Differential Geometry

Yury Holubeu *

February 25, 2026

This note is not intended for distribution. Differential geometry and applications are discussed in detail. Links below show contents of solved problems, summary of special topics. I used the following literature.

Contents

1 Preface and main motivation **18**

I —— Typical Differential Geometry in a Nutshell —— **19**

2 Basic diffhem constructions in a nutshell **19**
2.1 Main Concepts and Their Properties . . . 19
2.1.1 Basic Concepts Differential Geometry . . . 19
2.1.2 Differential Forms: Main Properties . . . 24
2.1.3 Differential Forms: Integration . . . 28
2.1.4 Tangent Spaces . . . 28
2.1.5 Vectors, Covectors and Vector Fields . . . 29
2.1.6 Tensors, Tensor Fields . . . 30
2.1.7 Lie Derivative . . . 30
2.1.8 $\mathrm{div}, \mathrm{rot}, \mathrm{grad}, \Box$ . . . 31
2.1.9 rotations (!?) . . . 33
2.1.10 Manifolds: Basic Properties . . . 34
2.2 Shapes, Surfaces . . . 34
2.2.1 Curvilinear Coordinates . . . 34
2.2.2 Parametric Surfaces, First and Second Fundamental Forms . . . 37
2.2.3 Metric . . . 44
2.2.4 Connections, Covariant Derivative . . . 46
2.2.5 Geodesics, Geodesic Deviation . . . 50
2.2.6 Parallel Transport (!???) . . . 52
2.2.7 Killing Vectors and Tensors (!?) . . . 53
2.2.8 Riemann Tensor, Ricci Scalar, etc. . . . 53
2.2.9 More on the Riemann Tensor and Applications . . . 55
2.2.10 Other About Surfaces (!!?!???!?) . . . 55
2.3 Curves . . . 55
2.3.1 Flat Curves . . . 55
2.3.2 Curves in 3d, Frenet–Serret formulas . . . 58
2.3.3 Other About Curves . . . 59
2.4 Selected Models of DG . . . 59
2.4.1 Lobachevsky Geometry . . . 59
2.5 GD in Physics: Selected Topics . . . 63
2.6 Selected Special Topics . . . 63
2.6.1 vector Bundles . . . 63
2.6.2 Homotopy Groups . . . 63
2.6.3 Connectivity in Bundle . . . 63

II Fundamentals of Differential Geometry **64**

*yuri.holubev@gmail.com

3 Differential Forms **64**
3.1 Tangent Space and Differential . . . 64
3.2 Integration of Differential Forms . . . 64
3.2.1 Unit Breakdown . . . 64
3.2.2 Stokes Formula . . . 64
3.3 Vector Analysis . . . 71
3.3.1 vector Fields . . . 71
3.3.2 vector Operators Through Differential Shapes . . . 71
3.3.3 Divergence . . . 72
3.4 homological Analysis of Forms . . . 72
3.4.1 closed and Precise Shapes . . . 72
3.4.2 cohomology de Rama . . . 74

4 Fundamentals of Riemannian Geometry **74**
4.1 Metrics . . . 74
4.1.1 Intuitive Understanding of the Metric . . . 74
4.1.2 Induced Metric . . . 77
4.1.3 metric Isometrics . . . 77
4.1.4 Examples of Metrics . . . 77
4.1.5 Other Properties of Metric . . . 77
4.2 Affinity Connectivity (!!!) . . . 77
4.2.1 Main Structure . . . 77
4.2.2 Covariant Derivative Endorsements . . . 79
4.2.3 Covariant Derivative Endorsements . . . 79
4.2.4 christoffel Symbols . . . 80
4.3 Riemann Curvature . . . 80
4.3.1 essence of Torsion and Curvature . . . 80
4.3.2 Torsion . . . 84
4.3.3 Riemannian Manifolds . . . 84
4.3.4 riemannian Volumes . . . 84
4.3.5 Integral Curves. . . . 84
4.4 Li Derivative (????) . . . 85
4.4.1 Algebraic Properties of the Lie Derivative . . . 85
4.4.2 li Derivative from Different Objects . . . 88
4.4.3 Geometric Meaning of the Lie Derivative . . . 89
4.4.4 Other Properties of Li Derivative . . . 94
4.4.5 Some Typical Applications of the Li Derivative . . . 94
4.5 Forward and Reverse Images . . . 94
4.5.1 Design . . . 95
4.5.2 Overview of Applications and Relationships to Other Structures . . . 96
4.6 cohomologies and Similar Constructs . . . 97
4.6.1 On the Construction of Cohlomologies . . . 97
4.6.2 Examples of Typical Cohlomologies . . . 100
4.6.3 the Eulerian Characteristic . . . 100
4.6.4 Homologies . . . 100
4.6.5 the Fundamental Group . . . 100
4.6.6 Atypical Cohlomologies . . . 101
4.7 Killing Vector . . . 101
4.7.1 Idea of Killing Vector . . . 101
4.7.2 Overview of Applications of Killing Fields (???) . . . 101

5 Surfaces and curves **101**
5.1 Coordinate System . . . 101
5.2 Riemannian Manifolds . . . 104
5.3 Curve . . . 104
5.3.1 curve Definitions . . . 104
5.3.2 length and Characteristics . . . 105
5.3.3 Curvature . . . 105
5.3.4 flat Curves . . . 106
5.3.5 spatial Curves . . . 106
5.3.6 complements and Tasks About Flat Curves . . . 106
5.3.7 curves in Space . . . 106
5.4 Description of Surfaces . . . 106
5.4.1 second Fundamental Shape of Surfaces . . . 106
5.4.2 Riemann Curvature Tensor for Surfaces . . . 107
5.5 Representative Surfaces: . . . 108

6 1 Euclidean Spaces **108**
6.0.1 1 Euclidean Spaces . . . 108
6.1 §1 Smooth Functions on a Euclidean Space . . . 109
6.1.1 1.1C^∞ Versus Analytic Functions . . . 109
6.1.2 1.2 Taylor's Theorem with Remainder . . . 111
6.1.3 Problems . . . 113
6.2 §2 Tangent Vectors in r^n as Derivations . . . 115
6.2.1 2.1 The Directional Derivative . . . 115
6.2.2 2.2 Germs of Functions . . . 116
6.2.3 2.3 Derivations at a Point . . . 117
6.2.4 2.4 Vector Fields . . . 118
6.2.5 2.5 Vector Fields as Derivations . . . 120
6.2.6 Problems . . . 121
6.3 §3 the Exterior Algebra of Multicovectors . . . 121
6.3.1 3.1 Dual Space . . . 122
6.3.2 3.2 Permutations . . . 123
6.3.3 3.3 Multilinear Functions . . . 126
6.3.4 3.4 The Permutation Action on Multilinear Functions . . . 127
6.3.5 3.5 The Symmetrizing and Alternating Operators . . . 128
6.3.6 3.6 The Tensor Product . . . 128
6.3.7 3.7 The Wedge Product . . . 129
6.3.8 3.8 Anticommutativity of the Wedge Product . . . 130
6.3.9 3.9 Associativity of the Wedge Product . . . 131
6.3.10 3.10 A Basis for k-covectors . . . 133
6.3.11 Problems . . . 134
6.4 §4 Differential Forms on $\mathbb{R}^n$ . . . 136
6.4.1 4.1 Differential 1-forms and the Differential of a Function . . . 136
6.4.2 4.2 Differential k-forms . . . 138
6.4.3 4.3 Differential Forms as Multilinear Functions on Vector Fields . . . 139
6.4.4 4.4 The Exterior Derivative . . . 140
6.4.5 4.5 Closed Forms and Exact Forms . . . 142
6.4.6 4.6 Applications to Vector Calculus . . . 142
6.4.7 4.7 Convention on Subscripts and Superscripts . . . 145
6.4.8 Problems . . . 145

7 2 Manifolds **146**
7.1 §5 Manifolds . . . 147
7.1.1 5.1 Topological Manifolds . . . 148
7.1.2 5.2 Compatible Charts . . . 149
7.1.3 5.3 Smooth Manifolds . . . 151
7.1.4 5.4 Examples of Smooth Manifolds . . . 152
7.1.5 Problems . . . 154
7.2 §6 Smooth Maps on a Manifold . . . 156
7.2.1 6.1 Smooth Functions on a Manifold . . . 157
7.2.2 6.2 Smooth Maps between Manifolds . . . 158
7.2.3 6.3 Diffeomorphisms . . . 159
7.2.4 6.4 Smoothness in Terms of Components . . . 160
7.2.5 6.5 Examples of Smooth Maps . . . 161
7.2.6 Examples. . . . 162
7.2.7 6.6 Partial Derivatives . . . 163
7.2.8 6.7 The Inverse Function Theorem . . . 164
7.2.9 Problems . . . 165
7.3 §7 Quotients . . . 166
7.3.1 7.1 The Quotient Topology . . . 166
7.3.2 7.2 Continuity of a Map on a Quotient . . . 167
7.3.3 7.3 Identification of a Subset to a Point . . . 168
7.3.4 7.4 A Necessary Condition for a Hausdorff Quotient . . . 168
7.3.5 7.5 Open Equivalence Relations . . . 168
7.3.6 7.6 Real Projective Space . . . 170
7.3.7 7.7 The Standard c^∞ Atlas on a Real Projective Space . . . 174
7.3.8 Problems . . . 175

8 3 The Tangent Space **178**
8.1 §8 the Tangent Space . . . 178
8.1.1 8.1 The Tangent Space at a Point . . . 179
8.1.2 8.2 The Differential of a Map . . . 180
8.1.3 8.3 The Chain Rule . . . 180
8.1.4 8.4 Bases for the Tangent Space at a Point . . . 181
8.1.5 8.5 A Local Expression for the Differential . . . 182
8.1.6 8.6 Curves in a Manifold . . . 183
8.1.7 8.7 Computing the Differential Using Curves . . . 186
8.1.8 8.8 Immersions and Submersions . . . 186
8.1.9 8.9 Rank, and Critical and Regular Points . . . 187
8.1.10 Problems . . . 188
8.2 §9 Submanifolds . . . 190
8.2.1 9.1 Submanifolds . . . 190
8.2.2 9.2 Level Sets of a Function . . . 192
8.2.3 9.3 The Regular Level Set Theorem . . . 194
8.2.4 9.4 Examples of Regular Submanifolds . . . 195
8.2.5 Problems . . . 196
8.3 §10 Categories and Functors . . . 199
8.3.1 10.1 Categories . . . 199
8.3.2 10.2 Functors . . . 200
8.3.3 10.3 The Dual Functor and the Multicovector Functor . . . 201
8.3.4 Problems . . . 202
8.4 §11 the Rank of a Smooth Map . . . 203
8.4.1 11.1 Constant Rank Theorem . . . 203
8.4.2 11.2 The Immersion and Submersion Theorems . . . 205
8.4.3 11.3 Images of Smooth Maps . . . 207

8.4.4 11.4 Smooth Maps Into a Submanifold . . . 210
8.4.5 11.5 The Tangent Plane to a Surface in $\mathbb{R}^3$ . . . 212
8.4.6 Problems . . . 213
8.5 §12 the Tangent Bundle . . . 214
8.5.1 12.1 The Topology of the Tangent Bundle . . . 215
8.5.2 12.2 The Manifold Structure on the Tangent Bundle . . . 218
8.5.3 12.3 Vector Bundles . . . 219
8.5.4 12.4 Smooth Sections . . . 221
8.5.5 12.5 Smooth Frames . . . 223
8.5.6 Problems . . . 224
8.6 §13 Bump Functions and Partitions of Unity . . . 225
8.6.1 13.1 c^∞ Bump Functions . . . 225
8.6.2 13.2 Partitions of Unity . . . 229
8.6.3 13.3 Existence of a Partition of Unity . . . 230
8.6.4 Problems . . . 232
8.7 §14 Vector Fields . . . 233
8.7.1 14.1 Smoothness of a Vector Field . . . 233
8.7.2 14.2 Integral Curves . . . 235
8.7.3 14.3 Local Flows . . . 237
8.7.4 14.4 The Lie Bracket . . . 239
8.7.5 14.5 The Pushforward of Vector Fields . . . 241
8.7.6 14.6 Related Vector Fields . . . 242
8.7.7 Problems . . . 242

9 4 Lie Groups and Lie Algebras 245
9.1 §15 Lie Groups . . . 246
9.1.1 15.1 Examples of Lie Groups . . . 246
9.1.2 15.2 Lie Subgroups . . . 248
9.1.3 15.3 The Matrix Exponential . . . 250
9.1.4 15.4 The Trace of a Matrix . . . 252
9.1.5 15.5 The Differential of det at the Identity . . . 254
9.1.6 Problems . . . 255
9.2 §16 Lie Algebras . . . 258
9.2.1 16.1 Tangent Space at the Identity of a Lie Group . . . 258
9.2.2 16.2 Left-invariant Vector Fields on a Lie Group . . . 260
9.2.3 16.3 The Lie Algebra of a Lie Group . . . 262
9.2.4 16.4 The Lie Bracket on $\mathfrak{gl}(n, \mathbb{R})$ . . . 262
9.2.5 16.5 The Pushforward of Left-invariant Vector Fields . . . 264
9.2.6 16.6 The Differential as a Lie Algebra Homomorphism . . . 265
9.2.7 Problems . . . 266

10 5 Differential Forms 267
10.1 §17 Differential 1-forms . . . 269
10.1.1 17.1 The Differential of a Function . . . 269
10.1.2 17.2 Local Expression for a Differential 1-form . . . 270
10.1.3 17.3 The Cotangent Bundle . . . 270
10.1.4 17.4 Characterization of c^∞ 1-Forms . . . 271
10.1.5 17.5 Pullback of 1-forms . . . 273
10.1.6 17.6 Restriction of 1-forms to an Immersed Submanifold . . . 275
10.1.7 Problems . . . 276
10.2 §18 Differential k-forms . . . 277
10.2.1 18.1 Differential Forms . . . 277
10.2.2 18.2 Local Expression for $\boldsymbol{ak}$-Form . . . 279
10.2.3 18.3 The Bundle Point of View . . . 280

10.2.4 18.4 Smooth k-forms . . . 280
10.2.5 18.5 Pullback of $\boldsymbol{k}$-Forms . . . 281
10.2.6 18.6 The Wedge Product . . . 282
10.2.7 18.7 Differential Forms on a Circle . . . 283
10.2.8 18.8 Invariant Forms on a Lie Group . . . 283
10.2.9 Problems . . . 284
10.3 §19 the Exterior Derivative . . . 285
10.3.1 19.1 Exterior Derivative on a Coordinate Chart . . . 286
10.3.2 19.2 Local Operators . . . 287
10.3.3 19.3 Existence of an Exterior Derivative on a Manifold . . . 287
10.3.4 19.4 Uniqueness of the Exterior Derivative . . . 288
10.3.5 19.5 Exterior Differentiation under a Pullback . . . 289
10.3.6 19.6 Restriction of k-forms to a Submanifold . . . 291
10.3.7 19.7 A Nowhere-vanishing 1-form on the Circle . . . 291
10.3.8 Problems . . . 293
10.4 §20 the Lie Derivative and Interior Multiplication . . . 296
10.4.1 20.1 Families of Vector Fields and Differential Forms . . . 296
10.4.2 20.2 The Lie Derivative of a Vector Field . . . 298
10.4.3 20.3 The Lie Derivative of a Differential Form . . . 301
10.4.4 20.4 Interior Multiplication . . . 301
10.4.5 20.5 Properties of the Lie Derivative . . . 303
10.4.6 20.6 Global Formulas for the Lie and Exterior Derivatives . . . 306
10.4.7 Problems . . . 307
10.5 The Change into Non-Coordinate Language and Back (????) . . . 308

11 6 Integration **308**
11.1 §21 Orientations . . . 309
11.1.1 21.1 Orientations of a Vector Space . . . 309
11.1.2 21.2 Orientations and n-covectors . . . 311
11.1.3 21.3 Orientations on a Manifold . . . 312
11.1.4 21.4 Orientations and Differential Forms . . . 314
11.1.5 21.5 Orientations and Atlases . . . 317
11.1.6 Problems . . . 318
11.2 §22 Manifolds with Boundary . . . 319
11.2.1 22.1 Smooth Invariance of Domain in $\mathbb{R}^n$ . . . 320
11.2.2 22.2 Manifolds with Boundary . . . 321
11.2.3 22.3 The Boundary of a Manifold with Boundary . . . 323
11.2.4 22.4 Tangent Vectors, Differential Forms, and Orientations . . . 323
11.2.5 22.5 Outward-pointing Vector Fields . . . 324
11.2.6 22.6 Boundary Orientation . . . 325
11.2.7 Problems . . . 326
11.3 §23 Integration on Manifolds . . . 329
11.3.1 23.1 The Riemann Integral of a Function on $\mathbb{R}^{\boldsymbol{n}}$ . . . 329
11.3.2 23.2 Integrability Conditions . . . 331
11.3.3 23.3 The Integral of an $\boldsymbol{n}$-Form on $\mathbb{R}^{\boldsymbol{n}}$ . . . 332
11.3.4 23.5 Stokes's Theorem . . . 336
11.3.5 23.6 Line Integrals and Green's Theorem . . . 338
11.3.6 Problems . . . 339

12 7 De Rham Theory **339**
12.1 §24 De Rham Cohomology . . . 341
12.1.1 24.1 De Rham Cohomology . . . 341
12.1.2 24.2 Examples of de Rham Cohomology . . . 343
12.1.3 24.3 Diffeomorphism Invariance . . . 344

12.1.4 24.4 The Ring Structure on de Rham Cohomology . . . 345
12.1.5 Problems . . . 346
12.2 §25 the Long Exact Sequence in Cohomology . . . 346
12.2.1 25.1 Exact Sequences . . . 347
12.2.2 25.2 Cohomology of Cochain Complexes . . . 348
12.2.3 25.3 The Connecting Homomorphism . . . 349
12.2.4 25.4 The Zig-zag Lemma . . . 350
12.2.5 Problems . . . 352
12.3 §26 the Mayer-vietoris Sequence . . . 352
12.3.1 26.1 The Mayer-vietoris Sequence . . . 352
12.3.2 26.2 The Cohomology of the Circle . . . 357
12.3.3 26.3 The Euler Characteristic . . . 359
12.3.4 Problems . . . 359
12.4 §27 Homotopy Invariance . . . 359
12.4.1 27.1 Smooth Homotopy . . . 360
12.4.2 27.2 Homotopy Type . . . 361
12.4.3 27.3 Deformation Retractions . . . 362
12.4.4 27.4 The Homotopy Axiom for de Rham Cohomology . . . 363
12.4.5 Problems . . . 364
12.5 §28 Computation of de Rham Cohomology . . . 364
12.5.1 28.1 Cohomology Vector Space of a Torus . . . 365
12.5.2 28.2 The Cohomology Ring of a Torus . . . 366
12.5.3 28.3 The Cohomology of a Surface of Genus $\boldsymbol{g}$ . . . 368
12.5.4 Problems . . . 372
12.6 §29 Proof of Homotopy Invariance . . . 373
12.6.1 29.1 Reduction to Two Sections . . . 374
12.6.2 29.2 Cochain Homotopies . . . 374
12.6.3 29.3 Differential Forms on $m \times \setminus$ . . . 375
12.6.4 29.4 A Cochain Homotopy between i_0^* and i_1^* . . . 376
12.6.5 29.5 Verification of Cochain Homotopy . . . 377
12.6.6 Problems . . . 378

13 Appendices **379**
13.1 §A Point-set Topology . . . 379
13.1.1 A. 1 Topological Spaces . . . 379
13.1.2 A. 2 Subspace Topology . . . 381
13.1.3 A. 3 Bases . . . 382
13.1.4 A. 4 First and Second Countability . . . 383
13.1.5 A. 5 Separation Axioms . . . 385
13.1.6 A. 6 Product Topology . . . 386
13.1.7 A. 7 Continuity . . . 388
13.1.8 A. 8 Compactness . . . 389
13.1.9 A. 9 Boundedness in $\mathbb{R}^n$ . . . 391
13.1.10 A. 10 Connectedness . . . 392
13.1.11 A. 11 Connected Components . . . 393
13.1.12 A. 12 Closure . . . 393
13.1.13 A. 13 Convergence . . . 395
13.2 B. The Inverse Function Theorem on $\mathbb{R}^n$ and Related Results . . . 398
13.2.1 B. 1 the Inverse Function Theorem . . . 398
13.2.2 B. 2 the Implicit Function Theorem . . . 398
13.2.3 B. 3 Constant Rank Theorem . . . 402
13.2.4 Problems . . . 403
13.3 §C Existence of a Partition of Unity in General . . . 404
13.4 §D Linear Algebra . . . 406

13.4.1 D. 1 Quotient Vector Spaces . . . 407
13.4.2 D. 2 Linear Transformations . . . 407
13.4.3 D. 3 Direct Product and Direct Sum . . . 408
13.4.4 Problems . . . 410
13.5 §E Quaternions and the Symplectic Group . . . 410
13.5.1 E. 1 Representation of Linear Maps by Matrices . . . 411
13.5.2 E. 2 Quaternionic Conjugation . . . 412
13.5.3 E. 3 Quaternionic Inner Product . . . 413
13.5.4 E. 4 Representations of Quaternions by Complex Numbers . . . 413
13.5.5 E. 5 Quaternionic Inner Product in Terms of Complex Components . . . 413
13.5.6 E. 6 H-linearity in Terms of Complex Numbers . . . 414
13.5.7 E. 7 Symplectic Group . . . 415
13.5.8 Problems . . . 416
13.6 Solutions to Selected Exercises Within the Text . . . 416
13.7 Hints and Solutions to Selected End-of-section Problems . . . 422

14 Lectures on Classical Differential Geometry by Ivanov, A.A. Tuzhilin 449
14.1 1 Curves in Euclidean space . . . 449
14.1.1 1.1 Symbols and Reminders . . . 449
14.1.2 1.2 Parametric curves . . . 450
14.1.3 1.3 Graph Curves . . . 453
14.1.4 1.4 Implicit curves . . . 453
14.1.5 1.5 Local equivalence . . . 454
14.1.6 1.6 Curve length, natural parameter . . . 455
14.1.7 1.7 Regular Curve Curvature . . . 456
14.1.8 1.8 Flat curves . . . 457
14.1.9 1.9 Problems . . . 459
14.2 1.10 Curves in Euclidean space: Supplemental material . . . 460
14.2.1 1.10.1 Peano curve in a triangle . . . 460
14.2.2 1.10.2 Polyline as a smooth parametric curve . . . 461
14.2.3 1.10.3 The length of the curve as the limit of the lengths of inscribed polylines 462
14.2.4 1.10.4 Osculating Circle . . . 463
14.2.5 1.10.5 On explicit solutions of the natural equation . . . 464
14.2.6 1.10.6 Evolutes & Evolvents . . . 464
14.2.7 1.10.7 Huygens Pendulum . . . 467
14.3 2 Curves in three-dimensional space . . . 468
14.3.1 2.1 Frenet's Formulas . . . 468
14.3.2 2.2 Natural equations . . . 470
14.3.3 2.4 Slope lines . . . 472
14.4 3. Surfaces and First Fundamental Form . . . 473
14.4.1 3.1 Parametric Surfaces . . . 473
14.4.2 3.2 Graphic Surfaces . . . 474
14.4.3 3.3 Implicit surfaces . . . 475
14.4.4 3.4 Local equivalence . . . 476
14.4.5 3.5 Surfaces and Mappings . . . 477
14.4.6 3.6 Surface curves . . . 478
14.4.7 3.7 Tangent Space . . . 478
14.4.8 3.8 Differential . . . 480
14.4.9 3.9 First fundamental form . . . 481
14.4.10 3.10 Lengths and angles . . . 483
14.4.11 3.11 Surface isometries . . . 484
14.4.12 3.12 Problems . . . 485
14.4.13 3.13.1 Examples of calculation of induced metric . . . 486
14.4.14 3.13.2 Rhumb line . . . 487

14.4.15 3.13.3 First quadratic form and volume . . . 488
14.4.16 3.13.4 Euclidean Metric Isometries . . . 490
14.5 4 Surfaces. The second fundamental form . . . 491
14.5.1 4.1 Definition of the second form . . . 491
14.5.2 4.2 The geometric meaning of the second form is the curvatures of flat sections 493
14.5.3 4.3 Principal curvatures and principal directions . . . 495
14.5.4 4.4 Mean and Gaussian curvatures . . . 497
14.5.5 4.5 Bonn Theorem . . . 499
14.5.6 4.6 Problems . . . 499
14.5.7 4.7 Supplemental material . . . 500
14.5.8 4.7.1 Other proof of claim 4.7 . . . 500
14.5.9 4.7.2 Second cylinder shape and Meunier's theorem . . . 500
14.5.10 4.7.3 Main Cylinder Curvatures . . . 501
14.5.11 4.7.4 Laplace-Poisson theorem . . . 502
14.5.12 4.7.5 Mean and Gaussian curvatures of a two-dimensional surface . . . 504
14.5.13 4.7.6 On the choice of coordinates on the surface . . . 504
14.5.14 4.7.9 Rotation surfaces of constant Gaussian curvature . . . 507
14.5.15 4.7.10 Case of arbitrary codimensionality . . . 509
14.6 5 Elements of differential calculus on surfaces . . . 510
14.6.1 5.1 Weingarten-Gauss derivational formulas . . . 510
14.6.2 5.2 Gaussian Theorem . . . 513
14.6.3 5.3 Covariant derivative of tangent vector Field . . . 514
14.6.4 5.4 Supplemental material . . . 517
14.6.5 5.4.1 On multidimensional generalizations of the Gaussian theorem . . . 517
14.6.6 5.4.2 Other proof of claim 5.27 . . . 517
14.7 6 Geodetic on surfaces . . . 518
14.7.1 6.1 Definition and simplest properties of geodetic . . . 518
14.7.2 6.2 Examples, Claireaux's theorem . . . 519
14.8 7 Curvilinear coordinates in the area and on the surface . . . 525
14.8.1 7.1 Determination of curvilinear coordinates . . . 525
14.8.2 7.2 Examples of regular coordinate systems . . . 527
14.8.3 7.3 Tangent Space to Area . . . 528
14.8.4 7.4 Euclidean metric in curvilinear coordinates . . . 530
14.8.5 7.5 Regular coordinates on the surface . . . 531
14.8.6 7.6 Stereographic coordinates of the sphere . . . 532
14.8.7 7.7 Problems . . . 532
14.8.8 7.8 Supplemental material . . . 533
14.8.9 7.8.1 Stereographic Projection Properties . . . 533
14.8.10 7.8.2 Multidimensional stereographic projection . . . 533
14.8.11 7.8.3 Normal coordinates . . . 535
14.8.12 7.8.4 Semi-geodetic coordinates . . . 537
14.9 8 Riemann and pseudo-Riemann metrics . . . 538
14.9.1 8.1 Bilinear shape in the area . . . 539
14.9.2 8.2 Pseudo-Euclidean spaces . . . 539
14.9.3 8.3 Spacetime . . . 540
14.9.4 8.4 Pseudosphere geometry . . . 541
14.9.5 8.5 Problems . . . 543
14.9.6 8.6.1 Two-dimensional surfaces of constant Gaussian curvature . . . 544
14.109 Lobachevsky geometry . . . 548
14.10.1 9.1 Non-Euclidean geometries . . . 548
14.10.2 9.1.1 Elliptical geometry . . . 548
14.10.3 9.1.2 Lobachevsky geometry (hyperbolic geometry) . . . 549
14.10.4 9.2 Poincaré model in a circle . . . 549

14.10.5 9.3 Fractional-linear transformations . . . 551
14.10.6 9.4 Recording the metric in a comprehensive form . . . 554
14.10.7 9.5 Top half-plane model . . . 555
14.10.8 9.6 Lobachevsky plane isometries . . . 556
14.10.9 9.7 Problems . . . 557
14.11 9.8 Lobachevsky geometry: Supplemental material . . . 558
14.11.1 9.8.1 Stereographic Projection Properties . . . 558
14.11.2 9.8.2 Metric in the upper half-plane model . . . 559
14.11.3 9.8.3 Geodetic on the plane of Lobachevsky . . . 560
14.11.4 9.8.4 Lobachevsky plane motions in the Poincaré model in a circle . . . 560
14.11.5 9.8.5 Distance between points on the Lobachevsky plane . . . 561
14.11.6 9.8.6 Triangles on the Lobachevsky plane . . . 561
14.12 10 Topological Spaces and Continuous Mappings . . . 565
14.12.1 10.1 Topological spaces . . . 565
14.12.2 10.1.1 Topology and topological space . . . 565
14.12.3 10.1.2 Metric Spaces . . . 566
14.12.4 10.1.3 Topological interior and topological closure . . . 568
14.12.5 10.2 Continuous mappings . . . 568
14.12.6 10.3 Standard Structures . . . 570
14.12.7 10.4 Problems . . . 571
14.12.8 10.5.1 Examples of metric spaces . . . 572
14.12.9 10.5.2 Properties of metric spaces . . . 572
14.12.10 10.5.3 Examples of topological spaces . . . 573
14.13 11 Classes of topological spaces . . . 574
14.13.1 11.1 connectedness . . . 574
14.13.2 11.2 Separability axioms . . . 576
14.13.3 11.3 Compact . . . 577
14.13.4 11.4 Problems . . . 579
14.13.5 11.5.1 Functional separability . . . 580
14.13.6 11.5.2 Unit breakdown . . . 582
14.14 12 Diversity . . . 583
14.14.1 12.1 Topological manifolds . . . 584
14.14.2 12.2 Functions and displays . . . 585
14.14.3 12.3 Smooth manifolds . . . 586
14.14.4 12.4 Examples of smooth manifolds . . . 587
14.14.5 12.5 Smooth functions, smooth mappings, diffeomorphisms . . . 588
14.14.6 12.6 Setting manifolds by equations - the geometric meaning of the implicit function theorem . . . 590
14.14.7 12.7 Problems . . . 592
14.14.8 12.8 Supplemental material . . . 593
14.14.9 12.8.1 Pathological Examples . . . 593
14.14.10 12.8.2 Setting the structure of a smooth manifold on a set . . . 594
14.14.11 12.8.3 Homeomorphism $\mathbb{RP}(3)$ и $SO(3)$ . . . 594
14.15 13 Tangent space to diversity, differential . . . 595
14.15.1 13.1 Tangent vector definitions . . . 595
14.15.2 13.2 Tangential delamination . . . 599
14.15.3 13.3 Definition of differential . . . 600
14.15.4 13.4 Local Properties of Mappings . . . 601
14.15.5 13.5 Problems . . . 604
14.15.6 13.6.2 Local device of submersions and immersions . . . 606
14.16 14 Injections of manifolds into Euclidean space . . . 607
14.16.1 14.1 Existence of an attachment . . . 608
14.16.2 14.2 Sard's theorem . . . 610

14.16.3 14.3 Whitney's "weak" theorem . . . 611
14.16.4 14.4 Problems . . . 613
14.16.5 14.5 Supplemental material . . . 613
14.16.6 14.5.1 To prove Sard's theorem . . . 613
14.16.7 14.5.2 Examples of dives . . . 614
14.17 15 Additional structures: Riemannian metric, orientability . . . 615
14.17.1 15.1 Riemannian metric, Riemannian manifolds . . . 615
14.17.2 15.2 Isometries . . . 619
14.17.3 15.4 Problems . . . 626
14.17.4 15.5 Supplemental material . . . 626
14.17.5 15.5.1 Examples: Klein bottle . . . 626
14.17.6 15.5.2 Examples: projective space $\mathbb{RP}^3$ . . . 627
14.18 16 Classification of connected compact two-dimensional manifolds . . . 628
14.18.1 16.1 Polygon gluing . . . 628
14.18.2 16.2 Sphere sealing . . . 630
14.18.3 16.3 Classification theorem . . . 632
14.18.4 16.3.1 Triangulations . . . 633
14.18.5 16.3.2 Canonical polygon glues . . . 634
14.18.6 16.3.3 Last step, Euler characteristic . . . 634

15 Ivanov 2 **636**
15.1 1: Tensors and tensor fields . . . 638
15.1.1 1.1. The simplest examples of tensors . . . 638
15.1.2 1.2. General tensor definition . . . 641
15.1.3 1.3. Linear space of tensors . . . 645
15.1.4 Problems . . . 646
15.1.5 Extra Topics . . . 647
15.2 2: Algebraic operations on tensors and tensor fields . . . 649
15.2.1 2.1. Linear Combination . . . 649
15.2.2 2.2. Permutation of indices of the same type . . . 650
15.2.3 2.3. свертка . . . 651
15.2.4 2.4. Tensor product . . . 652
15.2.5 2.5. Lowering and raising the index . . . 653
15.2.6 2.6. Symmetry and alternation . . . 655
15.2.7 2.7. Partial alternation . . . 656
15.2.8 Problems . . . 657
15.2.9 Extra Topics . . . 657
15.3 3: Algebra of external differential forms . . . 660
15.3.1 3.1. The space of skew-symmetric tensors . . . 660
15.3.2 3.2. External differential shapes . . . 663
15.3.3 3.3. External Multiplication . . . 664
15.3.4 3.4. Shapes & Displays . . . 667
15.3.5 Problems . . . 668
15.3.6 Extra Topics . . . 669
15.4 4: Differentiating and Integrating Forms . . . 672
15.4.1 4.1. External differentiation . . . 672
15.4.2 4.3. Edge Manifolds . . . 677
15.4.3 Problems . . . 678
15.4.4 Extra Topics . . . 679
15.5 5: The Stokes Theorem . . . 680
15.5.1 5.1. The submanifold integral and the Stokes formula . . . 680
15.5.2 5.3. Green's formula . . . 682
15.5.3 5.4. Gauss-Ostrogradsky formula . . . 683
15.5.4 5.5. Stokes formula for surfaces . . . 685

15.5.5 Problems . . . 687
15.5.6 Extra Topics . . . 688
15.6 6: Cogomology de Rama . . . 691
15.6.1 6.1. Definition of de Rham cohomology groups . . . 692
15.6.2 6.2. Cohomology and mapping . . . 694
15.6.3 6.3. Homotopies and cohomology . . . 695
15.6.4 6.4. Cohomology and Stokes' general formula . . . 696
15.6.5 Problems . . . 697
15.6.6 Extra Topics . . . 697
15.7 7: Covariant differentiation . . . 701
15.7.1 7.1. Euclidean connectivity . . . 702
15.7.2 7.2. Affine Connectivity . . . 705
15.7.3 7.3. Covariant directional derivative . . . 706
15.7.4 7.4. Euclidean coordinates for connectivity . . . 708
15.7.5 Problems . . . 708
15.7.6 Extra Topics . . . 709
15.8 8: Properties of covariant differentiation . . . 709
15.8.1 8.1. Algebraic properties of covariant differentiation . . . 709
15.8.2 8.2. "Uniqueness" of the tensor differentiation operation . . . 711
15.8.3 8.3. Riemannian connectivity . . . 714
15.8.4 8.4. Euclidean coordinates for the metric . . . 716
15.8.5 Problems . . . 717
15.8.6 Extra Topics . . . 717
15.9 9: Parallel Transfer and Geodetic . . . 718
15.9.1 9.1. Translation (geometry) . . . 718
15.9.2 9.2. Parallel transfer in Riemann connectivity . . . 720
15.9.3 9.3. Definition and simplest properties of geodetic . . . 720
15.9.4 9.4. Normal coordinates . . . 722
15.9.5 Problems . . . 724
15.9.6 Extra Topics . . . 724
15.10 10. Extreme properties of geodetic . . . 728
15.10.1 10.1. Derivative of the curve length function when it varies . . . 728
15.10.2 10.2. Gaussian lemma and local minimality of geodetic . . . 731
15.10.3 Problems . . . 732
15.10.4 Extra Topics . . . 732
15.11 11: Curvature tensor . . . 734
15.11.1 11.1. Coordinate determination of the curvature tensor . . . 734
15.11.2 11.2. Vector field commutator . . . 735
15.11.3 11.3. Invariant determination of the curvature tensor for symmetric connectivity 739
15.11.4 Problems . . . 740
15.11.5 Extra Topics . . . 741
15.12 12: The case of Riemann connectivity (Riemann tensor) . . . 742
15.12.1 12.1. New symmetries of the Riemann tensor . . . 742
15.12.2 12.2. 2D surface curvature tensor . . . 745
15.12.3 12.3. Independent components of the Riemann tensor . . . 747
15.12.4 Problems . . . 748
15.12.5 Extra Topics . . . 749
15.13 13: Degree of Display . . . 749
15.13.1 13.1. Definition and main properties of the degree . . . 750
15.13.2 13.2. Fundamental theorem of algebra . . . 752
15.13.3 13.3. Theorem "about what" . . . 754
15.13.4 Problems . . . 755
15.13.5 Extra Topics . . . 756

15.1414: Other Applications of Display Degree . . . 757
15.14.1 14.1. Degree and Integral . . . 757
15.14.2 14.2. Gauss-Bonnet theorem . . . 758
15.14.3 14.3. Special points of vector fields . . . 760
15.14.4 14.4. Brauer's theorem . . . 760
15.14.5 Problems . . . 761
15.14.6 Extra Topics . . . 761
15.1515: Elements of the Calculus of Variations . . . 762
15.15.1 15.1. Classical variation problem . . . 762
15.15.2 15.2. Examples of Lagrangians . . . 764
15.15.3 15.3. Conservation laws . . . 765
15.15.4 15.4. Multivariate variational problems . . . 766
15.15.5 Problems . . . 768
15.15.6 Literature . . . 768

III Problems of Differential Geometry 771

16 Catalog of Problems 771
16.1 General Questions . . . 771
16.1.1 Questions on Understanding the Essence of the Diffhem . . . 771
16.1.2 Typical Knowledge Test Questions and Tasks . . . 771
16.1.3 Questions on Understanding Typical Details . . . 772
16.2 Typical Technical Tasks . . . 772
16.2.1 Problems About the Properties of Differential Shapes . . . 772
16.2.2 Problems About Defining Surface Parameters (???) . . . 772
16.2.3 Problems About Vector Operations in Curved Coordinates (??) . . . 772
16.2.4 Problems About Understanding Diversity . . . 773
16.2.5 Problems About Killing Vectors and Symmetries . . . 774
16.2.6 Problems About the Display . . . 776
16.2.7 Problems About the Topology of Manifolds . . . 778
16.3 Tasks of Other Topics . . . 779
16.3.1 Tasks About Rings and Differentials . . . 779
16.3.2 Problems of Homologous Algebra (??) . . . 779
16.3.3 Linear Algebra Problems for Diffhem . . . 779
16.4 Tasks About Differential Geometry Applications (???) . . . 781
16.4.1 Problems About Diffgem in Mechanics . . . 781
16.4.2 Problems About Diffhem in Field Theory . . . 781
16.4.3 Problems About Diffhem in String Theory . . . 781

IV —— Special Differential Geometry in a Nutshell —— 782

17 Deeper Understanding of Basics 782
17.1 Lectures on Classical Differential Geometry by Ivanov, A.A. Tuzhilin . . . 782

18 Other, Special Methods 783
18.0.1 Thetrads and Frames of Reference . . . 783
18.0.2 Characteristic Classes . . . 783
18.0.3 Other About Bundles, Homotopies . . . 784
18.1 Other Structures . . . 785
18.1.1 Manifolds in General . . . 785
18.1.2 Symplectic Structure and Geometry . . . 786
18.1.3 Spin Structure on Manifolds (!?!?) . . . 787

18.1.4 Topological Spaces . . . 787
18.1.5 Homologies . . . 787
18.1.6 Super Geometry . . . 787

19 Applications in Physics **787**
19.1 Gravity and Cosmology . . . 787
19.1.1 spacetimes in General Relativity . . . 787
19.1.2 Geometry of Friedmann's Universe . . . 787
19.2 Mechanics and Quantum Mechanics . . . 791
19.2.1 On the Basics of Mechanics . . . 791
19.2.2 Other Properties of Mechanics (!!!??) . . . 792
19.2.3 Quantum Mechanics . . . 799
19.3 Field Theories: Good Models . . . 799
19.3.1 Electrodynamics . . . 800
19.3.2 Field Theories . . . 800
19.3.3 On Gravity . . . 801
19.4 Field Theories: Mathematical Proposals . . . 801
19.4.1 CFT . . . 801
19.4.2 Supergravity . . . 801
19.5 Other Physics . . . 802
19.5.1 On Gravitational Lensing (??????) . . . 802
19.5.2 On Thermodynamics . . . 802

V Other Constructions of Differential Geometry **803**

20 Catalog of geometric objects (??) **803**

21 Some Other Models **803**
21.1 Kähler Geometry (????) . . . 803
21.2 Hyperbolic Geometry . . . 803
21.2.1 Hypercycles and Horocycles . . . 803
21.2.2 Poincaré Geometry . . . 803
21.2.3 Lobachevsky Geometry . . . 803
21.2.4 The Beltrami–klein Model . . . 803
21.2.5 The Poincare Half-plane Model . . . 803
21.2.6 The Hyperboloid Model . . . 803
21.2.7 On Other Models of Hyperbolic Spaces (??) . . . 803
21.2.8 Homogeneous Tilings on the Hyperbolic Plane . . . 803
21.3 Typical Surfaces and Manifolds . . . 804
21.3.1 (anti-) de Sitter Spaces . . . 804
21.3.2 Beltrami Surface . . . 804
21.3.3 Moebius Belt . . . 804
21.4 Symplectic and Poisson Manifolds . . . 804
21.4.1 Calabi-Yau Manifolds . . . 804
21.4.2 Klein Bottle . . . 805
21.5 Transformations. . . . 805
21.5.1 Transformations. . . . 805
21.5.2 homotopy Invariants . . . 805
21.5.3 Applications to Mechanics . . . 805
21.6 Tensor Fields . . . 805
21.7 Homotopy Invariants . . . 808
21.7.1 Theory . . . 808
21.7.2 Applications Overview . . . 808

22 Cohomology **808**
22.1 Theory . . . 808
22.1.1 Theory . . . 808
22.2 Other About Cohomologies . . . 808
22.2.1 Applications Overview . . . 808

23 Homology (???) **808**

24 Introduction to other sections **808**
24.0.1 Adding Curved Space to Physical Theories (??) . . . 808
24.1 Super Geometry . . . 808
24.2 Laminated Spaces . . . 808
24.2.1 Theory . . . 808
24.2.2 Applications Overview . . . 808
24.2.3 General Theory of Stratified Spaces . . . 809
24.3 K-theory . . . 809
24.3.1 Theory . . . 809
24.3.2 Revision . . . 809
24.4 Geometry Modeling and Numerical Methods . . . 809
24.5 Paradoxes of Geometry and Topology in a Nutshell . . . 809
24.5.1 Banach-Tarski paradox . . . 809
24.5.2 Paradoxical set . . . 810
24.5.3 Coastline paradox . . . 810
24.5.4 Coin rotation paradox . . . 810
24.5.5 Gabriel's Horn . . . 810
24.5.6 Hausdorff paradox . . . 810
24.5.7 Hooper's paradox . . . 810
24.5.8 Nikodym set . . . 810
24.5.9 Sphere eversion . . . 810

VI Applications of Differential Geometry **811**

25 Typical mathematical applications **811**
25.1 Geometry in Differential Equations . . . 811
25.2 Geometry in Functional Analysis . . . 811

26 Typical physical applications **811**
26.1 Geometry in Mechanics . . . 811
26.1.1 Mechanics in the Language of Forms . . . 811
26.1.2 Samples . . . 811
26.1.3 Definitions and Construction of Mechanics . . . 812
26.2 Geometry in Quantum Mechanics . . . 813
26.3 Geometry in Typical Field Theory . . . 813
26.3.1 Some Themes: . . . 813
26.3.2 Spinor Fields on a Riemannian Manifold . . . 814
26.3.3 covariant Derivative in Field Theory . . . 815
26.3.4 Electrodynamics in the Language of Forms . . . 815
26.3.5 Maxwell's Equations from a Geometric Point of View . . . 815
26.3.6 Electromagnetic Field in Differential Forms . . . 815
26.4 Geometry of Gravity and Relativity . . . 829
26.4.1 Essence of Oto . . . 830
26.4.2 Examples of Manifolds in Gr and Sr . . . 830

27 Other physical applications **830**
27.1 Geometry in Special Field Theories . . . 830
27.2 Geometry in Gravitational Lensing . . . 830

28 Other applications **830**
28.1 philosophical Applications . . . 830

VII Appendix **831**

A Introduction and overview of the diffem **831**
A.1 Another Motivation for Differential Geometry . . . 831
A.1.1 Other Common Motivation . . . 831
A.1.2 Motivation to Sections and to Study the Features of the Diffhem . . . 832
A.2 The Mindset of Geometry Professionals . . . 833
A.2.1 A Look at Differential Geometry . . . 833
A.2.2 Ways to Guess the Main Ideas of Geometry . . . 834
A.2.3 The Real Importance of Differential Geometry . . . 834
A.2.4 The Exceptional . . . 834
A.2.5 Thinking to Effectively Learn Geometry . . . 834
A.2.6 How one Should not Study Differential Geometry? (!!!) . . . 836
A.3 Acknowledgements . . . 836
A.4 Literature . . . 836
A.4.1 Main Literature . . . 836
A.4.2 Complementary and In-depth Literature . . . 837
A.4.3 Literature on Applications . . . 838
A.5 Overview of Differential Geometry . . . 838
A.5.1 Geometry in a Nutshell . . . 838
A.5.2 review of Theoretical Approaches . . . 839
A.5.3 Summary of Attachments . . . 839
A.5.4 Overview of Further Developments in Geometry . . . 839
A.5.5 Brief Historical Overview . . . 839
A.5.6 Links With Other Mathematical Sciences . . . 839
A.6 Description of Record . . . 840
A.6.1 Features of the Record as a Whole . . . 840
A.6.2 Instructions . . . 840
A.6.3 Legend . . . 840
A.6.4 About Terminology . . . 841
A.7 Differential Geometry Puzzles . . . 841
A.7.1 Selected Fun Puzzles . . . 841
A.7.2 Puzzles for Nerds . . . 841

B Mathematics for Diffhem **842**
B.1 Elements of Algebra for Diffhem . . . 842
B.1.1 About Duality . . . 842
B.1.2 Polyline Shapes . . . 842
B.1.3 Bilinear Shapes . . . 843
B.1.4 Tensor Algebra . . . 843
B.1.5 About Graduated Algebras . . . 845
B.2 Superalgebra . . . 845
B.3 Elements of Topology . . . 845
B.3.1 Fundamentals of Topology . . . 845
B.4 Elements of Mathematical Analysis . . . 845
B.4.1 Reverse Mapping Theorem . . . 845

B.4.2 different Operators??? 846
B.5 Other Mathematical Constructions 846
B.5.1 Category Theory Elements 846

C Bibliography **847**

1 Preface and main motivation

Let's discuss some minimum knowledge and motivation that would be good to understand for studying the subject.

В самой разной теоретической физике то дифференциальные формы, то кривизна встречаются периодически, так что без подготовки по ним будут постоянно тормоза

И не так сложно в этом разобраться, просто некоторое количество конструкций, которые можно прописать просто - и при случае не тормозить на разных свойствах.

(напишу с примерами, что в гравитации, в механике, в теориях поля они встречаются)

Basic preparation in differential geometry is needed for physics and personal development

(типа с точки зрения общей подготовки и саморазвития - это отличное занятие, сесть и нормально проработать диффгем на университетском уровне. вот и большая причина его проходить. раскрою потом эту мысль!)

Лучшие идейные головоломки (???)

(как-то соберу, пока хз)

Лучшие технические головоломки (???)

(как-то соберу, пока хз)

Part I
—— Typical Differential Geometry in a Nutshell ——

2 Basic diffhem constructions in a nutshell

2.1 Main Concepts and Their Properties

2.1.1 Basic Concepts Differential Geometry

Topological Spaces

(полстраницы определений коротко, чтобы не возвращаться к этому.)

Differentiable Manifolds

(полстраницы определений коротко, чтобы не возвращаться к этому.)

Definitions of Tensors and Forms

1-form

At each point $p \in M$, we have a vector space $T_p(M)$. The dual of this space, $T_p^*(M)$ is called the cotangent space at p, and an element of this space is called a cotangent vector, sometimes shortened to covector. Given a basis $\{e_\mu\}$ of $T_p(M)$, we can introduce the dual basis $\{f^\mu\}$ for $T_p^*(M)$ and expand any co-vector as $\omega = \omega_\mu f^\mu$.

(сокращу то, что ниже!)

We can also construct fields of cotangent vectors, by picking a member of $T_p^*(M)$ for each point p in a smooth manner. Such a cotangent field is better known as a one-form; they map vector fields to real numbers. The set of all one-forms on M is denoted $\Lambda^1(M)$.

There is a particularly simple way to construct a one-form. Take a function $f \in C^\infty(M)$ and define $df \in \Lambda^1(M)$ by

$$df(X) = X(f)$$

We can use this method to build a basis for $\Lambda^1(M)$. If we introduce coordinates x^μ on M with the corresponding coordinate basis $e_\mu = \partial/\partial x^\mu$ of vector fields, which we often write in shorthand as $\partial/\partial x^\mu \equiv \partial_\mu$. We then we simply take the functions $f = x^\mu$ which, from (2.16), gives

$$dx^\mu\left(\partial_\nu\right) = \partial_\nu\left(x^\mu\right) = \delta^\mu_\nu$$

This means that $f^\mu = dx^\mu$ provides a basis for $\Lambda^1(M)$, dual to the coordinate basis ∂/∂_μ. In general, an arbitrary one-form $\omega \in \Lambda^1(M)$ can then be expanded as

$$\omega = \omega_\mu dx^\mu$$

$$df = \frac{\partial f}{\partial x^\mu} dx^\mu$$

If we exchange the basis:

$$\omega = \omega_\mu dx^\mu = \tilde{\omega}_\mu d\tilde{x}^\mu \quad \text{with} \quad \tilde{\omega}_\mu = \frac{\partial x^\nu}{\partial \tilde{x}^\mu}\omega_\nu$$

(строчка док-ва)

$$df = \frac{\partial f}{\partial x^\alpha} dx^\alpha$$

$$df(v) = v^\beta df\left(\frac{\partial}{\partial x^\beta}\right) = v^\beta \frac{\partial f}{\partial x^\alpha} dx^\alpha\left(\frac{\partial}{\partial x^\beta}\right)$$

$$dx^\alpha\left(\frac{\partial}{\partial x^\beta}\right) = \delta^\alpha_\beta$$

$$df(v) = v^\alpha \frac{\partial f}{\partial x^\alpha}$$

$$df(v) = v(f)$$

$$\omega = \omega_1 dx^1 + \ldots + \omega_n dx^n = \omega_\alpha dx^\alpha$$

$$\omega = \omega_\alpha(x) dx^\alpha = \omega'_\beta\left(x'\right) dx'^\beta = \omega'_\beta\left(x'\right) \frac{\partial x'^\beta}{\partial x^\alpha} dx^\alpha, \quad \omega_\alpha(x) = \omega'_\beta\left(x'\right) \frac{\partial x'^\beta}{\partial x^\alpha}$$

Tensors

A tensor of rank(r, s) at a point $p \in M$ is defined to be a multi-linear map

$$T : T^*_{p_1}(M) \times \ldots \times T^*_{p_r}(M) \times T_{p_1}(M) \times \ldots \times T_{p_s}(M) \to \mathbf{R}$$

Such a tensor is said to have total rank $r + s$. We've seen some examples already. A cotangent vector in $T^*_p(M)$ is a tensor of type $(0, 1)$, while a tangent vector in $T_p(M)$ is a tensor of type $(1, 0)$ (using the fact that $T_p(M) = T^{**}_p(M)$).

Given a basis $\{e_\mu\}$ for vector fields and a dual basis $\{f^\mu\}$ for one-forms, the components of the tensor are defined to be

$$T^{\mu_1 \ldots \mu_r}_{\nu_1 \ldots \nu_s} = T\left(f^{\mu_1}, \ldots, f^{\mu_r}, e_{\nu_1}, \ldots, e_{\mu_s}\right)$$

Given a tensor S of rank (p, q) and a tensor T of rank (r, s), we can form the tensor product, $S \otimes T$ which a new tensor of rank $(p + r, q + s)$, defined by

$$\begin{aligned} S \otimes T\left(\omega_1, \ldots, \omega_p, \eta_1, \ldots, \eta_r, X_1, \ldots X_q, Y_1, \ldots, Y_s\right) \\ = S\left(\omega_1, \ldots, \omega_p, X_1, \ldots X_q\right) T\left(\eta_1, \ldots, \eta_r, Y_1, \ldots, Y_s\right) \end{aligned}$$

In terms of components:

$$(S \otimes T)^{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_r}{}_{\rho_1 \ldots \rho_q \sigma_1 \ldots \sigma_s} = S^{\mu_1 \ldots \mu_p}{}_{\rho_1 \ldots \rho_q} T^{\nu_1 \ldots \nu_r}{}_{\sigma_1 \ldots \sigma_s}$$

(тут обозначения из линала добавлю, сперва там их напишу, тоже мб по Тонгу, типичные тензорные обозначения.)

(маленький абзац про базисы, какой и почему где базисы, тут!!!)

p-forms and wedge

Totally anti-symmetric $(0,p)$ tensors fields are called p-forms. The set of all p-forms over a manifold M is denoted $\Lambda^p(M)$.

Given a p-form ω and a q-form η, we can take the tensor product to construct a $(p+q)$-tensor. If we anti-symmetrise this, we then get a $(p+q)$-form. This construction is called the wedge product, and is defined by

$$(\omega \wedge \eta)_{\mu_1 \ldots \mu_p \nu_1 \ldots \nu_q} = \frac{(p+q)!}{p!q!} \omega_{[\mu_1 \ldots \mu_p} \eta_{\nu_1 \ldots \nu_q]}$$

$$(\omega \wedge \eta)_{\mu\nu} = \omega_\mu \eta_\nu - \omega_\nu \eta_\mu$$

$$\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$$

$$\omega \wedge (\eta \wedge \lambda) = (\omega \wedge \eta) \wedge \lambda.$$

$$\omega_p \wedge \omega_q = (-1)^{pq} \omega_q \wedge \omega_p$$

$$d(\omega_p \wedge \omega_q) = d\omega_p \wedge \omega_q + (-1)^p \omega_p \wedge d\omega_q$$

$$\wedge : \Omega^k(U) \times \Omega^m(U) \to \Omega^{k+m}(U)$$

$$(\omega \wedge \lambda)(v_1, \ldots, v_{k+m}) = \frac{1}{k!m!} \sum_{\sigma \in \hat{\zeta}_{k+m}} (\operatorname{sgn} \sigma) \omega\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) \cdot \lambda\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+m)}\right)$$

$$(f\omega)(v_1, \ldots, v_k) = f \cdot \omega(v_1, \ldots, v_k)$$

$$\omega = \sum_{\alpha_1 < \ldots < \alpha_k} \omega_{\alpha_1 \ldots \alpha_k} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}$$

$$\omega(v_1, \ldots, v_i + w_i, \ldots, v_k) = \omega(v_1, \ldots, v_i, \ldots, v_k) + \omega(v_1, \ldots, w_i, \ldots, v_k)$$

$$\omega(f_1 v_1, \ldots, f_k v_k) = f_1 \ldots f_k \omega(v_1, \ldots, v_k)$$

Поднятие индекса: создание дифференциальной формы по вектору

Наличие полуримановой структуры позволяет преобразовать вектор в дифференциальную форму первой степени по формуле

$$\xi^b(\eta) = \mathfrak{g}(\xi, \eta)$$

Например, для вектора с координатами ξ^α получается форма с координатами

$$\xi^b_\alpha = g_{\alpha\beta} \xi^\beta$$

Опускание индекса: создание вектора по дифференциальной форме

Возможна и обратная операция, делающая из формы первой степени вектор в соответствии с формулой

$$\omega(\eta) = \mathfrak{g}\left(\omega^\#, \eta\right)$$

Если речь идёт о стандартной евклидовой структуре в $\mathbb{R}^3$ и работе в ортонормированной системе координат, то подняте и опускание индексов оставляют те же самые координаты векторов и форм, ведь $g_{\alpha\beta} = \delta_{\alpha\beta}$; в этом случае верхние и нижние индексы можно не различать.

Связь метрики и обратной метрики: $g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha$

Наличие (полу)римановой структуры порождает билинейные симметрические формы на всех связанных с касательным пространством в точке пространствах. Так, форма (полу)римановой структуры $\mathfrak{g}$ может рассматриваться как отображение

$$\mathfrak{g} : T_pM \otimes T_pM \to \mathbb{R}$$

а также как элемент $T^\star_pM \otimes T^\star_pM$, или как отображение $T_pM \to T^\star_pM$ (опускание индексов) с обратным отображением $T^\star_pM \to T_pM$ (поднятие индексов). Композиция $\mathfrak{g}$ и двух поднятий индексов на её аргументах даёт билинейное отображение

$$\overline{\mathfrak{g}} : T^\star_pM \otimes T^\star_pM \to \mathbb{R}$$

то есть симметричную билинейную форму на кокасательном пространстве.

Определим, как связаны матрицы $g_{\alpha\beta}$ и $\bar{g}_{\alpha\beta}$ (последнюю в теоретической физике пишут как $g^{\alpha\beta}$), представляющие (полу)риманову структуру на исходном касательном пространстве и соответствующую билинейную форму на двойственном к нему кокасательном пространстве.

По построению очевидно, что

$$\overline{\mathfrak{g}}\left(\xi^b, \eta^b\right) = \mathfrak{g}(\xi, \eta) \qquad \Leftrightarrow \qquad g^{\alpha\beta}\xi^b_\alpha\eta^b_\beta = g_{\mu\nu}\xi^\mu\eta^\nu$$

Подставляя в последнее выражение формулу $\xi^b(\eta) = \mathfrak{g}(\xi, \eta)$, имеем $g^{\alpha\beta}g_{\alpha\mu}g_{\beta\nu} = g_{\mu\nu}$. Отсюда

$$g_{\alpha\beta}g^{\beta\gamma} = \delta^\gamma_\alpha$$

то есть матрица $\bar{g}_{\alpha\beta} := g^{\alpha\beta}$ является обратной к $g_{\alpha\beta}$, в силу чего $g^{\alpha\beta}$ часто называют обратной метрикой.

$\overline{\mathfrak{g}}$ даёт изоморфизм между $\Omega^{n-k}(M)$ в точке и его двойственным.

(?? это пока не так уж понимаю, потом лучше подумаю, почему. перенесу в раздел про метрикук)

Метрика на $\Omega^k_p(M)$ (?)

Тензорно перемножая $\overline{\mathfrak{g}}$ на себя и, группируя тензорные множители, можно расширить её до отображения

$$\overline{\mathfrak{g}} : \underbrace{T^\star_pM \otimes \ldots \otimes T^\star_pM}_{k} \otimes \underbrace{T^\star_pM \otimes \ldots \otimes T^\star_pM}_{k} \to \mathbb{R}$$

Вкладывая пространство $\Omega^k_p(M)$ дифференциальных форм степени k в точке p в тензорное произведение

$$\Omega^k_p(M) \subseteq \underbrace{T^\star_pM \otimes \ldots \otimes T^\star_pM}_{k}$$

мы таким образом можем рассмотреть $\overline{\mathfrak{g}}$ как билинейную форму на $\Omega^k_p(M)$. Поскольку кососимметричные формы по разному вкладываются в полилинейные формы (с точностью до умножения на зависящую от k постоянную), то нам надо внести определённость и нормировать произведение $\overline{\mathfrak{g}}_p(\omega, \lambda)$.

Мы будем нормировать так, что если линейные формы $\omega_1, \ldots, \omega_n \in \Omega^1(M)$ образуют (почти) ортонормированный базис в точке p, то есть

$$\overline{\mathfrak{g}}_p\left(\omega_i, \omega_j\right) = \pm\delta_{ij}$$

то формы $\omega_{i_1} \wedge \ldots \wedge \omega_{i_k}$ будут образовывать (почти) ортонормированный базис относительно $\overline{\mathfrak{g}}_p$.

Невырожденное спаривание

Рассмотрим многообразие M размерности n. Внешнее умножение между $\Omega^k(M)$ и $\Omega^{n-k}(M)$ в точке p является невырожденным спариванием, то есть для всякой ненулевой $\omega \in \Omega^k(M)$ можно найти $\lambda \in \Omega^{n-k}(M)$ так, что $\omega \wedge \lambda$ не будет равна нулю в точке p.

(?? почему?? потом подробнее напишу)

(?? как это на англ назвается??)

Maps Between Manifolds

Push-Foward and Pull-Back

Function: $f : N \to \mathbb{R}$. Map, diffeomorphism: $\varphi : M \to N$. M, N – manifolds

The pull-back drags objects originally defined on N onto M:

$$\forall f : N \to \mathbb{R} \qquad (\varphi^* f)(p) = f(\varphi(p)), \qquad \text{so} \quad (\varphi^* f) : M \to \mathbb{R}$$

Basically it is like φ^{-1}, but for functions. In coordinates x^μ on M and y^α on N, then the map is $\varphi(x) = y^\alpha(x)$, and the pull-back is $(\varphi^* f)(x) = f(y(x))$.

The push-forward maps a vector field Y on M to a new vector field $(\varphi_* Y)$ on N.

$$f : N \to \mathbb{R} \qquad (\varphi_* Y)(f) = Y(\varphi^* f). \qquad \text{or} \ [(\varphi_* Y)(f)](\varphi(p)) = [Y(\varphi^* f)](p)$$

Basically we are just plugging in functions-coordinates of a new manifold by the inverse map. If $Y = Y^\mu \partial/\partial x^\mu$ is the vector field on M,

$$(\varphi_* Y)(f) = Y^\mu \frac{\partial f(y(x))}{\partial x^\mu} = Y^\mu \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial f(y)}{\partial y^\alpha} \qquad \text{on} N$$

Written in components, $(\varphi_* Y) = (\varphi_* Y)^\alpha \, \partial/\partial y^\alpha$, we then have

$$(\varphi_* Y)^\alpha = Y^\mu \frac{\partial y^\alpha}{\partial x^\mu}$$

The pushforward is always denoted as φ_* and goes in the same way as the original map, the pull-back is always denoted as φ^* and goes in the opposite direction.

(???? wiki. later I'll learn also the below) The differential of a smooth map φ induces, in an obvious manner, a bundle map (in fact a vector bundle homomorphism) from the tangent bundle of M to the tangent bundle of N, denoted by $d\varphi$, which fits into the following commutative diagram:

$$\begin{tikzcd} TM \arrow[r, "\mathrm{d}\varphi"] \arrow[d, "\pi_M"'] & TN \arrow[d, "\pi_N"] \\ M \arrow[r, "\varphi"'] & N \end{tikzcd}$$

where π_M and π_N denote the bundle projections of the tangent bundles of M and N respectively. $\mathrm{d}\varphi$ induces a bundle map from TM to the pullback bundle $\varphi^* TN$ over M via

$$(m, v_m) \mapsto (m, \ \mathrm{d}\varphi(m, v_m)),$$

where $m \in M$ and $v_m \in T_m M$. The latter map may in turn be viewed as a section of the vector bundle $\mathrm{Hom}\,(TM, \varphi^* TN)$ over M. The bundle map $\mathrm{d}\varphi$ is also denoted by $T\varphi$ and called the tangent map. In this way, T is a functor.

Under a map $\varphi : M \to N$, we saw that a vector field X on M can be pushed forwards to a vector field $\varphi_* X$ on N. In contrast, one-forms go the other way: given a one-form ω on N, we can pull this back to a one-form $(\varphi^*\omega)$ on M, defined by

$$(\varphi^*\omega)\,(X) = \omega\,(\varphi_* X)$$

(??? додумаю) If we introduce coordinates x^μ on M and y^α on N then the components of the pull-back are given by

$$(\varphi^*\omega)_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}$$

The push-forward of a tensor T from M to N acts on one-forms $\omega \in \Lambda^1(N)$ and vector fields $X \in \mathfrak{X}(N)$ and is given by

$$(\varphi_* T)\,(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) = T\left(\varphi^*\omega_1, \ldots, \varphi^*\omega_r, \left(\varphi_*^{-1} X_1\right), \ldots, \left(\varphi_*^{-1} X_s\right)\right)$$

Here $\varphi^*\omega$ are the pull-backs of ω from N to M, while $\varphi_*^{-1} X$ are the push-forwards of X from N to M.

2.1.2 Differential Forms: Main Properties

Основные операции

Внешний дифференциал d

$$d : \Omega^k(U) \ \to \Omega^{k+1}(U)$$

у него три свойства:
(1) на функциях он является обычными дифференциалом
(2) нильпотентность $d^2 = 0$
(3) $d(\omega \wedge \lambda) = d\omega \wedge \lambda$

$$d\omega = \sum_{\alpha_1 < \ldots < \alpha_k} d\omega_{\alpha_1 \ldots \alpha_k} \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k} + \sum_{\alpha_1 < \ldots < \alpha_k} \omega_{\alpha_1 \ldots \alpha_k} d^2 x^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k} + \ldots$$

$$d\omega = \sum_{\alpha_1 < \ldots < \alpha_k} d\omega_{\alpha_1 \ldots \alpha_k} \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}$$

$$d\omega = \sum_{\alpha_1 < \ldots < \alpha_k, \beta \neq \{\alpha_1, \ldots, \alpha_k\}} \frac{\partial \omega_{\alpha_1 \ldots \alpha_k}}{\partial x^\beta} dx^\beta \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}$$

$$= \sum_{\alpha_1 < \ldots < \alpha_k, \alpha < \beta} \left(\frac{\partial^2 \omega_{\alpha_1 \ldots \alpha_k}}{\partial x^\alpha \partial x^\beta} dx^\alpha \wedge dx^\beta + \frac{\partial^2 \omega_{\alpha_1 \ldots \alpha_k}}{\partial x^\beta \partial x^\alpha} dx^\beta \wedge dx^\alpha \right) \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k}$$

$$\frac{\partial^2 \omega_{\alpha_1 \ldots \alpha_k}}{\partial x^\alpha \partial x^\beta} = \frac{\partial^2 \omega_{\alpha_1 \ldots \alpha_k}}{\partial x^\beta \partial x^\alpha}$$

Второй дифференциал от любой формы - всегда ноль.

$$d : \Lambda^p(M) \to \Lambda^{p+1}(M)$$

$$(d\omega) = \frac{1}{p!}\frac{\partial \omega_{\mu_1 \ldots \mu_p}}{\partial x^\nu} dx^\nu \wedge dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}$$

$$(d\omega)_{\mu_1 \ldots \mu_{p+1}} = (p+1)\partial_{[\mu_1} \omega_{\mu_2 \ldots \mu_{p+1}]}$$

$d^2 = 0$

$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$, where $\omega \in \Lambda^p(M)$

$d(\varphi^* \omega) = \varphi^*(d\omega)$ where φ^* is the pull-back associated to the map between manifolds, $\varphi : M \to N$

Because the exterior derivative commutes with the pull-back, it also commutes with the Lie derivative. This ensures that we have $d(\mathcal{L}_X \omega) = \mathcal{L}_X(d\omega)$.

Example: for a one-form $\omega = \omega_\mu dx^\mu$, the exterior derivative gives a 2-form:

$$(d\omega)_{\mu\nu} = \partial_\mu \omega_\nu - \partial_\nu \omega_\mu \Rightarrow d\omega = \frac{1}{2}\left(\partial_\mu \omega_\nu - \partial_\nu \omega_\mu\right) dx^\mu \wedge dx^\nu$$

Внутренее умножение

(суть этого приведу тут)

Внутреннее умножение (внутреннее дифференцирование). определяется как отображение $\imath_X : \Omega^k(M) \to \Omega^{k-1}(M)$ по формуле

$$\imath_X \omega\left(X_2, \ldots, X_k\right) = \omega\left(X, X_2, \ldots, X_k\right)$$

(?? тут в явном виде сразу пример напишу!) При умножении X или ω на функцию f выражение $\imath_X \omega$ просто умножается на функцию f.

Внутреннее умножение по определению линейно по вектору и по форме.

$$\imath_X(\omega \wedge \lambda) = \imath_X \omega \wedge \lambda + (-1)^{\deg \omega} \omega \wedge \imath_X \lambda \qquad \text{(Leibnitz rule)}$$

(???? напишу ниже док-во короче, додумаю его!!!) Из определения внутреннего умножения и внешнего умножения форм следует, что должна выполняться какая-то такая формула с точностью до знаков и множителей. Проверить знаки и множители по линейности достаточно на базисных элементах.

$$\imath_{\partial\beta} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k} = (-1)^{\ell-1} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{\ell-1}} \wedge dx^{\alpha_{\ell+1}} \wedge \ldots \wedge dx^{\alpha_k}, \quad \beta = \alpha_\ell$$

и

$$\imath_{\partial_\beta} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k} = 0$$

если β не встречается среди α_ℓ. После этого проверка правила Лейбница становится понятной: внутреннему умножению на ∂_β необходимо «добраться» до своего dx^β, причём при «перепрыгивании» через неподходящие dx^β знак выражения меняется.

$$[\imath_X, l_Y]_+ \omega = \imath_X^2 Y \omega + \imath_Y \imath_X \omega = 0$$

Ибо $[\imath_X, l_Y]_+ \omega = \omega(X, Y) + \omega(Y, X) = 0$.

Об операторе Ходжа

(мб в другой раздел передвину)

Суть и основные свойства

Звезда Ходжа превращает форму, двойственную к -форме, в $(n-p)$-форму в n-мерном пространстве:

$$*\left(dx^{\mu_1} \wedge \ldots \wedge dx^{\mu_p}\right) = \frac{|g|^{1/2}}{(n-p)!} \varepsilon^{\mu_1}_{\mu_{p+1}} \mu_{p_n} dx^{\mu_{p+1}} \wedge \ldots \wedge dx^{\mu_*}$$

$$* * \omega_p = (-1)^{p(n-p)} \omega_p$$
$$\omega_p \wedge *\omega_q = \omega_q \wedge *\omega_p$$

Определяется так. В присутствии (полу)римановой структуры g на ориентированном многообразии M^n (чтобы vol g можно было считать элементом $\Omega^n(M)$) для любой формы $\beta \in \Omega^k(M)$ формула

$$\alpha \wedge *\beta = \widetilde{g}(\alpha, \beta) \operatorname{vol}_g, \forall \alpha \in \Omega^k(M),$$

определяет форму $*\beta \in \Omega^{n-k}(M)$ и определяет линейное преобразование $* : \Omega^k(M) \to \Omega^{n-k}(M)$, звёздочку Ходжа.

Точнее можно сказать, что $*$ определяется как композиция изоморфизмов в каждой точке

$$\Omega^k_p(M) \longrightarrow \left(\Omega^k_p(M)\right)^* \longrightarrow \Omega^{n-k}_p(M),$$

в которой первый возникает из невырожденного спаривания $\widetilde{g}$ между формами k степени, а второй - из спаривания, заданного внешним умножением с делением на vol_g.

