\left(\gamma^\mu\right), D\left(\gamma^\nu\right)\right\} = 2\eta^{\mu\nu}$.

The three basic facts that we need are discussed in many mathematical texts such as [13, 14]. Consider the set of all finite-dimensional irreducible representations and choose one representative within each class of equivalent representations. The set so formed, which may be called the set of all inequivalent irreducible representations, has the following properties:

1. The sum of the squares of the dimensions of these representations is equal to the order of the group.

2. The number of inequivalent irreducible representations is equal to the number of conjugacy classes in the group.

3. The number of inequivalent one-dimensional representations is equal to the index of the commutator subgroup G_c. The index of a subgroup is the ratio of the order of the group divided by the order of the subgroup.

The conjugacy classes of the group are sets of products $\pm\Gamma^B\Gamma^A\left(\Gamma^B\right)^{-1}$ (with no sum on B).

Exercise 3.43 Show that for rank $r \geq 1$ there is a conjugacy class containing the pair $\left(\Gamma^A, -\Gamma^A\right)$ for each distinct Γ^A, and that $\mathbb{1}$ and $-\mathbb{1}$ belong to different conjugacy classes.

Thus there are a total of $2^{2m} + 1$ conjugacy classes.

The commutator subgroup is generated by all products of the form $\pm\Gamma^B\Gamma^A\left(\Gamma^B\right)^{-1}\left(\Gamma^A\right)^{-1}$. But in our case this subgroup contains only $\pm\mathbb{1}$, so the order of the subgroup is 2 and its index is 2^{2m}.

These facts establish that the group has exactly one irreducible representation of dimension 2^m plus 2^{2m} inequivalent one-dimensional representations. We must now show that the 2^m-dimensional representation of the group is also a representation of the algebra. We use the fact that any finite-dimensional algebra has a (reducible) representation called the regular representation for which the algebra itself is the carrier space. The dimension is thus the dimension of the algebra, 2^{2m} in our case. The regular representation $\Gamma^A \to T\left(\Gamma^A\right)$ is defined by $T\left(\Gamma^A\right)\Gamma^B \equiv \Gamma^A\Gamma^B$. This algebra representation, in which $T\left(-\Gamma^A\right) = -T\left(\Gamma^A\right)$ is necessarily satisfied, is also a group representation. Its decomposition into irreducible components thus cannot contain any one-dimensional group representations in which $D\left(-\Gamma^A\right) = +D\left(\Gamma^A\right)$. Thus the only possibility is that the regular representation decomposes into 2^m copies of the 2^m-dimensional irreducible representation. This proves the essential fact that there is exactly one irreducible representation of the Clifford algebra for even spacetime dimension. For dimension $D = 2m$, the dimension of the Clifford representation is 2^m.

Another fact from finite group theory is helpful at this point. Any representation of a finite group is equivalent to a representation by unitary matrices. We can and therefore will choose a representation in which the spatial γ-matrices $\gamma^i, i = 1, ..., D-1$, which satisfy $\left(\gamma^i\right)^2 = \mathbb{1}$, are hermitian, and γ^0, which satisfies $\left(\gamma^0\right)^2 = -\mathbb{1}$, is anti-hermitian.

8.2.16 A.3 The Clifford algebra for odd spacetime dimensions

We gave in (3.40) two different sets of γ-matrices for odd dimensions. They are inequivalent as representations of the generating elements. Indeed it is easily seen that $S{\gamma_+}^\mu S^{-1} = {\gamma_-}^\mu$ cannot be satisfied. This requires $S\gamma^\mu S^{-1} = \gamma^\mu$ for the first $2m$ components. But then, from the product form in (3.6) and (3.30), we obtain $S\gamma^{2m}S^{-1} = +\gamma^{2m}$, rather than the opposite sign needed.

It follows from Ex. 2.8 that the two sets of second rank elements constructed from the generating elements above, namely

$$\begin{aligned}\Sigma_{\pm}\mu\nu &= \frac{1}{4}\left[\gamma^{\mu},\gamma^{\nu}\right], \quad \mu, v=0,...,2m-1,\\ &= \frac{1}{4}\left[\gamma^{\mu},\pm\gamma_{*}\right], \quad \mu=0,...,2m-1, \quad v=2m\end{aligned}$$

are each representations of the Lie algebra $\mathfrak{so}(2m,1)$. The two representations are equivalent, however, since $\gamma_{*}\Sigma_{+}{}^{\mu v}\gamma_{*}=\Sigma_{-}{}^{\mu\nu}$. This representation is irreducible; indeed it is a copy of the unique 2^{2m}-dimensional fundamental irreducible representation with Dynkin designation $(0,0,...,0,1)$. It is associated with the short simple root of the Dynkin diagram for B_m.

We refer readers to [9][10, 11, 12, 15, 16] for alternative discussions of γ-matrices and Majorana spinors.

8.2.17 A.4 Determination of symmetries of γ-matrices

We will determine the possible symmetries of γ-matrices for each spacetime dimension $D=2m$ by showing that each matrix $C\Gamma_A$ formed from the basis (3.25) has a definite symmetry that depends only on the tensor rank r. Then we will count the number of symmetric and antisymmetric matrices in the list $\{C\Gamma_A\}$, which must be equal to $2^{m-1}\left(2^m\pm 1\right)$ for $D=2$ m. For given values of t_0 and t_1, the number of antisymmetric matrices in the list $\{C\Gamma_A\}$ is given, using (3.44), by

$$\begin{aligned}N_{-} &= \sum_{r=0}^{2m}\frac{1}{2}\left[1+t_r\right]C_r^{2m}\\ &= 2^{2m-1}+\frac{1}{2}t_0\sum_{s=0}^{m}(-)^s C_{2s}^{2m}+\frac{1}{2}t_1\sum_{s=0}^{m-1}(-)^s C_{2s+1}^{2m}\\ &= 2^{2m-1}+t_0 2^{m-1}\cos\frac{m\pi}{2}+t_1 2^{m-1}\sin\frac{m\pi}{2}\\ &= 2^{m-1}\left(2^m-1\right).\end{aligned}$$

We thus find

$$t_0\cos\frac{m\pi}{2}+t_1 2^{m-1}\sin\frac{m\pi}{2}=-1$$

9 In [15], the discussion of Majorana spinors is in Sec. 4, pp. 843-851. which leads to the solutions that are in Table 3.1 for even dimensions.

To understand the situation in odd $D=2m+1$ we note that the highest rank Clifford element γ_{*} in (3.30) has the symmetry determined by t_{2m}. Since we attach $\pm\gamma_{*}=\gamma^{2m}$ as the last generating element in (3.40) we must require it to have the same symmetry as the other generating γ^{μ}, and thus t_{2m} should be equal to t_1. This determines which of the two possibilities for even dimensions in Table 3.1 is valid in the next odd dimension.

8.2.18 A.5 Friendly representations

8.2.19 General construction

In this section we present an explicit recursive construction of the generating γ^{μ} for any even dimension $D=2m$. In this representation each generating matrix will be either pure real or pure imaginary. A representation of this type will be called a friendly representation. [10] Using this representation it is also possible to prove the existence of Majorana (and pseudo-Majorana) spinors in a quite simple way [17, 12] (see Appendix B in [17]).

We already know that the γ-matrices in dimension $D = 2m$ are $2^m \times 2^m$ matrices. In the recursive construction the generating matrices γ^μ for dimension $D = 2m$ will be written as direct products of the $\tilde{\gamma}^\mu$ and $\tilde{\gamma}_*$ for dimension $D = 2m - 2$ with the Pauli matrices σ_i.

We start in $D = 2$ and write

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathrm{i}\sigma_2, \quad \gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma_1$$

which is a really real, hermitian, and friendly representation. The matrix γ_* is also real:

$$\gamma_* = -\gamma_0\gamma_1 = \sigma_3.$$

Adding it to (3.107) as γ^2 gives a real representation in $D = 3$.

The recursion relation for moving from a $D = 2m - 2$ representation with $\tilde{\gamma}$ to $D = 2m$ is

$$\begin{aligned} \gamma^\mu &= \tilde{\gamma}^\mu \otimes \mathbb{1}, \quad \mu = 0, ..., 2m-3, \\ \gamma^{2m-2} &= \tilde{\gamma}_* \otimes \sigma_1, \quad \gamma^{2m-1} = \tilde{\gamma}_* \otimes \sigma_3. \end{aligned}$$

This gives

$$\gamma_* = -\tilde{\gamma}_* \otimes \sigma_2.$$

This matrix γ_* can be used as γ^{2m} to define a representation in $D = 2m + 1$ dimensions.

This construction gives a real representation in four dimensions, which is explicitly given in (3.81). This one has an imaginary γ_* and hence this construction will not give real representations for higher dimensions. The matrix B is obtained as the product of all the imaginary γ-matrices.

We thus obtained representations for all dimensions, and really real for $D = 2, 3, 4$. The latter can be extended to any $D = 10, 11, 12$ or any other dimension that differs from it modulo 8. To see this, consider the following 16×16 matrices:

$$\begin{aligned} E_1 &= \sigma_1 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\ E_2 &= \sigma_3 \otimes \mathbb{1} \otimes \mathbb{1} \otimes \mathbb{1}, \\ E_3 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_1 \otimes \mathbb{1}, \\ E_4 &= \sigma_2 \otimes \sigma_2 \otimes \sigma_3 \otimes \mathbb{1}, \\ E_5 &= \sigma_2 \otimes \sigma_1 \otimes \mathbb{1} \otimes \sigma_2, \\ E_6 &= \sigma_2 \otimes \sigma_3 \otimes \mathbb{1} \otimes \sigma_2, \\ E_7 &= \sigma_2 \otimes \mathbb{1} \otimes \sigma_2 \otimes \sigma_1, \\ E_8 &= \sigma_2 \otimes \mathbb{1} \otimes \sigma_2 \otimes \sigma_3, \\ E_* &= E_1 ... E_8 = \sigma_2 \otimes \sigma_2 \otimes \sigma_2 \otimes \sigma_2. \end{aligned}$$

This is a real representation for Euclidean γ-matrices in $D = 8$ (or $D = 9$ if one includes E_*). Using this and a representation $\tilde{\gamma}^\mu$ in any D, one can construct a representation γ^μ in $D + 8$ dimensions by

$$\begin{aligned} \gamma^\mu &= \tilde{\gamma}^\mu \otimes E_*, \quad \mu = 0, ..., D-1, \\ \gamma^{D-1+i} &= \mathbb{1} \otimes E_i, \quad i = 1, ..., 8. \end{aligned}$$

10 All of our friends use friendly representations.

When the $\tilde{\gamma}^\mu$ are real, the γ-matrices in $D+8$ are also real. Hence this gives explicitly real representations in $D = 2, 3, 4 \bmod 8$. For even dimensions, one obtains

$$\gamma_* = \tilde{\gamma}_* \otimes E_*.$$

Hence this is real if $\tilde{\gamma}_*$ is real. For the real representations we saw that it is real for $D = 2$ and not in $D = 4$. This shows explicitly that we can define real projections P_L and P_R on real spinors if and only if $D = 2 \bmod 8$. These are called Majorana-Weyl representations.

Exercise 3.44 We denote a Clifford algebra in s space-like and t time-like directions as $\mathcal{C}(s,t)$ (the ones discussed above are thus of the form $\mathcal{C}(D-1,1)$, apart from the E_i that correspond to $\mathcal{C}(8,0)$). See that the above construction proves that the reality properties of $\mathcal{C}(s+8,t)$ are the same as $\mathcal{C}(s,t)$. Further, show that the analogous construction starting with (3.107) shows that also $\mathcal{C}(s+1,t+1)$ has the same properties as $\mathcal{C}(s,t)$.

8.3 The Maxwell and Yang-Mills gauge fields

In this chapter we discuss the classical abelian and non-abelian gauge fields. Although our treatment is self-contained, it is best taken as a review for readers who have previously studied the role of the vector potential as the gauge field in Maxwell's electromagnetism and also have some acquaintance with Yang-Mills theory.

We will again take a general dimensional viewpoint, but let's begin the discussion in four dimensions with some remarks about the particle representations of the Poincare group and the fields usually used to describe elementary particles. A particle is classified by its mass m and spin s, and a massive particle of spin s has $2s+1$ helicity states. Massless particles of spin $s = 0$ or $s = 1/2$ have one or two helicity states, respectively, in agreement with the counting for massive particles. However, massless particles of spin $s \geq 1/2$ have two helicity states, for all values of s.

Helicity is defined as the eigenvalue of the component of angular momentum in the direction of motion. For a massless particle of spin s, the two helicity states have eigenvalues $\pm s$. For a massive particle of spin s the helicity eigenvalues, called λ, range in integer steps from $\lambda = s$ to $\lambda = -s$.

Let us compare the count of the helicity states with the number of independent functions that must be specified as initial data for the Cauchy initial value problem of the associated field. The first number can be considered to be the number of on-shell degrees of freedom, or number of quantum degrees of freedom, while the second is the number of classical degrees of freedom.

Let's do the counting for massless particles that are identified with their anti-particles. The associated fields are real for bosons and satisfy the Majorana condition for fermions. The counting is similar for complex fields. We assume that the equations of motion are second order in time for bosons and first order for fermions. A unique solution of the Cauchy problem for the scalar $\phi(x)$ requires the initial data $\phi(\vec{x},0)$ and $\dot{\phi}(\vec{x},0)$, the time derivative. For $\Psi_\alpha(x)$, we must specify the initial values $\Psi_\alpha(\vec{x},0)$ of all four components, and the first order Dirac equation then determines the future evolution of $\Psi_\alpha(\vec{x},t)$ and thus the time derivatives $\dot{\Psi}_\alpha(\vec{x},0)$. The number of helicity states (number of on-shell degrees of freedom) is 1 for $\phi(x)$ and 2 for $\Psi_\alpha(x)$. The number of classical degrees of freedom is twice the number of helicity states.

We continue this counting, in a naive fashion, for vector $A_\mu(x)$, vector-spinor $\psi_{\mu\alpha}(x)$, and symmetric tensor $h_{\mu\nu}(x)$ fields, the latter describing gravitons in Minkowski space. Following the earlier pattern we would expect to need 8, 16, and 20 functions, respectively, as initial data. These numbers greatly exceed the two helicity states for spin-1, spin-3/2 and spin-2 particles. Something new is required to resolve this mismatch.

The lessons from quantum electrodynamics, Yang-Mills theory, general relativity and supergravity teach us that the only way to proceed is to use very special field equations with gauge invariance. Gauge invariance accomplishes the following goals:

(a) Relativistic covariance is maintained.

(b) The field equations do not determine certain 'longitudinal' field components (such as $\partial^\mu A_\mu$ for vector fields).

(c) A subset of the field equations are constraints on the initial data rather than time evolution equations. The independent initial data are contained in four real functions, thus again two for each helicity state.

(d) The field describes a pure spin-s particle with no lower spin admixtures. Otherwise there would be some negative metric ghosts.

(e) Most important, for $s = 1, 3/2, 2$, gauge invariant interactions can be introduced. [1] Classical dynamics is consistent at the nonlinear level and the theories can be quantized (although power-counting renormalizability is expected to fail except for spin 1).

The dynamics of the gauge fields $A_\mu, \psi_{\mu\alpha}$ and $h_{\mu\nu}$ is analyzed in this and subsequent chapters. In every case the purpose is to separate the Euler-Lagrange equations into time evolution equations and constraints and determine the initial data required for a unique solution of the former. In the process we will find that certain field components are harmonic functions in Minkowski space; they satisfy the Laplace equation $\nabla^2\phi(\vec{x}) = 0$, which is time independent. Any combination of gauge field components that satisfies this equation is simply eliminated because the Laplace equation has no normalizable solutions in flat space $\mathbb{R}^{D-1}$. The relevance of the normalizability criterion can be seen by transforming the Laplace equation to momentum space where it becomes $\vec{k}^2\hat{\phi}(\vec{k}) = 0$. The only smooth solution vanishes identically. A smooth solution is one that contains no δ-function-type terms.

8.3.1 The abelian gauge field $A_\mu(x)$

We now review the elementary features of gauge invariance for spin 1. One purpose is to set the stage for spin 3/2 in the next chapter.

8.3.2 Gauge invariance and fields with electric charge

In Chs. 1 and 2 we discussed the global U(1) symmetry of complex scalar and spinor fields. The abelian gauge symmetry of quantum electrodynamics is an extension in which the phase parameter $\theta(x)$ becomes an arbitrary function in Minkowski spacetime. We generalize the previous discussion slightly and assign an electric charge q, an arbitrary real

1 There are gauge invariant free fields for massless particles of any spin (see [18], for example). It appears to be impossible to introduce consistent interactions for any finite subset of these, but remarkably one can make progress for certain infinite sets of fields and for background spacetimes different from Minkowski space [19, 20, 21]. number at this stage, to each complex field in the system. For a Dirac spinor field of charge q, the gauge transformation, a local change of the phase of the complex field, is [2]

$$\Psi(x) \rightarrow \Psi'(x) \equiv \mathrm{e}^{\mathrm{i}q\theta(x)}\Psi(x).$$

The goal is to formulate field equations that transform covariantly under the gauge transformation. This requires the introduction of a new field, the vector gauge field or vector potential $A_\mu(x)$, which is defined to transform as

$$A_\mu(x) \rightarrow A'_\mu(x) \equiv A_\mu(x) + \partial_\mu\theta(x)$$

One then defines the covariant derivative

$$D_\mu \Psi(x) \equiv (\partial_\mu - \mathrm{i}qA_\mu(x))\,\Psi(x)$$

which transforms with the same phase factor as $\Psi(x)$, namely $D_\mu\Psi(x) \to \mathrm{e}^{\mathrm{i}q\theta(x)}D_\mu\Psi(x)$. The desired field equation is obtained by replacing $\partial_\mu\Psi \to D_\mu\Psi$ in the free Dirac equation (2.16), viz.

$$[\gamma^\mu D_\mu - m]\,\Psi \equiv [\gamma^\mu\,(\partial_\mu - \mathrm{i}qA_\mu) - m]\,\Psi = 0.$$

This equation is gauge covariant; if $\Psi(x)$ satisfies (4.4) with gauge potential $A_\mu(x)$, then $\Psi'(x)$ satisfies the same equation with gauge potential $A'_\mu(x)$.

The same procedure can be applied to a complex scalar field $\phi(x)$, to which we assign an electric charge q (which may differ from the charge of Ψ). We extend the global U(1) symmetry discussed in Ch. 1 to the local gauge symmetry $\phi(x) \to \phi'(x) = \mathrm{e}^{\mathrm{i}q\theta(x)}\phi(x)$ by defining the covariant derivative $D_\mu\phi = (\partial_\mu - \mathrm{i}qA_\mu)\,\phi$ and modifying the Klein-Gordon equation to the form

$$\left[D^\mu D_\mu - m^2\right]\phi = 0.$$

The procedure of promoting the global U(1) symmetry of the Dirac or Klein-Gordon equation to a local or gauge symmetry through the introduction of the vector potential in the covariant derivative is called the principle of minimal coupling. Another part of standard vocabulary is to say that fields with electric charge, such as ϕ or Ψ, are charged 'matter fields', which are minimally coupled to the gauge field A_μ.

On-shell degrees of freedom = number of helicity states.

Off-shell degrees of freedom = number of field components - gauge transformations.

2 In the notation of Ch. 1, the 'matrix' generator is $t = -\mathrm{i}q$. The U(1) transformation in Sec. 2.7.1 corresponds to the choice $q = 1$.

8.3.3 The free gauge field

It is quite remarkable that the promotion of global to gauge symmetry requires a new field $A_\mu(x)$. In some cases one may wish to consider (4.4) or (4.5) in a fixed external background gauge potential, but it is far more interesting to think of $A_\mu(x)$ as a field that is itself determined dynamically by its coupling to charged matter in a gauge invariant fashion. The resulting theory is quantum electrodynamics, the quantum and Lorentz covariant version of Maxwell's theory of electromagnetism. The predictions of this theory, both classical and quantum, are well confirmed by experiment. There can be no doubt that Nature knows about gauge principles.

Although we expect that readers are familiar with classical electromagnetism, we review the construction because there are similar patterns in Yang-Mills theory, gravity, and supergravity. The first step is the observation that the antisymmetric derivative of the gauge potential, called the field strength

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

is invariant under the gauge transformation, a fact that is trivial to verify. In four dimensions $F_{\mu\nu}$ has six components, which split into the electric $E_i = F_{i0}$ and magnetic $B_i = \frac{1}{2}\varepsilon_{ijk}F_{jk}$ fields.

Since A_μ is a bosonic field, we expect it to satisfy a second order wave equation. The only Lorentz covariant and gauge invariant quantity available is $\partial^\mu F_{\mu\nu}$, so the free electromagnetic field satisfies

$$\partial^\mu F_{\mu\nu} = 0$$

Since $\partial^\nu \partial^\mu F_{\mu\nu}$ vanishes identically, [3] (4.7) comprises $D-1$ independent components in D-dimensional Minkowski spacetime. This is not enough to determine the D components of A_μ, which is not surprising because of the gauge symmetry. One can change $A_\mu \rightarrow A_\mu + \partial_\mu \theta$ without affecting $F_{\mu\nu}$. So far, we did not yet use the field equations. Therefore, we will call this number, i.e. $(D-1)$ for the gauge vectors, the number of off-shell degrees of freedom.

One deals with this situation by 'fixing the gauge'. This means that one imposes one condition on the D components of A_μ, which eliminates the freedom to change gauge. Different gauge conditions illuminate different physical features of the theory. We will look first at the condition $\partial^i A_i(\vec{x}, t) = 0$, which is called the Coulomb gauge condition. Although non-covariant it is a useful gauge to explore the structure of the initial value problem and determine the true degrees of freedom of the system. Note that the time-space split implicit in the initial value problem is also non-covariant.

First let's show that this condition does eliminate the gauge freedom. We check whether there are gauge functions $\theta(x)$ with the property that $\partial^i A_i' = \partial^i\left(A_i + \partial_i \theta\right) = 0$ when $\partial^i A_i = 0$. This requires that $\nabla^2 \theta = 0$. As explained above, the only smooth solution is $\theta(x) \equiv 0$, so the gauge freedom has been completely fixed.

3 This is the 'Noether identity', a relation between the field equations that is a consequence of the gauge symmetry.

Let's write out the time $(\mu \rightarrow 0)$ and space $(\mu \rightarrow i)$ components of the Maxwell equation (4.7). Using (4.6) and lowering all indices with the Minkowski metric, one finds

$$\begin{aligned} \nabla^2 A_0 - \partial_0\left(\partial^i A_i\right) &= 0, \\ \square A_i - \partial_i \partial^0 A_0 - \partial_i\left(\partial^j A_j\right) &= 0. \end{aligned}$$

In the Coulomb gauge, the first equation simplifies to $\nabla^2 A_0 = 0$, and we see that [4] A_0 vanishes. The second equation in (4.8) then becomes $\square A_i = 0$, so the spatial components A_i satisfy the massless scalar wave equation.

We can now count the classical degrees of freedom, which are the initial data $A_i(\vec{x}, 0)$ and $\dot{A}_i(\vec{x}, 0)$ required for a unique solution of $\square A_i = 0$. There is a total of $2(D-2)$ independent degrees of freedom, because the initial data must be constrained to obey the Coulomb gauge condition.

This number thus agrees for $D=4$ with the rule that the classical degrees of freedom are twice the number of on-shell degrees of freedom counted as helicity states. In general, we find for the gauge vectors $(D-1)$ off-shell degrees of freedom and $(D-2)$ on-shell degrees of freedom. These numbers are the dimension of the vector representation of $\mathrm{SO}(D-1)$ off-shell and $\mathrm{SO}(D-2)$ on-shell.

It is instructive to write the solution of $\square A_i = 0$ as the Fourier transform

$$A_i(x) = \int \frac{\mathrm{d}^{(D-1)} k}{(2\pi)^{(D-1)} 2k^0} \sum_\lambda \left[\mathrm{e}^{\mathrm{i}k\cdot x} \epsilon_i(\vec{k}, \lambda) a(\vec{k}, \lambda) + \mathrm{e}^{-\mathrm{i}k\cdot x} \epsilon_i^*(\vec{k}, \lambda) a^*(\vec{k}, \lambda)\right]$$

where $\vec{k}, k^0 = |\vec{k}|$, is the on-shell energy-momentum vector. The $\epsilon_i(\vec{k}, \lambda)$ are called polarization vectors, which are constrained by the Coulomb gauge condition to satisfy $k^i \epsilon_i(\vec{k}, \lambda) = 0$. So there are $(D-2)$ independent polarization vectors, indexed by λ, and there are $2(D-2)$ independent real degrees of freedom contained in the complex quantities $a(\vec{k}, \lambda)$. As in the case of the plane wave expansions of the free Klein-Gordon and Dirac fields, $a(\vec{k}, \lambda)$ and $a^*(\vec{k}, \lambda)$ are interpreted as Fourier amplitudes in the classical theory and as annihilation and creation operators for particle states after quantization. There are $D-2$ particle states.

To understand these particle states better, we discuss the case $D=4$ and assume that the spatial momentum is in the 3 -direction, i.e. $\vec{k} = (0,0,k)$ with $k > 0$. The two polarization vectors may be taken to be $\epsilon_i((0,0,k), \pm) = (1/\sqrt{2})(1, \pm \mathrm{i}, 0)$. We formally add the 0 -component

$\epsilon_0 = 0$ to form 4-vectors $\epsilon_\mu((0,0,k),\pm)$, which are eigenvectors of the rotation generator $J_3 = -m_{[12]}$, about the 3 -axis (see text above (1.93)), with angular momentum $\lambda = \pm 1$. For general spatial momentum $\vec{k} = k(\sin\beta\cos\alpha, \sin\beta\sin\alpha, \cos\beta)$, we define polarization vectors $\epsilon_\mu(\vec{k},\pm)$ by applying the spatial rotation with Euler angles α, β, which rotates the 3 -axis to the direction of $\vec{k}$. The associated particle states are photons with helicity \pm 1.

The same ideas determine the properties of particle states of the gauge field for $D \geq 5$. For spatial momentum in the direction $D-1$, i.e. $\vec{k} = (0,0,\dots,k)$, there are $D-2$ independent polarization vectors. We need not specify these in detail; the important point to

4 When a source current is added to the Maxwell equation (4.7), A_0 no longer vanishes, but it is determined by the source. Thus it is not a degree of freedom of the gauge field system. note is that these vectors are a basis of the vector representation of the Lie group $\mathrm{SO}(D-2)$, which is the group that 'fixes' the vector $\vec{k}$. The associated particle states also transform in this representation. On the other hand it is clear from (4.9) that the Coulomb gauge vector potential transforms in the vector representation of $\mathrm{SO}(D-1)$.

It should be noted that the equations of the free electromagnetic field can be formulated as conditions involving only the field strength components $F_{\mu\nu}$, with the gauge potential A_μ appearing as a derived quantity. In this form of the theory one has the pair of equations

$$\partial^\mu F_{\mu\nu} = 0$$
$$\partial_\mu F_{\nu\rho} + \partial_\nu F_{\rho\mu} + \partial_\rho F_{\mu\nu} = 0.$$

The second equation is called the Bianchi identity. It is easy to see that it is automatically satisfied if $F_{\mu\nu}$ is expressed in terms of A_μ as in (4.6). In a topologically trivial spacetime such as Minkowski space, this is the general solution. This is a consequence of the Poincaré lemma in the theory of differential forms discussed in Ch. 7. Although the manifestly gauge invariant formalism (4.10) and (4.11) is available for the free gauge field, the vector potential is required *ab* initio to describe the minimal coupling to charged matter fields.

This chapter has progressed too far without exercises for readers, so we must now try to remedy this deficiency.

Exercise 4.1 Derive from (4.3) that

$$[D_\mu, D_\nu]\,\Psi \equiv (D_\mu D_\nu - D_\nu D_\mu)\,\Psi = -\mathrm{i} q F_{\mu\nu}\Psi$$

Derive from (4.4) that the charged Dirac field also satisfies the second order equation

$$\left[D^\mu D_\mu - \frac{1}{2}\mathrm{i} q\gamma^{\mu\nu}F_{\mu\nu} - m^2\right]\Psi = 0$$

Exercise 4.2 Using only (4.10) and (4.11), show that the field strength tensor satisfies the equation $\square F_{\mu\nu} = 0$. This is a gauge invariant derivation of the fact that the free electromagnetic field describes massless particles.

8.3.4 Sources and Green's function

Let us now discuss sources for the electromagnetic field. Conventionally one takes an electric source that appears only in (4.10), which is modified to read

$$\partial^\mu F_{\mu\nu} = -J_\nu.$$

The Bianchi identity (4.11) is unchanged, so that $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Lorentz covariance requires that the source is a vector, which is called the electric current vector. Since $\partial^\nu\partial^\mu F_{\mu\nu}$ vanishes identically, the current must be conserved. The theory is inconsistent unless the current satisfies $\partial^v J_v = 0$. This condition simply reflects the conventional idea that electric charge cannot be created or destroyed. It is also possible to include sources that carry magnetic charge and appear on the right-hand side of (4.11). However, this requires more sophisticated considerations, which we postpone to Sec. 4.2.3, so we will confine our attention to electric sources.

Exercise 4.3 Repeat Ex. 4.2 when there is an electric source. Show that

$$\Box F_{v\rho} = -\left(\partial_\nu J_\rho - \partial_\rho J_v\right)$$

Consider first the analogous problem of the scalar field coupled to a source $J(x)$:

$$\left(\Box - m^2\right)\phi(x) = -J(x).$$

The response to the source is determined by the Green's function $G(x-y)$, which satisfies the equation

$$\left(\Box - m^2\right)G(x-y) = -\delta(x-y).$$

The translation symmetry of Minkowski spacetime implies that the Green's function depends only on the coordinate difference $x^\mu - y^\mu$ between observation point x^μ and source point y^μ. Lorentz symmetry implies that it depends only on the invariant quantities $(x-y)^2 = \eta^{\mu\nu}(x-y)^\mu(x-y)^v$ and $\operatorname{sgn}\left(x^0 - y^0\right)$. In Euclidean space $\mathcal{R}^D$, there is a unique solution of the equation analogous to (4.17), which is damped in the limit of large separation of observation and source points. In Lorentzian signature Minkowski space, there are several choices, which differ in their causal structure, that is in the dependence on $\operatorname{sgn}\left(x^0 - y^0\right)$. Many texts on quantum field theory, such as [22, 23, 9], discuss these choices.

The Euclidean Green's function is simplest and sufficient for the purposes of this book. The solution of (4.17) can be written as the Fourier transform

$$G(x-y) = \int \frac{\mathrm{d}^D k}{(2\pi)^D}\frac{\mathrm{e}^{\mathrm{i}k\cdot(x-y)}}{k^2+m^2}.$$

The integral can be expressed in terms of modified Bessel functions. In the massless case the result simplifies to the power law

$$G(x-y) = \frac{\Gamma\left(\frac{1}{2}(D-2)\right)}{4\pi^{\frac{1}{2}D}(x-y)^{(D-2)}}.$$

Here $(x-y)^2 = \delta_{\mu\nu}(x-y)^\mu(x-y)^v$ is the Euclidean distance between source point y and observation point x. Given $G(x-y)$, the solution of (4.16) can be expressed as the integral

$$\phi(x) = \int \mathrm{d}^D y G(x-y)J(y)$$

One may note that the Green's function is formally the inverse of the wave operator, i.e. $G = -\left(\Box - m^2\right)^{-1}$. In Euclidean space $\Box = \nabla^2$ which is the D-dimensional Laplacian.

Let's continue in Euclidean space and find the Green's function for the gauge field. We might expect to solve (4.14) using a Green's function $G_{\mu\nu}(x-y)$, which is a tensor. However, we run into the immediate difficulty that there is no solution to the equation

$$\left(\delta^{\mu\rho}\Box - \partial^\mu\partial^\rho\right)G_{\rho v}(x,y) = -\delta^\mu_v\delta(x-y).$$

The Maxwell wave operator $\delta^{\mu\nu}\Box - \partial^\mu\partial^\nu$ is not invertible since any pure gradient $\partial_\nu f(x)$ is a zero mode. This problem is easily resolved. Since the source J_v is conserved, we can replace (4.21) by the weaker condition

$$(\delta^{\mu\rho}\Box - \partial^\mu\partial^\rho)\, G_{\rho v}(x,y) = -\delta^\mu_v \delta(x-y) + \frac{\partial}{\partial y^\nu}\Omega^\mu(x,y)$$

where $\Omega^\mu(x,y)$ is an arbitrary vector function. If $\Omega^\mu(x,y)$ and $J_v(y)$ are suitably damped at large distance, the effect of the second term in (4.22) cancels (after partial integration) in the formula

$$A_\mu(x) = \int \mathrm{d}^D y G_{\mu\nu}(x,y) J^\nu(y)$$

which is the analogue of (4.20).

We now derive the precise form of $G_{\mu\nu}(x,y)$. By Euclidean symmetry, we can assume the tensor form

$$G_{\mu\nu}(x,y) = \delta_{\mu\nu} F(\sigma) + (x-y)_\mu (x-y)_\nu \hat{S}(\sigma)$$

where $\sigma = \frac{1}{2}(x-y)^2$. It is more useful, but equivalent, to take advantage of gauge invariance and rewrite this ansatz as

$$G_{\mu\nu}(x,y) = \delta_{\mu\nu} F(\sigma) + \partial_\mu \partial_\nu S(\sigma)$$

because the pure gauge term involving $S(\sigma)$ has no effect in (4.23) and cancels in (4.22). We may also assume that the gauge term in (4.22) has the Euclidean invariant form $\partial^\mu\partial_\nu\Omega(\sigma)$. Substituting (4.25) in (4.22) we find the two independent tensors δ^μ_ν and $(x-y)^\mu(x-y)_\nu$ and thus two independent differential equations involving F and Ω, namely

$$\begin{aligned} 2\sigma F''(\sigma) + (D-1)F'(\sigma) &= \Omega'(\sigma) \\ F''(\sigma) &= -\Omega''(\sigma) \end{aligned}$$

Note that $F'(\sigma) = \mathrm{d}F(\sigma)/\mathrm{d}\sigma$, etc. We have dropped the δ-function term in (4.22), because we will first solve these equations for $\sigma \neq 0$. The second equation in (4.26) may be integrated immediately, giving $F'(\sigma) = -\Omega'(\sigma)$; a possible integration constant is chosen to vanish, so that $F'(\sigma)$ vanishes at large distance. The first equation then becomes $2\sigma F''(\sigma) + DF'(\sigma) = 0$, which has the power-law solution $F(\sigma) \sim \sigma^{1-\frac{1}{2}D}$. However, on any function of σ, the Laplacian acts as $\Box F(\sigma) = 2\sigma F''(\sigma) + DF'(\sigma)$. In our case there is a hidden δ-function in $\Box F(\sigma)$ because the power law is singular. The effect of the δ-function in (4.22) is automatically incorporated if we take $F(\sigma) = G(x-y)$ where G is the massless scalar Green's function in (4.19). The result of this analysis is the gauge field Green's function

$$G_{\mu\nu}(x,y) = \delta_{\mu\nu} G(x-y) + \partial_\mu \partial_\nu S(\sigma)$$

The gauge function $S(\sigma)$ is arbitrary and may be taken to vanish. Then (4.27) becomes the usual Feynman gauge propagator.

The gauge field propagators found above are usually derived after gauge fixing in the path integral formalism in quantum field theory texts. The derivation here is purely classical, as appropriate since the response of the gauge field to a conserved current source is a purely classical phenomenon.

It may not be obvious why this method works. To see why, apply $\partial/\partial x^\mu$ to both sides of (4.22), obtaining

$$0 = -\partial_\nu \delta(x-y) - \partial_\nu \Box \Omega(\sigma)$$

in which $\partial_\nu = \partial/\partial x^\nu$. This consistency condition is satisfied because the analysis above led us the result $\Omega(\sigma) = -F(\sigma) = -G\left((x-y)^2\right)$.

Exercise 4.4 In $D=4$ dimensions, consider a point charge at rest, i.e. $J^{\mu}(x)=\delta_0^{\mu} q \delta(\vec{x})$. Obtain, using (4.23), that the resulting value of A^0, and therefore of the electric field, is

$$A^0(x)=\frac{q}{4\pi}\frac{1}{|\vec{x}|}, \quad \vec{E}=\frac{q}{4\pi}\frac{\vec{x}}{|x|^3}$$

8.3.5 Quantum electrodynamics

The current vector J_v in (4.14) may describe a piece of laboratory apparatus, such as a magnetic solenoid. However, we are more interested in the case where the source is the field of a charged elementary particle, such as the Dirac spinor Ψ. This is the theory of quantum electrodynamics, which contains equations that determine both the electromagnetic field A_μ, with $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, and Ψ. In dealing with coupled fields it is generally best to package the dynamics in a Lorentz invariant action. The equations of motion then emerge as the condition for a critical point of the action functional and are guaranteed to be mutually consistent.

It is also advantageous to change notation from that of Sec. 4.1.1 by scaling the vector potential, $A_\mu \rightarrow e A_\mu$, where e is the conventional coupling constant of the electromagnetic field to charged fields; $e^2/4\pi \approx 1/137$ is called the fine structure constant. In this notation the relevant equations of Sec. 4.1 read:

$$\begin{aligned}
F_{\mu\nu} &\equiv \partial_\mu A_\nu-\partial_\nu A_\mu, \\
A_\mu \rightarrow A_\mu^{\prime} &\equiv A_\mu+\frac{1}{e}\partial_\mu \theta \\
D_\mu \Psi &\equiv \left(\partial_\mu-\mathrm{i} e q A_\mu\right)\Psi \\
\left[D_\mu, D_\nu\right]\Psi &= -\mathrm{i} e q F_{\mu\nu}\Psi .
\end{aligned}$$

The electric charges q of the various charged fields are then simple rational numbers, for example $q=1$ for the electron. [5]

The action functional for the electromagnetic field interacting with a field of charge q, which we take to be a massive Dirac field, is the sum of two terms, each gauge invariant,

5 It is an interesting question why the electric charges of elementary particles in Nature are quantized; that they appear to be integer multiples of a lowest fundamental charge. Two reasons have been found. The first is that quantum theory requires quantization of electric charge if a magnetic monopole exists. Second, electric charge can emerge as an unbroken U(1) generator of a larger non-abelian gauge theory with spontaneous gauge symmetry breaking. These reasons are not independent since monopoles solutions exist when gauge symmetry is broken with residual U(1) symmetry. See Sec. 17A.1 and [24] for discussion of these ideas.

$$S\left[A_\mu, \bar{\Psi}, \Psi\right]=\int \mathrm{d}^D x\left[-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\bar{\Psi}\left(\gamma^\mu D_\mu-m\right)\Psi\right]$$

The Euler variation of (4.31) with respect to the gauge potential A_v is

$$\frac{\delta \mathcal{L}}{\delta A^\nu}=\partial^\mu F_{\mu\nu}+\mathrm{i} e q \bar{\Psi}\gamma_\nu \Psi=0$$

This is equivalent to (4.14) with the electric current a multiple of the Noether current of the global U(1) phase symmetry discussed in Sec. 2.7.1. It is a typical feature of the various fundamental gauge symmetries in physics that the Noether current of a system with global symmetry becomes the source for the gauge field introduced when the symmetry is gauged. The Euler variation with respect to $\bar{\Psi}$ gives the gauge covariant Dirac equation (4.4), with $D_\mu \Psi$ given in (4.30).

8.3.6 The stress tensor and gauge covariant translations

The situation of the stress tensor of this system is quite curious. The canonical stress tensor, calculated from the Noether formula (1.67) with $\Delta_A\phi^i \to \partial_\nu A_\rho, \partial_\nu\bar{\Psi}, \partial_\nu\Psi$ for the three independent fields, is

$$T^{\mu}{}_{\nu} = F^{\mu\rho}\partial_\nu A_\rho + \bar{\Psi}\gamma^\mu\partial_\nu\Psi + \delta^\mu_v\mathcal{L}$$

It is conserved on the index μ, but not on v, not symmetric and not gauge invariant. The situation can be improved by treating fermion terms as in Sec. 2.7.2 and then adding $\Delta T^{\mu}{}_{\nu} = -\partial_\rho\left(F^{\mu\rho}A_v\right)$ in accord with the discussion in Sec. 1.3. The final result is the gauge invariant symmetric stress tensor

$$\Theta_{\mu\nu} = F_{\mu\rho}F_\nu{}^\rho + \frac{1}{4}\bar{\Psi}\left(\gamma_\mu \overset{\leftrightarrow}{D}_\nu + \gamma_\nu \overset{\leftrightarrow}{D}_\mu\right)\Psi + \eta_{\mu\nu}\mathcal{L}$$

Exercise 4.5 Consider the gauge covariant translation, defined by $\delta A_\mu = a^\nu F_{v\mu}$ and $\delta\Psi = a^v D_v\Psi$. Show that they differ from a conventional translation by a gauge transformation with gauge dependent parameter $\theta = -ea^vA_v$. Gauge covariant translations are a symmetry of the action (4.31). What is the Noether current for this symmetry? How is it related to the stress tensor (4.34)?

8.3.7 Electromagnetic duality

The subject of electromagnetic duality has several interesting applications in supergravity theories. For example, the symmetry group of black hole solutions of matter-coupled supergravity theories generally contains duality transformations. We recommend that all readers study Secs. 4.2.1 and 4.2.2. However, because the applications of duality are somewhat advanced, the rest of the section can be omitted in the first reading of the book.

8.3.8 Dual tensors

We begin by discussing the duality property of second rank antisymmetric tensors $H_{\mu\nu}$ in four-dimensional Minkowski spacetime. We use the Levi-Civita tensor introduced in Sec. 3.1.3 to define the dual tensor

$$\tilde{H}^{\mu\nu} \equiv -\frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}H_{\rho\sigma}$$

In our conventions the dual tensor is imaginary. The indices of $\tilde{H}$ can be raised and lowered with the Minkowski [6] metric $\eta_{\mu\nu}$. It is also useful to define the linear combinations

$$H^\pm_{\mu\nu} = \frac{1}{2}\left(H_{\mu\nu} \pm \tilde{H}_{\mu\nu}\right), \quad H^\pm_{\mu\nu} = \left(H^\mp_{\mu\nu}\right)^*$$

Exercise 4.6 Prove that the dual of the dual is the identity, specifically that

$$-\frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}\tilde{H}_{\rho\sigma} = H^{\mu\nu}$$

You will need (3.9). The validity of this property is the reason for the i in the definition (4.35).

Show that $H^+_{\mu\nu}$and $H^-_{\mu\nu}$are, respectively, self-dual and anti-self-dual, i.e.

$$-\frac{1}{2}\mathrm{i}\varepsilon_{\mu\nu}{}^{\rho\sigma}H^\pm_{\rho\sigma} = \pm H^\pm_{\mu\nu}$$

Let $G_{\mu\nu}$ be another antisymmetric tensor with $G^{\pm}_{\mu\nu}$defined as in (4.36). Prove the following relations (where $(\mu\nu)$ means symmetrization between the indices):

$$G^{+\mu\nu}H^{-}_{\mu\nu}=0,\quad G^{\pm\rho(\mu}H^{\pm\nu)}_{\rho}=-\frac{1}{4}\eta^{\mu\nu}G^{\pm\rho\sigma}H^{\pm}_{\rho\sigma},\quad G^{+}_{\rho[\mu}H^{-}_{\nu]}\rho$$

Hint: you could first prove

$$\tilde{G}^{\rho\mu}\tilde{H}^{\nu}_{\rho}=-\frac{1}{2}\eta^{\mu\nu}G^{\rho\sigma}H_{\rho\sigma}-G^{\rho v}H^{\mu}_{\rho}.$$

Exercise 4.7 The duality operation can also be applied to matrices of the Clifford algebra. Define the quantity $L_{\mu\nu}=\gamma_{\mu\nu}P_L$. Show that this is anti-self-dual. Hint: check first that $\gamma_{\mu\nu}\gamma_*=\frac{1}{2}\mathrm{i}\varepsilon_{\mu\nu\rho\sigma}\gamma^{\rho\sigma}$.

8.3.9 Duality for one free electromagnetic field

Duality operates as an interesting symmetry of field theories containing one or more abelian gauge fields which may interact with other fields, principally scalars. In this section we discuss the simplest case, namely a single free gauge field. First note that, after contraction with the ε-tensor, the Bianchi identity (4.11) can be expressed as $\partial_\mu\tilde{F}^{\mu\nu}=0$.

6 The definition (4.35) is valid in Minkowski space, but must be modified in curved spacetimes as we will discuss in Ch. 7.

So we can temporarily ignore the vector potential and regard $F_{\mu\nu}$ as the basic field variable which must satisfy both the Maxwell and Bianchi equations:

$$\partial_\mu F^{\mu\nu}=0,\quad \partial_\mu\tilde{F}^{\mu\nu}=0.$$

We can now consider the change of variables (the i is included to make the transformation real):

$$F^{\mu\nu}\to F'^{\mu\nu}=\mathrm{i}\tilde{F}^{\mu\nu}$$

Since $F'^{\mu\nu}$ also obeys both equations of (4.41) we have defined a symmetry of the free electromagnetic field.

Exercise 4.8 Show that the symmetry (4.42) exchanges the electric and magnetic fields: $E_i\to E'_i=-B_i$ and $B_i\to B'_i=E_i$.

It is not possible to extend the symmetry to the vector potentials $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ and $F'_{\mu\nu}=\partial_\mu A'_\nu-\partial_\nu A'_\mu$ because A_μ and A'_μ are not related by any local transformation.

Here are some basic exercises involving the duality transform of the field strength tensor $F_{\mu\nu}$.

Exercise 4.9 Show that the self-dual combinations $F^{\pm}_{\mu\nu}$contain only photons of one polarization in their plane wave expansions:

$$F^{\pm}_{\mu\nu}=2\mathrm{i}\int\frac{\mathrm{d}^3k}{(2\pi)^3 2k^0}\left[\mathrm{e}^{\mathrm{i}k\cdot x}k_{[\mu}\epsilon_{\nu]}(\vec{k},\pm)a(\vec{k},\pm)-\mathrm{e}^{-\mathrm{i}k\cdot x}k_{[\mu}\epsilon^*_{\nu]}(\vec{k},\mp)a^*(\vec{k},\mp)\right]$$

To perform this exercise, check first that with the polarization vectors given in Sec. 4.1.2, one has

$$-\frac{1}{2}\mathrm{i}\varepsilon^{\mu\nu\rho\sigma}k_\rho\epsilon_\sigma(\vec{k},\pm)=\pm k^{[\mu}\epsilon^{\nu]}(\vec{k},\pm).$$

Exercise 4.10 Show that the quantity $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total derivative, i.e.

$$F_{\mu\nu}\tilde{F}^{\mu\nu}=-\mathrm{i}\partial_\mu\left(\varepsilon^{\mu\nu\rho\sigma}A_\nu F_{\rho\sigma}\right)$$

Show, using (1.45), that under a Lorentz transformation

$$\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)(x) \rightarrow \det \Lambda^{-1}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)(\Lambda x)$$

Thus $F_{\mu\nu}\tilde{F}^{\mu\nu}$ transforms as a scalar under proper Lorentz transformations but changes sign under space or time reflections. Use the Schouten identity (3.11) to prove that

$$F_{\mu\rho}\tilde{F}_{\nu}^{\rho} = \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}\tilde{F}^{\rho\sigma}$$

8.3.10 Duality for gauge field and complex scalar

The simplest case of electromagnetic duality in an interacting field theory occurs with one abelian gauge field $A_\mu(x)$ and a complex scalar field $Z(x)$. The electromagnetic part of the Lagrangian is

$$\mathcal{L} = -\frac{1}{4}(\operatorname{Im} Z)F_{\mu\nu}F^{\mu\nu} - \frac{1}{8}(\operatorname{Re} Z)\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}.$$

Actions in which the gauge field kinetic term is multiplied by a function of complex scalar fields are quite common in supersymmetry and supergravity. We now define an extension of the duality transformation (4.42) which gives a non-abelian global SL(2, $\mathbb{R}$) symmetry of the gauge field equations of this theory. In Sec. 7.12.2 we will discuss a generalized scalar kinetic term that is invariant under SL$(2,\mathbb{R})$. The field $Z(x)$ carries dynamics, and the equations of motion of the combined vector and scalar theory are also invariant.

The gauge Bianchi identity and equation of motion of our theory are

$$\partial_\mu \tilde{F}^{\mu\nu} = 0, \quad \partial_\mu\left[(\operatorname{Im} Z)F^{\mu\nu} + \mathrm{i}(\operatorname{Re} Z)\tilde{F}^{\mu\nu}\right] = 0.$$

It is convenient to define the real tensor

$$G^{\mu\nu} \equiv \varepsilon^{\mu\nu\rho\sigma}\frac{\delta S}{\delta F^{\rho\sigma}} = -\mathrm{i}(\operatorname{Im} Z)\tilde{F}^{\mu\nu} + (\operatorname{Re} Z)F^{\mu\nu}$$

and to consider the self-dual combinations $F^{\mu\nu\pm}$ and $G^{\mu\nu\pm}$. Note that these are related by

$$G^{\mu\nu-} = ZF^{\mu\nu-}, \quad G^{\mu\nu+} = \bar{Z}F^{\mu\nu+}.$$

The information in (4.49) can then be reexpressed as

$$\partial_\mu \operatorname{Im} F^{\mu\nu-} = 0, \quad \partial_\mu \operatorname{Im} G^{\mu\nu-} = 0$$

We define a matrix of the group SL$(2,\mathbb{R})$ by

$$\mathcal{S} \equiv \begin{pmatrix} d & c \\ b & a \end{pmatrix}, \quad ad - bc = 1$$

The group SL$(2,\mathbb{R})$ acts on the tensors $F^-$and G^-as follows:

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S}\begin{pmatrix} F^- \\ G^- \end{pmatrix}$$

Since $\mathcal{S}$ is real, the conjugate tensors $F^+$and G^+also transform in the same way.

Exercise 4.11 Assume that $\operatorname{Im} F^-$and $\operatorname{Im} G^-$satisfy (4.52), and show that $\operatorname{Im} F'^-$ and $\operatorname{Im} G'^-$ also obey the same equations. Show that G'^- and a transformed scalar Z' satisfy $G'^{\mu\nu-} = Z'F'^{\mu\nu-}$, if Z' is defined as the following nonlinear transform of Z:

$$Z' = \frac{aZ+b}{cZ+d}$$

The two equations (4.54) and (4.55) specify the $\mathrm{SL}(2,\mathbb{R})$ duality transformation on the field strength and complex scalar of our system. The exercise shows that the Bianchi identity and generalized Maxwell equations are duality invariant. In general the duality transform is not a symmetry of the Lagrangian or the action integral. The following exercise illustrates this.

Exercise 4.12 Show that the Lagrangian (4.48) can be rewritten as

$$\mathcal{L}(F,Z) = -\frac{1}{2}\operatorname{Im}\left(ZF^{-}_{\mu\nu}F^{\mu\nu-}\right).$$

Consider the $\mathrm{SL}(2,\mathbb{R})$ transformation with parameters $a=d=1$ and $b=0$. Show that

$$\mathcal{L}\left(F',Z'\right) = -\frac{1}{2}\operatorname{Im}\left(Z(1+cZ)F^{-}_{\mu\nu}F^{\mu\nu-}\right) \neq \mathcal{L}(F,Z)$$

The symmetric gauge invariant stress tensor of this theory is

$$\Theta^{\mu\nu} = (\operatorname{Im} Z)\left(F^{\mu\rho}F^{\nu}_{\ \rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right)$$

As we will see in Ch. 8, when the theory is coupled to gravity, it is this stress tensor that is the source of the gravitational field; see (8.4). It is then important that $\operatorname{Im} Z$ is positive, which restricts the domain of Z to the upper half-plane. It is also important that the stress tensor is invariant under the duality transformations (4.54) and (4.55). This is the reason for the duality symmetry of many black hole solutions of supergravity,

Exercise 4.13 Prove that the energy-momentum tensor (4.58) is invariant under duality. Here are some helpful relations which you will need:

$$\operatorname{Im} Z' = \frac{\operatorname{Im} Z}{(cZ+d)(c\bar{Z}+d)}$$

Further you need again (4.47) and a similar identity (proven by contracting ε-tensors)

$$\tilde{F}_{\mu\rho}\tilde{F}^{\rho}_{\nu} = -F_{\mu\rho}F_{\nu}{}^{\rho} + \frac{1}{2}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}.$$

This leads to

$$F'_{\mu\rho}F'^{\rho}_{\nu} - \frac{1}{4}\eta_{\mu\nu}F'_{\rho\sigma}F'^{\rho\sigma} = |cZ+d|^2\left[F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}\eta_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\right].$$

When the $\mathrm{SL}(2,\mathbb{R})$ duality transformation appears in supergravity, there is also a scalar kinetic term in the Lagrangian which is invariant under the symmetry, specifically under the transformation (4.55). The prototype Lagrangian with this symmetry is the nonlinear σ-model whose target space is the Poincaré plane. This model and its $\mathrm{SL}(2,\mathbb{R})$ symmetry group will be discussed in Sec. 7.12; see (7.151) and (7.152). The Poincaré plane is the upper half-plane $\operatorname{Im} Z > 0$. The relation (4.59) shows that duality transformations map the upper half-plane into itself. The positive sign is preserved by $\mathrm{SL}(2,\mathbb{R})$ transformations and the energy density obtained from the stress tensor Θ^{00} above will be positive!

Exercise 4.14 The free Maxwell theory is the special case of (4.48) with fixed $Z=\mathrm{i}$. Suppose that the gauge field is coupled to a conserved current as in (4.14). Check that the electric charge can be expressed in terms of F or G by

$$q \equiv \int \mathrm{d}^3\vec{x}J^0 = \int \mathrm{d}^3\vec{x}\partial_i F^{0i} = -\frac{1}{2}\int \mathrm{d}^3\vec{x}\varepsilon^{ijk}\partial_i G_{jk}$$

A magnetic charge can be introduced in Maxwell theory as the divergence of $\vec{B}$ (recall $E^i = F^{0i}$ and $B^i = \frac{1}{2}\varepsilon^{ijk}F_{jk}$). This leads to a definition [7]

$$p \equiv -\frac{1}{2}\int \mathrm{d}^3\vec{x}\varepsilon^{ijk}\partial_i F_{jk}$$

Show that $\begin{pmatrix} p \\ q \end{pmatrix}$ is a vector that transforms under $\mathrm{SL}(2,\mathbb{R})$ in the same way as the tensors $F^-$and G^-in (4.54).

In many applications of electromagnetic duality, magnetic and electric charges appear as sources for the Bianchi 'identity' and generalized Maxwell equations of (4.49). As exemplified in Ex. 4.14 this leads to an $\mathrm{SL}(2,\mathbb{R})$ vector of charges. Particles that carry both electric and magnetic charge are called dyons. In quantum mechanics, dyon charges must obey the Schwinger-Zwanziger quantization condition. If a theory contains two dyons with charges (p_1, q_1) and (p_2, q_2), these charges must satisfy $p_1q_2 - p_2q_1 = 2\pi n$, where n is an integer. [8] This condition is invariant under $\mathrm{SL}(2,\mathbb{R})$ transformations of the charges. However, one can show [25] that there is a lowest non-zero value of the electric charge and that all allowed charges are restricted to an infinite discrete set of points called the charge lattice. The allowed SL(2) transformations must take one lattice point to another, and this restricts the group parameters in (4.53) to be integers. This restriction defines the subgroup $\mathrm{SL}(2,\mathbb{Z})$, often called the modular group. [9] One can show that this subgroup is generated by the following choices of $\mathcal{S}$:

$$\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$
$$Z' = Z + 1, \quad Z' = -\frac{1}{Z}.$$

This means that one can express any element of $\mathrm{SL}(2,\mathbb{Z})$ as the product of (finitely many) factors of the two generators above and their inverses.

Exercise 4.15 In (4.48), the kinetic terms of the electromagnetic fields are determined by a variable Z that was treated as a scalar field. Z can also be replaced by a coupling constant, and typically one takes Z to be the imaginary number [10]i/g^2, where g is a coupling constant. Observe that the first transformation of (4.64) does not preserve the restriction that Z is imaginary. However, the second one does. Prove that this transformation is of the type (4.42), interchanging the electric and magnetic fields. It transforms g to its inverse, and thus relates the strong and weak coupling descriptions of the theory. In

7 In order to obtain a symplectic vector (p, q) and not $(-p, q)$, we changed the sign of the magnetic charge with respect to some classical works. This implies that we have $\vec{\nabla}\cdot\vec{B} = -j^0_{\mathrm{m}}$, where j^0_{m} is the magnetic charge density.

8 For the case $(p_1, q_1) = (p, 0)$ and $(p_2, q_2) = (0, q)$, this reduces to condition $pq = 2\pi n$ found by Dirac in 1933.

9 The modular group generated by the matrices (4.64) is in fact PSL(2, $\mathbb{Z}$). In PSL(2, $\mathbb{Z}$), the elements M and $-M$ of $\mathrm{SL}(2,\mathbb{Z})$ are identified. Both these elements give in fact the same transformation $Z'(Z)$.

10 One often adds an extra term that is a real so-called θ-parameter, but we will omit this here.

Secs. 4.1 and 4.2.2 we considered $Z = \mathrm{i}g = \mathrm{i}$. Check that general duality transformations in this case are of the form

$$F'^{-}_{\mu\nu} = (d + \mathrm{i}c)F^{-}_{\mu\nu}, \qquad \text{i.e.} \qquad F'_{\mu\nu} = dF_{\mu\nu} - \mathrm{i}c\tilde{F}_{\mu\nu}.$$

8.3.11 Electromagnetic duality for coupled Maxwell fields

In this section we explore how the duality symmetry is extended to systems containing a set of abelian gauge fields $A_\mu^A(x)$, indexed by $A = 1, 2, ..., m$ together with scalar fields ϕ^i. Scalars enter the theory through complex functions $f_{AB}(\phi) = f_{BA}(\phi)$. We consider the action

$$S = \int \mathrm{d}^4 x \mathcal{L}, \quad \mathcal{L} = -\frac{1}{4}\left(\operatorname{Re} f_{AB}\right) F_{\mu\nu}^A F^{\mu\nu B} + \frac{1}{4}\mathrm{i}\left(\operatorname{Im} f_{AB}\right) F_{\mu\nu}^A \tilde{F}^{\mu\nu B}$$

which is real since $\tilde{F}^{\mu\nu}$ is pure imaginary, as defined in (4.35). The first term is a generalized kinetic Lagrangian for the gauge fields, so we usually require that $\operatorname{Re} f_{AB}$ is a positive definite matrix. This ensures that gauge field kinetic energies are positive. Although $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is a total derivative, the second term does contribute to the equations of motion when $\operatorname{Im} f_{AB}$ is a function of the scalars ϕ^i. Our discussion will not involve the scalars directly. However, as in Sec. 4.2.3, additional terms to specify the scalar dynamics will appear when theories of this type are encountered in extended $D = 4$ supergravity. The treatment that follows is modeled on Sec. 4.2.3 (where f_{AB} was taken to be $-\mathrm{i}Z$).

Using the self-dual tensors of (4.36), we then rewrite the Lagrangian (4.66) as

$$\begin{aligned}
\mathcal{L}\left(F^+, F^-\right) &= -\frac{1}{2}\operatorname{Re}\left(f_{AB} F_{\mu\nu}^{-A} F^{\mu\nu -B}\right) \\
&= -\frac{1}{4}\left(f_{AB} F_{\mu\nu}^{-A} F^{\mu\nu -B} + f_{AB}^* F_{\mu\nu}^{+A} F^{\mu\nu +B}\right),
\end{aligned}$$

and define the new tensors

$$\begin{aligned}
G_A^{\mu\nu} &= \varepsilon^{\mu\nu\rho\sigma}\frac{\delta S}{\delta F^{\rho\sigma A}} = -\left(\operatorname{Im} f_{AB}\right) F^{\mu\nu B} - \mathrm{i}\left(\operatorname{Re} f_{AB}\right)\tilde{F}^{\mu\nu B} = G_A^{\mu\nu +} + G_A^{\mu\nu -}, \\
G_A^{\mu\nu -} &= -2\mathrm{i}\frac{\delta S\left(F^+, F^-\right)}{\delta F_{\mu\nu}^{-A}} = \mathrm{i} f_{AB} F^{\mu\nu -B}, \\
G_A^{\mu\nu +} &= 2\mathrm{i}\frac{\delta S\left(F^+, F^-\right)}{\delta F_{\mu\nu}^{+A}} = -\mathrm{i} f_{AB}^* F^{\mu\nu +B}.
\end{aligned}$$

Since the field equation for the action containing (4.67) is

$$0 = \frac{\delta S}{\delta A_v^A} = -2\partial_\mu \frac{\delta S}{\delta F_{\mu\nu}^A}$$

the Bianchi identity and the equation of motion can be expressed in the concise form

$$\begin{aligned}
\partial^\mu \operatorname{Im} F_{\mu\nu}^{A-} &= 0 \qquad \text{Bianchi identities} \\
\partial_\mu \operatorname{Im} G_A^{\mu\nu -} &= 0 \qquad \text{equations of motion.}
\end{aligned}$$

(The same equations hold for $\operatorname{Im} F^{A+}$ and $\operatorname{Im} {G_A}^+$.)

Duality transformations are linear transformations of the $2m$ tensors $F^{A\mu\nu}$ and $G_A^{\mu\nu}$ (accompanied by transformations of the f_{AB}) which mix Bianchi identities and equations of motion, but preserve the structure that led to (4.70). Since the equations (4.70) are real, we can mix them by a real $2m \times 2m$ matrix. We extend these transformations to the (anti-)self-dual tensors, and consider

$$\begin{pmatrix} F'^- \\ G'^- \end{pmatrix} = \mathcal{S}\begin{pmatrix} F^- \\ G^- \end{pmatrix} \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix}\begin{pmatrix} F^- \\ G^- \end{pmatrix}$$

with real $m \times m$ submatrices A, B, C, D. Owing to the reality of these matrices, the same relations hold for the self-dual tensors $F^+$and G^+. In Sec. 4.2.3, these matrices were just numbers:

$$A = d, \quad B = c, \quad C = b, \quad D = a.$$

We require that the transformed field tensors F'^A and G'_A are also related by the definitions (4.68), with appropriately transformed f_{AB}. We work out this requirement in the following steps:

$$G'^- = (C + \mathrm{i}Df)F^- = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1}F^-,$$

such that we conclude that

$$\mathrm{i}f' = (C + \mathrm{i}Df)(A + \mathrm{i}Bf)^{-1}$$

The last equation gives the symmetry transformation relating f'_{AB} to f_{AB}. If $G'^-_{\mu\nu}$ is to be the variational derivative of a transformed action, as (4.68) requires, then the matrix f' must be symmetric. For a generic [11] symmetric f, this requires that the matrices A, B, C, D satisfy

$$A^T C - C^T A = 0, \quad B^T D - D^T B = 0, \quad A^T D - C^T B = \mathbb{1}.$$

These relations among A, B, C, D are the defining conditions of a matrix of the symplectic group in dimension $2m$ so we reach the conclusion that

$$\mathcal{S} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}(2m, \mathbb{R})$$

The conditions (4.75) may be summarized as

$$\mathcal{S}^T \Omega \mathcal{S} = \Omega \quad \text{where} \quad \Omega = \begin{pmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{pmatrix}$$

11 If the initial f_{AB} is non-generic, then the matrix $\mathbb{1}$ in the last equation can be replaced by any matrix which commutes with f_{AB}. For generic f_{AB}, this must be a constant multiple of the unit matrix. The constant, which should be positive to preserve the sign of the kinetic energy of the vectors, can be absorbed by rescaling the matrices A, B, C, D.

The duality transformations in four dimensions are transformations in the symplectic group $\mathrm{Sp}(2m, \mathbb{R})$.

The matrix Ω is often called the symplectic metric, and the transformations (4.71) are then called symplectic transformations. This is the main result originally derived in [26]. Duality transformations in four spacetime dimensions are transformations of the group $\mathrm{Sp}(2m, \mathbb{R})$, which is a non-compact group.

Exercise 4.16 The dimension of the group $\mathrm{Sp}(2m, \mathbb{R})$ is the number of elements of the matrix $\mathcal{S}$, namely $4m^2$ minus the number of independent conditions contained in (4.77). Show that the dimension is $m(2m + 1)$.

Duality transformations have two types of applications: they can describe symmetries of one theory and they can describe transformations from one theory to another. In the first case, the symmetries concerned form a subgroup of the 'maximal' duality group $\mathrm{Sp}(2m, \mathbb{R})$ discussed above. The subgroup consists of transformations (4.74) of $f_{AB}\,(\phi^i)$ induced by the symmetry transformations of the elementary scalars ϕ^i. These scalar transformations must be symmetries of the scalar kinetic term and other parts of the Lagrangian. The model of Sec. 4.2.3 is one example. The transformation of Z defined in (4.55) is the standard $\mathrm{SL}(2, \mathbb{R})$ symmetry of the Poincaré plane. This could be part of the full symmetry group of all the scalar fields of the theory. In extended supergravities it turns out that all the symmetry transformations that act on the scalars appear also as transformations of the vector kinetic matrix. Hence, the symmetry group is then a subgroup of the 'maximal' group $\mathrm{Sp}(2m, \mathbb{R})$ discussed above.

However, another application is of the type that we encountered in Ex. 4.15. In that case constants that specify the theory under consideration change under the duality transformations. The constants that transform are sometimes called 'spurionic quantities'. The transformations thus relate two different theories. Solutions of one theory are mapped into solutions of the other one. This is the basic idea of dualities in M-theory.

Symplectic transformations always transform solutions of (4.70) into other solutions. However, they are not always invariances of the action. Indeed, writing

$$\mathcal{L} = -\frac{1}{2}\operatorname{Re}\left(f_{AB}F_{\mu\nu}^{-A}F^{\mu\nu-B}\right) = -\frac{1}{2}\operatorname{Im}\left(F_{\mu\nu}^{-A}G_A^{\mu\nu-}\right)$$

we obtain

$$\operatorname{Im}F'^{-}G'^{-} = \operatorname{Im}\left(F^{-}G^{-}\right) + \operatorname{Im}\left[2F^{-}\left(C^TB\right)G^{-} + F^{-}\left(C^TA\right)F^{-} + G^{-}\left(D^TB\right)G^{-}\right].$$

If $C \neq 0, B = 0$ the Lagrangian is invariant up to a 4-divergence, since $\operatorname{Im}F^{-}F^{-} = -\frac{1}{4}\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}$ and the matrices A and C are real constants. For $B \neq 0$ neither the Lagrangian nor the action is invariant.

Electromagnetic duality has important applications to black hole solutions of extended supergravity theories. Supergravity is also very relevant to the analysis of black hole solutions of string theory. Many black holes are dyons; they carry both magnetic and electric charges for the gauge fields of the system. The general situation is a generalization of what was discussed at the end of Sec. 4.2.3. The charges form a symplectic vector $\begin{pmatrix} q_{\mathrm{m}}^A \\ q_{\mathrm{e}A} \end{pmatrix}$ which must transform as in (4.71). The Dirac-Schwinger-Zwanziger quantization condition restricts these charges to a lattice. Invariance of this lattice restricts the symplectic transformations of (4.71) to a discrete subgroup $\mathrm{Sp}(2m,\mathbb{Z})$, which is analogous to the $\mathrm{SL}(2,\mathbb{Z})$ group discussed previously.

Finally, we comment that symplectic transformations with $B \neq 0$ should be considered as non-perturbative for the following reasons. A system with no magnetic charges as in classical electromagnetism is transformed to a system with magnetic charges. The elements of f_{AB} may be regarded as coupling constants (see Ex. 4.15), and a system with weak coupling is transformed to one with strong coupling. A duality transformation which mixes electric and magnetic fields cannot be realized by transformation of the vector potential A_μ. One would need a 'magnetic' partner of A_μ to reexpress the $F'_{\mu\nu}$ and $G'_{\mu\nu}$ in terms of potentials.

The important properties of the matrix f_{AB} are that it is symmetric and that $\operatorname{Re}f_{AB}$ define a positive definite quadratic form in order to have positive gauge field energy. These properties are preserved under symplectic transformations defined by (4.74).

8.3.12 Non-abelian gauge symmetry

Yang-Mills theory is based on a non-abelian generalization of the U(1) gauge symmetry. It is the fundamental idea underlying the standard model of elementary particle interactions. We follow the pattern of Sec. 4.1.1, starting with the global symmetry and then gauging it. The focus of our discussion is the derivation of the basic formulas of the classical gauge theory. Readers may need more information on the underlying geometric ideas and the structure and stunning applications of the quantized theory. They are referred to a modern text in quantum field theory. [12]

8.3.13 Global internal symmetry

Suppose that G is a compact simple Lie group of dimension$^{\dim_G}$. Closely associated with the group is its Lie algebra, denoted by $\mathfrak{g}$, which is a real algebra of dimension$_{\dim_G}$. The theory of Lie algebras and Lie groups is an important subject of mathematics with many applications to physics. With some oversimplification we review only the most essential features required by Yang-Mills theory for compact simple groups.

Each compact simple Lie algebra has an infinite number of inequivalent finitedimensional irreducible representations R of dimension $\dim_R$. In each representation, there is a basis of matrix generators $t_A, A=1, \ldots, \dim_G$, which are anti-hermitian in the case of a compact gauge group. The commutator of the generators determines the local geometrical structure of the group:

12 See, for example, Ch. 15 of [9]. This text also reviews aspects of group theory needed in physical applications.

$$[t_A, t_B] = f_{AB}{}^C t_C$$

The array of real numbers $f_{AB}{}^C$ are structure constants of the algebra (the same in all representations). They obey the Jacobi identity

$$f_{AD}^E f_{BC}{}^D + f_{BD}{}^E f_{CA}^D + f_{CD}^E f_{AB}^D = 0.$$

The indices can be lowered by the Cartan-Killing metric defined in Appendix B (see (B.6)), and then the f_{ABC} are totally antisymmetric. For simple algebras, the generators can be chosen to be trace orthogonal, $\operatorname{Tr}(t_A t_B) = -c\delta_{AB}$, with c positive for compact groups, and the Cartan-Killing metric is then proportional to this expression.

One important representation is the adjoint representation of dimension$_{\dim} = \dim_G$, in which the representation matrices are closely related to the structure constants by $(t_A)^D{}_E = f_{AE}{}^D$. Note that the labels DE denote row and column indices of the matrix t_A. The adjoint representation is a real representation; the representation matrices are real and antisymmetric for compact algebras. For complex representations we will use the notation $(t_A)^\alpha{}_\beta$. Anti-hermiticity then requires $(t_A^*)^\alpha{}_\beta = -(t_A)^\beta$. The row and column indices will often be suppressed when no ambiguity arises.

Exercise 4.17 Use (4.81) to show that the matrices $(t_A)^D{}_E = f_{AE}{}^D$ satisfy (4.80) and therefore give a representation.

The general element of $\mathfrak{g}$ is represented by a superposition of generators $\theta^A t_A$ where the θ^A are $\dim_G$ real parameters. The relation between G and $\mathfrak{g}$ is given by exponentiation, namely $\mathrm{e}^{-\theta^A t_A}$ is an element of G in the representation R.

A theory with global non-abelian internal symmetry contains scalar and spinor fields, each of which transforms in an irreducible representation R. For example, there may be a Dirac spinor [13] field $\Psi^\alpha(x), \alpha=1, \ldots, \dim_R$, that transforms in the complex representation R as

$$\Psi^\alpha(x) \rightarrow \left(\mathrm{e}^{-\theta^A t_A}\right)^\alpha{}_\beta \Psi^\beta(x).$$

The conjugate spinor [14] is denoted by $\bar{\Psi}_\alpha$ and transforms as

$$\bar{\Psi}_\alpha \rightarrow \bar{\Psi}_\beta\left(\mathrm{e}^{\theta^A t_A}\right)^\beta_\alpha$$

For most of our discussion it is sufficient to restrict attention to the infinitesimal transformations,

$$\begin{aligned}
\delta \Psi &= -\theta^A t_A \Psi \\
\delta \bar{\Psi} &= \bar{\Psi} \theta^A t_A, \\
\delta \phi^A &= \theta^C f_{BC}{}^A \phi^B.
\end{aligned}$$

13 Note that we use here indices $\alpha, \ldots$ for the representation of the gauge group. They should not be confused with spinor indices, which we usually omit.

14 The Dirac conjugate (2.30) is used here rather than the Majorana conjugate (3.50).

The first two relations are just the terms of (4.82) and (4.83) that are first order in θ^A. The last relation is the infinitesimal transformation of a field in the adjoint representation, taken here as the set of $\dim_G$ real scalars ϕ^A. Of course, scalars could be assigned to any representation R.

Actions, such as the kinetic action for massive fermion fields,

$$S[\bar{\Psi}, \Psi] = -\int \mathrm{d}^D x \bar{\Psi} \left[\gamma^\mu \partial_\mu - m\right] \Psi$$

are required to be invariant under (4.82).

Exercise 4.18 Show that (4.85) is invariant under the transformation (4.82) and (4.83). Consider an infinitesimal transformation and derive the conserved current

$$J_{A\mu} = -\bar{\Psi} t_A \gamma_\mu \Psi, \quad A = 1, \ldots, \dim_G$$

Show that the current transforms as a field in the adjoint representation, i.e.

$$\delta J_{A\mu} = \theta^C f_{CA}^B J_{B\mu}.$$

Show that $\delta\left(\phi^A J_{A\mu}\right) = 0$.

8.3.14 Gauging the symmetry

In gauged non-abelian internal symmetry, the group parameter $\theta^A(x)$ is promoted to an arbitrary function of x^μ. The first step in the systematic formulation of gauge invariant field equations is to introduce the gauge potentials, namely a set of vectors $A_\mu^A(x)$ whose infinitesimal transformation rule is

$$\delta A_\mu^A(x) = \frac{1}{g} \partial_\mu \theta^A + \theta^C(x) A_\mu^B(x) f_{BC}^A$$

The first term is the gradient term similar to that for the abelian gauge field in (4.2), and the second is exactly the transformation of a field in the adjoint representation, as one can see from the third equation in (4.84). The constant g is the Yang-Mills coupling, which replaces the electromagnetic coupling e of Sec. 4.1.4.

Following the pattern of Sec. 4.1.1, we next define the covariant derivative of a field in the representation R with matrix generators t_A. For the fields Ψ^α, $\bar{\Psi}_\alpha$, and ϕ^A of (4.84) we write

$$\begin{aligned} D_\mu \Psi &= \left(\partial_\mu + g t_A A_\mu^A\right) \Psi, \\ D_\mu \bar{\Psi} &= \partial_\mu \bar{\Psi} - g \bar{\Psi} t_A A_\mu^A, \\ D_\mu \phi^A &= \partial_\mu \phi^A + g f_{BC}{}^A A_\mu^B \phi^C. \end{aligned}$$

Note that the gauge transformation (4.88) can be written as $\delta A_\mu^A(x) = (1/g) D_\mu \theta^A$ using the covariant derivative for the adjoint representation.

Exercise 4.19 Show that the covariant derivatives of the three fields in (4.89) transform in the same way as the fields themselves, and with no derivatives of the gauge parameters. For example $\delta D_\mu \Psi = -\theta^A t_A D_\mu \Psi$.

Given this result it is easy to see that any globally symmetric action for scalar and spinor matter fields becomes gauge invariant if one replaces $\partial_\mu \to D_\mu$ for all fields. If this is done in (4.85), one obtains the equation of motion

$$\frac{\delta S}{\delta \bar{\Psi}_{\alpha}} = -\left[\gamma^{\mu} D_{\mu} - m\right] \Psi^{\alpha} = 0$$

8.3.15 Yang-Mills field strength and action

The next step in the development is to define the quantities that determine the dynamics of the gauge field itself. The simplest way to proceed is to compute the commutator of two covariant derivatives acting on a field in the representation R. We would get the same information, no matter which representation, so we will study just the case $[D_{\mu}, D_{v}]\,\Psi \equiv (D_{\mu}D_{v} - D_{v}D_{\mu})\,\Psi$. A careful computation gives

$$[D_{\mu}, D_{\nu}]\,\Psi = gF^{A}_{\mu\nu} t_{A} \Psi$$

where

$$F^{A}_{\mu\nu} = \partial_{\mu}A^{A}_{\nu} - \partial_{\nu}A^{A}_{\mu} + gf_{BC}{}^{A}A^{B}_{\mu}A^{C}_{\nu}.$$

The properties of the covariant derivative guarantee that the right-hand side of (4.91) transforms as a field in the same representation as Ψ. Thus $F^{A}_{\mu\nu}$ should have simple transformation properties. Indeed, one can derive

$$\delta F^{A}_{\mu\nu} = \theta^{C} F^{B}_{\mu\nu} f_{BC}{}^{A}.$$

We see that $F^{A}_{\mu\nu}$ is an antisymmetric tensor in spacetime, which transforms as a field in the adjoint representation of $\mathfrak{g}$; $F^{A}_{\mu\nu}$ is the non-abelian generalization of the electromagnetic field strength (4.6). The principal differences between abelian and non-abelian gauge symmetry are that the non-abelian field strength is not gauge invariant, but transforms in the adjoint representation, and that it is nonlinear in the gauge potential A^{A}_{μ}.

Exercise 4.20 Derive (4.93).

Despite these significant differences, it is quite straightforward to formulate the YangMills equations by following the ideas of the electromagnetic case. Since both the current and field strength transform in the adjoint representation, and the covariant derivative does not change the transformation properties, the equation

$$D^{\mu} F^{A}_{\mu\nu} = -J^{A}_{v}$$

is both gauge and Lorentz covariant. It is the basic dynamical equation of classical YangMills theory and the analogue of (4.14) for electromagnetism. One important difference, however, is that in the absence of matter sources, when the right-hand side of (4.94) vanishes, that equation is still a (much studied!) nonlinear equation for A^{A}_{μ}.

There is also a non-abelian analogue of the Bianchi identity (4.11), which takes the form

$$D_{\mu}F^{A}_{\nu\rho} + D_{\nu}F^{A}_{\rho\mu} + D_{\rho}F^{A}_{\mu\nu} = 0,$$

where $D_{\mu}F^{A}_{v\rho} = \partial_{\mu}F^{A}_{\nu\rho} + gf_{BC}{}^{A}A^{B}_{\mu}F^{C}_{\nu\rho}$.

Exercise 4.21 Show that (4.95) is satisfied identically if $F^{A}_{v\rho}$ is written in the form (4.92).

Exercise 4.22 Show that $D^{\nu}D^{\mu}F^{A}_{\mu\nu}$ vanishes identically (despite the nonlinearity). This is again a Noether identity: a relation between field equations that follows from the gauge symmetry.

As in the electromagnetic case, this means that the equation of motion (4.94) is consistent only if the current is covariantly conserved, i.e. only if $D^{v}J^{A}_{v} = 0$. It also means that (4.94) contains $(D-1)\dim_G$ independent equations, which is enough to determine the $D\dim_G$ components of A_μ^A up to a gauge transformation. It is usually convenient to "fix the gauge' by specifying $\dim_G$ conditions on the components of A_μ^A.

Note that, in the limit $g\to 0$, equations (4.92), (4.94), and (4.95) reduce to linear equations, which are $\dim_G$ copies of the corresponding equations for the free electromagnetic field. The count of degrees of freedom of Sec. 4.1.2 can be repeated in the Coulomb gauge $\partial^i A_i^A(\vec{x},t)=0$. For each component $A=1,\ldots,\dim_G, 2(D-2)$ functions must be specified as initial data, and each $A_i^A(x)$ has a Fourier transform identical to (4.9). In this free limit, the gauge field thus describes a particle with $D-2$ polarization states transforming in the adjoint representation of $\mathfrak{g}$.

The equations of motion of the Yang-Mills field A^A coupled to the Dirac field Ψ^α can be obtained from an action functional that is a natural generalization of (4.31):

$$S\left[A_\mu^A,\bar{\Psi}_\alpha,\Psi^\alpha\right]=\int \mathrm{d}^D x\left[-\frac{1}{4}F^{A\mu\nu}F_{\mu\nu}^A-\bar{\Psi}_\alpha\left(\gamma^\mu D_\mu-m\right)\Psi^\alpha\right]$$

The action is gauge invariant. The Euler variation with respect to A_v^A gives (4.94) with current source (4.86), and the variation with respect to $\bar{\Psi}_\alpha$ gives (4.90).

8.3.16 Yang-Mills theory for $G=\mathrm{SU}(N)$

The most commonly studied gauge group for Yang-Mills theory is $\mathrm{SU}(N)$. The generators of the fundamental representation of its Lie algebra are a set of N^2-1 traceless antihermitian $N\times N$ matrices t_A, which are normalized by the bilinear trace relation

$$\operatorname{Tr}\left(t_A t_B\right)=-\frac{1}{2}\delta_{AB}$$

In this section we discuss the special notation that has been developed for this case and is frequently used in the literature. In this notation gauge transformations are explicitly realized at the level of the group $\mathrm{SU}(N)$ rather than just at the level of its Lie algebra $\mathfrak{su}(N)$ as in the previous sections.

We will use the notation $U(x)=\mathrm{e}^{-\Theta(x)}$, with $\Theta(x)=\theta^A(x)t_A$, to denote an element of the gauge group in the fundamental representation. This may be viewed as a map $x^\mu\to U\left(x^\mu\right)$ from Minkowski spacetime into the group $\mathrm{SU}(N)$. In this notation the gauge transformation of a spinor field Ψ in the fundamental representation can be written (see (4.82))

$$\Psi(x)\to U(x)\Psi(x).$$

Row and column indices of the fundamental representation are consistently omitted in this notation. Usually we will omit the spacetime argument x^μ also, unless useful for special emphasis.

Given any matrix generator t_A, the unitary transformation $U(x)t_AU(x)^{-1}$ gives another traceless anti-hermitian matrix, which must then be a linear combination of the t_B. Therefore we can write

$$U(x)t_AU(x)^{-1}=t_BR(x)_A^B,$$

where $R(x)^B{}_A$ is a real $(N^2-1)\times(N^2-1)$ matrix.

Exercise 4.23 Consider the product of two gauge group elements U_1 and U_2, which gives a third via $U_1U_2=U_3$. For each element U_i, there is an associated matrix $\left(R_i\right)^B{}_A$, defined by $U_it_AU_i^{-1}=t_B\left(R_i\right)^B{}_A$. Prove that $\left(R_3\right)^B{}_A=\left(R_1\right)^B{}_C\left(R_2\right)^C{}_A$, which shows that the matrices $R^B{}_A$ defined by (4.99) are the matrices of an (N^2-1)-dimensional representation of $\mathrm{SU}(N)$.

Use (4.99) to show that, to first order in the gauge parameters θ^C, $R_A^B = \delta_A^B + \theta^C f_{AC}{}^B + \ldots$ This shows that the matrices $R^B{}_A$ are exactly those of the adjoint representation. [15]

Given any set of $N^2 - 1$ real quantities X^A, that is any element of the vector space $\mathbb{R}^{N^2-1}$, we can form the matrix $\mathbf{X} = t_A X^A$. For any group element U, we have $U\mathbf{X}U^{-1} = t_B R^B{}_A X^A$. Thus the unitary transformation of the matrix $\mathbf{X}$ contains the information that the quantities $X^A = -2\delta^{AB}\operatorname{Tr}(t_B\mathbf{X})$ transform in the adjoint representation, that is as $X^A \to R^A{}_B X^B$. Thus, given any field in the adjoint representation, such as $\phi^A(x)$, we can form the matrix $\Phi(x) = t_A\phi^A(x)$. Gauge transformations can then be implemented as

$$\Phi(x) \to U(x)\Phi(x)U(x)^{-1}$$

One can also form the matrix $\mathbf{A}_\mu(x) = t_A A_\mu^A(x)$ for the gauge potential. Quite remarkably, the gauge transformation of the potential can be expressed in matrix form if we define the transformation by

$$\mathbf{A}_\mu(x) \to \mathbf{A}'_\mu(x) \equiv \frac{1}{g}U(x)\partial_\mu U(x)^{-1} + U(x)\mathbf{A}_\mu(x)U(x)^{-1}$$

15 The equation (4.99) is true for the generators t_A of any representation of any Lie algebra $\mathfrak{g}$ and the associated group element $U = \mathrm{e}^{-\theta^C t_C}$. It follows that the matrices $R^B{}_A$ are those of the adjoint representation of G. A matrix description of Yang-Mills theory for a general gauge group can then be constructed by following the procedure discussed below for the fundamental representation of $\mathrm{SU}(N)$.

For infinitesimal transformations this becomes

$$\delta A_\mu(x) = \frac{1}{g}\partial_\mu\Theta(x) + [A_\mu(x), \Theta(x)]$$

which agrees with (4.88).

Exercise 4.24 Suppose that $\mathbf{A}_\mu \to \mathbf{A}'_\mu$ by the gauge transformation $U_2(x)$ followed by $\mathbf{A}'_\mu \to \mathbf{A}''_\mu$ by the gauge transformation $U_1(x)$. Show that the combined transformation $\mathbf{A}_\mu \to \mathbf{A}''_\mu$ is correctly described by the definition (4.101) for the product matrix $U_2(x)U_1(x)$. This result is compatible with (1.23) and with the implementation of gauge transformations by unitary operators in the quantum theory.

It is easy to define covariant derivatives in which the gauge potential appears in matrix form. For fields Ψ in the fundamental and $\bar\Psi$ in the anti-fundamental representation (and transforming as $\bar\Psi \to \bar\Psi U^{-1}$), the previous definitions in (4.89) can simply be rewritten as

$$\begin{aligned} D_\mu\Psi &\equiv (\partial_\mu + g\mathbf{A}_\mu)\Psi \\ D_\mu\bar\Psi &\equiv \partial_\mu\bar\Psi - g\bar\Psi\mathbf{A}_\mu \end{aligned}$$

For a field in the adjoint representation, such as Φ, we define

$$D_\mu\Phi = \partial_\mu\Phi + g[\mathbf{A}_\mu, \Phi],$$

which involves the matrix commutator.

Exercise 4.25 Demonstrate that these covariant derivatives transform correctly, specifically that

$$D_\mu\Psi \to U(x)D_\mu\Psi, \quad D_\mu\bar\Psi \to D_\mu\bar\Psi U(x)^{-1}, \quad D_\mu\Phi \to U(x)D_\mu\Phi U(x)^{-1}$$

The non-abelian field strength can also be converted to matrix form as

$$\mathbf{F}_{\mu\nu} = t_A F_{\mu\nu}^A = \partial_\mu\mathbf{A}_\nu - \partial_\nu\mathbf{A}_\mu + g[\mathbf{A}_\mu, \mathbf{A}_\nu]$$

8.3.17 Exercise 4.26 Prove this.

The matrix formalism is a convenient way to express quantities of interest in the theory. For example the Yang-Mills action (4.96) can be written as

$$S\left[\mathbf{A}_\mu, \bar{\Psi}, \Psi\right] = \int \mathrm{d}^D x \left[\frac{1}{2} \operatorname{Tr}\left(\mathbf{F}^{\mu v} \mathbf{F}_{\mu\nu}\right) - \bar{\Psi}\left(\gamma^\mu \mathrm{D}_\mu - \mathrm{m}\right) \Psi\right]$$

The N^2-1 matrix generators $(t_A)^\alpha{}_\beta$ of the fundamental representation, normalized as in (4.97), together with the matrix $\mathrm{i}\delta^\alpha_\beta$ form a complete set of $N \times N$ anti-hermitian matrices, which are orthogonal in the trace norm. Therefore one can expand any $N \times N$ anti-hermitian matrix $H^\alpha{}_\beta$ in this set as

$$\begin{aligned} H^\alpha_\beta &= \mathrm{i} h_0 \delta^\alpha_\beta + h^A \left(t_A\right)^\alpha_\beta, \\ h_0 = -\tfrac{\mathrm{i}}{N} \operatorname{Tr} H, \quad h^A &= -2\delta^{AB} \operatorname{Tr}\left(H t_B\right). \end{aligned}$$

Note that there is a sum over the N^2-1 values of the repeated indices A, B in (4.108) and in the exercise below.

Exercise 4.27 Use the completeness property (perhaps with Sec. 3.2.3 as a guide) to derive the rearrangement relation

$$\delta^\alpha_\beta \delta^\gamma_\delta = \frac{1}{N} \delta^\alpha_\delta \delta^\gamma_\beta - 2\left(t_A\right)^\alpha_\delta \delta^{AB} \left(t_B\right)^\gamma_\beta$$

8.3.18 Internal symmetry for Majorana spinors

Majorana spinors play a central role in supersymmetric field theories. In many applications they transform in a representation of a non-abelian internal symmetry group. For example, the spinor fields of super-Yang-Mills theory are denoted as λ^A and transform in the adjoint representation of the gauge group. In the notation of Sec. 4.3.4, we have $\lambda^A \to \lambda'^A = R^A_B \lambda^B$. Since the matrix R^A_B is real, this transformation rule is consistent with the fact that Majorana spinors obey a reality constraint. Indeed, as shown in Sec. 3A.5, there are really real representations of the Clifford algebra in which the spinors are explicitly real. One can consider the more general situation of a set of Majorana spinors Ψ^α transforming as $\Psi^\alpha \to \Psi'^\alpha = \left(\mathrm{e}^{-\theta^A t_A}\right)^\alpha{}_\beta \Psi^\beta$. The transformed Ψ'^α must also satisfy the Majorana condition, and this requires that the matrices $\mathrm{e}^{-\theta^A t_A}$ are those of a really real representation of the group G. (Obviously there is a similar requirement on the symmetry transformation of a set of real scalars, such as the ϕ^A of Sec. 4.3.1.)

In $D=4$ dimensions, the requirement that Majorana spinors transform in a real representation of the gauge group can be bypassed because internal symmetries can include chiral transformations, which involve the highest rank element $\gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$ of the Clifford algebra discussed in Sec. 3.1.6. This matrix is imaginary in a Majorana representation, or in general under the C-operation; see (3.78). We use the chiral projectors P_L and P_R as in (3.38). Suppose that the matrices t_A are generators of a complex representation of the Lie algebra. Then the complex conjugate matrices t^*_A are generators of the conjugate representation. Let χ^α denote a set of Majorana spinors to which we assign the group transformation rule

$$\chi^\alpha \to \chi'^\alpha \equiv \left(\mathrm{e}^{-\theta^A\left(t_A P_L + t^*_A P_R\right)}\right)^\alpha{}_\beta \chi^\beta$$

The matrices $t_A P_L + t^*_A P_R$ are generators of a representation of an explicitly real representation of the Lie algebra, so the transformed spinors χ'^α also satisfy the Majorana condition. This is the transformation rule used for Majorana spinors in supersymmetric gauge theories in Ch. 6.

By applying the projectors to (4.110), one can see that the chiral and anti-chiral projections of χ transform as

$$P_L\chi \to P_L\chi' \equiv \left(\mathrm{e}^{-\theta^A t_A}\right) P_L\chi,$$
$$P_R\chi \to P_R\chi' \equiv \left(\mathrm{e}^{-\theta^A t_A^*}\right) P_R\chi.$$

Exercise 4.28 What is the covariant derivative $D_\mu\chi$? We now use $\bar{\chi}$ for a Majorana conjugate (3.50), where the transpose includes a transpose in the representation space. When representation indices are needed, $\bar{\chi}$ carries a lower index. Show that then the kinetic Lagrangian density $\bar{\chi}\gamma^\mu D_\mu\chi$ is invariant under the infinitesimal limit of the transformation (4.110) for anti-hermitian t_A, and that the variation of the mass term is

$$\delta(\bar{\chi}\chi) = -\theta^A\bar{\chi}\left(t_A + t_A^T\right)\gamma_*\chi$$

The mass term is invariant only for the subset of generators that are antisymmetric, and thus real. This condition defines a subalgebra of the original Lie algebra $\mathfrak{g}$ of the theory, specifically the subalgebra that contains only parity conserving vector-like gauge transformations. For the case $\mathfrak{g} = \mathfrak{su}(N)$, the subalgebra is isomorphic to $\mathfrak{so}(N)$. Non-invariance of the Majorana mass term is a special case of the general idea that chiral symmetry requires massless fermions.

Exercise 4.29 Show that

$$\frac{1}{2}\int \mathrm{d}^4x\bar{\chi}\gamma^\mu D_\mu\chi = \int \mathrm{d}^4x\bar{\chi}\gamma^\mu P_L D_\mu\chi = \int \mathrm{d}^4x\bar{\chi}\gamma^\mu P_R D_\mu\chi$$

Note that $P_{L,R}D_\mu\chi = D_\mu P_{L,R}\chi$.

8.4 The free Rarita-Schwinger field

In this chapter we begin to assemble the ingredients of supergravity by studying the free spin-3/2 field. Supergravity is the gauge theory of global supersymmetry, which we will usually abbreviate as SUSY. The key feature is that the symmetry parameter of global SUSY transformations is a constant spinor ϵ_α. In supergravity it becomes a general function in spacetime, $\epsilon_\alpha(x)$. The associated gauge field is a vector-spinor $\Psi_{\mu\alpha}(x)$. This field and the corresponding particle have acquired the name 'gravitino'.

Supergravity theories necessarily contain the gauge multiplet, the set of fields required to gauge the symmetry in a consistent interacting theory, and may contain matter multiplets, sets of fields on which global SUSY is realized. The gauge multiplet contains the gravitational field, one or more vector-spinors, and sometimes other fields. This structure is derived from representations of the SUSY algebras in Sec. 6.4.2. In this chapter we are concerned with the free limit, in which the various fields do not interact, and we can consider them separately. In particular we consider $\Psi_\mu(x)$ (omitting the spinor index α) as a free field, subject to the gauge transformation

$$\Psi_\mu(x) \to \Psi_\mu(x) + \partial_\mu\epsilon(x)$$

Furthermore we will assume that Ψ_μ and ϵ are complex spinors with $2^{[D/2]}$ spinor components for spacetime dimension D. This is fine for the free theory in any dimension D, but interacting supergravity theories are more restrictive as to the spinor type permitted in a given spacetime dimension (and such theories exist only for $D \leq 11$). We will need to use the required Majorana and/or Weyl spinors when we study these theories in later chapters (and the number $2^{[D/2]}$ must be adjusted to agree with the number of components of each type of spinor).

It is consistent with the pattern set in the previous chapter that the gauge field $\Psi_\mu(x)$ is a field with one more vector index than the gauge parameter $\epsilon(x)$. Furthermore, as in the case of electromagnetism, the antisymmetric derivative $\partial_\mu\Psi_\nu - \partial_\nu\Psi_\mu$ is gauge invariant. An important difference arises because we now seek a gauge invariant first order wave equation for the fermion field. It is advantageous to start with the action, which must be (a) Lorentz invariant, (b) first order in spacetime derivatives, (c) invariant under the gauge transformation (5.1) and the simultaneous conjugate transformation of $\bar\Psi_\mu$, and (d) hermitian, so that the Euler-Lagrange equation for $\bar\Psi_\mu$ is the Dirac conjugate of that for Ψ_μ. It is easy to see that the expression

$$S = -\int \mathrm{d}^D x \bar\Psi_\mu \gamma^{\mu\nu\rho}\partial_\nu \Psi_\rho$$

which contains the third rank Clifford algebra element $\gamma^{\mu\nu\rho}$, has all these properties. Note that the action is gauge invariant but the Lagrangian density is not. Instead its variation is the total derivative $\delta\mathcal{L} = -\partial_\mu\left(\bar\epsilon\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho\right)$. The reason is that the fermionic gauge symmetry is the remnant of supersymmetry, and the anti-commutator of two SUSY transformations is a spacetime symmetry.

It should be noted that a physically equivalent theory can be obtained by rewriting (5.2) in terms of the new field variable $\Psi'_\mu \equiv \Psi_\mu + a\gamma_\mu\gamma\cdot\Psi$ where a is an arbitrary parameter. [1] The gauge transformation is modified accordingly. The presentation in (5.1) and (5.2) is universally used in the modern literature, because the gauge transformation is simplest and closely resembles that of electromagnetism. Historically, Rarita and Schwinger invented a wave equation for a massive spin-3/2 particle in 1941. The massless limit of the action is a transformed version of (5.2), and Rarita and Schwinger simply noted that it possesses a fermionic gauge symmetry. [2]

The equation of motion obtained from (5.2) reads

$$\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho = 0.$$

One can immediately see that it shares some of the properties of the analogous electromagnetic equation (4.7), which is $\partial^\mu F_{\mu\nu} = 0$. Gauge invariance is manifest, and the lefthand side vanishes identically when the derivative ∂_μ is applied. Thus (5.3) comprises $(D-1)2^{[D/2]}$ independent equations, which is enough to determine the $2^{[D/2]}D$ components of Ψ_ρ up to the freedom of a gauge transformation. The difference between the number of components of the gauge field and those of the gauge parameter, in this case $(D-1)2^{[D/2]}$, is called the number of off-shell degrees of freedom.

Exercise 5.1 Show directly that for $D = 3$, the field equation (5.3) implies that $\partial_\nu\Psi_\rho$ - $\partial_\rho\Psi_v = 0$. This means that the field has no gauge invariant degrees of freedom and thus no propagating particle modes. This is the supersymmetric counterpart of the situation in gravity for $D = 3$, where the field equation $R_{\mu\nu} = 0$ implies that the full curvature tensor $R_{\mu\nu\rho\sigma} = 0$. Hence no degrees of freedom.

We notice that (5.3) can be rewritten in an equivalent but simpler form. For this purpose, we use the γ-matrix relation $\gamma_\mu\gamma^{\mu\nu\rho} = (D-2)\gamma^{\nu\rho}$, which implies that $\gamma^{\nu\rho}\partial_\nu\Psi_\rho = 0$ in spacetime dimension $D > 2$. We also note that $\gamma^{\mu\nu\rho} = \gamma^\mu\gamma^{\nu\rho} - 2\eta^{\mu[\nu}\gamma^{\rho]}$. Using this information, it is easy to see that (5.3) implies that

$$\gamma^\mu\left(\partial_\mu\Psi_\nu - \partial_\nu\Psi_\mu\right) = 0$$

1 The case $a = -1/D$ requires special treatment since $\gamma\cdot\Psi' = 0$.

2 One of the present authors met Prof. Schwinger at a cocktail party in the early 1980s. Supergravity came up in the conversation, and Schwinger remarked lightheartedly 'I should have discovered supergravity.'

This is an alternative form of the equation of motion, equivalent to (5.3), but which cannot be obtained directly from an action. To see that (5.4) is equivalent, note that one can apply γ^ν and obtain $\gamma^{\nu\rho}\partial_\nu\Psi_\rho = 0$. The previous steps can then be reversed to obtain (5.3) from (5.4). One can also show that the left-hand side of (5.4) vanishes identically if $\gamma^\nu\partial\!\!\!/$ is applied. Finally, let's apply ∂_ρ to (5.4) and antisymmetrize in ρv to obtain

$$\partial\!\!\!/ \left(\partial_\rho\Psi_\nu - \partial_\nu\Psi_\rho\right) = 0.$$

This is a gauge invariant derivation of the fact that the wave equations, either (5.3) or (5.4), describe massless particles.

Exercise 5.2 Do all the manipulations in the preceding paragraph. Do them backwards and forwards.

8.4.1 The initial value problem

Let's now study the initial value problem for (5.3) and thus count the number of on-shell degrees of freedom. We must untangle constraints on the initial data from time evolution equations. For this purpose we need to fix the gauge, so we impose the non-covariant condition

$$\gamma^i\Psi_i = 0$$

which will play the same role as the Coulomb gauge condition we used in Sec. 4.1.2.

Exercise 5.3 Show by an argument analogous to that in Sec. 4.1.2 that this condition does fix the gauge uniquely.

We use the equivalent form (5.4) of the field equations. The $v = 0$ and $v \to i$ components are

$$\begin{aligned}\gamma^i\partial_i\Psi_0 - \partial_0\gamma^i\Psi_i &= 0,\\ \gamma\cdot\partial\Psi_i - \partial_i\gamma\cdot\Psi &= 0.\end{aligned}$$

Using the gauge condition one can see that $\nabla^2\Psi_0 = 0$, so $\Psi_0 = 0$ according to the discussion on p. 69. The spatial components Ψ_i then satisfy the Dirac equation

$$\gamma\cdot\partial\Psi_i = 0$$

which is a time evolution equation. However, there is an additional constraint, $\partial^i\Psi_i = 0$, obtained by contracting (5.8) with γ^i. Thus from the gauge condition and the equation of motion, we find $3\times 2^{[D/2]}$ independent constraints on the initial data, namely

$$\begin{aligned}\gamma^i\Psi_i(\vec{x},0) &= 0,\\ \Psi_0(\vec{x},0) &= 0,\\ \partial^i\Psi_i(\vec{x},0) &= 0.\end{aligned}$$

On-shell degrees of freedom $= \frac{1}{2}(D-3)2^{[D/2]}$.

Off-shell degrees of freedom $= (D-1)2^{[D/2]}$.

These constraints imply that there are only $2^{[D/2]}(D-3)$ initial components of Ψ_i to be specified. The time derivatives are already determined by the Dirac equation (5.8). Hence there are $2^{[D/2]}(D-3)$ classical degrees of freedom for the Rarita-Schwinger gauge field in D-dimensional Minkowski space. The on-shell degrees of freedom are half this number. In dimension $D = 4$, with Majorana conditions, we find the two states expected for a massless particle for any spin $s > 0$. We will show below that these states carry helicity $\pm 3/2$. In general dimension, it should be a representation of $SO(D-2)$ as discussed in Sec. 4.1.2. Indeed, the vector-spinor representation is an irreducible representation after subtraction of the γ-trace. It then contains $\frac{1}{2}(D-3)2^{[D/2]}$ components.

Exercise 5.4 Analyze the degrees of freedom using the original equation of motion (5.3).

According to the discussion for $D=4$ at the beginning of Ch. 4, we would expect the Fourier expansion of the field to contain annihilation and creation operators for states of helicity $\lambda = \pm 3/2$. Let's derive this fact starting from the plane wave

$$\Psi_i(x) = \mathrm{e}^{\mathrm{i}p\cdot x} v_i(\vec{p}) u(\vec{p})$$

for a positive null energy-momentum vector $p^\mu = (|\vec{p}|, \vec{p})$. Since $\Psi_i(x)$ satisfies the Dirac equation (5.8), the four-component spinor $u(\vec{p})$ must be a superposition of the massless helicity spinors $u(\vec{p}, \pm)$ given in (2.44). Thus we use the Weyl representation (2.19) of the γ-matrices. The vector $v_i(\vec{p})$ may be expanded in the complete set

$$v_i(\vec{p}) = a p_i + b\epsilon_i(\vec{p}, +) + c\epsilon_i(\vec{p}, -)$$

where $\epsilon_i(\vec{p}, \pm)$ are the transverse polarization vectors of Sec. 4.1.2, i.e. they satisfy $p^i \epsilon_i(\vec{p}, \pm) = 0$. The constraint (5.11) requires that $a = 0$. Thus (5.12) is reduced to the form

$$\begin{aligned}\Psi_i(x) = \mathrm{e}^{\mathrm{i}p\cdot x}[&b_+\epsilon_i(\vec{p}, +)u(\vec{p}, +) + c_+\epsilon_i(\vec{p}, -)u(\vec{p}, +)\\ &+ b_-\epsilon_i(\vec{p}, +)u(\vec{p}, -) + c_-\epsilon_i(\vec{p}, -)u(\vec{p}, -)]\end{aligned}$$

We must still enforce the constraint $\gamma^i\Psi_i = 0$. Some detailed algebra is needed, which we leave to the reader; the result is that $c_+ = b_- = 0$, while $b_+$and $c_-$are arbitrary. Thus there are two independent physical wave functions $\epsilon_i(\vec{p}, \pm)u(\vec{p}, \pm)$ for each p^μ.

Exercise 5.5 Do the algebra that was just left for the reader. Show that the resulting vector-spinor wave functions $\epsilon_i(\vec{p}, \pm)u(\vec{p}, \pm)$ carry helicity $\pm 3/2$. Show that the spinor wave functions for the conjugate plane wave are $\epsilon_i^*(\vec{p}, \pm)v(\vec{p}, \pm)$, where $v(\vec{p}, \pm) = B^{-1}u(\vec{p}, \pm)^*$ are the massless v spinors of (2.45).

The net result of this analysis is that the Rarita-Schwinger field that satisfies the equation of motion and constraints above has the Fourier expansion (we add the trivial 0-components $\epsilon_0 = 0$ to polarization vectors)

$$\Psi_\mu(x) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0} \sum_\lambda \left[\mathrm{e}^{\mathrm{i}p\cdot x}\epsilon_\mu(\vec{p}, \lambda)u(\vec{p}, \lambda)c(\vec{p}, \lambda) + \mathrm{e}^{-\mathrm{i}p\cdot x}\epsilon_\mu^*(\vec{p}, \lambda)v(\vec{p}, \lambda)d^*(\vec{p}, \lambda)\right].$$

The sum extends over the two physical wave functions of helicity $\pm 3/2$. In the quantum theory the Fourier amplitude $c(\vec{p}, \lambda)$ becomes the annihilation operator for helicity $\pm 3/2$ particles, and $d^*(\vec{p}, \lambda)$ becomes the creation operator for anti-particles. The situation is similar to that for the Dirac field in (2.24). A Majorana gravitino has the same expansion, with $d^*(\vec{p}, \lambda) = c^*(\vec{p}, \lambda)$, since there is no distinction between particles and anti-particles.

In dimension $D > 4$ the allowed gravitino modes are obtained by starting with products of the $D-2$ transverse polarization vectors $\epsilon_i(\vec{p}, j)$ and the $\frac{1}{2}2^{[D/2]}$ massless Dirac spinors $u(\vec{p}, s)$. The gauge fixing constraint $\gamma^i\Psi_i = 0$ must then be enforced on linear combinations of these products as was done in (5.14). This leads to $\frac{1}{2}2^{[D/2]}(D-3)$ independent wave functions, which describe the on-shell states of the gravitino.

The canonical stress tensor obtained from (5.2) is

$$T_{\mu\nu} = \bar{\Psi}_\rho \gamma_\mu^{\rho\sigma} \partial_\nu \Psi_\sigma - \eta_{\mu\nu}\mathcal{L}.$$

It is neither symmetric nor gauge invariant under (5.1) (and its Dirac conjugate). It can be made symmetric (see [27]), but gauge non-invariance is intrinsic and cannot be restored by adding terms of the form $\partial_\sigma S^{\sigma\mu\nu}$. The reason is that the gravitino must be joined with gravity in the gauge multiplet of SUSY. In a gravitational theory there is no well-defined energy density.