The Hodge dual allows us to define an inner product on each $\Lambda^p(M)$. If $\omega, \eta \in \Lambda^p(M)$, we define

$$\langle \eta, \omega \rangle = \int\limits_M \eta \wedge \star\omega$$

which makes sense because $\star\omega \in \Lambda^{n-p}(M)$ and so $\eta \wedge \star\omega$ is a top form that can be integrated over the manifold.

(??? доучивать нужно!!)

Оператор звёздочки является поточечным, то есть линейным относительно умножения на функцию, $*(f\alpha) = f(*\alpha)$. Например, для $\mathbb{R}^3$ со стандартной евклидовой струк-

На n-мерном (полу)римановом многообразии для звёздочки $* : \Omega^k(M) \to \Omega^{n-k}(M)$ выполняется

$$* * \alpha = (-1)^{k(n-k)} \operatorname{sgn} \det g \cdot \alpha.$$

Для положительно определённой g звёздочка Ходжа является изометрией для скалярного произведения $\widetilde{g}$ на внешних формах касательного пространства в точке.

Связь формы с полиномами (?)

(чет я хз в этом)

Всякий инвариантный полином является одновременно замкнутой и точной формой

(укажу в абзац док-во, пока не особо думал)

Определение. В присутствии (полу)римановой структуры $\mathfrak{g}$ на ориентированном многообразии M формула

$$\omega \wedge \star\lambda = \overline{\mathfrak{g}}(\omega, \lambda)\tau_{\mathfrak{g}}$$

корректно определяет линейный оператор $\star : \Omega^k(M) \to \Omega^{n-k}(M)$, т.н. звёздочку Ходжа.

(?????) Это определение корректно, потому что $\overline{\mathfrak{g}}$ даёт изоморфизм между $\Omega^{n-k}(M)$ в точке и его двойственным, мы можем обосновать корректность следующего определения.

Точнее можно сказать, что $\star$ определяется как композиция изоморфизмов в каждой точке

$$\Omega^k_p(M) \to \left(\Omega^k_p(M)\right)^\star \to \Omega^{n-k}_p(M),$$

в которой первый возникает из невырожденного спаривания $\overline{\mathfrak{g}}$ между формами k-степени, а второй - из спаривания, заданного внешним умножением с делением на форму риманова объёма $\tau_{\mathfrak{g}}$.

(?? где там моя схема отображений, которую уже я рисовал в этом файле?)

Например, для $\mathbb{R}^3$ со стандартной евклидовой структурой и координатами x, y, z мы получим

$$\star 1 = dx \wedge dy \wedge dz, \quad \star(dx \wedge dy \wedge dz) = 1 \quad \star\, dx = dy \wedge dz, \quad \star dy = dz \wedge dx,$$
$$\star dz = dx \wedge dy \quad \star\,(dx \wedge dy) = dz, \quad \star(dy \wedge dz) = dx, \quad \star(dz \wedge dx) = dy$$

Действительно, по определению

$$dx \wedge \star dx = \overline{\mathfrak{g}}(dx, dx) dx \wedge dy \wedge dz = g^{11} dx \wedge dy \wedge dz = dx \wedge dy \wedge dz$$
$$\star dx = dy \wedge dz$$

Остальные тождества получаются аналогичными вычислениями.

(потренируюсь позже!)

На n-мерном римановом многообразии M с полуримановой структурой $\mathfrak{g}$ для звёздочки $\star : \Omega^k(M) \to \Omega^{n-k}(M)$ выполняется

$$\star \star \omega = (-1)^{k(n-k)} \operatorname{sgn} \det \mathfrak{g} \cdot \omega$$

(тут предлагается доказать это самому, пока не до этого, потом потренируюсь)

[Проверьте в координатах, в которых $\mathfrak{g}$ имеет диагональный вид $c \pm 1$ на диагонали.]

Обзор применений оператора Ходжа

(по идее много где она нужна, 2/3 страницы планируется отсылок к разным формулам в физике.)

О дуальности Ходжа

(просто утверждение и суть о простом, подробнее - скорее всего во 2й 1й части будет. примерно на 1/2 страницы планируется)

Hodge's Theorem: There is an isomorphism

$$\operatorname{Harm}^p(M) \cong H^p(M)$$

where $H^p(M)$ is the de Rham cohomology group (????). In particular, the Betti numbers can be computed by counting the number of linearly independent harmonic forms,

$$B_p = \dim \operatorname{Harm}^p(M)$$

(идею док-ва укажу тоже)

внутреннее умножение

также известно как внутреннее дифференцирование

вкратце о внутреннем умножении

примеры применения

$$i_{\frac{\partial}{\partial z}} \omega_E = E_y dx + E_x dy$$

если бы наше поле умножалось на любую функцию - то внутреннее

случаи применения внутреннее умножения (??)

cohomology and Harmonic Forms (??)

(just write main staement, it is not a big topic if we don't ask many questions)

2.1.3 Differential Forms: Integration

On Stokes' Theorem

$$\int_M d\omega = \int_{\partial M} \omega,$$

где ∂M есть граница многообразия M.

2.1.4 Tangent Spaces

Consider a function $f : M \to \mathbb{R}$. To differentiate the function at some point p, we introduce a chart $\phi = (x^1, \ldots, x^n)$ in a neighbourhood of p. So construct the map $f \circ \phi^{-1} : U \to \mathbb{R}$ with $U \subset \mathbb{R}^n$ and by knowing how to differentiate functions on $\mathbb{R}^n$ and this gives us a way to differentiate functions on M, namely

$$\left.\frac{\partial f}{\partial x^\mu}\right|_p := \left.\frac{\partial \left(f \circ \phi^{-1}\right)}{\partial x^\mu}\right|_{\phi(p)}$$

(тут много определений и формул потом будет)

Tangent Vectors

A tangent vector X_p is an object that differentiates functions at a point $p \in M$. Specifically, $X_p : C^\infty(M) \to \mathbb{R}$ satisfying

i) Linearity: $X_p(f+g) = X_p(f) + X_p(g)$ for all $f, g \in C^\infty(M)$.

ii) $X_p(f) = 0$ when f is the constant function.

iii) Leibnizarity: $X_p(fg) = f(p)X_p(g) + X_p(f)g(p)$ for all $f, g \in C^\infty(M)$. This, of course, is the product rule.

The set of all tangent vectors at point p forms an n-dimensional vector space. We call this the tangent space $T_p(M)$. The tangent vectors $\partial_\mu|_p$ provide a basis for $T_p(M)$. This means that we can write any tangent vector as

$$X_p = X^\mu \partial_\mu|_p$$

with $X^\mu = X_p\left(x^\mu\right)$ the components of the tangent vector in this basis. (??? потом док-во изучу, но это в теории, пока не понимаю, почему это не очевидно? мб предложение про суть док-ва добавлю.)

Vector Fields

A vector field X is defined to be a smooth assignment of a tangent vector X_p to each point $p \in M$. So if you feed a function to a vector field, then it spits back another function, which is the differentiation of the first. In symbols, a vector field is therefore a map $X : C^\infty(M) \to C^\infty(M)$. The function $X(f)$ is defined by

$$(X(f))(p) = X_p(f)$$

The space of all vector fields on M is denoted $\mathfrak{X}(M)$.

$$[X,Y](f) = X(Y(f)) - Y(X(f))$$
$$[X,Y] = \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\right) \frac{\partial}{\partial x^\nu}$$

Integral Curves

We can then define a vector field by taking the tangent to the streamlines at each point. In a given coordinate system, the components of the vector field are

$$X^\mu\left(x^\mu(t)\right) = \frac{dx^\mu(t)}{dt}$$

Vector fields which generate a flow for all $t \in \mathbf{R}$ are called complete. It turns out that all vector fields on a manifold M are complete if M is compact. Roughly speaking, "compact" means that M doesn't "stretch to infinity". More precisely, a topological space M is compact if, for any family of open sets covering M there always exists a finite sub-family which also cover M. So $\mathbf{R}$ is not compact because the family of sets $\{(-n,n), n \in \mathbf{Z}^+\}$covers $\mathbf{R}$ but has no finite sub-family. Similarly, $\mathbf{R}^n$ is non-compact. However, $\mathbf{S}^n$ and $\mathbf{T}^n$ are compact manifolds.)

2.1.5 Vectors, Covectors and Vector Fields

Определения и основные свойства вектора

(тут потом куча слов будет)

$$\frac{\partial f}{\partial \mathbf{v}} = \lim_{\epsilon \to 0} \frac{f(p+\epsilon \mathbf{v}) - f(p)}{\epsilon}$$
$$\frac{\partial (fg)}{\partial \mathbf{v}} = \frac{\partial f}{\partial \mathbf{v}} g + f \frac{\partial g}{\partial \mathbf{v}}$$

$$v(f) \equiv \frac{\partial f}{\partial \mathbf{v}} \equiv \lim_{\epsilon \to 0} \frac{f\left(p + \epsilon v^\alpha \mathbf{e}_\alpha\right) - f(p)}{\epsilon} = \lim_{\epsilon \to 0} \frac{f(p) + \epsilon v^\alpha \frac{\partial f}{\partial x^\alpha} + \mathcal{O}\left(\epsilon^2\right) - f(p)}{\epsilon} = v^\alpha \frac{\partial f}{\partial x^\alpha}$$

$$v = v^\beta(x) \frac{\partial}{\partial x^\beta}$$

$$v = v'^\alpha\left(x'\right) \frac{\partial}{\partial x'^\alpha}$$

$$v'^\alpha\left(x'\right) \frac{\partial}{\partial x'^\alpha} = v^\beta(x) \frac{\partial}{\partial x^\beta} = v^\beta(x) \frac{\partial x'^\alpha}{\partial x^\beta} \frac{\partial}{\partial x'^\alpha}$$
$$v'^\alpha\left(x'\right) = \frac{\partial x'^\alpha}{\partial x^\beta} v^\beta(x) = \Lambda^\alpha_\beta v^\beta(x)$$

Действие вектора на функцию есть

$$X(f) = df(X)$$

(???)

(тут четко пишу, почему так)

О ковекторном поле

(хз, что это и почему, вообще такой язык не нужен)
ковекторное поле - элемент пространства $\mathrm{Hom}_{C^\infty)\mathbb{R}^n)}(\mathrm{Vect})\mathbb{R}^n), C^\infty)\mathbb{R}^n))$
то есть $\forall\alpha$ и $\forall v \in \mathrm{Vect})\mathbb{R}^n)$ определено $\alpha(v) \in C^\infty)\mathbb{R}^n)$ так что $\forall \in C^\infty)\mathbb{R}^n)$, $\alpha(fv) = f\alpha(v)$

2.1.6 Tensors, Tensor Fields

2.3.1 Covectors and One-forms

2.3.3 Tensors and Tensor Fields

2.1.7 Lie Derivative

The Lie Derivative

Suppose that we are given a vector field X on M. This generates a flow $\sigma_t : M \to M$, which is a map between manifolds, now with $N = M$. This means that we can use $(\varphi_* Y)^\alpha = Y^\mu \frac{\partial y^\alpha}{\partial x^\mu}$ to generate a pushforward map from $T_p(M)$ to $T_{\sigma_t(p)}(M)$. But this is exactly what we need if we want to compare tangent vectors at neighbouring points. The resulting differential operator is called the Lie derivative and is denoted $\mathcal{L}_X$.

For functions:

$$\mathcal{L}_X f = \lim_{t\to 0} \frac{f(\sigma_t(x)) - f(x)}{t} = \left.\frac{df(\sigma_t(x))}{dt}\right|_{t=0} = \left.\frac{\partial f}{\partial x^\mu}\frac{dx^\mu}{dt}\right|_{t=0} = X^\mu(x)\frac{\partial f}{\partial x^\mu} = X(f). \quad \frac{dx^\mu}{dt} \equiv X^\mu.$$

For a vector field Y

$$\mathcal{L}_X Y = [X, Y]$$

Proof: $\mathcal{L}_X Y = \lim_{t\to 0} \frac{\left((\sigma_{-t})_* Y\right)_p - Y_p}{t}$ For a coordinate basis $\partial_\mu = \partial/\partial x^\mu$ we have $\mathcal{L}_X \partial_\mu = \lim_{t\to 0} \frac{(\sigma_{-t})_*\partial_\mu - \partial_\mu}{t}$,

$$(\sigma_{-t})_* \partial_\mu = \left(\delta^\nu_\mu - t\frac{\partial X^\nu}{\partial x^\mu} + \dots\right)\partial_\nu$$

Acting on a coordinate basis, we then have

$$\mathcal{L}_X \partial_\mu = -\frac{\partial X^\nu}{\partial x^\mu}\partial_\nu$$

$$\mathcal{L}_X\left(Y^\mu \partial_\mu\right) = \left(\mathcal{L}_X Y^\mu\right)\partial_\mu + Y^\mu\left(\mathcal{L}_X \partial_\mu\right)$$

$$\mathcal{L}_X\left(Y^\mu \partial_\mu\right) = X^\nu \frac{\partial Y^\mu}{\partial x^\nu}\partial_\mu - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\partial_\nu$$

(?? досмотрю и додумаю позже это еще!)

The Lie derivative act on a general one-form $\omega = \omega_\mu dx^\mu$ by

$$\begin{aligned}\mathcal{L}_X\omega &= (\mathcal{L}_X\omega_\mu)\, dx^\mu + \omega_\nu \mathcal{L}_X\left(dx^\nu\right)\\ &= \left(X^\nu\partial_\nu\omega_\mu + \omega_\nu\partial_\mu X^\nu\right) dx^\mu\end{aligned}$$

(?? пара строчке док-ва, подумаю потом, в принципе похожая идея на Ли от векторов. в теории напишу подробно.) Remember $(\varphi^*\omega)(X) = \omega(\varphi_* X)$ or $(\varphi^*\omega)_\mu = \omega_\alpha \frac{\partial y^\alpha}{\partial x^\mu}$

Свойства

Формула Картана:

$$L_X = i_X d + d i_X$$

2.1.8 div, rot, grad, □

О градиенте grad

(!!! все указания о нем именно тут! выпишу многое уже очень скоро!!!)

$$\operatorname{grad} f = (df)^\sharp$$

где

О дивергенции div

(!!! все указания о ней именно тут! выпишу многое уже очень скоро!!!)

Конструкция дивергенции

$$d\imath_X\tau = (\operatorname{div} X)\cdot\tau \qquad \text{−def of div}$$

Это оператор, который берет форму наибольшей степени, делает ее степенью меньше, потом снова больше.

$$\operatorname{div} X = *d\left(*X^b\right)$$

$*1 = \mathrm{vol}_g$

Определение дивергенции для риманова многообразия согласовано с определением дивергенции относительно формы объёма как $\operatorname{div} X\, \mathrm{vol}_g = L_X(\mathrm{vol}_g)$.

Смысл следующий. На $\mathbb{R}^3$ определена нигде не нулевая форма высшей степени - форма объёма $\tau = dx\wedge dy\wedge dz$. Пусть на $\mathbb{R}^3$ задано векторное поле $X = X_x\partial_x + X_y\partial_y + X_z\partial_z$ По векторному полю $X \in \operatorname{Vect}(\mathbb{R}^3)$ можно построить 2-форму посредством применения внутреннего умножения к форме объёма: $\imath_X\tau$. Из неё мы получим 3-форму (форму высшей степени на $\mathbb{R}^3$) путём взятия внешнего дифференциала d. Но, пространство 3-форм на $\mathbb{R}^3$ имеет единственный базисный вектор - форму объёма τ. Значит, полученная нами 3-форма $d_X\tau$ есть некоторая гладкая функция, помноженная на форму объёма. Эту гладкую функцию и назовём дивергенцией $\operatorname{div} X$ векторного поля X, то есть

$$d\imath_X\tau = (\operatorname{div} X)\cdot\tau$$

она не зависит от системы координат, потому что определяли мы составляющие дивиргенции в бескоординатном способе

форма потока

$$i_x\tau = X_x dy\wedge dz + X_y dz\wedge dx + X_z dx\wedge dy$$

О другом про дивергенцию

дивергенция на искривленном многообразии (???)
через звездочку ходжа он попробует там.
В координатах:

$$\begin{aligned}\imath_X\tau =&\tau(X) = dx\wedge dy\wedge dz\left(X_x\partial_x + X_y\partial_y + X_z\partial_z\right) = \\ =&X_x dx\wedge dy\wedge dz\left(\partial_x\right) + X_y dx\wedge dy\wedge dz\left(\partial_y\right) + X_z dx\wedge dy\wedge dz\left(\partial_z\right) = \\ =&X_x dy\wedge dz + X_y dz\wedge dx + X_z dx\wedge dy\end{aligned}$$

Дифференциал получившейся 2-формы мы уже находили (см. формулу для $d\omega_E$). Значит

$$d\tau_X\tau = \left(\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z}\right)\tau \quad\Rightarrow\quad \operatorname{div} X = \partial_x X_x + \partial_y X_y + \partial_z X_z \text{ or } \operatorname{div} X = \partial_\alpha X^\alpha$$

Инвариантность дивергенции только при заменах координат с постоянным якобианом (??)

Заметим, что внутреннее произведение и дифференциал по построению никак не зависят от выбора системы координат. Значит, от выбора системы координат зависит только форма объёма. (????)

Выясним, при каких заменах координат дивергенция не меняется.

Как мы покажем в параграфе 3.4, форма высшей степени на $\mathbb{R}^n$ при криволинейной замене координат преобразуется как

$$\tau' = \det\left(\frac{\partial x'^\alpha}{\partial x^\beta}\right) dx^1\wedge\ldots\wedge dx^n \equiv \det\Lambda^\alpha_\beta dx^1\wedge\ldots\wedge dx^n$$

где $\tau = dx^1\wedge\ldots\wedge dx^n, \det\Lambda^\alpha_\beta = \mathcal{J}_\varphi$ - якобиан замены.

Поэтому

$$d\imath_X\tau' = d\left(\mathcal{J}_\varphi\imath_X\tau\right) = d\mathcal{J}_\varphi\wedge\imath_X\tau + \mathcal{J}_\varphi d_X\tau$$

В силу произвольности векторного поля X получаем, что дивергенция не меняется лишь при таких заменах координат, для которых

$$d\mathcal{J}_\varphi = 0, \quad \mathcal{J}_\varphi = \text{ const.}$$

(?? пример тут неплохо бы вставить.)

При инверсии координатных осей $x^\alpha \mapsto x'^\alpha = -x^\alpha$, дивергенция векторного поля X не изменяется.

Как это согласуется с формулой (2.52)?

(я хз, но и не интересно в этом копаться)

О роторе rot

$$\operatorname{rot} X = \left(*d\left(X^b\right)\right)^\sharp,$$

(тут еще раз пояснение операций!)

(потом допишу, многое можно указать будет, пока не до него)

On Laplacian $\triangle$

We can combine d and $d^\dagger$ to construct the Laplacian, $\triangle : \Lambda^p(M) \to \Lambda^p(M)$, defined as

$$\triangle = \left(d + d^\dagger\right)^2 = dd^\dagger + d^\dagger d$$

(??? что за даггер???)

Example: by acting on functions f, we have $d^\dagger f = 0$ (because $\star f$ is a top form so $d \star f = 0$) so,

$$\begin{aligned}
\triangle(f) &= - \star d \star (\partial_\mu f dx^\mu) \\
&= -\frac{1}{(n-1)!} \star d \left((\partial_\mu f)\, g^{\mu\nu} \sqrt{|g|} \epsilon_{\nu\rho_1 \ldots \rho_{n-1}} dx^{\rho_1} \wedge \ldots \wedge dx^{\rho_{n-1}} \right) \\
&= -\frac{1}{(n-1)!} \star \partial_\sigma \left(\sqrt{|g|} g^{\mu\nu} \partial_\mu f \right) \epsilon_{\nu\rho_1 \ldots \rho_{n-1}} dx^\sigma \wedge dx^{\rho_1} \wedge \ldots \wedge dx^{\rho_{n-1}} \\
&= - \star \partial_\nu \left(\sqrt{|g|} g^{\mu\nu} \partial_\mu f \right) dx^1 \wedge \ldots \wedge dx^n \\
&= -\frac{1}{\sqrt{|g|}} \partial_\nu \left(\sqrt{|g|} g^{\mu\nu} \partial_\mu f \right)
\end{aligned}$$

There is a particularly nice story involving p-forms γ that obey

$$\triangle \gamma = 0$$

Such forms are said to be harmonic. An harmonic form is necessarily closed, meaning $d\gamma = 0$, and co-closed, meaning $d^\dagger \gamma = 0$. This follows by writing $\langle \gamma, \Delta\gamma \rangle = \langle d\gamma, d\gamma \rangle + \left\langle d^\dagger \gamma, d^\dagger \gamma \right\rangle = 0$ and noting that the inner product is positive-definite.

2.1.9 rotations (!?)

(соберу потом про это всё тут, мб еще какая-то теория добавится.)

Конструкция вращений

(потом пропишу нормально описание)

Рассмотрим типичные вращения

$$x'^\alpha = \Lambda^\alpha_\beta x^\beta$$

$$\Lambda(\phi) = \mathbf{1} - \mathrm{i} s \cdot \phi, \quad \phi \to 0$$

$$s^\alpha = \mathrm{i} \cdot \lim_{\phi \to 0} \frac{\partial \Lambda(\phi)}{\partial \phi^\alpha}$$

$$v'^\alpha(x) = \frac{\partial x^\alpha}{\partial x'^\beta} v^\beta(x') = \left[\delta^\alpha_\beta + \mathrm{i} \left(s^\gamma\right)^\alpha_\beta \phi_\gamma \right] v^\beta(x') = \left[\delta^\alpha_\beta + \mathrm{i} \left(s^\gamma\right)^\alpha_\beta \phi_\gamma \right] \cdot \left[v^\beta(x) + \frac{\partial v^\beta}{\partial x^\lambda} \delta x^\lambda \right]$$

$$\left(\hat{\ell}^\gamma\right)^\alpha_\beta = -\mathrm{i} \delta^\alpha_\beta \epsilon_{\lambda\mu\nu} x^\mu \frac{\partial}{\partial x^\nu}$$

$$v'^\alpha(x) = v^\alpha(x) + \mathrm{i} \left(\hat{\ell}^\gamma + s^\gamma\right)^\alpha_\beta \phi_\gamma v^\beta(x)$$

$$\delta v^\alpha = \mathrm{i} \left(\hat{j}^\gamma\right)^\alpha_\beta \phi_\gamma v^\beta(x), \quad \hat{j} = \hat{\ell} + s$$

Вращения в пространстве Минковского

(потом соберу их)

О применении вращений векторных полей

(указание на 1/2 страницы)

2.1.10 Manifolds: Basic Properties

(all from Ivanov, that basic useless topics)

Топологические многообразия. Гладкие многообразия. Гладкие структуры. Гладкие отображения, диффеоморфизмы.

Размерность многообразия. Инвариантность размерности при диффеоморфизмах.

Задание многообразий уравнениями.

Три определения касательного вектора к многообразию. Их эквивалентность.

Касательное пространство к многообразию, касательное расслоение.

Регулярные точки и регулярные значения отображения. Теорема о прообразе регулярного значения. Погружения и вложения. Подмногообразия.

Существование вложения компактного гладкого многообразия в евклидово пространство.

Теорема Уитни.

2.2 Shapes, Surfaces

2.2.1 Curvilinear Coordinates

(пока раздел, а там посмотрим)

7.1 Определение криволинейной системы координат

Пусть Ω - некоторая область в $\mathbb{R}^n$, и $(x^1, \ldots, x^n)$ - стандартные координаты в $\mathbb{R}^n$. Рассмотрим еще один экземпляр $\mathbb{R}^n$ (мы его обозначим через $\mathbb{R}^n_1$) со стандартными координатами $(y^1, \ldots, y^n)$. Система из n функций $y^i = y^i(x^1, \ldots, x^n), i = 1, \ldots, n$, заданных на Ω, называется непрерьвной системой координат, если отображение $\psi : \Omega \to \mathbb{R}^n_1$, заданное в виде

$$\begin{cases} y^1 = y^1(x^1, \ldots, x^n), \\ \ldots \\ y^n = y^1(x^1, \ldots, x^n) \end{cases}$$

Если все функции $x^i(y^1, \ldots, y^n)$ - гладкие, то координаты y^i называются гладкими. Отметим, что из гладкости системы координат не вытекает гладкость отображения ψ (достаточно рассмотреть пример замены координат на прямой $\mathbb{R}^1$, заданное в виде $x = y^3$; в точке $y = 0$ обратное отображение, т.е. отображение ϕ, гладким не является).

Для гладкой системы координат определена матрица Якоби J, составленная из частных производных $\left(\frac{\partial x^i}{\partial y^j}\right)$, и якобиан $|J|$, равный определителю матрицы Якоби.

Гладкая система координат называется регулярной, если ее якобиан всюду отличен от нуля, иными словами, матрица Якоби всюду невырождена. Из теоремы об обратной функции вытекает, что для регулярной криволинейной системы координат отображение ψ также гладкое, и его матрица Якоби равна J^{-1}, поэтому, в частности, матрица Якоби отображения ψ невырождена.

Пусть $(y_0^1, \ldots, y_0^n)$ - криволинейные координаты некоторой точки $P \in \Omega$. Для каждого i рассмотрим кривую $\delta_i(t)$, определенную для t, близких к y_0^i так:

$$\begin{cases} x^1(t) = x^1\left(y_0^1, \ldots, y_0^{i-1}, t, y_0^{i+1}, \ldots, y_0^n\right), \\ \ldots \\ x^n(t) = x^n\left(y_0^1, \ldots, y_0^{i-1}, t, y_0^{i+1}, \ldots, y_0^n\right). \end{cases}$$

Кривая $\delta_i(t)$ называется i-ой координатной кривой, проходящей через точку P. Отметим, что если координаты регулярны, то кривая $\delta_i(t)$ также регулярна (докажите).

Аналогично определим k-мерные координатные поверхности. Для этого выберем $k < n$ индексов $(i_1, \ldots, i_k)$ и для t^{i_p}, близких к $y_0^{i_p}$, зададим параметрически k-мерную поверхность так:

$$\begin{cases} x^1\left(t^{i_1}, \ldots, t^{i_p}\right) = x^1\left(y_0^1, \ldots, t^{i_1}, \ldots, t^{i_p}, \ldots, y^n\right), \\ \ldots \\ x^n\left(t^{i_1}, \ldots, t^{i_p}\right) = x^n\left(y_0^1, \ldots, t^{i_1}, \ldots, t^{i_p}, \ldots, y^n\right). \end{cases}$$

Если $k = n - 1$, то k-мерные координатные поверхности называются ординатными гиперповерхностями.

7.4.1 Закон изменения компонент метрики при замене координат

Посмотрим, как меняются компоненты метрики g_{ij} при замене регулярных координат y^i на регулярные координаты z^i. Обозначим через h_{ij} компоненты евклидовой метрики в координатах z^i. По определению, $h_{ij} = \langle \partial_{z^i}, \partial_{z^j} \rangle$. Имеем

$$h_{ij} = \langle \partial_{z^i}, \partial_{z^j} \rangle = \left\langle \sum_k \frac{\partial y^k}{\partial z^i} \partial_{y^k}, \sum_l \frac{\partial y^l}{\partial z^j} \partial_{y^l} \right\rangle =$$
$$\sum_{k,l} \frac{\partial y^k}{\partial z^i} \frac{\partial y^l}{\partial z^j} \left\langle \partial_{y^k}, \partial_{y^l} \right\rangle = \sum_{k,l} \frac{\partial y^k}{\partial z^i} \frac{\partial y^l}{\partial z^j} g_{kl}.$$

Иными словами, если $G = (g_{ij}), H = (h_{ij})$, то $H = J(y,z)^T G J(y,z)$, где $(\cdot)^T$ обозначает транспонирование матрицы. Снова отметим, что мы получили точно такую же формулу, как в теории поверхностей.

7.6 Стереографические координаты на сфере

Для простоты, рассмотрим двумерную сферу радиуса R. Пусть (x, y, z) - стандартные координаты в $\mathbb{R}^3$, тогда сфера S^2 задается уравнением $x^2 + y^2 + z^2 = R^2$. Обозначим через N северный полюс сферы S^2, т.е. точку с координатами $(0, 0, R)$, а через S - южный полюс, т.е. точку с координатами $(0, 0, -R)$. Пусть Π - координатная плоскость $z = 0$. Зададим отображение

$$\nu : S^2 \backslash \{N\} \to \Pi$$

следующим образом. Для каждой точки $P \in S^2 \backslash \{N\}$ рассмотрим точку P' пересечения прямой NP с плоскостью Π. Положим по определению $\nu(P) = P'$. Отображение ν называется стереографической проекиией из северного полюса N. Отметим, что аналогично можно определить стереографическую проекцию из южного полюса.

Стереографическая проекция ν задает координаты на сфере S^2 без северного полюса. Эти координаты называются стереографическими. Вычислим в явном виде стереографические координаты.

Итак, в стереографических координатах (u, v) сфера S^2 радиуса R без северного полюса N записывается так:

$$\left(x = \frac{2R^2u}{u^2+v^2+R^2}, y = \frac{2R^2v}{u^2+v^2+R^2}, z = R\frac{u^2+v^2-R^2}{u^2+v^2+R^2}\right),$$

или, в полярных координатах (ρ, φ), так:

$$\left(x = \frac{2R^2\rho\cos\varphi}{\rho^2+R^2}, y = \frac{2R^2\rho\sin\varphi}{\rho^2+R^2}, z = R\frac{\rho^2-R^2}{\rho^2+R^2}.\right)$$

Теорема 7.1 Индуцированнал метрика $d\sigma^2$ на сфере S^2 радиуса R в стереографических координатах (u, v) или (ρ, φ) имеет вид:

$$d\sigma^2 = \frac{4R^4}{\left(u^2+v^2+R^2\right)^2}\left(du^2+dv^2\right) = \frac{4R^4}{\left(\rho^2+R^2\right)^2}\left(d\rho^2+\rho^2d\varphi^2\right)$$

Утверждение 7.1 Предположим, ито в области $\Omega \subset \mathbb{R}^n$ заданы две римановых метрики ds^2 и $d\sigma^2$, отличающиеся на положительную функиию λ^2, т.е. $d\sigma^2 = \lambda^2 ds^2$. Тогда угль между парой пересекающихся кривых, вычисленные по отношению к обеим этим метрикам, одинаковы.

Следствие 7.1 Стереографическая проекиия сохраняет угль между кривылми на сфере S^2.

Добавление 7.3. Нормальные координаты. С помощью геодезических мы определим на поверхности так называемые нормальные координаты, которые оказываются удобными для многих вычислений. Итак, пусть $M = \{r : U \to \mathbb{R}^{n+1}\}$ - регулярная поверхность, заданная в параметрическом виде, P - некоторая точка на M, и T_PM - касательное пространство к поверхности M в точке P. Пусть V - произвольный касательный вектор из T_PM. Тогда, в силу следствия 6.1, существует и единственна геодезическая $\xi_V(s)$, удовлетворяющая начальным условиям $\xi_V(0) = P$, и $\dot{\xi}_V(0) = V$ (здесь точкой обозначено дифференцирование по s). Пусть a - произвольное вещественное число. Тогда для вектора aV существует своя такая геодезическая $\xi_{aV}(s)$. Оказывается геодезические $\xi_V(s)$ и $\xi_{aV}(s)$ отличаются на перепараметризацию. Лемма 7.1 В сделанных выше обозначениях,

$$\xi_{aV}(s) = \xi_V(as)$$

в общей области определения.

Пусть в некоторой нормальной окрестности точки P фиксированы нормальные координаты. Рассмотрим в касательном пространстве T_PM сферу $S^n(\varepsilon)$ радиуса ε, целиком лежащую в нормальной окрестности точки $0 \in T_PM$. Образ $\exp_P\left(S^n(\varepsilon)\right)$ сферы $S^n(\varepsilon)$ при экспоненциальном отображении называется геодезической сферой с иентром в точке P и радиуса ε. Имеет место следующее интересное утверждение.

Предложение 7.2 (Лемма Гаусса) Пусть Q - произвольная точка геодезической сферы Σ с иентром в точке P. Тогда геодезическая, соединяющая точки P и Qu челиком лежащая в нормальной окрестности точки P, - единственна. Более того, эта геодезическая приходит на геодезическую сферу Σ под прямым углом. Последнее означает, ито геодезическая перпендикулярна любой регулярной кривой, проходящей через Q и лежащей в Σ.

Добавление 7.4. Полугеодезические координаты на двумерной поверхности. Нормальные координаты, которые мы определили в предыдущем разделе, являются весьма частным случаем так называемых координат Ферми. Координаты Ферми мы определим в следующем семестре для подмногообразия в римановом многообразии, а сейчас ограничимся еще одним частным случаем - так называемыми полугеодезическими координатами на двумерной поверхности.

Итак, пусть $M = \{r : U \to \mathbb{R}^3\}$ - регулярная двумерная поверхность, и $\gamma : I \to \mathbb{R}^3$ регулярная кривая на M. Пусть $E(t)$ - гладкое касательное векторное поле единичной длины, заданное в точках кривой γ и перпендикулярное вектору скорости $\dot{\gamma}(t)$ в точке $\gamma(t)$. Построим отображение некоторой окрестности отрезка $I \times \{0\}$ в прямом произведении $I \times \mathbb{R}^1$ в поверхность M. В каждой точке $\gamma(t)$ кривой γ мы построим геодезическую $\nu_t(s)$, определенную на некотором интервале $[-s_0(t), s_0(t)]$, и удовлетворяющую начальным условиям $\nu_t(0) = \gamma(t), \nu_t'(0) = E(t)$. Точке $(t, s) \in I \times \mathbb{R}^1$ мы поставим в соответствие точку $\nu_t(s)$ поверхности M. Построенное отображение мы обозначим через $\exp_\gamma$ и назовем экспоненциальным отображением нормалей вдоль γ.

Предложение 7.3 (Обобщенная лемма Гаусса) Полугеодезические координаты в окрестности регулярной натурально параметризованной кривой на неособой двумерной поверхности ортогональнl.

2.2.2 Parametric Surfaces, First and Second Fundamental Forms

Parametric Surfaces

Определение 3.1. Непрерывной параметрической поверхностью размерности k в n мерном евклидовом пространстве $\mathbb{R}^n, n \geq k$, называется произвольное непрерывное отображение $r : \Omega \to \mathbb{R}^n$ из некоторой области $\Omega \subset \mathbb{R}^k$ в $\mathbb{R}^n$.

Лемма 3.14. Для любой регулярной параметрической поверхности $r : \Omega \to \mathbb{R}^n u$ любой точки $P \in \Omega$ существует такая окрестность $U \subset \Omega$ этой точки, что образ отображения $r|_U$ является графиком некоторой гладкой функции (возможно, после перенумерации координат в $\mathbb{R}^n$).

Theorem 3.20.

С локальной точки зрения три способа задания регулярной поверхности эквивалентны в следующем смысле:

- если $r : \Omega \to \mathbb{R}^n$ - регулярная параметрическая поверхность, то для каждого $u \in \Omega$ существует такая окрестность U точки и в области Ω, что ее образ $r(U)$ является поверхностью-графиком для некоторой гладкой векторнозначной функи, ии;
- для каждой точки P неявной поверхности M_c существует такая окрестность $U \subset \mathbb{R}^n$, что $U \cap M_c$ есть поверхность-график.

Конструкция 3.24. Пусть $M \subset \mathbb{R}^n$ - регулярная поверхность, и $f : M \to \mathbb{R}^m$ - (векторнозначная) функция на поверхности. Пусть $r_M : \Omega_M \to \mathbb{R}^n$ - (взаимно-однозначное с образом) параметризующее отображение поверхности M. Положим $\tilde{f} = f \circ r_M$. Обратно, так как отображение r_M взаимно-однозначно с образом, то для каждой функции $\tilde{f} : \Omega_M \to \mathbb{R}^m$ однозначно определена функция $f : M \to \mathbb{R}^m$, такая, что $\tilde{f} = f \circ r_M$, а именно $f = \tilde{f} \circ r_M^{-1}$, где $r_M^{-1} : M \to \Omega_M$.

Аналогично, пусть задано отображение $\gamma : I \to M$ промежутка I в регулярную поверхность M. Тогда определено отображение $\tilde{\gamma} = r_M^{-1} \circ \gamma : I \to \Omega_M$, и наоборот, для каждого отображения $\tilde{\gamma} : I \to \Omega_M$ из промежутка I в параметризующую область Ω_M однозначно определено отображение $\gamma = r_M \circ \tilde{\gamma} : I \to M$, причем $\tilde{\gamma} = r_M^{-1} \circ \gamma$.

Наконец, пусть $N \subset \mathbb{R}^n$ - ещё одна регулярная поверхность, и $r_N : \Omega_N \to \mathbb{R}^n$ - ее (взаимно-однозначное с образом) параметризующее отображение. Пусть $F : M \to N$ отображение поверхности M в поверхность N. Тогда в силу взаимной однозначности параметризующих отображений определено отображение $\tilde{F} : \Omega_M \to \Omega_N$ следующего вида:

$\tilde{F} = r_N^{-1} \circ F \circ r_M$. Обратно, для каждого отображения $\tilde{F} : \Omega_M \to \Omega_N$ однозначно определено отображение поверхностей $F : M \to N$, такое, что $\tilde{F} = r_N^{-1} \circ F \circ r_M$.

Определение 3.25. Функция $\tilde{f}$ и отображения $\tilde{F}$ и $\tilde{\gamma}$ называются координатными представлениями функции f и отображений F и γ соответственно.

Определение 3.28. Говорят, что кривая γ лежит на M, если $\gamma(I) \subset M$. Лежащая на поверхности параметрическая кривая называется гладкой, если она гладкая как отображение промежутка в поверхность, т.е., если её координатное представление в смысле определения 3.25 - гладкое.

Определение 3.32. Вектор скорости $\dot{\gamma}(t_0) \in \mathbb{R}^n$ кривой γ в точке P назовем касательным вектором к поверхности M в точке P. Совокупность касательных векторов в точке P, построенных по всем гладким кривым на поверхности M, проходящим через эту точку, называется касательным пространством к поверхности M в точке P и обозначается через $T_P M$.

Утверждение 3.37. При замене координат на регулярной поверхности, векторы канонического базиса в каждой точке поверхности преобразуются с помощью матрицы Якоби этой замены координат, которая играет роль матрицы перехода от одного базиса к другому. Компоненты одного и того же касательного вектора в разных системах координат связаны между собой с помощью этой матрицы перехода.

Конструкция 3.39. Пусть $F : M \to N$ - гладкое отображение регулярных поверхностей, $P \in M$ - произвольная точка поверхности M и $v \in T_P M$ - произвольный касательный вектор. Напомним, что каждый такой вектор является по определению вектором скорости некоторой кривой γ, проходящей через точку P. Пусть $v = \dot{\gamma}(t_0)$, где γ - гладкая кривая на M и $\gamma(t_0) = P$. Построим отображение $dF|_P : T_P M \to T_{F(P)} N$, положив

$$dF|_P(v) = \left.\frac{d}{dt}\right|_{t=t_0} F(\gamma(t))$$

Заметим, что $F \circ \gamma$ - гладкая кривая на поверхности N, проходящая через точку $F(P)$, а именно, $F(\gamma(t_0)) = F(P)$, поэтому отображение $dF|_P$ определено корректно.

Определение 3.40. Отображение $dF|_P : T_P M \to T_{F(P)} N$ называется дифференциалом отображения F в точке P.

the First Fundamental Form = Индуцированная метрика

Пусть, как и выше, P - произвольная точка регулярной k-мерной поверхности $M \subset \mathbb{R}^n$. Скалярное произведение векторов в $\mathbb{R}^n$ порождает скалярное произведение касательных векторов из $T_P M$.

Конструкция 3.43. Пусть ξ и η - касательные векторы к поверхности M в точке P. Определим функцию $\mathfrak{G}(P) : T_P M \times T_P M \to \mathbb{R}$ так: $\mathfrak{G}(\xi, \eta) = \langle \xi, \eta \rangle$, где справа стоит стандартное скалярное произведение в $\mathbb{R}^n$. Другими словами, функция $\mathfrak{G}(P)$ есть ограничение стандартного скалярного произведения на подпространство $T_P M$.

Утверждение 3.44. В произвольной точке P регулярной поверхности M функция $\mathfrak{G}(P)$ - это положительно определенная симметричная билинейная форма на касательном пространстве $T_P M$.

Определение 3.45. Билинейная форма $\mathfrak{G}(P)$ называется первой фундаментальной формой поверхности в точке P или индуцированной метрикой.

Утверждение 3.47. Пусть $x^i\left(u^1,\ldots,u^k\right)$ - координатные функции регулярной поверхности. Тогда компоненты g_{ij} матрицы первой фундаментальной формы имеют вид

$$g_{ij}\left(u^1,\ldots,u^k\right)=\sum_{p=1}^{n}\frac{\partial x^p}{\partial u^i}\frac{\partial x^p}{\partial u^j}$$

B частности, g_{ij} - гладкие функции координат u^i, т.е. гладко зависят от точки P.

Other basic about surfaces

Isometries of surfaces

Определение 3.56. Пусть заданы две k-мерные регулярные поверхности $M_i, i=1,2$. Будем говорить, что поверхности M_1 и M_2 изометричны, если существует гладкое, взаимно-однозначное, регулярное отображение $\psi: M_1\to M_2$, сохраняющее длины всех переходящих друг в друга гладких кривых.

Утверждение 3.58. Гладкое взаимно-однозначное регулярное отображение $\psi: M_1\to M_2$ задает изометрию регулярных поверхностей $M_i, i=1,2$, если и только если отображение ψ "сохраняет первую фундаментальную форму" в следующем смысле:

$$G_1=J^TG_2J,$$

где G_i - матрица первой фундаментальной формы поверхности M_i, aJ - матрица Якоби отображения ψ в соответствующих координатах.

The Second Fundamental Form

Конструкция 4.1. Обозначим через ($x^1,\ldots,x^n$) стандартные координаты $\mathbb{R}^n$, а через $(u^1,\ldots,u^{n-1})$ - координаты на поверхности. В каждой точке P на M определен канонический базис $\{\partial_{u^i}\}$ касательного пространства T_PM. Поэтому в каждой точке P поверхности однозначно определен единичный вектор $N(P)$, ортогональный всем векторам ∂_{u^i} и дополняющий систему векторов $\{\partial_{u^i}\}$ до положительно ориентированного базиса в $\mathbb{R}^n$. Другими словами, вектор $N(P)$ однозначно определяется из соотношений $\langle\partial_{u^i},N(P)\rangle=0, \langle N(P),N(P)\rangle=1$ и, наконец,

$$\det\left(\partial_{u^1},\ldots,\partial_{u^{n-1}},N(P)\right)>0.$$

Так как векторы ∂_{u^i} гладко зависят от точки поверхности, то и вектор $N(P)$ тоже гладко зависит от точки.

Определение 4.2. Вектор $N(P)$ называется (ориентированной) поле нормалью к поверхности M, соответствующей координатам (u^i). Также будем говорить, что N это поле ориентированных нормалей, имея ввиду, что вектор $N(P)$ задан в каждой точке $P\in M$.

4.1 Определение второй фундаментальной формы регулярной поверхности

Предложение 4.1 Соотношение $\mathfrak{q}(\xi)=\langle\ddot{\gamma},N\rangle, \xi\in T_PM$, корректно определяет в произвольной точке P гиперповерхности M некоторую квадратичную форму q на

касательном пространстве T_PM. Если $(u^1, \ldots, u^{n-1})$ - координаты на M, то матрица квадратичной формы $\mathfrak{q}$ в базисе (∂_{u^i}) имеет вид $q_{ij} = \langle r_{u^i u^j}, N\rangle$.

Определение. Квадратичная форма q на касательном пространстве T_PM называется второй фундаментальной формой или второй квадратичной формой гиперповерхности M в точке P (по отношению к нормали N).

Замечание. В отличие от первой фундаментальной формы поверхности, вторая фундаментальная форма, вообще говоря, не обязана быть ни невырожденной, ни положительно определенной.

Замечание. Вторую фундаментальную форму также часто записывают в дифференциальном виде так:

$$dq^2 = \sum_{ij} q_{ij} du^i du^j$$

Пример. Пусть двумерная гиперповерхность задана в виде графика $z = f(x, y)$. Тогда соответствующее параметрическое представление имеет вид

$$\begin{gathered} r = (x, y, f(x, y)), \quad r_x = (1, 0, f_x), \quad r_y = (0, 1, f_y), \\ r_{xx} = (0, 0, f_{xx}), \quad r_{xy} = (0, 0, f_{xy}), r_{yy} = (0, 0, f_{yy}), \\ N = \frac{[r_x, r_y]}{\|[r_x, r_y]\|} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}, \end{gathered}$$

и, значит, матрица второй фундаментальной формы в координатах (x, y) выглядит так:

$$\frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}$$

Конструкция 4.6. Пусть M - регулярная гиперповерхность в $\mathbb{R}^n$ и $P \in M$ - некоторая ее точка. Фиксируем одно из двух полей нормалей к поверхности M и обозначим его через N. Пусть $\gamma(t)$ - произвольная кривая, лежащая на M и проходящая через P, $\gamma(t_0) = P$, и $\xi \in T_PM$ - вектор скорости $\dot{\gamma}(t_0)$ кривой γ в точке P. Построим функцию $\mathfrak{q} : T_PM \to \mathbb{R}$, положив $\mathfrak{q}(\xi) = \langle \ddot{\gamma}(t_0), N\rangle$.

Утверждение 4.7. Пусть на регулярной гиперповерхности M фиксировано поле нормалей N. Функция $\mathfrak{q}$ является квадратичной формой на линейном пространстве T_PM. Если $(u^1, \ldots, u^{n-1})$ - координаты на M, то матрица квадратичной формы $\mathfrak{q}$ в соответствующем каноническом базисе (∂_{u^i}) имеет вид $q_{ij} = \langle r_{u^i u^j}, N\rangle$.

Пример 4.14. Пусть двумерная гиперповерхность задана в виде графика $z = f(x, y)$. Тогда соответствующее параметрическое представление имеет вид

$$\begin{gathered} r = (x, y, f(x, y)), \quad r_x = (1, 0, f_x), \quad r_y = (0, 1, f_y), \\ r_{xx} = (0, 0, f_{xx}), \quad r_{xy} = (0, 0, f_{xy}), r_{yy} = (0, 0, f_{yy}), \\ N = \frac{[r_x, r_y]}{\|[r_x, r_y]\|} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}} \end{gathered}$$

и, значит, матрица второй фундаментальной формы в координатах (x, y) выглядит следующим образом:

$$\frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

4.2 Геометрический смысл второй формы - кривизны плоских сечений

Пусть $\Pi \subset \mathbb{R}^n$ - произвольная проходящая через $P \in M$ двумерная аффинная плоскость, не лежащая в $T_P M$. Несложно показать, что Π пересекает поверхность M около точки P по некоторой регулярной кривой γ. Каждая такая кривая называется плоским сечением, проходящим через P. Если плоскость Π проходит через нормаль N к поверхности M в точке P, то плоское сечение γ называется нормальным. Если t - произвольный параметр на γ, такой что $\gamma(t_0) = P$, то будем говорить, что сечение γ проведено в направлении вектора $\dot{\gamma}(t_0)$.

Теорема 4.1 (Об отношении форм) Пусть $\xi \in T_P M$ - произвольный ненулевой вектор, γ - плоское сечение, проведенное через P в направлении ξ,, k - кривизна сечения γ в точке P. Тогда или k и $\mathfrak{q}(\xi)$ одновременно равнь нулю, или

$$k \cos\theta = \frac{\mathfrak{q}(\xi)}{\mathfrak{G}(\xi)},$$

где θ - угол между главной нормалью $m\kappa$ сечению γ и нормалью $N\kappa$ поверхности M в точке P (напомним, ито вторая фундаментальная форма вычисляется по отношению κN).

Следствие 4.2 Значение q (ξ) второй фундаментальной формь поверхности M на единичном векторе $\xi \in T_P M$ равно плюс или минус кривизне k плоского нормального сечения γ, проведенного через P в направлении ξ. При этом знак "плюс" вьбирается тогда и только тогда, когда N является главной нормалью сечения γ. В частности, $k = |\mathfrak{q}(\xi)|$.

Обозначим через γ_n плоское нормальное сечение, проведенное через P в направлении ξ, и пусть k_n - кривизна сечения γ_n в точке P. Приводимое ниже следствие из теоремы 4.1 называется теоремой Менье.

Следствие 4.3 (Теорема Менье) Пусть $\xi \in T_P M$ - ненулевой вектор, а γ и γ_n - некоторое плоское сечение и плоское нормальное сечение, проведеннье через P в одном и том же направлении ξ. Тогда или одновременно $k = 0$ и $k_n = 0$, или $k_n = k\cos\theta$, где θ - угол между главными нормалями и к сечениям γ и γ_n соответственно.

4.3 Главные кривизны и главные направления

Предложение 4.2 Пусть в линейном пространстве $\mathbb{R}^n$, в котором фиксирован произвольный базис, задано скалярное произведение $\langle \cdot, \cdot \rangle$ с помощью симметричной невьрожденной положительно определенной матриць G. Пусть, кроме того, задана квадратичнал форма q. Тогда в $\mathbb{R}^n$ сущеествует базис (e_i), ортонормальньй относительно указанного скалярного произведения, и такой что форма q имеет в нем диагональный вид. При этом, если Q - эта матрица бормы q в исходном базисе, то базис (e_i) и соответствующие собственнье числа формь Q относительно формь G могут быть найдень из следующего характеристического уравнения:

$$\det(Q - \lambda G) = 0$$

Определение. Пусть G и Q - матрицы первой и второй фундаментальных форм регулярной гиперповерхности M в некоторой ее точке P. Корни уравнения $\det(Q - \lambda G) = 0$ называются главными кривизнами поверхности M в точке P. Если λ_0 - главная кривизна, то векторы из $T_P M$ соответствующе нетривиальным решениям линейного уравнения $(Q - \lambda_0 G)\, X = 0$, которые, очевидно, существуют, называются главными направлениями поверхности M в точке P.

Следствие 4.4 Пусть P - произвольная точка регулярной гиперповерхности M. Тогда в касательном пространстве T_PM можно выбрать такой базис, что первал и вторал фундаментальные формы поверхности в этом базисе будут иметь вид

$$ds^2 = \sum_i \left(du^i\right)^2, \quad dq^2 = \sum_i \lambda_i \left(du^i\right)^2$$

соответственно. При этом, указанный базис состоит из векторов главных направлений, а числа λ_i являются главньли кривизнами поверхности M в точке P.

Из следствия 4.2 вытекает следующий результат, поясняющий геометрический смысл главных кривизн.

Следствие 4.5 Главная кривизна λ_i поверхности M в точке P равна, *с* точностью до знака, кривизне нормального сечения поверхности в точке P вдоль соответствующего главного направления.

Кривизна $k_n(V)$ нормального сечения поверхности в точке P в направлении V может быть вычислена следующим образом:

$$\pm k_n(V) = dq^2(V, V) = \sum_i \lambda_i \left(\cos\varphi_i\right)^2$$

Например, в R3:

Следствие 4.6 (Формула Эйлера) Пусть M^2 - двумерная регулярная поверхность в $\mathbb{R}^3$, и P - точка из M. Тогда кривизна $k_n(V)$ нормального сечения в точке P в направлении $V \in T_PM$ может быть вычислена так:

$$\pm k_n(V) = \lambda_1(\cos\varphi)^2 + \lambda_2(\sin\varphi)^2$$

где λ_1, λ_2 - главнье кривизны поверхности в точке M, а φ - - уголмежду вектором V и главным направлением, соответствующим главной кривизне λ_1.

Следствие 4.7 I лавнье кривизны и главнье направления двумерной регулярной поверхности в точке P соответствуют наибольшему и наименьшему значениям функиии $\lambda_1(\cos\varphi)^2 + \lambda_2(\sin\varphi)^2$, значение которой равно кривизне нормального сечения, взятой со знаком $\langle m, N >$, где m - главная нормаль этого сечения, а N - та нормаль κ поверхности, относительно которой определена ее вторая фундаментальная форма.

Следствие 4.25 (Теорема Менье).

Пусть $\xi \in T_PM$ - ненулевой вектор, $\gamma u \gamma_n$ некоторое плоское и плоское нормальное сечения, проведенные через P в одном и том же направлении ξ, а kuk_n - их кривизны. Тогда или одновременно $k = 0uk_n = 0$, или $k_n = k\cos\alpha$, где α - угол между главными нормалями κ этим сечениям.

4.4 Средняя и гауссова кривизна гиперповерхности

В предыдущем пункте мы в каждой точке регулярной гиперповерхности $M \subset \mathbb{R}^n$ определили базис из главных направлений и главные кривизны. Однако, как и в линейной алгебре, иногда бывает полезно изучать инвариантные функции главных кривизн, а не сами главные кривизны. Мы рассмотрим две такие функции - так называемые среднюю и гауссову кривизны.

Определение. Сумма главных кривизн поверхности M в точке P называется средней кривизной поверхности M в точке P и обозначается через $H(P)$. Произведение главных кривизн называется гауссовой кривизной и обозначается через $K(P)$.

Утверждение 4.2 Средняя кривизна и гауссова кривизна поверхности не зависят от выбора параметризачии поверхности. При этом, если G и - матриць первой и второй квадратичных форм поверхности в точке P, то средняя и гауссова кривизны могут быть вычислены так:

$$H(P) = \operatorname{trace}\left(G^{-1}Q\right), \quad K(P) = \det\left(G^{-1}Q\right) = \frac{\det Q}{\det G}$$

Знак гауссовой кривизны двумерной поверхности имеет простой геометрический смысл.

Предложение 4.3 Пусть P - произвольная точка двумерной регулярной поверхности $M \subset \mathbb{R}^3$. Если в точке P гауссова кривизна $K(P)$ поверхности M положительна, то достаточно малая окрестность точки P поверхности челиком лежит в одном из двух полупространств, определяемьх касательной плоскостью T_PM, рассматриваемой как аффинная гиперплоскость, проходящал через точку P пространства $\mathbb{R}^3$. Если же $K(P) < 0$, то это не так, а именно, любая окрестность точки P пересекается с внутренностью обоих полупространств.

Геом смысл средней кривизны: Предложение 4.4 Пусть P - произвольная точка гиперповерхности $M \subset \mathbb{R}^n, uN$ - нормаль κM в точке P. Пусть $v_1, \ldots, v_{n-1}$ - произвольный набор попарно ортогональных единичных векторов из T_PM. Обозначим через k_i кривизну нормального сечения поверхности M вдоль направления v_i, умноженную на знак $\langle m_i, N\rangle$, где m_i - главная нормаль этого сечения. Тогда сумма чисел k_i равна средней кривизне поверхности M в точке P.

Основные примеры

Добавление 4.5. Средняя и гауссова кривизны двумерной поверхности. Пусть двумерная поверхность задана параметрически, (u, v) - координаты на этой поверхности, и пусть метрика ds^2 и вторая форма $d\sigma^2$ имеют вид:

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2, \quad d\sigma^2 = Ldu^2 + 2Mdudv + Ndv^2.$$

Отметим, что такие буквенные обозначения для компонент первой и второй фундаментальных форм являются традиционными в классической теории поверхностей.

Вернемся к нашим вычислениям. Матрица, обратная к матрице метрики, имеет вид:

$$\frac{1}{EG - F^2}\begin{pmatrix} G & -F \\ -F & E \end{pmatrix}$$

поэтому

$$H = \frac{GL - 2FM + EN}{EG - F^2}, \quad K = \frac{LN - M^2}{EG - F^2}$$

Пример. Пусть поверхность задана графиком функции $z = f(x, y)$. Тогда, как мы уже вычисляли, первая и вторая фундаментальные формы имеют вид:

$$G = \begin{pmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{pmatrix}, \quad Q = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}}\begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix},$$

поэтому

$$G^{-1} = \frac{1}{1 + f_x^2 + f_y^2}\begin{pmatrix} 1 + f_y^2 & -f_x f_y \\ -f_x f_y & 1 + f_x^2 \end{pmatrix}$$

и, значит, средняя и гауссова кривизны, в соответствие с утверждением 4.2, имеют вид:

$$H = \frac{\left(1+f_y^2\right) f_{xx} - 2f_x f_y f_{xy} + \left(1+f_x^2\right) f_{yy}}{\left(1+f_x^2+f_y^2\right)^{3/2}}, \quad K = \frac{f_{xx}f_{yy} - f_{xy}^2}{\left(1+f_x^2+f_y^2\right)^2}.$$

Пример. Пусть $M = S^2$ - стандартная сфера радиуса R. В качестве N выберем внутренние нормали к S^2, т.е. нормали, направленные в противоположную сторону по отношению к соответствующим радиус-векторам. Так как каждое плоское нормальное сечение является окружностью радиуса R, кривизна каждого такого сечения равна $1/R$. Поэтому кривизны главных нормальных сечений также равны $1/R$, и, значит, средняя кривизна равна $2/R$, а гауссова - равна $1/R^2$.

О другом про вторую форму и с ней связанное

4.5 О теореме Бонне

Теорема 4.2 (Бонне, 1867) Если заданнье на Ω положительно определеннал квадратичнал форма g и произвольная квадратичнал форма q удовлетворяют условиям Петерсона-Кодачии, то существует единственная (с точностью до движения объемлющего пространства) регулярная поверхность на Ω, такая что ее первая квадратичная форма совпадает с g, а вторая квадратичная форма совпадает с q.

(там еще есть отсылки на это)

О теореме Лапласа-Пуассона

Добавление 4.4. Теорема Лапласа-Пуассона. Понятие средней кривизны двумерной поверхности было впервые определено Томасом Юнгом (1805), а затем независимо Пьером Симоном Лапласом (1806). Лаплас изучал форму поверхности раздела двух физических сред, находящихся в равновесии, и выяснил что средняя кривизна поверхности раздела пропорциональна разности давлений в разделяемых средах. Этот результат в механике сплошной среды называется теоремой Лапласа. Коэффициент пропорциональности называется коэффиииентом поверхностного натяэения.

Теорема 4.3 (Лаплас-Пуассон) Средняя кривизна поверхности раздела двух равновесных физических сред пропориионалъна разности давлений в этих средах вблизи поверхности раздела.

2.2.3 Metric

О метрике по Иванову Тужилину

$$\langle v, w\rangle = g_{ij} v^i w^j$$

Утверждение 8.1 Определенное выше скалярное произведение не зависит от выбора координат.

Рассмотрим пример псевдоримановой метрики в $\mathbb{R}^n$, заданной в стандартных координатах в виде

$$ds^2 = -\sum_{i=1}^{s} \left(dx^i\right)^2 + \sum_{j=s+1}^{n} \left(dx^j\right)^2.$$

Такая метрика называется псевдоевклидовой метрикой индекса s. Эта метрика порождает в каждом касательном пространстве $T_P\mathbb{R}^n$ псевдоскалярное произведение $\langle\cdot,\cdot\rangle_s$: если $v =$

$(v^1, \ldots, v^n)$ и $w = (w^1, \ldots, w^n)$ - два вектора из $T_P\mathbb{R}^n$, то

$$\langle v, w\rangle_s = -\sum_{i=1}^{s} v^i w^i + \sum_{j=s+1}^{n} v^j w^j$$

О пространстве Минковского

(пока по Иванову Тужилину, вообще, важное указание, с теорией поля потом согласую)

Псевдоевклидово расстояние между точками P и Q, т.е. величина, равная $\sqrt{\langle P-Q, P-Q\rangle_1}$, называется пространственно-временныл интервалом между событиями P u Q; нулевые векторы, т.е. векторы V, такие что $\langle V, V\rangle_1 = 0$, называются световыли (поэтому изотропный конус называется световым), векторы P, для которых $\langle P, P\rangle_1 < 0$, называются времениподобными, а векторы P, для которых $\langle P, P\rangle_1 > 0$ - пространственноподобньми.

Пример. Рассмотрим в пространстве Минковского $\mathbb{R}^{n+1}_1$ псевдосферу S^n_1 чисто мнимого радиуса iR. По определению, псевдосфера S^n_1 задается уравнением

$$-\left(x^1\right)^2 + \left(x^2\right)^2 + \cdots + \left(x^{n+1}\right)^2 = -R^2.$$

Ясно, что псевдосфера S^n_1 состоит из двух компонент, разделенных координатной гиперплоскостью $x^1 = 0$ (скажем, при $n = 2$ это обычный двуполостный гиперболоид). Точка $N = (R, 0, \ldots, 0)$ называется севернылм полюсом псевдосферы, а точка $S = (-R, 0, \ldots, 0)$ - южным полюсом. Определим стереографическую проекиию σ половинки псевдосферы S^n_1, однозначно определенной неравенством $x^1 > 0$, в координатную плоскость $x^1 = 0$ так. Произвольную точку P из S^n_1 соединим с южным полюсом отрезком SP. Этот отрезок, очевидно, пересекает плоскость $x^1 = 0$ в некоторой точке, которую мы обозначим через $\sigma(P)$. Определим отображение σ, положив $\sigma : P \mapsto \sigma(P)$. Если мы обозначим через $(u^1, \ldots, u^n)$ стандартные координаты в плоскости $x^1 = 0$ (для точек плоскости $x^1 = 0$ по определению положим $x^i = u^{i-1}, i = 2, \ldots, n+1$). В этих координатах стереографическая проекция псевдосферы выглядит так.

Лемма 8.1 B сделанных выше обозначениях, если $(u^1, \ldots, u^n)$ - точки $\sigma(P)$, то координаты $(x^1, \ldots, x^{n+1})$ точки P могут быть вычислены так:

$$x^1 = R\frac{R^2 + \sum\limits_i \left(u^i\right)^2}{R^2 - \sum\limits_i \left(u^i\right)^2},$$

$$x^{i+1} = \frac{2R^2 u^i}{R^2 - \sum\limits_i \left(u^i\right)^2}, \quad i = 1, \ldots, n$$

Утверждение 8.2 Псевдосфера чисто мнимого радиуса iR с центром в начале координат в пространстве Минковского $\mathbb{R}^{n+1}_1$ является пространственно подобной поверхностью. Индуцированная на ней первая квадратичная форма в координатах стереографической проекиии (соответствующая риманова метрика в шаре радиуса R) имеет вид:

$$ds^2 = \frac{4R^4}{\left(R^2 - \sum\limits_i \left(u^i\right)^2\right)^2} \sum_i \left(du^i\right)^2.$$

Открытый шар радиуса R в $\mathbb{R}^n$ с метрикой

$$ds^2 = \frac{4R^4}{\left(R^2 - \sum\limits_i (u^i)^2\right)^2} \sum_i \left(du^i\right)^2$$

называется шаром Пуанкаре (при $n = 2$ - кругом Пуанкаре).

3.1.1 Riemannian Manifolds

3.1.2 Lorentzian Manifolds

3.1.3 The Joys of a Metric

3.1.4 A Sniff of Hodge Theory

2.2.4 Connections, Covariant Derivative

3.2.1 The Covariant Derivative

(абзац про суть ее!)

Компоненты связности - по определению это коэффициенты в выражении

$$\nabla_{e_\rho} e_\nu = \Gamma^\mu_{\rho\nu} e_\mu.$$

В случае связности Леви-Чивиты они называются символами Кристоффеля.

$$\begin{aligned} A^i{}_{;l} =& \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl} A^k \\ A_{i;l} =& \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il} A_k \end{aligned}$$

$$A^{ik}{}_{;l} = \frac{\partial A^{ik}}{\partial x^l} + \Gamma^i_{ml} A^{mk} + \Gamma^k_{ml} A^{im}$$

$$\begin{aligned} A^i_{k;l} =& \frac{\partial A^i_k}{\partial x^l} - \Gamma^m_{kl} A^i_m + \Gamma^i_{ml} A^m_k \\ A_{ik;l} =& \frac{\partial A_{ik}}{\partial x^l} - \Gamma^m_{il} A_{mk} - \Gamma^m_{kl} A_{im} \end{aligned}$$

Или в общем виде

$$\begin{aligned} T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q;\rho} =& T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q,\rho} + \Gamma^{\mu_1}_{\rho\sigma} T^{\sigma\mu_2\ldots\mu_p}{}_{\nu_1\ldots\nu_q} + \ldots + \Gamma^{\mu_p}_{\rho\sigma} T^{\mu_1\ldots\mu_{p-1}\sigma}{}_{\nu_1\ldots\nu_q} - \\ & - \Gamma^\sigma_{\rho\nu_1} T^{\mu_1\ldots\mu_p}{}_{\sigma\nu_2\ldots\nu_q} - \ldots - \Gamma^\sigma_{\rho\nu_q} T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_{q-1}\sigma} \end{aligned}$$

Ковариантная производная хорошо написана у Дэвида Тонга и по запросу мы можем обсудить её после прохождения курса базовой электродинамики.

5.1 Деривационные формулы Вейнгартена-Гаусса

Предложение 5.1 Для вторых производных радиус вектора неособой поверхности M имеют место следующие деривационные формулы Гаусса-Вейнгартена:

$$\frac{\partial}{\partial u^j} \partial_{u^i} = r_{ij} = \sum_\alpha \Gamma^\alpha_{ij} r_\alpha + q_{ij} N,$$

где q_{ij} - матрица второй квадратичной формы поверхности, а Γ_{ij}^k символь Кристоффеля поверхности M. Символь Кристоффеля вычисляются в терминах первой квадратичной формы поверхности и ее первых производных так:

$$\Gamma_{ij}^k = \frac{1}{2}\sum_\alpha \left(g^{k\alpha} \left(\frac{\partial g_{j\alpha}}{\partial u^i} + \frac{\partial g_{\alpha i}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\alpha} \right) \right).$$

Производные нормального векторного поля N вдоль на поверхности имеют вид

$$\frac{\partial N}{\partial u^i} = \sum_\alpha b_i^\alpha \partial_{u^\alpha}$$

где коэффичиенты b_i^α выражаются через первую и вторую квадратичные формы поверхности так:

$$b_i^\alpha = -\sum_k q_{ik} g^{k\alpha}$$

Введение: В практических задачах часто приходится рассматривать на поверхностях семейства векторов, зависящих от точки поверхности. Такие семейства называются векторными полями. Особый интерес представляют так называемые касательные векторные поля - семейства векторов $X = X(P)$ из касательных плоскостей $T_P M$ к поверхности.

Если на поверхности M фиксированы координаты $\left(u^1, \ldots, u^k\right)$ то, каждое касательное векторное поле X может быть задано как набор функций $X^i\left(u^1, \ldots, u^k\right), i = 1, \ldots, k$. По этим функциям однозначно восстанавливается вектор их $T_P M$ как линейная комбинация векторов канонического базиса с коэффициентами X^i:

$$X = X^1 \partial_{u^1} + \cdots + X^k \partial_{u^k}$$

Функции X^i называются компонентами касательного векторного поля X в координатах $\left(u^1, \ldots, u^k\right)$. Ясно, что если изменить координаты, то изменятся и функции, задающие компоненты данного векторного поля.

Наша ближайшая цель - научиться дифференцировать векторные поля на поверхности. Для простоты изложения мы вновь ограничимся случаем гиперповерхностей, т.е. $k = n - 1$. Пусть поверхность задана параметрически в виде $r : \Omega \to \mathbb{R}^n$. Чтобы продифференцировать поле X записанное в виде $X = \sum\limits_i X^i \partial_{u^i}$, нужно, очевидно, научиться дифференцировать векторы канонического репера. Напомним, что вектор ∂_{u^i} канонического репера в точке P - это производная $\partial r / \partial u^i$ радиус вектора поверхности по координате u^i в точке P. Поэтому производная вектора ∂_{u^i} по координате

u^j - это вектор, равный второй производной радиус вектора:

$$\frac{\partial}{\partial u^j} \partial_{u^i} = \frac{\partial^2 r}{\partial u^i \partial u^j}.$$

Для краткости, мы будем обозначать дифференцирование по u^i нижним индексом i, например, $\frac{\partial^2 r}{\partial u^i \partial u^j} = r_{ij}$, а $\partial_{u^i} = r_i$. Как и любой другой вектор в точке P, вектор r_{ij} раскладывается по базису объемлющего пространства $\mathbb{R}^n$, составленного из векторов r_i канонического репера и вектора N нормали к гиперповерхности. Соответствующие формулы и называются деривачионными формулами Гаусса-Вейнгартена. Выведем их. Разложим вектор r_{ij} по векторам базиса $r_1, \ldots, r_{n-1}, N$:

$$r_{ij} = \sum_\alpha \Gamma_{ij}^\alpha r_\alpha + q_{ij} N$$

(преобразования в страницу)

$$\Gamma_{ij}^k = \frac{1}{2}\sum_\alpha \left(g^{k\alpha} \left(\frac{\partial g_{j\alpha}}{\partial u^i} + \frac{\partial g_{\alpha i}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\alpha} \right) \right)$$

Определение. Набор чисел Γ_{ij}^k, заданный только что выписанными формулами, называется символами Кристоффеля поверхности M в точке P в системе координат $(u^1, \ldots, u^{n-1})$.

5.2 Теорема Гаусса

Теорема 5.1 (C. F. Gauss) Гауссова кривизна двумерной поверхности не меняется при изометрии. Другими словами, двумерные изометричные поверхности имеют в соответствующих точках одинаковую гауссову кривизну.

Идея доказательства состоит в том, чтобы получить из деривационных формул уравнения Гаусса, приравняв коэффициенты в выражениях для третьих производных радиус вектора поверхности. Уравнения Гаусса позволяют выразить гауссову кривизну поверхности через элементы ее первой квадратичной формы и их производные, откуда и вытекает утверждение теоремы. Действительно, напомним, что на изометричных поверхностях можно выбрать такие координаты, что их первые квадратичные формы будут просто совпадать, т.е. будут совпадать элементы соответствующих матриц. Поэтому, в таких координатах, будут совпадать и соответствующие производные первых квадратичных форм. Таким образом, в этих координатах совпадают все функции, входящие в выражения для гауссовой кривизны, а значит совпадают и сами гауссовы кривизны.

Следствие 5.1 Никакую часть стандартной двумерной сферь нельзя изометрично отобразить на плоскость.

Доказательство. Действительно, в противном случае их гауссовы кривизны совпадали бы. Однако, гауссова кривизна сферы радиуса R равна $1/R^2$, а гауссова кривизна плоскости равна нулю. Доказательство закончено.

5.3 Ковариантная производная касательного векторного поля

Основная теория по Иванову

Определение. Ковариантной производной $\nabla_i X$ касательного векторного поля X по координате u^i называется касательное векторное поле, полученное в каждой точке из вектора $\frac{\partial X}{\partial u^i}$ ортогональной проекцией на касательную плоскость:

$$\nabla_i X = \left(\frac{\partial X}{\partial u^i} \right)^T,$$

где через $(\cdot)^T$ обозначена операция ортогонального проектирования на касательную плоскость.

Непосредственно из определения получаем следующее выражение для координат векторного поля $\nabla_i X$.