Exercise 5.6 Show that the total energy-momentum $P^v = \int \mathrm{d}^3\vec{x}T^{0v}(\vec{x},t)$ is gauge invariant and given (for $D=4$) by

$$P^v = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3 2p^0} p^\nu \sum_\lambda \left[c^*(\vec{p},\lambda)c(\vec{p},\lambda) - d(\vec{p},\lambda)d^*(\vec{p},\lambda)\right]$$

8.4.2 Sources and Green's function

Let's follow the pattern of Sec. 4.1.3 and couple the Rarita-Schwinger field to a vectorspinor source via

$$\gamma^{\mu\nu\rho}\partial_\nu\Psi_\rho = J^\mu$$

The contraction of ∂_μ with the left-hand side vanishes identically, which indicates that (5.18) is a consistent equation only if the source current is conserved, i.e. $\partial_\mu J^\mu = 0$. This is the exact analogue of what happens in electromagnetism and Yang-Mills theory. In those theories, the gauge field was later coupled to matter systems, and the source was the Noether current of the global symmetry. Supergravity theories are more complicated.

The same phenomenon occurs, but only as an approximation valid to lowest order in the gravitational coupling. The current J^μ is the Noether supercurrent of the matter multiplets in the theory.

Let's now apply the method of Sec. 4.1.3 to find the Green's function that determines the response of the field to the source. We first solve the simpler problem for the Dirac field,

$$(\not{\partial} - m)\Psi(x) = J(x).$$

Given a Green's function $S(x-y)$ that satisfies

$$(\not{x}_x - m)\, S(x-y) = -\delta(x-y),$$

the solution of (5.19) is given by

$$\Psi(x) = -\int \mathrm{d}^D y S(x-y)J(y)$$

Let's solve this problem using the Fourier transform. The symmetries of Minkowski spacetime allow us to assume the Fourier representation

$$S(x-y) = \int \frac{\mathrm{d}^D p}{(2\pi)^D} \mathrm{e}^{\mathrm{i}p\cdot(x-y)} S(p)$$

In momentum space, (5.20) reads

$$(\mathrm{i}\not{p} - m)S(p) = -1$$

and the solution (with Feynman's causal structure) is

$$S(p) = -\frac{1}{\mathrm{i}\not{p} - m} = \frac{\mathrm{i}\not{p} + m}{p^2 + m^2 - \mathrm{i}\epsilon}.$$

Comparing with (4.18), we see that we can express $S(x-y)$ in terms of the scalar Green's function as

$$S(x-y) = (\not{\partial}x + m)G(x-y).$$

This result satisfies (5.20) by inspection and could have been guessed at the start. However, the Fourier transform method is useful as a warmup for the more complicated case of the Rarita-Schwinger field.

We expect the Green's function solution of (5.18) to take the form

$$\Psi_\mu(x) = -\int \mathrm{d}^D y S_{\mu\nu}(x-y) J^\nu(y)$$

where $S_{\mu\nu}(x-y)$ is a tensor bispinor. A bispinor has two spinor indices, which are suppressed in our notation, and it can be regarded as a matrix of the Clifford algebra. As in the electromagnetic case, the Rarita-Schwinger operator is not invertible, but we can assume that the Green's function satisfies

$$\gamma^{\mu\sigma\rho}\frac{\partial}{\partial x^\sigma} S_{\rho\nu}(x-y) = -\delta^\mu_v \delta(x-y) + \frac{\partial}{\partial y^\nu}\Omega^\mu(x-y).$$

The last term on the right is a 'pure gauge' in the source point index. In momentum space (5.27) reads

$$\mathrm{i}\gamma^{\mu\sigma\rho} p_\sigma S_{\rho v}(p) = -\delta^\mu_v - \mathrm{i} p_v \Omega^\mu(p)$$

We will solve (5.28) by writing an appropriate ansatz for $S_{\rho\nu}(p)$ and then find the unknown functions in the ansatz. The matrix $\gamma^{\mu\sigma\rho}p_\sigma$ in (5.28) contains an odd rank element of the Clifford algebra and it is odd under the reflection $p_\sigma \to -p_\sigma$. It is reasonable to guess that the ansatz we need should also involve odd rank Clifford elements and be odd under the reflection. We would also expect that terms that contain the momentum vectors p_ρ or p_v are 'pure gauges' and thus arbitrary additions to the propagator, which would not be determined by the equation (5.28). So we omit such terms and postulate the ansatz

$$\mathrm{i}\ S_{\rho\nu}(p) = A\left(p^2\right)\eta_{\rho\nu}\not{p} + B\left(p^2\right)\gamma_\rho \not{p}\gamma_\nu.$$

The next step is to substitute the ansatz in (5.28) and simplify the products of γ-matrices that appear. This process yields

$$\begin{aligned}
\mathrm{i}\gamma^{\mu\sigma\rho} p_\sigma S_{\rho\nu}(p) =& A\gamma^{\mu\sigma}{}_\nu \not{p} p_\sigma + (D-2)B\gamma^{\mu\sigma}\not{p}\gamma_\nu p_\sigma \\
=& A\left(p^\mu\gamma^\sigma_\nu - p^\sigma\gamma^\mu_\nu\right)p_\sigma + (D-2)B\left(-p^\mu\gamma^\sigma + p^\sigma\gamma^\mu\right)\gamma_\nu p_\sigma \\
&+ \ldots \\
=& [A-(D-2)B]\left(p^\mu\gamma^\sigma{}_\nu - p^\sigma\gamma^\mu{}_\nu\right)p_\sigma + (D-2)Bp^2\delta^\mu_v \\
&+ \ldots
\end{aligned}$$

We have omitted terms that are proportional to the vector p_ν, because such terms will be 'matched' in (5.28) by $\Omega^\mu(p)$ rather than by δ^μ_v. It is now easy to see that the δ^μ_v term in (5.28) determines the values $A = -1/p^2$ and $B = -1/\left((D-2)p^2\right)$. Thus we have found the gravitino propagator

$$S_{\mu\nu}(p) = \mathrm{i}\frac{1}{p^2}\left[\eta_{\mu\nu}\not{p} + \frac{1}{D-2}\gamma_\mu\not{p}\gamma_\nu + Cp_\mu\gamma_\nu + E\gamma_\mu p_v + Fp_\mu\not{p}p_v\right]$$

in which we have added possible gauge terms that are not determined by this procedure. In position space the propagator is

$$S_{\mu\nu}(x-y) = \left[\eta_{\mu\nu}\not{\partial} + \frac{1}{D-2}\gamma_\mu\not{\gamma}_\nu + C\partial_\mu\gamma_\nu + E\gamma_\mu\partial_\nu - F\partial_\mu\not{\partial}\partial_\nu\right]G(x-y),$$

where $G(x-y)$ is the massless scalar propagator (4.19), and all derivatives are with respect to x.

Exercise 5.7 Include the omitted p_v terms in (5.30) and $\Omega(p)$ in the analysis and verify that the gauge terms in the propagator are arbitrary. Show that, for the choice $E = -1/(D-2)$, and arbitrary C and F, the propagator satisfies

$$\mathrm{i}\gamma^{\mu\sigma\rho}p_\sigma S_{\rho v}(p) = -\left(\delta^\mu_v - \frac{p^\mu p_v}{p^2}\right)$$

Show that, for $D = 4$, the propagator, with $C = -1$, takes the 'reverse index' form $S_{\mu\nu}(p) = -\mathrm{i}\frac{1}{2}\gamma_\nu \not{p} \gamma_\mu$, which is the form used in most of the literature on perturbative studies in supergravity [28].

8.4.3 Massive gravitinos from dimensional reduction

Our aim in this section is quite narrow, but the approach will be broad. The narrow goal is to extend the Rarita-Schwinger equation to describe massive gravitinos, but we wish to do it by introducing the important technique of dimensional reduction, which is also called Kaluza-Klein theory. The main idea is that a fundamental theory, perhaps supergravity or string theory, that is formulated in D' spacetime dimensions can lead to an observable spacetime of dimension $D < D'$. In the most common variant of this scenario, there is a stable solution of the equations of the fundamental theory that describes a manifold of the structure $M_{D'} = M_D \times X_d$ with $d = D' - D$. The factor M_D is the spacetime in which we might live, thus non-compact with small curvature, while X_d is a tiny compact manifold of spatial extent L. The compact space X_d can be thought of as hidden dimensions of spacetime that are not accessible to direct observation because of basic properties of wave physics that are coded in quantum mechanics as the uncertainty principle. This principle asserts that it would take wave excitations of energy $E \approx 1/L$ to explore structures of spatial scale L. If L is sufficiently small, this energy scale cannot be achieved by available apparatus. Nevertheless, the dimensional reduction might be confirmed since the presence of X_d has important indirect effects on physics in M_D.

In this section we study an elementary version of dimensional reduction, which still has interesting physics to teach. Instead of obtaining the structure $M_D \times X_d$ from a fundamental theory including gravity, we will simply explore the physics of the various free fields we have studied, assuming that the (D+1)-dimensional spacetime is Minkowski $_D \otimes S^1$. The main feature is that Fourier modes of fields on S^1 are observed as infinite 'towers' of massive particles by an observer in Minkowski i_D. The reduction of the free massless gravitino equation in $D+1$ dimensions will then tell us the correct description of massive gravitinos. Massive gravitinos appear in the physical spectrum of $D = 4$ supergravity when SUSY is spontaneously broken.

8.4.4 Dimensional reduction for scalar fields

Let's change to a more convenient notation and rename the coordinates of the $(D+1)$ dimensional product spacetime $x^0 = t, x^1, ..., x^{D-1}, y$, where y is the coordinate of S^1 with range $0 \le y \le 2\pi L$. We consider a massive complex scalar field $\phi\,(x^\mu, y)$ that obeys the Klein-Gordon equation

$$\left[\square_{D+1} - m^2\right]\phi = \left[\square_D + \left(\frac{\partial}{\partial y}\right)^2 - m^2\right]\phi = 0$$

Acceptable solutions must be single-valued on S^1 and thus have a Fourier series expansion

$$\phi\,(x^\mu, y) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}ky/L}\phi_k\,(x^\mu)$$

It is immediate that the spacetime function associated with the k th Fourier mode, namely $\phi_k(x^\mu)$, satisfies

$$\left[\Box_D - \left(\frac{k}{L}\right)^2 - m^2\right]\phi_k = 0$$

Thus it describes a particle of mass $m_k^2 = (k/L)^2 + m^2$. So the spectrum of the theory, as viewed in Minkowski $_D$, contains an infinite tower of massive scalars!

There is an even simpler way to find the mass spectrum. Just substitute the plane wave $\mathrm{e}^{\mathrm{i}p^\mu x_\mu}\mathrm{e}^{\mathrm{i}ky/L}$ directly in the $(D+1)$-dimensional equation (5.34). The D-component energy-momentum vector p^μ must satisfy $p^\mu p_\mu = (k/L)^2 + m^2$. The mass shift due to the Fourier wave on S^1 is immediately visible.

8.4.5 Dimensional reduction for spinor fields

We will consider the dimensional reduction process for a complex spinor $\Psi(x^\mu, y)$ for even $D = 2m$ (so that the spinors in $D+1$ dimensions have the same number of components). Two new ideas enter the game. The first just involves the Dirac equation in D dimensions. We remark that if $\Psi(x)$ satisfies

$$[\partial\!\!\!/_D - m]\,\Psi(x) = 0,$$

then the new field $\tilde{\Psi} \equiv \mathrm{e}^{-\mathrm{i}\gamma_*\beta}\Psi$, obtained by applying a chiral phase factor, satisfies

$$[\partial\!\!\!/_D - m(\cos 2\beta + \mathrm{i}\gamma_* \sin 2\beta)]\,\tilde{\Psi} = 0.$$

Physical quantities are unchanged by the field redefinition, so both equations describe particles of mass m. One simple implication is that the sign of m in (5.37) has no physical significance, since it can be changed by field redefinition with $\beta = \pi/2$.

The second new idea is that a fermion field can be either periodic or anti-periodic $\Psi(x^\mu, y) = \pm\Psi(x^\mu, y + 2\pi L))$. Anti-periodic behavior is permitted because a fermion field is not observable. Rather, bilinear quantities such as the energy density $T^{00} = -\bar{\Psi}\gamma^0\partial^0\Psi$ are observables and they are periodic even when Ψ is anti-periodic. Thus we consider the Fourier series

$$\Psi(x^\mu, y) = \sum_k \mathrm{e}^{\mathrm{i}ky/L}\Psi_k(x^\mu)$$

where the mode number k is integer or half-integer for periodic or anti-periodic fields, respectively. In either case when we substitute (5.39) in the $(D+1)$-dimensional Dirac equation $[\partial\!\!\!/_{+1} - m]\,\Psi((x^\mu, y) = 0$, we find that $\Psi_k(x^\mu)$ satisfies [3]

$$\left[\partial\!\!\!/ - \left(m - \mathrm{i}\gamma_*\frac{k}{L}\right)\right]\Psi_k(x^\mu) = 0$$

By applying a chiral transformation with phase $\tan 2\beta = k/(mL)$, we see that $\Psi_k(x^\mu)$ describes particles of mass ${m_k}^2 = (k/L)^2 + m^2$. Again we would observe an infinite tower of massive spinor particles with distinct spectra for the periodic and anti-periodic cases.

3 Recall from Ch. 3 that for odd spacetime dimension $D = 2m+1, \gamma^D = \pm\gamma_*$, where γ_* is the highest rank Clifford element in $D = 2m$ dimensions.

8.4.6 Dimensional reduction for the vector gauge field

We now apply circular dimensional reduction to Maxwell's equation

$$\partial^v F_{\nu\mu} = \Box_{D+1} A_\mu - \partial_\mu \left(\partial^v A_\nu\right) = 0$$

in $D+1$ dimensions, and we assume a periodic Fourier series representation

$$A_\mu(x,y) = \sum_k \mathrm{e}^{\mathrm{i}ky/L} A_{\mu k}(x), \quad A_D(x,y) = \sum_k \mathrm{e}^{\mathrm{i}ky/L} A_{Dk}(x)$$

with k an integer. The analysis simplifies greatly if we assume the gauge conditions $A_{Dk}(x) = 0$ for $k \neq 0$ and vector component D tangent to S^1. It is easy to see that this gauge can be achieved and uniquely fixes the Fourier modes $\theta_k(x), k \neq 0$, of the gauge function. The gauge invariant Fourier mode $A_{D0}(x)$ remains a physical field in the dimensionally reduced theory. A quick examination of the $\mu \to D$ component of (5.41) shows that it reduces to

$$\begin{aligned} k = 0: &\quad \Box_{D+1} A_{D0} = \Box_D A_{D0} = 0, \\ k \neq 0: &\quad \partial^\mu A_{\mu k} = 0, \end{aligned}$$

so the mode $A_{D0}(x)$ simply describes a massless scalar in D dimensions. For $\mu \leq D-1$, the wave equation (5.41) implies that the vector modes $A_{\mu k}(x)$ satisfy

$$\left[\Box_D - \frac{k^2}{L^2}\right] A_{\mu k} - \partial_\mu \left(\partial^v A_{\nu k}\right) = 0$$

For mode number $k = 0$ this is just the Maxwell equation in D dimensions with its gauge symmetry under $A_{\mu 0} \to A_{\mu 0} + \partial_\mu \theta_0$ intact, since the Fourier mode $\theta_0(x)$ remained unfixed in the process above. For mode number $k \neq 0$, (5.44) is the standard equation [4] for a massive vector field with mass $m_k^2 = k^2/L^2$, namely the equation of motion of the action

$$S = \int \mathrm{d}^D x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2} m^2 A_\mu A^\mu\right]$$

A counting argument similar to that for the massless case in Ch. 4 shows that we have the D-component field $A_{\mu k}$ subject to the single constraint (5.43) and thus giving $D-1$ quantum degrees of freedom for each Fourier mode $k \neq 0$. The $D-1$ particle states for each fixed energy-momentum p^μ transform in the vector representation of $\mathrm{SO}(D-1)$ as appropriate for a massive particle. Note that there are three states for $D = 4$, which agrees with $2s+1$ for spin $s = 1$. The count of states is the same in the massless $k = 0$ sector also, where we have the gauge vector $A_{\mu 0}$ plus the scalar A_{D0} with $(D-2)+1$ on-shell degrees of freedom.

8.4.7 Finally $\Psi_\mu(x,y)$

Let's apply dimensional reduction to the massless Rarita-Schwinger field in $D+1$ dimensions with $D = 2m$. We will assume that the field $\Psi_\mu(x,y)$ is anti-periodic in y so that its Fourier series involves modes $\exp(\mathrm{i}ky/L)\Psi_{\mu k}(x)$ with half-integral k. This assumption simplifies the analysis, since only $k \neq 0$ occurs, and all modes will be massive.

We would like to start with (5.3) in dimension $D+1$ and derive the wave equation of a massive gravitino in Minkowski $_D$. A gauge choice makes this task much easier. All Fourier modes have $k \neq 0$, so we can impose the gauge condition $\Psi_{Dk}(x) = 0$ on all modes and completely eliminate the field component $\Psi_D(x,y)$.

4 Note that the result (5.43) can be obtained by applying ∂^μ to (5.44) and is thus consistent with that equation.

Let's write out the $\mu = D$ and $\mu \leq D-1$ components of (5.3) with $\Psi_D = 0$ (using $\gamma^D = \gamma_*$)

$$\begin{aligned}
&\gamma^{\nu\rho}\partial_\nu\Psi_{\rho k} = 0 \\
&\left[\gamma^{\mu\nu\rho}\partial_\nu - \mathrm{i}\frac{k}{L}\gamma_*\gamma^{\mu\rho}\right]\Psi_{\rho k} = 0.
\end{aligned}$$

Note that the first equation of (5.46) follows by application of ∂^μ to the second one.

Exercise 5.8 Show that the chiral transformation $\Psi_{\rho k} = \mathrm{e}^{(-\mathrm{i}\pi\gamma_*/4)}\Psi'_{\rho k}$ leads, after replacing $\Psi' \rightarrow \Psi$, to the equation of motion

$$\left(\gamma^{\mu\nu\rho}\partial_v - m\gamma^{\mu\rho}\right)\Psi_\rho = 0$$

The last equation is the Euler-Lagrange equation of the action

$$S = -\int \mathrm{d}^D x \bar{\Psi}_\mu \left[\gamma^{\mu\nu\rho}\partial_v - m\gamma^{\mu\rho}\right]\Psi_\rho$$

Exercise 5.9 The equation of motion (5.47) also contains constraints on the initial data. Obtain $\gamma^{\mu\nu}\partial_\mu\Psi_\nu = 0$, which is not a constraint, by contracting the equation with ∂_μ. Then find the constraint $\gamma^\mu\Psi_\mu = 0$ by contracting with γ_μ. Show that the $\mu = 0$ component of the equation of motion gives the constraint $\left(\gamma^{ij}\partial_i - m\gamma^j\right)\Psi_j = 0$.

Exercise 5.10 By analysis similar to that which led from (5.3) to (5.4) in the massless case, derive $(\not{\partial} + m)\Psi_\mu = 0$, which closely resembles the Dirac equation. The constraints of Ex. 5.9 must still be applied to the initial data, but the new equation clearly shows that the field has definite mass m.

It is useful to recapitulate the equations that we obtained during the analysis (or directly from (5.47)) that determine the counting of the number of initial data and thus the number of degrees of freedom:

$$\begin{aligned}
&\gamma^\mu\Psi_\mu = 0 \\
&\left(\gamma^{ij}\partial_i - m\gamma^j\right)\Psi_j = 0, \\
&[\not{\partial} + m]\Psi_\mu = 0
\end{aligned}$$

As in the massless case, the time derivatives are determined by the Dirac equation (last equation of (5.49)). The initial data are thus the values at $t = 0$ of the Ψ_μ restricted by the first two equations of (5.49). Hence, the complex field $\Psi_\mu(x)$ with $D \times 2^{[D/2]}$ degrees of freedom contains $(D-2) \times 2^{[D/2]}$ independent classical degrees of freedom and thus $\frac{1}{2}(D-2) \times 2^{[D/2]}$ on-shell physical states. For $D = 4$ these are the four helicity states required for a massive particle of spin $s = 3/2$. In the situation of dimensional reduction, there is a massive gravitino with $m = |k|/L$ for every Fourier mode k, each with $\frac{1}{2}(D-2)2^{[D/2]}$ states. Note that this is the same as the number of states of a massless gravitino in $D+1$ dimensions.

Exercise 5.11 Study the Kaluza-Klein reduction for the Rarita-Schwinger field assuming periodicity $\Psi_\mu(x, y + 2\pi) = \Psi_\mu(x, y)$ in y. Show that the spectrum seen in Minkowski i_D consists of a massive gravitino for each Fourier mode $k \neq 0$ plus a massless gravitino and massless Dirac particle for the zero mode.

The dimensional reduction process has thus taught us the correct action for a massive gravitino. In particular the mass term is $m\bar{\Psi}_\mu\gamma^{\mu v}\Psi_v$. There is a more general action, namely

$$S = -\int \mathrm{d}^D x \bar{\Psi}_\mu \left[\gamma^{\mu\nu\rho}\partial_\nu - m\gamma^{\mu\rho} - m'\eta^{\mu\rho}\right]\Psi_\rho$$

which contains an additional Lorentz invariant term with a coefficient m' with the dimension of mass. It is curious that this does not give the correct description of a massive gravitino, because it contains too many degrees of freedom. In the following exercise we ask readers to verify this.

Exercise 5.12 Derive the equation of motion for the action (5.50). Analyze this equation as in Ex. 5.9, and show that the previous constraint $\gamma \cdot \Psi = 0$ does not hold if $m' \neq 0$. The field components $\gamma \cdot \Psi$ then describe additional degrees of freedom (which propagate as negative Hilbert space metric 'ghosts'). See [28] for an analysis in terms of projection operators.

8.5 $\mathcal{N} = 1$ global supersymmetry in $D = 4$

In global SUSY the scope of symmetries included in quantum field theory is extended from Poincaré and internal symmetry transformations, with charges $M_{[\mu\nu]}, P_\mu$, and T_A, to include spinor supercharges Q^i_α, where α is a spacetime spinor index, and $i = 1, ..., \mathcal{N}$ is an index labeling the distinct supercharges. We will assume that the Q^i_α are four-component Majorana spinors, although an equivalent formulation using two-component Weyl spinors is also commonly used. In this chapter we will mostly study the simplest case $\mathcal{N} = 1$ where there is a single spinor charge Q_α. This case is called $\mathcal{N} = 1$ SUSY or simple SUSY. Some features of theories with $\mathcal{N} > 1$ spinor charges, called extended SUSY theories, are discussed in Sec. 6.4 and Appendix 6A.

In $\mathcal{N} = 1$ global SUSY the Poincaré generators and Q_α join in a new algebraic structure, that of a superalgebra. A superalgebra contains two classes of elements, even and odd. From the physics viewpoint, they can be called bosonic (B) and fermionic (F). The structure relations include both commutators and anti-commutators in the pattern $[B, B] = B$, $[B, F] = F, \{F, F\} = B$. The bosonic charges span a Lie algebra. In SUSY the subalgebra of the bosonic charges $M_{[\mu\nu]}$ and P_μ is the Lie algebra of the Poincaré group discussed in Ch. 1, while the new structure relations involving Q_α are

$$\begin{aligned} \left\{Q_\alpha, \bar{Q}^\beta\right\} &= -\frac{1}{2} \left(\gamma_\mu\right)_\alpha{}^\beta P^\mu, \\ \left[M_{[\mu\nu]}, Q_\alpha\right] &= -\frac{1}{2} \left(\gamma_{\mu\nu}\right)_\alpha{}^\beta Q_\beta, \\ \left[P_\mu, Q_\alpha\right] &= 0. \end{aligned}$$

Note that these are the classical (anti-)commutator relations; see Secs. 1.4 and 1.5. We will discuss this further in Ch. 11.

Exercise 6.1 Use (2.30) to reexpress the supercharge anti-commutator in terms of Q and $Q^\dagger$. Then use the correspondence principle, that is multiply by the imaginary i, to obtain the quantum anti-commutator from the classical relation. This procedure gives the operator relation

$$\left\{Q_\alpha, \left(Q^\dagger\right)^\beta\right\}_{\text{qu}} = \frac{1}{2} \left(\gamma_\mu \gamma^0\right)_\alpha{}^\beta P^\mu$$

Trace on the spinor indices to obtain the positivity condition

$$\text{Tr}\left(QQ^\dagger + Q^\dagger Q\right) = 2P^0$$

The energy $E = P^0$ of any state in the Hilbert space of a global supersymmetric field theory must be positive.

Many SUSY theories, but not all, are invariant under a chiral U(1) symmetry called R-symmetry. We denote the generator by T_R. This acts on Q_α via

$$\left[T_R, Q_\alpha\right] = -\mathrm{i} \left(\gamma_*\right)_\alpha^\beta Q_\beta$$

but this generator T_R is not required. Other internal symmetries, which commute with Q_α and are frequently called outside charges, can also be included.

There are two important theorems that severely limit the type of charges and algebras that can be realized in an interacting relativistic quantum field theory in $D = 4$ (strictly speaking in a theory with a non-trivial S-matrix in flat space). According to the ColemanMandula (CM) theorem [29, 30], in the presence of massive particles, bosonic charges are limited to $M_{[\mu\nu]}$ and P_μ plus (optional) scalar internal symmetry charges, and the Lie algebra is the direct sum of the Poincaré algebra and a (finite-dimensional) compact Lie algebra for internal symmetry.

If superalgebras are admitted, the situation is governed by the Haag-ŁopuszańskiSohnius (HLS) theorem [31, 30], and the algebra of symmetries admits spinor charges Q^i_α. If there is only one Q_α, then the superalgebra must agree with the $\mathcal{N} = 1$ Poincaré SUSY algebra in (6.1). When $\mathcal{N} > 1$, the possibilities are restricted to the extended SUSY algebras discussed in Appendix 6A. [1] The main thought that we wish to convey is that SUSY theories realize the most general symmetry possible within the framework of the few assumptions made in the hypotheses of the CM and HLS theorems. [2] They also unify bosons and fermions, the two broad classes of particles found in Nature.

The parameters of global SUSY transformations are constant anti-commuting Majorana spinors ϵ_α. In supergravity SUSY is gauged, necessarily with the Poincaré generators, since they are joined in the superalgebra (6.1). This means that gravity is included, so the spinor parameters become arbitrary functions $\epsilon_\alpha(x)$ on a curved spacetime manifold. It is logically possible to skip ahead to Ch.9 where $\mathcal{N} = 1, D = 4$ supergravity is presented. But much important background will be missed, and we encourage only readers quite familiar with global SUSY to do this. We endeavor to give a succinct, pedagogical treatment of classical aspects of SUSY field theories. This material is certainly elegant, and part of the reason that SUSY is so appealing. However, there is much more in the deep results that have been discovered in perturbative and non-perturbative quantum supersymmetry that we cannot include.

The purpose of this chapter is to move as quickly as possible to an understanding of the structure of the major interacting SUSY field theories. At the classical level an interacting field theory is simply one in which the equations of motion are nonlinear. In Sec. 6.4, we give a short survey of the massless particle representations of extended Poincaré SUSY algebras.

1 They also found the extension with central charges, which will be discussed in Sec. 12.6.2.

2 In theories that contain only massless fields and are scale invariant at the quantum level, there are the additional possibilities of conformal and superconformal symmetries. The superconformal algebras contain the Poincaré SUSY algebras as subalgebras. They will be discussed later.

8.5.1 Basic SUSY field theory

SUSY theories contain both bosons and fermions, which are the basis states of a particle representation of the SUSY algebra (6.1)-(6.4). We give a systematic treatment of these representations in Sec. 6.4, but start with an informal discussion here. The states of particles with momentum $\vec{p}$ and energy $E(\vec{p}) = \sqrt{\vec{p}^2 + m^2_{B,F}}$ are denoted by $|\vec{p}, B\rangle$ and $|\vec{p}, F\rangle$, where the labels B and F include particle helicity. SUSY transformations connect these states. Since the spinor Q_α carries angular momentum $1/2$, it transforms bosons into fermions and fermions into bosons. Hence $Q_\alpha|\vec{p}, B\rangle = |\vec{p}, F\rangle$ and $Q_\alpha|\vec{p}, F\rangle \propto |\vec{p}, B\rangle$. Since $[P^\mu, Q_\alpha] = 0$, the transformed states have the same momentum and energy, hence the same mass, so $m^2_B = m^2_F$. We show in Sec. 6.4.1 that a representation of the algebra contains the same number of boson and fermion states.

The simplest representations of the algebra that lead to the most basic SUSY field theories are:

(i) the chiral multiplet, which contains a self-conjugate spin- 1/2 fermion described by the Majorana field $\chi(x)$ plus a complex spin- 0 boson described by the scalar field $Z(x)$. Alternatively, $\chi(x)$ may be replaced by the Weyl spinor $P_L\chi$ and/or $Z(x)$ by the combination $Z(x)=(A(x)+\mathrm{i}B(x))/\sqrt{2}$ where A and B are a real scalar and pseudo-scalar, respectively. A chiral multiplet can be either massless or massive.

(ii) the gauge multiplet consisting of a massless spin-1 particle, described by a vector gauge field $A_\mu(x)$, plus its spin- 1/2 fermionic partner, the gaugino, described by a Majorana spinor $\lambda(x)$ (or the corresponding Weyl field $P_L\lambda$).

8.5.2 Conserved supercurrents

It follows from our discussion of the Noether formalism for symmetries that the spinor charge should be the integral of a conserved vector-spinor current, the supercurrent $\mathcal{J}_\alpha^\mu$, hence

$$Q_\alpha=\int \mathrm{d}^3x\mathcal{J}_\alpha^0(\vec{x},t)$$

If the current is conserved for all solutions of the equations of motion of a theory, then the theory has a fermionic symmetry. By the HLS theorem this symmetry must be supersymmetry!

Therefore we begin the technical discussion of SUSY in quantum field theory by displaying such conserved currents, [3] first for free fields and then for one non-trivial interacting system. Consider a free scalar field $\phi(x)$ satisfying the Klein-Gordon equation $(\square-m^2)\,\phi=0$ and a spinor field $\Psi(x)$ satisfying the Dirac equation $(\partial\!\!\!/-m)\Psi=0$.

Exercise 6.2 Show that the current $\mathcal{J}^\mu=(\partial\!\!\!/\phi-m\phi)\gamma^\mu\Psi$ is conserved for all field configurations satisfying the Klein-Gordon and Dirac equations.

3 The spinor index α on the current and on most spinorial quantities will normally be suppressed.

It is no surprise to find unusual conserved currents in a free theory. In fact the current $\mathcal{J}^\mu{}_\nu=(\partial\!\!\!/\phi\phi-m\phi)\gamma^\mu\partial_\nu\Psi$, which gives a charge that violates the HLS theorem, is conserved. Such currents cannot be conserved in an interacting theory. For similar reasons conservation of $\mathcal{J}^\mu$ at the free level holds whether ϕ and Ψ are real or complex. To extend to interactions we will have to take $\phi\to Z$, a complex scalar, and Ψ a Majorana spinor. Note also that the current in Ex. 6.2 is conserved for any spacetime dimension D; this is another property that fails with interactions.

As the second example let's look at the free gauge multiplet with vector potential A_μ and field strength $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$ satisfying the Maxwell equation $\partial^\mu F_{\mu\nu}=0$ and a spinor λ satisfying $\partial\!\!\!/\lambda=0$. Let's show that the current $\mathcal{J}^\mu=\gamma^{\nu\rho}F_{\nu\rho}\gamma^\mu\lambda$ is conserved. We have

$$\partial_\mu\mathcal{J}^\mu=\partial_\mu F_{\nu\rho}\gamma^{\nu\rho}\gamma^\mu\lambda+\gamma^{\nu\rho}F_{\nu\rho}\partial\!\!\!/\lambda.$$

The last term vanishes. To treat the first term we manipulate the γ-matrices as discussed in Sec. 3.1.4:

$$\gamma^{\nu\rho}\gamma^\mu=\gamma^{\nu\rho\mu}+2\gamma^{[v}\eta^{\rho]\mu}.$$

When inserted in the first term of (6.6) we see that the first term vanishes by the gauge field Bianchi identity (4.11), and the second one by the Maxwell equation.

8.5.3 SUSY Yang-Mills theory

With little more work we can now exhibit an important interacting theory, $\mathcal{N}=1$ SUSY Yang-Mills theory and its conserved supercurrent. The theory contains the gauge boson $A_\mu^A(x)$

and its SUSY partner, the Majorana spinor gaugino $\lambda^A(x)$ in the adjoint representation of a simple, compact, non-abelian gauge group G. The action is [4]

$$S = \int d^4x \left[-\frac{1}{4} F^{\mu\nu A} F^A_{\mu\nu} - \frac{1}{2} \bar{\lambda}^A \gamma^\mu D_\mu \lambda^A \right]$$

For details of the notation see Secs. 3.4.1 and 4.3. Note that the gaugino action vanishes unless $\lambda^A(x)$ is anti-commuting! The Euler-Lagrange equations (and gauge field Bianchi identity) are

$$\begin{aligned} D^\mu F^A_{\mu\nu} &= -\frac{1}{2} g f_{BC}{}^A \bar{\lambda}^B \gamma_\nu \lambda^C \\ D_\mu F^A_{\nu\rho} + D_\nu F^A_{\rho\mu} &+ D_\rho F^A_{\mu\nu} = 0 \\ \gamma^\mu D_\mu \lambda^A &= 0 \end{aligned}$$

The supercurrent is

$$\mathcal{J}^\mu = \gamma^{\nu\rho} F^A_{\upsilon\rho} \gamma^\mu \lambda^A$$

4 We assume in this chapter that the Lie algebra has an invariant metric δ_{AB}, so that two 'upper' indices can be contracted.

The proof that it is conserved begins as in the free (abelian) case:

$$\begin{aligned} \partial_\mu \mathcal{J}^\mu &= D_\mu F^A_{\nu\rho} \gamma^{\nu\rho} \gamma^\mu \lambda^A + \gamma^{\upsilon\rho} F^A_{\nu\rho} \gamma^\mu D_\mu \lambda^A \\ &= -2 D^\mu F^A_{\mu\nu} \gamma^\nu \lambda^A \\ &= g f_{ABC} \gamma^\nu \lambda^A \bar{\lambda}^B \gamma_\nu \lambda^C. \end{aligned}$$

The right-hand side vanishes due to (3.68) and the supercurrent (6.10) is conserved!

Thus we have established the existence of our first interacting SUSY field theory. Notice how the basic relations of non-abelian gauge symmetry such as the Bianchi identity, the relativistic description of spin by the Dirac-Clifford algebra, and the anti-commutativity of fermion fields are all blended in the proof. Readers whose intellectual curiosity is not excited by this are advised to put this book aside permanently and watch television instead of reading it.

The two main approaches to SUSY field theories are the approach of this chapter, in which we deal with the separate field components describing each physical particle in the theory, and the superspace approach, in which the separate fields are grouped in superfields. The latter approach is not used in this book, but is briefly discussed in Appendix 14A (see references there). A Fierz relation is always required to establish supersymmetry in the 'component' approach to any interacting SUSY theory. This is one reason why the existence and field content of SUSY field theories depend so markedly on the spacetime dimension. The Fierz relation also restricts the type of fermion required in the theory.

Here is an exercise in which readers are asked to show that super-Yang-Mills (SYM) theories with gauge field A_μ plus a specific type of spinor ψ^A and the supercurrent $\mathcal{J}^\mu = \gamma^{\nu\rho} F^A_{\nu\rho} \gamma^\mu \psi^A$ do exist in certain spacetime dimensions [32].

Exercise 6.3 Study the appropriate Fierz rearrangement and, using the results of Ex. 3.27, show that the supercurrent is conserved in the following cases:

(i) Majorana spinors in $D = 3$,

(ii) Majorana (or Weyl) spinors in $D = 4$, which is the case analyzed above,

(iii) symplectic Weyl spinors in $D = 6$, and

(iv) Majorana-Weyl spinors in $D = 10$.

Notice that, in every case, the number of on-shell degrees of freedom of the gauge field, namely $D - 2$, matches those of the fermion, which are $2 \times 2^{[(D-2)/2]}/k$ (real), where $k = 1$ for a complex Dirac fermion, $k=2$ for a Majorana, a Weyl or a symplectic Weyl fermion, and $k=4$ for a Majorana-Weyl fermion. Equality of the total number of boson and fermion states is a necessary condition for SUSY. This basic fact will be proved for massive and massless physical states in Sec. 6.4.1 and in general (on- or off-shell) in Appendix 6B.

8.5.4 SUSY transformation rules

Although global SUSY can be formulated using conserved supercurrents as the primary vehicle, as was done above, it is usually more convenient to emphasize the idea of SUSY field variations involving spinor parameters ϵ_α under which actions must be invariant. The field variations are also called SUSY transformation rules. Given the conserved current one can form the supercharge using (6.5) and use the canonical formalism to compute the field variations, i.e.

$$\delta\Phi(x) = \{\bar{\epsilon}^\alpha Q_\alpha, \Phi(x)\}_{\mathrm{PB}} = -\mathrm{i}\,[\bar{\epsilon}^\alpha Q_\alpha, \Phi(x)]_{\mathrm{qu}}$$

where Φ denotes any field of the system under study. A brief description of Poisson brackets (PB) and commutation relations in the canonical formalism is given in Secs. 1.4 and 1.5. A link in the opposite direction is provided by the Noether formalism, which produces a conserved supercurrent given field variations under which the action is invariant. One reason to emphasize the field variations, *ab* initio, is that this avoids some subtleties in the canonical formalism for gauge theories and for Majorana spinors.

The next exercise illustrates the link between the supercurrent and field variations. It involves the free scalar-spinor $\phi-\Psi$ system of Ex. 6.2. The spinors Ψ, the supersymmetry parameters ϵ and the supersymmetry generator Q are Majorana spinors. They all mutually anti-commute. For the canonical formalism, one can either treat Ψ and $\bar{\Psi}$ as independent variables, or use Dirac brackets to obtain

$$\begin{aligned}\{\phi(x), \partial_0\phi(y)\}_{\mathrm{PB}} &= -\{\partial_0\phi(x), \phi(y)\}_{\mathrm{PB}} = \delta^3(\vec{x}-\vec{y})\\ \left\{\Psi_\alpha(x), \bar{\Psi}^\beta(y)\right\}_{\mathrm{PB}} &= \left\{\bar{\Psi}^\beta(x), \Psi_\alpha(y)\right\}_{\mathrm{PB}} = \left(\gamma^0\right)_\alpha{}^\beta \delta^3(\vec{x}-\vec{y})\end{aligned}$$

Exercise 6.4 Use $\bar{Q} = (1/\sqrt{2})\int \mathrm{d}^3\vec{x}\bar{\Psi}\gamma^0(\not{\partial}+m)\phi$ or $Q = (1/\sqrt{2})\int \mathrm{d}^3\vec{x}\ (\not{\partial}\phi - m\phi)\gamma^0\Psi$ to obtain

$$\begin{aligned}\delta\phi(x) &= \{\bar{\epsilon}Q, \phi(x)\}_{\mathrm{PB}} = \frac{1}{\sqrt{2}}\bar{\epsilon}\Psi(x)\\ \delta\Psi(x) &= \{\bar{\epsilon}Q, \Psi(x)\}_{\mathrm{PB}} = \frac{1}{\sqrt{2}}(\not{\partial}+m)\phi\epsilon.\end{aligned}$$

Note that $\left[\bar{Q}\epsilon, \Psi_\alpha(x)\right]_{\mathrm{PB}} = -\left\{\bar{Q}^\beta, \Psi_\alpha(x)\right\}_{\mathrm{PB}}\epsilon_\beta$.

8.5.5 SUSY field theories of the chiral multiplet

The physical fields of the chiral multiplet are a complex scalar $Z(x)$ and the Majorana spinor $\chi(x)$. It simplifies the structure in several ways to bring in a complex scalar auxiliary field $F(x)$. The field equations of F are algebraic, so F can be eliminated from the system at a later stage. The set of fields $Z, P_L\chi, F$ constitute a chiral multiplet, and their conjugates [5] $\bar{Z}, P_R\chi, \bar{F}$ are an anti-chiral multiplet. The treatment is streamlined because we use the chiral projections $P_L\chi$ and $P_R\chi$, but can still regard χ as a Majorana spinor; see Sec. 3.4.2.

Our program is to present the SUSY transformation rules of these multiplets, discuss invariant actions, and then study the SUSY algebra (6.1)-(6.4). The spinor parameter ϵ

5 Here we use $\bar{Z}$ to denote the complex conjugate rather than *, which was used earlier. Both notations will be used later in the book. is a Majorana spinor, whose spinor components anti-commute with each other and with components of χ and $\bar{\chi}$.

The transformation rules of the chiral multiplet are

$$\begin{aligned}\delta Z &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_L\chi,\\ \delta P_L\chi &= \frac{1}{\sqrt{2}}P_L(\not{\partial}Z+F)\epsilon,\\ \delta F &= \frac{1}{\sqrt{2}}\bar{\epsilon}\not{\partial}P_L\chi.\end{aligned}$$

The anti-chiral multiplet transformation rules are

$$\begin{aligned}\delta \bar{Z} &= \frac{1}{\sqrt{2}}\bar{\epsilon}P_R\chi,\\ \delta P_R\chi &= \frac{1}{\sqrt{2}}P_R(\not{\partial}\bar{Z}+\bar{F})\epsilon,\\ \delta \bar{F} &= \frac{1}{\sqrt{2}}\bar{\epsilon}\not{\partial}P_R\chi.\end{aligned}$$

Note that the form of the transformation rules for the physical components is similar to those of the 'toy model' in Ex. 6.4.

Exercise 6.5 Show that the variations $\delta\bar{Z}, \delta P_R\chi, \delta\bar{F}$ are the complex conjugates of δZ, $\delta P_L\chi, \delta F$.

There are two basic actions, which are separately invariant under the transformation rules above. The first is the free kinetic action

$$S_{\text{kin}} = \int \mathrm{d}^4x\left[-\partial^\mu\bar{Z}\partial_\mu Z - \bar{\chi}\not{\partial}P_L\chi + \bar{F}F\right]$$

in which we have presented the spinor term in chiral form. The interaction is determined by an arbitrary holomorphic function, the superpotential $W(Z)$. Given this we define the action

$$S_F = \int \mathrm{d}^4x\left[FW'(Z) - \frac{1}{2}\bar{\chi}P_L W''(Z)\chi\right]$$

(The reason for the apparent extra derivative will be explained shortly.) Note that S_F involves only the fields of the chiral multiplet and no anti-chiral components. Thus the action S_F is not hermitian, so we must also consider the conjugate action $S_{\bar{F}} = (S_F)^\dagger$. The complete action of the chiral multiplet is the sum

$$S = S_{\text{kin}} + S_F + S_{\bar{F}}.$$

Exercise 6.6 Consider the superpotential $W = \frac{1}{2}mZ^2 + \frac{1}{3}gZ^3$, which gives the SUSY theory first considered by Wess and Zumino in 1973 [3]. Obtain the equations of motion for all fields, then eliminate F and $\bar{F}$ and show that the correct equations of motion for the physical fields are obtained if you first eliminate F and $\bar{F}$ by solving their algebraic equations of motion and substituting the result in the action. Substitute $Z = (A+\mathrm{i}B)/\sqrt{2}$ and show that the action (after elimination of auxiliary fields) takes the form

$$\begin{aligned}S_{WZ} = \int \mathrm{d}^4x\Big[&-\frac{1}{2}\left(\partial_\mu A\partial^\mu A + \partial_\mu B\partial^\mu B\right) - \frac{1}{2}m^2\left(A^2+B^2\right) - \frac{1}{2}\bar{\chi}(\not{\partial}+m)\chi\Big)\\ &-\frac{g}{\sqrt{2}}\bar{\chi}\left(A+\mathrm{i}\gamma_* B\right)\chi - \frac{mg}{\sqrt{2}}\left(A^3+AB^2\right) - \frac{g^2}{4}\left(A^2+B^2\right)^2\Big].\end{aligned}$$

From the viewpoint of a particle theorist this is a parity conserving, renormalizable theory with equal-mass fields and Yukawa plus quartic interactions.

Let's outline the proof that the actions S_{kin} and S_F are invariant under the SUSY transformation (6.15) and (6.16). It is rather intricate, so trusting readers may wish to move ahead. For S_{kin} the work is simplified by an observation that is correct in any representation of the Clifford algebra, but clearest in the Weyl representation (2.19). The projections $P_L\epsilon$ and $\bar{\epsilon}P_L$ involve the same half of the components of the Majorana ϵ, while $P_R\epsilon$ and $\bar{\epsilon}P_R$ involve the conjugate components. We write the total variation $\delta S = \delta_{P_L\epsilon}S + \delta_{P_R\epsilon}S$, temporarily separating the two chiral projections of ϵ in the transformation rules. Since S_{kin} is hermitian, it is sufficient to compute $\delta_{P_L\epsilon}S_{\text{kin}}$; then $\delta_{P_R\epsilon}S$ is its adjoint. In the calculation we temporarily allow $\epsilon(x)$ to be an arbitrary function in Minkowski spacetime since that provides a simple way [33] to obtain the Noether current for SUSY. We also need $\delta\bar{\chi}P_R = -(1/\sqrt{2})\bar{\epsilon}(\partial\!\!\!/\bar{Z} - \bar{F})P_R$. Either the Dirac conjugate (2.30) or the Majorana conjugate relations (3.56) and (3.54) can be used. We suggest practice with the latter. (Note that $t_0 = -t_1 = 1$ in four dimensions.)

Now that we have prepared the way, let's calculate

$$\begin{aligned}\delta_{P_L\epsilon}S_{\text{kin}} = -\frac{1}{\sqrt{2}}\int \mathrm{d}^4x\big[\partial^\mu\bar{Z}\partial_\mu\left(\bar{\epsilon}P_L\chi\right) - \bar{\epsilon}(\partial\!\!\!/\bar{Z})\partial\!\!\!/P_L\chi \\ +\bar{\chi}\partial\!\!\!/\left(P_LF\epsilon\right) - \left(\bar{\epsilon}\partial\!\!\!/P_R\chi\right)F\big]\,.\end{aligned}$$

We have included all $P_L\epsilon$ and $\bar{\epsilon}P_L$ terms and dropped others. The $\bar{Z}\chi$ and $F\chi$ terms are independent and must vanish separately if we are to have a symmetry (when ϵ is constant). After a Majorana flip in the last term, we find that the $F\chi$ terms combine to (even for $\epsilon(x)$)

$$-\frac{1}{\sqrt{2}}\int \mathrm{d}^4x\partial_\mu\left(\bar{\chi}\gamma^\mu P_LF\epsilon\right)$$

which vanishes. The $\bar{Z}\chi$ terms can then be processed by substituting

$$\begin{aligned}\partial_\mu\left(\bar{\epsilon}P_L\chi\right) = \left(\partial_\mu\bar{\epsilon}\right)P_L\chi + \bar{\epsilon}P_L\partial_\mu\chi \\ \bar{\epsilon}P_L\gamma^\mu\gamma^\nu\left(\partial_\mu\bar{Z}\right)\partial_\nu\chi = \bar{\epsilon}P_L\left[\left(\partial^\mu\bar{Z}\right)\partial_\mu\chi + \gamma^{\mu\nu}\partial_\nu\left(\partial_\mu\bar{Z}\chi\right)\right]\end{aligned}$$

in (6.21). Two of the four terms cancel. After partial integration and use of $\eta^{\mu\nu} - \gamma^{\mu\nu} = \gamma^v\gamma^\mu$, we find the net result

$$\delta_{P_L\epsilon}S_{\text{kin}} \;=-\frac{1}{\sqrt{2}}\int \mathrm{d}^4x\partial_\mu\bar{\epsilon}P_L(\partial\!\!\!/\bar{Z})\gamma^\mu\chi$$

This shows that δS_{kin} vanishes for constant ϵ, which is enough to prove SUSY. The remaining term is a contribution to the supercurrent of the complete theory in (6.19), and we will include it below.

Since SUSY for the free action S_{kin} is not worth celebrating, we move on to discuss the interaction term S_F. The variation δS_F under the transformations (6.15) has the structure

$$\begin{aligned}\delta S_F = \int \mathrm{d}^4x[\delta FW'(Z) + \delta ZFW''(Z) \\ \left.-\delta\bar{\chi}P_L\chi W''(Z) - \frac{1}{2}\delta Z\bar{\chi}P_L\chi W'''(Z)\right],\end{aligned}$$

where we have taken the derivatives of $W(Z)$ required to include all sources of the δZ variation. After use of (6.15) we combine terms. Two $FP_L\chi$ terms cancel and we are left with the net result

$$\delta S_F = \frac{1}{\sqrt{2}}\int \mathrm{d}^4x\left[\bar{\epsilon}\partial\!\!\!/\left(W'P_L\chi\right) - \frac{1}{2}W'''\bar{\epsilon}P_L\chi\bar{\chi}P_L\chi\right]$$

The last term vanishes, since $P_L\chi$ has two independent components and any cubic expression vanishes by anti-commutativity! Thus δS_F vanishes for constant ϵ and is supersymmetric. It is clear that $\delta S_{\bar{F}}$ is just the conjugate of (6.26). At last SUSY is established at the interacting level!

The remaining $\partial_\mu\bar{\epsilon}$ terms in (6.24) and (6.26) plus their conjugates combine to give the Noether supercurrent of the interacting theory. This can be written as

$$\mathcal{J}^\mu = \frac{1}{\sqrt{2}}\left[P_L(\not{\partial}\bar{Z} - F) + P_R(\not{\partial}Z - \bar{F})\right]\gamma^\mu\chi$$

in which one must use the auxiliary field equations of motion $F = -\bar{W}'(\bar{Z})$ and $\bar{F} = -W'(Z)$.

Exercise 6.7 Show that the current is conserved for all solutions of the equations of motion of the theory (6.19).

Exercise 6.8 Given the component fields $\bar{Z}, P_R\chi, \bar{F}$ of an anti-chiral multiplet, show that $\bar{F}, P_L\chi\chi, \Box\bar{Z}$ transform in the same way as the $Z, P_L\chi, F$ components of a chiral multiplet; see (6.15) and (6.16).

8.5.6 U(1)$_R$ symmetry

The R-symmetry is a phase transformation of the fields of the chiral and anti-chiral multiplets. The fields $Z, P_L\chi, F$ carry R-charges $r, r_\chi = r-1, r_F = r-2$ respectively. U(1)$_R$ is a chiral symmetry, so the charges of component fields of the conjugate anti-chiral multiplet are the opposite of those above. We will discuss below how r is determined.

Infinitesimal transformations with parameter ρ are written as

$$\begin{aligned}\delta_R Z &= \mathrm{i}\rho r Z\\ \delta_R P_L\chi &= \mathrm{i}\rho(r-1)P_L\chi\\ \delta_R F &= \mathrm{i}\rho(r-2)F\end{aligned}$$

There are similar variations, with opposite charges, for $\bar{Z}, P_R\chi, \bar{F}$. The relations $r_\chi = r-1$ and $r_F = r-2$ are required by the commutator (6.4),

$$[\delta_R(\rho), \delta(\epsilon)] = \rho\bar{\epsilon}^\alpha\,[T_R, Q_\alpha] = -\mathrm{i}\rho\bar{\epsilon}^\alpha\,(\gamma_*)_\alpha^{\ \beta}\,Q_\beta$$

where $\delta(\epsilon)$ are the supersymmetry transformations (6.15).

Exercise 6.9 Calculate the variation $\delta_R\mathcal{J}^\mu$ of the supercurrent (6.27) using $\delta_R Z = \mathrm{i}\rho r Z, \delta_R P_L\chi = \mathrm{i}\rho r_\chi P_L\chi, \delta_R F = \mathrm{i}\rho r_F F$. Show that the result agrees with (6.4) if and only if $r_\chi = r-1$ and $r_F = r-2$.

The kinetic action S_{kin} is invariant under (6.28), so it is the interaction S_F that controls the situation. We now study the conditions for invariance of S_F. Clearly FW' and $\bar{\chi}W''\chi$ must be separately invariant. The condition for the vanishing of $\delta\,(FW')$ is

$$\delta_R\,(FW') = \mathrm{i}\rho F\,[(r-2)W' + rZW''] = 0.$$

This must hold for all field configurations, which means that the superpotential must satisfy

$$r\,(W' + ZW'') = 2W' \Rightarrow ZW' = \frac{2}{r}W$$

(since a constant can be absorbed in the definition of W without changing the physics). Similarly the condition for the vanishing of $\delta_R\,(\bar{\chi}W''\chi)$ is

$$\delta_R\,(\bar{\chi}W''\chi) = \mathrm{i}\rho\bar{\chi}\,[2(r-1)W'' + rZW''']\,\chi = 0.$$

However, the quantity in square brackets is the derivative of (6.31) and thus vanishes if $W(Z)$ satisfies (6.31).

The conclusion is that $W(Z)$ must be a homogeneous function of order $2/r$. This means that a theory with monomial superpotential $W(Z) = Z^k$ is $\mathrm{U}(1)_R$ invariant provided we assign $r = 2/k$ as the R-charge of the elementary scalar field Z. In the Wess-Zumino model, $W(Z) = mZ^2/2 + gZ^3/3$. If $g = 0$, then we have $\mathrm{U}(1)_R$ symmetry with $r = 1$. If $m = 0$ we have $\mathrm{U}(1)_R$ symmetry with $r = 2/3$. For general values of m and g, the symmetry is absent.

R-symmetry plays an important role in phenomenological applications of global supersymmetry. For example (see Sec. 28.1 of [30] or Sec. 5.2 of [34]), the related discrete R-parity is used to rule out undesired terms in the minimally supersymmetric standard model. $\mathrm{U}(1)_R$ plays an important role in models with supersymmetry breaking.

8.5.7 The SUSY algebra

In this section we will study the realization of the SUSY algebra on the components of a chiral multiplet. It is convenient and interesting that the $\{Q, \bar{Q}\}$ anti-commutator in (6.1) is realized in classical manipulations as the commutator of two successive variations of the fields with distinct (anti-commuting) parameters ϵ_1, ϵ_2.

The variation of a generic field $\Phi(x)$ is given in (6.12). The commutator of successive variations δ_1, δ_2 of $\Phi(x)$, with parameters ϵ_1, ϵ_2, respectively, is (recall that $\bar{\epsilon}Q = \bar{Q}\epsilon$ for Majorana spinors)

$$\begin{aligned}
[\delta_1, \delta_2]\, \Phi(x) &= \left[\bar{\epsilon}_1 Q, \left[\bar{Q}\epsilon_2, \Phi(x)\right]\right] - (\epsilon_1 \leftrightarrow \epsilon_2) \\
&= \bar{\epsilon}_1^\alpha \left[\left\{Q_\alpha, \bar{Q}^\beta\right\}, \Phi(x)\right] \epsilon_{2\beta} \\
&= -\frac{1}{2}\bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu \Phi(x)
\end{aligned}$$

The standard Jacobi identity has been used to reach the second line and the first relation in (6.1) to obtain the last line. The key result is that the commutator of two SUSY variations is an infinitesimal spacetime translation with parameter $-\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2$.

Let's carry out the computation of $[\delta_1, \delta_2]\, Z(x)$ on the scalar field of a chiral multiplet. Using (6.15) we write

$$\begin{aligned}
[\delta_1, \delta_2]\, Z &= \frac{1}{\sqrt{2}}\delta_1\left(\bar{\epsilon}_2 P_L \chi\right) - [1 \leftrightarrow 2] \\
&= \frac{1}{2}\bar{\epsilon}_2 P_L (\not{\partial} Z + F)\epsilon_1 - [1 \leftrightarrow 2] \\
&= -\frac{1}{2}\bar{\epsilon}_1 \gamma^\mu \epsilon_2 \partial_\mu Z
\end{aligned}$$

The symmetry properties of Majorana spinor bilinears (see (3.51)) have been used to reach the final result, which clearly shows the promised infinitesimal translation.

The analogous computation of $[\delta_1, \delta_2]\, P_L\chi(x)$ is more complex because a Fierz rearrangement is required. We outline it here:

$$\begin{aligned}
[\delta_1, \delta_2]\, P_L\chi &= \frac{1}{\sqrt{2}} P_L\left(\not{\partial}\delta_1 Z + \delta_1 F\right)\epsilon_2 - [1 \leftrightarrow 2] \\
&= \frac{1}{2} P_L \gamma^\mu \epsilon_2 \bar{\epsilon}_1 P_L \partial_\mu \chi + \frac{1}{2} P_L \epsilon_2 \bar{\epsilon}_1 \not{\partial} P_L \chi - [1 \leftrightarrow 2] \\
&= -\frac{1}{8}\left(\bar{\epsilon}_1 \Gamma_A \epsilon_2\right) P_L \left(\gamma^\mu \Gamma^A + \Gamma^A \gamma^\mu\right) P_L \partial_\mu \chi - [1 \leftrightarrow 2]
\end{aligned}$$

Each term in the second line was reordered as in Ex. 3.28 (with $\bar{\lambda}_1$ of (3.70) removed). We now find a great deal of simplification. Because of the antisymmetrization in $\epsilon_1 \leftrightarrow \epsilon_2$ the only non-vanishing bilinears are $\Gamma_A \to \gamma_\nu$ or $\gamma_{\nu\rho}$. However, only γ^ν survives the chiral projection in the last factor. Thus we find the expected result

$$[\delta_1, \delta_2]\, P_L\chi = -\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2 P_L\partial_\mu\chi$$

as the just reward for our labor.

Exercise 6.10 It is quite simple to demonstrate that

$$[\delta_1, \delta_2]\, F = -\frac{1}{2}\bar{\epsilon}_1\gamma^\mu\epsilon_2\partial_\mu F.$$

Zealous readers should do it.

The auxiliary field F can be eliminated from the action by substituting the value $F = -\bar{W}'(\bar{Z})$, which is the solution of its equation of motion from (6.19), without affecting the classical (or quantum) dynamics. Here is an exercise to show that SUSY is also maintained after elimination.

Exercise 6.11 Consider the theory after elimination of F and $\bar{F}$. Show that the action

$$S = \int \mathrm{d}^4x \left[-\partial^\mu\bar{Z}\partial_\mu Z - \bar{\chi}\not{\partial}P_L\chi - \bar{W}'W' - \frac{1}{2}\bar{\chi}\left(P_L W'' + P_R\bar{W}''\right)\chi\right]$$

is invariant under the transformation rules (6.15) and their conjugates (6.16). Show that $[\delta_1, \delta_2]\, Z$ is exactly the same as in (6.34), but $[\delta_1, \delta_2]\, P_L\chi$ is modified as follows:

$$[\delta_1, \delta_2]\, P_L\chi = \bar{\epsilon}_1\gamma^\mu\epsilon_2 P_L\left[-\frac{1}{2}\partial_\mu\chi + \frac{1}{4}\gamma_\mu\left(\not{\partial} + \bar{W}''\right)\chi\right]$$

We find the spacetime translation plus an extra term that vanishes for any solution of the equations of motion.

Since the commutator of symmetries must give a symmetry of the action [6] and translations are a known symmetry, the remaining transformation, namely

$$\begin{aligned}
\delta Z &= 0 \\
\delta\chi &= v^\mu\gamma_\mu\left(\not{\partial} + P_L W'' + P_R\bar{W}''\right)\chi,
\end{aligned}$$

is itself a symmetry for any constant vector v^μ. However, its Noether charge vanishes when the fermion equation of motion is satisfied, so it has no physical effect. Such a symmetry is sometimes called a 'zilch symmetry'.

Although nothing physically essential is changed by eliminating auxiliary fields, we can nevertheless see that they play a useful role:

(i) It is only with F and $\bar{F}$ included that the form of the SUSY transformation rules (6.15) and (6.16) is independent of the superpotential $W(Z)$.

(ii) The SUSY algebra is also universal on all components of the chiral multiplet when F is included. The phrase used in the literature is that the SUSY algebra is 'closed off-shell' when auxiliary fields are included and 'closed only on-shell' when they are eliminated.

(iii) Auxiliary fields are very useful in determining the terms in a SUSY Lagrangian describing couplings between different multiplets. An example is the general SUSY gauge theory described in the next section.

(iv) In local supersymmetry auxiliary fields simplify the couplings of Faddeev-Popov ghost fields.

6 The argument is easy: a symmetry is a transformation such that $S_{,i}\delta(\epsilon)\phi^i = 0$, where $S_{,i}$ is the functional derivative with respect to the field ϕ^i. Applying a second transformation gives

$S_{,ij}\delta\left(\epsilon_1\right)\phi^i\delta\left(\epsilon_2\right)\phi^j + S_{,i}\delta\left(\epsilon_2\right)\delta\left(\epsilon_1\right)\phi^i = 0$. Taking the commutator, the first term vanishes by symmetry, and the second term says that the commutator defines a symmetry.

It is also the case that auxiliary fields are known only for a few extended SUSY theories in four-dimensional spacetime and unavailable for many theories in dimension $D > 4$. Indeed many of the most interesting SUSY theories have no known auxiliary fields.

Although we hope that some readers enjoy the detailed manipulations needed to study SUSY theories, we suspect that many are fed up with Fierz rearrangement and would like a more systematic approach. To a large extent the superspace formalism does exactly that and has many advantages. Unfortunately, complete superspace methods are also unavailable when auxiliary fields are not known.

8.5.8 More chiral multiplets

We conclude this section with a discussion to establish a more general SUSY theory containing several chiral multiplets and their conjugates. We present this theory in the same 'blended' notation used above in which fermions always appear as the chiral projections $P_L\chi$ and $P_R\chi$ of Majorana spinors and the symmetry properties of Majorana bilinears can be used in all manipulations. To make the notation compatible with gauge symmetry in the next section, we denote chiral multiplets [7] by $Z^\alpha, P_L\chi^\alpha, F^\alpha$ and their anti-chiral adjoints by $\bar{Z}_\alpha, P_R\chi_\alpha, \bar{F}_\alpha$. Note that we use lower indices α for the fields of the anti-chiral multiplets.

The interactions of the general theory are determined by an arbitrary holomorphic superpotential $W\left(Z^\alpha\right)$. We denote derivatives of W by $W_\alpha = \partial W/\partial Z^\alpha, W_{\alpha\beta} = \partial^2 W/\partial Z^\alpha \partial Z^\beta$, etc. The kinetic action S_{kin} is the obvious generalization of (6.17) to include a sum over the index α, while the chiral interaction term becomes

$$S_F = \int \mathrm{d}^4x \left[F^\alpha W_\alpha - \frac{1}{2}\bar{\chi}^\alpha P_L W_{\alpha\beta}\chi^\beta\right]$$

and one must add the conjugate action $S_{\bar{F}}$. The transformation rules of each multiplet are unmodified, but the index α is required, e.g. $\delta Z^\alpha = (1/\sqrt{2})\bar{\epsilon}P_L\chi^\alpha$. This general form of S_F explains why W' and W'' appear in (6.18).

Exercise 6.12 Show that the new actions $S_{\text{kin}}, S_F, S_{\bar{F}}$ are each invariant. The only essential new feature is that a Fierz rearrangement argument is required to show that the cubic term $W_{\alpha\beta\gamma}\bar{\epsilon}P_L\chi^\alpha\bar{\chi}^\beta P_L\chi^\gamma$, which is the analogue of the last term in (6.26), vanishes.

After elimination of the auxiliary field using $F^\alpha = \partial\bar{W}/\partial\bar{Z}_\alpha \equiv \bar{W}^\alpha$ one finds the physically equivalent action

$$\begin{aligned} S = \int \mathrm{d}^4x \Big[& -\partial^\mu\bar{Z}_\alpha\partial_\mu Z^\alpha - \bar{\chi}_\alpha \not{\partial} P_L\chi^\alpha - \bar{W}^\alpha W_\alpha \\ & -\frac{1}{2}\bar{\chi}^\alpha P_L W_{\alpha\beta}\chi^\beta - \frac{1}{2}\bar{\chi}_\alpha P_R\bar{W}^{\alpha\beta}\chi_\beta\Big]. \end{aligned}$$

7 We do not use spinor indices any more, such that the use of $\alpha, \ldots$ to indicate the multiplets should not create confusion.

The $\mathrm{U}(1)_R$ symmetry discussed in Sec. 6.2.1 may be extended to the general chiral multiplet theory provided that the superpotential $W\left(Z^\alpha\right)$ satisfies an appropriate condition. To investigate this we assign charges $r_\alpha, r_\alpha - 1, r_\alpha - 2$ to the fields $Z^\alpha, P_L\chi^\alpha, F^\alpha$, so that infinitesimal transformations with parameter ρ are written as

$$\begin{aligned} \delta_R Z^\alpha &= \mathrm{i}\rho r_\alpha Z^\alpha, \\ \delta_R P_L\chi^\alpha &= \mathrm{i}\rho\left(r_\alpha - 1\right)P_L\chi^\alpha, \\ \delta_R F^\alpha &= \mathrm{i}\rho\left(r_\alpha - 2\right)F^\alpha. \end{aligned}$$

(No sum on α.)

Exercise 6.13 Show that the general S_F of (6.41) is $\mathrm{U}(1)_R$ invariant for any set of charges r_α such that the superpotential satisfies the homogeneity condition

$$\sum_\alpha r_\alpha Z^\alpha W_\alpha = 2W$$

To prove this, you must generalize the argument of Sec. 6.2.1. The condition (6.44) is equivalent to the statement that W has definite R-charge $r_W = 2$.

For each specific theory with superpotential $W(Z^\alpha)$ there are several possibilities. There may not be any choice of the r_α for which (6.44) holds. This is the case in the Wess-Zumino model with $m \neq 0$ and $g \neq 0$ discussed at the end of Sec. 6.2.1. In some theories there is a unique set of R-charges, and in others many choices.

8.5.9 SUSY gauge theories

The basic SUSY gauge theory is the $\mathcal{N} = 1$SYM theory containing the gauge multiplet A_μ^A, λ^A, where A is the index of the adjoint representation of a compact, non-abelian gauge group G. This theory was described in Sec. 6.1.2. The discussion there focused on the conserved supercurrent and will now be extended to include field variations, auxiliary fields, and the SUSY algebra.

We assume that the group has an invariant metric that can be chosen as δ_{AB}. This is the case for 'reductive groups', i.e. products of compact simple groups and abelian factors, i.e. $G = G_1 \otimes G_2 \otimes ...$, where each factor is a simple group or U(1). The normalization of the generators is fixed in each factor, which can lead to different gauge coupling constants $g_1, g_2, ...$ for each of these factors. We have taken here the normalizations of the generators where these coupling constants do not appear explicitly. One can replace everywhere t_A with $g_i t_A$ and $f_{AB}{}^C$ with $g_i f_{AB}{}^C$, where g_i can be chosen independently in each simple factor, to re-install these coupling constants. Usually one also redefines then the parameters θ^A to $(1/g_i)\,\theta^A$. This leads to the formulas with coupling constant g in Sec. 4.3. Further note that for these algebras, the structure constants can be written as f_{ABC}, which are completely antisymmetric.

8.5.10 SUSY Yang-Mills vector multiplet

Our first objective is to obtain the SUSY variations δA_μ^A and $\delta\lambda^A$ under which the action (6.8) is invariant. This will give 'on-shell' supersymmetry; then we will add the auxiliary field. We will organize the presentation to make use of previous work in Secs. 6.1.1 and 6.1.2, which established that the supercurrent (6.10) is conserved.

The variation of (6.8) is

$$\delta S = \int \mathrm{d}^4 x \left[\delta A_\nu^A D^\mu F_{\mu\nu}^A - \delta\bar{\lambda}^A \gamma^\mu D_\mu \lambda^A + \frac{1}{2} f_{ABC} \delta A_\mu^A \bar{\lambda}^B \gamma^\mu \lambda^C\right]$$

We first note that the forms

$$\delta A_\mu^A = -\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A, \quad \delta\lambda^A = \frac{1}{4}\gamma^{\rho\sigma} F_{\rho\sigma}^A \epsilon$$

are determined, up to constant factors, by Lorentz and parity symmetry and by the dimensions (in units of l^{-1}) of the quantities involved. Denoting the dimension of any quantity x by $[x]$, we have $[\epsilon] = -1/2, [A_\mu] = 1, [\lambda] = 3/2$. Note that if we use the assumed form for δA_μ, then the last term in (6.45) vanishes by the Fierz rearrangement identity (3.68). We substitute both assumed variations, assuming that $\epsilon(x)$ is a general function, and integrate by parts in the second term of (6.45) to obtain

$$\begin{aligned} \delta S &= -\frac{1}{2}\int \mathrm{d}^4x \left[\bar{\epsilon}\gamma^\nu\lambda^A D^\mu F^A_{\mu\nu} + \frac{1}{2}\bar{\epsilon}\gamma^{\rho\sigma}\gamma^\mu\lambda^A D_\mu F^A_{\rho\sigma} + \frac{1}{2}\partial_\mu\bar{\epsilon}\gamma^{\rho\sigma}\gamma^\mu F^A_{\rho\sigma}\lambda^A\right] \\ &= -\frac{1}{2}\int \mathrm{d}^4x \left[\bar{\epsilon}\gamma^\nu\lambda^A D^\mu F^A_{\mu\nu} - \bar{\epsilon}\gamma^\nu\lambda^A D^\mu F^A_{\mu\nu} + \frac{1}{2}\partial_\mu\bar{\epsilon}\gamma^{\rho\sigma}\gamma^\mu F^A_{\rho\sigma}\lambda^A\right] \end{aligned}$$

where (6.7) and the gauge field Bianchi identity were used to reach the final line. Thus δS vanishes for constant ϵ, establishing supersymmetry, while the supercurrent $\mathcal{J}^\mu$ of (6.10) appears in the last term! ![8]

The auxiliary field required for the gauge multiplet is a real pseudo-scalar field D^A in the adjoint representation of G. This fact follows from the superspace formulation. The auxiliary field enters the action and transformation rules in the quite simple fashion

$$\begin{aligned} S &= \int \mathrm{d}^4x \left[-\frac{1}{4}F^{\mu\nu A}F^A_{\mu\nu} - \frac{1}{2}\bar{\lambda}^A\gamma^\mu D_\mu\lambda^A + \frac{1}{2}D^A D^A\right], \\ \delta A^A_\mu &= -\frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A, \\ \delta\lambda^A &= \left[\frac{1}{4}\gamma^{\mu\nu}F^A_{\mu\nu} + \frac{1}{2}\mathrm{i}\gamma_* D^A\right]\epsilon, \\ \delta D^A &= \frac{1}{2}\mathrm{i}\bar{\epsilon}\gamma_*\gamma^\mu D_\mu\lambda^A, \quad D_\mu\lambda^A \equiv \partial_\mu\lambda^A + \lambda^C A^B_\mu f_{BC}{}^A. \end{aligned}$$

Exercise 6.14 Show that $\delta S = 0$. Only terms involving D^A need to be examined.

8 This term indicates that the true supercurrent should have been written in (6.10) with a factor $\frac{1}{4}$ included. $\frac{1}{4}\mathcal{J}^\mu$ generates correctly normalized SUSY variations.

The fields of the gauge multiplet transform also under an internal gauge symmetry:

$$\begin{aligned} \delta(\theta)A^A_\mu &= \partial_\mu\theta^A + \theta^C A^B_\mu f_{BC}{}^A, \\ \delta(\theta)\lambda^A &= \theta^C\lambda^B f_{BC}{}^A \\ \delta(\theta)D^A &= \theta^C D^B f_{BC}{}^A. \end{aligned}$$

Let us first remark that the commutator of these internal gauge transformations and supersymmetry vanishes.

Exercise 6.15 Use the transformation rules above to derive the SUSY commutator algebra of the gauge multiplet

$$\begin{aligned} [\delta_1, \delta_2]\, A^A_\mu &= -\frac{1}{2}\bar{\epsilon}_1\gamma^\nu\epsilon_2 F^A_{\nu\mu}, \\ [\delta_1, \delta_2]\, \lambda^A &= -\frac{1}{2}\bar{\epsilon}_1\gamma^\nu\epsilon_2 D_\nu\lambda^A \\ [\delta_1, \delta_2]\, D^A &= -\frac{1}{2}\bar{\epsilon}_1\gamma^\nu\epsilon_2 D_\nu D^A \end{aligned}$$

It is no surprise that the commutator of two gauge covariant variations from (6.49) is gauge covariant, but at first glance the result seems to disagree with the spacetime translation required by (6.1) and (6.33). Note that in all three cases in Ex. 6.15 the difference between the covariant result in (6.51) and a translation is a gauge transformation by the field dependent gauge parameter $\theta^A = \frac{1}{2}\bar{\epsilon}_1\gamma^v\epsilon_2 A^A_v$. The covariant forms that occur in (6.51) are called gauge covariant translations. The conclusion is that on the fields of a gauge theory the SUSY algebra closes on gauge covariant translations. See [35] and Sec. 4.1.5 for further information on this issue.

8.5.11 Chiral multiplets in SUSY gauge theories

We now present and briefly discuss the class of SUSY theories in which the gauge multiplet A_μ^A, λ^A, D^A is coupled to a chiral matter multiplet $Z^\alpha, P_L\chi^\alpha, F^\alpha$ in an arbitrary finitedimensional representation $\mathbf{R}$ of G with matrix generators $(t_A)^\alpha{}_\beta$. The representation may be either reducible or irreducible. A reducible representation may be decomposed into a direct sum of irreducible components $\mathbf{R}_i$. The matrix generators in each $\mathbf{R}_i$ are denoted by t_{Ai}. Formally the decomposition is expressed by

$$\begin{aligned} \mathbf{R} &= \bigoplus_i \mathbf{R}_i \\ t_A &= \bigoplus_i t_{Ai} \end{aligned}$$

For most purposes and in most formulas below, the decomposition into irreducible representations need not be indicated explicitly, and we use it only where a more detailed notation is required.

The theory necessarily contains the conjugate anti-chiral multiplet $\bar{Z}_\alpha, P_R\chi_\alpha, \bar{F}_\alpha$, and we use lower indices to indicate that these fields transform in the conjugate representation $\overline{\mathbf{R}}$. Under an infinitesimal gauge transformation with parameters $\theta^A(x)$ the fermions transform as

$$\begin{aligned} \delta P_L\chi^\alpha &= -\theta^A (t_A)^\alpha{}_\beta P_L\chi^\beta, \\ \delta P_R\chi_\alpha &= -\theta^A (t_A)_\alpha^{*\beta} P_R\chi_\beta, \end{aligned}$$

with similar rules for the other fields. Representation indices are suppressed in most formulas. Thus we can write covariant derivatives of the various fields as

$$\begin{aligned} D_\mu\lambda^A &= \partial_\mu\lambda^A + f_{BC}^A A_\mu^B \lambda^C, \\ D_\mu Z &= \partial_\mu Z + t_A A_\mu^A Z, \\ D_\mu P_L\chi &= \partial_\mu P_L\chi + t_A A_\mu^A P_L\chi, \\ D_\mu P_R\chi &= \partial_\mu P_R\chi + t_A^* A_\mu^A P_R\chi. \end{aligned}$$

The system need not contain a superpotential, but superpotentials $W(Z^\alpha)$, which must be both holomorphic and gauge invariant, are optional. It is useful to express the condition of gauge invariance of $W(Z^\alpha)$ as

$$\delta_{\text{gauge}} W = W_\alpha \delta_{\text{gauge}} Z^\alpha = -W_\alpha \theta^A (t_A)^\alpha{}_\beta Z^\beta = 0.$$

The action of the general theory is the sum of several terms

$$S = S_{\text{gauge}} + S_{\text{matter}} + S_{\text{coupling}} + S_W + S_{\bar{W}}.$$

The form of some terms agrees with expressions given earlier in this chapter. Since convenience is a virtue and repetition is no sin, we shall write everything here:

$$\begin{aligned}
S_{\text{gauge}} &= \int \mathrm{d}^4x \left[-\frac{1}{4}F^{\mu\nu A}F^A_{\mu\nu} - \frac{1}{2}\bar{\lambda}^A\gamma^\mu D_\mu\lambda^A + \frac{1}{2}D^A D^A \right], \\
S_{\text{matter}} &= \int \mathrm{d}^4x \left[-D^\mu \bar{Z} D_\mu Z - \bar{\chi}\gamma^\mu P_L D_\mu \chi + \bar{F}F \right], \\
S_{\text{coupling}} &= \int \mathrm{d}^4x \left[-\sqrt{2}\left(\bar{\lambda}^A \bar{Z} t_A P_L \chi - \bar{\chi} P_R t_A Z \lambda^A\right) + \mathrm{i} D^A \bar{Z} t_A Z \right], \\
S_F &= \int \mathrm{d}^4x \left[F^\alpha W_\alpha + \frac{1}{2}\bar{\chi}^\alpha P_L W_{\alpha\beta}\chi^\beta \right], \\
S_{\bar{F}} &= \int \mathrm{d}^4x \left[\bar{F}_\alpha \bar{W}^\alpha + \frac{1}{2}\bar{\chi}_\alpha P_R \bar{W}^{\alpha\beta}\chi_\beta \right].
\end{aligned}$$

The full action is invariant under the SUSY transformation rules given in (6.49) for the gauge multiplet and the following modified gauge covariant transformation rules for the chiral and anti-chiral multiplets:

$$\begin{aligned}
\delta Z &= \frac{1}{\sqrt{2}}\bar{\epsilon} P_L \chi \\
\delta P_L \chi &= \frac{1}{\sqrt{2}} P_L \left(\gamma^\mu D_\mu Z + F\right)\epsilon \\
\delta F &= \frac{1}{\sqrt{2}}\bar{\epsilon} P_R \gamma^\mu D_\mu \chi - \bar{\epsilon} P_R \lambda^A t_A Z
\end{aligned}$$

and

$$\begin{aligned}
\delta \bar{Z} &= \frac{1}{\sqrt{2}}\bar{\epsilon} P_R \chi, \\
\delta P_R \chi &= \frac{1}{\sqrt{2}} P_R \left(\gamma^\mu D_\mu \bar{Z} + \bar{F}\right)\epsilon, \\
\delta \bar{F} &= \frac{1}{\sqrt{2}}\bar{\epsilon} P_L \gamma^\mu D_\mu \chi - \bar{\epsilon} P_L \lambda^A \left(t_A\right)^* \bar{Z}.
\end{aligned}$$

Some modifications in the action and transformation rules above, notably the introduction of gauge covariant derivatives, are clearly required in a gauge theory, but other additions such as the form of the action S_{coupling} are surely not obvious. The best explanation is that they are dictated by the superspace formalism. However, all features can be explained from the component viewpoint. For example, in the SUSY variation δS_{matter} many terms cancel by the same manipulations required to show that δS_{kin} of (6.17) vanishes by simply replacing ∂_μ by D_μ. But there are extra terms due to the variation δA^A_μ,

$$\delta S_{\text{matter}} = \int \mathrm{d}^4x \frac{1}{2}\bar{\epsilon}\gamma_\mu\lambda^A \left(\bar{\chi} t_A \gamma^\mu P_L \chi - \bar{Z} t_A D^\mu Z + D^\mu \bar{Z} t_A Z\right)$$

which involves the gauge current of the matter fields, and there is a correction to (6.23) due to the gauge Ricci identity (4.91) applied to $\bar{Z}$. The correction is proportional to $F^A_{\mu\nu}\bar{\epsilon} P_L \gamma^{\mu\nu} \bar{Z} t_A \chi$. These terms are canceled by the variations of Z and χ in $\delta S_{\text{coupling.}}$ A complete demonstration that the total action (6.56) is invariant under the variations (6.49) and (6.62) requires quite delicate calculations, which we recommend only for sufficiently diligent readers. The reader will also be invited to explain the extra terms in (6.62) from an algebraic viewpoint below in Ex. 14.2

Exercise 6.16 Show that the action (6.56) is supersymmetric. How does the variation $\delta F^\alpha W_\alpha(Z)$ induced by the last term in δF cancel?

The $\mathrm{U}(1)_R$ symmetry extends to SUSY gauge theories as a global symmetry which commutes with gauge transformations. Therefore the R-charges r_α that appear in the transformation (6.43) of chiral fields must be the same for all fields in a given irreducible component $\mathbf{R}_i$ of the full group representation $\mathbf{R}$. Invariance of the Yukawa terms in (6.59) determines the gaugino transformation

$$\delta_R \lambda^A = \mathrm{i}\rho\gamma_* \lambda^A$$

If the superpotential satisfies (6.44), then the full gauge theory is also $\mathrm{U}(1)_R$ invariant at the classical level. However, conservation of the Noether current R^μ is typically violated by the quantum level axial anomaly, and this has important consequences. We refer the reader to Sec. 29.3 of [30] and Sec. 2.C. of [36].

Exercise 6.17 Show that the Noether current R^μ and the gauge currents of the general theory are given by

$$\begin{aligned} R^\mu &= -\frac{1}{2}\mathrm{i}\bar{\lambda}^A\gamma^\mu\gamma_*\lambda^A + \mathrm{i}\sum_\alpha \left(r_\alpha\left(\bar{Z}_\alpha D_\mu Z^\alpha - D_\mu \bar{Z}_\alpha Z^\alpha\right) - \mathrm{i}\left(r_\alpha - 1\right)\bar{\chi}_\alpha\gamma^\mu P_L\chi\right) \\ J_A^\mu &= -f_{ABC}\bar{\lambda}^B\gamma^\mu\lambda^C - \left(\bar{Z}t_A D_\mu Z - D_\mu\bar{Z}t_A Z\right) + \mathrm{i}\bar{\chi}\gamma^\mu P_L t_A\chi \end{aligned}$$

8.5.12 Massless representations of $\mathcal{N}$-extended supersymmetry

Up till now we have considered the supersymmetry generated by one Majorana spinor Q_α, subject to the structure relations of the superalgebra (6.1). The field theories that realize this algebra are said to possess simple supersymmetry. We now consider superalgebras containing $\mathcal{N} > 1$ Majorana spinor charges $Q_{i\alpha}, i = 1, ..., \mathcal{N}$. These algebras are called $\mathcal{N}$-extended supersymmetry algebras.

In the minimal extension, the different supersymmetry generators anti-commute, and each of them separately satisfies (6.1). We will rewrite (6.1) using the chiral components of the Majorana spinors (as justified in Box 3.3). We define $Q_{i\alpha}$ as the left-handed chiral components. They are simplest in the Weyl representation, in which the spinors have the form of (3.93). Then $Q_{i\alpha} = (Q_{i1}, Q_{i2}, 0, 0)$. We write the index i up for their hermitian conjugates, i.e. in Weyl representation: $Q^{\dagger i\alpha} = \left((Q_{i1})^*, (Q_{i2})^*, 0, 0\right)$. Therefore, we can effectively only use the index range $\alpha = 1, 2$. We will use the quantum expression, which according to Sec. 1.5 implies a multiplication with i; see (1.84). This gives the algebra

$$\begin{aligned} \left\{Q_{i\alpha}, Q^{\dagger j\beta}\right\}_{\mathrm{qu}} &= \frac{1}{2}\delta_i^j\left(\gamma_\mu\gamma^0\right)_\alpha{}^\beta P^\mu, \quad \alpha = 1,2 \\ \left[M_{[\mu\nu]}, Q_{i\alpha}\right]_{\mathrm{qu}} &= -\frac{1}{2}\mathrm{i}\left(\gamma_{\mu\nu}\right)_\alpha{}^\beta Q_{i\beta} \\ \left[P_\mu, Q_{i\alpha}\right]_{\mathrm{qu}} &= 0 \end{aligned}$$

We refer to Appendix 6A for a more detailed, and representation independent, definition of $Q_{i\alpha}$ and $Q^{\dagger j\beta}$ and the explanation of the anti-commutator in (6.87).

In this chapter, we will restrict attention to the extended supersymmetry algebra in (6.67). More general algebras, e.g. including the concept of 'central charges', will be deferred to Ch. 12.

8.5.13 Particle representations of $\mathcal{N}$-extended supersymmetry

We now discuss the particle representations of the superalgebras (6.67), that is, representations whose carrier space is the Hilbert space of a relativistic quantum field theory. Therefore we use a basis in which particles of energy-momentum $p^\mu = \left(p^0 = E = \sqrt{m^2 + \vec{p}^2}, \vec{p}\right)$ and spin s are described by states $|p^\mu, s, h\rangle$. For massive particles the helicity h takes $2s+1$ equally spaced values in the range $-s \le h \le s$. For a massless particle, there are two helicity values $h = \pm s$ if $s > 0$, and the unique value $h = 0$ if $s = 0$.

The carrier space of a particle representation of supersymmetry consists of the states $|p^\mu, s, h\rangle$ of a set of bosons, $s = 0, 1, 2, ...$, and fermions, $s = 1/2, 3/2, 5/2, ...$ The basic observation needed to study these representations is that $[Q, P] = 0$. Supersymmetry preserves the energy-momentum p^μ and thus the mass m of any particle state. Thus all that is needed to find the representations of the SUSY algebra is to consider finite sets of Bose and Fermi particles with fixed 4-momentum and various spins s and determine those sets on which the basic anti-commutator of supercharges can be realized irreducibly.

Let's first prove a very general result, namely that any irreducible representation of the SUSY algebra, whether it involves massive or massless particles, contains equal numbers of boson and fermion states. Taking a sum on spinor indices in (6.67) gives

$$Q_{i\alpha} Q^{\dagger j\alpha} + Q^{\dagger j\alpha} Q_{i\alpha} = \delta_i^j P^0$$

Consider the operator $\mathrm{e}^{-2\pi \mathrm{i} J_3}$ which implements rotations by angle 2π. Clearly its effect on boson and fermion states and on the supercharges is

$$\begin{aligned} \mathrm{e}^{-2\pi \mathrm{i} J_3} |p^\mu, s, h\rangle &= (-)^{2s} |p^\mu, s, h\rangle \\ \left\{Q_{i\alpha}, \mathrm{e}^{-2\pi \mathrm{i} J_3}\right\} &= 0 \end{aligned}$$

Multiply (6.68) on the right by $\mathrm{e}^{-2\pi \mathrm{i} J_3}$ and form the Hilbert space expectation value in a particle state $|p^\mu, s, h\rangle$. Finally sum over the spins and helicities of the particles in the carrier space of a representation. One finds

$$\begin{aligned} &\sum_{s,h} \left\langle p^\mu, s, h \,\middle|\, \left(Q_{i\alpha} Q^{\dagger j\alpha} + Q^{\dagger j\alpha} Q_{i\alpha}\right) \mathrm{e}^{-2\pi \mathrm{i} J_3} \,\middle|\, p^\mu, s, h\right\rangle \\ &\qquad = \delta_i^j \sum_{s,h} \left\langle p^\mu, s, h \,\middle|\, P^0 \mathrm{e}^{-2\pi \mathrm{i} J_3} \,\middle|\, p^\mu, s, h\right\rangle . \end{aligned}$$

The sum over spins and helicities with fixed p^μ is equivalent to a matrix trace in a finite-dimensional subspace of the Hilbert space and can be manipulated as a conventional matrix trace. Hence we can rewrite (6.70) as

$$\begin{aligned} \mathrm{Tr}\left(Q_{i\alpha} Q^{\dagger j\alpha} \mathrm{e}^{-2\pi \mathrm{i} J_3} + Q^{\dagger j\alpha} Q_{i\alpha} \mathrm{e}^{-2\pi \mathrm{i} J_3}\right) &= \delta_i^j E \,\mathrm{Tr}\, \mathrm{e}^{-2\pi \mathrm{i} J_3} \\ \mathrm{Tr}\left(Q_{i\alpha} Q^{\dagger j\alpha} \mathrm{e}^{-2\pi \mathrm{i} J_3} - Q^{\dagger j\alpha} \mathrm{e}^{-2\pi \mathrm{i} J_3} Q_{i\alpha}\right) &= \delta_i^j E \,\mathrm{Tr}\, \mathrm{e}^{-2\pi \mathrm{i} J_3} \end{aligned}$$

The left-hand side vanishes by cyclicity of the trace! The trace on the right-hand side can be rewritten as a sum over spins $s = 0, 1/2, 1, ...$ weighted by the number of particles n_s of spin s in the representation and the number of helicity states for each s. We thus obtain separate sum rules for massive and massless representations:

$$\begin{aligned} m > 0, &\quad \sum_{s \ge 0} (-)^{2s} n_s (2s+1) = 0, \\ m = 0, &\quad 2 \sum_{s > 0} (-)^{2s} n_s + n_0 = 0. \end{aligned}$$

This is the desired result since $(-)^{2s}$ is equal to +1 for bosons and -1 for fermions.

There is a small subtlety in the interpretation of (6.73). Lorentz transformations do not change the helicity of a massless particle, so an irreducible representation of the Poincaré group involves the momentum states for a single value of the helicity h. However, the CPT reflection symmetry requires that both helicity states $h = \pm s$ are present in the quantum field theory of a massless particle with spin $s > 0$. This doubling is incorporated in (6.73).

8.5.14 Structure of massless representations

In this section we derive the particle content of unitary irreducible representations involving massless particles. Similar techniques apply to both massless and massive representations, but we focus on the massless representations because they are simpler. The review of Sohnius [37] contains a more complete treatment.

We now use the Weyl representation (2.19) of the γ-matrices explicitly and (6.67) then gives

$$\begin{aligned}\left\{Q_{i\alpha}, Q^{\dagger j\beta}\right\}_{\text{qu}} &= \frac{1}{2}\delta_i^j\left(\mathbb{1}P^0 - \vec{\sigma}\cdot\vec{P}\right)_\alpha^{\ \beta} \\ \left\{Q_{i\alpha}, Q_{j\beta}\right\}_{\text{qu}} = 0, &\quad \left\{Q^{\dagger i\alpha}, Q^{\dagger j\beta}\right\}_{\text{qu}} = 0 \\ \left[\vec{J}, Q_{i\alpha}\right]_{\text{qu}} &= -\frac{1}{2}(\vec{\sigma})_\alpha^{\ \beta}Q_{i\beta}\end{aligned}$$

$\vec{J}$ stands for the space components $J^i = -\frac{1}{2}\varepsilon^{ijk}M_{jk}$.

Since SUSY transformations do not change the 4-momentum, it is sufficient to consider the action of the supercharges on a set of particle states $|\bar{p}, h\rangle$ of fixed energy-momentum $\bar{p}^\mu = (E, 0, 0, E)$. On states of 4-momentum $\bar{p}^\mu$, we find from (6.74) that

$$\begin{aligned}\left\{Q_{i1}, Q^{\dagger j1}\right\}_{\text{qu}} &= 0, \\ \left\{Q_{i2}, Q^{\dagger j2}\right\}_{\text{qu}} &= E\delta_i^j.\end{aligned}$$

We want a unitary representation, one in which the Hilbert space norm of all states is positive. The positivity properties of the anti-commutator then require that Q_{i1} and its adjoint must be represented trivially. They have vanishing action on all states of 4-momentum $\bar{p}^\mu$. The remaining non-trivial anti-commutator in our basis involves the $\mathcal{N}$ supercharge components Q_{i2}. Physicists can immediately recognize that this anti-commutator describes the creation and annihilation operators for $\mathcal{N}$ independent fermions. Equivalently the Q_{i2} anticommutator defines a Clifford algebra with $\mathcal{N}$ complex generators. This is equivalent to the real $2\mathcal{N}$-dimensional Clifford algebra we discussed in Sec. 3.1. The unique irreducible representation of this algebra has dimension $2^{\mathcal{N}}$, so the massless SUSY representation must also have $2^{\mathcal{N}}$ particle states.

The standard Fock space techniques of physics tells us that the unique unitary representation has the following structure. We choose the $Q^{\dagger i2}$ as the creation operators and the Q_{i2} as the annihilators. Note that (6.76) implies that $\left[J^3, Q^{\dagger i2}\right] = -\frac{1}{2}Q^{\dagger i2}$. Thus $Q^{\dagger i2}$ lowers the helicity of a state by 1/2. To specify the representation we define its 'Fock vacuum' $|\bar{p}, h_0\rangle$, with h_0 any positive or negative integer or half-integer, as the state that satisfies

$$Q_{i2}\,|\bar{p}, h_0\rangle = 0, \quad J^3\,|\bar{p}, h_0\rangle = h_0\,|\bar{p}, h_0\rangle\,, \quad \forall i = 1, ..., \mathcal{N}$$

The basis [9] of the representation consists of the vacuum state together with all states obtained by applying products of creators. Such products are automatically antisymmetric due to (6.75). In more detail the basis is

9 The particle spin $s = |h|$ is a redundant label, so it is omitted on all states of the Fock basis.

$$
\begin{gathered}
|\bar{p}, h_0\rangle \\
\left|\bar{p}, h_0 - \frac{1}{2}, i\right\rangle = Q^{\dagger i2}\,|\bar{p}, h_0\rangle \\
|\bar{p}, h_0 - 1, [ij]\rangle = Q^{\dagger i2} Q^{\dagger j2}\,|\bar{p}, h_0\rangle
\end{gathered}
$$

etc.

States of helicity $h_0 - \frac{1}{2}m$ have multiplicity $\left(\begin{array}{c}\mathcal{N}\\ m\end{array}\right) = \mathcal{N}!/[m!(\mathcal{N}-m)!]$, and the sequence stops at the multiplicity 1 state of lowest helicity $h_0 - \frac{1}{2}\mathcal{N}$. The total number of states is $2^{\mathcal{N}}$ as is required by the representation theory of Clifford algebras. One can check that half the states are bosons and half are fermions.

Thus a massless irreducible representation of supersymmetry contains a 'tower' of helicity states of maximum helicity h_0 and minimum helicity $h_0 - \frac{1}{2}\mathcal{N}$. A local field theory always contains particles in CPT conjugate pairs with helicities $h = \pm s$. Therefore a supersymmetric field theory typically describes a reducible representation of the algebra in which the CPT conjugate states are added to the states of the basis (6.79). These states are obtained by starting from the CPT conjugate Clifford vacuum $|\bar{p}, -h_0\rangle$ and applying products of the operator Q_{i2}, which raises helicity. When $\mathcal{N} = 4\,|h_0|$, the initial sequence (6.79) is already self-conjugate so nothing needs to be added.

Because helicities are always paired, it is simplest to describe the field theory representations in terms of the number of particle states of a given spin. In Table 6.1 [38] we list the spin content [10] of all representations whose maximum spin satisfies $s_{\max} \leq 2$. It is this set of field theories that can incorporate nonlinear interactions.

Exercise 6.18 Show that the spin content of representations with $\mathcal{N} = 4s_{\max}$ and $\mathcal{N} = 4s_{\max} - 1$ is the same (see footnote 10).

The $\mathcal{N} = 1$ multiplets with maximum spin $s_{\max} = 1/2$ and $s_{\max} = 1$ are the chiral and gauge multiplets whose interactions are discussed earlier in this chapter. In principle the next $\mathcal{N} = 1$ multiplet has spins $(1, 3/2)$. There is a corresponding free field theory, but no interacting field theory is known for this multiplet without supergravity. The reason, discussed in Ch. 5, is that field theories for spin-3/2 fields involve a local supersymmetry. The supersymmetry algebra would contain local translations, and hence general relativity. Therefore, we find the spin- 3/2 particle only in the multiplet $(3/2, 2)$. This is the supergravity multiplet that we will consider in Ch. 9.

The table is limited to $\mathcal{N} \leq 8$ because of the great difficulty of consistent higher spin interactions. For $\mathcal{N} \geq 9$ massless representations necessarily contain some particles of higher spin $s \geq 5/2$. Despite much effort, no interacting field theories in Minkowski spacetime exist. The content of the table may be summarized by the statements:

1. For $\mathcal{N} \leq 4$ there are interacting theories with global SUSY and $s_{\max} \leq 1$.

2. For $\mathcal{N} \leq 8$, there are theories with local SUSY, which involve one spin-2 graviton, $\mathcal{N}$ spin-3/2 gravitinos, and, for $\mathcal{N} \geq 2$, lower spin particles. These are supergravity theories.

3. For $\mathcal{N} \geq 9$ there is the higher spin desert.

10 There is a subtle hermiticity requirement for $\mathcal{N} = 2$, which requires that the multiplet $(-1/2, 0, 0, 1/2)$ must be doubled although it is self-conjugate.

		$s=2$	$s=3/2$	$s=1$	$s=1/2$	$s=0$
$\mathcal{N}=1$	$s_{\max}=2$	1	1			
	$s_{\max}=3/2$		1	1		
	$s_{\max}=1$			1	1	
	$s_{\max}=1/2$				1	$1+1$
$\mathcal{N}=2$	$s_{\max}=2$	1	2	1		
	$s_{\max}=3/2$		1	2	1	
	$s_{\max}=1$			1	2	$1+1$
	$s_{\max}=1/2$				2	$2+2$
$\mathcal{N}=3$	$s_{\max}=2$	1	3	3	1	
	$s_{\max}=3/2$		1	3	3	$1+1$
	$s_{\max}=1$			1	$3+1$	$3+3$
$\mathcal{N}=4$	$s_{\max}=2$	1	4	6	4	$1+1$
	$s_{\max}=3/2$		1	4	$6+1$	$4+4$
	$s_{\max}=1$			1	4	6
$\mathcal{N}=5$	$s_{\max}=2$	1	5	10	$10+1$	$5+5$
	$s_{\max}=3/2$		1	$5+1$	$10+5$	$10+10$
$\mathcal{N}=6$	$s_{\max}=2$	1	6	$15+1$	$20+6$	$15+15$
	$s_{\max}=3/2$		1	6	15	20
$\mathcal{N}=7$	$s_{\max}=2$	1	$7+1$	$21+7$	$35+21$	$35+35$
$\mathcal{N}=8$	$s_{\max}=2$	1	8	28	56	70

Exercise 6.19 The reader should check Table 6.1.

8.5.15 Appendix 6A Extended supersymmetry and Weyl spinors

It is often useful to discuss extended supersymmetry using Weyl spinors and their conjugates rather than Majorana spinors. The equivalence is discussed in Box 3.3, where we showed that one can represent Majorana spinors in terms of Weyl spinors and their conjugates, i.e. $Q = P_L Q + P_R Q$. The chirality of the (Majorana) conjugate spinors can be obtained from (3.56), which implies that

$$\overline{(P_R Q)} \equiv (P_R Q)^T C = \bar{Q} P_R$$

Hence applying a chiral projector to (6.1) teaches us that the two supercharges should have opposite chirality in order to have a non-vanishing anti-commutator:

$$\left\{ (P_L Q)_\alpha \, , \overline{(P_R Q)}^\beta \right\} = -\frac{1}{2} \left(P_L \gamma^\mu\right)_\alpha^{\ \beta} P_\mu .$$

It is convenient to use the up or down position of the index $i = 1, ..., \mathcal{N}$ to indicate at once the chiral projections of the supersymmetry generators for extended supersymmetry:

$$Q_i = P_L Q_i, \quad Q^i = P_R Q^i$$

The Majorana spinors are thus $Q^i + Q_i$, and Q_i is the charge conjugate of Q^i. From (6.80) we obtain

$$\bar{Q}_i = \overline{(P_L Q_i)} = \bar{Q}_i P_L, \quad \bar{Q}^i = \overline{(P_R Q^i)} = \bar{Q}^i P_R$$

The minimal extended algebra is then

$$\begin{aligned}
\left\{Q_{i\alpha}, \bar{Q}^{j\beta}\right\} &= -\frac{1}{2}\delta_i^j \left(P_L\gamma_\mu\right)_\alpha^{\ \beta} P^\mu, & \left\{Q^i_\alpha, \bar{Q}_j^\beta\right\} &= -\frac{1}{2}\delta_j^i \left(P_R\gamma_\mu\right)_\alpha^{\ \beta} P^\mu, \\
\left\{Q_{i\alpha}, \bar{Q}_j^\beta\right\} &= 0, & \left\{Q^i_\alpha, \bar{Q}^{j\beta}\right\} &= 0, \\
\left[M_{[\mu\nu]}, Q_{i\alpha}\right] &= -\frac{1}{2}\left(\gamma_{\mu\nu}\right)_\alpha^{\ \beta} Q_{i\beta}, & \left[M_{[\mu\nu]}, Q^i_\alpha\right] &= -\frac{1}{2}\left(\gamma_{\mu\nu}\right)_\alpha^{\ \beta} Q^i_\beta, \\
\left[P_\mu, Q_{i\alpha}\right] &= 0, & \left[P_\mu, Q^i_\alpha\right] &= 0.
\end{aligned}$$

Exercise 6.20 Lower the spinor indices in the first anti-commutator of (6.84) and check that the equation is consistent with the one obtained by taking the charge conjugate.

In Sec. 6.4 we used the complex conjugate spinors. First, remark that Q is Majorana, and we can thus also use $\bar{Q} = \mathrm{i}Q^\dagger\gamma^0$ in (6.80) to write

$$\overline{(P_RQ)} = \mathrm{i}Q^\dagger\gamma^0 P_R = \mathrm{i}Q^\dagger P_L\gamma^0 = \mathrm{i}\left(P_LQ\right)^\dagger\gamma^0,$$

which illustrates again that (P_RQ) is not Majorana. Using the notation (6.82), this gives

$$\bar{Q}^i = \mathrm{i}\left(Q_i\right)^\dagger\gamma^0 = \mathrm{i}Q^{\dagger i}\gamma^0, \quad Q^{\dagger i\alpha} \equiv \left(Q_{i\alpha}\right)^\dagger.$$

In the last equation we define $Q^{\dagger i}$, with upper i index, as the hermitian conjugate of Q_i, which implies (omitting again spinor indices) $Q^{\dagger i} = Q^{\dagger i}P_L$. This leads to

$$\left\{Q_{i\alpha}, Q^{\dagger j\beta}\right\} = -\frac{1}{2}\mathrm{i}\delta_i^j\left(P_L\gamma_\mu\gamma^0\right)_\alpha^{\ \beta} P^\mu, \quad \alpha = 1, 2$$

In the Weyl representation, $Q_{i\alpha} = (Q_{i1}, Q_{i2}, 0, 0)$, and $\bar{Q}^i$ are their right-handed conjugates, i.e. $\bar{Q}^{i\alpha} = \left(0, 0, (\mathrm{i}Q_{i1})^*, \mathrm{i}\left(Q_{i2}\right)^*\right)$. The extra factor i to go to the quantum bracket thus leads to (6.67).

8.5.16 Appendix 6B On- and off-shell multiplets and degrees of freedom

It has been shown in Sec. 6.4.1 that on-shell multiplets have equal numbers of bosonic and fermionic degrees of freedom. But the reader may have noticed in the explicit examples in

There are equal numbers of bosonic and fermionic degrees of freedom in any realization of a supersymmetry algebra of the form $\{Q, Q\} = P$.

this chapter that also off-shell the multiplets have equal numbers of bosonic and fermionic degrees of freedom. More strictly stated, the theorem of Box 6.1 holds. The theorem is proven in [37]. Off-shell equality of bosonic and fermionic degrees of freedom holds for some extended supersymmetry and higher dimensional theories, but it is not always true. It is valid when the algebra (6.1) holds. In practice this holds when the theory has auxiliary fields which 'close the algebra off-shell'.

Consider first the example of the chiral multiplet. We have discussed the chiral multiplet first with the fields $\{Z, \chi, F\}$, and satisfying this algebra. Z and F are complex fields, and thus there are four real off-shell (since we did not use field equations) bosonic degrees of freedom. These are balanced by the four components of the Majorana spinor χ. We say that the chiral multiplet is a $4 + 4$ off-shell multiplet.

On the other hand, we have seen in Sec. 6.2.2 that the algebra is also valid when the equations of motion are used. Then F is no longer an independent field, we count two bosonic degrees of freedom for the complex Z, and also the fermions have two on-shell degrees of freedom. So the chiral multiplet is also a $2 + 2$ on-shell multiplet. These two ways of counting are called on-shell counting and off-shell counting.

To interpret this theorem we remind the reader of the terminology of on-shell and offshell degrees of freedom that we introduced in Box 4.1. To illustrate that the relevant definition of off-shell degrees of freedom indeed should contain the subtraction of gauge transformations, we consider the gauge multiplet. The off-shell counting is easily established: the gauge field A_μ and the gaugino λ both describe two on-shell degrees of freedom. To apply the theorem in the off-shell case, we have to remember that at the end of Sec. 6.3.1 we saw that the anti-commutator of two supersymmetries involves also a gauge transformation. Therefore, we can only apply the theorem on gauge invariant states (or identify states that differ by a gauge transformation), i.e. we have to subtract the gauge transformations

field	off-shell		on-shell	
		$D=4$		$D=4$
ϕ	1	1	1	1
λ	$2^{[D/2]}$	4	$\frac{1}{2}2^{[D/2]}$	2
A_μ	$D-1$	3	$D-2$	2
ψ_μ	$(D-1)2^{[D/2]}$	12	$\frac{1}{2}(D-3)2^{[D/2]}$	2
$g_{\mu\nu}$	$\frac{1}{2}D(D-1)$	6	$\frac{1}{2}D(D-3)$	2

in the counting. Thus, the gauge vector A_μ counts off-shell for three degrees of freedom, which together with the one real degree of freedom of the auxiliary field D balance the four off-shell ones of the gaugino.

Since this counting can be used to understand the structure of many realizations of supersymmetry, we end with Table 6.2 that summarizes the results for the degrees of freedom for the scalar field ϕ, a Majorana fermion λ, the gauge field A_μ, the Majorana RaritaSchwinger field ψ_μ and the graviton field $g_{\mu\nu}$.

Exercise 6.21 Check that the entries of the table correspond with the results on degrees of freedom obtained in previous chapters. The results for the graviton will be obtained in Sec. 8.2.

9 Solitons Instantons, Monopoles, Vortices, Kinks by Tong

These TASI lectures cover aspects of solitons with focus on applications to the quantum dynamics of supersymmetric gauge theories and string theory. The lectures consist of four sections, each dealing with a different soliton. We start with instantons and work down in co-dimension to monopoles, vortices and, eventually, domain walls. Emphasis is placed on the moduli space of solitons and, in particular, on the web of connections that links solitons of different types. The D-brane realization of the ADHM and Nahm construction for instantons and monopoles is reviewed, together with related constructions for vortices and domain walls. Each lecture ends with a series of vignettes detailing the roles solitons play in the quantum dynamics of supersymmetric gauge theories in various dimensions. This includes applications to the AdS/CFT correspondence, little string theory, S-duality, cosmic strings, and the quantitative correspondence between 2d sigma models and 4d gauge theories.

9.1 Introduction

170 years ago, a Scotsman on horseback watched a wave travelling down Edinburgh's Union canal. He was so impressed that he followed the wave for several miles, described the day of observation as the happiest of his life, and later attempted to recreate the experience in his

own garden. The man's name was John Scott Russell and he is generally credited as the first person to develop an unhealthy obsession with the "singular and beautiful phenomenon" that we now call a soliton.

Russell was ahead of his time. The features of stability and persistence that so impressed him were not appreciated by his contemporaries, with Airy arguing that the "great primary wave" was neither great nor primary[1]. It wasn't until the following century that solitons were understood to play an important role in areas ranging from engineering to biology, from condensed matter to cosmology.

The purpose of these lectures is to explore the properties of solitons in gauge theories. There are four leading characters: the instanton, the monopole, the vortex, and the domain wall (also known as the kink). Most reviews of solitons start with kinks and work their way up to the more complicated instantons. Here we're going to do things backwards and follow the natural path: instantons are great and primary, other solitons follow. A major theme of these lectures is to flesh out this claim by describing the web of inter-relationships connecting our four solitonic characters.

Each lecture will follow a similar pattern. We start by deriving the soliton equations and examining the basic features of the simplest solution. We then move on to discuss the interactions of multiple solitons, phrased in terms of the moduli space. For each type of soliton, D-brane techniques are employed to gain a better understanding of the relevant geometry. Along the way, we shall discuss various issues including fermionic zero modes, dyonic excitations and non-commutative solitons. We shall also see the earlier solitons reappearing in surprising places, often nestling within the worldvolume of a larger soliton, with interesting consequences. Each lecture concludes with a few brief descriptions of the roles solitons play in supersymmetric gauge theories in various dimensions.

These notes are aimed at advanced graduate students who have some previous awareness of solitons. The basics will be covered, but only very briefly. A useful primer on solitons can be found in most modern field theory textbooks (see for example [1]). More details are contained in the recent book by Manton and Sutcliffe [2]. There are also a number of good reviews dedicated to solitons of a particular type and these will be mentioned at the beginning of the relevant lecture. Other background material that will be required for certain sections includes a basic knowledge of the structure of supersymmetric gauge theories and D-brane dynamics. Good reviews of these subjects can be found in [3, 4, 5].

[1] More background on Russell and his wave can be found at http://www.ma.hw.ac.uk/~chris/scott_russell.html and http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Russell_Scott.html.

10 Instantons

30 years after the discovery of Yang-Mils instantons [6], they continue to fascinate both physicists and mathematicians alike. They have lead to new insights into a wide range of phenomenon, from the structure of the Yang-Mills vacuum [7, 8, 9] to the classification of four-manifolds [10]. One of the most powerful uses of instantons in recent years is in the analysis of supersymmetric gauge dynamics where they play a key role in unravelling the plexus of entangled dualities that relates different theories. The purpose of this lecture is to review the classical properties of instantons, ending with some applications to the quantum dynamics of supersymmetric gauge theories.

There exist many good reviews on the subject of instantons. The canonical reference for basics of the subject remains the beautiful lecture by Coleman [11]. More recent applications to supersymmetric theories are covered in detail in reviews by Shifman and Vainshtein [12] and by Dorey, Hollowood, Khoze and Mattis [13]. This latter review describes the ADHM construction of instantons and overlaps with the current lecture.

10.1 The Basics

The starting point for our journey is four-dimensional, pure $SU(N)$ Yang-Mills theory with action[2]

$$S = \frac{1}{2e^2}\int d^4x \; \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \tag{10.1}$$

Motivated by the semi-classical evaluation of the path integral, we search for finite action solutions to the Euclidean equations of motion,

$$\mathcal{D}_\mu F^{\mu\nu} = 0 \tag{10.2}$$

which, in the imaginary time formulation of the theory, have the interpretation of mediating quantum mechanical tunnelling events.

The requirement of finite action means that the potential A_μ must become pure gauge as we head towards the boundary $r \to \infty$ of spatial $\mathbf{R}^4$,

$$A_\mu \to i g^{-1}\,\partial_\mu g \tag{10.3}$$

with $g(x) = e^{iT(x)} \in SU(N)$. In this way, any finite action configuration provides a map from $\partial\mathbf{R}^4 \cong \mathbf{S}^3_\infty$ into the group $SU(N)$. As is well known, such maps are classified by homotopy theory. Two maps are said to lie in the same homotopy class if they can be continuously deformed into each other, with different classes labelled by the third homotopy group,

$$\Pi_3(SU(N)) \cong \mathbf{Z} \tag{10.4}$$

[2]Conventions: We pick Hemitian generators T^m with Killing form $\mathrm{Tr}\, T^mT^n = \frac{1}{2}\delta^{mn}$. We write $A_\mu = A_\mu^m T^m$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$. Adjoint covariant derivatives are $\mathcal{D}_\mu X = \partial_\mu X - i[A_\mu, X]$. In this section alone we work with Euclidean signature and indices will wander from top to bottom with impunity; in the following sections we will return to Minkowski space with signature $(+,-,-,-)$.

The integer $k \in \mathbf{Z}$ counts how many times the group wraps itself around spatial $\mathbf{S}^3_\infty$ and is known as the Pontryagin number, or second Chern class. We will sometimes speak simply of the "charge" k of the instanton. It is measured by the surface integral

$$k = \frac{1}{24\pi^2} \int_{\mathbf{S}^3_\infty} d^3S_\mu \ \mathrm{Tr}\,(\partial_\nu g)g^{-1}\,(\partial_\rho g)g^{-1}\,(\partial_\sigma g)g^{-1}\,\epsilon^{\mu\nu\rho\sigma} \tag{10.5}$$

The charge k splits the space of field configurations into different sectors. Viewing $\mathbf{R}^4$ as a foliation of concentric $\mathbf{S}^3$'s, the homotopy classification tells us that we cannot transform a configuration with non-trivial winding $k \neq 0$ at infinity into one with trivial winding on an interior $\mathbf{S}^3$ while remaining in the pure gauge ansatz (10.3). Yet, at the origin, obviously the gauge field must be single valued, independent of the direction from which we approach. To reconcile these two facts, a configuration with $k \neq 0$ cannot remain in the pure gauge form (10.3) throughout all of $\mathbf{R}^4$: it must have non-zero action.

10.1.0An Example: $SU(2)$

The simplest case to discuss is the gauge group $SU(2)$ since, as a manifold, $SU(2) \cong \mathbf{S}^3$ and it's almost possible to visualize the fact that $\Pi_3(\mathbf{S}^3) \cong \mathbf{Z}$. (Ok, maybe $\mathbf{S}^3$ is a bit of a stretch, but it is possible to visualize $\Pi_1(\mathbf{S}^1) \cong \mathbf{Z}$ and $\Pi_2(\mathbf{S}^2) \cong \mathbf{Z}$ and it's not the greatest leap to accept that, in general, $\Pi_n(\mathbf{S}^n) \cong \mathbf{Z}$). Examples of maps in the different sectors are

- $g^{(0)} = 1$, the identity map has winding $k = 0$
- $g^{(1)} = (x_4 + i x_i \sigma^i)/r$ has winding number $k = 1$. Here $i = 1, 2, 3$, and the σ^i are the Pauli matrices
- $g^{(k)} = [g^{(1)}]^k$ has winding number k.

To create a non-trivial configuration in $SU(N)$, we could try to embed the maps above into a suitable $SU(2)$ subgroup, say the upper left-hand corner of the $N \times N$ matrix. It's not obvious that if we do this they continue to be a maps with non-trivial winding since one could envisage that they now have space to slip off. However, it turns out that this doesn't happen and the above maps retain their winding number when embedded in higher rank gauge groups.

10.1.1 The Instanton Equations

We have learnt that the space of configurations splits into different sectors, labelled by their winding $k \in \mathbf{Z}$ at infinity. The next question we want to ask is whether solutions actually exist for different k. Obviously for $k = 0$ the usual vacuum $A_\mu = 0$ (or gauge transformations thereof) is a solution. But what about higher winding with $k \neq 0$? The first step to constructing solutions is to derive a new set of equations that the instantons will obey, equations that are first order rather than second order as in (10.2). The trick for doing this is usually referred to as the Bogomoln'yi bound [14] although, in the case of instantons, it was actually introduced in the original paper [6]. From the above considerations, we have seen that any configuration with $k \neq 0$ must have some non-zero action. The Bogomoln'yi bound quantifies this. We rewrite the action by completing the square,

$$\begin{aligned} S_{\rm inst} &= \frac{1}{2e^2}\int d^4x\ \mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu} \\ &= \frac{1}{4e^2}\int d^4x\ \mathrm{Tr}\,(F_{\mu\nu} \mp {}^\star F^{\mu\nu})^2 \pm 2\mathrm{Tr}\ F_{\mu\nu}{}^\star F^{\mu\nu} \\ &\geq \pm\frac{1}{2e^2}\int d^4x\ \partial_\mu\left(A_\nu F_{\rho\sigma} + \tfrac{2i}{3}A_\nu A_\rho A_\sigma\right)\epsilon^{\mu\nu\rho\sigma} \end{aligned} \tag{10.6}$$

where the dual field strength is defined as ${}^\star F_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}$ and, in the final line, we've used the fact that $F_{\mu\nu}{}^\star F^{\mu\nu}$ can be expressed as a total derivative. The final expression is a surface term which measures some property of the field configuration on the boundary $\mathbf{S}^3_\infty$. Inserting the asymptotic form $A_\nu \to ig^{-1}\partial_\nu g$ into the above expression and comparing with (10.5), we learn that the action of the instanton in a topological sector k is bounded by

$$S_{\text{inst}} \geq \frac{8\pi^2}{e^2}\,|k| \tag{10.7}$$

with equality if and only if

$$\begin{aligned} F_{\mu\nu} &= {}^\star F_{\mu\nu} \qquad\qquad (k>0) \\ F_{\mu\nu} &= -{}^\star F_{\mu\nu} \qquad\qquad (k<0) \end{aligned}$$

Since parity maps $k \to -k$, we can focus on the self-dual equations $F = {}^\star F$. The Bogomoln'yi argument (which we shall see several more times in later sections) says that a solution to the self-duality equations must necessarily solve the full equations of motion since it minimizes the action in a given topological sector. In fact, in the case of instantons, it's trivial to see that this is the case since we have

$$\mathcal{D}_\mu F^{\mu\nu} = \mathcal{D}_\mu{}^\star F^{\mu\nu} = 0 \tag{10.8}$$

by the Bianchi identity.

10.1.2 Collective Coordinates

So we now know the equations we should be solving to minimize the action. But do solutions exist? The answer, of course, is yes! Let's start by giving an example, before we move on to examine some of its properties, deferring discussion of the general solutions to the next subsection.

The simplest solution is the $k=1$ instanton in $SU(2)$ gauge theory. In singular gauge, the connection is given by

$$A_\mu = \frac{\rho^2 (x-X)_\nu}{(x-X)^2((x-X)^2+\rho^2)}\,\bar\eta^i_{\mu\nu}\,(g\sigma^i g^{-1}) \tag{10.9}$$

The σ^i, $i=1,2,3$ are the Pauli matrices and carry the $su(2)$ Lie algebra indices of A_μ. The $\bar\eta^i$ are three 4×4 anti-self-dual 't Hooft matrices which intertwine the group structure of the index i with the spacetime structure of the indices μ,ν. They are given by

$$\bar\eta^1 = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad \bar\eta^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \bar\eta^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \tag{10.10}$$

It's a useful exercise to compute the field strength to see how it inherits its self-duality from the anti-self-duality of the $\bar\eta$ matrices. To build an anti-self-dual field strength, we need to simply exchange the $\bar\eta$ matrices in (10.9) for their self-dual counterparts,

$$\eta^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \qquad \eta^2 = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad \eta^3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \tag{10.11}$$

For our immediate purposes, the most important feature of the solution (10.9) is that it is not unique: it contains a number of parameters. In the context of solitons, these are known as *collective coordinates*. The solution (10.9) has eight such parameters. They are of three different types:

i) 4 translations X_μ: The instanton is an object localized in $\mathbf{R}^4$, centered around the point $x_\mu = X_\mu$.

ii) 1 scale size ρ: The interpretation of ρ as the size of the instanton can be seen by rescaling x and X in the above solution to demote ρ to an overall constant.

iii) 3 global gauge transformations $g \in SU(2)$: This determines how the instanton is embedded in the gauge group.

At this point it's worth making several comments about the solution and its collective coordinates.

- For the $k = 1$ instanton, each of the collective coordinates described above is a Goldstone mode, arising because the instanton configuration breaks a symmetry of the Lagrangian (10.1). In the case of X_μ and g it is clear that the symmetry is translational invariance and $SU(2)$ gauge invariance respectively. The parameter ρ arises from broken conformal invariance. It's rather common that all the collective coordinates of a single soliton are Goldstone modes. It's not true for higher k.

- The apparent singularity at $x_\mu = X_\mu$ is merely a gauge artifact (hence the name "singular gauge"). A plot of a gauge invariant quantity, such as the action density, reveals a smooth solution. The exception is when the instanton shrinks to zero size $\rho \to 0$. This singular configuration is known as the small instanton. Despite its singular nature, it plays an important role in computing the contribution to correlation functions in supersymmetric theories. The small instanton lies at finite distance in the space of classical field configurations (in a way which will be made precise in Section 1.2).

- You may be surprised that we are counting the gauge modes g as physical parameters of the solution. The key point is that they arise from the *global* part of the gauge symmetry, meaning transformations that don't die off asymptotically. These are physical symmetries of the system rather than redundancies. In the early days of studying instantons the 3 gauge modes weren't included, but it soon became apparent that many of the nicer mathematical properties of instantons (for example, hyperKählerity of the moduli space) require us to include them, as do certain physical properties (for example, dyonic instantons in five dimensions)

The $SU(2)$ solution (10.9) has 8 collective coordinates. What about $SU(N)$ solutions? Of course, we should keep the $4 + 1$ translational and scale parameters but we would expect more orientation parameters telling us how the instanton sits in the larger $SU(N)$ gauge group. How many? Suppose we embed the above $SU(2)$ solution in the upper left-hand corner of an $N \times N$ matrix. We can then rotate this into other embeddings by acting with $SU(N)$, modulo the stabilizer which leaves the configuration untouched. We have

$$SU(N)/S[U(N-2) \times U(2)] \tag{10.12}$$

where the $U(N-2)$ hits the lower-right-hand corner and doesn't see our solution, while the $U(2)$ is included in the denominator since it acts like g in the original solution (10.9) and we don't want to overcount. Finally, the notation $S[U(p) \times U(q)]$ means that we lose the overall central $U(1) \subset U(p) \times U(q)$. The coset space above has dimension $4N - 8$. So, within the ansatz (10.9) embedded in $SU(N)$, we see that the $k = 1$ solution has $4N$ collective coordinates. In fact, it turns out that this is all of them and the solution (10.9), suitably embedded, is the most general $k = 1$ solution in an $SU(N)$ gauge group. But what about solutions with higher k? To discuss this, it's useful to introduce the idea of the moduli space.

10.2 The Moduli Space

We now come to one of the most important concepts of these lectures: the *moduli space*. This is defined to be the space of all solutions to $F = {}^\star F$, modulo gauge transformations, in a given winding sector k and gauge group $SU(N)$. Let's denote this space as $\mathcal{I}_{k,N}$. We will define similar moduli spaces for the other solitons and much of these lectures will be devoted to understanding the different roles these moduli spaces play and the relationships between them.

Coordinates on $\mathcal{I}_{k,N}$ are given by the collective coordinates of the solution. We've seen above that the $k = 1$ solution has $4N$ collective coordinates or, in other words, $\dim(\mathcal{I}_{1,N}) = 4N$. For higher k, the number of collective coordinates can be determined by index theorem techniques. I won't give all the details, but will instead simply tell you the answer.

$$\dim(\mathcal{I}_{k,N}) = 4kN \tag{10.13}$$

This has a very simple interpretation. The charge k instanton can be thought of as k charge 1 instantons, each with its own position, scale, and gauge orientation. When the instantons are well separated, the solution does indeed look like this. But when instantons start to overlap, the interpretation of the collective coordinates can become more subtle.

Strictly speaking, the index theorem which tells us the result (10.13) doesn't count the number of collective coordinates, but rather related quantities known as *zero modes*. It works as follows. Suppose we have a solution A_μ satisfying $F = {}^\star F$. Then we can perturb this solution $A_\mu \to A_\mu + \delta A_\mu$ and ask how many other solutions are nearby. We require the perturbation δA_μ to satisfy the linearized self-duality equations,

$$\mathcal{D}_\mu \delta A_\nu - \mathcal{D}_\nu \delta A_\mu = \epsilon_{\mu\nu\rho\sigma} \mathcal{D}^\rho \delta A^\sigma \tag{10.14}$$

where the covariant derivative $\mathcal{D}_\mu$ is evaluated on the background solution. Solutions to (10.14) are called zero modes. The idea of zero modes is that if we have a general solution $A_\mu = A_\mu(x_\mu, X^\alpha)$, where X^α denote all the collective coordinates, then for each collective coordinate we can define the zero mode $\delta_\alpha A_\mu = \partial A_\mu / \partial X^\alpha$ which will satisfy (10.14). In general however, it is not guaranteed that any zero mode can be successfully integrated to give a corresponding collective coordinate. But it will turn out that all the solitons discussed in these lectures do have this property (at least this is true for bosonic collective coordinates; there is a subtlety with the Grassmannian collective coordinates arising from fermions which we'll come to shortly).

Of course, any local gauge transformation will also solve the linearized equations (10.14) so we require a suitable gauge fixing condition. We'll write each zero mode to include an infinitesimal gauge transformation Ω_α,

$$\delta_\alpha A_\mu = \frac{\partial A_\mu}{\partial X^\alpha} + \mathcal{D}_\mu \Omega_\alpha \tag{10.15}$$

and choose Ω_α so that $\delta_\alpha A_\mu$ is orthogonal to any other gauge transformation, meaning

$$\int d^4x \; \mathrm{Tr}\,(\delta_\alpha A_\mu)\, \mathcal{D}_\mu \eta = 0 \qquad \forall\; \eta \tag{10.16}$$

which, integrating by parts, gives us our gauge fixing condition

$$\mathcal{D}_\mu \left(\delta_\alpha A_\mu\right) = 0 \tag{10.17}$$

This gauge fixing condition does not eliminate the collective coordinates arising from global gauge transformations which, on an operational level, gives perhaps the clearest reason why we must include them. The Atiyah-Singer index theorem counts the number of solutions to (10.14) and (10.17) and gives the answer (10.13).

So what does the most general solution, with its $4kN$ parameters, look like? The general explicit form of the solution is not known. However, there are rather clever ansatzë which give rise to various subsets of the solutions. Details can be found in the original literature [15, 16] but, for now, we head in a different, and ultimately more important, direction and study the geometry of the moduli space.

10.2.1 The Moduli Space Metric

A priori, it is not obvious that $\mathcal{I}_{k,N}$ is a manifold. In fact, it does turn out to be a smooth space apart from certain localized singularities corresponding to small instantons at $\rho \to 0$ where the field configuration itself also becomes singular.

The moduli space $\mathcal{I}_{k,N}$ inherits a natural metric from the field theory, defined by the overlap of zero modes. In the coordinates X^α, $\alpha = 1, ..., 4kN$, the metric is given by

$$g_{\alpha\beta} = \frac{1}{2e^2} \int d^4x \ \mathrm{Tr} \ \left(\delta_\alpha A_\mu\right) \left(\delta_\beta A_\mu\right) \tag{10.18}$$

It's hard to overstate the importance of this metric. It distills the information contained in the solutions to $F = {}^\star F$ into a more manageable geometric form. It turns out that for many applications, everything we need to know about the instantons is contained in the metric $g_{\alpha\beta}$, and this remains true of similar metrics that we will define for other solitons. Moreover, it is often much simpler to determine the metric (10.18) than it is to determine the explicit solutions.

The metric has a few rather special properties. Firstly, it inherits certain isometries from the symmetries of the field theory. For example, both the $SO(4)$ rotation symmetry of spacetime and the $SU(N)$ gauge action will descend to give corresponding isometries of the metric $g_{\alpha\beta}$ on $\mathcal{I}_{k,N}$.

Another important property of the metric (10.18) is that it is *hyperKähler*, meaning that the manifold has reduced holonomy $Sp(kN) \subset SO(4kN)$. Heuristically, this means that the manifold admits something akin to a quaternionic structure[3]. More precisely, a hyperKählermanifold admits three complex structures J^i, $i = 1, 2, 3$ which obey the relation

$$J^i \, J^j = -\delta^{ij} + \epsilon^{ijk} \, J^k \tag{10.19}$$

[3]Warning: there is also something called a quaternionic manifold which arises in $\mathcal{N} = 2$ supergravity theories [17] and is different from a hyperKähler manifold. For a discussion on the relationship see [18].

The simplest example of a hyperKählermanifold is $\mathbf{R}^4$, viewed as the quaternions. The three complex structures can be taken to be the anti-self-dual 't Hooft matrices $\bar{\eta}^i$ that we defined in (10.10), each of which gives a different complex pairing of $\mathbf{R}^4$. For example, from $\bar{\eta}^3$ we get $z^1 = x^1 + ix^2$ and $z^2 = x^3 - ix^4$.

10.2.1An Aside: HyperKähler Metrics

The instanton moduli space $\mathcal{I}_{k,N}$ inherits its complex structures J^i from those of $\mathbf{R}^4$. To see this, note if δA_μ is a zero mode, then we may immediately write down three other zero modes $\bar{\eta}^i_{\nu\mu}\,\delta A_\mu$, each of which satisfy the equations (10.14) and (10.17). It must be possible to express these three new zero modes as a linear combination of the original ones, allowing us to define three matrices J^i,

$$\bar{\eta}^i_{\mu\nu}\;\delta_\beta A_\nu = (J^i)^\alpha_{\;\;\beta}\;[\delta_\alpha A_\mu] \tag{10.20}$$

These matrices J^i then descend to three complex structures on the moduli space $\mathcal{I}_{k,N}$ itself which are given by

$$(J^i)^\alpha_{\;\;\beta} = g^{\alpha\gamma}\int d^4x\;\bar{\eta}^i_{\mu\nu}\;\mathrm{Tr}\;\delta_\beta A_\mu\;\delta_\gamma A_\nu \tag{10.21}$$

So far we have shown only that J^i define almost complex structures. To prove hyperKählerity, one must also show integrability which, after some gymnastics, is possible using the formulae above. A more detailed discussion of the geometry of the moduli space in this language can be found in [19, 20] and more generally in [21, 22]. For physicists the simplest proof of hyperKählerity follows from supersymmetry as we shall review in section 1.3.

It will prove useful to return briefly to discuss the isometries. In Kähler and hyperKählermanifolds, it's often important to state whether isometries are compatible with the complex structure J. If the complex structure doesn't change as we move along the isometry, so that the Lie derivative $\mathcal{L}_k J = 0$, with k the Killing vector, then the isometry is said to be *holomorphic*. In the instanton moduli space $\mathcal{I}_{k,N}$, the $SU(N)$ gauge group action is tri-holomorphic, meaning it preserves all three complex structures. Of the $SO(4) \cong SU(2)_L \times SU(2)_R$ rotational symmetry, one half, $SU(2)_L$, is tri-holomorphic, while the three complex structures are rotated under the remaining $SU(2)_R$ symmetry.

10.2.2 An Example: A Single Instanton in $SU(2)$

In the following subsection we shall show how to derive metrics on $\mathcal{I}_{k,N}$ using the powerful ADHM technique. But first, to get a flavor for the ideas, let's take a more pedestrian route for the simplest case of a $k = 1$ instanton in $SU(2)$. As we saw above, there are three types of collective coordinates.

i) The four translational modes are $\delta_{(\nu)}A_\mu = \partial A_\mu/\partial X^\nu + \mathcal{D}_\mu\Omega_\nu$ where Ω_ν must be chosen to satisfy (10.17). Using the fact that $\partial/\partial X^\nu = -\partial/\partial x^\nu$, it is simple to see that the correct choice of gauge is $\Omega_\nu = A_\nu$, so that the zero mode is simply given by $\delta_\nu A_\mu = F_{\mu\nu}$, which satisfies the gauge fixing condition by virtue of the original equations of motion (10.2). Computing the overlap of these translational zero modes then gives

$$\int d^4x\;\mathrm{Tr}\;(\delta_{(\nu)}A_\mu\,\delta_{(\rho)}A_\mu) = S_{\mathrm{inst}}\;\delta_{\nu\rho} \tag{10.22}$$

ii) One can check that the scale zero mode $\delta A_\mu = \partial A_\mu/\partial\rho$ already satisfies the gauge fixing condition (10.17) when the solution is taken in singular gauge (10.9). The overlap integral in this case is simple to perform, yielding

$$\int d^4x \ \mathrm{Tr}\ (\delta A_\mu\, \delta A_\mu) = 2S_{\mathrm{inst}} \tag{10.23}$$

iii) Finally, we have the gauge orientations. These are simply of the form $\delta A_\mu = \mathcal{D}_\mu\Lambda$, but where Λ does not vanish at infinity, so that it corresponds to a global gauge transformation. In singular gauge it can be checked that the three $SU(2)$ rotations $\Lambda^i = [(x-X)^2/((x-X)^2+\rho^2)]\sigma^i$ satisfy the gauge fixing constraint. These give rise to an $SU(2) \cong \mathbf{S}^3$ component of the moduli space with radius given by the norm of any one mode, say, Λ^3

$$\int d^4x \ \mathrm{Tr}\ (\delta A_\mu\, \delta A_\mu) = 2S_{\mathrm{inst}}\, \rho^2 \tag{10.24}$$

Note that, unlike the others, this component of the metric depends on the collective coordinate ρ, growing as ρ^2. This dependance means that the $\mathbf{S}^3$ arising from $SU(2)$ gauge rotations combines with the $\mathbf{R}^+$ from scale transformations to form the space $\mathbf{R}^4$. However, there is a discrete subtlety. Fields in the adjoint representation are left invariant under the center $Z_2 \subset SU(2)$, meaning that the gauge rotations give rise to $\mathbf{S}^3/\mathbf{Z}_2$ rather than $\mathbf{S}^3$. Putting all this together, we learn that the moduli space of a single instanton is

$$\mathcal{I}_{1,2} \cong \mathbf{R}^4 \times \mathbf{R}^4/\mathbf{Z}_2 \tag{10.25}$$

where the first factor corresponds to the position of the instanton, and the second factor determines its scale size and $SU(2)$ orientation. The normalization of the flat metrics on the two $\mathbf{R}^4$ factors is given by (10.22) and (10.23). In this case, the hyperKählerstructure on $\mathcal{I}_{1,2}$ comes simply by viewing each $\mathbf{R}^4 \cong \mathbb{H}$, the quaternions. As is clear from our derivation, the singularity at the origin of the orbifold $\mathbf{R}^4/\mathbf{Z}_2$ corresponds to the small instanton $\rho \to 0$.

10.3 Fermi Zero Modes

So far we've only concentrated on the pure Yang-Mills theory (10.1). It is natural to wonder about the possibility of other fields in the theory: could they also have non-trivial solutions in the background of an instanton, leading to further collective coordinates? It turns out that this doesn't happen for bosonic fields (although they do have an important impact if they gain a vacuum expectation value as we shall review in later sections). Importantly, the fermions do contribute zero modes.

Consider a single Weyl fermion λ transforming in the adjoint representation of $SU(N)$, with kinetic term $i\mathrm{Tr}\,\bar\lambda\,\bar{\not{\mathcal{D}}}\lambda$. In Euclidean space, we treat λ and $\bar\lambda$ as independent variables, a fact which leads to difficulties in defining a real action. (For the purposes of this lecture, we simply ignore the issue - a summary of the problem and its resolutions can be found in [13]). The equations of motion are

$$\bar{\not{\mathcal{D}}}\lambda \equiv \bar\sigma^\mu \mathcal{D}_\mu\lambda = 0 \qquad\qquad \not{\mathcal{D}}\bar\lambda \equiv \sigma^\mu\mathcal{D}_\mu\bar\lambda = 0 \tag{10.26}$$

where $\not{\mathcal{D}} = \sigma^\mu\mathcal{D}_\mu$ and the 2×2 matrices are $\sigma^\mu = (\sigma^i, -i1_2)$. In the background of an instanton $F = {}^\star F$, only λ picks up zero modes. $\bar\lambda$ has none. This situation is reversed in the background of an anti-instanton $F = -{}^\star F$. To see that $\bar{\lambda}$ has no zero modes in the background of an instanton, we look at

$$\bar{\not{D}}\,\not{D} = \bar{\sigma}^\mu \sigma^\nu D_\mu \mathcal{D}_\nu = \mathcal{D}^2\,1_2 + F^{\mu\nu}\bar{\eta}^i_{\mu\nu}\sigma^i \qquad (10.27)$$

where $\bar{\eta}^i$ are the anti-self-dual 't Hooft matrices defined in (10.10). But a self-dual matrix $F_{\mu\nu}$ contracted with an anti-self-dual matrix $\bar{\eta}_{\mu\nu}$ vanishes, leaving us with $\bar{\not{D}}\,\not{D} = \mathcal{D}^2$. And the positive definite operator $\mathcal{D}^2$ has no zero modes. In contrast, if we try to repeat the calculation for λ, we find

$$\not{D}\,\bar{\not{D}} = \mathcal{D}^2\,1_2 + F^{\mu\nu}\eta^i_{\mu\nu}\sigma^i \qquad (10.28)$$

where η^i are the self-dual 't Hooft matrices (10.11). Since we cannot express the operator $\not{D}\,\bar{\not{D}}$ as a total square, there's a chance that it has zero modes. The index theorem tells us that each Weyl fermion λ picks up $4kN$ zero modes in the background of a charge k instanton. There are corresponding Grassmann collective coordinates, which we shall denote as χ, associated to the most general solution for the gauge field and fermions. But these Grassmann collective coordinates occasionally have subtle properties. The quick way to understand this is in terms of supersymmetry. And often the quick way to understand the full power of supersymmetry is to think in higher dimensions.

10.3.1 Dimension Hopping

It will prove useful to take a quick break in order to make a few simple remarks about instantons in higher dimensions. So far we've concentrated on solutions to the self-duality equations in four-dimensional theories, which are objects localized in Euclidean spacetime. However, it is a simple matter to embed the solutions in higher dimensions simply by insisting that all fields are independent of the new coordinates. For example, in $d = 4 + 1$ dimensional theories one can set $\partial_0 = A_0 = 0$, with the spatial part of the gauge field satisfying $F = {}^\star F$. Such configurations have finite energy and the interpretation of particle like solitons. We shall describe some of their properties when we come to applications. Similarly, in $d = 5 + 1$, the instantons are string like objects, while in $d = 9+1$, instantons are five-branes. While this isn't a particularly deep insight, it's a useful trick to keep in mind when considering the fermionic zero modes of the soliton in supersymmetric theories as we shall discuss shortly.

When solitons have a finite dimensional worldvolume, we can promote the collective coordinates to fields which depend on the worldvolume directions. These correspond to massless excitations living on the solitons. For example, allowing the translational modes to vary along the instanton string simply corresponds to waves propagating along the string. Again, this simple observation will become rather powerful when viewed in the context of supersymmetric theories.

A note on terminology: Originally the term "instanton" referred to solutions to the self-dual Yang-Mills equations $F = {}^\star F$. (At least this was true once Physical Review lifted its censorship of the term!). However, when working with theories in spacetime dimensions other than four, people often refer to the relevant finite action configuration as an instanton. For example, kinks in quantum mechanics are called instantons. Usually this doesn't lead to any ambiguity but in this review we'll consider a variety of solitons in a variety of dimensions. I'll try to keep the phrase "instanton" to refer to (anti)-self-dual Yang-Mills instantons.

10.3.2 Instantons and Supersymmetry

Instantons share an intimate relationship with supersymmetry. Let's consider an instanton in a $d = 3 + 1$ supersymmetric theory which could be either $\mathcal{N} = 1$, $\mathcal{N} = 2$ or $\mathcal{N} = 4$ super Yang-Mills. The supersymmetry transformation for any adjoint Weyl fermion takes the form

$$\delta\lambda = F^{\mu\nu}\sigma_\mu\bar{\sigma}_\nu\epsilon \quad , \quad \delta\bar{\lambda} = F^{\mu\nu}\bar{\sigma}_\mu\sigma_\nu\bar{\epsilon} \tag{10.29}$$

where, again, we treat the infinitesimal supersymmetry parameters ϵ and $\bar{\epsilon}$ as independent. But we've seen above that in the background of a self-dual solution $F = {}^\star F$ the combination $F^{\mu\nu}\bar{\sigma}_\mu\sigma_\nu = 0$. This means that the instanton is annihilated by half of the supersymmetry transformations $\bar{\epsilon}$, while the other half, ϵ, turn on the fermions λ. We say that the supersymmetries arising from ϵ are broken by the soliton, while those arising from $\bar{\epsilon}$ are preserved. Configurations in supersymmetric theories which are annihilated by some fraction of the supersymmetries are known as BPS states (although the term Witten-Olive state would be more appropriate [23]).

Both the broken and preserved supersymmetries play an important role for solitons. The broken ones are the simplest to describe, for they generate fermion zero modes $\lambda = F^{\mu\nu}\sigma_\mu\bar{\sigma}_\nu\epsilon$. These "Goldstino" modes are a subset of the $4kN$ fermion zero modes that exist for each Weyl fermion λ. Further modes can also be generated by acting on the instanton with superconformal transformations.

The unbroken supersymmetries $\bar{\epsilon}$ play a more important role: they descend to a supersymmetry on the soliton worldvolume, pairing up bosonic collective coordinates X with Grassmannian collective coordinates χ. There's nothing surprising here. It's simply the statement that if a symmetry is preserved in a vacuum (where, in this case, the "vacuum" is the soliton itself) then all excitations above the vacuum fall into representations of this symmetry. However, since supersymmetry in $d = 0 + 0$ dimensions is a little subtle, and the concept of "excitations above the vacuum" in $d = 0 + 0$ dimensions even more so, this is one of the places where it will pay to lift the instantons to higher dimensional objects. For example, instantons in theories with 8 supercharges (equivalent to $\mathcal{N} = 2$ in four dimensions) can be lifted to instanton strings in six dimensions, which is the maximum dimension in which Yang-Mills theory with eight supercharges exists. Similarly, instantons in theories with 16 supercharges (equivalent to $\mathcal{N} = 4$ in four dimensions) can be lifted to instanton five-branes in ten dimensions. Instantons in $\mathcal{N} = 1$ theories are stuck in their four dimensional world.

Considering Yang-Mills instantons as solitons in higher dimensions allows us to see this relationship between bosonic and fermionic collective coordinates. Consider exciting a long-wavelength mode of the soliton in which a bosonic collective coordinate X depends on the worldvolume coordinate of the instanton s, so $X = X(s)$. Then if we hit this configuration with the unbroken supersymmetry $\bar{\epsilon}$, it will no longer annihilate the configuration, but will turn on a fermionic mode proportional to $\partial_s X$. Similarly, any fermionic excitation will be related to a bosonic excitation.

The observation that the unbroken supersymmetries descend to supersymmetries on the worldvolume of the soliton saves us a lot of work in analyzing fermionic zero modes: if we understand the bosonic collective coordinates and the preserved supersymmetry, then the fermionic modes pretty much come for free. This includes some rather subtle interaction terms.

For example, consider instanton five-branes in ten-dimensional super Yang-Mills. The worldvolume theory must preserve 8 of the 16 supercharges. The only such theory in 5+1 dimensions is a sigma-model on a hyperKählertarget space [24] which, for instantons, is the manifold $\mathcal{I}_{k,N}$. The Lagrangian is

$$\mathcal{L} = g_{\alpha\beta}\partial X^\alpha \partial X^\beta + i\bar{\chi}^\alpha D_{\alpha\beta}\chi^\beta + \tfrac{1}{4}R_{\alpha\beta\gamma\delta}\bar{\chi}^\alpha\chi^\beta\bar{\chi}^\gamma\chi^\delta \tag{10.30}$$

where ∂ denotes derivatives along the soliton worldvolume and the covariant derivative is $D_{\alpha\beta} = g_{\alpha\beta}\partial + \Gamma^\gamma_{\alpha\beta}(\partial X_\gamma)$. This is the slick proof that the instanton moduli space metric must be hyperKähler: it is dictated by the 8 preserved supercharges.

The final four-fermi term couples the fermionic collective coordinates to the Riemann tensor. Suppose we now want to go back down to instantons in four dimensional $\mathcal{N} = 4$ super Yang-Mills. We can simply dimensionally reduce the above action. Since there are no longer worldvolume directions for the instantons, the first two terms vanish, but we're left with the term

$$S_{\rm inst} = \tfrac{1}{4}R_{\alpha\beta\gamma\delta}\bar{\chi}^\alpha\chi^\beta\bar{\chi}^\gamma\chi^\delta \tag{10.31}$$

This term reflects the point we made earlier: zero modes cannot necessarily be lifted to collective coordinates. Here we see this phenomenon for fermionic zero modes. Although each such mode doesn't change the action of the instanton, if we turn on four Grassmannian collective coordinates at the same time then the action does increase! One can derive this term without recourse to supersymmetry but it's a bit of a pain [25]. The term is very important in applications of instantons.

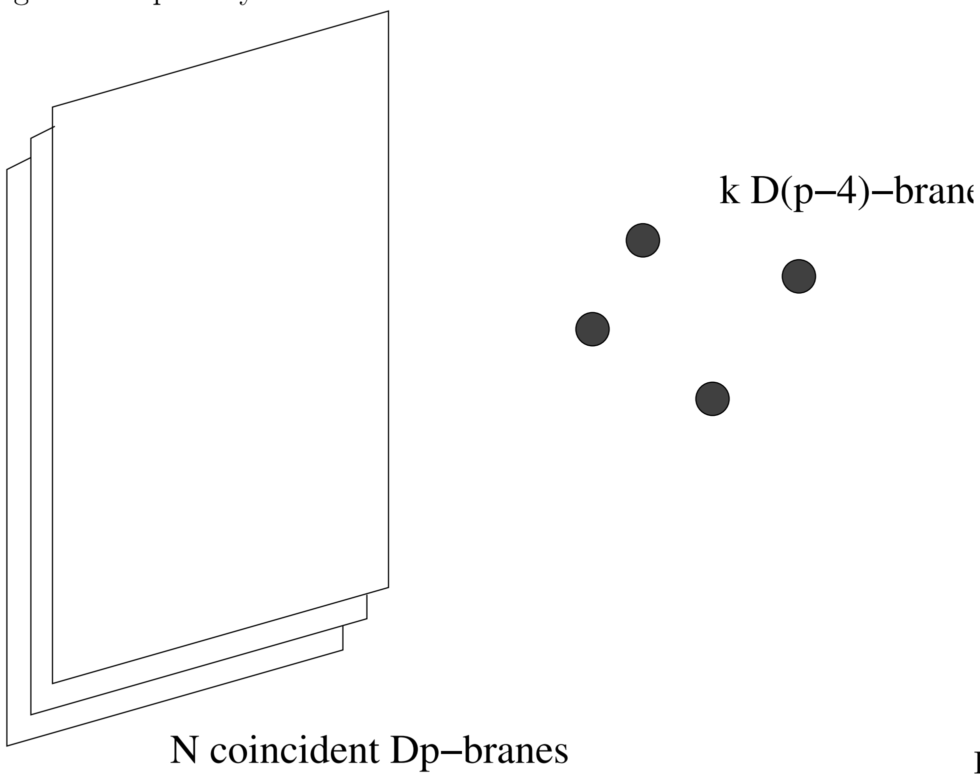
Instantons in four-dimensional $\mathcal{N} = 2$ theories can be lifted to instanton strings in six dimensions. The worldvolume theory must preserve half of the 8 supercharges. There are two such super-algebras in two dimensions, a non-chiral $(2,2)$ theory and a chiral $(0,4)$ theory, where the two entries correspond to left and right moving fermions respectively. By analyzing the fermionic zero modes one can show that the instanton string preserves $(0,4)$ supersymmetry. The corresponding sigma-model doesn't contain the term (10.31). (Basically because the $\bar{\chi}$ zero modes are missing). However, similar terms can be generated if we also consider fermions in the fundamental representation.

Finally, instantons in $\mathcal{N} = 1$ super Yang-Mills preserve $(0,2)$ supersymmetry on their worldvolume.

In the following sections, we shall pay scant attention to the fermionic zero modes, simply stating the fraction of supersymmetry that is preserved in different theories. In many cases this is sufficient to fix the fermions completely: the beauty of supersymmetry is that we rarely have to talk about fermions!

10.4 The ADHM Construction

In this section we describe a powerful method to solve the self-dual Yang-Mills equations $F = {}^\star F$ due to Atiyah, Drinfeld, Hitchin and Manin and known as the ADHM construction [26]. This will also give us a new way to understand the moduli space $\mathcal{I}_{k,N}$ and its metric. The natural place to view the ADHM construction is twistor space. But, for a physicist, the simplest place to view the ADHM construction is type II string theory [27, 28, 29]. We'll do things the simple way.



Dp-branes as instantons.

The brane construction is another place where it's useful to consider Yang-Mills instantons embedded as solitons in a $p+1$ dimensional theory with $p \geq 3$. With this in mind, let's consider a configuration of N Dp-branes, with k D$(p-4)$-branes in type II string theory (Type IIB for p odd; type IIA for p even). A typical configuration is drawn in figure 1. We place all N Dp-branes on top of each other so that, at low-energies, their worldvolume dynamics is described by

$$d = p + 1 \ U(N) \text{ Super Yang-Mills with 16 Supercharges}$$

For example, if $p = 3$ we have the familiar $\mathcal{N} = 4$ theory in $d = 3 + 1$ dimensions. The worldvolume theory of the Dp-branes also includes couplings to the various RR-fields in the bulk. This includes the term

$$\mathrm{Tr}\int_{Dp} d^{p+1}x\ \ C_{p-3}\wedge F\wedge F \tag{10.32}$$

where F is the $U(N)$ gauge field, and C_{p-3} is the RR-form that couples to D$(p-4)$-branes. The importance of this term lies in the fact that it relates instantons on the Dp-branes to D$(p-4)$ branes. To see this, note that an instanton with non-zero $F\wedge F$ gives rise to a source $(8\pi^2/e^2)\int d^{p-3}x\ C_{p-3}$ for the RR-form. This is the same source induced by a D$(p-4)$-brane. If you're careful in comparing the factors of 2 and π and such like, it's not hard to show that the instanton has precisely the mass and charge of the D$(p-4)$-brane [3, 5]. They are the same object! We have the important result that

$$\text{Instanton in D}p\text{-Brane}\ \equiv\ \text{D}(p-4)\text{-Brane} \tag{10.33}$$

The strategy to derive the ADHM construction from branes is to view this whole story from the perspective of the D$(p-4)$-branes [27, 28, 29]. For definiteness, let's revert back to $p=3$, so that we're considering D-instantons interacting with $D3$-branes. This means that we have to write down the $d=0+0$ dimensional theory on the D-instantons. Since supersymmetric theories in no dimensions may not be very familiar, it will help to keep in mind that the whole thing can be lifted to higher p.

Suppose firstly that we don't have the D3-branes. The theory on the D-instantons in flat space is simply the dimensional reduction of $d=3+1$ $\mathcal{N}=4$ $U(k)$ super Yang-Mills to zero dimensions. We will focus on the bosonic sector, with the fermions dictated by supersymmetry as explained in the previous section. We have 10 scalar fields, each of which is a $k\times k$ Hermitian matrix. For later convenience, we split them into two batches:

$$(X^\mu,\hat{X}^m)\qquad \mu=1,2,3,4;\quad m=5,...,10 \tag{10.34}$$

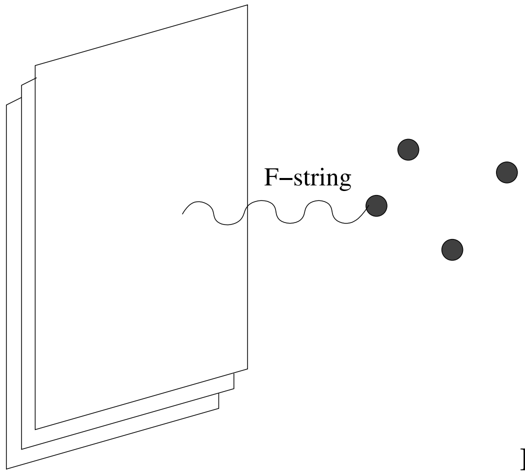
where we've put hats on directions transverse to the D3-brane. We'll use the index notation $(X^\mu)^\alpha{}_\beta$ to denote the fact that each of these is a $k\times k$ matrix. Note that this is a slight abuse of notation since, in the previous section, $\alpha=1,...,4k$ rather than $1,...,k$ here. We'll also introduce the complex notation

$$Z=X_1+iX_2\qquad\qquad W=X_3-iX_4 \tag{10.35}$$

When X_μ and $\hat{X}_m$ are all mutually commuting, their $10k$ eigenvalues have the interpretation of the positions of the k D-instantons in flat ten-dimensional space.

What effect does the presence of the $D3$-branes have? The answer is well known. Firstly, they reduce the supersymmetry on the lower dimensional brane by half, to eight supercharges (equivalent to $\mathcal{N}=2$ in $d=3+1$). The decomposition (10.34) reflects this, with the $\hat{X}_m$ lying in a vector multiplet and the X_μ forming an adjoint hypermultiplet. The new fields which reduce the supersymmetry are N hypermultiplets, arising from quantizing strings stretched between the Dp-branes and D$(p-4)$-branes. Each hypermultiplet carries an $\alpha=1,...k$ index, corresponding to the D$(p-4)$-brane on which the string ends, and an $a=1,...,N$ index corresponding to the Dp-brane on which the other end of the string sits.. Again we ignore fermions. The two complex scalars in each hypermultiplet are denoted

$$\psi^\alpha{}_a\qquad\qquad \tilde{\psi}^a{}_\alpha \tag{10.36}$$



F-strings give rise to hypermultiplets. where the index structure reflects the fact that ψ transforms in the $\mathbf{k}$ of the $U(k)$ gauge symmetry, and the $\bar{\mathbf{N}}$ of a $SU(N)$ flavor symmetry. In contrast $\tilde{\psi}$ transforms in the $(\bar{\mathbf{k}}, \mathbf{N})$ of $U(k) \times SU(N)$. (One may wonder about the difference between a gauge and flavor symmetry in zero dimensions; again the reader is invited to lift the configuration to higher dimensions where such nasty questions evaporate. But the basic point will be that we treat configurations related by $U(k)$ transformations as physically equivalent). These hypermultiplets can be thought of as the dimensional reduction of $\mathcal{N} = 2$ hypermultiplets in $d = 3 + 1$ dimensions which, in turn, are composed of two chiral multiplets ψ and $\tilde{\psi}$.

The scalar potential for these fields is fixed by supersymmetry (Actually, supersymmetry in $d = 0 + 0$ dimensions is rather weak; at this stage we should lift up to, say $p = 7$, where so we can figure out the familiar $\mathcal{N} = 2$ theory on the D$(p-3)$=D3-branes, and then dimensionally reduce back down to zero dimensions). We have

$$\begin{aligned} V &= \frac{1}{g^2}\sum_{m,n=5}^{10}[\hat{X}_m, \hat{X}_n]^2 + \sum_{m=5}^{10}\sum_{\mu=1}^{4}[\hat{X}_m, X_\mu]^2 + \sum_{a=1}^{N}(\psi^{a\dagger}\hat{X}_m^2\psi_a + \tilde{\psi}^a\hat{X}_m^2\tilde{\psi}_a^\dagger) \\ &\quad + g^2\,\mathrm{Tr}\,(\sum_{a=1}^{N}\psi_a\psi^{a\dagger} - \tilde{\psi}_a^\dagger\tilde{\psi}^a + [Z, Z^\dagger] + [W, W^\dagger])^2 + g^2\,\mathrm{Tr}\,|\sum_{a=1}^{N}\psi_a\tilde{\psi}^a + [Z, W]|^2 \end{aligned} \tag{10.37}$$

The terms in the second line are usually referred to as D-terms and F-terms respectively (although, as we shall shall review shortly, they are actually on the same footing in theories with eight supercharges). Each is a $k \times k$ matrix. The third term in the first line ensures that the hypermultiplets get a mass if the $\hat{X}_m$ get a vacuum expectation value. This reflects the fact that, as is clear from the picture, the Dp-D$(p-4)$ strings become stretched if the branes are separated in the $\hat{X}^m$, $m = 5, ..., 10$ directions. In contrast, there is no mass for the hypermultiplets if the D$(p-4)$ branes are separated in the X_μ, $\mu = 1, 2, 3, 4$ directions. Finally, note that we've included an auxiliary coupling constant g^2 in (10.37). Strictly speaking we should take the limit $g^2 \to \infty$.

We are interested in the ground states of the D-instantons, determined by the solutions to $V = 0$. There are two possibilities

1. The second line vanishes if $\psi = \tilde{\psi} = 0$ and X_μ are diagonal. The first two terms vanish if $\hat{X}_m$ are also diagonal. The eigenvalues of X_μ and $\hat{X}_m$ tell us where the k D-instantons are

placed in flat space. They are unaffected by the existence of the D3-branes whose presence is only felt at the one-loop level when the hypermultiplets are integrated out. This is known as the "*Coulomb branch*", a name inherited from the structure of gauge symmetry breaking: $U(k) \to U(1)^k$. (The name is, of course, more appropriate in dimensions higher than zero where particles charged under $U(1)^k$ experience a Coulomb interaction).

2. The first line vanishes if $\hat{X}_m = 0$, $m = 5, ..., 10$. This corresponds to the D$(p-4)$ branes lying on top of the Dp-branes. The remaining fields ψ, $\tilde{\psi}$, Z and W are constrained by the second line in (10.37). Since these solutions allow $\psi, \tilde{\psi} \neq 0$ we will generically have the $U(k)$ gauge group broken completely, giving the name "*Higgs branch*" to this class of solutions. More precisely, the Higgs branch is defined to be the space of solutions

$$\mathcal{M}_{\text{Higgs}} \cong \{\hat{X}_m = 0, V = 0\}/U(k) \tag{10.38}$$

where we divide out by $U(k)$ gauge transformations. The Higgs branch describes the D$(p-4)$ branes nestling inside the larger Dp-branes. But this is exactly where they appear as instantons. So we might expect that the Higgs branch knows something about this. Let's start by computing its dimension. We have $4kN$ real degrees of freedom in ψ and $\tilde{\psi}$ and a further $4k^2$ in Z and W. The D-term imposes k^2 real constraints, while the F-term imposes k^2 complex constraints. Finally we lose a further k^2 degrees of freedom when dividing by $U(k)$ gauge transformations. Adding, subtracting, we have

$$\dim(\mathcal{M}_{\text{Higgs}}) = 4kN \tag{10.39}$$

Which should look familiar (10.13). The first claim of the ADHM construction is that we have an isomorphism between manifolds,

$$\mathcal{M}_{\text{Higgs}} \cong \mathcal{I}_{k,N} \tag{10.40}$$

10.4.1 The Metric on the Higgs Branch

To summarize, the D-brane construction has lead us to identify the instanton moduli space $\mathcal{I}_{k,N}$ with the Higgs branch of a gauge theory with 8 supercharges (equivalent to $\mathcal{N} = 2$ in $d = 3 + 1$). The field content of this gauge theory is

$$\begin{aligned} U(k) \text{ Gauge Theory } \;&+\; \text{Adjoint Hypermultiplet } Z, W \\ &+\; N \text{ Fundamental Hypermultiplets } \psi_a, \tilde{\psi}^a \end{aligned} \tag{10.41}$$

This auxiliary $U(k)$ gauge theory defines its own metric on the Higgs branch. This metric arises in the following manner: we start with the flat metric on $\mathbf{R}^{4k(N+k)}$, parameterized by ψ, $\tilde{\psi}$, Z and W. Schematically,

$$ds^2 = |d\psi|^2 + |d\tilde{\psi}|^2 + |dZ|^2 + |dW|^2 \tag{10.42}$$

This metric looks somewhat more natural if we consider higher dimensional D-branes where it arises from the canonical kinetic terms for the hypermultiplets. We now pull back this metric to the hypersurface $V = 0$, and subsequently quotient by the $U(k)$ gauge symmetry, meaning that we only consider tangent vectors to $V = 0$ that are orthogonal to the $U(k)$ action. This procedure defines a metric on $\mathcal{M}_{\text{Higgs}}$. The second important result of the ADHM construction is that this metric coincides with the one defined in terms of solitons in (10.18).

I haven't included a proof of the equivalence between the metrics here, although it's not too hard to show (for example, using Macocia's hyperKähler potential [22] as reviewed in [13]). However, we will take time to show that the isometries of the metrics defined in these two different ways coincide. From the perspective of the auxiliary $U(k)$ gauge theory, all isometries appear as flavor symmetries. We have the $SU(N)$ flavor symmetry rotating the hypermultiplets; this is identified with the $SU(N)$ gauge symmetry in four dimensions. The theory also contains an $SU(2)_R$ R-symmetry, in which $(\psi, \tilde{\psi}^\dagger)$ and $(Z, W^\dagger)$ both transform as doublets (this will become more apparent in the following section in equation (10.44)). This coincides with the $SU(2)_R \subset SO(4)$ rotational symmetry in four dimensions. Finally, there exists an independent $SU(2)_L$ symmetry rotating just the X_μ.

The method described above for constructing hyperKählermetrics is an example of a technique known as the hyperKählerquotient [30]. As we have seen, it arises naturally in gauge theories with 8 supercharges. The D- and F-terms of the potential (10.37) give what are called the triplet of "moment-maps" for the $U(k)$ action.

10.4.2 Constructing the Solutions

As presented so far, the ADHM construction relates the moduli space of instantons $\mathcal{I}_{k,N}$ to the Higgs branch of an auxiliary gauge theory. In fact, we've omitted the most impressive part of the story: the construction can also be used to give solutions to the self-duality equations. What's more, it's really very easy! Just a question of multiplying a few matrices together. Let's see how it works.

Firstly, we need to rewrite the vacuum conditions in a more symmetric fashion. Define

$$\omega_a = \begin{pmatrix} \psi^\alpha{}_a \\ \tilde{\psi}^{\dagger\alpha}{}_a \end{pmatrix} \tag{10.43}$$

Then the real D-term and complex F-term which lie in the second line of (10.37) and define the Higgs branch can be combined in to the triplet of constraints,

$$\sum_{a=1}^{N} \omega_a^\dagger \sigma^i \, \omega_a - i[X_\mu, X_\mu]\bar{\eta}^i_{\mu\nu} = 0 \tag{10.44}$$

where σ^i are, as usual, the Pauli matrices and $\bar{\eta}^i$ the 't Hooft matrices (10.10). These give three $k \times k$ matrix equations. The magic of the ADHM construction is that for each solution to the algebraic equations (10.44), we can build a solution to the set of non-linear partial differential equations $F = {}^\star F$. Moreover, solutions to (10.44) related by $U(k)$ gauge transformations give rise to the same field configuration in four dimensions. Let's see how this remarkable result is achieved.

The first step is to build the $(N + 2k) \times 2k$ matrix Δ,

$$\Delta = \begin{pmatrix} \omega^T \\ X_\mu \sigma^\mu \end{pmatrix} + \begin{pmatrix} 0 \\ x_\mu \sigma^\mu \end{pmatrix} \tag{10.45}$$

where $\sigma_\mu = (\sigma^i, -i\mathbf{1}_2)$. These have the important property that $\sigma_{[\mu}\bar{\sigma}_{\nu]}$ is self-dual, while $\bar{\sigma}_{[\mu}\sigma_{\nu]}$ is anti-self-dual, facts that we also used in Section 1.3 when discussing fermions. In the second matrix we've re-introduced the spacetime coordinate x_μ which, here, is to be thought of as multiplying the $k \times k$ unit matrix. Before proceeding, we need a quick lemma:

Lemma: $\Delta^\dagger \Delta = f^{-1} \otimes 1_2$

where f is a $k \times k$ matrix, and 1_2 is the unit 2×2 matrix. In other words, $\Delta^\dagger\Delta$ factorizes and is invertible.

Proof: Expanding out, we have (suppressing the various indices)

$$\Delta^\dagger \Delta = \omega^\dagger \omega + X^\dagger X + (X^\dagger x + x^\dagger X) + x^\dagger x 1_k \tag{10.46}$$

Since the factorization happens for all $x \equiv x_\mu \sigma^\mu$, we can look at three terms separately. The last is $x^\dagger x = x_\mu \bar{\sigma}^\mu x_\nu \sigma^\nu = x^2\, 1_2$. So that works. For the term linear in x, we simply need the fact that $X_\mu = X_\mu^\dagger$ to see that it works. What's more tricky is the term that doesn't depend on x. This is where the triplet of D-terms (10.44) comes in. Let's write the relevant term from (10.46) with all the indices, including an $m, n = 1, 2$ index to denote the two components we introduced in (10.43). We require

$$\begin{aligned} & \omega^{\dagger\alpha}_{ma}\, \omega_{a\beta n} + (X_\mu)^\alpha{}_\gamma (X_\nu)^\gamma{}_\beta\, \bar{\sigma}^{\mu\, mp} \sigma^\nu{}_{pn} \sim \delta^m_n \\ \Leftrightarrow \quad & \mathrm{tr}_2\ \sigma^i \left[\omega\omega^\dagger + X^\dagger X\right] = 0 \qquad i = 1,2,3 \\ \Leftrightarrow \quad & \omega^\dagger \sigma^i \omega + X_\mu X_\nu \bar{\sigma}^\mu \sigma^i \sigma^\nu = 0 \end{aligned} \tag{10.47}$$

But, using the identity $\bar{\sigma}^\mu\sigma^i\sigma^\nu = 2i\bar{\eta}^i_{\mu\nu}$, we see that this last condition is simply the D-terms (10.44). This concludes our proof of the lemma. $\square$

The rest is now plain sailing. Consider the matrix Δ as defining $2k$ linearly independent vectors in $\mathbf{C}^{N+2k}$. We define U to be the $(N+2k) \times N$ matrix containing the N normalized, orthogonal vectors. i.e

$$\Delta^\dagger U = 0 \qquad U^\dagger U = 1_N \tag{10.48}$$

Then the potential for a charge k instanton in $SU(N)$ gauge theory is given by

$$A_\mu = iU^\dagger\, \partial_\mu U \tag{10.49}$$

Note firstly that if U were an $N \times N$ matrix, this would be pure gauge. But it's not, and it's not. Note also that A_μ is left unchanged by auxiliary $U(k)$ gauge transformations.

We need to show that A_μ so defined gives rise to a self-dual field strength with winding number k. We'll do the former, but the latter isn't hard either: it just requires more matrix multiplication. To help us in this, it will be useful to construct the projection operator $P = UU^\dagger$ and notice that this can also be written as $P = 1 - \Delta f \Delta^\dagger$. To see that these expression indeed coincide, we can check that $PU = U$ and $P\Delta = 0$ for both. Now we're almost there:

$$\begin{aligned}
F_{\mu\nu} &= \partial_{[\mu} A_{\nu]} - iA_{[\mu} A_{\nu]} \\
&= \partial_{[\mu}\, iU^\dagger \partial_{\nu]} U + iU^\dagger (\partial_{[\mu} U) U^\dagger (\partial_{\nu]} U) \\
&= i(\partial_{[\mu} U^\dagger)(\partial_{\nu]} U) - i(\partial_{[\mu} U^\dagger) U U^\dagger (\partial_{\nu]} U) \\
&= i(\partial_{[\mu} U^\dagger)(1 - UU^\dagger)(\partial_{\nu]} U) \\
&= i(\partial_{[\mu} U^\dagger) \Delta\, f\, \Delta^\dagger (\partial_{\nu]} U) \\
&= iU^\dagger (\partial_{[\mu} \Delta)\, f\, (\partial_{\nu]} U) \\
&= iU^\dagger \sigma_{[\mu} f \bar{\sigma}_{\nu]} U
\end{aligned}$$

At this point we use our lemma. Because $\Delta^\dagger \Delta$ factorizes, we may commute f past σ_μ. And that's it! We can then write

$$F_{\mu\nu} = iU^\dagger f \sigma_{[\mu} \bar{\sigma}_{\nu]} U = {}^\star F_{\mu\nu} \tag{10.50}$$

since, as we mentioned above, $\sigma_{\mu\nu} = \sigma_{[\mu} \bar{\sigma}_{\nu]}$ is self-dual. Nice huh! What's harder to show is that the ADHM construction gives all solutions to the self-dualily equations. Counting parameters, we see that we have the right number and it turns out that we can indeed get all solutions in this manner.

The construction described above was first described in ADHM's original paper, which weighs in at a whopping 2 pages. Elaborations and extensions to include, among other things, $SO(N)$ and $Sp(N)$ gauge groups, fermionic zero modes, supersymmetry and constrained instantons, can be found in [31, 32, 33, 34].

10.4.2An Example: The Single $SU(2)$ Instanton Revisited

Let's see how to re-derive the $k = 1$ $SU(2)$ solution (10.9) from the ADHM method. We'll set $X_\mu = 0$ to get a solution centered around the origin. We then have the 4×2 matrix

$$\Delta = \begin{pmatrix} \omega^T \\ x_\mu \sigma^\mu \end{pmatrix} \tag{10.51}$$

where the D-term constraints (10.44) tell us that $\omega^{\dagger a}{}_m (\sigma^i)^m{}_n \omega^n{}_b = 0$. We can use our $SU(2)$ flavor rotation, acting on the indices $a, b = 1, 2$, to choose the solution

$$\omega^{\dagger a}{}_m \omega^m{}_b = \rho^2 \delta^a{}_b \tag{10.52}$$

in which case the matrix Δ becomes $\Delta^T = (\rho 1_2, x_\mu \sigma^\mu)$. Then solving for the normalized zero eigenvectors $\Delta^\dagger U = 0$, and $U^\dagger U = 1$, we have

$$U = \begin{pmatrix} \sqrt{x^2/(x^2+\rho^2)}\, 1_2 \\ -\sqrt{\rho^2/x^2(x^2+\rho^2)}\, x_\mu \bar{\sigma}^\mu \end{pmatrix} \tag{10.53}$$

From which we calculate

$$A_\mu = iU^\dagger \partial_\mu U = \frac{\rho^2 x_\nu}{x^2(x^2+\rho^2)}\, \bar{\eta}^i_{\mu\nu}\, \sigma^i \tag{10.54}$$

which is indeed the solution (10.9) as promised.

10.4.3 Non-Commutative Instantons

There's an interesting deformation of the ADHM construction arising from studying instantons on a non-commutative space, defined by

$$[x_\mu, x_\nu] = i\theta_{\mu\nu} \tag{10.55}$$

The most simple realization of this deformation arises by considering functions on the space $\mathbf{R}^4_\theta$, with multiplication given by the $\star$-product

$$f(x) \star g(x) = \exp\left(\frac{i}{2}\theta_{\mu\nu}\frac{\partial}{\partial y^\mu}\frac{\partial}{\partial x^\nu}\right) f(y)g(x)\bigg|_{x=y} \tag{10.56}$$

so that we indeed recover the commutator $x_\mu \star x_\nu - x_\nu \star x_\mu = i\theta_{\mu\nu}$. To define gauge theories on such a non-commutative space, one must extend the gauge symmetry from $SU(N)$ to $U(N)$. When studying instantons, it is also useful to decompose the non-commutivity parameter into self-dual and anti-self-dual pieces:

$$\theta_{\mu\nu} = \xi^i\,\eta^i_{\mu\nu} + \zeta^i\,\bar{\eta}^i_{\mu\nu} \tag{10.57}$$

where η^i and $\bar{\eta}^i$ are defined in (10.11) and (10.10) respectively. At the level of solutions, both ξ and ζ affect the configuration. However, at the level of the moduli space, we shall see that the self-dual instantons $F = {}^\star F$ are only affected by the anti-self-dual part of the non-commutivity, namely ζ^i. (A similar statement holds for $F = -{}^\star F$ solutions and ξ). This change to the moduli space appears in a beautifully simple fashion in the ADHM construction: we need only add a constant term to the right hand-side of the constraints (10.44), which now read

$$\sum_{a=1}^{N} \omega_a^\dagger \sigma^i\,\omega_a - i[X_\mu, X_\mu]\bar{\eta}^i_{\mu\nu} = \zeta^i\,1_k \tag{10.58}$$

From the perspective of the auxiliary $U(k)$ gauge theory, the ζ^i are Fayet-Iliopoulous (FI) parameters.

The observation that the FI parameters ζ^i appearing in the D-term give the correct deformation for non-commutative instantons is due to Nekrasov and Schwarz [35]. To see how this works, we can repeat the calculation above, now in non-commutative space. The key point in constructing the solutions is once again the requirement that we have the factorization

$$\Delta^\dagger \star \Delta = f^{-1}\,1_2 \tag{10.59}$$

The one small difference from the previous derivation is that in the expansion (10.46), the $\star$-product means we have

$$x^\dagger \star x = x^2\,1_2 - \zeta^i\sigma^i \tag{10.60}$$

Notice that only the anti-self-dual part contributes. This extra term combines with the constant terms (10.47) to give the necessary factorization if the D-term with FI parameters (10.58) is satisfied. It is simple to check that the rest of the derivation proceeds as before, with $\star$-products in the place of the usual commutative multiplication.

The addition of the FI parameters in (10.58) have an important effect on the moduli space $\mathcal{I}_{k,N}$: they resolve the small instanton singularities. From the ADHM perspective, these arise when $\psi = \tilde{\psi} = 0$, where the $U(k)$ gauge symmetry does not act freely. The FI parameters remove these points from the moduli space, $U(k)$ acts freely everywhere on the Higgs branch, and the deformed instanton moduli space $\mathcal{I}_{k,N}$ is smooth. This resolution of the instanton moduli space was considered by Nakajima some years before the relationship to non-commutivity was known [36]. A related fact is that non-commutative instantons occur even for $U(1)$ gauge theories. Previously such solutions were always singular, but the addition of the FI parameter stabilizes them at a fixed size of order $\sqrt{\theta}$. Reviews of instantons and other solitons on non-commutative spaces can be found in [37, 38].

10.4.4 Examples of Instanton Moduli Spaces

10.4.4A Single Instanton

Consider a single $k = 1$ instanton in a $U(N)$ gauge theory, with non-commutivity turned on. Let us choose $\theta_{\mu\nu} = \zeta\bar{\eta}^3_{\mu\nu}$. Then the ADHM gauge theory consists of a $U(1)$ gauge theory with N charged hypermultiplets, and a decoupled neutral hypermultiplet parameterizing the center of the instanton. The D-term constraints read

$$\sum_{a=1}^N |\psi_a|^2 - |\tilde{\psi}_a|^2 = \zeta \qquad \sum_{a=1}^N \tilde{\psi}_a\psi_a \;=\; 0 \tag{10.61}$$

To get the moduli space we must also divide out by the $U(1)$ action $\psi_a \to e^{i\alpha}\psi_a$ and $\tilde{\psi}_a \to e^{-i\alpha}\tilde{\psi}_a$. To see what the resulting space is, first consider setting $\tilde{\psi}_a = 0$. Then we have the space

$$\sum_{a=1}^N |\psi_a|^2 = \zeta \tag{10.62}$$

which is simply $\mathbf{S}^{2N-1}$. Dividing out by the $U(1)$ action then gives us the complex projective space $\mathbb{CP}^{N-1}$ with size (or Kähler class) ζ. Now let's add the $\tilde{\psi}$ back. We can turn them on but the F-term insists that they lie orthogonal to ψ, thus defining the co-tangent bundle of $\mathbb{CP}^{N-1}$, denoted $T^\star\mathbb{CP}^{N-1}$. Including the decoupled $\mathbf{R}^4$, we have [39]

$$\mathcal{I}_{1,N} \cong \mathbf{R}^4 \times T^\star\mathbb{CP}^{N-1} \tag{10.63}$$

where the size of the zero section $\mathbb{CP}^{N-1}$ is ζ. As $\zeta \to 0$, this cycle lying in the center of the space shrinks and $\mathcal{I}_{1,N}$ becomes singular at this point.

For a single instanton in $U(2)$, the relative moduli space is $T^\star\mathbf{S}^2$. This is the smooth resolution of the A_1 singularity $\mathbf{C}^2/\mathbf{Z}_2$ which we found to be the moduli space in the absence of non-commutivity. It inherits a well-known hyperKähler metric known as the Eguchi-Hanson metric [40],

$$ds^2_{EH} = \left(1 - 4\zeta^2/\rho^4\right)^{-1} d\rho^2 + \frac{\rho^2}{4}\left(\sigma_1^2 + \sigma_2^2 + \left(1 - 4\zeta^2/\rho^4\right)\sigma_3^2\right) \tag{10.64}$$

Here the σ_i are the three left-invariant $SU(2)$ one-forms which, in terms of polar angles $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 2\pi$, take the form

$$\begin{aligned} \sigma_1 &= -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi \\ \sigma_2 &= \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi \\ \sigma_3 &= d\psi + \cos\theta\, d\phi \end{aligned} \tag{10.65}$$

As $\rho \to \infty$, this metric tends towards the cone over $\mathbf{S}^3/\mathbf{Z}_2$. However, as we approach the origin, the scale size is truncated at $\rho^2 = 2\zeta$, where the apparent singularity is merely due to the choice of coordinates and hides the zero section $\mathbf{S}^2$.

10.4.4Two $U(1)$ Instantons

Before resolving by a non-commutative deformation, there is no topology to support a $U(1)$ instanton. However, it is perhaps better to think of the $U(1)$ theory as admitting small, singular, instantons with moduli space given by the symmetric product $\mathrm{Sym}^k(\mathbf{C}^2)$, describing the positions of k points. Upon the addition of a non-commutivity parameter, smooth $U(1)$ instantons exist with moduli space given by a resolution of $\mathrm{Sym}^k(\mathbf{C}^2)$. To my knowledge, no explicit metric is known for $k \geq 3$ $U(1)$ instantons, but in the case of two $U(1)$ instantons, the metric is something rather familiar, since $\mathrm{Sym}^2\mathbf{C}^2 \cong \mathbf{C}^2 \times \mathbf{C}^2/Z_2$ and we have already met the resolution of this space above. It is

$$\mathcal{I}_{k=2,N=1} \cong \mathbf{R}^4 \times T^\star\mathbf{S}^2 \tag{10.66}$$

endowed with the Eguchi-Hanson metric (10.64) where ρ now has the interpretation of the separation of two instantons rather than the scale size of one. This can be checked explicitly by computing the metric on the ADHM Higgs branch using the hyperKählerquotient technique [41]. Scattering of these instantons was studied in [42]. So, in this particular case we have $\mathcal{I}_{1,2} \cong \mathcal{I}_{2,1}$. We shouldn't get carried away though as this equivalence doesn't hold for higher k and N (for example, the isometries of the two spaces are different).

10.5 Applications

Until now we've focussed exclusively on classical aspects of the instanton configurations. But, what we're really interested in is the role they play in various quantum field theories. Here we sketch a two examples which reveal the importance of instantons in different dimensions.

10.5.1 Instantons and the AdS/CFT Correspondence

We start by considering instantons where they were meant to be: in four dimensional gauge theories. In a semi-classical regime, instantons give rise to non-perturbative contributions to correlation functions and there exists a host of results in the literature, including exact results in both $\mathcal{N} = 1$ [43, 44] and $\mathcal{N} = 2$ [45, 34, 37] supersymmetric gauge theories. Here we describe the role instantons play in $\mathcal{N} = 4$ super Yang-Mills and, in particular, their relationship to the AdS/CFT correspondence [47]. Instantons were first considered in this context in [48, 49]. Below we provide only a sketchy description of the material covered in the paper of Dorey et al [50]. Full details can be found in that paper or in the review [13].

In any instanton computation, there's a number of things we need to calculate [7]. The first is to count the zero modes of the instanton to determine both the bosonic collective coordinates

X and their fermionic counterparts χ. We've described this in detail above. The next step is to perform the leading order Gaussian integral over all modes in the path integral. The massive (i.e. non-zero) modes around the background of the instanton leads to the usual determinant operators which we'll denote as $\det \Delta_B$ for the bosons, and $\det \Delta_F$ for the fermions. These are to be evaluated on the background of the instanton solution. However, zero modes must be treated separately. The integration over the associated collective coordinates is left unperformed, at the price of introducing a Jacobian arising from the transformation between field variables and collective coordinates. For the bosonic fields, the Jacobian is simply $J_B = \sqrt{\det\, g_{\alpha\beta}}$, where $g_{\alpha\beta}$ is the metric on the instanton moduli space defined in (10.18). This is the role played by the instanton moduli space metric in four dimensions: it appears in the measure when performing the path integral. A related factor J_F occurs for fermionic zero modes. The final ingredient in an instanton calculation is the action $S_{\rm inst}$ which includes both the constant piece $8\pi k/g^2$, together with terms quartic in the fermions (10.31). The end result is summarized in the instanton measure

$$d\mu_{\rm inst} = d^{n_B}X\, d^{n_F}\chi\ J_B J_F\ \frac{\det\Delta_F}{\det^{1/2}\Delta_B}\, e^{-S_{\rm inst}} \tag{10.67}$$

where there are $n_B = 4kN$ bosonic and n_F fermionic collective coordinates. In supersymmetric theories in four dimensions, the determinants famously cancel [7] and we're left only with the challenge of evaluating the Jacobians and the action. In this section, we'll sketch how to calculate these objects for $\mathcal{N} = 4$ super Yang-Mills.

As is well known, in the limit of strong 't Hooft coupling, $\mathcal{N} = 4$ super Yang-Mills is dual to type IIB supergravity on $AdS_5 \times \mathbf{S}^5$. An astonishing fact, which we shall now show, is that we can see this geometry even at weak 't Hooft coupling by studying the $d = 0 + 0$ ADHM gauge theory describing instantons. Essentially, in the large N limit, the instantons live in $AdS_5 \times \mathbf{S}^5$. At first glance this looks rather unlikely! We've seen that if the instantons live anywhere it is in $\mathcal{I}_{k,N}$, a $4kN$ dimensional space that doesn't look anything like $AdS_5 \times \mathbf{S}^5$. So how does it work?

While the calculation can be performed for an arbitrary number of k instantons, here we'll just stick with a single instanton as a probe of the geometry. To see the AdS_5 part is pretty easy and, in fact, we can do it even for an instanton in $SU(2)$ gauge theory. The trick is to integrate over the orientation modes of the instanton, leaving us with a five-dimensional space parameterized by X_μ and ρ. The rationale for doing this is that if we want to compute gauge invariant correlation functions, the $SU(N)$ orientation modes will only give an overall normalization. We calculated the metric for a single instanton in equations (10.22)-(10.24), giving us $J_B \sim \rho^3$ (where we've dropped some numerical factors and factors of e^2). So integrating over the $SU(2)$ orientation to pick up an overall volume factor, we get the bosonic measure for the instanton to be

$$d\mu_{\rm inst} \sim \rho^3\, d^4X d\rho \tag{10.68}$$

We want to interpret this measure as a five-dimensional space in which the instanton moves, which means thinking of it in the form $d\mu = \sqrt{G}\, d^4X\, d\rho$ where G is the metric on the five-dimensional space. It would be nice if it was the metric on AdS_5. But it's not! In the appropriate coordinates, the AdS_5 metric is,

$$ds^2_{AdS} = \frac{R^2}{\rho^2}(d^4X + d\rho^2) \tag{10.69}$$

giving rise to a measure $d\mu_{AdS} = (R/\rho)^5 d^4X d\rho$. However, we haven't finished with the instanton yet since we still have to consider the fermionic zero modes. The fermions are crucial for quantum conformal invariance so we may suspect that their zero modes are equally crucial in revealing the AdS structure, and this is indeed the case. A single $k = 1$ instanton in the $\mathcal{N} = 4$ $SU(2)$ gauge theory has 16 fermionic zero modes. 8 of these, which we'll denote as ξ are from broken supersymmetry while the remaining 8, which we'll call ζ arise from broken superconformal transformations. Explicitly each of the four Weyl fermions λ of the theory has a profile,

$$\lambda = \sigma^{\mu\nu} F_{\mu\nu}(\xi - \sigma^{\rho}\zeta\,(x_\rho - X_\rho)) \tag{10.70}$$

One can compute the overlap of these fermionic zero modes in the same way as we did for bosons. Suppressing indices, we have

$$\int d^4x\, \frac{\partial\lambda}{\partial\xi}\frac{\partial\lambda}{\partial\xi} = \frac{32\pi^2}{e^2} \qquad \int d^4x\, \frac{\partial\lambda}{\partial\zeta}\frac{\partial\lambda}{\partial\zeta} = \frac{64\pi^2\rho^2}{e^2} \tag{10.71}$$

So, recalling that Grassmannian integration is more like differentiation, the fermionic Jacobian is $J_F \sim 1/\rho^8$. Combining this with the bosonic contribution above, the final instanton measure is

$$d\mu_{\text{inst}} = \left(\frac{1}{\rho^5}\, d^4X d\rho\right)\, d^8\xi d^8\zeta = d\mu_{Ads}\, d^8\xi d^8\zeta \tag{10.72}$$

So the bosonic part does now look like AdS_5. The presence of the 16 Grassmannian variables reflects the fact that the instanton only contributes to a 16 fermion correlation function. The counterpart in the AdS/CFT correspondence is that D-instantons contribute to R^4 terms and their 16 fermion superpartners and one can match the supergravity and gauge theory correlators exactly.

So we see how to get AdS_5 for $SU(2)$ gauge theory. For $SU(N)$, one has $4N - 8$ further orientation modes and $8N - 16$ further fermi zero modes. The factors of ρ cancel in their Jacobians, leaving the AdS_5 interpretation intact. But there's a problem with these extra fermionic zero modes since we must saturate them in the path integral in some way even though we still want to compute a 16 fermionic correlator. This is achieved by the four-fermi term in the instanton action (10.31). However, when looked at in the right way, in the large N limit these extra fermionic zero modes will generate the $\mathbf{S}^5$ for us. I'll now sketch how this occurs.

The important step in reforming these fermionic zero modes is to introduce auxiliary variables $\hat{X}$ which allows us to split up the four-fermi term (10.31) into terms quadratic in the fermions. To get the index structure right, it turns out that we need six such auxiliary fields, let's call them $\hat{X}^m$, with $m = 1, ..., 6$. In fact we've met these guys before: they're the scalar fields in the vector multiplet of the ADHM gauge theory. To see that they give rise to the promised four fermi term, let's look at how they appear in the ADHM Lagrangian. There's already a term quadratic in $\hat{X}$ in (10.37), and another couples this to the surplus fermionic collective coordinates χ so that, schematically,

$$\mathcal{L}_{\hat{X}} \sim \hat{X}^2\omega^\dagger\omega + \bar{\chi}\hat{X}\chi \tag{10.73}$$

where, as we saw in Section 1.4, the field ω contains the scale and orientation collective coordinates, with $\omega^\dagger\omega \sim \rho^2$. Integrating out $\hat{X}$ in the ADHM Lagrangian does indeed result in a four-fermi term which is identified with (10.31). However, now we perform a famous trick: we integrate out the variables we thought we were interested in, namely the χ fields, and focus on the ones we thought were unimportant, the $\hat{X}$'s. After dealing correctly with all the indices we've been dropping, we find that this results in the contribution to the measure

$$d\mu_{\text{auxiliary}} = d^6\hat{X}\ (\hat{X}^m\hat{X}^m)^{2N-4}\ \exp\left(-2\rho^2\hat{X}^m\hat{X}^m\right) \tag{10.74}$$

In the large N limit, the integration over the radial variable $|\hat{X}|$ may be performed using the saddle point approximation evaluated at $|\hat{X}| = \rho$. The resulting powers of ρ are precisely those mentioned above that are needed to cancel the powers of ρ appearing in the bosonic Jacobian. Meanwhile, the integration over the angular coordinates in $\hat{X}^m$ have been left untouched. The final result for the instanton measure becomes

$$d\mu_{\text{inst}} = \left(\frac{1}{\rho^5}\,d^4X\,d\rho\,d^5\hat{\Omega}\right)\ d^8\xi d^8\zeta \tag{10.75}$$

And the instanton indeed appears as if its moving in $AdS_5 \times \mathbf{S}^5$ as promised.

The above discussion is a little glib. The invariant meaning of the measure alone is not clear: the real meaning is that when integrated against correlators, it gives results in agreement with gravity calculations in $\text{AdS}_5 \times \mathbf{S}^5$. This, and several further results, were shown in [50]. Calculations of this type were later performed for instantons in other four-dimensional gauge theories, both conformal and otherwise [51, 52, 53, 54, 55]. Curiously, there appears to be an unresolved problem with performing the calculation for instantons in non-commutative gauge theories.

10.5.2 Instanton Particles and the $(2,0)$ Theory

There exists a rather special superconformal quantum field theory in six dimensions known as the $(2,0)$ theory. It is the theory with 16 supercharges which lives on N M5-branes in M-theory and it has some intriguing and poorly understood properties. Not least of these is the fact that it appears to have N^3 degrees of freedom. While it's not clear what these degrees of freedom are, or even if it makes sense to talk about "degrees of freedom" in a strongly coupled theory, the N^3 behavior is seen when computing the free energy $F \sim N^3T^6$ [56], or anomalies whose leading coefficient also scales as N^3 [57].

If the $(2,0)$ theory is compactified on a circle of radius R, it descends to $U(N)$ $d = 4+1$ super Yang-Mills with 16 supercharges, which can be thought of as living on D4-branes in Type IIA string theory. The gauge coupling e^2, which has dimension of length in five dimensions, is given by

$$e^2 = 8\pi^2R \tag{10.76}$$

As in any theory compactified on a spatial circle, we expect to find Kaluza-Klein modes, corresponding to momentum modes around the circle with mass $M_{\text{KK}} = 1/R$. Comparison with the gauge coupling constant (10.76) gives a strong hint what these particles should be, since

$$M_{\text{kk}} = M_{\text{inst}} \tag{10.77}$$

and, as we discussed in section 1.3.1, instantons are particle-like objects in $d = 4+1$ dimensions. The observation that instantons are Kaluza-Klein modes is clear from the IIA perspective: the instantons in the D4-brane theory are D0-branes which are known to be the Kaluza-Klein modes for the lift to M-theory.

The upshot of this analysis is a remarkable conjecture: the maximally supersymmetric $U(N)$ Yang-Mills theory in five dimensions is really a six-dimensional theory in disguise, with the size of the hidden dimension given by $R \sim e^2$ [58, 59, 60]. As $e^2 \to \infty$, the instantons become light. Usually as solitons become light, they also become large floppy objects, losing their interpretation as particle excitations of the theory. But this isn't necessarily true for instantons because, as we've seen, their scale size is arbitrary and, in particular, independent of the gauge coupling.

Of course, the five-dimensional theory is non-renormalizable and we can only study questions that do not require the introduction of new UV degrees of freedom. With this caveat, let's see how we can test the conjecture using instantons. If they're really Kaluza-Klein modes, they should exhibit Kaluza-Klein-eqsue behavior which includes a characteristic spectrum of threshold bound state of particles with k units of momentum going around the circle. This means that if the five-dimensional theory contains the information about its six dimensional origin, it should exhibit a threshold bound state of k instantons for each k. But this is something we can test in the semi-classical regime by solving the low-energy dynamics of k interacting instantons. As we have seen, this is given by supersymmetric quantum mechanics on $\mathcal{I}_{k,N}$, with the Lagrangian given by (10.30) where $\partial = \partial_t$ in this equation.

Let's review how to solve the ground states of $d = 0+1$ dimensional supersymmetric sigma models of the form (10.30). As explained by Witten, a beautiful connection to de Rahm cohomology emerges after quantization [61]. Canonical quantization of the fermions leads to operators satisfying the algebra

$$\{\chi_\alpha, \chi_\beta\} = \{\bar{\chi}_\alpha, \bar{\chi}_\beta\} = 0 \quad \text{and} \quad \{\chi_\alpha, \bar{\chi}_\beta\} = g_{\alpha\beta} \tag{10.78}$$

which tells us that we may regard $\bar{\chi}_\alpha$ and χ_β as creation and annihilation operators respectively. The states of the theory are described by wavefunctions $\varphi(X)$ over the moduli space $\mathcal{I}_{k,N}$, acted upon by some number p of fermion creation operators. We write $\varphi_{\alpha_1,\ldots,\alpha_p}(X) \equiv \bar{\chi}_{\alpha_1} \cdots \bar{\chi}_{\alpha_p}\, \varphi(X)$. By the Grassmann nature of the fermions, these states are anti-symmetric in their p indices, ensuring that the tower stops when $p = \dim(\mathcal{I}_{k,N})$. In this manner, the states can be identified with the space of all p-forms on $\mathcal{I}_{k,N}$.

The Hamiltonian of the theory has a similarly natural geometric interpretation. One can check that the Hamiltonian arising from (10.30) can be written as

$$H = QQ^\dagger + Q^\dagger Q \tag{10.79}$$

where Q is the supercharge which takes the form $Q = -i\bar{\chi}_\alpha p_\alpha$ and $Q^\dagger = -i\chi_\alpha p_\alpha$, and p_α is the momentum conjugate to X^α. Studying the action of Q on the states above, we find that $Q = d$, the exterior derivative on forms, while $Q^\dagger = d^\dagger$, the adjoint operator. We can therefore write the Hamiltonian is the Laplacian acting on all p-forms,

$$H = dd^\dagger + d^\dagger d \tag{10.80}$$

We learn that the space of ground states $H = 0$ coincide with the harmonic forms on the target space.

There are two subtleties in applying this analysis to instantons. The first is that the instanton moduli space $\mathcal{I}_{k,N}$ is singular. At these points, corresponding to small instantons, new UV degrees of freedom are needed. Presumably this reflects the non-renormalizability of the five-dimensional gauge theory. However, as we have seen, one can resolve the singularity by turning on non-commutivity. The interpretation of the instantons as KK modes only survives if there is a similar non-commutative deformation of the $(2,0)$ theory which appears to be the case.

The second subtlety is an infra-red effect: the instanton moduli space is non-compact. For compact target spaces, the ground states of the sigma-model coincide with the space of harmonic forms or, in other words, the cohomology. For non-compact target spaces such as $\mathcal{I}_{k,N}$, we have the further requirement that any putative ground state wavefunction must be normalizable and we need to study cohomology with compact support. With this in mind, the relationship between the five-dimensional theory and the six-dimensional $(2,0)$ theory therefore translates into the conjecture

There is a unique normalizable harmonic form on $\mathcal{I}_{k,N}$ for each k and N

Note that even for a single instanton, this is non-trivial. As we have seen above, after resolving the small instanton singularity, the moduli space for a $k = 1$ instanton in $U(N)$ theory is $T^\star(\mathbf{CP}^{N-1})$, which has Euler character $\chi = N$. Yet, there should be only a single groundstate. Indeed, it can be shown explicitly that of these N putative ground states, only a single one has sufficiently compact support to provide an L^2 normalizable wavefunction [62]. For an arbitrary number of k instantons in $U(N)$ gauge theory, there is an index theorem argument that this unique bound state exists [63].

So much for the ground states. What about the N^3 degrees of freedom. Is it possible to see this from the five-dimensional gauge theory? Unfortunately, so far, no one has managed this. Five dimensional gauge theories become strongly coupled in the ultra-violet where their non-renormalizability becomes an issue and we have to introduce new degrees of freedom. This occurs at an energy scale $E \sim 1/e^2N$, where the 't Hooft coupling becomes strong. This is parametrically lower than the KK scale $E \sim 1/R \sim 1/e^2$. Supergravity calculations reveal that the N^3 degrees of freedom should also become apparent at the lower scale $E \sim 1/e^2N$ [64]. This suggests that perhaps the true degrees of freedom of the theory are "fractional instantons", each with mass $M_{\rm inst}/N$. Let me end this section with some rampant speculation along these lines. It seems possible that the $4kN$ moduli of the instanton may rearrange themselves into the positions of kN objects, each living in $\mathbf{R}^4$ and each, presumably, carrying the requisite mass $1/e^2N$. We shall see a similar phenomenon occurring for vortices in Section 3.8.2. If this speculation is true, it would also explain why a naive quantization of the instanton leads to a continuous spectrum, rather strange behavior for a single particle: it's because the instanton is really a multi-particle state. However, to make sense of this idea we would need to understand why the fractional instantons are confined to lie within the instanton yet, at the same time, are also able to wander freely as evinced by the $4kN$ moduli. Which, let's face it, is odd! A possible explanation for this strange behavior may lie in the issues of non-normalizability of non-abelian modes discussed above, and related issues described in [65].

While it's not entirely clear what a fractional instanton means on $\mathbf{R}^4$, one can make rigorous sense of the idea when the theory is compactified on a further $\mathbf{S}^1$ with a Wilson line [66, 67]. Moreover, there's evidence from string dualities [68, 39] that the moduli space of instantons on compact spaces $\mathbf{M} = \mathbf{T}^4$ or $K3$ has the same topology as the symmetric product $\mathrm{Sym}^{kN}(\mathbf{M})$, suggesting an interpretation in terms of kN entities (strictly speaking, one needs to resolve these spaces into an object known as the Hilbert scheme of points over $\mathbf{M}$).

11 Monopoles

The tale of magnetic monopoles is well known. They are postulated particles with long-range, radial, magnetic field B_i, $i = 1, 2, 3$,

$$B_i = \frac{g\,\hat{r}_i}{4\pi r^2} \tag{11.1}$$

where g is the magnetic charge. Monopoles have never been observed and one of Maxwell's equations, $\nabla \cdot B = 0$, insists they never will be. Yet they have been a recurrent theme in high energy particle physics for the past 30 years! Why?

The study of monopoles began with Dirac [69] who showed how one could formulate a theory of monopoles consistent with a gauge potential A_μ. The requirement that the electron doesn't see the inevitable singularities in A_μ leads to the famed quantization condition

$$eg = 2\pi n \qquad n \in \mathbf{Z} \tag{11.2}$$

However, the key step in the rehabilitation of magnetic monopoles was the observation of 't Hooft [70] and Polyakov [71] that monopoles naturally occur in non-abelian gauge theories, making them a robust prediction of grand unified theories based on semi-simple groups. In this lecture we'll review the formalism of 't Hooft-Polyakov monopoles in $SU(N)$ gauge groups, including the properties of the solutions and the D-brane realization of the Nahm construction. At the end we'll cover several applications to quantum gauge theories in various dimensions.

There are a number of nice reviews on monopoles in the literature. Aspects of the classical solutions are dealt with by Sutcliffe [72] and Shnir [73]; the mathematics of monopole scattering can be found in the book by Atiyah and Hitchin [74]; the application to S-duality of quantum field theories is covered in the review by Harvey [75]. A comprehensive review of magnetic monopoles by Weinberg and Yi will appear shortly [76].

11.1 The Basics

To find monopoles, we first need to change our theory from that of Lecture 1. We add a single real scalar field $\phi \equiv \phi^a{}_b$, transforming in the adjoint representation of $SU(N)$. The action now reads

$$S = \mathrm{Tr} \int d^4x \, \frac{1}{2e^2} F_{\mu\nu} F^{\mu\nu} + \frac{1}{e^2} (\mathcal{D}_\mu \phi)^2 \tag{11.3}$$

where we're back in Minkowski signature $(+,-,-,-)$. The spacetime index runs over $\mu = 0,1,2,3$ and we'll also use the purely spatial index $i = 1,2,3$. Actions of this type occur naturally as a subsector of $\mathcal{N}=4$ and $\mathcal{N}=2$ super Yang-Mills theories. There is no potential for ϕ so, classically, we are free to chose the vacuum expectation value (vev) as we see fit. Gauge inequivalent choices correspond to different ground states of the theory. By use of a suitable gauge transformation, we may set

$$\langle \phi \rangle = \mathrm{diag}(\phi_1, ..., \phi_N) = \vec{\phi} \cdot \vec{H} \tag{11.4}$$

where the fact we're working in $SU(N)$ means that $\sum_{a=1}^{N} \phi_a = 0$. We've also introduced the notation of the root vector $\vec{\phi}$, with $\vec{H}$ a basis for the $(N-1)$-dimensional Cartan subalgebra of $su(N)$. If you're not familiar with roots of Lie algebras and the Cartn-Weyl basis then you can simply think of $\vec{H}$ as the set of N matrices, each with a single entry 1 along the diagonal. (This is actually the Cartan subalgebra for $u(N)$ rather than $su(N)$ but this will take care of itself if we remember that $\sum_a \phi_a = 0$). Under the requirement that $\phi_a \neq \phi_b$ for $a \neq b$ the gauge symmetry breaks to the maximal torus,

$$SU(N) \to U(1)^{N-1} \tag{11.5}$$

The spectrum of the theory consists of $(N-1)$ massless photons and scalars, together with $\frac{1}{2}N(N-1)$ massive W-bosons with mass $M_W^2 = (\phi_a - \phi_b)^2$. In the following we will use the Weyl symmetry to order $\phi_a < \phi_{a+1}$.

In the previous lecture, instantons arose from the possibility of winding field configurations non-trivially around the $\mathbf{S}^3_\infty$ infinity of Euclidean spacetime. Today we're interested in particle-like solitons, localized in space rather than spacetime. These objects are supported by the vev (11.4) twisting along its gauge orbit as we circumvent the spatial boundary $\mathbf{S}^2_\infty$. If we denote the two coordinates on $\mathbf{S}^2_\infty$ as θ and φ, then solitons are supported by configurations with $\langle \phi \rangle = \langle \phi(\theta, \varphi) \rangle$. Let's classify the possible windings. A vev of the form (11.4) is one point in a space of gauge equivalent vacua, given by $SU(N)/U(1)^{N-1}$ where the stabilizing group in the denominator is the unbroken symmetry group (11.5) which leaves (11.4) untouched. We're therefore left to consider maps: $\mathbf{S}^2_\infty \to SU(N)/U(1)^{N-1}$, characterized by

$$\Pi_2 \left(SU(N)/U(1)^{N-1} \right) \cong \Pi_1 \left(U(1)^{N-1} \right) \cong \mathbf{Z}^{N-1} \tag{11.6}$$

This classification suggests that we should be looking for $(N-1)$ different types of topological objects. As we shall see, these objects are monopoles carrying magnetic charge in each of the $(N-1)$ unbroken abelian gauge fields (11.5).

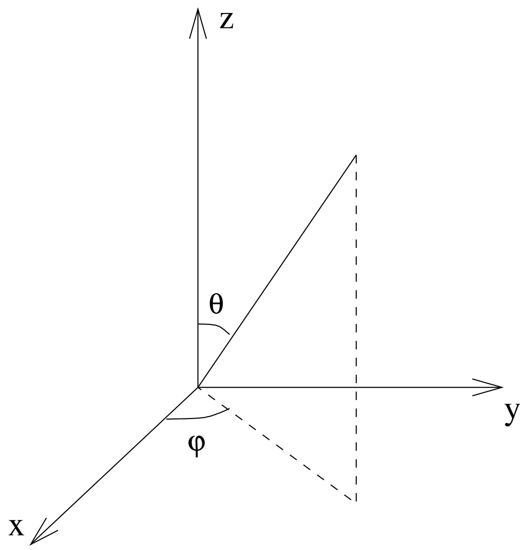
Why is winding of the scalar field ϕ at infinity associated with magnetic charge? To see the precise connection is actually a little tricky — details can be found in [70, 71] and in [77] for $SU(N)$ monopoles — but there is a simple heuristic argument to see why the two are related. The important point is that if a configuration is to have finite energy, the scalar kinetic term $\mathcal{D}_\mu\phi$ must decay at least as fast as $1/r^2$ as we approach the boundary $r \to \infty$. But if $\langle\phi\rangle$ varies asymptotically as we move around $\mathbf{S}^2_\infty$, we have $\partial\phi \sim 1/r$. To cancel the resulting infrared divergence we must turn on a corresponding gauge potential $A_\theta \sim 1/r$, leading to a magnetic field of the form $B \sim 1/r^2$.

Physically, we would expect any long range magnetic field to propagate through the massless $U(1)$ photons. This is indeed the case. If $\mathcal{D}_i\phi \to 0$ as $r \to \infty$ then $[\mathcal{D}_i, \mathcal{D}_j]\phi = -i[F_{ij}, \phi] \to 0$ as $r \to \infty$. Combining these two facts, we learn that the non-abelian magnetic field carried by the soliton is of the form,

$$B_i = \vec{g} \cdot \vec{H}(\theta, \varphi)\, \frac{\hat{r}_i}{4\pi r^2} \tag{11.7}$$

Here the notation $\vec{H}(\theta, \varphi)$ reminds us that the unbroken Cartan subalgebra twists within the $su(N)$ Lie algebra as we move around the $\mathbf{S}^2_\infty$.

11.1.1 Dirac Quantization Condition



The allowed magnetic charge vectors $\vec{g}$ may be determined by studying the winding of the scalar field ϕ around $\mathbf{S}^2_\infty$. However, since the winding is related to the magnetic charge, and the latter is a characteristic of the long range behavior of the monopole, it's somewhat easier to neglect the non-abelian structure completely and study just the $U(1)$ fields. The equivalence between the two methods is reflected in the equality between first and second homotopy groups in (11.6).

For this purpose, it is notationally simpler to work in unitary, or singular, gauge in which the vev $\langle\phi\rangle = \vec{\phi} \cdot \vec{H}$ is fixed to be constant at infinity. This necessarily re-introduces Dirac string-like singularities for any single-valued gauge potential, but allows us to globally write the magnetic field in diagonal form,

$$B_i = \mathrm{diag}(g_1, \dots, g_N)\, \frac{\hat{r}_i}{4\pi r^2} \tag{11.8}$$

where $\sum_{a=1}^{N} g_a = 0$ since the magnetic field lies in $su(N)$ rather than $u(N)$.

What values of g_a are allowed? A variant of Dirac's original argument, due to Wu and Yang [78], derives the magnetic field (11.8) from two gauge potentials defined respectively on the northern and southern hemispheres of $\mathbf{S}^2_\infty$:

$$\begin{aligned} A^N_\varphi &= \frac{1-\cos\theta}{4\pi r \sin\theta}\, \vec{g}\cdot\vec{H} \\ A^S_\varphi &= -\frac{1+\cos\theta}{4\pi r\sin\theta}\, \vec{g}\cdot\vec{H} \end{aligned} \tag{11.9}$$

where A^N goes bad at the south pole $\theta=\pi$, while A^S sucks at the north pole $\theta=0$. To define a consistent field strength we require that on the overlap $\theta\neq 0,\pi$, the two differ by a gauge transformation which, indeed, they do:

$$A^N_i = U(\partial_i + A^S_i)U^{-1} \tag{11.10}$$

with $U(\theta,\varphi) = \exp(-i\vec{g}\cdot\vec{H}\varphi/2\pi)$. Notice that as we've written it, this relationship only holds in unitary gauge where $\vec{H}$ doesn't depend on θ or φ, requiring that we work in singular gauge. The final requirement is that our gauge transformation is single valued, so $U(\varphi) = U(\varphi+2\pi)$ or, in other words, $\exp(i\,\vec{g}\cdot\vec{H}) = 1$. This requirement is simply solved by

$$g_a \in 2\pi\mathbf{Z} \tag{11.11}$$

This is the Dirac quantization condition (11.2) in units in which the electric charge $e=1$, a convention which arises from scaling the coupling outside the action in (11.3). In fact, in our theory the W-bosons have charge 2 under any $U(1)$ while matter in the fundamental representation would have charge 1.

There's another notation for the magnetic charge vector $\vec{g}$ that will prove useful. We write

$$\vec{g} = 2\pi \sum_{a=1}^{N-1} n_a\,\vec{\alpha}_a \tag{11.12}$$

where $n_a \in \mathbf{Z}$ by the Dirac quantization condition[4] and $\vec{\alpha}_a$ are the simple roots of $su(N)$. The choice of simple roots is determined by defining $\vec{\phi}$ to lie in a positive Weyl chamber. What this means in practice, with our chosen ordering $\phi_a < \phi_{a+1}$, is that we can write each root as an N-vector, with

$$\begin{aligned} \vec{\alpha}_1 &= (1,-1,0,...,0) \\ \vec{\alpha}_2 &= (0,1,-1,...,0) \end{aligned} \tag{11.13}$$

[4] For monopoles in a general gauge group, the Dirac quantization condition becomes $\vec{g} = 4\pi\sum_a n_a\vec{\alpha}^\star_a$ where $\vec{\alpha}^\star_a$ are simple co-roots.

through to

$$\vec{\alpha}_{N-1} = (0, 0, ..., 1, -1) \tag{11.14}$$

Then translating between two different notations for the magnetic charge vector we have

$$\begin{aligned} \vec{g} &= \mathrm{diag}(g_1, ..., g_N) \\ &= 2\pi \, \mathrm{diag}(n_1 \, , \, n_2 - n_1, ..., n_{N-1} - n_{N-2} \, , \, -n_{N-1}) \end{aligned} \tag{11.15}$$

The advantage of working with the integers n_a, $a = 1, ..., N-1$ will become apparent shortly.

11.1.2 The Monopole Equations

As in Lecture 1, we've learnt that the space of field configurations decomposes into different topological sectors, this time labelled by the vector $\vec{g}$ or, equivalently, the $N-1$ integers n_a. We're now presented with the challenge of finding solutions in the non-trivial sectors. We can again employ a Bogomoln'yi bound argument (this time actually due to Bogomoln'yi [14]) to derive first order equations for the monopoles. We first set $\partial_0 = A_0 = 0$, so we are looking for time independent configurations with vanishing electric field. Then the energy functional of the theory gives us the mass of a magnetic monopole,

$$\begin{aligned} M_{\mathrm{mono}} &= \mathrm{Tr} \int d^3x \; \frac{1}{e^2} B_i^2 + \frac{1}{e^2} (\mathcal{D}_i \phi)^2 \\ &= \mathrm{Tr} \int d^3x \; \frac{1}{e^2} \left(B_i \mp \mathcal{D}_i \phi \right)^2 \pm \frac{2}{e^2} \, B_i \mathcal{D}_i \phi \\ &\geq \frac{2}{e^2} \int d^3x \; \partial_i \; \mathrm{Tr}(B_i \phi) \end{aligned} \tag{11.16}$$

where we've used the Bianchi identity $\mathcal{D}_i B_i = 0$ when integrating by parts to get the final line. As in the case of instantons, we've succeeded in bounding the energy by a surface term which measures a topological charge. Comparing with the expressions above we have

$$M_{\mathrm{mono}} \geq \frac{|\vec{g} \cdot \vec{\phi}|}{e^2} = \frac{2\pi}{e^2} \sum_{a=1}^{N-1} n_a \phi_a \tag{11.17}$$

with equality if and only if the monopole equations (often called the Bogomoln'yi equations) are obeyed,

$$\begin{aligned} B_i &= \mathcal{D}_i \phi \qquad\qquad \text{if } \vec{g} \cdot \vec{\phi} > 0 \\ B_i &= -\mathcal{D}_i \phi \qquad\qquad \text{if } \vec{g} \cdot \vec{\phi} < 0 \end{aligned} \tag{11.18}$$

For the rest of this lecture we'll work with $\vec{g} \cdot \vec{\phi} > 0$ and the first of these equations. Our path will be the same as in lecture 1: we'll first examine the simplest solution to these equations and then study its properties before moving on to the most general solutions. So first:

11.1.3 Solutions and Collective Coordinates

The original magnetic monopole described by 't Hooft and Polyakov occurs in $SU(2)$ theory broken to $U(1)$. We have $SU(2)/U(1) \cong \mathbf{S}^2$ and $\Pi_2(\mathbf{S}^2) \cong \mathbf{Z}$. Here we'll describe the simplest such monopole with charge one. To better reveal the topology supporting this monopole (as well as to demonstrate explicitly that the solution is smooth) we'll momentarily revert back to a gauge where the vev winds asymptotically. The solution to the monopole equation (11.18) was found by Prasad and Sommerfield [79]

$$\begin{aligned} \phi &= \frac{\hat{r}_i \sigma^i}{r}\left(vr\coth(vr) - 1\right) \\ A_\mu &= -\epsilon_{i\mu j}\frac{\hat{r}^j\sigma^i}{r}\left(1 - \frac{vr}{\sinh vr}\right) \end{aligned} \tag{11.19}$$

This solution asymptotes to $\langle\phi\rangle = v\sigma^i\hat{r}^i$, where σ^i are the Pauli matrices (i.e. comparing notation with (11.4) in, say, the $\hat{r}^3$ direction, we have $v = -\phi_1 = \phi_2$). The $SU(2)$ solution presented above has 4 collective coordinates, although none of them are written explicitly. Most obviously, there are the three center of mass coordinates. As with instantons, there is a further collective coordinate arising from acting on the soliton with the unbroken gauge symmetry which, in this case, is simply $U(1)$.

For monopoles in $SU(N)$ we can always generate solutions by embedding the configuration (11.19) above into a suitable $SU(2)$ subgroup. Note however that, unlike the situation for instantons, we can't rotate from one $SU(2)$ embedding to another since the $SU(N)$ gauge symmetry is not preserved in the vacuum. Each $SU(2)$ embedding will give rise to a different monopole with different properties — for example, they will have magnetic charges under different $U(1)$ factors.

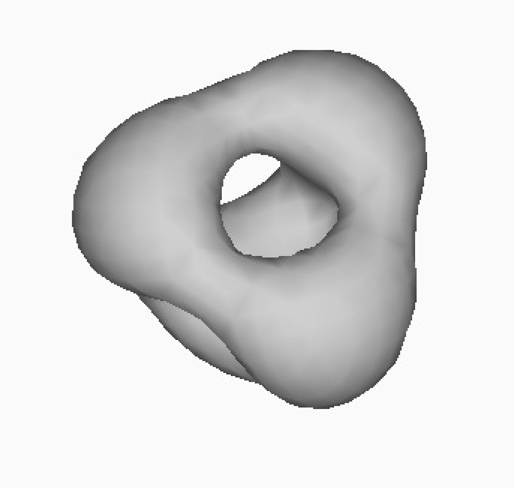
Of the many inequivalent embeddings of $SU(2)$ into $SU(N)$, there are $(N-1)$ special ones. These have generators given in the Cartan-Weyl basis by $\vec{\alpha}\cdot\vec{H}$ and $E_{\pm\vec{\alpha}}$ where $\vec{\alpha}$ is one of the simple roots (11.13). In a less sophisticated language, these are simply the $(N-1)$ contiguous 2×2 blocks which lie along the diagonal of an $N\times N$ matrix. Embedding the monopole in the a^{th} such block gives rise to the magnetic charge $\vec{g} = \vec{\alpha}_a$.