Предложение 5.2 Пусть в окрестности точки P реаулярной аиперповерхности M задано касательное векторное поле X, имеющее в координатах $(u^1, \ldots, u^{n-1})$ вид $X = \sum\limits_\alpha X^\alpha \partial_{u^\alpha}$. Тогда координаты касательного векторного поля $\nabla_i X$ относительно $(u^1, \ldots, u^{n-1})$ могут быть вьчислены так:

$$(\nabla_i X)^j = \frac{\partial X^j}{\partial u^i} + \sum_\alpha \Gamma_{i\alpha}^j X^\alpha.$$

Здесь Γ_{ij}^k - символь Кристоффеля поверхности M.

Замечание. Операция ковариантного дифференцирования определяется только первой квадратичной формой поверхности.

Предложение 5.3 Ковариантное дифферениирование сохраняет первую квадратичную форму поверхности в следующем смысле: если X и Y произвольные касательные векторные поля на поверхности, то

$$\nabla_i \langle X, Y \rangle = \langle \nabla_i X, Y \rangle + \langle X, \nabla_i Y \rangle .$$

Утверждение 5.1 Пусть X и Y - касательные векторные поля на регулярной гиперповерхности M с координатами $(u^1, \ldots, u^{n-1})$. Тогда $\nabla_X Y$ - это векторное поле, i-ая координата которого имеет вид

$$(\nabla_X Y)^i = \sum_\alpha \frac{\partial Y^i}{\partial u^\alpha} X^\alpha + \sum_{k\alpha} \Gamma_{\alpha k}^i X^\alpha Y^k$$

Следствие 5.2 Пусть на поверхности M задано три касательньх векторных поля X_1, X_2 и Y. Тогда имеет место следующее соотношение:

$$\nabla_Y \langle X_1, X_2 \rangle = \langle \nabla_Y X_1, X_2 \rangle + \langle X_1, \nabla_Y X_2 \rangle .$$

В дальнейшем нам также понадобятся следующие обозначения. Пусть X - векторное поле на поверхности M, и γ - регулярная кривая на M, параметризованная параметром t. Рассмотрим ограничение $X(t)$ векторного поля X на кривую γ. Обозначим через $\nabla_t X$ ковариантную производную поля X вдоль касательного к кривой поля $\dot{\gamma}$. Геометрический смысл этой операции ясен: мы дифференцируем поле $X(t)$ по t и результат дифференцирования ортогонально проектируем на M. Если $(u^1, \ldots, u^{n-1})$ - регулярные координаты на M, то поле $\nabla_t X$ имеет вид:

$$(\nabla_t X)^k = \sum_i \frac{\partial X^k}{\partial u^i} \dot{u}^i + \sum_{ip} \Gamma_{ip}^k \dot{u}^i X^p.$$

On geometrical meaning of tursion (!???!!)

On physical meaning of tursion in some models (!???!!)

(указание, очень интересное)

7.13. Ковариантная производная по Карасеву

Мы продолжаем рассматривать многообразие (M, g) с (полу)римановой структурой g (невырожденной, но необязательно положительно определённой). Мы хотим установить существование операции ковариантного дифференцирования векторных полей и, аналогично внешнему дифференцированию, мы сначала перечислим требуемые свойства этой операции $\nabla_X Y$, которая из двух векторных полей делает третье (в этих формулах f - любая гладкая функция): - $\nabla_{fX} Y = f \nabla_X Y$ (линейность по первому аргументу); - $\nabla_X fY = X(f)Y + f\nabla_X Y$ (правило Λ ейбница для второго аргумента); - $\nabla_X Y - \nabla_Y X = [X, Y]$ (отсутствие кручения); - $X(g(Y, Z)) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$ (совместимость с (полу)римановой структурой g). Существование и единственность ковариантной производной для данной (полу)римановой структуры устанавливает

следующая теорема. Теорема 7.127 (Формула Козюля и существование ковариантной производной). Из условий на ковариантное дифференцирование следует формула

$$2g\left(\nabla_X Y, Z\right) = \\ = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y).$$

Если сиитать правую иасть формуль определением левой иасти, то построенное так ковариантное дифферениирование будет иметь требуемьге свойства.

Отличие ковариантной производной $\nabla_X Y$ от производной Λ и $L_X Y$ заключается в том, что ковариантная производна C^∞-линейна по вектору X,

$$\nabla_{fX} Y = f \nabla_X Y$$

иначе говоря значение $\nabla_X Y$ в точке зависит только от значения X в той же точке, но не от производных X. Свойство отсутствия кручения при этом показывает, что некоторая связь с производной Аи у ковариантной производной всё же имеется.

3.2.3 The Levi-civita Connection

3.2.4 The Divergence Theorem

3.2.5 The Maxwell Action

2.2.5 Geodesics, Geodesic Deviation

6.1 Определение и простейшие свойства геодезических

Пусть M - регулярная гиперповерхность заданная параметрически в виде $r : \Omega \to \mathbb{R}^n$, и пусть γ - регулярная кривая на M, параметризованная параметром s. Фиксируем на M какие-нибудь координаты $(u^1, \ldots, u^{n-1})$, и пусть $u^i(s), i = 1, \ldots, n$, - координатные функции кривой γ. Обозначим через $x^k\left(u^1, \ldots, u^{n-1}\right)$ координатные функции поверхности M. Как и выше, будем обозначать дифференцирование по u^i нижним индексом i, и точкой - дифференцирование по параметру s. Тогда вектор скорости $\dot\gamma$ кривой γ как элемент касательного пространства к поверхности имеет вид:

$$\dot\gamma = \sum_\alpha r_\alpha \dot u^\alpha$$

$$\ddot\gamma = \nabla_{\dot\gamma}\dot\gamma + \mathfrak{q}(\dot\gamma, \dot\gamma)N$$

где через q обозначена вторая квадратичная форма поверхности.

Утверждение 6.1 Вектор ускорения кривой γ на гиперповерхности в каждой точке раскладьвается в сумму касательного вектора $\nabla_{\dot\gamma}\dot\gamma$ и нормального вектора $\mathfrak{q}(\dot\gamma, \dot\gamma)$.

Утверждение 6.2 Регулярная кривая γ на поверхности M является геодезической, если и только если в каждой точке из γ имеет место равенство

$$\nabla_{\dot\gamma}\dot\gamma = 0$$

Если на поверхности фиксировань координаты $(u^1, \ldots, u^{n-1})$, то последнее равенство эквивалентно следующей системе уравнений:

$$\ddot u^k + \sum_{\alpha\beta} \Gamma^k_{\alpha\beta} \dot u^\alpha \dot u^\beta = 0, \quad k = 1, \ldots, n-1.$$

Следствие 6.1 Пусть M - регулярная гиперповерхность, P - ее произвольная точка, и $v \in T_P M$ - произвольный касательный вектор κM в точке P. Тогда на M существует

геодезическая γ, выходящая из P и такая, что ее вектор скорости в точке P равен v. Более того, геодезическая γ единственна в том смысле, что любые две такие геодезических совпадают в пересечении областей определения.

Вернемся к выражению для вектора $\ddot{\gamma}$ ускорения произвольной натурально параметризованной кривой γ на поверхности. Напомним, что модуль вектора $\ddot{\gamma}$ называется кривизной кривой. Из утверждения 6.1 и теоремы Пифагора вытекает, что кривизна $k(P)$ натурально параметризованной кривой γ в произвольной точке P может быть вычислена так:

$$k(P) = \sqrt{\|\nabla_{\dot{\gamma}}\dot{\gamma}\|^2 + (\mathfrak{q}(\dot{\gamma},\dot{\gamma}))^2}.$$

Поскольку вектор $\dot{\gamma}$ имеет единичную длину, то $\mathfrak{q}(\dot{\gamma},\dot{\gamma})$ - это в точности кривизна $k_N(P)$ нормального сечения в направлении $\dot{\gamma}$, см. следствие 4.5. Величину $\|\nabla_{\dot{\gamma}}\dot{\gamma}\|$ обычно обозначают через $k_g(P)$ и называют геодезической кривизной кривой γ в точке P. Непосредственно из определений и утверждения 6.2 вытекает следующий результат.

Утверждение 6.3 Регулярная кривая γ на поверхности является геодезической если и только если ее геодезическая кривизна равна нулю в каждой точке.

Утверждение 6.4 Кривизна k регулярной кривой $\gamma \subset \mathbb{R}^n$, лежащей на поверхности M, может быть вычислена так:

$$k = \sqrt{k_g^2 + k_N^2},$$

где k_g - геодезическая кривизна кривой γ, а k_N - кривизна нормального сечения в направлении вектора $\dot{\gamma}$.

6.2 Примеры геодезических, теорема Клеро

Утверждение 6.6 Геодезические на стандартной евклидовой плоскости - это всевозможные прямье линии.

Утверждение 6.7 Пусть $M \subset \mathbb{R}^3$ - поверхность вращения графика положительной гладкой функции $f(z)$. Тогда каждый меридиан на M является геодезической. Среди параллелей геодезическими являются те и только те, которые соответствуют критическим точкам функиии f.

Следствие 6.2 Геодезические на стандартной двумерной сфере - это большие круги и только они.

Для доказательства вводим метрику, считаем кристоффеля, В итоге, уравнения геодезических имеют вид:

$$\begin{cases} 0 = \ddot{\varphi} + 2\frac{\rho'(\tau)}{\rho(\tau)}\dot{\varphi}\dot{\tau}, \\ 0 = \ddot{\tau} + \frac{\rho'(\tau)}{\rho(\tau)}\left(\dot{\tau}^2 - \dot{\varphi}^2\right). \end{cases}$$

$$\dot{c}(s) = \frac{d}{ds}\left(\frac{\rho(\tau)\dot{\varphi}}{\sqrt{\dot{\varphi}^2 + \dot{\tau}^2}}\right) = \frac{\left(\rho'\dot{\tau}\dot{\varphi} + \rho\ddot{\varphi}\right)\left(\dot{\varphi}^2 + \dot{\tau}^2\right) - \rho\dot{\varphi}(\dot{\varphi}\ddot{\varphi} + \dot{\tau}\ddot{\tau})}{\left(\dot{\varphi}^2 + \dot{\tau}^2\right)^{3/2}}$$

Перепишем числитель последнего выражения в виде

$$\rho\dot{\tau}^2\left(\ddot{\varphi} + 2\frac{\rho'}{\rho}\dot{\varphi}\dot{\tau}\right) - \rho\dot{\varphi}\dot{\tau}\left(\ddot{\tau} + \frac{\rho'}{\rho}\left(\dot{\tau}^2 - \dot{\varphi}^2\right)\right).$$

Так как выражения в больших скобках равны нулю в силу уравнений геодезических, заключаем, что $c(s) = \text{const}$ вдоль произвольной геодезической. Этот факт носит название теоремы Клеро.

Теорема 6.1 (Клеро) Пусть $\gamma(s)$ - произвольная геодезическая на поверхности, полученной вращением плоской регулярной кривой вокруг прямой, лежащей в той же плоскости, причем ось вращения и кривая не пересекаются. Тогда произведение c(s)

расстояния от оси врашения до точки $\gamma(s)$ на синус угла между $\gamma(s)$ и соответствующим меридианом есть величина постоянная вдоль γ:

$$c(s) = \rho(\tau(s)) \sin \alpha(s) = \text{ const}$$

С помощью теоремы Клеро можно решить, например, следующую задачу.

Упражнение 6.4. Описать все поверхности вращения, на которых имеются замкнутые геодезические.

Main Formulas of Geodesics

A geodesic is a curve tangent to a vector field X that obeys

$$\nabla_X X = 0$$

Along the curve C, we can substitute the expression (3.25) into (3.27) to find

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma^\mu_{\rho\nu} \frac{dx^\rho}{d\tau} \frac{dx^\nu}{d\tau} = 0$$

This is precisely the geodesic equation (1.30) that we derived in Section 1 by considering the action for a particle moving in spacetime. In fact, we find that the condition (3.28) results in geodesics with affine parameterisation.

For the Levi-Civita connection, we have $\nabla_X g = 0$. This ensures that for any vector field Y parallely transported along a geodesic X, so $\nabla_X Y = \nabla_X X = 0$, we have

$$\frac{d}{d\tau} g(X, Y) = 0$$

This tells us that the vector field Y makes the same angle with the tangent vector along each point of the geodesic.

3.3.1-tong Geodesics Revisited

Normal Coordinates

(!!??!? это нужно переучивать!!!)

Geodesics lend themselves to the construction of a particularly useful coordinate system. On a Riemannian manifold, in the neighbourhood of a point $p \in M$, we can always find coordinates such that

$$g_{\mu\nu}(p) = \delta_{\mu\nu} \quad \text{and} \quad g_{\mu\nu,\rho}(p) = 0$$

The same holds for Lorentzian manifolds, now with $g_{\mu\nu}(p) = \eta_{\mu\nu}$. These are referred to as normal coordinates.

We can also see why we shouldn't expect to set the second derivative of the metric to zero. Requiring $g_{\mu\nu,\rho\sigma} = 0$ is $\frac{1}{4}n^2(n+1)^2$ constraints. Meanwhile, the next term in the Taylor expansion is $\partial^3 \tilde{x}^\rho / \partial x^\mu \partial x^\nu \partial x^\lambda$ which has $\frac{1}{6}n^2(n+1)(n+2)$ independent coefficients. We see that the numbers no longer match. This time we fall short, leaving

$$\frac{1}{4}n^2(n+1)^2 - \frac{1}{6}n^2(n+1)(n+2) = \frac{1}{12}n^2\left(n^2 - 1\right)$$

3.3.4 Geodesic Deviation

Some Typical Examples of Geodesics

3.3.3-tong Path Dependence: Curvature and Torsion

2.2.6 Parallel Transport (!???)

(!! напишу уже скоро, нужно просто сесть и подумать про это!)

Construction

$$\Delta Z^\mu_r = Z^\mu_r - Z^\mu_r = -\left(\Gamma^\mu_{\rho\nu,\sigma} - \Gamma^\mu_{\rho\sigma,\nu}\right)_p \left(Y^\nu Z^\rho X^\sigma\right)_p \delta\sigma\delta\tau + \dots$$
$$= \left(R^\mu{}_{\rho\sigma\nu} Y^\nu Z^\rho X^\sigma\right)_p \delta\sigma\delta\tau + \dots$$

(Тут примерно компактно указания на док-ва.)

2.2.7 Killing Vectors and Tensors (!?)

Main Properties of Killing Vectors

(тут опр и обоснование)

$$\nabla_{(\sigma}\xi_{\mu\nu)} = 0$$

(тут про тензора Киллинга)

$$\nabla_\sigma \nabla_\rho \xi_\mu = R^\lambda{}_{\sigma\rho\mu}\xi_\lambda$$

Это следует из применения свойств вектора Киллинга к уравнению на тензор Римана, небольших преобразований, а также свойств Бьянки.

For $U_\mu = (1, 0, 0, 0)$

$$\xi_{\mu\nu} = a^2 \left(g_{\mu\nu} + U_\mu U_\nu\right) \qquad \text{(in mostly plus signature)}$$

is a Killing tensor in a cosmological spacetime, i.e., $\nabla_{(\sigma}\xi_{\mu\nu)} = 0$.

On Applications of Killing Vectors

(тут абзац как по геодезической и векторам Киллинга находить отклонение света, типичное явление)

2.2.8 Riemann Tensor, Ricci Scalar, etc.

Riemann Tensor

Ricci Scalar

Other properties

О двумерных поверхностях постоянной гауссовой кривизны.

Теорема Гаусса, доказанная нами выше, утверждает, что изометричные двумерные поверхности имеют одинаковую гауссову кривизну. В данном разделе мы покажем, что

если гауссова кривизна поверхности постоянна, то верно и обратное, т.е. две поверхности одинаковой постоянной гауссовой кривизны локально изометричны.

Лемма 8.2 Пусть $\exp_\gamma : (t,s) \to M$ - полугеодезические координаты на регулярной поверхности M, ассочиированнье с натурально параметризованной геодезической $\gamma(t)$. Первая квадратичная форма поверхности в этих координатах имеет вид:

$$ds^2 + G(t,s)dt^2$$

причем функиия G удовлетворяет следующим соотношения.м:

$$G(t,0) = 1, \quad \frac{\partial G(t,0)}{\partial s} = 0.$$

Вычислим теперь гауссову кривизну поверхности в координатах (t,s). Для этого воспользуемся формулой Гаусса:

$$\sum_\beta g_{\beta 2} \left\{ \frac{\partial \Gamma^\beta_{11}}{\partial u^2} - \frac{\partial \Gamma^\beta_{12}}{\partial u^1} + \sum_k \left(\Gamma^k_{11}\Gamma^\beta_{k2} - \Gamma^k_{12}\Gamma^\beta_{k1} \right) \right\} = \det Q,$$

где Q - матрица второй квадратичной формы поверхности, g_{ij} - матрица первой квадратичной формы, и Γ^k_{ij} - символы Кристоффеля. В силу диагональности матрицы первой квадратичной формы, получаем для гауссовой кривизны K следующее выражение:

$$K = \frac{g_{22}}{\det(g_{ij})} \left\{ \frac{\partial \Gamma^2_{11}}{\partial u^2} - \frac{\partial \Gamma^2_{12}}{\partial u^1} + \sum_k \left(\Gamma^k_{11}\Gamma^2_{k2} - \Gamma^k_{12}\Gamma^2_{k1} \right) \right\}$$

Лемма 8.3 Пусть метрика двумерной поверхности M в регулярных координатах (u,v) имеет вид $du^2 + G(u,v)dv^2$. Тогда гауссова кривизна K поверхности M может быть вычислена так:

$$K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}$$

Теорема 8.1 Пусть M_1 и M_2 - двумерные регулярные поверхности постоянной гауссовой кривизны K, и пусть P_i - произвольная точка на $M_i, i = 1,2$. Тогда у точек P_i на $M_i, i = 1,2$, существуют изометричные окрестности. Другими словами, поверхности M_1 и M_2 локально изометричны.

Поверхности нулевой и положительной постоянной гауссовой кривизны нам уже встречались - это, соответственно, плоскость и сфера.

Следствие 8.1 Двумерная поверхность нулевой гауссовой кривизны локально изометрична плоскости.

Двумерная поверхность положительной гауссовой кривизны a^2 локально изометрична стандартной сфере радиуса $1/a$.

(???? не плоскость???) Для псевдосферы в трехмерном пространстве Минковского $\mathbb{R}^3_1$ метрика в открытом круге с центром в нуле радиуса R в стандартных координатах (u^1, u^2) имеет вид

$$ds^2 = \frac{4R^4}{\left(R^2 - (u^1)^2 - (u^2)^2\right)^2} \left(\left(du^1\right)^2 + \left(du^2\right)^2 \right)$$

и называется метрикой Лобачевского в координатах стереографической проекиии. Иногда удобно записывать эту метрику в полярных координатах (r,φ) на плоскости, в которых она, очевидно, имеет вид

$$ds^2 = \frac{4R^4}{(R^2 - r^2)^2} \left((dr)^2 + r^2 (d\varphi)^2 \right)$$

Круг радиуса R с введенной на нем метрикой Лобачевского называется моделью Пуанкаре геометрии Лобачевского, или, как мы уже говорили, кругом Пуанкаре, см. комментарии в следующем разделе.

Нам, однако, будет удобно переписать эту метрику в других координатах. Для этого мы сначала введем на пространстве Минковского так называемые псевдосферические координатьи, положив:

$$x^1(\rho,\chi,\varphi) = \rho\cosh\chi,$$
$$x^2(\rho,\chi,\varphi) = \rho\sinh\chi\cos\varphi,$$
$$x^3(\rho,\chi,\varphi) = \rho\sinh\chi\sin\varphi$$

Предложение 8.1 Гауссова кривизна псевдосферы S^2_1 радиуса iR в $\mathbb{R}^3_1$ постоянна равна $-1/R^2$.

Следствие 8.2 Двумерная поверхность отричательной гауссовой кривизны $-a^2$ локально изометрична псевдосфере радиуса i/a.

2.2.9 More on the Riemann Tensor and Applications

(это перенесу в приложения скорее всего, пока тут. по Тонгу конечно же все.)

3.4.1 The Ricci and Einstein Tensors

3.4.2 Connection 1-forms and Curvature 2-forms

3.4.3 An Example: the Schwarzschild Metric

3.4.4 The Relation to Yang-mills Theory

2.2.10 Other About Surfaces (!!?!???!?)

(то же, что и в матане тут, как считать площади?)

General properties of non-Euclidean geometry

(???)

2.3 Curves

2.3.1 Flat Curves

1.1 Параметрические кривые

(there are such curves and there are some definitions connected to them)

Theorem 1.27.

С локальной точки зрения три способа задания регулярной кривой эквивалентны в следующем смысле:

- если $\gamma : (a,b) \to \mathbb{R}^n$ - регулярная параметрическая кривая, то для каждого $t \in (a,b)$ существует такая окрестность U точки t в интервале (a,b), что ее образ $\gamma(U)$ является и гладкой кривой-графиком, и неявной кривой;
- для каждой точки P кривой-графика $\mathrm{Gr}(f)$ произвольной гладкой функции f, заданной на некотором интервале (a,b), существует такая окрестность $U \subset \mathbb{R}^n$, что $U \cap \mathrm{Gr}(f)$ есть и образ регулярной параметрической кривой, и неявная кривая;

- для каждой точки P неявной кривой M_c существует такая окрестность $U \subset \mathbb{R}^n$, что $U \cap M_c$ есть и образ регулярной параметрической кривой, и криваяграфик в $\mathbb{R}^n$.

Определение 1.31. Длиной $\ell(\gamma)$ гладкой параметрической кривой $\gamma : [a, b] \to \mathbb{R}^n$ называется величина интеграла $\int_a^b \|\dot{\gamma}(t)\| dt$.

Предложение 1.32. Длина гладкой параметрической кривой не меняется при замене параметризации.

Statement 1.36. Переход к натуральному параметру сохраняет гладкость параметрической кривой, если и только если эта кривая регулярна.

Statement 1.39. Пусть s - натуральный параметр на регулярной кривой γ. Тогда в каждой точке кривой γ вектор ускорения $\ddot{\gamma}$ перпендикулярен вектору скорости $\dot{\gamma}$.

1.2 Кривые-графики и неявные кривые

1.3 Определение регулярной кривой

1.4 Длина кривой, натуральный параметр

Определение 1.31. Длиной $\ell(\gamma)$ гладкой параметрической кривой $\gamma : [a, b] \to \mathbb{R}^n$ называется величина интеграла $\int_a^b \|\dot{\gamma}(t)\| dt$.

Предложение 1.32. Длина гладкой параметрической кривой не меняется при замене параметризации.

Определение 1.33. Параметр $s, s \in [a, b]$, гладкой параметрической кривой $\gamma(s)$ называется натуральным, если величина $s - a$ равна длине части кривой γ между точками $\gamma(a)$ и $\gamma(s)$ для любого $s \in [a, b]$.

Statement 1.34. Параметр s на гладкой кривой $\gamma(s)$ является натуральным, если и только если $\|\dot{\gamma}(s)\| = 1$.

Statement 1.36. Переход к натуральному параметру сохраняет гладкость параметрической кривой, если и только если эта кривая регулярна.

Statement 1.39. Пусть s - натуральный параметр на регулярной кривой γ. Тогда в каждой точке кривой γ вектор ускорения $\ddot{\gamma}$ перпендикулярен вектору скорости $\dot{\gamma}$.

1.5 Кривизна регулярной кривой

Утверждение 1.42. Если t - произвольный параметр на регулярной кривой γ, то кривизна в точке $\gamma(t)$ может быть вычислена по формуле

$$k(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3},$$

где через $\|v \times w\|$ обозначена площадь параллелограмма, натянутого на векторы v и w из $\mathbb{R}^n$.

В случае кривой на плоскости $\mathbb{R}^2$ формула для кривизны в явном виде имеет вид

$$k(t) = \frac{|\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)|}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{3/2}}$$

где (x, y) - стандартнье координаты на плоскости, а в случае пространства $\mathbb{R}^3$ в числителе стоит модуль векторного произведения векторов $\dot{\gamma}(t)$ и $\ddot{\gamma}(t)$.

(add here also an idea of the proof)

1.6 Плоские кривые

Theorem 1.48 (Формулы Френе).

Пусть γ - натурально параметризованная плоская бирегулярная кривая, us - натуральный параметр. Тогда справедливы следующие соотношения:

$$\begin{cases} \dot{\tau} = k\nu, \\ \dot{\nu} = -k\tau, \end{cases}$$

где $k = k(s)$ - кривизна кривой γ в точке $\gamma(s)$.

Theorem1.58. Пусть $f(s)$ - произвольная положительная гладкая функиия, заданная на промежутке I. Тогда существует плоская бирегулярная кривая γ, натурально параметризованная параметром s, такая, что ее кривизна $k(s)$ в каждой точке $\gamma(s)$ равна $f(s)$, причем такая кривая единственна с точностью до движения плоскости.

Добавление 1.8. Эволюты и эвольвенты.

Рассмотрим следующую известную задачу. Пусть γ - регулярная кривая на плоскости, и $\varepsilon \in \mathbb{R}$ - некоторое число. Построим новую кривую γ_ε, заменив каждую точку $\gamma(s)$ точкой, полученной из $\gamma(s)$ смещением на вектор $\varepsilon\nu_o(s)$, т.е. $\gamma_\varepsilon(s) = \gamma(s) + \varepsilon\nu_o(s)$. Эта кривая называется ε-эквидистантной кривой γ. Ясно, что γ_ε при малых ε по-прежнему является регулярной кривой. Однако, когда ε возрастает, у эквидистант могут появляться особые точки. Опишем эти точки. Пусть кривая γ параметризована натуральным параметром s. Тогда вектор скорости ε-эквидистанты, в силу формул Френе, имеет вид:

$$\dot{\gamma}_\varepsilon(s) = \dot{\gamma}(s) + \varepsilon\dot{\nu}_o(s) = (1 - \varepsilon k_o(s))\,\tau(s),$$

Statement 1.87. Вектор скорости ε-эквидистанты обращается в нуль, если и только если в соответствующей точке ориентированная кривизна исходной кривой равна $1/\varepsilon$. Множество особых точек эквидистант - это в точности множество центров кривизны исходной кривой.

Определение 1.88. Множество центров кривизны регулярной кривой γ называется каустикой, или эволютой кривой γ.

Рис. 1.6: Циклоида и ее эволюта.

Следствие 1.89. Эволюта - это множество всех особых точек всех эквидистант.

Statement 1.91. Эволюта циклоиды - это снова циклоида.

Рис. 1.7: Построение эволюты эллипса: жирная точка - центр кривизны, тонкий прямолинейный отрезок - радиус соответствующей соприкасающейся окружности.

2.3.2 Curves in 3d, Frenet–Serret formulas

(Иванов Тужилин глава 2)

2.1 Формулы Френе

Итак, пусть $\gamma(s)$ - натурально параметризованная бирегулярная кривая в $\mathbb{R}^3$. Обозначим через τ вектор скорости $\dot\gamma(s)$; через ν - нормированный вектор ускорения $\ddot\gamma(s)/\|\ddot\gamma(s)\|$, который также называется элавной нормалью $\kappa\gamma$ в точке $\gamma(s)$; а через β - векторное произведение $[\tau,\nu]$, называемое бинормалью $\kappa\gamma$ в точке $\gamma(s)$. Тройка (τ,ν,β) образует т.н. (ортонормальный) репер Френе.

Продифференцируем бинормаль β. По формуле Лейбница, имеем

$$\dot\beta = [\tau,\nu]\cdot = [\dot\tau,\nu] + [\tau,\dot\nu] = [\tau,\dot\nu],$$

где последнее равенство справедливо, так как вектор $\dot\tau$, равный $\ddot\gamma$, колинеарен ν, и, значит, $[\dot\tau,\nu] = 0$. Отсюда, в частности, вытекает, что вектор $\dot\beta$ ортогонален τ. Далее, так как вектор $\beta(s)$ - единичный при каждом s, то $\dot\beta$ перпендикулярен β. Следовательно, $\dot\beta$ колинеарен вектору ν. Положим $\varkappa = -\langle\dot\beta,\nu\rangle$, т.е. $\dot\beta = -\varkappa\nu$. Величина $\varkappa$ называется кручением кривой γ в точке $\gamma(s)$. Ясно, что кручение $\varkappa$ является гладкой функцией параметра s.

Теорема 2.1 (Формулы Френе) В сделанных предположениях, имеет место следующая система уравнений:

$$\begin{cases} \dot\tau = k\nu, \\ \dot\nu = -k\tau + \varkappa\beta, \\ \dot\beta = -\varkappa\nu. \end{cases}$$

аде k и обозначают соответственно кривизну и кручение кривой γ в точке $\gamma(s)$.

Теорема 2.2 Пусть $\gamma(t)$ - бирегулярная кривая. Тогда

$$k = \frac{\|[\dot\gamma,\ddot\gamma]\|}{\|\dot\gamma\|^3}, \quad \varkappa = \frac{(\dot\gamma,\ddot\gamma,\dddot\gamma)}{\|[\dot\gamma,\ddot\gamma]\|^2},$$

где через (u,v,w) мы обозначили смешенное произведение векторов u,v и w из $\mathbb{R}^3$.

2.2 Натуральные уравнения

Теорема 2.3 Пусть $f(s)$ и $g(s)$ - две гладкие функиии на отрезке $[a,b]$, причем функиия $f(s)$ везде положительна. Тогда, с точностью до сохраняющего ориентаиию движения пространства, существует единственная натурально параметризованная кривая $\gamma(s)$, такая ито ее кривизна $k(s)$ и кручение $\varkappa(s)$ равны соответственно $f(s)$ и $g(s)$.

Лемма 2.10. При каждом s тройка $(\tau(s),\nu(s),\beta(s))$ образует ортонормальный положительно ориентированный репер.

2.4 Slope lines

Бирегулярная кривая называется линией откоса, если вдоль нее отношение кручения к кривизне не меняется, т.е. $k(s)/\varkappa(s) = \text{const}$. Опишем класс таких кривых.

Statement 2.24. Всякая линия откоса есть такая линия на подходящей ицилиндрической поверхности, которая переходит в прямую после разворачивания поверхности в плоскость. Обратно: всякая кривая, обладающая указанным свойством, есть линия откоса.

Определение 3.7. Гладкая параметрическая поверхность называется регулярной, если ее базисные касательные векторы линейно независимы в каждой точке параметризующей области.

3.10 Lengths and angles

Определение 3.53. Пусть γ_1 и γ_2 - две кривые на поверхности, пересекающиеся в точке P. Тогда углом между кривыми $\gamma_1 u \gamma_2$ в точке P называется меньший из двух углов между касательными векторами к этим кривым в точке P.

Напомним, что угол между векторами определяется через скалярное произведение. А именно, если a и b - произвольные векторы, то косинус угла между ними, по определению, равен

$$\frac{\langle a, b\rangle}{\|a\|\|b\|}$$

Утверждение 3.54. Пусть P - точка регулярной поверхности, u кривые $\gamma_p, p = 1{,}2$, лежат на ней u пересекаются в точке P. Пусть $u^1, \ldots, u^k$ - координаты на этой поверхности, $g_{ij}(u)$ - компоненты её первой фундаментальной формы в этих координатах, а $u_p^i = u_p^i(t), p = 1, 2,$ - координатные функции кривых γ_p. Тогда косинус угла между кривыми γ_p может быть вычислен по формуле

$$\frac{\sum g_{\alpha\beta}(P)\dot{u}_1^\alpha \dot{u}_2^\beta}{\sqrt{\sum g_{\alpha\beta}(P)\dot{u}_1^\alpha \dot{u}_1^\beta}\sqrt{\sum g_{\alpha\beta}(P)\dot{u}_2^\alpha \dot{u}_2^\beta}}.$$

2.3.3 Other About Curves

(I'll add some selected exotic topics here, for now, see theory)

2.4 Selected Models of DG

2.4.1 Lobachevsky Geometry

Школьная геометрия, или евклидова геометрия, была аксиоматизирована в знаменитых "Началах" Евклида (якобы 3 век до н.э.) Свои построения Евклид базировал на пяти основных постулатах.

1) через две точки можно провести прямую,

2) любой отрезок можно неограниченно продолжить,

3) данным радиусом из данной точки можно провести окружность,

4) все прямые углы равны между собой (обеспечивается единственность продолжения прямой),

5) если две прямые, лежащие в одной плоскости, пересечены третьей, и если сумма внутренних односторонних углов меньше суммы двух прямых, то прямые пересекаются при неограниченном продолжении с той стороны, с которой эта сумма меньше.

У Евклида были, впрочем, и другие аксиомы (например, аксиомы равенства и порядка).

Обычно пятый постулат формулируют в следующем (эквивалентном) виде:

5') через данную точку вне прямой можно провести единственную параллельную ей (т.е. не пересекающуюся с ней) прямую.

Начнем с примера геометрии, в которой пятый постулат не выполнен, а из остальных аксиом выполняются почти все. Рассмотрим стандартную сферу $S^2 \subset \mathbb{R}^3$ единичного радиуса с центром в нуле. Точками будут точки из S^2, прямыми - большие круги (пересечение сферы S^2 с плоскостями, проходящими через ее центр), а движениями - преобразования сферы, индуцированные ортогональными преобразованиями пространства $\mathbb{R}^3$. Ясно, что в такой геометрии вообще нет параллельных прямых. Недостаток через диаметрально противоположные точки сферы проходит больше одной (фактически, бесконечно много) прямых. От этого неприятного обстоятельства можно легко избавиться, отождествив противоположные точки x и $-x$. В результате мы получим проективную плоскость $\mathbb{RP}^2$ и естественную проекцию $\pi : S^2 \to \mathbb{RP}^2$. В качестве прямых на $\mathbb{R}\mathrm{P}^2$ рассмотрим образы больших кругов при проекции π. Теперь через две разные точки проходит ровно одна прямая, однако имеется еще одно неприятное обстоятельство: как на сфере, так и на $\mathbb{RP}^2$ не определено корректно понятие "между". Можно показать, что на $\mathbb{R}\mathrm{P}^2$ выполняются все аксиомы евклидовой геометрии, за исключением аксиомы параллельности и аксиом порядка. Построенная только что геометрия на $\mathbb{R}\mathrm{P}^2$ называется эллиптической или проективной геометрией.

Рассмотрим пространство $\mathbb{R}^3_1$. В координатах (x, y, z) псевдосфера мнимого радиуса iR (двуполостный гиперболоид) задается уравнением

$$-x^2 + y^2 + z^2 = -R^2.$$

Обозначим половинку этой псевдосферы, выделенную условием $x > 0$, через $L^2(R)$, и рассмотрим на $L^2(R)$ геометрию, в которой прямье - это пересечения $L^2(R)$ с плоскостями, проходящими через начало координат в $\mathbb{R}^3_1$, а движения - преобразования $L^2(R)$, индуцированные преобразованиями пространства $\mathbb{R}^3_1$, сохраняющими псевдоскалярное произведение и переводящими $L^2(R)$ в себя. Поверхность $L^2(R)$ называется плоскостью Лобачевского, а определенная на ней геометрия - геометрией Лобачевского или гиперболической геометрией.

9.2 Модель Пуанкаре плоскости Лобачевского

Следствие 9.1 Угль между кривьли на плоскости Лобачевского в модели Пуанкаре совпадают с евклидовьми углами, т.е. углами, измеренньими в евклидовой метрике.

Утверждение 9.1 Стереографическая проекиия $\sigma : L^2(R) \to \Pi$ переводит каждую прямую Лобачевского в дугу окружности, перпендикулярную абсолюту, или в диаметр диска $D^2(R)$.

Теперь невыполнимость аксиомы параллельных становится еще более наглядной, см. рис. 20.

Через точку P проходит бесконечно много прямых, параллельных прямой 1.

9.3 Дробно линейные преобразования плоскости

Пусть теперь a, b, c и d - комплексные числа. Определим дробнолинейное преобразование так:

$$z \mapsto \frac{az + b}{cz + d}.$$

Каждому дробно-линейному преобразованию поставим в соответствие комплексную матрицу $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Легко проверяется, что это отображение есть отображение в точку если и только если матрица $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ вырождена (это в точности условие того, что пары комплексных чисел (a, b) и (c, b) пропорциональны с некоторым комплексным коэффициентом, поэтому отношение числителя к знаменателю дробно-линейного отображения постоянно).

Предложение 9.1 Каждое дробно-линейное преобразование представимо в одном из двух видов:

$$z \mapsto Az + B \quad \text{или } z \mapsto \frac{A}{z+B} + C,$$

где A, B, C и D - некоторье комплексные числа. Поэтому каждое дробно-линейное преобразование есть или композиииия поворота, растяжения и сдвига (в первом случае), или композичия сдвига, инверсии, осевой симметрии, поворота, растяжения и сдвига (во втором случае).

Предложение 9.2 Каждое дробно-линейное преобразование переводит окружности в окружности.

Предложение 9.3 Отображение ψ из группы комплексных невырожденных 2×2-матрии в группу дробно-линейных преобразований плоскости является гомоморфизмом на всю группу преобразований, т.е. является эпиморфизмом.

Предложение 9.4 Группа дробно-линейных отображений плоскости изоморфна аруппе $\mathrm{GL}(2, \mathbb{C})/\mathbb{C}^*$.

9.4 О метрики в комплексной форме

В комплексном виде удобно записывать метрику и работать с ней. Особенно удобно работать с конформно-евклидовыми метриками, т.е. метриками, заданными в виде $ds^2 = \lambda(u, v)\,(du^2 + dv^2)$. Для того, чтобы записать метрику в комплексной форме, введем следующие комплексные дифференциалы:

$$dz = du + idv, \quad d\bar{z} = du - idv.$$

Легко видеть, что $du^2 + dv^2 = dzd\bar{z}$, поэтому конформно-евклидова метрика $ds^2 = \lambda(u, v)\,(du^2 + dv^2)$ в комплексном виде выглядит так: $ds^2 = \lambda(z, \bar{z})dzd\bar{z}$

Пусть $z = \frac{aw+b}{cw+d}$ - дробно-линейное преобразование. Тогда

$$dz = \frac{a(cw+d) - c(aw+b)}{(cw+d)^2}dw = \frac{ad-cb}{(cw+d)^2}dw, \quad d\bar{z} = \frac{\overline{ad-cb}}{\overline{(cw+d)^2}}d\bar{w}$$

поэтому

$$ds^2 = \lambda dzd\bar{z} = \lambda\frac{|ad-cb|^2}{|cw+d|^4}dwd\bar{w},$$

т.е. метрика по-прежнему имеет конформно-евклидов вид. Итак, мы доказали следующий результат. Предложение 9.6 Дробно-линейнье преобразования сохраняют конформноевклидов вид метрики. В частности, для конформно-евклидовых метрик, дробно-линейнье преобразования сохраняют усльи между кривьли.

Приведем пример комплексной записи метрики евклидовой плоскости, сферы и плоскости Лобачевского: 1) $ds^2 = dzd\bar{z}$ (метрика плоскости), $z = u + iv, (u, v)$ - стандартные евклидова координаты; 2) $ds^2 = \frac{4R^4 dzd\bar{z}}{(R^2+|z|^2)^2}$ (метрика сферы радиуса $R), z = u + iv, (u, v)$ стереографические координаты; 3) $ds^2 = \frac{4R^4 dzd\bar{z}}{(R^2-|z|^2)^2}$ (метрика плоскости Лобачевского $L^2(R)$ в модели Пуанкаре), $z = u + iv, (u, v)$ - стереографические координаты.

9.5 Модель верхней полуплоскости

Построим теперь новую модель плоскости Лобачевского. Рассмотрим дробно-линейное преобразование плоскости, заданное так: $w = -iR\frac{z+iR}{z-iR}$. Это дробно-линейное преобразование задает новые координаты на круге Пуанкаре

Предложение 9.7 В модели верхней полуплоскости метрика Лобачевского имеет вид:

$$ds^2 = R^2\frac{du^2+dv^2}{v^2} = -\frac{4R^2dwd\bar{w}}{(w-\bar{w})^2},$$

аде $w = u + iv$ - соответствующая комплексная координата.

9.6 Изометрии плоскости Лобачевского

Для понимания геометрии Лобачевского, а также для конкретных вычислений в этой геометрии очень полезно выяснить, как устроены движения в этой геометрии. Для этого мы воспользуемся моделью верхней полуплоскости. Напомним, что в этой модели

$$ds^2 = -\frac{4R^2dzd\bar{z}}{(z-\bar{z})^2},$$

где $z = u + iv$. Будем искать движения в виде дробно-линейных преобразований:

$$z = \frac{aw+b}{cw+d}.$$

Для простоты, в дальнейших вычислениях положим $R = 1$.

Теорема 9.1 Вещественное дробно-линейное преобразование $z = \frac{aw+b}{cw+d}$ сохраняет верхнюю полуплоскость если и только если матрица $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ этого преобразования имеет положительный определитель.

Предложение 9.8 Каждое вещественное дробно-линейное преобразование, сохраняющее верхнюю полуплоскость, можно представить матричей $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ из $\mathrm{SL}(2,\mathbb{R})$. Обратно, каждое дробно-линейное преобразование вида $z = \frac{aw+b}{cw+d}$, где $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$-матрица из $\mathrm{SL}(2,\mathbb{R})$, сохранлет верхнюю полуплоскость.

Предложение 9.9 Вещественные дробно-линейные преобразования, сохраняющие верхнюю полуплоскость, являются движениями плоскости Лобачевского.

Далее, легко видеть, что вещественные дробно-линейные преобразования, сохраняющие верхнюю полуплоскость, образуют группу.

Лемма 9.1 Отображение ν, ставящее в соответствие каждой матрице $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ из $\mathrm{SL}(2,\mathbb{R})$ дробно-линейное преобразование $z = \frac{aw+b}{cw+d}$, является эпиморфизмом групп.

Теорема 9.2 Подгруппа группь движений плоскости Лобачевского, составленная из всех вещественных дробно-линейных преобразований, изоморфна $\mathrm{SL}(2,\mathbb{R})/\{E,-E\}$.

О движениях плоскости Лобачевского

Дробно-линейные преобразования, сохраняющие единичный круг, являются движениями метрики Лобачевского в модели Пуанкаре. Матрицы $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ таких преобразований удовлетворяют следующим условиям:

$$|a|^2 - |c|^2 = 1, \quad |b|^2 - |d|^2 = -1, \quad a\bar{b} - c\bar{d} = 0,$$

и, поэтому, записываются в виде

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \text{ где } |a|^2 - |b|^2 = 1.$$

Группа таких матриц обозначается через $\mathrm{SU}(1,1)$. Другое удобное представление дробнолинейных преобразований, сохраняющих единичный круг:

$$z \mapsto e^{i\varphi}\frac{z - z_0}{1 - \bar{z}_0 z}, \quad \varphi \in \mathbb{R}, \quad |z_0| < 1.$$

Чтобы получить полную группу движений плоскости Лобачевского, необходимо добавить сопряжение $z \mapsto \bar{z}$.

О треугольниках

Предложение 9.10 (Неравенство треугольника) Пусть BC - произвольный треугольник на плоскости Лобачевского. Тогда имеет место следующее неравенство:

$$|AB|_L + |BC|_L \geq |AC|_L$$

Утверждение 9.2 Косинус отношения длинь дуги AB псевдоокружности к ее радиусу равен псевдоскалярному произведению радиус-векторов точек A и B, деленному на квадрат радиуса окружности. В этом смысле, отношение длины дуги окружности κ радиусу может играть роль меры угла между радиус-векторами кониевых точек этой дуги.

Предложение 9.11 Сумма углов произвольного треугольника ABC на плоскости Лобачевского строго меньше чем π.

Предложение 9.12 Пусть ABC - произвольный треугольник на плоскости Лобатреугольника BC, то

$$S = R^2(\pi - \alpha - \beta - \gamma)$$

2.5 GD in Physics: Selected Topics

(see for now part 1-1)

2.6 Selected Special Topics

2.6.1 vector Bundles

(Лекции 1-2 Рослый)

2.6.2 Homotopy Groups

(Лекции 3-4 Рослый)

2.6.3 Connectivity in Bundle

(Лекции 5-6 Рослый)

Part II
Fundamentals of Differential Geometry

(!! эту часть переписывать просто полностью буду, пока над 1 частью работать буду)

3 Differential Forms

3.1 Tangent Space and Differential

(я удалил кучу теории, потому что потом лучше выгружу из нормальных лекций, это гораздо проще, чем переписывать то, что было.)

3.2 Integration of Differential Forms

(скорее всего по катанаеву писать буду)

3.2.1 Unit Breakdown

(у иванова она вообще в топологии кстати.)

3.2.2 Stokes Formula

Теорема Стокса в двух словах

(тут же и укажу идею док-ва, пока про неё хз)

Определение интеграла от формы

(тут давно вопрос был, типа мы так выходим к многообразию от евклидового пространства, да? то есть изначально всё в евклидовом задается, да?)

Интеграл дифференциальной формы $\nu \in \Omega_c^n(M)$ с компактным носителем по ориентированному n-мерному многообразию M определяется с помощью разбиения единицы в окрестности носителя ν

$$\rho_1 + \cdots + \rho_m = 1,$$

подчинённого некоторому набору положительно ориентированных карт как

$$\int\limits_M \nu = \sum_i \int\limits_M \rho_i \nu$$

где интегралы справа рассматриваются в координатных картах, содержащих носители соответствующих ρ_i.

что при сравнении интеграмов каждого слагаемого $\rho_i \sigma_j \nu$ в карте U_i и в карте V_j мы используем независимость интеграла от замены одних координат на другие (отображением $\psi_j \circ \varphi_i^{-1}$) с положительным якобианом, таким образом подтверждая корректность выражения.

Для ориентированного многообразия с краем $(M, \partial M)$ (край рассматривается с согласованной ориентацией) размерности п и формы $\alpha \in \Omega_c^{n-1}(M)$ выполняется

$$\int\limits_M d\alpha = \int\limits_{\partial M} \alpha.$$

В этой формуле мы рассматриваем естественное гладкое вложение многообразий $i : \partial M \to M$ (см.

также задачу 6.107) и, строго говоря, в правой части равенства интегрируем $i^*\alpha$ по краю ∂M. Напомним, что в координатном представлении мы просто требуем продолжимости α на окрестность края и таким образом $i^*\alpha$ корректно опремелена.

Такое же соглашение применяется всякий раз, когда мы хотим интегрировать дифференциальную форму, заданную на одном многообразии, по его подмногообразию, то есть рассматривать образ некоторого гладкого инъективного (или даже необязательно инъективного) отображения $i : N \to M$ одного многообразия в другое и брать форму $i^*\alpha$ На N

идея замены переменных в интеграле от формы

Правила замены переменных в интеграле от формы говорят, что если мы хотим проинтегрировать форму в карте, отрицательной относительно данной ориентации многообразия, то мы можем это сделать, но нам нужно будет поменять знак получившегося выражения.

Также уточним, что такое интеграм формы нулевой степени с компактным носителем, то есть на самом деле функций с конечным носителем, по 0мерному многообразию.

Это просто сумма значений этой функции в точках 0-мерного многообразия, взятых со знаками, происходяшими из ориентации точек данного многообразия.

На практике форму по многообразию можно интегрировать без разбиения единицы, если если интегрировать её в карте, которая покрывает всё многообразие кроме множества меры нуль.

Например, сферу радиуса R в $\mathbb{R}^3$ можно параметризовать почти всю как

$$x = R\cos\varphi\sin\vartheta, \quad y = R\sin\varphi\sin\vartheta, \quad z = R\cos\vartheta$$

при $0 < \varphi < 2\pi, 0 < \vartheta < \pi$ и интегрировать в указанном диапазоне. С учётом определения интеграла формы по многообразию и сделанных замечаний мы готовы доказать основную формулу про интегрирование дифференциальных форм.

Определение интеграла не зависит от выбора системы положительных карт в данной ориентации и подчинённого им разбиения единицы

есть лемма о том, что определение интеграла не зависит от выбора системы положительных карт в данной ориентации и подчинённого им разбиения единицы.

Действительно.

Пусть у нас были карты $\varphi_i : U_i \to \mathbb{R}^n$ с подчинённым разбиением единицы $\rho_i : M \to \mathbb{R}^+$ и новые карты $\psi_j : V_j \to \mathbb{R}^n$ с подчинённым разбиением единицы $\sigma_j : M \to \mathbb{R}^+$.

Если мы выбрали другую систему с той же ориентацией, то якобианы переходов от старых к новым должны быть положительными.

Выпишем формулу

$$\sum_i \rho_i \left(\sum_j \sigma_j \right) = \sum_{i,j} \rho_i \sigma_j = \sum_j \sigma_j \left(\sum_i \rho_i \right)$$

которая позволяет от Δ вух разбиений единицы перейти к их обшему «подразбиению» $\{\rho_i\sigma_j\}_{i,j}$, и умножим её на ν. Тогда мы можем написать

$$\sum_i \int\limits_M \rho_i\nu = \sum_i \int\limits_M \rho_i \left(\sum_j \sigma_j\right)\nu = \sum_i\sum_j \int\limits_M \rho_i\sigma_j\nu = \sum_j \int\limits_M \left(\sum_i \rho_i\right)\sigma_j\nu = \sum_j \int\limits_M \sigma_j\nu$$

Внутренние знаки равенства интерпретируются как уже известная нам аддитивность интеграла в пределах координатной карты.

Выражение посередине подразумевает,

Доказательство формулы Стокса

Используя разбиение единицы в окрестности носитемя α по лемме 6.123, подчинённое покрытию носителя α координатными окрестностями, мы сведём задачу к вопросу в коорАинатах, то есть к случаю $M = (-\infty, 0] \times \mathbb{R}^{n-1}$ и $\partial M = \{0\} \times \mathbb{R}^{n-1}$ Рассмотрим случай $\alpha = f dx_2 \wedge \cdots \wedge dx_n$ И $d\alpha = \frac{\partial f}{\partial x_1} dx_1 \wedge dx_2 \wedge \cdots \wedge dx_n$. Начав интегри- ровать по x_1, мы получим по теореме Фубини и формуле Ньютона- Λ ейбница

$$\int\limits_M \frac{\partial f}{\partial x_1} dx_1 \wedge \cdots \wedge dx_n = \int\limits_{\mathbb{R}^{n-1}} \left(\int\limits_{-\infty}^{0} \frac{\partial f}{\partial x_1} dx_1\right) dx_2 \wedge \cdots \wedge dx_n = \int\limits_{\partial M} f\left(0, x_2, \ldots, x_n\right) dx_2 \wedge \cdots \wedge dx_n$$

что доказывает теорему в этом случае.

Мы можем считать, что мы рассматривали связные координатные карты некоторого знака.

Отрицательность координатной карты могла внести знак «-» в левую часть формулы, но по определению ориентации на

краю в правой части тоже появится «-», так что формула в любом случае остаётся верной.

Если же $\alpha = f dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ при $i > 1$, то уже интегрирование по x_i обнуляет мевую часть (по формуле Ньютона- $\wedge$ ейбница), а в правой части будет стоять нуль по определению, так как ограничение формы α на ∂M окажется нулевым в силу того, что $dx_1 \equiv 0$ на краю. Также заметим, что в координатной окрестности, в которой края не видно, левая часть равенства обрашается в нуль, как мы уже видели в доказательстве независимости определения интеграла дифференциальной формы от замены координат с положительным якобианом.

Правая часть в таких координатных окрестностях очевидно нулевая.

формула Стокса для многообразий с углами (???)

карасев тупо упомянул, ничего не знаю)

Задача 6.140.

Распространите формулу Стокса на парамлелепипед в $\mathbb{R}^n$ и на симплекс

$$\{(x_1, \ldots, x_n) : x_1, \ldots, x_n \geqslant 0, x_1 + \cdots + x_n \leqslant 1\}$$

считая интеграл по краю как сумму интеграмов по граням с правимьной ориентацией. [[Можно обобшить рассуждение в основном доказательстве на случай, когда локамьно множество, по которому интегрируют $d\alpha$, устроено как $(-\infty, 0]^k \times \mathbb{R}^{n-k}$.]]

формулировка для мнгообразий (??? хз, да чего пока что)

от мб Катанаева
формулу Стокса можно записать в виде

$$\int\limits_{\partial\mathbb{U}} X^\alpha ds_\alpha = \int\limits_{\mathbb{U}} dx^1 \wedge \cdots \wedge dx^n \sqrt{|g|}\widetilde{\nabla}_\alpha X^\alpha$$

где ориентированный элемент площади гиперповерхности:

$$ds_\alpha := \frac{1}{(n-1)!} dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_n} \varepsilon_{\alpha\alpha_2\ldots\alpha_n}$$

Действительно
Пусть на многообразии задано векторное поле $X = X^\alpha \partial_\alpha$.
Тогда ему можно взаимно однозначно поставить в соответствие $(n-1)$ -форму:

$$A := *X = \frac{1}{(n-1)!} dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_n} \varepsilon_{\beta\alpha_2\ldots\alpha_n} X^\beta$$

Внешний дифференциал этой формы равен

$$\begin{aligned} dA &= \frac{1}{(n-1)!} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_n} \partial_{\alpha_1} \left(\varepsilon_{\beta\alpha_2\ldots\alpha_n} X^\beta\right) = \\ &= dx^1 \wedge \cdots \wedge dx^n \sqrt{|g|}\widetilde{\nabla}_\alpha X^\alpha \end{aligned}$$

Отсюда следует,
что формулу Стокса можно записать в виде

$$\int\limits_{\partial\mathbb{U}} X^\alpha ds_\alpha = \int\limits_{\mathbb{U}} dx^1 \wedge \cdots \wedge dx^n \sqrt{|g|}\widetilde{\nabla}_\alpha X^\alpha$$

где ориентированный элемент площади гиперповерхности:

$$ds_\alpha := \frac{1}{(n-1)!} dx^{\alpha_2} \wedge \cdots \wedge dx^{\alpha_n} \varepsilon_{\alpha\alpha_2\ldots\alpha_n}$$

частные случаи формулы Стокса

Рассмотрим частные случаи формулы Стокса.

обзор частных случаев

Пример 3.7.7.
Пусть задана 1 -форма $A = dx^\alpha A_\alpha$ в трехмерном евклидовом про- сочно гладкой границей ∂ U.
Тогда обшая формула Стокса (3.86) принимает вид

$$\begin{aligned} &\oint_{\partial\mathbb{U}} dx^\alpha A_\alpha = \\ &\qquad = \int\limits_{\mathbb{U}} \left[dx^1 \wedge dx^2 \left(\partial_1 A_2 - \partial_2 A_1\right) + dx^1 \wedge dx^3 \left(\partial_1 A_3 - \partial_3 A_1\right) + dx^2 \wedge dx^3 \left(\partial_2 A_3 - \partial_3 A_2\right)\right] \end{aligned}$$

Если поверхность задана параметрически $x^\alpha = x^\alpha(u, v)$, то последний интеграл можно преобразовать к виду

$$\oint_{\partial\mathbb{U}} dx^\alpha A_\alpha = \int\limits_{\mathbb{U}} du \wedge dv \sqrt{|g|}(\operatorname{rot} A, N)$$

где введен роток (3.65) 1-формы: $[\operatorname{rot} A]^\alpha := \varepsilon^{\alpha\beta\gamma}\partial_\beta A_\gamma$ и $N = N^\alpha \partial_\alpha$ - единичный вектор нормали.

Таким образом,

получена формула Стокса.

обзор вычислительных формул

формула Ньютона-Лейбница

хз

6.20.

Частные случаи формулы Стокса и её применения.

Обсудим интеграл по многообразию и формулу Стокса в малых размерностях.

Одномерное компактное многообразие с краем (см.

задачу 6.111) - это просто набор кривых, диффеоморфных отрезкам или окружностям.

Δ мя определённости мы будем рассматривать кривую, параметризованную отрезком.

Формула Стокса для ориентированной кривой с начамом в точке p и концом в точке q сводится к утверждению

$$\int\limits_{\gamma} df = f(q) - f(p)$$

которое конечно можно было бы доказать и непосредственно, рассмотрев интеграл в параметризации кривой и применив формулу Ньютона- Λ ейбница.

понятия ориентации нормальные

Задача 6.141.

Проверьте, что ориентация кривой, которую мы определяли при изучении кривых, и ориентация вложенной кривой как одномерного многообразия дают по сути одно и то же понятие.

формула Грина

Отметим, что формально кривая не доджна иметь самопересечений, чтобы считаться вложенным многообразием в евклиовом пространстве, но выписанная формула очевидно верна без этого предположения, если мы интегрируем обратный образ df по отрезку в параметризации.

Также кривую можно считать не бесконечно гладкой, а всего мишь кусочно непрерывно дифференцируемой, формула всё равно остаётся верной.

Перейдём к менее тривиамьному случаю.

$\Delta_{\text{ля компактной области}}$ $G \subset \mathbb{R}^2$ с гладкой границей (представляющей из себя вложенное одномерное гладкое многообразие в $\mathbb{R}^2$), ориентированной так, что при движении по ∂G область G оказывается слева (это и есть правильная ориентация края G в двумерном случае), верна формула Грина

$$\int\limits_{\partial G} Pdx + Qdy = \int\limits_{G} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

С помощью аккуратного предельного перехода можно обобшить это утверждение на области с кусочно-гладкой (или даже кусочно один раз непрерывно дифференцируемой) границей, многие полезные для практики обобшения можно получить в духе задачи 6.140. $\Delta_{1\text{ я компактной области}}$ $G \subset \mathbb{R}^3$ с гладкой границей (то есть представляюшей из себя вложенное трёхмерное гладкое многообразие с краем в $\mathbb{R}^3$) верна формула ГауссаОстроградского

$$\int\limits_{\partial G} Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy = \int\limits_{G} \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz$$

Задача 6.142.

Проверьте, что ориентация ∂G в формуле Гаусса-Остроградского должна быть такая, что в положительной карте на ∂G с координатами (u, v) вектор $[r'_u \times r'_v]$ торчит вовне G. для компактной двумерной поверхности с краем (то есть вложенного двумерного многообразия с краем) $S \subset \mathbb{R}^3$ верна формула Стокса в узком смысле

$$\int\limits_{\partial S} Pdx + Qdy + Rdz = \int\limits_{S} \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right) dy \wedge dz + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right) dz \wedge dx + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dx \wedge dy$$

Ориентация в данном случае в простых терминах может быть описана аналогично предыдущему случаю:

Задача 6.143.

Проверьте, что при согласованной ориентации поверхности и её края в трёхмерной формуле Стокса ориентирующий касательный вектор ∂S, ориентирующая нормамь S (вида $[r'_u \times r'_v]$ для некоторых положительных координат (u, v)), и касательный вектор к S, направленный перпендикулярно ∂S от S, должны образовывать правую тройку.

о поверхностях и применимости формулы Стокса

Заметим, что на самом деле формула Стокса в $\mathbb{R}^3$ (а на самом деле и в больших размерностях) верна не только для вложенных двумерных многообразий, но и вообще $_\triangle$ всякого образа гладкого отображения $f : D \to \mathbb{R}^3$ области $D \subset \mathbb{R}^2$ с кусочно гладкой границей (даже не инъективного), если интеграмы мы понимаем как интеграмы обратных образов $f^*(\alpha)$ и $f^*(d\alpha)$ по ∂D и D соответственно.

Собственно, в такой формулировке вопрос сводится к формуле Грина на пмоскости.

$\Delta_{\text{ия практических применений}}$ полезно ослабить условия гладкости $f_{\Delta 0}$ непрерывной дифференцируемости, тогда f надо равномерно приближать вместе с его первыми производными последовательностью гладких отображений (f_k) по теореме 6.19 и переходить к пределу в интеграле, замечая, что в его координатном представлении используются производные f не более чем первого порядка.

Поверхность $S \subset \mathbb{R}^3$ в формуле Стокса в узком смысле может быть кусочно-гладкой, то есть «склеенной» из ориентированных кусков S_i с кусочно гладкими краями так, что все минии склейки участвуют в двух из частей ∂S_i с разными ориентациями.

Краем склеенной поверхности тогда считаются те части из ∂S_i, которые не участвовали в склейке.

При таком определении при сложении формул Стокса будут складываться и правые части, и левые части, с сокрашением интегралов по линиям склейки.

Формула Гаусса-Остроградского тоже может быть с некоторыми усилиями обобшена на случай, когда ∂G является кусочно-гладкой поверхностью в описанном смысле, читатель может поразмышлять над этим самостоятельно, а также проверить, что у поверхности ∂G не будет края в описанном выше кусочно-гладком смысле и убедиться,

что бесконечную гладкость поверхности S можно заменить на её однократную непрерывную дифференцируемость в параметрическом представлении.

площадь области (??)

Задача 6.144.

Δ окажите, что площадь области, ограниченной замкнутой гладкой кривой без самопересечений $C \subset \mathbb{R}^2$, можно посчитать по формуле:

$$A = \pm \int\limits_C x dy$$

где знак выбирается в зависимости от ориентации кривой. Задача 6.145.

* Четыре замкнутые кривые на плоскости

$$\alpha, \beta, \gamma, \delta : [0, 1] \to \mathbb{R}^2$$

не имеют самопересечений и ограничивают области площадей $S_\alpha, S_\beta, S_\gamma, S_\delta$. Также оказалось, что в любой момент времени точки $\alpha(t), \beta(t), \gamma(t), \delta(t)$ образуют квадрат в указанном порядке, ориентированный по часовой стрелке.

Как найти площадь одной из этих областей, зная площади трёх других? [[Выпишите плошади по формуле Грина.

]]

формула объема области

Задача 6.146.

Δ окажите, что объём области в $\mathbb{R}^3$, ограниченной гладкой связной вложенной поверхностью без края $S \subset \mathbb{R}^3$, можно посчитать по формуле:

$$A = \pm \int\limits_S x dy \wedge dz$$

где знак выбирается в зависимости от ориентации поверхности, не забывая про положительность объёма.

При определении дифференциамьных форм на многообразии с краем мы требовами, чтобы в координатных картах дифференциамьная форма «чуть-чуть» продолжамась за край.

Иногда это ограничение может быть чрезмерным, в следуюшей задаче мы намечаем способ ослабить такое ограничение.

Теорема Гаусса-Остароградского с точки зрения теоремы Стокса

Пример 3.7.8.

Пусть в трехмерном евклидовом пространстве $\mathbb{R}^3$ с декартовыми координатами x^α задана 2 -форма $A = \frac{1}{2} dx^\alpha \wedge dx^\beta A_{\alpha\beta}$.

Рассмотрим двумерную компактную поверхность $\mathbb{U} \hookrightarrow \mathbb{R}^3$ с кусочно гладкой границей $\partial\mathbb{U}$.

Будем считать,

что поверхность задана параметрически $x^\alpha = x^\alpha(u, v)$.

Тогда общая формула Стокса (3.86) принимает вид

$$\frac{1}{2}\int\limits_{\partial\mathbb{U}} dx^\alpha \wedge dx^\beta A_{\alpha\beta} = \int\limits_{\mathbb{U}} dx^1 \wedge dx^2 \wedge dx^3 \left(\partial_1 A_{23} + \partial_2 A_{31} + \partial_3 A_{12}\right)$$

Как уже отмечалось 2-форма в трехмерном пространстве взаимно однозначно определяется векторным полем $A_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} X^\gamma$.

Интеграл в левой части равенства сводится к интегралу по поверхности от скалярного произведения вектора X^α на единичную нормаль $N = N^\alpha \partial_\alpha$ к поверхности (3.85).

Подынтегральное выражение в правой части (3.91) представляет собой дивергенцию от векторного поля

$$\partial_1 A_{23} + \partial_2 A_{31} + \partial_3 A_{12} = \partial_\alpha X^\alpha$$

Таким образом, окончательно получаем равенство

$$\int\limits_{\partial\mathrm{U}} du \wedge dv \sqrt{|g|} X^\alpha N_\alpha = \int\limits_{\mathbb{U}} dx^1 \wedge dx^2 \wedge dx^3 \partial_\alpha X^\alpha$$

где $\sqrt{|g|}$ - квадратный корень из индуцированной метрики (см.

пример [3.7.4).

Это есть формула Гаусса-Остроградского в трехмерном евклидовом пространстве.

примеры работы с $\mathbb{R}^4$

типа как раз с механикой много связи и настоящие приложения

круто, на этом этапе можно радоваться, так как теория кратного интеграла сделана, вот, научились ее применять и вводить.

основа на формах, ну и на общих определениях.

так что на этом моменте как раз приложения прорабатываем.

3.3 Vector Analysis

3.3.1 vector Fields

Pull-back

Push-forward

3.3.2 vector Operators Through Differential Shapes

сама конструкция

Имеется два отображения: d - внешнее дифференцирование, которое переводит форму в форму степенью выше, и $*$ - звездочка Ходжа, которая из формы $n-$ой степени сделает форму $(N-n)$-ой.

$$\begin{tikzcd} 0 \arrow[r, "d"] & \Lambda^1 \arrow[r, "d"] \arrow[rr, bend left, "*"] & \Lambda^2 \arrow[r, "d"] & \Lambda^3 \arrow[r, "d"] \arrow[ll, bend left, "*"] & 0 \end{tikzcd} \tag{1}$$

дивергенция на языке форм

Функции - это формы нулевой степени.

Попробуем обратно к ней же и прийти, погуляв по нашей схеме.

Сделать это можно следующей последовательностью, которую обозначим оператором Лапласа:

$$\Delta f = *d(*df)$$

3.3.3 Divergence

Пусть имеется форма третьей степени

$$\tau = dx \wedge dy \wedge dz$$

$$A = A_x\partial_x + A_y\partial_y + A_z\partial_z$$

$$d_{\imath A}\tau = (\operatorname{div} A) \cdot \tau$$

$$^2A^\top = \tau(A) = dx \wedge dy \wedge dz\,(A_x\partial_x + A_y\partial_y + A_z\partial_z) = A_x dx \wedge dy \wedge dz\,(\partial_x) +$$
$$+A_y dx \wedge dy \wedge dz\,(\partial_y) + A_z dx \wedge dy \wedge dz\,(\partial_z) = A_x dy \wedge dz + A_y dz \wedge dx + A_z dx \wedge dy$$

$$dz_A\tau = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}\right)\tau, \quad \operatorname{div} A = \partial_x A_x + \partial_y A_y + \partial_z A_z$$

$$\tau' = \det\left(\frac{\partial x'^\alpha}{\partial x^\beta}\right) dx_1 \wedge \ldots \wedge dx_n \equiv \det \Lambda^\alpha_\beta dx_1 \wedge \ldots \wedge dx_n$$

3.4 homological Analysis of Forms

3.4.1 closed and Precise Shapes

Всякий инвариантный полином является одновременно замкнутой и точной формой

(?? не вникал особо)

В оставшейся части этого раздела мы докажем утверждение о том, что всякий инвариантный полином является одновременно замкнутой и точной формой.

Определим инвариантный полином как такой полином, который удовлетворяет условию $P(\alpha) = P\left(g^{-1}\alpha g\right)$ Начнем с определения однородного инвариантного полинома степени r, зависяшего от форм α_i: $P = P\left(\alpha_1, \alpha_2, \ldots, \alpha_r\right)$

Продифференцируем этот полином, тщательно выписывая производные каждой формы, содержашейся в P:

$$dP = \sum_{1\leq i\leqslant r} (-1)^{d_1+\ldots+d_i-1} P\left(\alpha_1, \ldots, d\alpha_i, \ldots, \alpha_r\right)$$

Всякий раз, когда оператор производной d проходит над формой α_i, он приобретает соответствующий множитель:

$$d\alpha_i = (d\alpha_i) + \alpha_i d(-1)^{d_i}$$

Теперь предположим, что

$$\alpha \to g^{-1}\alpha g$$

Для g, близких к единице, всегда можно записать $g = 1 + \omega$

$$\delta\alpha = -\omega \wedge \alpha + \alpha \wedge \omega$$

Вычислим вариацию однородного полинома P степени r при таком сДвиге:

$$\delta P\left(\alpha_1, \alpha_2, \ldots, \alpha_r\right) = 0 = \sum_{1 \leqslant i \leqslant r} (-1)^{d_1 + \ldots d_{i-1}} \left[P\left(\alpha_1 \alpha_2, \ldots, \omega \wedge \alpha_i, \ldots, \alpha_r\right)\right.$$
$$\left. -(-1)^{d_i} P\left(\alpha_1, \alpha_2, \ldots, \alpha_i \wedge \omega, \ldots, \alpha_r\right)\right] (\text{П.3.}$$

Фокус теперь будет состоять в том, что мы сложим вклады от dP и от δP:

$$dP + \delta P = \sum_{1 \leq i \leqslant r} (-1)^{d_1 + \ldots + d_i} P\left(\alpha_1, \alpha_2, \ldots, D\alpha_i, \ldots, \alpha_r\right)$$

Теперь положим, что каждая α_i является формой кривизны, подчиняюшейся тождеству Бьянки $D\alpha_i = 0$. Таким образом, получаем $dP(\Omega) = 0$ что и утверждалось.

Вторая часть доказательства несколько сложнее.

Определим $\Omega = d\omega + \omega \wedge \omega$ $\Omega = d\omega' + \omega \wedge \omega'$ Теперь наш план состоит в том, чтобы для некоторой формы показать выполнение равенства

$$P\left(\Omega'\right) - P(\Omega) = dQ$$

Сначала мы хотим выписать форму кривизны, позволяюшую непрерывно интерполировать между Ω и Ω'. Пусть

$$\omega_t = \omega + t\eta$$

$\eta^- = \omega' - \omega$.

Заметим, что переменная t позволяет интерполировать между этими двумя формами:

$$t = 0 \to \omega_t = \omega$$
$$t = 1 \to \omega_t = \omega'$$

Теперь легко найти форму кривизны, осуществляюшую эту интерполяцию, как функцию t:

$$\Omega_t = d\omega_t + \omega_t \wedge \omega_t$$

где

$$t = 0 \to \Omega_t = \Omega$$

Эта форма изменяется между Ω и Ω', когда t изменяется от нуля до единицы.

Теперь выпишем форму q, такую что

$$q(\beta, \alpha) = rP(\beta, \alpha, \alpha \ldots (r - 1\text{раз}) \ldots \alpha)$$

где форма α повторяется $r - 1$ раз в инвариантном полиноме.

Эта форма q будет играть ключевую роль в доказательстве того, что P является точной формой. В силу приведенного выше рассуждения при дифференцировании

q получим

$$dq\left(\eta, \Omega_t\right) = rdP\left(\eta, \Omega_t \ldots (r - 1\text{раз}) \ldots \Omega_t\right) = q\left(D\eta, \Omega_t\right) -$$
$$- r(r-1)tP\left(\eta, \Omega_t \wedge \eta - \eta \wedge \Omega_t, \Omega_t \ldots, \Omega_t\right).$$

Но в силу того факта, что q является инвариантным полиномом, мы получаем также тождество

$$2q\left(\eta\wedge\eta,\Omega_t\right)+r(r-1)P\left(\eta,\Omega_t\wedge\eta-\eta\wedge\Omega_t,\ldots,\Omega_t\right)=0$$

В (П.3.43) мы использовали

$$q=1+\eta$$

Собирая вместе оба эти уравнения, находим

$$dq\left(\eta,\Omega_t\right)=q\left(D\eta,\Omega_t\right)+2tq\left(\eta\wedge\eta,\Omega_t\right)=\frac{dP\left(\Omega_t\right)}{dt}$$

Таким образом, мы приходим к следующему результату:

$$\int\limits_0^1\frac{dP\left(\Omega_t\right)}{dt}=P\left(\Omega'\right)-P(\Omega)=d\int\limits_0^1 q\left(\omega'-\omega,\Omega_t\right)dt=dQ$$

Итак, мы получили наш окончательный результат:

$$P\left(\Omega'\right)-P(\Omega)=dQ$$

$$Q=\int\limits_0^1 q\left(\omega'-\omega,\Omega_t\right)dt+\text{замкнутые формы}$$

Таким образом, инвариантный полином, основанный на два-формах кривизны, является одновременно замкнутой и точной формой.