11.2 The Moduli Space

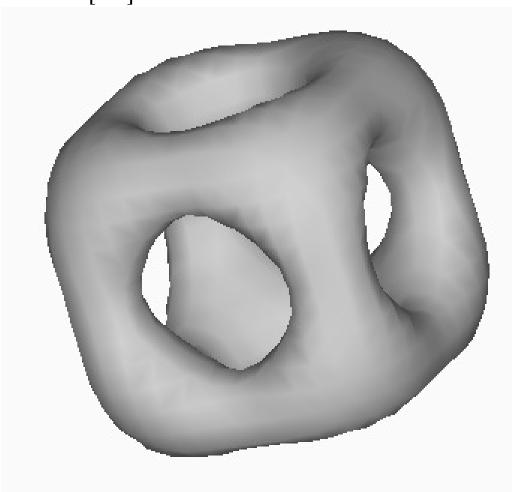
For a monopole with magnetic charge $\vec{g}$, we want to know how many collective coordinates are contained within the most general solution. The answer was given by E. Weinberg [80]. There are subtleties that don't occur in the instanton calculation, and a variant of the Atiyah-Singer index theorem due to Callias is required [81]. But the result is very simple. Define the moduli space of monopoles with magnetic charge $\vec{g}$ to be $\mathcal{M}_{\vec{g}}$. Then the number of collective coordinates is

$$\dim(\mathcal{M}_{\vec{g}}) = 4\sum_{a=1}^{N-1} n_a \tag{11.20}$$

The interpretation of this is as follows. There exist $(N-1)$ "elementary" monopoles, each associated to a simple root $\vec{\alpha}_a$, carrying magnetic charge under exactly one of the $(N-1)$ surviving $U(1)$ factors of (11.5). Each of these elementary monopoles has 4 collective coordinates. A monopole with general charge $\vec{g}$ can be decomposed into $\sum_a n_a$ elementary monopoles, described by three position coordinates and a phase arising from $U(1)$ gauge rotations.



You should be surprised by the existence of this large class of solutions since it implies that monopoles can be placed at arbitrary separation and feel no force. But this doesn't happen for electrons! Any objects carrying the same charge, whether electric or magnetic, repel. So what's special about monopoles? The point is that monopoles also experience a second long range force due to the massless components of the scalar field ϕ. This gives rise to an attraction between the monopoles that precisely cancels the electromagnetic repulsion [82]. Such cancellation of forces only occurs when there is no potential for ϕ as in (11.3).



The interpretation of the collective coordinates as positions of particle-like objects holds only when the monopoles are more widely separated than their core size. As the monopoles approach, weird things happen! Two monopoles form a torus. Three monopoles form a tetrahedron, seemingly splitting into four lumps of energy as seen in figure 4. Four monopoles form a cube as in figure 5. (Both of these figures are taken from [83]). We see that monopoles really lose their individual identities as the approach and merge into each other. Higher monopoles form platonic solids, or buckyball like objects.

11.2.1 The Moduli Space Metric

The metric on $\mathcal{M}_{\vec{g}}$ is defined in a similar fashion to that on the instanton moduli space $\mathcal{I}_{k,N}$. To be more precise, it's defined in an identical fashion. Literally! The key point is that the monopole equations $B = \mathcal{D}\phi$ and the instanton equations $F = {}^\star F$ are really the same: the difference between the two lies in the boundary conditions. To see this, consider instantons with $\partial_4 = 0$ and endow the component of the gauge field $A_4 \equiv \phi$ with a vev $\langle\phi\rangle$. We end up with the monopole equations. So using the notation $\delta\phi = \delta A_4$, we can reuse the linearized self-dual equations (1.14) and the gauge fixing condition (1.17) from the Lecture 1 to define the monopole zero modes. The metric on the monopole moduli space $\mathcal{M}_{\vec{g}}$ is again given by the overlap of zero modes,

$$g_{\alpha\beta} = \frac{1}{e^2}\,\mathrm{Tr}\int d^3x\ (\delta_\alpha A_i\,\delta_\beta A_i + \delta_\alpha\phi\,\delta_\beta\phi) \tag{11.21}$$

The metric on the monopole moduli space has the following properties:

- The metric is hyperKähler.
- The metric enjoys an $SO(3) \times U(1)^{N-1}$ isometry. The former descends from physical rotations of the monopoles in space. The latter arise from the unbroken gauge group. The $U(1)^{N-1}$ isometries are tri-holomorphic, while the $SO(3)$ isometry rotates the three complex structures.
- The metric is smooth. There are no singular points analogous to the small instanton singularities of $\mathcal{I}_{k,N}$ because, as we have seen, the scale of the monopole isn't a collective coordinate. It is fixed to be $L_{\mathrm{mono}} \sim 1/M_W$, the Compton wavelength of the W-bosons.
- Since the metrics on $\mathcal{I}_{k,N}$ and $\mathcal{M}_{\vec{g}}$ arise from the same equations, merely endowed with different boundary conditions, one might wonder if we can interpolate between them. In fact we can. In the study of instantons on $\mathbf{R}^3\times\mathbf{S}^1$, with a non-zero Wilson line around the $\mathbf{S}^1$, the $4N$ collective coordinates of the instanton gain the interpretation of the positions of N "fractional instantons" [66, 67]. These are often referred to as calorons and are identified as the monopoles discussed above. By taking the radius of the circle to zero, and some calorons to infinity, we can interpolate between the metrics on $\mathcal{M}_{\vec{g}}$ and $\mathcal{I}_{k,N}$ [62].

11.2.2 The Physical Interpretation of the Metric

For particles such as monopoles in $d = 3+1$ dimensions, the metric on the moduli space has a beautiful physical interpretation first described by Manton [84]. Suppose that the monopoles move slowly through space. We approximate the motion by assuming that the field configurations remain close to the static solutions, but endow the collective coordinates X^α with time dependence: $X^\alpha \to X^\alpha(t)$. If monopoles collide at very high energies this approximation will not be valid. As the monopoles hit they will spew out massive W-bosons and, on occasion, even monopole-anti-monopole pairs. The resulting field configurations will look nothing like the static monopole solutions. Even for very low-energy scattering it's not completely clear that the approximation is valid since the theory doesn't have a mass gap and the monopoles can emit very soft photons. Nevertheless, there is much evidence that this procedure, known as the *moduli space approximation*, does capture the true physics of monopole scattering at low energies. The time dependence of the fields is

$$A_\mu = A_\mu(X^\alpha(t)) \qquad \phi = \phi(X^\alpha(t)) \tag{11.22}$$

which reduces the dynamics of an infinite number of field theory degrees of freedom to a finite number of collective coordinates. We must still satisfy Gauss' law,

$$\mathcal{D}_i E_i - i[\phi, \mathcal{D}_0\phi] = 0 \tag{11.23}$$

which can be achieved by setting $A_0 = \Omega_\alpha \dot{X}^\alpha$, where the Ω_α are the extra gauge rotations that we introduced in (1.15) to ensure that the zero modes satisfy the background gauge fixing condition. This means that the time dependence of the fields is given in terms of the zero modes,

$$\begin{aligned} E_i = F_{0i} &= \delta_\alpha A_i \, \dot{X}^\alpha \\ \mathcal{D}_0\phi &= \delta_\alpha \phi \, \dot{X}^\alpha \end{aligned} \qquad (11.24)$$

Plugging this into the action (11.3) we find

$$\begin{aligned} S &= \mathrm{Tr} \int d^4x \, \frac{1}{e^2} \left(E_i^2 + B_i^2 + (\mathcal{D}_0\phi)^2 + (\mathcal{D}_i\phi)^2 \right) \\ &= \int dt \, \left(M_{\rm mono} + \tfrac{1}{2} g_{\alpha\beta} \dot{X}^\alpha \dot{X}^\beta \right) \end{aligned} \qquad (11.25)$$

The upshot of this is analysis is that the low-energy dynamics of monopoles is given by the $d = 0 + 1$ sigma model on the monopole moduli space. The equations of motion following from (11.25) are simply the geodesic equations for the metric $g_{\alpha\beta}$. We learn that the moduli space metric captures the velocity-dependent forces felt by monopoles, such that low-energy scattering is given by geodesic motion.

In fact, this logic can be reversed. In certain circumstances it's possible to figure out the trajectories followed by several moving monopoles. From this one can construct a metric on the configuration space of monopoles such that the geodesics reconstruct the known motion. This metric agrees with that defined above in a very different way. This procedure has been carried out for a number of examples [85, 86, 87].

11.2.3 Examples of Monopole Moduli Spaces

Let's now give a few examples of monopole moduli spaces. We start with the simple case of a single monopole where the metric may be explicitly computed.

11.2.3One Monopole

Consider the $\vec{g} = \vec{\alpha}_1$ monopole, which is nothing more that the charge one $SU(2)$ solution we saw previously (11.19). In this case we can compute the metric directly. We have two different types of collective coordinates:

i) The three translational modes. The linearized monopole equation and gauge fixing equation are solved by $\delta_{(i)} A_j = -F_{ij}$ and $\delta_{(i)}\phi = -\mathcal{D}_i\phi$, so that the overlap of zero modes is

$$\mathrm{Tr} \, \frac{1}{e^2} \int d^3x \, \left(\delta_{(i)} A_k \, \delta_{(j)} A_k + \delta_{(i)}\phi \, \delta_{(j)}\phi \right) = M_{\rm mono} \, \delta_{ij} \qquad (11.26)$$

ii) The gauge mode arises from transformation $U = \exp(i\phi\chi/v)$, where the normalization has been chosen so that the collective coordinate χ has periodicity 2π. This gauge transformation leaves ϕ untouched while the transformation on the gauge field is $\delta A_i = (\mathcal{D}_i\phi)/v$.

Putting these two together, we find that single monopole moduli space is

$$\mathcal{M}_{\vec{\alpha}} \cong \mathbf{R}^3 \times \mathbf{S}^1 \tag{11.27}$$

with metric

$$ds^2 = M_{\rm mono} \left(dX^i dX^i + \frac{1}{v^2}\, d\chi^2 \right) \tag{11.28}$$

where $M_{\rm mono} = 4\pi v/e^2$ in the notation used in the solution (11.19).

11.2.3Two Monopoles

Two monopoles in $SU(2)$ have magnetic charge $\vec{g} = 2\alpha_1$. The direct approach to compute the metric that we have just described becomes impossible since the most general analytic solution for the two monopole configuration is not available. Nonetheless, Atiyah and Hitchin were able to determine the two monopole moduli space using symmetry considerations alone, most notably the constraints imposed by hyperKählerity [74, 88]. It is

$$\mathcal{M}_{2\vec{\alpha}} \cong \mathbf{R}^3 \times \frac{\mathbf{S}^1 \times \mathcal{M}_{AH}}{\mathbf{Z}_2} \tag{11.29}$$

where $\mathbf{R}^3$ describes the center of mass of the pair of monopoles, while $\mathbf{S}^1$ determines the overall phase $0 \leq \chi \leq 2\pi$. The four-dimensional hyperKählerspace $\mathcal{M}_{AH}$ is the famous Atiyah-Hitchin manifold. Its metric can be written as

$$ds^2 = f(r)^2 dr^2 + a(r)^2 \sigma_1^2 + b(r)^2 \sigma_2^2 + c(r)^2 \sigma_3^2 \tag{11.30}$$

Here the radial coordinate r measures the separation between the monopoles in units of the monopole mass. The σ_i are the three left-invariant $SU(2)$ one-forms which, in terms of polar angles $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$ and $0 \leq \psi \leq 2\pi$, take the form

$$\begin{aligned} \sigma_1 &= -\sin\psi\; d\theta + \cos\psi \sin\theta\; d\phi \\ \sigma_2 &= \cos\psi\; d\theta + \sin\psi \sin\theta\; d\phi \\ \sigma_3 &= d\psi + \cos\theta\; d\phi \end{aligned} \tag{11.31}$$

For far separated monopoles, θ and ϕ determine the angular separation while ψ is the relative phase. The $\mathbf{Z}_2$ quotient in (11.29) acts as

$$\mathbf{Z}_2 : \chi \rightarrow \chi + \pi \qquad \psi \rightarrow \psi + \pi \tag{11.32}$$

The hyperKählercondition can be shown to relate the four functions f, a, b and c through the differential equation

$$\frac{2bc}{f} \frac{da}{dr} = (b-c)^2 - a^2 \tag{11.33}$$

together with two further equations obtained by cyclically permuting a, b and c. The solutions can be obtained in terms of elliptic integrals but it will prove more illuminating to present the asymptotic expansion of these functions. Choosing coordinates such that $f(r) = -b(r)/r$, we have

$$\begin{aligned} a^2 &= r^2 \left(1 - \frac{2}{r}\right) - 8r^3 e^{-r} + \ldots \\ b^2 &= r^2 \left(1 - \frac{2}{r}\right) + 8r^3 e^{-r} + \ldots \\ c^2 &= 4 \left(1 - \frac{2}{r}\right)^{-1} + \ldots \end{aligned} \tag{11.34}$$

If we suppress the exponential corrections, the metric describes the velocity dependant forces between two monopoles interacting through their long range fields. In fact, this asymptotic metric can be derived by treating the monopoles as point particles and considering their Liénard-Wiechert potentials. Note that in this limit there is an isometry associated to the relative phase ψ. However, the minus sign before the $2/r$ terms means that the metric is singular. The exponential corrections to the metric resolve this singularity and contain the information about the behavior of the monopoles as their non-abelian cores overlap.

The Atiyah-Hitchin metric appears in several places in string theory and supersymmetric gauge theories, including the M-theory lift of the type IIA O6-plane [89], the solution of the quantum dynamics of 3d gauge theories [90], in intersecting brane configurations [91], the heterotic string compactified on ALE spaces [92, 93] and NS5-branes on orientifold 8-planes [94]. In each of these places, there is often a relationship to magnetic monopoles underlying the appearance of this metric.

For higher charge monopoles of the same type $\vec{g} = n\vec{\alpha}$, the leading order terms in the asymptotic expansion of the metric, associated with the long-range fields of the monopoles, have been computed. The result is known as the Gibbons-Manton metric [86]. The full metric on the monopole moduli space remains an open problem.

11.2.3Two Monopoles of Different Types

As we have seen, higher rank gauge groups $SU(N)$ for $N \geq 3$ admit monopoles of different types. If a $\vec{g} = \vec{\alpha}_a$ monopole and a $\vec{g} = \vec{\alpha}_b$ monopole live in entirely different places in the gauge group, so that $\vec{\alpha}_a \cdot \vec{\alpha}_b = 0$, then they don't see each other and their moduli space is simply the product $(\mathbf{R}^3 \times \mathbf{S}^1)^2$. However, if they live in neighboring subgroups so that $\vec{\alpha}_a \cdot \vec{\alpha}_b = -1$, then they do interact non-trivially.

The metric on the moduli space of two neighboring monopoles, sometimes referred to as the $(1,1)$ monopole, was first computed by Connell [95]. But he chose not to publish. It was rediscovered some years later by two groups when the connection with electro-magnetic duality made the study of monopoles more pressing [96, 97]. It is simplest to describe if the two monopoles have the same mass, so $\vec{\phi} \cdot \vec{\alpha}_a = \vec{\phi} \cdot \vec{\alpha}_b$. The moduli space is then

$$\mathcal{M}_{\vec{\alpha}_1+\vec{\alpha}_2} \cong \mathbf{R}^3 \times \frac{\mathbf{S}^1 \times \mathcal{M}_{TN}}{\mathbf{Z}_2} \tag{11.35}$$

where the interpretation of the $\mathbf{R}^3$ factor and $\mathbf{S}^1$ factor are the same as before. The relative moduli space is the Taub-NUT manifold, which has metric

$$ds^2 = \left(1 + \frac{2}{r}\right)(dr^2 + r^2(\sigma_1^2 + \sigma_2^2)) + \left(1 + \frac{2}{r}\right)^{-1} \sigma_3^2 \tag{11.36}$$

The $+2/r$ in the metric, rather than the $-2/r$ of Atiyah-Hitchin, means that the metric is smooth. The apparent singularity at $r = 0$ is merely a coordinate artifact, as you can check by transforming to the variables $R = \sqrt{r}$. Once again, the $1/r$ terms capture the long range interactions of the monopoles, with the minus sign traced to the fact that each sees the other with opposite magnetic charge (essentially because $\vec{\alpha}_1 \cdot \vec{\alpha}_2 = -1$). There are no exponential corrections to this metric. The non-abelian cores of the two monopoles do not interact.

The exact moduli space metric for a string of neighboring monopoles, $\vec{g} = \sum_a \vec{\alpha}_a$ has been determined. Known as the Lee-Weinberg-Yi metric, it is a higher dimensional generalization of the Taub-NUT metric [87]. It is smooth and has no exponential corrections.

11.3 Dyons

Consider the one-monopole moduli space $\mathbf{R}^3 \times \mathbf{S}^1$. Motion in $\mathbf{R}^3$ is obvious. But what does motion along the $\mathbf{S}^1$ correspond to?

We can answer this by returning to our specific $SU(2)$ solution (11.19). We determined that the zero mode for the $U(1)$ action is $\delta A_i = \mathcal{D}_i\phi$ and $\delta\phi = 0$. Translating to the time dependence of the fields (11.24), we find

$$E_i = \frac{(\mathcal{D}_i\phi)}{v}\,\dot{\chi} = \frac{B_i}{v}\; e^2\dot{\chi} \tag{11.37}$$

We see that motion along the $\mathbf{S}^1$ induces an electric field for the monopole, proportional to its magnetic field. In the unbroken $U(1)$, this gives rise to a long range electric field,

$$\mathrm{Tr}(E_i\phi) = \frac{qve^2\hat{r}_i}{2\pi r^2} \tag{11.38}$$

where, comparing with the normalization above, the electric charge q is given by

$$q = \frac{2\pi\dot{\chi}}{ve^2} \tag{11.39}$$

Note that motion in $\mathbf{R}^3$ also gives rise to an electric field, but this is the dual to the familiar statement that a moving electric charge produces a magnetic field. Motion in $\mathbf{S}^1$, on the other hand, only has the effect of producing an electric field [98].

A particle with both electric and magnetic charges is called a *dyon*, a term first coined by Schwinger [99]. Since we have understood this property from the perspective of the monopole worldline, can we return to our original theory (11.3) and find the corresponding solution there? The answer is yes. We relax the condition $E_i = 0$ when completing the Bogomoln'yi square in (11.16) and write

$$\begin{aligned} M_{\text{dyon}} \;&=\; \mathrm{Tr}\int d^3x\;\frac{1}{e^2}\left(E_i - \cos\alpha\,\mathcal{D}_i\phi\right)^2 + \frac{1}{e^2}\left(B_i - \sin\alpha\,\mathcal{D}_i\phi\right)^2 \\ &\quad + \frac{2}{e^2}\mathrm{Tr}\int d^3x\;\partial_i\left(\cos\alpha\,E_i\phi + \sin\alpha\,B_i\phi\right) \end{aligned} \tag{11.40}$$

which holds for all α. We write the long range magnetic field as $E_i = \vec{q}\cdot\vec{H}\,e^2\hat{r}^i/4\pi r^2$. Then by adjusting α to make the bound as tight as possible, we have

$$M_{\text{dyon}} \geq \sqrt{\left(\vec{q}\cdot\vec{\phi}\right)^2 + \left(\frac{\vec{g}\cdot\vec{\phi}}{e^2}\right)^2} \tag{11.41}$$

and, given a solution to the monopole, it is easy to find a corresponding solution for the dyon for which this bound is saturated, with the fields satisfying

$$B_i = \sin\alpha\, \mathcal{D}_i\phi \quad \text{and} \quad E_i = \cos\alpha\, \mathcal{D}_i\phi \tag{11.42}$$

This method of finding solutions in the worldvolume theory of a soliton, and subsequently finding corresponding solutions in the parent 4d theory, will be something we'll see several more times in later sections.

I have two further comments on dyons.

- We could add a theta term $\theta F \wedge F$ to the 4d theory. Careful calculation of the electric Noether charges shows that this induces an electric charge $\vec{q} = \theta\vec{g}/2\pi$ on the monopole. In the presence of the theta term, monopoles become dyons. This is known as the Witten effect [100].
- Both the dyons arising from (11.42), and those arising from the Witten effect, have $\vec{q} \sim \vec{g}$. One can create dyons whose electric charge vector is not parallel to the magnetic charge by turning on a vev for a second, adjoint scalar field [101, 102]. These states are 1/4-BPS in $\mathcal{N} = 4$ super Yang-Mills and correspond to (p,q)-string webs stretched between D3-branes. From the field theory perspective, the dynamics of these dyons is described by motion on the monopole moduli space with a potential induced by the second scalar vev [103, 104, 105].

11.4 Fermi Zero Modes

As with instantons, when the theory includes fermions they may be turned on in the background of the monopole without raising the energy of the configuration. A Dirac fermion λ in the adjoint representation satisfies

$$i\gamma^\mu \mathcal{D}_\mu \lambda - i[\phi, \lambda] = 0 \tag{11.43}$$

Each such fermion carried $4\sum_a n_a$ zero modes.

Rather than describing this in detail, we can instead resort again to supersymmetry. In $\mathcal{N} = 4$ super Yang-Mills, the monopoles preserve one-half the supersymmetry, corresponding to $\mathcal{N} = (4,4)$ on the monopole worldvolume. While, monopoles in $\mathcal{N} = 2$ supersymmetric theories preserve $\mathcal{N} = (0,4)$ on their worldvolume. Monopoles in $\mathcal{N} = 1$ theories are not BPS; they preserve no supersymmetry on their worldvolume.

There is also an interesting story with fermions in the fundamental representation, leading to the phenomenon of solitons carrying fractional fermion number [106]. A nice description of this can be found in [75].

11.5 Nahm's Equations

In the previous section we saw that the ADHM construction gave a powerful method for understanding instantons, and that it was useful to view this from the perspective of D-branes in string theory. You'll be pleased to learn that there exists a related method for studying monopoles. It's known as the Nahm construction [107]. It was further developed for arbitrary classical gauge group in [108], while the presentation in terms of D-branes was given by Diaconescu [109].

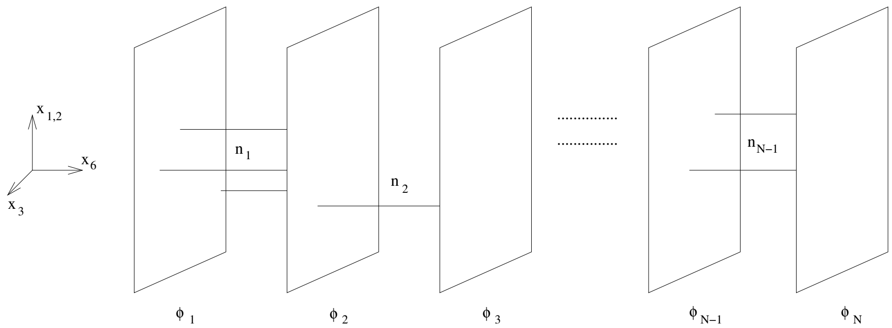


Figure 11.1: The D-brane set-up for monopoles of charge $\vec{g} = \sum_a n_a \vec{\alpha}_a$.

We start with $\mathcal{N} = 4$ $U(N)$ super Yang-Mills, realized on the worldvolume of D3-branes. To reflect the vev $\langle \phi \rangle = \mathrm{diag}(\phi_1 ..., \phi_N)$, we separate the D3-branes in a transverse direction, say the x^6 direction. The a^{th} D3-brane is placed at position $x_6 = \phi_a$.

As is well known, the W-bosons correspond to fundamental strings stretched between the D3-branes. The monopoles are their magnetic duals, the D-strings. At this point our notation for the magnetic charge vector $\vec{g} = \sum_a n_a \vec{\alpha}_a$ becomes more visual. This monopole in sector $\vec{g}$ is depicted by stretching n_a D-strings between the a^{th} and $(a+1)^{\text{th}}$ D3-branes.

Our task now is to repeat the analysis of lecture 1 that led to the ADHM construction: we must read off the theory on the D1-branes, which we expect give us a new perspective on the dynamics of magnetic monopoles. From the picture it looks like the dynamics of the D-strings will be governed by something like a $\prod_a U(n_a)$ gauge theory, with each group living on the interval $\phi_a \leq x_6 \leq \phi_{a+1}$. And this is essentially correct. But what are the relevant equations dictating the dynamics? And what happens at the boundaries?

To get some insight into this, let's start by considering n infinite D-strings, with worldvolume x_0, x_6, and with D3-brane impurities inserted at particular points $x_6 = \phi_a$, as shown below.

The theory on the D-strings is a $d = 1 + 1$ $U(n)$ gauge theory with 16 supercharges (known as $\mathcal{N} = (8, 8)$). Each D3-brane impurity donates a hypermultiplet to the theory, breaking

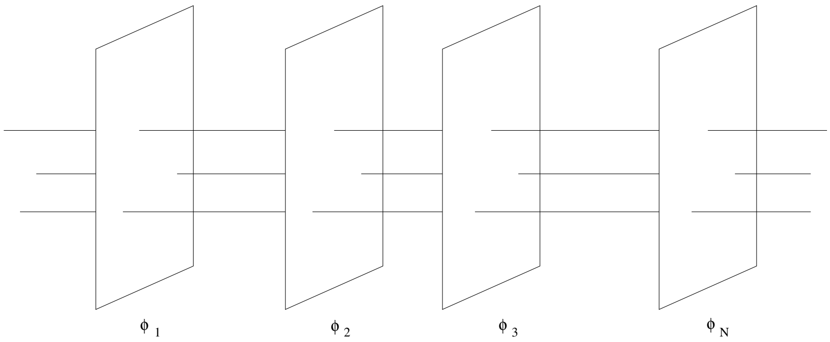


Figure 11.2: The D3-branes give rise to impurities on the worldvolume of the D1-branes.

supersymmetry by half to $\mathcal{N} = (4,4)$. As in lecture 1, we write the hypermultiplets as

$$\omega_a = \begin{pmatrix} \psi_a \\ \tilde{\psi}_a^\dagger \end{pmatrix} \qquad a = 1, ..., N \tag{11.44}$$

where ψ_a transforms in the $\mathbf{n}$ of $U(n)$, while $\tilde{\psi}_a$ transforms in the $\bar{\mathbf{n}}$. The coupling of these impurities (or defects as they're also known) is uniquely determined by supersymmetry, and again occurs in a triplet of D-terms (or, equivalently, a D-term and an F-term). In lecture 1, I unapologetically quoted the D-term and F-term arising in the ADHM construction (equation (1.44)) since they can be found in any supersymmetry text book. However, now we have an impurity theory which is a little less familiar. Nonetheless, I'm still going to quote the result, but this time I'll apologize. We could derive this interaction by examining the supersymmetry in more detail, but it's easier to simply tell you the answer and then give a couple of remarks to try and convince you that it's right. It turns out that the (admittedly rather strange) triplet of D-terms occurring in the Lagrangian is

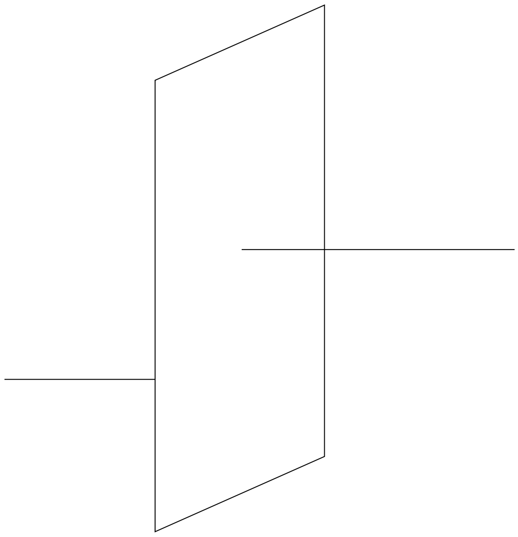
$$\mathrm{Tr}\,\left(\frac{\partial X^i}{\partial x^6} - i[A_6, X^i] - \frac{i}{2}\epsilon_{ijk}[X^j, X^k] + \sum_{a=1}^{N} \omega_a^\dagger \sigma^i \omega_a\ \delta(x^6 - \phi_a)\right)^2 \tag{11.45}$$

In the ground state of the D-strings, this term must vanish. Some motivating remarks:

- The configuration shown in figure 7 arises from T-dualizing the D0-D4 system. This viewpoint makes it clear that A_6 is the right bosonic field to partner X^i in a hypermultiplet.
- Set $\partial_6 = 0$. Then, relabelling $A_6 = X^4$, this term is almost the same as the triplet of D-terms appearing in the ADHM construction. The only difference is the appearance of the delta-functions.
- We know that D-strings can end on D3-branes. The delta-function sources in the D-term are what allow this to happen. For example, consider a single $n = 1$ D-string, so that all commutators above vanish. We choose $\tilde{\psi} = 0$, to find the triplet of D-terms

$$\partial_6 X^1 = 0 \qquad \partial_6 X^2 = 0 \qquad \partial_6 X^3 = |\psi|^2 \delta(0) \tag{11.46}$$

  which allows the D-string profile to take the necessary step (function) to split on the D3-brane as shown below.



If that wasn't enough motivation, one can find the full supersymmetry analysis in the original papers [110, 111] and, in most detail, in [112]. Accepting (11.45) we can make progress in understanding monopole dynamics by studying the limit in which several D-string segments, including the semi-infinite end segments, move off to infinity, leaving us back with the picture of figure 6.

The upshot of this is that the dynamics of the $\vec{g} = \sum_a n_a \vec{\alpha}_a$ monopoles are described as follows: In the interval $\phi_a \leq x_6 \leq \phi_{a+1}$, we have a $U(n_a)$ gauge theory, with three adjoint scalars X^i, $i = 1, 2, 3$ satisfying

$$\frac{dX^i}{dx^6} - i[A_6, X^i] - \tfrac{1}{2}\epsilon^{ijk}[X^i, X^j] = 0 \tag{11.47}$$

These are Nahm's equations. The boundary conditions imposed at the end of the interval depend on the number of monopoles in the neighbouring segment. (Set $n_0 = n_N = 0$ in what follows)

$\underline{n_a = n_{a+1}}$: The $U(n_a)$ gauge symmetry is extended to the interval $\phi_a \leq x^6 \leq \phi_{a+2}$ and an impurity is added to the right-hand-side of Nahm's equations

$$\frac{dX^i}{dx^6} - i[A_6, X^i] - \tfrac{1}{2}\epsilon^{ijk}[X^i, X^j] = \omega^\dagger_{a+1}\sigma^i\omega_{a+1}\ \delta(x^6 - \phi_{a+1}) \tag{11.48}$$

This, of course, follows immediately from (11.45).

$\underline{n_a = n_{a+1} - 1}$: In this case, $X^i \to (X_i)_-$, a set of three constant $n_a \times n_a$ matrices as $x^6 \to (\phi_{a+1})_-$. To the right of the impurity, the X^i are $(n_a + 1) \times (n_a + 1)$ matrices. They are required to satisfy the boundary condition

$$X^i \to \begin{pmatrix} y^i & a^{i\dagger} \\ a^i & (X^i)_- \end{pmatrix} \qquad \text{as } x_6 \to (\phi_{a+1})_+ \tag{11.49}$$

where $y^i \in \mathbf{R}$ and each a^i is a complex n_a-vector. One can derive this boundary condition without too much trouble by starting with (11.48) and taking $|\omega| \to \infty$ to remove one of the monopoles [113].

$\underline{n_a \leq n_{a+1} - 2}$ Once again $X^i \to (X^i)_-$ as $x_6 \to (\phi_{a+1})_-$ but, from the other side, the matrices X_μ now have a simple pole at the boundary,

$$X^i \to \left(\begin{array}{cc} J^i/s + Y^i & \mathcal{O}(s^\gamma) \\ \mathcal{O}(s^\gamma) & (X^i)_- \end{array} \right) \qquad \text{as } x_6 \to (\phi_{a+1})_+ \tag{11.50}$$

Here $s = (x^6 - \phi_{a+1})$ is the distance to the impurity. The matrices J^i are the irreducible $(n_{a+1}-n_a)\times(n_{a+1}-n_a)$ representation of $su(2)$, and Y^i are now constant $(n_{a+1}-n_a)\times(n_{a+1}-n_a)$ matrices. Note that the simple pole structure is compatible with Nahm's equations, with both the derivative and the commutator term going like $1/s^2$. Finally, $\gamma = \frac{1}{2}(n_{a+1} - n_a - 1)$, so the off-diagonal terms vanish as we approach the boundary. The boundary condition (11.50) can also be derived from (11.49) by removing a monopole to infinity [113].

When $n_a > n_{a+1}$, the obvious parity flipped version of the above conditions holds.

11.5.1 Constructing the Solutions

Just as in the case of ADHM construction, Nahm's equations capture information about both the monopole solutions and the monopole moduli space. The space of solutions to Nahm's equations (11.47), subject to the boundary conditions detailed above, is isomorphic to the monopole moduli space $\mathcal{M}_{\vec{g}}$. The phases of each monopole arise from the gauge field A_6, while X^i carry the information about the positions of the monopoles. Moreover, there is a natural metric on the solutions to Nahm's equations which coincides with the metric on the monopole moduli space. I don't know if anyone has calculated the Atiyah-Hitchin metric using Nahm data, but a derivation of the Lee-Weinberg-Yi metric was given in [114].

Given a solution to Nahm's equations, one can explicitly construct the corresponding solution to the monopole equation. The procedure is analogous to the construction of instantons in 1.4.2, although its a little harder in practice as its not entirely algebraic. We now explain how to do this. The first step is to build a Dirac-like operator from the solution to (11.47). In the segment $\phi_a \leq x^6 \leq \phi_{a+1}$, we construct the Dirac operator

$$\Delta = \frac{d}{dx^6} - iA_6 - i(X^i + r^i)\sigma^i \tag{11.51}$$

where we've reintroduced the spatial coordinates r^i into the game. We then look for normalizable zero modes U which are solutions to the equation

$$\Delta U = 0 \tag{11.52}$$

One can show that there are N such solutions, and so we consider U as a $2n_a \times N$-dimensional matrix. Note that the dimension of U jumps as we move from one interval to the next. We want to appropriately normalize U, and to do so choose to integrate over all intervals, so that

$$\int_{\phi_1}^{\phi_N} dx^6 \; U^\dagger U = \mathbf{1}_N \tag{11.53}$$

Once we've figured out the expression for U, a Higgs field ϕ and a gauge field A_i which satisfy the monopole equation are given by,

$$\phi = \int_{\phi_1}^{\phi_N} dx^6 \; x^6 \, U^\dagger U \qquad\qquad A_i = \int_{\phi_1}^{\phi_N} dx^6 \; U^\dagger \partial_6 U \tag{11.54}$$

The similarity between this procedure and that described in section 1.4.2 for instantons should be apparent.

In fact, there's a slight complication that I've brushed under the rug. The above construction only really holds when $n_a \neq n_{a+1}$. If we're in a situation where $n_a = n_{a+1}$ for some a, then we have to take the hypermultiplets ω_a into account, since their value affects the monopole solution. This isn't too hard — it just amounts to adding some extra discrete pieces to the Dirac operator Δ. Details can be found in [108].

A string theory derivation of the construction part of the Nahm construction was recently given in [115].

11.5.1An Example: The Single $SU(2)$ Monopole Revisited

It's very cute to see the single $n = 1$ solution (11.19) for the $SU(2)$ monopole drop out of this construction. This is especially true since the Nahm data is trivial in this case: $X^i = A_6 = 0$!

To see how this arises, we look for solutions to

$$\Delta U = \left(\frac{d}{dx^6} - r^i \sigma^i \right) U = 0 \tag{11.55}$$

where $U = U(x^6)$ is a 2×2 matrix. This is trivially solved by

$$U = \sqrt{\frac{r}{\sinh(2vr)}} \left(\cosh(rx^6) \, \mathbf{1}_2 + \sinh(rx^6) \, \hat{r}^i \sigma^i \right) \tag{11.56}$$

which has been designed to satisfy the normalizability condition $\int_{-v}^{+v} U^\dagger U \; dx^6 = \mathbf{1}_2$. Armed with this, we can easily reproduce the monopole solution (11.19). For example, the Higgs field is given by

$$\phi = \int_{-v}^{+v} dx^6 \; x^6 U^\dagger U = \frac{\hat{r}^i \sigma^i}{r} \left(vr \coth(vr) - 1 \right) \tag{11.57}$$

as promised. And the gauge field A_i drops out just as easily. See — told you it was cute! Monopole solutions with charge of the type $(1, 1, ..., 1)$ were constructed using this method in [116].

11.6 What Became of Instantons

In the last lecture we saw that pure Yang-Mills theory contains instanton solutions. Now we've added a scalar field, where have they gone?! The key point to note is that the theory was conformal before ϕ gained its vev. As we saw in Lecture 1, this led to a collective coordinate ρ, the scale size of the instanton. Now with $\langle\phi\rangle \neq 0$ we have introduced a mass scale into the game and no longer expect ρ to correspond to an exact collective coordinate. This turns out to be true: in the presence of a non-zero vev $\langle\phi\rangle$, the instanton minimizes its action by shrinking to zero size $\rho \to 0$. Although, strictly speaking, no instanton exists in the theory with $\langle\phi\rangle \neq 0$, they still play a crucial role. For example, the famed Seiberg-Witten solution can be thought of as summing these small instanton corrections.

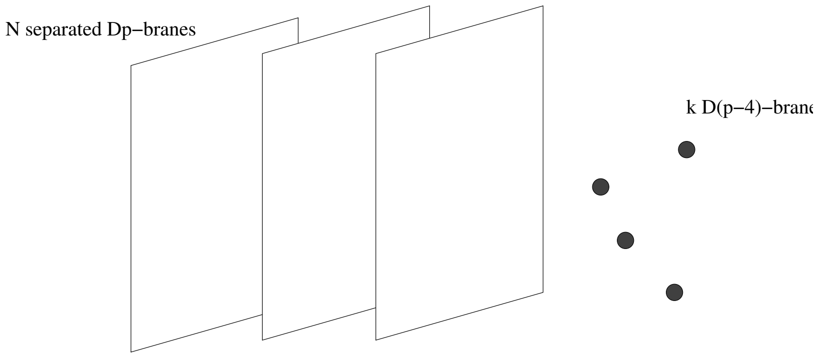


Figure 11.3: Separating the Dp-branes gives rise to a mass for the hypermultiplets

How can we see this behavior from the perspective of the worldvolume theory? We can return to the D-brane set-up, now with the Dp-branes separated in one direction, say x_6, to mimic the vev $\langle\phi\rangle$. Each Dp-D$(p-4)$ string is now stretched by a different amount, reflecting the fact that each hypermultiplet has a different mass. The potential on the worldvolume theory of the D-instantons is now

$$\begin{aligned} V &= \frac{1}{g^2}\sum_{m,n=5}^{10}[X_m,X_n]^2 + \sum_{m,\mu}[X_m,X_\mu]^2 + \sum_{a=1}^{N}\psi^{a\dagger}(X_m-\phi^m{}_a)^2\psi_a + \tilde{\psi}^a(X_m-\phi^m{}_a)^2\tilde{\psi}^\dagger_a \\ &\quad + g^2\,\mathrm{Tr}\,(\sum_{a=1}^{N}\psi_a\psi^{a\dagger} - \tilde{\psi}^\dagger_a\tilde{\psi}^a + [Z,Z^\dagger]+[W,W^\dagger])^2 + g^2\,\mathrm{Tr}\,|\sum_{a=1}^{N}\psi_a\tilde{\psi}^a + [Z,W]|^2 \end{aligned}$$

We've actually introduced more new parameters here than we need, since the D3-branes can be separated in 6 different dimensions, so we have the corresponding positions $\phi^m{}_a$, $m = 4, ..., 9$ and $a = 1, ...N$. Since we have been dealing with just a single scalar field ϕ in this section, we will set $\phi^m_i = 0$ except for $m = 6$ (I know...why 6?!). The parameters $\phi^6{}_a = \phi_a$ are the components of the vev (11.4).

We can now re-examine the vacuum condition for the Higgs branch. If we wish to try to turn on ψ and $\tilde{\psi}$, we must first set $X_m = \phi_a$, for some a. Then the all ψ_b and $\tilde{\psi}^b$ must vanish except for $b = a$. But, taking the trace of the D- and F-term conditions tells us that even ψ_a and $\tilde{\psi}_a$ vanish. We have lost our Higgs branch completely. The interpretation is that the instantons have shrunk to zero size. Note that in the case of non-commutativity, the instantons don't vanish but are pushed to the $U(1)$ instantons with, schematically, $|\psi|^2 \sim \zeta$.

Although the instantons shrink to zero size, there's still important information to be gleaned from the potential above. One can continue to think of the instanton moduli space $\mathcal{I}_{k,N} \cong \mathcal{M}_{\text{Higgs}}$ as before, but now with a potential over it. This potential arises after integrating out the X_m and it is not hard to show that it is of a very specific form: it is the length-squared of a triholomorphic Killing vector on $\mathcal{I}_{k,N}$ associated with the $SU(N)$ isometry.

This potential on $\mathcal{I}_{k,N}$ can be derived directly within field theory without recourse to D-branes or the ADHM construction [117]. This is the route we follow here. The question we want to ask is: given an instanton solution, how does the presence of the ϕ vev affect its action? This gives the potential on the instanton moduli space which is simply

$$V = \int d^4x \ \text{Tr}\,(\mathcal{D}_\mu \phi)^2 \tag{11.58}$$

where $\mathcal{D}_\mu$ is evaluated on the background instanton solution. We are allowed to vary ϕ so it minimizes the potential so that, for each solution to the instanton equations, we want to find ϕ such that

$$\mathcal{D}^2 \phi = 0 \tag{11.59}$$

with the boundary condition that $\phi \to \langle \phi \rangle$. But we've seen an equation of this form, evaluated on the instanton background, before. When we were discussing the instanton zero modes in section 1.2, we saw that the zero modes arising from the overall $SU(N)$ gauge orientation were of the form $\delta A_\mu = \mathcal{D}_\mu \Lambda$, where Λ tends to a constant at infinity and satisfies the gauge fixing condition $\mathcal{D}_\mu \delta A_\mu = 0$. This means that we can re-write the potential in terms of the overlap of zero modes

$$V = \int d^4x \ \text{Tr}\ \delta A_\mu \delta A_\mu \tag{11.60}$$

for the particular zero mode $\delta A_\mu = \mathcal{D}_\mu \phi$ associated to the gauge orientation of the instanton. We can give a nicer geometrical interpretation to this. Consider the action of the Cartan subalgebra $\vec{H}$ on $\mathcal{I}_{k,N}$ and denote the corresponding Killing vector as $\vec{k} = \vec{k}^\alpha \partial_\alpha$. Then, since ϕ generates the transformation $\vec{\phi} \cdot \vec{H}$, we can express our zero mode in terms of the basis $\delta A_\mu = (\vec{\phi} \cdot \vec{k}^\alpha)\, \delta_\alpha A_\mu$. Putting this in our potential and performing the integral over the zero modes, we have the final expression

$$V = g_{\alpha\beta}\, (\vec{\phi} \cdot \vec{k}^\alpha)\, (\vec{\phi} \cdot \vec{k}^\beta) \tag{11.61}$$

The potential vanishes at the fixed points of the $U(1)^{N-1}$ action. This is the small instanton singularity (or related points on the blown-up cycles in the resolved instanton moduli space). Potentials of the form (11.61) were first discussed by Alvarez-Gaume and Freedman who showed that, for tri-holomorphic Killing vectors k, they are the unique form allowed in a sigma-model preserving eight supercharges [118].

The concept of a potential on the instanton moduli space $\mathcal{I}_{k,N}$ is the modern way of viewing what used to known as the "constrained instanton", that is an approximate instanton-like solution to the theory with $\langle \phi \rangle \neq 0$ [119]. These potentials play an important role in Nekrasov's first-principles computation of the Seiberg-Witten prepotential [46]. Another application occurs in the five-dimensional theory, where instantons are particles. Here the motion on the moduli space may avoid the fate of falling to the zeroes of (11.61) by spinning around the potential like a motorcyclist on the wall of death. These solutions of the low-energy dynamics are dyonic instantons which carry electric charge in five dimensions [117, 120, 121].

11.7 Applications

Time now for the interesting applications, examining the role that monopoles play in the quantum dynamics of supersymmetric gauge theories in various dimensions. We'll look at monopoles in 3, 4, 5 and 6 dimensions in turn.

11.7.1 Monopoles in Three Dimensions

In $d = 2+1$ dimensions, monopoles are finite action solutions to the Euclidean equations of motion and the role they play is the same as that of instantons in $d = 3+1$ dimensions: in a semi-classical evaluation of the path-integral, one must sum over these monopole configurations. In 1975, Polyakov famously showed how a gas of these monopoles leads to linear confinement in non-supersymmetric Georgi-Glashow model [122] (that is, an $SU(2)$ gauge theory broken to $U(1)$ by an adjoint scalar field).

In supersymmetric theories, monopoles give rise to somewhat different physics. The key point is that they now have fermionic zero modes, ensuring that they can only contribute to correlation functions with a suitable number of fermionic insertions to soak up the integrals over the Grassmannian collective coordinates. In $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories[5] in $d = 2+1$ dimensions, instantons generate superpotentials, lifting moduli spaces of vacua [123]. In $\mathcal{N} = 8$ theories, instantons contribute to particular 8 fermi correlation functions which have a beautiful interpretation in terms of membrane scattering in M-theory [124, 125]. In this section, I'd like to describe one of the nicest applications of monopoles in three dimensions which occurs in theories with $\mathcal{N} = 4$ supersymmetry, or 8 supercharges.

We'll consider $\mathcal{N} = 4$ $SU(2)$ super Yang-Mills. The superpartners of the gauge field include 3 adjoint scalar fields, ϕ^α, $\alpha = 1,2,3$ and 2 adjoint Dirac fermions. When the scalars gain an expectation value $\langle\phi^\alpha\rangle \neq 0$, the gauge group is broken $SU(2) \to U(1)$ and the surviving, massless, bosonic fields are 3 scalars and a photon. However, in $d = 2+1$ dimensions, the photon has only a single polarization and can be exchanged in favor of another scalar σ. We achieve this by a duality transformation:

$$F_{ij} = \frac{e^2}{2\pi}\epsilon_{ijk}\partial^k \sigma \tag{11.62}$$

[5] A (foot)note on nomenclature. In any dimension, the number of supersymmetries $\mathcal{N}$ counts the number of supersymmetry generators in units of the minimal spinor. In $d = 2+1$ the minimal Majorana spinor has 2 real components. This is in contrast to $d = 3+1$ dimensions where the minimal Majorana (or equivalently Weyl) spinor has 4 real components. This leads to the unfortunate fact that $\mathcal{N} = 1$ in $d = 3+1$ is equivalent to $\mathcal{N} = 2$ in $d = 2+1$. It's annoying. The invariant way to count is in terms of supercharges. Four supercharges means $\mathcal{N} = 1$ in four dimensions or $\mathcal{N} = 2$ in three dimensions.

where we have chosen normalization so that the scalar σ is periodic: $\sigma = \sigma + 2\pi$. Since supersymmetry protects these four scalars against becoming massive, the most general low-energy effective action we can write down is the sigma-model

$$L_{\mathrm{low-energy}} = \frac{1}{2e^2}\, g_{\alpha\beta}\, \partial_i \phi^\alpha\, \partial^i \phi^\beta \tag{11.63}$$

where $\phi^\alpha = (\phi^1, \phi^2, \phi^3, \sigma)$. Remarkably, as shown by Seiberg and Witten, the metric $g_{\alpha\beta}$ can be determined uniquely [90]. It turns out to be an old friend: it is the Atiyah-Hitchin metric (11.30)! The dictionary is $\phi^i = e^2 r^i$ and $\sigma = \psi$. Comparing with the functions a, b and c listed in (11.34), the leading constant term comes from tree level in our 3d gauge theory, and the $1/r$ terms arise from a one-loop correction. Most interesting is the e^{-r} term in (11.34). This comes from a semi-classical monopole computation in $d = 2+1$ which can be computed exactly [128]. So we find monopoles arising in two very different ways: firstly as an instanton-like configuration in the 3d theory, and secondly in an auxiliary role as the description of the low-energy dynamics. The underlying reason for this was explained by Hanany and Witten [91], and we shall see a related perspective in section 2.7.4.

So the low-energy dynamics of $\mathcal{N} = 4$ $SU(2)$ gauge theory is dictated by the two monopole moduli space. It can also be shown that the low-energy dynamics of the $\mathcal{N} = 4$ $SU(N)$ gauge theory in $d = 2+1$ is governed by a sigma-model on the moduli space of N magnetic monopoles in an $SU(2)$ gauge group [126]. There are 3d quiver gauge theories related to monopoles in higher rank, simply laced (i.e. ADE) gauge groups [91, 127] but, to my knowledge, there is no such correspondence for monopoles in non-simply laced groups.

11.7.2 Monopoles and Duality

Perhaps the most important application of monopoles is the role they play in uncovering the web of dualities relating various theories. Most famous is the S-duality of $\mathcal{N} = 4$ super Yang-Mills in four dimensions. The idea is that we can re write the gauge theory treating magnetic monopoles as elementary particles rather than solitons [129]. The following is a lightening review of this large subject. Many more details can be found in [75].

The conjecture of S-duality states that we may re-express the theory, treating monopoles as the fundamental objects, at the price of inverting the coupling $e \to 4\pi/e$. Since this is a strong-weak coupling duality, we need to have some control over the strong coupling behavior of the theory to test the conjecture. The window on this regime is provided by the BPS states [23], whose mass is not renormalized in the maximally supersymmetric $\mathcal{N} = 4$ theory which, among other reasons, makes it a likely place to look for S-duality [130]. In fact, this theory exhibits a more general $SL(2, \mathbf{Z})$ group of duality transformations which acts on the complexified coupling $\tau = \theta/2\pi + 4\pi i/e^2$ by

$$\tau \longrightarrow \frac{a\tau + b}{c\tau + d} \qquad \text{with } a, b, c, d \in \mathbf{Z} \ \text{ and } ad - bc = 1 \tag{11.64}$$

A transformation of this type mixes up what we mean by electric and magnetic charges. Let's work in the $SU(2)$ gauge theory for simplicity so that electric and magnetic charges in the unbroken $U(1)$ are each specified by an integer (n_e, n_m). Then under the $SL(2, \mathbf{Z})$ transformation,

$$\begin{pmatrix} n_e \\ n_m \end{pmatrix} \longrightarrow \begin{pmatrix} a & -b \\ c & -d \end{pmatrix} \begin{pmatrix} n_e \\ n_m \end{pmatrix} \tag{11.65}$$

The conjecture of S-duality has an important prediction that can be tested semi-classically: the spectrum must form multiplets under the $SL(2,Z)$ transformation above. In particular, if S-duality holds, the existence of the W-boson state $(n_e, n_m) = (1,0)$ implies the existence of a slew of further states with quantum numbers $(n_e, n_m) = (a, c)$ where a and c are relatively prime. The states with magnetic charge $n_m = c = 1$ are the dyons that we described in Section 2.3 and can be shown to exist in the quantum spectrum. But we have to work much harder to find the states with magnetic charge $n_m = c > 1$. To do so we must examine the low-energy dynamics of n_m monopoles, described by supersymmetric quantum mechanics on the monopole moduli space. Bound states saturating the Bogomoln'yi bound correspond to ground states of the quantum mechanics. But, as we described in section 1.5.2, this questions translates into the more geometrical search for normalizable harmonic forms on the monopole moduli space.

In the $n_m = 2$ monopole sector, the bound states were explicitly demonstrated to exist by Sen [131]. S-duality predicts the existence of a tower of dyon states with charges $(n_e, 2)$ for all n_e odd which translates into the requirement that there is a unique harmonic form ω on the Atiyah-Hitchin manifold. The electric charge still comes from motion in the $\mathbf{S}^1$ factor of the monopole moduli space (11.29), but the need for only odd charges n_e to exist requires that the form ω is odd under the $\mathbf{Z}_2$ action (11.32). Uniqueness requires that ω is either self-dual or anti-self-dual. In fact, it is the latter. The ansatz,

$$\omega = F(r)(d\sigma_1 - \frac{fa}{bc} dr \wedge \sigma_1) \tag{11.66}$$

is harmonic provided that $F(r)$ satisfies

$$\frac{dF}{dr} = -\frac{fa}{bc} F \tag{11.67}$$

One can show that this form is normalizable, well behaved at the center of the moduli space and, moreover, unique. Historically, the existence of this form was one the compelling pieces of evidence in favor of S-duality, leading ultimately to an understanding of strong coupling behavior of many supersymmetric field theories and string theories.

The discussion above is for $\mathcal{N} = 4$ theories. In $\mathcal{N} = 2$ theories, the bound state described above does not exist (a study of the $\mathcal{N} = (0,4)$ supersymmetric quantum mechanics reveals that the Hilbert space is identified with holomorphic forms and ω is not holomorphic). Nevertheless, there exists a somewhat more subtle duality between electrically and magnetically charged states, captured by the Seiberg-Witten solution [132]. Once again, there is a semi-classical test of these ideas along the lines described above [133]. There is also an interesting story in this system regarding quantum corrections to the monopole mass [134].

11.7.3 Monopole Strings and the $(2,0)$ Theory

We've seen that the moduli space of a single monopole is $\mathcal{M} \cong \mathbf{R}^3 \times \mathbf{S}^1$ with metric,

$$ds^2 = M_{\text{mono}} \left(dX^i dX^i + \frac{1}{v^2} d\chi^2 \right) \tag{11.68}$$

where $\chi \in [0, 2\pi)$. It looks as if, at low-energies, the monopole is moving in a higher dimensional space. Is there any situation where we can actually interpret this motion in the $\mathbf{S}^1$ as motion in an extra, hidden dimension of space?

One problem with interpreting internal degrees of freedom, such as χ, in terms of an extra dimension is that there is no guarantee that motion in these directions will be Lorentz covariant. For example, Einstein's speed limit tells us that the motion of the monopole in $\mathbf{R}^3$ is bounded by the speed of light: i.e. $\dot{X} \leq 1$. But is there a similar bound on $\dot{\chi}$? This is an question which goes beyond the moduli space approximation, which keeps only the lowest velocities, but is easily answered since we know the exact spectrum of the dyons. The energy of a relativistically moving dyon is $E^2 = M_{\rm dyon}^2 + p_i p_i$, where p_i is the momentum conjugate to the center of mass X_i. Using the mass formula (11.41), we have the full Hamiltonian

$$H_{\rm dyon} = \sqrt{M_{\rm mono}^2 + v^2 p_\chi^2 + p_i p_i} \tag{11.69}$$

where $p_\chi = 2q$ is the momentum conjugate to χ. This gives rise to the Lagrangian,

$$L_{\rm dyon} = -M_{\rm mono}\sqrt{1 - \dot{\chi}^2/v^2 - \dot{X}^i\dot{X}^i} \tag{11.70}$$

which, at second order in velocities, agrees with the motion on the moduli space (11.68). So, surprisingly, the internal direction χ does appear in a Lorentz covariant manner in this Lagrangian and is therefore a candidate for an extra, hidden, dimension.

However, looking more closely, our hopes are dashed. From (11.70) (or, indeed, from (11.68)), we see that the radius of the extra dimension is proportional to $1/v$. But the width of the monopole core is also $1/v$. This makes it a little hard to convincingly argue that the monopole can happily move in this putative extra dimension since there's no way the dimension can be parametrically larger than the monopole itself. It appears that χ is stuck in the auxiliary role of endowing monopoles with electric charge, rather than being promoted to a physical dimension of space.

Things change somewhat if we consider the monopole as a string-like object in a $d = 4 + 1$ dimensional gauge theory. Now the low-energy effective action for a single monopole is simply the action (11.70) lifted to the two dimensional worldsheet of the string, yielding the familiar Nambu-Goto action

$$S_{\rm string} = -T_{\rm mono}\int d^2y\ \sqrt{1 - (\partial\chi)^2/v^2 - (\partial X^i)^2} \tag{11.71}$$

where ∂ denotes derivatives with respect to both worldsheet coordinates, σ and τ. We've rewritten $M_{\rm mono} = T_{\rm mono} = 4\pi v/e^2$ to stress the fact that it is a tension, with dimension 2 (recall that e^2 has dimension -1 in $d = 4 + 1$). As it stands, we're in no better shape. The size of the circle is still $1/v$, the same as the width of the monopole string. However, now we have a two dimensional worldsheet we may apply T-duality. This means exchanging momentum modes around $\mathbf{S}^1$ for winding modes so that

$$\partial\chi = {}^\star\partial\tilde{\chi} \tag{11.72}$$

We need to be careful with the normalization. A careful study reveals that,

$$\frac{1}{4\pi}\int d^2y\, R^2\,(\partial\chi)^2 \ \rightarrow\ \frac{1}{4\pi}\int d^2y\,\frac{1}{R^2}\,(\partial\tilde{\chi})^2 \tag{11.73}$$

where, up to that important factor of 4π, R is the radius of the circle measured in string units. Comparing with our normalization, we have $R^2 = 8\pi^2/ve^2$, and the dual Lagrangian is

$$S_{\rm string} = -T_{\rm mono}\int d^2y\ \sqrt{1-(e^2/8\pi^2)^2(\partial\tilde{\chi})^2-(\partial X^i)^2} \tag{11.74}$$

We see that the physical radius of this dual circle is now $e^2/8\pi^2$. This can be arbitrarily large and, in particular, much larger than the width of the monopole string. It's a prime candidate to be interpreted as a real, honest, extra dimension. In fact, in the maximally supersymmetric Yang-Mills theory in five dimensions, it is known that this extra dimension is real. It is precisely the hidden circle that takes us up to the six-dimensional $(2,0)$ theory that we discussed in section 1.5.2. The monopole even tells us that the instantons must be the Kaluza-Klein modes since the inverse radius of the dual circle is exactly $M_{\rm inst}$. Once again, we see that solitons allow us to probe important features of the quantum physics where myopic perturbation theory fails. Note that the derivation above does rely on supersymmetry since, for the Hamiltonian (11.69) to be exact, we need the masses of the dyons to saturate the Bogomoln'yi bound (11.41).

11.7.4 D-Branes in Little String Theory

Little string theories are strongly interacting string theories without gravity in $d = 5+1$ dimensions. For a review see [136]. The maximally supersymmetric variety can be thought of as the decoupled theory living on NS5-branes. They come in two flavors: the type iia little string theory is a $(2,0)$ supersymmetric theory which reduces at low-energies to the conformal field theory discussed in sections 1.5.2 and 2.7.3. In contrast, the type iib little string has $(1,1)$ non-chiral supersymmetry and reduces at low-energies to $d = 5+1$ Yang-Mills theory. When this theory sits on the Coulomb branch it admits monopole solutions which, in six dimensions, are membranes. Let's discuss some of the properties of these monopoles in the $SU(2)$ theory.

The low-energy dynamics of a single monopole is the $d = 2+1$ dimensional sigma model with target space $\mathbf{R}^3\times\mathbf{S}^1$ and metric (11.68). But, as we already discussed, in $d = 2+1$ we can exchange the periodic scalar χ for a $U(1)$ gauge field living on the monopole. Taking care of the normalization, we find

$$F_{mn} = \frac{8\pi^2}{e^2}\epsilon_{mnp}\,\partial^p\chi \tag{11.75}$$

with $m,n = 0,1,2$ denoting the worldvolume dimensions of the monopole 2-brane. The low-energy dynamics of this brane can therefore be written as

$$\begin{aligned} S_{\rm brane} &= \int d^3x\ \frac{1}{2}T_{\rm mono}\left((\partial_m X^i)^2+\frac{1}{v^2}(\partial_m\chi)^2\right) &(11.76)\\ &= \int d^3x\ \frac{1}{2g^2}\left((\partial_m\varphi^i)^2+\frac{1}{2}F_{mn}F^{mn}\right) &(11.77)\end{aligned}$$

where $g^2 = 4\pi^2T_{\rm mono}/v^2$ is fixed by the duality (11.75) and insisting that the scalar has canonical kinetic term dictates $\varphi^i = (8\pi^2/e^2)\,X^i = T_{\rm inst}X^i$. This normalization will prove important. Including the fermions, we therefore find the low-energy dynamics of a monopole membrane to be free $U(1)$ gauge theory with 8 supercharges (called $\mathcal{N} = 4$ in three dimensions), containing a photon and three real scalars.

Six dimensional gauge theories also contain instanton strings. These are the "little strings" of little string theory. We will now show that strings can end on the monopole 2-brane. This is simplest to see from the worldvolume perspective in terms of the original variable χ. Defining the complex coordinate on the membrane worldvolume $z = x^4 + ix^5$, we have the BPS "BIon" spike [137, 138] solution of the theory (11.76)

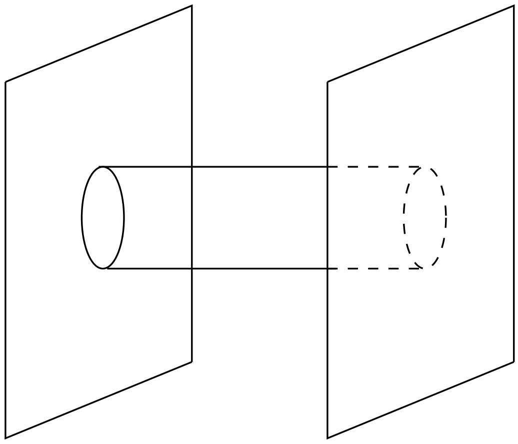
$$X^1 + \frac{i}{v}\chi = \frac{1}{v}\log(vz) \tag{11.78}$$

Plotting the value of the transverse position X^1 as a function of $|z|$, we see that this solution indeed has the profile of a string ending on the monopole 2-brane. Since χ winds once as we circumvent the origin $z = 0$, after the duality transformation we see that this string sources a radial electric field. In other words, the end of the string is charged under the $U(1)$ gauge field on the brane (11.75). We have found a D-brane in the six-dimensional little string theory.

Having found the string solution from the perspective of the monopole worldvolume theory, we can ask whether we can find a solution in the full $d = 5 + 1$ dimensional theory. In fact, as far as I know, no one has done this. But it is possible to write down the first order equations that this solution must solve [139]. They are the dimensional reduction of equations found in [140] and read

$$\begin{gathered} F_{23} + F_{45} = \mathcal{D}_1\phi \qquad F_{35} = -F_{42} \qquad F_{34} = -F_{25} \\ F_{31} = \mathcal{D}_2\phi \qquad F_{12} = \mathcal{D}_3\phi \qquad F_{51} = \mathcal{D}_4\phi \qquad F_{14} = \mathcal{D}_5\phi \end{gathered} \tag{11.79}$$

Notice that among the solutions to these equations are instanton strings stretched in the x^1 directions, and monopole 2-branes with spatial worldvolume (x^4, x^5). It would be interesting to find an explicit solution describing the instanton string ending on the monopole brane.



We find ourselves in a rather familiar situation. We have string-like objects which can terminate on D-brane objects, where their end is electrically charged. Yet all this is within the context of a gauge theory, with no reference to string theory or gravity. Let's remind ourselves about some further properties of D-branes in string theory to see if the analogy can be pushed further. For example, there are two methods to understand the dynamics of D-branes in string theory, using either closed or open strings. The first method — the closed string description — uses the supergravity solution for D-branes to compute their scattering. In contrast, in the second method — the open string description — the back-reaction on the bulk is ignored. Instead the strings stretched between two branes are integrated in, giving rise to new, light fields of the worldvolume theory as the branes approach. In flat space, this enhances $U(1)^n$ worldvolume gauge symmetry to $U(n)$ [141]. The quantum effects from these non-abelian fields capture the scattering of the D-branes. The equivalence of these two methods is assured by open-closed string duality, where the diagram drawn in figure 10 can be interpreted as tree-level closed string or one-loop open string exchange. Generically the two methods have different regimes of validity.

Is there an analogous treatment for our monopole D-branes? The analogy of the supergravity description is simply the Manton moduli space approximation described in section 2.2. What about the open string description? Can we integrate in the light states arising from instanton strings stretched between two D-branes? They have charge $(+1,-1)$ under the two branes and, by the normalization described above, mass $T_{\rm inst}|X^i_1 - X^i_2| = |\varphi^i_1 - \varphi^i_2|$. Let's make the simplest assumption that quantization of these strings gives rise to W-bosons, enhancing the worldvolume symmetry of n branes to $U(n)$. Do the quantum effects of these open strings mimic the classical scattering of monopoles? Of course they do! This is precisely the calculation we described in section 2.7.1: the Coulomb branch of the $U(n)$ $\mathcal{N}=4$ super Yang-Mills in $d=2+1$ dimensions is the n monopole moduli space.

The above discussion is not really new. It is nothing more than the "Hanany-Witten" story [91], with attention focussed on the NS5-brane worldvolume rather than the usual 10-dimensional perspective. Nevertheless, it's interesting that one can formulate the story without reference to 10-dimensional string theory. In particular, if we interpret our results in terms of open-closed string duality summarized in figure 10, it strongly suggests that the bulk six-dimensional Yang-Mills fields can be thought of as quantized loops of instanton strings.

To finish, let me confess that, as one might expect, the closed and open string descriptions have different regimes of validity. The bulk calculation is valid in the full quantum theory only if we can ignore higher derivative corrections to the six-dimensional action. These scale as $e^{2n}\partial^{2n}$. Since the size of the monopole is $\partial \sim v^{-1}$, we have the requirement $v^2e^2 \ll 1$ for the "closed string" description to be valid. What about the open string description? We integrate in an object of energy $E = T_{\rm inst}\Delta X$, where ΔX is the separation between branes. We do not want to include higher excitations of the string which scale as v. So we have $E \ll v$. At the same time, we want $\Delta X > 1/v$, the width of the branes, in order to make sense of the discussion. These two requirements tell us that $v^2e^2 \gg 1$. The reason that the two calculations yield the same result, despite their different regimes of validity, is due to a non-renormalization theorem, which essentially boils down the restrictions imposed by the hyperKähler nature of the metric.

12 Vortices

In this lecture, we're going to discuss vortices. The motivation for studying vortices should be obvious: they are one of the most ubiquitous objects in physics. On table-tops, vortices appear as magnetic flux tubes in superconductors and fractionally charged quasi-excitations in quantum Hall fluids. In the sky, vortices in the guise of cosmic strings have been one one of the most enduring themes in cosmology research. With new gravitational wave detectors coming on line, there is hope that we may be able to see the distinctive signatures of these strings as the twist and whip. Finally, and more formally, vortices play a crucial role in determining the phases of low-dimensional quantum systems: from the phase-slip of superconducting wires, to the physics of strings propagating on Calabi-Yau manifolds, the vortex is key.

As we shall see in detail below, in four dimensional theories vortices are string like objects, carrying magnetic flux threaded through their core. They are the semi-classical cousins of the more elusive QCD flux tubes. In what follows we will primarily be interested in the dynamics of infinitely long, parallel vortex strings and the long-wavelength modes they support. There are a number of reviews on the dynamics of vortices in four dimensions, mostly in the context of cosmic strings [142, 143, 144].

12.1 The Basics

In order for our theory to support vortices, we must add a further field to our Lagrangian. In fact we must make two deformations

- We increase the gauge group from $SU(N)$ to $U(N)$. We could have done this before now, but as we have considered only fields in the adjoint representation the central $U(1)$ would have simply decoupled.
- We add matter in the fundamental representation of $U(N)$. We'll add N_f scalar fields q_i, $i = 1..., N_f$.

The action that we'll work with throughout this lecture is

$$\begin{aligned} S &= \int d^4x \ \mathrm{Tr}\left(\frac{1}{2e^2}F^{\mu\nu}F_{\mu\nu} + \frac{1}{e^2}(\mathcal{D}_\mu\phi)^2\right) + \sum_{i=1}^{N_f}|\mathcal{D}_\mu q_i|^2 \\ & \qquad - \sum_{i=1}^{N_f} q_i^\dagger\phi^2 q_i - \frac{e^2}{4}\,\mathrm{Tr}\,(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\,1_N)^2 \end{aligned} \tag{12.1}$$

The potential is of the type admitting a completion to $\mathcal{N}=1$ or $\mathcal{N}=2$ supersymmetry. In this context, the final term is called the D-term. Note that everything in the bracket of the D-term is an $N \times N$ matrix. Note also that the couplings in front of the potential are not arbitrary: they have been tuned to critical values.

We've included a new parameter, v^2, in the potential. Obviously this will induce a vev for q. In the context of supersymmetric gauge theories, this parameter is known as a Fayet-Iliopoulos term.

We are interested in ground states of the theory with vanishing potential. For $N_f < N$, one can't set the D-term to zero since the first term is, at most, rank N_f, while the v^2 term is rank N. In the context of supersymmetric theories, this leads to spontaneous supersymmetry breaking. In what follows we'll only consider $N_f \geq N$. In fact, for the first half of this section we'll restrict ourselves to the simplest case:

$$N_f = N \tag{12.2}$$

With this choice, we can view q as an $N \times N$ matrix $q^a{}_i$, where a is the color index and i the flavor index. Up to gauge transformations, there is a unique ground state of the theory,

$$\phi = 0 \qquad q^a{}_i = v\delta^a{}_i \tag{12.3}$$

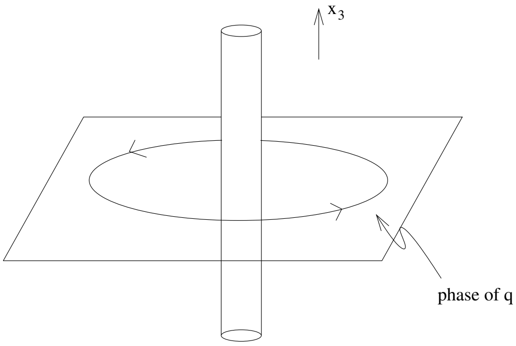
Studying small fluctuations around this vacuum, we find that all gauge fields and scalars are massive, and all have the same mass $M^2 = e^2v^2$. The fact that all masses are equal is a consequence of tuning the coefficients of the potential.

The theory has a $U(N)_G \times SU(N)_F$ gauge and flavor symmetry. On the quark fields q this acts as

$$q \to UqV \qquad U \in U(N)_G, \quad V \in SU(N)_F \tag{12.4}$$

The vacuum expectation value (12.3) is preserved only for transformations of the form $U = V$, meaning that we have the pattern of spontaneous symmetry breaking

$$U(N)_G \times SU(N)_F \to SU(N)_{\mathrm{diag}} \tag{12.5}$$



This is known as the color-flavor-locked phase in the high-density QCD literature [145].

When $N = 1$, our theory is the well-studied abelian Higgs model, which has been known for many years to support vortex strings [146, 147]. These vortex strings also exist in the non-abelian theory and enjoy rather rich properties, as we shall now see. Let's choose the strings to lie in the x^3 direction. To support such objects, the scalar fields q must wind around $\mathbf{S}^1_\infty$ at spatial infinity in the (x^1, x^2) plane, transverse to the string. As we're used to by now, such winding is characterized by the homotopy group, this time

$$\Pi_1\left(U(N) \times SU(N)/SU(N)_{\mathrm{diag}}\right) \cong \mathbf{Z} \tag{12.6}$$

Which means that we can expect vortex strings supported by a single winding number $k \in \mathbf{Z}$. To see that this winding of the scalar is associated with magnetic flux, we use the same trick as for monopoles. Finiteness of the quark kinetic term requires that $\mathcal{D}q \sim 1/r^2$ as $r \to \infty$. But a winding around $\mathbf{S}^1_\infty$ necessarily means that $\partial q \sim 1/r$. To cancel this, we must turn on

$A \to i\partial q\, q^{-1}$ asymptotically. The winding of the scalar at infinity is determined by an integer k, defined by

$$2\pi k = \mathrm{Tr} \oint_{\mathbf{S}^1_\infty} i\partial_\theta q\ q^{-1} = \mathrm{Tr} \oint_{\mathbf{S}^1_\infty} A_\theta = \mathrm{Tr} \int dx^1 dx^2\ B_3 \tag{12.7}$$

This time however, in contrast to the case of magnetic monopoles, there is no long range magnetic flux. Physically this is because the theory has a mass gap, ensuring any excitations die exponentially. The result, as we shall, is that the magnetic flux is confined in the center of the vortex string.

The Lagrangian of equation (12.1) is very special, and far from the only theory admitting vortex solutions. Indeed, the vortex zoo is well populated with different objects, many exhibiting curious properties. Particularly interesting examples include Alice strings [148, 149], and vortices in Chern-Simons theories [150]. In this lecture we shall stick with the vortices arising from (12.1) since, as we shall see, they are closely related to the instantons and monopoles described in the previous lectures. To my knowledge, the properties of non-abelian vortices in this model were studied only quite recently in [151] (a related model, sharing similar properties, appeared at the same time [152]).

12.2 The Vortex Equations

To derive the vortex equations we once again perform the Bogomoln'yi completing the square trick (due, once again, to Bogomoln'yi [14]). We look for static strings in the x^3 direction, so make the ansatz $\partial_0 = \partial_3 = 0$ and $A_0 = A_3 = 0$. We also set $\phi = 0$. In fact ϕ will not play a role for the remainder of this lecture, although it will be resurrected in the following lecture. The tension (energy per unit length) of the string is

$$\begin{aligned} T_{\rm vortex} &= \int dx^1 dx^2\ \mathrm{Tr}\left(\frac{1}{e^2}B_3^2 + \frac{e^2}{4}(\sum_{i=1}^N q_i q_i^\dagger - v^2\, 1_N)^2\right) + \sum_{i=1}^N |\mathcal{D}_1 q_i|^2 + |\mathcal{D}_2 q_i|^2 \\ &= \int dx^1 dx^2\ \frac{1}{e^2}\mathrm{Tr}\left(B_3 \mp \frac{e^2}{2}(\sum_{i=1}^N q_i q_i^\dagger - v^2\, 1_N)\right)^2 + \sum_{i=1}^N |\mathcal{D}_1 q_i \mp i\mathcal{D}_2 q_i|^2 \\ &\quad \mp v^2 \int dx^1 dx^2\ \mathrm{Tr}\, B_3 \end{aligned} \tag{12.8}$$

To get from the first line to the second, we need to use the fact that $[D_1, D_2] = -iB_3$, to cancel the cross terms from the two squares. Using (12.7), we find that the tension of the charge $|k|$ vortex is bounded by

$$T_{\rm vortex} \geq 2\pi v^2\, |k| \tag{12.9}$$

In what follows we focus on vortex solutions with winding $k < 0$. (These are mapped into $k > 0$ vortices by a parity transformation, so there is no loss of generality). The inequality is then saturated for configurations obeying the vortex equations

$$B_3 = \frac{e^2}{2}(\sum_i q_i q_i^\dagger - v^2\, 1_N) \qquad \mathcal{D}_z q_i = 0 \tag{12.10}$$

where we've introduced the complex coordinate $z = x^1 + ix^2$ on the plane transverse to the vortex string, so $\partial_z = \frac{1}{2}(\partial_1 - i\partial_2)$. If we choose $N = 1$, then the Lagrangian (12.1) reduces to the abelian-Higgs model and, until recently, attention mostly focussed on this abelian variety of the equations (12.10). However, as we shall see below, when the vortex equations are non-abelian, so each side of the first equation (12.10) is an $N \times N$ matrix, they have a much more interesting structure.

Unlike monopoles and instantons, no analytic solution to the vortex equations is known. This is true even for a single $k = 1$ vortex in the $U(1)$ theory. There's nothing sinister about this. It's just that differential equations are hard and no one has decided to call the vortex solution a special function and give it a name! However, it's not difficult to plot the solution numerically and the profile of the fields is sketched below. The energy density is localized within a core of the vortex of size $L = 1/ev$, outside of which all fields return exponentially to their vacuum.

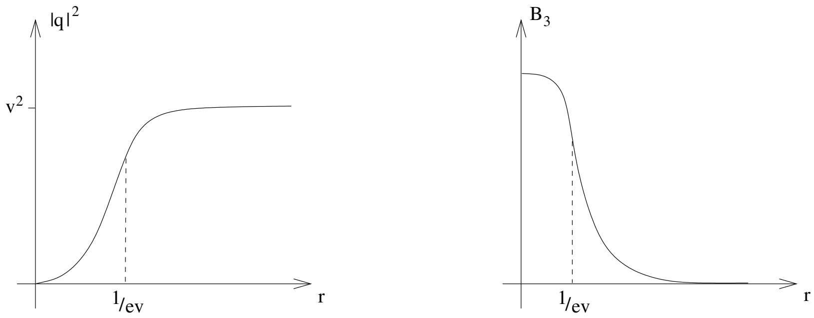


Figure 12.1: A sketch of the vortex profile.

The simplest $k = 1$ vortex in the abelian $N = 1$ theory has just two collective coordinates, corresponding to its position on the z-plane. But what are the collective coordinates of a vortex in $U(N)$. We can use the same idea we saw in the instanton lecture, and embed the abelian vortex — let's denote it $q^\star$ and $A_z^\star$ — in the $N \times N$ matrices of the non-abelian theory. We have

$$A_z = \begin{pmatrix} A_z^\star & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{pmatrix} \qquad q = \begin{pmatrix} q^\star & & & \\ & v & & \\ & & \ddots & \\ & & & v \end{pmatrix} \tag{12.11}$$

where the columns of the q matrix carry the color charge, while the rows carry the flavor charge. We have chosen the embedding above to lie in the upper left-hand corner but this isn't unique. We can rotate into other embeddings by acting with the $SU(N)_{\rm diag}$ symmetry preserved in the vacuum. Dividing by the stabilizer, we find the internal moduli space of the single non-abelian vortex to be

$$SU(N)_{\rm diag}/S[U(N-1) \times U(1)] \cong \mathbb{CP}^{N-1} \tag{12.12}$$

The appearance of $\mathbb{CP}^{N-1}$ as the internal space of the vortex is interesting: it tells us that the low-energy dynamics of a vortex string is the much studied quantum $\mathbb{CP}^{N-1}$ sigma model. We'll see the significance of this in the following lecture. For now, let's look more closely at the moduli of the vortices.

12.3 The Moduli Space

We've seen that a single vortex has $2N$ collective coordinates: 2 translations, and $2(N-1)$ internal modes, dictating the orientation of the vortex in color and flavor space. We denote the moduli space of charge k vortices in the $U(N)$ gauge theory as $\mathcal{V}_{k,N}$. We've learnt above that

$$\mathcal{V}_{1,N} \cong \mathbf{C} \times \mathbb{CP}^{N-1} \tag{12.13}$$

What about higher k? An index theorem [154, 151] tells us that the number of collective coordinates is

$$\dim(\mathcal{V}_{k,N}) = 2kN \tag{12.14}$$

Look familiar? Remember the result for k instantons in $U(N)$ that we found in lecture 1: $\dim(\mathcal{I}_{k,N}) = 4kN$. We'll see more of this similarity between instantons and vortices in the following.

As for previous solitons, the counting (12.14) has a natural interpretation: k parallel vortex strings may be placed at arbitrary position, each carrying $2(N-1)$ independent orientational modes. Thinking physically in terms of forces between vortices, this is a consequence of tuning the coefficient $e^2/4$ in front of the D-term in (12.1) so that the mass of the gauge bosons equals the mass of the q scalars. If this coupling is turned up, the scalar mass increases and so mediates a force with shorter range than the gauge bosons, causing the vortices to repel. (Recall the general rule: spin 0 particles give rise to attractive forces; spin 1 repulsive). This is a type II non-abelian superconductor. If the coupling decreases, the mass of the scalar decreases and the vortices attract. This is a non-abelian type I superconductor. In the following, we keep with the critically coupled case (12.1) for which the first order equations (12.10) yield solutions with vortices at arbitrary position.

12.3.1 The Moduli Space Metric

There is again a natural metric on $\mathcal{V}_{k,N}$ arising from taking the overlap of zero modes. These zero modes must solve the linearized vortex equations together with a suitable background gauge fixing condition. The linearized vortex equations read

$$\mathcal{D}_z \delta A_{\bar{z}} - \mathcal{D}_{\bar{z}} \delta A_z = \frac{ie^2}{4}(\delta q\, q^\dagger + q\, \delta q^\dagger) \quad \text{ and } \quad \mathcal{D}_z \delta q = i\delta A_z q \tag{12.15}$$

where q is to be viewed as an $N \times N$ matrix in these equations. The gauge fixing condition is

$$\mathcal{D}_z \delta A_{\bar{z}} + \mathcal{D}_{\bar{z}} \delta A_z = -\frac{ie^2}{4}(\delta q\, q^\dagger - q\, \delta q^\dagger) \tag{12.16}$$

which combines with the first equation in (12.15) to give

$$\mathcal{D}_{\bar{z}} \delta A_z = -\frac{ie^2}{4} \delta q\, q^\dagger \tag{12.17}$$

Then, from the index theorem, we know that there are $2kN$ zero modes $(\delta_\alpha A_z, \delta_\alpha q)$, $\alpha, \beta = 1, ..., 2kN$ solving these equations, providing a metric on $\mathcal{V}_{k,N}$ defined by

$$g_{\alpha\beta} = \text{Tr} \int dx^1 dx^2\; \frac{1}{e^2} \delta_\alpha A_a \delta_\beta A_{\bar{z}} + \frac{1}{2} \delta_\alpha q \delta_\beta q^\dagger \; + \; \text{h.c.} \tag{12.18}$$

The metric has the following properties [155, 156]

- The metric is Kähler. This follows from similar arguments to those given for hyperKählerity of the instanton moduli space, the complex structure now descending from that on the plane $\mathbf{R}^2$, together with the obvious complex structure on q.
- The metric is smooth. It has no singularities as the vortices approach each other. Strictly speaking this statement has been proven only for abelian vortices. For non-abelian vortices, we shall show this using branes in the following section.
- The metric inherits a $U(1)\times SU(N)$ holomorphic isometry from the rotational and internal symmetry of the Lagrangian.
- The metric is unknown for $k \geq 2$. The leading order, exponentially suppressed, corrections to the flat metric were computed recently [157].

12.3.2 Examples of Vortex Moduli Spaces

12.3.2A Single $U(N)$ Vortex

We've already seen above that the moduli space for a single $k = 1$ vortex in $U(N)$ is

$$\mathcal{V}_{1,N} \cong \mathbf{C} \times \mathbb{CP}^{N-1} \tag{12.19}$$

where the isometry group $SU(N)$ ensures that $\mathbb{CP}^{N-1}$ is endowed with the round, Fubini-Study metric. The only question remaining is the size, or Kähler class, of the $\mathbb{CP}^{N-1}$. This can be computed either from a D-brane construction [151] or, more conventionally, from the overlap of zero modes [158]. We'll see the former in the following section. Here let's sketch the latter. The orientational zero modes of the vortex take the form

$$\delta A_z = \mathcal{D}_z \Omega \qquad\qquad \delta q = i(\Omega q - q\Omega_0) \tag{12.20}$$

where the gauge transformation asymptotes to $\Omega \to \Omega_0$, and Ω_0 is the flavor transformation. The gauge fixing condition requires

$$\mathcal{D}^2\Omega = \frac{e^2}{2}\{\Omega, qq^\dagger\} - 2qq^\dagger\Omega_0 \tag{12.21}$$

By explicitly computing the overlap of these zero modes, it can be shown that the size of the $\mathbb{CP}^{N-1}$ is

$$r = \frac{4\pi}{e^2} \tag{12.22}$$

This important equation will play a crucial role in the correspondence between 2d sigma models and 4d gauge theories that we'll meet in the following lecture.

12.3.2Two $U(1)$ Vortices

The moduli space of two vortices in a $U(1)$ gauge theory is topologically

$$\mathcal{V}_{k=2,N=1} \cong \mathbf{C} \times \mathbf{C}/\mathbf{Z}_2 \tag{12.23}$$

where the $\mathbf{Z}_2$ reflects the fact that the two solitons are indistinguishable. Note that the notation we used above actually describes more than the topology of the manifold because, topologically, $\mathbf{C}^k/\mathbf{Z}_k \cong \mathbf{C}^k$ (as any polynomial will tell you). So when I write $\mathbf{C}/\mathbf{Z}_2$ in (12.23), I mean that asymptotically the space is endowed with the flat metric on $\mathbf{C}/\mathbf{Z}_2$. Of course, this can't be true closer to the origin since we know the vortex moduli space is complete. The cone must be smooth at the tip, as shown in figure 13. The metric on the cone has been computed numerically [159], no analytic form is known. The deviations from the flat, singular, metric on the cone are exponentially suppressed and parameterized by the size of the vortex $L \sim 1/ev$.

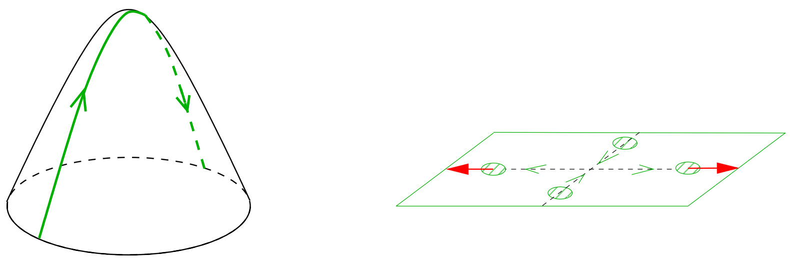


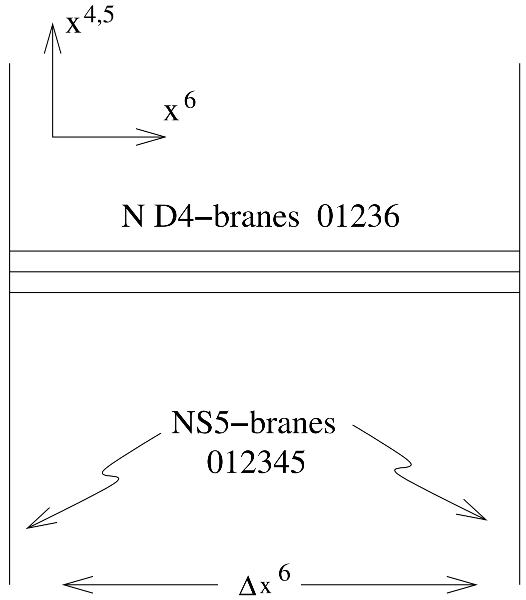
Figure 12.2: Right-angle scattering from the moduli space of two vortices.

Even without the exact form of the metric, we learn something very important about vortices. Consider two vortices colliding head on. This corresponds to the trajectory in moduli space that goes up and over the tip of the cone, as shown in the figure. What does this correspond to in real space? One might think that the vortices collide and rebound. But that's wrong: it would correspond to the trajectory going to the tip of the cone, and returning down the same side. Instead, the trajectory corresponds to vortices scattering at right angles [160]. The key point is that the $\mathbf{Z}_2$ action in (12.23), arising because the vortices are identical, means that the single valued coordinate on the moduli space is z^2 rather than z, the separation between the vortices. The collision sends $z^2 \to -z^2$ or $z \to iz$. This result doesn't depend on the details of the metric on the vortex moduli space, but follows simply from the fact that, near the origin, the space is smooth. Right-angle scattering of this type is characteristic of soliton collisions, occurring also for magnetic monopoles.

For $k \geq 3$ $U(1)$ vortices, the moduli space is topologically and asymptotically $\mathbf{C}^k/\mathbf{Z}_k$. The leading order exponential corrections to the flat metric on this space are known, although the full metric is not [157].

12.4 Brane Construction

For both instantons and monopoles, it was fruitful to examine the solitons from the perspective of D-branes. This allowed us to re-derive the ADHM and Nahm constructions respectively. What about for vortices? Here we present a D-brane constuction of vortices [151] that will reveal interesting information about the moduli space of solutions although, ultimately, won't be as powerful as the ADHM and Nahm constructions described in previous sections.



We use the brane set-ups of Hanany and Witten [91], consisting of D-branes suspended between a pair of NS5-branes. We work in type IIA string theory, and build the $d = 3 + 1$, $U(N)$ gauge theory[6] with $\mathcal{N} = 2$ supersymmetry. The D-brane set-up is shown in figure 14, and consists of N D4-branes with worldvolume 01236, stretched between two NS5-branes, each with worldvolume 012345, and separated in the x^6 direction. The gauge coupling e^2 is determined by the separation between the NS5-branes,

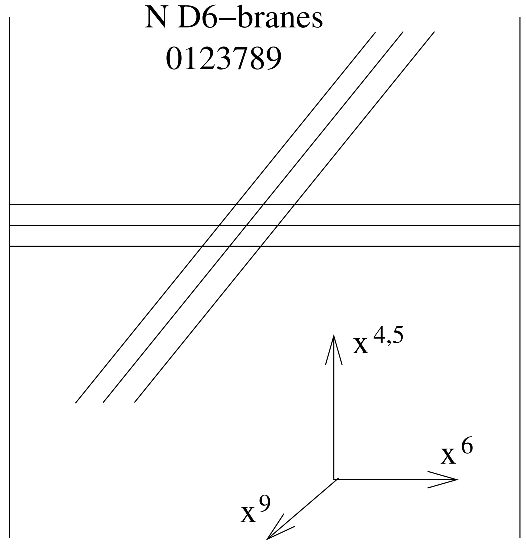
$$\frac{1}{e^2} = \frac{\Delta x^6\, l_s}{2g_s} \tag{12.24}$$

where l_s is the string length, and g_s the string coupling. The D4-branes may slide up and down between the NS5-branes in the x^4 and x^5 direction. This corresponds to turning on a vev for the complex adjoint scalar in the $\mathcal{N} = 2$ vector multiplet. Since we consider only a real adjoint scalar ϕ in our theory, we have

$$\phi_a = \left.\frac{x^4}{l_s^2}\right|_{D4_a} \tag{12.25}$$

and we'll take all D4-branes to lie coincident in the x^5 direction.

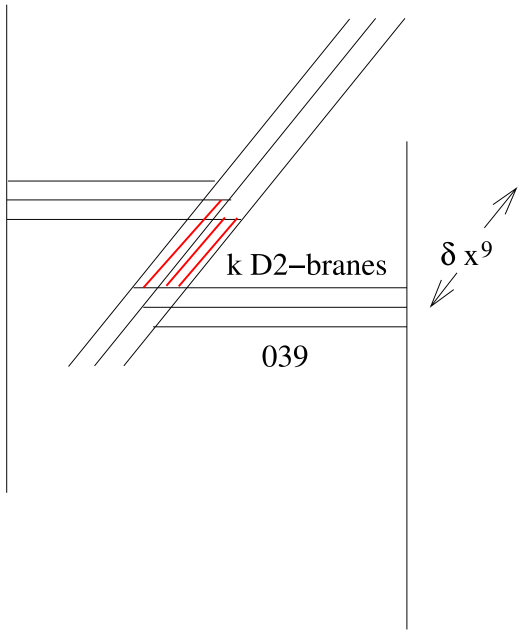
[6]In fact, for four-dimensional theories the overall $U(1)$ decouples in the brane set-up, and we have only $SU(N)$ gauge theory [161]. This doesn't affect our study of the vortex moduli space; if you're bothered by this, simply T-dualize the problem to type IIB where you can study vortices in $d = 2 + 1$ dimensions.



The hypermultiplets arise in the form of N D6-branes with worldvolume 0123789. The positions of the D6-branes in the $x^4 + ix^5$ directions will correspond to complex masses for the hypermultiplets. We shall consider these in the following section, but for now we set all D6-branes to lie at the origin of the x^4 and x^5 plane.

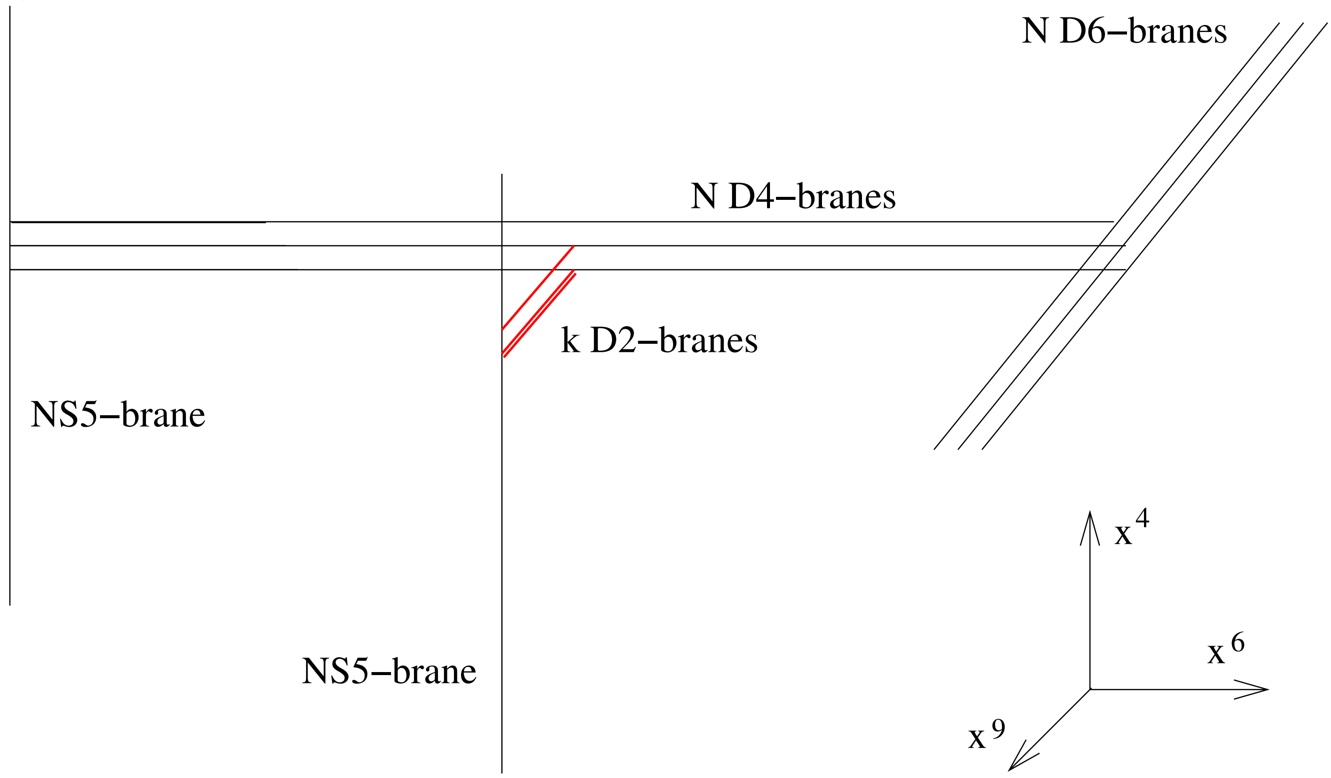
We also need to turn on the FI parameter v^2. This is achieved by taking the right-hand NS5-brane and pulling it out of the page in the x^9 direction. In order to remain in the ground state, the D4-branes are not allowed to tilt into the x^9 direction: this would break supersymmetry and increase their length, reflecting a corresponding increase in the ground state energy of the theory. Instead, they must split on the D6-branes. Something known as the S-rule [91, 162] tells us that only one D4-brane can end on a given D6-brane while preserving supersymmetry, ensuring that we need at least N D6-branes to find a zero-energy ground state. The final configuration is drawn in the figure 16, with the field theory dictionary given by

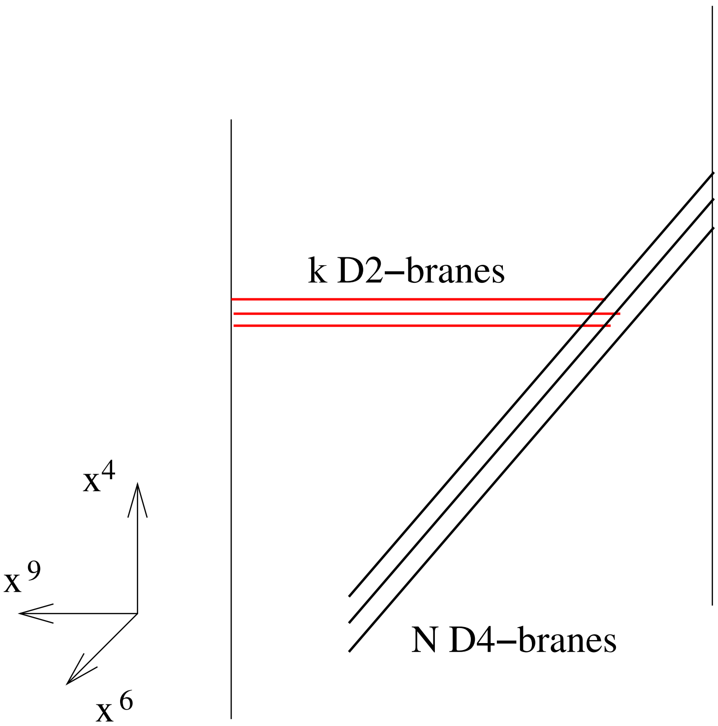
$$v^2 = \frac{\Delta x^9}{(2\pi)^3 g_s l_s^3} \tag{12.26}$$



Now we've built our theory, we can look to find the vortices. We expect them to appear as other D-branes in the configuration. There is a unique BPS D-brane with the correct mass: it is a D2-brane, lying coincident with the D6-branes, with worldvolume 039, as shown in figure 16 [163]. The x^3 direction here is the direction of the vortex string.

The problem is: what is the worldvolume theory on the D2-branes. It's hard to read off the theory directly because of the boundary conditions where the D2-branes end on the D4-branes. But, already by inspection, we might expect that it's related to the Dp-D$(p-4)$ system described in Lecture 1 in the context of instantons. To make progress we play some brane games. Move the D6-branes to the right. As they pass the NS5-brane, the Hanany-Witten transition occurs and the right-hand D4-branes disappear [91]. We get the configuration shown in figure 17.



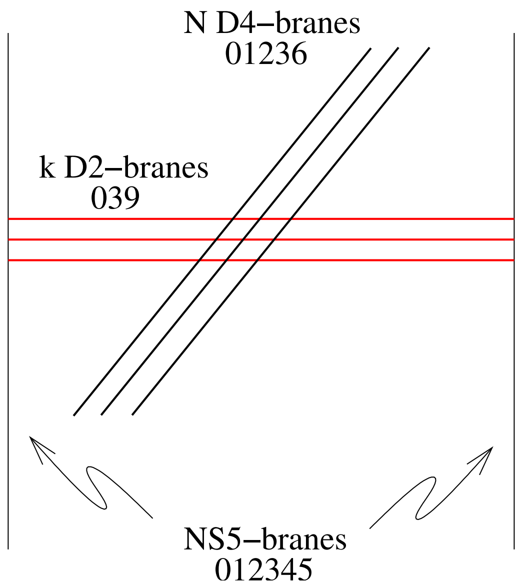


Moving the D6-branesRotating our viewpoint

Let's keep the D6-branes moving. Off to infinity. Finally, we rotate our perspective a little, viewing the D-branes from a different angle, shown in figure 18. This is our final D-brane configuration and we can now read off the dynamics.

We want to determine the theory on the D2-branes in figure 18. Let's start with the easier problem in figure 19. Here the D4-branes extend to infinity in both $x^6 \to \pm\infty$ directions, and the D2-branes end on the other NS5. The theory on the D2-branes is simple to determine: it is a $U(k)$ gauge theory with 4 real adjoint scalars, or two complex scalars

$$\sigma = X^4 + iX^5 \qquad Z = X^1 + iX^2 \tag{12.27}$$



which combine to give the $\mathcal{N} = (4,4)$ theory in $d = 1+1$. The D4-branes contribute hypermultiplets $(\psi_a, \tilde{\psi}_a)$ with $a = 1, ..., N$. These hypermultiplets get a mass only when the D2-branes and D6-branes are separated in the X^4 and X^5 directions. This means we have a coupling like

$$\sum_{a=1}^{N} \psi_a^\dagger \{\sigma^\dagger, \sigma\} \psi_a + \tilde{\psi}_a \{\sigma^\dagger, \sigma\} \tilde{\psi}_a^\dagger \tag{12.28}$$

But there is no such coupling between the hypermultiplets and Z. The coupling (12.28) breaks supersymmetry to $\mathcal{N} = (2,2)$. So we now understand the D2-brane theory of figure 19. However, the D2-brane theory that we're really interested in, shown in figure 18, differs from this in two ways

- The right-hand NS5-brane is moved out of the page. But we already saw in the manoeuvres around figure 16 that this induces a FI parameter on brane theory. Except this this time the FI parameter is for the D2-brane theory. It's given by

$$r = \frac{\Delta x^6}{2\pi g_s l_s} = \frac{4\pi}{e^2} \tag{12.29}$$

- We only have half of the D4-branes, not all of them. If a full D4-brane gives rise to a hypermultiplet, one might guess that half a D4-brane should give rise to half a hypermultiplet, otherwise known as a chiral multiplet. Although the argument is a little glib, it turns out that this is the correct answer [164].

We end up with the gauge theory in $d = 1+1$ dimensions with $\mathcal{N} = (2,2)$ supersymmetry

$$\begin{array}{rcl} U(k) \text{ Gauge Theory} & + & \text{Adjoint Chiral Multiplet } Z \\ & + & N \text{ Fundamental Chiral Multiplets } \psi_a \end{array}$$

This theory has a FI parameter $r = 4\pi/e^2$. Now this should be looking very familiar — it's very similar to the instanton theory we described in Lecture 1. We'll return to this shortly. For now let's keep examining our vortex theory. The potential for the various scalars is dictated by supersymmetry and is given by

$$\begin{aligned} V &= \frac{1}{g^2}\text{Tr}\,|[\sigma,\sigma^\dagger]|^2 + \text{Tr}\,|[\sigma,Z]|^2 + \text{Tr}\,|[\sigma,Z^\dagger]|^2 + \sum_{a=1}^{N} \psi_a^\dagger \sigma^\dagger \sigma \psi_a \\ &\quad + \frac{g^2}{2}\,\text{Tr}\left(\sum_a \psi_a \psi_a^\dagger + [Z,Z^\dagger] - r\,1_k\right)^2 \end{aligned} \tag{12.30}$$

Here g^2 is an auxiliary gauge coupling which we take to infinity $g^2 \to \infty$ to restrict us to the Higgs branch, the vacuum moduli space defined by

$$\mathcal{M}_{\text{Higgs}} \cong \{\sigma = 0, V = 0\}/U(k) \tag{12.31}$$

Counting the various degrees of freedom, the Higgs branch has real dimension $2kN$. From the analogy with the instanton case, it is natural to conjecture that this is the vortex moduli space [151]

$$\mathcal{V}_{k,N} \cong \mathcal{M}_{\text{Higgs}} \tag{12.32}$$

While the ADHM construction has a field theoretic underpinning, I know of no field theory derivation of the above result for vortices. So what evidence do we have that the Higgs branch indeed coincides with the vortex moduli space? Because of the FI parameter, $\mathcal{M}_{\text{Higgs}}$ is a smooth manifold, as is $\mathcal{V}_{k,N}$ and, obviously the dimensions work out. Both spaces have a $SU(N) \times U(1)$ isometry which, in the above construction, act upon ψ and Z respectively. Finally, in all cases we can check, the two spaces agree (as, indeed, do their Kähler classes). Let's look at some examples.

12.4.1 Examples of Vortex Moduli Spaces Revisited

12.4.1One Vortex in $U(N)$

The gauge theory for a single $k = 1$ vortex in $U(N)$ is a $U(1)$ gauge theory. The adjoint scalar Z decouples, parameterizing the complex plane $\mathbf{C}$, leaving us with the N charged scalars satisfying

$$\sum_{a=1}^{N} |\psi_a|^2 = r \tag{12.33}$$

modulo the $U(1)$ action $\psi_a \to e^{i\alpha}\psi_a$. This gives us the moduli space

$$\mathcal{V}_{1,N} \cong \mathbf{C} \times \mathbb{CP}^{N-1} \tag{12.34}$$

where the $\mathbb{CP}^{N-1}$ has the correct Kähler class $r = 4\pi/e^2$ in agreement with (12.22). The metric on $\mathbb{CP}^{N-1}$ is, again, the round Fubini-Study metric.

12.4.1k Vortices in $U(1)$

The Higgs branch corresponding to the k vortex moduli space is

$$\{\psi\psi^\dagger + [Z, Z^\dagger] = r\, 1_k\}/U(k) \tag{12.35}$$

which is asymptotic to the cone $\mathbf{C}^k/\mathbf{Z}_k$, with the singularities resolved. This is in agreement with the vortex moduli space. *However*, the metric on $\mathcal{M}_{\text{Higgs}}$ differs by power law corrections from the flat metric on the orbifold $\mathbf{C}^k/\mathbf{Z}_k$. But, as we've discussed, $\mathcal{V}_{k,N}$ differs from the flat metric by exponential corrections.

More recently, the moduli space of two vortices in $U(N)$ was studied in some detail and shown to possess interesting and non-trivial topology [165], with certain expected features of $\mathcal{V}_{2,N}$ reproduced by the Higgs branch.

In summary, it is conjectured that the vortex moduli space $\mathcal{V}_{k,N}$ is isomorphic to the Higgs branch (12.34). But, except for the case $k = 1$ where the metric is determined by the isometry, the metrics do not agree. A direct field theory proof of this correspondence remains to be found.

12.4.2 The Relationship to Instantons

As we've mentioned a few times, the vortex theory bears a striking resemblance to the ADHM instanton theory we met in Lecture 1. In fact, the gauge theoretic construction of vortex moduli space $\mathcal{V}_{k,N}$ involves exactly half the fields of the ADHM construction. Or, put another way, the vortex moduli space is half of the instanton moduli space. We can state this more precisely: $\mathcal{V}_{k,N}$ is a complex, middle dimensional submanifold of $\mathcal{I}_{k,N}$. It can be defined by looking at the action of the isometry rotating the instantons in the $x^3 - x^4$ plane. Denote the corresponding Killing vector as h. Then

$$\mathcal{V}_{k,N} \cong \mathcal{I}_{k,N}|_{h=0} \tag{12.36}$$

where $\mathcal{I}_{k,N}$ is the resolved instanton moduli space with non-commutativity parameter $\theta_{\mu\nu} = r\bar{\eta}^3_{\mu\nu}$. We'll see a physical reason for this relationship shortly.

An open question: The ADHM construction is constructive. As we have seen, it allows us to build solutions to $F = {}^\star F$ from the variables of the Higgs branch. Does a similar construction exist for vortices?

Relationships between the instanton and vortex equations have been noted in the past. In particular, a twisted reduction of instantons in $SU(2)$ Yang-Mills on $\mathbf{R}^2 \times \mathbf{S}^2$ gives rise to the $U(1)$ vortex equations [166]. While this relationship appears to share several characteristics to the correspondence described above, it differs in many important details. It don't understand the relationship between the two approaches.

12.5 Adding Flavors

Let's now look at vortices in a $U(N_c)$ gauge theory with $N_f \geq N_c$ flavors. Note that we've added subscripts to denote color and flavor. In theories with $N_c = 1$ and $N_f > 1$, these were called semi-local vortices [167, 168, 169, 170]. The name derives from the fact the theory has both a gauge (local) group and a flavor (global) group. But for us, it's not a great name as all our theories have both types of symmetries, but it's only when $N_f > N_c$ that the extra properties of "semi-local" vortices become apparent.

The Lagrangian (12.1) remains but, unlike before, the theory no longer has a mass gap in vacuum. Instead there are N_c^2 massive scalar fields and scalars, and $2N_c(N_f - N_c)$ massless scalars. At low-energies, the theory reduces to a σ-model on the Higgs branch of the gauge theory (12.1),

$$\mathcal{M}_{\text{Higgs}} \cong \{\sum_{i=1}^{N_f} q_i q_i^\dagger = v^2\, 1_{N_c}\}/U(N_c) \cong G(N_c, N_f) \tag{12.37}$$

When we have an abelian $N_c = 1$ theory, this Higgs branch is the projective space $G(1, N_f) \cong \mathbb{CP}^{N_f - 1}$. For non-abelian theories, the Higgs branch is the Grassmannian $G(N_c, N_f)$, the space of $\mathbf{C}^{N_c}$ planes in $\mathbf{C}^{N_f}$. In a given vacuum, the symmetry breaking pattern is $U(N_c) \times SU(N_f) \to S[U(N_c) \times U(N_f - N_c)]$.

The first order vortex equations (12.10) still give solutions to the full Lagrangian, now with the flavor index running over values $i = 1..., N_f$. Let's denote the corresponding vortex moduli space as $\hat{\mathcal{V}}_{k,N_c,N_f}$, so our previous notation becomes $\mathcal{V}_{k,N} \cong \hat{\mathcal{V}}_{k,N,N}$. The index theorem now tells us the dimension of the vortex moduli space

$$\dim(\hat{\mathcal{V}}_{k,N_c,N_f}) = 2kN_f \tag{12.38}$$

The dimension depends only on the number of flavors, and the semi-local vortices inherit new modes. These modes are related to scaling modes of the vortex — the size of the vortex becomes a parameter, just as it was for instantons [171].

These vortices arising in the theory with extra flavors are related to other solitons, known as a sigma-model lumps. (These solitons have other names, depending on the context, sometimes referred to as "textures", "Skyrmions" or, in the context of string theory, "worldsheet instantons"). Let's see how this works. At low-energies (or, equivalently, in the strong coupling limit $e^2 \to \infty$) our gauge theory flows to the sigma-model on the Higgs branch $\mathcal{M}_{\rm Higgs} \cong G(N_c, N_f)$. In this limit our vortices descend to lumps, objects which gain their topological support once we compactify the $(x^1 - x^2)$-plane at infinity, and wrap this sphere around $\mathcal{M}_{\rm Higgs} \cong G(N_c, N_f)$ [172, 173]

$$\Pi_2(G(N_c, N_f)) \cong \mathbf{Z} \tag{12.39}$$

When $N_f = N_c$ there is no Higgs branch, the vortices have size $L = 1/ev$ and become singular as $e^2 \to \infty$. In contrast, when $N_f > N_c$, the vortices may have arbitrary size and survive the strong coupling limit. However, while the vortex moduli space is smooth, the lump moduli space has singularities, akin to the small instanton singularities we saw in Lecture 1. We see that the gauge coupling $1/e^2$ plays the same role for lumps as θ plays for Yang-Mills instantons.

The brane construction for these vortices is much like the previous section - we just need more $D6$ branes. By performing the same series of manoeuvres, we can deduce the worldvolume theory. It is again a $d = 1 + 1$ dimensional, $\mathcal{N} = (2, 2)$ theory with

$$\begin{array}{lcl} U(k) \text{ Gauge Theory} & + & \text{Adjoint Chiral Multiplet } Z \\ & + & N_c \text{ Fundamental Chiral Multiplets } \psi_a \\ & + & (N_f - N_c) \text{ Anti-Fundamental Chiral Multiplets } \tilde{\psi}_a \end{array}$$

Once more, the FI parameter is $r = 4\pi/e^2$. The D-term constraint of this theory is

$$\sum_{a=1}^{N_c} \psi_a \psi_a^\dagger - \sum_{b=1}^{N_f - N_c} \tilde{\psi}_b^\dagger \tilde{\psi}_b + [Z, Z^\dagger] = r\, 1_k \tag{12.40}$$

A few comments

- Unlike the moduli space $\mathcal{V}_{k,N}$, the presence of the $\tilde{\psi}$ means that this space doesn't collapse as we send $r \to 0$. Instead, in this limit it develops singularities at $\psi = \tilde{\psi} = 0$ where the $U(k)$ gauge group doesn't act freely. This is the manifestation of the discussion above.
- The metric inherited from the D-term (12.40) again doesn't coincide with the metric on the vortex moduli space $\hat{\mathcal{V}}_{k,N_c,N_f}$. In fact, here the discrepancy is more pronounced, since the metric on $\hat{\mathcal{V}}_{k,N_c,N_f}$ has non-normalizable modes: the directions in moduli space corresponding to the scaling the solution are suffer an infra-red logarithmic divergence [174, 171]. The vortex theory arising from branes doesn't capture this.

12.5.1 Non-Commutative Vortices

As for instantons, we can consider vortices on the non-commutative plane

$$[x^1, x^2] = i\vartheta \tag{12.41}$$

These objects were first studied in [175]. How does this affect the moduli space? In the ADHM construction for instantons, we saw that non-commutivity added a FI parameter to the D-term constraints. But, for vortices, we already have a FI parameter: $r = 4\pi/e^2$. It's not hard to show using D-branes [151], that the effect of non-commutivity is to deform,

$$r = \frac{4\pi}{e^2} + 2\pi v^2\vartheta \tag{12.42}$$

This has some interesting consequences. Note that for $N_f = N_c$, there is a critical FI parameter $\vartheta_c = -v^2/e^2$ for which $r = 0$. At this point the vortex moduli space becomes singular. For $\vartheta < \vartheta_c$, no solutions to the D-term equations exist. Indeed, it can be shown that in this region, no solutions to the vortex equations exist either [176]. We see that the Higgs branch correctly captures the physics of the vortices.

For $N_f > N_c$, the Higgs branch makes an interesting prediction: the vortex moduli space should undergo a topology changing transition as $\vartheta \to \vartheta_c$. For example, in the case of a single $k = 1$ vortex in $U(2)$ with $N_f = 4$, this is the well-known flop transition of the conifold. To my knowledge, no one has confirmed this behavior of the vortex moduli space from field theory. Nor has anyone found a use for it!

12.6 What Became Of.......

Let's now look at what became of the other solitons we studied in the past two lectures.

12.6.1 Monopoles

Well, we've set $\phi = 0$ throughout this lecture and, as we saw, the monopoles live on the vev of ϕ. So we shouldn't be surprised if they don't exist in our theory (12.1). We'll see them reappear in the following section.

12.6.2 Instantons

These are more interesting. Firstly the vev $q \neq 0$ breaks conformal invariance, causing the instantons to collapse. This is the same behavior that we saw in Section 2.6. But recall that in the middle of the vortex string, $q \to 0$. So maybe it's possible for the instanton to live inside the vortex string, where the non-abelian gauge symmetry is restored. To see that this can indeed occur, we can look at the worldsheet of the vortex string. As we've seen, the low-energy dynamics for a single string is

$U(1)$ with N charged chiral multiplets and FI parameter $r = 4\pi/e^2$

But this falls into the class of theories we discussed in section 3.5. So if the worldsheet is Euclidean, the theory on the vortex string itself admits a vortex solution: a vortex in a vortex. The action of this vortex is [177]

$$S_{\text{vortex in vortex}} = 2\pi r = \frac{8\pi^2}{e^2} = S_{\text{inst}} \tag{12.43}$$

which is precisely the action of the Yang-Mills instanton. Such a vortex has $2N$ zero modes which include scaling modes but, as we mentioned previously, not all are normalizable.

There is also a 4d story for these instantons buried in the vortex string. This arises by completing the square in the Lagrangian in a different way to (12.8). We still set $\phi = 0$, but now allow for all fields to vary in all four dimensions [177]. We write $z = x^1 + ix^2$ and $w = x^3 - ix^4$,

$$\begin{aligned} S &= \int d^4x \; \frac{1}{2e^2}\mathrm{Tr}\; F_{\mu\nu}F^{\mu\nu} + \sum_{i=1}^{N_f} |\mathcal{D}_\mu q_i|^2 + \frac{e^2}{4}\mathrm{Tr}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\,\mathbf{1}_{N_c})^2 \\ &= \int d^4x \; \frac{1}{2e^2}\mathrm{Tr}\left(F_{12} - F_{34} - \frac{e^2}{2}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\,\mathbf{1}_{N_c})\right)^2 \\ &\quad + \sum_{i=1}^{N_f} |\mathcal{D}_z q_i|^2 + |\mathcal{D}_\omega q_i|^2 + \frac{1}{e^2}\mathrm{Tr}\left((F_{14} - F_{23})^2 + (F_{13} + F_{24})^2\right) \\ &\quad + \frac{1}{e^2}\mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu} + F_{12}v^2 + F_{34}v^2 \\ &\geq \int d^4x \; \frac{1}{e^2}\mathrm{Tr}\, F_{\mu\nu}{}^\star F^{\mu\nu} + \mathrm{Tr}\,(F_{12}v^2 + F_{34}v^2) \end{aligned} \tag{12.44}$$

The last line includes three topological charges, corresponding to instantons, vortex strings in the $(x^1 - x^2)$ plane, and further vortex strings in the $(x^3 - x^4)$ plane. The Bogomoln'yi equations describing these composite solutions are

$$F_{14} = F_{23} \quad F_{13} = F_{24} \quad , \quad F_{12} - F_{34} = \frac{e^2}{2}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\mathbf{1}_{N_c}) \quad \mathcal{D}_z q_i = \mathcal{D}_w q_i = 0$$

It is not known if solutions exist, but the previous argument strongly suggests that there should be solutions describing an instanton trapped inside a vortex string. Some properties of this configuration were studied in [178].

The observation that a vortex in the vortex string is a Yang-Mills instanton gives some rationale to the fact that $\mathcal{V}_{k,N} \subset \mathcal{I}_{k,N}$.

12.7 Fermi Zero Modes

In this section, I'd like to describe an important feature of fermionic zero modes on the vortex string: they are chiral. This means that a Weyl fermion in four dimensions will give rise to a purely left-moving (or right-moving) mode on the (anti-) vortex worldsheet. In fact, a similar behavior occurs for instantons and monopoles, but since this is the first lecture where the solitons are string-like in four-dimensions, it makes sense to discuss this phenomenon here.

The exact nature of the fermionic zero modes depends on the fermion content in four dimensions. Let's stick with the supersymmetric generalization of the Lagrangian (12.1). Then we have the gaugino λ, an adjoint valued Weyl fermion which is the superpartner of the gauge field. We also have fermions in the fundamental representation, χ_i with $i = 1, ..., N$, which are the superpartners of the scalars q_i. These two fermions mix through Yukawa couplings of the form $q_i^\dagger \lambda \chi_i$, and the Dirac equations read

$$-i\,\bar{\not{\mathcal{D}}}\lambda + i\sqrt{2}\sum_{i=1}^{N} q_i \bar{\chi}_i = 0 \qquad \text{and} \qquad -i\,\not{\mathcal{D}}\bar{\chi}_i - i\sqrt{2}q_i^\dagger \lambda = 0 \tag{12.45}$$

where the Dirac operators take the form,

$$\not{\mathcal{D}} \equiv \sigma^\mu \mathcal{D}_\mu = \begin{pmatrix} \mathcal{D}_+ & \mathcal{D}_z \\ \mathcal{D}_{\bar{z}} & \mathcal{D}_- \end{pmatrix} \qquad \text{and} \qquad \bar{\not{\mathcal{D}}} \equiv \bar{\sigma}^\mu \mathcal{D}_\mu = \begin{pmatrix} \mathcal{D}_- & -\mathcal{D}_z \\ -\mathcal{D}_{\bar{z}} & \mathcal{D}_+ \end{pmatrix} \tag{12.46}$$

which, as we can see, nicely split into $\mathcal{D}_\pm = \mathcal{D}_0 \pm \mathcal{D}_3$ and $\mathcal{D}_z = \mathcal{D}_1 - i\mathcal{D}_2$ and $\mathcal{D}_{\bar{z}} = \mathcal{D}_1 + i\mathcal{D}_2$. The bosonic fields in (12.45) are evaluated on the vortex solution which, crucially, includes $\mathcal{D}_z q_i = 0$ for the vortex (or $\mathcal{D}_{\bar{z}} q_i = 0$ for the anti-vortex). We see the importance of this if we take the first equation in (12.45) and hit it with $\not{\mathcal{D}}$, while hitting the second equation with $\bar{\not{\mathcal{D}}}$. In each equation terms of the form $\mathcal{D}_z q_i$ will appear, and subsequently vanish as we evaluate them on the vortex background. Let's do the calculation. We split up the spinors into their components λ_α and $(\chi_\alpha)_i$ with $\alpha = 1, 2$ and, for now, look for zero modes that don't propagate along the string, so $\partial_+ = \partial_- = 0$. Then the Dirac equations in component form become

$$\begin{aligned} (-\mathcal{D}_z\mathcal{D}_{\bar{z}} + 2q_i q_i^\dagger)\lambda_1 = 0 \qquad &\text{and} \qquad (-\mathcal{D}_{\bar{z}}\mathcal{D}_z + 2q_i q_i^\dagger)\lambda_2 - \sqrt{2}(\mathcal{D}_{\bar{z}} q_i)\bar{\chi}_{1i} = 0 \\ (-\mathcal{D}_{\bar{z}}\mathcal{D}_z {\delta^j}_i + 2q_i^\dagger q_j)\bar{\chi}_{2j} = 0 \qquad &\text{and} \qquad (-\mathcal{D}_z\mathcal{D}_{\bar{z}} {\delta^j}_i + 2q_i q_j^\dagger)\bar{\chi}_{1j} - \sqrt{2}(\mathcal{D}_z q_i^\dagger)\lambda_2 = 0 \end{aligned}$$

The key point is that the operators appearing in the first column are positive definite, ensuring that λ_1 and χ_{2i} have no zero modes. In contrast, the equations for λ_2 and $\bar{\chi}_{1i}$ do have zero modes, guaranteed by the index. We therefore know that any zero modes of the vortex are of the form,

$$\lambda = \begin{pmatrix} 0 \\ \lambda \end{pmatrix} \qquad \text{and} \qquad \bar{\chi}_i = \begin{pmatrix} \bar{\chi}_i \\ 0 \end{pmatrix} \tag{12.47}$$

If we repeat the analysis for the anti-vortex, we find that the other components turn on. To see the relationship to the chirality on the worldsheet, we now allow the zero modes to propagate along the string, so that $\lambda = \lambda(x^0, x^3)$ and $\bar{\chi}_i = \bar{\chi}_i(x^0, x^3)$. Plugging this ansatz back into the Dirac equation, now taking into account the derivatives $\mathcal{D}_\pm$ in (12.46), we find the equations of motion

$$\partial_+ \lambda = 0 \qquad \text{and} \qquad \partial_+ \bar{\chi}_i = 0 \tag{12.48}$$

Or, in other words, $\lambda = \lambda(x_-)$ and $\bar{\chi} = \bar{\chi}(x_-)$: both are right movers.

In fact, the four-dimensional theory with only fundamental fermions χ_i is anomalous. Happily, so is the $\mathbb{CP}^{N-1}$ theory on the string with only right-moving fermions, suffering from the sigma-model anomaly [179]. To rectify this, one may add four dimensional Weyl fermions $\tilde{\chi}_i$ in the anti-fundamental representation, which provide left movers on the worldsheet. If the four-dimensional theory has $\mathcal{N} = 2$ supersymmetry, the worldsheet theory preserves $\mathcal{N} = (2, 2)$ [180]. Alternatively, one may work with a chiral, non-anomalous $\mathcal{N} = 1$ theory in four-dimensions, resulting in a chiral non-anomalous $\mathcal{N} = (0, 2)$ theory on the worldsheet.

12.8 Applications

Let's now turn to discussion of applications of vortices in various field theoretic contexts. We review some of the roles vortices play as finite action, instanton-like, objects in two dimensions, as particles in three dimensions, and as strings in four dimensions.

12.8.1 Vortices and Mirror Symmetry

Perhaps the most important application of vortices in string theory is in the context of the $d = 1 + 1$ dimensional theory on the string itself. You might protest that the string worldsheet theory doesn't involve a gauge field, so why would it contain vortices?! The trick, as described by Witten [172], is to view sigma-models in terms of an auxiliary gauge theory known as a *gauged linear sigma model*. We've already met this trick several times in these lectures: the sigma-model target space is the Higgs branch of the gauge theory. Witten showed how to construct gauge theories that have compact Calabi-Yau manifolds as their Higgs branch.

In $d = 1 + 1$ dimensions, vortices are finite action solutions to the Euclidean equations of motion. In other words, they play the role of instantons in the theory. As we explained Section 3.5 above, the vortices are related to worldsheet instantons wrapping the 2-cycles of the Calabi-Yau Higgs branch. It turns out that it is much easier to deal with vortices than directly with worldsheet instantons (essentially because their moduli space is free from singularities). Indeed, in a beautiful paper, Morrison and Plesser succeeded in summing the contribution of all vortices in the topological A-model on certain Calabi-Yau manifolds, showing that it agreed with the classical prepotential derived from the B-model on the mirror Calabi-Yau [181].

More recently, Hori and Vafa used vortices to give a proof of $\mathcal{N} = (2, 2)$ mirror symmetry for all Calabi-Yau which can be realized as complete intersections in toric varieties [135]. Hori and Vafa work with dual variables, performing the so-called Rocek-Verlinde transformation to twisted chiral superfields [182]. They show that vortices contribute to a two fermi correlation function which, in terms of these dual variables, is cooked up by a superpotential. This superpotential then captures the relevant quantum information about the original theory. Similar methods can be used in $\mathcal{N} = (4, 4)$ theories to derive the T-duality between NS5-branes and ALE spaces [183, 184, 185, 186], with the instantons providing the necessary ingredient to break translational symmetry after T-duality, leading to localized, rather than smeared, NS5-branes.

12.8.2 Swapping Vortices and Electrons

In lecture 2, we saw that it was possible to rephrase four-dimensional field theories, treating the monopoles as elementary particles instead of solitons. This trick, called electric-magnetic duality, gives key insight into the strong coupling behavior of four-dimensional field theories. In three dimensions, vortices are particle like objects and one can ask the same question: is it possible to rewrite a quantum field theory, treating the vortices as fundamental degrees of freedom?

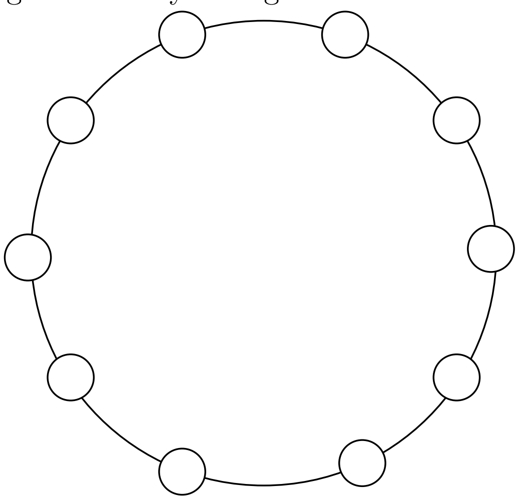
The answer is yes. In fact, condensed matter theorists have been using this trick for a number of years (see for example [187]). Things can be put on a much more precise footing in the supersymmetric context, with the first examples given by Intriligator and Seiberg [188]. They called this phenomenon "mirror symmetry" in three dimensions as it had some connection to the mirror symmetry of Calabi-Yau manifolds described above.

Let's describe the basic idea. Following Intriligator and Seiberg, we'll work with a theory with eight supercharges (which is $\mathcal{N} = 4$ supersymmetry in three dimensions). Each gauge field comes with three real scalars and four Majorana fermions. The charged matter, which we'll refer to as "electrons", lives in a hypermultiplet, containing two complex scalars together with two Dirac fermions. The theory we start with is:

Theory A: $U(1)$ with N charged hypermultiplets

The vortices in this theory fall into the class described in Section 3.5. Each vortex has $2N$ zero modes but, as we discussed, not all of these zero modes are normalizable. The overall center of mass is, of course, normalizable (the vortex has mass $M = 2\pi v^2$) but the remaining $2(N-1)$ modes of a single vortex are logarithmically divergent.

We now wish to rewrite this theory, treating the vortices as fundamental objects. What properties must the theory have in order to mimic the behavior of the vortex? It will prove useful to think of each vortex as containing N individual "fractional vortices". We postulate that these fractional vortices suffer a logarithmic confining potential, so that any number $n < N$ have a logarithmically divergent mass, but N together form a state with finite mass. Such a system would exhibit the properties of the vortex zero modes described above: the $2N$ zero modes correspond to the positions of the N fractional vortices. They can move happily as a whole, but one pays a logarithmically divergent cost to move these objects individually. (Note: a logarithmically divergent cost isn't really that much!)



In fact, it's very easy to cook up a theory with these properties. In $d = 2+1$, an electron experiences logarithmic confinement, since its electric field goes as $E \sim 1/r$ so its energy $\int d^2x\, E^2$ suffers a logarithmic infra-red divergence. These electrons will be our "fractional vortices". We will introduce N different types of electrons and, in order to assure that only bound states of all N are gauge singlets, we introduce $N-1$ gauge fields with couplings dictated by the quiver diagram shown in the figure. Recall that quiver diagrams are read in the following way: the nodes of the quiver are gauge groups, each giving a $U(1)$ factor in this case. Meanwhile, the links denote hypermultiplets with charge $(+1, -1)$ under the gauge groups to which it is attached. Although there are N nodes in the quiver, the overall $U(1)$ decouples, leaving us with the theory

Theory B: $U(1)^{N-1}$ with N hypermultiplets

This is the Seiberg-Intriligator mirror theory, capturing the same physics as Theory A. The duality also works the other way, with the electrons of Theory A mapping to the vortices of Theory B. It can be shown that the low-energy dynamics of these two theories exactly agree. This statement can be made precise at the two-derivative level. The Higgs branch of Theory A coincides with the Coulomb branch of Theory B: both are $T^\star(\mathbb{CP}^{N-1})$. Similarly, the Coulomb branch of Theory A coincides with the Higgs branch of Theory A: both are the A_{N-1} ALE space.

There are now many mirror pairs of theories known in three dimensions. In particular, it's possible to tinker with the mirror theories so that they actually coincide at all length scales, rather than simply at low-energies [189]. Mirror pairs for non-abelian gauge theories are known, but are somewhat more complicated to due to presence of instanton corrections (which, recall, are monopoles in three dimensions) [190, 191, 192, 193, 194]. Finally, one can find mirror pairs with less supersymmetry [195, 196], including mirrors for interesting Chern-Simons theories [197, 198, 199]. Finite quantum correction to the vortex mass in $\mathcal{N} = 2$ theories was described in [200]. The Chern-Simons mirrors reduce to Hori-Vafa duality under compactification to two dimensions [201].

12.8.3 Vortex Strings

In $d = 3 + 1$ dimensions, vortices are string like objects. There is a very interesting story to be told about how we quantize vortex worldsheet theory, which is a sigma-model on $\mathcal{V}_{k,N}$. But this will have to wait for the next lecture.

Here let me mention an application of vortices in the context of cosmic strings which shows that reconnection of vortices in gauge theories is inevitable at low-energies. Reconnection of strings means that they swap partners as they intersect as shown in the figure. In general, it's a difficult problem to determine whether reconnection occurs and requires numerical study. However, at low-energies we may reliably employ the techniques of the moduli space approximation that we learnt above [202, 203, 204].

The first step is to reduce the dynamics of cosmic strings to that of particles by considering one of two spatial slices shown in the figure. The vertical slice cuts the strings to reveal a vortex-anti-vortex pair. After reconnection, this slice no longer intersects the strings, implying the annihilation of this pair. Alternatively, one can slice horizontally to reveal two vortices. Here the smoking gun for reconnection is the right-angle scattering of the vortices at (or near) the interaction point. Such 90° degree scattering is a requirement since, as is clear from the figure, the two ends of each string are travelling in opposite directions after the collision. By varying the slicing along the string, one can reconstruct the entire dynamics of the two strings in this manner and show the inevitability of reconnection at low-energies.

Hence, reconnection of cosmic strings requires both the annihilation of vortex-anti-vortex pairs and the right-angle scattering of two vortices. The former is expected (at least for suitably slow collisions). And we saw in Section 3.3.2 that the latter occurs for abelian vortices in the

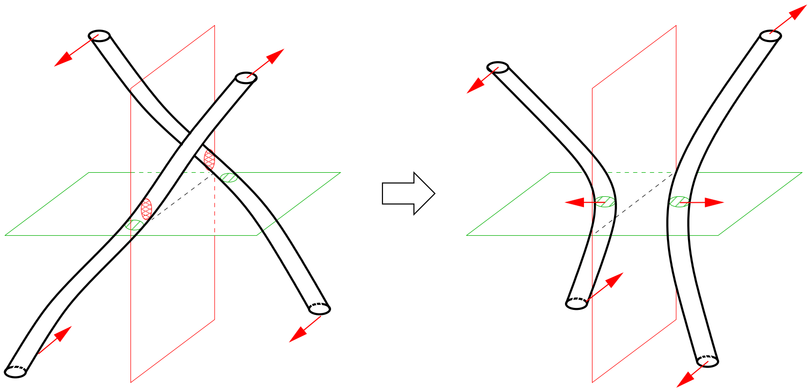


Figure 12.3: The reconnection cosmic strings. Slicing vertically, one sees a vortex-anti-vortex pair annihilate. Slicing horizontally, one sees two vortices scattering at right angles.

moduli space approximation. We conclude that abelian cosmic strings do reconnect at low energies. Numerical simulations reveal that these results are robust, holding for very high energy collisions [205].

For cosmic strings in non-abelian theories this result continues to hold, with strings reconnecting except for very finely tuned initial conditions [165]. However, in this case there exist mechanisms to push the strings to these finely tuned conditions, resulting in a probability for reconnection less than 1.

Recently, there has been renewed interest in the reconnection of cosmic strings, with the realization that cosmic strings may be fundamental strings, stretched across the sky [206]. These objects differ from abelian cosmic strings as they have a reduced probability of reconnection, proportional to the string coupling g_s^2 [207, 208]. If cosmic strings are ever discovered, it may be possible to determine their probability of reconnection, giving a vital clue to their microscopic origin. The recent developments of this story have been nicely summarized in the review [144].

13 Domain Walls

So far we've considered co-dimension 4 instantons, co-dimension 3 monopoles and co-dimension 2 vortices. We now come to co-dimension 1 domain walls, or kinks as they're also known. While BPS domain walls exist in many supersymmetric theories (for example, in Wess-Zumino models [209]), there exists a special class of domain walls that live in gauge theories with 8 supercharges. They were first studied by Abraham and Townsend [210] and have rather special properties. These will be the focus of this lecture. As we shall explain below, the features of these domain walls are inherited from the other solitons we've met, most notably the monopoles.

13.1 The Basics

To find domain walls, we need to deform our theory one last time. We add masses m_i for the fundamental scalars q_i. Our Lagrangian is that of a $U(N_c)$ gauge theory, coupled to a real adjoint scalar field ϕ and N_f fundamental scalars q_i

$$\begin{aligned} S &= \int d^4x \; \mathrm{Tr}\left(\frac{1}{2e^2}F^{\mu\nu}F_{\mu\nu} + \frac{1}{e^2}(\mathcal{D}_\mu\phi)^2\right) + \sum_{i=1}^{N_f}|\mathcal{D}_\mu q_i|^2 \\ &\qquad - \sum_{i=1}^{N_f} q_i^\dagger(\phi - m_i)^2 q_i - \frac{e^2}{4}\,\mathrm{Tr}\,(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2\,1_N)^2 \end{aligned} \tag{13.1}$$

Notice the way the masses mix with ϕ, so that the true mass of each scalar is $|\phi - m_i|$. Adding masses in this way is consistent with $\mathcal{N} = 2$ supersymmetry. We'll pick all masses to be distinct and, without loss of generality, choose

$$m_i < m_{i+1} \tag{13.2}$$

As in Lecture 3, there are vacua with $V = 0$ only if $N_f \geq N_c$. The novelty here is that, for $N_f > N_c$, we have multiple isolated vacua. Each vacuum is determined by a choice of N_c distinct elements from a set of N_f

$$\Xi = \{\xi(a) : \; \xi(a) \neq \xi(b) \text{ for } a \neq b\} \tag{13.3}$$

where $a = 1, ..., N_c$ runs over the color index, and $\xi(a) \in \{1, ..., N_f\}$. Let's set $\xi(a) < \xi(a+1)$. Then, up to a Weyl transformation, we can set the first term in the potential to vanish by

$$\phi = \mathrm{diag}(m_{\xi(1)}, ..., m_{\xi(N_c)}) \tag{13.4}$$

This allows us to turn on the particular components $q^a{}_i \sim \delta^a{}_{i=\xi(a)}$ without increasing the energy. To cancel the second term in the potential, we require

$$q^a{}_i = v\delta^a{}_{i=\xi(a)} \tag{13.5}$$

The number of vacua of this type is

$$N_{\rm vac} = \left(\begin{array}{c} N_f \\ N_c \end{array}\right) = \frac{N_f!}{N_c!(N_f - N_c)!} \tag{13.6}$$

Each vacuum has a mass gap in which there are N_c^2 gauge bosons with $M_\gamma^2 = e^2v^2 + |m_{\xi(a)} - m_{\xi(b)}|^2$, and $N_c(N_f - N_c)$ quark fields with mass $M_q^2 = |m_{\xi(a)} - m_i|^2$ with $i \notin \Xi$.

Turning on the masses has explicitly broken the $SU(N_f)$ flavor symmetry to

$$SU(N_f) \rightarrow U(1)_F^{N_f-1} \tag{13.7}$$

while the $U(N_c)$ gauge group is also broken completely in the vacuum. (Strictly speaking it is a combination of the $U(N_c)$ gauge group and $U(1)_F^{N_f-1}$ that survives in the vacuum).

13.2 Domain Wall Equations

The existence of isolated vacua implies the existence of a domain wall, a configuration that interpolates from a given vacuum Ξ_- at $x^3 \rightarrow -\infty$ to a distinct vacuum Ξ_+ at $x^3 \rightarrow +\infty$. As in each previous lecture, we can derive the first order equations satisfied by the domain wall using the Bogomoln'yi trick. We'll chose x^3 to be the direction transverse to the wall, and set

$\partial_0 = \partial_1 = \partial_3 = 0$ as well as $A_0 = A_1 = A_2 = 0$. The tension of the domain wall can be written as [246]

$$\begin{aligned} T_{\text{wall}} &= \int dx^3 \, \frac{1}{e^2}\text{Tr}\left(\mathcal{D}_3\phi + \frac{e^2}{2}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2)\right)^2 - \mathcal{D}_3\phi\,(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2) \\ &\quad + \sum_{i=1}^{N_f}\Big(|\mathcal{D}_3 q_i + (\phi - m_i)q_i|^2 - q_i^\dagger(\phi - m_i)\mathcal{D}_3 q_i - \mathcal{D}_3 q_i^\dagger(\phi - m_i)q_i\Big) \\ &\geq v^2\ [\text{Tr}\phi]^{+\infty}_{-\infty} \end{aligned} \tag{13.8}$$

With our vacua Ξ_- and Ξ_+ at left and right infinity, we have the tension of the domain wall bounded by

$$T_{\text{wall}} \geq v^2[\text{Tr}\phi]^{+\infty}_{-\infty} = v^2 \sum_{i\in\Xi_+} m_i - v^2 \sum_{i\in\Xi_-} m_i \tag{13.9}$$

and the minus signs have been chosen so that this quantity is positive (if this isn't the case we must swap left and right infinity and consider the anti-wall). The bound is saturated when the domain wall equations are satisfied,

$$\mathcal{D}_3\phi = -\frac{e^2}{2}(\sum_{i=1}^{N_f} q_i q_i^\dagger - v^2) \qquad \mathcal{D}_3 q_i = -(\phi - m_i)q_i \tag{13.10}$$

Just as the monopole equations $\mathcal{D}\phi = B$ arise as the dimensional reduction of the instanton equations $F = {}^\star F$, so the domain wall equations (13.10) arise from the dimensional reduction of the vortex equations. To see this, we look for solutions to the vortex equations with $\partial_2 = 0$ and relabel $x^1 \to x^3$ and $(A_1, A_2) \to (A_3, \phi)$. Finally, the analogue of turning on the vev in going from the instanton to the monopole, is to turn on the masses m_i in going from the vortex to the domain wall. These can be thought of as a "vev" for $SU(N_f)$ the flavor symmetry.

13.2.1 An Example

The simplest theory admitting a domain wall is $U(1)$ with $N_f = 2$ scalars q_i. The domain wall equations are

$$\partial_3\phi = -\frac{e^2}{2}(|q_1|^2 + |q_2|^2 - v^2) \qquad \mathcal{D}_3 q_i = -(\phi - m_i)q_i \tag{13.11}$$

We'll chose $m_2 = -m_1 = m$. The general solution to these equations is not known. The profile of the wall depends on the value of the dimensionless constant $\gamma = e^2v^2/m^2$. For $\gamma \ll 1$, the wall can be shown to have a three layer structure, in which the q_i fields decrease to zero in the outer layers, while ϕ interpolates between its two expectation values at a more leisurely pace [211]. The result is a domain wall with width $L_{\text{wall}} \sim m/e^2v^2$. Outside of the wall, the fields asymptote exponentially to their vacuum values.

In the opposite limit $\gamma \gg 1$, the inner segment collapses and the two outer layers coalesce, leaving us with a domain wall of width $L_{\text{wall}} \sim 1/m$. In fact, if we take the limit $e^2 \to \infty$, the first equation (13.11) becomes algebraic while the second is trivially solved. We find the profile of the domain wall to be [212]

$$q_1 = \frac{v}{A}\, e^{-m(x_3 - X) + i\theta} \qquad q_2 = \frac{v}{A}\, e^{+m(x_3 - X) - i\theta} \tag{13.12}$$

where $A^2 = e^{-2m(x_3 - X)} + e^{+2m(x_3 - X)}$.

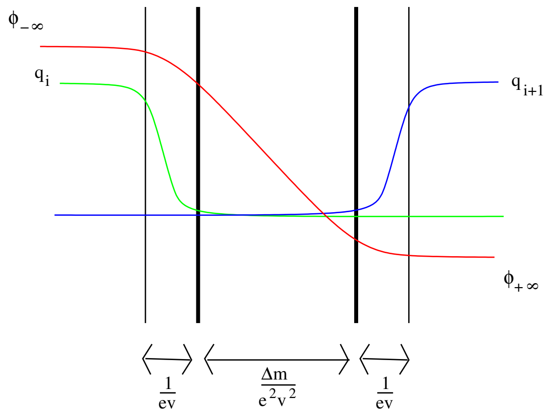


Figure 13.1: The three layer structure of the domain wall when $e^2v^2 \ll m^2$.

The solution (13.12) that we've found in the $e^2 \to \infty$ limit has two collective coordinates, X and θ. The former is simply the position of the domain wall in the transverse x^3 direction. The latter is also easy to see: it arises from acting on the domain wall with the $U(1)_F$ flavor symmetry of the theory [210]:

$$U(1)_F : q_1 \to e^{i\theta} q_1 \quad q_2 \to e^{-i\theta} q_2 \tag{13.13}$$

In each vacuum, this coincides with the $U(1)$ gauge symmetry. However, in the interior of the domain wall, it acts non-trivially, giving rise to a phase collective coordinate θ for the solution. It can be shown that X and θ remain the only two collective coordinates of the domain wall when we return to finite e^2 [213].

13.2.2 Classification of Domain Walls

So we see above that the simplest domain wall has two collective coordinates. What about the most general domain wall, characterized by the choice of vauca Ξ_- and Ξ_+ at left and right infinity. At first sight it appears a little daunting to classify these objects. After all, a strict classification of the topological charge requires a statement of the vacuum at left and right infinity, and the number of vacua increases exponentially with N_f. To ameliorate this sense of confusion, it will help to introduce a coarser classification of domain walls which will capture some information about the topological sector, without specifying the vacua completely. This classification, introduced in [214], will prove most useful when relating our domain walls to the other solitons we've met previously. To this end, define the N_f-vector

$$\vec{m} = (m_1, ..., m_{N_f}) \tag{13.14}$$

We can then write the tension of the domain wall as

$$T_{\text{wall}} = v^2 \vec{g} \cdot \vec{m} \tag{13.15}$$

which defines a vector $\vec{g}$ that contains entries 0 and ± 1 only. Following the classification of monopoles in Lecture 2, let's decompose this vector as

$$\vec{g} = \sum_{i=1}^{N_f} n_i \vec{\alpha}_i \tag{13.16}$$

with $n_i \in \mathbf{Z}$ and the $\vec{\alpha}_i$ the simple roots of $su(N_f)$,

$$\begin{aligned}\vec{\alpha}_1 &= (1,-1,0,...,0)\\ \vec{\alpha}_2 &= (0,1,-1,...,0)\\ \vec{\alpha}_{N_f-1} &= (0,...,0,1,-1)\end{aligned}$$

Since the vector $\vec{g}$ can only contain 0's, 1's and -1's, the integers n_i cannot be arbitrary. It's not hard to see that this restriction means that neighboring n_i's are either equal or differ by one: $n_i = n_{i+1}$ or $n_i = n_{i+1} \pm 1$.

13.3 The Moduli Space

A choice of $\vec{g}$ does not uniquely determine a choice of vacua at left and right infinity. Nevertheless, domain wall configurations which share the same $\vec{g}$ share certain characteristics, including the number of collective coordinates. The collective coordinates carried by a given domain wall was calculated in a number of situations in [215, 216, 217]. Using our classification, the index theorem tells us that there are solutions to the domain wall equations (13.10) only if $n_i \geq 0$ for all i. Then the number of collective coordinates is given by [214],

$$\dim \mathcal{W}_{\vec{g}} = 2\sum_{i=1}^{N_f-1} n_i \tag{13.17}$$

where $\mathcal{W}_{\vec{g}}$ denotes the moduli space of any set of domain walls with charge $\vec{g}$. Again, this should be looking familiar! Recall the result for monopoles with charge $\vec{g}$ was $\dim(\mathcal{M}_{\vec{g}}) = 4\sum_a n_a$. The interpretation of the result (13.17) is, as for monopoles, that there are $N_f - 1$ elementary types of domain walls associated to the simple roots $\vec{g} = \vec{\alpha}_i$. A domain wall sector in sector $\vec{g}$ then splits up into $\sum_i n_i$ elementary domain walls, each with its own position and phase collective coordinate.

13.3.1 The Moduli Space Metric

The low-energy dynamics of multiple, parallel, domain walls is described, in the usual fashion, by a sigma-model from the domain wall worldvolume to the target space is $\mathcal{W}_{\vec{g}}$. As with other solitons, the domain walls moduli space $\mathcal{W}_{\vec{g}}$ inherits a metric from the zero modes of the solution. In notation such that $q = q^a{}_i$ is an $N_c \times N_f$ matrix, the linearized domain wall equations (13.10)

$$\begin{aligned}\mathcal{D}_3\delta\phi - i[\delta A_3, \phi] &= -\frac{e^2}{2}(\delta q\, q^\dagger + q\delta q^\dagger)\\ \mathcal{D}_3\delta q - i\delta A_3\, q &= -(\phi\delta q + \delta\phi\, q - \delta q\, m)\end{aligned} \tag{13.18}$$

where $m = \mathrm{diag}(m_1, ..., m_{N_f})$ is an $N_f \times N_f$ matrix. Again, these are to be supplemented by a background gauge fixing condition,

$$\mathcal{D}_3\delta A_3 - i[\phi, \delta\phi] = i\frac{e^2}{2}(q\delta q^\dagger - \delta q\, q^\dagger) \tag{13.19}$$

and the metric on the moduli space $\mathcal{W}_{\vec{g}}$ is defined by the overlap of these zero modes,

$$g_{\alpha\beta} = \int dx^3\ \mathrm{Tr}\left(\frac{1}{e^2}\left[\delta_\alpha A_3\,\delta_\beta A_3 + \delta_\alpha\phi\,\delta_\beta\phi\right] + \delta_\alpha q\,\delta_\beta q^\dagger + \delta_\beta q\,\delta_\alpha q^\dagger\right) \tag{13.20}$$

By this stage, the properties of the metric on the soliton moduli space should be familiar. They include.

- The metric is Kähler.
- The metric is smooth. There is no singularity as two domain walls approach each other.
- The metric inherits a $U(1)^{N-1}$ isometry from the action of the unbroken flavor symmetry (13.7) acting on the domain wall.

13.3.2 Examples of Domain Wall Moduli Spaces

Let's give some simple examples of domain wall moduli spaces.

13.3.2One Domain Wall

We've seen that a single elementary domain wall $\vec{g} = \vec{\alpha}_1$ (for example, the domain wall described above in the theory with $N_c = 1$ and $N_f = 2$) has two collective coordinates: its center of mass X and a phase θ. The moduli space is

$$\mathcal{W}_\alpha \cong \mathbf{R} \times \mathbf{S}^1 \tag{13.21}$$

The metric on this space is simple to calculate. It is

$$ds^2 = (v^2 \vec{m} \cdot \vec{g})\, dX^2 + v^2\, d\theta^2 \tag{13.22}$$

with the phase collective coordinate living in $\theta \in [0, 2\pi)$.

13.3.2Two Domain Walls

We can't have two domain walls of the same type, say $\vec{g} = 2\vec{\alpha}_1$, since there is no choice of vacua that leads to this charge. Two elementary domain walls must necessarily be of different types, $\vec{g} = \vec{\alpha}_i + \vec{\alpha}_j$ for $i \neq j$. Let's consider $\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_2$.

The moduli space is simplest to describe if the two domain walls have the same mass, so $\vec{m} \cdot \vec{\alpha}_a = \vec{m} \cdot \vec{\alpha}_b$. The moduli space is

$$\mathcal{W}_{\vec{\alpha}_1 + \vec{\alpha}_2} \cong \mathbf{R} \times \frac{\mathbf{S}^1 \times \mathcal{M}_{\text{cigar}}}{\mathbf{Z}_2} \tag{13.23}$$

where the interpretation of the $\mathbf{R}$ factor and $\mathbf{S}^1$ factor are the same as before. The relative moduli space has the topology and asymptotic form of a cigar, The relative separation between

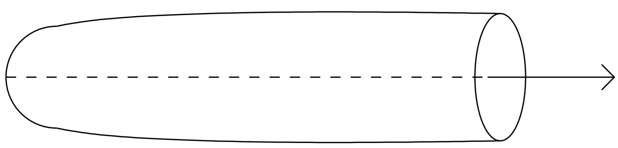


Figure 13.2: The relative moduli space of two domain walls is a cigar.

domain walls is denoted by R. The tip of the cigar, $R = 0$, corresponds to the two domain walls sitting on top of each other. At this point the relative phase of the two domain walls degenerates, resulting in a smooth manifold. The metric on this space has been computed in the $e^2 \to \infty$ limit, although it's not particularly illuminating [212] and gives a good approximation to the metric at large finite e^2 [218]. Asymptotically, it deviates from the flat metric on the cylinder by exponentially suppressed corrections e^{-R}, as one might expect since the profile of the domain walls is exponentially localized.

13.4 Dyonic Domain Walls

You will have noticed that, rather like monopoles, the domain wall moduli space includes a phase collective coordinate $\mathbf{S}^1$ for each domain wall. For the monopole, excitations along this $\mathbf{S}^1$ give rise to dyons, objects with both magnetic and electric charges. For domain walls, excitations along this $\mathbf{S}^1$ also give rise to dyonic objects, now carrying both topological (kink) charge and flavor charge. Abraham and Townsend called these objects "Q-kinks" [210].

First order equations of motion for these dyonic domain walls may be obtained by completing the square in the Lagrangian (13.1), now looking for configurations that depend on both x^0 and x^3, allowing for a non-zero electric field F_{03}. We have

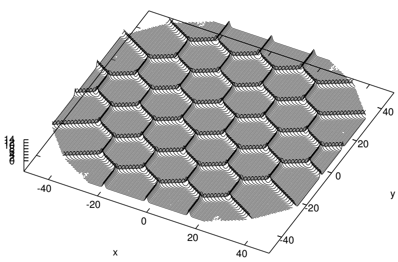
$$\begin{aligned} T_{\rm wall} \;=\; & \int dx^3\, \frac{1}{e^2}\mathrm{Tr}\left(\cos\alpha\,\mathcal{D}_3\phi + \frac{e^2}{2}(\sum_{i=1}^{N_f}|q_iq_i^\dagger - v^2)\right)^2 - \cos\alpha\,\mathcal{D}_3\phi\,(\sum_{i=1}^{N_f} q_iq_i^\dagger - v^2) \\ & + \sum_{i=1}^{N_f}\Big(|\mathcal{D}_3 q_i + \cos\alpha\,(\phi - m_i)q_i|^2 - \cos\alpha\,(q_i^\dagger(\phi-m_i)\mathcal{D}_3 q_i + \text{ h.c.})\Big) \\ & + \frac{1}{e^2}\mathrm{Tr}(F_{03} - \sin\alpha\,\mathcal{D}_3\phi)^2 + \frac{1}{e^2}\sin\alpha\,F_{03}\mathcal{D}_3\phi \\ & + \sum_{i=1}^{N_f}\Big(|\mathcal{D}_0 q_i + i\sin\alpha(\phi - m_i)q_i|^2 - \sin\alpha(iq_i^\dagger(\phi - m_i)\mathcal{D}_0 q_i + \text{ h.c})\Big) \end{aligned}$$

As usual, insisting upon the vanishing of the total squares yields the Bogomoln'yi equations. These are now to augmented with Gauss' law,

$$\mathcal{D}_3 F_{03} = ie^2\sum_{i=1}^{N_f}(q_i\mathcal{D}_0 q_i^\dagger - (\mathcal{D}_0 q_i)q_i^\dagger) \tag{13.24}$$

Using this, we may re-write the cross terms in the energy-density to find the Bogomoln'yi bound,

$$T_{\rm wall} \geq \pm v^2[\mathrm{Tr}\phi]^{+\infty}_{-\infty}\,\cos\alpha + (\vec{m}\cdot\vec{S})\sin\alpha \tag{13.25}$$



where $\vec{S}$ is the Noether charge associated to the surviving $U(1)^{N_f-1}$ flavor symmetry, an N_f-vector with $i^{\rm th}$ component given by

$$S_i = i(q_i\mathcal{D}_0 q_i^\dagger - (\mathcal{D}_0 q_i)q_i^\dagger) \tag{13.26}$$

Maximizing with respect to α results in the Bogomoln'yi bound for dyonic domain walls,

$$\mathcal{H} \geq \sqrt{v^4(\vec{m}\cdot\vec{g})^2 + (\vec{m}\cdot\vec{S})^2} \tag{13.27}$$

This square-root form is familiar from the spectrum of dyonic monopoles that we saw in Lecture 2. More on this soon. For now, some further comments, highlighting the some similarities between dyonic domain walls and monopoles.

- There is an analog of the Witten effect. In two dimensions, where the domain walls are particle-like objects, one may add a theta term of the form θF_{01}. This induces a flavor charge on the domain wall, proportional to its topological charge, $\vec{S} \sim \vec{g}$ [219].
- One can construct dyonic domain walls with $\vec{g}$ and $\vec{S}$ not parallel if we turn on complex masses and, correspondingly, consider a complex adjoint scalar ϕ [220, 221]. The resulting 1/4 and 1/8-BPS states are the analogs of the 1/4-BPS monopoles we briefly mentioned in Lecture 2.
- The theory with complex masses also admits interesting domain wall junction configurations [222, 223]. Most notably, Eto et al. have recently found beautiful webs of domain walls, reminiscent of (p, q)5-brane webs of IIB string theory, with complicated moduli as the strands of the web shift, causing cycles to collapse and grow [224, 225]. Examples include the intricate honeycomb structure shown in figure 24 (taken from [224]).

Other aspects of these domain walls were discussed in [226, 227, 228, 229, 230]. A detailed discussed of the mass renormalization of supersymmetric kinks in $d = 1 + 1$ dimensions can be found in [231, 232].

13.5 The Ordering of Domain Walls

The cigar moduli space for two domain walls illustrates an important point: domain walls cannot pass each other. In contrast to other solitons, they must satisfy a particular ordering on the line. This is apparent in the moduli space of two domain walls since the relative separation takes values in $R \in \mathbf{R}^+$ rather than $\mathbf{R}$. The picture in spacetime shown in figure 25.

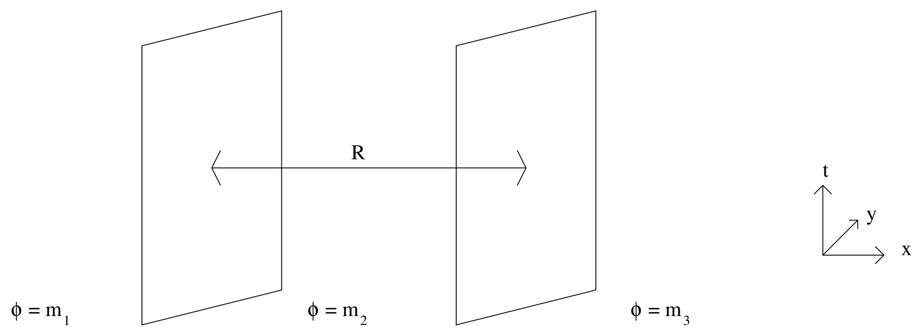


Figure 13.3: Two interacting domain walls cannot pass through each other. The $\vec{\alpha}_1$ domain wall is always to the left of the $\vec{\alpha}_2$ domain wall.

However, it's not always true that domain walls cannot pass through each other. Domain walls which live in different parts of the flavor group, so have $\vec{\alpha}_i \cdot \vec{\alpha}_j = 0$, do not interact so can happily move through each other. When these domain walls are two of many in a topological sector $\vec{g}$, an interesting pattern of interlaced walls arises, determined by which walls bump into each other, and which pass through each other. This pattern was first explored in [217]. Let's see how the ordering emerges. Start at left infinity in a particular vacuum Ξ_-. Then each elementary domain wall shifts the vacuum by increasing a single element $\xi(a) \in \Xi$ by one. The restriction that the N_c elements are distinct means that only certain domain walls can occur. This point is one that is best illustrated by a simple example:

13.5.0An Example: $N_c = 2$, $N_f = 4$

Consider the domain walls in the $U(2)$ theory with $N_f = 4$ flavors. We'll start at left infinity in the vacuum $\Xi_- = \{1,2\}$ and end at right infinity in the vacuum $\Xi_+ = \{3,4\}$. There are two different possibilities for the intermediate vacua. They are:

$$\Xi_- = \{1,2\} \longrightarrow \{1,3\} \longrightarrow \{1,4\} \longrightarrow \{2,4\} \longrightarrow \{3,4\} = \Xi_+$$
$$\Xi_- = \{1,2\} \longrightarrow \{1,3\} \longrightarrow \{2,3\} \longrightarrow \{2,4\} \longrightarrow \{3,4\} = \Xi_+$$

In terms of domain walls, these two ordering become,

$$\begin{array}{c} \vec{\alpha}_2 \longrightarrow \vec{\alpha}_3 \longrightarrow \vec{\alpha}_1 \longrightarrow \vec{\alpha}_2 \\ \vec{\alpha}_2 \longrightarrow \vec{\alpha}_1 \longrightarrow \vec{\alpha}_3 \longrightarrow \vec{\alpha}_2 \end{array} \tag{13.28}$$

We see that the two $\vec{\alpha}_2$ domain walls must play bookends to the $\vec{\alpha}_1$ and $\vec{\alpha}_3$ domain walls. However, one expects that these middle two walls are able to pass through each other.

13.5.0The General Ordering of Domain Walls

We may generalize the discussion above to deduce the rule for ordering of general domain walls [217]. One finds that the n_i elementary $\vec{\alpha}_i$ domain walls must be interlaced between the $\vec{\alpha}_{i-1}$ and $\vec{\alpha}_{i+1}$ domain walls. (Recall that $n_i = n_{i+1}$ or $n_i = n_{i+1} \pm 1$ so the concept of interlacing is well defined). The final pattern of domain walls is captured in figure 26, where x^3 is now plotted horizontally and the vertical position of the domain wall simply denotes its type. We shall see this vertical position take on life in the D-brane set-up we describe shortly.

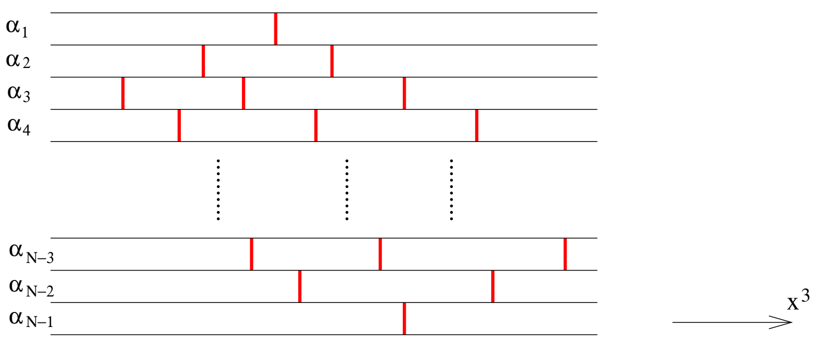


Figure 13.4: The ordering of many domain walls. The horizontal direction is their position, while the vertical denotes the type of domain wall.

Notice that the $\vec{\alpha}_1$ domain wall is trapped between the two $\vec{\alpha}_2$ domain walls. These in turn are trapped between the three $\vec{\alpha}_3$ domain walls. However, the relative positions of the $\vec{\alpha}_1$ and middle $\vec{\alpha}_3$ domain walls are not fixed: these objects can pass through each other.

13.6 What Became Of......

Now let's play our favorite game and ask what happened to the other solitons now that we've turned on the masses. We start with.....

13.6.1 Vortices

The vortices described in the previous lecture enjoyed zero modes arising from their embedding in $SU(N)_{\rm diag} \subset U(N_c) \times SU(N_f)$. Let go back to the situation with $N_f = N_c = N$, but with the extra terms from (13.1) added to the Lagrangian,

$$V = \frac{1}{e^2}{\rm Tr}\,(\mathcal{D}_\mu\phi)^2 + \sum_{i=1}^{N} q_i^\dagger(\phi - m_i)^2 q_i \tag{13.29}$$

As we've seen, this mass term breaks $SU(N)_{\rm diag} \rightarrow U(1)^{N-1}_{\rm diag}$, which means we can no longer rotate the orientation of the vortices within the gauge and flavor groups. We learn that the masses are likely to lift some zero modes of the vortex moduli space [233, 158, 177].

The vortex solutions that survive are those whose energy isn't increased by the extra terms in V above. Or, in other words, those vortex configurations which vanish when evaluated on V above. If we don't want the vortex to pick up extra energy from the kinetic terms $\mathcal{D}\phi^2$, we need to keep ϕ in its vacuum,

$$\phi = {\rm diag}(\phi_1, ..., \phi_N) \tag{13.30}$$

which means that only the components $q^a_{\ i} \sim \delta^a_{\ i}$ can turn on keeping $V = 0$.

For the single vortex $k = 1$ in $U(N)$, this means that the internal moduli space $\mathbb{CP}^{N-1}$ is lifted, leaving behind N different vortex strings, each with magnetic field in a different diagonal component of the gauge group,

$$\begin{aligned} B_3 &= {\rm diag}(0, ..., B_3^\star, ..., 0) \\ q &= {\rm diag}(v, ..., q^\star, ..., v) \end{aligned} \tag{13.31}$$

In summary, rather than having a moduli space of vortex strings, we are left with N different vortex strings, each carrying magnetic flux in a different $U(1) \subset U(N)$.

How do we see this from the perspective of the vortex worldsheet? We can re-derive the vortex theory using the brane construction of the previous lecture, but now with the D6-branes separated in the x^4 direction, providing masses m_i for the hypermultiplets q_i. After performing the relevant brane-game manipulations, we find that these translate into masses m_i for the chiral multiplets in the vortex theory. The potential for the vortex theory (3.30) is replaced by,

$$\begin{aligned} V &= \frac{1}{g^2}{\rm Tr}\,|[\sigma,\sigma^\dagger]|^2 + {\rm Tr}\,|[\sigma, Z]|^2 + {\rm Tr}\,|[\sigma, Z^\dagger]|^2 \\ &\quad + \sum_{a=1}^{N}\psi_a^\dagger(\sigma - m_a)^2\psi_a + \frac{g^2}{2}\,{\rm Tr}\,\left(\sum_a \psi_a\psi_a^\dagger + [Z, Z^\dagger] - r\,1_k\right)^2 \end{aligned} \tag{13.32}$$

where $r = 2\pi/e^2$ as before. The masses m_i of the four-dimensional theory have descended to masses m_a on the vortex worldsheet.

To see the implications of this, consider the theory on a single $k = 1$ vortex. The potential is simply,

$$V_{k=1} = \sum_{a=1}^{N} (\sigma - m_a)^2 |\psi_a|^2 + \frac{g^2}{2} (\sum_{a=1}^{N} |\psi_a|^2 - r)^2 \tag{13.33}$$

Whereas before we could set $\sigma = 0$, leaving ψ_a to parameterize $\mathbb{CP}^{N-1}$, now the Higgs branch is lifted. We have instead N isolated vacua,

$$\sigma = m_a \qquad |\psi_b|^2 = r\delta_{ab} \tag{13.34}$$

These correspond to the N different vortex strings we saw above.

13.6.1A Potential on the Vortex Moduli Space

We can view the masses m_i as inducing a potential on the Higgs branch of the vortex theory after integrating out σ. This potential is equal to the length of Killing vectors on the Higgs branch associated to the $U(1)^{N-1} \subset SU(N)_{\rm diag}$ isometry. This is the same story we saw in Lecture 2.6, where the a vev for ϕ induced a potential on the instanton moduli space.

In fact, just as we saw for instantons, this result can also be derived directly within the field theory itself [177]. Suppose we fix a vortex configuration (A_z, q) that solves the vortex equations before we introduce masses. We want to determine how much the new terms (13.29) lift the energy of this vortex. We minimize V by solving the equation of motion for ϕ in the background of the vortex,

$$\mathcal{D}^2\phi = \frac{e^2}{2} \sum_{i=1}^{N} \{\phi, q_i q_i^\dagger\} - 2 q_i q_i^\dagger m_i \tag{13.35}$$

subject to the vev boundary condition $\phi \to \mathrm{diag}(m_1, ..., m_N)$ as $r \to \infty$. But we have seen this equation before! It is precisely the equation (3.21) that an orientational zero mode of the vortex must satisfy. This means that we can write the excess energy of the vortex in terms of the relevant orientational zero mode

$$V = \int d^2x \; \frac{2}{e^2} \mathrm{Tr}\, \delta A_z \, \delta A_{\bar z} + \frac{1}{2} \sum_{i=1}^{N} \delta q_i \, \delta q_i^\dagger \tag{13.36}$$

for the particular orientation zero mode $\delta A_z = \mathcal{D}_z \phi$ and $\delta q_i = i(\phi q_i - q_i m_i)$. We can give a nicer geometrical interpretation to this following the discussion in Section 2.6. Denote the Cartan subalgebra of $SU(N)_{\rm diag}$ as $\vec{H}$, and the associated Killing vectors on $\mathcal{V}_{k,N}$ as $\vec{k}_\alpha$. Then, since ϕ generates the transformation $\vec{m} \cdot \vec{H}$, we can express our zero mode in terms of the basis $\delta A_z = (\vec{m} \cdot \vec{k}^\alpha)\, \delta_\alpha A_z$ and $\delta q_i = (\vec{m} \cdot \vec{k}^\alpha)\, \delta_\alpha q_i$. Putting this in our potential and performing the integral over the zero modes, we have the final expression

$$V = g_{\alpha\beta}\, (\vec{m} \cdot \vec{k}^\alpha)\, (\vec{m} \cdot \vec{k}^\beta) \tag{13.37}$$

This potential vanishes at the fixed points of the $U(1)^{N-1}$ action. For the one-vortex moduli space $\mathbb{CP}^{N-1}$, it's not hard to see that this gives rise to the N vacuum states described above (13.34).

13.6.2 Monopoles

To see where the monopoles have gone, it's best if we first look at the vortex worldsheet theory [233]. This is now have a $d = 1+1$ dimensional theory with isolated vacua, guaranteeing the existence of domain wall, or kink, in the worldsheet. In fact, for a single $k = 1$ vortex, the theory on the worldsheet is precisely of the form (13.1) that we started with at the beginning of this lecture. (For $k > 1$, the presence of the adjoint scalar Z means that isn't precisely the same action, but is closely related). The equations describing kinks on the worldsheet are the same as (13.10),

$$\partial_3\sigma = g^2(\sum_{a=1}^{N}|\psi_a|^2 - r) \qquad \mathcal{D}_3\psi_a = (\sigma - m_a)\psi_a \tag{13.38}$$

where we should take the limit $g^2 \to \infty$, in which the first equation becomes algebraic. What's the interpretation of this kink on the worldsheet? We can start by examining its mass,

$$M_{\rm kink} = (\vec{m}\cdot\vec{g})\,r = \frac{2\pi}{e^2}\,(\vec{\phi}\cdot\vec{g}) = M_{\rm mono} \tag{13.39}$$

So the kink has the same mass as the monopole! In fact, it also has the same quantum numbers. To see this, recall that the different vacua on the vortex string correspond to flux tubes lying in different $U(1) \subset U(N)$ subgroups. For example, for $N = 2$, the kink must take the form shown in figure 27. So whatever the kink is, it must soak up magnetic field $B = \mathrm{diag}(0,1)$

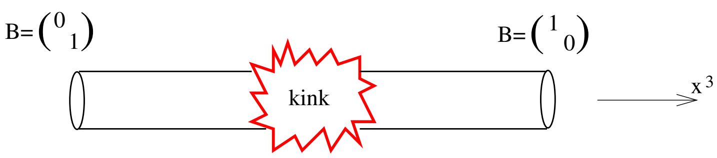


Figure 13.5: The kink on the vortex string.

and spit out magnetic field $B = \mathrm{diag}(1,0)$. In other words, it is a source for the magnetic field $B = \mathrm{diag}(1,-1)$. This is precisely the magnetic field sourced by an $SU(2)$ 't Hooft-Polyakov monopole.

What's happening here? We are dealing with a theory with a mass gap, so any magnetic monopole that lives in the bulk can't emit a long-range radial magnetic field since the photon can't propagate. We're witnessing the Meissner effect in a non-abelian superconductor. The monopole is confined, its magnetic field departing in two semi-classical flux tubes. This effect is, of course, well known and it is conjectured that a dual effect leads to the confinement of quarks in QCD. Here we have a simple, semi-classical realization in which to explore this scenario.

Can we find the monopole in the $d = 3+1$ dimensional bulk? Although no solution is known, it turns out that we can write down the Bogomoln'yi equations describing the configuration [233]. Let's go back to our action (13.1) and complete the square in a different way. We now insist only that $\partial_0 = A_0 = 0$, and write the Hamiltonian as,

$$\begin{aligned}\mathcal{H} &= \int d^3x\, \frac{1}{e^2}\mathrm{Tr}\, \left[(\mathcal{D}_1\phi + B_1)^2 + (\mathcal{D}_2 + B_2)^2 + (\mathcal{D}_3\phi + B_3 - \frac{e^2}{2}(\sum_{i=1}^N q_i q_i^\dagger - v^2))^2\right] \\ &\quad + \sum_{i=1}^N |(\mathcal{D}_1 - i\mathcal{D}_2)q_i|^2 + \sum_{i=1}^N |\mathcal{D}_3 q_i - (\phi - m_i)q_i|^2 + \mathrm{Tr}\,[-v^2 B_3 - \frac{2}{e^2}\partial_i(\phi B_i)] \\ &\geq \left(\int dx^3\, T_{\rm vortex}\right) + M_{\rm mono} \end{aligned} \tag{13.40}$$

where the inequality is saturated when the terms in the brackets vanish,

$$\begin{aligned} \mathcal{D}_1\phi + B_1 = 0 \quad &, \quad \mathcal{D}_1 q_i = i\mathcal{D}_2 q_i \\ \mathcal{D}_2\phi + B_2 = 0 \quad &, \quad \mathcal{D}_3 q_i = (\phi - m_i)q_i \\ \mathcal{D}_3\phi + B_3 \ &= \ \frac{e^2}{2}(\sum_{i=1}^N q_i q_i^\dagger - v^2) \end{aligned} \tag{13.41}$$

As you can see, these are an interesting mix of the monopole equations and the vortex equations. In fact, they also include the domain wall equations — we'll see the meaning of this when we come to discuss the applications. These equations should be thought of as the master equations for BPS solitons, reducing to the other equations in various limits. Notice moreover that these equations are over-determined, but it's simple to check that they satisfy the necessary integrability conditions to admit solutions. However, no non-trivial solutions are known analytically. (Recall that even the solution for a single vortex is not known in closed form). We expect that there exist solutions that look like figure 28.

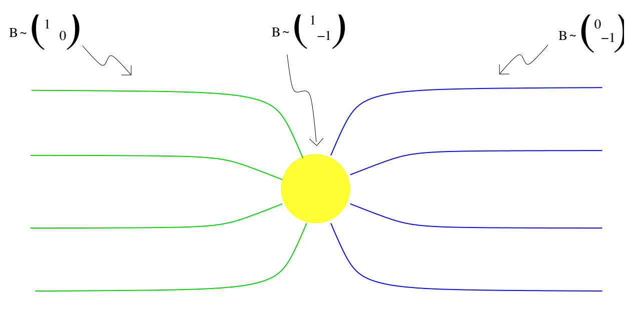


Figure 13.6: The confined magnetic monopole.

The above discussion was for $k = 1$ and $N_f = N_c$. Extensions to $k \geq 2$ and also to $N_f \geq N_c$ also exist, although the presence of the adjoint scalar Z on the vortex worldvolume means that the kinks on the string aren't quite the same as the domain wall equations (13.10). But if we set $Z = 0$, so that the strings lie on top of each other, then the discussion of domain walls in four-dimensions carries over to kinks on the string. In fact, it's not hard to check that we've chosen our notation wisely: magnetic monopoles of charge $\vec{g}$ descend to kinks on the vortex strings with topological charge $\vec{g}$.

In summary,

Kink on the Vortex String = Confined Magnetic Monopole

The BPS confined monopole was first described in [233], but the idea that kinks on string should be interpreted as confined monopoles arose previously in [234] in the context of Z_N flux tubes. More recently, confined monopoles have been explored in several different theories [235, 236, 237, 238]. We'll devote Section 4.7 to more discussion on this topic.

13.6.3 Instantons

We now ask what became of instantons. At first glance, it doesn't look promising for the instanton! In the bulk, the FI term v^2 breaks the gauge group, causing the instanton to shrink. And the presence of the masses means that even in the center of various solitons, there's only a $U(1)$ restored, not enough to support an instanton. For example, an instanton wishing to nestle within the core of the vortex string shrinks to vanishing size and it looks as if the theory (13.1) admits only singular, small instantons.

While the above paragraph is true, it also tells us how we should change our theory to allow the instantons to return: we should consider non-generic mass parameters, so that the $SU(N_f)$ flavor symmetry isn't broken to the maximal torus, but to some non-abelian subgroup. Let's return to the example discussed in Section 4.5: $U(2)$ gauge theory with $N_f = 4$ flavors. Rather than setting all masses to be different, we chose $m_1 = m_2 = m$ and $m_3 = m_4 = -m$. In this limit, the breaking of the flavor symmetry is $SU(4) \to S[U(2) \times U(2)]$, and this has interesting consequences.

To find our instantons, we look at the domain wall which interpolates between the two vacua $\phi = m\mathbf{1}_2$ and $\phi = -m\mathbf{1}_2$. When all masses were distinct, this domain wall had 8 collective coordinates which had the interpretation of the position and phase of 4 elementary domain walls (13.28). Now that we have non-generic masses, the domain wall retains all 8 collective coordinates, but some develop a rather different interpretation: they correspond to new orientation modes in the unbroken flavor group. In this way, part of the domain wall theory becomes the $SU(2)$ chiral Lagrangian [239].

Inside the domain wall, the non-abelian gauge symmetry is restored, and the instantons may safely nestle there, finding refuge from the symmetry breaking of the bulk. One can show that, from the perspective of the domain wall worldvolume theory, they appear as Skyrmions [240]. Indeed, closer inspection reveals that the low-energy dynamics of the domain wall also includes a four derivative term necessary to stabilize the Skyrmion, and one can successfully compare the action of the instanton and Skyrmion. The relationship between instantons and Skyrmions was first noted long ago by Atiyah and Manton [241], and has been studied recently in the context of deconstruction [242, 243, 244].

13.7 The Quantum Vortex String

So far our discussion has been entirely classical. Let's now turn to the quantum theory. We have already covered all the necessary material to explain the main result. The basic idea is that $d = 1 + 1$ worldsheet theory on the vortex string captures quantum information about the $d = 3 + 1$ dimensional theory in which it's embedded. If we want certain information about the 4d theory, we can extract it using much simpler calculations in the 2d worldsheet theory.

I won't present all the calculations here, but instead simply give a flavor of the results [158, 177]. The precise relationship here holds for $\mathcal{N}=2$ theories in $d=3+1$, corresponding to $\mathcal{N}=(2,2)$ theories on the vortex worldsheet. The first hint that the 2d theory contains some information about the 4d theory in which its embedded comes from looking at the relationship between the 2d FI parameter and the 4d gauge coupling,

$$r = \frac{4\pi}{e^2} \tag{13.42}$$

This is a statement about the classical vortex solution. Both e^2 in 4d and r in 2d run at one-loop. However, the relationship (13.42) is preserved under RG flow since the beta functions computed in 2d and 4d coincide,

$$r(\mu) = r_0 - \frac{N_c}{2\pi}\log\left(\frac{\mu_{UV}}{\mu}\right) \tag{13.43}$$

This ensures that both 4d and 2d theories hit strong coupling at the same scale $\Lambda = \mu\exp(-2\pi r/N_c)$.

Exact results about the 4d theory can be extracted using the Seiberg-Witten solution [132]. In particular, this allows us to determine the spectrum of BPS states in the theory. Similarly, the exact spectrum of the 2d theory can also be determined by computing the twisted superpotential [219, 245]. The punchline is that the spectrum of the two theories coincide. Let's see what this means. We saw in (13.39) that the classical kink mass coincides with the classical monopole mass

$$M_{\rm kink} = M_{\rm mono} \tag{13.44}$$

This equality is preserved at the quantum level. Let me stress the meaning of this. The left-hand side is computed in the $d=1+1$ dimensional theory. When $(m_i - m_j) \gg \Lambda$, this theory is weakly coupled and $M_{\rm kink}$ receives a one-loop correction (with, obviously, two-dimensional momenta flowing in the loop). Although supersymmetry forbids higher loop corrections, there are an infinite series of worldsheet instanton contributions. The final expression for the mass of the kink schematically of the form,

$$M = M_{\rm clas} + M_{\rm one-loop} + \sum_{n=1}^{\infty} M_{\rm n-inst} \tag{13.45}$$

The right-hand-side of (13.44) is computed in the $d=3+1$ dimensional theory, which is also weakly coupled for $(m_i - m_J) \gg \Lambda$. The monopole mass $M_{\rm mono}$ receives corrections at one-loop (now integrating over four-dimensional momenta), followed by an infinite series of Yang-Mills instanton corrections. *And term by term these two series agree*!

The agreement of the worldsheet and Yang-Mills instanton expansions apparently has its microscopic origin in the results if the previous lecture. Recall that performing an instanton computation requires integration over the moduli space ($\mathcal{V}$ for the worldsheet instantons; $\mathcal{I}$ for Yang-Mills). Localization theorems hold when performing the integrals over $\mathcal{I}_{k,N}$ in $\mathcal{N}=2$ super Yang-Mills, and the final answer contains contributions from only a finite number of points in $\mathcal{I}_{k,N}$ [46]. It is simple to check that all of these points lie on $\mathcal{V}_{k,N}$ which, as we have seen, is a submanifold of $\mathcal{I}_{k,N}$.

The equation (13.44) also holds in strong coupling regimes of the 2d and 4d theories where no perturbative expansion is available. Nevertheless, exact results allow the masses of BPS states to be computed and successfully compared. Moreover, the quantum correspondence between the masses of kinks and monopoles is not the only agreement between the two theories. Other results include:

- The elementary internal excitations of the string can be identified with W-bosons of the 4d theory. When in the bulk, away from the string, these W-bosons are non-BPS. But they can reduce their mass by taking refuge in the core of the vortex whereupon they regain their BPS status.

This highlights an important point: the spectrum of the 4d theory, both for monopoles and W-bosons, is calculated in the Coulomb phase, when the FI parameter $v^2 = 0$. However, the vortex string exists only in the Higgs phase $v^2 \neq 0$. What's going on? A heuristic explanation is as follows: inside the vortex, the Higgs field q dips to zero and the gauge symmetry is restored. The vortex theory captures information about the 4d theory on its Coulomb branch.

- As we saw in Sections 2.3 and 4.4, both the 4d theory and the 2d theory contain dyons. We've already seen that the spectrum of both these objects is given by the "square-root" formula (2.41) and (13.27). Again, these agree at the quantum level.
- Both theories manifest the Witten effect: adding a theta angle to the 4d theory induces an electric charge on the monopole, shifting its mass. This also induces a theta angle on the vortex worldsheet and, hence, turns the kinks into dyons.
- We have here described the theory with $N_f = N_c$. For $N_f > N_C$, the story can be repeated and again the spectrum of the vortex string coincides with the spectrum of the 4d theory in which it's embedded.

In summary, we have known for over 20 years that gauge theories in 4d share many qualitative features with sigma models in 2d, including asymptotic freedom, a dynamically generated mass gap, large N expansions, anomalies and the presences of instantons. However, the vortex string provides a quantitative relationship between the two: in this case, they share the same quantum spectrum.

13.8 The Brane Construction

In Lecture 3, we derived the brane construction for $U(N_c)$ gauge theory with N_f hypermultiplets. To add masses, one must separate the hypermultiplets in the x^4 direction. One can now see the number of vacua (13.6) since each of the N_c D4-branes must end on one of the N_f D6-branes.

To describe a domain wall, the D4-branes must start in one vacua, Ξ_- at $x_3 \to -\infty$, and interpolate to the final vacua Ξ_+ as $x^3 \to +\infty$. Viewing this integrated over all x^3, we have the picture shown in figure 29. To extract the dynamics of domain walls, we need to understand the worldvolume theory of the curved D4-brane. This isn't at all clear. Related issues have troubled previous attempts to extract domain wall dynamics from D-brane set-up [246, 247],

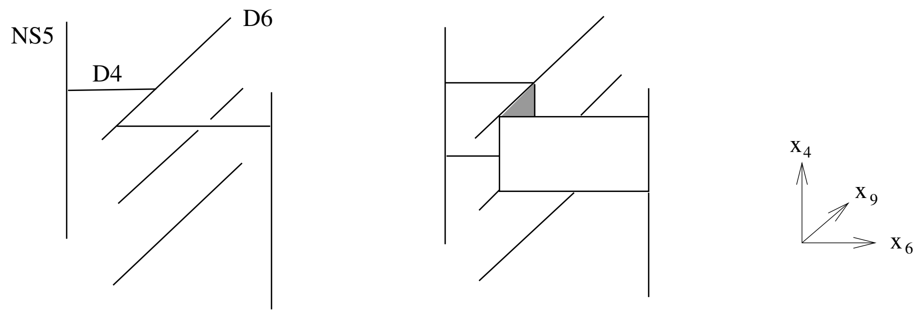


Figure 13.7: The D-brane configuration for an elementary $\vec{g} = \vec{\alpha}_1$ domain wall when $N_c = 1$ and $N_f = 3$.

although some qualitative features can be seen. However, we can make progress by studying this system in the limit $e^2 \to \infty$, so that the two NS5-branes and the N_f D6-branes lie coincident in the x^6 direction [248]. The portions of the D4-branes stretched in x^6 vanish, and we're left with D4-branes with worldvolume 01249, trapped in squares in the 49 directions where they are sandwiched between the NS5 and D6-branes. Returning to the system of domain walls in an arbitrary topological sector $\vec{g} = \sum_i n_i \vec{\alpha}_i$, we have the system drawn in figure 30.

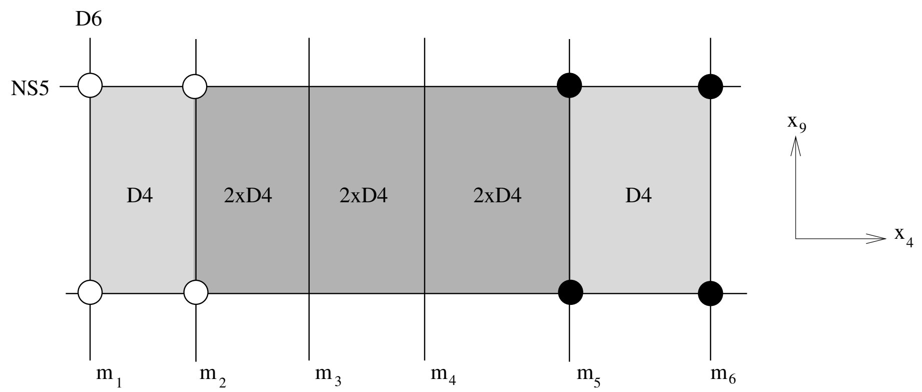


Figure 13.8: The D-brane configuration in the $e^2 \to \infty$ limit.

We can now read off the gauge theory living on the D4-branes. One might expect that it is of the form $\prod_i U(n_i)$. This is essentially correct. The NS5-branes project out the A_9 component of the gauge field, however the A_4 component survives and each $U(n_a)$ gauge theory lives in the interval $m_i \leq x_4 \leq m_{i+1}$. In each segment, we have A_4 and X_3, each an $n_i \times n_i$ matrix. These fields satisfy

$$\frac{dX_3}{dx^4} - i[A_4, X_3] = 0 \tag{13.46}$$

modulo $U(n_i)$ gauge transformations acting on the interval $m_i \leq x_4 \leq m_{i+1}$, and vanishing at the boundaries. These equations are kind of trivial: the interesting details lie in the boundary conditions. As in the case of monopoles, the interactions between neighbouring segments depends on the relative size of the matrices:

$\underline{n_i = n_{i+1}}$: The $U(n_i)$ gauge symmetry is extended to the interval $m_i \leq x_4 \leq m_{i+2}$ and an impurity is added to the right-hand-side of Nahm's equations, which now read

$$\frac{dX_3}{dx_4} - i[A_4, X_3] = \psi\psi^\dagger \delta(x_4 - m_{i+1}) \tag{13.47}$$

where the impurity degree of freedom ψ transforms in the fundamental representation of the $U(n_i)$ gauge group, ensuring the combination $\psi\psi^\dagger$ is a $n_i \times n_i$ matrix transforming, like X_1, in the adjoint representation. These ψ degrees of freedom are chiral multiplets which survive the NS5-brane projection.

$\underline{n_i = n_{i+1} - 1}$: In this case $X_3 \to (X_3)_-$, a $n_i \times n_i$ matrix, as $x_4 \to (m_i)_-$ from the left. To the right of m_i, X_3 is a $(n_i + 1) \times (n_i + 1)$ matrix obeying

$$X_3 \to \left(\begin{array}{cc} y & a^\dagger \\ a & (X)_- \end{array} \right) \quad \quad \text{as } x_4 \to (m_i)_+ \tag{13.48}$$

where $y_\mu \in \mathbf{R}$ and each a_μ is a complex n_i-vector. The obvious analog of this boundary condition holds when $n_i = n_{i+1} + 1$.

These boundary conditions are obviously related to the Nahm boundary conditions for monopoles that we met in Lecture 2.

13.8.1 The Ordering of Domain Walls Revisited

We now come to the important point: the ordering of domain walls. Let's see how the brane construction captures this. We can use the gauge transformations to make A_4 constant over the interval $m_i \leq x^4 \leq m_{i+1}$. Then (13.46) can be trivially integrated in each segment to give

$$X_3(x^4) = e^{iA_4 x^4} \hat{X}_3 e^{-iA_4 x^4} \tag{13.49}$$

Then the positions of the $\vec{\alpha}_i$ domain walls are given by the eigenvalues of X_3 restricted to the interval $m_i \leq x_4 \leq m_{i+1}$. Let us denote this matrix as $X_3^{(i)}$ and the eigenvalues as $\lambda_m^{(i)}$, where $m = 1, ...n_i$. We have similar notation for the $\vec{\alpha}_{i+1}$ domain walls. Suppose first that $n_i = n_{i+1}$. Then the impurity (13.47) relates the two sets of eigenvalues by the jumping condition

$$X_1^{(i+1)} = X_1^{(i)} + \psi\psi^\dagger \tag{13.50}$$

We will now show that this jumping condition (13.50) correctly captures the interlacing nature of neighboring domain walls.

To see this, consider firstly the situation in which $\psi^\dagger\psi \ll \Delta\lambda_m^{(i)}$ so that the matrix $\psi\psi^\dagger$ may be treated as a small perturbation of $X_1^{(i)}$. The positivity of $\psi\psi^\dagger$ ensures that each $\lambda_m^{(i+1)} \geq \lambda_m^{(i)}$. Moreover, it is simple to show that the $\lambda_m^{(i+1)}$ increase monotonically with $\psi^\dagger\psi$. This leaves us to consider the other extreme, in which $\psi^\dagger\psi \to \infty$. It this limit ψ becomes one of the eigenvectors of $X_1^{(i+1)}$ with corresponding eigenvalue $\lambda_{n_i}^{(i+1)} = \psi^\dagger\psi$, corresponding to the limit in which the last domain wall is taken to infinity. What we want to show is that the remaining $n_i - 1$ $\vec{\alpha}_{i+1}$ domain walls are trapped between the n_i $\vec{\alpha}_i$ domain walls as depicted in figure 26. Define the $n_i \times n_i$ projection operator

$$P = 1 - \hat{\psi}\hat{\psi}^\dagger \tag{13.51}$$

where $\hat{\psi} = \psi/\sqrt{\psi^\dagger\psi}$. The positions of the remaining $(n_i - 1)$ $\vec{\alpha}_{i+1}$ domain walls are given by the (non-zero) eigenvalues of $PX_1^{(i)}P$. We must show that, given a rank n hermitian matrix X, the eigenvalues of PXP are trapped between the eigenvalues of X. This well known property of hermitian matrices is simple to show:

$$\begin{aligned}\det(PXP - \mu) &= \det(XP - \mu) \\ &= \det(X - \mu - X\hat{\psi}\hat{\psi}^\dagger) \\ &= \det(X - \mu)\det(1 - (X - \mu)^{-1}X\hat{\psi}\hat{\psi}^\dagger)\end{aligned}$$

Since $\hat{\psi}\hat{\psi}^\dagger$ is rank one, we can write this as

$$\begin{aligned}\det(PXP - \mu) &= \det(X - \mu)\,[1 - \mathrm{Tr}((X - \mu)^{-1}X\hat{\psi}\hat{\psi}^\dagger)] \\ &= -\mu\,\det(X - \mu)\,\mathrm{Tr}((X - \mu)^{-1}\hat{\psi}\hat{\psi}^\dagger) \\ &= -\mu\left[\prod_{m=1}^{n}(\lambda_m - \mu)\right]\left[\sum_{m=1}^{n}\frac{|\hat{\psi}_m|^2}{\lambda_m - \mu}\right]\end{aligned} \tag{13.52}$$

where $\hat{\psi}_m$ is the m^{th} component of the vector ψ. We learn that PXP has one zero eigenvalue while, if the eigenvalues λ_m of X are distinct, then the eigenvalues of PXP lie at the roots the function

$$R(\mu) = \sum_{m=1}^{n}\frac{|\hat{\psi}_m|^2}{\lambda_m - \mu} \tag{13.53}$$

The roots of $R(\mu)$ indeed lie between the eigenvalues λ_m. This completes the proof that the impurities (13.47) capture the correct ordering of the domain walls.

The same argument shows that the boundary condition (13.48) gives rise to the correct ordering of domain walls when $n_{i+1} = n_i + 1$, with the $\vec{\alpha}_i$ domain walls interlaced between the $\vec{\alpha}_{i+1}$ domains walls. Indeed, it is not hard to show that (13.48) arises from (13.47) in the limit that one of the domain walls is taken to infinity.

13.8.2 The Relationship to Monopoles

You will have noticed that the brane construction above is closely related to the Nahm construction we discussed in Lecture 2. In fact, just as the vortex moduli space $\mathcal{V}_{k,N}$ is related to the instanton moduli space $\mathcal{I}_{k,N}$, so the domain wall moduli space $\mathcal{W}_{\vec{g}}$ is related to the monopole moduli space $\mathcal{M}_{\vec{g}}$. The domain wall theory is roughly a subset of the monopole theory. Correspondingly, the domain wall moduli space is a complex submanifold of the monopole moduli space. To make this more precise, consider the isometry rotating the monopoles in the $x^1 - x^2$ plane (mixed with a suitable $U(1)$ gauge action). It we denote the corresponding Killing vector as h, then

$$\mathcal{W}_{\vec{g}} \cong \mathcal{M}_{\vec{g}}\big|_{h=0} \tag{13.54}$$

This is the analog of equation (3.36), relating the vortex and instanton moduli spaces.

Nahm's equations have appeared previously in describing domain walls in the $\mathcal{N} = 1^\star$ theory [249]. I don't know how those domain walls are related to the ones discussed here.

13.9 Applications

We've already seen one application of kinks in section 4.7, deriving a relationship between 2d sigma models and 4d gauge theories. I'll end with a couple of further interesting applications.

13.9.1 Domain Walls and the 2d Black Hole

Recall that we saw in Section 4.3.2 that the relative moduli space of a two domain walls with charge $\vec{g} = \vec{\alpha}_1 + \vec{\alpha}_2$ is the cigar shown in figure 22. Suppose we consider domain walls as strings in a $d = 2+1$ dimensional theory, so that the worldvolume of the domain walls is $d = 1+1$ dimensional. Then the low-energy dynamics of two domain walls is described by a sigma-model on the cigar.

There is a very famous conformal field theory with a cigar target space. It is known as the two-dimensional black hole [250]. It has metric,

$$ds^2_{BH} = k^2[dR^2 + \tanh^2 R\, d\theta^2] \tag{13.55}$$

The non-trivial curvature at the tip of the cigar is cancelled by a dilaton which has the profile

$$\Phi = \Phi_0 - 2\cosh R \tag{13.56}$$

So is the dynamics of the domain wall system determined by this conformal field theory? Well, not so obviously: the metric on the domain wall moduli space $\mathcal{W}_{\vec{\alpha}_1+\vec{\alpha}_2}$ does not coincide with (13.55). However, $d = 1+1$ dimensional theory is not conformal and the metric flows as we move towards the infra-red. There is a subtlety with the dilaton which one can evade by endowing the coordinate R with a suitable anomalous transformation under RG flow. With this caveat, it can be shown that the theory on two domain walls in $d = 2+1$ dimensions does indeed flow towards the conformal theory of the black hole with the identification $k = 2v^2/m$ [251].

The conformal field theory of the 2d black hole is dual to Liouville theory [252, 253]. If we deal with supersymmetric theories, this $\mathcal{N} = (2,2)$ conformal field theory has Lagrangian

$$L_{Liouville} = \int d^4\theta\, \frac{1}{2k}|Y|^2 + \frac{\mu}{2}\int d^2\theta\, e^{-Y} + \text{h.c.} \tag{13.57}$$

and the equivalence between the two theories was proven using the techniques of mirror symmetry in [254]. In fact, one can also prove this duality by studying the dynamics of domain walls. Which is rather cute. We work with the $\mathcal{N} = 4$ (eight supercharges) $U(1)$ gauge theory in $d = 2+1$ with N_f charged hypermultiplets. As we sketched above, if we quantize the low-energy dynamics of the domain walls, we find the $\mathcal{N} = (2,2)$ conformal theory on the cigar. However, there is an alternative way to proceed: we could choose first to integrate out some of the matter in three dimensions. Let's get rid of the charged hypermultiplets to leave a low-energy effective action for the vector multiplet. As well as the gauge field, the vector multiplet contains a triplet of real scalars $\boldsymbol{\phi}$, the first of which is identified with the ϕ we met in (13.1). The low-energy dynamics of this effective theory in $d = 2+1$ dimensions can be shown to be

$$L_{eff} = H(\boldsymbol{\phi})\, \partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} + H^{-1}(\boldsymbol{\phi})(\partial_\mu \sigma + \boldsymbol{\omega} \cdot \partial_\mu \boldsymbol{\phi})^2 - v^4 H^{-1} \tag{13.58}$$

Here σ is the dual photon (see (2.62)) and $\nabla \times \boldsymbol{\omega} = \nabla H$, while the harmonic function H includes the corrections from integrating out the N_f hypermultiplets,

$$H(\boldsymbol{\phi}) = \frac{1}{e^2} + \sum_{i=1}^{N_f} \frac{1}{|\boldsymbol{\phi} - \mathbf{m}_i|} \tag{13.59}$$

where each triplet $\mathbf{m}_i$ is given by $\mathbf{m}_i = (m_i, 0, 0)$. We can now look for domain walls in this $d = 2+1$ effective theory. Since we want to study two domain walls, let's set $N_f = 3$. We see that the theory then has three, isolated vacua, at $\boldsymbol{\phi} = (\phi, 0, 0) = (m_i, 0, 0)$.

We now want to study the domain wall that interpolates between the two outer vacua $\phi = m_1$ and $\phi = m_3$. It's not hard to show that, in contrast to the microscopic theory (13.1), there is no domain wall solutions interpolating between these vacua. One can find a $\vec{\alpha}_1$ domain wall interpolating between $\phi = m_1$ and $\phi = m_2$. There is also a $\vec{\alpha}_2$ domain wall interpolating between $\phi = m_2$ and $\phi = m_3$. But no $\vec{\alpha}_1 + \vec{\alpha}_2$ domain wall between the two extremal vacua $\phi = m_1$ and $\phi = m_3$. The reason is essentially that only a single scalar, ϕ, changes in the domain wall profile, with equations of motion given by flow equations,

$$\partial_3 \phi = v^2 H^{-1}(\phi) \tag{13.60}$$

But since we have only a single scalar field, it must actually pass through the middle vacuum (as opposed to merely getting close) at which point the flow equations tell us $\partial_3 \phi = 0$ and it doesn't move anymore.

Although there is no solution interpolating between $\phi = m_1$ and $\phi = m_2$, one can always write down an approximate solution simply by superposing the $\vec{\alpha}_1$ and $\vec{\alpha}_2$ domain walls in such a way that they are well separated. One can then watch the evolution of this configuration under the equations of motion and, from this, extract an effective force between the domain wall [255]. For the case in hand, this calculation was performed in [251], where it was shown that the force is precisely that arising from the Liouville Lagrangian (13.57). In this way, we can use the dynamics of domain walls to derive the mirror symmetry between the cigar and Liouville theory.

13.9.2 Field Theory D-Branes

As we saw in Section 4.3.2 of this lecture, the moduli space of a single domain wall is $\mathcal{W} \cong \mathbf{R} \times \mathbf{S}^1$. This means that the theory living on the $d = 2+1$ dimensional worldvolume of the domain wall contains a scalar X, corresponding to fluctuations of the domain wall in the x^3 direction, together with a periodic scalar θ determining the phase of the wall. But in $d = 2+1$ dimensions, a periodic scalar can be dualized in favor of a photon living on the wall $4\pi v^2 \partial_\mu \theta = \epsilon_{\mu\nu\rho} F^{\nu\rho}$. Thus the low-energy dynamics of the wall can alternatively be described by a free $U(1)$ gauge theory with a neutral scalar X,

$$L_{\rm wall} = \tfrac{1}{2} T_{\rm wall} \left((\partial_\mu X)^2 + \frac{1}{16\pi^2 v^4} F_{\mu\nu} F^{\mu\nu} \right) \tag{13.61}$$

This is related to the mechanism for gauge field localization described in [256].

As we have seen above, the theory also contains vortex strings. These vortex strings can end on the domain wall, where their ends are electrically charged. In other words, the domain walls are semi-classical D-branes for the vortex strings. These D-branes were first studied in [257, 211, 239]. (Semi-classical D-brane configurations in other theories have been studied in [258, 259, 260] in situations without the worldvolume gauge field). The simplest way to see that the domain wall is D-brane is using the BIon spike described in Section 2.7.4, where we described monopole as D-branes in $d = 5 + 1$ dimensions.

We can also see this D-brane solution from the perspective of the bulk theory. In fact, the solution obeys the equations (13.41) that we wrote down before. To see this, let's complete the square again but we should be more careful in keeping total derivatives. In a theory with multiple vacua, we have

$$\begin{aligned}
\mathcal{H} \;&=\; \int d^3x\, \frac{1}{2e^2}\mathrm{Tr}\left[(\mathcal{D}_1\phi + B_1)^2 + (\mathcal{D}_2 + B_2)^2 + (\mathcal{D}_3\phi + B_3 - e^2(\sum_{i=1}^{N} q_i q_i^\dagger - v^2))^2\right] \\
&\quad + \sum_{i=1}^{N} |(\mathcal{D}_1 - i\mathcal{D}_2)q_i|^2 + \sum_{i=1}^{N} |\mathcal{D}_3 q_i - (\phi - m_i)q_i|^2 + \mathrm{Tr}\left[-v^2 B_3 - \frac{1}{e^2}\partial_i(\phi B_i) + v^2\partial_3\phi\right] \\
&\geq \left(\int dx^1 dx^2\, T_{\text{wall}}\right) + \left(\int dx^3\, T_{\text{vortex}}\right) + M_{\text{mono}} \qquad (13.62)
\end{aligned}$$

and we indeed find the central charge appropriate for the domain wall. In fact these equations were first discovered in abelian theories to describe D-brane objects [211].

These equations have been solved analytically in the limit $e^2 \to \infty$ [257, 229]. Moreover, when multiple domain walls are placed in parallel along the line, one can construct solutions with many vortex strings stretched between them as figure 31, taken from [229], graphically illustrates.

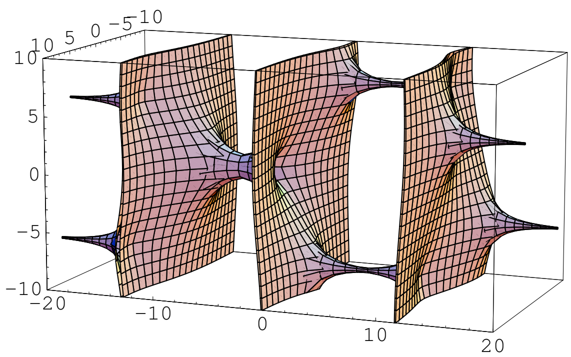


Figure 13.9: Plot of a field theoretic D-brane configuration [229].

Some final points on the field theoretic D-branes

- In each vacuum there are N_c different vortex strings. Not all of them can end on the bordering domain walls. There exist selection rules describing which vortex string can end on a given wall. For the $\vec{\alpha}_i$ domain wall, the string associated to q_i can end from the left, while the string associated to q_{i+1} can end from the right [214].
- For finite e^2, there is a negative binding energy when the string attaches itself to the domain wall, arising from the monopole central charge in (13.62). Known as a boojum, it was studied in this context in [214, 261]. (The name boojum was given by Mermin to a related configuration in superfluid ^{3}He [262]).
- One can develop an open string description of the domain wall dynamics, in which the motion of the walls is governed by the quantum effects of new light states that appear as the walls approach. Chern-Simons interactions on the domain wall worldvolume are responsible for stopping the walls from passing. Details can be found in [263].

Part V
Modified Gravity Theories

14 Differential geometry and gravity by Proeyen

14.1 Differential geometry

In this chapter we collect the ideas of differential geometry that are required to formulate general relativity and supergravity. There are several books, written for physicists, which explore this subject at greater length and greater depth.

In general relativity spacetime is viewed as a differentiable manifold of dimension $D \geq 4$ with a metric of Lorentzian signature $(-, +, +, ..., +)$ indicating one time dimension and $D-1$ space dimensions. We assume that readers of this book are not intimidated by the idea of $D-4$ hidden dimensions that are not directly observed. We will also consider manifolds of purely Euclidean signature $(+, +, ..., +)$, which may appear in the hidden extra dimensions and as the target space of nonlinear σ-models.

We will give a reasonably rigorous definition of a manifold and then introduce the various quantities that 'live on it' in a less formal manner, emphasizing the way that the quantities transform under changes of coordinates. Invariance under coordinate transformations is one of the key principles that underlie general relativity. The most important structures we need are the metric, connection, and curvature. But other quantities such as vector and tensor fields and differential forms are also very useful. We will discuss them first since they require only the manifold structure.

It would be good if readers have already encountered some of the more elementary ideas before, perhaps in a course on general relativity. Our primary purpose is to collect the necessary ideas and explain them, hopefully clearly albeit non-rigorously, and thus to prepare readers for later chapters where the ideas are applied. Readers who do the suggested exercises will achieve the most thorough preparation.

14.1.1 7.1 Manifolds

A D-dimensional manifold is a topological space M together with a family of open sets M_i that cover it, i.e. $M = \cup_i M_i$. The M_i are called coordinate patches. On each patch there is a $1:1$ map ϕ_i, called a chart, from $M_i \to \mathbb{R}^D$. In more concrete language a point $p \in M_i \subset M$ is mapped to $\phi_i(p) = \left(x^1, x^2, ..., x^D\right)$. We say that the set $\left(x^1, x^2, ..., x^D\right)$ are the local coordinates of the point p in the patch M_i. If $p \in M_i \cap M_j$, then the map $\phi_j(p) = \left(x'^1, x'^2, ..., x'^D\right)$ specifies a second set of coordinates for the point p. The compound map $\phi_j \circ \phi_i^{-1}$ from $\mathbb{R}^D \to \mathbb{R}^D$ is then specified by the set of functions $x'^\mu\left(x^\nu\right)$. These functions, and their inverses $x^\nu\left(x'^\mu\right)$, are required to be smooth, usually C^∞. See Fig. 7.1 for an illustration of the ideas just discussed.

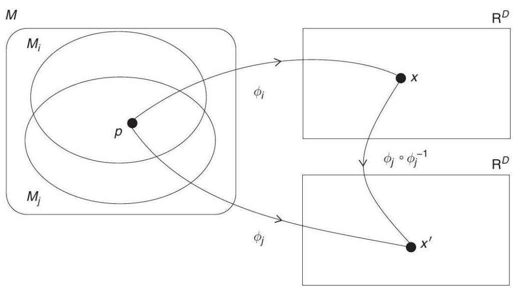


Fig. 7.1 Two charts in $\mathbb{R}^D$ for subsets M_i and M_j of the space M, and the compound map.

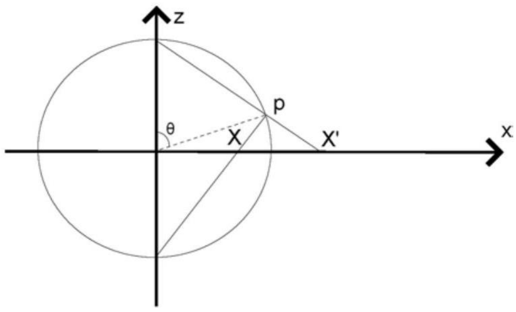


Fig. 7.2 Stereographic projection of the $x-z$ plane of the 2-sphere. The coordinates of the point p are $(x, y, z)=(\sin \theta, 0, \cos \theta)$.

We now describe the unit 2-sphere S^2 as an interesting and useful example of a manifold. Initially it may be defined as the surface $x^2+y^2+z^2=1$ embedded in $\mathbb{R}^3$. It is common to use the usual spherical polar coordinates θ, ϕ with $z=\cos \theta, x=\sin \theta \cos \phi, y=\sin \theta \sin \phi$. This is fine for some purposes, but it does not define a good coordinate chart at the poles $\theta=0, \pi$, since these points have no unique values of ϕ.

There are many ways to introduce coordinate charts to define a manifold structure. One useful way is to use the stereographic projection illustrated in Fig. 7.2. There are two patches whose union covers the sphere, namely M_1, consisting of the sphere with south pole deleted, and M_2, which is the sphere with north pole deleted. From the plane geometry of the triangles in Fig. 7.2, one defines the maps ϕ_1 and ϕ_2 to the central plane in the figure. These maps take the point with polar coordinates θ, ϕ to points X, Y and X', Y' respectively. The maps are given by

$$
\begin{aligned}
& \phi_1: X+\mathrm{i} Y=\mathrm{e}^{\mathrm{i} \phi} \tan (\theta / 2), \\
& \phi_2: X'+\mathrm{i} Y'=\mathrm{e}^{\mathrm{i} \phi} \cot (\theta / 2)
\end{aligned}
$$

On the overlap, we see that

$$
\phi_2 \circ \phi_1^{-1}(X, Y)=X'+\mathrm{i} Y'=1 /(X-\mathrm{i} Y)
$$

Exercise 7.1 Derive (7.1) and (7.2).

14.1.2 7.2 Scalars, vectors, tensors, etc.

The simplest objects to define on a manifold M are scalar functions f that map $M \rightarrow \mathbb{R}$. We say that the point p maps to $f(p)=z \in \mathbb{R}$. On each coordinate patch M_i we can define the compound map $f \circ \phi_i^{-1}$ from $\mathbb{R}^D \rightarrow \mathbb{R}$ as $f_i(x) \equiv f \circ \phi_i^{-1}(x)=z$, where x stands for $\{x^\mu\}$. On the overlap $M_i \cap M_j$ of two patches with local coordinates x^μ and x'^ν of the point p, the two descriptions of f must agree. Thus $f_i(x)=f_j(x')$.

We now define the properties of scalar functions in the less formal way we will use for most of the objects that live on M. We no longer refer to a covering by coordinate patches. Instead we conceive of the manifold as a set whose points may be described by many different coordinate systems, say $\left(x^0, x^1, \ldots, x^{D-1}\right)$ and $\left(x'^0, x'^1, \ldots, x'^{D-1}\right)$. Any two sets of coordinates are related by a set of C^{∞} functions, e.g. $x'^\mu\left(x^\nu\right)$ with non-singular Jacobian $\partial x'^\mu / \partial x^\nu$. We refer to such a change of coordinates as a general coordinate transformation. A scalar function, also called a scalar field, is described by $f(x)$ in one set of coordinates and $f'\left(x'\right)$ in the second set. The two functions must be pointwise equal, i.e.

$$
f'\left(x'\right)=f(x) .
$$

Locally, at least, the informal definition agrees with the more formal one above.

In the same fashion, a contravariant vector field is described by D functions $V^\mu(x)$ in one coordinate system and D functions $V'^\mu(x')$ in the second. They are related by

$$V'^\mu(x') = \frac{\partial x'^\mu}{\partial x^\nu} V^\nu(x)$$

with a summation convention on the repeated index v. We go on to define covariant vector fields $\omega_\mu(x)$ and (mixed) tensors $T^\mu_\nu(x)$ by their behavior under coordinate transformations, namely

$$\begin{aligned} \omega'_\mu(x') &= \frac{\partial x^\nu}{\partial x'^\mu} \omega_\nu(x), \\ T'^\mu_\nu(x') &= \frac{\partial x'^\mu}{\partial x^\sigma} \frac{\partial x^\rho}{\partial x'^\nu} T^\sigma_\rho(x). \end{aligned}$$

We leave it to the reader to devise the analogous definitions of higher rank tensors such as $T^{\mu\nu}(x), S_{\mu\nu\rho}$, etc. A tensor field with p contravariant and q covariant indices is called a tensor of type (p, q) and rank $p+q$.

At this point in the development, contravariant and covariant quantities are unrelated objects, which transform differently. However a contravariant and covariant index can be contracted (i.e. summed) to define tensorial quantities of lower rank.

Exercise 7.2 Given $V^\mu(x), \omega_\mu(x), T^\mu_\nu(x)$, show that $V^\mu(x)\omega_\mu(x)$ transforms as a scalar field and that $T^\mu_\nu(x)V^\nu(x)$ transforms as a contravariant vector.

One can proceed with concrete local definitions of this type to obtain a physically satisfactory formulation of general relativity. However, there is much richness to be gained, and considerable practical advantage, if we develop the ideas further and incorporate some of the concepts of a more mathematical treatment of differential geometry.

Given a contravariant vector field $V^\mu(x)$, one can consider the system of differential equations

$$\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda} = V^\mu(x)$$

A solution $x^\mu(\lambda)$ is a map from $\mathbb{R} \to M$, which is a curve on M, called an integral curve of the vector field. There is an integral curve through every point of any open subset of M in which the vector field does not vanish. If the manifold is $\mathbb{R}^D$, then we know that the vector $\mathrm{d}x^\mu/\mathrm{d}\lambda$ is tangent to the curve $x^\mu(\lambda)$, and we make the same interpretation for a general manifold.

Let $x^\mu(\lambda)$ be the integral curve through the point p of M with coordinates $x^\mu(\lambda_0)$. Then $\left.\mathrm{d}x^\mu/\mathrm{d}\lambda\right|_{\lambda_0} = V^\mu(x(\lambda_0))$ is the tangent vector to the curve $x^\mu(\lambda)$ at p. We can now consider $D-1$ other vector fields $\tilde{V}^\mu(x)$ whose values $\tilde{V}^\mu(x(\lambda_0))$, together with the first $V^\mu(x(\lambda_0))$, fill out a basis of $\mathbb{R}^D$. Each $\tilde{V}^\mu(x(\lambda_0))$ is the tangent vector of an integral curve $\tilde{x}(\lambda)$ through p. Thus the vector fields evaluated at p determine the D-dimensional vector space $T_p(M)$, the tangent space to the manifold at point p. A vector field $V^\mu(x)$ may then be thought of as a smooth assignment of a tangent vector in each $T_p(M)$ as p varies over M. We shall use the notation $T(M)$ to denote the space of contravariant vector fields on M.

One important structure that one can form using the components $V^\mu(x)$ of a contravariant vector field is the differential operator $V = V^\mu(x)\partial/\partial x^\mu$. It follows from the transformation property (7.4) and the chain rule that V is constructed in the same way in all coordinate systems, e.g. $V = V'^\mu(x')\partial/\partial x'^\mu$. In this sense it is invariant under coordinate transformations. The differential operator V acts naturally on a scalar field $f(x)$, yielding another scalar field

$$\mathcal{L}_V f(x) \equiv V^\mu(x) \frac{\partial f}{\partial x^\mu}$$

On the manifold $\mathbb{R}^D$, this operation is just the directional derivative $V \cdot \nabla f$, and it has the same interpretation on a general manifold M. At each point p with coordinates x^m, $\mathcal{L}_V f(x)$ is the derivative of $f(x)$ in the direction of the tangent of the integral curve of $V^\mu(x)$ through p.