3.4.2 cohomology de Rama

4 Fundamentals of Riemannian Geometry

(потом изменю деление на главы)

4.1 Metrics

4.1.1 Intuitive Understanding of the Metric

(потом напишу четко, книги по диффгему проверить лучше бы)

Иллюстрация метрики

Fifth Instruction:
0.208... = (1.5) (0.8) cos(θxz)
θxz = 80°
Sixth Instruction:
0.24 = (0.6) (0.8) cos(θyz)
θyz = 60°
60°

time
z
x
y
10
tx
ty
tz
xy
xz
yz

Geometric View
List of Resizing Instructions
1. dx → a dx
2. dt → b dt
3. Angle → Ψ...
Coordinative View
time
space

Euclidean
Angles
Hyperbolic
Angles
Ottawa
Toronto
NEW YO
time
space

Euclean Trigonometry
Distance Element: $dx^2 + dy^2$
Trigonometry: $\cos\theta$; $\sin\theta$; $\tan\theta$
Curves of constant distance: Circles
Hyperbolic Trigonomtry
Distance Element: $dt^2 - dx^2$
Trigonometry: $\cosh\Psi$; $\sinh\Psi$; $\tanh\Psi$
Curves of constant distance: Hyperbolas

$ds^2 = t^2 - x^2 - y^2 - z^2$
$+ 2tx\sinh(\Psi_{tx}) + 2ty\sinh(\Psi_{ty}) + 2tz\sinh(\Psi_{tz})$
$- 2xy\cos(\theta_{xy}) - 2xz\cos(\theta_{xz}) - 2yz\cos(\theta_{yz})$
ds
t

4.1.2 Induced Metric

типичные способы получения индуцированной метрики

индуцированная метрика с помощью разбиения единицы (???)

4.1.3 metric Isometrics

определение

легко написать, потом напишу.

обзор применений

4.1.4 Examples of Metrics

метрика на сфере

обзор метрик в общей теории относительности

4.1.5 Other Properties of Metric

связь метрики с топологией

4.2 Affinity Connectivity (!!!)

(конспект Володи тут)

4.2.1 Main Structure

определение аффинной связности

Аффинной связностью ∇ называется билинейное отображение векторных полей $\kappa(M)$ на дифференцируемом многообразии M

$$\nabla : \kappa(M) \times \kappa(M) \to \kappa(M),$$

такое, что для любых гладких функций $f, g \in C^\infty(M)$ и для любых векторных полей $X, Y, Z \in \kappa(M)$ справедливо:

(1) f-линейность по первому аргументу: $\nabla(fX + gY, Z) = f\nabla(X, Z) + g\nabla(Y, Z)$,

(2) правило Лейбница по второму аргументу: $\nabla(X, fY) = f\nabla(X, Y) + X(f)Y$.

Эти правила естественные, потому что это мы и видим в производной (? тут хз, я не настолько хорошо разбираюсь в производных, чтобы эти вещи были естественными)

Далее ковариантную производную будем обозначать посредством $\nabla(X, Y) := \nabla_X Y$.

(?? что там про симметричность ков. пр?)

О тензорности ковариантной производной

ковариантная производная функции есть простое дифференцирование по направлению, но в случае тензорных полей это не так

(????)

Из правила Лейбница ясно, что ковариантная производная функции есть простое дифференцирование по направлению, то есть

$$\nabla_X f = X(f) = df(X) = Ł_X f$$

(? а что такое дифференцирование по направлению?)

(????) В общем случае произвольных тензорных полей ковариантная производная и производная Ли не совпадают. (???) что видно уже из того, что ковариантная производная векторного поля $\nabla_X Y$ является f -линейной по вектору X, в отличие от производной Ли $Ł_X Y$.

Иначе говоря, значение $\nabla_X Y$ в точке зависит только от значения X в той же точке, но не от производных X.

(?? тут подробнее пропишу, когда умным стану)

Кроме того, видно, что хотя символы аффинной связности не являются компонентами тензора, но их антисимметричная часть $\Gamma^{\alpha}_{[\mu\nu]}$ образует смешанный анти-симметричный по перестановке нижних индексов тензор.

Компоненты аффинной связности или кристоффеля

(?? это ведь самое настоящее их определение??? не через геодезические?)

Часто используются так называемые символы Кристоффеля $\Gamma^{\gamma}_{\alpha\beta}$, определяемые как:

$$\nabla_\alpha e_\beta \equiv \Gamma^{\gamma}_{\alpha\beta} e_\gamma$$

Кристоффеля полезны для записи ковариантной производной векторного поля $\nabla_X Y$ в координатах. Запишем его, помня, что базис в пространстве векторный полей $\kappa(M)$, как известно, имеет вид $\{e_\alpha\} = \{\partial_\alpha\}$:

$$\nabla_X Y = \nabla_X \left(Y^\beta e_\beta\right) = \nabla_X \left(Y^\beta\right) e_\beta + Y^\beta \left(\nabla_X e_\beta\right) = X^\alpha \partial_\alpha Y^\beta e_\beta + X^\alpha Y^\beta \nabla_{e_\alpha} e_\beta$$

введем $\nabla_{e_\alpha} := \nabla_\alpha$, и получаем

$$\nabla_X Y = X^\alpha \nabla_\alpha Y = X^\alpha \left(\partial_\alpha Y^\gamma + \Gamma^{\gamma}_{\alpha\beta} Y^\beta\right) e_\gamma$$

или

$$\left(\nabla_\alpha Y\right)^\gamma := \nabla_\alpha Y^\gamma = \partial_\alpha Y^\gamma + \Gamma^{\gamma}_{\alpha\beta} Y^\beta$$

(тут пара слов про то, что с того, что так записали?)

Кристоффеля не тензор (?)

Символы Кристоффеля не являются компонентами какого-нибудь смешанного тензора третьего ранга. Действительно, в новых координатах ковариантную производную базисного вектора можно записать как

$$\nabla'_\alpha e'_\beta = \Gamma'^{\gamma}_{\alpha\beta} e'_\gamma$$

Имея в виду, что $e'_\alpha = \frac{\partial x^\beta}{\partial x'^\alpha} e_\beta$, получаем

$$\Gamma'^{\gamma}_{\alpha\beta} \frac{\partial x^\lambda}{\partial x'^\gamma} e_\lambda = \nabla'_\alpha e'_\beta = \frac{\partial x^\mu}{\partial x'^\alpha} \frac{\partial x^\nu}{\partial x'^\beta} \Gamma^{\lambda}_{\mu\nu} e_\lambda + \frac{\partial^2 x^\lambda}{\partial x'^\alpha \partial x'^\beta} e_\lambda$$

Ковариантная производная дифференциальной формы

Ковариантная производная от дифференциальной формы ω первой степени на M имеет вид:

???.

Действие ковариантной производной ∇_X на дифференциальных формах первой степени определяется из требования выполнения правила Лейбница для канонического умножения 1-форм на векторные поля:

$$\left(\nabla_X \omega\right)(Y) = \nabla_X(\omega(Y)) - \omega\left(\nabla_X Y\right) = X(\omega(Y)) - \omega\left(\nabla_X Y\right)$$

Перепишем последнюю формулу в терминах локальных координат.

$$X^\mu \nabla_\mu \omega_\nu Y^\nu = X^\mu \partial_\mu \left(Y^\nu \omega_\nu\right) - X^\mu \omega_\nu \left(\nabla_\mu Y^\nu\right) = X^\mu \left(\partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda\right) Y^\nu,$$

откуда

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda.$$

Нетрудно заметить, что эта формула отличается от ковариантной производной векторного поля знаком при компонентах аффинной связности.

На формы высшей степени коваринатную производную тоже можно распространить, потребовав выполнения правила Лейбница для внешнего умножения форм.

Ковариантная производная тензорных полей

Также можно распространить определение ковариантной производной по правилу Лейбница на тензорные поля:

$$\nabla_\alpha t^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} = \partial_\alpha t^{\mu_1 \dots \mu_p}{}_{\nu_1 \dots \nu_q} + \gamma^{\mu_1}_{\alpha\beta_1} t^{\beta_1 \mu_2 \dots \mu_p}{}_{\nu_1 \dots \nu_q} + \dots + \gamma^{\mu_p}_{\alpha\beta_p} t^{\mu_1 \dots \mu_{p-1} \beta_p} \nu_1 \dots \nu_q -$$
$$- \Gamma^{\gamma_1}_{\alpha\nu_1} t^{\mu_1 \dots \mu_p} \Gamma_1 \nu_2 \dots \nu_q - \dots - \Gamma^{\gamma_q}_{\alpha\nu_q} t^{\mu_1 \dots \mu_p} \nu_1 \dots \nu_{q-1} \Gamma_q{}'$$

(??? я хз, зачем это нужно и что тут происходит, не до этого)

Сама по себе ковариантная производная не тензор

Замечание. Несмотря на то, что значение оператора ковариантной производной на каждом тензорном поле является тензором, (???? что есть сам по себе оператор?) сам по себе оператор ковариантной производной тензором не является.

я не понял, почему так.

Обзор обобщений аффинной связности

(?? говорят, есть обобщения,)

а может и нет, узнаю что-то - пропишу сюда.

4.2.2 Covariant Derivative Endorsements

(тут разные идеи, которые пока не смотрел, ибо они не актуальные)

Ковариантная производная спин-тензорных полей

хз вообще.

О заблуждениях про ковариантную производную

(Соберу их, наверняка их много)

4.2.3 Covariant Derivative Endorsements

(то, что пока подвисло, как сделаю основное, поищу, что же это)

упорядоченная экспонента

Группа голономии, локальная и глобальная

Плоская связность

Ковариантная производная связывает два многообразия

прослушал

4.2.4 christoffel Symbols

Приведем короткую теорию, которая их описывает.

мотивация

очень важно взять и пройти их наконец-то!

основные формулы

примеры кристоффелей

4.3 Riemann Curvature

4.3.1 essence of Torsion and Curvature

(?? пока плаваю в этом)

Определение кривизны

Кривизной аффинной связности называется отображение $R : x(M) \times x(M) \to x(M)$, определяемое формулой

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

Для некоторой формы ω первой степени и векторных полей X, Y, Z определим

$$R(\omega, X, Y, Z) = \omega \left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \right)$$

О евклидовой связности

Аффинная связность с нулевой кривизной называется евклидовой.
и что с ними?

Алгебраически определенные кручение и кривизна действительно тензоры

(!!! см ниже нормальную верстку Тонга этого вопроса!!!)
Явными вычислениями проверим, что T и R действительно являются тензорами.
Для этого необходимо показать, что они линейны по всем аргументам.
$\mathbb{R}$ -линейность T и R по всем аргументам ясна из последних формул.
Посмотрим, что происходит при умножении на гладкую функцию f.
Для кручения имеем

$$T(f\omega, X, Y) = f\omega \left(\nabla_X Y - \nabla_Y X - [X,Y] \right) = fT(\omega, X, Y)$$
$$T(\omega, fX, Y) = \omega \left(f\nabla_X Y - f\nabla_Y X - Y(f)X - f[X,Y] + Y(f)X \right) = fT(\omega, X, Y).$$

В последнем выражении учтено, что $[fX, Y] = f[X,Y] - Y(f)X$.
Аналогично

$$T(\omega, X, fY) = \omega \left(f\nabla_X Y + X(f)Y - f\nabla_Y X - f[X,Y] - X(f)Y \right) = fT(\omega, X, Y)$$

Для кривизны получаем

$$R(f\omega, X, Y, Z) = f\omega \left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \right) = fR(\omega, X, Y, Z)$$
$$R(\omega, fX, Y, Z) = R(\omega, X, fY, Z) = \omega \left(f\nabla_X \nabla_Y Z - \nabla_Y \left(f\nabla_X Z \right) - \nabla_{(f[X,Y] - Y(f)X)} Z \right) =$$
$$= \omega \left(f\nabla_X \nabla_Y Z - f\nabla_Y \nabla_X Z - Y(f)\nabla_X Z - f\nabla_{[X,Y]} Z + Y(f)\nabla_X Z \right) = fR(\omega, X, Y, Z)$$
$$R(\omega, X, Y, fZ) = \omega \left(f\nabla_X \nabla_Y Z + X(f)\nabla_Y Z + Y(f)\nabla_X Z + X(Y(f))Z - \right.$$
$$-f\nabla_Y \nabla_X Z - Y(f)\nabla_X Z - Y(X(f))Z - X(f)\nabla_Y Z -$$
$$\left. -f\nabla_{[X,Y]} Z - X(Y(f))Z + Y(X(f)) \right) = fR(\omega, X, Y, Z).$$

Таким образом, и кручение, и кривизна являются тензорами.

To demonstrate that T and R are indeed tensors, we need to show that they are linear in all arguments. Linearity in ω is straightforward. For the others, there are some small calculations to do. For example, we must show that $T(\omega; fX, Y) = fT(\omega; X, Y)$. To see this, we just run through the definitions of the various objects,

$$T(\omega; fX, Y) = \omega\left(\nabla_{fX}Y - \nabla_Y(fX) - [fX, Y]\right)$$

We then use $\nabla_{fX}Y = f\nabla_X Y$ and $\nabla_Y(fX) = f\nabla_Y X + Y(f)X$ and $[fX, Y] = f[X, Y]$ - $Y(f)X$. The two $Y(f)X$ terms cancel, leaving us with

$$\begin{aligned} T(\omega; fX, Y) &= f\omega\left(\nabla_X Y - \nabla_Y X - [X, Y]\right) \\ &= fT(\omega; X, Y) \end{aligned}$$

Similarly, for the curvature tensor we have

$$\begin{aligned} R(\omega; fX, Y, Z) &= \omega\left(\nabla_{fX}\nabla_Y Z - \nabla_Y\nabla_{fX}Z - \nabla_{[fX,Y]}Z\right. \\ &= \omega\left(f\nabla_X\nabla_Y Z - \nabla_Y\left(f\nabla_X Z\right) - \nabla_{(f[X,Y]-Y(f)X)}Z\right) \\ &= \omega\left(f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - \nabla_{f[X,Y]}Z + \nabla_{Y(f)X}Z\right) \\ &= \omega\left(f\nabla_X\nabla_Y Z - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - f\nabla_{[X,Y]}Z + Y(f)\nabla_X Z\right) \\ &= f\omega\left(\nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z\right) \\ &= fR(\omega; X, Y, Z) \end{aligned}$$

Linearity in Y follows from linearity in X. But we still need to check linearity in Z,

$$\begin{aligned} R(\omega; X, Y, fZ) =& \omega\left(\nabla_X\nabla_Y(fZ) - \nabla_Y\nabla_X(fZ) - \nabla_{[X,Y]}(fZ)\right) \\ =& \omega\left(\nabla_X\left(f\nabla_Y Z + Y(f)Z\right) - \nabla_Y\left(f\nabla_X Z + X(f)Z\right)\right. \\ & \left. -f\nabla_{[X,Y]}Z - [X, Y](f)Z\right) \\ =& \omega\left(f\nabla_X\nabla_Y + X(f)\nabla_Y Z + Y(f)\nabla_X Z + X(Y(f))Z\right. \\ & - f\nabla_Y\nabla_X Z - Y(f)\nabla_X Z - X(f)\nabla_Y Z - Y(X(f))Z \\ & \left. -f\nabla_{[X,Y]}Z - [X, Y](f)Z\right) \\ =& fR(\omega; X, Y, Z) \end{aligned}$$

Thus, both torsion and curvature define new tensors on our manifold.

Компоненты кривизны и кручения

Вы пишем компоненты этих тензоров в координатном базисе.

Компонентами кручения являются

$$T^\alpha_{\mu\nu} = T\left(dx^\alpha, e_\mu, e_\nu\right) = dx^\alpha\left(\nabla_\mu e_\nu - \nabla_\nu e_\mu - [e_\mu, e_\nu]\right) = \gamma^\alpha_{\mu\nu} - \gamma^\alpha_{\nu\mu} = 2\gamma^\alpha_{[\mu\nu]} = -T^\alpha_{\nu\mu}$$

В последнем выражении учтено, что $[e_\mu, e_\nu] = [\partial_\mu, \partial_\nu] = 0$.

Кроме того, видно, что хотя символы аффинной связности не являются компонентами тензора, но их антисимметричная часть $\Gamma^\alpha_{[\mu\nu]}$ образует смешанный анти-симметричный по перестановке нижних индексов тензор.

Связности, свободные от кручения

Аффинные связности, компоненты которых симметричны по перестановкам нижних индексов: $\gamma^\alpha_{\mu\nu} = \gamma^\alpha_{\nu\mu}$, имеют нулевое кручение $T^\alpha_{\mu\nu} = 0$.

Такие аффинные связности называются связностями, свободными от кручения.

Тензор Римана через производные (????)

откуда формула ниже: (???)

$$R^{\alpha}_{\beta\mu\nu} = 2\partial_{[\mu}\Gamma^{\alpha}_{\nu]\beta} + 2\Gamma^{\alpha}_{[\mu|\lambda|}\Gamma^{\lambda}_{\nu]\beta}$$

Компоненты тензора кривизны в координатном базисе определяются согласно

$$\begin{aligned}
R^{\alpha}_{\beta\mu\nu} =& R\left(dx^{\alpha}, e_{\mu}, e_{\nu}, e_{\beta}\right) = \\
=& dx^{\alpha}\left(\nabla_{\mu}\nabla_{\nu}e_{\beta} - \nabla_{\nu}\nabla_{\mu}e_{\beta}\right) = dx^{\alpha}\left(\nabla_{\mu}\left(\Gamma^{\sigma}_{\nu\beta}e_{\sigma}\right) - \nabla_{\nu}\left(\Gamma^{\sigma}_{\mu\beta}e_{\sigma}\right)\right) = \\
=& dx^{\alpha}\left(\partial_{\mu}\Gamma^{\lambda}_{\nu\beta}e_{\lambda} + \Gamma^{\lambda}_{\nu\beta}\Gamma^{\sigma}_{\mu\lambda}e_{\sigma} - \partial_{\nu}\Gamma^{\lambda}_{\mu\beta}e_{\lambda} - \Gamma^{\lambda}_{\mu\beta}\Gamma^{\sigma}_{\nu\lambda}e_{\sigma}\right) = \\
=& \partial_{\mu}\Gamma^{\alpha}_{\nu\beta} - \partial_{\nu}\Gamma^{\alpha}_{\mu\beta} + \Gamma^{\lambda}_{\nu\beta}\Gamma^{\alpha}_{\mu\lambda} - \Gamma^{\lambda}_{\mu\beta}\Gamma^{\alpha}_{\nu\lambda} = \\
=& -R^{\alpha}_{\beta\nu\mu}
\end{aligned}$$

Из последнего выражения видно, что $R^{\alpha}_{\beta\mu\nu} = R^{\alpha}_{\beta[\mu\nu]^*}$.

О тождестве Риччи в общем случае

$$\left[\nabla_{\mu}, \nabla_{\nu}\right] Z^{\sigma} = R^{\sigma}_{\rho\mu\nu}Z^{\rho} - T^{\rho}_{\mu\nu}\nabla_{\rho}Z^{\sigma}$$

зачем оно - да хз пока что.

Вывод тождества Риччи

Для вывода тождества Риччи, раскроем коммутатор $\left[\nabla_{\mu}, \nabla_{\nu}\right] Z^{\sigma} = 2\nabla_{[\mu}\nabla_{\nu]}Z^{\sigma}$: (?? проверю, если будет нужно)

$$\begin{aligned}
\nabla_{[\mu}\nabla_{\nu]}Z^{\sigma} =& \partial_{[\mu}\left(\nabla_{\nu]}Z^{\sigma}\right) + \Gamma^{\sigma}_{[\mu|\lambda|}\nabla_{\nu]}Z^{\lambda} - \Gamma^{\rho}_{[\mu\nu]}\nabla_{\rho}Z^{\sigma} = \\
=& \partial_{[\mu}\partial_{\nu]}Z^{\sigma} + \left(\partial_{[\mu}\Gamma^{\sigma}_{\nu]\rho}\right)Z^{\rho} + \left(\partial_{[\mu}Z^{\rho}\right)\Gamma^{\sigma}_{\nu]\rho} + \Gamma^{\sigma}_{[\mu|\lambda|}\partial_{\nu]}Z^{\lambda} + \Gamma^{\sigma}_{[\mu|\lambda|}\Gamma^{\lambda}_{\nu]\rho}Z^{\rho} - \Gamma^{\rho}_{[\mu\nu]}\nabla_{\rho}Z^{\sigma},
\end{aligned}$$

и в этом выражении

$$R^{\alpha}_{\beta\mu\nu} = 2\partial_{[\mu}\Gamma^{\alpha}_{\nu]\beta} + 2\Gamma^{\alpha}_{[\mu|\lambda|}\Gamma^{\lambda}_{\nu]\beta}, \quad T^{\alpha}_{\mu\nu} = 2\Gamma^{\alpha}_{[\mu\nu]},$$

так что имеем тождество Риччи

$$\left[\nabla_{\mu}, \nabla_{\nu}\right] Z^{\sigma} = R^{\sigma}_{\rho\mu\nu}Z^{\rho} - T^{\rho}_{\mu\nu}\nabla_{\rho}Z^{\sigma}.$$

О фундаментальной теоремы римановой геометрии, утверждающей существование связности Леви-Чивита

Фундаментальная теорема римановой геометрии утверждает следующее.

Теорема 1. *В случае связности Леви-Чивита при наличии (полу)римановой структуры g на многообразии M существует единственная аффинная связность, свободная от кручения и совместимая в метрикой g, то есть*

$$\nabla_X g = 0$$

для любого векторного поля X на римановом многообразии M.

(о деталях этой теоремы потом подумаю)

Такая аффинная связность называется связностью Леви-Чивита.

доказательство фундаментальной теоремы римановой геометрии

(чет пока максимально лень смотреть)

Доказательство фундаментальной теоремы римановой геометрии начнем с записи ковариантных производных

$$\nabla_X(g(Y,Z)) = (\nabla_X g)(Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = g(\nabla_X Y, Z) + g(Y, \nabla_X Z)$$

Совершая циклическую перестановку векторных полей, получаем

$$\begin{aligned} X(g(Y,Z)) &= g(\nabla_X Y, Z) + g(Y, \nabla_X Z) \\ Y(g(Z,X)) &= g(\nabla_Y Z, X) + g(Z, \nabla_Y X) \\ Z(g(X,Y)) &= g(\nabla_Z X, Y) + g(X, \nabla_Z Y) \end{aligned}$$

Поскольку кручение отсутствует, то $\nabla_X Y - \nabla_Y X = [X,Y] \quad \forall X, Y \in \kappa(M)$, а значит

$$\begin{aligned} X(g(Y,Z)) =& g(\nabla_Y X, Z) + g(Y, \nabla_X Z) + g([X,Y], Z) \\ Y(g(Z,X)) =& g(\nabla_Z Y, X) + g(Z, \nabla_Y X) + g([Y,Z], X) \\ Z(g(X,Y)) =& g(\nabla_X Z, Y) + g(X, \nabla_Z Y) + g([Z,X], Y) \end{aligned}$$

Отсюда

$$g(\nabla_Y X, Z) = \frac{1}{2}\big[X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g([X,Y],Z) - g([Y,Z],X) + g([Z,X],Y)\big].$$

В силу невырожденности (полу)римановой метрики последняя формула однозначно определяет аффинную связность на римановом многообразии и называется формулой Козюля.

(?? зачем формула Козюля???)

Действительно, проверим, например, свойство f-линейности ковариантной производной по первому аргументу:

$$\begin{aligned} g(\nabla_{fY} X, Z) =& \frac{1}{2}[fX(g(Y,Z)) + X(f)g(Y,Z) + fY(g(Z,X)) - \\ & - fZ(g(X,Y)) - Z(f)g(X,Y) - fg([X,Y],Z) - X(f)g(Y,Z) - \\ & - fg([Y,Z],X) + Z(f)g(X,Y) + fg([Z,X],Y)] = \\ =& g(f\nabla_Y X, Z) \end{aligned}$$

Остальные свойства аффинной связности проверяются аналогично.

Запишем формулу Козюля для базисных векторных полей, то есть вычислим

$$g(\nabla_\mu e_\nu, e_\alpha) = \Gamma^\lambda_{\mu\nu} g_{\lambda\alpha} := \Gamma_{\alpha|\mu\nu}$$

Учитывая, что $[e_\alpha, e_\beta] = [\partial_\alpha, \partial_\beta] = 0$, получаем

$$\Gamma_{\alpha|\mu\nu} = \frac{1}{2}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu})$$

Поднимая индексы обратной метрикой $g^{\lambda\alpha}$ находим, что

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\alpha}(\partial_\mu g_{\nu\alpha} + \partial_\nu g_{\mu\alpha} - \partial_\alpha g_{\mu\nu})$$

Компоненты связности Леви- Чивита $\Gamma^\lambda_{\mu\nu}$ или $\Gamma_{\lambda|\mu\nu}$ называются символами Кристоффеля.

Они, очевидно, симметричны по перестановке индексов μ и ν.

Тождество Риччи для связности Леви-Чивита

Тождество Риччи для связности Леви-Чивита имеет вид

$$[\nabla_\mu, \nabla_\nu] Z^\sigma = R^\sigma_{\rho\mu\nu} Z^\rho$$

О проверке многообразия на евклидовость

Чтобы узнать, является ли пространство евклидовым в заданных координатах, необходимо вычислить тензор кривизны Римана: риманова кривизна евклидова пространства тождественно равна нулю, во всех точках.

Если многообразие M являет собой просто евклидово пространство $\mathbb{R}^n$, то существует глобальный базис, в котором риманова метрика имеет диагональный вид: $g_{\mu\nu} = \delta_{\alpha\beta}$, а символы Кристоффеля и тензор кривизны Римана обращаются в нуль во всех точках пространства; ковариантная производная в этом случае сводится к частной производной $\nabla_\mu \mapsto \partial_\mu$. значит

$$[\nabla_\mu, \nabla_\nu] Z^\sigma \mapsto [\partial_\mu, \partial_\nu] Z^\sigma \equiv 0,$$

поскольку частное дифференцирование перестановочно.

обзор применений кривизны

4.3.2 Torsion

(тут всё про него)

Определение кручения

Кручением аффинной связности ∇ называется отображение $T : x(M) \times x(M) \to x(M)$, определяемое формулой

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

где через $[\bullet, \bullet]$ обозначается скобка Ли векторных полей.

Для неё необходимо, чтобы была задана связность.

Для некоторой формы ω первой степени и векторных полей X, Y, Z определим

$$T(\omega, X, Y) = \omega\left(\nabla_X Y - \nabla_Y X - [X, Y]\right)$$

обзор применений кручения

а хз, где нужна она.

4.3.3 Riemannian Manifolds

4.3.4 riemannian Volumes

риманов объем шара

риманов объем трехмерной сферы

4.3.5 Integral Curves.

(пока с этим связка совсем слабая, знаю, что они только есть и всё)

4.4 Li Derivative (????)

(??? почему она второй раз написана уже???)

4.4.1 Algebraic Properties of the Lie Derivative

Основные свойства производной Ли

Эта операция, очевидно, $\mathbb{R}$-линейна по векторным полям и дифференциальным формам, то есть $\forall \lambda_1, \lambda_2 \in \mathbb{R}$ выносятся:

$$\begin{aligned} Ł_\xi (\lambda_1\omega_1 + \lambda_2\omega_2) &= \lambda_1 Ł_\xi \omega_1 + \lambda_2 Ł_\xi \omega_2, \\ Ł_{(\lambda_1\xi_1+\lambda_2\xi_2)}\omega &= \lambda_1 Ł_{\xi_1}\omega + \lambda_2 Ł_{\xi_2}\omega, \end{aligned}$$

(проверю, как овладею d и i)

Однако, она не f-линейна: при умножении векторного поля ξ или формы ω на гладкую функцию f, эта функция не просто выносится за скобки, а возникает дополнительный член $df \wedge \omega(\xi)$. Действительно,

$$\begin{aligned} Ł_{f\xi}\omega &= \imath_{f\xi} d\omega + d\imath_{f\xi}\omega \\ &= d\omega(f\xi) + d(\omega(f\xi)) = \\ &= f d\omega(\xi) + d(f\omega(\xi)) = \\ &= f d\imath_\xi\omega + f d\imath_\xi\omega + df \wedge \omega(\xi). \end{aligned}$$

Правило Лейбница для производной Ли для обычного умножения (?)

(?? в общем виде??)

Правило Лейбница в частности есть способ действия производной Ли на произведение функций и форму:

$$Ł_\xi f\omega = f Ł_\xi \omega + \omega \cdot \xi(f)$$

Действительно, (????)

$$\begin{aligned} Ł_{f\xi}\omega &= f Ł_\xi \omega + df \wedge \imath_\xi \omega \\ Ł_\xi f\omega &= \imath_\xi d(f\omega) + d\imath_\xi(f\omega) = \\ &= \imath_\xi(df \wedge \omega + f d\omega) + d(f\imath\xi\omega) = \\ &= df(\xi) \cdot \omega - df \wedge \imath_\xi\omega + f \cdot \imath_\xi d\omega + df \wedge \tau_\xi\omega + f dv_\xi\omega = \\ &= f \cdot Ł_\xi\omega + \xi(f) \cdot \omega, \end{aligned}$$

то есть

$$Ł_\xi f\omega = f Ł_\xi \omega + \omega \cdot \xi(f)$$

Формула (1.73) выражает частный случай формулы Лейбница для производной Ли вдоль векторного поля ξ от внешнего умножения дифференциальнх форм (здесь, 0-формы f и дифференциальной формы ω).

Производная Ли гладкой функции (?)

Действительно, производная Ли гладкой функции по определению

$$Ł_\xi f = \imath_\xi df + di_\xi f = \imath_\xi df = df(\xi) = \xi(f)$$

поэтому формула (1.73) принимает вид

$$Ł_\xi(f \wedge \omega) = f \wedge Ł_\xi\omega + Ł_\xi f \wedge \omega$$

Правило Лейбница для производной Ли в общем случае

В общем случае правило Лейбница для дифференциальных форм следует из соответствующих правил для внутреннего умножения и внешнего дифференцирования:

$$\begin{aligned}
\imath_\xi(\omega \wedge \lambda) =& \imath_\xi \omega \wedge \lambda + (-1)^{\deg \omega} \omega \wedge z_\xi \lambda, \\
d(\omega \wedge \lambda) =& d\omega \wedge \lambda + (-1)^{\deg \omega} \omega \wedge d\lambda \\
Ł_\xi(\omega \wedge \lambda) =& Ł_\xi \omega \wedge \lambda + \omega \wedge Ł_\xi \lambda
\end{aligned}$$

Производная Ли от дифференциала df

Полезно также посмотреть действие производной Ли на дифференциал df:

$$Ł_\xi df = \imath_\xi d^2 f + d\imath_\xi df = dv_\xi df = d(df(\xi)) = dŁ_\xi f$$

(??? зачем это??)
(вроде тут как раз все свойства, их не так вроде много)

Коммутирование производной Ли с внешним дифференцированием

Равенство (1.76) можно описать как коммутирование операции $Ł_\xi$ с внешним дифференцированием d для функции, то есть

$$[Ł_\xi, d]\, f = Ł_\xi df - dŁ_\xi f = 0$$

Нетрудно показать равенство нулю коммутатора $[Ł_\xi, d]$ для форм любой степени.
Действительно,

$$[Ł_\xi, d]\, \omega = Ł_\xi d\omega - dŁ_\xi \omega = \imath_\xi d^2 \omega + d\imath_\xi d\omega - di_\xi d\omega - d^2 \imath_\xi \omega = 0$$

(потом посмотрю)

Коммутирование производной Ли с внутренним умножением

Производная Ли $Ł_\xi$ коммутирует с операцией внутреннего умножения $\imath_\xi$ для форм любой степени, то есть:

$$[Ł_\xi, \iota_\xi]\, \omega = Ł_\xi \imath_\xi \omega - \imath_\xi Ł_\xi \omega = \imath_\xi d\imath_\xi \omega + di^\imath_\xi \omega\omega - i_\xi \imath_\xi d\omega - \imath_\xi d\imath_\xi \omega = 0.$$

Найдём выражение для производной Ли дифференциальной 1-формы $\lambda = \lambda_\mu dx^\mu$ вдоль векторного поля $\xi = \xi^\mu \partial_\mu$ в координатах.

$$Ł_\xi \lambda = Ł_{\xi^\nu \partial_\nu} \left(\lambda_\mu dx^\mu\right)$$

Из (???) (1.72) и (1.73) получаем

$$\begin{aligned}
Ł_\xi \lambda =& \lambda_\mu Ł_\xi dx^\mu + \xi\left(\lambda_\mu\right) dx^\mu = \\
=& \lambda_\mu \xi^\nu Ł_{\partial_\nu} dx^\mu + \lambda_\mu d\xi^\nu \wedge dx^\mu\left(\partial_\nu\right) + \xi^\nu \frac{\partial \lambda_\mu}{\partial x^\nu} dx^\mu = \\
=& \lambda_\mu \frac{\partial \xi^\mu}{\partial x^\nu} dx^\nu + \xi^\mu \frac{\partial \lambda_\nu}{\partial x^\mu} dx^\nu = \\
=& \left(\lambda_\nu \partial_\mu \xi^\nu + \xi^\nu \partial_\nu \lambda_\mu\right) dx^\mu = \\
=& Ł_\xi \lambda_\mu dx^\mu
\end{aligned}$$

$$Ł_\xi \lambda_\mu = \lambda_\nu \partial_\mu \xi^\nu + \xi^\nu \partial_\nu \lambda_\mu$$

(хз, потом посмотрю это)

пример: производная Ли от 1-формы

$$L_X\omega_\gamma = \omega_\beta \frac{\partial x^\beta}{\partial x^\gamma} + X^\beta \frac{\partial \omega_\gamma}{\partial x^\beta}$$

производная Ли одного векторного поля вдоль другого

(??? тут я пока вообще хз)

Определим теперь производную Ли одного векторного поля вдоль другого векторного поля.

Для этого воспользуемся двойственностью между касательным и кокасательным пространствами в точке и определим векторное поле $Ł_\xi\eta$ (производная Ли η по ξ) через его каноническое произведение на некоторую линейную 1-форму ω: по сути потребуем выполнения формулы Лейбница

$$Ł_\xi(\omega(\eta)) = \xi(\omega(\eta)) = (Ł_\xi\omega)(\eta) + \omega(Ł_\xi\eta).$$

Из формулы (1.78) получаем

$$\omega(Ł_\xi\eta) = \imath_\xi d\imath_\eta\omega - \imath_\eta d\imath_\xi\omega - \imath_\eta\imath_\xi d\omega$$

Важно отметить, что при умножении 1-формы ω на функцию f в выражении (1.79) правая часть равенства также умножается на функцию f. Для того, чтобы проверить данное утверждение, совершим в (1.79) замену $\omega \mapsto f\omega$:

$$f\omega(Ł_\xi\eta) = \imath_\xi d\imath_\eta(f\omega) - \imath_\eta d\imath_\xi(f\omega) - \imath_\eta^\imath\xi d(f\omega) = f\imath_\xi\imath_\xi d\imath_\eta\omega - f\imath_\eta d\imath_\xi\omega - f\imath_\eta^l\xi d\omega +$$

$$+ (\imath_\xi(df))(\imath_\eta\omega) - (\imath_\eta(df))(\imath_\xi\omega) - \imath_\eta\imath_\xi(df\wedge\omega) = f\imath_\xi d\imath_\eta\omega - f\imath_\eta d\imath_\xi\omega -$$

$$-f\imath_\eta^2\xi d\omega + (\imath_\xi(df))(\imath_\eta\omega) - (\imath_\eta(df))(\imath_\xi\omega) - (\imath_\eta\omega)(\imath_\xi(df)) + (\imath_\eta(df))(u_\xi\omega)$$

откуда, очевидно, получаем, что и требовалось.

Значит, зная выражение в правой части (1.79) для базисных форм $dx^\mu(Ł_\xi\eta)$, мы для произвольной 1-формы $\omega = \omega_\mu dx^\mu$ запишем его в виде

$$\omega(Ł_\xi\eta) = \omega_\mu dx^\mu(Ł_\xi\eta)$$

Это означает, что (1.80) корректно определяет производную Ли $Ł_\xi\eta$ в каждой точке $p \in M$ многообразия M как линейный функционал на $T_p^\star M$ Координаты векторного поля $Ł_\xi\eta_{\text{в}}$ данной системе координат равны

$$(Ł_\xi\eta)^\mu = dx^\mu(Ł_\xi\eta) = \imath_\xi d(dx^\mu(\eta)) - \imath_\eta d(dx^\mu(\xi))$$
$$\text{Ес}\, \eta = \eta^\nu\partial_\nu, \xi = \xi^\nu\partial_\nu, \text{ то}$$
$$Ł_\xi\eta^\mu = \imath_\xi d\eta^\mu - \imath_\eta d\xi^\mu = \partial_\nu\eta^\mu \cdot dx^\nu(\xi) - \partial_\nu\xi^\mu \cdot dx^\nu(\eta) = \xi^\nu\partial_\nu\eta^\mu - \eta^\nu\partial_\nu\xi^\mu,$$

окончательно,

$$Ł_\xi\eta^\mu = \xi^\nu\partial_\nu\eta^\mu - \eta^\nu\partial_\nu\xi^\mu$$

Если применить векторное поле $Ł_\xi\eta$ как дифференцирование функции f, то получим

$$(Ł_\xi\eta)f = df(Ł_\xi\eta) = \imath_\xi d(\imath_\eta df) - \imath_\eta d(\imath_\xi df) = \xi(\eta(f)) - \eta(\xi(f))$$

Видно, что с точки зрения дифференциальных операторов на функциях, производная Ли $Ł_\xi\eta$ - это коммутатор векторных полей $[\xi, \eta]$, то есть коммутатор соответствующих им дифференциальных операторов.

кососимметричность и тождество Якоби (???)

Несложно проверить справедливость следующих свойств производной Ли $[\xi, \eta]$: кососимметричность:

$$[\xi, \eta] = -[\eta, \xi],$$

тождество Якоби:

$$[\xi, [\eta, \lambda]] + [\lambda, [\xi, \eta]] + [\eta, [\lambda, \xi]] = 0.$$

(????) Поэтому векторные поля образуют алгебру Ли относительно коммутатора (скобки Ли)

$$Ł_\xi \eta = [\xi, \eta]$$

Скобка Ли от умножения векторных полей на функции (???)

Посмотрим теперь как ведёт себя скобка Ли векторных полей $[\xi, \eta]$ при умножении их на функции f и g, то есть исследуем выражение $[f\xi, g\eta]$.

Для этого подействуем этой скобкой Ли на пробную функцию φ:

$$\begin{aligned}[f\xi, g\eta](\varphi) = f\xi(g\eta(\varphi)) - g\eta(f\xi(\varphi)) = f\xi(g)\eta(\varphi) + fg\xi(\eta(\varphi)) - \\ -g\eta(f)\xi(\varphi) - gf\eta(\xi(\varphi)) = fg[\xi, \eta](\varphi) - g\eta(f)\xi(\varphi) + f\xi(g)\eta(\varphi),\end{aligned}$$

откуда находим, что

$$[f\xi, g\eta] = fg[\xi, \eta] + f\xi(g)\eta - g\eta(f)\xi.$$

(??? и что с этого???)

4.4.2 li Derivative from Different Objects

Производная Ли скалярного поля $Ł_\xi f$ есть производная по направлению $\xi(f)$

Производная Ли скалярного поля $Ł_\xi f$ есть производная по направлению $\xi(f)$.

(тут напишу, почему это так и мб комментарии)

производная Ли от функций

производная Ли от векторного поля

.....

$$(L_X Y)f = [X, Y]f$$

окажется, что она определяет алгебру Ли (?)

Производная Ли от тензорного поля

Для тензорного поля у производной Ли есть “тензорное правило Лейбница” по аналогии с случаем форм и векторов. То есть для произвольных тензорных полей $\mathbf{S}$ и $\mathbf{T}$ должно выполняться равенство

$$Ł_\xi(\mathbf{S} \otimes \mathbf{T}) = Ł_\xi \mathbf{S} \otimes \mathbf{T} + \mathbf{S} \otimes Ł_\xi \mathbf{T}.$$

Таким образом, производная Ли тензорных полей полностью определяется следующими свойствами:

- Производная Ли скалярного поля $Ł_\xi f$ есть производная по направлению $\xi(f)$.

Производная Ли векторного поля $Ł_\xi \eta$ есть скобка Ли $Ł_\xi \eta = [\xi, \eta]$. - Для произвольных векторных полей и 1-формы выполняется тождество

$$(Ł_\xi \omega)(\eta) = (d\omega)(\xi, \eta) + \eta\omega(\xi)$$

- Для произвольных тензорных полей S и T выполняется

$$Ł_\xi(\mathbf{S}\otimes\mathbf{T}) = Ł_\xi\mathbf{S}\otimes\mathbf{T} + \mathbf{S}\otimes Ł_\xi\mathbf{T}$$

Найдём выражение для производной Ли $Ł_\xi\mathbf{T}$ произвольного тензорного поля типа (p,q) вдоль векторного поля ξ:

$$\mathbf{T} = T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q} e_{\mu_1}\otimes\ldots\otimes e_{\mu_p}\otimes\mathfrak{h}^{\nu_1}\otimes\ldots\otimes\mathfrak{h}^{\nu_q}$$

в координатах.

производная Ли от тензорного поля типа $(1,1)$

Рассмотрим случай смешанного тензорного поля $\mathbf{W}$ типа $(1,1)$:

$$\mathbf{W} = W^\alpha_\beta e_\alpha\otimes\mathfrak{h}^\beta = W^\alpha_\beta\partial_\alpha\otimes dx^\beta$$

Распишем ее по правилу Лейбница:

$$Ł_\xi\mathbf{W} = Ł_\xi\left(W^\alpha_\beta\partial_\alpha\right)\otimes dx^\beta + W^\alpha_\beta\partial_\alpha\otimes Ł_\xi dx^\beta$$

Для вычисления второго слагаемого воспользуемся формулой Картана (1.71), а с первым слагаемым разберёмся с помощью (1.84).

Тогда

$$\begin{aligned} Ł_\xi\mathbf{W} = W^\alpha_\beta\xi^\gamma\left[\partial_\gamma,\partial_\alpha\right]\otimes dx^\beta + \xi^\gamma\left(\partial_\gamma W^\alpha_\beta\right)\partial_\alpha\otimes dx^\beta - \\ -W^\gamma_\beta\left(\partial_\gamma\xi^\alpha\right)\partial_\alpha\otimes dx^\beta + W^\alpha_\gamma\left(\partial_\beta\xi^\gamma\right)\partial_\alpha\otimes dx^\beta \end{aligned}$$

откуда окончательно получаем

$$Ł_\xi\mathbf{W} = (Ł_\xi\mathbf{W})^\alpha{}_\beta\partial_\alpha\otimes dx^\beta, \quad Ł_\xi W^\alpha_\beta = \xi^\gamma\partial_\gamma W^\alpha_\beta - W^\gamma_\beta\partial_\gamma\xi^\alpha + W^\alpha_\gamma\partial_\beta\xi^\gamma$$

Теперь нетрудно понять, что для произвольного тензорного поля (1.86) производная Ли в координатах будет иметь следующий вид:

$$\begin{aligned} Ł_\xi T^{\mu_1\ldots\mu_p}{}_{\nu_1\ldots\nu_q} = \xi^\gamma\partial_\gamma T^{\mu_1\ldots\mu_p}_1{}_{\nu_1\ldots\nu_q} - \partial_{\alpha_1}\xi^{\mu_1}T^{\alpha_1\mu_2\ldots\mu_p}_1\nu_1\ldots\nu_q - \\ -\ldots-\partial_{\alpha_p}\xi^{\mu_p}T^{\mu_1\ldots\mu_{p-1}\alpha_p}\nu_1\ldots\nu_q + \partial_{\nu_1}\xi^{\beta_1}T^{\mu_1\ldots\mu_p}\beta_1\nu_2\ldots\nu_q + \ldots + \partial_{\nu_q}\xi^{\beta_q}T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_{q-1}\beta_q} \end{aligned}$$

Если вам эта формула не очевидна, проверьте её самостоятельно (по определению).

Ближайшей нашей целью будет дать другое (геометрическое) определение производной Ли.

Для этого необходимо посмотреть на интегрирование векторных полей.

Начнём с того, что определим прямой и обратный образы тензорных полей при гладком отображении φ между многообразиями.

4.4.3 Geometric Meaning of the Lie Derivative

(пока это разбирать лень, потому что не актуально)

Суть геометрического смысла

хз

Обзор применений геометрического определения производной Ли

(пока хз, потом узнаю)

интегральные кривые для векторного поля

Пусть на многообразии M задано векторное поле ξ.

Найдём на M интегральные кривые $\gamma(t)$ векторного поля ξ, т.е. кривые, касательные векторы к которым в каждой точке совпадают с векторным полем ξ.

При этом нас будет интересовать задача о нахождении интегральной кривой $\gamma : [a, b] \to M$, проходящей через точку $p \in M$.

По сути речь идёт о решении дифференциального уравнения

$$\frac{d\gamma}{dt} = \xi_{\gamma(t)}, \quad \gamma\left(t_0\right) = p, \quad t_0 \in [a, b]$$

Здесь производную кривой по t можно определить как прямой образ вектора ∂_t в точке $\gamma(t)$, то есть

$$\frac{d\gamma}{dt} = \gamma_\star\left(\partial_t\right)$$

или, что то же самое, как дифференцирование гладких функций на M в точке $\gamma(t)$

$$\gamma_\star \partial_t(f) = \partial_t(f \circ \gamma(t)) = \frac{d}{dt} f(\gamma(t))$$

Вводя координаты на многообразии нетрудно понять, что локально это и есть нахождение решения автономного дифференциального уравнения

$$\frac{dx^\mu(t)}{dt} = \xi^\mu(\gamma(t)), \quad \gamma\left(t_0\right) = p, \quad t_0 \in [a, b]$$

Предполагается, что ξ не зависит явно от t.

Если векторному полю ξ разрешить зависеть от t явно, то это уже будет нахождение решения неавтономного уравнения.

Далее нас будет интересовать только автономный случай, когда векторное поле не зависит явно от t.

ξ в точке $p \in M$ не равно нулю, то в некоторой криволинейной системе координат $x_1, \ldots, x_n$ в окрестности точки p оно может быть выпрямлено, то есть приведено к виду $\xi = \partial_{x_1}$.

Эта теорема позволила бы нам в нужный момент (к примеру) быстро вывести формулу Картана (1.71) для производной Ли из её геометрического определения.

Хотя мы в напем курсе это сделаем другим путём, теорема о выпрямлении векторного поля остаётся весьма полезным вычислительным инструментом дифференциальной геометрии.

Вернёмся к глобальному интегрированию векторных полей на многообразиях и рассмотрим вопрос о возможности продолжения интегральной кривой векторного поля.

конструкция непродолжаемого решения - максимальную кривую векторного поля (???)

Пусть на многообразии M (без края) задано векторное поле ξ. Тогда $\forall t_0$ и точки $p \in M$ существует интегральная кривая γ, определённая на некотором интервале (a, b), содержащем t_0 и удовлетворяющая условию $\gamma\left(t_0\right) = p$. Она максимальная, то есть любая другая интегральная кривая γ' векторного поля ξ, удовлетворяющая тому же условию $\gamma'\left(t_0\right) = p$, является ограничением γ на некоторый интервал $(a', b') \subseteq (a, b)$, то есть $\gamma' = \gamma|_{(a', b')}$

(??? и что?? потом разберусь)

Теорема о непродолжаемости для максимальной интегральной кривой (???)

Пусть на многообразии M (без края) задано векторное поле $\xi, \text{а}\gamma : (a, b) \to M$ - его максимальная интегральная кривая, не продолжающаяся за пределы интервала (a, b). Если b конечно, то кривая γ покидает любой компакт при $t \to b - 0$ в следующем смысле: $\forall$ компактного множества $K \subseteq M \exists t_K \in (a, b) : \gamma(t) \notin K$ при $t > t_K$ Аналогичное утверждение формулируется для конечного a (сделайте ЭТО camh).

следствие.

Пусть на многообразии M (без края) задано векторное поле ξ с компактным носителем.

Тогда все интегральные кривые векторного поля ξ продолжаются по времени неограниченно в обе стороны. По следствию, решая задачу Коши, можно сопоставить векторному полю ξ семейство диффеоморфизмов $\varphi_{t,t_0} : M \to M$, удовлетворяющее соотношению

$$\frac{d}{dt}\varphi_{t,t_0}(x) = \xi_{\varphi_{t,t_0}}, \quad \varphi_{t_0,t_0} = \mathrm{id}_M$$

Смысл $\varphi_{t,t_0}(x)$ можно объяснить как нахождение интегральной кривой $\gamma(t)$ векторного поля ξ с начальным условием $\gamma(t_0) = x$ и определение

$\varphi_{t,t_0}(x) = \gamma(t)$; причём верна следующая формула композиции

$$\varphi_{t_2,t_1} \circ \varphi_{t_1,t_0} = \varphi_{t_2,t_0}$$

В автономном случае, когда наше векторное поле не зависит явно от времени, нетрудно заметить, что если $\gamma(t)$ является решением дифференциального уравнения, то и $\gamma(t + \tau)$ тоже является решением.

Значит,

$$\varphi_{t,t_0} = \varphi_{t+\tau,t_0+\tau}, \quad \forall \tau$$

Без потери общности веберем τ таким образом, чтобы $\varphi_{t,t_0} = \varphi_{t,0} \equiv \varphi_t$; тогда $\varphi_{t_1} \circ \varphi_{t_2} = \varphi_{t_1+t_2}$, в частности $\varphi_t^{-1} = \varphi_{-t}$. Таким образом век- торное поле порождает на многообразии однопараметрическую группу диффеоморфизмов. В терминах диффеоморфизмов φ_t можно дать другое, т.н.

Геометрическое определение производной Ли от гладких функций

Построим его последовательно, начиная с производной Ли вдоль векторного поля от гладких функций.

Определим производную Ли вдоль векторного поля ξ гладкой функции f через обратный образ функции при отображении φ_t по формуле

$$Ł_\xi f = \lim_{t\to 0} \frac{\varphi_t^\star f - f}{t} = \frac{d}{dt}\varphi_t^\star f\Big|_{t=0}$$

В локальных координатах (1.101) примет вид

$$Ł_\xi f = \frac{df\left(\varphi_t(x)\right)}{dt}\Big|_{t=0} = \frac{\partial f}{\partial x^\alpha}\frac{dx^\alpha}{dt} = \xi^\alpha \frac{\partial f}{\partial x^\alpha} = \xi(f) = df(\xi)$$

Здесь мы воспользовались координатной записью уравнения (1.97) и учли, что $\dot{x}^\alpha = \xi^\alpha$.

Из формулы (1.102) видно, что производная Ли от функции есть простое дифференцирование f по направлению ξ, что в точности совпадает с выражением для производной Ли от 0-формы (гладкой функции), полученным из формулы Картана - алгебраического определения производной Ли.

Производная Ли векторного поля η вдоль векторного поля ξ определяется аналогично

$$Ł_\xi \eta = \frac{d}{dt}\varphi_t^\star \eta\Big|_{t=0}, \quad \varphi_t^\star \eta = (\varphi_{-t})_\star \eta$$

Рис.

8: Кривая, на которой лежат точки p и $q = \varphi_t(p)$, как нетрудно понять, являет собой интегральную кривую векторного поля ξ, проходяшую через точку p. Кривые, пересекающие интегральную кривую векторного поля ξ - суть интегральные кривые векторного поля η, проходящие через точки p и q соответственно.

Чтобы иметь возможность сравнения векторов в разных точках многообразия M (хоть и расположенных вдоль одной интегральной кривой), в результате которого мы снова получим векторное поле на M, необходимо совершить параллельный перенос в точку p посредством $(\varphi_{-t})_\star$. При этом, результат параллельного переноса, вообще говоря, не совпадает с исходным вектором в этой точке, а различие определяется производной Ли. Геометрическое определение (1.103) производной Ли $Ł_\xi \eta$ наглядней всего расшифровывает рисунок 1. Вычислим производную Ли векторного поля η вдоль другого $-\xi$ в координатной карте по определению (1.103).

Для начала найдём производную Ли векторного поля $\eta = \partial_\alpha$ вдоль ξ:

$$Ł_\xi \partial_\alpha = \frac{d}{dt}(\varphi_{-t})_\star \partial_\alpha\Big|_{t=0}$$

Прямой образ $(\varphi_{-t})_\star \partial_\alpha$ определяется, как известно, по формуле

$$[(\varphi_{-t})_\star \partial_\alpha] f = \frac{\partial}{\partial x^\alpha}(f \circ \varphi_{-t})$$

Согласно уравнению (1.99), записанному в локальных координатах,

$$\varphi_t\left(x^\beta\right) = x^\beta(t) = x^\beta(0) + t\xi^\beta + \mathcal{O}\left(t^2\right), \quad t \to 0$$

Значит,

$$\varphi_{-t}\left(x^\beta\right) = x^\beta(0) - t\xi^\beta + \mathcal{O}\left(t^2\right), \quad t \to 0$$

Отсюда при $t \to 0$ находим

$$[(\varphi_{-t})_\star \partial_\alpha] f = \partial_\beta f\left(\delta_\alpha^\beta - t\partial_\alpha \xi^\beta + \mathcal{O}\left(t^2\right)\right)$$

Из формулы (1.106) окончательно получаем, что производная Ли

$$Ł_\xi \partial_\alpha = \frac{d}{dt}(\varphi_{-t})_\star \partial_\alpha\Big|_{t=0} = -\partial_\alpha \xi^\beta \partial_\beta$$

Заметим, что согласно геометрическому определению производной Ли от функций и векторных полей должно выполняться обычное правило Лейбница (если вам это не очевидно, будет полезно доказать данное утверждение самостоятельно)

$$Ł_\xi(f\eta) = f Ł_\xi \eta + \eta Ł_\xi f.$$

В силу (1.108) имеем

$$Ł_\xi(f \cdot \partial_\alpha) = f \cdot Ł_\xi \partial_\alpha + (Ł_\xi f)\partial_\alpha = f \cdot Ł_\xi \partial_\alpha + \xi(f)\partial_\alpha$$

Из последнего тождества можно легко выписать выражение для производной Ли произвольного векторного поля $\eta = \eta^\alpha \frac{\partial}{\partial x^\alpha}$ вдоль векторного поля $\xi = \xi^\beta \frac{\partial}{\partial x^\beta}$:

$$Ł_\xi \eta = -\eta^\alpha \frac{\partial \xi^\beta}{\partial x^\alpha} \frac{\partial}{\partial x^\beta} + \xi^\alpha \frac{\partial \eta^\beta}{\partial x^\alpha} \frac{\partial}{\partial x^\beta}$$

то есть

$$Ł_\xi \eta^\alpha = \xi^\beta \partial_\beta \eta^\alpha - \eta^\beta \partial_\beta \xi^\alpha$$

что в точности совпадает с выражением (1.82) для производной Ли одного векторного поля вдоль другого, которое мы получили из алгебраического определения производной Ли.

Геометрическое определение производной Ли вдоль векторного поля ξ от дифференциальных форм строится абсолютно аналогично.

Так, для произвольной дифференциальной формы ω имеем

$$Ł_\xi \omega = \frac{d}{dt} \varphi_t^\star \omega \Big|_{t=0}$$

Непосредственно из определения (1.109) следует справедливость обычного правила Лейбница для внешнего умножения форм (если это не очевидно, докажите данное утверждение самостоятельно):

$$Ł_\xi (\omega \wedge \lambda) = Ł_\xi \omega \wedge \lambda + \omega \wedge Ł_\xi \lambda$$

Найдём для примера производную Ли от дифференциальной 1-формы $\omega = \omega_\alpha dx^\alpha$ из её геометрического определения.

Для этого поступим также, как для случая векторного поля и вычислим сначала

$$Ł_\xi dx^\alpha = \frac{d}{dt} \varphi_t^\star dx^\alpha \Big|_{t=0}$$

Как мы уже показывали ранее (когда рассматривали производную Ли от векторных полей),

$$\varphi_t \left(x^\alpha\right) = x^\alpha(t) = x^\alpha(0) + t\xi^\alpha + \mathcal{O}\left(t^2\right), \quad t \to 0$$

Значит, при $t \to 0$

$$\varphi_t^\star dx^\alpha = \left(\delta^\alpha_\beta + t\partial_\beta \xi^\alpha + \mathcal{O}\left(t^2\right)\right) dx^\beta$$

Подставляя полученное выражение в (1.111), получаем

$$Ł_\xi dx^\alpha = \frac{d}{dt} \varphi_t^\star dx^\alpha \Big|_{t=0} = \partial_\beta \xi^\alpha dx^\beta$$

Имея ввиду справедливость правила Лейбница (1.110), находим

$$Ł_\xi \left(f dx^\alpha\right) = f \cdot Ł_\xi dx^\alpha + \xi(f) dx^\alpha$$

Тогда производная Ли произвольной 1-формы $\omega = \omega_\alpha dx^\alpha$

$$Ł_\xi \omega = \xi^\beta \frac{\partial \omega_\alpha}{\partial x^\beta} dx^\alpha + \omega_\beta \frac{\partial \xi^\beta}{\partial x^\alpha} dx^\alpha$$

то есть

$$Ł_\xi \omega_\alpha = \xi^\beta \partial_\beta \omega_\alpha + \omega_\beta \partial_\alpha \xi^\beta$$

что в точности совпадает с выражением для производной Ли для 1формы, полученной из формулы Картана.

Таким образом мы показали, что геометрическое определение производной Ли эквивалентно её алгебраическому определению - формуле Картана, ведь для доказательства этого факта достаточно было рассматривать только формы нулевой и первой степени, так как обе части равенства удовлетворяют правилу Лейбница без знаков относительно внешнего умножения.

(?!!!) Эквивалентность геометрического и алгебраического определений производной Ли

(тут пока что не порешал, усвою - обязательно четко пропишу.)

Геометрическое определение производной Ли от тензорного поля

Производная Ли произвольного тензорного поля $\mathbf{T}$ вдоль векторного поля ξ определяется следующим образом. Пусть на многообразии M задано векторное поле ξ и тензорное поле $\mathbf{T}$. Тогда производная Ли от $\mathbf{T}$ вдоль векторного поля ξ есть

$$Ł_\xi \mathbf{T} = \lim_{t\to 0} \frac{\varphi_t^\star \mathbf{T} - \mathbf{T}}{t} = \frac{d}{dt}\varphi_t^\star \mathbf{T}\bigg|_{t=0},$$

причём из этого определения понятно, что должно выполняться правило Лейбница для тензорного произведения

$$Ł_\xi(\mathbf{S}\otimes\mathbf{T}) = Ł_\xi\mathbf{S}\otimes\mathbf{T} + \mathbf{S}\otimes Ł_\xi\mathbf{T}$$

иллюстрация (?!!)

(хз)

доказательство эквивалентности определений

4.4.4 Other Properties of Li Derivative

(соберем их тут, ибо для этого её и изучаем)

Алгебраическое определение производной Ли

Производная Ли $Ł_\xi$ вдоль векторного поля ξ на дифференциальных формах определяется как антикоммутатор

$$Ł_\xi \equiv \{\imath_\xi, d\} = \imath_\xi d + d\imath_\xi$$

Связь производной Ли и ковариантной производной (???)

Володя говорил, что в общем случае не существует, потом подробнее пропишу.
явное отличие следущее:
?? тут формула
Совпадают они тогда, когда (???)

4.4.5 Some Typical Applications of the Li Derivative

(соберем их тут, ибо для этого её и изучаем)

4.5 Forward and Reverse Images

(пока как подраздел, потом мб компактнее сделаю, но по идее тут всё и раскрою сполна про это!)

4.5.1 Design

суть и применения прямого и обратного образа (????!!!)

!! тут самое главное, что я хочу узнать, главные формулы!!!

Суть обратного образа, что имеются два многообразия и мы в функцию, действующую на втором многообразии, подставляем координаты первого.

определение гладкого многообразия

Гладким отображением между многообразиями $\varphi : M \to N$ размерностей m и n называется непрерывное отображение, которое в окрестности каждой точки, в достаточно малых координатных картах, выглядит как гладкое отображение области из $\mathbb{R}^m$ в область в $\mathbb{R}^n$.

обратный образ функции

суть обратного образа в том, что он подставляет в функции на многообразии-образе координаты многообразия-прообраза (?).

Рассмотрим два многообразиями M и N, между которыми задано гладкое отображение $\varphi : M \to N$.

Обратный образ $\varphi^\star f : M \to \mathbb{R}$ можно определить для функции $f : N \to \mathbb{R}$ при отображении $\varphi : M \to N$ согласно формуле

$$(\varphi^\star f)(p) = f(q), \quad \text{где} \quad p \in M,\ q = \varphi(p) \in N.$$

В локальных координатах x^α на многообразии M и координатах y^β на многообразии N, получаем

$$(\varphi^\star f)(x) = f(\varphi(x)) = f(y(x))$$

Формально, действие $\varphi^\star$ на функцию f можно записать в виде композиции

$$\varphi^\star f = f \circ \varphi$$

прямой образ векторного поля

Суть прямого образа в том, что он отображает поля вместе с самим многообразием (???)

Пусть $\varphi = N$ является гладким отображением между многообразиями M и N с обратным гладким, т.е. является диффеоморфизмом.

для функции $f : N \to \mathbb{R}$

Рассмотрим векторное поле v на M, $v \in \mathrm{Vect}(M)$.

Тогда можно определить прямой образ векторного поля $\varphi_\star v \in \mathrm{Vect}(N)$ при отображении φ как

$$(\varphi_\star v)(f)\big|_{\varphi(p)} = v\,(\varphi^\star f)\big|_p$$

Формально это можно записать как

$$(\varphi_\star v)(f) = v(f \circ \varphi) = v \circ \varphi^\star f, \quad \varphi_\star v = v \circ \varphi^\star$$

В координатной карте прямой образ векторного поля

$$(\varphi_\star v)(f)\big|_{y(x)} = v^\alpha \frac{\partial}{\partial x^\alpha}(f \circ \varphi) = v^\alpha \frac{\partial y^\beta}{\partial x^\alpha}\frac{\partial f}{\partial y^\beta}$$

где

$$v = v^\alpha \frac{\partial}{\partial x^\alpha}, \quad \varphi_\star v = (\varphi_\star v)^\alpha \frac{\partial}{\partial y^\alpha}$$

Значит, координаты векторного поля $\varphi_\star v$ представляются в виде

$$(\varphi_\star v)^\alpha = \frac{\partial y^\alpha}{\partial x^\beta} v^\beta.$$

Определим обратный образ дифференциальной формы $\varphi^\star \omega$ при гладком отображении многообразий $\varphi : M \to N$.

Для всякого гладкого отображения $\varphi : M \to N$ между гладкими многообразиями определено отображение пространств дифференциальных форм

$$\varphi^\star : \Omega^k(N) \to \Omega^k(M)$$

действующее по формуле (для некоторых векторов $v_1, \ldots, v_k$ в одной и той же точке $p \in M$)

$$\varphi^\star \omega \left(v_1, \ldots, v_k\right) = \omega \left(\varphi_\star v_1, \ldots, \varphi_\star v_k\right)$$

В этом определении следует понимать, что когда левая часть вычисляется в точке $p \in M$, то правая вычисляется в точке $q = \varphi(p) \in N$.

Важно понимать, что в отличие от векторных полей, для дифференциальных форм (в частности, для функций) любое гладкое отображение порождает отображение пространства дифференциальных форм в обратную сторону.

взятие обратного образа дифференциальных форм коммутирует с внешним умножением и внешним дифференцированием

Покажем, что взятие обратного образа дифференциальных форм коммутирует с внешним умножением и внешним дифференцированием.

(???) то

4.5.2 Overview of Applications and Relationships to Other Structures

(пока не применял, хз, какие применения)

связь обратного образа и детерминанта

Связь обратного образа и детерминанта имеет вид:

$$\varphi^\star f(y) dy_1 \wedge \ldots \wedge dy_n = f(y(x)) \cdot \det \left(\frac{\partial y^\alpha}{\partial x^\beta} \right) dx_1 \wedge \ldots \wedge dx_n$$

то есть в общем случае просто вылезает якобиан замены.

Чтобы лучше понять абстрактную формулу????(1.94) для обратного образа дифференциальной формы при гладком отображении многообразий,

Выпишем в явном виде действие $\varphi^\star$ на дифференциальные формы высшей степени ν в областях $\mathbb{R}^n$, где φ криволинейная замена координат.

Сделаем это, решив задачу для $\mathbb{R}^2$, где общий вид формы высшей степени в координатах имеет вид $\nu = f(y) dy_1 \wedge dy_2$

Запишем согласно (???) (1.92) и (1.94):

$$\begin{aligned}\varphi^\star\nu =& f(y(x))\cdot\frac{\partial y_1}{\partial x^\alpha}dx^\alpha\wedge\frac{\partial y_2}{\partial x^\beta}dx^\beta = \\ =& f(y(x))\left(\frac{\partial y_1}{\partial x_1}dx_1+\frac{\partial y_1}{\partial x_2}dx_2\right)\wedge\left(\frac{\partial y_2}{\partial x_1}dx_1+\frac{\partial y_2}{\partial x_2}dx_2\right) = \\ =& f(y(x))\cdot\left(\frac{\partial y_1}{\partial x_1}\frac{\partial y_2}{\partial x_2}-\frac{\partial y_1}{\partial x_2}\frac{\partial y_2}{\partial x_1}\right)dx_1\wedge dx_2 = \\ =& f(y(x))\cdot\det\left(\frac{\partial y^\alpha}{\partial x^\beta}\right)dx_1\wedge dx_2\end{aligned}$$

то есть просто по определению замены координат, подставляем, приводим слагаемые, вот и все.

В случае произвольного n, как нетрудно понять, в правой части также появится якобиан замены $J_\varphi = \det\left(\frac{\partial y^\alpha}{\partial x^\beta}\right)$, то есть

$$\varphi^\star f(y)dy_1\wedge\ldots\wedge dy_n = f(y(x))\cdot\det\left(\frac{\partial y^\alpha}{\partial x^\beta}\right)dx_1\wedge\ldots\wedge dx_n$$

Связь прямого и обратного образа с производной Ли (???)

(хз прямо пока что, Володя точно знает)

4.6 cohomologies and Similar Constructs

(потом буду разбирать, пока не актуально. мб тут слишком подробно для первой части, скорее всего в другие части перекину эту теорию, тут самую суть оставлю только!)

4.6.1 On the Construction of Cohlomologies

(пока выгрузки)

Суть (??)

Лемма Пуанкаре

(тут про неё подробненько)

Комплекс де Рама

Важным свойством внешнего дифференциала d является его нильпотентность:

$$d^2 = 0.$$

Рассмотрим дифференциальную форму $\omega\in\Omega^k(M)$ на некотором многообразии M.

Форма ω называется замкнутой, если $d\omega = 0$.

Форма ω называется точной, если для дифференциальной формы $\omega\in\Omega^k(M)$ найдётся форма $\lambda\in\Omega^{k-1}(M)$, для которой $d\lambda = \omega$.

Очевидно, что всякая точная форма является замкнутой (это мгновенно следует из свойства нильпотентности оператора внешнего дифференцирования).

Однако не всякая точна. Для того, чтобы понять это, рассмотрим 1-форму

$$\omega = \frac{xdy - ydx}{x^2+y^2}$$

на её естественной области определения: $\mathbb{R}^2\backslash\{0,0\}$. Покажем, что ω замкнута:

$$d\omega = \frac{y^2 - x^2}{\left(x^2+y^2\right)^2} dx \wedge dy - \frac{x^2 - y^2}{\left(x^2+y^2\right)^2} dy \wedge dx = 0$$

Если бы 1-форма ω была точна, то существовала бы функция f, такая что $\omega = df$. В этом случае интеграл от df по любой замкнутой кривой γ, лежащей в $\mathbb{R}^2\backslash\{0,0\}$ согласно формуле Ньютона-Лейбница равнялся бы нулю:

$$\int\limits_{\gamma} \omega = \int\limits_{\gamma} df = 0$$

Возьмём в качестве замкнутой кривой положительно ориентированную единичную окружность $\gamma = \mathbb{S}^1$с центром в начале координат и найдём интеграл от дифференциальной 1-формы ω. Ограничение формы ω на $\mathbb{S}^1$, очевидно, равно

$$\omega|_{\mathbb{S}^1} = xdy - ydx$$

Перепишем последнюю формулу в полярных координатах:

$$\omega|_{\mathbb{S}^1} = \left(\cos^2\phi + \sin^2\phi\right) d\phi = d\phi$$

Тогда интеграл (2.54):

$$\int\limits_{\mathbb{S}^1} \omega = \int\limits_{\mathbb{S}^1} d\phi = 2\pi \neq 0 = \int\limits_{\mathbb{S}^1} df$$

То есть, мы пришли к противоречию. Значит, 1-форма ω не является точной на своей естественной области определения: $\mathbb{R}^2\backslash\{0,0\}$.

Этот пример показывает нам, что не всякая замкнутая форма является точпой па мпогообразии M. Причём, всё зависит пе только от самой формы, но и от топологии многообразия. Если многообразие M представляет собой просто евклидово пространство $M = \mathbb{R}^n$, то согласно лемме Пуанкаре любая замкнутая форма на $\mathbb{R}^n$ будет обязательно точной. Если же топология многообразия более интересная и сложная, то такого однозначного утверждения (типа леммы Пуанкаре) сделать уже нельзя.

Рассмотрим гладкое многообразие M. Множество замкнутых дифференциальных форм степени k на M обозначим через $\mathrm{Z}^k(M)$. Формально можно записать, что

$$\mathrm{Z}^k(M) = \operatorname{Ker} d, \quad d : \Omega^k(M) \to \Omega^{k+1}(M)$$

Множество точных форм обозначим через $\mathrm{B}^k(M)$:

$$\mathrm{B}^k(M) = \operatorname{Im} d, \quad d : \Omega^{k-1}(M) \to \Omega^k(M)$$

Отклассифицируем замкнутые формы на многообразии M следующим образом: две замкнутые формы $\omega, \omega' \in \mathrm{Z}^k(M)$ называются когомологичными, если они отличаются на точную форму, то есть их разность $\omega - \omega' \in \mathrm{B}^k(M)$. Это определение порождает отношение эквивалентности на множестве замкнутых форм в $\Omega^k(M)$:

$$\omega \sim \omega' \Longleftrightarrow \omega = \omega' + \lambda, \quad \lambda \in \mathrm{B}^k(M)$$

Когомологичный класс

Когомологичным классом $[\omega]$ формы ω называется множество всех замкнутых форм, отличающихся от ω на точную форму.