Locally, there is a $1:1$ correspondence between contravariant vector fields $V^\mu(x)$ and differential operators. In mathematical treatments a vector field is viewed as a smooth assignment of a differential operator at each point p. The set of elementary operators $\{\partial/\partial x^\mu, \mu = 1, ..., D\}$ are a basis in this view of the tangent space $T_p(M)$. This is consistent with our discussion since $\partial f/\partial x^\mu$ for a given value of μ is the derivative in the direction of the tangent to the curve on which the single coordinate x^μ changes, but the other coordinates x^ν for $v \neq \mu$ are constant. The basis $\{\partial/\partial x^\mu, \mu = 1, ..., D\}$ is called a coordinate basis because these operators differentiate along such coordinate curves at each p.

The derivative $\mathcal{L}_V f(x)$ defined in (7.7) may be extended to vector and tensor fields of any type (p, q), always yielding another tensor of the same type. For the vectors and tensors in (7.4) and (7.5), the precise definition is

$$\begin{aligned}
\mathcal{L}_V U^\mu &= V^\rho \partial_\rho U^\mu - (\partial_\rho V^\mu)\, U^\rho, \\
\mathcal{L}_V \omega_\mu &= V^\rho \partial_\rho \omega_\mu + (\partial_\mu V^\rho)\, \omega_\rho, \\
\mathcal{L}_V T^\mu_\nu &= V^\rho \partial_\rho T^\mu_\nu - (\partial_\rho V^\mu)\, T^\rho_\nu + (\partial_\nu V^\rho)\, T^\mu_\rho
\end{aligned}$$

The derivative defined in this way is called the Lie derivative. Its definition requires a vector field, but not a connection; yet it preserves the tensor transformation property.

Exercise 7.3 Show explicitly that $\mathcal{L}_V U^\mu, \mathcal{L}_V \omega_\mu$, and $\mathcal{L}_V T^\mu_\nu$ defined in (7.8) do transform under coordinate transformations as required by (7.4) and (7.5).

The Lie derivative of a contravariant vector field has special significance because it occurs in the commutator of the corresponding differential operators $U = U^\mu(x)\partial/\partial x^\mu$ and $V = V^\mu(x)\partial/\partial x^\mu$. An elementary calculation gives

$$[U, V] = W = W^\mu(x)\frac{\partial}{\partial x^\mu}$$

with $W^\mu = \mathcal{L}_U V^\mu = -\mathcal{L}_V U^\mu$. The new vector field W^μ is called the Lie bracket of U^μ and V^μ. This discussion also shows that the contravariant tensor fields on M naturally form a Lie algebra.

Let us consider the transformation properties of (7.3)-(7.5) for infinitesimal coordinate transformations, namely those for which $x'^\mu = x^\mu - \xi^\mu(x)$. To first order in $\xi^\mu(x)$, the previous transformation rules can be expressed in terms of Lie derivatives as

$$\begin{aligned}
\delta\phi(x) &\equiv \phi'(x) - \phi(x) = \mathcal{L}_\xi \phi, \\
\delta U^\mu(x) &\equiv U'^\mu(x) - U^\mu(x) = \mathcal{L}_\xi U^\mu, \\
\delta\omega_\mu(x) &\equiv \omega'_\mu(x) - \omega_\mu(x) = \mathcal{L}_\xi \omega_\mu, \\
\delta T^\mu_v(x) &\equiv T'^\mu_v(x) - T^\mu_v(x) = \mathcal{L}_\xi T^\mu_v.
\end{aligned}$$

Thus one of the useful roles of Lie derivatives is in the description of infinitesimal coordinate transformations.

Exercise 7.4 Show that the transformations (7.10) follow from (7.3)-(7.5).

Next we focus attention on covariant vector fields, such as $\omega_\mu(x)$. We already noted in Ex. 7.2 that the contraction $\omega_\mu(x)V^\mu(x)$ with any contravariant vector field gives a scalar field. Thus at any point p with coordinates $x^\nu, \omega_\mu(x)$ can be regarded as an element of the dual space $T^*_p(M)$, a linear functional that maps $T_p(M) \to \mathbb{R}$. The space $T^*_p(M)$ is usually called the cotangent space at p.

In parallel to the way in which we associated contravariant vector fields $V^\mu(x)$ with differential operators $V = V^\mu(x)\partial/\partial x^\mu$, we use the coordinate differentials $\mathrm{d}x^\mu$ to write $\Omega = \omega_\mu(x)\mathrm{d}x^\mu$.

Note that both $\omega_\mu(x)$ and $\mathrm{d}x^\mu$ transform under coordinate transformations, but $\Omega = \omega'_\mu(x')\,\mathrm{d}x'^\mu$ is constructed in the same way in any coordinate system. Ω is called a differential 1-form on M. Note that the gradient $\partial_\mu \phi(x)$ of any scalar transforms as a covariant vector and that the associated differential 1-form $\mathrm{d}\phi = \partial_\mu \phi \mathrm{d}x^\mu$ is just the differential of calculus. We can think of the set of coordinate differentials $\{\mathrm{d}x^\mu, \mu = 1, ..., D\}$ as a basis of the space of 1 -forms.

The notion of the cotangent space $T_p^*(M)$ of linear functionals on $T_p(M)$ is naturally extended to the level of 1-forms and differential operators. We define the pairing of basis elements as $\langle \mathrm{d}x^\mu | \partial/\partial x^\nu \rangle \equiv \delta_\nu^\mu$. This is extended using linearity to any general 1-form Ω and differential operator V, so that we then have $\langle \Omega | V \rangle = \omega_\mu(x) V^\mu(x)$. This agrees with the initial definition as the contraction of component indices.

14.1.3 7.3 The algebra and calculus of differential forms

Among the various fields defined on M, the scalars ϕ, covariant vectors ω_μ, and totally antisymmetric tensors such as $\omega_{\mu\nu} = -\omega_{\nu\mu}$ have a particularly useful structure when considered together. Note that antisymmetry is preserved under coordinate transformations so it is a tensorial property. Using the coordinate differentials $\mathrm{d}x^\mu$, we can construct differential p-forms for $p = 1, 2, ..., D$ as

$$
\begin{aligned}
\omega^{(1)} &= \omega_\mu(x) \mathrm{d}x^\mu \\
\omega^{(2)} &= \frac{1}{2} \omega_{\mu\nu}(x) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&\vdots \\
\omega^{(p)} &= \frac{1}{p!} \omega_{\mu_1 \mu_2 \cdots \mu_p}\, \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}
\end{aligned}
$$

The wedge product is defined as antisymmetric; that is, $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = -\mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$, $\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho = -\mathrm{d}x^\rho \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\mu$, etc. At each point we have an element of the p-fold antisymmetric tensor product of the cotangent space, so the differential form $\omega^{(p)}$ is a smooth assignment of an element of this tensor product as the point varies over M. The space of p-forms is denoted by $\Lambda^p(M)$. By convention the scalars are considered to be 0 -forms.

There is an exterior algebra and calculus of p-forms, which we will not develop in detail. See [40, 41, 42, 44, 45] for more complete discussions. Rather we will state some key properties without proof and write the specific examples needed later to discuss frames, connections, and curvature. In the exterior algebra, a p-form $\omega^{(p)}$ and a q-form $\omega^{(q)}$ can be multiplied to give a $(p+q)$-form if $p + q \leq D$. The product vanishes if $p + q > D$. The product satisfies $\omega^{(p)} \wedge \omega^{(q)} = (-)^{pq} \omega^{(q)} \wedge \omega^{(p)}$ and it is associative. Some examples are

$$
\begin{aligned}
\omega^{(1)} \wedge \tilde{\omega}^{(1)} &= \omega_\mu \mathrm{d}x^\mu \wedge \tilde{\omega}_\nu \mathrm{d}x^\nu \\
&= \frac{1}{2} \left(\omega_\mu \tilde{\omega}_\nu - \omega_\nu \tilde{\omega}_\mu \right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
\omega^{(1)} \wedge \omega^{(2)} &= \omega_\mu \mathrm{d}x^\mu \wedge \frac{1}{2} \omega_{\nu\rho}(x) \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho \\
&= \frac{1}{6} \left(\omega_\mu \omega_{\nu\rho} + \omega_\nu \omega_{\rho\mu} + \omega_\rho \omega_{\mu\nu} \right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho
\end{aligned}
$$

The explicit antisymmetrization in the second line of each example is not necessary, since it is implicit in the wedge products of the $\mathrm{d}x^\mu$. But it is convenient to indicate that the covariant tensor field associated with each form is antisymmetric.

The exterior calculus is based on the exterior derivative, which maps p-forms into $(p+1)$-forms as follows:

$$\mathrm{d}\omega^{(p)} = \frac{1}{p!}\partial_\mu \omega_{\mu_1\mu_2\cdots\mu_p}\ \mathrm{d}x^\mu \wedge \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}$$

Exercise 7.5 Show that the operation d is nilpotent, i.e. $\mathrm{d}\left(\mathrm{d}\omega^{(p)}\right) = 0$ on any p-form, and that it satisfies the distributive property

$$\mathrm{d}\left(\omega^{(p)} \wedge \omega^{(q)}\right) = \mathrm{d}\omega^{(p)} \wedge \omega^{(q)} + (-)^p \omega^{(p)} \wedge \mathrm{d}\omega^{(q)}$$

On forms of degree $0, 1, 2$

$$\begin{aligned}
\mathrm{d}\phi &= \partial_\mu \phi \mathrm{d}x^\mu \\
\mathrm{d}\omega^{(1)} &= \frac{1}{2}\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
\mathrm{d}\omega^{(2)} &= \frac{1}{6}\left(\partial_\mu \omega_{\nu\rho} + \partial_\nu \omega_{\rho\mu} + \partial_\rho \omega_{\mu\nu}\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \wedge \mathrm{d}x^\rho
\end{aligned}$$

A p-form that satisfies $\mathrm{d}\omega^{(p)} = 0$ is called closed. A p-form $\omega^{(p)}$ that can be expressed as $\omega^{(p)} = \mathrm{d}\omega^{(p-1)}$ is called exact. The Poincaré lemma implies that locally any closed p-form can be expressed as $\mathrm{d}\omega^{(p-1)}$, but $\omega^{(p-1)}$ may not be well defined globally on M.

We saw that the exterior derivative is a map from p-forms into $(p+1)$-forms. There is also an interior derivative, which maps p-forms into $(p-1)$-forms. The latter depends on a vector V and is denoted as i_V. It is defined as follows:

$$\begin{aligned}
\left(i_V \omega^{(p)}\right) &= \frac{1}{(p-1)!} V^\mu \omega_{\mu\mu_1\ldots\mu_{p-1}}\ \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_{p-1}} \\
i_V\left(\ \mathrm{d}x^{\mu_1}\right. & \left.\wedge \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}\right) \\
&= V^{\mu_1}\ \mathrm{d}x^{\mu_2} \wedge \cdots \wedge \mathrm{d}x^{\mu_p} - V^{\mu_2}\ \mathrm{d}x^{\mu_1} \wedge \mathrm{d}x^{\mu_3} \cdots \wedge \mathrm{d}x^{\mu_p} + \ldots
\end{aligned}$$

Exercise 7.6 Prove that the interior derivative is also nilpotent, i.e. $i_V i_V = 0$.

Like the internal and external derivatives, the Lie derivative, introduced in Sec. 7.2 as a derivative on tensor fields, has a simple action on p-forms. It maps p-forms to p-forms via the formula

$$\mathcal{L}_V = \mathrm{d}i_V + i_V\ \mathrm{d}$$

It is instructive to work out the example $\mathcal{L}_V \omega^{(1)} = \left(\mathrm{d}i_V + i_V\ \mathrm{d}\right) \omega^{(1)}$:

$$\begin{aligned}
\left(\mathrm{d}i_V + i_V\ \mathrm{d}\right) \omega^{(1)} &= \mathrm{d}\left(V^\mu \omega_\mu\right) + i_V \frac{1}{2}\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu \\
&= \left(\partial_\nu V^\mu \omega_\mu + V^\mu \partial_\nu \omega_\mu\right) \mathrm{d}x^\nu + V^\mu\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) \mathrm{d}x^\nu \\
&= \left(V^\mu \partial_\mu \omega_\nu + \partial_\nu V^\mu \omega_\mu\right) \mathrm{d}x^\nu = \left(\mathcal{L}_V \omega\right)_\nu \mathrm{d}x^\nu
\end{aligned}$$

The Lie derivative of the covariant vector field ω_μ, which contains the components of the 1 -form $\omega^{(1)}$, was defined in (7.8) and appears in the final result. [1]

Exercise 7.7 Use the formula (7.17) to calculate the Lie derivative of a 0-form (where the first term vanishes by definition) and a 2-form. The final result should again contain the components of the Lie derivative as defined in Sec. 7.2.

Differential forms have a natural application to the theories of electromagnetism, to Yang-Mills theory, and to the antisymmetric tensor gauge theories that appear in higher dimensional supergravity. However, we need to bring in some other ideas in the next section before discussing these physical applications.

14.1.4 7.4 The metric and frame field on a manifold

We now introduce the additional structure of a metric on a manifold M. In general relativity the metric is of primary importance in describing the geometry of spacetime and the dynamics of gravity. In theories such as supergravity where there are fermions coupled to gravity, one must use an auxiliary quantity, the frame field (more commonly called the vierbein or vielbein), which we discuss in detail. The metric tensor is quadratically related to the frame field.

14.1.5 7.4.1 The metric

A metric or inner product on a real vector space V is a non-degenerate bilinear map from $V \otimes V \rightarrow \mathbb{R}$. The inner product of two vectors $u, v \in V$ is a real number denoted by (u, v). The inner product must satisfy the following properties:

(i) bilinearity, $(u, c_1 v_1 + c_2 v_2) = c_1 (u, v_1) + c_2 (u, v_2)$ and $(c_1 v_1 + c_2 v_2, u) = c_1 (v_1, u) + c_2 (v_2, u)$

(ii) non-degeneracy, if $(u, v) = 0$ for all $v \in V$, then $u = 0$;

(iii) symmetry, $(u, v) = (v, u)$.

1 The distributive formula $\mathcal{L}_V (\omega_\mu \mathrm{d}x^\mu) = (\mathcal{L}_V \omega_\mu) \mathrm{d}x^\mu + \omega_\mu \mathcal{L}_V \, \mathrm{d}x^\mu$ can be used if it interpreted carefully. Both terms are non-vanishing and can be calculated using (7.17) and (7.16). The latter equation requires that each component of $\mathcal{L}_V \omega_\mu = i_V \, \mathrm{d}\omega_\mu$ is calculated as the Lie derivative of a 0 -form.

The metric on a manifold is a smooth assignment of an inner product map on each $T_p(M) \otimes T_p(M) \rightarrow \mathbb{R}$. In local coordinates the metric is specified by a covariant second rank symmetric tensor field $g_{\mu\nu}(x)$, and the inner product of two contravariant vectors $U^\mu(x)$ and $V^\mu(x)$ is $g_{\mu\nu}(x) U^\mu(x) V^\nu(x)$, which is a scalar field. In particular the metric gives a formula for the length s of a curve $x^\mu(\lambda)$ with tangent vector $\mathrm{d}x^\mu / \mathrm{d}\lambda$:

$$s_{12} = \int_{\lambda_1}^{\lambda_2} \mathrm{d}\lambda \sqrt{g_{\mu\nu}(x(\lambda)) (\mathrm{d}x^\mu / \mathrm{d}\lambda) (\mathrm{d}x^\nu / \mathrm{d}\lambda)}$$

Thus it is most convenient to summarize the properties of a given metric by the line element

$$\mathrm{d}s^2 = g_{\mu\nu}(x) \mathrm{d}x^\mu \mathrm{d}x^\nu .$$

Non-degeneracy means that $\det g_{\mu\nu} \neq 0$, so the inverse metric $g^{\mu\nu}(x)$ exists as a rank 2 symmetric contravariant tensor, which satisfies

$$g^{\mu\rho} g_{\rho v} = g_{v\rho} g^{\rho\mu} = \delta_v^\mu .$$

The metric tensor and its inverse may be used to lower and raise indices, e.g. $V_\mu(x) = g_{\mu\nu} V^\nu(x)$ and $\omega^\mu(x) = g^{\mu\nu}(x) \omega_\nu(x)$, thus providing a natural isomorphism between the spaces of contravariant and covariant vectors and tensors.

In a gravity theory in spacetime, the metric has signature $- + + \cdots +$. Concretely this means that the metric tensor $g_{\mu\nu}$ may be diagonalized by an orthogonal transformation, i.e. $(O^{-1})_\mu^a = O_\mu^a$ and

$$g_{\mu\nu} = O_\mu^a D_{ab} O_v^b ,$$

with positive eigenvalues λ^a in $D_{ab} = \operatorname{diag}\left(-\lambda^0, \lambda^1, \ldots, \lambda^{D-1}\right)$.

Exercise 7.8 Show that $\lambda^a(x) > 0$ holds throughout M if the metric is non-degenerate. In another coordinate system the transformed metric $g'_{\rho\sigma} = (\mathrm{d}x^\mu / \mathrm{d}x'^\rho)(\mathrm{d}x^\nu / \mathrm{d}x'^\sigma) g_{\mu\nu}$ may be diagonalized giving another set of eigenvalues λ'^a, in general different from the λ^a. Show that the $\lambda'^a > 0$. Thus the signature of a metric is a global invariant.

14.1.6 7.4.2 The frame field

The construction above, which involved only matrix linear algebra, allows us to define an important auxiliary quantity in a theory of gravity, namely

$$e_\mu^a(x) \equiv \sqrt{\lambda^a(x)}O_\mu^a(x)$$

In four dimensions this quantity is commonly called the tetrad or vierbein. In general dimension the term vielbein is frequently used, but we prefer the term frame field for reasons that will become clear as we discuss its properties.

Note that

$$g_{\mu\nu}(x) = e_\mu^a(x)\eta_{ab}e_\nu^b(x),$$

where $\eta_{ab} = \mathrm{diag}(-1, 1, ..., 1)$ is the metric of flat D-dimensional Minkowski spacetime. It is (7.24) that states the general relation between metric and frame field. For a given metric tensor $g_{\mu\nu}(x)$, the frame field $e_\mu^a(x)$ of (7.23), obtained by diagonalization, is not the only solution. Given any x-dependent matrix $\Lambda_b^a(x)$ which leaves η_{ab} invariant, in other words, given a local Lorentz transformation, we can construct another solution of (7.24), namely

$$e_\mu^{\prime a}(x) = \Lambda^{-1a}{}_b(x)e_\mu^b(x).$$

All choices of frame fields related by local Lorentz transformations are viewed as equivalent. So we require that the frame field and geometrical quantities derived from it must be used in a way that is covariant with respect to the transformation (7.25).

Local Lorentz transformations in curved spacetime differ from the global Lorentz transformations of Minkowski space discussed in Ch. 1. Only frame indices $a, b, ...$ of a quantity transform, coordinate indices $\mu, v, ...$ are inert, and the spacetime coordinate does not change. Instead, (7.24) requires that the frame field e_μ^a transforms as a covariant vector under diffeomorphisms (coordinate transformations), viz.

$$e_\mu^{\prime a}(x') = \frac{\partial x^\rho}{\partial x'^\mu}e_\rho^a(x)$$

while the frame index is inert. [2]

Since e_μ^a is a non-singular $D \times D$ matrix, with $\det e_\mu^a = \sqrt{-\det g} \neq 0$, there is an inverse frame field $e_a^\mu(x)$, which satisfies $e_\mu^a e_b^\mu = \delta_b^a$ and $e_a^\mu e_v^a = \delta_v^\mu$.

Exercise 7.9 Show that

$$e_a^\mu = g^{\mu\nu}\eta_{ab}e_v^b, \quad e_a^\mu g_{\mu\nu}e_b^v = \eta_{ab}$$

The last relation shows that the (inverse) frame field can be used to relate a general metric of signature $- + + \cdots +$ to the Minkowski metric. Show that, under local Lorentz and coordinate transformations,

$$e_a^{\prime\mu}(x) = \Lambda_a^{-1b}{}_b e_b^\mu(x), \quad e_a^{\prime\mu}(x') = \frac{\partial x'^\mu}{\partial x^\rho}e_a^\rho(x)$$

Frame indices are raised and lowered using the Minkowski metric.

The second relation of (7.27) indicates that the e_a^μ form an orthonormal set of vectors in the tangent space of M at each point. Since $\det e_a^\mu \neq 0$, we have a basis of each tangent space. Any contravariant vector field has a unique expansion in the new basis, i.e. $V^\mu(x) = V^a(x)e_a^\mu(x)$ with $V^a(x) = V^\mu(x)e_\mu^a(x)$. The $V^a(x)$ are the frame components of the original vector field $V^\mu(x)$. They transform as a set of D scalar fields under coordinate transformations, and as a vector under Lorentz transformations, i.e. $V'^a(x) = \Lambda^{-1a}{}_b(x)V^b(x)$. The same may be done for covariant vectors, i.e. $\omega_\mu(x) = \omega_a(x)e^a_\mu(x)$ with $\omega_a(x) = \omega_\mu(x)e^\mu_a(x)$. These constructions may be extended to tensor fields of any rank in a straightforward way.

Thus we may use e^a_μ and e^μ_a to transform vector and tensor fields back and forth between a coordinate basis with indices $\mu, v, \ldots$ and a local Lorentz basis with indices $a, b, \ldots$ in

2 The relation between local and global Lorentz transformations is discussed further in Sec. 11.3.1. which the metric is η_{ab}. Invariants such as the inner product may be calculated in either basis.

Exercise 7.10 Show that

$$U^\mu(x)V_\mu(x) = g_{\mu\nu}(x)U^\mu(x)V^v(x) = \eta_{ab}U^a(x)V^b(x) = U^a(x)V_a(x).$$

At the level of differential operators the change of basis in the tangent space is expressed as

$$E_a \equiv e^\mu_a(x)\frac{\partial}{\partial x^\mu}$$

This makes it clear that the local Lorentz basis is a non-coordinate basis. If there were local coordinates y^a such that $E_a = \partial/\partial y^a$, these differential operators would commute. However, the commutator

$$[E_a, E_b] = -\Omega^c_{ab}E_c,$$

where $\Omega_{ab}{}^c = -e^c_\mu \mathcal{L}_{e_a} e^\mu_b = e^c_\mu \mathcal{L}_{e_b} e^\mu_a$ are the frame components of the Lie bracket, which do not vanish in a general manifold, and are called 'anholonomy coefficients'.

Exercise 7.11 Show that $\Omega_{ab}{}^c = e^\mu_a e^v_b \left(\partial_\mu e^c_v - \partial_\nu e^c_\mu\right)$.

We can also use the frame field e^a_μ to define a new basis in the spaces $\Lambda^p(M)$ of differential forms. The local Lorentz basis of 1 -forms is

$$e^a \equiv e^a_\mu(x)\mathrm{d}x^\mu$$

This is the dual basis to (7.30), since the pairing is given by $\langle e^a | E_b \rangle = \delta^a_b$. For 2-forms the basis consists of the wedge products $e^a \wedge e^b$, and so on.

In a field theory containing only bosonic fields, which are always vectors or tensors, the use of local frames is unnecessary, although it is an option that is convenient for some purposes. Local frames are a necessity to treat the coupling of fermion fields to gravity, because spinors are defined by their special transformation properties under Lorentz transformations.

14.1.7 7.4.3 Induced metrics

In many applications of differential geometry one encounters a manifold of dimension D which can be viewed as a surface embedded in flat Minkowski or Euclidean space of dimension $D+1$. We discuss the Euclidean case for $D=2$. Suppose that our surface is described by the equation

$$f(x, y, z) = 0.$$

On the surface the differential vanishes, viz.

$$\mathrm{d}f = \frac{\partial f}{\partial x}\,\mathrm{d}x + \frac{\partial f}{\partial y}\,\mathrm{d}y + \frac{\partial f}{\partial z}\,\mathrm{d}z = 0$$

The intrinsic geometry of the surface is determined by the Euclidean metric

$$\mathrm{d}s^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2.$$

To find it one can, in principle, solve (7.33) to eliminate one variable and then use (7.34) to find a relation among the coordinate differentials. When this information is inserted in (7.35), one has the induced metric. Voila!

This is often easier said than done, so we confine our discussion to the solvable and instructive example of the unit 2-sphere for which the embedding equation (7.33) is

$$x^2 + y^2 + z^2 = 1.$$

Let's proceed using spherical coordinates:

$$z = r\cos\theta, \quad x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi$$

The embedding equation becomes simply $r^2 = 1$, so we can eliminate the coordinate r and write the differentials:

$$\begin{aligned} \mathrm{d}z &= -\sin\theta \mathrm{d}\theta \\ \mathrm{d}x &= \cos\theta\cos\varphi \mathrm{d}\theta - \sin\theta\sin\varphi \mathrm{d}\varphi \\ \mathrm{d}y &= \cos\theta\sin\varphi \mathrm{d}\theta + \sin\theta\cos\varphi \mathrm{d}\varphi \end{aligned}$$

Upon substitution in (7.35) one finds the induced metric

$$\mathrm{d}s^2 = \mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2$$

This is a commonly used and quite useful metric on S_2, but it is evidently singular at the north and south poles where the metric tensor is not invertible. One can do somewhat better using one of the two sets of coordinates defined by the stereographic projection in Sec. 7.1, and this is the subject of the following exercise.

Exercise 7.12 Reexpress the metric (7.39) in the coordinates $X = \cos\varphi\tan(\theta/2), Y = \sin\varphi\tan(\theta/2)$. Show that the new metric is

$$\mathrm{d}s^2 = \frac{4\left(\mathrm{d}X^2 + \mathrm{d}Y^2\right)}{\left(1 + X^2 + Y^2\right)^2}$$

14.1.8 7.5 Volume forms and integration

The equations of motion in any field theory are most conveniently packaged in the action integral. In a gravitational theory this requires integration over the curved spacetime manifold. We thus need a procedure for integration that is invariant under coordinate transformations. The volume form is the key to this procedure.

On a D-dimensional manifold, one may choose any top degree D-form $\omega^{(D)}$ as a volume form and define the integral

$$\begin{aligned} I &= \int \omega^{(D)} \\ &= \frac{1}{D!}\int \omega_{\mu_1\cdots\mu_D}(x)\mathrm{d}x^{\mu_1}\wedge\cdots\wedge \mathrm{d}x^{\mu_D} \\ &= \int \omega_{01\cdots D-1}\ \mathrm{d}x^0\ \mathrm{d}x^1 \ldots \mathrm{d}x^{D-1}. \end{aligned}$$

The antisymmetric tensor $\omega_{\mu_1\cdots\mu_D}(x)$ has only one independent component, and we have used this fact in the last line above to write the integral so that it may be performed by the rules of multi-variable calculus. For the same reason any two D-forms $\tilde{\omega}(D)$ and $\omega^{(D)}$ must be related by $\tilde{\omega}(D) = f\omega^{(D)}$, where $f(x)$ is a scalar field. Thus the definition (7.41) includes $\int f\omega^{(D)}$.

Exercise 7.13 Show that in a new coordinate system with coordinates $x'^{\mu}\left(x^{\nu}\right)$ the integral I in (7.41) takes the form

$$I=\frac{1}{D!}\int\omega'_{\mu_1\cdots\mu_D}\left(x'\right)\mathrm{d}x'^{\mu_1}\wedge\cdots\wedge\mathrm{d}x'^{\mu_D}$$

and is thus coordinate invariant.

Although there are many possible volume forms, there are two types that usually appear in the context of physics. The first, which is the more specialized, occurs when the physical theory contains form fields. As an example, on a 3-manifold the wedge product $\omega^{(1)}\wedge\omega^{(2)}$ can be chosen as a volume form. Using (7.12) we see that

$$\begin{aligned}I&=\int\omega^{(1)}\wedge\omega^{(2)}\\&=\frac{1}{6}\int\left(\omega_\mu\omega_{\nu\rho}+\omega_\nu\omega_{\rho\mu}+\omega_\rho\omega_{\mu\nu}\right)\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu\wedge\mathrm{d}x^\rho\\&=\int\left(\omega_0\omega_{12}+\omega_1\omega_{20}+\omega_2\omega_{01}\right)\mathrm{d}x^0\ \mathrm{d}x^1\ \mathrm{d}x^2\end{aligned}$$

The integral is coordinate invariant, and it does not involve the metric on M. The action integral of the simplest Chern-Simons field theory, in which $\omega^{(2)}=\mathrm{d}\omega^{(1)}$, takes this form.

The second type of volume form is far more common in physics and we call it the canonical volume form. There are several ways to introduce it, and we will use the frame field $e_\mu^a(x)$ and the basis of frame 1-forms e^a for this purpose. As a preliminary we define the Levi-Civita alternating symbol in local frame components:

$$\varepsilon_{a_1a_2\cdots a_D}=\begin{cases}+1,a_1a_2\cdots a_D\text{ an even permutation of }01\ldots(D-1)\\-1,a_1a_2\cdots a_D\text{ an odd permutation of }01\ldots(D-1)\\\ \ 0,\text{ otherwise.}\end{cases}$$

Under (proper) Lorentz transformations, i.e. $\det\Lambda^a{}_b=1$, this is an invariant tensor that takes the same form in any Lorentz frame. As usual Lorentz indices are raised with η^{ab}. Note that $\varepsilon^{01\cdots(D-1)}=-1$.

Note that the Levi-Civita symbol provides a useful formula for the determinant of any $D\times D$ matrix A_b^a, namely

$$\det A\varepsilon_{b_1b_2\cdots b_D}=\varepsilon_{a_1a_2\cdots a_D}A_{b_1}^{a_1}A_{b_2}^{a_2}\cdots A_{b_D}^{a_D},$$

and that there are systematic identities for the contraction of p of the $D=p+q$ indices, as we saw in (3.9).

The Levi-Civita form in the coordinate basis is defined by contracting with frame fields and inserting factors of $e=\det e_\mu^a$ or e^{-1}:

$$\begin{aligned}\varepsilon_{\mu_1\mu_2\cdots\mu_D}&\equiv e^{-1}\varepsilon_{a_1a_2\cdots a_D}e_{\mu_1}^{a_1}e_{\mu_2}^{a_2}\cdots e_{\mu_D}^{a_D}\\\varepsilon^{\mu_1\mu_2\cdots\mu_D}&\equiv e\varepsilon^{a_1a_2\cdots a_D}e_{a_1}^{\mu_1}e_{a_2}^{\mu_2}\cdots e_{a_D}^{\mu_D}\end{aligned}$$

Note that these definitions ensure that $\varepsilon^{\mu_1\cdots\mu_D}$ and $\varepsilon_{\mu_1\cdots\mu_D}$ take the constant values given on the right-hand side of (7.44). This can be seen using (7.45). The quantities defined in (7.46) are called tensor densities. It is important to recognize that $\varepsilon^{\mu_1\mu_2\cdots\mu_D}$ cannot be obtained by raising the indices of $\varepsilon_{\mu_1\mu_2\cdots\mu_D}$ in the usual way using the inverse of the metric. Therefore expressions like $\varepsilon^{\mu_1\cdots\mu_p}\mu_{p+1}\cdots\mu_D$ are not well defined. There is no such problem for $\varepsilon^{a_1\cdots a_p}b_{p+1}\cdots b_D$.

Exercise 7.14 Prove, using (7.45), that both $\varepsilon^{\mu_1\mu_2\cdots\mu_D}$ and $\varepsilon_{\mu_1\mu_2\cdots\mu_D}$ take values \pm 1 for any choice of frame field e_μ^a. This guarantees that they are invariant under infinitesimal changes of the frame field. Show also directly that $\delta\varepsilon^{\mu_1\mu_2\cdots\mu_D} = 0$ for any δe_a^μ using the general matrix formula

$$\delta \det M = (\det M) \operatorname{Tr}\left(M^{-1}\delta M\right),$$

and the Schouten identity; see (3.11).

With these preliminaries, the canonical volume form is defined as

$$\begin{aligned} \mathrm{d}V &\equiv e^0 \wedge e^1 \wedge \cdots \wedge e^{D-1} \\ &= \frac{1}{D!}\varepsilon_{a_1\cdots a_D} e^{a_1} \wedge \cdots \wedge e^{a_D} \\ &= \frac{1}{D!} e\varepsilon_{\mu_1\cdots\mu_D}\, \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_D} \\ &= e\ \mathrm{d}x^0 ... \mathrm{d}x^{D-1} \\ &= \mathrm{d}^D x\sqrt{-\det g} \end{aligned}$$

Note that the determinant of the frame field e_μ^a appears in a natural fashion. In the last line we give the abbreviated notation we will use in most applications. For example, given the Lagrangian of a system of fields, such as the kinetic Lagrangian $L = \frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$ of a scalar field, the action integral is written as

$$S = \int \mathrm{d}V L = \int \mathrm{d}^D x\sqrt{-\det g} L$$

14.1.9 7.6 Hodge duality of forms

Since p - and q-forms have the same number of components when $p+q = D$, it is possible to define a $1:1$ map between them. This map is the Hodge duality map from $\Lambda^p(M) \to \Lambda^q(M)$, and it is quite useful in the physics of supergravity. The map is denoted by $\Omega^{(q)} = *\omega^{(p)}$

Since the map is linear we can define it on a basis of p-forms and then extend to a general form. It is convenient to use the local frame basis initially and define

$$*e^{a_1} \wedge \cdots \wedge e^{a_p} = \frac{1}{q!} e^{b_1} \wedge \cdots \wedge e^{b_q} \varepsilon_{b_1\cdots b_q}{}^{a_1\cdots a_p}.$$

A general p-form can be expressed in this basis, and we can proceed to define its dual via

$$\begin{aligned} \Omega^{(q)} = {}^*\omega^{(p)} &= * \left(\frac{1}{p!}\omega_{a_1\cdots a_p} e^{a_1} \wedge \cdots \wedge e^{a_p}\right) \\ &= \frac{1}{p!}\omega_{a_1\cdots a_p}{}^* e^{a_1} \wedge \cdots \wedge e^{a_p} \end{aligned}$$

Exercise 7.15 Show that the frame components of $\Omega^{(q)}$ are given by

$$\Omega_{b_1\cdots b_q} = \left({}^*\omega\right)_{b_1\cdots b_q} = \frac{1}{p!}\varepsilon_{b_1\cdots b_q}{}^{a_1\cdots a_p}\omega_{a_1\cdots a_p}$$

These formulas are far less complicated than they look since there is only one independent term in each sum. For example, for $D = 4$ the dual of a 3-form is a 1-form. For basis elements we have $*e^1 \wedge e^2 \wedge e^3 = e^0$ and ${}^*e^0 \wedge e^1 \wedge e^2 = e^3$. For components, $\left({}^*\omega\right)_0 = \omega_{123}$ and $\left({}^*\omega\right)_3 = \omega_{012}$.

The duality has an important involutive property, which can be inferred from the following sequence of operations on basis elements:

$$
\begin{aligned}
*{}^{*}e^{a_1} \wedge \cdots \wedge e^{a_p}) &= \frac{1}{q!} e^{b_1} \wedge \cdots \wedge e^{b_q} \varepsilon_{b_1 \cdots b_q}{}^{a_1 \cdots a_p} \\
&= \frac{1}{p!q!} e^{c_1} \wedge \cdots \wedge e^{c_p} \varepsilon_{c_1 \cdots c_p}{}^{b_1 \cdots b_q} \varepsilon_{b_1 \cdots b_q}{}^{a_1 \cdots a_p} \\
&= -(-)^{pq} e^{c_1} \wedge \cdots \wedge e^{c_p} \delta^{a_1 \cdots a_p}_{c_1 \cdots c_p} \\
&= -(-)^{pq} e^{a_1} \wedge \cdots \wedge e^{a_p}
\end{aligned}
$$

This leads to the general relation ${}^{*}\left({}^{*}\omega^{(p)}\right) = -(-)^{pq}\omega^{(p)}$. This is the correct relation for a Lorentzian signature manifold. For Euclidean signature the involution property is $*\left({}^{*}\omega^{(p)}\right) = (-)^{pq}\omega^{(p)}$.

For even dimension $D = 2m$, it is possible to impose the constraint of self-duality (or anti-self-duality) on forms of degree m, i.e. $\Omega^{(m)} = \pm{}^{*}\Omega^{(m)}$. In a given dimension this condition is consistent only if duality is a strict involution, i.e. $-(-)^{m^2} = -(-)^m = +1$ for Lorentzian signature and $(-)^m = +1$ for Euclidean signature. Thus it is possible to have self-dual Yang-Mills instantons in four Euclidean dimensions. A self-dual $F^{(5)}$ is possible in $D = 10$ Lorentzian signature, and it indeed appears in Type IIB supergravity.

The duality relations defined above in a frame basis are easily transformed to a coordinate basis using the relations $e^a = e^a_\mu(x)\mathrm{d}x^\mu$ and $\mathrm{d}x^\mu = e^\mu_a(x)e^a$. For coordinate basis elements the duality map is

$$
*(\ \mathrm{d}x^{\mu_1} \wedge \cdots \wedge \mathrm{d}x^{\mu_p}) = \frac{1}{q!} e g^{\mu_1\rho_1} \cdots g^{\mu_p\rho_p}\ \mathrm{d}x^{\nu_1} \wedge \cdots \wedge \mathrm{d}x^{\nu_p} \varepsilon_{\nu_1 \cdots \nu_q \rho_1 \cdots \rho_p}.
$$

For antisymmetric tensor components, we have

$$
({}^{*}\omega)_{\mu_1 \cdots \mu_q} = \frac{1}{p!} e \varepsilon_{\mu_1 \cdots \mu_q \rho_1 \cdots \rho_p} g^{\nu_1\rho_1} \ldots g^{v_p\rho_p} \omega_{\nu_1 \cdots v_p}
$$

Following the discussion in Sec. 7.5, we may take as a volume form on M the wedge product ${}^{*}\omega^{(p)} \wedge \omega^{(p)}$ of any p-form and its Hodge dual. The integral of this volume form is simply the standard invariant norm of the tensor components of $\omega^{(p)}$, i.e.

$$
\int {}^{*}\omega^{(p)} \wedge \omega^{(p)} = \frac{1}{p!} \int \mathrm{d}^D x \sqrt{-g}\, \omega^{\mu_1 \cdots \mu_p} \omega_{\mu_1 \cdots \mu_p}
$$

Exercise 7.16 Prove (7.56). Use the definitions above and those in Sec. 7.5 and the fact that

$$
e^{a_1} \wedge \cdots \wedge e^{a_q} \wedge e^{b_1} \wedge \cdots \wedge e^{b_p} = -\varepsilon^{a_1 \cdots a_q b_1 \cdots b_p}\ \mathrm{d}V
$$

where $\mathrm{d}V$ is the canonical volume element of (7.48).

Exercise 7.17 Show that the volume form $\mathrm{d}V$ can also be written as $*1$.

Exercise 7.18 Compare these definitions with Sec. 4.2.1, to obtain

$$
\tilde{F}_{\mu\nu} = -\mathrm{i}\left({}^{*}F\right)_{\mu\nu}.
$$

Show that the factor i ensures that the tilde operation squares to the identity. Self-duality is then possible for complex 2-forms.

Exercise 7.19 For applications to gauge field theories it is useful to record the relation between the components of the field strength 2-form and its dual:

$$
{}^{*}F_{\mu\nu} \equiv \frac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \qquad {}^{*}F^{\mu\nu} = \frac{1}{2\sqrt{-g}}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}
$$

Verify the second relation. Since both $F_{\mu\nu}$ and ${}^{*}F_{\mu\nu}$ are tensors, their indices are raised by $g^{\mu\nu}$.

14.1.10 7.7 Stokes' theorem and electromagnetic charges

Suppose that M is a manifold of dimension D, and that Σ_p with boundary $\Sigma_{p-1} = \partial\Sigma_p$ is a submanifold of dimension $p \le D$. Suppose further that ω^{p-1} is a $(p-1)$-form that satisfies certain smoothness properties which we omit here. [3] Stokes' theorem asserts that

$$\int\limits_{\Sigma_p} \mathrm{d}\omega^{p-1} = \int\limits_{\Sigma_{p-1}} \omega^{p-1}$$

The integrals can be evaluated using any choice of coordinates.

In Ex. 4.14 electric and magnetic charges in three-dimensional flat spacetime were expressed as volume integrals. We will use Stokes' theorem to convert these to surface integrals. However, we generalize the discussion and consider a spacetime metric $g_{\mu\nu}(x)$ on M and a conserved current J^v to which the gauge field is coupled. The coordinate invariant action that describes this coupling is

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[-\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} + A_\nu J^\nu\right]$$

The gauge field equation of motion (4.49) and the tensor $G^{\mu v}$ of (4.50) generalize to

$$\begin{aligned}
&\frac{\delta S}{\delta A_\nu} = \partial_\mu\left(\sqrt{-g}F^{\mu\nu}\right) + \sqrt{-g}J^\nu = 0 \\
&G_{\mu\nu} \equiv \varepsilon_{\mu\nu\rho\sigma}\frac{\delta S}{\delta F_{\rho\sigma}} = -\frac{1}{2}\sqrt{-g}\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma} = -{}^*F_{\mu\nu}.
\end{aligned}$$

The volume integrals (4.62) and (4.63) for the electric and magnetic charge contained in a region Σ_3 with boundary Σ_2 now become

$$\begin{aligned}
q &= \int\limits_{\Sigma_3} \mathrm{d}^3x\sqrt{-g}J^0 = \int\limits_{\Sigma_3} \mathrm{d}^3x\partial_i\sqrt{-g}F^{0i} = -\frac{1}{2}\int\limits_{\Sigma_3} \mathrm{d}^3x\varepsilon^{ijk}\partial_i G_{jk}, \\
p &= -\frac{1}{2}\int\limits_{\Sigma_3} \mathrm{d}^3x\varepsilon^{ijk}\partial_i F_{jk}.
\end{aligned}$$

The integrands in the final expressions are each the exterior derivatives of 2-forms on Σ_3, so we can apply Stokes' theorem and rewrite them as

$$\begin{pmatrix} p \\ q \end{pmatrix} = -\frac{1}{2}\int\limits_{\Sigma_3} \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k\partial_i\begin{pmatrix} F_{jk} \\ G_{jk} \end{pmatrix} = -\frac{1}{2}\int\limits_{\Sigma^2} \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu\begin{pmatrix} F_{\mu\nu} \\ G_{\mu\nu} \end{pmatrix}$$

The detailed form of $G_{\mu\nu} = -{}^*F_{\mu\nu}$ in terms of the components of $F_{\mu\nu}$ is given in (7.59).

Exercise 7.20 The components of $F_{\mu\nu}$ which describe point charges are quite basic quantities. Derive them for flat spacetime using polar coordinates with metric

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right)$$

3 See [43] for details and proof of the theorem. Stokes' theorem dates from 1850 and 1854.

Show that a field configuration whose only non-vanishing component is

$$F_{\theta\phi} = -\frac{p}{4\pi}\sin\theta$$

is a solution of Maxwell's equation (7.62) and has magnetic charge p. Show that a field configuration whose only non-vanishing component is

$$E_r = F_{rt} = \frac{q}{4\pi r^2}$$

is a solution of Maxwell's equation (7.62) which describes an electric point charge q (use $\varepsilon_{\mathrm{tr}\,\theta\phi} = 1$).

14.1.11 7.8 p-form gauge fields

Using the ideas of Sec. 7.6, we can rewrite the simplest kinetic actions of scalars and gauge vectors as integrals of differential forms:

$$\begin{aligned} S_0 &= -\tfrac{1}{2} \int {}^*F^{(1)} \wedge F^{(1)}, \quad F^{(1)} \equiv \mathrm{d}\phi, \\ S_1 &= -\tfrac{1}{2} \int {}^*F^{(2)} \wedge F^{(2)}, \quad F^{(2)} \equiv \mathrm{d}A^{(1)}. \end{aligned}$$

In each case there is a Bianchi identity, $\mathrm{d}F^{(1)} = 0$ and $\mathrm{d}F^{(2)} = 0$, which implies that the field strengths can be written as differentials of a lower form. For the form $A^{(1)}$, which describes the photon, there is a gauge transformation that can be written as $\delta A^{(1)} = \mathrm{d}\Lambda^{(0)}$. We can interpret the actions of (7.69) as the definition of field theories for 0 -form and 1 -form 'potentials'.

This suggests a generalization. We can describe a p-form 'potential' in terms of a $(p+1)$-form 'field strength' and write the action

$$S_p = -\frac{1}{2} \int *F^{(p+1)} \wedge F^{(p+1)}, \quad F^{(p+1)} \equiv \mathrm{d}A^{(p)}$$

Again there is a gauge transformation $\delta A^{(p)} = \mathrm{d}\Lambda^{(p-1)}$, and these transformations of the p-form gauge potential leave $F^{(p+1)}$ and the action invariant.

Exercise 7.21 Show that the action (7.70) can be expressed in form components as

$$\begin{aligned} S_p &= -\frac{1}{2(p+1)!} \int \mathrm{d}^D x \sqrt{-g} F^{\mu_1 \cdots \mu_{p+1}} F_{\mu_1 \cdots \mu_{p+1}}, \\ F_{\mu_1 \cdots \mu_{p+1}} &= (p+1) \partial_{[\mu_1} A_{\mu_2 \ldots \mu_{p+1}]} \end{aligned}$$

We now determine the number of degrees of freedom of a p-form gauge field. The number of independent components in $\Lambda^{(p-1)}$ is $\begin{pmatrix} D \\ p-1 \end{pmatrix}$. However, not all the components of

15 Supergravity

15.1 Basics of supergravity

15.1.1 Idea of supergravity

(подрбно, в чем идея этого?)

Idea

Analogy

How to visualize supergravity? (????????????)

(don't know)

16 Другие альтернативные теории гравитации

пока без структуры, я просто знаю, что они есть, пусть вот и будут.

16.1 Purely affine gravity

(крутая статья от Nikodem J. Pop lawski, потом разберу её, очень много отсылок на разные теории гравитации)

16.1.1 Gravitation, electromagnetism and cosmological constant

(тут нужно будет перечитать статью, потому что там гоитческий лагранжиан, который матпикс не берет, так что переписвать всё нужно будет!!)

Обзор (???)

The Ferraris-Kijowski purely affine Lagrangian for the electromagnetic field, that has the form of the Maxwell Lagrangian with the metric tensor replaced by the symmetrized Ricci tensor, is dynamically equivalent to the metric Einstein-Maxwell Lagrangian, except the zero-field limit, for which the metric tensor is not well-defined. This feature indicates that, for the Ferraris-Kijowski model to be physical, there must exist a background field that depends on the Ricci tensor. The simplest possibility, supported by recent astronomical observations, is the cosmological constant, generated in the purely affine formulation of gravity by the Eddington Lagrangian. In this paper we combine the electromagnetic field and the cosmological constant in the purely affine formulation. We show that the sum of the two affine (Eddington and Ferraris-Kijowski) Lagrangians is dynamically inequivalent to the sum of the analogous (ΛCDM and Einstein-Maxwell) Lagrangians in the metricaffine/metric formulation. We also show that such a construction is valid, like the affine EinsteinBorn-Infeld formulation, only for weak electromagnetic fields, on the order of the magnetic field in outer space of the Solar System. Therefore the purely affine formulation that combines gravity, electromagnetism and cosmological constant cannot be a simple sum of affine terms corresponding separately to these fields. A quite complicated form of the affine equivalent of the metric EinsteinMaxwell-Λ Lagrangian suggests that Nature can be described by a simpler affine Lagrangian, leading to modifications of the Einstein-Maxwell-ΛCDM theory for electromagnetic fields that contribute to the spacetime curvature on the same order as the cosmological constant.

Суть (??)

(вообще хз)

16.1.2 FIELD EQUATIONS IN PURELY AFFINE GRAVITY

В двух словах

The condition for a Lagrangian density to be covariant is that it must be a product of a scalar and the square root of the determinant of a covariant tensor of rank two, or a linear combination of such products [2, 3]. A general purely affine Lagrangian density $\mathfrak{L}$ depends on the affine connection $\Gamma^{\rho}_{\mu\nu}$ (not restricted to be symmetric in the lower indices), the curvature tensor $R^{\rho}_{\mu\sigma\nu} = \Gamma^{\rho}_{\mu\nu,\sigma} - \Gamma^{\rho}_{\mu\sigma,\nu} + \Gamma^{\kappa}_{\mu\nu}\Gamma^{\rho}_{\kappa\sigma} - \Gamma^{\kappa}_{\mu\sigma}\Gamma^{\rho}_{\kappa\nu}$, and their covariant derivatives (with respect to $\Gamma^{\rho}_{\mu\nu}$). In order to be generally covariant, the Lagrangian density $\mathfrak{L}$ may depend on $\Gamma^{\rho}_{\mu\nu}$ only through the covariant derivatives of tensors. If we assume that is of the first differential order with respect to the connection, as usually are Lagrangians in classical mechanics with respect to the configuration, and that derivatives of the connection appear in? only through the curvature $R^{\rho}_{\mu\sigma\nu}$, then $=\mathbb{L}(S,R)$.

We assume that the dependence of $\mathfrak{L}$ on the curvature is restricted to the symmetric part $P_{\mu\nu}=R_{(\mu\nu)}$ of the Ricci tensor $R_{\mu\nu}=R^{\rho}_{\mu\rho\nu}$, as in general relativity: $\boldsymbol{\mathcal{L}}=\boldsymbol{Z}(S,P)$.[2] We also assume that $\boldsymbol{x}$ depends on, in addition to the torsion tensor and the symmetrized Ricci tensor, a matter field ϕ and its covariant derivatives $\nabla\phi$: $\boldsymbol{L}=\mathfrak{L}(S,P,\phi,\nabla\phi)$. We denote this Lagrangian density as $\mathfrak{L}(\Gamma,P,\phi,\partial\phi)$, bearing in mind that its dependence on the connection Γ and ordinary derivatives $\partial\phi$ is not arbitrary, but such a Lagrangian is a covariant function of the torsion S and $\nabla\phi$. The variation of the corresponding action $I=\frac{1}{c}\int d^4x\mathfrak{L}$ is given by

$$\delta I=\frac{1}{c}\int d^4x\left(\frac{\partial\mathbf{1}}{\partial\Gamma^{\rho}_{\mu\nu}}\delta\Gamma^{\rho}_{\mu\nu}+\frac{\partial\mathbf{2}}{\partial P_{\mu\nu}}\delta P_{\mu\nu}+\frac{\partial}{\partial\phi}\delta\phi+\frac{\partial\mathbf{L}}{\partial\phi_{,\mu}}\delta\left(\phi_{,\mu}\right)\right).$$

The fundamental tensor density $\mathrm{g}^{\mu\nu}$ associated with a purely affine Lagrangian is obtained using [4, 14, 15, 17, 31, 32]

$$\mathrm{g}^{\mu\nu}\equiv-\mathfrak{L}\kappa\frac{\partial\mathbf{2}}{\partial P_{\mu\nu}},$$

where $\kappa=\frac{8\pi G}{c^4}$ (for purely affine Lagrangians that depend on the symmetric part of the Ricci tensor, this definition is equivalent to that in Refs. [2, 3, 42] : $\mathrm{g}^{\mu\nu}=-2\kappa\frac{\partial\boldsymbol{L}}{\partial R_{\mu\nu}}\Big)$. This density introduces the metric structure in purely affine gravity by defining the symmetric contravariant metric tensor [4]:

$$g^{\mu\nu}\equiv\frac{\mathrm{g}^{\mu\nu}}{\sqrt{-\det\mathrm{g}^{\rho\sigma}}}.$$

To make this definition meaningful, we must assume $\det\mathrm{g}^{\mu\nu}\neq0$. The physical signature requirement for $g_{\mu\nu}$ implies that we must take into account only those configurations with $\det\mathrm{g}^{\mu\nu}<0$, which guarantees that $g_{\mu\nu}$ has the Lorentzian signature $(+,-,-,-)$ or $(-,+,+,+)$[16]. The symmetric covariant metric tensor $g_{\mu\nu}$ is related to the contravariant metric tensor by $g^{\mu\nu}g_{\rho\nu}=\delta^{\mu}_{\rho}$.[3] The tensors $g^{\mu\nu}$ and $g_{\mu\nu}$ are used for raising and lowering indices.

We also define the hypermomentum density conjugate to the affine connection [47, 48, 49] :[4]

$$\Pi^{\mu\nu}_{\rho}\equiv-2\kappa\frac{\partial}{\partial\Gamma^{\rho}_{\mu\nu}}$$

which has the same dimension as the connection. Consequently, the variation of the action (1) can be written as

$$\delta I=-\frac{1}{2\kappa c}\int d^4x\left(\Pi^{\mu}_{\rho}\delta\Gamma^{\rho}_{\mu\nu}+\mathrm{g}^{\mu\nu}\delta R_{\mu\nu}\right)+\frac{1}{c}\int d^4x\left(\frac{\partial\mathbf{I}}{\partial\phi}\delta\phi+\frac{\partial\mathbf{I}}{\partial\phi_{,\mu}}\delta\left(\phi_{,\mu}\right)\right).$$

If we assume that the field ϕ vanishes at the boundary of integration, then the field equation for ϕ is $\frac{\delta}{\delta\phi}=0$, where

$$\delta I=\frac{1}{c}\int d^4x\left[-\frac{1}{2\kappa}\left(\Pi^{\mu}_{\rho}\delta\Gamma^{\rho}_{\mu\nu}+\mathrm{g}^{\mu\nu}\delta P_{\mu\nu}\right)+\frac{\delta\mathbf{f}}{\delta\phi}\delta\phi\right]$$

For a general affine connection, the variation of the Ricci tensor is given by the Palatini formula [3, 43, 47]: $\delta R_{\mu\nu}=\delta\Gamma^{\rho}_{\mu\nu;\rho}-\delta\Gamma^{\rho}_{\mu\rho;\nu}-2S^{\sigma}_{\rho\nu}\delta\Gamma^{\rho}_{\mu\sigma}$, where the semicolon denotes the covariant differentiation with respect to $\Gamma^{\rho}_{\mu\nu}$. Using the identity $\int d^4x\left(\mathrm{V}^{\mu}\right)_{;\mu}=2\int d^4xS_{\mu}\mathrm{V}^{\mu}$, where V^{μ} is an arbitrary contravariant vector density that vanishes at the boundary of the integration and $S_\mu = S^\nu_{\mu\nu}$ is the torsion vector [3, 43], and applying the principle of least action $\delta S = 0$ for arbitrary variations $\delta\Gamma^\rho_{\mu\nu}$, we obtain

$$\mathrm{g}^{\mu\nu};\rho - \mathrm{g}^{\mu\sigma}{}_{;\sigma}\delta^\nu_\rho - 2\ \mathrm{g}^{\mu\nu}S_\rho + 2\ \mathrm{g}^{\mu\sigma}S_\sigma\delta^\nu_\rho + 2\ \mathrm{g}^{\mu\sigma}S^\nu_{\rho\sigma} = \Pi^{\mu\nu}_\rho.$$

This equation is equivalent to

$$\mathrm{g}^{\mu\nu}{}_{,\rho} + {}^*\Gamma^\mu_{\sigma\rho}\mathrm{g}^{\sigma\nu} + {}^*\Gamma^\nu_{\rho\sigma}\mathrm{g}^{\mu\sigma} - {}^*\Gamma^\sigma_{\sigma\rho}\mathrm{g}^{\mu\nu} = \Pi^{\mu\nu}_\rho - \frac{1}{3}\Pi^\mu_\sigma\sigma^\nu_\rho,$$

where ${}^*\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu} + \frac{2}{3}\delta^\rho_\mu S_\nu$[3, 24]. Contracting the indices μ and ρ in Eq. (7) yields [5]

$$\Pi^{\sigma\nu}_\sigma = 0,$$

which is a constraint on how a purely affine Lagrangian depends on the connection. This unphysical constraint is related to the fact that the gravitational part of this Lagrangian (proportional to $\mathrm{g}^{\mu\nu}P_{\mu\nu}$, see the next section) is invariant under projective transformations of the connection while the matter part, that can depend explicitly on the connection, is generally not invariant [48, 49, 50]. We cannot assume that any form of matter will comply with this condition. Therefore the field equations (7) seem to be inconsistent. To overcome this constraint we can restrict the torsion tensor to be traceless: $S_\mu = 0$[50]. Consequently, ${}^*\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}$. This condition enters the Lagrangian density as a Lagrange multiplier term $-\frac{1}{2\kappa}\mathrm{B}^\mu S_\mu$, where the Lagrange multiplier B $^\mu$ is a vector density. Consequently, there is an extra term $\mathrm{B}^{[\mu}\delta^{\nu]}_\rho$ on the right-hand side of Eq. (7) and Eq. (9) becomes $\frac{3}{2}$ $\mathrm{B}^\nu = \Pi^\sigma{}_\sigma{}^\nu$. Setting this equation to be satisfied identically removes the constraint (9) and brings Eq. (8) into

$$\mathrm{g}^{\mu\nu}{}_{,\rho} + \Gamma^\mu_{\sigma\rho}\mathrm{g}^{\sigma\nu} + \Gamma^\nu_{\rho\sigma}\mathrm{g}^{\mu\sigma} - \Gamma^\sigma_{\sigma\rho}\mathrm{g}^{\mu\nu} = \Pi^{\mu\nu}_\rho - \frac{1}{3}\Pi^{\mu\sigma}_\sigma\delta^\nu_\rho - \frac{1}{3}\Pi^\sigma_{\sigma\rho}\delta^\mu_\rho$$

Equation (10) is an algebraic equation for $\Gamma^\rho_{\mu\nu}$ (with $S_\mu = 0$) as a function of the metric tensor, its first derivatives and the density $\Pi^\mu_\rho 2$. We seek its solution in the form:

$$\Gamma^\rho_{\mu\nu} = \left\{{}^{\rho}_{\mu\nu}\right\}_g + V^\rho_{\mu\nu},$$

which is a constraint on how a purely affine Lagrangian depends on the connection. This unphysical constraint is related to the fact that the gravitational part of this Lagrangian (proportional to $\mathrm{g}^{\mu\nu}P_{\mu\nu}$, see the next section) is invariant under projective transformations of the connection while the matter part, that can depend explicitly on the connection, is generally not invariant [48, 49, 50]. We cannot assume that any form of matter will comply with this condition. Therefore the field equations (7) seem to be inconsistent. To overcome this constraint we can restrict the torsion tensor to be traceless: $S_\mu = 0$[50]. Consequently, ${}^*\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}$. This condition enters the Lagrangian density as a Lagrange multiplier term $-\frac{1}{2\kappa}\mathrm{B}^\mu S_\mu$, where the Lagrange multiplier B $^\mu$ is a vector density. Consequently, there is an extra term $\mathrm{B}^{[\mu}\delta^{\nu]}_\rho$ on the right-hand side of Eq. (7) and Eq. (9) becomes $\frac{3}{2}$ $\mathrm{B}^\nu = \Pi^\sigma{}_\sigma{}^\nu$. Setting this equation to be satisfied identically removes the constraint (9) and brings Eq. (8) into

$$\mathrm{g}^{\mu\nu}{}_{,\rho} + \Gamma^\mu_{\sigma\rho}\mathrm{g}^{\sigma\nu} + \Gamma^\nu_{\rho\sigma}\mathrm{g}^{\mu\sigma} - \Gamma^\sigma_{\sigma\rho}\mathrm{g}^{\mu\nu} = \Pi^{\mu\nu}_\rho - \frac{1}{3}\Pi^{\mu\sigma}_\sigma\delta^\nu_\rho - \frac{1}{3}\Pi^\sigma_{\sigma\rho}\delta^\mu_\rho$$

Equation (10) is an algebraic equation for $\Gamma^\rho_{\mu\nu}$ (with $S_\mu = 0$) as a function of the metric tensor, its first derivatives and the density $\Pi^\mu_\rho 2$. We seek its solution in the form:

$$\Gamma^\rho_{\mu\nu} = \left\{{}^{\rho}_{\mu\nu}\right\}_g + V^\rho_{\mu\nu},$$

where $\left\{{\rho \atop \mu\nu}\right\}_g = \frac{1}{2}g^{\rho\sigma}\left(g_{\nu\sigma,\mu} + g_{\mu\sigma,\nu} - g_{\mu\nu,\sigma}\right)$ is the Christoffel connection of the metric tensor $g_{\mu\nu}$. Consequently, the Ricci tensor of the affine connection $\Gamma^\rho_{\mu\nu}$ is given by [43]

$$R_{\mu\nu}(\Gamma) = R_{\mu\nu}(g) + V^\rho_{\mu\nu:\rho} - V^\rho_{\mu\rho:\nu} + V^\sigma_{\mu\nu}V^\rho_{\sigma\rho} - V^\sigma_{\mu\rho}V^\rho_{\sigma\nu},$$

where $R_{\mu\nu}(g)$ is the Riemannian Ricci tensor of the metric tensor $g_{\mu\nu}$ and the colon denotes the covariant differentiation with respect to $\left\{{\rho \atop \mu\nu}\right\}_g$. Substituting Eq. (11) to Eq. (10) gives

$$V^\mu_{\sigma\rho}\mathrm{g}^{\sigma\nu} + V^\nu_{\rho\sigma}\mathrm{g}^{\mu\sigma} - V^\sigma_{\sigma\rho}\mathrm{g}^{\mu\nu} = \Pi^{\mu\nu}_{\ \ \rho} - \frac{1}{3}\Pi^\mu_\sigma\sigma^\nu_\rho - \frac{1}{3}\Pi^\sigma_{\sigma\sigma}\delta^\mu_\rho,$$

which is a linear relation between $V^\rho_{\mu\nu}$ and $\Pi^{\mu\nu}_{\ \ \rho}$ and can be solved [51, 52]. If a purely affine Lagrangian does not depend explicitly on the connection (such Lagrangians are studied later in this paper) then $\Pi^{\mu\nu}_{\ \ \rho} = 0$. In this case, we do not need to introduce the condition $S_\mu = 0$ (or any constraint on four degrees of freedom of the connection) and Eq. (8) becomes

$$\mathrm{g}^{\mu\nu}{}_{,\rho} + {}^*\Gamma^\mu_{\sigma\rho}\mathrm{g}^{\sigma\nu} + {}^*\Gamma^\nu_{\rho\sigma}\mathrm{g}^{\mu\sigma} - {}^*\Gamma^\sigma_{\sigma\rho}\mathrm{g}^{\mu\nu} = 0.$$

The tensor $P_{\mu\nu}$ is invariant under a projective transformation $\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu} + \delta^\rho_\mu W_\nu$. We can use this transformation, with $W_\mu = \frac{2}{3}S_\mu$, to bring the torsion vector S_μ to zero and make ${}^*\Gamma^\rho_{\mu\nu} = \Gamma^\rho_{\mu\nu}$. From Eq. (14) it follows that the affine connection is the Christoffel connection of the metric tensor:

$$\Gamma^\rho_{\mu\nu} = \left\{ {\rho \atop \mu\nu} \right\}_g,$$

which is the special case of Eq. (11) with $V^\rho_{\mu\nu} = 0$. The theory based on a general Lagrangian density $\boldsymbol{x}(S, P, \phi, \nabla\phi)$ without any constraints on the affine connection does not determine the connection uniquely because the tensor $P_{\mu\nu}$ is invariant under projective transformations of the connection, $\Gamma^\rho_{\mu\nu} \to \Gamma^\rho_{\mu\nu} + \delta^\rho_\mu V_\nu$, where V_ν is a vector function of the coordinates. Therefore at least four degrees of freedom must be constrained to make such a theory consistent from a physical point of view [48, 49]. The condition $S_\mu = 0$ is not the only way to impose such a constraint; other possibilities include vanishing of the Weyl vector $W_\nu = \frac{1}{2}\left(\Gamma^\rho_{\rho\nu} - \left\{ {\rho \atop \rho\nu} \right\}_g\right) = 0$[48, 49] or adding the dependence on the segmental curvature tensor, which is not projectively invariant, to the Lagrangian [48, 49, 51, 52].[6]

If we assume that the affine connection in the purely affine variational principle is symmetric, as in the original formulation of purely affine gravity [4, 14, 15], then instead of Eq. (7) we find [16]:

$$\mathrm{g}^{\mu\nu};\rho - \mathrm{g}^{\sigma(\mu};\sigma^{\nu)}_\rho = \Pi^{\mu\nu}_{\ \ \rho}.$$

The hypermomentum density $\Pi^{\mu\nu}_{\ \ \rho}$ is, because of the definition (4), symmetric in the upper indices. Contracting the indices μ and ρ in Eq. (18) does not lead to any algebraic constraint on $\Pi^\mu_\rho 2$. Therefore, for purely affine Lagrangians that depend only on the connection and the symmetric part of the Ricci tensor, the relation $S_\mu = 0$ appears either as a remedy for the unphysical constraint on the hypermomentum density, or results simply from using a symmetric affine connection as a dynamical variable. In this paper we study purely affine Lagrangians that depend on the connection only via the symmetrized Ricci tensor: $\boldsymbol{x} = \boldsymbol{x}(P, \phi, \partial\phi)$, so S_μ decouples from the Einstein equations and can be brought to zero by a projective transformation, leading to Eq. (15).

16.1.3 EQUIVALENCE OF AFFINE, METRIC-AFFINE AND METRIC PICTURES

В двух словах

If we apply to a purely affine Lagrangian $x(\Gamma, P, \phi, \partial\phi)$ the Legendre transformation with respect to $P_{\mu\nu}$[4, 14, 15], defining the Hamiltonian density $\underset{3}{3}$ (or rather the Routhian density [54], since we can also apply the Legendre transformation with respect to $\Gamma^\rho_{\mu\nu}$): we find for the differential $d \leqq$:

$$d\sharp\boldsymbol{\xi} = \frac{\partial \mathbf{x}}{\partial\Gamma^\rho_{\mu\nu}} d\Gamma^\rho_{\mu\nu} + \frac{1}{2\kappa}P_{\mu\nu} d\ \mathrm{g}^{\mu\nu} + \frac{\partial \mathbf{z}}{\partial\phi}d\phi + \frac{\partial\mathbf{x}}{\partial\phi_{,\mu}} d\phi_{,\mu^*}$$

Accordingly, the Hamiltonian density is a covariant function of $\Gamma^\rho_{\mu\nu}, \mathrm{g}^{\mu\nu}, \phi$ and $\partial\phi$: $\exists\underset{3}{\mathfrak{j}}(S,\ \mathrm{g}, \phi, \nabla\phi)$, and the action variation (6) takes the form:

$$\delta I = \frac{1}{c}\delta\int d^4x\left(\mathfrak{j}(\Gamma, g, \phi, \partial\phi) - \frac{1}{2\kappa}\mathrm{g}^{\mu\nu}P_{\mu\nu}\right)$$
$$= \frac{1}{c}\int d^4x\left(\frac{\partial\mathfrak{j}}{\partial\Gamma^\rho_{\mu\nu}}\delta\Gamma^\rho_{\mu\nu} + \frac{\partial\mathfrak{j}}{\partial\mathrm{g}^{\mu\nu}}\delta\mathrm{g}^{\mu\nu} - \frac{1}{2\kappa}\mathrm{g}^{\mu\nu}\delta P_{\mu\nu} - \frac{1}{2\kappa}P_{\mu\nu}\delta\mathrm{g}^{\mu\nu} + \frac{\delta\mathfrak{z}}{\delta\phi}\delta\phi\right)$$

The variation with respect to $\mathrm{g}^{\mu\nu}$ yields the first Hamilton equation [4, 14, 15]:

$$P_{\mu\nu} = 2\kappa\frac{\partial\jmath}{\partial\mathrm{g}\mu\nu}$$

16.1.4 EDDINGTON LAGRANGIAN

В двух словах

The simplest purely affine Lagrangian density $\boldsymbol{x} = \boldsymbol{x}\,(P_{\mu\nu})$ was introduced by Eddington [2, 17]:

$$\mathbf{e}_{\mathrm{Edd}} = \frac{1}{\kappa\Lambda}\sqrt{-\det P_{\mu\nu}}$$

To make this Lagrangian density meaningful, we assume $\det P_{\mu\nu} < 0$, that is, the symmetrized Ricci tensor $P_{\mu\nu}$ has the Lorentzian signature. The Eddington Lagrangian does not depend explicitly on the affine connection, which is analogous in classical mechanics to free Lagrangians that depend only on generalized velocities: $L = L\,(\dot{q}^i)$. Accordingly, the Lagrangian density (27) describes a free gravitational field. Substituting Eq. (27) into Eq. (2) yields [3]

$$\mathrm{g}^{\mu\nu} = -\frac{1}{\Lambda}\sqrt{-\det P_{\rho\sigma}}P^{\mu\nu},$$

where the symmetric tensor $P^{\mu\nu}$ is reciprocal to the symmetrized Ricci tensor $P_{\mu\nu} : P^{\mu\nu}P_{\rho\nu} = \delta^\mu_\rho$. Equation (28) is equivalent to

$$P_{\mu\nu} = -\Lambda g_{\mu\nu}.$$

Since the Lagrangian density (27) does not depend explicitly on the connection, the field equations are given by Eq. (15). As a result, Eq. (29) becomes

$$R_{\mu\nu}(g) = -\Lambda g_{\mu\nu},$$

16.1.5 FERRARIS-KIJOWSKI LAGRANGIAN

В двух словах

The purely affine Lagrangian density of Ferraris and Kijowski [17]:

$$_{\mathrm{FK}} = -\frac{1}{4}\sqrt{-\det P_{\mu\nu}}F_{\alpha\beta}F_{\rho\sigma}P^{\alpha\rho}P^{\beta\sigma},$$

where $\det P_{\mu\nu} < 0$, has the form of the metric-affine (or metric, since the connection does not appear explicitly) Maxwell Lagrangian density for the electromagnetic field $F_{\mu\nu}$:[8]

$$\ggg_{\mathrm{Max}} = -\frac{1}{4}\sqrt{-g}F_{\alpha\beta}F_{\rho\sigma}g^{\alpha\rho}g^{\beta\sigma},$$

in which the covariant metric tensor is replaced by the symmetrized Ricci tensor $P_{\mu\nu}$ and the contravariant metric tensor by the tensor $P^{\mu\nu}$ reciprocal to $P_{\mu\nu}$. Substituting Eq. (32) to Eq. (2) gives (in purely affine picture)

$$\mathrm{g}^{\mu\nu} = \kappa\sqrt{-\det P_{\rho\sigma}}P^{\beta\sigma}F_{\alpha\beta}F_{\rho\sigma}\left(\frac{1}{4}P^{\mu\nu}P^{\alpha\rho} - P^{\mu\alpha}P^{\nu\rho}\right).$$

From Eqs. (22) and (33) it follows that (in metric-affine/metric picture)

$$P_{\mu\nu} - \frac{1}{2}Pg_{\mu\nu} = \kappa\left(\frac{1}{4}F_{\alpha\beta}F_{\rho\sigma}g^{\alpha\rho}g^{\beta\sigma}g_{\mu\nu} - F_{\mu\alpha}F_{\nu\beta}g^{\alpha\beta}\right)$$

which yields $P = 0$. Consequently, Eq. (19) reads $_{\mathrm{Max}} = \underset{\mathrm{Max}}{\mathbf{M}}$, where that is dynamically equivalent to the Maxwell Lagrangian density (33). Similarly, $\bar{\beta}_{\mathrm{FK}} = \boldsymbol{z}_{\mathrm{FK}}$, where metric-affine density that is dynamically equivalent to the Ferraris-Kijowski Lagrangian density (33).

16.1.6 AFFINE EINSTEIN-BORN-INFELD FORMULATION

В двух словах

Since the metric structure in the purely affine Ferraris-Kijowski model of electromagnetism is not well-defined in the zero-field limit, we need to combine the electromagnetic field and the cosmological constant. In the Lagrangian density (32) we used the determinant of the symmetrized Ricci tensor $P_{\mu\nu}$, multiplied by the simplest scalar containing the electromagnetic field tensor and $P_{\mu\nu}$. As an alternative way to add the electromagnetic field into purely affine gravity, we can include the tensor $F_{\mu\nu}$ inside this determinant, [12] constructing a purely affine version of the EinsteinBorn-Infeld theory [38, 39, 40]. For $F_{\mu\nu} = 0$, this construction reduces to the Eddington Lagrangian so the metric structure in the zero-field limit is well-defined. Therefore it describes both the electromagnetic field and cosmological constant. Let us consider the following Lagrangian density:

$$? = \frac{1}{\kappa\Lambda}\sqrt{-\det\left(P_{\mu\nu} + B_{\mu\nu}\right)}$$

where

$$B_{\mu\nu} = i\sqrt{\kappa\Lambda}F_{\mu\nu}$$

and $\Lambda > 0$. Let us also assume

$$|B_{\mu\nu}| \ll |P_{\mu\nu}|$$

where the bars denote the order of the largest (in magnitude) component of the corresponding tensor. [13] Consequently, we can expand the Lagrangian density (47) in small terms $B_{\mu\nu}$. If $s_{\mu\nu}$ is a symmetric tensor and $a_{\mu\nu}$ is an antisymmetric
tensor, the determinant of their sum is given by [43, 58]

$$\det\left(s_{\mu\nu}+a_{\mu\nu}\right)=\det s_{\mu\nu}\left(1+\frac{1}{2}a_{\alpha\beta}a_{\rho\sigma}s^{\alpha\rho}s^{\beta\sigma}+\frac{\det a_{\mu\nu}}{\det s_{\mu\nu}}\right),$$

where the tensor $s^{\mu\nu}$ is reciprocal to $s_{\mu\nu}$. If we associate $s_{\mu\nu}$ with $P_{\mu\nu}$ and $a_{\mu\nu}$ with $B_{\mu\nu}$, and neglect the last term in Eq. (50), we obtain

$$\det\left(P_{\mu\nu}+B_{\mu\nu}\right)=\det P_{\mu\nu}\left(1+\frac{1}{2}B_{\alpha\beta}B_{\rho\sigma}P^{\alpha\rho}P^{\beta\sigma}\right)$$

In the same approximation, the Lagrangian density (47) becomes

$$\boldsymbol{x}=\frac{1}{\kappa\Lambda}\sqrt{-\det P_{\mu\nu}}\left(1+\frac{1}{4}B_{\alpha\beta}B_{\rho\sigma}P^{\alpha\rho}P^{\beta\sigma}\right)$$

which is equal to the sum of the Lagrangian densities (27) and (32). Equations (2) and (3) define the contravariant metric tensor, [14] for which we find [15]

$$\sqrt{-g}g^{\mu\nu}=-\frac{1}{\Lambda}\Big[P^{\mu\nu}\left(1+\frac{1}{4}B_{\alpha\beta}B_{\rho\sigma}P^{\alpha\rho}P^{\beta\sigma}\right)$$
$$-P^{\alpha\beta}B_{\alpha\rho}B_{\beta\sigma}P^{\mu\rho}P^{\nu\sigma}\Big]\sqrt{-\det P_{\mu\nu}}.$$

In the terms containing $B_{\mu\nu}$ and in the determinant [16] we can use the relation $P^{\mu\nu}=-\Lambda^{-1}g^{\mu\nu}$ (equivalent to Eq. (30)) valid for $B_{\mu\nu}=0$. As a result, we obtain

$$g^{\mu\nu}=-\Lambda P^{\mu\nu}+\Lambda^{-2}\left(\frac{1}{4}g^{\mu\nu}B_{\rho\sigma}B^{\rho\sigma}-B^{\mu\rho}B^{\nu}_{\ \rho}\right).$$

Introducing the energy-momentum tensor for the electromagnetic field:

$$T^{\mu\nu}=\frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}-F^{\mu\rho}F^{\nu}_{\ \rho}$$

turns Eq. (54) into

$$P^{\mu\nu}=-\Lambda^{-1}g^{\mu\nu}-\kappa\Lambda^{-2}T^{\mu\nu},$$

which is equivalent, in the approximation (49), to the Einstein equations of general relativity with the cosmological constant in the presence of the electromagnetic field:

$$P_{\mu\nu}=-\Lambda g_{\mu\nu}+\kappa T_{\mu\nu}.$$

As in the case for the gravitational field only, $P_{\mu\nu}=R_{\mu\nu}(g)$. The tensor (55) is traceless, from which it follows that

$$R_{\mu\nu}(g)-\frac{1}{2}R(g)g_{\mu\nu}=\Lambda g_{\mu\nu}+\kappa T_{\mu\nu}.$$

Since the tensor $R_{\mu\nu}(g)$ satisfies the contracted Bianchi identities:

$$\left(R_{\mu\nu}(g)-\frac{1}{2}R(g)g_{\mu\nu}\right)^{;\nu}=0$$

16.2 Многомерные теории гравитации

Такие есть, к ним у меня есть заготовки, ибо Лашкевича проходил, так что мб потом пару слов впишу.

16.2.1 Заготовки для многомерных теорий гравитаций

(выписка из лекций лашкевича, он в общем виде что-то пробовал решать)

16.2.2 Мотивация заниматься многомерными теориями гравитации

Лашкевич вот занимается
я вот хз, зачем, вот правда.

а вдруг мир не 4-х мерный

фиговая мотивация, другого я придумать не могу.

16.2.3 особенности многомерных теорий гравитаций

выделим, чтобы не тупить.

16.2.4 Двумерная гравитация

подумаем про это

16.3 $f(R)$ гравитация

Она есть, она может объяснить темную материю, я ей заниматься пока что не буду.

16.3.1 Уравнения движения

(!!! потренируюсь сам когда-то, хорошая тренировка!! все ленюсь)

Вывод по Шмидту (!!!??)

1.3. Уравнения движения $f(R)$-гравитации. Хотя изучение модифицированных теорий гравитации в текущем семестре не входит в наши планы, получим полевые уравнения движения в рамках модифицированной $f(R)$-гравитации, которую мы упомянули в конце первого параграфа. Вариационная производная функционала действия $S_\mathrm{f}[g]$ имеет вид

$$\delta_g S_\mathrm{f} = -\frac{1}{2}m_\mathrm{pl}^2 \int\limits_M d_4x[\delta\sqrt{-\det g}f(R) + \sqrt{-\det g}F(R)\delta R]$$

где под функцией $F(R)$ понимается производная

$$F(R) = \frac{df}{dR}.$$

Подставляя в последнее выражение полученные ранее формулы для вариаций скалярной кривизны и корня из детерминанта метрики, находим

$$\delta_g S_{\mathrm{f}} = -\frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g \left[1/2 g^{\mu\nu}\delta g_{\mu\nu} f(R) + F(R)\delta g^{\beta\nu} R_{\beta\nu} + F(R) g^{\beta\nu}\delta R_{\beta\nu}\right] =$$

$$= -\frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g \left[1/2 g^{\mu\nu}\delta g_{\mu\nu} f(R) - F(R) R^{\mu\nu}\delta g_{\mu\nu} + g^{\alpha\beta} F(R)\left[\nabla_\sigma \delta\Gamma^\sigma_{\alpha\beta} - \nabla_\beta \delta\Gamma^\sigma_{\alpha\sigma}\right]\right]$$

Преобразуем немного полученное выражение, а именно, проинтегрируем по частям последнее слагаемое

$$-\frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g\, g^{\alpha\beta} F(R)\left[\nabla_\sigma \delta\Gamma^\sigma_{\alpha\beta} - \nabla_\beta \delta\Gamma^\sigma_{\alpha\sigma}\right] =$$

$$= -\frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g \nabla_\sigma \left[g^{\alpha\beta} F(R)\delta\Gamma^\sigma_{\alpha\beta}\right] + \frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g \nabla_\beta \left[g^{\alpha\beta} F(R)\delta\Gamma^\sigma_{\alpha\sigma}\right] +$$

$$+\frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g\, g^{\alpha\beta}\delta\Gamma^\sigma_{\alpha\beta}\nabla_\sigma F(R) - \frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g\, g^{\alpha\beta}\delta\Gamma^\sigma_{\alpha\sigma}\nabla_\beta F(R) =$$

$$= -\frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_{\partial M} d\Sigma_\sigma \left[\sqrt{-\det g}\, g^{\alpha\beta} F(R)\delta\Gamma^\sigma_{\alpha\beta}\right] + \frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_{\partial M} d\Sigma_\beta \left[\sqrt{-\det g}\, g^{\alpha\beta} F(R)\delta\Gamma^\sigma_{\alpha\sigma}\right] +$$

$$+\frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g\, g^{\alpha\beta}\delta\Gamma^\sigma_{\alpha\beta}\nabla_\sigma F(R) - \frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g\, g^{\alpha\beta}\delta\Gamma^\sigma_{\alpha\sigma}\nabla_\beta F(R)$$

В этом выражении мы воспользовались формулой Остроградского-Гаусса. Поверхностные интегралы

$$\int\limits_{\partial M} d\Sigma_\sigma \left[\sqrt{-\det g}\, g^{\alpha\beta} F(R)\delta\Gamma^\sigma_{\alpha\beta}\right] = 0$$

и

$$\int\limits_{\partial M} d\Sigma_\beta \left[\sqrt{-\det g}\, g^{\alpha\beta} F(R)\delta\Gamma^\sigma_{\alpha\sigma}\right] = 0$$

в силу соответствующих граничных условий: вариации полей $\delta g_{\mu\nu}$ и их производных $\partial_\alpha \delta g_{\mu\nu}$ на поверхности интегрирования обращаются к нуль. С учетом этого вариация действия

$$\delta_g S_{\mathrm{f}} = \frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g \left[F(R) R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} f(R)\right]\delta g_{\mu\nu} +$$

$$+\frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g\, g^{\alpha\beta}\delta\Gamma^\sigma_{\alpha\beta}\nabla_\sigma F(R) - \frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g\, g^{\alpha\beta}\delta\Gamma^\sigma_{\alpha\sigma}\nabla_\beta F(R) =$$

$$= \frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g \left[F(R) R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} f(R)\right]\delta g_{\mu\nu} + \frac{1}{2}m_{\mathrm{pl}}^2 \int\limits_M \mathrm{vol}_g \nabla^\mu F(R)\left[\nabla^\nu \delta g_{\mu\nu} - \nabla_\mu\left(g^{\alpha\beta}\delta g_{\alpha\beta}\right)\right]$$

Повторное интегрирование по частям соответствующего слагаемого в последнем выражении дает

$$\int\limits_M \mathrm{vol}_g \nabla^\mu F(R)\left[\nabla^\nu \delta g_{\mu\nu} - \nabla_\mu\left(g^{\alpha\beta}\delta g_{\alpha\beta}\right)\right] = \int\limits_M \mathrm{vol}_g \left[g^{\mu\nu}\Box_g F(R) - \nabla^\mu\nabla^\nu F(R)\right]\delta g_{\mu\nu}$$

Здесь вновь была использована теорема Остроградского-Гаусса. Как и раньше, она позволила занулить соответствующие поверхностные интегралы. Дифференциальный оператор $\square_g : C^\infty(M) \to C^\infty(M)$ представляет собой т.н. оператор Бельтрами-Лапласа, связанный с метрикой g:

$$\square_g = g^{\mu\nu}\nabla_\mu\nabla_\nu = \frac{1}{\sqrt{-\det g}}\partial_\mu\left(\sqrt{-\det g}g^{\mu\nu}\partial_\nu\right)$$

Подставляя полученное выше уравнение в выражение для вариации действия, после применения принципа экстремального действия находим динамические уравнения модифицированной $f(R)$-гравитации [1] :

$$F(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) = \nabla_\mu\nabla_\nu F(R) - g_{\mu\nu}\square_g F(R)$$

Отсюда видно, что теории гравитации с действием S_{f} не содержат высших производных в уравнениях движения при любом допустимом выборе функции $f(R)$.

Другие вопросы

(там потом создам структуру с обзором приложений, тем, как догадаться, моделями, но пока не до этого.)

16.4 Теории с кручением

ОТО - теория без кручения, а если бы она была - что бы было?
вот тут подумаем, если захочу.

16.5 теория Эйнштейна-Картана

хз, но слышал о ней, мб в космологии нужно.

16.6 метрическая аффинная гравитация

хз.

16.7 Гравитация на бране

(потом мб поизучаю. см. Classical tests of general relativity in brane world models Christian G Bohmer 1, Giuseppe De Risi2, Tiberiu Harko3 and Francisco S N Lobo. по ней многое напишу мб, пока не до этого.)

16.7.1 Gravitational field equations on the brane

(пока лишь пара выгрузок, изучать не скоро буду.)

Описание

(?? тут вообще хз.)

We start by considering a 5D spacetime (the bulk), with a single 4D brane, on which matter is confined. The 4D brane world ${}^{(4)}M, g_{\mu v})$ is located at a hypersurface $\left(B\left(X^A\right) = 0\right)$ in the 5D bulk spacetime $\left({}^{(5)}M, g_{AB}\right)$, where the coordinates are described by $X^A, A = 0, 1, ..., 4$. The induced 4D coordinates on the brane are $x^\mu, \mu = 0, 1, 2, 3$. The action of the system is given by [2]

$$S = S_{\text{bulk}} + S_{\text{brane}},$$

where

$$S_{\text{bulk}} = \int_{(5)M} \sqrt{-{}^{(5)}g} \left[\frac{1}{2k_5^2} {}^{(5)}R + {}^{(5)}L_m + \Lambda_5 \right] \mathrm{d}^5 X,$$

and

$$S_{\text{brane}} = \int_{(4)M} \sqrt{-{}^{(5)}g} \left[\frac{1}{k_5^2} K^\pm + L_{\text{brane}} \left(g_{\alpha\beta}, \psi \right) + \lambda_b \right] \mathrm{d}^4 x,$$

where $k_5^2 = 8\pi G_5$ is the 5D gravitational constant, ${}^{(5)}R$ and ${}^{(5)}L_m$ are the 5D scalar curvature and the matter Lagrangian in the bulk $L_{\text{brane}} \left(g_{\alpha\beta}, \psi \right)$ is the 4D Lagrangian, which is given by a generic functional of the brane metric $g_{\alpha\beta}$ and of the matter fields ψ, $K^\pm$is the trace of the extrinsic curvature on either side of the brane and Λ_5 and λ_b (the constant brane tension) are the negative vacuum energy densities in the bulk and on the brane, respectively. The energy-momentum tensor of bulk matter fields is defined as

$${}^{(5)}\tilde{T}_{IJ} \equiv -2 \frac{\delta^{(5)} L_m}{\delta^{(5)} g^{IJ}} + {}^{(5)}g_{IJ} {}^{(5)}L_m,$$

while $T_{\mu\nu}$ is the energy-momentum tensor localized on the brane and is given by

16.7.2 The DMPR brane world vacuum solution

(выгрузки из статьи, потом мб займусь.)

Описание

The first brane solution we consider is a solution of the vacuum field equations, obtained by Dadhich, Maartens, Papadopoulos and Rezania (DMPR) in [6], which represent the simplest generalization of the Schwarzschild solution of GR. We call this type of brane black hole the DMPR black hole. The Solar System tests for the DMPR solutions were extensively analyzed in [29], but we use the general and novel formalism developed above as a consistency check. For this solution the metric tensor components are given by

$$\mathrm{e}^v = \mathrm{e}^{-\lambda} = 1 - \frac{2m}{r} + \frac{Q}{r^2},$$

where Q is the so-called tidal charge parameter. In the limit $Q \to 0$ we recover the usual general relativistic case. In terms of the general equations discussed in section 2, this class of brane world spherical solution is characterized by an equation of state relating dark energy and pressure: $P = -2U$. The metric is asymptotically flat, with $\lim_{r\to\infty} \exp(\nu) = \lim_{r\to\infty} \exp(\lambda) = 1$. There are two horizons, given by

$$r_{\mathrm{h}}^\pm = m \pm \sqrt{m^2 - Q}.$$

Both horizons lie inside the Schwarzschild horizon $r_{\mathrm{s}} = 2m, 0 \leqslant r_{\mathrm{h}}^- \leqslant r_{\mathrm{h}}^+ \leqslant r_{\mathrm{s}}$. In the brane world models there is also the possibility of a negative $Q < 0$, which leads to only one horizon $r_{\mathrm{h}+}$ lying outside the Schwarzschild horizon,

$$r_{\mathrm{h}+} = m + \sqrt{m^2 + Q} > r_{\mathrm{s}}$$

Part VI
Modified Quantum Fields and Gravities

17 title

17.0.1 title

Part VII
Problems

18 General Problems

(от Проена многое напишу, странно, что еще не написал)

18.0.1 Questions about understanding nature of fields and gravity

18.0.2 Questions about understanding typical modifications

19 Typical technical questions

(!! напишу там мотивацию в каждую задачу, почему важно ее решать. часто они что-то дают и где-то используются)

19.0.1 Problems in general gauge theories

Problem from slides around sect. 11.1

1. Use the general rule for gauge fields to find the supersymmetry transformation of the frame field e_μ^a as being the gauge field of translations and the gravitino ψ_μ^α as the gauge field of supersymmetry.

$$\delta(\epsilon)e_\mu{}^a = \epsilon^\gamma \psi_\mu{}^\beta f_{\beta\gamma}{}^a = -\frac{1}{2}\epsilon^\gamma \left(\gamma^a\right)_{\beta\gamma} \psi_\mu{}^\beta = \frac{1}{2}\bar{\epsilon}\gamma^a\psi_\mu$$

(There are no other commutators of the form $[T_A, Q] = ...P^a$)

Solution

Take for A only a, and for the components of ϵ^A only supersymmetry. First term is absent, and

$$\{Q_\alpha, Q_\beta\} = -\frac{1}{2}\left(\gamma^a\right)_{\alpha\beta} P_a, \quad \rightarrow \quad f_{\alpha\beta}{}^a = -\frac{1}{2}\left(\gamma^a\right)_{\alpha\beta} = f_{\beta\alpha}{}^a$$

Problem 2 from slides around sect. 11.1

Find the supersymmetry transformation of the gravitino, using that in the algebra Q appears only at the r.h.s. in $[M_{ab}, Q_\alpha] = -\frac{1}{2}\left(\gamma_{ab}\right)_\alpha{}^\beta Q_\beta$ and in case of AdS algebra: $P_a, Q_\alpha] = \frac{1}{2L}\left(\gamma_a Q\right)_\alpha$

Solution:

Take for A the susy index α

$$M_{ab}, Q_\alpha] = -\frac{1}{2}\left(\gamma_{ab}\right)_\alpha{}^\beta Q_\beta \rightarrow f_{[ab]\alpha}^\beta = -\frac{1}{2}\left(\gamma^\beta\right)$$
$$[P_a, Q_\alpha] = \frac{1}{2L}\left(\gamma_a Q\right)_\alpha \rightarrow f_{a\alpha}^\beta = \frac{1}{2L}\left(\gamma_a\right)_\alpha^\beta$$

$$\delta(\epsilon)\psi_\mu^\alpha = \partial_\mu\epsilon^\alpha - \frac{1}{4}\epsilon^\gamma\omega_\mu^{ab}\left(\gamma_{ab}\right)\gamma^\alpha + \frac{1}{2L}\epsilon^\gamma e_\mu^a\left(\gamma_a\right)_\gamma{}^\alpha$$
$$\delta\bar\psi_\mu = \partial_\mu\bar\epsilon - \frac{1}{4}\bar\epsilon\omega_\mu^{ab}\gamma_{ab} + \frac{1}{2L}\bar\epsilon\gamma_\mu$$
$$\hat D_\mu\epsilon \equiv \left(D_\mu - \frac{1}{2L}\gamma_\mu\right)\epsilon = \left(\partial_\mu + \frac{1}{4}\omega_{\mu ab}\gamma^{ab} - \frac{1}{2L}\gamma_\mu\right)\epsilon$$

20 Advanced research questions for practice

Here are questions, that one would need at least a weak to do (being generally prepared to work in field theory).

(I'll structure them later by types.)

1 Duality transformations and their currents

In the appendix A there is a text on duality transformation, which is for a large part based on [2]:
M.K. Gaillard and B. Zumino, "Duality rotations for interacting fields", Nucl.
Phys. B 193 (1981) 221-244

This is an extension of Section 4.2.4 of the book. In the book, we did not treat the interaction with other fields than Abelian gauge fields (though implicitly it is present in Sec. 21.2). Here these are considered systematically. Read the attached text, and make the expressions explicit using the example in my text at the end.

Read Sec. 2.2 and 2.3 of Gaillard-Zumino and obtain the current and energy-momentum tensor for the example. Why is the current different from what we learned about currents in the first chapter?

[1] D. Z. Freedman and A. Van Proeyen, Supergravity. Cambridge Univ. Press, Cambridge, UK, 2012.
http://www.cambridge.org/mw/academic/subjects/physics/theoretical-physics-and-mathematical-physics/supergravity?format=AR

[2] M. K. Gaillard and B. Zumino, Duality rotations for interacting fields, Nucl. Phys. B193 (1981) 221

A Dualities

The historical standard reference on dualities in field theories is [2], though they appeared before in [13, 14, 15] and some extensions are in [16. Duality transformations were explained in [1, Sec. 4.2] for coupled Maxwell fields in $D = 4$. We want to extend these transformations to the other parts of the action, including all other fields ϕ (bosons and fermions). Here is first a summary of the main concepts that were in [1, Sec. 4.2]. We considered there actions $S(F)$ that depend on field strengths $F_{\mu\nu}^A$, which are determined in terms of (abelian) vectors A_μ^A. We consider actions at most quadratic in spacetime derivatives, and thus also at most quadratic in $F_{\mu\nu}^A$. Having introduced the dual and self-dual combinations in [1, (4.35-36)][1]

$$\tilde F_{\mu\nu} = -\frac{1}{2}\mathrm{i}e\varepsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}, \quad F_{\mu\nu}^\pm \equiv \frac{1}{2}\left(F_{\mu\nu} \pm \tilde F_{\mu\nu}\right) \tag{A.1}$$

the Bianchi identities and equation of motions for the vectors can be written as

$$\begin{aligned} \nabla^{\mu} \operatorname{Im} F_{\mu\nu}^{A+} &= 0 \quad \text{Bianchi identities} \\ \nabla_{\mu} \operatorname{Im} G_{A}^{\mu\nu+} &= 0 \quad \text{Equations of motion of } A_{\nu}^{A} \end{aligned} \tag{A.2}$$

where

$$G_{A}^{+\mu\nu} \equiv 2\mathrm{i}e^{-1} \frac{\delta S\left(F^{+}, F^{-}, \phi\right)}{\delta F_{\mu\nu}^{+A}}, \quad \text{i.e.} \quad \tilde{G}_{A}^{\mu\nu} = 2\mathrm{i}e^{-1} \frac{\delta S(F, \phi)}{F_{\mu\nu}^{A}} \tag{A.3}$$

In the first part above, the action $S(F, \phi) = S\left(F^{+}, F^{-}, \phi\right)$ is considered as a function of the self-dual and anti-self-dual parts of F. In [1, Sec. 4.2] only the part of the action quadratic in F was considered, and thus G was linear in F. Now we consider that there can also be parts independent of F (terms in the action linear in F) and thus (here $\mathcal{N}_{AB} = -\mathrm{i}\bar{f}_{AB}$ in terms of f_{AB} in chapter 4 (as also done in ch. 21))

$$G_{A\mu\nu}^{+} = \mathcal{N}_{AB}(\phi) F_{\mu\nu}^{+B} + H_{A\mu\nu}^{+}(\phi) = G_{\mathrm{b}A\mu\nu}^{+} + H_{A\mu\nu}^{+}(\phi) \tag{A.4}$$

Since $\mathcal{N}$ is by the last two equations the second derivative of the action w.r.t. F, it is a symmetric tensor. Since the indices μ, ν should go somewhere and we do not consider higher derivative actions, the H will in practice depend on fermion bilinears, but we just write that they are functions of all the fields ϕ.

The dynamical equations A.2 are then invariant under real symplectic transformations [2]

$$\begin{gathered} \delta_d \begin{pmatrix} F_{\mu\nu}^{A} \\ G_{A\mu\nu} \end{pmatrix} = \begin{pmatrix} A_{B}^{A} & B^{AB} \\ C_{AB} & D_{A}^{B} \end{pmatrix} \begin{pmatrix} F_{\mu\nu}^{B} \\ G_{B\mu\nu} \end{pmatrix} \\ B^{AB} = B^{BA}, \quad C_{AB} = C_{BA}, \quad D_{A}^{B} = -A_{B}^{A} \end{gathered} \tag{A.5}$$

Now we can consider also the field equations for the other fields, which we denote as ϕ^{i}. We can omit the frame fields e_{μ}^{a}, which are inert under the duality transformations. We write

$$E_{i} \equiv \frac{\delta S}{\delta \phi^{i}} \tag{A.6}$$

using DeWitt notation, which means for an action that is function of fields and derivatives of fields (and adding a total derivative)

$$S = \int \mathrm{d}^{4} x e L\left(\phi, \partial_{\mu} \phi\right), \quad \frac{\delta S}{\delta \phi^{i}} = e\left[\frac{\delta L}{\delta \phi^{i}} - \nabla_{\mu} \frac{\delta L}{\delta \partial_{\mu} \phi^{i}}\right] \tag{A.7}$$

We will use the notation that a derivative w.r.t. a field is a left derivative. For bosons left or right derivative makes no difference.

We consider transformations of these fields under the duality transformations:

$$\delta_d \phi^{i} = \xi^{i}(\phi) \tag{A.8}$$

where ξ^{i} is not dependent on $F_{\mu\nu}$.
Full duality transformation can then be written as

[61] We insert here factors e as in [1, (7.59)] to take into account that we can be in curved spacetime, though this is not important for the duality transformations. After the first definitions, we will omit the factors e.
[2] We write here the infinitesimal transformations, while in [1, (4.71)], A, B, C, D were used for global transformations

$$\delta_d = \xi^i \frac{\delta}{\delta\phi^i} + \left(A^A_B F^B_{\mu\nu} + B^{AB} G_{B\mu\nu}\right) \frac{\delta}{\delta F^A_{\mu\nu}} \tag{A.9}$$

E.g. the total transformation of the action $S(F,\phi)$ is

$$\begin{aligned} \delta_d S &= \left(\xi^i \frac{\delta}{\delta\phi^i} + \left(F^T A^T + G^T B\right) \frac{\delta}{\delta F}\right) S \\ &= \xi^i E_i - \frac{1}{2}\left(\mathrm{i}\left(F^{+T} A^T + G^{+T} B\right) G^+ + \text{h.c. }\right) \end{aligned} \tag{A.10}$$

Check this equation, and also (A.3), using properties of dual tensors explained in [1, Sec. 4.2.1 and Ex. 4.6]. From here onwards we use simplifications in the equations. We consider matrix multiplication to write the expression in the brackets in (A.9). We omit the indices $[\mu\nu]$ on F^A and G_A and they are summed over in A.10). We use DeWitt notation [3] i.e. $\xi^i E_i$ contains an integral over spacetime, and derivatives w.r.t. spacetime can be treated as in A.7). Furthermore we will omit dependence on the frame. fields. They can be reinserted in an obvious way such that they reinstall general coordinate transformations. It is easier to work with self-dual combinations as in the last expression of A.10). Some of the above relations are then simpler written as

$$\frac{\delta}{\delta F^+} S = -\frac{1}{2}\mathrm{i} G^+, \quad \frac{\partial G^+_A}{\partial F^{+B}} = \mathcal{N}_{AB} \tag{A.11}$$

It is useful to write the commutator of field derivatives with δ_d (with $\partial_i = \frac{\delta}{\delta\phi^2}$ and using that ξ^i does not depend on F^A)

$$\begin{aligned} \frac{\delta}{\delta F^+}\delta_d &= \delta_d \frac{\delta}{\delta F^+} + \left(A^T + \mathcal{N} B\right) \frac{\delta}{\delta F^+} \\ \partial_i \delta_d &= \delta_d \partial_i + \left(\partial_i \xi^j\right) \partial_j + \left(\partial_i G^{+T} B \frac{\delta}{\delta F^+} + \text{h.c. }\right) \end{aligned} \tag{A.12}$$

On the action we thus obtain

$$\begin{aligned} \frac{\delta}{\delta F^+}\delta_d S &= -\frac{1}{2}\mathrm{i}\left(\delta_d G^+ + \left(A^T + \mathcal{N} B\right) G^+\right) = -\frac{1}{2}\mathrm{i}\left[C F^+ + \mathcal{N} B\right) G^+\Big] \\ &= -\frac{1}{4}\mathrm{i}\frac{\delta}{\delta F^+}\left[F^{+T} C F^+ + G^{+T} B G^+\right] \\ \partial_i \delta_d S &= \delta_d E_i + \left(\partial_i \xi^j\right) E_j - \frac{1}{2}\left(\mathrm{i}\left(\partial_i G^{+T} B\right) G^+ + \text{h.c. }\right) \end{aligned} \tag{A.13}$$

Thus we find

$$\delta_d S = -\frac{1}{4}\mathrm{i}\left[F^{+T} C F^+ + G^{+T} B G^+\right] + \text{h.c. }, \quad \delta_d E_i = -\left(\partial_i \xi^j\right) E_j \tag{A.14}$$

For the first equation, note that this is, reinserting all notations,

$$\delta_d S = -\frac{1}{8}\int \mathrm{d}^4 x \varepsilon^{\mu\nu\rho\sigma}\left(F^A_{\mu\nu} C_{AB} F^B_{\rho\sigma} + G_{A\mu\nu} D^{AB} G_{B\rho\sigma}\right) \tag{A.15}$$

[63] This simplifies many expressions in [2, where the first term would be written as $\int \mathrm{d}^4 x \left[\xi^i \frac{\delta L}{\delta\phi^2} + \partial_\mu \xi^i \frac{\delta L}{\delta\partial_\mu\phi^2}\right]$.

and the first term is a total derivative. The second equation in A.14 then implies that field equations transform to field equations, hence preserving the dynamics.

Note that the transformation $\delta_d S$ is the transformation of

$$S_{\text{non-inv}} = -\frac{1}{4}\mathrm{i}F^{+A}G_A^+ + \text{h.c.} = S_2(F,\phi) - \frac{1}{4}\left(\mathrm{i}F^{+A}H_A^+ + \text{h.c.}\right) \tag{A.16}$$

where $S_2(F,\phi)$ is the part of the action that is quadratic in F and can be written in various ways:

$$\begin{aligned} S_2(F,\phi) &= -\frac{1}{4}\mathrm{i}F^{+A}G_{\mathrm{b}A}^+ + \text{h.c.} = -\frac{1}{4}\mathrm{i}F^{+A}\mathcal{N}_{AB}F^{+B} + \text{ h.c.} \\ &= \frac{1}{4}\int \mathrm{d}^4x\left[e\left(\operatorname{Im}\mathcal{N}_{AB}\right)F_{\mu\nu}^A F^{B\mu\nu} - \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}\left(\operatorname{Im}\mathcal{N}_{AB}\right)F_{\mu\nu}^A F_{\rho\sigma}^B\right] \end{aligned} \tag{A.17}$$

where we gave also the full expression in curved space. The linear part in F is

$$\begin{aligned} S_1(F,\phi) &= -\frac{1}{2}\mathrm{i}F^{+A}H_A^+ + \text{h.c.} \\ &= -\frac{1}{4}\int \mathrm{d}^4x\varepsilon^{\mu\nu\rho\sigma}F_{\mu\nu}^A H_{A\rho\sigma} \end{aligned} \tag{A.18}$$

This implies that one half of this $S_1(F,\phi)$ sits in $S_{\text{non-inv}}$ and the other half is in the invariant part

$$S_{\text{inv}} = \frac{1}{2}S_1(F,\phi) + S_0(\phi) \tag{A.19}$$

where $S_0(\phi)$ is the part of the action without gauge fields: Note that this is also the remaining part when $G_{\mu\nu} = 0$, i.e. when the equations of motion of the vector fields are satisfied.

We can obtain aspects of the duality transformations more in detail. First we consider the consistency of A.4 with A.5). This implies that

$$\begin{aligned} \delta_d\mathcal{N}_{AB}(\phi) &= \xi^i\partial_i\mathcal{N}_{AB} = \left(C - \mathcal{N}A - A^T\mathcal{N} - \mathcal{N}B\mathcal{N}\right)_{AB} \\ \delta_d H^+(\phi) &= \xi^i\partial_i H^+ = -\left(A^T + \mathcal{N}B\right)H^+ \end{aligned} \tag{A.20}$$

The transformation of the first term of the invariant part A.19) under the duality is

$$\delta_d\frac{1}{2}S_1(F,\phi) = -\frac{1}{4}\mathrm{i}H^{+T}BH^+ \tag{A.21}$$

which is only function of the ϕ^i and should thus be compensated by a transformation of $S_0(\phi)$.

The way in which a H^+in agreement with the transformation in appears, is from a symplectic vector $\left(Q^{+A}, P_A^+\right)$with $P_A^+ = \overline{\mathcal{N}}_{AB}Q^{+B}$. Check that this is then a 'symplectic vector', which means that the vector transforms as $\left(F^A, G_A\right)$ in A.5. The $H^+$from this symplectic vector is

$$H_{\mu\nu}^+ = \left(\mathcal{N}Q^+ - P^+\right)_{\mu\nu} = 2\mathrm{i}(\operatorname{Im}\mathcal{N})Q_{\mu\nu}^+ \tag{A.22}$$

Then

$$\frac{1}{2}S_1(F,\phi) = -\frac{1}{4}\mathrm{i}F^{+A}\left(\mathcal{N}Q^+ - P^+\right) = \frac{1}{4}\mathrm{i}\left(F^{+A}P_A^+ - G_{\mathrm{b}}^{+A}Q_A^+\right) + \text{h.c.} \tag{A.23}$$

If we replace G_b by the full G, this is a symplectic invariant. Hence this determines which parts of $S_0(\phi)$ are separately invariant:

$$S_0(\phi) = \frac{1}{4}\mathrm{i} H^+_{A\mu\nu} Q^{+A\mu\nu} + \text{ h.c. } + S_{0,\text{ inv}} \tag{A.24}$$

Note that by the remarks after A.4 the first term is in practice a 4 -fermion term.
The scalar action is thus in the invariant part, which means that the scalar transformations should be isometries. This will determine a subalgebra of the symplectic algebra that is the symmetry of the theory. Hopefully this will be clarified with the example.

Example

The example is (a part of) a theory in $\mathcal{N}=2$ supergravity with one vector multiplet $\frac{4}{4}$ Hence there are two vectors, one from the gravity multiplet and one from the vector multiplet, but they will be mixed. We thus use $A=1,2$. We will neglect the frame field and the gravitinos, but still consider the complex scalar z, and two fermions $\chi^i, i=1,2$. The Lagrangian is

$$\begin{aligned}
\mathcal{L} =& \frac{3}{(z-\bar{z})^2}\left[\partial_\mu z \partial^\mu \bar{z} + \frac{1}{2}\left(\bar{\chi}^1 P_L \not{\partial}\chi^1 + \bar{\chi}^2 P_L \not{\chi}^2\right)\right] \\
&+ \left\{-\frac{1}{4}\mathrm{i}\left[\frac{1}{2}z^2(3\bar{z}+z)F^{+1}_{\mu\nu}F^{+1\mu\nu} - 3z(z+\bar{z})F^{+1}_{\mu\nu}F^{+2\mu\nu} + \frac{3}{2}(3z+\bar{z})F^{+2}_{\mu\nu}F^{+2\mu\nu}\right]\right. \\
&\left.-\frac{3}{8}\bar{\chi}^1 P_R \gamma^{\mu\nu}\chi^2 \bar{y}\left(-zF^{+1}_{\mu\nu} + F^{+2}_{\mu\nu}\right)\right\} + \text{ h.c. } + \ldots
\end{aligned} \tag{A.25}$$

Here appears also a variable y. In terms of z only its modulus is determined by

$$|y|^2 = (\mathrm{i}(z-\bar{z}))^{-3} \tag{A.26}$$

Choosing the phase of y is in fact a choice of a phase symmetry, that could also act on the fermions, but we will fix this below by the form of the transformations of y. First look at the scalar part, which determines the isometries.
Obtain all the quantities of the main text, where the matrices in the duality transformations are in terms of the parameters for the isometries:

$$A = \begin{pmatrix} -\frac{3}{2}\theta^2 & -3\theta^3 \\ \theta^1 & -\frac{1}{2}\theta^2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{2}{3}\theta^3 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 0 \\ 0 & 6\theta^1 \end{pmatrix} \tag{A.27}$$

The variable y transforms consistently with A.26) as (here a phase has been chosen for y)

$$\delta_d \bar{y} = -3\left(\frac{1}{2}\theta^2 + \bar{z}\theta^3\right)\bar{y} \tag{A.28}$$

Since $P_L\chi^i$ is the supersymmetry transform of z under the two supersymmetries:

$$\delta_d z = k(z) \rightarrow \delta_d P_L \chi^i = (\partial_z k)\, P_L \chi^i \tag{A.29}$$

and the complex conjugate for $\delta_d P_R \chi^i$.
Are the .... terms in A.25 invariant, or what should we still add?

[64] Technically, in terms of the concepts in [1, Chs. 20 and 21] it is the model with prepotential $F = \left(X^1\right)^3/X^0$, but you do not need that.

Solution

(see my note about diality, I have to change a little there and then I'll copy it here)

2 Compactification from 5D: action and manifold with boundary

In [1, Section 5.3] we discussed the dimensional reduction from odd dimensions to the lower even dimension by an extra dimension that is a circle. We did not consider the action. In appendix B the reduction of the action and also the reduction of the gravity action is reviewed. You should first study this and then consider [3]

Aspects of compactification on a linear dilaton background, Ignatios Antoniadis, Chrysoula Markou and François Rondeau,
JHEP 21 (2021) 137, e-Print: https://arxiv.org/abs/2106.15184

Here the 5th dimension is a line with two boundaries, and there is a scalar field (dilaton) whose value is a linear function on that line ('linear dilaton background'). Study section 2 and 3.1 of this paper and formulate it similar as in appendix B (where the 5 th dimension is a circle).

PS: At the end of Sec. 2.1 of the paper appears the Gibbons-Hawking term. You do not need much about this. Try just to understand why such a term is added. You may find this e.g in "Extras on chapter 8" of online information related to the book, or in Wikipedia pages Gibbons-Hawking-York boundary term.

More in detail on what you have to do: after studying Sec. 2 of the review, you should be able to extend it to the case with a linear dilaton. You can check their equations in their sections 2 and 3.1 using the methods that you have learned. But you should thus perform new calculations. Start with their action (2.1) and obtain the equations of motion. For gravity you can see this as an extension of [1, Ex. 8.3]. The difference is that you cannot use the last sentence in that exercise since there is a function of ϕ in front of the Einstein term. That gives new contributions, which you can find back in (2.3). An alternative, maybe easier, way would be to go to other variables using Ex. B. 3 to get rid of the function of ϕ in front of the Einstein term. This is called 'going to Einstein frame'. Then you can take the Einstein equations as we know them, and then go back to the old variables.

Then you can continue in the paper and see what you can use from the review and what you should adopt for the new situations. See how terms from the branes on the border are added, the eom's are modified and split and a solution is valid. A new term for a massless scalar is added. Consider the solution. Tell your colleagues in the presentation especially what is different when you reduce from 5 to 4 dimensions with an interval, rather than with a circle as extra dimension.

B Review on compactification of extra dimensions

In this section we will present the basic steps how Lagrangian field theories formulated in D' spacetime dimensions lead to a theory in an observable spacetime of dimension $D < D'$. We will start by considering $D' = D + 1$ and consider a manifold of the structure $M_{D+1} = M_D \times S^1$ where S^1 is a circle of radius R, and M_D is a flat spacetime of signature $(- + \ldots +)$. Though many steps could be done for general D, we will for pedagogical reasons consider $D = 4$. We will split the dependence of the fields on spacetime coordinates in 5 dimensions, as positions in M_4 and the position in the circle direction, y,

$$\left\{x^M\right\} = \{x^\mu, y\} = \left\{x^0, \ldots, x^3, y\right\}, \quad 0 \le y \le 2\pi R \tag{B.1}$$

and identify the end points $y = 0$ and $y = 2\pi R$. The structure of the circle implies that we can expand the behaviour in the fifth dimension as Fourier modes of fields on S^1,

$$\Phi(x,y) = \sum_{n\in\mathbb{Z}} \phi_n(x) \mathrm{e}^{\mathrm{i}yn/R} \tag{B.2}$$

The fields $\phi_n(x)$ will be observed as infinite 'towers' of massive particles by an observer in Minkowski $_4$. Considering real fields Φ (we will specify the case of fermions below), this implies complex fields ϕ_n with

$$\phi_n^* = \phi_{-n} \tag{B.3}$$

We will explore the physics of the various fields in 5 dimensions: scalar fields, fermions, gauge fields and gravitons.

B. 1 Scalar field

For a free scalar field in 5 dimensions we have the Klein-Gordon equation

$$\begin{aligned} 0 = \left[\Box_5 - M^2\right]\Phi &= \left[\Box_4 + \left(\frac{\partial}{\partial y}\right)^2 - M^2\right]\Phi \\ &= \sum_{n\in\mathbb{Z}} \left[\Box_4 - \left(\frac{n}{R}\right)^2 - M^2\right]\phi_n \end{aligned} \tag{B.4}$$

Since the different terms in the Fourier expansion are independent, the 4 D observer sees a Kaluza-Klein (KK) spectrum of particles with masses

$$M_n^2 = M^2 + \frac{n^2}{R^2} \tag{B.5}$$

Let us also consider the action. A canonically normalized real scalar field in 5 dimensions, has the action

$$\begin{aligned} S &= -\frac{1}{2}\int \mathrm{d}^4x \int_0^{2\pi R} \mathrm{d}y \left\{(\partial_M\Phi)\left(\partial^M\Phi\right) + M^2\Phi^2\right\} \\ &= -\frac{1}{2}\int \mathrm{d}^4x \int_0^{2\pi R} \mathrm{d}y \sum_{m,n\in\mathbb{Z}} \mathrm{e}^{\mathrm{i}(m+n)y/R} \left\{(\partial_\mu\phi_m)\left(\partial^\mu\phi_n\right) + \left(-\frac{mn}{R^2} + M^2\right)\phi_m\phi_n\right\} \end{aligned} \tag{B.6}$$

The y integral is only non-vanishing for $m+n=0$ and using the condition (B.3) we obtain

$$S = -\frac{2\pi R}{2}\int \mathrm{d}^4x \sum_{n\in\mathbb{Z}} \left\{(\partial_\mu\phi_n)\left(\partial^\mu\phi_n^*\right) + \left(\frac{n^2}{R^2} + M^2\right)|\phi_n|^2\right\} \tag{B.7}$$

The modes in the expansion will thus have the canonical normalization for redefined scalars, and masses as in (B.5)

$$\tilde{\phi}_n = \sqrt{2\pi R}\phi_n, \quad M_n^2 = M^2 + \frac{n^2}{R^2} \tag{B.8}$$

As an example how interactions reduce to lower dimensions, let us consider in 5 dimensions the interaction Lagrangian

$$S_I = -\lambda_5 \int \mathrm{d}^4x \int_0^{2\pi R} \mathrm{d}y \Phi^4(x,y) \tag{B.9}$$

The same integral over y gives again KK-momentum conservation, a factor $2\pi R$ and four times the redefinition as in (B.8) gives

$$S_I = -\frac{\lambda_5}{2\pi R} \sum_{n_1+n_2+n_3+n_4=0} \int \mathrm{d}^4x \tilde{\phi}_{n_1}\tilde{\phi}_{n_2}\tilde{\phi}_{n_3}\tilde{\phi}_{n_4} \tag{B.10}$$

This creates thus vertices

$$\tilde{\phi}_{n_1}^{\tilde{\phi}_{n_2}} \lambda_{\tilde{\phi}_{n_4}}^{\tilde{\phi}_{n_3}} n_1 + n_2 + n_3 + n_4 = 0, \quad \lambda_4 = \frac{\lambda_5}{2\pi R} \tag{B.11}$$

B. 2 Abelian gauge fields in 5D

We consider an abelian gauge field with local gauge transformation in 5D

$$\delta A_M(x,y) = \partial_M \Lambda(x,y) \tag{B.12}$$

We start by splitting the components of the 5D gauge field in

$$A_M(x,y) = \begin{pmatrix} A_\mu(x,y) \\ A_4(x,y) \end{pmatrix}, \quad \phi(x,y) = A_4(x,y), \quad \begin{matrix} \delta A_\mu(x,y) = \partial_\mu \Lambda(x,y) \\ \delta\phi(x,y) = \partial_y \Lambda(x,y) \end{matrix} \tag{B.13}$$

Each component and the gauge parameter $\Lambda(x,y)$ is then again divided in Fourier modes, and the transformations reduce to

$$\delta A_{\mu,n}(x) = \partial_\mu \Lambda_n(x), \quad \delta\phi_n(x) = \mathrm{i}\frac{n}{R}\Lambda_n(x) \tag{B.14}$$

This says that $\Lambda_0(x)$ acts as an ordinary gauge transformation on the 4 D abelian gauge field $A_{\mu,0}$, and ϕ_0 is a neutral real scalar field. However, the ϕ_n for $n \neq 0$ shift with constant local parameters Λ_n. Hence, we can take a gauge choice

$$\Lambda_{n\neq 0} \text{ gauge:} \quad \phi_n(x) = 0 \text{ for } n \neq 0 \tag{B.15}$$

This implies that $A_{\mu,n}$ are now vectors without a gauge invariance. They eat the scalars, and the meal makes them massive.5

Observe that one degrees of freedom (dof) in 5D gives a priori under the decomposition (B.2) one dof for every Fourier mode. In [1, Sec.5.3.3] it is shown explicitly how the Maxwell field equations lead to the expected redistribution of the on-shell dof:

$$\begin{aligned} 5D :& A_M \text{ has } D-2 = 3 \text{ on-shell dof} \\ &\to 4D : A_{\mu,0} \text{ has } D-2 = 2 \text{ on-shell dof, } \phi_0 \text{ has 1 on-shell dof} \\ &A_{\mu,n\neq 0}, \text{ has } D-1 = 3 \text{ on-shell dof,} \end{aligned} \tag{B.16}$$

in agreement with the fact that $A_{\mu,0}$ is a massless gauge vector, and the $A_{\mu,n\neq 0}$ are massive vector fields.

Ex. B. 1 Check that the off-shell dof (components - gauge transformations) also reduce consistently from $D = 5$ to $D = 4$.

We now consider the dynamics from the point of view of the Lagrangian. The latter contains the field strengths $F_{MN} = 2\partial_{[M}A_{N]}$:

$$F_{MN}F^{MN} = F_{\mu\nu}F^{\mu\nu} + 2F_{\mu 4}F^{\mu 4}, \quad F_{\mu 4} = \partial_\mu \phi - \partial_y A_\mu$$
$$F_{\mu 4,0} = \partial_\mu \phi_0, \quad F_{\mu 4, n\neq 0} = -\frac{\mathrm{i}n}{R}A_{\mu,n} \tag{B.17}$$

where in the last line we used the gauge (B.15). We can then again use the integral over y to write[6]

$$\begin{aligned} S_A &= -\frac{1}{4}\int \mathrm{d}^4x \int_0^{2\pi R} \mathrm{d}y F_{MN}F^{MN} \\ &= -2\pi R \int \mathrm{d}^4x \left\{ \sum_{n\in\mathbb{Z}} \left[\frac{1}{4}\left|F_{\mu\nu,n}\right|^2 + \frac{1}{2}\frac{n^2}{R^2}\left|A_{\mu,n}\right|^2 \right] + \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 \right\} \end{aligned} \tag{B.18}$$

This contains the actions of massive vectors $A_{\mu,n}$ with $n = \pm 1, \pm 2, ...$ (or complex ones with $n = 1, 2, ...$), the U(1) massless gauge field $A_{\mu,0}$ and real scalar ϕ_0.

To obtain a canonical normalization, we define as before

$$\tilde{A}_{\mu,n} = \sqrt{2\pi R}A_{\mu,n}, \quad \tilde{\phi}_0 = \sqrt{2\pi R}\phi_0 \tag{B.19}$$

removing the prefactor in (B.18).

Ex. B. 2 Generalize all this to non-abelian gauge fields.

B. 3 Fermions

We explore fermions in 5 D][7] where the γ_M matrices have the same 4×4 size as those of $D = 4$. We can take $\gamma_4 = \gamma_* = \mathrm{i}\gamma_0\gamma_1\gamma_2\gamma_3$. The minimal spinors are symplectic spinors: a doublet of complex spinors $\chi^i, i = 1, 2$ with a reality condition. Thus these represent 8 real components, as a Dirac spinor in 4D. The symplectic reality condition can be expressed either as

$$\left(\chi^i\right)^* = -\tilde{B}\chi^j\varepsilon_{ji}, \quad \overline{\chi^i} = -\mathrm{i}\left(\chi^j\right)^\dagger \gamma^0\varepsilon_{ji} \tag{B.20}$$

We write here $\tilde{B}$ to indicate the corresponding matrix in 5 D, and the bar in the second equation is the Majorana conjugate $\overline{\chi^i} = \left(\chi^i\right)^T\tilde{C}$. The matrices $\tilde{C}, \tilde{B}$ are related to those in 4D, C and B by

$$\tilde{B} = \mathrm{i}\tilde{C}\gamma^0, \quad B = \mathrm{i}C\gamma^0, \quad \tilde{C} = C\gamma_*, \quad \tilde{B} = -B\gamma_* \tag{B.21}$$

A kinetic action of the fermion in 5 D is typically

$$S_\chi = \frac{1}{2}\int \mathrm{d}^5x\overline{\chi^i}\ \partial\!\!\!/\chi^j\varepsilon_{ji} = \int \mathrm{d}^5x\overline{\chi^2}\ \partial\!\!\!/\chi^1 \tag{B.22}$$
$$= \int \mathrm{d}^5x\mathrm{i}\left(\chi^1\right)^\dagger\gamma^0\ \partial\!\!\!/\chi^1 \tag{B.23}$$

[65] Equivalently to taking the gauge choice, we could define $\tilde{A}_{\mu,n} = A_{\mu,n} - \mathrm{i}\frac{R}{n}\partial_\mu\phi_n$, which are inert under Λ_n transformations, and the invariance under Λ_n transformations then implies that in the basis $\left\{\tilde{A}_{\mu,n}, \phi_n\right\}$ the fields ϕ_n will drop from any gauge-invariant quantity such as the action.

In the last line appears the Dirac conjugate and only χ^1. Therefore, we will now further only use the complex $\chi = \chi^1$ and write[8]

$$S_\chi = \int \mathrm{d}^5 x \bar{\chi}\ \partial\!\!\!/ \chi, \quad \bar{\chi} \equiv \mathrm{i}\chi^\dagger \gamma^0 \tag{B.24}$$

We will thus further use here the Dirac conjugate, defined in (B.24) for all D. Furthermore, we will assume that the fermion field transforms under the $U(1)$ of the gauge field in the previous subsection, with coupling constant g_5 and define

$$D_M \chi = \left(\partial_M + \mathrm{i} g_5 A_M\right) \chi \tag{B.25}$$

We furthermore expand χ and A_M in modes as before (with the gauge (B.15):

$$\chi = \sum_n \chi_n \mathrm{e}^{\mathrm{i}yn/R}, \quad \bar{\chi} = \sum_n \bar{\chi}_n \mathrm{e}^{-\mathrm{i}yn/R} \tag{B.26}$$

This gives

$$\begin{aligned} \left(D_\mu \chi\right)_n &= \partial_\mu \chi_n + \mathrm{i} g_5 \sum_m A_{\mu,n-m} \chi_m \\ \left(D_y \chi\right)_n &= \frac{\mathrm{i}n}{R} \chi_n + \mathrm{i} g_5 \phi_0 \chi_n \end{aligned} \tag{B.27}$$

We consider the action of the fermion field interacting with the abelian gauge field as in (B.18), and thus the action is ?[9]

$$\begin{aligned} S_{\chi,A} &= \int \mathrm{d}^4 x \int_0^{2\pi R} \mathrm{d}y \left\{ \bar{\chi}\ D\!\!\!\!/ \chi - \frac{1}{4} F_{MN} F^{MN} \right\} \\ &= \int \mathrm{d}^4 x \left\{ \sum_{n\in\mathbb{Z}} \tilde{\chi}_n \left[\partial\!\!\!/ + \mathrm{i}\gamma_* M_n\right] \tilde{\chi}_n + \mathrm{i} g_4 \bar{\chi}_n \gamma^\mu \sum_m \tilde{A}_{\mu,n-m} \tilde{\chi}_m \right. \\ &\left. - \left[\frac{1}{4} \left| \tilde{F}_{\mu\nu,n} \right|^2 + \frac{1}{2} \frac{n^2}{R^2} \left| \tilde{A}_{\mu,n} \right|^2 \right] + \frac{1}{2} \partial_\mu \tilde{\phi}_0 \partial^\mu \tilde{\phi}_0 \right\}, \end{aligned} \tag{B.28}$$

with, together with (B.19):

$$\tilde{\chi}_n = \sqrt{2\pi R}\chi_n, , \quad M_n = \frac{n}{R} + g_4 \tilde{\phi}_0, \quad g_4 = \frac{g_5}{\sqrt{2\pi R}}, \quad g_4 \tilde{\phi}_0 = g_5 \phi_0 \tag{B.29}$$

Note that the masses of the fermion fields are still as in (B.5): $M_n^2 = (n/R)^2$ if $A_4 = \phi_0$ has zero expectation value. In fact, as explained in detail in [1, Sec. 5.3.2], the factors $\mathrm{i}\gamma_*$ in the mass and Yukawa term can be removed by a redefinition of χ_n to $\left(1 + \mathrm{i}\gamma_*\right) \chi_n / \sqrt{2}$, which does not change the kinetic and interaction with the photon field. If the scalar ϕ_0 has a non-trivial

[66] Here and below we write for simplicity expressions like $\left|F_{\mu\nu}\right|^2$ to indicate $F_{\mu\nu} F^{*\mu\nu}$

[7] We follow the notations of spinors and gamma matrices of [1, Ch. 3].

[68] In terms of the Majorana conjugate and the doublet Majorana-Weyl spinors $\bar{\chi}\ \partial\!\!\!/ \chi = \frac{1}{2} \bar{\chi}^i\ \partial\!\!\!/ \chi^j \varepsilon_{ij}$ up to a total derivative. On the other hand, in terminology of $D = 4, \chi$ is complex, and can be written as 2 Majorana spinors $\chi = \chi_1 + \mathrm{i}\chi_2$, not confusing χ_i with the χ^i above, which were complex. then $\bar{\chi}\ \partial\!\!\!/ \chi = \bar{\chi}_i\ \partial\!\!\!/ \chi_j \delta^{ij}$ in terms of Majorana conjugates, which are for the χ_i the same as Dirac conjugates.

[9] Note that $D\!\!\!\!/$ in the first line contains a sum over 5 dimensions, while in the second line $\partial\!\!\!/$ is only over 4 dimensions.

vacuum expectation value $<\phi_0>$ then the effective mass term of the fermion modes χ_n is M_n. If it varies over the circle then we have a massless fermion when the Wilson line

$$\int \mathrm{d} y g_5 \phi_0=-2 \pi n \tag{B.30}$$

In the non-abelian case, the term in $F_{MN} F^{MN}$ that is fourth order in the gauge fields, gives rise to a potential for the scalar $V\left(\phi_0\right)$.

B. 4 Gravity in 5D

Denoting the metric in 5 D as G_{MN}, it decomposes in

$$G_{MN}(x, y)=\left(\begin{array}{cc}G_{\mu \nu} & G_{\mu 4} \\ G_{4 \mu} & G_{44}\end{array}\right)(x, y) \tag{B.31}$$

The general coordinate transformations of $5D$ act as

$$\delta(\xi) G_{MN}=\xi^P \partial_P G_{MN}+2 \partial_{(M} \xi^P G_{N) P} \tag{B.32}$$

Consider in particular

$$\delta(\xi) G_{44}=\left(\xi^\mu \partial_\mu+\xi^4 \partial_y\right) G_{44}+2 G_{4 \mu} \partial_y \xi^\mu+2 G_{44} \partial_y \xi^4 \tag{B.33}$$

Since the flat manifold is a product space, $G_{44} \neq 0$, and we parametrize it as $\square$[10]

$$G_{44}=\mathrm{e}^{2 \phi} \tag{B.34}$$

The field ϕ then transforms under the ξ^4 transformations as

$$\begin{aligned} \delta\left(\xi^4\right) \phi & =\partial_y \xi^4+\xi^4 \partial_y \phi \\ \delta\left(\xi^4\right) \phi_n & =\frac{i n}{R} \xi_n^4+\sum_m \frac{i m}{R} \phi_m \xi_{n-m}^4\end{aligned} \tag{B.35}$$

where in the second line we introduced the modes as before. This implies that we can gauge-fix the $\xi_{n \neq 0}^4$ transformations by taking

$$\xi_{n \neq 0}^4 \text { gauge: } \quad \phi_{n \neq 0}=0 \tag{B.36}$$

\footnotetext{
[10] We may later still rescale ϕ arbitrary, e.g. to obtain canonical kinetic terms for this scalar.
Similarly we consider the ξ^ν transformations of the off-diagonal components

$$\begin{aligned} \delta\left(\xi^\nu\right) G_{\mu 4} & =\xi^\nu \partial_\nu G_{\mu 4}+G_{\nu 4} \partial_\mu \xi^\nu+G_{\mu \nu} \partial_y \xi^\nu \\ \delta\left(\xi^\nu\right) G_{\mu 4, n} & =\sum_m\left(\xi_m^\nu \partial_\nu G_{\mu 4, n-m}+G_{\nu 4, n-m} \partial_\mu \xi_m^\nu+G_{\mu \nu, n-m} \frac{i m}{R} \xi_m^\nu\right)\end{aligned} \tag{B.37}$$

Since $G_{\mu \nu, 0}$, being proportional to the metric in the 4 D space, should be invertible, we can also fix the $\xi_{n \neq 0}^\mu$ transformations

$$\xi_{n \neq 0}^\mu \text { gauge: } \quad G_{\mu 4, n \neq 0}=0 \tag{B.38}$$

The remaining gauge transformations are thus only the 0 -modes $\xi_0^\nu(x)$ and $\xi_0^4(x)$, and $\partial_y \xi^M=0$. One thus easily checks from B.32 that the remaining $\xi^\nu(x)=\xi_0^\nu$ acts as general coordinate transformations in $4D$. We will still consider the $\xi_0^4(x)$ transformations below, but we can already consider the dof similar to what we did for the vectors in (B.16):

$$\begin{aligned}
&5D: G_{MN} \text{ has } D(D-3)/2 = 5 \text{ on-shell dof}\\
&\to 4D: G_{\mu\nu,0} \text{ has } D(D-3)/2 = 2 \text{ on-shell dof: the 4D graviton,}\\
&G_{\mu,0} \text{ has } D-2 = 2 \text{ on-shell dof: a 4D gauge vector,}\\
&\phi \text{ has 1 on-shell dof: a 4D scalar,}\\
&G_{\mu\nu,n\neq 0}, \text{ has 5 on-shell dof: massive spin 2 fields.}
\end{aligned} \tag{B.39}$$

The latter 5 are the helicity states $0, \pm 1, \pm 2$ of the massive spin 2.
Having eliminated the non-zero modes of the symmetries and of ϕ, we see from (B.35) that $\phi(x)$ is invariant under $\xi^4(x)$. Considering the ξ^4 transformations of $G_{\mu 4}$ we find from (B.32), using also B.38

$$\delta\left(\xi^4\right) G_{\mu 4} = \left(\partial_\mu \xi^4\right) G_{44} = \mathrm{e}^{2\phi}\partial_\mu \xi^4 \tag{B.40}$$

Therefore (these fields only depend on x due to (B.36) and (B.38))

$$\mathcal{A}_\mu \equiv \mathrm{e}^{-2\phi} G_{\mu 4} \tag{B.41}$$

is the canonically normalized gauge field for the $\xi^4(x)$ transformations. Finally considering

$$\delta\left(\xi^4\right) G_{\mu\nu}(x,y) = \xi^4(x)\partial_y G_{\mu\nu}(x,y) + 2\partial_{(\mu}\xi^4(x) G_{\nu)4}(x) \tag{B.42}$$

The first term represent the usual y coordinate transformation. The second can be eliminated by considering $G_{\mu\nu}(x,y) - \mathcal{A}_\mu\mathcal{A}_\nu \mathrm{e}^{2\phi}$, whose ξ^4 transformation has only the $\xi^4\partial_y$-term. For convenience we define

$$\hat{G}_{\mu\nu}(x,y) = \mathrm{e}^{-2\phi} G_{\mu\nu}(x,y), \quad \hat{g}_{\mu\nu}(x,y) = \hat{G}_{\mu\nu}(x,y) - \mathcal{A}_\mu(x)\mathcal{A}_\nu(x) \tag{B.43}$$

The y-independent part of $\hat{g}_{\mu\nu}$ is then invariant under ξ^4, and will be proportional to the metric in 4D. We can later take another overall factor with the scalar field in order to obtain the Einstein frame, i.e. where the kinetic terms of $g_{\mu\nu} \propto \hat{g}_{\mu\nu}(x,0)$ and ϕ are separated.
To summarize we can write

$$\begin{aligned}
G_{MN} &= \mathrm{e}^{2\phi}\hat{G}_{MN}\\
\hat{G}_{MN} &= \begin{pmatrix} \hat{G}_{\mu\nu}(x,y) & \mathcal{A}_\mu(x)\\ \mathcal{A}_\nu(x) & 1 \end{pmatrix} = \begin{pmatrix} \hat{g}_{\mu\nu} + \mathcal{A}_\mu\mathcal{A}_\nu & \mathcal{A}_\mu(x)\\ \mathcal{A}_\nu(x) & 1 \end{pmatrix}
\end{aligned} \tag{B.44}$$

or

$$\mathrm{d}s^2 \equiv G_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N = \mathrm{e}^{2\phi}\mathrm{d}\hat{s}^2, \qquad \mathrm{d}\hat{s}^2 = \hat{G}_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N$$

$$\mathrm{d}\hat{s}^2 = \hat{G}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + 2\mathcal{A}_\mu\mathrm{d}x^\mu\mathrm{d}y + \mathrm{d}y\,\mathrm{d}y \tag{B.45}$$

$$= \hat{g}_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \left(\mathrm{d}y + \mathcal{A}_\mu\mathrm{d}x^\mu\right)^2 \tag{B.46}$$

The matrix $\hat{G}_{MN}$ can also be decomposed as

$$\hat{G}_{MN} = \begin{pmatrix} \delta_\mu^\rho & \mathcal{A}_\mu\\ 0 & 1 \end{pmatrix}\begin{pmatrix} \hat{g}_{\rho\sigma} & 0\\ 0 & 1 \end{pmatrix}\begin{pmatrix} \delta_\nu^\sigma & 0\\ \mathcal{A}_\nu & 1 \end{pmatrix}, \quad \to \quad \det\left(\hat{G}_{MN}\right) = \det\left(\hat{g}_{\mu\nu}\right) \tag{B.47}$$

In preparation of the calculation of the connections, we also obtain the inverse of $\hat{G}_{MN}$, which we denote here as $\hat{G}^{MN}$, in terms of the inverse of $\hat{g}_{\mu\nu}$, which we denote as $\hat{g}^{\mu\nu}$:

$$\hat{G}^{MN} = \begin{pmatrix} \hat{g}^{\mu\nu} & -\hat{\mathcal{A}}^{\mu} \\ -\hat{\mathcal{A}}^{\nu} & 1+\hat{\mathcal{A}}^{\rho}\mathcal{A}_{\rho} \end{pmatrix}, \quad \hat{\mathcal{A}}^{\mu} \equiv \hat{g}^{\mu\nu}\mathcal{A}_{\nu} \tag{B.48}$$

For calculating curvatures, we will often have to define metrics that are proportional to each other by a scalar factor. The following exercise is convenient for that

Ex. B. 3 This exercise is useful in many different contexts. Therefore we look at it for general dimension D. Consider the rescaling

$$\tilde{g}_{\mu\nu} = \mathrm{e}^{2\Phi} g_{\mu\nu} \tag{B.49}$$

Then obtain torsionless metric compatible connections and curvatures (see e.g. [1, Sec. 7.9, 7.10])

$$\begin{aligned}
\Gamma^{\rho}_{\mu\nu}(\tilde{g}) =& \Gamma^{\rho}_{\mu\nu}(g) + 2\delta^{\rho}_{(\mu}\partial_{\nu)}\Phi - g_{\mu\nu}\partial^{\rho}\Phi \\
R_{\mu\nu}{}^{\rho}(\tilde{g}) =& R_{\mu\nu}{}^{\rho}{}_{\sigma}(g) - 2\delta^{\rho}_{[\mu}\nabla_{\nu]}\partial_{\sigma}\Phi + 2g_{\sigma[\mu}\nabla_{\nu]}\partial^{\rho}\Phi \\
&+ 2\delta^{\rho}_{[\mu^{\nu}}\left(\partial_{\nu]}\Phi\right)\left(\partial_{\sigma}\Phi\right) - 2g_{\sigma[\mu}\left(\partial_{\nu]}\Phi\right)\left(\partial^{\rho}\Phi\right) - 2\delta^{\rho}_{[\mu}g_{\nu]\sigma}(\partial\Phi\cdot\partial\Phi), \\
R_{\nu\sigma}(\tilde{g}) =& R_{\nu\sigma}(g) + (2-D)\nabla_{\nu}\partial_{\sigma}\Phi - g_{\nu\sigma}\nabla^2\Phi + (D-2)\left(\left(\partial_{\nu}\Phi\right)\left(\partial_{\sigma}\Phi\right) - g_{\nu\sigma}(\partial\Phi\cdot\partial\Phi)\right), \\
R(\tilde{g}) =& \mathrm{e}^{-2\Phi}\left[R(g) - 2(D-1)\nabla^2\Phi - (D-1)(D-2)\partial\Phi\cdot\partial\Phi\right].
\end{aligned} \tag{B.50}$$

The ∇ derivative uses the connection $\Gamma^{\rho}_{\mu\nu}(g)$ and all inner products use $g^{\mu\nu}$. Further remark that $\sqrt{\det|\tilde{g}|} = \mathrm{e}^{D\Phi}\sqrt{\det|g|}$, and that you can rewrite the result for the action as

$$\sqrt{-\tilde{g}}R(\tilde{g}) = \mathrm{e}^{(D-2)\Phi}\sqrt{-g}[R(g) + (D-1)(D-2)\partial\Phi\cdot\partial\Phi] - 2\frac{D-1}{D-2}\sqrt{-g}\nabla^2\mathrm{e}^{(D-2)\Phi} \tag{B.51}$$

The last term is a total derivative, which can be omitted.

We now consider the connections related to the metric $\hat{G}$, for the lowest modes (no ∂_y derivatives) and obtain

$$\begin{array}{lll}
\Gamma^{\rho}_{\mu\nu}(\hat{G}) = \Gamma^{\rho}_{\mu\nu}(\hat{g}) + \hat{g}^{\rho\sigma}\mathcal{A}_{(\mu}\mathcal{F}_{\nu)\sigma}, & \Gamma^{\rho}_{\mu 4}(\hat{G}) = \frac{1}{2}\hat{g}^{\rho\sigma}\mathcal{F}_{\mu\sigma}, & \Gamma^{\rho}_{44}(\hat{G}) = 0 \\
\Gamma^{4}_{\mu\nu}(\hat{G}) = \nabla_{(\mu}\mathcal{A}_{\nu)} - \hat{\mathcal{A}}^{\rho}\mathcal{A}_{(\mu}\mathcal{F}_{\nu)\rho}, & \Gamma^{4}_{\mu 4}(\hat{G}) = -\frac{1}{2}\hat{\mathcal{A}}^{\sigma}\mathcal{F}_{\mu\sigma}, & \Gamma^{4}_{44}(\hat{G}) = 0
\end{array} \tag{B.52}$$

with $\mathcal{F}_{\mu\nu} = 2\partial_{[\mu}\mathcal{A}_{\nu]}$ and ∇ is compatible with $\hat{g}_{\mu\nu}$. With a straightforward but tedious calculation (not necessary for this task) you find [11]

$$\begin{aligned}
R_{\mu\nu}(\hat{G}) &= R_{\mu\nu}(\hat{g}) + \mathcal{A}_{(\mu}\nabla^{\rho}\mathcal{F}_{\nu)\rho} - \frac{1}{2}\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu\sigma}\hat{g}^{\rho\sigma} + \frac{1}{4}\mathcal{A}_{\mu}\mathcal{A}_{\nu}\hat{\mathcal{F}}^2 \\
R_{\mu 4}(\hat{G}) &= \frac{1}{2}\nabla^{\nu}\mathcal{F}_{\mu\nu} + \frac{1}{4}\mathcal{A}_{\mu}\hat{\mathcal{F}}^2, \quad \hat{R}_{44}(\hat{G}) = \frac{1}{4}\hat{\mathcal{F}}^2, \quad \hat{\mathcal{F}}^2 \equiv \mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}
\end{aligned} \tag{B.53}$$

We then obtain for the scalar curvature

$$\begin{aligned}
R(\hat{G}) &= \hat{g}^{\mu\nu}R_{\mu\nu}(\hat{G}) - 2\hat{\mathcal{A}}^{\mu}R_{\mu 4}(\hat{G}) + \left(1 + \hat{\mathcal{A}}^{\rho}\mathcal{A}_{\rho}\right)\Big) R_{44}(\hat{G}) \\
&= R(\hat{g}) - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}
\end{aligned} \tag{B.54}$$

Using the result of the above exercise for metrics related as in (B.44), we take

$$\tilde{g}_{\mu\nu} \to G_{MN}, \quad g_{\mu\nu} \to \hat{G}_{MN}, \quad \Phi \to \phi, \quad D = 5 \tag{B.55}$$

and obtain (we put a hat on curvatures calculated in $D = 5$ and indicate the metric used to define them)

$$\begin{aligned}\sqrt{-G}R(G) &= \mathrm{e}^{3\phi}\sqrt{-\hat{G}}\left[R(\hat{G}) + 4.3\partial_\mu\phi\partial_\nu\phi\hat{G}^{\mu\nu}\right] + \text{ total der.} \\ &= \mathrm{e}^{3\phi}\sqrt{-\hat{g}}\left(R(\hat{g}) - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma} + 12\partial_\mu\phi\partial_\nu\phi\hat{g}^{\mu\nu}\right) + \text{ total der.}\end{aligned} \tag{B.56}$$

where we inserted B.54 and used that the determinant of $\hat{G}_{MN}$ is the determinant of $\hat{g}_{\mu\nu}$ and $\hat{G}^{\mu\nu} = \hat{g}^{\mu\nu}$;

The factor $\mathrm{e}^{3\phi}$ in front of the Einstein term defined from $\hat{g}_{\mu\nu}$ can be eliminated using again (B.51), now for $D = 4$, taking

$$g_{\mu\nu} \to \hat{g}_{\mu\nu}, \quad \tilde{g}_{\mu\nu} \to g_{\mu\nu}, \quad \Phi \to \frac{3}{2}\phi, \quad D = 4 \tag{B.57}$$

This leads to the equation

$$\sqrt{-g}R(g) = \mathrm{e}^{3\phi}\sqrt{-\hat{g}}\left[R(\hat{g}) + 3\cdot 2\cdot\left(\frac{3}{2}\right)^2\partial_\mu\phi\partial_\nu\phi\hat{g}^{\mu\nu}\right] + \text{ total der..} \tag{B.58}$$

\footnotetext{

[11] To simplify your calculations you can use that $\Gamma^\rho_{\mu\nu}(\hat{g})$ appears only in $4D$ covariant derivatives.

Thus the variable $g_{\mu\nu}$ chosen like this defines the Einstein frame in $D = 4$. Using (B.58) in (B.56) we obtain for the $D = 5$ Einstein Lagrangian as (up to total derivatives)

$$\begin{aligned}\sqrt{-G}R(G) &= \sqrt{-g}R(g) + \sqrt{-\hat{g}}\mathrm{e}^{3\phi}\left[\left(12 - \frac{27}{2}\right)\partial_\mu\phi\partial_\nu\phi\hat{g}^{\mu\nu} - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}\right] \\ &= \sqrt{-g}\left[R(g) - \frac{3}{2}\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} - \frac{1}{4}\mathrm{e}^{3\phi}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\right]\end{aligned} \tag{B.59}$$

where in the last line we used

$$\hat{g}_{\mu\nu} = \mathrm{e}^{-3\phi}g_{\mu\nu}, \quad \hat{g}^{\mu\nu} = \mathrm{e}^{3\phi}g^{\mu\nu}, \quad \sqrt{-\hat{g}} = \mathrm{e}^{-6\phi}\sqrt{-g} \tag{B.60}$$

We may still normalize ϕ for a canonical kinetic energy:

$$\begin{aligned}&\sqrt{3}\phi = \varphi \\ &\sqrt{-G}R(G) = \sqrt{-g}\left[R(g) - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu} - \frac{1}{4}\mathrm{e}^{\sqrt{3}\varphi}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\right]\end{aligned} \tag{B.61}$$

We thus find positive kinetic energies for the 3 fields, and an interaction of the scalar field with the gauge field. This is the Kaluza-Klein unification of gravity with U(1) gauge theory. Together they describe gravity in $D = 5$. To summarize the definitions, we can write

$$G_{MN} = \mathrm{e}^{2\varphi/\sqrt{3}}\begin{pmatrix}\mathrm{e}^{-\sqrt{3}\varphi}g_{\mu\nu} + \mathcal{A}_\mu\mathcal{A}_\nu & \mathcal{A}_\mu \\ \mathcal{A}_\nu & 1\end{pmatrix} \tag{B.62}$$

An alternative proof of (B.61) using a flat frame and spin connections is in 17.

Ex. B. 4 Rather than going to Einstein frame with $\sqrt{B.57}$, we may also normalize $g^S_{\mu\nu}$ such that $G_{\mu\nu} = g^S_{\mu\nu} + ...$ That is done by choosing $\Phi = \phi$. Check that in this way the explicit kinetic terms cancel, and we obtain

$$\sqrt{-G}R(G) = \sqrt{-g^S}\left[\mathrm{e}^{\phi}R\left(g^S\right) - \frac{1}{4}\mathrm{e}^{3\phi}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\right] \tag{B.63}$$

You probably will use B.51) multiplied by e^{ϕ}, such that the last term of that equation is not anymore a total derivative. Instead you will need that

$$\mathrm{e}^{\phi}\nabla^2\mathrm{e}^{2\phi} = \frac{2}{3}\nabla^2\mathrm{e}^{3\phi} - 2\mathrm{e}^{3\phi}\nabla\phi\cdot\nabla\phi \tag{B.64}$$

The physical kinetic energy for the scalar is still present by its interaction with the graviton. The result is valid for any reduction $D+1 \to D$. A proof using the Palatini identity is in [18, Sec. 15.2].

This is called the string frame. See the comparison

$$g^S_{\mu\nu} = \mathrm{e}^{2\phi}\hat{g}_{\mu\nu} = \mathrm{e}^{-\phi}g_{\mu\nu} \tag{B.65}$$

Finally we discuss the normalization of the Einstein term. Since the metric and related fields have no mass dimension, the Lagrangians as in (B.61) have mass dimension 2 due to the two spacetime derivatives. The action is then

$$S_E = \frac{1}{2\kappa_D^2}\int \mathrm{d}x^D\sqrt{-g}R(g) \tag{B.66}$$

where the mass dimension of κ_D^{-2} is $D-2$. In $D=4$: $\kappa_4^2 = \kappa^2 = 8\pi G$, where G is the Newton constant and the reduced Planck scale is

$$\kappa^2 = 8\pi G = (m_{\mathrm{P}})^{-2}, \quad m_{\mathrm{P}} = 2.4\times 10^{18}\mathrm{GeV} \tag{B.67}$$

The y-integral for compactification over a circle gives as in previous cases a factor $2\pi R$, and we obtain

$$\kappa^{-2} = 2\pi R\kappa_5^{-2} \tag{B.68}$$

This is similar to the relation between the Yang-Mills coupling constants, see (B.29).
So far the scalars and vectors were part of the metric and therefore dimensionless. To give them canonical dimensions and canonically normalize the terms in the action, we may redefine

$$\varphi \to \sqrt{2}\kappa\varphi, \quad \mathcal{A}_\mu \to \sqrt{2}\kappa\mathcal{A}_\mu \tag{B.69}$$

and end up with

$$\begin{aligned} S =& \sqrt{-g}\left[\frac{1}{2\kappa^2}R(g) - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu} - \frac{1}{4}\mathrm{e}^{\sqrt{6}\kappa\varphi}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\right] \\ &+ \sum_{n\neq 0}(\text{ massive KK spin-2 fields }). \end{aligned} \tag{B.70}$$

The field φ is called the 'radion'. Using the metric in the y direction $\mathrm{d}s^2 = (\mathrm{d}y)^2 G_{44} = (\mathrm{d}y)^2\exp\left(2\sqrt{\frac{2}{3}}\kappa\varphi\right)$, the length of the extra dimension is measured as

$$\int \mathrm{d}s = \int_0^{2\pi R} \mathrm{d}y\exp\left(\sqrt{\frac{2}{3}}\kappa\varphi\right) \tag{B.71}$$

B. 5 Gravitinos

The reduction of the gravitino $\Psi_M(x,y)$ action in $D=5$ is similar to the reduction of the fermions in Sec. B.3. The gravitino has a gauge symmetry with a local parameter $\epsilon(x,y)$.

As explained in detail in [1, Sec. 5.3.4], the y-dependent part can be gauge fixed by fixing the y-dependent part of $\Psi_4(x,y)$, i.e. reducing this component to $\Psi_4(x)$. We use again only one, but complex, component of the symplectic spinor Ψ^i_M, and with the Dirac conjugate we have then in 5D (for simplicity we restrict here to flat spacetime) [12]

$$S_\Psi = \int \mathrm{d}^5x \bar{\Psi}_M \gamma^{MNP} \partial_N \Psi_P \tag{B.72}$$

The y-dependent modes lead to massive gravitinos (the N index above can then lead to ∂_y leading to massive terms with masses as in (B.5)) as in previous subsections or in 1, Sec. 5.3.4]. We will now consider the massless (y-independent) part. Hence the N-index is in 4D, and we obtain up to total derivatives

$$S_\Psi = 2\pi R \int \mathrm{d}^4x \left[\bar{\Psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \Psi_\rho + 2\mathrm{i}\bar{\Psi}_\mu\gamma^{\mu\nu}\gamma_*\partial_\nu\lambda\right] \tag{B.73}$$

where we defined $\Psi_4 = \mathrm{i}\lambda$. The factor i is necessary in order that the contribution from $M=4$ and $P=4$ in B.72 can be written in the same way. It then also implies that $\mathrm{i}\bar{\Psi}_\mu\gamma^{\mu\nu}\gamma_*\partial_\nu\lambda$ can be written as $\mathrm{i}\Psi_{\mu i}\gamma^{\mu\nu}\gamma_*\partial_\nu\lambda_j\delta^{ij}$ in terms of 4 D Majorana spinors $\Psi_\mu = \Psi_{\mu 1} + \mathrm{i}\Psi_{\mu 2}$ and $\lambda = \lambda_1 + \mathrm{i}\lambda_2$

To diagonalize the Lagrangian we redefine

$$\Psi_\mu = \Psi'_\mu - \frac{1}{2}\mathrm{i}\gamma_\mu\gamma_*\lambda, \quad \Psi_4 = \mathrm{i}\lambda \tag{B.74}$$

and obtain

$$S_\Psi = 2\pi R \int \mathrm{d}^4x \left[\bar{\Psi}_\mu \gamma^{\mu\nu\rho}\partial_\nu \Psi_\rho - \frac{3}{2}\bar{\lambda}\ \partial\!\!\!/\lambda\right] \tag{B.75}$$

the expected kinetic action of a gravitino and an extra fermion (both complex here).
You can now also construct the massive action.

\footnotetext{

[12] The footnote 8 can also be applied here, reducing the 5 D gravitino to two Majorana gravitinos in $D=4$.

3 Superfields

Superspace is a much-used mechanism for supersymmetry. The classical book is [4]:
J. Wess and J. Bagger, "Supersymmetry and Supergravity".

Consider also the original paper [5.
Study chapters IV, V and VI of the book and solve the exercises in that part.
You may also use [1, Appendix 14.A] to make the translations between superspace expressions and formulas in the lectures.

Explain in your presentation how the transformations of the chiral multiplet as written in [1, (6.15)] in our book follow from superspace concepts.

Explain the translation between this formalism and our book. You will also have to choose a particular representation of gamma matrices to make the link.

4 Group manifold approach

In the group manifold approach one considers a supermanifold such that the local symmetries of supergravity (gct, Lorentz and supersymmetry) are described as superdiffeomorphisms on this manifold. A year ago a short review has been written on this method, explaining in particular how the symmetries and the action of $\mathcal{N}=1, D=4$ supergravity are obtained using this method [6]:
L. Castellani (Torino U.), "Group manifold approach to supergravity".

The exercise here consists in understanding the equations and connecting them with equations in the Supergravity book, whenever possible. Unfortunately, they use mostly minus metric, and some other normalizations (Lorentz algebra) are different. Thus you will also have to find translation rules to make the connection. But anyway, you should be able to recover results from [1, Chapter 7 and 9] from this approach. You can skip section 4 from the review.

5 Gravity actions of higher order in the curvature.

One might generalize the Hilbert action of pure gravity in [1, Ch.8] to other actions that contain higher order terms in the curvature. An original reference for this is [7, Sec. 5]. But I made a review in Appendix E. There are 3 types of actions that are quadratic in curvatures. We consider respectively Gauss-Bonnet gravity, Weyl gravity and R^2 gravity. The first one is special that it does not lead to high order field equations, and in $D=4$ even trivial field equations. You will have to prove that following the text in Appendix E. Weyl gravity, as the name suggests, is invariant under conformal transformations. R^2 gravity can be translated to an action with a scalar field and you get a particular type of potential, used in cosmology (Starobinsky model, see e.g. https://en.wikipedia.org/wiki/Starobinsky_inflation. Another application is in the holography context, and a year ago there was a paper in our group: https://arxiv.org/abs/2211.05907. You may certainly recognize (2.1) there, but the paper goes in another direction as this task.

You should still study the physical content of the actions (in $D=4$): physical or non-physical degrees of freedom and masses. All is explained, and with questions to you in Appendix E.
P.S. Read carefully the hints and use the suggested methods. Otherwise you will enter in very long calculations.

E. 1 Gauss-Bonnet gravity

This uses the Lagrangian $\sqrt{-g}E^2$, where E^2 is given in [1, (23.103)]. Use the way of writing

$$S_{\mathrm{GB}} = \int \mathrm{d}^D x\sqrt{-g}E^2 = 6\int \mathrm{d}^D x\sqrt{-g}R_{\mu\nu}{}^{\alpha\beta}R_{\rho\sigma}{}^{\gamma\delta}\delta^{\mu\nu\rho\sigma}_{\alpha\beta\gamma\delta} \tag{E.1}$$

and prove that this is the same as the one with the 3 terms with different contractions.

1. Show that the field equations do not lead to terms containing more than two spacetime derivatives on the metric (i.e. not containing derivatives on the curvature).
   hint: use the exercises of [1, Ch. 7] (and we have no torsion here) to prove that

$$\delta R_{\mu\nu\rho\sigma} = -2\nabla_{[\mu}\nabla_{[\rho}\delta g_{\sigma]\nu]} + \text{ terms proportional to curvatures} \tag{E.2}$$

where the latter have also no derivatives on $\delta g_{\mu\nu}$. This should allow you to prove the absence of higher derivatives in the equations of motion using also the Bianchi identity [1, (7.122)]. Why are the not-explicitly written terms in the above equation not relevant?

2. For $D = 4$ prove that the field equations are trivial. For this, use a rewriting of the δ in the expression for $S_{\rm GB}$ as the product of two Levi-Civita tensors ε. Be careful about the $\sqrt{-g}$. Now you need the second part of the above expression of $\delta R_{\mu\nu\rho\sigma}$.

E. 2 Weyl gravity

This uses $\sqrt{-g}W^2$ as Lagrangian. Prove that it can be written in terms of the square of the Weyl tensor

$$C_{\mu\nu}{}^{\rho\sigma} = R_{\mu\nu}{}^{\rho\sigma} + x\delta_{[\mu}^{[\rho}R_{\nu]}^{\sigma]} + y\delta_{[\mu}^{[\rho}\delta_{\nu]}^{\sigma]}R \tag{E.3}$$

Find the numbers x and y for general dimension using the requirement that C is traceless. Show that for $D = 4$ the action has a local symmetry $\delta g_{\mu\nu} = g_{\mu\nu}\Lambda$, where Λ is an arbitrary x-dependent parameter (written as $\exp(2f(x))$ at the end of [1, Sec. 8.2] by proving that C (with how many upper-lower indices?) is invariant for the same values of x and y. You can prove the invariance of C (with a number of upper and lower indices) for all D, but you should then find that the action is invariant only for $D = 4$.
hint: you may re-use expressions proven for the previous part. In your calculation, split the contributions with derivatives on Λ from those which are just proportional to Λ without derivatives.

E. 3R^2-gravity

(You may restrict yourself to $D = 4$). Prove, using a solvable algebraic field equation, that $R + \alpha R^2$ gravity is equivalent to the action

$$S_2 = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\sqrt{-g}\left[(1+2\alpha\phi)R - \alpha\phi^2\right] \tag{E.4}$$

where α is a dimensionful constant and ϕ is a scalar. On the other hand, by making a rescaling $g'_{\mu\nu} = (1+2\alpha\phi)g_{\mu\nu}$ (and look at Ex. B.3) show that this is equivalent to a scalar (with positive kinetic energy) interacting with gravity and having specific self-interactions. Choose the variable $\varphi = \sqrt{3/2}\ln(1+2\alpha\phi)$ to bring the scalar action in standard form.

E. 4 Physical content

We now work in $D = 4$. For pure gravity, omitting total derivatives, we write (with $\kappa = 1$ for simplicity)

$$S = \int \mathrm{d}^4x\frac{1}{2}\sqrt{-g}\left[R + \alpha R^2 + \beta\left(R_{\mu\nu}R^{\mu\nu} - \frac{1}{3}R^2\right)\right] \tag{E.5}$$

You already know that the Einstein-Hilbert action describes the massless graviton field, using the field e_μ^a, which has 6 off-shell degrees of freedom (dof), counting 16 components minus the 10 gauge dof of the Poincaré algebra. One can describe these also as 5 dof of a conformal frame field, and a scalar compensator field, ϕ, whose value determines the gravitational constant κ.

For the α term you can use your results of R^2-gravity. It shows that this makes the scalar also propagating and by expanding you show that it is a massive physical scalar with $m_0{}^{-2} = 6\alpha$.

You may do a similar procedure for understanding the physics of the β term. You may rewrite the action with the β term (E.5) as

$$S_w = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[R + 2G_{\mu\nu}w^{\mu\nu} - \beta^{-1}\left(w_{\mu\nu}w^{\mu\nu} - w^2\right)\right], \quad w \equiv w_{\mu\nu}g^{\mu\nu} \tag{E.6}$$

where $G_{\mu\nu}$ is the Einstein tensor, for which you can check that it can be defined as

$$G_{\mu\nu} = \frac{\delta}{\sqrt{-g}\delta g^{\mu\nu}} \int \mathrm{d}^4x\sqrt{-g}R \tag{E.7}$$

Prove that S_w is the same action as S after elimination of auxiliary variables by their algebraic field equations.

To understand the physics, you can use expansions as in [1, Sec. 8.2] and we look to the action quadratic the physical fluctuations $\delta g_{\mu\nu} = h_{\mu\nu}$. Note that with the gauges as chosen there, only the first term in [1, (8.16)] is relevant, since $\partial^\mu h_{\mu\nu} = 0$ and $h^\mu{}_\mu = 0$. Understand that for these fluctuations $G^{\rm lin}_{\mu\nu} = R^{\rm lin}_{\mu\nu}$ and $\sqrt{-g} = 1 + \mathcal{O}\left(h^2\right)$. The relevant part of the Einstein action should then be such that $\sqrt{-g}R$ is replaced by an $R^{\rm eff}$ such that (indices now raised or lowered with $\eta_{\mu\nu}$)

$$G^{\rm in}_{\mu\nu} = \frac{\delta}{\delta g^{\mu\nu}} \int \mathrm{d}^4x R^{\rm eff} = -\frac{\delta}{\delta h^{\mu\nu}} \int \mathrm{d}^4x R^{\rm eff}$$

Signs are important to understand the nature of kinetic energy!
Thus we can in S_w (apart from putting $\sqrt{-g} = 1$) replace R by

$$R^{\rm eff} = \frac{1}{4} h^{\mu\nu}\Box h_{\mu\nu}. \tag{E.8}$$

See that this gives positive kinetic energy to the gravity fields.
Then define a new $\bar{h}_{\mu\nu}$, a linear combination of $h_{\mu\nu}$ and $w_{\mu\nu}$, such that you eliminate the mixing between the graviton and the w-field. See then that after this separation the w-field has the opposite sign for its kinetic energy than the graviton. Hence it is a ghost (and with its index structure you can believe that it is a spin 2 ghost). Find its squared mass. The massive spin 2 ghost can be identified with the conformal part of e_μ^a. In this way the 6 off-shell dof from the Poincaré theory describe the massive states in the above action.

6 Anti-de Sitter supergravity

After discussing $\mathcal{N} = 1, D = 4$ supergravity in [1, Ch. 9], we shortly discussed in [1, Sec. 9.6] the modifications for Anti-de Sitter supergravity. A first step was written down there. Remark: In the book some equations were indeed given for general D, but for the full theory at the end one needs $D = 4$. In other dimensions more fields should be included (already for Minkowski supergravity). Only the 'universal steps' are valid for general D.

First solve already [1, Ex. 9.6]. Then consider again the proof of the invariance of the full supergravity action, explained earlier in that chapter, using these new covariant derivatives. Check that everything still applies (all the details of that chapter with the torsion corrections) and show the modifications. At the end, you come to the soft algebra of local supersymmetry (possible modification of [1, Sec. 9.5]). Make contact with [1, Sec. 12.6.1] for a solution with zero fermions. You can read [1, Secs. 22.2-3] to understand the importance of the solution. Try to understand how the algebra of [1, Sec. 12.6.1] appears for the solution. It suffices to understand the main features. In particular: explain how the Lorentz transformation appears now in the commutator of two supersymmetries as in [1, (12.17)].

7 The type IIA supergravity

Using the methods of dimensional reduction from an odd to the lower even dimension (discussed shortly in [1, Ch.5], consider the construction of IIA supergravity in 10 dimensions. We are only interested in the massless sector ($k = 0$ in the notation of that chapter). You start from the formulation of the $D = 11$ supergravity as written in [1, Ch. 10] and define components of the fields that can be used in the lower dimension. In the book, we did not explain how to choose exactly the fields in the lower dimension such that they are transforming as a tensor in that dimension. You may learn that from section 1.1 and 1.2 in the lecture notes of Chris Pope 'Lectures on Kaluza-Klein'. See his home page http://people.physics.tamu.edu/pope/. You can also consider Sec. B.4, but generalize this to a reduction of $D+1$ to D dimensions, see Appendix C.1, which you will have to check.

What I expect from you is the action and the transformation rules. However, you may neglect in the action all terms quartic in the fermions, and in the transformation laws terms quadratic in fermion fields, i.e. terms of the form δ fermion $= \epsilon$ (fermion)(fermion). You may use the original papers [8, Sec.I], 99, Sec. 2], [10, Sec. 2.1 and 3] However, all these papers use different notations, and you have to obtain something that is consistent with the notations adopted in the [1]. But you can use their way of working.

B. 4 Gravity in 5D

Denoting the metric in 5 D as G_{MN}, it decomposes in

$$G_{MN}(x,y)=\left(\begin{array}{cc} G_{\mu\nu} & G_{\mu 4} \\ G_{4\mu} & G_{44} \end{array}\right)(x,y) \tag{B.31}$$

The general coordinate transformations of $5D$ act as

$$\delta(\xi) G_{MN}=\xi^{P}\partial_{P} G_{MN}+2\partial_{(M}\xi^{P} G_{N)P} \tag{B.32}$$

Consider in particular

$$\delta(\xi) G_{44}=\left(\xi^{\mu}\partial_{\mu}+\xi^{4}\partial_{y}\right) G_{44}+2 G_{4\mu}\partial_{y}\xi^{\mu}+2 G_{44}\partial_{y}\xi^{4} \tag{B.33}$$

Since the flat manifold is a product space, $G_{44}\neq 0$, and we parametrize it as $\square$[10]

$$G_{44}=\mathrm{e}^{2\phi} \tag{B.34}$$

The field ϕ then transforms under the ξ^{4} transformations as

$$\begin{aligned} \delta\left(\xi^{4}\right)\phi &=\partial_{y}\xi^{4}+\xi^{4}\partial_{y}\phi \\ \delta\left(\xi^{4}\right)\phi_{n} &=\frac{\mathrm{i}n}{R}\xi_{n}^{4}+\sum_{m}\frac{\mathrm{i}m}{R}\phi_{m}\xi_{n-m}^{4} \end{aligned} \tag{B.35}$$

where in the second line we introduced the modes as before. This implies that we can gauge-fix the $\xi_{n\neq 0}^{4}$ transformations by taking

$$\xi_{n\neq 0}^{4}\text{ gauge:}\quad \phi_{n\neq 0}=0 \tag{B.36}$$

\footnotetext{
[10] We may later still rescale ϕ arbitrary, e.g. to obtain canonical kinetic terms for this scalar.

Similarly we consider the ξ^{ν} transformations of the off-diagonal components

$$\begin{aligned} \delta\left(\xi^{\nu}\right) G_{\mu 4} &=\xi^{\nu}\partial_{\nu} G_{\mu 4}+G_{\nu 4}\partial_{\mu}\xi^{\nu}+G_{\mu\nu}\partial_{y}\xi^{\nu} \\ \delta\left(\xi^{\nu}\right) G_{\mu 4, n} &=\sum_{m}\left(\xi_{m}^{\nu}\partial_{\nu} G_{\mu 4, n-m}+G_{\nu 4, n-m}\partial_{\mu}\xi_{m}^{\nu}+G_{\mu\nu, n-m}\frac{\mathrm{i}m}{R}\xi_{m}^{\nu}\right) \end{aligned} \tag{B.37}$$

Since $G_{\mu\nu,0}$, being proportional to the metric in the 4 D space, should be invertible, we can also fix the $\xi^{\mu}_{n\neq 0}$ transformations

$$\xi^{\mu}_{n\neq 0} \text{ gauge:} \quad G_{\mu 4, n\neq 0} = 0 \tag{B.38}$$

The remaining gauge transformations are thus only the 0 -modes $\xi_0^{\nu}(x)$ and $\xi_0^4(x)$, and $\partial_y \xi^M = 0$. One thus easily checks from B.32 that the remaining $\xi^{\nu}(x) = \xi_0^{\nu}$ acts as general coordinate transformations in $4D$. We will still consider the $\xi_0^4(x)$ transformations below, but we can already consider the dof similar to what we did for the vectors in (B.16):

$$\begin{aligned}
&5D : G_{MN} \text{ has } D(D-3)/2 = 5 \text{ on-shell dof} \\
&\to 4D : G_{\mu\nu,0} \text{ has } D(D-3)/2 = 2 \text{ on-shell dof: the 4D graviton,} \\
&G_{\mu,0} \text{ has } D-2 = 2 \text{ on-shell dof: a 4D gauge vector,} \\
&\phi \text{ has 1 on-shell dof: a 4D scalar,} \\
&G_{\mu\nu,n\neq 0}, \text{ has 5 on-shell dof: massive spin 2 fields.}
\end{aligned} \tag{B.39}$$

The latter 5 are the helicity states $0, \pm 1, \pm 2$ of the massive spin 2.
Having eliminated the non-zero modes of the symmetries and of ϕ, we see from (B.35) that $\phi(x)$ is invariant under $\xi^4(x)$. Considering the ξ^4 transformations of $G_{\mu 4}$ we find from (B.32), using also B.38

$$\delta\left(\xi^4\right) G_{\mu 4} = \left(\partial_\mu \xi^4\right) G_{44} = \mathrm{e}^{2\phi} \partial_\mu \xi^4 \tag{B.40}$$

Therefore (these fields only depend on x due to (B.36) and (B.38))

$$\mathcal{A}_\mu \equiv \mathrm{e}^{-2\phi} G_{\mu 4} \tag{B.41}$$

is the canonically normalized gauge field for the $\xi^4(x)$ transformations. Finally considering

$$\delta\left(\xi^4\right) G_{\mu\nu}(x,y) = \xi^4(x) \partial_y G_{\mu\nu}(x,y) + 2\partial_{(\mu} \xi^4(x) G_{\nu)4}(x) \tag{B.42}$$

The first term represent the usual y coordinate transformation. The second can be eliminated by considering $G_{\mu\nu}(x,y) - \mathcal{A}_\mu \mathcal{A}_\nu \mathrm{e}^{2\phi}$, whose ξ^4 transformation has only the $\xi^4 \partial_y$-term. For convenience we define

$$\hat{G}_{\mu\nu}(x,y) = \mathrm{e}^{-2\phi} G_{\mu\nu}(x,y), \quad \hat{g}_{\mu\nu}(x,y) = \hat{G}_{\mu\nu}(x,y) - \mathcal{A}_\mu(x) \mathcal{A}_\nu(x) \tag{B.43}$$

The y-independent part of $\hat{g}_{\mu\nu}$ is then invariant under ξ^4, and will be proportional to the metric in 4D. We can later take another overall factor with the scalar field in order to obtain the Einstein frame, i.e. where the kinetic terms of $g_{\mu\nu} \propto \hat{g}_{\mu\nu}(x,0)$ and ϕ are separated.

To summarize we can write

$$\begin{aligned}
&G_{MN} = \mathrm{e}^{2\phi} \hat{G}_{MN} \\
&\hat{G}_{MN} = \begin{pmatrix} \hat{G}_{\mu\nu}(x,y) & \mathcal{A}_\mu(x) \\ \mathcal{A}_\nu(x) & 1 \end{pmatrix} = \begin{pmatrix} \hat{g}_{\mu\nu} + \mathcal{A}_\mu \mathcal{A}_\nu & \mathcal{A}_\mu(x) \\ \mathcal{A}_\nu(x) & 1 \end{pmatrix}
\end{aligned} \tag{B.44}$$

or

$$\mathrm{d}s^2 \equiv G_{MN}\, \mathrm{d}x^M\, \mathrm{d}x^N = \mathrm{e}^{2\phi} \mathrm{d}\hat{s}^2, \quad \mathrm{d}\hat{s}^2 = \hat{G}_{MN}\, \mathrm{d}x^M\, \mathrm{d}x^N$$

$$\mathrm{d}\hat{s}^2 = \hat{G}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + 2\mathcal{A}_\mu \mathrm{d}x^\mu \mathrm{d}y + \mathrm{d}y\ \mathrm{d}y \tag{B.45}$$

$$= \hat{g}_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu + \left(\mathrm{d}y + \mathcal{A}_\mu \mathrm{d}x^\mu\right)^2 \tag{B.46}$$

The matrix $\hat{G}_{MN}$ can also be decomposed as

$$\hat{G}_{MN} = \begin{pmatrix} \delta^\rho_\mu & \mathcal{A}_\mu \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{g}_{\rho\sigma} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \delta^\sigma_\nu & 0 \\ \mathcal{A}_\nu & 1 \end{pmatrix}, \quad \to \quad \det\left(\hat{G}_{MN}\right) = \det\left(\hat{g}_{\mu\nu}\right) \tag{B.47}$$

In preparation of the calculation of the connections, we also obtain the inverse of $\hat{G}_{MN}$, which we denote here as $\hat{G}^{MN}$, in terms of the inverse of $\hat{g}_{\mu\nu}$, which we denote as $\hat{g}^{\mu\nu}$:

$$\hat{G}^{MN} = \begin{pmatrix} \hat{g}^{\mu\nu} & -\hat{\mathcal{A}}^\mu \\ -\hat{\mathcal{A}}^\nu & 1 + \hat{\mathcal{A}}^\rho \mathcal{A}_\rho \end{pmatrix}, \quad \hat{\mathcal{A}}^\mu \equiv \hat{g}^{\mu\nu}\mathcal{A}_\nu \tag{B.48}$$

For calculating curvatures, we will often have to define metrics that are proportional to each other by a scalar factor. The following exercise is convenient for that

Ex. B. 3 This exercise is useful in many different contexts. Therefore we look at it for general dimension D. Consider the rescaling

$$\tilde{g}_{\mu\nu} = \mathrm{e}^{2\Phi} g_{\mu\nu} \tag{B.49}$$

Then obtain torsionless metric compatible connections and curvatures (see e.g. [1, Sec. 7.9, 7.10])

$$\begin{aligned}
\Gamma^\rho_{\mu\nu}(\tilde{g}) =& \Gamma^\rho_{\mu\nu}(g) + 2\delta^\rho_{(\mu}\partial_{\nu)}\Phi - g_{\mu\nu}\partial^\rho\Phi \\
R_{\mu\nu}{}^\rho(\tilde{g}) =& R_{\mu\nu}{}^\rho{}_\sigma(g) - 2\delta^\rho_{[\mu}\nabla_{\nu]}\partial_\sigma\Phi + 2g_{\sigma[\mu}\nabla_{\nu]}\partial^\rho\Phi \\
& + 2\delta^\rho_{[\mu^\nu}\left(\partial_{\nu]}\Phi\right)\left(\partial_\sigma\Phi\right) - 2g_{\sigma[\mu}\left(\partial_{\nu]}\Phi\right)\left(\partial^\rho\Phi\right) - 2\delta^\rho_{[\mu}g_{\nu]\sigma}(\partial\Phi\cdot\partial\Phi), \\
R_{\nu\sigma}(\tilde{g}) =& R_{\nu\sigma}(g) + (2-D)\nabla_\nu\partial_\sigma\Phi - g_{\nu\sigma}\nabla^2\Phi + (D-2)\left((\partial_\nu\Phi)\left(\partial_\sigma\Phi\right) - g_{\nu\sigma}(\partial\Phi\cdot\partial\Phi)\right), \\
R(\tilde{g}) =& \mathrm{e}^{-2\Phi}\left[R(g) - 2(D-1)\nabla^2\Phi - (D-1)(D-2)\partial\Phi\cdot\partial\Phi\right].
\end{aligned} \tag{B.50}$$

The ∇ derivative uses the connection $\Gamma^\rho_{\mu\nu}(g)$ and all inner products use $g^{\mu\nu}$. Further remark that $\sqrt{\det|\tilde{g}|} = \mathrm{e}^{D\Phi}\sqrt{\det|g|}$, and that you can rewrite the result for the action as

$$\sqrt{-\tilde{g}}R(\tilde{g}) = \mathrm{e}^{(D-2)\Phi}\sqrt{-g}[R(g) + (D-1)(D-2)\partial\Phi\cdot\partial\Phi] - 2\frac{D-1}{D-2}\sqrt{-g}\nabla^2\mathrm{e}^{(D-2)\Phi} \tag{B.51}$$

The last term is a total derivative, which can be omitted.

We now consider the connections related to the metric $\hat{G}$, for the lowest modes (no ∂_y derivatives) and obtain

$$\begin{array}{lll}
\Gamma^\rho_{\mu\nu}(\hat{G}) = \Gamma^\rho_{\mu\nu}(\hat{g}) + \hat{g}^{\rho\sigma}\mathcal{A}_{(\mu}\mathcal{F}_{\nu)\sigma}, & \Gamma^\rho_{\mu 4}(\hat{G}) = \frac{1}{2}\hat{g}^{\rho\sigma}\mathcal{F}_{\mu\sigma}, & \Gamma^\rho_{44}(\hat{G}) = 0 \\
\Gamma^4_{\mu\nu}(\hat{G}) = \nabla_{(\mu}\mathcal{A}_{\nu)} - \hat{\mathcal{A}}^\rho\mathcal{A}_{(\mu}\mathcal{F}_{\nu)\rho}, & \Gamma^4_{\mu 4}(\hat{G}) = -\frac{1}{2}\hat{\mathcal{A}}^\sigma\mathcal{F}_{\mu\sigma}, & \Gamma^4_{44}(\hat{G}) = 0
\end{array} \tag{B.52}$$

with $\mathcal{F}_{\mu\nu} = 2\partial_{[\mu}\mathcal{A}_{\nu]}$ and ∇ is compatible with $\hat{g}_{\mu\nu}$. With a straightforward but tedious calculation (not necessary for this task) you find [11]

$$\begin{aligned}
R_{\mu\nu}(\hat{G}) &= R_{\mu\nu}(\hat{g}) + \mathcal{A}_{(\mu}\nabla^\rho\mathcal{F}_{\nu)\rho} - \frac{1}{2}\mathcal{F}_{\mu\rho}\mathcal{F}_{\nu\sigma}\hat{g}^{\rho\sigma} + \frac{1}{4}\mathcal{A}_\mu\mathcal{A}_\nu\hat{\mathcal{F}}^2 \\
R_{\mu 4}(\hat{G}) &= \frac{1}{2}\nabla^\nu\mathcal{F}_{\mu\nu} + \frac{1}{4}\mathcal{A}_\mu\hat{\mathcal{F}}^2, \quad \hat{R}_{44}(\hat{G}) = \frac{1}{4}\hat{\mathcal{F}}^2, \quad \hat{\mathcal{F}}^2 \equiv \mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}
\end{aligned} \tag{B.53}$$

We then obtain for the scalar curvature

$$
\begin{aligned}
R(\hat{G}) &= \hat{g}^{\mu\nu} R_{\mu\nu}(\hat{G}) - 2\hat{\mathcal{A}}^{\mu} R_{\mu 4}(\hat{G}) + \left(1 + \hat{\mathcal{A}}^{\rho}\mathcal{A}_{\rho}\right)\Big) R_{44}(\hat{G}) \\
&= R(\hat{g}) - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}
\end{aligned}
\tag{B.54}
$$

Using the result of the above exercise for metrics related as in (B.44), we take

$$
\tilde{g}_{\mu\nu} \to G_{MN}, \quad g_{\mu\nu} \to \hat{G}_{MN}, \quad \Phi \to \phi, \quad D = 5 \tag{B.55}
$$

and obtain (we put a hat on curvatures calculated in $D = 5$ and indicate the metric used to define them)

$$
\begin{aligned}
\sqrt{-G}R(G) &= \mathrm{e}^{3\phi}\sqrt{-\hat{G}}\left[R(\hat{G}) + 4.3\partial_{\mu}\phi\partial_{\nu}\phi\hat{G}^{\mu\nu}\right] + \text{ total der.} \\
&= \mathrm{e}^{3\phi}\sqrt{-\hat{g}}\left(R(\hat{g}) - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma} + 12\partial_{\mu}\phi\partial_{\nu}\phi\hat{g}^{\mu\nu}\right) + \text{ total der.}
\end{aligned}
\tag{B.56}
$$

where we inserted B.54 and used that the determinant of $\hat{G}_{MN}$ is the determinant of $\hat{g}_{\mu\nu}$ and $\hat{G}^{\mu\nu} = \hat{g}^{\mu\nu}$;

The factor $\mathrm{e}^{3\phi}$ in front of the Einstein term defined from $\hat{g}_{\mu\nu}$ can be eliminated using again (B.51), now for $D = 4$, taking

$$
g_{\mu\nu} \to \hat{g}_{\mu\nu}, \quad \tilde{g}_{\mu\nu} \to g_{\mu\nu}, \quad \Phi \to \frac{3}{2}\phi, \quad D = 4 \tag{B.57}
$$

This leads to the equation

$$
\sqrt{-g}R(g) = \mathrm{e}^{3\phi}\sqrt{-\hat{g}}\left[R(\hat{g}) + 3 \cdot 2 \cdot \left(\frac{3}{2}\right)^{2} \partial_{\mu}\phi\partial_{\nu}\phi\hat{g}^{\mu\nu}\right] + \text{ total der..} \tag{B.58}
$$

\footnotetext{

[11] To simplify your calculations you can use that $\Gamma^{\rho}_{\mu\nu}(\hat{g})$ appears only in $4D$ covariant derivatives.

Thus the variable $g_{\mu\nu}$ chosen like this defines the Einstein frame in $D = 4$. Using (B.58) in (B.56) we obtain for the $D = 5$ Einstein Lagrangian as (up to total derivatives)

$$
\begin{aligned}
\sqrt{-G}R(G) &= \sqrt{-g}R(g) + \sqrt{-\hat{g}}\mathrm{e}^{3\phi}\left[\left(12 - \frac{27}{2}\right)\partial_{\mu}\phi\partial_{\nu}\phi\hat{g}^{\mu\nu} - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}\hat{g}^{\mu\rho}\hat{g}^{\nu\sigma}\right] \\
&= \sqrt{-g}\left[R(g) - \frac{3}{2}\partial_{\mu}\phi\partial_{\nu}\phi g^{\mu\nu} - \frac{1}{4}\mathrm{e}^{3\phi}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\right]
\end{aligned}
\tag{B.59}
$$

where in the last line we used

$$
\hat{g}_{\mu\nu} = \mathrm{e}^{-3\phi}g_{\mu\nu}, \quad \hat{g}^{\mu\nu} = \mathrm{e}^{3\phi}g^{\mu\nu}, \quad \sqrt{-\hat{g}} = \mathrm{e}^{-6\phi}\sqrt{-g} \tag{B.60}
$$

We may still normalize ϕ for a canonical kinetic energy:

$$
\begin{aligned}
\sqrt{3}\phi &= \varphi \\
\sqrt{-G}R(G) &= \sqrt{-g}\left[R(g) - \frac{1}{2}\partial_{\mu}\varphi\partial_{\nu}\varphi g^{\mu\nu} - \frac{1}{4}\mathrm{e}^{\sqrt{3}\varphi}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\right]
\end{aligned}
\tag{B.61}
$$

We thus find positive kinetic energies for the 3 fields, and an interaction of the scalar field with the gauge field. This is the Kaluza-Klein unification of gravity with U(1) gauge theory. Together they describe gravity in $D=5$. To summarize the definitions, we can write

$$G_{MN}=\mathrm{e}^{2\varphi/\sqrt{3}}\left(\begin{array}{cc}\mathrm{e}^{-\sqrt{3}\varphi}g_{\mu\nu}+\mathcal{A}_\mu\mathcal{A}_\nu & \mathcal{A}_\mu\\ \mathcal{A}_\nu & 1\end{array}\right)\tag{B.62}$$

An alternative proof of (B.61) using a flat frame and spin connections is in 17.

Ex. B. 4 Rather than going to Einstein frame with $\sqrt{B.57}$, we may also normalize $g^S_{\mu\nu}$ such that $G_{\mu\nu}=g^S_{\mu\nu}+\ldots$. That is done by choosing $\Phi=\phi$. Check that in this way the explicit kinetic terms cancel, and we obtain

$$\sqrt{-G}R(G)=\sqrt{-g^S}\left[\mathrm{e}^{\phi}R\left(g^S\right)-\frac{1}{4}\mathrm{e}^{3\phi}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\right]\tag{B.63}$$

You probably will use B.51) multiplied by e^{ϕ}, such that the last term of that equation is not anymore a total derivative. Instead you will need that

$$\mathrm{e}^{\phi}\nabla^2\mathrm{e}^{2\phi}=\frac{2}{3}\nabla^2\mathrm{e}^{3\phi}-2\mathrm{e}^{3\phi}\nabla\phi\cdot\nabla\phi\tag{B.64}$$

The physical kinetic energy for the scalar is still present by its interaction with the graviton. The result is valid for any reduction $D+1\rightarrow D$. A proof using the Palatini identity is in [18, Sec. 15.2].

This is called the string frame. See the comparison

$$g^S_{\mu\nu}=\mathrm{e}^{2\phi}\hat{g}_{\mu\nu}=\mathrm{e}^{-\phi}g_{\mu\nu}\tag{B.65}$$

Finally we discuss the normalization of the Einstein term. Since the metric and related fields have no mass dimension, the Lagrangians as in (B.61) have mass dimension 2 due to the two spacetime derivatives. The action is then

$$S_E=\frac{1}{2\kappa_D^2}\int\mathrm{d}x^D\sqrt{-g}R(g)\tag{B.66}$$

where the mass dimension of κ_D^{-2} is $D-2$. In $D=4:\kappa_4^2=\kappa^2=8\pi G$, where G is the Newton constant and the reduced Planck scale is

$$\kappa^2=8\pi G=\left(m_{\mathrm{P}}\right)^{-2},\quad m_{\mathrm{P}}=2.4\times10^{18}\mathrm{GeV}\tag{B.67}$$

The y-integral for compactification over a circle gives as in previous cases a factor $2\pi R$, and we obtain

$$\kappa^{-2}=2\pi R\kappa_5^{-2}\tag{B.68}$$

This is similar to the relation between the Yang-Mills coupling constants, see (B.29).
So far the scalars and vectors were part of the metric and therefore dimensionless. To give them canonical dimensions and canonically normalize the terms in the action, we may redefine

$$\varphi\rightarrow\sqrt{2}\kappa\varphi,\quad\mathcal{A}_\mu\rightarrow\sqrt{2}\kappa\mathcal{A}_\mu\tag{B.69}$$

and end up with

$$
\begin{aligned}
S = & \sqrt{-g}\left[\frac{1}{2\kappa^2}R(g) - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi g^{\mu\nu} - \frac{1}{4}\mathrm{e}^{\sqrt{6}\kappa\varphi}\mathcal{F}_{\mu\nu}\mathcal{F}_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}\right] \\
& + \sum_{n\neq 0}(\text{ massive KK spin-2 fields }).
\end{aligned} \tag{B.70}
$$

The field φ is called the 'radion'. Using the metric in the y direction $\mathrm{d}s^2 = (\mathrm{d}y)^2 G_{44} = (\mathrm{d}y)^2 \exp\left(2\sqrt{\frac{2}{3}}\kappa\varphi\right)$, the length of the extra dimension is measured as

$$
\int \mathrm{d}s = \int_0^{2\pi R} \mathrm{d}y \exp\left(\sqrt{\frac{2}{3}}\kappa\varphi\right) \tag{B.71}
$$

C. 1 Reduction from $D+1$ to D dimensions

We treated this already for $D=4$ in Sec. B.4. Consider that section starting from (B.44) until (B.62) for general D and prove that, rather than (B.61), we arrive at

$$
\int \mathrm{d}y \frac{1}{2\kappa_{D+1}^2}\sqrt{-G}R(G) = \frac{1}{2\kappa_D^2}\sqrt{-g}\left[R(g) - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\mathrm{e}^{\varphi/\beta(D)}\right] \tag{C.1}
$$

using rescalings

$$
g_{\mu\nu} = \mathrm{e}^{\varphi/\beta(D)}\hat{g}_{\mu\nu}, \quad \beta(D) = -\sqrt{\frac{D-2}{2(D-1)}}, \quad \phi = \beta(D)\varphi \tag{C.2}
$$

such that the action remains in Einstein frame. After an appropriate scaling the dilaton field has a standard kinetic term [14] and we have

$$
\int \mathrm{d}y \frac{1}{2\kappa_{D+1}^2}\sqrt{-G}R(G) = \frac{1}{2\kappa_D^2}\sqrt{-g}\left[R(g) - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\mathrm{e}^{\varphi/\beta(D)}\right] \tag{C.3}
$$

Here we use $\kappa_D^2 = \kappa_{D+1}^2 2\pi L$. In general: $\kappa_D^2 = \kappa^2(2\pi L)^{D-4}$, where $\kappa^2 = \kappa_4^2 = 8\pi G$ has dimension of length ^{2}. One can check that $\mathcal{A}_\mu$ is the gauge field of the part of the gct of $D+1$ dimensions ξ^M in the direction of the extra dimension:

$$
\delta\left(\xi^D\right)\mathcal{A}_\mu = \partial_\mu\xi^D(x) - c \tag{C.4}
$$

Another result from the previous part is that the length element is now
Check also that the length element is now

$$
\begin{aligned}
\mathrm{d}s_{D+1}^2 = & G_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N = \mathrm{e}^{s(D)\varphi}g_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{e}^{2\beta(D)}\left(\mathrm{d}y + \mathcal{A}^{(1)}\right)^2 \\
s(D) = & -\frac{2\beta(D)}{D-2} = -\frac{1}{(D-1)\beta(D)} = \sqrt{\frac{2}{(D-1)(D-2)}}
\end{aligned} \tag{C.5}
$$

[613] In the preparation of this part, I used a lot 19, 20, 17. An earlier review of the methods was 21].
[14] In principle to have the standard kinetic terms for the scalar and the vector, we should still redefine them by a factor $\sqrt{2}\kappa_D$. But we prefer to keep working with the fields without a mass dimension.

For p-form fields in $D+1$ dimensions $\hat{A}^{(p)} = \frac{1}{p!}\hat{A}^{(p)}_{M_1\ldots M_p}\mathrm{d}x^{M_1}\ldots\mathrm{d}x^{M_P}$, we reduce them to D dimensions as

$$\hat{A}^{(p)}_{\mu_1\ldots\mu_p} = A^{(p)}_{\mu_1\ldots\mu_p}, \quad \hat{A}^{(p)}_{\mu_1\ldots\mu_{p-1}D} = A^{(p-1)}_{\mu_1\ldots\mu_{p-1}}, \quad \hat{A}^{(p)} = A^{(p)} + A^{(p-1)} \wedge \mathrm{d}y \tag{C.6}$$

where the last expression is the same as the first two in form language.
The field strength in $D+1$ dimensions are divided in D dimensions in two parts using the 1-form that appears in (C.5):

$$\begin{aligned}\hat{F}^{(p+1)} &= \mathrm{d}\hat{A}^{(p)} = A^{(p+1)} + F^{(p)} \wedge \left(\mathrm{d}y + \mathcal{A}^{(1)}\right)\\ F^{(p+1)} &= \mathrm{d}A^{(p)} - \mathrm{d}A^{(p-1)} \wedge \mathcal{A}^{(1)}, \quad F^{(p)} = \mathrm{d}A^{(p-1)}\end{aligned} \tag{C.7}$$

In component language, the first line is written as

$$\hat{F}^{(p+1)}_{\mu_1\ldots\mu_{p+1}} = F^{(p+1)}_{\mu_1\ldots\mu_{p+1}} + (p+1)F^{(p)}_{[\mu_1\ldots\mu_p}\mathcal{A}_{\mu_{p+1}]}, \quad \hat{F}^{(p+1)}_{\mu_1\ldots\mu_p D} = F^{(p)}_{\mu_1\ldots\mu_{p+1}}, \tag{C.8}$$

The $\mathcal{A}^{(1)} = \mathcal{A}_\mu \mathrm{d}x^\mu$ terms have been introduced in the latter equations such that $F^{(p+1)}$ and $F^{(p)}$ are inert under ξ^D gct. But moreover, these definitions with (B.44) imply that

$$\begin{aligned}&\hat{F}^{(p+1)}_{M_1\ldots M_{p+1}}\hat{F}^{(p+1)}_{N_1\ldots N_{p+1}}\hat{G}^{M_1N_1}\ldots\hat{G}^{M_{p+1}N_{p+1}} = \\ &= F^{(p+1)}_{\mu_1\ldots\mu_{p+1}}F^{(p+1)}_{\nu_1\ldots\nu_{p+1}}\hat{g}^{\mu_1\nu_1}\ldots\hat{g}^{\mu_{p+1}\nu_{p+1}} + (p+1)F^{(p)}_{\mu_1\ldots\mu_p}F^{(p)}_{\nu_1\ldots\nu_p}\hat{g}^{\mu_1\nu_1}\ldots\hat{g}^{\mu_p\nu_p}\end{aligned} \tag{C.9}$$

Inserting the factors of the dilaton field φ from (C.2) and (B.44):

$$\sqrt{-G} = \sqrt{-g}\mathrm{e}^{s(D)\varphi}, \quad G^{\mu\nu} = g^{\mu\nu}\mathrm{e}^{-s(D)\varphi}, \quad \hat{g}^{\mu\nu} = \mathrm{e}^{\varphi/\beta(D)}g^{\mu\nu} \tag{C.10}$$

we can write (in $D+1$ dimensions in the left-hand side and D dimensions in the right-hand side):

$$\begin{aligned}*\hat{F}^{(p+1)} \wedge \hat{F}^{(p+1)} &= \mathrm{d}^D x\ \mathrm{d}y\frac{1}{(p+1)!}\sqrt{-G}\hat{F}^{(p+1)}_{M_1\ldots M_{p+1}}\hat{F}^{(p+1)}_{N_1\ldots N_{p+1}}G^{M_1N_1}\ldots G^{M_{p+1}N_{p+1}}\\ &= \mathrm{e}^{-ps(D)\varphi}\left[*F^{(p+1)} \wedge \hat{F}^{(p+1)} + \mathrm{e}^{(D-1)s(D)\varphi} * F^{(p)} \wedge \hat{F}^{(p)}\right]\end{aligned} \tag{C.11}$$

8 Non-relativistic gravity

When we consider gravity in a non-relativistic setting, one can still apply techniques of gauge symmetries (similar to what we learned in [1, Ch. 11]). But one has to use other algebras, and related gauge fields. A review of last year is [11]. Compare [11, Sec. 2.1] with what we learned in [1, Ch. 7] about geometry. Then go to the new algebra. You can consider [11, (14)] as a rewriting of the basic equations for frame fields, using indices a only for the space directions in the local frame and defining

$$e_\mu^0 = c\tau_\mu, \quad e_0^\mu = c^{-1}\tau^\mu \tag{8.1}$$

In this way c^2 appears in the line element where you expect this. The non-relativistic limit is obtained by $c \to 0$, keeping τ_μ constant. See e.g. how [11, (15)] is obtained in this way. Can you also generalise this to what is called the 'Stringy Newton-Cartan geometry'?

Look further how many parts are a direct consequence of what we studied in [1, Ch. 11] using another algebra.

For [11, Sec.3] it suffices that you go through it without calculations. But make a table of the fields and symmetries. Which fields disappear as independent fields by the gauge fixings, and which symmetries remain. Which fields remain independent? Can you count the number of dof? [11, Sec. 4 and 5] are not necessary for this task.

PS: To get a general idea of these developments, you can also look to the slides of a talk of Eric Bergshoeff: https://fys.kuleuven.be/itf/events/francqui-antoniadis-2021/ericbleuven0607.pdf

9 Coset manifold

In supergravity, the scalars define a manifold that is in many cases a symmetric space. Such symmetric spaces are coset manifolds G/H for G the isometry group and H the isotropy group. A magical case is the structure of the scalar part of the $\mathcal{N}=8$ theory. The geometry of that manifold is determined by the Lie algebra structure. At the end even the metric (kinetic term in the Lagrangian) can be obtained from the structure of the algebra. The $\mathcal{N}=8$ theory is of course complicated, but we will use a simple example. A text on coset manifolds is in Appendix D. It contains the example already known from the lectures, but now defined from the algebra structure.

Read this text with the questions to you and make the theory explicit for the example. Though it is not absolutely necessary for this text, a knowledge of basics of Lie algebras as in my course of last year or a similar course can be helpful.

D Coset manifold

Coset manifolds appear in many supergravity theories. In fact, the previous sections of this text lead to examples. We study their geometry and at the end the relation with the action.

D. 1 Coset representatives

We first summarize the main ideas for coset manifolds G/H, and then give a simple example [19] For such manifolds, the operators of G transform a point to another point, and every point can be reached starting from any base point by an element of G. The coset elements are the equivalence classes of elements of G defined as $g \sim g'$ if $g' = gh$ for an $h \in H \subset G$. The manifold of these coset elements is described by fields ϕ^I, the coordinates of the manifold. For each point $\phi = \left\{\phi^I\right\}$, there is a certainly an $L(\phi) \in G$: an element of G that transforms the base point to ϕ. We will call this set $L(\phi)$ 'coset representatives' if it is unique element for the equivalence class: there is no $h \in H$ such that $L(\phi) = L\left(\phi'\right) h$ if $\phi \neq \phi'$.

To set the idea for $\mathcal{N}=8$ supergravity in 4 dimensions, the ϕ^I are the 70 scalar fields and we can introduce coset representatives as

$$L(\phi) = \exp\left(\phi^I T_I\right) \tag{D.1}$$

where T_I are 70 generators of E_7: the 7 generators of the Cartan subalgebra, and the 63 positive roots. This subalgebra is called the Borel subalgebra of $G = E_{7,7}$. The subalgebra H is in this case $H = \mathrm{SU}(8)$. Any object that is of the form (D.1) is then a coset representative. One can use a matrix representation of the group, so that $L(\phi)$ is a matrix (or work abstractly with the generators). The set of matrices $\{L(\phi)\}$ is only a subset of all matrices representing G. The multiplication with $g \in G$ on $L(\phi)$ should lead to another point in the manifold. A priori the formal product $gL(\phi)$ may lead to a matrix that is not in the set $\{L(\phi)\}$, but acting with an element $h \in H$ should bring it to a $L\left(\phi'\right)$: i.e.

$$L(\phi) \to L(\phi') : gL(\phi) = L(\phi')\, h(\phi, g) \tag{D.2}$$

where the $h(\phi, g)$ is chosen such that $L(\phi')$ is a coset representative. That the $L(\phi)$ uniquely represent the coset equivalence class means that this determines the $h(\phi, g)$ uniquely.

We analyze this equation now for the algebra of the groups: elements close to the identity. In this way we will find equations for the relations between the generators and the Killing

vectors. We define the algebra $\mathbb{G}$ corresponding to G as generated by operators T_A, and the algebra $\mathbb{H}$ corresponding to H as generated by M_i. Thus, for the example for $\mathcal{N} = 8$: the I run over 70 values as above, the A run over the 133 generators of E_7, and the i run over the 63 generators of $\mathfrak{su}(8)$. For g close to the identity, the $h(\phi, g)$ will be close to the identity and the infinitesimal part for any T_A can be expanded in the M_i, depending on the point in the coset manifold. This defines an embedding matrix $w_A{}^i(\phi)$, also called H-compensators:

$$g = 1 + \theta^A T_A, \quad \delta_\theta \phi^I = \phi^I - \phi'^I = \theta^A k_A{}^I(\phi), \quad h(\phi, g) = 1 - \theta^A w_A{}^i(\phi) M_i \tag{D.3}$$

Inserting this in first order in θ^A in (D.2), we get [21]

$$k_A L(\phi) = -T_A L(\phi) - L(\phi) w_A{}^i(\phi) M_i, \quad k_A \equiv k_A{}^I(\phi)\partial_I \tag{D.4}$$

Calculating the algebra using

$$[T_A, T_B] = f_{AB}^C T_C, \quad [M_i, M_j] = f_{ij}^k M_k \tag{D.5}$$

we get

$$\begin{aligned}
k_A k_B L(\phi) =& - T_B\left(-T_A L(\phi) - L(\phi) w_A^i(\phi) M_i\right) \\
& - \left(-T_A L(\phi) - L(\phi) w_A^i(\phi) M_i\right) w_B^j(\phi) M_j - L(\phi) k_A w_B^i(\phi) M_i \\
[k_A, k_B]\, L(\phi) =& - f_{AB}{}^C T_C L(\phi) \\
& + L(\phi) w_A{}^i(\phi) w_B{}^j(\phi) f_{ij}{}^k M_k - 2L(\phi) k_{[A} w_{B]}{}^i(\phi) M_i
\end{aligned} \tag{D.6}$$

The statement that L uniquely represents the coset equivalence class means that L and LM_i should be independent, we can split this relation in

$$\begin{aligned}
[k_A, k_B] &= f_{AB}{}^C k_C \\
2k_{[A} w_{B]}{}^k(\phi) &= f_{AB}{}^C w_C^k + w_A{}^i(\phi) w_B{}^j(\phi) f_{ij}{}^k.
\end{aligned} \tag{D.7}$$

One can define the one-forms

$$\Omega(\phi) = -L^{-1}(\phi) \mathrm{d} L(\phi) \tag{D.8}$$

which satisfy the 'Maurer-Cartan equation'

$$\mathrm{d}\Omega(\phi) - \Omega(\phi) \wedge \Omega(\phi) = 0 \tag{D.9}$$

From (D.2) and the definition D.8 it follows immediately that

[619] There are many reviews. An old one that clearly explains the ingredients in detail is 22. More recent treatments are e.g. in [18, Appendix A.4], [23, Sec. 2.1], [24, Sec. 11.3] [25, Sec. 9.4].

[20] Though we will write below the exponential as a product of exponentials.

$$\Omega\left(\phi'\right)=h(\phi,g)\Omega(\phi)h^{-1}(\phi,g)-h(\phi,g)\mathrm{d}h^{-1}(\phi,g) \tag{D.10}$$

Inserting in here the infinitesimal transformations (D.3) we find

$$\theta^A\mathcal{L}_{k_A}\Omega(\phi)=\Omega(\phi)-\Omega\left(\phi'\right)=\theta^A w_A{}^i\left[M_i,\Omega(\phi)\right]+\theta^A\;\mathrm{d}w_A{}^i M_i \tag{D.11}$$

We identified the difference of the 1-forms with the Lie derivativ [222] of Ω w.r.t. the Killing vector k^A. Indeed, for a form, taking into account also the translation of $\mathrm{d}\phi'$ in $\mathrm{d}\phi$, the difference $\omega^{(p)}(\phi)-\omega^{(p)}\left(\phi'\right)$ is in first order the Lie derivative $\mathcal{L}_k\omega^{(p)}(\phi)$. In fact, for scalar functions the Lie derivative reduces to $\mathcal{L}_k f(\phi)=i_k\;\mathrm{d}f(\phi)=k^I\partial_I f(\phi)$ and this is the operation that we for simplicity defined as k in (D.4).

Exercise: We can rewrite (D.4) by multiplying from the left with $L^{-1}(\phi)$ also as

$$i_{k_A}\Omega(\phi)=L^{-1}(\phi)T_A L(\phi)+w_A{}^i(\phi)M_i=0 \tag{D.12}$$

Apply d and use also (D.9) to find back (D.11) in this way.
Now you should read the above again considering the simple example that follows.

D. 2 A simple example: the coset $S\ell(2,\mathbb{R})/U(1)$

In [1, Sec. 7.12.2] we found the symmetries of the Poincaré plane, whose points are the z with positive imaginary parts by solving the Killing equation. That lead to the algebra $\mathfrak{s}\ell(2)$. Any point in the plane, is invariant under a one dimensional compact subalgebra. The action and 3 symmetries written there are (up to a renormalization of θ^2 for later convenience)

$$\mathcal{L}=2\frac{\partial_\mu Z\partial^\mu\bar{Z}}{(Z-\bar{Z})^2},\quad \delta_\theta Z=\theta^1-2\theta^2 Z+\theta^3 Z^2=\theta,\quad \operatorname{Im} Z>0 \tag{D.13}$$

Exercise: write for any point the combination of the θ^A that forms the isotropy generator. We write with real coordinates

$$Z=\chi+\mathrm{i}\mathrm{e}^{-\varphi},\quad \mathcal{L}=-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi-\frac{1}{2}\mathrm{e}^{2\varphi}\partial_\mu\chi\partial^\mu\chi \tag{D.14}$$

The obvious symmetries are [23]

$$\delta_{\theta,c}\varphi=2\theta,\quad \delta_{\theta,c}\chi=c-2\theta\chi,\quad [\delta_\theta,\delta_c]=\delta_{c'},\quad c'=2\theta c \tag{D.15}$$

We define generators H and E by declaring

$$\delta_{\theta,c}=\theta H+cE \tag{D.16}$$

Exercise: check that the commutator in (D.15) then translates to a commutator equal to the one between the matrices

$$H=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad E=\begin{pmatrix}0&1\\0&0\end{pmatrix} \tag{D.17}$$

[621] We define in (D.3) the Killing vector k_A and $\delta\phi$ as $\phi-\phi'$, rather than $\phi'=\phi+\delta\phi$ in [1, Sec. 7.12]. The role of ϕ and ϕ' are interchanged. In this way we obtain in (D.4) the minus sign between multiplication by matrices T_A and the action of the Killing vectors, which is similar to 1, (7.148)] and cares for the same algebra for Killing vectors and multiplication by matrices, see [1, Ex. 7.48].

[622] Definitions and a short introduction on the Lie derivative and its expression in terms of the insertion operator i is given in [1, Sec.7.3].

[23] The -2 is for convenience to get to standard normalizations of the algebra.

Together they form the 'Borel algebra' of $\mathfrak{sl}(2)$. We will describe the Poincaré plane as $S\ell(2,\mathbb{R})/SO(2))$.

Exercise: Relate θ^1 and θ^2 of (D.13) to the parameters in (D.16).
We write for the coset representative [24]

$$L(\varphi,\chi) = e^{-\chi E} e^{-\frac{1}{2}\varphi H} \tag{D.18}$$

Exercise: Visualize L as a 2×2 matrix. You can either work with the matrices, or use the general BCH formula

$$e^X Y e^{-X} = \exp(\mathrm{ad}\, X)Y = Y + [X,Y] + \frac{1}{2}[X,[X,Y]] + \ldots \tag{D.19}$$

to calculate Ω, expanding in $\mathrm{d}\chi$ and $\mathrm{d}\varphi$. Check the Maurer-Cartan equation.
Exercise: We first check that with these normalizations of the fields, the transformation (D.4) with $\phi = \{\varphi,\chi\}$ reproduces (D.15). We define the matrices T_1 and T_2 and the transformations (D.15) define 2 Killing vectors:

$$T_1 = H, \quad T_2 = E, \quad k_1 = 2\partial_\varphi - 2\chi\partial_\chi, \quad k_2 = \partial_\chi \tag{D.20}$$

Exercise: Check that they are related by (D.4), For these, we and that we do not need a compensating w_A:

$$w_1 = w_2 = 0 \tag{D.21}$$

Note that we obtained this agreement by writing in $L(\varphi,\chi)$, (D.18) the exponential with the root E in front of the one with H. Then (D.19) leads to the transformations as in (D.15).

We now complete the group to the full $S\ell(2,\mathbb{R})$, and introduce its maximal compact subgroup SO(2). For the algebra, this involves the introduction of the operator E^T, with root -2:

$$T_3 = E^T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \left[H, E^T\right] = -2E^T, \quad \left[E, E^T\right] = H \tag{D.22}$$

This is the negative root. The transpose can be defined independent of the matrix realization with declaring $H^T = H$, and defining the commutation relations of a step operator and its conjugate as in (D.22). The full algebra $\mathfrak{sl}(2)$ is thus generated by $\{T_A\} = \left\{H, E, E^T\right\}$. The compact subgroup is generated by just one M_i

$$M_1 = E - E^T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{D.23}$$

To find the action of E^T on the manifold we impose (D.4) for $A = 3$.
Exercise: Find $w_3^1(\varphi,\chi)$ and k_3 by imposing (D.4). Check that on Z this is proportional to the transformation that was missing from (D.13).

D. 3 Frame fields, metric and curvature

In the algebra $\mathbb{G}$ we can take a basis splitting the set $\{T_A\}$ in the basis of the subalgebra $\mathbb{H}$ $\{M_i\}$ and remaining $\{K_a\}$, spanning $\mathbb{K}$:

[624] Note that we split the exponential in (D.1) in two exponentials. If we would have written $\exp(\varphi H + \chi E)$ that would by the Baker-Campbell-Hausdorff (BCH) formulae just amount to a redefinition of the field χ. But with D.18) each exponential contains only one operator (below: only commuting operators), such that $\exp(-X)\mathrm{d}\exp X = \mathrm{d}X$

$$\mathbb{G} = \mathbb{H} \oplus \mathbb{K} \tag{D.24}$$

If this can be done such that their mutual commutators are in $\{K_a\}$, i.e.

$$[\mathbb{H}, \mathbb{K}] \subset \mathbb{K}, \quad [M_i, K_a] = f_{ia}{}^b K_b \tag{D.25}$$

i.e. $f_{ia}{}^j = 0$, this is called a 'reductive split'. In that case $\mathbb{K}$ is a representation of $\mathbb{H}$. We will further always assume this. In general

$$[K_a, K_b] = f_{ab}{}^i M_i + f_{ab}{}^c K_c \tag{D.26}$$

If the last term is zero, i.e.

$$f_{ab}{}^c = 0, \quad [\mathbb{K}, \mathbb{K}] \subset \mathbb{H} \tag{D.27}$$

then the coset is not only reductive but also called 'symmetric', as mentioned in [1, Sec. 12.5]. For all the supergravities with more than 8 supersymmetries, this is the case, see [1. Table 12.3].

Since L is an element of the group G, its derivative is in the algebra $\mathbb{G}$, and Ω is a one-form in the algebra of the group. For any reductive split, we write

$$\Omega(\phi) = e^a(\phi) K_a + \omega^i(\phi) M_i \tag{D.28}$$

where $e^a(\phi)$ and $\omega^i(\phi)$ are one-forms.
Exercise: Check that in our example of $\mathfrak{s}\ell(2)$, for a reductive split we should take

$$\{K_a\} = \left\{H, E + E^T\right\}, \quad M_1 = E - E^T, \quad \{T_A\} = \left\{H, E, E^T\right\} \tag{D.29}$$

From your earlier calculation of the one-form Ω identify e^a and ω^1. Consider [1, (7.81)] and identify the $\omega^a{}_b$ with a multiple of ω^1 here.

To find the transformations of these one-forms under the symmetries, one can split (D.11) using the commutators (D.25) and (D.5) and we get

$$\begin{aligned} \mathcal{L}_{k_A} e^a(\phi) &= e^b(\phi) w_A{}^i(\phi) f_{ib}{}^a \\ \mathcal{L}_{k_A} \omega_i(\phi) &= \mathrm{d}w_A{}^i + w_A{}^j(\phi)\omega^k(\phi) f_{jk}{}^i \end{aligned} \tag{D.30}$$

Using a symmetric tensor B_{ab} we can then define a metric

$$\mathrm{d}s^2 = B_{ab} e^a \otimes e^b \tag{D.31}$$

which is invariant if

$$f_{i(a}{}^c B_{b)c} = 0 \tag{D.32}$$

This condition is expressed in words as ' B_{ab} should be an $\mathbb{H}$-invariant metric on $\mathbb{K}$ '. With a reductive split, i.e. D.25), the part of the Cartan-Killing metric

$$K_{AB} = f_{AC}{}^D f_{BD}{}^C \to K_{ab} = -2 f_{c(a}{}^i f_{b)i}{}^c + f_{ac}{}^d f_{bd}{}^c \tag{D.33}$$

satisfies this requirement. Note that the last term in (D.33) is absent when the coset is symmetric.

Exercise: Check that for our $\mathfrak{s}l(2)$ example a B_{ab} proportional to δ_{ab} satisfies (D.32). Recognize the Lagrangian (D.14)

$$\mathcal{L} = -2\left(e^1 \otimes e^1 + e^2 \otimes e^2\right) \tag{D.34}$$

It is thus obtained in this way from the coset construction.
The Maurer-Cartan equation (D.9) can also be split and leads to differential equations on the frame field and the H-connection:

$$\begin{aligned} \mathrm{d}e^a - \frac{1}{2}e^b \wedge e^c f_{bc}{}^a + e^b \wedge \omega^i f_{ib}{}^a = 0 \\ \mathrm{d}\omega^i - \frac{1}{2}e^b \wedge e^c f_{bc}{}^i - \frac{1}{2}\omega^j \wedge \omega^k f_{jk}{}^i = 0 \end{aligned} \tag{D.35}$$

The first equation looks like [1, (7.81)], when we identify

$$\omega^a{}_b = -\omega^i f_{ib}{}^a, \quad T^a = \frac{1}{2}e^b \wedge e^c f_{bc}{}^a \tag{D.36}$$

Note that $f_{ic}{}^a B^{bc}$, with B^{ab} the inverse of B_{ab}, is antisymmetric. Hence this corresponds indeed to the situation in [1, Sec. 7.9]: an antisymmetric ω^{ab}. For the symmetric manifolds, $f_{bc}{}^a = 0$, we have no torsion. We consider now such symmetric manifolds. Then one can calculate the 2-form curvature [1, (7.117)]. The second term of the latter with the contribution of the last term in (D.35) lead to a Jacobi identity and vanishes for symmetric manifolds. We therefore obtain

$$\rho^a{}_b \equiv \mathrm{d}\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = -\frac{1}{2}e^c \wedge e^d f_{cd}{}^i f_{ib}{}^a. \tag{D.37}$$

The curvature tensor in flat indices is thus

$$R_{cd}{}^a{}_b = -f_{cd}{}^i f_{ib}{}^a. \tag{D.38}$$

Without torsion, the constancy of $R_{cd}{}^a{}_b$ implies that the curvature is covariantly constant, which is in fact another definition of symmetric spaces. The Ricci tensor is then

$$R_{ab} = f_{ac}{}^i f_{ib}{}^c = -\frac{1}{2}K_{ab} \tag{D.39}$$

which proves that they are Einstein spaces when we use the metric B_{ab} proportional to the Cartan-Killing metric.

10 Complex and Kähler manifolds

The Kähler geometry is the one that is defined by the scalar fields in $\mathcal{N} = 1$ supersymmetry and supergravity. This geometry is explained in the next chapter of the book that follows what we studied in the lectures (chapter 13).