В связи с вышесказанным, естественно дать следующее определение. Когомологии де Рама H^k_{dR} гладкого многообразия M - это факторпространства

$$H^k_{\mathrm{dR}}(M) = \left(\operatorname{Ker} d : \Omega^k(M) \to \Omega^{k|1}(M)\right) / d\Omega^{kk}(M)$$

При $k = 0$ в этом определении мы считаем $d\Omega^{-1}(M)$ нулевым пространством и на самом деле рассматриваем функции $f \in \Omega^0(M) = C^\infty(M)$, такие что $df = 0$ - то есть локально постоянные функции.

О градуировке когомологий

Когомологии де Рама градуированны, то есть распадаются в прямую сумму k-мерных когомологий

$$H^\star(M) = \bigoplus_{k \geqslant 0} H^k_{\mathrm{dR}}(M)$$

Кроме того, пространство когомологий является абелевой группой с законом сложения

$$[\omega_1] + [\omega_2] = [\omega_1 + \omega_2]$$

причём k-мерные когомологии де Рама, очевидно, образуют подгруппу $H^k_{\mathrm{dR}}(M)$, а операция внешнего умножения дифференциальных форм порождает умножение на группе когомологий:

$$[\omega_1] \smile [\omega_2] = [\omega_1 \wedge \omega_2]$$

которое ассоциативно и обладает следующими свойствами:

$$H^k_{\mathrm{dR}}(M) \smile H^m_{\mathrm{dR}}(M) \subset H^{k+m}_{\mathrm{dR}}(M)$$

$$a \smile b = (-1)^{km} b \smile a, \quad a \in H^k_{\mathrm{dR}}(M), b \in H^m_{\mathrm{dR}}(M)$$

Действительно, докажем корректность определённой таким образом операции умножения (алгебраические свойства автоматически вытекают из свойств внешнего умножения дифференциальных форм). Для этого необходимо показать, что класс эквивалентности $[\omega_1] \smile [\omega_2]$ не зависит от выбора замкнутых форм ω_1 и ω_2

$$\begin{aligned}(\omega_1 + d\alpha_1) \wedge (\omega_2 + d\alpha_2) &= \omega_1 \wedge \omega_2 + \omega_1 \wedge d\alpha_2 + d\alpha_1 \wedge \omega_2 + d\alpha_1 \wedge d\alpha_2 = \\ &= \omega_1 \wedge \omega_2 + d\left((-1)^k \omega_1 \wedge \alpha_2 + \alpha_1 \wedge \omega_2 + \alpha_1 \wedge d\alpha_2\right) := \omega_1 \wedge \omega_2 + d\lambda\end{aligned}$$

Отсюда получаем требуемое, то есть

$$[(\omega_1 + d\alpha_1) \wedge (\omega_2 + d\alpha_2)] = [\omega_1 \wedge \omega_2] = [\omega_1] \smile [\omega_2]$$

(???? и зачем это нужно????)

О применениях когомологий (!!?)

(пока хз, потом узнаю лучше.)

О когомологиях не де Рама (??)

(хз, есть ли такие вообще)

4.6.2 Examples of Typical Cohlomologies

Когомологии на окружности

хз
и аналогичные вопросы - тоже хз.
потом когда-то буду досматривать.

Когомологии Бетти

(хз, зачем это)
Числа Бетти многообразия M определяются по формуле

$$B_k = \dim H^k_{\mathrm{dR}}(M)$$

Число Бетти может принимать неотрицательные целые значения или бесконечность. Для разумно устроенного конечномерного пространства все числа Бетти конечны и, начиная с некоторого номера, равны нулю. Нетрудно заметить, что нулевое число Бетти

$$B_0 = \dim H^0_{\mathrm{dR}}(M)$$

совпадает с числом компонент связности многообразия M. Так, для любого связного многообразия $B_0 = 1$.

4.6.3 the Eulerian Characteristic

Основная теория

Наконец, удобно ввести эйлерову характеристику многообразия M, которая является гомотопическим инвариантом и определяется как знакопеременная (альтернированная) сумма чисел Бетти, то есть

$$\chi(M) = \sum_k (-1)^k B_k$$

и что?

Характеристика окружности

Найти эйлерову характеристику стандартной единичной окружносmu $\mathbb{S}^1$.
хз

4.6.4 Homologies

пока я не знаю, куда это вставить, так что просто тут эти темы будут висеть и всё.

Гомологии

Двойственность Пуанкаре и потоки

4.6.5 the Fundamental Group

пока буду считать, что они связаны с когомологиями, может, это не так.

Фундаментальная группа Накрытия

Связь фундаментальной группы и одномерных гомологий

4.6.6 Atypical Cohlomologies

(вообще, хз, что это, но заниматься этим я и не буду, ибо пока основы слишком слабые, просто знаю, что какие-то там еще есть)

4.7 Killing Vector

(пока вообще не до них, есть - и хорошо)

4.7.1 Idea of Killing Vector

(???)

4.7.2 Overview of Applications of Killing Fields (???)

(хз)

5 Surfaces and curves

Приведем полную теорию поверхностей и кривых.
(у иванова с мехмата скорее всего возьму потом)

5.1 Coordinate System

(мб в первую часть перекину, пока тут)

Сферические координаты

Обсудим в деталях сферические координаты.

Основные формулы сферической системы координат

Базисные векторы имеют вид:

$$\mathbf{e}_r = \frac{\partial \mathbf{r}}{\partial r}, \quad \mathbf{e}_\vartheta = \frac{1}{r}\frac{\partial \mathbf{r}}{\partial \vartheta}, \quad \mathbf{e}_\varphi = \frac{1}{r\sin\vartheta}\frac{\partial \mathbf{r}}{\partial \varphi}$$

$$v_r = (\mathbf{v}, \mathbf{e}_r) = \dot{x}\sin\vartheta\cos\varphi + \dot{y}\sin\vartheta\sin\varphi + \dot{z}\cos\vartheta$$

$$\begin{cases} \dot{x} = \dot{r}\sin\vartheta\cos\varphi + r\dot{\vartheta}\cos\vartheta\cos\varphi - r\dot{\varphi}\sin\vartheta\sin\varphi \\ \dot{y} = \dot{r}\sin\vartheta\sin\varphi + r\vartheta\cos\vartheta\sin\varphi + r\dot{\varphi}\sin\vartheta\cos\varphi \\ \dot{z} = \dot{r}\cos\vartheta - r\dot{\vartheta}\sin\vartheta \end{cases}$$

$$v_r = (\mathbf{v}\cdot\mathbf{e}_r) = (\dot{r}\sin\vartheta\cos\varphi + r\dot{\vartheta}\cos\vartheta\cos\varphi - r\dot{\varphi}\sin\vartheta\sin\varphi)\sin\vartheta\cos\varphi + (\dot{r}\sin\vartheta\sin\varphi + \\ + r\dot{\vartheta}\cos\vartheta\sin\varphi + r\dot{\varphi}\sin\vartheta\cos\varphi)\sin\vartheta\sin\varphi + (\dot{r}\cos\vartheta - r\dot{\vartheta}\sin\vartheta)\cos\vartheta = \dot{r}$$

$$v_r = \dot{r}, v_\varphi = r\dot{\varphi}\sin\vartheta, v_\vartheta = r\dot{\vartheta}$$

$$\ddot{x} = \left(\ddot{r} - r\left(\dot{\vartheta}^2 + \dot{\varphi}^2\right)\right)\sin\vartheta\cos\varphi + (2\dot{r}\dot{\vartheta} + r\ddot{\vartheta})\cos\vartheta\cos\varphi - (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\sin\vartheta\sin\varphi - 2r\dot{\varphi}\dot{v}\cos\vartheta\sin\varphi$$

$$rd^2\omega = \sum_{\alpha_1<\ldots<\alpha_k,\alpha\neq\beta} \frac{\partial^2\omega_{\alpha_1\ldots\alpha_k}}{\partial x^\alpha \partial x^\beta} dx^\alpha \wedge dx^\beta \wedge dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_k} =$$

$$\ddot{y} = \left(\ddot{r} - r\left(\dot{\vartheta}^2 + \dot{\varphi}^2\right)\right)\sin\vartheta\sin\varphi + (2\dot{r}\dot{\vartheta} + r\ddot{\vartheta})\cos\vartheta\sin\varphi + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\sin\vartheta\cos\varphi +$$
$$+2r\dot{\varphi}\dot{v}\cos\vartheta\cos\varphi$$

$$\ddot{z} = \left(\ddot{r} - r\dot{\vartheta}^2\right)\cos\vartheta - (2\dot{r}\dot{\vartheta} + r\ddot{\vartheta})\sin\vartheta$$
$$w_r = \ddot{r} - r\dot{\vartheta}^2 - r\dot{\varphi}^2\sin^2\vartheta,$$
$$w_\vartheta = (2\dot{r}\dot{\vartheta} + r\ddot{\vartheta}) - r\dot{\varphi}^2\sin\vartheta\cos\vartheta$$

$$w_r = \ddot{r} - r\dot{\varphi}^2, \quad w_\varphi = 2\dot{r}\dot{\varphi} + r\ddot{\varphi}$$

Признаки применения сферической системы координат

(?? пока не могу четко сказать, где нужны они)

Параболические координаты

Основные формулы параболической системы координат

Параболические координаты через декартовы имеют вид:

$$\begin{cases} x = \sqrt{\kappa\eta}\cos\varphi \\ y = \sqrt{\kappa\eta}\sin\varphi \\ z = \dfrac{1}{2}(\kappa - \eta) \end{cases}$$

Из цилиндрических в параболические переход имеет вид:

$$\begin{cases} \rho = \sqrt{\kappa\eta} \\ \varphi = \varphi \\ z = \dfrac{1}{2}(\kappa - \eta) \end{cases}$$

$$\mathbf{e}_\kappa = \frac{1}{H_\kappa}\frac{\partial\mathbf{r}}{\partial\kappa}, \quad \mathbf{e}_\eta = \frac{1}{H_\eta}\frac{\partial\mathbf{r}}{\partial\eta}, \quad \mathbf{e}_\varphi = \frac{1}{H_\varphi}\frac{\partial\mathbf{r}}{\partial\varphi}$$

$$\frac{\partial\mathbf{r}}{\partial\kappa} = \frac{\partial\mathbf{r}}{\partial x}\frac{\partial x}{\partial\kappa} + \frac{\partial\mathbf{r}}{\partial y}\frac{\partial y}{\partial\kappa} + \frac{\partial\mathbf{r}}{\partial z}\frac{\partial z}{\partial\kappa} = \mathbf{e}_x\frac{\partial x}{\partial\kappa} + \mathbf{e}_y\frac{\partial y}{\partial\kappa} + \mathbf{e}_z\frac{\partial z}{\partial\kappa}$$

$$\frac{\partial x}{\partial\kappa} = \frac{x}{2\kappa}, \quad \frac{\partial y}{\partial\kappa} = \frac{y}{2\kappa}, \quad \frac{\partial z}{\partial\kappa} = \frac{1}{2}$$

$$\frac{\partial\mathbf{r}}{\partial\kappa} = \frac{1}{2}\left[\mathbf{e}_x\frac{x}{\kappa} + \mathbf{e}_y\frac{y}{\kappa} + \mathbf{e}_z\right] = \frac{1}{2}\left[\mathbf{e}_x\sqrt{\frac{\eta}{\kappa}}\cos\varphi + \mathbf{e}_y\sqrt{\frac{\eta}{\kappa}}\sin\varphi + \mathbf{e}_z\right]$$

$$\frac{\partial x}{\partial\eta} = \frac{x}{2\eta}, \frac{\partial y}{\partial\eta} = \frac{y}{2\eta}, \frac{\partial z}{\partial\eta} = -\frac{1}{2}, \quad \frac{\partial\mathrm{r}}{\partial\eta} = \frac{1}{2}\left[\mathbf{e}_x\sqrt{\frac{\kappa}{\eta}}\cos\varphi + \mathbf{e}_y\sqrt{\frac{\kappa}{\eta}}\sin\varphi - \mathbf{e}_z\right]$$

Коэффициенты Ламе параболической системы координат

$$
\begin{aligned}
H_\kappa =& \sqrt{\left(\frac{\partial x}{\partial \kappa}\right)^2 + \left(\frac{\partial y}{\partial \kappa}\right)^2 + \left(\frac{\partial z}{\partial \kappa}\right)^2} = \frac{1}{2}\sqrt{1+\frac{\eta}{\kappa}} \\
H_\eta =& \sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2 + \left(\frac{\partial z}{\partial \eta}\right)^2} = \frac{1}{2}\sqrt{1+\frac{\kappa}{\eta}} \\
H_\varphi =& \sqrt{\left(\frac{\partial x}{\partial \varphi}\right)^2 + \left(\frac{\partial y}{\partial \varphi}\right)^2 + \left(\frac{\partial z}{\partial \varphi}\right)^2} = \sqrt{\kappa\eta}
\end{aligned}
$$

(?? откуда??)

$$
\begin{aligned}
\mathbf{e}_x =& e_x\sqrt{\frac{\eta}{\kappa+\eta}}\cos\varphi + \mathbf{e}_y\sqrt{\frac{\eta}{\kappa+\eta}}\sin\varphi + \mathbf{e}_z\sqrt{\frac{\kappa}{\kappa+\eta}} \\
\mathbf{e}_\eta =& \mathbf{e}_x\sqrt{\frac{\kappa}{\kappa+\eta}}\cos\varphi + \mathbf{e}_y\sqrt{\frac{\kappa}{\kappa+\eta}}\sin\varphi - \mathbf{e}_z\sqrt{\frac{\eta}{\kappa+\eta}} \\
\mathbf{e}_\varphi =& -\mathbf{e}_x\sin\varphi + \mathbf{e}_y\cos\varphi
\end{aligned}
$$

$$
A = \begin{pmatrix} \sqrt{\frac{\eta}{\kappa+\eta}}\cos\varphi & \sqrt{\frac{\kappa}{\kappa+\eta}}\cos\varphi & -\sin\varphi \\ \sqrt{\frac{\eta}{\kappa+\eta}}\sin\varphi & \sqrt{\frac{\kappa}{\kappa+\eta}}\sin\varphi & \cos\varphi \\ \sqrt{\frac{\kappa}{\kappa+\eta}} & -\sqrt{\frac{\eta}{\kappa+\eta}} & 0 \end{pmatrix}
$$

$$
AA^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \mathbf{1}
$$

$$
\begin{cases}
\dot{x} = \frac{\dot{k}}{2}\sqrt{\frac{\eta}{\kappa}}\cos\varphi + \frac{\dot{\eta}}{2}\sqrt{\frac{\kappa}{\eta}}\cos\varphi - \dot{\varphi}\sqrt{\kappa\eta}\sin\varphi \\
\dot{y} = \frac{\dot{r}}{2}\sqrt{\frac{\eta}{\kappa}}\sin\varphi + \frac{\eta}{2}\sqrt{\frac{\kappa}{\eta}}\sin\varphi + \dot{\varphi}\sqrt{\kappa\eta}\cos\varphi \\
\dot{z} = \frac{1}{2}(\dot{\kappa} - \dot{\eta})
\end{cases}
$$

$$
\begin{aligned}
\mathbf{e}_\kappa =& \left(\sqrt{\frac{\eta}{\kappa+\eta}}\cos\varphi \quad \sqrt{\frac{\eta}{\kappa+\eta}}\sin\varphi \quad \sqrt{\frac{\kappa}{\kappa+\eta}}\right) \\
\mathbf{e}_\eta =& \left(\sqrt{\frac{\kappa}{\kappa+\eta}\cos\varphi}\sqrt{\frac{\kappa}{\kappa+\eta}\sin\varphi} - \sqrt{\frac{\eta}{\kappa+\eta}}\right), \\
\mathbf{e}_\varphi =& (-\sin\varphi \quad \cos\varphi \quad 0)
\end{aligned}
$$

(?? зачем то, что ниже??)

$$
\begin{aligned}
v_\kappa =& (\mathbf{v}\cdot\mathbf{e}_\kappa) = \frac{\dot{\kappa}}{2}\sqrt{1+\frac{\eta}{\kappa}}, \\
v_\eta =& \frac{\dot{\eta}}{2}\sqrt{1+\frac{\kappa}{\eta}}, \\
v_\varphi =& \dot{\varphi}\sqrt{\kappa\eta}
\end{aligned}
$$

Признаки применения параболической системы координат (??)

(?? пока хз, где нужны они)
где-то мб нужны в механике.

5.2 Riemannian Manifolds

(вообще хз, что о них известно)

определение риманово многообразия

5.3 Curve

параметры кривых

способы задания кривых - обзор

$$\begin{cases} x = f_1\left(x^1, x^2, x^3\right) \\ y = f_2\left(x^1, x^2, x^3\right) \\ z = f_3\left(x^1, x^2, x^3\right) \end{cases}$$

локальный базис

то се, коэффициенты Ламе
$\mathbf{e}_\mu = \frac{1}{H_\mu}\frac{\partial \mathbf{r}}{\partial x^\mu}$,

коэффициенты Ламе - обзор

$H_\mu = \left|\frac{\partial \mathbf{r}}{\partial x^\mu}\right|$

5.3.1 curve Definitions

мотивация изучать кривые

(????)
Кривые нужны для метрик и работы с формами, и вообще по мехмату начинается все с этой темы.

кривые в $\mathbb{R}^n$ (??)

работа идет в $\mathbb{R}^n$.
в этом весь интерес. поэтому нужно вводить то, что для $\mathbb{R}^3$ было очевидно.

определение $\mathbb{R}^n$

начинаем с определения $\mathbb{R}^n$ - н мерное евклидово пространство, то есть множество упорядоченных н вещественных чисел, на котором введена
1) естественная структура линейного пространства и
2) стандартное скалярное произведение.
говорят, что определенные векторы образуют в нем базис.

а еще, что скалярное произведение порождает норму, и далее определяется функция расстояния между двумя элементами и далее можно определить шары, сферы.

определение параметрической кривой

Определяется параметрическая кривая.

Определение 5.1. *регулярная кривая, сингулярные точки, гладкая кривая*

способы задания кривых

если у нас зависимость от параметра, то это множество называется кривой-графиком.
в общем, тут 2 варианта, как задавать это все можно, второй вариант - через неявные кривые.
и типа множество решений такой-то системы уравнений - неявная кривая.
в общем, на этом этапе нужно понимать, что у нас 3 типа задания кривой, и нужно сказать пару слов об эквивалентности.

натуральная параметризация

5.3.2 length and Characteristics

определение длины и формулы для разных случаев

Определение 5.2. *длина - это* $\int\limits_a^b \|\dot{\gamma(t)}\|dt$ $\gamma : [a, b] - \mathbb{R}^n$.

натуральный параметр

вектор ускорения - свойства

Свойства вектора ускорения следующие:

Теорема 2. *в каждой точке вектор ускорения перпендикуляен скорости.*

(я пока не пришел. как это связано с ускорением и скоростью в физике?)
возможно, тут дело в том, что именно от параметризации зависит перпендикулярность.
но в общем, простые разделы, научились мы длину кривой считать.

5.3.3 Curvature

после длины логично перейти к форме.
Определим кривизну.

Определение 5.3. *кривизна* $k() = \|\ddot{\gamma(s)}\|$

Определение 5.4. *радиус кривизны* - $R(s) = 1/k$

свойства кривизны (???)

дальше прикольные формулы про кривизну, которые позже обязательно напишу и пойму
ну и явный вид кривизны для $\mathbb{R}^2$.

5.3.4 flat Curves

репер Френе

формулы френе

Определение 5.5. *формулы френе*

5.3.5 spatial Curves

и это самое интересное.

Формулы Френеля

5.3.6 complements and Tasks About Flat Curves

я позже порешаю их.
и про механику тут можно на самом деле многое написать.
и там связка с интегралом френеля, вообще интересное очень место!!!!!
Также в этом разделе много примеров.

5.3.7 curves in Space

вот тут спешить вообще не нужно, нужно четко усвоить формулы френе, потому что они главные при работе с кривыми, а кривые прямо много где появляются.

5.4 Description of Surfaces

параметрическая поверхность - определение

Определение 5.6. *Параметрическая поверхность размерности к в н мерном пространстве* $\mathbb{R}^n$ *- это непрерывное отображение из области в к мерном подпространстве в* $\mathbb{R}^n$.

что кстати заставляет задуматься, что поверхность - отображение.
(прямо такое место задуматься кстати, тут как в многообразиях, только координатно, ну и интересно конечно очень)

касательный вектор - определение

Определение 5.7. *касательный вектор определяем*

5.4.1 second Fundamental Shape of Surfaces

суть второй фундаментальной формы

???

конструкция

???

5.4.2 Riemann Curvature Tensor for Surfaces

(?? хз, почему он в этом разделе, но пока выгрузки про него тут будут)
судя по [**?**] это частный случай тензора кривизны.

смысл тензора кривизны Римана (???)

получение тензора кривизны Римана

Главная идея в том, что нам нужно получить значение коммутатора двух ковариантных производных:

$$[\nabla_\alpha, \nabla_\beta]a_\gamma = a_\lambda R^\lambda_{\alpha\gamma\beta}$$

в декартовых координатах ковариантная производная сводится к частной производной: $\nabla_\alpha \to \partial_\alpha$

Поэтому в Евклидовом пространстве: $\nabla_\alpha a_\gamma \to \partial_\alpha a_\gamma$ и $\nabla_\alpha \nabla_\beta a_\Gamma \to \partial_\alpha \partial_\beta a_\gamma$

поэтому если построить: $(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha)a_\Gamma = 0$ так как частное дифференцирование перестановочно.

И если так происходит во всем пространстве тождественно, то значит, мы имеем дело с Евклидовым пространством.

А если где-то отлично от нуля, то значит силы инерции будут отличны от нуля.

Посмотрим в криволинейном пространстве:

$$\nabla_\beta = \partial_\beta a_\gamma - a_\lambda \Gamma^\lambda_{\beta\gamma}$$

посмотрим, как действует $(\nabla_\alpha \nabla_\beta a_\gamma)$

так как?????? (?? тут произведение кристоффелей нужно бы знать судя по всему) $\nabla_\alpha t_{\beta\gamma} = \partial_\alpha t_{\beta\gamma} - \Gamma^{\lambda'}_{\alpha\beta} t_{jj} - \Gamma^\lambda_{\gamma\alpha} t_{\beta\lambda'}$, то

$$(\nabla_\alpha \nabla_\beta a_\gamma) = \partial_\alpha(\partial_\beta a_\Gamma - a_\lambda \Gamma^\lambda_{\beta\Gamma}) - \Gamma^{\lambda'}_{\alpha\beta}(\partial_{\lambda'} a_\gamma - a_\lambda \Gamma^\lambda_{\lambda'\gamma}) - \Gamma^{\lambda'}_{\alpha\gamma}(\partial_\beta a_{\lambda'} - a_\lambda \Gamma^\lambda_{\beta\lambda'})$$

и то же самое:

$$(\nabla_\beta \nabla_\alpha a_\gamma) = \partial_\beta(\partial_\alpha a_\gamma - a_\lambda \Gamma^\lambda_{\alpha\gamma}) - \Gamma^{\lambda'}_{\alpha\beta}(\partial_{\lambda'} a_\gamma - a_\lambda \Gamma^\lambda_{\lambda'\gamma}) - \Gamma^{\lambda'}_{\beta\gamma}(\partial_\alpha a_{\lambda'} - a_\lambda \Gamma^\lambda_{\alpha\lambda'})$$

поэтому после разницы выражений выше у нас каждое слагаемое с каждым сокращается, и в итоге получаем 0.

Поэтому

$$[\nabla_\alpha, \nabla_\beta]a_\Gamma = a_\lambda) - \partial_\alpha \Gamma^\lambda_{\beta\Gamma} + \partial_\beta \Gamma^\lambda_{\alpha\Gamma} + \Gamma^{\lambda'}_{\alpha\Gamma}\Gamma^\lambda_{\beta\lambda'} - ....)$$

и это ковариантный тензор 3го ранга.

введем тензор кривизны Римана:

$$R^\lambda_{\alpha\gamma\beta}$$

таким образом, если тензор Римана равен нулю, то это Евклидово пространство, а если отличен от нуля, то

$$[\nabla_\alpha, \nabla_\beta]a_\Gamma = a_\lambda R^\lambda_{\alpha\Gamma\beta}$$

и мы получаем Риманово пространство.

Свойства тензора кривизны Римана

потом впишу

геометрический смысл тензора кривизны Римана

рассмторим параллельны перенос...
см семинар Киселева, потом.

5.5 Representative Surfaces:

Для полноты приведем описание поверхностей, которые мне встречались по жизни.
(тут решаю задачи и вписываю свойства того, что наисследовал)

Многомерные параллелограммы (?!!!)

(тут потренирую формулы выше. пока опыта слишком мало для того, чтобы зарешать задачи эти. что-то с внешним произведением связано..)

Площадь 3-мерной поверхности в 3-мерном пространстве

давно интересно это было бы научиться считать.

Площадь 2-мерного параллелограмма в 3-мерном пространстве

Площадь 3-мерной поверхности в 4-мерном пространстве

давно интересно это было бы научиться считать.

Площадь 2-мерного параллелограмма в 4-мерном пространстве

Многомерные сферы

Площадь 3-мерной сферы в 4-мерном пространстве

Поверхность Бельтрами

(потом потренируюсь)

Свойства

Приведем свойства, которые ниже будет доказывать.

Обзор применений поверхности Бельтрами

хз

6 1 Euclidean Spaces

6.0.1 1 Euclidean Spaces

The Euclidean space $\mathbb{R}^n$ is the prototype of all manifolds. Not only is it the simplest, but locally every manifold looks like $\mathbb{R}^n$. A good understanding of $\mathbb{R}^n$ is essential in generalizing differential and integral calculus to a manifold.

Euclidean space is special in having a set of standard global coordinates. This is both a blessing and a handicap. It is a blessing because all constructions on $\mathbb{R}^n$ can be defined in

terms of the standard coordinates and all computations carried out explicitly. It is a handicap because, defined in terms of coordinates, it is often not obvious which concepts are intrinsic, i.e., independent of coordinates. Since a manifold in general does not have standard coordinates, only coordinate-independent concepts will make sense on a manifold. For example, it turns out that on a manifold of dimension n, it is not possible to integrate functions, because the integral of a function depends on a set of coordinates. The objects that can be integrated are differential forms. It is only because the existence of global coordinates permits an identification of functions with differential n-forms on $\mathbb{R}^n$ that integration of functions becomes possible on $\mathbb{R}^n$.

Our goal in this chapter is to recast calculus on $\mathbb{R}^n$ in a coordinate-free way suitable for generalization to manifolds. To this end, we view a tangent vector not as an arrow or as a column of numbers, but as a derivation on functions. This is followed by an exposition of Hermann Grassmann's formalism of alternating multilinear functions on a vector space, which lays the foundation for the theory of differential forms. Finally, we introduce differential forms on $\mathbb{R}^n$, together with two of their basic operations, the wedge product and the exterior derivative, and show how they generalize and simplify vector calculus in $\mathbb{R}^3$.

6.1 §1 Smooth Functions on a Euclidean Space

The calculus of C^∞ functions will be our primary tool for studying higher-dimensional manifolds. For this reason, we begin with a review of C^∞ functions on $\mathbb{R}^n$.

6.1.1 1.1C^∞ Versus Analytic Functions

Write the coordinates on $\mathbb{R}^n$ as $x^1, \ldots, x^n$ and let $p = (p^1, \ldots, p^n)$ be a point in an open set U in $\mathbb{R}^n$. In keeping with the conventions of differential geometry, the indices on coordinates are superscripts, not subscripts. An explanation of the rules for superscripts and subscripts is given in Subsection 4.7.

Definition 1.1. Let k be a nonnegative integer. A real-valued function $f : U \to \mathbb{R}$ is said to be C^k at $p \in U$ if its partial derivatives

$$\frac{\partial^j f}{\partial x^{i_1} \cdots \partial x^{i_j}}$$

of all orders $j \leq k$ exist and are continuous at p. The function $f : U \to \mathbb{R}$ is C^∞ at p if it is C^k for all $k \geq 0$; in other words, its partial derivatives $\partial^j f / \partial x^{i_1} \cdots \partial x^{i_j}$ of all orders exist and are continuous at p. A vector-valued function $f : U \to \mathbb{R}^m$ is said to be C^k at p if all of its component functions $f^1, \ldots, f^m$ are C^k at p. We say that $f : U \to \mathbb{R}^m$ is C^k on U if it is C^k at every point in U. A similar definition holds for a C^∞ function on an open set U. We treat the terms " C^∞ " and "smooth" as synonymous.

Example 1.2.

(i) A C^0 function on U is a continuous function on U.

(ii) Let $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x^{1/3}$. Then

$$f'(x) = \begin{cases} \frac{1}{3}x^{-2/3} & \text{for } x \neq 0 \\ \text{undefined} & \text{for } x = 0 \end{cases}$$

Thus the function f is C^0 but not C^1 at $x = 0$.

(iii) Let $g : \mathbb{R} \to \mathbb{R}$ be defined by

$$g(x)=\int_0^x f(t)dt=\int_0^x t^{1/3}dt=\frac{3}{4}x^{4/3}$$

Then $g'(x)=f(x)=x^{1/3}$, so $g(x)$ is C^1 but not C^2 at $x=0$. In the same way one can construct a function that is C^k but not C^{k+1} at a given point.
(iv) The polynomial, sine, cosine, and exponential functions on the real line are all C^∞.

A neighborhood of a point in $\mathbb{R}^n$ is an open set containing the point. The function f is real-analytic at p if in some neighborhood of p it is equal to its Taylor series at p :

$$\begin{aligned} f(x)=f(p)+\sum_i \frac{\partial f}{\partial x^i}(p)\left(x^i-p^i\right)+\frac{1}{2!}\sum_{i,j}\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\left(x^i-p^i\right)\left(x^j-p^j\right) \\ +\cdots+\frac{1}{k!}\sum_{i_1,\ldots,i_k}\frac{\partial^k f}{\partial x^{i_1}\cdots\partial x^{i_k}}(p)\left(x^{i_1}-p^{i_1}\right)\cdots\left(x^{i_k}-p^{i_k}\right)+\cdots, \end{aligned}$$

in which the general term is summed over all $1\le i_1,\ldots,i_k\le n$.
A real-analytic function is necessarily C^∞, because as one learns in real analysis, a convergent power series can be differentiated term by term in its region of convergence. For example, if

$$f(x)=\sin x=x-\frac{1}{3!}x^3+\frac{1}{5!}x^5-\cdots,$$

then term-by-term differentiation gives

$$f'(x)=\cos x=1-\frac{1}{2!}x^2+\frac{1}{4!}x^4-\cdots.$$

The following example shows that a C^∞ function need not be real-analytic. The idea is to construct a C^∞ function $f(x)$ on $\mathbb{R}$ whose graph, though not horizontal, is "very flat" near 0 in the sense that all of its derivatives vanish at 0 .

Fig. 1.1. A $C^{\infty}
$ function all of whose derivatives vanish at 0 .}

Example 1.3 (A C^∞ function very flat at 0). Define $f(x)$ on $\mathbb{R}$ by

$$f(x)=\begin{cases} e^{-1/x} & \text{for } x>0, \\ 0 & \text{for } x\le 0. \end{cases}$$

(See Figure 1.1.) By induction, one can show that f is C^∞ on $\mathbb{R}$ and that the derivatives $f^{(k)}(0)$ are equal to 0 for all $k\ge 0$ (Problem 1.2).

The Taylor series of this function at the origin is identically zero in any neighborhood of the origin, since all derivatives $f^{(k)}(0)$ equal 0 . Therefore, $f(x)$ cannot be equal to its Taylor series and $f(x)$ is not real-analytic at 0 .

6.1.2 1.2 Taylor's Theorem with Remainder

Although a C^∞ function need not be equal to its Taylor series, there is a Taylor's theorem with remainder for C^∞ functions that is often good enough for our purposes. In the lemma below, we prove the very first case, in which the Taylor series consists of only the constant term $f(p)$.

We say that a subset S of $\mathbb{R}^n$ is star-shaped with respect to a point p in S if for every x in S, the line segment from p to x lies in S (Figure 1.2).

Fig. 1.2. Star-shaped with respect to p, but not with respect to q.

Lemma 1.4 (Taylor's theorem with remainder). Let f be a C^∞ function on an open subset U of $\mathbb{R}^n$ star-shaped with respect to a point $p = (p^1, \dots, p^n)$ in U. Then there are functions $g_1(x), \dots, g_n(x) \in C^\infty(U)$ such that

$$f(x) = f(p) + \sum_{i=1}^{n} \left(x^i - p^i\right) g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Proof. Since U is star-shaped with respect to p, for any x in U the line segment $p + t(x - p), 0 \le t \le 1$, lies in U (Figure 1.3). So $f(p + t(x - p))$ is defined for $0 \le t \le 1$.

Fig. 1.3. The line segment from p to x.

By the chain rule,

$$\frac{d}{dt} f(p+t(x-p)) = \sum \left(x^i - p^i\right) \frac{\partial f}{\partial x^i}(p+t(x-p)).$$

If we integrate both sides with respect to t from 0 to 1 , we get

$$f(p+t(x-p))]_0^1 = \sum \left(x^i - p^i\right) \int_0^1 \frac{\partial f}{\partial x^i}(p+t(x-p))dt. \tag{1.1}$$

Let

$$g_i(x) = \int_0^1 \frac{\partial f}{\partial x^i}(p+t(x-p))dt$$

Then $g_i(x)$ is C^∞ and (1.1) becomes

$$f(x) - f(p) = \sum \left(x^i - p^i\right) g_i(x).$$

Moreover,

$$g_i(p) = \int_0^1 \frac{\partial f}{\partial x^i}(p)dt = \frac{\partial f}{\partial x^i}(p).$$

□

In case $n = 1$ and $p = 0$, this lemma says that

$$f(x) = f(0) + xg_1(x)$$

for some C^∞ function $g_1(x)$. Applying the lemma repeatedly gives

$$g_i(x) = g_i(0) + xg_{i+1}(x),$$

where g_i, g_{i+1} are C^∞ functions. Hence,

$$\begin{aligned} f(x) =& f(0) + x\left(g_1(0) + xg_2(x)\right) \\ =& f(0) + xg_1(0) + x^2\left(g_2(0) + xg_3(x)\right) \\ & \vdots \\ =& f(0) + g_1(0)x + g_2(0)x^2 + \cdots + g_i(0)x^i + g_{i+1}(x)x^{i+1}. \end{aligned} \tag{1.2}$$

Differentiating (1.2) repeatedly and evaluating at 0 , we get

$$g_k(0) = \frac{1}{k!} f^{(k)}(0), \quad k = 1, 2, \ldots, i.$$

So (1.2) is a polynomial expansion of $f(x)$ whose terms up to the last term agree with the Taylor series of $f(x)$ at 0 .

Remark. Being star-shaped is not such a restrictive condition, since any open ball

$$B(p, \varepsilon) = \{x \in \mathbb{R}^n \mid \|x - p\| < \varepsilon\}$$

is star-shaped with respect to p. If f is a C^∞ function defined on an open set U containing p, then there is an $\varepsilon > 0$ such that

$$p \in B(p, \varepsilon) \subset U.$$

When its domain is restricted to $B(p, \varepsilon)$, the function f is defined on a star-shaped neighborhood of p and Taylor's theorem with remainder applies.

NOTATION. It is customary to write the standard coordinates on $\mathbb{R}^2$ as x, y, and the standard coordinates on $\mathbb{R}^3$ as x, y, z.

6.1.3 Problems

1.1. A Function That is c^2 but not C^3

Let $g : \mathbb{R} \to \mathbb{R}$ be the function in Example 1.2(iii). Show that the function $h(x) = \int_0^x g(t)dt$ is C^2 but not C^3 at $x = 0$.

1.2.* a c^∞ function very flat at 0

Let $f(x)$ be the function on $\mathbb{R}$ defined in Example 1.3.
(a) Show by induction that for $x > 0$ and $k \geq 0$, the k th derivative $f^{(k)}(x)$ is of the form $p_{2k}(1/x)e^{-1/x}$ for some polynomial $p_{2k}(y)$ of degree $2k$ in y.
(b) Prove that f is C^∞ on $\mathbb{R}$ and that $f^{(k)}(0) = 0$ for all $k \geq 0$.

1.3. A Diffeomorphism of an Open Interval with $\diagdown$

Let $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ be open subsets. A C^∞ map $F : U \to V$ is called a diffeomorphism if it is bijective and has a C^∞ inverse $F^{-1} : V \to U$.
(a) Show that the function $f :]-\pi/2, \pi/2[\to \mathbb{R}, f(x) = \tan x$, is a diffeomorphism.
(b) Let a, b be real numbers with $a < b$. Find a linear function $h :]a, b[\to]-1, 1[$, thus proving that any two finite open intervals are diffeomorphic.

The composite $f \circ h :]a, b[\to \mathbb{R}$ is then a diffeomorphism of an open interval with $\mathbb{R}$.
(c) The exponential function $\exp : \mathbb{R} \to]0, \infty[$ is a diffeomorphism. Use it to show that for any real numbers a and b, the intervals $\mathbb{R},]a, \infty[$, and $]-\infty, b[$ are diffeomorphic.

1.4. A Diffeomorphism of an Open Cube with $\mathbb{R}^n$

Show that the map

$$f :]-\frac{\pi}{2}, \frac{\pi}{2}[^n \to \mathbb{R}^n, \quad f(x_1, \ldots, x_n) = (\tan x_1, \ldots, \tan x_n)$$

is a diffeomorphism.

1.5. A Diffeomorphism of an Open Ball with $\mathbb{R}^n$

Let $\mathbf{0} = (0, 0)$ be the origin and $B(\mathbf{0}, 1)$ the open unit disk in $\mathbb{R}^2$. To find a diffeomorphism between $B(\mathbf{0}, 1)$ and $\mathbb{R}^2$, we identify $\mathbb{R}^2$ with the xy-plane in $\mathbb{R}^3$ and introduce the lower open hemisphere

$$S : x^2 + y^2 + (z - 1)^2 = 1, \quad z < 1$$

in $\mathbb{R}^3$ as an intermediate space (Figure 1.4). First note that the map

$$f : B(\mathbf{0}, 1) \to S, \quad (a, b) \mapsto \left(a, b, 1 - \sqrt{1 - a^2 - b^2}\right),$$

is a bijection.
(a) The stereographic projection $g : S \to \mathbb{R}^2$ from $(0, 0, 1)$ is the map that sends a point $(a, b, c) \in S$ to the intersection of the line through $(0, 0, 1)$ and (a, b, c) with the xy-plane. Show that it is given by

$$(a,b,c) \mapsto (u,v) = \left(\frac{a}{1-c}, \frac{b}{1-c}\right), \quad c = 1 - \sqrt{1-a^2-b^2}$$

with inverse

$$(u,v) \mapsto \left(\frac{u}{\sqrt{1+u^2+v^2}}, \frac{v}{\sqrt{1+u^2+v^2}}, 1 - \frac{1}{\sqrt{1+u^2+v^2}}\right)$$

Fig. 1.4. A diffeomorphism of an open disk with $\mathbb{R ^{2}$.}

(b) Composing the two maps f and g gives the map

$$h = g \circ f : B(\mathbf{0},1) \to \mathbb{R}^2, \quad h(a,b) = \left(\frac{a}{\sqrt{1-a^2-b^2}}, \frac{b}{\sqrt{1-a^2-b^2}}\right).$$

Find a formula for $h^{-1}(u,v) = (f^{-1} \circ g^{-1})(u,v)$ and conclude that h is a diffeomorphism of the open disk $B(\mathbf{0},1)$ with $\mathbb{R}^2$.
(c) Generalize part (b) to $\mathbb{R}^n$.

1.6* Taylor's Theorem with Remainder to Order 2

Prove that if $f : \mathbb{R}^2 \to \mathbb{R}$ is C^∞, then there exist C^∞ functions g_{11}, g_{12}, g_{22} on $\mathbb{R}^2$ such that

$$\begin{aligned} f(x,y) = f(0,0) &+ \frac{\partial f}{\partial x}(0,0)x + \frac{\partial f}{\partial y}(0,0)y \\ &+ x^2 g_{11}(x,y) + xy g_{12}(x,y) + y^2 g_{22}(x,y). \end{aligned}$$

1.7.* a Function with a Removable Singularity

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a C^∞ function with $f(0,0) = \partial f/\partial x(0,0) = \partial f/\partial y(0,0) = 0$. Define

$$g(t,u) = \begin{cases} \frac{f(t,tu)}{t} & \text{for } t \neq 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Prove that $g(t,u)$ is C^∞ for $(t,u) \in \mathbb{R}^2$. (Hint: Apply Problem 1.6.)

1.8. Bijective c^∞ maps

Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = x^3$. Show that f is a bijective C^∞ map, but that f^{-1} is not C^∞. (This example shows that a bijective C^∞ map need not have a C^∞ inverse. In complex analysis, the situation is quite different: a bijective holomorphic map $f : \mathbb{C} \to \mathbb{C}$ necessarily has a holomorphic inverse.)

6.2 §2 Tangent Vectors in r^n as Derivations

In elementary calculus we normally represent a vector at a point p in $\mathbb{R}^3$ algebraically as a column of numbers

$$v = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix}$$

or geometrically as an arrow emanating from p (Figure 2.1).

Fig. 2.1. A vector v at p.

Recall that a secant plane to a surface in $\mathbb{R}^3$ is a plane determined by three points of the surface. As the three points approach a point p on the surface, if the corresponding secant planes approach a limiting position, then the plane that is the limiting position of the secant planes is called the tangent plane to the surface at p. Intuitively, the tangent plane to a surface at p is the plane in $\mathbb{R}^3$ that just "touches" the surface at p. A vector at p is tangent to a surface in $\mathbb{R}^3$ if it lies in the tangent plane at p (Figure 2.2).

Fig. 2.2. A tangent vector v to a surface at p.

Such a definition of a tangent vector to a surface presupposes that the surface is embedded in a Euclidean space, and so would not apply to the projective plane, for example, which does not sit inside an $\mathbb{R}^n$ in any natural way.

Our goal in this section is to find a characterization of tangent vectors in $\mathbb{R}^n$ that will generalize to manifolds.

6.2.1 2.1 The Directional Derivative

In calculus we visualize the tangent space $T_p(\mathbb{R}^n)$ at p in $\mathbb{R}^n$ as the vector space of all arrows emanating from p. By the correspondence between arrows and column vectors, the vector space $\mathbb{R}^n$ can be identified with this column space. To distinguish between points and vectors, we write a point in $\mathbb{R}^n$ as $p = (p^1, \ldots, p^n)$ and a vector in the tangent space $T_p(\mathbb{R}^n)$ as

$$v = \begin{bmatrix} v^1 \\ \vdots \\ v^n \end{bmatrix} \quad \text{or} \quad \langle v^1, \ldots, v^n \rangle$$

We usually denote the standard basis for $\mathbb{R}^n$ or $T_p(\mathbb{R}^n)$ by $e_1, \ldots, e_n$. Then $v = \sum v^i e_i$ for some $v^i \in \mathbb{R}$. Elements of $T_p(\mathbb{R}^n)$ are called tangent vectors (or simply vectors) at p in $\mathbb{R}^n$. We sometimes drop the parentheses and write $T_p\mathbb{R}^n$ for $T_p(\mathbb{R}^n)$.

The line through a point $p = (p^1, \ldots, p^n)$ with direction $v = \langle v^1, \ldots, v^n \rangle$ in $\mathbb{R}^n$ has parametrization

$$c(t) = \left(p^1 + tv^1, \ldots, p^n + tv^n\right).$$

Its i th component $c^i(t)$ is $p^i + tv^i$. If f is C^∞ in a neighborhood of p in $\mathbb{R}^n$ and v is a tangent vector at p, the directional derivative of f in the direction v at p is defined to be

$$D_v f = \lim_{t \to 0} \frac{f(c(t)) - f(p)}{t} = \left.\frac{d}{dt}\right|_{t=0} f(c(t)).$$

By the chain rule,

$$D_v f = \sum_{i=1}^n \frac{dc^i}{dt}(0) \frac{\partial f}{\partial x^i}(p) = \sum_{i=1}^n v^i \frac{\partial f}{\partial x^i}(p). \tag{2.1}$$

In the notation $D_v f$, it is understood that the partial derivatives are to be evaluated at p, since v is a vector at p. So $D_v f$ is a number, not a function. We write

$$D_v = \sum v^i \left.\frac{\partial}{\partial x^i}\right|_p$$

for the map that sends a function f to the number $D_v f$. To simplify the notation we often omit the subscript p if it is clear from the context.

The association $v \mapsto D_v$ of the directional derivative D_v to a tangent vector v offers a way to characterize tangent vectors as certain operators on functions. To make this precise, in the next two subsections we study in greater detail the directional derivative D_v as an operator on functions.

6.2.2 2.2 Germs of Functions

A relation on a set S is a subset R of $S \times S$. Given x, y in S, we write $x \sim y$ if and only if $(x, y) \in R$. The relation R is an equivalence relation if it satisfies the following three properties for all $x, y, z \in S$:
(i) (reflexivity) $x \sim x$,
(ii) (symmetry) if $x \sim y$, then $y \sim x$,
(iii) (transitivity) if $x \sim y$ and $y \sim z$, then $x \sim z$.

As long as two functions agree on some neighborhood of a point p, they will have the same directional derivatives at p. This suggests that we introduce an equivalence relation on the C^∞ functions defined in some neighborhood of p. Consider the set of all pairs (f, U), where U is a neighborhood of p and $f : U \to \mathbb{R}$ is a C^∞ function. We say that (f, U) is equivalent to (g, V) if there is an open set $W \subset U \cap V$ containing p such that $f = g$ when restricted to W. This is clearly an equivalence relation because it is reflexive, symmetric, and transitive. The equivalence class of (f, U) is called the germ of f at p. We write $C_p^\infty(\mathbb{R}^n)$, or simply C_p^∞ if there is no possibility of confusion, for the set of all germs of C^∞ functions on $\mathbb{R}^n$ at p.

Example. The functions

$$f(x) = \frac{1}{1 - x}$$

with domain $\mathbb{R} - \{1\}$ and

$$g(x) = 1 + x + x^2 + x^3 + \cdots$$

with domain the open interval $]-1,1[$ have the same germ at any point p in the open interval $]-1,1[$.

An algebra over a field K is a vector space A over K with a multiplication map

$$\mu : A \times A \to A,$$

usually written $\mu(a,b) = a \cdot b$, such that for all $a, b, c \in A$ and $r \in K$,
(i) (associativity) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$,
(ii) (distributivity) $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot (b + c) = a \cdot b + a \cdot c$,
(iii) (homogeneity) $r(a \cdot b) = (ra) \cdot b = a \cdot (rb)$.

Equivalently, an algebra over a field K is a ring A (with or without multiplicative identity) that is also a vector space over K such that the ring multiplication satisfies the homogeneity condition (iii). Thus, an algebra has three operations: the addition and multiplication of a ring and the scalar multiplication of a vector space. Usually we omit the multiplication sign and write ab instead of $a \cdot b$.

A map $L : V \to W$ between vector spaces over a field K is called a linear map or a linear operator if for any $r \in K$ and $u, v \in V$,
(i) $L(u + v) = L(u) + L(v)$;
(ii) $L(rv) = rL(v)$.

To emphasize the fact that the scalars are in the field K, such a map is also said to be K-linear.

If A and A' are algebras over a field K, then an algebra homomorphism is a linear map $L : A \to A'$ that preserves the algebra multiplication: $L(ab) = L(a)L(b)$ for all $a, b \in A$.

The addition and multiplication of functions induce corresponding operations on C_p^∞, making it into an algebra over $\mathbb{R}$ (Problem 2.2).

6.2.3 2.3 Derivations at a Point

For each tangent vector v at a point p in $\mathbb{R}^n$, the directional derivative at p gives a map of real vector spaces

$$D_v : C_p^\infty \to \mathbb{R}.$$

By (2.1), D_v is $\mathbb{R}$-linear and satisfies the Leibniz rule

$$D_v(fg) = (D_v f)\, g(p) + f(p) D_v g, \tag{2.2}$$

precisely because the partial derivatives $\partial/\,\partial x^i\big|_p$ have these properties.
In general, any linear map $D : C_p^\infty \to \mathbb{R}$ satisfying the Leibniz rule (2.2) is called a derivation at p or a point-derivation of C_p^∞. Denote the set of all derivations at p by $\mathcal{D}_p(\mathbb{R}^n)$. This set is in fact a real vector space, since the sum of two derivations at p and a scalar multiple of a derivation at p are again derivations at p (Problem 2.3).

Thus far, we know that directional derivatives at p are all derivations at p, so there is a map

$$\begin{aligned} \phi : T_p(\mathbb{R}^n) &\to \mathcal{D}_p(\mathbb{R}^n), \\ v &\mapsto D_v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p . \end{aligned} \tag{2.3}$$

Since D_v is clearly linear in v, the map ϕ is a linear map of vector spaces.
Lemma 2.1. If D is a point-derivation of C_p^∞, then $D(c) = 0$ for any constant function c.

Proof. Since we do not know whether every derivation at p is a directional derivative, we need to prove this lemma using only the defining properties of a derivation at p.

By $\mathbb{R}$-linearity, $D(c) = cD(1)$. So it suffices to prove that $D(1) = 0$. By the Leibniz rule (2.2),

$$D(1) = D(1 \cdot 1) = D(1) \cdot 1 + 1 \cdot D(1) = 2D(1).$$

Subtracting $D(1)$ from both sides gives $0 = D(1)$. $\square$

The Kronecker delta δ is a useful notation that we frequently call upon:

$$\delta_j^i = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Theorem 2.2. The linear map $\phi : T_p(\mathbb{R}^n) \to \mathcal{D}_p(\mathbb{R}^n)$ defined in (2.3) is an isomorphism of vector spaces.

Proof. To prove injectivity, suppose $D_v = 0$ for $v \in T_p(\mathbb{R}^n)$. Applying D_v to the coordinate function x^j gives

$$0 = D_v\left(x^j\right) = \sum_i v^i \frac{\partial}{\partial x^i}\bigg|_p x^j = \sum_i v^i \delta_i^j = v^j.$$

Hence, $v = 0$ and ϕ is injective.

To prove surjectivity, let D be a derivation at p and let (f, V) be a representative of a germ in C_p^∞. Making V smaller if necessary, we may assume that V is an open ball, hence star-shaped. By Taylor's theorem with remainder (Lemma 1.4) there are C^∞ functions $g_i(x)$ in a neighborhood of p such that

$$f(x) = f(p) + \sum \left(x^i - p^i\right) g_i(x), \quad g_i(p) = \frac{\partial f}{\partial x^i}(p).$$

Applying D to both sides and noting that $D(f(p)) = 0$ and $D\left(p^i\right) = 0$ by Lemma 2.1, we get by the Leibniz rule (2.2)

$$Df(x) = \sum \left(Dx^i\right) g_i(p) + \sum \left(p^i - p^i\right) Dg_i(x) = \sum \left(Dx^i\right) \frac{\partial f}{\partial x^i}(p).$$

This proves that $D = D_v$ for $v = \langle Dx^1, \ldots, Dx^n \rangle$.

This theorem shows that one may identify the tangent vectors at p with the derivations at p. Under the vector space isomorphism $T_p(\mathbb{R}^n) \simeq \mathcal{D}_p(\mathbb{R}^n)$, the standard basis $e_1, \ldots, e_n$ for $T_p(\mathbb{R}^n)$ corresponds to the set $\partial/\,\partial x^1|_p, \ldots, \partial/\,\partial x^n|_p$ of partial derivatives. From now on, we will make this identification and write a tangent vector $v = \langle v^1, \ldots, v^n \rangle = \sum v^i e_i$ as

$$v = \sum v^i \frac{\partial}{\partial x^i}\bigg|_p. \tag{2.4}$$

The vector space $D_p(\mathbb{R}^n)$ of derivations at p, although not as geometric as arrows, turns out to be more suitable for generalization to manifolds.

6.2.4 2.4 Vector Fields

A vector field X on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U a tangent vector X_p in $T_p(\mathbb{R}^n)$. Since $T_p(\mathbb{R}^n)$ has basis $\left\{\partial/\,\partial x^i|_p\right\}$, the vector X_p is a linear combination

$$X_p = \sum a^i(p) \frac{\partial}{\partial x^i}\bigg|_p, \quad p \in U, \quad a^i(p) \in \mathbb{R}$$

Omitting p, we may write $X = \sum a^i \partial/\partial x^i$, where the a^i are now functions on U. We say that the vector field X is C^∞ on U if the coefficient functions a^i are all C^∞ on U.

Example 2.3. On $\mathbb{R}^2 - \{\mathbf{0}\}$, let $p = (x, y)$. Then

$$X = \frac{-y}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{x}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y} = \left\langle \frac{-y}{\sqrt{x^2+y^2}}, \frac{x}{\sqrt{x^2+y^2}} \right\rangle$$

is the vector field in Figure 2.3(a). As is customary, we draw a vector at p as an arrow emanating from p. The vector field $Y = x\partial/\partial x - y\partial/\partial y = \langle x, -y \rangle$, suitably rescaled, is sketched in Figure 2.3(b).

(a) The vector field X on $\mathbb{R}^2 - \{\mathbf{0}\}$ (b) The vector field $\langle x, -y \rangle$ on $\mathbb{R}^2$

Fig. 2.3. Vector fields on open subsets of $\mathbb{R
^{2}$.}

One can identify vector fields on U with column vectors of C^∞ functions on U :

$$X = \sum a^i \frac{\partial}{\partial x^i} \longleftrightarrow \begin{bmatrix} a^1 \\ \vdots \\ a^n \end{bmatrix}.$$

This is the same identification as (2.4), but now we are allowing the point p to move in U.

The ring of C^∞ functions on an open set U is commonly denoted by $C^\infty(U)$ or $\mathcal{F}(U)$. Multiplication of vector fields by functions on U is defined pointwise:

$$(fX)_p = f(p)X_p, \quad p \in U.$$

Clearly, if $X = \sum a^i \partial/\partial x^i$ is a C^∞ vector field and f is a C^∞ function on U, then $fX = \sum (fa^i)\, \partial/\partial x^i$ is a C^∞ vector field on U. Thus, the set of all C^∞ vector fields on U, denoted by $\mathfrak{X}(U)$, is not only a vector space over $\mathbb{R}$, but also a module over the ring $C^\infty(U)$. We recall the definition of a module.

Definition 2.4. If R is a commutative ring with identity, then a (left) R-module is an abelian group A with a scalar multiplication map

$$\mu : R \times A \to A,$$

usually written $\mu(r, a) = ra$, such that for all $r, s \in R$ and $a, b \in A$,
(i) (associativity) $(rs)a = r(sa)$,
(ii) (identity) if 1 is the multiplicative identity in R, then $1a = a$,
(iii) (distributivity) $(r + s)a = ra + sa, r(a + b) = ra + rb$.

If R is a field, then an R-module is precisely a vector space over R. In this sense, a module generalizes a vector space by allowing scalars in a ring rather than a field.

Definition 2.5. Let A and A' be R-modules. An R-module homomorphism from A to A' is a map $f : A \to A'$ that preserves both addition and scalar multiplication: for all $a, b \in A$ and $r \in R$,
(i) $f(a+b) = f(a) + f(b)$,
(ii) $f(ra) = rf(a)$.

6.2.5 2.5 Vector Fields as Derivations

If X is a C^∞ vector field on an open subset U of $\mathbb{R}^n$ and f is a C^∞ function on U, we define a new function Xf on U by

$$(Xf)(p) = X_p f \quad \text{for any } p \in U.$$

Writing $X = \sum a^i \partial/\partial x^i$, we get

$$(Xf)(p) = \sum a^i(p) \frac{\partial f}{\partial x^i}(p),$$

or

$$Xf = \sum a^i \frac{\partial f}{\partial x^i}$$

which shows that Xf is a C^∞ function on U. Thus, a C^∞ vector field X gives rise to an $\mathbb{R}$-linear map

$$\begin{aligned} C^\infty(U) &\to C^\infty(U), \\ f &\mapsto Xf. \end{aligned}$$

Proposition 2.6 (Leibniz rule for a vector field). If X is a C^∞ vector field and f and g are C^∞ functions on an open subset U of $\mathbb{R}^n$, then $X(fg)$ satisfies the product rule (Leibniz rule):

$$X(fg) = (Xf)g + fXg$$

Proof. At each point $p \in U$, the vector X_p satisfies the Leibniz rule:

$$X_p(fg) = (X_p f)\, g(p) + f(p) X_p g.$$

As p varies over U, this becomes an equality of functions:

$$X(fg) = (Xf)g + fXg$$

If A is an algebra over a field K, a derivation of A is a K-linear map $D : A \to A$ such that

$$D(ab) = (Da)b + aDb \quad \text{for all } a, b \in A.$$

The set of all derivations of A is closed under addition and scalar multiplication and forms a vector space, denoted by $\operatorname{Der}(A)$. As noted above, a C^∞ vector field on an open set U gives rise to a derivation of the algebra $C^\infty(U)$. We therefore have a map

$$\begin{aligned} \varphi : \mathfrak{X}(U) &\to \operatorname{Der}\left(C^\infty(U)\right), \\ X &\mapsto (f \mapsto Xf). \end{aligned}$$

Just as the tangent vectors at a point p can be identified with the point-derivations of C_p^∞, so the vector fields on an open set U can be identified with the derivations of the algebra

$C^\infty(U)$; i.e., the map φ is an isomorphism of vector spaces. The injectivity of φ is easy to establish, but the surjectivity of φ takes some work (see Problem 19.12).

Note that a derivation at p is not a derivation of the algebra C_p^∞. A derivation at p is a map from C_p^∞ to $\mathbb{R}$, while a derivation of the algebra C_p^∞ is a map from C_p^∞ to C_p^∞.

6.2.6 Problems

2.1. Vector Fields

Let X be the vector field $x\partial/\partial x + y\partial/\partial y$ and $f(x, y, z)$ the function $x^2 + y^2 + z^2$ on $\mathbb{R}^3$. Compute Xf.

2.2. Algebra Structure on C_p^∞

Define carefully addition, multiplication, and scalar multiplication in C_p^∞. Prove that addition in C_p^∞ is commutative.

2.3. Vector Space Structure on Derivations at a Point

Let D and D' be derivations at p in $\mathbb{R}^n$, and $c \in \mathbb{R}$. Prove that
(a) the sum $D + D'$ is a derivation at p.
(b) the scalar multiple cD is a derivation at p.

2.4. Product of Derivations

Let A be an algebra over a field K. If D_1 and D_2 are derivations of A, show that $D_1 \circ D_2$ is not necessarily a derivation (it is if D_1 or $D_2 = 0$), but $D_1 \circ D_2 - D_2 \circ D_1$ is always a derivation of A.

6.3 §3 the Exterior Algebra of Multicovectors

As noted in the introduction, manifolds are higher-dimensional analogues of curves and surfaces. As such, they are usually not linear spaces. Nonetheless, a basic principle in manifold theory is the linearization principle, according to which every manifold can be locally approximated by its tangent space at a point, a linear object. In this way linear algebra enters into manifold theory.

Instead of working with tangent vectors, it turns out to be more fruitful to adopt the dual point of view and work with linear functions on a tangent space. After all, there is only so much that one can do with tangent vectors, which are essentially arrows, but functions, far more flexible, can be added, multiplied, scalar-multiplied, and composed with other maps. Once one admits linear functions on a tangent space, it is but a small step to consider functions of several arguments linear in each argument. These are the multilinear functions on a vector space. The determinant of a matrix, viewed as a function of the column vectors of the matrix, is an example of a multilinear function. Among the multilinear functions, certain ones such as the determinant and the cross product have an antisymmetric or alternating property: they change sign if two arguments are switched. The alternating multilinear functions with k arguments on a vector space are called multicovectors of degree k, or k-covectors for short.

It took the genius of Hermann Grassmann, a nineteenth-century German mathematician, linguist, and high-school teacher, to recognize the importance of multicovectors. He constructed a vast edifice based on multicovectors, now called the exterior algebra, that generalizes parts of vector calculus from $\mathbb{R}^3$ to $\mathbb{R}^n$. For example, the wedge product of two multicovectors on an n-dimensional vector space is a generalization of the cross product in $\mathbb{R}^3$ (see Problem 4.6).

Grassmann's work was little appreciated in his lifetime. In fact, he was turned down for a university position and his Ph.D. thesis rejected, because the leading mathematicians of his day such as Möbius and Kummer failed to understand his work. It was only at the turn of the twentieth century, in the hands of the great differential geometer Élie Cartan (1869-1951), that Grassmann's

Hermann Grassmann (1809-1877)

exterior algebra found its just recognition as the algebraic basis of the theory of differential forms. This section is an exposition, using modern terminology, of some of Grassmann's ideas.

6.3.1 3.1 Dual Space

If V and W are real vector spaces, we denote by $\operatorname{Hom}(V, W)$ the vector space of all linear maps $f : V \to W$. Define the dual space $V^\vee$ of V to be the vector space of all real-valued linear functions on V :

$$V^\vee = \operatorname{Hom}(V, \mathbb{R}).$$

The elements of $V^\vee$ are called covectors or 1-covectors on V.

In the rest of this section, assume V to be a finite-dimensional vector space. Let $e_1, \ldots, e_n$ be a basis for V. Then every v in V is uniquely a linear combination $v = \sum v^i e_i$ with $v^i \in \mathbb{R}$. Let $\alpha^i : V \to \mathbb{R}$ be the linear function that picks out the i th coordinate, $\alpha^i(v) = v^i$. Note that α^i is characterized by

$$\alpha^i\left(e_j\right) = \delta_j^i = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

Proposition 3.1. The functions $\alpha^1, \ldots, \alpha^n$ form a basis for $V^\vee$.

Proof. We first prove that $\alpha^1, \ldots, \alpha^n$ span $V^\vee$. If $f \in V^\vee$ and $v = \sum v^i e_i \in V$, then

$$f(v) = \sum v^i f\left(e_i\right) = \sum f\left(e_i\right) \alpha^i(v).$$

Hence,

$$f = \sum f(e_i)\,\alpha^i$$

which shows that $\alpha^1, \ldots, \alpha^n$ span $V^\vee$.
To show linear independence, suppose $\sum c_i \alpha^i = 0$ for some $c_i \in \mathbb{R}$. Applying both sides to the vector e_j gives

$$0 = \sum_i c_i \alpha^i(e_j) = \sum_i c_i \delta^i_j = c_j, \quad j = 1, \ldots, n$$

Hence, $\alpha^1, \ldots, \alpha^n$ are linearly independent.
This basis $\alpha^1, \ldots, \alpha^n$ for $V^\vee$ is said to be dual to the basis $e_1, \ldots, e_n$ for V.
Corollary 3.2. The dual space $V^\vee$ of a finite-dimensional vector space V has the same dimension as V.

Example 3.3 (Coordinate functions). With respect to a basis $e_1, \ldots, e_n$ for a vector space V, every $v \in V$ can be written uniquely as a linear combination $v = \sum b^i(v) e_i$, where $b^i(v) \in \mathbb{R}$. Let $\alpha^1, \ldots, \alpha^n$ be the basis of $V^\vee$ dual to $e_1, \ldots, e_n$. Then

$$\alpha^i(v) = \alpha^i\left(\sum_j b^j(v) e_j\right) = \sum_j b^j(v) \alpha^i(e_j) = \sum_j b^j(v) \delta^i_j = b^i(v).$$

Thus, the dual basis to $e_1, \ldots, e_n$ is precisely the set of coordinate functions $b^1, \ldots, b^n$ with respect to the basis $e_1, \ldots, e_n$.

6.3.2 3.2 Permutations

Fix a positive integer k. A permutation of the set $A = \{1, \ldots, k\}$ is a bijection $\sigma : A \to A$. More concretely, σ may be thought of as a reordering of the list $1, 2, \ldots, k$ from its natural increasing order to a new order $\sigma(1), \sigma(2), \ldots, \sigma(k)$. The cyclic permutation, $(a_1 a_2 \cdots a_r)$ where the a_i are distinct, is the permutation σ such that $\sigma(a_1) = a_2$, $\sigma(a_2) = a_3, \ldots, \sigma(a_{r-1}) = (a_r), \sigma(a_r) = a_1$, and σ fixes all the other elements of A. A cyclic permutation $(a_1 a_2 \cdots a_r)$ is also called a cycle of length r or an r-cycle. A transposition is a 2-cycle, that is, a cycle of the form (ab) that interchanges a and b, leaving all other elements of A fixed. Two cycles $(a_1 \cdots a_r)$ and $(b_1 \cdots b_s)$ are said to be disjoint if the sets $\{a_1, \ldots, a_r\}$ and $\{b_1, \ldots, b_s\}$ have no elements in common. The product $\tau\sigma$ of two permutations τ and σ of A is the composition $\tau \circ \sigma : A \to A$, in that order; first apply σ, then τ.

A simple way to describe a permutation $\sigma : A \to A$ is by its matrix

$$\begin{bmatrix} 1 & 2 & \cdots & k \\ \sigma(1) & \sigma(2) & \cdots & \sigma(k) \end{bmatrix}.$$

Example 3.4. Suppose the permutation $\sigma : \{1, 2, 3, 4, 5\} \to \{1, 2, 3, 4, 5\}$ maps 1,2 , $3, 4, 5$ to $2, 4, 5, 1, 3$ in that order. As a matrix,

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix}. \tag{3.1}$$

To write σ as a product of disjoint cycles, start with any element in $\{1, 2, 3, 4, 5\}$, say 1 , and apply σ to it repeatedly until we return to the initial element; this gives a cycle: $1 \mapsto 2 \mapsto 4 \to 1$. Next, repeat the procedure beginning with any of the remaining elements, say 3 , to get a second cycle: $3 \mapsto 5 \mapsto 3$. Since all elements of $\{1, 2, 3, 4, 5\}$ are now accounted for, σ equals (124)(35):

From this example, it is easy to see that any permutation can be written as a product of disjoint cycles $(a_1 \cdots a_r)(b_1 \cdots b_s) \cdots$.

Let S_k be the group of all permutations of the set $\{1, \ldots, k\}$. A permutation is even or odd depending on whether it is the product of an even or an odd number of transpositions. From the theory of permutations we know that this is a well-defined concept: an even permutation can never be written as the product of an odd number of transpositions and vice versa. The sign of a permutation σ, denoted by $\operatorname{sgn}(\sigma)$ or $\operatorname{sgn}\sigma$, is defined to be +1 or -1 depending on whether the permutation is even or odd. Clearly, the sign of a permutation satisfies

$$\operatorname{sgn}(\sigma\tau) = \operatorname{sgn}(\sigma)\operatorname{sgn}(\tau) \tag{3.2}$$

for $\sigma, \tau \in S_k$.

Example 3.5. The decomposition

$$(12345) = (15)(14)(13)(12)$$

shows that the 5-cycle (1 2345) is an even permutation.
More generally, the decomposition

$$(a_1a_2\cdots a_r) = (a_1a_r)\,(a_1a_{r-1})\cdots(a_1a_3)\,(a_1a_2)$$

shows that an r-cycle is an even permutation if and only if r is odd, and an odd permutation if and only if r is even. Thus one way to compute the sign of a permutation is to decompose it into a product of cycles and to count the number of cycles of even length. For example, the permutation $\sigma = (124)\ (35)$ in Example 3.4 is odd because (124) is even and (35) is odd.

An inversion in a permutation σ is an ordered pair $(\sigma(i), \sigma(j))$ such that $i < j$ but $\sigma(i) > \sigma(j)$. To find all the inversions in a permutation σ, it suffices to scan the second row of the matrix of σ from left to right; the inversions are the pairs (a, b) with $a > b$ and a to the left of b. For the permutation σ in Example 3.4, from its matrix (3.1) we can read off its five inversions: $(2, 1), (4, 1), (5, 1), (4, 3)$, and $(5, 3)$.

Exercise 3.6 (Inversions).* Find the inversions in the permutation $\tau = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \end{pmatrix}$ of Example 3.5.

A second way to compute the sign of a permutation is to count the number of inversions, as we illustrate in the following example.

Example 3.7. Let σ be the permutation of Example 3.4. Our goal is to turn σ into the identity permutation $\mathbb{1}$ by multiplying it on the left by transpositions.
(i) To move 1 to its natural position at the beginning of the second row of the matrix of σ, we need to move it across the three elements $2, 4, 5$. This can be accomplished by multiplying σ on the left by three transpositions: first (51), then (41), and finally (2 1):

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 5 & 1 & 3 \end{bmatrix} \xrightarrow{(5\ 1)} \begin{bmatrix} 2 & 4 & 1 & 5 & 3 \end{bmatrix} \xrightarrow{(4\ 3)} / \begin{bmatrix} 2 & 1 & 4 & 5 & 3 \end{bmatrix} \xrightarrow{(2\ 1)} / \begin{bmatrix} 2 & & & & \\ 1 & 2 & 4 & 5 & 3 \end{bmatrix}.$$

The three transpositions (5 1), (4 1), and (2 1) correspond precisely to the three inversions of σ ending in 1 .
(ii) The element 2 is already in its natural position in the second row of the matrix.
(iii) To move 3 to its natural position in the second row, we need to move it across two elements 4, 5. This can be accomplished by

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 4 & 5 & 3 \end{bmatrix} \xrightarrow{(5\ 3)} \begin{bmatrix} 1 & 2 & 4 & 3 & 5 \end{bmatrix} \xrightarrow{(4\ 3)} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \end{bmatrix} = \mathbb{1}.$$

Thus,

$$(43)(53)(21)(41)(51)\sigma = \mathbb{1}. \tag{3.3}$$

Note that the two transpositions (53) and (43) correspond to the two inversions ending in 3. Multiplying both sides of (3.3) on the left by the transpositions (43), then (53), then (21), and so on eventually yields

$$\sigma = \begin{pmatrix} 5 & 1 \end{pmatrix} (4\ 1\ ) \begin{pmatrix} 2 & 1 \end{pmatrix} (53)(43).$$

This shows that σ can be written as a product of as many transpositions as the number of inversions in it.

With this example in mind, we prove the following proposition.
Proposition 3.8. A permutation is even if and only if it has an even number of inversions.