Study this chapter and make the exercises.

11 General actions with $\mathcal{N} = 1$ supersymmetry

Study Chapter 14 of the book until and including Sec. 14.5.4. Explain in your talk what is the general possibility for an $\mathcal{N} = 1$ supersymmetric theory. In Sec. 14.3 some results of Chapter 13 are used. The only information that you need, is that if you consider $g_{\alpha\beta}$, obtained in (14.16) as a metric, then the geometry is described by a metric and connections as in $(13.17 - 22)$.

Solve the 10 exercises in that part.
PS: Look to the errata website of ch.14:
https://itf.fys.kuleuven.be/supergravity/index.php?id=14&type=ErrataCh14.html.

12 Supergravity from conformal methods

Read sections 15.1-6 of the book. to get acquainted with the conformal algebra and how it is used. It suffices to do Exercises 15.2 - 15.4.

Then go to Chapter 16. You may skip exercises $16.3, 16.5$ and 16.9 (level 3 exercises). Explain the construction of supergravity, and compare with the action that we obtained during the course (chapter 9).

Why do we get now a formulation with auxiliary fields? Count the bosonic and fermionic off-shell components and prove that they match.

13 Kaluza-Klein reduction of $D = 11$ on tori.

In [1, Table 12.3] appear the exceptional algebras. In particular for $\mathcal{N} = 8$ in $D = 4$ the isometry group E_7 appears. How do we get to that group is the subject of this task.

The text is in Appendix C (related to parts of the lecture notes of Chris Pope 'Lectures on Kaluza-Klein'. See his home page http://people.physics.tamu.edu/pope//. It is explained how the action of $D = 11$, which we constructed in the course, can be reduced on tori to lower dimensions. We concentrate on the bosonic sector. It is explained how the scalar sector is related to a coset in any dimension, and in case of $D = 4$ this is $E_{7,7}/\mathrm{SU}(8)$. There are some questions in the text. You should try to explain to the other students how this coset arises.

This subject should only be taken by someone who followed my lectures on Lie algebras, or a similar course where Dynkin diagrams and related subjects were explained.

C Reduction of scalar manifold from $D = 11$

We consider here, in the terminology of [1, Sec. 5.3] the consistent truncation to $k = 0$ for $D = 4$. This is consistent since any mode with $k \neq 0$ appears in the action after integrating over y multiplied by another mode with $k \neq 0$, and their field equations are thus consistently zero after the truncation. [13] Reduction is the consistent truncation mentioned above such that the D-dimensional fields will only depend on x^μ and not on y.

C. 1 Reduction from $D + 1$ to D dimensions

We treated this already for $D = 4$ in Sec. B.4. Consider that section starting from (B.44) until (B.62) for general D and prove that, rather than (B.61), we arrive at

$$\int \mathrm{d}y \frac{1}{2\kappa_{D+1}^2}\sqrt{-G}R(G) = \frac{1}{2\kappa_D^2}\sqrt{-g}\left[R(g) - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\mathrm{e}^{\varphi/\beta(D)}\right] \tag{C.1}$$

using rescalings

$$g_{\mu\nu} = \mathrm{e}^{\varphi/\beta(D)}\hat{g}_{\mu\nu}, \quad \beta(D) = -\sqrt{\frac{D-2}{2(D-1)}}, \quad \phi = \beta(D)\varphi \tag{C.2}$$

such that the action remains in Einstein frame. After an appropriate scaling the dilaton field has a standard kinetic term [14] and we have

$$\int \mathrm{d}y \frac{1}{2\kappa_{D+1}^2}\sqrt{-G}R(G) = \frac{1}{2\kappa_D^2}\sqrt{-g}\left[R(g) - \frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{4}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\mathrm{e}^{\varphi/\beta(D)}\right] \tag{C.3}$$

Here we use $\kappa_D^2 = \kappa_{D+1}^2 2\pi L$. In general: $\kappa_D^2 = \kappa^2(2\pi L)^{D-4}$, where $\kappa^2 = \kappa_4^2 = 8\pi G$ has dimension of length ^{2}. One can check that $\mathcal{A}_\mu$ is the gauge field of the part of the gct of $D+1$ dimensions ξ^M in the direction of the extra dimension:

$$\delta\left(\xi^D\right)\mathcal{A}_\mu = \partial_\mu \xi^D(x) - c \tag{C.4}$$

Another result from the previous part is that the length element is now
Check also that the length element is now

$$\begin{aligned} \mathrm{d}s_{D+1}^2 =& G_{MN}\,\mathrm{d}x^M\,\mathrm{d}x^N = \mathrm{e}^{s(D)\varphi} g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathrm{e}^{2\beta(D)}\left(\mathrm{d}y + \mathcal{A}^{(1)}\right)^2 \\ s(D) =& -\frac{2\beta(D)}{D-2} = -\frac{1}{(D-1)\beta(D)} = \sqrt{\frac{2}{(D-1)(D-2)}} \end{aligned} \tag{C.5}$$

For p-form fields in $D+1$ dimensions $\hat{A}^{(p)} = \frac{1}{p!}\hat{A}^{(p)}_{M_1\ldots M_p}\mathrm{d}x^{M_1}\ldots\mathrm{d}x^{M_P}$, we reduce them to D dimensions as

$$\hat{A}^{(p)}_{\mu_1\ldots\mu_p} = A^{(p)}_{\mu_1\ldots\mu_p}, \quad \hat{A}^{(p)}_{\mu_1\ldots\mu_{p-1}D} = A^{(p-1)}_{\mu_1\ldots\mu_{p-1}}, \quad \hat{A}^{(p)} = A^{(p)} + A^{(p-1)} \wedge \mathrm{d}y \tag{C.6}$$

where the last expression is the same as the first two in form language.
The field strength in $D+1$ dimensions are divided in D dimensions in two parts using the 1-form that appears in (C.5):

$$\begin{aligned} \hat{F}^{(p+1)} &= \mathrm{d}\hat{A}^{(p)} = A^{(p+1)} + F^{(p)} \wedge \left(\mathrm{d}y + \mathcal{A}^{(1)}\right) \\ F^{(p+1)} &= \mathrm{d}A^{(p)} - \mathrm{d}A^{(p-1)} \wedge \mathcal{A}^{(1)}, \quad F^{(p)} = \mathrm{d}A^{(p-1)} \end{aligned} \tag{C.7}$$

In component language, the first line is written as

$$\hat{F}^{(p+1)}_{\mu_1\ldots\mu_{p+1}} = F^{(p+1)}_{\mu_1\ldots\mu_{p+1}} + (p+1)F^{(p)}_{[\mu_1\ldots\mu_p}\mathcal{A}_{\mu_{p+1}]}, \quad \hat{F}^{(p+1)}_{\mu_1\ldots\mu_p D} = F^{(p)}_{\mu_1\ldots\mu_{p+1}}, \tag{C.8}$$

The $\mathcal{A}^{(1)} = \mathcal{A}_\mu \mathrm{d}x^\mu$ terms have been introduced in the latter equations such that $F^{(p+1)}$ and $F^{(p)}$ are inert under ξ^D gct. But moreover, these definitions with (B.44) imply that

$$\begin{aligned} &\hat{F}^{(p+1)}_{M_1\ldots M_{p+1}}\hat{F}^{(p+1)}_{N_1\ldots N_{p+1}}\hat{G}^{M_1N_1}\ldots\hat{G}^{M_{p+1}N_{p+1}} = \\ &= F^{(p+1)}_{\mu_1\ldots\mu_{p+1}}F^{(p+1)}_{\nu_1\ldots\nu_{p+1}}\hat{g}^{\mu_1\nu_1}\ldots\hat{g}^{\mu_{p+1}\nu_{p+1}} + (p+1)F^{(p)}_{\mu_1\ldots\mu_p}F^{(p)}_{\nu_1\ldots\nu_p}\hat{g}^{\mu_1\nu_1}\ldots\hat{g}^{\mu_p\nu_p} \end{aligned} \tag{C.9}$$

Inserting the factors of the dilaton field φ from (C.2) and (B.44):

$$\sqrt{-G} = \sqrt{-g}\mathrm{e}^{s(D)\varphi}, \quad G^{\mu\nu} = g^{\mu\nu}\mathrm{e}^{-s(D)\varphi}, \quad \hat{g}^{\mu\nu} = \mathrm{e}^{\varphi/\beta(D)}g^{\mu\nu} \tag{C.10}$$

we can write (in $D+1$ dimensions in the left-hand side and D dimensions in the right-hand side):

$$\begin{aligned} *\hat{F}^{(p+1)} \wedge \hat{F}^{(p+1)} &= \mathrm{d}^D x\,\mathrm{d}y\frac{1}{(p+1)!}\sqrt{-G}\hat{F}^{(p+1)}_{M_1\ldots M_{p+1}}\hat{F}^{(p+1)}_{N_1\ldots N_{p+1}}G^{M_1N_1}\ldots G^{M_{p+1}N_{p+1}} \\ &= \mathrm{e}^{-ps(D)\varphi}\left[*F^{(p+1)} \wedge \hat{F}^{(p+1)} + \mathrm{e}^{(D-1)s(D)\varphi} * F^{(p)} \wedge \hat{F}^{(p)}\right] \end{aligned} \tag{C.11}$$

[613] In the preparation of this part, I used a lot 19, 20, 17. An earlier review of the methods was 21].
[14] In principle to have the standard kinetic terms for the scalar and the vector, we should still redefine them by a factor $\sqrt{2}\kappa_D$. But we prefer to keep working with the fields without a mass dimension.

C. 2 Reduction from $D=11$ to $D=4$

We start from the bosonic part of the action in [1, (10.26)], but will eliminate the $\sqrt{2}$ factors by rescaling the 3 -form there to $\sqrt{2}$ times the 3 -form here. We iterate the previous procedure. Starting from the reduction of $D=11$ to $D=10$ and iterating, we use a counting number $i=1,...,7$ to go to 4 dimensions. Fields that appear for the first time in $D=11-i$ get an extra index i.

We have seen that the metric in $D+1$ dimensions gives rise to a scalar φ and the 1-form $\mathcal{A}^{(1)}$, which, if we do this in step i we will call φ_i and $\mathcal{A}^{(1)i}$. We saw that a p-form gives rise to a p-form and a $p-1$ form in the next (lower) dimension. When we iterate this, e.g. the 1 -form $\mathcal{A}^{(1)i}$ gives rise to a zero form (scalar) $\mathcal{A}^{(0)i}{}_j$ in the further steps $j>i$. Thus we write that the 1 -form that is obtained from the metric at step i is

$$\hat{\mathcal{A}}^{(1)i}=\mathcal{A}^{(1)i}+\mathcal{A}^{(0)i}{}_j\,\mathrm{d}y^j. \tag{C.12}$$

To clarify, we give the first two steps explicitly:

$$\begin{array}{r|llllllllll}
D=11 & g_{\mu\nu}, & & & & & & A^{(3)}, & & & \\
D=10 & g_{\mu\nu}, & & & \varphi_1, & \mathcal{A}^{(1)1}, & & A^{(3)}, & & A^{(2)}_1, & \\
D=9 & g_{\mu\nu}, & \varphi_2, & \mathcal{A}^{(1)2}, & \varphi_1, & \mathcal{A}^{(1)1}, & \mathcal{A}^{(0)1}{}_2, & A^{(3)}, & A^{(2)}_2, & A^{(2)}_1, & A^{(1)}_{12}
\end{array}$$

We thus obtain the full set of fields for $D=4$:

$$\begin{array}{l}
g_{\mu\nu}\\
\varphi_i \text{ for } i=1,...,7, \text{ and further often written as } \vec{\varphi},\\
\mathcal{A}^{(1)i} \text{ for } i=1,...,7, \text{ with field strenghts } \mathcal{F}^{(2)}_i,\\
\mathcal{A}^{(0)i}{}_j \text{ for } i<j, 21 \text{ scalars with } \mathcal{F}^{(1)i}{}_j=\mathrm{d}\mathcal{A}^{(0)i}{}_j,\\
A^{(3)} \text{ which in } D=4 \text{ does not correspond to a propagating field,}\\
A^{(2)}_i \text{ which in } D=4 \text{ are dualized to 7 scalars, which we will call } \chi^i,\\
A^{(1)}_{ij}, \text{ defining also } A^{(1)}_{ji}=-A^{(1)}_{ij}, 21 \text{ vectors,}\\
A^{(0)}_{ijk}, \text{ defining also } A^{(1)}_{ijk}=A^{(1)}_{[ijk]}, \text{ another set of 35 scalars.}
\end{array} \tag{C.14}$$

These thus give indeed the fields that we expect for $\mathcal{N}=8$ as in [1, Table 6.1]. We want to write the bosonic action in detail.

Note the position of the i index on $\mathcal{A}$ fields. In the reduction of the metric they appear in the form $g_{i\mu}\propto g_{ij}\mathcal{A}^j_\mu$. When forms are reduced to lower forms there is a factor $\mathrm{d}y^i$ and the field thus gets a lower i index. In fact, the consistency of the reduction to y-independent fields, now implies that components of the general coordinate transformations from higher D in direction i reduce to $\xi^i(x,y)=\xi^i(x)+c^i{}_jy^j$ (compare with (C.4) where the first term is the gauge transformation of $\mathcal{A}^{(1)i}$, and the second defines a rigid $\mathrm{g}\ell(7,\mathbb{R})$ transformation.

The 3 -form $\hat{A}^{(3)}$ from $D=11$ is then written as

$$\hat{A}^{(3)}=A^{(3)}+A^{(2)}_i\wedge\mathrm{d}y^i+\frac{1}{2}A^{(1)}_{ij}\wedge\mathrm{d}y^i\wedge\mathrm{d}y^j+\frac{1}{6}A^{(0)}_{ijk}\mathrm{d}y^i\wedge\mathrm{d}y^j\wedge\mathrm{d}y^k \tag{C.15}$$

From the transformation $\delta\hat{A}^{(p)}=\mathrm{d}\Lambda^{(p-1)}(x,y)$, the part that is compatible with the truncation to 4 dimensions are the transformations like $\delta A^{(3)}=\mathrm{d}\Lambda^{(2)}(x)$, and for the zero forms $\delta A^{(0)}_{ijk}=c_{ijk}$,

$$\delta A^{(0)}_{ijk} = c_{ijk}, \quad \delta \mathcal{A}^{(0)i}{}_j = c^i{}_j, \tag{C.16}$$

where $c^i{}_j$ (with $i > j$) and c_{ijk} (antisymmetric) are constants.
In order to reduce the action iteratively with equations similar to (C.11) we define the components of forms in lower dimension similar to (C.7). However, the 1-form $\mathcal{A}^{(1)}$ is in each lower dimension redefined as in C.6). We therefore write

$$h^i = \mathrm{d}y^i + \mathcal{A}^{(1)i} + \mathcal{A}^{(0)i}{}_j \; \mathrm{d}y^j = \tilde{\gamma}^i{}_j \; \mathrm{d}y^j + \mathcal{A}^{(1)i}, \quad \tilde{\gamma}^i{}_j = \delta^i_j + \mathcal{A}^{(0)i}{}_j, \tag{C.17}$$

or the inverse

$$\begin{aligned} &\mathrm{d}y^i = \gamma^i{}_j h^j - \mathcal{A}^{(1)i}, \quad \gamma^i{}_j \tilde{\gamma}^j{}_k = \delta^i_k, \\ &\gamma^i{}_j = \delta^i{}_j - \mathcal{A}^{(0)i}{}_j + \mathcal{A}^{(0)i}{}_k \mathcal{A}^{(0)k}{}_j + ..., \end{aligned} \tag{C.18}$$

where the series is finite since e.g. the last term appears only for k such that $i < k < j$. We define the forms in lower D from the 4 -form $\hat{F}^{(4)}$ of $D = 11$ as

$$\begin{aligned} \hat{F}^{(4)} &= \mathrm{d}\hat{A}^{(3)} = \mathrm{d}A^{(3)} + \mathrm{d}A^{(2)}_i \wedge \mathrm{d}y^i + \frac{1}{2}\; \mathrm{d}A^{(1)}_{ij} \wedge \mathrm{d}y^i \wedge \mathrm{d}y^j + \frac{1}{6}\; \mathrm{d}A^{(0)}_{ijk} \mathrm{d}y^i \wedge \mathrm{d}y^j \wedge \mathrm{d}y^k \\ &= F^{(4)} + F^{(3)}_i \wedge h^i + \frac{1}{2} F^{(2)}_{ij} \wedge h^i \wedge h^j + \frac{1}{6} F^{(1)}_{ijk} \wedge h^i \wedge h^j \wedge h^k \end{aligned} \tag{C.19}$$

Similarly we define the forms $\mathcal{F}$:

$$\begin{aligned} \hat{\mathcal{F}}^{(2)i} &= \mathrm{d}\hat{\mathcal{A}^{(1)i}} = \mathrm{d}\mathcal{A}^{(1)i} + \mathrm{d}\mathcal{A}^{(0)i}{}_j \; \mathrm{d}y^j \\ &= \mathcal{F}^{(2)i} + \mathcal{F}^{(1)i}{}_j h^j. \end{aligned} \tag{C.20}$$

Some examples (a full list of forms is given in [19, 20, 17]), related to the scalars that we will discuss below, are

$$\begin{aligned} &F^{(1)}_{ijk} = \left(\mathrm{d}A^{(0)}_{\ell mn}\right) \gamma^\ell{}_i \gamma^m{}_j \gamma^n{}_k, \quad \mathcal{F}^{(1)i}{}_j = \left(\mathrm{d}\mathcal{A}^{(0)i}{}_k\right) \gamma^k{}_j, \\ &\text{e.g. } \mathcal{F}^{(1)i}{}_{i+1} = \mathrm{d}\mathcal{A}^{(0)i}{}_{i+1}, \quad \mathcal{F}^{(1)i}{}_{i+2} = \mathrm{d}\mathcal{A}^{(0)i}{}_{i+2} - \left(\mathrm{d}\mathcal{A}^{(0)i}{}_{i+i}\right) \mathcal{A}^{(0)i+1}{}_{i+2}. \end{aligned} \tag{C.21}$$

With these definitions, the actions in lower dimensions depend (apart from the Chern-Simons term) only on the forms F and $\mathcal{F}$ as in (C.3). We obtain the dilaton factors from (C.3) and (C.11) with $D = 11 - i$. Defining

$$\vec{s} = \{s_i\}, \quad s_i = s(11-i) = \sqrt{2/(10-i)(9-i)}, \quad \vec{\varphi} = \{\varphi_i\} \tag{C.22}$$

we see from C.11 that a p-form reduced to a lower dimension gets a factor $\mathrm{e}^{-p\vec{s}\cdot\vec{\varphi}}$. Going to the lower form (adding an index i) gives immediately a factor $\mathrm{e}^{(D-1)s(D)\varphi}$, which at level i is $\mathrm{e}^{(10-i)s_i\varphi_i}$. For the further reductions we have to take into account that p has diminished by 1. This is thus a correction with a factor $\mathrm{e}^{\overrightarrow{f_i}\cdot\varphi}$ with

$$\overrightarrow{f_i} = \{0, 0, ..., 0, (10-i)s_i, s_{i+1}, s_{i+2}, ...\} \text{ with the first nonzero entry in position } i \tag{C.23}$$

On the other hand, for the $\mathcal{A}$ fields, there is first in (C.3) a factor for the $*\mathcal{F}^{(2)i} \wedge \mathcal{F}^{(2)i}$:

$$e^{\varphi/\beta(D)} = e^{-(D-1)s(D)\varphi_{11-D}} \to e^{-(f_i)i\varphi_i} \tag{C.24}$$

When reduced to the $*\mathcal{F}^{(1)i}{}_j \wedge \mathcal{F}^{(1)i}{}_j$ there are the extra factors as above. We can thus write the bosonic Lagrangian [15] as [19, 20, 17]

$$\begin{aligned}
\mathcal{L} = R * 1 - \frac{1}{2}\Bigg[& \mathrm{d}\vec{\varphi} \cdot \wedge \mathrm{d}\vec{\varphi} + e^{\vec{a}\cdot\vec{\varphi}} * F^{(4)} \wedge F^{(4)} + \sum_i e^{\vec{a}_i\cdot\vec{\varphi}} * F_i^{(3)} \wedge F_i^{(3)} \\
& + \sum_{i<j} e^{\vec{a}_{ij}\cdot\vec{\varphi}} * F_{ij}^{(2)} \wedge F_{ij}^{(2)} + \sum_{i<j<k} e^{\vec{a}_{ijk}\cdot\vec{\varphi}} * F_{ijk}^{(1)} \wedge F_{ijk}^{(1)} \\
& + \sum_i e^{\vec{b}^i\cdot\vec{\varphi}} * \mathcal{F}^{(2)i} \wedge \mathcal{F}^{(2)i} + \sum_{i<j} e^{\vec{b}^i\cdot\cdot\vec{\varphi}} * \mathcal{F}^{(1)i}{}_j \wedge \mathcal{F}^{(1)i}{}_j \Bigg] \\
& + \mathcal{L}_{FFA}.
\end{aligned} \tag{C.25}$$

where

$$\begin{aligned}
&\vec{a} = -3\vec{s}, \quad \vec{a}_i = -3\vec{s} + \vec{f}_i, \quad \vec{a}_{ij} = -3\vec{s} + \vec{f}_i + \vec{f}_j \\
&\vec{a}_{ijk} = -3\vec{s} + \vec{f}_i + \vec{f}_j + \vec{f}_k, \quad \vec{b}^i = -\vec{f}_i, \quad \vec{b}_{jj} = -\vec{f}_i + \vec{f}_j
\end{aligned} \tag{C.26}$$

There are a few identities that will be used below 16

$$\vec{b^i}_j + \vec{b^j}_k = \vec{b^i}_k, \tag{C.27}$$

$$\sum_{i=1}^{9} \vec{f}_i = 9\vec{s} \tag{C.28}$$

$$\vec{a}_{k\ell m} + \vec{a}_{npq} = -\vec{a}_j \varepsilon_{jk\ell mnpq} \text{ for } k, \ell, m, n, p, q \text{ all different.} \tag{C.29}$$

$$\vec{s}\cdot\vec{s} = \frac{7}{9}, \quad \vec{s}\cdot\vec{f}_i = 1, \quad \vec{f}_i\cdot\vec{f}_j = 2\delta_{ij} + 1 \tag{C.30}$$

Exercise: Consider the vectors that appear in a reduction from $D = 11$ to $D = 8$, i.e. on T^3, and are related to a scalar sector (1-form field strength). Consider their inner products, and recognize them as positive roots of an algebra.

For the Chern-Simons term, defining

$$\mathrm{d}y^i \wedge \mathrm{d}y^j \wedge \mathrm{d}y^k \wedge \mathrm{d}y^\ell \wedge \mathrm{d}y^m \wedge \mathrm{d}y^n \wedge \mathrm{d}y^p = \varepsilon^{ijk\ell mnp}\, \mathrm{d}^7 y \tag{C.31}$$

we obtain that [1, (10.24)] gives rise in $D = 4$ to

$$\begin{aligned}
\mathcal{L}_{FFA} &= -\frac{1}{6}\, \mathrm{d}\hat{A}^{(3)} \wedge \mathrm{d}\hat{A}^{(3)} \wedge \hat{A}^{(3)} \\
&= \left(-\frac{1}{48}\, \mathrm{d}A_{ij}^{(1)} \wedge \mathrm{d}A_{k\ell}^{(1)} A_{mnp}^{(0)} - \frac{1}{72}\, \mathrm{d}A_i^{(2)} A_{jk\ell}^{0} \wedge \mathrm{d}A_{mnp}^{(0)} \right) \varepsilon^{ijk\ell mnp}\, \mathrm{d}^7 y
\end{aligned} \tag{C.32}$$

We will further insert the $\int \mathrm{d}^7 y$ in the normalization from the κ_{11} to the κ_4.

[615] We write $S = \frac{1}{2\kappa^2}\int \mathcal{L}$.

[16] In 19, 20, 17] the generalization of all these equations has been written down when one reduces to $D > 4$ or even $D = 3$.

C. 3 Scalars and their isometries

We will now first consider the scalar sector and its isometries. Let us first consider the scalars that are already scalars without dualization. Thus we consider the Lagrangian

$$-2\mathcal{L}_0 = *\,\mathrm{d}\vec{\varphi}\cdot\wedge\mathrm{d}\vec{\varphi} + \sum_{i<j<k} \mathrm{e}^{\vec{a}_{ijk}\cdot\vec{\varphi}} * F^{(1)}_{ijk}\wedge F^{(1)}_{ijk} + \sum_{i<j} \mathrm{e}^{\vec{b}_i}\cdot\vec{\varphi} * \mathcal{F}^{(1)i}{}_j \wedge \mathcal{F}^{(1)i}{}_j \tag{C.33}$$

First remark that the symmetries C.16 have to be adapted with more transformations of the c's on other scalars such that the forms are invariant. See e.g. from (C.21) that $c^{i+1}{}_{i+2}$ would not leave $\mathcal{F}^{(1)i}{}_{i+2}$ invariant. Hence this transformation should also act on $\mathcal{A}^{(0)i}{}_{i+2}$. This is solved by defining (for simplicity, for the discussion of the scalars we omit the indication (0) on $\mathcal{A}^{(0)}{}_j$ and $A^{(0)}_{ijk}$)

$$\begin{aligned}
\delta_c\tilde{\gamma}^i{}_j = \delta_c\mathcal{A}^i{}_j = \tilde{\gamma}^i{}_k c^k{}_j = c^i{}_j + \mathcal{A}^i{}_k c^k{}_j,&\\
\rightarrow \delta_c\gamma^i{}_j = -c^i{}_k\gamma^k{}_j, \quad \delta_c\,\mathrm{d}\mathcal{A}^i{}_j = \left(\mathrm{d}\mathcal{A}^i{}_k\right)c^k{}_j,&\\
\delta_c A_{ijk} = c_{ijk} + 3A_{\ell[jk}c^\ell{}_{i]},&
\end{aligned} \tag{C.34}$$

which then leave the forms invariant. These $c^i{}_j$ thus act as the upper triangular part of $g\ell(7,\mathbb{R})$, mentioned above as remaining part of the general coordinate transformations of $D=11$.

Similarly the kinetic term of the scalars $\vec{\varphi}$ is invariant under translations, but because of the exponential factors in C.33 that transformation should also act on the other scalars. We can write (with no sums over i indices) [17]

$$\delta_\theta\vec{\varphi} = 2\vec{\theta}, \quad \delta_\theta\mathcal{A}^i{}_j = -\vec{b}^i{}_j\cdot\vec{\theta}\mathcal{A}^i{}_j, \quad \delta_\theta A_{ijk} = -\vec{a}_{ijk}\cdot\vec{\theta}A_{ijk}. \tag{C.35}$$

The construction of $\gamma^i{}_j$ in C.18 and C.27) imply that $\delta_\theta\gamma^i{}_j = -\vec{bi_j}\cdot\vec{\theta}\gamma^i{}_j$ and $\delta_\theta\mathcal{F}^{(1)i}{}_j = -\vec{b^i}_j\cdot\vec{\theta}\mathcal{F}^{(1)i}{}_j$.

As mentioned in (C.14) there are 7 more scalars to obtain from dualization of the two-forms $A^{(2)}_i$. Indeed these fields appear in the action only as $\mathrm{d}A^{(2)}_i$ (we wrote C.32 in such a way that this is obvious). In the 3 -form according to (C.18 we have

$$\mathrm{d}A^{(2)}_i = F^{(3)}_j\tilde{\gamma}^j{}_i + \text{ parts depending on the vectors } \mathcal{A}^{(1)} \tag{C.36}$$

Therefore we dualize these fields by inserting Lagrange multipliers χ^i to impose that the first line above is a closed form. Copying the last term on the first line in (C.25) and adding the second term of C.32 writing $G^{(1)}_i = -*F^{(3)}_i$

$$\mathcal{L}_\chi = -\frac{1}{2}\sum_i \mathrm{e}^{\vec{a}_i\cdot\vec{\varphi}}G^{(1)}_i\wedge *G^{(1)}_i + \frac{1}{72} * G^{(1)}_q\tilde{\gamma}^q{}_i A_{jk\ell}\wedge\mathrm{d}A_{mnp}\varepsilon^{ijk\ell mnp} + *G^{(1)}_j\tilde{\gamma}^j{}_i\,\mathrm{d}\chi^i \tag{C.37}$$

Then we consider $G^{(1)}_i$ as an independent field whose field equation gives

$$\mathrm{e}^{\vec{a}_i\cdot\vec{\varphi}}G^{(1)}_i = \tilde{\gamma}^i{}_j\left(\,\mathrm{d}\chi^j + \frac{1}{72}A_{k\ell m}\wedge\mathrm{d}A_{npq}\varepsilon^{jk\ell mnpq}\right) \tag{C.38}$$

[617] So far, the implicit summation convention was always used or sums were explicitly indicated as in C.33). Below there will be various formulas where this is not the case. Sums are still assumed when the indices are consistently appearing as up-down, but not if they appear in other ways.

and C.37) reduces to

$$\mathcal{L}_\chi = -\frac{1}{2}\sum_i \mathrm{e}^{-\vec{a}_i\cdot\vec{\varphi}} * G_i^{(1)} \wedge G_i^{(1)} \tag{C.39}$$

There are shift symmetries acting on the χ^i (we take parameters c^i) and to respect the transformations (C.34 keeping $G_i^{(1)}$ invariant, the χ^i should transform also under these:

$$\delta_c\chi^i = c^i - \frac{1}{72}c_{k\ell m}A_{npq}\varepsilon^{ik\ell mnpq} - c^i{}_j\chi^j \tag{C.40}$$

Also the θ transformations from C.35 are still respected due to C.28). Therefore the second term in (C.38) gets also a θ transformation proportional to $\vec{a}_{k\ell m} + \vec{a}_{npq} = -\vec{a}_j$, (see (C.29) and we can define

$$\delta_\theta\chi_i = \vec{\theta}\cdot\vec{a}_i\chi_i, \rightarrow \delta_\theta G_i^{(1)} = \vec{\theta}\cdot a_i G_i^{(1)} \tag{C.41}$$

such that (C.39) is invariant.
Let us summarize: we have the scalars and isometries: (each time with $i<j<k$)

$$\begin{aligned} \text{scalars} &: \vec{\varphi}, \mathcal{A}^i{}_j, A_{ijk}, \chi^i \\ \text{parameters} &: \vec{\theta}, c^i{}_j, c_{ijk}, c^i \\ \text{generators} &: \vec{H}, E_i{}^j, E^{ijk}, E_i \end{aligned} \tag{C.42}$$

where the generators are defined such that the transformations are

$$\delta_{c,\theta} = \vec{\theta}\cdot\vec{H} + \sum_{i>j} c^i{}_j E_i{}^j + \sum_{i>j>k} c_{ijk}E^{ijk} + \sum_i c^i E_i \tag{C.43}$$

Note that we could write this with the implicit summation convention as

$$\delta_{c,\theta} = \vec{\theta}\cdot\vec{H} + c^i{}_j E_i{}^j + \frac{1}{6}c_{ijk}E^{ijk} + c^i E_i \tag{C.44}$$

since $E_i{}^j$ and $c^i{}_j$ exist only for $i>j$, while c_{ijk} is defined completely antisymmetric.
We have thus identified translation symmetries for all the scalar fields. This implies that the manifold of scalars is a homogeneous space. We first obtain the algebra of the symmetries. The $\vec{\theta}$ transformations mutually commute. The c transformations have the algebra

$$\begin{aligned} &[\delta_c, \delta_{c'}] = \delta_{c''}, \\ &\quad c''{}_j = c^i{}_k c'^k{}_j - (c \leftrightarrow c')\,, \quad c''_{ijk} = 3c_{\ell[ij}c'^\ell{}_{k]} - (c \leftrightarrow c')\,, \\ &\quad c''^i = c^i{}_j c'^j + \frac{1}{72}\varepsilon^{iklmnpq}c_{k\ell m}c'_{npq} - (c \leftrightarrow c')\,, \end{aligned} \tag{C.45}$$

and, with no sums over indices:

$$\begin{aligned} &[\delta_\theta, \delta_c] = \delta_{c'} \\ &c'^i{}_j = c^i{}_j\vec{\theta}\cdot\vec{b}^i{}_j, \quad c'_{ijk} = c_{ijk}\vec{\theta}\cdot\vec{a}_{ijk}, \quad c'^i = -c^i\vec{\theta}\cdot\vec{a}_i. \end{aligned} \tag{C.46}$$

We have the commutation relations of the operators (no sums over indices)

$$
\begin{gathered}
\left[\vec{H}, E_i{}^j\right] = \vec{b}^i{}_j E_i{}^j, \quad \left[\vec{H}, E^{ijk}\right] = \vec{a}_{ijk} E^{ijk}, \quad \left[\vec{H}, E_i\right] = -\vec{a}_i E_i \\
\left[E_i{}^j, E_k{}^\ell\right] = \delta^j_k E^\ell_i - \delta^\ell_i E_k{}^j, \quad \left[E^{ijk}, E_\ell{}^m\right] = \delta^i_\ell E^{jkm} + \delta^j_\ell E^{kim} + \delta^k_\ell E^{ijm} \\
\left[E_i{}^j, E_k\right] = \delta^j_k E_i, \quad \left[E^{ijk}, E^{\ell mp}\right] = \varepsilon^{ijk\ell mpq} E_q
\end{gathered} \tag{C.47}
$$

The $\vec{H}$ operators form a Cartan subalgebra: they mutually commute and the commutation relations with other generators are diagonalized. We have 63 roots, which we will call 'positive' [18]

$$
\alpha^A = \left\{\vec{b}^i{}_j (i < j), \vec{a}_{ijk} (i < j < k), -\vec{a}_i\right\}. \tag{C.48}
$$

We can find 7 simple roots: positive roots chosen such that the others are sums with positive integer coefficients of the simple roots. These are

$$
\vec{b}^i{}_{i+1} \text{ and } \vec{a}_{123}. \tag{C.49}
$$

Indeed e.g.

$$
\begin{gathered}
\vec{b}^i{}_{i+2} = \vec{b}^i{}_{i+1} + \vec{b}^{i+1}{}_{i+2}, \quad \vec{a}_{125} = \vec{a}_{123} + \vec{b}^3{}_4 + \vec{b}^4{}_5, \\
-\vec{a}_7 = 2\vec{a}_{123} + \vec{b}^1{}_2 + 2\vec{b}^2{}_3 + 3\vec{b}^3{}_4 + 2\vec{b}^4_5 + \vec{b}^5{}_6, \quad -\vec{a}_6 = -\vec{a}_7 + \vec{b}^6{}_7,
\end{gathered} \tag{C.50}
$$

\footnotetext{
[18] Here taken as: the last non-zero component is positive.

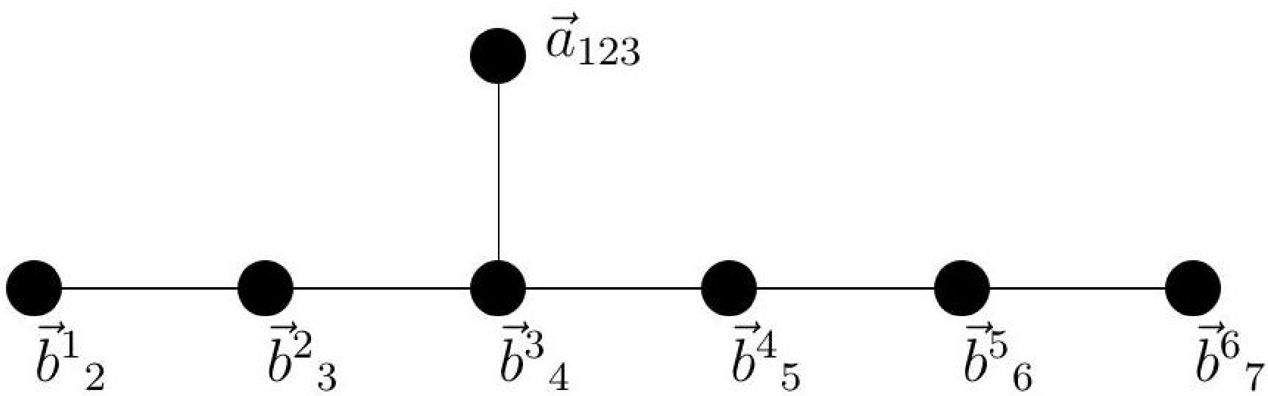
It also implies that in the full commutation rules the operators of the simple roots do not appear at the right-hand side apart from their commutation with $\vec{H}$.

Exercise: To be sure that the inner product - agrees with the Cartan-Killing metric check that the components of the roots in this basis satisfy

$$
\sum_A \left(\alpha^A\right)^i \left(\alpha^A\right)^j = x\delta^{ij} \tag{C.51}
$$

For which value of x? This implies that we can use inner products of the vectors to determine the relation between the simple roots. Check that all the roots have equal length.

Exercise: Calculate the inner products between the simple roots (C.49) and find that they can schematically be represented by the Dynkin diagram



This diagram defines the Lie algebra E_7. So far, we saw the generators of the Cartan subalgebra and the positive roots of that algebra, which is called the 'Borel subalgebra'. We are still missing the generators corresponding to the 63 negative roots. But it is known that the exponential of the Borel subalgebra defines a 70 -dimensional coset space, which should then be $E_{7,7}/\mathrm{SU}(8)$.

Exercise: can you see how this would have been if we reduced to dimensions bigger than 4? No details, just the facts that lead to understanding other parts of [1, Table 12.3].

14 The matter-coupled $\mathcal{N} = 1$ supergravity

In chapter 17 of the book, the couplings of the main $\mathcal{N} = 1$ multiplets (those that we saw in chapter 6) to supergravity has been derived. This involves methods that we did not study in the course, related to the superconformal approach. We will restrict ourselves to the result of that procedure (chapter 18) and some of its applications (chapter 19). Read these two chapters and make the exercises of chapter 18 (exercises concerning the potential for the scalars in these theories) and at least 3 exercises of chapter 19.
P.S.: Exercise 19.1 is a simpler version of what is done in [12, Sec. 2]. You may find help by looking into that paper.

15 Classical solutions of gravity and supergravity

Study the sections 22.1-4 and make the corresponding exercises, but you can skip Ex. 22.1, 22.8, 22.9 and 22.19

PS: In exercise 22.15 and 22.17 you may use $t_0 t_1 = -1$ as for the dimensions that we are mostly using.

16 Supersymmetric Black Holes

These are treated in sections $22.5-8$ in the book. Read these sections and solve the corresponding exercises. You may skip the exercises that are indicated with 3 bars ('level 3'): 22.21, 22.24, 22.26

The material is an application of what we saw in Section 4.2 and in Chapter 7 (until Sec. 7.10). For sections 22.6.1-2 also the basis knowledge of γ-matrices and spinors is needed. It can also be seen as an extension of Sec. 12.6.

Do not forget to check the errata on
http://itf.fys.kuleuven.be/supergravity/index.php?id=22&type=ErrataCh22.html
PS: In ex 22.24, note that 'is essentially...' is not 'is equal to...'

17 First steps in the AdS/CFT correspondence

Read from the beginning of Chapter 23 until and including Sec. 23.5. Then also Sec. 23.12. When reference is made to conformal operators ('relevant operators' and 'central charges'), you may neglect these statements. Make the corresponding exercises.

PS: in Ex. 23.2'... in Sec. 22.1.5' is a typo, and should be '... in Sec. 22.1.4' and other errata on
http://itf.fys.kuleuven.be/supergravity/index.php?id=23&type=ErrataCh23.html
Exercise 23.2 is difficult. If you can do it without mistakes you are really strong. Try some steps in any case to show that you understand what has to be done, but do not be too disappointed if you do not find all correct numbers and signs.

All References for questions above

[1] D. Z. Freedman and A. Van Proeyen, Supergravity. Cambridge Univ. Press, Cambridge, UK, 2012.
http://www.cambridge.org/mw/academic/subjects/physics/
theoretical-physics-and-mathematical-physics/supergravity?format=AR

[2] M. K. Gaillard and B. Zumino, Duality rotations for interacting fields, Nucl. Phys. B193 (1981) 221

[3] I. Antoniadis, C. Markou and F. Rondeau, Aspects of compactification on a linear dilaton background, JHEP 09 (2021) 137, arXiv:2106.15184 [hep-th]

[4] J. Wess and J. Bagger, Supersymmetry and supergravity. Princeton University Press, Princeton, NJ, USA, 1992.

[5] S. Ferrara, J. Wess and B. Zumino, Supergauge multiplets and superfields, Phys. Lett. B51 (1974) 239

[6] L. Castellani, Group manifold approach to supergravity, arXiv:2211.04318 [hep-th]

[7] K. S. Stelle, Classical gravity with higher derivatives, Gen.Rel.Grav. 9 (1978) 353-371

[8] F. Giani and M. Pernici, $N = 2$ supergravity in ten-dimensions, Phys. Rev. D30 (1984) $325 - 333$

[9] I. C. G. Campbell and P. C. West, $N = 2D = 10$ nonchiral supergravity and its spontaneous compactification, Nucl. Phys. B243 (1984) 112

[10] M. Huq and M. A. Namazie, Kaluza-Klein supergravity in ten-dimensions, Class. Quant. Grav. 2 (1985) 293, [Erratum-ibid. 2 (1985) 597]

[11] E. A. Bergshoeff and J. Rosseel, Non-Lorentzian supergravity. 2023.
arXiv:2211.02604 [hep-th].

[12] S. Ferrara and A. Van Proeyen, Mass formulae for broken supersymmetry in curved space-time, Fortsch. Phys. 64 (2016), no. 11-12, 896-902, arXiv: 1609.08480 [hep-th]

[13] S. Ferrara, J. Scherk and B. Zumino, Algebraic properties of extended supergravity theories, Nucl. Phys. B121 (1977) 393-402

[14] B. de Wit, Properties of SO(8) extended supergravity, Nucl. Phys. B158 (1979) $189 - 212$

[15] E. Cremmer and B. Julia, The SO(8) supergravity, Nucl. Phys. B159 (1979) 141-212

[16] A. Ceresole, R. D'Auria, S. Ferrara and A. Van Proeyen, Duality transformations in supersymmetric Yang-Mills theories coupled to supergravity, Nucl. Phys. B444 (1995) 92-124, arXiv:hep-th/9502072 [hep-th]

[17] C. Pope, Lectures on Kaluza-Klein, Available on the author's website http://people.physics.tamu.edu/pope/

[18] T. Ortín, Gravity and strings. Cambridge University Press, 2004.

[19] H. Lu and C. N. Pope, P-brane solitons in maximal supergravities, Nucl. Phys. B 465 (1996) 127-156, arXiv: hep-th/9512012

[20] E. Cremmer, B. Julia, H. Lu and C. N. Pope, Dualization of dualities. 1., Nucl. Phys. B 523 (1998) 73-144, arXiv:hep-th/9710119

[21] M. J. Duff, B. E. W. Nilsson and C. N. Pope, Kaluza-Klein Supergravity, Phys. Rept. 130 (1986) $1 - 142$

[22] P. van Nieuwenhuizen, General theory of coset manifolds and antisymmetric tensors applied to Kaluza-Klein supergravity, in Spring School on Supergravity and Supersymmetry. 10, 1984.

[23] M. Trigiante, Gauged Supergravities, Phys. Rept. 680 (2017) 1-175, arXiv:1609.09745 [hep-th]

[24] P. Fré and. Fedotov, Groups and manifolds. De Gruyter, 2018.

[25] G. Dall'Agata and M. Zagermann, Supergravity: from first principles to modern applications, vol. 991 of Lecture Notes in Physics. Springer, 7, 2021.

Part VIII
Другие теории гравитации

(что-то, что не вписывается в предыдущие деления - тут.)

21 Другие теории гравитации

Обсудим теории гравитации, использование которых я пока что нигде не встречал.

21.1 Теория взаимодействия полей с гравитацией

часто об этом заходит речь.

подробнее - в теории поля об этом, тем не менее, укажем некоторые главные моменты тут

(мб подниму куда-то выше главу эту)

21.1.1 Минимальная связь с гравитацией

конструкция минимальной связи

21.1.2 Неминимальная связь с гравитацией

я хз, есть ли такая

проверка на минимальность

как определить, она минимальная или нет?

а вот там теорема римановой геометрии.

потом напишу

21.2 Динамика спин-тензорных полей в гравитационном поле

(таким вопросом тоже можно задаться, потом и задамся. пока хз, не до этого мне абсолютно. мб потом как теоретик поисследую это подробно. типа как описывать в общем случае поля в гравитационных полях?????)

21.3 О дискретной теории гравитации на решетке

(вообще хз абсолютно, просто Вергелес упомянул и всё)

Part IX
Other Topics

(таких теорий много, так что если будет нечего делать, это раздел для этого создан. минимально актуально, конечно, тем не менее. мб когда-то займусь, все-таки полно самых разных тут теорий)

22 О неверных теориях гравитации

В жизни этот раздел может понадобится при общении с людьми, далекими от научного сообщества.

Для кругозора если не лень, можно почитать теории гравитации, которые признаны неверными, может быть, это даст понимание, как не нужно делать.

22.1 Обличение неверных наукообразных теории

22.1.1 теория Логунова

мб потом посмотрю, суслов говорил про нее

22.2 Обличение неверных псевдонаучных теорий

Существует много явно бреда про гравитацию, обличению которого посвящен этот раздел.

(мб когда-то почитаю разный бред, пропишу ответы на него тут, пока максимально не актуально)

23 Идеи вымерших теорий гравитации

Каждая неверная теория содержала в себе какую-то идею, поэтому интересно обсудить, что в этих идеях хорошего и плохого, чтобы знать, не повторяться, иметь кругозор.

(потом)

Part X
Appendix

A Введение и обзор предмета

Приведу общие важные соображения о том, как я вижу предмет.

(все это уже проявлено выше, тут лишь четкие указания для полноты и для тех, у кого вопросы в том, что мне итак очевидно.)

A.1 Другая мотивация

Обсудим все, что нас мотивирует для изучения предмета.

A.2 Мышление профессионала в модификациях теории поля

(потом раскрою)

A.2.1 Суть предмета

A.2.2 Как эффективно заниматься особыми теориями поля?

Теоретикам следует делать свою теорию и фокусироваться на ней, иначе ничего не сдвинется. Однако, только так прогресса, конечно, не достичь, потому что непонятно, какие теории и какие методы стоящие, а какие нет. Поэтому важно думать про фундаментальные законы природы, анализировать эксперименты. Это состовляет важную часть работы

(раскрою эту мысль, что очень плохо заниматься только теориями, потому что это математика, в физике же это не имеет практически никакой ценности. А также важно не заниматься одновременно и фундаментальной физикой и теоретическо-математической, потому что это большие, сложные, отличные друг от друга направления, на каждое нужна куча времени. Единственный разумный вариант - говорить, что сейчас вот такой-то один проект и только про него и думать, забить на другие важные вопросы. потом - это же в плане другого проекта. Лишь немного додумаывать общие методы во время этого проекта и немного заниматься соединениями всего в одну теорию. некоторые проекты - по фундаментальной физике, то есть придумывания, какие теории объясняли бы разные эксперименты. Поскольку такому я даже не знаю, где учат, возможно, нигде, то эти проекты очень много времени будут занимать, и пока нет уровня аспиранта 3го или 4го курса - нет смысла заниамться ими, и не из-за сложности, а из-за того, что просто нужны деньги на жизнь и ощутимое количество свободного времени. Это бессмысленное требование во время бакалавриата и магистратуры, а в первых годах аспирантуры будут свои проекты тоже.)

A.2.3 Отношение к особым теориям поля (!?)

(я может быть это уже писал? такое чувство, что писал это, ладно, когда-то еще напишу)

Просто 1 раз в 2 недели/2 месяца какую-то новую конструкцию изучаешь, это не так сложно, пройдет 3 года - наберется много понятых конструкций и понимание теории поля будет глубже. Такой подход имеет смысл.

Подход же, когда только сидишь на модифицированных теориях поля, имеет мало смысла, потому что одна модификация будет в природе, а 1000 других просто нет, так что никакой физики настоящей из этого не получится узнать. Время потратить тут легко как нигде в другом разделе физики. Будет повод - можно будет какую-то отдельную теорию или отдельный метод изучать. Не будет - можно и не заниматься.

Не скачешь по разным конструкциям! Иначе слишком мало будешь понимать! Спева додумай одну, потом иди к другой.

А то иначе можно и депрессию от скуки словить можно: куча формул и никаких физических указаний. А есть вообще физика в этих теориях или нет - дело большого анализа и размышлений.

Важно понимать связь с экспериментом, даже если многие теоретики даже в 2023 году это не понимают

Типичная теоретическая теория поля - просто какая-то теория, без указаний где это нужно и зачем. Как будто это не важно. Нет, это важно. Не нужно верить таким книгам, статьям, людям, хотя конкретно какой-то метод или теория могут быть очень интересными или полезными.

(подробнее допишу)

Очень раздражают теоретики, которые толкают свои теории, свою особую математику и не рассказывают вообще никакую физику за ними, как будто это нормально. Таких много было во второй половине 20го века, остается много в 1й половине 21, и скорее всего будет еще дальше много, пока человечество не осознает четко, что умничать в математике - не черта хорошего физика.

Это очень раздражает. Такие не ответят на вопросы, такие обожают свои теории крутить. Конечно, одни раздражают прямо сильно, другие же - неплохие люди, много знают, но тем не менее, они не научат никакой физике, потому что сами ее обходят стороной, и это неприятно. Такая вот особенность предмета, благо, им не нужно/не обязательно много заниматься.

Не следует изучать особые теории поля, если нет крепких знаний современной физики

Есть кафедры, которые дают уже на 4 курсе разные конформные теории поля, вводят студентов в теорию струн. Круто ли это? Нет. Даст ли это что-то? Скорее всего нет. Порадоваться следует тем студентам, которые не изучают это, потому что они не тратят время. Умничание очень мало что имеет общего с изучанием настоящей физики и познанием природы.

(раскрою эту мысль, типичное соображение про всё вообще направление. Кучу времени потратишь, ничего про природу не поймешь. Так что заниматься этим вообще есть смысл только если основные предметы по физике и математике доделаны до уровня, что уже скучновато и запросто все верные ответы уровня вуза выдаются. Иначе - да трата времени это все и ничего другого. Даже если статьи будут получаться, это не стоит того (если только нет возможности огромные гранты получать, тогда это стоит того, но редко есть такая возможность и вообще, зависит от страны, где изучающий живет))

A.2.4 Не идиоты ли те, кто заумные теориипридумают с кучей параметров? (!?!?)

(большое обсуждение, потому что очень даже кажется, что да. начинаешь изучать их теорию - сразу становится понятно, что нет, теория очень сложна к пониманию, следовательно, плохая. смысл дальше такую изучать? порассуждаю про это.)

На самом деле, посидев даже 1 час на всяких таких теориях становится понятно, что если с тобой не работает проверенный специалист, а просто какой-то желающий разобраться в этом или делающий вид, что разбирается, то не нужно с ним работать и вообще лучше сделать разбор в таких теориях хобби, а не основным направлением исследований. Потому что фигней только и будешь заниматься. Какой-то идиот написал какой-то бред, а тебе скажут сидеть и думать про это. Зачем? Итак за первый час понятно, что это крайне схоже с бредом и тратой времени.

Вывод: избегай работы с такими людьми. Лучше самому в свободное время о чем-то подумать и заниматься другими теориями, чем всерьез фокусироваться на всяком заумничестве. Так ничего не откроешь в науке, просто время на пустое потратишь, например, на обозначения кого-то или на “что вообще он имел в виду?”. Плохо написано - значит, скорее всего плохая теория, все, закрыли и забыли.

Я 1 раз поразбирался - спасибо, хватило, больше не буду без повода или большого желания этим заниматься. Конечно, интересно и приятно спустя пару попыток догадываться, как же там умные люди думали, содавая свои статьи, но это не стоит того.

A.2.5 Когда заниматься особыми теориями поля не трата времени? (!?!?)

(большое обсуждение, потому что там абстрактные теории, которые странным образом связаны с реальностью. так что большие вопросы. напишу потом.)

(именно исходя из этих соображений иногда я и начинаю про них думать, а по умолчанию - нет, лучше другими предметами позаниматься, а не страдать тут фигней)

A.2.6 Как меньше тратить времени на выод формул в особых теориях поля?

(потом напишу, уже частично это понимаю.)

Иногда полезно не писать индексы и не выписывать точно выражения, сохранить лишь главные члены и так прикинуть, чему вообще оно равно и равно ли тому, что написано в книгах/статьях?

Потому что с индексами и точностью обозначений возни полно.

(допишу потом мысль эту)

Очень полезно прописывать на черновике все формулы, а потом только заниматься латех-записями.

Потому что нужно больше тут думать. Меньше думаешь - латех записи только запутывать будут.

A.2.7 Способы заработать, зная особые теории поля

A.2.8 Актуальнейшие приложения

A.2.9 How we see the world after understanding profound theories of fields and gravity? (!(????)

(this is a very important question, currently I don't know an answer. But there should be a detailed answer! Later I'll think about it)

A.2.10 How understanding special theories helps us to live better? (!(????)

(this is a very important question, currently I don't know an answer. But there should be a detailed answer! Later I'll think about it)

A.2.11 Построение с нуля

(потом раскрою, еще я не профессионал, а вопрос этот самый профессиональный)

A.2.12 Способы догадаться до всех главных идей

незаменимая часть нормального понимания предмета.
(потом раскрою)

A.2.13 Мышление для эффективного изучения особых теорий поля

Осудим, какое мышление наиболее эффективное для усвоение предмета.

Как всегда, обычные соображения для изучения и тут наиболее полезны

Еще раз о важности структурного подхода и навыке фокусировать внимание, а не заниматься много чем

Структуры у меня всегда важны, потому что это изначально хорошая организация работы, которая потом постоянно будет проявляться. Без фокуса внимания тоже как всегда никуда не уйти.

Тут, кстати, все темы дополнительные, нет основных тем, потому что все теории гипотетические. Поэтому все не обязательно, и не нужно вообще менять темы, взял одну - вот и делаешь ее, потом уже другую когда-то начнешь, ну или не начнешь, если будут более приоритетные занятия.

Еще раз о важности в принципе не заниматься особыми теориями поля, если рядом одни как люди идиоты, пусть и в физике сильные

Встретил идиота физика-теоретика? Ух ты, как удивительно, кто бы мог подумать? Такого следует обойти стороной, даже если он что-то понимает в своих теориях. Работа с таким не сделает тебя счастливым, а вероятность, что будет открытие все равно крайне малая. Поэтому даже иногда заниматься исследованиями в особых теориях лучше не стоит, если не встретил приятного в общении и хорошего по жизни теоретика. В прочем, скорее всего такого и не сложно встретить, если заниматься наукой и общаться с многими людьми.

Еще раз о важности подумать самому прежде чем открывать новую литературу

Существует бесчетное число теоретических работ. Это не значит, что нужно их читать. Выбрал пару - изучаем их, потом когда-то что-то добавляешь. Все, иначе сойдешь с ума, только и будет заниматься чтением литературы. Вообще плевать, если ты не прочитаешь многие известные статьи, лучше понять 2 статьи, чем прочитать 10.

Необходимые темы для

(потом раскрою)

Дополнительные темы для

(потом раскрою)

A.3 Acknowledgements

Currently, no one except me has worked on the sections of this note (with the exception of sections taken from books).

A.4 Literature

A.4.1 Основная

Основная обучающая

[1] Freedman, Daniel Z. and Van Proeyen, Antoine Supergravity
Большая, концентрированная, хорошая для теоретиков книга, где много теоретических методов. Все написано компактно, можно даже использовать как конспект. Но приложений теорий, их проверки и настоящая физика за ними не обсуждаются, так что не считаю, что для нормального понимания ее хватит. Также немного странные обозначения есть, но это мелочь.

Books With Many Solved Problems

Литература крепкого минимума

A.4.2 Extra Literature

Дополнительная по теории

[2] Редьков В.М. Поля частиц в римановом пространстве и группа Лоренца
Большая теоретическая книга, где очень много обсуждаются спиноры, формализмы, уравнения Дирака. Очень много особых уравнений и спиноров. Есть ли от них польза, используются ли они? Пока не знаю.

В помощь

Статьи о теоретических методах

О приложениях

A.5 Обзор

(потом раскрою)

A.5.1 предмет в двух словах

Обсудим, что из себя представляет предмет наиболее кратко, выделяя самую суть.

появление Предмет в нашей картине мира

один подход

второй подход

один большой раздел

такой-то набор следствий

A.5.2 Итоговые формулы и закономерности

A.5.3 обзор теоретических подходов

такие-то есть, такие полезные, такие - нет.

A.5.4 Обзор дальнейших развитий

A.5.5 Связи с другими науками

Обсудим связи с разделами
(потом раскрою)

A.5.6 Описание записи

Общее описание записи

Общие особенности записи

Особенности глав и разделов

Первая часть про предмет в двух словах

Вторая часть

Часть про приложения

какие вообще приложения я разбирал?

Обозначения и константы

A.6 History of special field theories overview

(там много интересного, см. пока Вики, или пару страниц в записи Зу, потом буду прописывать.)

А.7 Головоломки

Обсудим в порядке интересности задачки и вопросы

А.7.1 Типичные головоломки

А.7.2 Бытовые головоломки

А.7.3 Принципиальные головоломки

А.7.4 Головоломки о деталях

А.7.5 Головоломки для освоения типичных понятий

Допустим, видим человека, который в принципе хотел бы понять суть, опишем, какие в какой последовательности будем задавать, чтобы в типичном случае ему было бы интереснее въезжать. В частности это могут быть младшие студенты или редко школьники.

Головоломки для освоения школьного уровня

Головоломки для освоения уровня теормина

Головоломки для освоения других особенностей

В Дополнения

В.0.1 title

C Bibliography

References

[1] Freedman, Daniel Z. and Van Proeyen, Antoine: *Supergravity*. Cambridge Univ. Press, Cambridge, UK, May 2012, ISBN 978-1-139-36806-3, 978-0-521-19401-3.

[2] В.М., Редьков: *Поля частиц в римановом пространстве и группа Лоренца*, 2008, ISBN 978-985-08-1003-8.

[3] Poplawski, Nikodem J.: *Gravitation, electromagnetism and cosmological constant in purely affine gravity*. Foundations of Physics, 39(3):307–330, Feb 2009, ISSN 1572-9516. http://dx.doi.org/10.1007/s10701-009-9284-y.

References

[1] S. Weinberg, “*The quantum theory of fields. Vol. 2: Modern applications*,” Cambridge University Press.

[2] N. S. Manton and P. Sutcliffe, “*Topological solitons*,”, Cambridge University Press.

[3] J. Polchinski, “*Lectures on D-branes*,” arXiv:hep-th/9611050; "*String Theory, Vols 1 and 2*", Cambridge University Press.

[4] A. Giveon and D. Kutasov, “*Brane dynamics and gauge theory*,” Rev. Mod. Phys. **71**, 983 (1999) [arXiv:hep-th/9802067].

[5] C. V. Johnson, “D-brane primer,” arXiv:hep-th/0007170; "*D-Branes*", Cambridge University Press.

[6] A. A. Belavin, A. M. Polyakov, A. S. Schwartz and Y. S. Tyupkin, “*Pseudoparticle Solutions Of The Yang-Mills Equations*,” Phys. Lett. B **59**, 85 (1975).

[7] G. ’t Hooft, “*Computation Of The Quantum Effects Due To A Four-Dimensional Pseudoparticle*,” Phys. Rev. D **14**, 3432 (1976) [Erratum-ibid. D **18**, 2199 (1978)].

[8] R. Jackiw and C. Rebbi, “*Vacuum Periodicity In A Yang-Mills Quantum Theory*,” Phys. Rev. Lett. **37**, 172 (1976).

[9] C. G. . Callan, R. F. Dashen and D. J. Gross, “*Toward A Theory Of The Strong Interactions*,” Phys. Rev. D **17**, 2717 (1978).

[10] S.K. Donaldson and P.B. Kronheimer, "The Geometry of Four Manifolds", Oxford University Press (1990).

[11] S. Coleman, "*Uses of Instantons*" in Aspects of Symmetry, Cambridge University Press (1985)

[12] M. A. Shifman and A. I. Vainshtein, “*Instantons versus supersymmetry: Fifteen years later*,” arXiv:hep-th/9902018.

[13] N. Dorey, T. J. Hollowood, V. V. Khoze and M. P. Mattis, “*The calculus of many instantons*,” Phys. Rept. **371**, 231 (2002) [arXiv:hep-th/0206063].

[14] E. B. Bogomoln'yi, "*The Stability of Classical Solutions,*" Sov. J. Nucl. Phys, **24** (1976) 49.

[15] E. Witten, "*Some Exact Multipseudoparticle Solutions Of Classical Yang-Mills Theory*," Phys. Rev. Lett. **38**, 121 (1977).

[16] G. 't Hooft, Unpublished; R. Jackiw, C. Nohl and C. Rebbi, "*Conformal Properties Of Pseudoparticle Configurations*," Phys. Rev. D **15**, 1642 (1977).

[17] J. Bagger and E. Witten, "*Matter Couplings In N=2 Supergravity*," Nucl. Phys. B **222**, 1 (1983).

[18] B. de Wit, M. Rocek and S. Vandoren, "*Hypermultiplets, hyperkaehler cones and quaternion-Kaehler geometry*" JHEP **0102**, 039 (2001) [arXiv:hep-th/0101161].

[19] J. A. Harvey and A. Strominger, "*String theory and the Donaldson polynomial*," Commun. Math. Phys. **151**, 221 (1993) [arXiv:hep-th/9108020].

[20] J. P. Gauntlett, "*Low-energy dynamics of N=2 supersymmetric monopoles*" Nucl. Phys. B **411**, 443 (1994) [arXiv:hep-th/9305068].

[21] M. F. Atiyah and I. M. Singer, "*Dirac Operators Coupled To Vector Potentials*," Proc. Nat. Acad. Sci. **81**, 2597 (1984).

[22] A. Maciocia, "*Metrics on the moduli spaces of instantons over Euclidean four space*," Commun. Math. Phys. **135**, 467 (1991).

[23] E. Witten and D. I. Olive, "*Supersymmetry Algebras That Include Topological Charges*," Phys. Lett. B **78**, 97 (1978).

[24] G. Sierra and P. K. Townsend, "*The Hyperkahler Supersymmetric Sigma Model In Six-Dimensions*," Phys. Lett. B **124**, 497 (1983).

[25] J. D. Blum, "*Supersymmetric quantum mechanics of monopoles in N=4 Yang-Mills theory*," Phys. Lett. B **333**, 92 (1994) [arXiv:hep-th/9401133].

[26] M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, "*Construction Of Instantons*," Phys. Lett. A **65** (1978) 185.

[27] M. R. Douglas, "*Branes within branes*," arXiv:hep-th/9512077.

[28] M. R. Douglas, "*Gauge Fields and D-branes*," J. Geom. Phys. **28**, 255 (1998) [arXiv:hep-th/9604198].

[29] E. Witten, "*Sigma models and the ADHM construction of instantons*," J. Geom. Phys. **15**, 215 (1995) [arXiv:hep-th/9410052].

[30] N. J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, "*Hyperkahler Metrics And Supersymmetry*," Commun. Math. Phys. **108**, 535 (1987).

[31] N. H. Christ, E. J. Weinberg and N. K. Stanton, "*General Self-Dual Yang-Mills Solutions*," Phys. Rev. D **18**, 2013 (1978).

[32] E. Corrigan and P. Goddard, "*Some Aspects Of Instantons*," DAMTP 79/18, Lectures given at Canadian Mathematical Society Meeting, Montreal, Canada, Sep 3-8, 1979

[33] H. Osborn, "*Semiclassical Functional Integrals For Selfdual Gauge Fields*," Annals Phys. **135**, 373 (1981).

[34] N. Dorey, V. V. Khoze and M. P. Mattis, "*Multi-Instanton Calculus in N=2 Supersymmetric Gauge Theory*," Phys. Rev. D **54**, 2921 (1996) [arXiv:hep-th/ 9603136]; "*Supersymmetry and the multi-instanton measure*," Nucl. Phys. B **513**, 681 (1998) [arXiv:hep-th/9708036].

[35] N. Nekrasov and A. Schwarz, "*Instantons on noncommutative R**4 and (2,0) superconformal six dimensional theory*," Commun. Math. Phys. **198**, 689 (1998) [arXiv:hep-th/9802068].

[36] H. Nakajima, "*Resolutions of Moduli Spaces of Ideal Instantons on R^4*", in Topology, Geometry and Field Theory, World Scientific 1994.

[37] N. A. Nekrasov, "*Trieste lectures on solitons in noncommutative gauge theories*," arXiv:hep-th/0011095.

[38] J. A. Harvey, "*Komaba lectures on noncommutative solitons and D-branes*," arXiv:hep-th/0102076.

[39] N. Seiberg and E. Witten, "*The D1/D5 system and singular CFT*," JHEP **9904**, 017 (1999) [arXiv:hep-th/9903224].

[40] T. Eguchi and A. J. Hanson, "*Asymptotically Flat Selfdual Solutions To Euclidean Gravity*," Phys. Lett. B **74**, 249 (1978).

[41] K. M. Lee, D. Tong and S. Yi, "*The moduli space of two U(1) instantons on noncommutative R^4 and $R^3 \times S^1$*," Phys. Rev. D **63**, 065017 (2001) [arXiv:hep-th/0008092].

[42] M. Hamanaka, Y. Imaizumi and N. Ohta, "*Moduli space and scattering of D0-branes in noncommutative super Yang-Mills theory*," Phys. Lett. B **529**, 163 (2002) [arXiv:hep-th/0112050].

[43] I. Affleck, M. Dine and N. Seiberg, "*Supersymmetry Breaking By Instantons*," Phys. Rev. Lett. **51**, 1026 (1983); "*Dynamical Supersymmetry Breaking In Supersymmetric QCD*," Nucl. Phys. B **241**, 493 (1984).

[44] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, "*Instanton Effects In Supersymmetric Theories*," Nucl. Phys. B **229**, 407 (1983); "*Exact Gell-Mann-Low Function Of Supersymmetric Yang-Mills Theories From Instanton Calculus*," Nucl. Phys. B **229**, 381 (1983); "*Supersymmetric Instanton Calculus: Gauge Theories With Matter*," Nucl. Phys. B **260**, 157 (1985) [Yad. Fiz. **42**, 1499 (1985)].

[45] D. Finnell and P. Pouliot, "*Instanton calculations versus exact results in four-dimensional SUSY gauge theories*," Nucl. Phys. B **453**, 225 (1995) [arXiv:hep-th/9503115].

[46] N. A. Nekrasov, "*Seiberg-Witten prepotential from instanton counting*," Adv. Theor. Math. Phys. **7**, 831 (2004) [arXiv:hep-th/0206161]; arXiv:hep-th/0306211.

[47] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, "*Large N field theories, string theory and gravity*," Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[48] T. Banks and M. B. Green, "*Non-perturbative effects in $AdS_\times S^5$ string theory and $d = 4$ SUSY Yang-Mills*," JHEP **9805**, 002 (1998) [arXiv:hep-th/9804170].

[49] M. Bianchi, M. B. Green, S. Kovacs and G. Rossi, "*Instantons in supersymmetric Yang-Mills and D-instantons in IIB superstring theory*," JHEP **9808**, 013 (1998) [arXiv:hep-th/9807033].

[50] N. Dorey, T. J. Hollowood, V. V. Khoze, M. P. Mattis and S. Vandoren, "*Multi instanton calculus and the AdS/CFT correspondence in N=4 superconformal field theory*," Nucl. Phys. B **552**, 88 (1999) [arXiv:hep-th/9901128].

[51] T. J. Hollowood, V. V. Khoze and M. P. Mattis, "*Summing the instanton series in $N = 2$ superconformal large-N QCD*," JHEP **9910**, 019 (1999) [arXiv:hep-th/9905209].

[52] T. J. Hollowood and V. V. Khoze, "*ADHM and D-instantons in orbifold AdS/CFT duality*," Nucl. Phys. B **575**, 78 (2000) [arXiv:hep-th/9908035].

[53] E. Gava, K. S. Narain and M. H. Sarmadi, "*Instantons in $N = 2$ Sp(N) superconformal gauge theories and the AdS/CFT correspondence*," Nucl. Phys. B **569**, 183 (2000) [arXiv:hep-th/9908125].

[54] T. J. Hollowood, "*Instantons, finite $N = 2$ Sp(N) theories and the AdS/CFT correspondence*," JHEP **9911**, 012 (1999) [arXiv:hep-th/9908201].

[55] M. B. Green and C. Stahn, "*D3-branes on the Coulomb branch and instantons*," JHEP **0309**, 052 (2003) [arXiv:hep-th/0308061].

[56] I. R. Klebanov and A. A. Tseytlin, "*Entropy of Near-Extremal Black p-branes*," Nucl. Phys. B **475**, 164 (1996) [arXiv:hep-th/9604089].

[57] J. A. Harvey, R. Minasian and G. W. Moore, "*Non-abelian tensor-multiplet anomalies*," JHEP **9809**, 004 (1998) [arXiv:hep-th/9808060].

[58] M. Rozali, "*Matrix theory and U-duality in seven dimensions*," Phys. Lett. B **400**, 260 (1997) [arXiv:hep-th/9702136].

[59] M. Berkooz, M. Rozali and N. Seiberg, "*Matrix description of M theory on T^4 and T^5*," Phys. Lett. B **408**, 105 (1997) [arXiv:hep-th/9704089].

[60] N. Seiberg, "*Notes on theories with 16 supercharges*," Nucl. Phys. Proc. Suppl. **67**, 158 (1998) [arXiv:hep-th/9705117].

[61] E. Witten, "*Constraints On Supersymmetry Breaking*," Nucl. Phys. B **202**, 253 (1982).

[62] K. M. Lee and P. Yi, "*Quantum spectrum of instanton solitons in five dimensional noncommutative $U(N)$ theories*," Phys. Rev. D **61**, 125015 (2000) [arXiv:hep-th/9911186].

[63] N. Dorey, T. J. Hollowood and V. V. Khoze, "*Notes on soliton bound-state problems in gauge theory and string theory*," arXiv:hep-th/0105090.

[64] N. Itzhaki, J. M. Maldacena, J. Sonnenschein and S. Yankielowicz, "*Supergravity and the large N limit of theories with sixteen supercharges*," Phys. Rev. D **58**, 046004 (1998) [arXiv:hep-th/9802042].

[65] P. Nelson and S. R. Coleman, "*What Becomes Of Global Color*," Nucl. Phys. B **237**, 1 (1984).

[66] K. M. Lee and P. Yi, "*Monopoles and instantons on partially compactified D-branes*," Phys. Rev. D **56**, 3711 (1997) [arXiv:hep-th/9702107].

[67] T. C. Kraan and P. van Baal, "*Periodic instantons with non-trivial holonomy*," Nucl. Phys. B **533**, 627 (1998) [arXiv:hep-th/9805168].

[68] R. Dijkgraaf, "*Instanton strings and hyperKaehler geometry*," Nucl. Phys. B **543**, 545 (1999) [arXiv:hep-th/9810210].

[69] P. A. M. Dirac, "*Quantised Singularities In The Electromagnetic Field,*" Proc. Roy. Soc. Lond. A **133** (1931) 60; "*The Theory Of Magnetic Poles*," Phys. Rev. **74** (1948) 817.

[70] G. 't Hooft, "*Magnetic Monopoles In Unified Gauge Theories*," Nucl. Phys. B **79**, 276 (1974).

[71] A. M. Polyakov, "*Particle Spectrum In Quantum Field Theory*," JETP Lett. **20** (1974) 194 [Pisma Zh. Eksp. Teor. Fiz. **20** (1974) 430].

[72] P. M. Sutcliffe, "*BPS monopoles*," Int. J. Mod. Phys. A **12**, 4663 (1997) [arXiv:hep-th/9707009].

[73] Y. Shnir, "*SU(N) monopoles with and without SUSY*," arXiv:hep-th/0508210; "*Magnetic Monopoles*", Springer (2005).

[74] M. F. Atiyah and N. J. Hitchin, "*The Geometry And Dynamics Of Magnetic Monopoles.*", M.B. Porter Lectures, Princeton University Press (1988)

[75] J. A. Harvey, "*Magnetic monopoles, duality, and supersymmetry*," arXiv:hep-th/9603086.

[76] E. J Weinberg and P. Yi, to appear.

[77] P. Goddard, J. Nuyts and D. I. Olive, "*Gauge Theories And Magnetic Charge*," Nucl. Phys. B **125**, 1 (1977).

[78] T. T. Wu and C. N. Yang, "*Dirac Monopole Without Strings: Monopole Harmonics*," Nucl. Phys. B **107**, 365 (1976).

[79] M. K. Prasad and C. M. Sommerfield, "*An Exact Classical Solution For The 't Hooft Monopole And The Julia-Zee Dyon*," Phys. Rev. Lett. **35**, 760 (1975).

[80] E. J. Weinberg, "*Parameter Counting For Multi Monopole Solutions*," Phys. Rev. D **20**, 936 (1979); "*Fundamental Monopoles And Multi Monopole Solutions For Arbitrary Simple Gauge Groups*," Nucl. Phys. B **167**, 500 (1980).

[81] C. Callias, "*Index Theorems On Open Spaces*," Commun. Math. Phys. **62**, 213 (1978).

[82] N. S. Manton, "*The Force Between 'T Hooft-Polyakov Monopoles*," Nucl. Phys. B **126** (1977) 525.

[83] C. J. Houghton and P. M. Sutcliffe, "*Tetrahedral and cubic monopoles*," Commun. Math. Phys. **180**, 343 (1996) [arXiv:hep-th/9601146].

[84] N. S. Manton, "*A Remark On The Scattering Of Bps Monopoles*," Phys. Lett. B **110**, 54 (1982).

[85] N. S. Manton, "*Monopole Interactions At Long Range*," Phys. Lett. B **154**, 397 (1985) [Erratum-ibid. **157B**, 475 (1985)].

[86] G. W. Gibbons and N. S. Manton, "*The Moduli space metric for well separated BPS monopoles*," Phys. Lett. B **356**, 32 (1995) [arXiv:hep-th/9506052].

[87] K. M. Lee, E. J. Weinberg and P. Yi, "*The Moduli Space of Many BPS Monopoles for Arbitrary Gauge Groups*," Phys. Rev. D **54**, 1633 (1996) [arXiv:hep-th/9602167].

[88] G. W. Gibbons and N. S. Manton, "*Classical And Quantum Dynamics Of BPS Monopoles*," Nucl. Phys. B **274**, 183 (1986).

[89] A. Sen, "*Strong coupling dynamics of branes from M-theory*," JHEP **9710**, 002 (1997) [arXiv:hep-th/9708002].

[90] N. Seiberg and E. Witten, "*Gauge dynamics and compactification to three dimensions*," arXiv:hep-th/9607163.

[91] A. Hanany and E. Witten, "*Type IIB superstrings, BPS monopoles, and three-dimensional gauge dynamics*," Nucl. Phys. B **492**, 152 (1997) [arXiv:hep-th/9611230].

[92] A. Sen, "*Dynamics of multiple Kaluza-Klein monopoles in M and string theory*," Adv. Theor. Math. Phys. **1**, 115 (1998) [arXiv:hep-th/9707042].

[93] E. Witten, "*Heterotic string conformal field theory and A-D-E singularities*," JHEP **0002**, 025 (2000) [arXiv:hep-th/9909229].

[94] A. Hanany and A. Zaffaroni, "*Monopoles in string theory*," JHEP **9912**, 014 (1999) [arXiv:hep-th/9911113].

[95] S. A. Connell, "*The Dynamics of the $SU(3)$ Charge $(1,1)$ Magnetic Monopole*", University of South Australia Preprint.

[96] J. P. Gauntlett and D. A. Lowe, "*Dyons and S-Duality in N=4 Supersymmetric Gauge Theory*," Nucl. Phys. B **472**, 194 (1996) [arXiv:hep-th/9601085].

[97] K. M. Lee, E. J. Weinberg and P. Yi, "*Electromagnetic Duality and $SU(3)$ Monopoles*," Phys. Lett. B **376**, 97 (1996) [arXiv:hep-th/9601097].

[98] B. Julia and A. Zee, "*Poles With Both Magnetic And Electric Charges In Nonabelian Gauge Theory*," Phys. Rev. D **11**, 2227 (1975).

[99] J. S. Schwinger, "*A Magnetic Model Of Matter*," Science **165** (1969) 757.

[100] E. Witten, "*Dyons Of Charge $e\theta/2\pi$*," Phys. Lett. B **86**, 283 (1979).

[101] O. Bergman, "*Three-pronged strings and 1/4 BPS states in $N = 4$ super-Yang-Mills theory*," Nucl. Phys. B **525**, 104 (1998) [arXiv:hep-th/9712211].

[102] K. M. Lee and P. Yi, "*Dyons in $N = 4$ supersymmetric theories and three - pronged strings*," Phys. Rev. D **58**, 066005 (1998) [arXiv:hep-th/9804174].

[103] D. Tong, "*A note on 1/4-BPS states*," Phys. Lett. B **460**, 295 (1999) [arXiv:hep-th/9902005].

[104] D. Bak, C. k. Lee, K. M. Lee and P. Yi, "*Low energy dynamics for 1/4 BPS dyons*," Phys. Rev. D **61**, 025001 (2000) [arXiv:hep-th/9906119].

[105] J. P. Gauntlett, C. j. Kim, K. M. Lee and P. Yi, "*General low energy dynamics of supersymmetric monopoles*," Phys. Rev. D **63**, 065020 (2001) [arXiv:hep-th/0008031].

[106] R. Jackiw and C. Rebbi, "*Solitons With Fermion Number 1/2*," Phys. Rev. D **13**, 3398 (1976).

[107] W. Nahm, "*The Construction of all Self-Dual Multi-Monopoles by the ADHM Method*", in "Monopoles in Quantum Field Theory", eds. N. S. Craigie, P. Goddard and W. Nahm, Singapore, World Scientific, 1982.

[108] J. Hurtubise and M. K. Murray, "*On The Construction Of Monopoles For The Classical Groups*" Commun. Math. Phys. **122**, 35 (1989).

[109] D. E. Diaconescu, "*D-branes, monopoles and Nahm equations*," Nucl. Phys. B **503**, 220 (1997) [arXiv:hep-th/9608163].

[110] A. Kapustin and S. Sethi, "*The Higgs branch of impurity theories*," Adv. Theor. Math. Phys. **2**, 571 (1998) [arXiv:hep-th/9804027].

[111] D. Tsimpis, "*Nahm equations and boundary conditions*," Phys. Lett. B **433**, 287 (1998) [arXiv:hep-th/9804081].

[112] O. DeWolfe, D. Z. Freedman and H. Ooguri, "*Holography and defect conformal field theories*," Phys. Rev. D **66**, 025009 (2002) [arXiv:hep-th/0111135].

[113] X. G. Chen and E. J. Weinberg, "*ADHMN boundary conditions from removing monopoles*," Phys. Rev. D **67**, 065020 (2003) [arXiv:hep-th/0212328].

[114] M. K. Murray, "*A note on the (1, 1,..., 1) monopole metric*," J. Geom. Phys. **23**, 31 (1997) [arXiv:hep-th/9605054].

[115] K. Hashimoto and S. Terashima, "*Stringy derivation of Nahm construction of monopoles*," arXiv:hep-th/0507078.

[116] E. J. Weinberg and P. Yi, "*Explicit multimonopole solutions in SU(N) gauge theory*," Phys. Rev. D **58**, 046001 (1998) [arXiv:hep-th/9803164].

[117] N. D. Lambert and D. Tong, "*Dyonic instantons in five-dimensional gauge theories*," Phys. Lett. B **462**, 89 (1999) [arXiv:hep-th/9907014].

[118] L. Alvarez-Gaume and D. Z. Freedman, "*Potentials For The Supersymmetric Nonlinear Sigma Model*," Commun. Math. Phys. **91**, 87 (1983).

[119] I. Affleck, "*On Constrained Instantons*," Nucl. Phys. B **191**, 429 (1981).

[120] E. Eyras, P. K. Townsend and M. Zamaklar, "*The heterotic dyonic instanton*," JHEP **0105**, 046 (2001) [arXiv:hep-th/0012016].

[121] K. Peeters and M. Zamaklar, "*Motion on moduli spaces with potentials*," JHEP **0112**, 032 (2001) [arXiv:hep-th/0107164].

[122] A. M. Polyakov, "*Quark Confinement And Topology Of Gauge Groups*," Nucl. Phys. B **120**, 429 (1977).

[123] I. Affleck, J. A. Harvey and E. Witten, "*Instantons And (Super)Symmetry Breaking In (2+1)-Dimensions*," Nucl. Phys. B **206**, 413 (1982).

[124] J. Polchinski and P. Pouliot, "*Membrane scattering with M-momentum transfer*," Phys. Rev. D **56**, 6601 (1997) [arXiv:hep-th/9704029].

[125] N. Dorey, V. V. Khoze and M. P. Mattis, "*Multi-instantons, three dimensional gauge theory, and the Gauss-Bonnet-Chern theorem*" Nucl. Phys. B **502**, 94 (1997) [arXiv:hep-th/9704197].

[126] G. Chalmers and A. Hanany, "*Three dimensional gauge theories and monopoles*," Nucl. Phys. B **489**, 223 (1997) [arXiv:hep-th/9608105].

[127] D. Tong, "*Three-dimensional gauge theories and ADE monopoles*," Phys. Lett. B **448**, 33 (1999) [arXiv:hep-th/9803148].

[128] N. Dorey, V. V. Khoze, M. P. Mattis, D. Tong and S. Vandoren, "*Instantons, three-dimensional gauge theory, and the Atiyah-Hitchin manifold*," Nucl. Phys. B **502**, 59 (1997) [arXiv:hep-th/9703228].

[129] C. Montonen and D. I. Olive, "*Magnetic Monopoles As Gauge Particles?*," Phys. Lett. B **72**, 117 (1977).

[130] H. Osborn, "*Topological Charges For N=4 Supersymmetric Gauge Theories And Monopoles Of Spin 1*," Phys. Lett. B **83**, 321 (1979).

[131] A. Sen, "*Dyon - monopole bound states, selfdual harmonic forms on the multi - monopole moduli space, and SL(2,Z) invariance in string theory*," Phys. Lett. B **329**, 217 (1994) [arXiv:hep-th/9402032].

[132] N. Seiberg and E. Witten, "*Electric magnetic duality, monopole condensation, and confinement in N=2 supersymmetric Yang-Mills theory*," Nucl. Phys. B **426**, 19 (1994) [Erratum-ibid. B **430**, 485 (1994)] [arXiv:hep-th/9407087]; "*Monopoles, duality and chiral symmetry breaking in N=2 supersymmetric QCD*," Nucl. Phys. B **431**, 484 (1994) [arXiv:hep-th/9408099].

[133] J. P. Gauntlett and J. A. Harvey, "*S duality and the dyon spectrum in N=2 superYang-Mills theory*," Nucl. Phys. B **463**, 287 (1996) [arXiv:hep-th/9508156].

[134] A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, "*A new anomaly in the central charge of the $N = 2$ monopole*," Phys. Lett. B **594**, 234 (2004) [arXiv:hep-th/0401116].

[135] K. Hori and C. Vafa, "*Mirror symmetry*," arXiv:hep-th/0002222.

[136] O. Aharony, "*A brief review of little string theories*," Class. Quant. Grav. **17**, 929 (2000) [arXiv:hep-th/9911147].

[137] C. G. . Callan and J. M. Maldacena, "*Brane dynamics from the Born-Infeld action*," Nucl. Phys. B **513**, 198 (1998) [arXiv:hep-th/9708147].

[138] G. W. Gibbons, "*Born-Infeld particles and Dirichlet p-branes*," Nucl. Phys. B **514**, 603 (1998) [arXiv:hep-th/9709027].

[139] J. Gauntlett, D. Tong and P. Townsend, unpublished (2000)

[140] E. Corrigan, C. Devchand, D. B. Fairlie and J. Nuyts, "*First Order Equations For Gauge Fields In Spaces Of Dimension Greater Than Four*," Nucl. Phys. B **214**, 452 (1983).

[141] E. Witten, "*Bound states of strings and p-branes*," Nucl. Phys. B **460**, 335 (1996) [arXiv:hep-th/9510135].

[142] A. Vilenkin and E. P. S. Shellard, "*Cosmic Strings and other Topological Defects*", Cambridge University Press, 1994.

[143] M. B. Hindmarsh and T. W. B. Kibble, "*Cosmic strings*," Rept. Prog. Phys. **58**, 477 (1995) [arXiv:hep-ph/9411342].

[144] J. Polchinski, "*Introduction to cosmic F- and D-strings*," arXiv:hep-th/0412244.

[145] K. Rajagopal and F. Wilczek, "*The condensed matter physics of QCD*," arXiv:hep-ph/0011333.

[146] A. A. Abrikosov, "*On The Magnetic Properties Of Superconductors Of The Second Group*," Sov. Phys. JETP **5**, 1174 (1957) [Zh. Eksp. Teor. Fiz. **32**, 1442 (1957)].

[147] H. B. Nielsen and P. Olesen, "*Vortex-Line Models For Dual Strings*," Nucl. Phys. B **61**, 45 (1973).

[148] A. S. Schwarz, "*Field Theories With No Local Conservation Of The Electric Charge*," Nucl. Phys. B **208**, 141 (1982).

[149] M. G. Alford, K. Benson, S. R. Coleman, J. March-Russell and F. Wilczek, "*The Interactions And Excitations Of Nonabelian Vortices*," Phys. Rev. Lett. **64**, 1632 (1990) [Erratum-ibid. **65**, 668 (1990)]. Nucl. Phys. B **349**, 414 (1991).

[150] G. V. Dunne, "*Aspects of Chern-Simons theory*," arXiv:hep-th/9902115.

[151] A. Hanany and D. Tong, "*Vortices, instantons and branes*," JHEP **0307**, 037 (2003) [arXiv:hep-th/0306150].

[152] R. Auzzi, S. Bolognesi, J. Evslin, K. Konishi and A. Yung, "*Nonabelian superconductors: Vortices and confinement in N=2 SQCD*," Nucl. Phys. B **673**, 187 (2003) [arXiv:hep-th/0307287].

[153] H. J. de Vega and F. A. Schaposnik, "*A Classical Vortex Solution Of The Abelian Higgs Model*," Phys. Rev. D **14**, 1100 (1976).

[154] E. J. Weinberg, "*Multivortex Solutions Of The Ginzburg-Landau Equations*," Phys. Rev. D **19**, 3008 (1979); "*Index Calculations For The Fermion-Vortex System*," Phys. Rev. D **24**, 2669 (1981).

[155] C. H. Taubes, "*Arbitrary N - Vortex Solutions To The First Order Landau-Ginzburg Equations*," Commun. Math. Phys. **72**, 277 (1980).

[156] T. M. Samols, "*Vortex scattering*," Commun. Math. Phys. **145**, 149 (1992).

[157] N. S. Manton and J. M. Speight, "*Asymptotic interactions of critically coupled vortices*," Commun. Math. Phys. **236**, 535 (2003) [arXiv:hep-th/0205307].

[158] M. Shifman and A. Yung, "*Non-Abelian string junctions as confined monopoles*," Phys. Rev. D **70**, 045004 (2004) [arXiv:hep-th/0403149].

[159] E. P. S. Shellard and P. J. Ruback, "*Vortex Scattering In Two-Dimensions*," Phys. Lett. B **209**, 262 (1988).

[160] P. J. Ruback, "*Vortex String Motion In The Abelian Higgs Model*," Nucl. Phys. B **296**, 669 (1988).

[161] E. Witten, "*Solutions of four-dimensional field theories via M-theory*," Nucl. Phys. B **500**, 3 (1997) [arXiv:hep-th/9703166].

[162] C. Bachas and M. B. Green, "*A classical manifestation of the Pauli exclusion principle*," JHEP **9801**, 015 (1998) [arXiv:hep-th/9712187].

[163] B. H. Lee, H. j. Lee, N. Ohta and H. S. Yang, "*Maxwell Chern-Simons solitons from type IIB string theory*," Phys. Rev. D **60**, 106003 (1999) [arXiv:hep-th/9904181].

[164] A. Hanany and K. Hori, "*Branes and $N = 2$ theories in two dimensions*," Nucl. Phys. B **513**, 119 (1998) [arXiv:hep-th/9707192].

[165] K. Hashimoto and D. Tong, "*Reconnection of non-abelian cosmic strings*," arXiv:hep-th/0506022.

[166] P. Forgacs and N. S. Manton, "*Space-Time Symmetries In Gauge Theories*," Commun. Math. Phys. **72**, 15 (1980).

[167] T. Vachaspati and A. Achucarro, "*Semilocal cosmic strings*," Phys. Rev. D **44**, 3067 (1991).

[168] A. Achucarro and T. Vachaspati, "*Semilocal and electroweak strings*," Phys. Rept. **327**, 347 (2000) [Phys. Rept. **327**, 427 (2000)] [arXiv:hep-ph/9904229].

[169] M. Hindmarsh, R. Holman, T. W. Kephart and T. Vachaspati, "*Generalized semilocal theories and higher Hopf maps*," Nucl. Phys. B **404**, 794 (1993) [arXiv:hep-th/9209088].

[170] M. Hindmarsh, "*Existence and stability of semilocal strings*," Phys. Rev. Lett. **68**, 1263 (1992).

[171] R. A. Leese and T. M. Samols, "*Interaction of semilocal vortices*," Nucl. Phys. B **396**, 639 (1993).

[172] E. Witten, "*Phases of $N = 2$ theories in two dimensions*," Nucl. Phys. B **403**, 159 (1993) [arXiv:hep-th/9301042].

[173] B. J. Schroers, "*The Spectrum of Bogomol'nyi Solitons in Gauged Linear Sigma Models*," Nucl. Phys. B **475**, 440 (1996) [arXiv:hep-th/9603101].

[174] R. S. Ward, "*Slowly Moving Lumps In The Cp**1 Model In (2+1)-Dimensions*," Phys. Lett. B **158**, 424 (1985).

[175] D. P. Jatkar, G. Mandal and S. R. Wadia, "*Nielsen-Olesen vortices in noncommutative Abelian Higgs model*," JHEP **0009**, 018 (2000) [arXiv:hep-th/0007078].

[176] D. Bak, K. M. Lee and J. H. Park, "*Noncommutative vortex solitons*," Phys. Rev. D **63**, 125010 (2001) [arXiv:hep-th/0011099].

[177] A. Hanany and D. Tong, "*Vortex strings and four-dimensional gauge dynamics*," JHEP **0404**, 066 (2004) [arXiv:hep-th/0403158].

[178] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, "*Instantons in the Higgs phase*," Phys. Rev. D **72**, 025011 (2005) [arXiv:hep-th/0412048].

[179] G. W. Moore and P. Nelson, "*The Etiology Of Sigma Model Anomalies*," Commun. Math. Phys. **100**, 83 (1985).

[180] J. D. Edelstein, C. Nunez and F. Schaposnik, "*Supersymmetry and Bogomolny equations in the Abelian Higgs model*," Phys. Lett. B **329**, 39 (1994) [arXiv:hep-th/9311055].

[181] D. R. Morrison and M. Ronen Plesser, "*Summing the instantons: Quantum cohomology and mirror symmetry in toric varieties*," Nucl. Phys. B **440**, 279 (1995) [arXiv:hep-th/9412236].

[182] M. Rocek and E. P. Verlinde, "*Duality, quotients, and currents*," Nucl. Phys. B **373**, 630 (1992) [arXiv:hep-th/9110053].

[183] R. Gregory, J. A. Harvey and G. W. Moore, "*Unwinding strings and T-duality of Kaluza-Klein and H-monopoles*," Adv. Theor. Math. Phys. **1**, 283 (1997) [arXiv:hep-th/9708086].

[184] D. Tong, "*NS5-branes, T-duality and worldsheet instantons*," JHEP **0207**, 013 (2002) [arXiv:hep-th/0204186].

[185] J. A. Harvey and S. Jensen, "*Worldsheet instanton corrections to the Kaluza-Klein monopole*," arXiv:hep-th/0507204.

[186] K. Okuyama, "*Linear sigma models of H and KK monopoles*," JHEP **0508**, 089 (2005) [arXiv:hep-th/0508097].

[187] A, Zee, "*Quantum Hall Fluids*", cond-mat/9501022.

[188] K. A. Intriligator and N. Seiberg, "*Mirror symmetry in three dimensional gauge theories*," Phys. Lett. B **387**, 513 (1996) [arXiv:hep-th/9607207].

[189] A. Kapustin and M. J. Strassler, "*On mirror symmetry in three dimensional Abelian gauge theories*," JHEP **9904**, 021 (1999) [arXiv:hep-th/9902033].

[190] J. de Boer, K. Hori, H. Ooguri and Y. Oz, "*Mirror symmetry in three-dimensional gauge theories, quivers and D-branes*," Nucl. Phys. B **493**, 101 (1997) [arXiv:hep-th/9611063].

[191] J. de Boer, K. Hori, H. Ooguri, Y. Oz and Z. Yin, "*Mirror symmetry in three-dimensional gauge theories, SL(2,Z) and D-brane moduli spaces*," Nucl. Phys. B **493**, 148 (1997) [arXiv:hep-th/9612131].

[192] B. Feng and A. Hanany, "*Mirror symmetry by O3-planes*," JHEP **0011**, 033 (2000) [arXiv:hep-th/0004092].

[193] V. Borokhov, A. Kapustin and X. k. Wu, "*Monopole operators and mirror symmetry in three dimensions*," JHEP **0212**, 044 (2002) [arXiv:hep-th/0207074].

[194] N. Ohta and D. Tomino, "*Noncommutative gauge dynamics from brane configurations with background B field*," Prog. Theor. Phys. **105**, 287 (2001) [arXiv:hep-th/0009021].

[195] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg and M. J. Strassler, "*Aspects of $N = 2$ supersymmetric gauge theories in three dimensions*," Nucl. Phys. B **499**, 67 (1997) [arXiv:hep-th/9703110].

[196] J. de Boer, K. Hori, Y. Oz and Z. Yin, "*Branes and mirror symmetry in $N = 2$ supersymmetric gauge theories in three dimensions*," Nucl. Phys. B **502**, 107 (1997) [arXiv:hep-th/9702154].

[197] T. Kitao, K. Ohta and N. Ohta, "*Three-dimensional gauge dynamics from brane configurations with (p,q)-fivebrane*," Nucl. Phys. B **539**, 79 (1999) [arXiv:hep-th/9808111].

[198] N. Dorey and D. Tong, "*Mirror symmetry and toric geometry in three dimensional gauge theories*," JHEP **0005**, 018 (2000) [arXiv:hep-th/9911094].

[199] D. Tong, "*Dynamics of $N = 2$ supersymmetric Chern-Simons theories*," JHEP **0007**, 019 (2000) [arXiv:hep-th/0005186].

[200] A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, "*Nonvanishing quantum corrections to the mass and central charge of the N=2 vortex and BPS saturation*," Nucl. Phys. B **679**, 382 (2004) [arXiv:hep-th/0307282].

[201] M. Aganagic, K. Hori, A. Karch and D. Tong, "*Mirror symmetry in 2+1 and 1+1 dimensions*," JHEP **0107**, 022 (2001) [arXiv:hep-th/0105075].

[202] E. J. Copeland and N. Turok, "*Cosmic String Interactions*," FERMILAB PUB 86-127-A

[203] E. P. S. Shellard, "*Understanding Intercommuting*," Proc. of Yale Workshop: Cosmic Strings: The Current Status (1988)

[204] A. Hanany and K. Hashimoto, "*Reconnection of colliding cosmic strings*," JHEP **0506**, 021 (2005) [arXiv:hep-th/0501031].

[205] R. A. Matzner, "*Interaction of $U(1)$ Cosmic Strings: Numerical Intercommutation*", Computers in Physics, **2**(5), 51 (1988)

[206] E. Witten, "*Cosmic Superstrings*," Phys. Lett. B **153**, 243 (1985).

[207] J. Polchinski, "*Collision Of Macroscopic Fundamental Strings*," Phys. Lett. B **209** (1988) 252.

[208] M. G. Jackson, N. T. Jones and J. Polchinski, "*Collisions of cosmic F- and D-strings*," arXiv:hep-th/0405229.

[209] S. Cecotti and C. Vafa, "*On classification of N=2 supersymmetric theories*," Commun. Math. Phys. **158**, 569 (1993) [arXiv:hep-th/9211097].

[210] E. R. Abraham and P. K. Townsend, "*Q Kinks*", Phys. Lett. B **291**, 85 (1992); '*'More On Q Kinks: A (1+1)-Dimensional Analog Of Dyons*", Phys. Lett. B **295**, 225 (1992).

[211] M. Shifman and A. Yung, "*Domain walls and flux tubes in $N = 2$ SQCD: D-brane prototypes*" Phys. Rev. D **67**, 125007 (2003) [arXiv:hep-th/0212293].

[212] D. Tong, "*The moduli space of BPS domain walls*" Phys. Rev. D **66**, 025013 (2002) [arXiv:hep-th/0202012].

[213] N. Sakai and Y. Yang, "*Moduli sapce of BPS walls in supersymmetric gauge theories*," arXiv:hep-th/0505136.

[214] N. Sakai and D. Tong, "*Monopoles, vortices, domain walls and D-branes: The rules of interaction*," JHEP **0503**, 019 (2005) [arXiv:hep-th/0501207].

[215] J. P. Gauntlett, D. Tong and P. K. Townsend, "*Multi Domain Walls in Massive Supersymmetric Sigma Models*", Phys. Rev. D **64**, 025010 (2001) [arXiv:hep-th/0012178].

[216] K. S. M. Lee, "*An index theorem for domain walls in supersymmetric gauge theories,*" Phys. Rev. D **67**, 045009 (2003) [arXiv:hep-th/0211058].

[217] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, "*Non-Abelian walls in supersymmetric gauge theories*," Phys. Rev. D **70**, 125014 (2004) [arXiv:hep-th/0405194].

[218] Y. Isozumi, K. Ohashi and N. Sakai, "*Exact wall solutions in 5-dimensional SUSY QED at finite coupling*" JHEP **0311**, 060 (2003) [arXiv:hep-th/0310189].

[219] N. Dorey, "*The BPS spectra of two-dimensional supersymmetric gauge theories with twisted mass terms*" JHEP **9811**, 005 (1998) [arXiv:hep-th/9806056].

[220] K. Lee and H. U. Yee, "*New BPS objects in N = 2 supersymmetric gauge theories*," arXiv:hep-th/0506256.

[221] M. Eto, Y. Isozumi, M. Nitta and K. Ohashi, "*1/2, 1/4 and 1/8 BPS equations in SUSY Yang-Mills-Higgs systems: Field theoretical brane configurations*," arXiv:hep-th/0506257.

[222] J. P. Gauntlett, D. Tong and P. K. Townsend, "*Supersymmetric intersecting domain walls in massive hyper-Kaehler sigma models*" Phys. Rev. D **63**, 085001 (2001) [arXiv:hep-th/0007124].

[223] K. Kakimoto and N. Sakai, "*Domain wall junction in N = 2 supersymmetric QED in four dimensions*" Phys. Rev. D **68**, 065005 (2003) [arXiv:hep-th/0306077].

[224] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, "*Webs of walls*," arXiv:hep-th/0506135.

[225] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, "*Non-abelian webs of walls*," arXiv:hep-th/0508241.

[226] M. Arai, M. Naganuma, M. Nitta and N. Sakai, "*BPS wall in N = 2 SUSY nonlinear sigma model with Eguchi-Hanson manifold,*" arXiv:hep-th/0302028; "*Manifest supersymmetry for BPS walls in N = 2 nonlinear sigma models,*" Nucl. Phys. B **652**, 35 (2003) [arXiv:hep-th/0211103].

[227] A. Losev and M. Shifman, "*N = 2 sigma model with twisted mass and superpotential: Central charges and solitons*" Phys. Rev. D **68**, 045006 (2003) [arXiv:hep-th/0304003].

[228] M. Arai, E. Ivanov and J. Niederle, "*Massive nonlinear sigma models and BPS domain walls in harmonic superspace*," Nucl. Phys. B **680**, 23 (2004) [arXiv:hep-th/0312037].

[229] Y. Isozumi, M. Nitta, K. Ohashi and N. Sakai, "*Construction of non-Abelian walls and their complete moduli space,*" Phys. Rev. Lett. **93**, 161601 (2004); [arXiv:hep-th/0404198]. "*All exact solutions of a 1/4 BPS equation,*" Phys. Rev. D **71**, 065018 (2005) [arXiv:hep-th/0405129].

[230] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta, N. Sakai and Y. Tachikawa, "*Global structure of moduli space for BPS walls*," Phys. Rev. D **71**, 105009 (2005) [arXiv:hep-th/0503033].

[231] A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, "*The anomaly in the central charge of the supersymmetric kink from dimensional regularization and reduction*," Nucl. Phys. B **648**, 174 (2003) [arXiv:hep-th/0207051].

[232] A. S. Goldhaber, A. Rebhan, P. van Nieuwenhuizen and R. Wimmer, "*Quantum corrections to mass and central charge of supersymmetric solitons*," Phys. Rept. **398**, 179 (2004) [arXiv:hep-th/0401152].

[233] D. Tong, "*Monopoles in the Higgs phase*" Phys. Rev. D **69**, 065003 (2004) [arXiv:hep-th/0307302].

[234] M. Hindmarsh and T. W. Kibble, "*Beads On Strings*" Phys. Rev. Lett. **55**, 2398 (1985).

[235] M. A. Kneipp, "*Z(k) string fluxes and monopole confinement in non-Abelian theories*" arXiv:hep-th/0211049.

[236] R. Auzzi, S. Bolognesi, J. Evslin and K. Konishi, "*Nonabelian monopoles and the vortices that confine them*" arXiv:hep-th/0312233.

[237] V. Markov, A. Marshakov and A. Yung, "*Non-Abelian vortices in N = 1* gauge theory*" arXiv:hep-th/0408235.

[238] R. Auzzi, S. Bolognesi and J. Evslin, "*Monopoles can be confined by 0, 1 or 2 vortices*", arXiv:hep-th/0411074.

[239] M. Shifman and A. Yung, "*Localization of non-Abelian gauge fields on domain walls at weak coupling (D-brane prototypes II)*" [arXiv:hep-th/0312257].

[240] M. Eto, M. Nitta, K. Ohashi and D. Tong, "*Skyrmions from Instantons inside Domain Walls*," arXiv:hep-th/0508130.

[241] M. F. Atiyah and N. S. Manton, "*Skyrmions From Instantons*," Phys. Lett. B **222**, 438 (1989).

[242] C. T. Hill, "*Topological solitons from deconstructed extra dimensions*," Phys. Rev. Lett. **88**, 041601 (2002) [arXiv:hep-th/0109068].

[243] C. T. Hill and C. K. Zachos, "*Dimensional deconstruction and Wess-Zumino-Witten terms*," Phys. Rev. D **71**, 046002 (2005) [arXiv:hep-th/0411157].

[244] D. T. Son and M. A. Stephanov, "*QCD and dimensional deconstruction*," Phys. Rev. D **69**, 065020 (2004) [arXiv:hep-ph/0304182].

[245] N. Dorey, T. J. Hollowood and D. Tong, "*The BPS spectra of gauge theories in two and four dimensions*" JHEP **9905**, 006 (1999) [arXiv:hep-th/9902134].

[246] N. D. Lambert and D. Tong, "*Kinky D-strings*" Nucl. Phys. B **569**, 606 (2000) [arXiv:hep-th/9907098].

[247] M. Eto, Y. Isozumi, M. Nitta, K. Ohashi, K. Ohta and N. Sakai, "*D-brane construction for non-Abelian walls*," Phys. Rev. D **71**, 125006 (2005) [arXiv:hep-th/0412024].

[248] A. Hanany and D. Tong, "*On monopoles and domain walls*," arXiv:hep-th/0507140.

[249] C. Bachas, J. Hoppe and B. Pioline, "*Nahm equations, $N = 1^\star$ domain walls, and D-strings in $AdS_5 \times S^5$*," JHEP **0107**, 041 (2001) [arXiv:hep-th/0007067].

[250] E. Witten, "*String Theory and Black Holes*", Phys. Rev. **D44** (1991) 314;

[251] D. Tong, "*Mirror mirror on the wall: On two-dimensional black holes and Liouville theory*," JHEP **0304**, 031 (2003) [arXiv:hep-th/0303151].

[252] V. Fateev, A. Zamalodchikov and Al. Zamalodchikov, unpublished, as cited in [253]

[253] V. Kazakov, I. Kostov, D. Kutasov, "*A Matrix Model for the Two-Dimensional Black Hole*", Nucl. Phys. **B622** (2002) 141, hep-th/0101011.

[254] K. Hori and A. Kapustin "*Duality of the Fermionic 2d Black Hole and N=2 Liouville Theory as Mirror Symmetry*", JHEP **0108**, 045 (2001), hep-th/0104202.

[255] N. S. Manton, "*An Effective Lagrangian For Solitons*," Nucl. Phys. B **150**, 397 (1979).

[256] G. R. Dvali and M. A. Shifman, "*Domain walls in strongly coupled theories*" Phys. Lett. B **396**, 64 (1997) [Erratum-ibid. B **407**, 452 (1997)] [arXiv:hep-th/9612128].

[257] J. P. Gauntlett, R. Portugues, D. Tong and P. K. Townsend, "*D-brane solitons in supersymmetric sigma models*" Phys. Rev. D **63**, 085002 (2001) [arXiv:hep-th/0008221].

[258] S. M. Carroll and M. Trodden, "*Dirichlet topological defects*" Phys. Rev. D **57**, 5189 (1998) [arXiv:hep-th/9711099].

[259] G. Dvali and A. Vilenkin, "*Solitonic D-branes and brane annihilation,*" Phys. Rev. D **67**, 046002 (2003) [arXiv:hep-th/0209217].

[260] M. Bowick, A. De Felice and M. Trodden, "*The shapes of Dirichlet defects,*" JHEP **0310**, 067 (2003) [arXiv:hep-th/0306224].

[261] R. Auzzi, M. Shifman and A. Yung, "*Studying boojums in $N = 2$ theory with walls and vortices*," Phys. Rev. D **72**, 025002 (2005) [arXiv:hep-th/0504148].

[262] N. D. Mermin, "*Surface Singularities and Superflow in ^{3}He-A*", in Quantum Fluids and Solids, eds. S. B. Trickey, E. D. Adams and J. W. Dufty, Plnum, New York, pp 3-22.

[263] D. Tong, "*D-Branes in Field Theory*", arXiv:hep-th/0512192.