Proof. We will obtain the identity permutation $\mathbb{1}$ by multiplying σ on the left by a number of transpositions. This can be achieved in k steps.
(i) First, look for the number 1 among $\sigma(1), \sigma(2), \ldots, \sigma(k)$. Every number preceding 1 in this list gives rise to an inversion, for if $1 = \sigma(i)$, then $(\sigma(1), 1), \ldots, (\sigma(i-1), 1)$ are inversions of

σ. Now move 1 to the beginning of the list across the $i-1$ elements $\sigma(1), \ldots, \sigma(i-1)$. This requires multiplying σ on the left by $i-1$ transpositions:

$$\sigma_1 = (\sigma(1)1) \cdots (\sigma(i-1)1)\sigma = \begin{bmatrix} 1 & \sigma(1) & \cdots & \sigma(i-1) \\ & \sigma(i+1) & \cdots & \sigma(k) \end{bmatrix}.$$

Note that the number of transpositions is the number of inversions ending in 1 .
(ii) Next look for the number 2 in the list: $1, \sigma(1), \ldots, \sigma(i-1), \sigma(i+1), \ldots, \sigma(k)$. Every number other than 1 preceding 2 in this list gives rise to an inversion $(\sigma(m), 2)$. Suppose there are i_2 such numbers. Then there are i_2 inversions ending in 2 . In moving 2 to its natural position $1, 2, \sigma(1), \sigma(2), \ldots$, we need to move it across i_2 numbers. This can be accomplished by multiplying σ_1 on the left by i_2 transpositions.

Repeating this procedure, we see that for each $j = 1, \ldots, k$, the number of transpositions required to move j to its natural position is the same as the number of inversions ending in j. In the end we achieve the identity permutation, i.e, the ordered list $1, 2, \ldots, k$, from $\sigma(1), \sigma(2), \ldots, \sigma(k)$ by multiplying σ by as many transpositions as the total number of inversions in σ. Therefore, $\operatorname{sgn}(\sigma) = (-1)^{\#\text{ inversions}}$,

6.3.3 3.3 Multilinear Functions

Denote by $V^k = V \times \cdots \times V$ the Cartesian product of k copies of a real vector space V. A function $f : V^k \to \mathbb{R}$ is k-linear if it is linear in each of its k arguments:

$$f(\ldots, av + bw, \ldots) = af(\ldots, v, \ldots) + bf(\ldots, w, \ldots)$$

for all $a, b \in \mathbb{R}$ and $v, w \in V$. Instead of 2-linear and 3-linear, it is customary to say "bilinear" and "trilinear." A k-linear function on V is also called a k-tensor on V. We will denote the vector space of all k-tensors on V by $L_k(V)$. If f is a k-tensor on V, we also call k the degree of f.

Example 3.9 (Dot product on $\mathbb{R}^n$). With respect to the standard basis $e_1, \ldots, e_n$ for $\mathbb{R}^n$, the dot product, defined by

$$f(v, w) = v \bullet w = \sum_i v^i w^i, \qquad \text{where } v = \sum v^i e_i, w = \sum w^i e_i,$$

is bilinear.
Example. The determinant $f(v_1, \ldots, v_n) = \det[v_1 \cdots v_n]$, viewed as a function of the n column vectors $v_1, \ldots, v_n$ in $\mathbb{R}^n$, is n-linear.

Definition 3.10. A k-linear function $f : V^k \to \mathbb{R}$ is symmetric if

$$f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) = f(v_1, \ldots, v_k)$$

for all permutations $\sigma \in S_k$; it is alternating if

$$f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) = (\operatorname{sgn}\sigma) f(v_1, \ldots, v_k)$$

for all $\sigma \in S_k$.

Examples.

(i) The dot product $f(v, w) = v \bullet w$ on $\mathbb{R}^n$ is symmetric.
(ii) The determinant $f(v_1, \ldots, v_n) = \det[v_1 \cdots v_n]$ on $\mathbb{R}^n$ is alternating.
(iii) The cross product $v \times w$ on $\mathbb{R}^3$ is alternating.

(iv) For any two linear functions $f, g : V \to \mathbb{R}$ on a vector space V, the function $f \wedge g : V \times V \to \mathbb{R}$ defined by

$$(f \wedge g)(u, v) = f(u)g(v) - f(v)g(u)$$

is alternating. This is a special case of the wedge product, which we will soon define.

We are especially interested in the space $A_k(V)$ of all alternating k-linear functions on a vector space V for $k > 0$. These are also called alternating k-tensors, k-covectors, or multicovectors of degree k on V. For $k = 0$, we define a 0 -covector to be a constant, so that $A_0(V)$ is the vector space $\mathbb{R}$. A 1 -covector is simply a covector.

6.3.4 3.4 The Permutation Action on Multilinear Functions

If f is a k-linear function on a vector space V and σ is a permutation in S_k, we define a new k-linear function σf by

$$(\sigma f)(v_1, \ldots, v_k) = f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right)$$

Thus, f is symmetric if and only if $\sigma f = f$ for all $\sigma \in S_k$ and f is alternating if and only if $\sigma f = (\operatorname{sgn} \sigma) f$ for all $\sigma \in S_k$.

When there is only one argument, the permutation group S_1 is the identity group and a 1 -linear function is both symmetric and alternating. In particular,

$$A_1(V) = L_1(V) = V^\vee.$$

Lemma 3.11. If $\sigma, \tau \in S_k$ and f is a k-linear function on V, then $\tau(\sigma f) = (\tau\sigma) f$.
Proof. For $v_1, \ldots, v_k \in V$,

$$\begin{aligned}
\tau(\sigma f)(v_1, \ldots, v_k) &= (\sigma f)\left(v_{\tau(1)}, \ldots, v_{\tau(k)}\right) \\
&= (\sigma f)(w_1, \ldots, w_k) \\
&= f\left(w_{\sigma(1)}, \ldots, w_{\sigma(k)}\right) \\
&= f\left(v_{\tau(\sigma(1))}, \ldots, v_{\tau(\sigma(k))}\right) = f\left(v_{(\tau\sigma)(1)}, \ldots, v_{(\tau\sigma)(k)}\right) \\
&= (\tau\sigma) f(v_1, \ldots, v_k)
\end{aligned}$$

In general, if G is a group and X is a set, a map

$$\begin{aligned}
G \times X &\to X \\
(\sigma, x) &\mapsto \sigma \cdot x
\end{aligned}$$

is called a left action of G on X if
(i) $e \cdot x = x$, where e is the identity element in G and x is any element in X, and
(ii) $\tau \cdot (\sigma \cdot x) = (\tau\sigma) \cdot x$ for all $\tau, \sigma \in G$ and $x \in X$.

The orbit of an element $x \in X$ is defined to be the set $Gx := \{\sigma \cdot x \in X \mid \sigma \in G\}$. In this terminology, we have defined a left action of the permutation group S_k on the space $L_k(V)$ of k-linear functions on V. Note that each permutation acts as a linear function on the vector space $L_k(V)$ since σf is $\mathbb{R}$-linear in f.

A right action of G on X is defined similarly; it is a map $X \times G \to X$ such that
(i) $x \cdot e = x$, and
(ii) $(x \cdot \sigma) \cdot \tau = x \cdot (\sigma\tau)$
for all $\sigma, \tau \in G$ and $x \in X$.
Remark. In some books the notation for σf is f^σ. In that notation, $(f^\sigma)^\tau = f^{\tau\sigma}$, not $f^{\sigma\tau}$.

6.3.5 3.5 The Symmetrizing and Alternating Operators

Given any k-linear function f on a vector space V, there is a way to make a symmetric k-linear function Sf from it:

$$(Sf)(v_1, \ldots, v_k) = \sum_{\sigma \in S_k} f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right)$$

or, in our new shorthand,

$$Sf = \sum_{\sigma \in S_k} \sigma f.$$

Similarly, there is a way to make an alternating k-linear function from f. Define

$$Af = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \sigma f$$

Proposition 3.12. If f is a k-linear function on a vector space V, then
(i) the k-linear function Sf is symmetric, and
(ii) the k-linear function Af is alternating.

Proof. We prove (ii) only, leaving (i) as an exercise. For $\tau \in S_k$,

$$\begin{aligned} \tau(Af) &= \textstyle\sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \tau(\sigma f) \\ &= \textstyle\sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)(\tau\sigma) f && (\text{ by Lemma 3.11 }) \\ &= (\operatorname{sgn} \tau) \textstyle\sum_{\sigma \in S_k} (\operatorname{sgn} \tau\sigma)(\tau\sigma) f && (\text{ by (3.2)}) \\ &= (\operatorname{sgn} \tau) Af \end{aligned}$$

since as σ runs through all permutations in S_k, so does $\tau\sigma$.

Exercise 3.13 (Symmetrizing operator).* Show that the k-linear function Sf is symmetric.

Lemma 3.14. If f is an alternating k-linear function on a vector space V, then $Af = (k!)f$.

Proof. Since for alternating f we have $\sigma f = (\operatorname{sgn} \sigma) f$, and $\operatorname{sgn} \sigma$ is ± 1, we must have

$$Af = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma) \sigma f = \sum_{\sigma \in S_k} (\operatorname{sgn} \sigma)(\operatorname{sgn} \sigma) f = (k!) f$$

Exercise 3.15 (Alternating operator).* If f is a 3-linear function on a vector space V and $v_1, v_2, v_3 \in V$, what is $(Af)(v_1, v_2, v_3)$?

6.3.6 3.6 The Tensor Product

Let f be a k-linear function and g an ℓ-linear function on a vector space V. Their tensor product is the ($k + \ell$)-linear function $f \otimes g$ defined by

$$(f \otimes g)(v_1, \ldots, v_{k+\ell}) = f(v_1, \ldots, v_k)\, g(v_{k+1}, \ldots, v_{k+\ell}).$$

Example 3.16 (Bilinear maps). Let $e_1, \ldots, e_n$ be a basis for a vector space $V, \alpha^1, \ldots, \alpha^n$ the dual basis in $V^\vee$, and $\langle\rangle :, V \times V \to \mathbb{R}$ a bilinear map on V. Set $g_{ij} = \langle e_i, e_j \rangle \in \mathbb{R}$. If $v = \sum v^i e_i$ and $w = \sum w^i e_i$, then as we observed in Example 3.3, $v^i = \alpha^i(v)$ and $w^j = \alpha^j(w)$. By bilinearity, we can express $\langle , \rangle intermsofthetensorproduct$:

$$\langle v, w \rangle = \sum v^i w^j \langle e_i, e_j \rangle = \sum \alpha^i(v) \alpha^j(w) g_{ij} = \sum g_{ij} \left(\alpha^i \otimes \alpha^j\right)(v, w).$$

Hence, $\langle\rangle =, \sum g_{ij} \alpha^i \otimes \alpha^j$. This notation is often used in differential geometry to describe an inner product on a vector space.

Exercise 3.17 (Associativity of the tensor product). Check that the tensor product of multilinear functions is associative: if f, g, and h are multilinear functions on V, then

$$(f \otimes g) \otimes h = f \otimes (g \otimes h)$$

6.3.7 3.7 The Wedge Product

If two multilinear functions f and g on a vector space V are alternating, then we would like to have a product that is alternating as well. This motivates the definition of the wedge product, also called the exterior product: for $f \in A_k(V)$ and $g \in A_\ell(V)$,

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g) \tag{3.4}$$

or explicitly,

$$\begin{aligned}
&(f \wedge g)(v_1, \ldots, v_{k+\ell}) \\
&\quad = \frac{1}{k!\ell!} \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\right).
\end{aligned} \tag{3.5}$$

By Proposition 3.12, $f \wedge g$ is alternating.
When $k = 0$, the element $f \in A_0(V)$ is simply a constant c. In this case, the wedge product $c \wedge g$ is scalar multiplication, since the right-hand side of (3.5) is

$$\frac{1}{\ell!} \sum_{\sigma \in S_\ell} (\operatorname{sgn} \sigma) \operatorname{cg}\left(v_{\sigma(1)}, \ldots, v_{\sigma(\ell)}\right) = \operatorname{cg}(v_1, \ldots, v_\ell).$$

Thus $c \wedge g = cg$ for $c \in \mathbb{R}$ and $g \in A_\ell(V)$.
The coefficient $1/k!\ell\,!$ in the definition of the wedge product compensates for repetitions in the sum: for every permutation $\sigma \in S_{k+\ell}$, there are $k\,!$ permutations τ in S_k that permute the first k arguments $v_{\sigma(1)}, \ldots, v_{\sigma(k)}$ and leave the arguments of g alone; for all τ in S_k, the resulting permutations $\sigma\tau$ in $S_{k+\ell}$ contribute the same term to the sum, since

$$\begin{aligned}
(\operatorname{sgn} \sigma\tau) f\left(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(k)}\right) &= (\operatorname{sgn} \sigma\tau)(\operatorname{sgn} \tau) f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) \\
&= (\operatorname{sgn} \sigma) f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right)
\end{aligned}$$

where the first equality follows from the fact that $(\tau(1), \ldots, \tau(k))$ is a permutation of $(1, \ldots, k)$. So we divide by $k!$ to get rid of the $k!$ repeating terms in the sum coming from permutations of the k arguments of f; similarly, we divide by $\ell\,!$ on account of the ℓ arguments of g.

Example 3.18. For $f \in A_2(V)$ and $g \in A_1(V)$,

$$\begin{aligned}
A(f \otimes g)(v_1, v_2, v_3) = &f(v_1, v_2) g(v_3) - f(v_1, v_3) g(v_2) + f(v_2, v_3) g(v_1) \\
&- f(v_2, v_1) g(v_3) + f(v_3, v_1) g(v_2) - f(v_3, v_2) g(v_1)
\end{aligned}$$

Among these six terms, there are three pairs of equal terms, which we have lined up vertically in the display above:

$$f(v_1, v_2) g(v_3) = -f(v_2, v_1) g(v_3), \qquad \text{and so on.}$$

Therefore, after dividing by 2 ,

$$(f \wedge g)(v_1, v_2, v_3) = f(v_1, v_2) g(v_3) - f(v_1, v_3) g(v_2) + f(v_2, v_3) g(v_1)$$

One way to avoid redundancies in the definition of $f \wedge g$ is to stipulate that in the sum (3.5), $\sigma(1), \ldots, \sigma(k)$ be in ascending order and $\sigma(k+1), \ldots, \sigma(k+\ell)$ also be in ascending order. We call a permutation $\sigma \in S_{k+\ell}$a(k, ℓ)-shuffle if

$$\sigma(1) < \cdots < \sigma(k) \quad \text{and} \quad \sigma(k+1) < \cdots < \sigma(k+\ell).$$

By the paragraph before Example 3.18, one may rewrite (3.5) as

$$\begin{aligned}
&(f \wedge g)\left(v_1, \ldots, v_{k+\ell}\right) \\
&\quad = \sum_{\substack{(k,\ell)\text{-shuffles} \\ \sigma}} (\operatorname{sgn} \sigma) f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\right). \qquad (3.6)
\end{aligned}$$

Written this way, the definition of $(f \wedge g)\left(v_1, \ldots, v_{k+\ell}\right)$ is a sum of $\binom{k+\ell}{k}$ terms, instead of $(\ k+\ell\ )!$ terms.

Example 3.19 (Wedge product of two covectors). * If f and g are covectors on a vector space V and $v_1, v_2 \in V$, then by (3.6),

$$(f \wedge g)\left(v_1, v_2\right) = f\left(v_1\right) g\left(v_2\right) - f\left(v_2\right) g\left(v_1\right).$$

Exercise 3.20 (Wedge product of two 2-covectors). For $f, g \in A_2(V)$, write out the definition of $f \wedge g$ using $(\ 2{,}2\ )$-shuffles.

6.3.8 3.8 Anticommutativity of the Wedge Product

It follows directly from the definition of the wedge product (3.5) that $f \wedge g$ is bilinear in f and in g.

Proposition 3.21. The wedge product is anticommutative: if $f \in A_k(V)$ and $g \in A_\ell(V)$, then

$$f \wedge g = (-1)^{k\ell} g \wedge f$$

Proof. Define $\tau \in S_{k+\ell}$ to be the permutation

$$\tau = \begin{bmatrix} 1 & \cdots & \ell & \ell+1 & \cdots & \ell+k \\ k+1 & \cdots & k+\ell & 1 & \cdots & k \end{bmatrix}.$$

This means that

$$\tau(1) = k+1, \ldots, \tau(\ell) = k+\ell, \tau(\ell+1) = 1, \ldots, \tau(\ell+k) = k.$$

Then

$$\begin{aligned}
\sigma(1) = \sigma\tau(\ell+1), \ldots, \sigma(k) = \sigma\tau(\ell+k), \\
\sigma(k+1) = \sigma\tau(1), \ldots, \sigma(k+\ell) = \sigma\tau(\ell).
\end{aligned}$$

For any $v_1, \ldots, v_{k+\ell} \in V$,

$$
\begin{aligned}
&A(f \otimes g)\left(v_1, \ldots, v_{k+\ell}\right) \\
&\quad= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) g\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\right) \\
&\quad= \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) f\left(v_{\sigma\tau(\ell+1)}, \ldots, v_{\sigma\tau(\ell+k)}\right) g\left(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(\ell)}\right) \\
&\quad= (\operatorname{sgn} \tau) \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma\tau) g\left(v_{\sigma\tau(1)}, \ldots, v_{\sigma\tau(\ell)}\right) f\left(v_{\sigma\tau(\ell+1)}, \ldots, v_{\sigma\tau(\ell+k)}\right) \\
&\quad= (\operatorname{sgn} \tau) A(g \otimes f)\left(v_1, \ldots, v_{k+\ell}\right)
\end{aligned}
$$

The last equality follows from the fact that as σ runs through all permutations in $S_{k+\ell}$, so does $\sigma\tau$.

We have proven

$$A(f \otimes g) = (\operatorname{sgn} \tau) A(g \otimes f).$$

Dividing by $k!\ell\,!$ gives

$$f \wedge g = (\operatorname{sgn} \tau) g \wedge f$$

Exercise 3.22 (Sign of a permutation).* Show that $\operatorname{sgn} \tau = (-1)^{k\ell}$. $\square$

Corollary 3.23. If f is a multicovector of odd degree on V, then $f \wedge f = 0$.

Proof. Let k be the degree of f. By anticommutativity,

$$f \wedge f = (-1)^{k^2} f \wedge f = -f \wedge f$$

since k is odd. Hence, $2f \wedge f = 0$. Dividing by 2 gives $f \wedge f = 0$. $\square$

6.3.9 3.9 Associativity of the Wedge Product

The wedge product of a k-covector f and an ℓ-covector g on a vector space V is by definition the $(\ k+\ell\ )$-covector

$$f \wedge g = \frac{1}{k!\ell!} A(f \otimes g)$$

To prove the associativity of the wedge product, we will follow Godbillon [14] by first proving a lemma on the alternating operator A.

Lemma 3.24. Suppose f is a k-linear function and g an ℓ-linear function on a vector space V. Then

(i) $A(A(f) \otimes g) = k! A(f \otimes g)$, and

(ii) $A(f \otimes A(g)) = \ell! A(f \otimes g)$.

Proof. (i) By definition,

$$A(A(f) \otimes g) = \sum_{\sigma \in S_{k+\ell}} (\operatorname{sgn} \sigma) \sigma \left(\sum_{\tau \in S_k} (\operatorname{sgn} \tau)(\tau f) \otimes g \right).$$

We can view $\tau \in S_k$ also as a permutation in $S_{k+\ell}$ fixing $k+1, \ldots, k+\ell$. Viewed this way, τ satisfies

$$(\tau f) \otimes g = \tau(f \otimes g)$$

Hence,

$$A(A(f)\otimes g)=\sum_{\sigma\in S_{k+\ell}}\sum_{\tau\in S_k}(\operatorname{sgn}\sigma)(\operatorname{sgn}\tau)(\sigma\tau)(f\otimes g) \tag{3.7}$$

For each $\mu\in S_{k+\ell}$ and each $\tau\in S_k$, there is a unique element $\sigma=\mu\tau^{-1}\in S_{k+\ell}$ such that $\mu=\sigma\tau$, so each $\mu\in S_{k+\ell}$ appears once in the double sum (3.7) for each $\tau\in S_k$, and hence $k!$ times in total. So the double sum (3.7) can be rewritten as

$$A(A(f)\otimes g)=k!\sum_{\mu\in S_{k+\ell}}(\operatorname{sgn}\mu)\mu(f\otimes g)=k!A(f\otimes g)$$

The equality in (ii) is proved in the same way.

Proposition 3.25 (Associativity of the wedge product). Let V be a real vector space and f,g,h alternating multilinear functions on V of degrees k,ℓ,m, respectively. Then

$$(f\wedge g)\wedge h=f\wedge(g\wedge h)$$

Proof. By the definition of the wedge product,

$$\begin{aligned}(f\wedge g)\wedge h&=\frac{1}{(k+\ell)!m!}A((f\wedge g)\otimes h)\\&=\frac{1}{(k+\ell)!m!}\frac{1}{k!\ell!}A(A(f\otimes g)\otimes h)\\&=\frac{(k+\ell)!}{(k+\ell)!m!k!\ell!}A((f\otimes g)\otimes h)\quad(\text{ by Lemma 3.24(i)})\\&=\frac{1}{k!\ell!m!}A((f\otimes g)\otimes h)\end{aligned}$$

Similarly,

$$\begin{aligned}f\wedge(g\wedge h)&=\frac{1}{k!(\ell+m)!}A\left(f\otimes\frac{1}{\ell!m!}A(g\otimes h)\right)\\&=\frac{1}{k!\ell!m!}A(f\otimes(g\otimes h))\end{aligned}$$

Since the tensor product is associative, we conclude that

$$(f\wedge g)\wedge h=f\wedge(g\wedge h)$$

By associativity, we can omit the parentheses in a multiple wedge product such as $(f\wedge g)\wedge h$ and write simply $f\wedge g\wedge h$.

Corollary 3.26. Under the hypotheses of the proposition,

$$f\wedge g\wedge h=\frac{1}{k!\ell!m!}A(f\otimes g\otimes h)$$

This corollary easily generalizes to an arbitrary number of factors: if $f_i\in A_{d_i}(V)$, then

$$f_1\wedge\cdots\wedge f_r=\frac{1}{(d_1)!\cdots(d_r)!}A\left(f_1\otimes\cdots\otimes f_r\right) \tag{3.8}$$

In particular, we have the following proposition. We use the notation $\left[b_j^i\right]$ to denote the matrix whose (i,j)-entry is b_j^i.

Proposition 3.27 (Wedge product of 1 -covectors). If $\alpha^1,\ldots,\alpha^k$ are linear functions on a vector space V and $v_1,\ldots,v_k\in V$, then

$$\left(\alpha^1\wedge\cdots\wedge\alpha^k\right)(v_1,\ldots,v_k)=\det\left[\alpha^i\left(v_j\right)\right].$$

Proof. By (3.8),

$$\begin{aligned}\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right)\left(v_1, \ldots, v_k\right) &= A\left(\alpha^1 \otimes \cdots \otimes \alpha^k\right)\left(v_1, \ldots, v_k\right) \\ &= \sum_{\sigma \in S_k}(\operatorname{sgn} \sigma) \alpha^1\left(v_{\sigma(1)}\right) \cdots \alpha^k\left(v_{\sigma(k)}\right) \\ &= \det\left[\alpha^i\left(v_j\right)\right]\end{aligned}$$

An algebra A over a field K is said to be graded if it can be written as a direct sum $A = \bigoplus_{k=0}^{\infty} A^k$ of vector spaces over K such that the multiplication map sends $A^k \times A^\ell$ to $A^{k+\ell}$. The notation $A = \bigoplus_{k=0}^{\infty} A^k$ means that each nonzero element of A is uniquely a finite sum

$$a = a_{i_1} + \cdots + a_{i_m}$$

where $a_{i_j} \neq 0 \in A^{i_j}$. A graded algebra $A = \oplus_{k=0}^{\infty} A^k$ is said to be anticommutative or graded commutative if for all $a \in A^k$ and $b \in A^\ell$,

$$ab = (-1)^{k\ell} ba.$$

A homomorphism of graded algebras is an algebra homomorphism that preserves the degree.

Example. The polynomial algebra $A = \mathbb{R}[x, y]$ is graded by degree; A^k consists of all homogeneous polynomials of total degree k in the variables x and y.

For a finite-dimensional vector space V, say of dimension n, define

$$A_*(V) = \bigoplus_{k=0}^{\infty} A_k(V) = \bigoplus_{k=0}^{n} A_k(V).$$

With the wedge product of multicovectors as multiplication, $A_*(V)$ becomes an anticommutative graded algebra, called the exterior algebra or the Grassmann algebra of multicovectors on the vector space V.

6.3.10 3.10 A Basis for k-covectors

Let $e_1, \ldots, e_n$ be a basis for a real vector space V, and let $\alpha^1, \ldots, \alpha^n$ be the dual basis for $V^\vee$. Introduce the multi-index notation

$$I = (i_1, \ldots, i_k)$$

and write e_I for $(\ e_{i_1}, \ldots, e_{i_k}\ )$ and α^I for $\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$.
A k-linear function f on V is completely determined by its values on all k-tuples $(e_{i_1}, \ldots, e_{i_k})$. If f is alternating, then it is completely determined by its values on $(e_{i_1}, \ldots, e_{i_k})$ with $1 \leq i_1 < \cdots < i_k \leq n$; that is, it suffices to consider e_I with I in strictly ascending order.

Lemma 3.28. Let $e_1, \ldots, e_n$ be a basis for a vector space V and let $\alpha^1, \ldots, \alpha^n$ be its dual basis in $V^\vee$. If $I = (1 \leq i_1 < \cdots < i_k \leq n)$ and $J = (1 \leq j_1 < \cdots < j_k \leq n)$ are strictly ascending multi-indices of length k, then

$$\alpha^I\left(e_J\right) = \delta_J^I = \begin{cases} 1 & \text{for } I = J, \\ 0 & \text{for } I \neq J. \end{cases}$$

Proof. By Proposition 3.27,

$$\alpha^I\left(e_J\right) = \det\left[\alpha^i\left(e_j\right)\right]_{i \in I, j \in J}$$

If $I = J$, then $[\alpha^i(e_j)]$ is the identity matrix and its determinant is 1 .
If $I \neq J$, we compare them term by term until the terms differ:

$$i_1 = j_1, \ldots, i_{\ell-1} = j_{\ell-1}, i_\ell \neq j_\ell, \ldots.$$

Without loss of generality, we may assume $i_\ell < j_\ell$. Then i_ℓ will be different from $j_1, \ldots, j_{\ell-1}$ (because these are the same as $i_1, \ldots, i_\ell$, and I is strictly ascending), and i_ℓ will also be different from $j_\ell, j_{\ell+1}, \ldots, j_k$ (because J is strictly ascending). Thus, i_ℓ will be different from $j_1, \ldots, j_k$, and the ℓ th row of the matrix $[a^i(e_j)]$ will be all zero. Hence, $\det[a^i(e_j)] = 0$.

Proposition 3.29. The alternating k-linear functions $\alpha^I, I = (i_1 < \cdots < i_k)$, form a basis for the space $A_k(V)$ of alternating k-linear functions on V.

Proof. First, we show linear independence. Suppose $\sum c_I \alpha^I = 0, c_I \in \mathbb{R}$, and I runs over all strictly ascending multi-indices of length k. Applying both sides to $e_J, J = (j_1 < \cdots < j_k)$, we get by Lemma 3.28,

$$0 = \sum_I c_I \alpha^I(e_J) = \sum_I c_I \delta_J^I = c_J,$$

since among all strictly ascending multi-indices I of length k, there is only one equal to J. This proves that the α^I are linearly independent.

To show that the α^I span $A_k(V)$, let $f \in A_k(V)$. We claim that

$$f = \sum f(e_I)\alpha^I,$$

where I runs over all strictly ascending multi-indices of length k. Let $g = \sum f(e_I)\alpha^I$. By k-linearity and the alternating property, if two k-covectors agree on all e_J, where $J = (j_1 < \cdots < j_k)$, then they are equal. But

$$g(e_J) = \sum f(e_I)\alpha^I(e_J) = \sum f(e_I)\delta_J^I = f(e_J).$$

Therefore, $f = g = \sum f(e_I)\alpha^I$.

Corollary 3.30. If the vector space V has dimension n, then the vector space $A_k(V)$ of k-covectors on V has dimension $\binom{n}{k}$.

Proof. A strictly ascending multi-index $I = (i_1 < \cdots < i_k)$ is obtained by choosing a subset of k numbers from $1, \ldots, n$. This can be done in $\binom{n}{k}$ ways.

Corollary 3.31. If $k > \dim V$, then $A_k(V) = 0$.

Proof. In $\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$, at least two of the factors must be the same, say $\alpha^j = \alpha^\ell = \alpha$. Because α is a 1 -covector, $\alpha \wedge \alpha = 0$ by Corollary 3.23, so $\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k} = 0$.

6.3.11 Problems

3.1. Tensor Product of Covectors

Let $e_1, \ldots, e_n$ be a basis for a vector space V and let $\alpha^1, \ldots, \alpha^n$ be its dual basis in $V^\vee$. Suppose $[g_{ij}] \in \mathbb{R}^{n \times n}$ is an $n \times n$ matrix. Define a bilinear function $f : V \times V \to \mathbb{R}$ by

$$f(v, w) = \sum_{1 \leq i,j \leq n} g_{ij} v^i w^j$$

for $v = \sum v^i e_i$ and $w = \sum w^j e_j$ in V. Describe f in terms of the tensor products of α^i and α^j, $1 \leq i, j \leq n$.

3.2. Hyperplanes

(a) Let V be a vector space of dimension n and $f : V \to \mathbb{R}$ a nonzero linear functional. Show that $\dim\ker f = n-1$. A linear subspace of V of dimension $n-1$ is called a hyperplane in V.
(b) Show that a nonzero linear functional on a vector space V is determined up to a multiplicative constant by its kernel, a hyperplane in V. In other words, if f and $g : V \to \mathbb{R}$ are nonzero linear functionals and $\ker f = \ker g$, then $g = cf$ for some constant $c \in \mathbb{R}$.

3.3. A Basis for k-tensors

Let V be a vector space of dimension n with basis $e_1, \ldots, e_n$. Let $\alpha^1, \ldots, \alpha^n$ be the dual basis for $V^\vee$. Show that a basis for the space $L_k(V)$ of k-linear functions on V is $\{\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}\}$ for all multi-indices $(i_1, \ldots, i_k)$ (not just the strictly ascending multi-indices as for $A_k(L)$). In particular, this shows that $\dim L_k(V) = n^k$. (This problem generalizes Problem 3.1.)

3.4. A Characterization of Alternating k-tensors

Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if f changes sign whenever two successive arguments are interchanged:

$$f(\ldots, v_{i+1}, v_i, \ldots) = -f(\ldots, v_i, v_{i+1}, \ldots)$$

for $i = 1, \ldots, k-1$.

3.5. Another Characterization of Alternating k-tensors

Let f be a k-tensor on a vector space V. Prove that f is alternating if and only if $f(v_1, \ldots, v_k) = 0$ whenever two of the vectors $v_1, \ldots, v_k$ are equal.

3.6. Wedge Product and Scalars

Let V be a vector space. For $a, b \in \mathbb{R}, f \in A_k(V)$, and $g \in A_\ell(V)$, show that $af \wedge bg = (ab) f \wedge g$.

3.7. Transformation Rule for a Wedge Product of Covectors

Suppose two sets of covectors on a vector space $V, \beta^1, \ldots, \beta^k$ and $\gamma^1, \ldots, \gamma^k$, are related by

$$\beta^i = \sum_{j=1}^k a_j^i \gamma^j, \quad i = 1, \ldots, k$$

for a $k \times k$ matrix $A = \left[a_j^i\right]$. Show that

$$\beta^1 \wedge \cdots \wedge \beta^k = (\det A) \gamma^1 \wedge \cdots \wedge \gamma^k.$$

3.8. Transformation Rule for k-covectors

Let f be a k-covector on a vector space V. Suppose two sets of vectors $u_1, \ldots, u_k$ and $v_1, \ldots, v_k$ in V are related by

$$u_j = \sum_{i=1}^k a_j^i v_i, \quad j = 1, \ldots, k$$

for a $k \times k$ matrix $A = \left[a_j^i\right]$. Show that

$$f\left(u_1, \ldots, u_k\right) = (\det A) f\left(v_1, \ldots, v_k\right)$$

3.9. Vanishing of a Covector of top Degree

Let V be a vector space of dimension n. Prove that if an n-covector ω vanishes on a basis $e_1, \ldots, e_n$ for V, then ω is the zero covector on V.

3.10.* Linear Independence of Covectors

Let $\alpha^1, \ldots, \alpha^k$ be 1 -covectors on a vector space V. Show that $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$ if and only if $\alpha^1, \ldots, \alpha^k$ are linearly independent in the dual space $V^\vee$.

3.11.* Exterior Multiplication

Let α be a nonzero 1 -covector and γ a k-covector on a finite-dimensional vector space V. Show that $\alpha \wedge \gamma = 0$ if and only if $\gamma = \alpha \wedge \beta$ for some ($k-1$)-covector β on V.

6.4 §4 Differential Forms on $\mathbb{R}^n$

Just as a vector field assigns a tangent vector to each point of an open subset U of $\mathbb{R}^n$, so dually a differential k-form assigns a k-covector on the tangent space to each point of U. The wedge product of differential forms is defined pointwise as the wedge product of multicovectors. Since differential forms exist on an open set, not just at a single point, there is a notion of differentiation for differential forms. In fact, there is a unique one, called the exterior derivative, characterized by three natural properties. Although we define it using the standard coordinates of $\mathbb{R}^n$, the exterior derivative turns out to be independent of coordinates, as we shall see later, and is therefore intrinsic to a manifold. It is the ultimate abstract extension to a manifold of the gradient, curl, and divergence of vector calculus in $\mathbb{R}^3$. Differential forms extend Grassmann's exterior algebra from the tangent space at a point globally to an entire manifold. Since its creation around the turn of the twentieth century, generally credited to É. Cartan [5] and H. Poincaré [34], the calculus of differential forms has had far-reaching consequences in geometry, topology, and physics. In fact, certain physical concepts such as electricity and magnetism are best formulated in terms of differential forms.

In this section we will study the simplest case, that of differential forms on an open subset of $\mathbb{R}^n$. Even in this setting, differential forms already provide a way to unify the main theorems of vector calculus in $\mathbb{R}^3$.

6.4.1 4.1 Differential 1-forms and the Differential of a Function

The cotangent space to $\mathbb{R}^n$ at p, denoted by $T_p^*\left(\mathbb{R}^n\right)$ or $T_p^*\mathbb{R}^n$, is defined to be the dual space $\left(T_p\mathbb{R}^n\right)^\vee$ of the tangent space $T_p\left(\mathbb{R}^n\right)$. Thus, an element of the cotangent space $T_p^*\left(\mathbb{R}^n\right)$ is a covector or a linear functional on the tangent space $T_p\left(\mathbb{R}^n\right)$. In parallel with the definition of a vector field, a covector field or a differential 1-form on an open subset U of $\mathbb{R}^n$ is a function ω that assigns to each point p in U a covector $\omega_p \in T_p^*\left(\mathbb{R}^n\right)$,

$$\begin{aligned} \omega : U &\to \bigcup_{p \in U} T_p^*\left(\mathbb{R}^n\right) \\ p &\mapsto \omega_p \in T_p^*\left(\mathbb{R}^n\right) \end{aligned}$$

Note that in the union $\bigcup_{p \in U} T_p^*(\mathbb{R}^n)$, the sets $T_p^*(\mathbb{R}^n)$ are all disjoint. We call a differential 1-form a 1-form for short.

From any C^∞ function $f : U \to \mathbb{R}$, we can construct a 1 -form df, called the differential of f, as follows. For $p \in U$ and $X_p \in T_pU$, define

$$(df)_p(X_p) = X_p f$$

A few words may be in order about the definition of the differential. The directional derivative of a function in the direction of a tangent vector at a point p sets up a bilinear pairing

$$\begin{aligned} T_p(\mathbb{R}^n) \times C_p^\infty(\mathbb{R}^n) &\to \mathbb{R}, \\ (X_p, f) &\mapsto \langle X_p, f \rangle = X_p f. \end{aligned}$$

One may think of a tangent vector as a function on the second argument of this pairing: $\langle X_p, \cdot \rangle$. The differential $(df)_p$ at p is a function on the first argument of the pairing:

$$(df)_p = \langle \cdot, f \rangle.$$

The value of the differential df at p is also written $df|_p$.
Let $x^1, \ldots, x^n$ be the standard coordinates on $\mathbb{R}^n$. We saw in Subsection 2.3 that the set $\left\{\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p\right\}$ is a basis for the tangent space $T_p(\mathbb{R}^n)$.
Proposition 4.1. If $x^1, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n, \left\{(dx^1)_p, \ldots, (dx^n)_p\right\}$ is the basis for the cotangent space $T_p^*(\mathbb{R}^n)$ dual to the basis $\left\{\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p\right\}$ for the tangent space $T_p(\mathbb{R}^n)$.

Proof. By definition,

$$\left(dx^i\right)_p \left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \left.\frac{\partial}{\partial x^j}\right|_p x^i = \delta_j^i.$$

If ω is a 1 -form on an open subset U of $\mathbb{R}^n$, then by Proposition 4.1, at each point p in U, ω can be written as a linear combination

$$\omega_p = \sum a_i(p)\left(dx^i\right)_p$$

for some $a_i(p) \in \mathbb{R}$. As p varies over U, the coefficients a_i become functions on U, and we may write $\omega = \sum a_i dx^i$. The covector field ω is said to be C^∞ on U if the coefficient functions a_i are all C^∞ on U.

If x, y, and z are the coordinates on $\mathbb{R}^3$, then dx, dy and dz are 1 -forms on $\mathbb{R}^3$. In this way, we give meaning to what was merely a notation in elementary calculus.

Proposition 4.2 (The differential in terms of coordinates). If $f : U \to \mathbb{R}$ is a C^∞ function on an open set U in $\mathbb{R}^n$, then

$$df = \sum \frac{\partial f}{\partial x^i} dx^i \tag{4.1}$$

Proof. By Proposition 4.1, at each point p in U,

$$(df)_p = \sum a_i(p)\left(dx^i\right)_p \tag{4.2}$$

for some real numbers $a_i(p)$ depending on p. Thus, $df = \sum a_i dx^i$ for some real functions a_i on U. To find a_j, apply both sides of (4.2) to the coordinate vector field $\partial/\partial x^j$:

$$df\left(\frac{\partial}{\partial x^j}\right)=\sum_i a_i dx^i\left(\frac{\partial}{\partial x^j}\right)=\sum_i a_i \delta_j^i=a_j.$$

On the other hand, by the definition of the differential,

$$df\left(\frac{\partial}{\partial x^j}\right)=\frac{\partial f}{\partial x^j}$$

Equation (4.1) shows that if f is a C^∞ function, then the 1 -form df is also C^∞.
Example. Differential 1-forms arise naturally even if one is interested only in tangent vectors. Every tangent vector $X_p \in T_p\left(\mathbb{R}^n\right)$ is a linear combination of the standard basis vectors:

$$X_p=\sum_i b^i\left(X_p\right)\left.\frac{\partial}{\partial x^i}\right|_p .$$

In Example 3.3 we saw that at each point $p \in \mathbb{R}^n$, we have $b^i\left(X_p\right)=\left(dx^i\right)_p\left(X_p\right)$. Hence, the coefficient b^i of a vector at p with respect to the standard basis $\left.\partial / \partial x^1\right|_p, \ldots,\left.\partial / \partial x^n\right|_p$ is none other than the dual covector $\left.dx^i\right|_p$ on $\mathbb{R}^n$. As p varies, $b^i=dx^i$.

6.4.2 4.2 Differential k-forms

More generally, a differential form ω of degree k or a k-form on an open subset U of $\mathbb{R}^n$ is a function that assigns to each point p in U an alternating k-linear function on the tangent space $T_p\left(\mathbb{R}^n\right)$, i.e., $\omega_p \in A_k\left(T_p \mathbb{R}^n\right)$. Since $A_1\left(T_p \mathbb{R}^n\right)=T_p^*\left(\mathbb{R}^n\right)$, the definition of a k-form generalizes that of a 1 -form in Subsection 4.1.

By Proposition 3.29, a basis for $A_k\left(T_p \mathbb{R}^n\right)$ is

$$dx_p^I=dx_p^{i_1} \wedge \cdots \wedge dx_p^{i_k}, \quad 1 \leq i_1<\cdots<i_k \leq n$$

Therefore, at each point p in U, ω_p is a linear combination

$$\omega_p=\sum a_I(p) dx_p^I, \quad 1 \leq i_1<\cdots<i_k \leq n$$

and a k-form ω on U is a linear combination

$$\omega=\sum a_I dx^I$$

with function coefficients $a_I: U \rightarrow \mathbb{R}$. We say that a k-form ω is C^∞ on U if all the coefficients a_I are C^∞ functions on U.

Denote by $\Omega^k(U)$ the vector space of $C^\infty k$-forms on U. A 0 -form on U assigns to each point p in U an element of $A_0\left(T_p \mathbb{R}^n\right)=\mathbb{R}$. Thus, a 0 -form on U is simply a function on U, and $\Omega^0(U)=C^\infty(U)$.

There are no nonzero differential forms of degree $>n$ on an open subset of $\mathbb{R}^n$. This is because if $\deg dx^I>n$, then in the expression dx^I at least two of the 1-forms $dx^i \alpha$ must be the same, forcing $dx^I=0$.

The wedge product of a k-form ω and an ℓ-form τ on an open set U is defined pointwise:

$$(\omega \wedge \tau)_p=\omega_p \wedge \tau_p, \quad p \in U$$

In terms of coordinates, if $\omega=\sum_I a_I dx^I$ and $\tau=\sum_J b_J dx^J$, then

$$\omega \wedge \tau=\sum_{I, J}\left(a_I b_J\right) dx^I \wedge dx^J$$

In this sum, if I and J are not disjoint on the right-hand side, then $dx^I \wedge dx^J = 0$. Hence, the sum is actually over disjoint multi-indices:

$$\omega \wedge \tau = \sum_{I,J \text{ disjoint}} (a_I b_J)\, dx^I \wedge dx^J$$

which shows that the wedge product of two C^∞ forms is C^∞. So the wedge product is a bilinear map

$$\wedge : \Omega^k(U) \times \Omega^\ell(U) \to \Omega^{k+\ell}(U)$$

By Propositions 3.21 and 3.25, the wedge product of differential forms is anticommutative and associative.

In case one of the factors has degree 0 , say $k = 0$, the wedge product

$$\wedge : \Omega^0(U) \times \Omega^\ell(U) \to \Omega^\ell(U)$$

is the pointwise multiplication of a $C^\infty \ell$-form by a C^∞ function:

$$(f \wedge \omega)_p = f(p) \wedge \omega_p = f(p)\omega_p$$

since as we noted in Subsection 3.7, the wedge product with a 0 -covector is scalar multiplication. Thus, if $f \in C^\infty(U)$ and $\omega \in \Omega^\ell(U)$, then $f \wedge \omega = f\omega$.
Example. Let x, y, z be the coordinates on $\mathbb{R}^3$. The C^∞1-forms on $\mathbb{R}^3$ are

$$f dx + g dy + h dz$$

where f, g, h range over all C^∞ functions on $\mathbb{R}^3$. The C^∞ 2-forms are

$$f dy \wedge dz + g dx \wedge dz + h dx \wedge dy$$

and the C^∞3-forms are

$$f dx \wedge dy \wedge dz$$

Exercise 4.3 (A basis for 3-covectors).* Let x^1, x^2, x^3, x^4 be the coordinates on $\mathbb{R}^4$ and p a point in $\mathbb{R}^4$. Write down a basis for the vector space $A_3\left(T_p\left(\mathbb{R}^4\right)\right)$.

With the wedge product as multiplication and the degree of a form as the grading, the direct sum $\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$ becomes an anticommutative graded algebra over $\mathbb{R}$. Since one can multiply $C^\infty k$-forms by C^∞ functions, the set $\Omega^k(U)$ of $C^\infty k$ forms on U is both a vector space over $\mathbb{R}$ and a module over $C^\infty(U)$, and so the direct sum $\Omega^*(U) = \bigoplus_{k=0}^n \Omega^k(U)$ is also a module over the ring $C^\infty(U)$ of C^∞ functions.

6.4.3 4.3 Differential Forms as Multilinear Functions on Vector Fields

If ω is a C^∞1-form and X is a C^∞ vector field on an open set U in $\mathbb{R}^n$, we define a function $\omega(X)$ on U by the formula

$$\omega(X)_p = \omega_p\left(X_p\right), \quad p \in U$$

Written out in coordinates,

$$\omega = \sum a_i dx^i, \quad X = \sum b^j \frac{\partial}{\partial x^j} \quad \text{for some } a_i, b^j \in C^\infty(U),$$

so

$$\omega(X)=\left(\sum a_i dx^i\right)\left(\sum b^j \frac{\partial}{\partial x^j}\right)=\sum a_i b^i,$$

which shows that $\omega(X)$ is C^∞ on U. Thus, a C^∞ 1-form on U gives rise to a map from $\mathfrak{X}(U)$ to $C^\infty(U)$.

This function is actually linear over the ring $C^\infty(U)$; i.e., if $f \in C^\infty(U)$, then $\omega(fX)=f\omega(X)$. To show this, it suffices to evaluate $\omega(fX)$ at an arbitrary point $p \in U$:

$$\begin{aligned}(\omega(fX))_p &= \omega_p\left(f(p)X_p\right) && (\text{ definition of } \omega(fX))\\ &= f(p)\omega_p\left(X_p\right) && (\omega_p \text{ is } \mathbb{R}\text{-linear })\\ &= (f\omega(X))_p && (\text{ definition of } f\omega(X))\end{aligned}$$

Let $\mathcal{F}(U)=C^\infty(U)$. In this notation, a 1-form ω on U gives rise to an $\mathcal{F}(U)$ linear map $\mathfrak{X}(U) \to \mathcal{F}(U), X \mapsto \omega(X)$. Similarly, a k-form ω on U gives rise to a k-linear map over $\mathcal{F}(U)$,

$$\begin{aligned}\underbrace{\mathfrak{X}(U) \times \cdots \times \mathfrak{X}(U)}_{k \text{ times}} &\to \mathcal{F}(U),\\ (X_1,\ldots,X_k) &\mapsto \omega\left(X_1,\ldots,X_k\right).\end{aligned}$$

Exercise 4.4 (Wedge product of a 2-form with a 1-form).* Let ω be a 2-form and τ a 1form on $\mathbb{R}^3$. If X, Y, Z are vector fields on M, find an explicit formula for $(\omega \wedge \tau)(X, Y, Z)$ in terms of the values of ω and τ on the vector fields X, Y, Z.

6.4.4 4.4 The Exterior Derivative

To define the exterior derivative of a $C^\infty k$-form on an open subset U of $\mathbb{R}^n$, we first define it on 0 -forms: the exterior derivative of a C^∞ function $f \in C^\infty(U)$ is defined to be its differential $df \in \Omega^1(U)$; in terms of coordinates, Proposition 4.2 gives

$$df=\sum \frac{\partial f}{\partial x^i} dx^i$$

Definition 4.5. For $k \geq 1$, if $\omega=\sum_I a_I dx^I \in \Omega^k(U)$, then

$$d\omega=\sum_I da_I \wedge dx^I=\sum_I\left(\sum_j \frac{\partial a_I}{\partial x^j} dx^j\right) \wedge dx^I \in \Omega^{k+1}(U).$$

Example. Let ω be the 1 -form $f dx+g dy$ on $\mathbb{R}^2$, where f and g are C^∞ functions on $\mathbb{R}^2$. To simplify the notation, write $f_x=\partial f / \partial x, f_y=\partial f / \partial y$. Then

$$\begin{aligned}d\omega &= df \wedge dx+dg \wedge dy\\ &= \left(f_x dx+f_y dy\right) \wedge dx+\left(g_x dx+g_y dy\right) \wedge dy\\ &= \left(g_x-f_y\right) dx \wedge dy\end{aligned}$$

In this computation $dy \wedge dx=-dx \wedge dy$ and $dx \wedge dx=dy \wedge dy=0$ by the anticommutative property of the wedge product (Proposition 3.21 and Corollary 3.23).

Definition 4.6. Let $A=\oplus_{k=0}^\infty A^k$ be a graded algebra over a field K. An antiderivation of the graded algebra A is a K-linear map $D: A \to A$ such that for $a \in A^k$ and $b \in A^\ell$,

$$D(ab)=(Da)b+(-1)^k a Db. \tag{4.3}$$

If there is an integer m such that the antiderivation D sends A^k to A^{k+m} for all k, then we say that it is an antiderivation of degree m. By defining $A_k=0$ for $k<0$, we can extend

the grading of a graded algebra A to negative integers. With this extension, the degree m of an antiderivation can be negative. (An example of an antiderivation of degree -1 is interior multiplication, to be discussed in Subsection 20.4.)

Proposition 4.7.

(i) The exterior differentiation $d : \Omega^*(U) \to \Omega^*(U)$ is an antiderivation of degree 1 :

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^{\deg \omega} \omega \wedge d\tau$$

(ii) $d^2 = 0$.

(iii) If $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, then $(df)(X) = Xf$.

Proof.

(i) Since both sides of (4.3) are linear in ω and in τ, it suffices to check the equality for $\omega = f dx^I$ and $\tau = g dx^J$. Then

$$\begin{aligned} d(\omega \wedge \tau) &= d\left(f g dx^I \wedge dx^J\right) \\ &= \sum \frac{\partial (fg)}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \\ &= \sum \frac{\partial f}{\partial x^i} g dx^i \wedge dx^I \wedge dx^J + \sum f \frac{\partial g}{\partial x^i} dx^i \wedge dx^I \wedge dx^J \end{aligned}$$

In the second sum, moving the 1 -form $(\partial g / \partial x^i)\, dx^i$ across the k-form dx^I results in the sign $(-1)^k$ by anticommutativity. Hence,

$$\begin{aligned} d(\omega \wedge \tau) &= \sum \frac{\partial f}{\partial x^i} dx^i \wedge dx^I \wedge g dx^J + (-1)^k \sum f dx^I \wedge \frac{\partial g}{\partial x^i} dx^i \wedge dx^J \\ &= d\omega \wedge \tau + (-1)^k \omega \wedge d\tau \end{aligned}$$

(ii) Again by the $\mathbb{R}$-linearity of d, it suffices to show that $d^2\omega = 0$ for $\omega = f dx^I$. We compute:

$$d^2\left(f dx^I\right) = d\left(\sum \frac{\partial f}{\partial x^i} dx^i \wedge dx^I\right) = \sum \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i \wedge dx^I$$

In this sum if $i = j$, then $dx^j \wedge dx^i = 0$; if $i \neq j$, then $\partial^2 f / \partial x^i \partial x^j$ is symmetric in i and j, but $dx^j \wedge dx^i$ is alternating in i and j, so the terms with $i \neq j$ pair up and cancel each other. For example,

$$\begin{aligned} &\frac{\partial^2 f}{\partial x^1 \partial x^2} dx^1 \wedge dx^2 + \frac{\partial^2 f}{\partial x^2 \partial x^1} dx^2 \wedge dx^1 \\ &\quad = \frac{\partial^2 f}{\partial x^1 \partial x^2} dx^1 \wedge dx^2 + \frac{\partial^2 f}{\partial x^1 \partial x^2}\left(-dx^1 \wedge dx^2\right) = 0 \end{aligned}$$

Therefore, $d^2\left(f dx^I\right) = 0$.

(iii) This is simply the definition of the exterior derivative of a function as the differential of the function.

Proposition 4.8 (Characterization of the exterior derivative). The three properties of Proposition 4.7 uniquely characterize exterior differentiation on an open set U in $\mathbb{R}^n$; that is, if $D : \Omega^*(U) \to \Omega^*(U)$ is (i) an antiderivation of degree 1 such that (ii) $D^2 = 0$ and (iii) $(Df)(X) = Xf$ for $f \in C^\infty(U)$ and $X \in \mathfrak{X}(U)$, then $D = d$.

Proof. Since every k-form on U is a sum of terms such as $f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, by linearity it suffices to show that $D = d$ on a k-form of this type. By (iii), $Df = df$ on C^∞ functions. It follows that $D dx^i = D D x^i = 0$ by (ii). A simple induction on k, using the antiderivation property of D, proves that for all k and all multi-indices I of length k,

$$D\left(dx^I\right) = D\left(dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) = 0 \tag{4.4}$$

Finally, for every k-form $f dx^I$,

$$\begin{aligned} D\left(f dx^I\right) &= (Df) \wedge dx^I + f D\left(dx^I\right) && (\text{ by (i)}) \\ &= (df) \wedge dx^I && (\text{ by (ii) and (4.4)}) \\ &= d\left(f dx^I\right) && (\text{ definition of } d). \end{aligned}$$

Hence, $D = d$ on $\Omega^*(U)$.

6.4.5 4.5 Closed Forms and Exact Forms

A k-form ω on U is closed if $d\omega = 0$; it is exact if there is a ($k-1$)-form τ such that $\omega = d\tau$ on U. Since $d(d\tau) = 0$, every exact form is closed. In the next section we will discuss the meaning of closed and exact forms in the context of vector calculus on $\mathbb{R}^3$.

Exercise 4.9 (A closed 1-form on the punctured plane). Define a 1-form ω on $\mathbb{R}^2 - \{\mathbf{0}\}$ by

$$\omega = \frac{1}{x^2 + y^2}(-y dx + x dy)$$

Show that ω is closed.

A collection of vector spaces $\left\{V^k\right\}_{k=0}^{\infty}$ with linear maps $d_k : V^k \to V^{k+1}$ such that $d_{k+1} \circ d_k = 0$ is called a differential complex or a cochain complex. For any open subset U of $\mathbb{R}^n$, the exterior derivative d makes the vector space $\Omega^*(U)$ of C^∞ forms on U into a cochain complex, called the de Rham complex of U :

$$0 \to \Omega^0(U) \xrightarrow{d} \Omega^1(U) \xrightarrow{d} \Omega^2(U) \to \cdots .$$

The closed forms are precisely the elements of the kernel of d, and the exact forms are the elements of the image of d.

6.4.6 4.6 Applications to Vector Calculus

The theory of differential forms unifies many theorems in vector calculus on $\mathbb{R}^3$. We summarize here some results from vector calculus and then show how they fit into the framework of differential forms.

By a vector-valued function on an open subset U of $\mathbb{R}^3$, we mean a function $\mathbf{F} = \langle P, Q, R \rangle : U \to \mathbb{R}^3$. Such a function assigns to each point $p \in U$ a vector $\mathbf{F}_p \in \mathbb{R}^3 \simeq T_p\left(\mathbb{R}^3\right)$. Hence, a vector-valued function on U is precisely a vector field on U. Recall the three operators gradient, curl, and divergence on scalar- and vector-valued functions on U :

$$\{ \text{ scalar func. } \} \xrightarrow{\text{grad}} \{ \text{ vector func. } \} \xrightarrow{\text{curl}} \{ \text{ vector func. } \} \xrightarrow{\text{div}} \{ \text{ scalar func. } \},$$

$$\operatorname{grad} f = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} f = \begin{bmatrix} f_x \\ f_y \\ f_z \end{bmatrix},$$

$$\operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \times \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} R_y - Q_z \\ -\left(R_x - P_z\right) \\ Q_x - P_y \end{bmatrix},$$

$$\operatorname{div} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{bmatrix} \cdot \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = P_x + Q_y + R_z.$$

Since every 1-form on U is a linear combination with function coefficients of dx, dy, and dz, we can identify 1 -forms with vector fields on U via

$$Pdx + Qdy + Rdz \longleftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix}.$$

Similarly, 2-forms on U can also be identified with vector fields on U :

$$Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \longleftrightarrow \begin{bmatrix} P \\ Q \\ R \end{bmatrix},$$

and 3-forms on U can be identified with functions on U :

$$f dx \wedge dy \wedge dz \longleftrightarrow f.$$

In terms of these identifications, the exterior derivative of a 0 -form f is

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz \longleftrightarrow \begin{bmatrix} \partial f / \partial x \\ \partial f / \partial y \\ \partial f / \partial x \end{bmatrix} = \operatorname{grad} f;$$

the exterior derivative of a 1-form is

$$\begin{aligned} & d(Pdx + Qdy + Rdz) \\ & \quad = (R_y - Q_z)\, dy \wedge dz - (R_x - P_z)\, dz \wedge dx + (Q_x - P_y)\, dx \wedge dy \end{aligned} \tag{4.5}$$

which corresponds to

$$\operatorname{curl} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = \begin{bmatrix} R_y - Q_z \\ -(R_x - P_z) \\ Q_x - P_y \end{bmatrix};$$

the exterior derivative of a 2 -form is

$$\begin{aligned} & d(Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy) \\ & \quad = (P_x + Q_y + R_z)\, dx \wedge dy \wedge dz \end{aligned} \tag{4.6}$$

which corresponds to

$$\operatorname{div} \begin{bmatrix} P \\ Q \\ R \end{bmatrix} = P_x + Q_y + R_z.$$

Thus, after appropriate identifications, the exterior derivatives d on 0 -forms, 1 forms, and 2-forms are simply the three operators grad, curl, and div. In summary, on an open subset U of $\mathbb{R}^3$, there are identifications

$$\begin{array}{ccccccccc} \Omega^0(U) & \xrightarrow{d} & \Omega^1(U) & \xrightarrow{d} & \Omega^2(U) & \xrightarrow{d} & \Omega^3(U) & & \\ & & \simeq\downarrow & & & & & & \\ C^\infty(U) & \xrightarrow[\text{grad}]{} & & & \mathfrak{X}(U) & \xrightarrow[\text{curl}]{} & \mathfrak{X}(U) & \xrightarrow[\text{div}]{} & C^\infty(U). \end{array} \quad \begin{array}{c} \simeq\downarrow \\ \simeq\downarrow \\ \simeq\downarrow \end{array}$$

Under these identifications, a vector field $\langle P, Q, R \rangle$ on $\mathbb{R}^3$ is the gradient of a C^∞ function f if and only if the corresponding 1-form $Pdx + Qdy + Rdz$ is df.

Next we recall three basic facts from calculus concerning grad, curl, and div.

Proposition A. $\operatorname{curl}(\operatorname{grad} f) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

Proposition B. $\operatorname{div}\left(\operatorname{curl}\begin{bmatrix} P \\ Q \\ R \end{bmatrix}\right) = 0$.

Proposition C. On $\mathbb{R}^3$, a vector field $\mathbf{F}$ is the gradient of some scalar function f if and only if $\operatorname{curl} \mathbf{F} = 0$.

Propositions A and B express the property $d^2 = 0$ of the exterior derivative on open subsets of $\mathbb{R}^3$; these are easy computations. Proposition C expresses the fact that a 1 -form on $\mathbb{R}^3$ is exact if and only if it is closed. Proposition C need not be true on a region other than $\mathbb{R}^3$, as the following well-known example from calculus shows.

Example. If $U = \mathbb{R}^3 - \{z\text{-axis }\}$, and $\mathbf{F}$ is the vector field

$$\mathbf{F} = \left\langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0 \right\rangle$$

on $\mathbb{R}^3$, then $\operatorname{curl} \mathbf{F} = \mathbf{0}$, but $\mathbf{F}$ is not the gradient of any C^∞ function on U. The reason is that if $\mathbf{F}$ were the gradient of a C^∞ function f on U, then by the fundamental theorem for line integrals, the line integral

$$\int_C -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

over any closed curve C would be zero. However, on the unit circle C in the (x, y) plane, with $x = \cos t$ and $y = \sin t$ for $0 \le t \le 2\pi$, this integral is

$$\int_C -ydx + xdy = \int_0^{2\pi} -(\sin t)d\cos t + (\cos t)d\sin t = 2\pi$$

In terms of differential forms, the 1-form

$$\omega = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

is closed but not exact on U. (This 1-form is defined by the same formula as the 1-form ω in Exercise 4.9, but is defined on a different space.)

It turns out that whether Proposition C is true for a region U depends only on the topology of U. One measure of the failure of a closed k-form to be exact is the quotient vector space

$$H^k(U) := \frac{\{\text{ closed } k\text{-forms on } U\}}{\{\text{ exact } k\text{-forms on } U\}},$$

called the k th de Rham cohomology of U.

The generalization of Proposition C to any differential form on $\mathbb{R}^n$ is called the Poincaré lemma: for $k \ge 1$, every closed k-form on $\mathbb{R}^n$ is exact. This is of course equivalent to the vanishing of the k th de Rham cohomology $H^k(\mathbb{R}^n)$ for $k \ge 1$. We will prove it in Section 27.

The theory of differential forms allows us to generalize vector calculus from $\mathbb{R}^3$ to $\mathbb{R}^n$ and indeed to a manifold of any dimension. The general Stokes theorem for a manifold that we will prove in Subsection 23.5 subsumes and unifies the fundamental theorem for line integrals, Green's theorem in the plane, the classical Stokes theorem for a surface in $\mathbb{R}^3$, and the divergence theorem. As a first step in this program, we begin the next chapter with the definition of a manifold.

6.4.7 4.7 Convention on Subscripts and Superscripts

In differential geometry it is customary to index vector fields with subscripts $e_1, \ldots, e_n$, and differential forms with superscripts $\omega^1, \ldots, \omega^n$. Being 0-forms, coordinate functions take superscripts: $x^1, \ldots, x^n$. Their differentials, being 1 -forms, should also have superscripts, and indeed they do: $dx^1, \ldots, dx^n$. Coordinate vector fields $\partial/\partial x^1, \ldots, \partial/\partial x^n$ are considered to have subscripts because the i in $\partial/\partial x^i$, although a superscript for x^i, is in the lower half of the fraction.

Coefficient functions can have superscripts or subscripts depending on whether they are the coefficient functions of a vector field or of a differential form. For a vector field $X = \sum a^i e_i$, the coefficient functions a^i have superscripts; the idea is that the superscript in a^i "cancels out" the subscript in e_i. For the same reason, the coefficient functions b_j in a differential form $\omega = \sum b_j dx^j$ have subscripts.

The beauty of this convention is that there is a "conservation of indices" on the two sides of an equality sign. For example, if $X = \sum a^i \partial/\partial x^i$, then

$$a^i = \left(dx^i\right)(X)$$

Here both sides have a net superscript i. As another example, if $\omega = \sum b_j dx^j$, then

$$\omega(X) = \left(\sum b_j dx^j\right)\left(\sum a^i \frac{\partial}{\partial x^i}\right) = \sum b_i a^i$$

after cancellation of superscripts and subscripts, both sides of the equality sign have zero net index. This convention is a useful mnemonic aid in some of the transformation formulas of differential geometry.

6.4.8 Problems

4.1. A 1-form on $\mathbb{R}^3$

Let ω be the 1 -form $zdx - dz$ and let X be the vector field $y\partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^3$. Compute $\omega(X)$ and $d\omega$.

4.2. A 2-form on $\mathbb{R}^3$

At each point $p \in \mathbb{R}^3$, define a bilinear function ω_p on $T_p\left(\mathbb{R}^3\right)$ by

$$\omega_p(\mathbf{a}, \mathbf{b}) = \omega_p\left(\begin{bmatrix} a^1 \\ a^2 \\ a^3 \end{bmatrix}, \begin{bmatrix} b^1 \\ b^2 \\ b^3 \end{bmatrix}\right) = p^3 \det \begin{bmatrix} a^1 & b^1 \\ a^2 & b^2 \end{bmatrix},$$

for tangent vectors $\mathbf{a}, \mathbf{b} \in T_p\left(\mathbb{R}^3\right)$, where p^3 is the third component of $p = (p^1, p^2, p^3)$. Since ω_p is an alternating bilinear function on $T_p\left(\mathbb{R}^3\right), \omega$ is a 2 -form on $\mathbb{R}^3$. Write ω in terms of the standard basis $dx^i \wedge dx^j$ at each point.

4.3. Exterior Calculus

Suppose the standard coordinates on $\mathbb{R}^2$ are called r and θ (this $\mathbb{R}^2$ is the (r, θ)-plane, not the (x, y)-plane). If $x = r\cos\theta$ and $y = r\sin\theta$, calculate dx, dy, and $dx \wedge dy$ in terms of dr and $d\theta$.

4.4. Exterior Calculus

Suppose the standard coordinates on $\mathbb{R}^3$ are called ρ, ϕ, and θ. If $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$, calculate dx, dy, dz, and $dx \wedge dy \wedge dz$ in terms of $d\rho, d\phi$, and $d\theta$.

4.5. Wedge Product

Let α be a 1 -form and β a 2 -form on $\mathbb{R}^3$. Then

$$\begin{aligned}\alpha &= a_1 dx^1 + a_2 dx^2 + a_3 dx^3 \\ \beta &= b_1 dx^2 \wedge dx^3 + b_2 dx^3 \wedge dx^1 + b_3 dx^1 \wedge dx^2\end{aligned}$$

Simplify the expression $\alpha \wedge \beta$ as much as possible.

4.6. Wedge Product and Cross Product

The correspondence between differential forms and vector fields on an open subset of $\mathbb{R}^3$ in Subsection 4.6 also makes sense pointwise. Let V be a vector space of dimension 3 with basis e_1, e_2, e_3, and dual basis $\alpha^1, \alpha^2, \alpha^3$. To a 1 -covector $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ on V, we associate the vector $\mathbf{v}_\alpha = \langle a_1, a_2, a_3\rangle \in \mathbb{R}^3$. To the 2 -covector

$$\gamma = c_1\alpha^2 \wedge \alpha^3 + c_2\alpha^3 \wedge \alpha^1 + c_3\alpha^1 \wedge \alpha^2$$

on V, we associate the vector $\mathbf{v}_\gamma = \langle c_1, c_2, c_3\rangle \in \mathbb{R}^3$. Show that under this correspondence, the wedge product of 1 -covectors corresponds to the cross product of vectors in $\mathbb{R}^3$: if $\alpha = a_1\alpha^1 + a_2\alpha^2 + a_3\alpha^3$ and $\beta = b_1\alpha^1 + b_2\alpha^2 + b_3\alpha^3$, then $\mathbf{v}_{\alpha\wedge\beta} = \mathbf{v}_\alpha \times \mathbf{v}_\beta$.

4.7. Commutator of Derivations and Antiderivations

Let $A = \oplus_{k=-\infty}^{\infty} A^k$ be a graded algebra over a field K with $A^k = 0$ for $k < 0$. Let m be an integer. A superderivation of A of degree m is a K-linear map $D : A \to A$ such that for all k, $D\left(A^k\right) \subset A^{k+m}$ and for all $a \in A^k$ and $b \in A^\ell$,

$$D(ab) = (Da)b + (-1)^{km} a(Db)$$

If D_1 and D_2 are two superderivations of A of respective degrees m_1 and m_2, define their commutator to be

$$[D_1, D_2] = D_1 \circ D_2 - (-1)^{m_1 m_2} D_2 \circ D_1$$

Show that $[D_1, D_2]$ is a superderivation of degree $m_1 + m_2$. (A superderivation is said to be even or odd depending on the parity of its degree. An even superderivation is a derivation; an odd superderivation is an antiderivation.)

7 2 Manifolds

Intuitively, a manifold is a generalization of curves and surfaces to higher dimensions. It is locally Euclidean in that every point has a neighborhood, called a chart, homeomorphic to an open subset of $\mathbb{R}^n$. The coordinates on a chart allow one to carry out computations as though in a Euclidean space, so that many concepts from $\mathbb{R}^n$, such as differentiability, point-derivations, tangent spaces, and differential forms, carry over to a manifold.

Like most fundamental mathematical concepts,

Bernhard Riemann
(1826-1866)

the idea of a manifold did not originate with a single person, but is rather the distillation of years of collective activity. In his masterpiece Disquisitiones generales circa superficies curvas ("General Investigations of Curved Surfaces") published in 1827, Carl Friedrich Gauss freely used local coordinates on a surface, and so he already had the idea of charts. Moreover, he appeared to be the first to consider a surface as an abstract space existing in its own right, independent of a particular embedding in a Euclidean space. Bernhard Riemann's inaugural lecture Über die Hypothesen, welche der Geometrie zu Grunde liegen ("On the hypotheses that underlie geometry") in Göttingen in 1854 laid the foundations of higher-dimensional differential geometry. Indeed, the word "manifold" is a direct translation of the German word "Mannigfaltigkeit," which Riemann used to describe the objects of his inquiry. This was followed by the work of Henri Poincaré in the late nineteenth century on homology, in which locally Euclidean spaces figured prominently. The late nineteenth and early twentieth centuries were also a period of feverish development in point-set topology. It was not until 1931 that one finds the modern definition of a manifold based on point-set topology and a group of transition functions [37].

In this chapter we give the basic definitions and properties of a smooth manifold and of smooth maps between manifolds. Initially, the only way we have to verify that a space is a manifold is to exhibit a collection of C^∞ compatible charts covering the space. In Section 7 we describe a set of sufficient conditions under which a quotient topological space becomes a manifold, giving us a second way to construct manifolds.

7.1 §5 Manifolds

While there are many kinds of manifolds-for example, topological manifolds, C^k manifolds, analytic manifolds, and complex manifolds-in this book we are concerned mainly with smooth manifolds. Starting with topological manifolds, which are Hausdorff, second countable, locally Euclidean spaces, we introduce the concept of a maximal C^∞ atlas, which makes a topological manifold into a smooth manifold. This is illustrated with a few simple examples.

7.1.1 5.1 Topological Manifolds

We first recall a few definitions from point-set topology. For more details, see Appendix A. A topological space is second countable if it has a countable basis. A neighborhood of a point p in a topological space M is any open set containing p. An open cover M is a collection $\{U_\alpha\}_{\alpha\in A}$ of open sets in M whose union $\bigcup_{\alpha\in A} U_\alpha$ is M.

Definition 5.1. A topological space M is locally Euclidean of dimension n if every point p in M has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi : U \to \mathbb{R}^n)$ a chart, U a coordinate neighborhood or a coordinate open set, and ϕ a coordinate map or a coordinate system on U. We say that a chart (U, ϕ) is centered at $p \in U$ if $\phi(p) = 0$.

Definition 5.2. A topological manifold is a Hausdorff, second countable, locally Euclidean space. It is said to be of dimension n if it is locally Euclidean of dimension n.

For the dimension of a topological manifold to be well defined, we need to know that for $n \neq m$ an open subset of $\mathbb{R}^n$ is not homeomorphic to an open subset of $\mathbb{R}^m$. This fact, called invariance of dimension, is indeed true, but is not easy to prove directly. We will not pursue this point, since we are mainly interested in smooth manifolds, for which the analogous result is easy to prove (Corollary 8.7). Of course, if a topological manifold has several connected components, it is possible for each component to have a different dimension.

Example. The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$, where $\mathbb{1}_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It is the prime example of a topological manifold. Every open subset of $\mathbb{R}^n$ is also a topological manifold, with chart $(U, \mathbb{1}_U)$.

Recall that the Hausdorff condition and second countability are "hereditary properties"; that is, they are inherited by subspaces: a subspace of a Hausdorff space is Hausdorff (Proposition A.19) and a subspace of a second-countable space is second countable (Proposition A.14). So any subspace of $\mathbb{R}^n$ is automatically Hausdorff and second countable.

Example 5.3 (A cusp). The graph of $y = x^{2/3}$ in $\mathbb{R}^2$ is a topological manifold (Figure 5.1 (a)). By virtue of being a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean, because it is homeomorphic to $\mathbb{R}$ via $\left(x, x^{2/3}\right) \mapsto x$.

Fig. 5.1.

Example 5.4 (A cross). Show that the cross in $\mathbb{R}^2$ in Figure 5.1 with the subspace topology is not locally Euclidean at the intersection p, and so cannot be a topological manifold.

Solution. Suppose the cross is locally Euclidean of dimension n at the point p. Then p has a neighborhood U homeomorphic to an open ball $B := B(0, \varepsilon) \subset \mathbb{R}^n$ with p mapping to 0 . The homeomorphism $U \to B$ restricts to a homeomorphism $U - \{p\} \to B - \{0\}$. Now $B - \{0\}$ is either connected if $n \geq 2$ or has two connected components if $n = 1$. Since $U - \{p\}$ has four connected components, there can be no homeomorphism from $U - \{p\}$ to $B - \{0\}$. This contradiction proves that the cross is not locally Euclidean at p. $\square$

7.1.2 5.2 Compatible Charts

Suppose ($U, \phi : U \to \mathbb{R}^n$) and ($V, \psi : V \to \mathbb{R}^n$) are two charts of a topological manifold. Since $U \cap V$ is open in U and $\phi : U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, the image $\phi(U \cap V)$ will also be an open subset of $\mathbb{R}^n$. Similarly, $\psi(U \cap V)$ is an open subset of $\mathbb{R}^n$.

Definition 5.5. Two charts $(U, \phi : U \to \mathbb{R}^n), (V, \psi : V \to \mathbb{R}^n)$ of a topological manifold are C^∞-compatible if the two maps

$$\phi \circ \psi^{-1} : \psi(U \cap V) \to \phi(U \cap V), \quad \psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞ (Figure 5.2). These two maps are called the transition functions between the charts. If $U \cap V$ is empty, then the two charts are automatically C^∞-compatible. To simplify the notation, we will sometimes write $U_{\alpha\beta}$ for $U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma}$ for $U_\alpha \cap U_\beta \cap U_\gamma$.

Fig. 5.2. The transition function $\psi \circ \phi^{-1} $ is defined on $\phi(U \cap V)$.}

Since we are interested only in C^∞-compatible charts, we often omit mention of " C^∞ " and speak simply of compatible charts.

Definition 5.6. A C^∞ atlas or simply an atlas on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞-compatible charts that cover M, i.e., such that $M = \bigcup_\alpha U_\alpha$.

Fig. 5.3. A $C^{\infty} $ atlas on a circle.}

Example 5.7 (AC^∞ atlas on a circle). The unit circle S^1 in the complex plane $\mathbb{C}$ may be described as the set of points $\{e^{it} \in \mathbb{C} \mid 0 \le t \le 2\pi\}$. Let U_1 and U_2 be the two open subsets of S^1 (see Figure 5.3)

$$U_1 = \left\{e^{it} \in \mathbb{C} \mid -\pi < t < \pi\right\},$$
$$U_2 = \left\{e^{it} \in \mathbb{C} \mid 0 < t < 2\pi\right\},$$

and define $\phi_\alpha : U_\alpha \to \mathbb{R}$ for $\alpha = 1, 2$ by

$$\phi_1\left(e^{it}\right) = t, \quad -\pi < t < \pi,$$
$$\phi_2\left(e^{it}\right) = t, \quad 0 < t < 2\pi.$$

Both ϕ_1 and ϕ_2 are branches of the complex log function $(1/i) \log z$ and are homeomorphisms onto their respective images. Thus, (U_1, ϕ_1) and (U_2, ϕ_2) are charts on S^1. The intersection $U_1 \cap U_2$ consists of two connected components,

$$A = \left\{ e^{it} \mid -\pi < t < 0 \right\},$$
$$B = \left\{ e^{it} \mid 0 < t < \pi \right\},$$

with

$$\phi_1 \left(U_1 \cap U_2 \right) = \phi_1(A \amalg B) = \phi_1(A) \amalg \phi_1(B) =]-\pi, 0[\amalg]0, \pi[,$$
$$\phi_2 \left(U_1 \cap U_2 \right) = \phi_2(A \amalg B) = \phi_2(A) \amalg \phi_2(B) =]\pi, 2\pi[???]0, \pi[.$$

Here we use the notation $A \amalg B$ to indicate a union in which the two subsets A and B are disjoint. The transition function $\phi_2 \circ \phi_1^{-1} : \phi_1(A \amalg B) \to \phi_2(A \amalg B)$ is given by

$$\left(\phi_2 \circ \phi_1^{-1} \right)(t) = \begin{cases} t + 2\pi & \text{for } t \in]-\pi, 0[\\ t & \text{for } t \in]0, \pi[\end{cases}$$

Similarly,

$$\left(\phi_1 \circ \phi_2^{-1} \right)(t) = \begin{cases} t - 2\pi & \text{for } t \in]\pi, 2\pi[\\ t & \text{for } t \in]0, \pi[\end{cases}$$

Therefore, (U_1, ϕ_1) and (U_2, ϕ_2) are C^∞-compatible charts and form a C^∞ atlas on S^1. Although the C^∞ compatibility of charts is clearly reflexive and symmetric, it is not transitive. The reason is as follows. Suppose (U_1, ϕ_1) is C^∞-compatible with (U_2, ϕ_2), and (U_2, ϕ_2) is C^∞-compatible with (U_3, ϕ_3). Note that the three coordinate functions are simultaneously defined only on the triple intersection U_{123}. Thus, the composite

$$\phi_3 \circ \phi_1^{-1} = \left(\phi_3 \circ \phi_2^{-1} \right) \circ \left(\phi_2 \circ \phi_1^{-1} \right)$$

is C^∞, but only on $\phi_1 \left(U_{123} \right)$, not necessarily on $\phi_1 \left(U_{13} \right)$ (Figure 5.4). A priori we know nothing about $\phi_3 \circ \phi_1^{-1}$ on $\phi_1 \left(U_{13} - U_{123} \right)$ and so we cannot conclude that (U_1, ϕ_1) and (U_3, ϕ_3) are C^∞-compatible.

We say that a chart (V, ψ) is compatible with an atlas $\{(U_\alpha, \phi_\alpha)\}$ if it is compatible with all the charts (U_α, ϕ_α) of the atlas.

Lemma 5.8. Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on a locally Euclidean space. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.

Fig. 5.4. The transition function $\phi_{3 \circ \phi_{1}^{-1}$ is C^∞ on $\phi_1 \left(U_{123} \right)$.}

Fig. 5.5. Two charts $(V, \psi), (W, \sigma)$ compatible with an atlas.

Proof. (See Figure 5.5.) Let $p \in V \cap W$. We need to show that $\sigma \circ \psi^{-1}$ is C^∞ at $\psi(p)$. Since $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for $M, p \in U_\alpha$ for some α. Then p is in the triple intersection $V \cap W \cap U_\alpha$.

By the remark above, $\sigma \circ \psi^{-1} = (\sigma \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \psi^{-1})$ is C^∞ on $\psi\,(V \cap W \cap U_\alpha)$, hence at $\psi(p)$. Since p was an arbitrary point of $V \cap W$, this proves that $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$. Similarly, $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. $\square$

Note that in an equality such as $\sigma \circ \psi^{-1} = (\sigma \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \psi^{-1})$ in the proof above, the maps on the two sides of the equality sign have different domains. What the equality means is that the two maps are equal on their common domain.

7.1.3 5.3 Smooth Manifolds

An atlas $\mathfrak{M}$ on a locally Euclidean space is said to be maximal if it is not contained in a larger atlas; in other words, if $\mathfrak{U}$ is any other atlas containing $\mathfrak{M}$, then $\mathfrak{U} = \mathfrak{M}$.

Definition 5.9. A smooth or C^∞ manifold is a topological manifold M together with a maximal atlas. The maximal atlas is also called a differentiable structure on M. A manifold is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is also called a curve, a 2-dimensional manifold a surface, and an n-dimensional manifold an n-manifold.

In Corollary 8.7 we will prove that if an open set $U \subset \mathbb{R}^n$ is diffeomorphic to an open set $V \subset \mathbb{R}^m$, then $n = m$. As a consequence, the dimension of a manifold at a point is well defined.

In practice, to check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of any atlas on M will do, because of the following proposition.

Proposition 5.10. Any atlas $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ on a locally Euclidean space is contained in a unique maximal atlas.

Proof. Adjoin to the atlas $\mathfrak{U}$ all charts (V_i, ψ_i) that are compatible with $\mathfrak{U}$. By Proposition 5.8 the charts (V_i, ψ_i) are compatible with one another. So the enlarged collection of charts is an atlas. Any chart compatible with the new atlas must be compatible with the original atlas $\mathfrak{U}$ and so by construction belongs to the new atlas. This proves that the new atlas is maximal.

Let $\mathfrak{M}$ be the maximal atlas containing $\mathfrak{U}$ that we have just constructed. If $\mathfrak{M}'$ is another maximal atlas containing $\mathfrak{U}$, then all the charts in $\mathfrak{M}'$ are compatible with $\mathfrak{U}$ and so by construction must belong to $\mathfrak{M}$. This proves that $\mathfrak{M}' \subset \mathfrak{M}$. Since both are maximal, $\mathfrak{M}' = \mathfrak{M}$. Therefore, the maximal atlas containing $\mathfrak{U}$ is unique.

In summary, to show that a topological space M is a C^∞ manifold, it suffices to check that
(i) M is Hausdorff and second countable,
(ii) M has a C^∞ atlas (not necessarily maximal).

From now on, a "manifold" will mean a C^∞ manifold. We use the terms "smooth" and " C^∞ " interchangeably. In the context of manifolds, we denote the standard coordinates on $\mathbb{R}^n$ by $r^1, \ldots, r^n$. If $(U, \phi : U \to \mathbb{R}^n)$ is a chart of a manifold, we let $x^i = r^i \circ \phi$ be the i th component of ϕ and write $\phi = (x^1, \ldots, x^n)$ and $(U, \phi) = (U, x^1, \ldots, x^n)$. Thus, for $p \in U, (x^1(p), \ldots, x^n(p))$ is a point in $\mathbb{R}^n$. The functions $x^1, \ldots, x^n$ are called coordinates or local coordinates on U. By abuse of notation, we sometimes omit the p. So the notation $(x^1, \ldots, x^n)$ stands alternately for local coordinates on the open set U and for a point in $\mathbb{R}^n$. By a chart (U, ϕ) about p in a manifold M, we will mean a chart in the differentiable structure of M such that $p \in U$.

7.1.4 5.4 Examples of Smooth Manifolds

Example 5.11 (Euclidean space). The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $(\mathbb{R}^n, r^1, \ldots, r^n)$, where $r^1, \ldots, r^n$ are the standard coordinates on $\mathbb{R}^n$.

Example 5.12 (Open subset of a manifold). Any open subset V of a manifold M is also a manifold. If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M, then $\left\{\left(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V}\right\}\right.$ is an atlas for V, where $\phi_\alpha|_{U_\alpha \cap V} : U_\alpha \cap V \to \mathbb{R}^n$ denotes the restriction of ϕ_α to the subset $U_\alpha \cap V$.

Example 5.13 (Manifolds of dimension zero). In a manifold of dimension zero, every singleton subset is homeomorphic to $\mathbb{R}^0$ and so is open. Thus, a zero-dimensional manifold is a discrete set. By second countability, this discrete set must be countable.

Example 5.14 (Graph of a smooth function). For a subset of $A \subset \mathbb{R}^n$ and a function $f : A \to \mathbb{R}^m$, the graph of f is defined to be the subset (Figure 5.6)

$$\Gamma(f) = \{(x, f(x)) \in A \times \mathbb{R}^m\}.$$

If U is an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ is C^∞, then the two maps

Fig. 5.6. The graph of a smooth function $f: \mathbb{R}^{n} \supset U \rightarrow \mathbb{R}^{m}$.

$$\phi : \Gamma(f) \to U, \quad (x, f(x)) \mapsto x,$$

and

$$(1, f) : U \to \Gamma(f), \quad x \mapsto (x, f(x)),$$

are continuous and inverse to each other, and so are homeomorphisms. The graph $\Gamma(f)$ of a C^∞ function $f : U \to \mathbb{R}^m$ has an atlas with a single chart $(\Gamma(f), \phi)$, and is therefore a C^∞ manifold. This shows that many of the familiar surfaces of calculus, for example an elliptic paraboloid or a hyperbolic paraboloid, are manifolds.

Example 5.15 (General linear groups). For any two positive integers m and n let $\mathbb{R}^{m \times n}$ be the vector space of all $m \times n$ matrices. Since $\mathbb{R}^{m \times n}$ is isomorphic to $\mathbb{R}^{mn}$, we give it the topology of $\mathbb{R}^{mn}$. The general linear group $\mathrm{GL}(n, \mathbb{R})$ is by definition

$$\mathrm{GL}(n,\mathbb{R}) := \left\{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\right\} = \det^{-1}(\mathbb{R} - \{0\}).$$

Since the determinant function

$$\det : \mathbb{R}^{n\times n} \to \mathbb{R}$$

is continuous, $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$ and is therefore a manifold.
The complex general linear group $\mathrm{GL}(n,\mathbb{C})$ is defined to be the group of nonsingular $n \times n$ complex matrices. Since an $n \times n$ matrix A is nonsingular if and only if $\det A \neq 0, \mathrm{GL}(n,\mathbb{C})$ is an open subset of $\mathbb{C}^{n\times n} \simeq \mathbb{R}^{2n^2}$, the vector space of $n \times n$ complex matrices. By the same reasoning as in the real case, $\mathrm{GL}(n,\mathbb{C})$ is a manifold of dimension $2n^2$.

Fig. 5.7. Charts on the unit circle.

Example 5.16 (Unit circle in the (x,y)-plane). In Example 5.7 we found a C^∞ atlas with two charts on the unit circle S^1 in the complex plane $\mathbb{C}$. It follows that S^1 is a manifold. We now view S^1 as the unit circle in the real plane $\mathbb{R}^2$ with defining equation $x^2 + y^2 = 1$, and describe a C^∞ atlas with four charts on it.

We can cover S^1 with four open sets: the upper and lower semicircles U_1, U_2, and the right and left semicircles U_3, U_4 (Figure 5.7). On U_1 and U_2, the coordinate function x is a homeomorphism onto the open interval $]-1,1[$ on the x-axis. Thus, $\phi_i(x,y) = x$ for $i = 1,2$. Similarly, on U_3 and U_4, y is a homeomorphism onto the open interval $]-1,1[$ on the y-axis, and so $\phi_i(x,y) = y$ for $i = 3,4$.

It is easy to check that on every nonempty pairwise intersection $U_\alpha \cap U_\beta, \phi_\beta \circ \phi_\alpha^{-1}$ is C^∞. For example, on $U_1 \cap U_3$,

$$\left(\phi_3 \circ \phi_1^{-1}\right)(x) = \phi_3\left(x, \sqrt{1-x^2}\right) = \sqrt{1-x^2},$$

which is C^∞. On $U_2 \cap U_4$,

$$\left(\phi_4 \circ \phi_2^{-1}\right)(x) = \phi_4\left(x, -\sqrt{1-x^2}\right) = -\sqrt{1-x^2},$$

which is also C^∞. Thus, $\{(U_i,\phi_i)\}_{i=1}^4$ is a C^∞ atlas on S^1.
Example 5.17 (Product manifold). If M and N are C^∞ manifolds, then $M \times N$ with its product topology is Hausdorff and second countable (Corollary A. 21 and Proposition A.22). To show that $M \times N$ is a manifold, it remains to exhibit an atlas on it. Recall that the product of two set maps $f : X \to X'$ and $g : Y \to Y'$ is

$$f \times g : X \times Y \to X' \times Y', \quad (f \times g)(x,y) = (f(x), g(y)).$$

Proposition 5.18 (An atlas for a product manifold). If $\{(U_\alpha, \boldsymbol{\phi}_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifolds M and N of dimensions m and n, respectively, then the collection

$$\{(U_\alpha \times V_i, \phi_\alpha \times \psi_i : U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\}$$

of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m+n$.

Proof. Problem 5.5. □

Example. It follows from Proposition 5.18 that the infinite cylinder $S^1 \times \mathbb{R}$ and the torus $S^1 \times S^1$ are manifolds (Figure 5.8).

Infinite cylinder. Torus.

Fig. 5.8.

Since $M \times N \times P = (M \times N) \times P$ is the successive product of pairs of spaces, if M, N, and P are manifolds, then so is $M \times N \times P$. Thus, the n-dimensional torus $S^1 \times \cdots \times S^1$ (n times) is a manifold.

Remark. Let S^n be the unit sphere

$$\left(x^1\right)^2 + \left(x^2\right)^2 + \cdots + \left(x^{n+1}\right)^2 = 1$$

in $\mathbb{R}^{n+1}$. Using Problem 5.3 as a guide, it is easy to write down a C^∞ atlas on S^n, showing that S^n has a differentiable structure. The manifold S^n with this differentiable structure is called the standard n-sphere.

One of the most surprising achievements in topology was John Milnor's discovery [27] in 1956 of exotic 7-spheres, smooth manifolds homeomorphic but not diffeomorphic to the standard 7 -sphere. In 1963, Michel Kervaire and John Milnor [24] determined that there are exactly 28 nondiffeomorphic differentiable structures on S^7.

It is known that in dimensions < 4 every topological manifold has a unique differentiable structure and in dimensions > 4 every compact topological manifold has a finite number of differentiable structures. Dimension 4 is a mystery. It is not known whether S^4 has a finite or infinite number of differentiable structures. The statement that S^4 has a unique differentiable structure is called the smooth Poincaré conjecture. As of this writing in 2010, the conjecture is still open.

There are topological manifolds with no differentiable structure. Michel Kervaire was the first to construct an example [23].

7.1.5 Problems

5.1. The Real Line with two Origins

Let A and B be two points not on the real line $\mathbb{R}$. Consider the set $S = (\mathbb{R} - \{0\}) \cup \{A, B\}$ (see Figure 5.9).

Fig. 5.9. Real line with two origins.

For any two positive real numbers c, d, define

$$I_A(-c,d) =]-c,0[\cup\{A\}\cup]0,d[$$

and similarly for $I_B(-c,d)$, with B instead of A. Define a topology on S as follows: On $(\mathbb{R}-\{0\})$, use the subspace topology inherited from $\mathbb{R}$, with open intervals as a basis. A basis of neighborhoods at A is the set $\{I_A(-c,d) \mid c,d>0\}$; similarly, a basis of neighborhoods at B is $\{I_B(-c,d) \mid c,d>0\}$.
(a) Prove that the map $h : I_A(-c,d) \to]-c,d[$ defined by

$$\begin{aligned} h(x) &= x \quad \text{for } x \in]-c,0[\cup]0,d[, \\ h(A) &= 0 \end{aligned}$$

is a homeomorphism.
(b) Show that S is locally Euclidean and second countable, but not Hausdorff.

5.2. A Sphere with a Hair

A fundamental theorem of topology, the theorem on invariance of dimension, states that if two nonempty open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are homeomorphic, then $n = m$ (for a proof, see [18, p. 126]). Use the idea of Example 5.4 as well as the theorem on invariance of dimension to prove that the sphere with a hair in $\mathbb{R}^3$ (Figure 5.10) is not locally Euclidean at q. Hence it cannot be a topological manifold.

Fig. 5.10. A sphere with a hair.

5.3. Charts on a Sphere

Let S^2 be the unit sphere

$$x^2 + y^2 + z^2 = 1$$

in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres-the front, rear, right, left, upper, and lower hemispheres (Figure 5.11):

$$\begin{aligned}
U_1 &= \{(x,y,z) \in S^2 \mid x > 0\}, \quad \phi_1(x,y,z) = (y,z),\\
U_2 &= \{(x,y,z) \in S^2 \mid x < 0\}, \quad \phi_2(x,y,z) = (y,z),\\
U_3 &= \{(x,y,z) \in S^2 \mid y > 0\}, \quad \phi_3(x,y,z) = (x,z),\\
U_4 &= \{(x,y,z) \in S^2 \mid y < 0\}, \quad \phi_4(x,y,z) = (x,z),\\
U_5 &= \{(x,y,z) \in S^2 \mid z > 0\}, \quad \phi_5(x,y,z) = (x,y),\\
U_6 &= \{(x,y,z) \in S^2 \mid z < 0\}, \quad \phi_6(x,y,z) = (x,y).
\end{aligned}$$

Describe the domain $\phi_4(U_{14})$ of $\phi_1 \circ \phi_4^{-1}$ and show that $\phi_1 \circ \phi_4^{-1}$ is C^∞ on $\phi_4(U_{14})$. Do the same for $\phi_6 \circ \phi_1^{-1}$.

Fig. 5.11. Charts on the unit sphere.

5.4.* Existence of a Coordinate Neighborhood

Let $\{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas on a manifold M. For any open set U in M and a point $p \in U$, prove the existence of a coordinate open set U_α such that $p \in U_\alpha \subset U$.

5.5. An Atlas for a Product Manifold

Prove Proposition 5.18.

7.2 §6 Smooth Maps on a Manifold

Now that we have defined smooth manifolds, it is time to consider maps between them. Using coordinate charts, one can transfer the notion of smooth maps from Euclidean spaces to manifolds. By the C^∞ compatibility of charts in an atlas, the smoothness of a map turns out to be independent of the choice of charts and is therefore well defined. We give various criteria for the smoothness of a map as well as examples of smooth maps.

Next we transfer the notion of partial derivatives from Euclidean space to a coordinate chart on a manifold. Partial derivatives relative to coordinate charts allow us to generalize the inverse function theorem to manifolds. Using the inverse function theorem, we formulate a criterion for a set of smooth functions to serve as local coordinates near a point.

7.2.1 6.1 Smooth Functions on a Manifold

Fig. 6.1. Checking that a function f is C^{∞} at p by pulling back to $\mathbb{R}^n$.

Definition 6.1. Let M be a smooth manifold of dimension n. A function $f : M \to \mathbb{R}$ is said to be C^∞ or smooth at a point p in M if there is a chart (U, ϕ) about p in M such that $f \circ \phi^{-1}$, a function defined on the open subset $\phi(U)$ of $\mathbb{R}^n$, is C^∞ at $\phi(p)$ (see Figure 6.1). The function f is said to be C^∞ on M if it is C^∞ at every point of M.

Remark 6.2. The definition of the smoothness of a function f at a point is independent of the chart (U, ϕ), for if $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$ and (V, ψ) is any other chart about p in M, then on $\psi(U \cap V)$,

$$f \circ \psi^{-1} = \left(f \circ \phi^{-1}\right) \circ \left(\phi \circ \psi^{-1}\right),$$

which is C^∞ at $\psi(p)$ (see Figure 6.2).

Fig. 6.2. Checking that a function f is C^{∞} at p via two charts.

In Definition 6.1, $f : M \to \mathbb{R}$ is not assumed to be continuous. However, if f is C^∞ at $p \in M$, then $f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$, being a C^∞ function at the point $\phi(p)$ in an open subset of $\mathbb{R}^n$, is continuous at $\phi(p)$. As a composite of continuous functions, $f = (f \circ \phi^{-1}) \circ \phi$ is continuous at p. Since we are interested only in functions that are smooth on an open set, there is no loss of generality in assuming at the outset that f is continuous.

Proposition 6.3 (Smoothness of a real-valued function). Let M be a manifold of dimension n, and $f : M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:
(i) The function $f : M \to \mathbb{R}$ is C^∞.
(ii) The manifold M has an atlas such that for every chart (U, ϕ) in the atlas, $f \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \to \mathbb{R}$ is C^∞.
(iii) For every chart (V, ψ) on M, the function $f \circ \psi^{-1} : \mathbb{R}^n \supset \psi(V) \to \mathbb{R}$ is C^∞.

Proof. We will prove the proposition as a cyclic chain of implications.
(ii) $\Rightarrow$ (i): This follows directly from the definition of a C^∞ function, since by (ii) every point $p \in M$ has a coordinate neighborhood (U, ϕ) such that $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

(i) $\Rightarrow$ (iii): Let (V, ψ) be an arbitrary chart on M and let $p \in V$. By Remark 6.2, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$. Since p was an arbitrary point of V, $f \circ \psi^{-1}$ is C^∞ on $\psi(V)$. (iii) $\Rightarrow$ (ii): Obvious. $\square$

The smoothness conditions of Proposition 6.3 will be a recurrent motif throughout the book: to prove the smoothness of an object, it is sufficient that a smoothness criterion hold on the charts of some atlas. Once the object is shown to be smooth, it then follows that the same smoothness criterion holds on every chart on the manifold.

Definition 6.4. Let $F : N \to M$ be a map and h a function on M. The pullback of h by F, denoted by F^*h, is the composite function $h \circ F$.

In this terminology, a function f on M is C^∞ on a chart (U, ϕ) if and only if its pullback $(\phi^{-1})^* f$ by ϕ^{-1} is C^∞ on the subset $\phi(U)$ of Euclidean space.

7.2.2 6.2 Smooth Maps between Manifolds

We emphasize again that unless otherwise specified, by a manifold we always mean a C^∞ manifold. We use the terms " C^∞ " and "smooth" interchangeably. An atlas or a chart on a smooth manifold means an atlas or a chart contained in the differentiable structure of the smooth manifold. We generally denote a manifold by M and its dimension by n. However, when speaking of two manifolds simultaneously, as in a map $f : N \to M$, we will let the dimension of N be n and that of M be m.

Definition 6.5. Let N and M be manifolds of dimension n and m, respectively. A continuous map $F : N \to M$ is C^∞ at a point p in N if there are charts (V, ψ) about $F(p)$ in M and (U, ϕ) about p in N such that the composition $\psi \circ F \circ \phi^{-1}$, a map from the open subset $\phi\left(F^{-1}(V) \cap U\right)$ of $\mathbb{R}^n$ to $\mathbb{R}^m$, is C^∞ at $\phi(p)$ (see Figure 6.3). The continuous map $F : N \to M$ is said to be C^∞ if it is C^∞ at every point of N.

Fig. 6.3. Checking that a map $F : N \to M$ is \$C^{\infty \$ at p.}

In Definition 6.5, we assume $F : N \to M$ continuous to ensure that $F^{-1}(V)$ is an open set in N. Thus, C^∞ maps between manifolds are by definition continuous.

Remark 6.6 (Smooth maps into $\mathbb{R}^m$). In case $M = \mathbb{R}^m$, we can take ($\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m}$) as a chart about $F(p)$ in $\mathbb{R}^m$. According to Definition 6.5, $F : N \to \mathbb{R}^m$ is C^∞ at $p \in N$ if and only if there is a chart (U, ϕ) about p in N such that $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ at $\phi(p)$. Letting $m = 1$, we recover the definition of a function being C^∞ at a point.

We show now that the definition of the smoothness of a map $F : N \to M$ at a point is independent of the choice of charts. This is analogous to how the smoothness of a function $N \to \mathbb{R}$ at $p \in N$ is independent of the choice of a chart on N about p.

Proposition 6.7. Suppose $F : N \to M$ is C^∞ at $p \in N$. If (U, ϕ) is any chart about p in N and (V, ψ) is any chart about $F(p)$ in M, then $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.

Proof. Since F is C^∞ at $p \in N$, there are charts (U_α, ϕ_α) about p in N and (V_β, ψ_β) about $F(p)$ in M such that $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is C^∞ at $\phi_\alpha(p)$. By the C^∞ compatibility

of charts in a differentiable structure, both $\phi_\alpha \circ \phi^{-1}$ and $\psi \circ \psi_\beta^{-1}$ are C^∞ on open subsets of Euclidean spaces. Hence, the composite

$$\psi \circ F \circ \phi^{-1} = \left(\psi \circ \psi_\beta^{-1}\right) \circ \left(\psi_\beta \circ F \circ \phi_\alpha^{-1}\right) \circ \left(\phi_\alpha \circ \phi^{-1}\right)$$

is C^∞ at $\phi(p)$.

The next proposition gives a way to check smoothness of a map without specifying a point in the domain.

Proposition 6.8 (Smoothness of a map in terms of charts). Let N and M be smooth manifolds, and $F : N \to M$ a continuous map. The following are equivalent:

(i) The map $F : N \to M$ is C^∞.

(ii) There are atlases $\mathfrak{U}$ for N and $\mathfrak{V}$ for M such that for every chart (U, ϕ) in $\mathfrak{U}$ and (V, ψ) in $\mathfrak{V}$, the map

$$\psi \circ F \circ \phi^{-1} : \phi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$

is C^∞.

(iii) For every chart (U, ϕ) on N and (V, ψ) on M, the map

$$\psi \circ F \circ \phi^{-1} : \phi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$

is C^∞.

Proof. (ii) $\Rightarrow$ (i): Let $p \in N$. Suppose $(U, \boldsymbol{\phi})$ is a chart about p in $\mathfrak{U}$ and $(V, \boldsymbol{\psi})$ is a chart about $F(p)$ in $\mathfrak{V}$. By (ii), $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. By the definition of a C^∞ map, $F : N \to M$ is C^∞ at p. Since p was an arbitrary point of N, the map $F : N \to M$ is C^∞.

(i) $\Rightarrow$ (iii): Suppose (U, ϕ) and (V, ψ) are charts on N and M respectively such that $U \cap F^{-1}(V) \neq \varnothing$. Let $p \in U \cap F^{-1}(V)$. Then (U, ϕ) is a chart about p and (V, ψ) is a chart about $F(p)$. By Proposition 6.7, $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. Since $\phi(p)$ was an arbitrary point of $\phi\left(U \cap F^{-1}(V)\right)$, the map $\psi \circ F \circ \phi^{-1} : \phi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$ is C^∞.

(iii) $\Rightarrow$ (ii): Clear.

Proposition 6.9 (Composition of C^∞ maps). If $F : N \to M$ and $G : M \to P$ are C^∞ maps of manifolds, then the composite $G \circ F : N \to P$ is C^∞.

Proof. Let $(U, \phi), (V, \psi)$, and (W, σ) be charts on N, M, and P respectively. Then

$$\sigma \circ (G \circ F) \circ \phi^{-1} = \left(\sigma \circ G \circ \psi^{-1}\right) \circ \left(\psi \circ F \circ \phi^{-1}\right)$$

Since F and G are C^∞, by Proposition 6.8(i) (iii), $\sigma \circ G \circ \psi^{-1}$ and $\psi \circ F \circ \phi^{-1}$ are C^∞. As a composite of C^∞ maps of open subsets of Euclidean spaces, $\sigma \circ (G \circ F) \circ \phi^{-1}$ is C^∞. By Proposition 6.8(iii) $\Rightarrow$ (i), $G \circ F$ is C^∞.

7.2.3 6.3 Diffeomorphisms

A diffeomorphism of manifolds is a bijective C^∞ map $F : N \to M$ whose inverse F^{-1} is also C^∞. According to the next two propositions, coordinate maps are diffeomorphisms, and conversely, every diffeomorphism of an open subset of a manifold with an open subset of a Euclidean space can serve as a coordinate map.

Proposition 6.10. If (U, ϕ) is a chart on a manifold M of dimension n, then the coordinate map $\phi : U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.

Proof. By definition, ϕ is a homeomorphism, so it suffices to check that both ϕ and ϕ^{-1} are smooth. To test the smoothness of $\phi : U \to \phi(U)$, we use the atlas $\{(U, \phi)\}$ with a single chart on U and the atlas $\left\{\left(\phi(U), \mathbb{1}_{\phi(U)}\right)\right\}$ with a single chart on $\phi(U)$. Since $\mathbb{1}_{\phi(U)} \circ \phi \circ \phi^{-1} : \phi(U) \to \phi(U)$ is the identity map, it is C^∞. By Proposition 6.8(ii) $\Rightarrow$ (i), ϕ is C^∞.

To test the smoothness of $\phi^{-1} : \phi(U) \to U$, we use the same atlases as above. Since $\phi \circ \phi^{-1} \circ \mathbb{1}_{\phi(U)} = \mathbb{1}_{\phi(U)} : \phi(U) \to \phi(U)$, the map ϕ^{-1} is also C^∞.

Proposition 6.11. Let U be an open subset of a manifold M of dimension n. If $F : U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then $(\ U, F\ )$ is a chart in the differentiable structure of M.

Proof. For any chart (U_α, ϕ_α) in the maximal atlas of M, both ϕ_α and ϕ_α^{-1} are C^∞ by Proposition 6.10. As composites of C^∞ maps, both $F \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ F^{-1}$ are C^∞. Hence, (U, F) is compatible with the maximal atlas. By the maximality of the atlas, the chart $(\ U, F\ )$ is in the atlas.

7.2.4 6.4 Smoothness in Terms of Components

In this subsection we derive a criterion that reduces the smoothness of a map to the smoothness of real-valued functions on open sets.

Proposition 6.12 (Smoothness of a vector-valued function). Let N be a manifold and $F : N \to \mathbb{R}^m$ a continuous map. The following are equivalent:
(i) The map $F : N \to \mathbb{R}^m$ is C^∞.
(ii) The manifold N has an atlas such that for every chart $(\ U, \phi\ )$ in the atlas, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.
(iii) For every chart $(\ U, \phi\ )$ on N, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞.

Proof. (ii) $\Rightarrow$ (i): In Proposition 6.8(ii), take $\mathfrak{V}$ to be the atlas with the single chart $(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})$ on $M = \mathbb{R}^m$.
(i) $\Rightarrow$ (iii): In Proposition 6.8(iii), let (V, ψ) be the chart $(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})$ on $M = \mathbb{R}^m$.
(iii) $\Rightarrow$ (ii): Obvious.

Proposition 6.13 (Smoothness in terms of components). Let N be a manifold. A vector-valued function $F : N \to \mathbb{R}^m$ is C^∞ if and only if its component functions $F^1, \dots, F^m : N \to \mathbb{R}$ are all C^∞.

Proof.
The map $F : N \to \mathbb{R}^m$ is C^∞
$\iff$ for every chart (U, ϕ) on N, the map $F \circ \phi^{-1} : \phi(U) \to \mathbb{R}^m$ is C^∞ (by Proposition 6.12)
$\iff$ for every chart (U, ϕ) on N, the functions $F^i \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ are all C^∞ (definition of smoothness for maps of Euclidean spaces)
$\iff$ the functions $F^i : N \to \mathbb{R}$ are all C^∞ (by Proposition 6.3).

Exercise 6.14 (Smoothness of a map to a circle).* Prove that the map $F : \mathbb{R} \to S^1, F(t) = (\cos t, \sin t)$ is C^∞.

Proposition 6.15 (Smoothness of a map in terms of vector-valued functions). Let $F : N \to M$ be a continuous map between two manifolds of dimensions n and m respectively. The following are equivalent:
(i) The map $F : N \to M$ is C^∞.
(ii) The manifold M has an atlas such that for every chart $(V, \psi) = (V, y^1, \dots, y^m)$ in the atlas, the vector-valued function $\psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is C^∞.
(iii) For every chart $(V, \psi) = (V, y^1, \dots, y^m)$ on M, the vector-valued function $\psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is C^∞.

Proof. (ii) $\Rightarrow$ (i): Let $\mathfrak{V}$ be the atlas for M in (ii), and let $\mathfrak{U} = \{(U, \boldsymbol{\phi})\}$ be an arbitrary atlas for N. For each chart (V, ψ) in the atlas $\mathfrak{V}$, the collection $\left\{\left(U \cap F^{-1}(V), \phi|_{U \cap F^{-1}(V)}\right)\right\}$ is an atlas for $F^{-1}(V)$. Since $\psi \circ F : F^{-1}(V) \to \mathbb{R}^m$ is C^∞, by Proposition 6.12(i) $\Rightarrow$ (iii),

$$\psi \circ F \circ \phi^{-1} : \phi\left(U \cap F^{-1}(V)\right) \to \mathbb{R}^m$$

is C^∞. It then follows from Proposition 6.8(ii) $\Rightarrow$ (i) that $F : N \to M$ is C^∞.
(i) $\Rightarrow$ (iii): Being a coordinate map, ψ is C^∞ (Proposition 6.10). As the composite of two C^∞

maps, $\psi \circ F$ is C^∞.
(iii) $\Rightarrow$ (ii): Obvious.

By Proposition 6.13, this smoothness criterion for a map translates into a smoothness criterion in terms of the components of the map.

Proposition 6.16 (Smoothness of a map in terms of components). Let $F : N \to M$ be a continuous map between two manifolds of dimensions n and m respectively. The following are equivalent:
(i) The map $F : N \to M$ is C^∞.
(ii) The manifold M has an atlas such that for every chart $(V, \psi) = (V, y^1, \ldots, y^m)$ in the atlas, the components $y^i \circ F : F^{-1}(V) \to \mathbb{R}$ of F relative to the chart are all C^∞.
(iii) For every chart $(V, \psi) = (V, y^1, \ldots, y^m)$ on M, the components $y^i \circ F : F^{-1}(V) \to \mathbb{R}$ of F relative to the chart are all C^∞.

7.2.5 6.5 Examples of Smooth Maps

We have seen that coordinate maps are smooth. In this subsection we look at a few more examples of smooth maps.

Example 6.17 (Smoothness of a projection map). Let M and N be manifolds and $\pi : M \times N \to M, \pi(p, q) = p$ the projection to the first factor. Prove that π is a C^∞ map.

Solution. Let $(\ p, q\ )$ be an arbitrary point of $M \times N$. Suppose $(U, \phi) = (U, x^1, \ldots, x^m)$ and $(V, \Psi) = (V, y^1, \ldots, y^n)$ are coordinate neighborhoods of p and q in M and N respectively. By Proposition 5.18, $(U \times V, \boldsymbol{\phi} \times \boldsymbol{\psi}) = (U \times V, x^1, \ldots, x^m, y^1, \ldots, y^n)$ is a coordinate neighborhood of (p, q). Then

$$\left(\phi \circ \pi \circ (\phi \times \psi)^{-1}\right)\left(a^1, \ldots, a^m, b^1, \ldots, b^n\right) = \left(a^1, \ldots, a^m\right),$$

which is a C^∞ map from $(\phi \times \psi)(U \times V)$ in $\mathbb{R}^{m+n}$ to $\phi(U)$ in $\mathbb{R}^m$, so π is C^∞ at (p, q). Since $(\ p, q\ )$ was an arbitrary point in $M \times N, \pi$ is C^∞ on $M \times N$.

Exercise 6.18 (Smoothness of a map to a Cartesian product).* Let M_1, M_2, and N be manifolds of dimensions m_1, m_2, and n respectively. Prove that a map $(f_1, f_2) : N \to M_1 \times M_2$ is C^∞ if and only if $f_i : N \to M_i, i = 1, 2$, are both C^∞.

Example 6.19. In Examples 5.7 and 5.16 we showed that the unit circle S^1 defined by $x^2 + y^2 = 1$ in $\mathbb{R}^2$ is a C^∞ manifold. Prove that a C^∞ function $f(x, y)$ on $\mathbb{R}^2$ restricts to a C^∞ function on S^1.

Solution. To avoid confusing functions with points, we will denote a point on S^1 as $p = (a, b)$ and use x, y to mean the standard coordinate functions on $\mathbb{R}^2$. Thus, $x(a, b) = a$ and $y(a, b) = b$. Suppose we can show that x and y restrict to C^∞ functions on S^1. By Exercise 6.18, the inclusion map $i : S^1 \to \mathbb{R}^2, i(p) = (x(p), y(p))$ is then C^∞ on S^1. As the composition of C^∞ maps, $f|_{S^1} = f \circ i$ will be C^∞ on S^1 (Proposition 6.9).

Consider first the function x. We use the atlas $(\ U_i, \phi_i\ )$ from Example 5.16. Since x is a coordinate function on U_1 and on U_2, by Proposition 6.10 it is C^∞ on $U_1 \cup U_2 = S^1 - \{(\pm 1, 0)\}$. To show that x is C^∞ on U_3, it suffices to check the smoothness of $x \circ \phi_3^{-1} : \phi_3(U_3) \to \mathbb{R}$:

$$\left(x \circ \phi_3^{-1}\right)(b) = x\left(\sqrt{1 - b^2}, b\right) = \sqrt{1 - b^2}$$

On U_3, we have $b \neq \pm 1$, so that $\sqrt{1 - b^2}$ is a C^∞ function of b. Hence, x is C^∞ on U_3. On U_4,

$$\left(x \circ \phi_4^{-1}\right)(b) = x\left(-\sqrt{1 - b^2}, b\right) = -\sqrt{1 - b^2}$$

which is C^∞ because b is not equal to ± 1. Since x is C^∞ on the four open sets U_1, U_2, U_3, and U_4, which cover S^1, x is C^∞ on S^1.

The proof that y is C^∞ on S^1 is similar.

Armed with the definition of a smooth map between manifolds, we can define a Lie group.

Definition 6.20. A Lie group [1] is a C^∞ manifold G having a group structure such that the multiplication map

$$\mu : G \times G \to G$$

and the inverse map

$$\imath : G \to G, \quad \imath(x) = x^{-1}$$

are both C^∞.

Similarly, a topological group is a topological space having a group structure such that the multiplication and inverse maps are both continuous. Note that a topological group is required to be a topological space, but not a topological manifold.

7.2.6 Examples.

(i) The Euclidean space $\mathbb{R}^n$ is a Lie group under addition.
(ii) The set $\mathbb{C}^\times$of nonzero complex numbers is a Lie group under multiplication.
(iii) The unit circle S^1 in $\mathbb{C}^\times$is a Lie group under multiplication.
(iv) The Cartesian product $G_1 \times G_2$ of two Lie groups (G_1, μ_1) and (G_2, μ_2) is a Lie group under coordinatewise multiplication $\mu_1 \times \mu_2$.

Example 6.21 (General linear group). In Example 5.15 we defined the general linear group

$$\mathrm{GL}(n, \mathbb{R}) = \left\{ A = [a_{ij}] \in \mathbb{R}^{n \times n} \mid \det A \neq 0 \right\}.$$

As an open subset of $\mathbb{R}^{n \times n}$, it is a manifold. Since the (i, j)-entry of the product of two matrices A and B in $\mathrm{GL}(n, \mathbb{R})$,

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

is a polynomial in the coordinates of A and B, matrix multiplication

$$\mu : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$$

is a C^∞ map.

Recall that the (i, j)-minor of a matrix A is the determinant of the submatrix of A obtained by deleting the i th row and the j th column of A. By Cramer's rule from linear algebra, the (i, j)-entry of A^{-1} is

$$\left(A^{-1}\right)_{ij} = \frac{1}{\det A} \cdot (-1)^{i+j}((j, i)\text{-minor of } A),$$

which is a C^∞ function of the a_{ij} 's provided $\det A \neq 0$. Therefore, the inverse map $\imath : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is also C^∞. This proves that $\mathrm{GL}(n, \mathbb{R})$ is a Lie group.

In Section 15 we will study less obvious examples of Lie groups.

Notation. The notation for matrices presents a special challenge. An $n \times n$ matrix A can represent a linear transformation $y = Ax$, with $x, y \in \mathbb{R}^n$. In this case, $y^i = \sum_j a^i_j x^j$, so $A = \left[a^i_j\right]$. An $n \times n$ matrix can also represent a bilinear form $\langle x, y \rangle = x^T A y$ with $x, y \in \mathbb{R}^n$. In this case, $\langle x, y \rangle = \sum_{i,j} x^i a_{ij} y^j$, so $A = [a_{ij}]$. In the absence of any context, we will write a

[1] Lie groups and Lie algebras are named after the Norwegian mathematician Sophus Lie (1842-1899). In this context, "Lie" is pronounced "lee," not "lye."

matrix as $A = [a_{ij}]$, using a lowercase letter a to denote an entry of a matrix A and using a double subscript $()_{ij}$ to denote the (i, j)-entry.

7.2.7 6.6 Partial Derivatives

On a manifold M of dimension n, let (U, ϕ) be a chart and f a C^∞ function As a function into $\mathbb{R}^n, \phi$ has n components $x^1, \ldots, x^n$. This means that if $r^1, \ldots, r^n$ are the standard coordinates on $\mathbb{R}^n$, then $x^i = r^i \circ \phi$. For $p \in U$, we define the partial derivative $\partial f/\partial x^i$ of f with respect to x^i at p to be

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \frac{\partial f}{\partial x^i}(p) := \frac{\partial\left(f \circ \phi^{-1}\right)}{\partial r^i}(\phi(p)) := \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)}\left(f \circ \phi^{-1}\right).$$

Since $p = \phi^{-1}(\phi(p))$, this equation may be rewritten in the form

$$\frac{\partial f}{\partial x^i}\left(\phi^{-1}(\phi(p))\right) = \frac{\partial\left(f \circ \phi^{-1}\right)}{\partial r^i}(\phi(p)).$$

Thus, as functions on $\phi(U)$,

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} = \frac{\partial\left(f \circ \phi^{-1}\right)}{\partial r^i}.$$

The partial derivative $\partial f/\partial x^i$ is C^∞ on U because its pullback $(\partial f/\partial x^i) \circ \phi^{-1}$ is C^∞ on $\phi(U)$.

In the next proposition we see that partial derivatives on a manifold satisfy the same duality property $\partial r^i/\partial r^j = \delta^i_j$ as the coordinate functions r^i on $\mathbb{R}^n$.

Proposition 6.22. Suppose $(U, x^1, \ldots, x^n)$ is a chart on a manifold. Then $\partial x^i/\partial x^j = \delta^i_j$.

Proof. At a point $p \in U$, by the definition of $\partial/\left.\partial x^j\right|_p$,

$$\frac{\partial x^i}{\partial x^j}(p) = \frac{\partial\left(x^i \circ \phi^{-1}\right)}{\partial r^j}(\phi(p)) = \frac{\partial\left(r^i \circ \phi \circ \phi^{-1}\right)}{\partial r^j}(\phi(p)) = \frac{\partial r^i}{\partial r^j}(\phi(p)) = \delta^i_j.$$

Definition 6.23. Let $F : N \to M$ be a smooth map, and let $(U, \boldsymbol{\phi}) = (U, x^1, \ldots, x^n)$ and $(V, \boldsymbol{\psi}) = (V, y^1, \ldots, y^m)$ be charts on N and M respectively such that $F(U) \subset V$. Denote by

$$F^i := y^i \circ F = r^i \circ \psi \circ F : U \to \mathbb{R}$$

the i th component of F in the chart (V, ψ). Then the matrix $[\partial F^i/\partial x^j]$ is called the Jacobian matrix of F relative to the charts (U, ϕ) and (V, ψ). In case N and M have the same dimension, the determinant $\det[\partial F^i/\partial x^j]$ is called the Jacobian determinant of F relative to the two charts. The Jacobian determinant is also written as $\partial\left(F^1, \ldots, F^n\right)/\partial\left(x^1, \ldots, x^n\right)$.

When M and N are open subsets of Euclidean spaces and the charts are $(U, r^1, \ldots, r^n)$ and $(V, r^1, \ldots, r^m)$, the Jacobian matrix $[\partial F^i/\partial r^j]$, where $F^i = r^i \circ F$, is the usual Jacobian matrix from calculus.

Example 6.24 (Jacobian matrix of a transition map). Let $(U, \phi) = (U, x^1, \ldots, x^n)$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ be overlapping charts on a manifold M. The transition map $\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$. Show that its Jacobian matrix $J\left(\psi \circ \phi^{-1}\right)$ at $\phi(p)$ is the matrix $[\partial y^i/\partial x^j]$ of partial derivatives at p.

Solution. By definition, $J\left(\psi \circ \phi^{-1}\right) = \left[\partial\left(\psi \circ \phi^{-1}\right)^i/\partial r^j\right]$, where

$$\frac{\partial\left(\psi \circ \phi^{-1}\right)^i}{\partial r^j}(\phi(p)) = \frac{\partial\left(r^i \circ \psi \circ \phi^{-1}\right)}{\partial r^j}(\phi(p)) = \frac{\partial\left(y^i \circ \phi^{-1}\right)}{\partial r^j}(\phi(p)) = \frac{\partial y^i}{\partial x^j}(p).$$

7.2.8 6.7 The Inverse Function Theorem

By Proposition 6.11, any diffeomorphism $F : U \to F(U) \subset \mathbb{R}^n$ of an open subset U of a manifold may be thought of as a coordinate system on U. We say that a C^∞ map $F : N \to M$ is locally invertible or a local diffeomorphism at $p \in N$ if p has a neighborhood U on which $F|_U : U \to F(U)$ is a diffeomorphism.

Given n smooth functions $F^1, \ldots, F^n$ in a neighborhood of a point p in a manifold N of dimension n, one would like to know whether they form a coordinate system, possibly on a smaller neighborhood of p. This is equivalent to whether $F = (F^1, \ldots, F^n) : N \to \mathbb{R}^n$ is a local diffeomorphism at p. The inverse function theorem provides an answer.

Theorem 6.25 (Inverse function theorem for $\mathbb{R}^n$). Let $F : W \to \mathbb{R}^n$ be a C^∞ map defined on an open subset W of $\mathbb{R}^n$. For any point p in W, the map F is locally invertible at p if and only if the Jacobian determinant $\det [\partial F^i / \partial r^j(p)]$ is not zero.

This theorem is usually proved in an undergraduate course on real analysis. See Appendix B for a discussion of this and related theorems. Because the inverse function theorem for $\mathbb{R}^n$ is a local result, it easily translates to manifolds.

Theorem 6.26 (Inverse function theorem for manifolds). Let $F : N \to M$ be a C^∞ map between two manifolds of the same dimension, and $p \in N$. Suppose for some charts $(U, \phi) = (U, x^1, \ldots, x^n)$ about p in N and $(V, \psi) = (V, y^1, \ldots, y^n)$ about $F(p)$ in $M, F(U) \subset V$. Set $F^i = y^i \circ F$. Then F is locally invertible at p if and only if its Jacobian determinant $\det [\partial F^i / \partial x^j(p)]$ is nonzero.

Fig. 6.4. The map F is locally invertible at p because $\psi \circ F \circ \phi^{-1}$ is locally invertible at $\phi(p)$.}

Proof. Since $F^i = y^i \circ F = r^i \circ \psi \circ F$, the Jacobian matrix of F relative to the charts (U, ϕ) and (V, ψ) is

$$\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \left[\frac{\partial \left(r^i \circ \psi \circ F\right)}{\partial x^j}(p)\right] = \left[\frac{\partial \left(r^i \circ \psi \circ F \circ \phi^{-1}\right)}{\partial r^j}(\phi(p))\right],$$

which is precisely the Jacobian matrix at $\phi(p)$ of the map

$$\psi \circ F \circ \phi^{-1} : \mathbb{R}^n \supset \phi(U) \to \psi(V) \subset \mathbb{R}^n$$

between two open subsets of $\mathbb{R}^n$. By the inverse function theorem for $\mathbb{R}^n$,

$$\det \left[\frac{\partial F^i}{\partial x^j}(p)\right] = \det \left[\frac{\partial r^i \circ (\psi \circ F \circ \phi^{-1})}{\partial r^j}(\phi(p))\right] \neq 0$$

if and only if $\psi \circ F \circ \phi^{-1}$ is locally invertible at $\phi(p)$. Since ψ and ϕ are diffeomorphisms (Proposition 6.10), this last statement is equivalent to the local invertibility of F at p (see Figure 6.4). $\square$

We usually apply the inverse function theorem in the following form.
Corollary 6.27. Let N be a manifold of dimension n. A set of n smooth functions $F^1, \ldots, F^n$ defined on a coordinate neighborhood $(U, x^1, \ldots, x^n)$ of a point $p \in N$ forms a coordinate system about p if and only if the Jacobian determinant $\det[\partial F^i/\partial x^j(p)]$ is nonzero.

Proof. Let $F = (F^1, \ldots, F^n) : U \to \mathbb{R}^n$. Then

$$\det\left[\partial F^i/\partial x^j(p)\right] \neq 0$$

$\iff F : U \to \mathbb{R}^n$ is locally invertible at p (by the inverse function theorem)
$\iff$ there is a neighborhood W of p in N such that $F : W \to F(W)$ is a diffeomorphism (by the definition of local invertibility)
$\iff (W, F^1, \ldots, F^n)$ is a coordinate chart about p in the differentiable structure of N (by Proposition 6.11).

Example. Find all points in $\mathbb{R}^2$ in a neighborhood of which the functions $x^2 + y^2 - 1, y$ can serve as a local coordinate system.
Solution. Define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x, y) = \left(x^2 + y^2 - 1, y\right)$$

The map F can serve as a coordinate map in a neighborhood of p if and only if it is a local diffeomorphism at p. The Jacobian determinant of F is

$$\frac{\partial\left(F^1, F^2\right)}{\partial(x, y)} = \det\begin{bmatrix} 2x & 2y \\ 0 & 1 \end{bmatrix} = 2x.$$

By the inverse function theorem, F is a local diffeomorphism at $p = (x, y)$ if and only if $x \neq 0$. Thus, F can serve as a coordinate system at any point p not on the y-axis.

7.2.9 Problems

6.1. Differentiable Structures on ⧵

Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \boldsymbol{\phi} = \mathbb{1} : \mathbb{R} \to \mathbb{R})$, and let $\mathbb{R}'$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \boldsymbol{\psi} : \mathbb{R} \to \mathbb{R})$, where $\boldsymbol{\psi}(x) = x^{1/3}$.
(a) Show that these two differentiable structures are distinct.
(b) Show that there is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. (Hint: The identity map $\mathbb{R} \to \mathbb{R}$ is not the desired diffeomorphism; in fact, this map is not smooth.)

6.2. The Smoothness of an Inclusion map

Let M and N be manifolds and let q_0 be a point in N. Prove that the inclusion map $i_{q_0} : M \to M \times N, i_{q_0}(p) = (p, q_0)$, is C^∞.

6.3.* Group of Automorphisms of a Vector Space

Let V be a finite-dimensional vector space over $\mathbb{R}$, and $\mathrm{GL}(V)$ the group of all linear automorphisms of V. Relative to an ordered basis $e = (e_1, \ldots, e_n)$ for V, a linear automorphism $L \in \mathrm{GL}(V)$ is represented by a matrix $\left[a_j^i\right]$ defined by

$$L\left(e_j\right) = \sum_i a_j^i e_i$$

The map

$$\phi_e : \mathrm{GL}(V) \to \mathrm{GL}(n, \mathbb{R}),$$
$$L \mapsto \left[a^i_j\right],$$

is a bijection with an open subset of $\mathbb{R}^{n\times n}$ that makes $\mathrm{GL}(V)$ into a C^∞ manifold, which we denote temporarily by $\mathrm{GL}(V)_e$. If $\mathrm{GL}(V)_u$ is the manifold structure induced from another ordered basis $u = (u_1, \ldots, u_n)$ for V, show that $\mathrm{GL}(V)_e$ is the same as $\mathrm{GL}(V)_u$.

6.4. Local Coordinate Systems

Find all points in $\mathbb{R}^3$ in a neighborhood of which the functions $x, x^2 + y^2 + z^2 - 1, z$ can serve as a local coordinate system.

7.3 §7 Quotients

Gluing the edges of a malleable square is one way to create new surfaces. For example, gluing together the top and bottom edges of a square gives a cylinder; gluing together the boundaries of the cylinder with matching orientations gives a torus (Figure 7.1). This gluing process is called an identification or a quotient construction.

Fig. 7.1. Gluing the edges of a malleable square.

The quotient construction is a process of simplification. Starting with an equivalence relation on a set, we identify each equivalence class to a point. Mathematics abounds in quotient constructions, for example, the quotient group, quotient ring, or quotient vector space in algebra. If the original set is a topological space, it is always possible to give the quotient set a topology so that the natural projection map becomes continuous. However, even if the original space is a manifold, a quotient space is often not a manifold. The main results of this section give conditions under which a quotient space remains second countable and Hausdorff. We then study real projective space as an example of a quotient manifold.

Real projective space can be interpreted as a quotient of a sphere with antipodal points identified, or as the set of lines through the origin in a vector space. These two interpretations give rise to two distinct generalizations-covering maps on the one hand and Grassmannians of k-dimensional subspaces of a vector space on the other. In one of the exercises, we carry out an extensive investigation of $G(2, 4)$, the Grassmannian of 2-dimensional subspaces of $\mathbb{R}^4$.

7.3.1 7.1 The Quotient Topology

Recall that an equivalence relation on a set S is a reflexive, symmetric, and transitive relation. The equivalence class $[x]$ of $x \in S$ is the set of all elements in S equivalent to x. An equivalence relation on S partitions S into disjoint equivalence classes. We denote the set of equivalence classes by $S/\sim$ and call this set the quotient of S by the equivalence relation $\sim$. There is a natural projection map $\pi : S \to S/\sim$ that sends $x \in S$ to its equivalence class $[x]$.

Assume now that S is a topological space. We define a topology on $S/\sim$ by declaring a set U in $S/\sim$ to be open if and only if $\pi^{-1}(U)$ is open in S. Clearly, both the empty set $\varnothing$ and the entire quotient $S/\sim$ are open. Further, since

$$\pi^{-1}\left(\bigcup_{\alpha} U_{\alpha}\right)=\bigcup_{\alpha} \pi^{-1}\left(U_{\alpha}\right)$$

and

$$\pi^{-1}\left(\bigcap_{i} U_{i}\right)=\bigcap_{i} \pi^{-1}\left(U_{i}\right),$$

the collection of open sets in $S/\sim$ is closed under arbitrary unions and finite intersections, and is therefore a topology. It is called the quotient topology on $S/\sim$. With this topology, $S/\sim$ is called the quotient space of S by the equivalence relation $\sim$. With the quotient topology on $S/\sim$, the projection map $\pi : S \to S/\sim$ is automatically continuous, because the inverse image of an open set in $S/\sim$ is by definition open in S.

7.3.2 7.2 Continuity of a Map on a Quotient

Let $\sim$ be an equivalence relation on the topological space S and give $S/\sim$ the quotient topology. Suppose a function $f : S \to Y$ from S to another topological space Y is constant on each equivalence class. Then it induces a map $\bar{f} : S/\sim \to Y$ by

$$\bar{f}([p]) = f(p) \quad \text{for } p \in S.$$

In other words, there is a commutative diagram

$$\begin{tikzcd} S \arrow[r, "f"] \arrow[d, "\pi"'] & Y. \\ S/\sim \arrow[ur, "\bar{f}"'] & \end{tikzcd}$$

Proposition 7.1. The induced map $\bar{f} : S/\sim \to Y$ is continuous if and only if the map $f : S \to Y$ is continuous.

Proof.

($\Rightarrow$) If $\bar{f}$ is continuous, then as the composite $\bar{f} \circ \pi$ of continuous functions, f is also continuous.
($\Leftarrow$) Suppose f is continuous. Let V be open in Y. Then $f^{-1}(V) = \pi^{-1}\left(\bar{f}^{-1}(V)\right)$ is open in S. By the definition of quotient topology, $\bar{f}^{-1}(V)$ is open in $S/\sim$. Since V was arbitrary, $\bar{f} : S/\sim \to Y$ is continuous. $\square$

This proposition gives a useful criterion for checking whether a function $\bar{f}$ on a quotient space $S/\sim$ is continuous: simply lift the function $\bar{f}$ to $f := f \circ \pi$ on S and check the continuity of the lifted map f on S. For examples of this, see Example 7.2 and Proposition 7.3.

7.3.3 7.3 Identification of a Subset to a Point

If A is a subspace of a topological space S, we can define a relation $\sim$ on S by declaring

$$x \sim x \quad \text{for all } x \in S$$

(so the relation is reflexive) and

$$x \sim y \quad \text{for all } x, y \in A.$$

This is an equivalence relation on S. We say that the quotient space $S/\sim$ is obtained from S by identifying A to a point.

Example 7.2. Let I be the unit interval $[0,1]$ and $I/\sim$ the quotient space obtained from I by identifying the two points $\{0,1\}$ to a point. Denote by S^1 the unit circle in the complex plane. The function $f : I \to S^1, f(x) = \exp(2\pi i x)$, assumes the same value at 0 and 1 (Figure 7.2), and so induces a function $\bar{f} : I/\sim \to S^1$.

Fig. 7.2. The unit circle as a quotient space of the unit interval.

Proposition 7.3. The function $\bar{f} : I/\sim \to S^1$ is a homeomorphism.
Proof. Since f is continuous, $\bar{f}$ is also continuous by Proposition 7.1. Clearly, $\bar{f}$ is a bijection. As the continuous image of the compact set I, the quotient $I/\sim$ is compact. Thus, $\bar{f}$ is a continuous bijection from the compact space $I/\sim$ to the Hausdorff space S^1. By Corollary A.36, $\bar{f}$ is a homeomorphism. □

7.3.4 7.4 A Necessary Condition for a Hausdorff Quotient

The quotient construction does not in general preserve the Hausdorff property or second countability. Indeed, since every singleton set in a Hausdorff space is closed, if $\pi : S \to S/\sim$ is the projection and the quotient $S/\sim$ is Hausdorff, then for any $p \in S$, its image $\{\pi(p)\}$ is closed in $S/\sim$. By the continuity of π, the inverse image $\pi^{-1}(\{\pi(p)\}) = [p]$ is closed in S. This gives a necessary condition for a quotient space to be Hausdorff.

Proposition 7.4. If the quotient space $S/\sim$ is Hausdorff, then the equivalence class $[p]$ of any point p in S is closed in S.

Example. Define an equivalence relation $\sim$ on $\mathbb{R}$ by identifying the open interval $]0,\infty[$ to a point. Then the quotient space $\mathbb{R}/\sim$ is not Hausdorff because the equivalence class $]0,\infty[$ of $\sim$ in $\mathbb{R}$ corresponding to the point $]0,\infty[$ in $\mathbb{R}/\sim$ is not a closed subset of $\mathbb{R}$.

7.3.5 7.5 Open Equivalence Relations

In this section we follow the treatment of Boothby [3] and derive conditions under which a quotient space is Hausdorff or second countable. Recall that a map $f : X \to Y$ of topological spaces is open if the image of any open set under f is open.

Definition 7.5. An equivalence relation $\sim$ on a topological space S is said to be open if the projection map $\pi : S \to S/\sim$ is open.

In other words, the equivalence relation $\sim$ on S is open if and only if for every open set U in S, the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x \in U} [x]$$

of all points equivalent to some point of U is open.

Example 7.6. The projection map to a quotient space is in general not open. For example, let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1 , and $\pi : \mathbb{R} \to \mathbb{R}/\sim$ the projection map.

Fig. 7.3. A projection map that is not open.

The map π is open if and only if for every open set V in $\mathbb{R}$, its image $\pi(V)$ is open in $\mathbb{R}/\sim$, which by the definition of the quotient topology means that $\pi^{-1}(\pi(V))$ is open in $\mathbb{R}$. Now let V be the open interval $]-2, 0[$ in $\mathbb{R}$. Then

$$\pi^{-1}(\pi(V)) =]-2, 0[\cup \{1\},$$

which is not open in $\mathbb{R}$ (Figure 7.3). Therefore, the projection map $\pi : \mathbb{R} \to \mathbb{R}/\sim$ is not an open map.

Given an equivalence relation $\sim$ on S, let R be the subset of $S \times S$ that defines the relation

$$R = \{(x, y) \in S \times S \mid x \sim y\}.$$

We call R the graph of the equivalence relation $\sim$.

Fig. 7.4. The graph R of an equivalence relation and an open set $U \times V$ disjoint from R.

Theorem 7.7. Suppose $\sim$ is an open equivalence relation on a topological space S. Then the quotient space $S/\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $S \times S$.

Proof. There is a sequence of equivalent statements:
R is closed in $S \times S$
$\Longleftrightarrow (S \times S) - R$ is open in $S \times S$
$\Longleftrightarrow$ for every $(x, y) \in S \times S - R$, there is a basic open set $U \times V$ containing (x, y) such that $(U \times V) \cap R = \varnothing$ (Figure 7.4)
$\Longleftrightarrow$ for every pair $x \nsim y$ in S, there exist neighborhoods U of x and V of y in S such that no element of U is equivalent to an element of V
$\Longleftrightarrow$ for any two points $[x] \neq [y]$ in $S/\sim$, there exist neighborhoods U of x and V of y in S such that $\pi(U) \cap \pi(V) = \varnothing$ in $S/\sim$.

We now show that this last statement (*) is equivalent to $S/\sim$ being Hausdorff. First assume ($*$). Since $\sim$ is an open equivalence relation, $\pi(U)$ and $\pi(V)$ are disjoint open sets in $S/\sim$ containing $[x]$ and $[y]$ respectively. Therefore, $S/\sim$ is Hausdorff.

Conversely, suppose $S/\sim$ is Hausdorff. Let $[x] \neq [y]$ in $S/\sim$. Then there exist disjoint open sets A and B in $S/\sim$ such that $[x] \in A$ and $[y] \in B$. By the surjectivity of π, we have $A = \pi(\pi^{-1}A)$ and $B = \pi(\pi^{-1}B)$ (see Problem 7.1). Let $U = \pi^{-1}A$ and $V = \pi^{-1}B$. Then $x \in U, y \in V$, and $A = \pi(U)$ and $B = \pi(V)$ are disjoint open sets in $S/\sim$. $\square$

If the equivalence relation $\sim$ is equality, then the quotient space $S/\sim$ is S itself and the graph R of $\sim$ is simply the diagonal

$$\Delta = \{(x, x) \in S \times S\}.$$

In this case, Theorem 7.7 becomes the following well-known characterization of a Hausdorff space by its diagonal (cf. Problem A.6).

Corollary 7.8. A topological space S is Hausdorff if and only if the diagonal Δ in $S \times S$ is closed.

Theorem 7.9. Let $\sim$ be an open equivalence relation on a topological space S with projection $\pi : S \to S/\sim$. If $\mathcal{B} = \{B_\alpha\}$ is a basis for S, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $S/\sim$.

Proof. Since π is an open map, $\{\pi(B_\alpha)\}$ is a collection of open sets in $S/\sim$. Let W be an open set in $S/\sim$ and $[x] \in W, x \in S$. Then $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open, there is a basic open set $B \in \mathcal{B}$ such that

$$x \in B \subset \pi^{-1}(W).$$

Then

$$[x] = \pi(x) \in \pi(B) \subset W,$$

which proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\sim$. $\square$

Corollary 7.10. If $\sim$ is an open equivalence relation on a second-countable space S, then the quotient space $S/\sim$ is second countable.

7.3.6 7.6 Real Projective Space

Define an equivalence relation on $\mathbb{R}^{n+1} - \{0\}$ by

$$x \sim y \quad \Longleftrightarrow \quad y = tx \text{ for some nonzero real number } t,$$

where $x, y \in \mathbb{R}^{n+1} - \{0\}$. The real projective space $\mathbb{R}P^n$ is the quotient space of $\mathbb{R}^{n+1} - \{0\}$ by this equivalence relation. We denote the equivalence class of a point $(a^0, \ldots, a^n) \in \mathbb{R}^{n+1} - \{0\}$ by

$[a^0, \ldots, a^n]$ and let $\pi : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$ be the projection. We call $[a^0, \ldots, a^n]$ homogeneous coordinates on $\mathbb{R}P^n$.

Geometrically, two nonzero points in $\mathbb{R}^{n+1}$ are equivalent if and only if they lie on the same line through the origin, so $\mathbb{R}P^n$ can be interpreted as the set of all lines through the origin in $\mathbb{R}^{n+1}$. Each line through the origin in $\mathbb{R}^{n+1}$ meets the unit

Fig. 7.5. A line through 0 in $\mathbb{R
^{3}$ corresponds to a pair of antipodal points on S^2.}

sphere S^n in a pair of antipodal points, and conversely, a pair of antipodal points on S^n determines a unique line through the origin (Figure 7.5). This suggests that we define an equivalence relation $\sim$ on S^n by identifying antipodal points:

$$x \sim y \Longleftrightarrow x = \pm y, \quad x, y \in S^n.$$

We then have a bijection $\mathbb{R}P^n \leftrightarrow S^n/\sim$.

Exercise 7.11 (Real projective space as a quotient of a sphere).* For $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, let $\|x\| = \sqrt{\sum_i (x^i)^2}$ be the modulus of x. Prove that the map $f : \mathbb{R}^{n+1} - \{0\} \to S^n$ given by

$$f(x) = \frac{x}{\|x\|}$$

induces a homeomorphism $\bar{f} : \mathbb{R}P^n \to S^n/\sim$. (Hint: Find an inverse map

$$\bar{g} : S^n/\sim \to \mathbb{R}P^n$$

and show that both $\bar{f}$ and $\bar{g}$ are continuous.)

Example 7.12 (The real projective line $\mathbb{R}P^1$).

Fig. 7.6. The real projective line $\mathbb{R P^{1}$ as the set of lines through 0 in $\mathbb{R}^2$.}

Each line through the origin in $\mathbb{R}^2$ meets the unit circle in a pair of antipodal points. By Exercise 7.11, $\mathbb{R}P^1$ is homeomorphic to the quotient $S^1/\sim$, which is in turn homeomorphic to the closed upper semicircle with the two endpoints identified (Figure 7.6). Thus, $\mathbb{R}P^1$ is homeomorphic to S^1.

Example 7.13 (The real projective plane $\mathbb{R}P^2$). By Exercise 7.11, there is a homeomorphism

$$\mathbb{R}P^2 \simeq S^2/\{\text{ antipodal points }\} = S^2/\sim .$$

For points not on the equator, each pair of antipodal points contains a unique point in the upper hemisphere. Thus, there is a bijection between $S^2/\sim$ and the quotient of the closed upper hemisphere in which each pair of antipodal points on the equator is identified. It is not difficult to show that this bijection is a homeomorphism (see Problem 7.2).

Let H^2 be the closed upper hemisphere

$$H^2 = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1, z \geq 0\right\}$$

and let D^2 be the closed unit disk

$$D^2 = \left\{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\right\}.$$

These two spaces are homeomorphic to each other via the continuous map

$$\begin{aligned} \varphi &: H^2 \to D^2 \\ \varphi(x, y, z) &= (x, y), \end{aligned}$$

and its inverse

$$\begin{aligned} \psi &: D^2 \to H^2 \\ \psi(x, y) &= \left(x, y, \sqrt{1 - x^2 - y^2}\right) \end{aligned}$$

On H^2, define an equivalence relation $\sim$ by identifying the antipodal points on the equator:

$$(x, y, 0) \sim (-x, -y, 0), \quad x^2 + y^2 = 1.$$

On D^2, define an equivalence relation $\sim$ by identifying the antipodal points on the boundary circle:

$$(x, y) \sim (-x, -y), \quad x^2 + y^2 = 1.$$

Then φ and ψ induce homeomorphisms

$$\bar{\varphi} : H^2/\sim \to D^2/\sim, \quad \bar{\psi} : D^2/\sim \to H^2/\sim .$$

In summary, there is a sequence of homeomorphisms

$$\mathbb{R}P^2 \xrightarrow{\sim} S^2/\sim \xrightarrow{\sim} H^2/\sim \xrightarrow{\sim} D^2/\sim$$

that identifies the real projective plane as the quotient of the closed disk D^2 with the antipodal points on its boundary identified. This may be the best way to picture $\mathbb{R}P^2$ (Figure 7.7).

Fig. 7.7. The real projective plane as the quotient of a disk.

The real projective plane $\mathbb{R}P^2$ cannot be embedded as a submanifold of $\mathbb{R}^3$. However, if we allow self-intersection, then we can map $\mathbb{R}P^2$ into $\mathbb{R}^3$ as a cross-cap (Figure 7.8). This map is not one-to-one.

Fig. 7.8. The real projective plane immersed as a cross-cap in $\mathbb{R
^{3}$.}

Proposition 7.14. The equivalence relation $\sim$ on $\mathbb{R}^{n+1} - \{0\}$ in the definition of $\mathbb{R}P^n$ is an open equivalence relation.

Proof. For an open set $U \subset \mathbb{R}^{n+1} - \{0\}$, the image $\pi(U)$ is open in $\mathbb{R}P^n$ if and only if $\pi^{-1}(\pi(U))$ is open in $\mathbb{R}^{n+1} - \{0\}$. But $\pi^{-1}(\pi(U))$ consists of all nonzero scalar multiples of points of U; that is,

$$\pi^{-1}(\pi(U)) = \bigcup_{t\in\mathbb{R}^\times} tU = \bigcup_{t\in\mathbb{R}^\times} \{tp \mid p \in U\}.$$

Since multiplication by $t \in \mathbb{R}^\times$is a homeomorphism of $\mathbb{R}^{n+1} - \{0\}$, the set tU is open for any t. Therefore, their union $\bigcup_{t\in\mathbb{R}^\times} tU = \pi^{-1}(\pi(U))$ is also open. $\square$

Corollary 7.15. The real projective space $\mathbb{R}P^n$ is second countable.
Proof. Apply Corollary 7.10. $\square$

Proposition 7.16. The real projective space $\mathbb{R}P^n$ is Hausdorff.
Proof. Let $S = \mathbb{R}^{n+1} - \{0\}$ and consider the set

$$R = \left\{(x, y) \in S \times S \mid y = tx \text{ for some } t \in \mathbb{R}^\times\right\}.$$

If we write x and y as column vectors, then $[xy]$ is an $(n+1) \times 2$ matrix, and R may be characterized as the set of matrices $[xy]$ in $S \times S$ of rank ≤ 1. By a standard fact from linear algebra, $\operatorname{rk}[xy] \leq 1$ is equivalent to the vanishing of all 2×2 minors of $[xy]$ (see Problem B.1). As the zero set of finitely many polynomials, R is a closed subset of $S \times S$. Since $\sim$ is an open equivalence relation on S, and R is closed in $S \times S$, by Theorem 7.7 the quotient $S/\sim \simeq \mathbb{R}P^n$ is Hausdorff. $\square$

7.3.7 7.7 The Standard c^∞ Atlas on a Real Projective Space

Let $[a^0, \ldots, a^n]$ be homogeneous coordinates on the projective space $\mathbb{R}P^n$. Although a^0 is not a well-defined function on $\mathbb{R}P^n$, the condition $a^0 \neq 0$ is independent of the choice of a representative for $[a^0, \ldots, a^n]$. Hence, the condition $a^0 \neq 0$ makes sense on $\mathbb{R}P^n$, and we may define

$$U_0 := \left\{\left[a^0, \ldots, a^n\right] \in \mathbb{R}P^n \mid a^0 \neq 0\right\}.$$

Similarly, for each $i = 1, \ldots, n$, let

$$U_i := \left\{\left[a^0, \ldots, a^n\right] \in \mathbb{R}P^n \mid a^i \neq 0\right\}.$$

Define

$$\phi_0 : U_0 \to \mathbb{R}^n$$

by

$$\left[a^0, \ldots, a^n\right] \mapsto \left(\frac{a^1}{a^0}, \ldots, \frac{a^n}{a^0}\right).$$

This map has a continuous inverse

$$\left(b^1, \ldots, b^n\right) \mapsto \left[1, b^1, \ldots, b^n\right]$$

and is therefore a homeomorphism. Similarly, there are homeomorphisms for each $i = 1, \ldots, n$:

$$\begin{aligned} \phi_i &: U_i \to \mathbb{R}^n, \\ \left[a^0, \ldots, a^n\right] &\mapsto \left(\frac{a^0}{a^i}, \ldots, \widehat{\frac{a^i}{a^i}}, \ldots, \frac{a^n}{a^i}\right), \end{aligned}$$

where the caret sign ^ over a^i/a^i means that that entry is to be omitted. This proves that $\mathbb{R}P^n$ is locally Euclidean with the $(\ U_i, \phi_i\ )$ as charts.

On the intersection $U_0 \cap U_1$, we have $a^0 \neq 0$ and $a^1 \neq 0$, and there are two coordinate systems

$$\overset{\phi_0}{\left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \ldots, \frac{a^n}{a^0}\right)} \phi_1 \left(\frac{a^0}{a^1}, \frac{a^2}{a^1}, \ldots, \frac{a^n}{a^1}\right).$$

We will refer to the coordinate functions on U_0 as $x^1, \ldots, x^n$, and the coordinate functions on U_1 as $y^1, \ldots, y^n$. On U_0,

$$x^i = \frac{a^i}{a^0}, \quad i = 1, \ldots, n,$$

and on U_1,

$$y^1 = \frac{a^0}{a^1}, \quad y^2 = \frac{a^2}{a^1}, \quad \ldots, \quad y^n = \frac{a^n}{a^1}.$$

Then on $U_0 \cap U_1$,

$$y^1 = \frac{1}{x^1}, \quad y^2 = \frac{x^2}{x^1}, \quad y^3 = \frac{x^3}{x^1}, \quad \ldots, \quad y^n = \frac{x^n}{x^1},$$

so

$$\left(\phi_1 \circ \phi_0^{-1}\right)(x) = \left(\frac{1}{x^1}, \frac{x^2}{x^1}, \frac{x^3}{x^1}, \ldots, \frac{x^n}{x^1}\right).$$

This is a C^∞ function because $x^1 \neq 0$ on $\phi_0\left(U_0 \cap U_1\right)$. On any other $U_i \cap U_j$ an analogous formula holds. Therefore, the collection $\{(U_i, \phi_i)\}_{i=0,\ldots,n}$ is a C^∞ atlas for $\mathbb{R}P^n$, called the standard atlas. This concludes the proof that $\mathbb{R}P^n$ is a C^∞ manifold.

7.3.8 Problems

7.1. Image of the Inverse Image of a map

Let $f : X \to Y$ be a map of sets, and let $B \subset Y$. Prove that $f\left(f^{-1}(B)\right) = B \cap f(X)$. Therefore, if f is surjective, then $f\left(f^{-1}(B)\right) = B$.

7.2. Real Projective Plane

Let H^2 be the closed upper hemisphere in the unit sphere S^2, and let $i : H^2 \to S^2$ be the inclusion map. In the notation of Example 7.13, prove that the induced map $f : H^2/\sim \to S^2/\sim$ is a homeomorphism. (Hint: Imitate Proposition 7.3.)

7.3. Closedness of the Diagonal of a Hausdorff Space

Deduce Theorem 7.7 from Corollary 7.8. (Hint: To prove that if $S/\sim$ is Hausdorff, then the graph R of $\sim$ is closed in $S \times S$, use the continuity of the projection map $\pi : S \to S/\sim$. To prove the reverse implication, use the openness of π.)

7.4.* Quotient of a Sphere with Antipodal Points Identified

Let S^n be the unit sphere centered at the origin in $\mathbb{R}^{n+1}$. Define an equivalence relation $\sim$ on S^n by identifying antipodal points:

$$x \sim y \Longleftrightarrow x = \pm y, \quad x, y \in S^n.$$

(a) Show that $\sim$ is an open equivalence relation.
(b) Apply Theorem 7.7 and Corollary 7.8 to prove that the quotient space $S^n/\sim$ is Hausdorff, without making use of the homeomorphism $\mathbb{R}P^n \simeq S^n/\sim$.

7.5.* Orbit Space of a Continuous Group Action

Suppose a right action of a topological group G on a topological space S is continuous; this simply means that the map $S \times G \to S$ describing the action is continuous. Define two points x, y of S to be equivalent if they are in the same orbit; i.e., there is an element $g \in G$ such that $y = xg$. Let S/G be the quotient space; it is called the orbit space of the action. Prove that the projection map $\pi : S \to S/G$ is an open map. (This problem generalizes Proposition 7.14, in which $G = R^\times = \mathbb{R} - \{0\}$ and $S = \mathbb{R}^{n+1} - \{0\}$. Because $\mathbb{R}^\times$is commutative, a left $\mathbb{R}^\times$-action becomes a right $\mathbb{R}^\times$-action if scalar multiplication is written on the right.)

7.6. Quotient of $\setminus$ by $2\pi\mathbb{Z}$

Let the additive group $2\pi\mathbb{Z}$ act on $\mathbb{R}$ on the right by $x \cdot 2\pi n = x + 2\pi n$, where n is an integer. Show that the orbit space $R/2\pi\mathbb{Z}$ is a smooth manifold.

7.7. The Circle as a Quotient Space

(a) Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha=1}^2$ be the atlas of the circle S^1 in Example 5.7, and let $\bar{\phi}_\alpha$ be the map ϕ_α followed by the projection $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$. On $U_1 \cap U_2 = A \amalg B$, since ϕ_1 and ϕ_2 differ by an integer multiple of 2π, $\bar{\phi}_1 = \bar{\phi}_2$. Therefore, $\bar{\phi}_1$ and $\bar{\phi}_2$ piece together to give a well-defined map $\bar{\phi} : S^1 \to \mathbb{R}/2\pi\mathbb{Z}$. Prove that $\bar{\phi}$ is C^∞.
(b) The complex exponential $\mathbb{R} \to S^1, t \mapsto e^{it}$, is constant on each orbit of the action of $2\pi\mathbb{Z}$ on $\mathbb{R}$. Therefore, there is an induced map $F : \mathbb{R}/2\pi\mathbb{Z} \to S^1, F([t]) = e^{it}$. Prove that F is C^∞.
(c) Prove that $F : \mathbb{R}/2\pi\mathbb{Z} \to S^1$ is a diffeomorphism.

7.8. The Grassmannian $G(k, n)$

The Grassmannian $G(k, n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k-plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $a_1, \ldots, a_k$ in $\mathbb{R}^n$. It is therefore completely specified by an $n \times k$ matrix $A = [a_1 \cdots a_k]$ of rank k, where the rank of a matrix A, denoted by $\operatorname{rk} A$, is defined to be the number of linearly independent columns of A. This matrix is called a matrix representative of the k-plane. (For properties of the rank, see the problems in Appendix B.)

Two bases $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$ determine the same k-plane if there is a change-ofbasis matrix $g = [g_{ij}] \in \mathrm{GL}(k, \mathbb{R})$ such that

$$b_j = \sum_i a_i g_{ij}, \quad 1 \le i, j \le k$$

In matrix notation, $B = Ag$.
Let $F(k, n)$ be the set of all $n \times k$ matrices of rank k, topologized as a subspace of $\mathbb{R}^{n\times k}$, and $\sim$ the equivalence relation

$$A \sim B \quad \text{iff} \quad \text{there is a matrix } g \in \mathrm{GL}(k, \mathbb{R}) \text{ such that } B = Ag.$$

In the notation of Problem B.3, $F(k, n)$ is the set $D_{\max}$ in $\mathbb{R}^{n\times k}$ and is therefore an open subset. There is a bijection between $G(k, n)$ and the quotient space $F(k, n)/\sim$. We give the Grassmannian $G(k, n)$ the quotient topology on $F(k, n)/\sim$.
(a) Show that $\sim$ is an open equivalence relation. (Hint: Either mimic the proof of Proposition 7.14 or apply Problem 7.5.)
(b) Prove that the Grassmannian $G(k, n)$ is second countable. (Hint: Apply Corollary 7.10.)
(c) Let $S = F(k, n)$. Prove that the graph R in $S \times S$ of the equivalence relation $\sim$ is closed. (Hint: Two matrices $A = [a_1 \cdots a_k]$ and $B = [b_1 \cdots b_k]$ in $F(k, n)$ are equivalent if and only if

every column of B is a linear combination of the columns of A if and only if $\operatorname{rk}[AB] \leq k$ if and only if all $(k+1) \times (k+1)$ minors of $[AB]$ are zero.)
(d) Prove that the Grassmannian $G(k,n)$ is Hausdorff. (Hint: Mimic the proof of Proposition 7.16.)

Next we want to find a C^∞ atlas on the Grassmannian $G(k,n)$. For simplicity, we specialize to $G(2,4)$. For any 4×2 matrix A, let A_{ij} be the 2×2 submatrix consisting of its i th row and j th row. Define

$$V_{ij} = \{A \in F(2,4) \mid A_{ij} \text{ is nonsingular } \}$$

Because the complement of V_{ij} in $F(2,4)$ is defined by the vanishing of $\det A_{ij}$, we conclude that V_{ij} is an open subset of $F(2,4)$.
(e) Prove that if $A \in V_{ij}$, then $Ag \in V_{ij}$ for any nonsingular matrix $g \in \mathrm{GL}(2,\mathbb{R})$.

Define $U_{ij} = V_{ij}/\sim$. Since $\sim$ is an open equivalence relation, $U_{ij} = V_{ij}/\sim$ is an open subset of $G(2,4)$.

For $A \in V_{12}$,

$$A \sim AA_{12}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} I \\ A_{34}A_{12}^{-1} \end{bmatrix}.$$

This shows that the matrix representatives of a 2-plane in U_{12} have a canonical form B in which B_{12} is the identity matrix.
(f) Show that the map $\tilde{\phi}_{12} : V_{12} \to \mathbb{R}^{2\times 2}$,

$$\tilde{\phi}_{12}(A) = A_{34}A_{12}^{-1},$$

induces a homeomorphism $\phi_{12} : U_{12} \to \mathbb{R}^{2\times 2}$.
(g) Define similarly homeomorphisms $\phi_{ij} : U_{ij} \to \mathbb{R}^{2\times 2}$. Compute $\phi_{12} \circ \phi_{23}^{-1}$, and show that it is C^∞.
(h) Show that $\{U_{ij} \mid 1 \leq i < j \leq 4\}$ is an open cover of $G(2,4)$ and that $G(2,4)$ is a smooth manifold.

Similar consideration shows that $F(k,n)$ has an open cover $\{V_I\}$, where I is a strictly ascending multi-index $1 \leq i_1 < \cdots < i_k \leq n$. For $A \in F(k,n)$, let A_I be the $k \times k$ submatrix of A consisting of i_1 th, $\ldots, i_k$ th rows of A. Define

$$V_I = \{A \in G(k,n) \mid \det A_I \neq 0\}$$

Next define $\tilde{\phi}_I : V_I \to \mathbb{R}^{(n-k)\times k}$ by

$$\tilde{\phi}_I(A) = \left(AA_I^{-1}\right)_{I'}$$

where $()_{I'}$ denotes the $(n-k) \times k$ submatrix obtained from the complement I' of the multiindex I. Let $U_I = V_I/\sim$. Then $\tilde{\phi}$ induces a homeomorphism $\phi : U_I \to \mathbb{R}^{(n-k)\times k}$. It is not difficult to show that $\{(U_I, \phi_I)\}$ is a C^∞ atlas for $G(k,n)$. Therefore the Grassmannian $G(k,n)$ is a C^∞ manifold of dimension $k(n-k)$.

7.9.* Compactness of Real Projective Space

Show that the real projective space $\mathbb{R}P^n$ is compact. (Hint: Use Exercise 7.11.)

8 3 The Tangent Space

By definition, the tangent space to a manifold at a point is the vector space of derivations at the point. A smooth map of manifolds induces a linear map, called its differential, of tangent spaces at corresponding points. In local coordinates, the differential is represented by the Jacobian matrix of partial derivatives of the map. In this sense, the differential of a map between manifolds is a generalization of the derivative of a map between Euclidean spaces.

A basic principle in manifold theory is the linearization principle, according to which a manifold can be approximated near a point by its tangent space at the point, and a smooth map can be approximated by the differential of the map. In this way, one turns a topological problem into a linear problem. A good example of the linearization principle is the inverse function theorem, which reduces the local invertibility of a smooth map to the invertibility of its differential at a point.

Using the differential, we classify maps having maximal rank at a point into immersions and submersions at the point, depending on whether the differential is injective or surjective there. A point where the differential is surjective is a regular point of the map. The regular level set theorem states that a level set all of whose points are regular is a regular submanifold, i.e., a subset that locally looks like a coordinate k-plane in $\mathbb{R}^n$. This theorem gives a powerful tool for proving that a topological space is a manifold.

We then introduce categories and functors, a framework for comparing structural similarities. After this interlude, we return to the study of maps via their differentials. From the rank of the differential, one obtains three local normal forms for smooth maps-the constant rank theorem, the immersion theorem, and the submersion theorem, corresponding to constant-rank differentials, injective differentials, and surjective differentials respectively. We give three proofs of the regular level set theorem, a first proof (Theorem 9.9), using the inverse function theorem, that actually produces explicit local coordinates, and two more proofs (p. 119) that are corollaries of the constant rank theorem and the submersion theorem.

The collection of tangent spaces to a manifold can be given the structure of a vector bundle; it is then called the tangent bundle of the manifold. Intuitively, a vector bundle over a manifold is a locally trivial family of vector spaces parametrized
by points of the manifold. A smooth map of manifolds induces, via its differential at each point, a bundle map of the corresponding tangent bundles. In this way we obtain a covariant functor from the category of smooth manifolds and smooth maps to the category of vector bundles and bundle maps. Vector fields, which manifest themselves in the physical world as velocity, force, electricity, magnetism, and so on, may be viewed as sections of the tangent bundle over a manifold.

Smooth C^∞ bump functions and partitions of unity are an indispensable technical tool in the theory of smooth manifolds. Using C^∞ bump functions, we give several criteria for a vector field to be smooth. The chapter ends with integral curves, flows, and the Lie bracket of smooth vector fields.

8.1 §8 the Tangent Space

In Section 2 we saw that for any point p in an open set U in $\mathbb{R}^n$ there are two equivalent ways to define a tangent vector at p :
(i) as an arrow (Figure 8.1), represented by a column vector;

Fig. 8.1. A tangent vector in $\mathbb{R ^{n}}$ as an arrow and as a column vector.}

(ii) as a point-derivation of C_p^∞, the algebra of germs of C^∞ functions at p.

Both definitions generalize to a manifold. In the arrow approach, one defines a tangent vector at p in a manifold M by first choosing a chart (U, ϕ) at p and then decreeing a tangent vector at p to be an arrow at $\phi(p)$ in $\phi(U)$. This approach, while more visual, is complicated to work with, since a different chart (V, ψ) at p would give rise to a different set of tangent vectors at p and one would have to decide how to identify the arrows at $\phi(p)$ in U with the arrows at $\psi(p)$ in $\psi(V)$.

The cleanest and most intrinsic definition of a tangent vector at p in M is as a point-derivation, and this is the approach we adopt.

8.1.1 8.1 The Tangent Space at a Point

Just as for $\mathbb{R}^n$, we define a germ of a C^∞ function at p in M to be an equivalence class of C^∞ functions defined in a neighborhood of p in M, two such functions being equivalent if they agree on some, possibly smaller, neighborhood of p. The set of germs of C^∞ real-valued functions at p in M is denoted by $C_p^\infty(M)$. The addition and multiplication of functions make $C_p^\infty(M)$ into a ring; with scalar multiplication by real numbers, $C_p^\infty(M)$ becomes an algebra over $\mathbb{R}$.

Generalizing a derivation at a point in $\mathbb{R}^n$, we define a derivation at a point in a manifold M, or a point-derivation of $C_p^\infty(M)$, to be a linear map $D : C_p^\infty(M) \to \mathbb{R}$ such that

$$D(fg) = (Df)g(p) + f(p)Dg.$$

Definition 8.1. A tangent vector at a point p in a manifold M is a derivation at p.

Just as for $\mathbb{R}^n$, the tangent vectors at p form a vector space $T_p(M)$, called the tangent space of M at p. We also write T_pM instead of $T_p(M)$.

Remark 8.2 (Tangent space to an open subset). If U is an open set containing p in M, then the algebra $C_p^\infty(U)$ of germs of C^∞ functions in U at p is the same as $C_p^\infty(M)$. Hence, $T_pU = T_pM$.

Given a coordinate neighborhood $(U, \phi) = (U, x^1, \ldots, x^n)$ about a point p in a manifold M, we recall the definition of the partial derivatives $\partial/\partial x^i$ first introduced in Section 6. Let $r^1, \ldots, r^n$ be the standard coordinates on $\mathbb{R}^n$. Then

$$x^i = r^i \circ \phi : U \to \mathbb{R}.$$

If f is a smooth function in a neighborhood of p, we set

$$\left.\frac{\partial}{\partial x^i}\right|_p f = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} \left(f \circ \phi^{-1}\right) \in \mathbb{R}.$$

It is easily checked that $\partial/\left.\partial x^i\right|_p$ satisfies the derivation property and so is a tangent vector at p.

When M is one-dimensional and t is a local coordinate, it is customary to write $d/\left.dt\right|_p$ instead of $\partial/\left.\partial t\right|_p$ for the coordinate vector at the point p. To simplify the notation, we will sometimes write $\partial/\partial x^i$ instead of $\partial/\left.\partial x^i\right|_p$ if it is understood at which point the tangent vector is located.

8.1.2 8.2 The Differential of a Map

Let $F : N \to M$ be a C^∞ map between two manifolds. At each point $p \in N$, the map F induces a linear map of tangent spaces, called its differential at p,

$$F_* : T_pN \to T_{F(p)}M$$

as follows. If $X_p \in T_pN$, then $F_*(X_p)$ is the tangent vector in $T_{F(p)}M$ defined by

$$(F_*(X_p))\, f = X_p(f \circ F) \in \mathbb{R} \quad \text{for } f \in C^\infty_{F(p)}(M). \tag{8.1}$$

Here f is a germ at $F(p)$, represented by a C^∞ function in a neighborhood of $F(p)$. Since (8.1) is independent of the representative of the germ, in practice we can be cavalier about the distinction between a germ and a representative function for the germ.

Exercise 8.3 (The differential of a map). Check that $F_*(X_p)$ is a derivation at $F(p)$ and that $F_* : T_pN \to T_{F(p)}M$ is a linear map.

To make the dependence on p explicit we sometimes write $F_{*,p}$ instead of F_*.

Example 8.4 (Differential of a map between Euclidean spaces). Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is smooth and p is a point in $\mathbb{R}^n$. Let $x^1, \ldots, x^n$ be the coordinates on $\mathbb{R}^n$ and $y^1, \ldots, y^m$ the coordinates on $\mathbb{R}^m$. Then the tangent vectors $\partial/\left.\partial x^1\right|_p, \ldots, \partial/\left.\partial x^n\right|_p$ form a basis for the tangent space $T_p(\mathbb{R}^n)$ and $\partial/\left.\partial y^1\right|_{F(p)}, \ldots, \partial/\left.\partial y^m\right|_{F(p)}$ form a basis for the tangent space $T_{F(p)}(\mathbb{R}^m)$. The linear map $F_* : T_p(\mathbb{R}^n) \to T_{F(p)}(\mathbb{R}^m)$ is described by a matrix $\left[a^i_j\right]$ relative to these two bases:

$$F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_k a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)}, \quad a^k_j \in \mathbb{R}. \tag{8.2}$$

Let $F^i = y^i \circ F$ be the i th component of F. We can find a^i_j by evaluating the righthand side (RHS) and left-hand side (LHS) of (8.2) on y^i :

$$\begin{aligned}
\text{RHS} &= \sum_k a^k_j \left.\frac{\partial}{\partial y^k}\right|_{F(p)} y^i = \sum_k a^k_j \delta^i_k = a^i_j, \\
\text{LHS} &= F_*\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) y^i = \left.\frac{\partial}{\partial x^j}\right|_p \left(y^i \circ F\right) = \frac{\partial F^i}{\partial x^j}(p).
\end{aligned}$$

So the matrix of F_* relative to the bases $\left\{\partial/\left.\partial x^j\right|_p\right\}$ and $\left\{\partial/\left.\partial y^i\right|_{F(p)}\right\}$ is $[\partial F^i/\partial x^j(p)]$. This is precisely the Jacobian matrix of the derivative of F at p. Thus, the differential of a map between manifolds generalizes the derivative of a map between Euclidean spaces.

8.1.3 8.3 The Chain Rule

Let $F : N \to M$ and $G : M \to P$ be smooth maps of manifolds, and $p \in N$. The differentials of F at p and G at $F(p)$ are linear maps

$$T_pN \xrightarrow{F_{*,p}} T_{F(p)}M \xrightarrow{G_{*,F(p)}} T_{G(F(p))}P.$$

Theorem 8.5 (The chain rule). If $F : N \to M$ and $G : M \to P$ are smooth maps of manifolds and $p \in N$, then

$$(G \circ F)_{*,p} = G_{*,F(p)} \circ F_{*,p}$$

Proof. Let $X_p \in T_pN$ and let f be a smooth function at $G(F(p))$ in P. Then

$$((G \circ F)_* X_p)\, f = X_p(f \circ G \circ F)$$

and

$$((G_* \circ F_*)\, X_p)\, f = (G_*\,(F_* X_p))\, f = (F_* X_p)\,(f \circ G) = X_p(f \circ G \circ F).$$

Example 8.13 shows that when written out in terms of matrices, the chain rule of Theorem 8.5 assumes a more familiar form as a sum of products of partial derivatives.

Remark. The differential of the identity map $\mathbb{1}_M : M \to M$ at any point p in M is the identity map

$$\mathbb{1}_{T_pM} : T_pM \to T_pM,$$

because

$$((\mathbb{1}_M)_*\, X_p)\, f = X_p\,(f \circ \mathbb{1}_M) = X_p f,$$

for any $X_p \in T_pM$ and $f \in C_p^\infty(M)$.

Corollary 8.6. If $F : N \to M$ is a diffeomorphism of manifolds and $p \in N$, then $F_* : T_pN \to T_{F(p)}M$ is an isomorphism of vector spaces.

Proof. To say that F is a diffeomorphism means that it has a differentiable inverse $G : M \to N$ such that $G \circ F = \mathbb{1}_N$ and $F \circ G = \mathbb{1}_M$. By the chain rule,

$$\begin{aligned}(G \circ F)_* &= G_* \circ F_* = (\mathbb{1}_N)_* = \mathbb{1}_{T_pN},\\ (F \circ G)_* &= F_* \circ G_* = (\mathbb{1}_M)_* = \mathbb{1}_{T_{F(p)}M}.\end{aligned}$$

Hence, F_* and G_* are isomorphisms. □

Corollary 8.7 (Invariance of dimension). If an open set $U \subset \mathbb{R}^n$ is diffeomorphic to an open set $V \subset \mathbb{R}^m$, then $n = m$.

Proof. Let $F : U \to V$ be a diffeomorphism and let $p \in U$. By Corollary 8.6, $F_{*,p} : T_pU \to T_{F(p)}V$ is an isomorphism of vector spaces. Since there are vector space isomorphisms $T_pU \simeq \mathbb{R}^n$ and $T_{F(p)} \simeq \mathbb{R}^m$, we must have that $n = m$. □

8.1.4 8.4 Bases for the Tangent Space at a Point

As usual, we denote by $r^1, \ldots, r^n$ the standard coordinates on $\mathbb{R}^n$, and if $(\,U, \phi\,)$ is a chart about a point p in a manifold M of dimension n, we set $x^i = r^i \circ \phi$. Since $\phi : U \to \mathbb{R}^n$ is a diffeomorphism onto its image (Proposition 6.10), by Corollary 8.6 the differential

$$\phi_* : T_pM \to T_{\phi(p)}\mathbb{R}^n$$

is a vector space isomorphism. In particular, the tangent space T_pM has the same dimension n as the manifold M.

Proposition 8.8. Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart about a point p in a manifold M. Then

$$\phi_*\left(\left.\frac{\partial}{\partial x^i}\right|_p\right) = \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)}.$$

Proof. For any $f \in C^{\infty}_{\phi(p)}(\mathbb{R}^n)$,

$$\begin{aligned}
\phi_*\left(\frac{\partial}{\partial x^i}\Big|_p\right) f &= \frac{\partial}{\partial x^i}\Big|_p (f \circ \phi) && (\text{ definition of } \phi_*) \\
&= \frac{\partial}{\partial r^i}\Big|_{\phi(p)} (f \circ \phi \circ \phi^{-1}) && \left(\text{ definition of } \partial/\,\partial x^i|_p\right) \\
&= \frac{\partial}{\partial r^i}\Big|_{\phi(p)} f.
\end{aligned}$$

Proposition 8.9. If $(U, \phi) = (U, x^1, \ldots, x^n)$ is a chart containing p, then the tangent space T_pM has basis

$$\frac{\partial}{\partial x^1}\bigg|_p, \ldots, \frac{\partial}{\partial x^n}\bigg|_p.$$

Proof. An isomorphism of vector spaces carries a basis to a basis. By Proposition 8.8 the isomorphism $\phi_* : T_pM \to T_{\phi(p)}(\mathbb{R}^n)$ maps $\partial/\,\partial x^1|_p, \ldots, \partial/\,\partial x^n|_p$ to $\partial/\,\partial r^1|_{\phi(p)}, \ldots, \partial/\,\partial r^n|_{\phi(p)}$, which is a basis for the tangent space $T_{\phi(p)}(\mathbb{R}^n)$. Therefore, $\partial/\,\partial x^1|_p, \ldots, \partial/\,\partial x^n|_p$ is a basis for T_pM.

Proposition 8.10 (Transition matrix for coordinate vectors). Suppose $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ are two coordinate charts on a manifold M. Then

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i}$$

on $U \cap V$.

Proof. At each point $p \in U \cap V$, the sets $\left\{\partial/\,\partial x^j|_p\right\}$ and $\left\{\partial/\,\partial y^i|_p\right\}$ are both bases for the tangent space T_pM, so there is a matrix $\left[a^i_j(p)\right]$ of real numbers such that on $U \cap V$,

$$\frac{\partial}{\partial x^j} = \sum_k a^k_j \frac{\partial}{\partial y^k}$$

Applying both sides of the equation to y^i, we get

$$\begin{aligned}
\frac{\partial y^i}{\partial x^j} &= \sum_k a^k_j \frac{\partial y^i}{\partial y^k} \\
&= \sum_k a^k_j \delta^i_k \quad (\text{ by Proposition 6.22}) \\
&= a^i_j
\end{aligned}$$

8.1.5 8.5 A Local Expression for the Differential

Given a smooth map $F : N \to M$ of manifolds and $p \in N$, let $(U, x^1, \ldots, x^n)$ be a chart about p in N and let $(V, y^1, \ldots, y^m)$ be a chart about $F(p)$ in M. We will find a local expression for the differential $F_{*,p} : T_pN \to T_{F(p)}M$ relative to the two charts.

By Proposition 8.9, $\left\{\partial/\,\partial x^j|_p\right\}_{j=1}^n$ is a basis for T_pN and $\left\{\partial/\,\partial y^i|_{F(p)}\right\}_{i=1}^m$ is a basis for $T_{F(p)}M$. Therefore, the differential $F_* = F_{*,p}$ is completely determined by the numbers a^i_j such that

$$F_*\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \sum_{k=1}^m a^k_j \frac{\partial}{\partial y^k}\bigg|_{F(p)}, \quad j = 1, \ldots, n.$$

Applying both sides to y^i, we find that

$$a_j^i = \left(\sum_{k=1}^m a_j^k \frac{\partial}{\partial y^k}\bigg|_{F(p)} \right) y^i = F_* \left(\frac{\partial}{\partial x^j}\bigg|_p \right) y^i = \frac{\partial}{\partial x^j}\bigg|_p \left(y^i \circ F\right) = \frac{\partial F^i}{\partial x^j}(p).$$

We state this result as a proposition.

Proposition 8.11. Given a smooth map $F : N \to M$ of manifolds and a point $p \in N$, let $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^m)$ be coordinate charts about p in N and $F(p)$ in M respectively. Relative to the bases $\left\{\partial/\,\partial x^j|_p\right\}$ for T_pN and $\left\{\partial/\,\partial y^i|_{F(p)}\right\}$ for $T_{F(p)}M$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is represented by the matrix $[\partial F^i/\partial x^j(p)]$, where $F^i = y^i \circ F$ is the ith component of F.

This proposition is in the spirit of the "arrow" approach to tangent vectors. Here each tangent vector in T_pN is represented by a column vector relative to the basis $\left\{\partial/\,\partial x^j|_p\right\}$, and the differential $F_{*,p}$ is represented by a matrix.

Remark 8.12 (Inverse function theorem). In terms of the differential, the inverse function theorem for manifolds (Theorem 6.26) has a coordinate-free description: a C^∞ map $F : N \to M$ between two manifolds of the same dimension is locally invertible at a point $p \in N$ if and only if its differential $F_{*,p} : T_pN \to T_{f(p)}M$ at p is an isomorphism.

Example 8.13 (The chain rule in calculus notation). Suppose $w = G(x, y, z)$ is a C^∞ function: $\mathbb{R}^3 \to \mathbb{R}$ and $(x, y, z) = F(t)$ is a C^∞ function: $\mathbb{R} \to \mathbb{R}^3$. Under composition,

$$w = (G \circ F)(t) = G(x(t), y(t), z(t))$$

becomes a C^∞ function of $t \in \mathbb{R}$. The differentials F_*, G_*, and $(G \circ F)_*$ are represented by the matrices

$$\begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix}, \quad \begin{bmatrix} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \end{bmatrix}, \quad \text{and} \quad \frac{dw}{dt},$$

respectively. Since composition of linear maps is represented by matrix multiplication, in terms of matrices the chain rule $(G \circ F)_* = G_* \circ F_*$ is equivalent to

$$\frac{dw}{dt} = \begin{bmatrix} \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} \frac{\partial w}{\partial z} \end{bmatrix} \begin{bmatrix} dx/dt \\ dy/dt \\ dz/dt \end{bmatrix} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$

This is the usual form of the chain rule taught in calculus.

8.1.6 8.6 Curves in a Manifold

A smooth curve in a manifold M is by definition a smooth map $c :]a, b[\to M$ from some open interval $]a, b[$ into M. Usually we assume $0 \in]a, b[$ and say that c is a curve starting at p if $c(0) = p$. The velocity vector $c'(t_0)$ of the curve c at time $t_0 \in]a, b[$ is defined to be

$$c'(t_0) := c_* \left(\frac{d}{dt}\bigg|_{t_0} \right) \in T_{c(t_0)}M$$

We also say that $c'(t_0)$ is the velocity of c at the point $c(t_0)$. Alternative notations for $c'(t_0)$ are

$$\frac{dc}{dt}(t_0) \quad \text{and} \quad \frac{d}{dt}\bigg|_{t_0} c.$$

Notation. When $c :]a, b[\to \mathbb{R}$ is a curve with target space $\mathbb{R}$, the notation $c'(t)$ can be a source of confusion. Here t is the standard coordinate on the domain $]a, b[$. Let x be the standard coordinate on the target space $\mathbb{R}$. By our definition, $c'(t)$ is a tangent vector at $c(t)$, hence a multiple of $d/\ dx|_{c(t)}$. On the other hand, in calculus notation $c'(t)$ is the derivative of a real-valued function and is therefore a scalar. If it is necessary to distinguish between these two meanings of $c'(t)$ when c maps into $\mathbb{R}$, we will write $\dot{c}(t)$ for the calculus derivative.

Exercise 8.14 (Velocity vector versus the calculus derivative).* Let $c :]a, b[\to \mathbb{R}$ be a curve with target space $\mathbb{R}$. Verify that $c'(t) = \dot{c}(t) d/\ dx|_{c(t)}$.

Example. Define $c : \mathbb{R} \to \mathbb{R}^2$ by

$$c(t) = \left(t^2, t^3\right)$$

(See Figure 8.2.)
Then $c'(t)$ is a linear combination of $\partial/\partial x$ and $\partial/\partial y$ at $c(t)$:

$$c'(t) = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}.$$

To compute a, we evaluate both sides on x :

Fig. 8.2. A cuspidal cubic.

$$a = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) x = c'(t)x = c_*\left(\frac{d}{dt}\right) x = \frac{d}{dt}(x \circ c) = \frac{d}{dt}t^2 = 2t.$$

Similarly,

$$b = \left(a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y}\right) y = c'(t)y = c_*\left(\frac{d}{dt}\right) y = \frac{d}{dt}(y \circ c) = \frac{d}{dt}t^3 = 3t^2.$$

Thus,

$$c'(t) = 2t\frac{\partial}{\partial x} + 3t^2\frac{\partial}{\partial y}.$$

In terms of the basis $\partial/\ \partial x|_{c(t)}, \partial/\ \partial y|_{c(t)}$ for $T_{c(t)}\left(\mathbb{R}^2\right)$,

$$c'(t) = \begin{bmatrix} 2t \\ 3t^2 \end{bmatrix}$$

More generally, as in this example, to compute the velocity vector of a smooth curve c in $\mathbb{R}^n$, one can simply differentiate the components of c. This shows that our definition of the velocity vector of a curve agrees with the usual definition in vector calculus.

Proposition 8.15 (Velocity of a curve in local coordinates). Let $c :]a, b[\to M$ be a smooth curve, and let $(U, x^1, \ldots, x^n)$ be a coordinate chart about $c(t)$. Write $c^i = x^i \circ c$ for the ith component of c in the chart. Then $c'(t)$ is given by

$$c'(t) = \sum_{i=1}^{n} \dot{c}^i(t) \frac{\partial}{\partial x^i}\bigg|_{c(t)} .$$

Thus, relative to the basis $\left\{\partial/\,\partial x^i|_p\right\}$ for $T_{c(t)}M$, the velocity $c'(t)$ is represented by the column vector

$$\begin{bmatrix} \dot{c}^1(t) \\ \vdots \\ \dot{c}^n(t) \end{bmatrix} .$$

Proof. Problem 8.5.

Every smooth curve c at p in a manifold M gives rise to a tangent vector $c'(0)$ in T_pM. Conversely, one can show that every tangent vector $X_p \in T_pM$ is the velocity vector of some curve at p, as follows.

Proposition 8.16 (Existence of a curve with a given initial vector). For any point p in a manifold M and any tangent vector $X_p \in T_pM$, there are $\varepsilon > 0$ and a smooth curve $c :]-\varepsilon, \varepsilon[\to M$ such that $c(0) = p$ and $c'(0) = X_p$.

Fig. 8.3. Existence of a curve through a point with a given initial vector.

Proof. Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart centered at p; i.e., $\phi(p) = \mathbf{0} \in \mathbb{R}^n$. Suppose $X_p = \sum a^i \partial/\,\partial x^i|_p$ at p. Let $r^1, \ldots, r^n$ be the standard coordinates on $\mathbb{R}^n$. Then $x^i = r^i \circ \phi$. To find a curve c at p with $c'(0) = X_p$, start with a curve α in $\mathbb{R}^n$ with $\alpha(0) = \mathbf{0}$ and $\alpha'(0) = \sum a^i \partial/\,\partial r^i|_{\mathbf{0}}$. We then map α to M via ϕ^{-1} (Figure 8.3). By Proposition 8.15, the simplest such α is

$$\alpha(t) = \left(a^1 t, \ldots, a^n t\right), \quad t \in]-\varepsilon, \varepsilon[,$$

where ε is sufficiently small that $\alpha(t)$ lies in $\phi(U)$. Define $c = \phi^{-1} \circ \alpha :]-\varepsilon, \varepsilon[\to M$. Then

$$c(0) = \phi^{-1}(\alpha(0)) = \phi^{-1}(\mathbf{0}) = p,$$

and by Proposition 8.8,

$$c'(0) = \left(\phi^{-1}\right)_* \alpha_* \left(\frac{d}{dt}\bigg|_{t=0}\right) = \left(\phi^{-1}\right)_* \left(\sum a^i \frac{\partial}{\partial r^i}\bigg|_0\right) = \sum a^i \frac{\partial}{\partial x^i}\bigg|_p = X_p.$$

□

In Definition 8.1 we defined a tangent vector at a point p of a manifold abstractly as a derivation at p. Using curves, we can now interpret a tangent vector geometrically as a directional derivative.

Proposition 8.17. Suppose X_p is a tangent vector at a point p of a manifold M and $f \in C_p^\infty(M)$. If $c :]-\varepsilon, \varepsilon[\to M$ is a smooth curve starting at p with $c'(0) = X_p$, then

$$X_p f = \left.\frac{d}{dt}\right|_0 (f \circ c).$$

Proof. By the definitions of $c'(0)$ and c_*,

$$X_p f = c'(0) f = c_* \left(\left.\frac{d}{dt}\right|_0 \right) f = \left.\frac{d}{dt}\right|_0 (f \circ c).$$

□

8.1.7 8.7 Computing the Differential Using Curves

We have introduced two ways of computing the differential of a smooth map, in terms of derivations at a point (equation (8.1)) and in terms of local coordinates (Proposition 8.11). The next proposition gives still another way of computing the differential $F_{*,p}$, this time using curves.

Proposition 8.18. Let $F : N \to M$ be a smooth map of manifolds, $p \in N$, and $X_p \in T_p N$. If c is a smooth curve starting at p in N with velocity X_p at p, then

$$F_{*,p}(X_p) = \left.\frac{d}{dt}\right|_0 (F \circ c)(t)$$

In other words, $F_{*,p}(X_p)$ is the velocity vector of the image curve $F \circ c$ at $F(p)$.
Proof. By hypothesis, $c(0) = p$ and $c'(0) = X_p$. Then

$$\begin{aligned} F_{*,p}(X_p) &= F_{*,p}(c'(0)) \\ &= (F_{*,p} \circ c_{*,0}) \left(\left.\frac{d}{dt}\right|_0 \right) \\ &= (F \circ c)_{*,0} \left(\left.\frac{d}{dt}\right|_0 \right) \quad \text{(by the chain rule, Theorem 8.5)} \\ &= \left.\frac{d}{dt}\right|_0 (F \circ c)(t) \end{aligned}$$

□

Example 8.19 (Differential of left multiplication). If g is a matrix in the general linear group $\mathrm{GL}(n, \mathbb{R})$, let $\ell_g : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ be left multiplication by g; thus, $\ell_g(B) = gB$ for any $B \in \mathrm{GL}(n, \mathbb{R})$. Since $\mathrm{GL}(n, \mathbb{R})$ is an open subset of the vector space $\mathbb{R}^{n \times n}$, the tangent space $T_g(\mathrm{GL}(n, \mathbb{R}))$ can be identified with $\mathbb{R}^{n \times n}$. Show that with this identification the differential $(\ell_g)_{*,I} : T_I(\mathrm{GL}(n, \mathbb{R})) \to T_g(\mathrm{GL}(n, \mathbb{R}))$ is also left multiplication by g.

Solution. Let $X \in T_I(\mathrm{GL}(n, \mathbb{R})) = \mathbb{R}^{n \times n}$. To compute $(\ell_g)_{*,I}(X)$, choose a curve $c(t)$ in $\mathrm{GL}(n, \mathbb{R})$ with $c(0) = I$ and $c'(0) = X$. Then $\ell_g(c(t)) = gc(t)$ is simply matrix multiplication. By Proposition 8.18,

$$(\ell_g)_{*,I}(X) = \left.\frac{d}{dt}\right|_{t=0} \ell_g(c(t)) = \left.\frac{d}{dt}\right|_{t=0} gc(t) = gc'(0) = gX$$

In this computation, $d/\,dt|_{t=0}\, gc(t) = gc'(0)$ by $\mathbb{R}$-linearity and Proposition 8.15. □

8.1.8 8.8 Immersions and Submersions

Just as the derivative of a map between Euclidean spaces is a linear map that best approximates the given map at a point, so the differential at a point serves the same purpose for a C^∞ map between manifolds. Two cases are especially important. A C^∞ map $F : N \to M$ is said to

be an immersion at $p \in N$ if its differential $F_{*,p} : T_pN \to T_{F(p)}M$ is injective, and a submersion at p if $F_{*,p}$ is surjective. We call F an immersion if it is an immersion at every $p \in N$ and a submersion if it is a submersion at every $p \in N$.

Remark 8.20. Suppose N and M are manifolds of dimensions n and m respectively. Then $\dim T_pN = n$ and $\dim T_{F(p)}M = m$. The injectivity of the differential $F_{*,p} : T_pN \to T_{F(p)}M$ implies immediately that $n \leq m$. Similarly, the surjectivity of the differential $F_{*,p}$ implies that $n \geq m$. Thus, if $F : N \to M$ is an immersion at a point of N, then $n \leq m$ and if F is a submersion a point of N, then $n \geq m$.

Example 8.21. The prototype of an immersion is the inclusion of $\mathbb{R}^n$ in a higherdimensional $\mathbb{R}^m$:

$$i\left(x^1, \ldots, x^n\right) = \left(x^1, \ldots, x^n, 0, \ldots, 0\right).$$

The prototype of a submersion is the projection of $\mathbb{R}^n$ onto a lower-dimensional $\mathbb{R}^m$:

$$\pi\left(x^1, \ldots, x^m, x^{m+1}, \ldots, x^n\right) = \left(x^1, \ldots, x^m\right).$$

Example. If U is an open subset of a manifold M, then the inclusion $i : U \to M$ is both an immersion and a submersion. This example shows in particular that a submersion need not be onto.

In Section 11, we will undertake a more in-depth analysis of immersions and submersions. According to the immersion and submersion theorems to be proven there, every immersion is locally an inclusion and every submersion is locally a projection.

8.1.9 8.9 Rank, and Critical and Regular Points

The rank of a linear transformation $L : V \to W$ between finite-dimensional vector spaces is the dimension of the image $L(V)$ as a subspace of W, while the rank of a matrix A is the dimension of its column space. If L is represented by a matrix A relative to a basis for V and a basis for W, then the rank of L is the same as the rank of A, because the image $L(V)$ is simply the column space of A.

Now consider a smooth map $F : N \to M$ of manifolds. Its rank at a point p in N, denoted by $\operatorname{rk} F(p)$, is defined as the rank of the differential $F_{*,p} : T_pN \to T_{F(p)}M$. Relative to the coordinate neighborhoods $(U, x^1, \ldots, x^n)$ at p and $(V, y^1, \ldots, y^m)$ at $F(p)$, the differential is represented by the Jacobian matrix $[\partial F^i / \partial x^j(p)]$ (Proposition 8.11), so

$$\operatorname{rk} F(p) = \operatorname{rk}\left[\frac{\partial F^i}{\partial x^j}(p)\right].$$

Since the differential of a map is independent of coordinate charts, so is the rank of a Jacobian matrix.

Definition 8.22. A point p in N is a critical point of F if the differential

$$F_{*,p} : T_pN \to T_{F(p)}M$$

fails to be surjective. It is a regular point of F if the differential $F_{*,p}$ is surjective. In other words, p is a regular point of the map F if and only if F is a submersion at p. A point in M is a critical value if it is the image of a critical point; otherwise it is a regular value (Figure 8.4).

Fig. 8.4. Critical points and critical values of the function $f(x, y, z) = z$.

Two aspects of this definition merit elaboration:
(i) We do not define a regular value to be the image of a regular point. In fact, a regular value need not be in the image of F at all. Any point of M not in the image of F is automatically a regular value because it is not the image of a critical point.
(ii) A point c in M is a critical value if and only if some point in the preimage $F^{-1}(\{c\})$ is a critical point. A point c in the image of F is a regular value if and only if every point in the preimage $F^{-1}(\{c\})$ is a regular point.

Proposition 8.23. For a real-valued function $f : M \to \mathbb{R}$, a point p in M is a critical point if and only if relative to some chart $(U, x^1, \ldots, x^n)$ containing p, all the partial derivatives satisfy

$$\frac{\partial f}{\partial x^j}(p) = 0, \quad j = 1, \ldots, n.$$

Proof. By Proposition 8.11 the differential $f_{*,p} : T_pM \to T_{f(p)}\mathbb{R} \simeq \mathbb{R}$ is represented by the matrix

$$\left[\frac{\partial f}{\partial x^1}(p) \cdots \frac{\partial f}{\partial x^n}(p)\right].$$

Since the image of $f_{*,p}$ is a linear subspace of $\mathbb{R}$, it is either zero-dimensional or one-dimensional. In other words, $f_{*,p}$ is either the zero map or a surjective map. Therefore, $f_{*,p}$ fails to be surjective if and only if all the partial derivatives $\partial f/\partial x^i(p)$ are zero. □

8.1.10 Problems

8.1.* Differential of a map

Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be the map

$$(u, v, w) = F(x, y) = (x, y, xy)$$

Let $p = (x, y) \in \mathbb{R}^2$. Compute $F_*\left(\partial/\,\partial x|_p\right)$ as a linear combination of $\partial/\partial u, \partial/\partial v$, and $\partial/\partial w$ at $F(p)$.

8.2. Differential of a Linear map

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. For any $p \in \mathbb{R}^n$, there is a canonical identification $T_p(\mathbb{R}^n) \xrightarrow{\sim} \mathbb{R}^n$ given by

$$\sum a^i \frac{\partial}{\partial x^i}\bigg|_p \mapsto \mathbf{a} = \langle a^1, \ldots, a^n \rangle.$$

Show that the differential $L_{*,p} : T_p(\mathbb{R}^n) \to T_{L(p)}(\mathbb{R}^m)$ is the map $L : \mathbb{R}^n \to \mathbb{R}^m$ itself, with the identification of the tangent spaces as above.

8.3. Differential of a map

Fix a real number α and define $F : \mathbb{R}^2 \to \mathbb{R}^2$ by

$$\begin{bmatrix} u \\ v \end{bmatrix} = (u, v) = F(x, y) = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Let $X = -y\partial/\partial x + x\partial/\partial y$ be a vector field on $\mathbb{R}^2$. If $p = (x, y) \in \mathbb{R}^2$ and $F_*(X_p) = (a\partial/\partial u + b\partial/\partial v)\,|_{F(p)}$, find a and b in terms of x, y, and α.

8.4. Transition Matrix for Coordinate Vectors

Let x, y be the standard coordinates on $\mathbb{R}^2$, and let U be the open set

$$U = \mathbb{R}^2 - \{(x, 0) \mid x \geq 0\}.$$

On U the polar coordinates r, θ are uniquely defined by

$$\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta, r > 0, 0 < \theta < 2\pi \end{aligned}$$

Find $\partial/\partial r$ and $\partial/\partial\theta$ in terms of $\partial/\partial x$ and $\partial/\partial y$.

8.5.* Velocity of a Curve in Local Coordinates

Prove Proposition 8.15.

8.6. Velocity Vector

Let $p = (x, y)$ be a point in $\mathbb{R}^2$. Then

$$c_p(t) = \begin{bmatrix} \cos 2t & -\sin 2t \\ \sin 2t & \cos 2t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad t \in \mathbb{R},$$

is a curve with initial point p in $\mathbb{R}^2$. Compute the velocity vector $c_p'(0)$.

8.7.* Tangent Space to a Product

If M and N are manifolds, let $\pi_1 : M \times N \to M$ and $\pi_2 : M \times N \to N$ be the two projections. Prove that for $(p, q) \in M \times N$,

$$(\pi_{1*}, \pi_{2*}) : T_{(p,q)}(M \times N) \to T_pM \times T_qN$$

is an isomorphism.

8.8. Differentials of Multiplication and Inverse

Let G be a Lie group with multiplication map $\mu : G \times G \to G$, inverse map $\imath : G \to G$, and identity element e.
(a) Show that the differential at the identity of the multiplication map μ is addition:

$$\begin{aligned} \mu_{*,(e,e)} &: T_eG \times T_eG \to T_eG \\ \mu_{*,(e,e)}(X_e, Y_e) &= X_e + Y_e \end{aligned}$$

(Hint: First, compute $\mu_{*,(e,e)}(X_e, 0)$ and $\mu_{*,(e,e)}(0, Y_e)$ using Proposition 8.18.)
(b) Show that the differential at the identity of ι is the negative:

$$\begin{aligned} \iota_{*,e} &: T_eG \to T_eG \\ \iota_{*,e}(X_e) &= -X_e \end{aligned}$$

(Hint: Take the differential of $\mu(c(t), (t \circ c)(t)) = e$.)

8.9.* Transforming Vectors to Coordinate Vectors

Let $X_1, \ldots, X_n$ be n vector fields on an open subset U of a manifold of dimension n. Suppose that at $p \in U$, the vectors $(X_1)_p, \ldots, (X_n)_p$ are linearly independent. Show that there is a chart $(V, x^1, \ldots, x^n)$ about p such that $(X_i)_p = (\partial/\partial x^i)_p$ for $i = 1, \ldots, n$.

8.10. Local Maxima

A real-valued function $f : M \to \mathbb{R}$ on a manifold is said to have a local maximum at $p \in M$ if there is a neighborhood U of p such that $f(p) \geq f(q)$ for all $q \in U$.
(a)* Prove that if a differentiable function $f : I \to \mathbb{R}$ defined on an open interval I has a local maximum at $p \in I$, then $f'(p) = 0$.
(b) Prove that a local maximum of a C^∞ function $f : M \to \mathbb{R}$ is a critical point of f. (Hint: Let X_p be a tangent vector in T_pM and let $c(t)$ be a curve in M starting at p with initial vector X_p. Then $f \circ c$ is a real-valued function with a local maximum at 0 . Apply (a).)

8.2 §9 Submanifolds

We now have two ways of showing that a given topological space is a manifold:
(a) by checking directly that the space is Hausdorff, second countable, and has a C^∞ atlas;
(b) by exhibiting it as an appropriate quotient space. Section 7 lists some conditions under which a quotient space is a manifold.

In this section we introduce the concept of a regular submanifold of a manifold, a subset that is locally defined by the vanishing of some of the coordinate functions. Using the inverse function theorem, we derive a criterion, called the regular level set theorem, that can often be used to show that a level set of a C^∞ map of manifolds is a regular submanifold and therefore a manifold.

Although the regular level set theorem is a simple consequence of the constant rank theorem and the submersion theorem to be discussed in Section 11, deducing it directly from the inverse function theorem has the advantage of producing explicit coordinate functions on the submanifold.

8.2.1 9.1 Submanifolds

The xy-plane in $\mathbb{R}^3$ is the prototype of a regular submanifold of a manifold. It is defined by the vanishing of the coordinate function z.

Definition 9.1. A subset S of a manifold N of dimension n is a regular submanifold of dimension k if for every $p \in S$ there is a coordinate neighborhood $(U, \phi) = (U, x^1, \ldots, x^n)$ of p in the maximal atlas of N such that $U \cap S$ is defined by the vanishing of $n-k$ of the coordinate functions. By renumbering the coordinates, we may assume that these $n-k$ coordinate functions are $x^{k+1}, \ldots, x^n$.

We call such a chart (U, ϕ) in N an adapted chart relative to S. On $U \cap S, \phi = \big(x^1, \ldots, x^k, 0, \ldots, 0\big)$. Let

$$\phi_S : U \cap S \to \mathbb{R}^k$$

be the restriction of the first k components of ϕ to $U \cap S$, that is, $\phi_S = \left(x^1, \ldots, x^k\right)$. Note that $(U \cap S, \phi_S)$ is a chart for S in the subspace topology.

Definition 9.2. If S is a regular submanifold of dimension k in a manifold N of dimension n, then $n-k$ is said to be the codimension of S in N.

Remark. As a topological space, a regular submanifold of N is required to have the subspace topology.

Example. In the definition of a regular submanifold, the dimension k of the submanifold may be equal to n, the dimension of the manifold. In this case, $U \cap S$ is defined by the vanishing of none of the coordinate functions and so $U \cap S = U$. Therefore, an open subset of a manifold is a regular submanifold of the same dimension.

Remark. There are other types of submanifolds, but unless otherwise specified, by a "submanifold" we will always mean a "regular submanifold."

Example. The interval $S :=]-1,1[$ on the x-axis is a regular submanifold of the xy-plane (Figure 9.1). As an adapted chart, we can take the open square $U =]-1,1[\times]-1,1[$ with coordinates x, y. Then $U \cap S$ is precisely the zero set of y on U.

Fig. 9.1.

Note that if $V =]-2,0[\times]-1,1[$, then (V, x, y) is not an adapted chart relative to S, since $V \cap S$ is the open interval $]-1,0[$ on the x-axis, while the zero set of y on V is the open interval $]-2,0[$ on the x-axis.

Fig. 9.2 The topologist's sine curve.

Example 9.3. Let Γ be the graph of the function $f(x) = \sin(1/x)$ on the interval $]0,1[$, and let S be the union of Γ and the open interval

$$I = \left\{(0,y) \in \mathbb{R}^2 \mid -1 < y < 1\right\}.$$

The subset S of $\mathbb{R}^2$ is not a regular submanifold for the following reason: if p is in the interval I, then there is no adapted chart containing p, since any sufficiently small neighborhood U of p in $\mathbb{R}^2$ intersects S in infinitely many components. (The closure of Γ in $\mathbb{R}^2$ is called the topologist's sine curve (Figure 9.2). It differs from S in including the endpoints $(1, \sin 1), (0,1)$, and $(0,-1)$.)

Proposition 9.4. Let S be a regular submanifold of N and $\mathfrak{U} = \{(U, \phi)\}$ a collection of compatible adapted charts of N that covers S. Then $\{(U \cap S, \phi_S)\}$ is an atlas for S. Therefore, a regular submanifold is itself a manifold. If N has dimension n and S is locally defined by the vanishing of $n-k$ coordinates, then $\dim S = k$.

Fig. 9.3. Overlapping adapted charts relative to a regular submanifold S.

Proof. Let $(U, \boldsymbol{\phi}) = (U, x^1, \ldots, x^n)$ and $(V, \boldsymbol{\psi}) = (V, y^1, \ldots, y^n)$ be two adapted charts in the given collection (Figure 9.3). Assume that they intersect. As we remarked in Definition 9.1, in any adapted chart relative to a submanifold S it is possible to renumber the coordinates so that the last $n-k$ coordinates vanish on points of S. Then for $p \in U \cap V \cap S$,

$$\phi(p) = \left(x^1, \ldots, x^k, 0, \ldots, 0\right) \quad \text{and} \quad \psi(p) = \left(y^1, \ldots, y^k, 0, \ldots, 0\right),$$

so

$$\phi_S(p) = \left(x^1, \ldots, x^k\right) \quad \text{and} \quad \psi_S(p) = \left(y^1, \ldots, y^k\right).$$

Therefore,

$$\left(\psi_S \circ \phi_S^{-1}\right)\left(x^1, \ldots, x^k\right) = \left(y^1, \ldots, y^k\right).$$

Since $y^1, \ldots, y^k$ are C^∞ functions of $x^1, \ldots, x^k$ (because $\psi \circ \phi^{-1}\left(x^1, \ldots, x^k, 0, \ldots, 0\right)$ is C^∞), the transition function $\psi_S \circ \phi_S^{-1}$ is C^∞. Similarly, since $x^1, \ldots, x^k$ are C^∞ functions of $y^1, \ldots, y^k, \phi_S \circ \psi_S^{-1}$ is also C^∞. Hence, any two charts in $\{(U \cap S, \phi_S)\}$ are C^∞ compatible. Since $\{U \cap S\}_{U \in \mathfrak{U}}$ covers S, the collection $\{(U \cap S, \phi_S)\}$ is a C^∞ atlas on S. $\square$

8.2.2 9.2 Level Sets of a Function

A level set of a map $F : N \to M$ is a subset

$$F^{-1}(\{c\}) = \{p \in N \mid F(p) = c\}$$

for some $c \in M$. The usual notation for a level set is $F^{-1}(c)$, rather than the more correct $F^{-1}(\{c\})$. The value $c \in M$ is called the level of the level set $F^{-1}(c)$. If $F : N \to \mathbb{R}^m$, then $Z(F) := F^{-1}(\mathbf{0})$ is the zero set of F. Recall that c is a regular value of F if and only if either c is not in the image of F or at every point $p \in F^{-1}(c)$, the differential $F_{*,p} : T_pN \to T_{F(p)}M$ is surjective. The inverse image $F^{-1}(c)$ of a regular value c is called a regular level set. If the zero set $F^{-1}(\mathbf{0})$ is a regular level set of $F : N \to \mathbb{R}^m$, it is called a regular zero set.

Remark 9.5. If a regular level set $F^{-1}(c)$ is nonempty, say $p \in F^{-1}(c)$, then the map $F : N \to M$ is a submersion at p. By Remark 8.20, $\dim N \geq \dim M$.

Example 9.6 (The 2-sphere in $\mathbb{R}^3$). The unit 2-sphere

$$S^2 = \left\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\right\}$$

is the level set $g^{-1}(1)$ of level 1 of the function $g(x, y, z) = x^2 + y^2 + z^2$. We will use the inverse function theorem to find adapted charts of $\mathbb{R}^3$ that cover S^2. As the proof will show, the process is easier for a zero set, mainly because a regular submanifold is defined locally as the zero set of coordinate functions. To express S^2 as a zero set, we rewrite its defining equation as

$$f(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$$

Then $S^2 = f^{-1}(0)$.
Since

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z,$$

the only critical point of f is $(0, 0, 0)$, which does not lie on the sphere S^2. Thus, all points on the sphere are regular points of f and 0 is a regular value of f.

Let p be a point of S^2 at which $(\partial f/\partial x)(p) = 2x(p) \neq 0$. Then the Jacobian matrix of the map $(f, y, z) : \mathbb{R}^3 \to \mathbb{R}^3$ is

$$\begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \\ \frac{\partial y}{\partial x} \\ \frac{\partial y}{\partial y} \\ \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial x} \\ \frac{\partial z}{\partial y} \\ \frac{\partial z}{\partial z} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

and the Jacobian determinant $\partial f/\partial x(p)$ is nonzero. By Corollary 6.27 of the inverse function theorem (Theorem 6.26), there is a neighborhood U_p of p in $\mathbb{R}^3$ such that
(U_p, f, y, z) is a chart in the atlas of $\mathbb{R}^3$. In this chart, the set $U_p \cap S^2$ is defined by the vanishing of the first coordinate f. Thus, (U_p, f, y, z) is an adapted chart relative to S^2, and ($U_p \cap S^2, y, z$) is a chart for S^2.

Similarly, if $(\partial f/\partial y)(p) \neq 0$, then there is an adapted chart (V_p, x, f, z) containing p in which the set $V_p \cap S^2$ is the zero set of the second coordinate f. If $(\partial f/\partial z)(p) \neq 0$, then there is an adapted chart (W_p, x, y, f) containing p. Since for every $p \in S^2$, at least one of the partial derivatives $\partial f/\partial x(p), \partial f/\partial y(p), \partial f/\partial z(p)$ is nonzero, as p varies over all points of the sphere we obtain a collection of adapted charts of $\mathbb{R}^3$ covering S^2. Therefore, S^2 is a regular submanifold of $\mathbb{R}^3$. By Proposition 9.4, S^2 is a manifold of dimension 2.

This is an important example because one can generalize its proof almost verbatim to prove that if the zero set of a function $f : N \to \mathbb{R}$ is a regular level set, then it is a regular submanifold of N. The idea is that in a coordinate neighborhood $(U, x^1, \ldots, x^n)$ if a partial derivative $\partial f/\partial x^i(p)$ is nonzero, then we can replace the coordinate x^i by f.

First we show that any regular level set $g^{-1}(c)$ of a C^∞ real function g on a manifold can be expressed as a regular zero set.

Lemma 9.7. Let $g : N \to \mathbb{R}$ be a C^∞ function. A regular level set $g^{-1}(c)$ of level c of the function g is the regular zero set $f^{-1}(0)$ of the function $f = g - c$.

Proof. For any $p \in N$,

$$g(p) = c \quad \Longleftrightarrow \quad f(p) = g(p) - c = 0.$$

Hence, $g^{-1}(c) = f^{-1}(0)$. Call this set S. Because the differential $f_{*,p}$ equals $g_{*,p}$ at every point $p \in N$, the functions f and g have exactly the same critical points. Since g has no critical points in S, neither does f.

Theorem 9.8. Let $g : N \to \mathbb{R}$ be a C^∞ function on the manifold N. Then a nonempty regular level set $S = g^{-1}(c)$ is a regular submanifold of N of codimension 1 .

Proof. Let $f = g - c$. By the preceding lemma, S equals $f^{-1}(0)$ and is a regular level set of f. Let $p \in S$. Since p is a regular point of f, relative to any chart $(U, x^1, \ldots, x^n)$ about $p, (\partial f/\partial x^i)(p) \neq 0$ for some i. By renumbering $x^1, \ldots, x^n$, we may assume that $(\partial f/\partial x^1)(p) \neq 0$.

The Jacobian matrix of the C^∞ map $(f, x^2, \ldots, x^n) : U \to \mathbb{R}^n$ is

$$\begin{bmatrix} \frac{\partial f}{\partial x^1} & \frac{\partial f}{\partial x^2} & \cdots & \frac{\partial f}{\partial x^n} \\ \frac{\partial x^2}{\partial x^1} & \frac{\partial x^2}{\partial x^2} & \cdots & \frac{\partial x^2}{\partial x^n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x^n}{\partial x^1} & \frac{\partial x^n}{\partial x^2} & \cdots & \frac{\partial x^n}{\partial x^n} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x^1} & * & \cdots & * \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

So the Jacobian determinant $\partial(f, x^2, \ldots, x^n)/\partial(x^1, x^2, \ldots, x^n)$ at p is $\partial f/\partial x^1(p) \neq 0$. By the inverse function theorem (Corollary 6.27), there is a neighborhood U_p of p on which $f, x^2, \ldots, x^n$ form a coordinate system. Relative to the chart $(U_p, f, x^2, \ldots, x^n)$ the level set $U_p \cap S$ is defined by setting the first coordinate f equal to 0 , so $(U_p, f, x^2, \ldots, x^n)$ is an adapted chart relative to S. Since p was arbitrary, S is a regular submanifold of dimension $n-1$ in N. □

8.2.3 9.3 The Regular Level Set Theorem

The next step is to extend Theorem 9.8 to a regular level set of a map between smooth manifolds. This very useful theorem does not seem to have an agreed-upon name in the literature. It is known variously as the implicit function theorem, the preimage theorem [17], and the regular level set theorem [25], among other nomenclatures. We will follow [25] and call it the regular level set theorem.

Theorem 9.9 (Regular level set theorem). Let $F : N \to M$ be a C^∞ map of manifolds, with $\dim N = n$ and $\dim M = m$. Then a nonempty regular level set $F^{-1}(c)$, where $c \in M$, is a regular submanifold of N of dimension equal to $n - m$.

Proof. Choose a chart $(V, \psi) = (V, y^1, \ldots, y^m)$ of M centered at c, i.e., such that $\psi(c) = \mathbf{0}$ in $\mathbb{R}^m$. Then $F^{-1}(V)$ is an open set in N that contains $F^{-1}(c)$. Moreover, in $F^{-1}(V), F^{-1}(c) = (\psi \circ F)^{-1}(\mathbf{0})$. So the level set $F^{-1}(c)$ is the zero set of $\psi \circ F$. If $F^i = y^i \circ F = r^i \circ (\psi \circ F)$, then $F^{-1}(c)$ is also the common zero set of the functions $F^1, \ldots, F^m$ on $F^{-1}(V)$.

Fig. 9.4. The level set $F^{-1}(c)$ of F is the zero set of $\psi \circ F$.

Because the regular level set is assumed nonempty, $n \geq m$ (Remark 9.5). Fix a point $p \in F^{-1}(c)$ and let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a coordinate neighborhood of p in N contained

in $F^{-1}(V)$ (Figure 9.4). Since $F^{-1}(c)$ is a regular level set, $p \in F^{-1}(c)$ is a regular point of F. Therefore, the $m \times n$ Jacobian matrix $[\partial F^i/\partial x^j(p)]$ has rank m. By renumbering the F^i and x^j s, we may assume that the first $m \times m$ block $[\partial F^i/\partial x^j(p)]_{1\le i,j\le m}$ is nonsingular.

Replace the first m coordinates $x^1, \ldots, x^m$ of the chart (U, ϕ) by $F^1, \ldots, F^m$. We claim that there is a neighborhood U_p of p such that $(U_p, F^1, \ldots, F^m, x^{m+1}, \ldots, x^n)$ is a chart in the atlas of N. It suffices to compute its Jacobian matrix at p :

$$\begin{bmatrix} \frac{\partial F^i}{\partial x^j} & \frac{\partial F^i}{\partial x^\beta} \\ \frac{\partial x^\alpha}{\partial x^j} & \frac{\partial x^\alpha}{\partial x^\beta} \end{bmatrix} = \begin{bmatrix} \frac{\partial F^i}{\partial x^j} & * \\ 0 & I \end{bmatrix},$$

where $1 \le i, j \le m$ and $m+1 \le \alpha, \beta \le n$. Since this matrix has determinant

$$\det \left[\frac{\partial F^i}{\partial x^j}(p)\right]_{1\le i,j\le m} \neq 0$$

the inverse function theorem in the form of Corollary 6.27 implies the claim.
In the chart $(U_p, F^1, \ldots, F^m, x^{m+1}, \ldots, x^n)$, the set $S := f^{-1}(c)$ is obtained by setting the first m coordinate functions $F^1, \ldots, F^m$ equal to 0 . So $(U_p, F^1, \ldots, F^m, x^{m+1}, \ldots, x^n)$ is an adapted chart for N relative to S. Since this is true about every point $p \in S, S$ is a regular submanifold of N of dimension $n-m$.

The proof of the regular level set theorem gives the following useful lemma.
Lemma 9.10. Let $F : N \to \mathbb{R}^m$ be a C^∞ map on a manifold N of dimension n and let S be the level set $F^{-1}(\mathbf{0})$. If relative to some coordinate chart $(U, x^1, \ldots, x^n)$ about $p \in S$, the Jacobian determinant $\partial(F^1, \ldots, F^m)/\partial(x^{j_1}, \ldots, x^{j_m})(p)$ is nonzero, then in some neighborhood of p one may replace $x^{j_1}, \ldots, x^{j_m}$ by $F^1, \ldots, F^m$ to obtain an adapted chart for N relative to S.

Remark. The regular level set theorem gives a sufficient but not necessary condition for a level set to be a regular submanifold. For example, if $f : \mathbb{R}^2 \to \mathbb{R}$ is the map $f(x, y) = y^2$, then the zero set $Z(f) = Z(y^2)$ is the x-axis, a regular submanifold of $\mathbb{R}^2$. However, since $\partial f/\partial x = 0$ and $\partial f/\partial y = 2y = 0$ on the x-axis, every point in $Z(f)$ is a critical point of f. Thus, although $Z(f)$ is a regular submanifold of $\mathbb{R}^2$, it is not a regular level set of f.

8.2.4 9.4 Examples of Regular Submanifolds

Example 9.11 (Hypersurface). Show that the solution set S of $x^3 + y^3 + z^3 = 1$ in $\mathbb{R}^3$ is a manifold of dimension 2 .

Solution. Let $f(x, y, z) = x^3 + y^3 + z^3$. Then $S = f^{-1}(1)$. Since $\partial f/\partial x = 3x^2$, $\partial f/\partial y = 3y^2$, and $\partial f/\partial z = 3z^2$, the only critical point of f is $(0, 0, 0)$, which is not in S. Thus, 1 is a regular value of $f : \mathbb{R}^3 \to \mathbb{R}$. By the regular level set theorem (Theorem 9.9), S is a regular submanifold of $\mathbb{R}^3$ of dimension 2. So S is a manifold (Proposition 9.4).

Example 9.12 (Solution set of two polynomial equations). Decide whether the subset S of $\mathbb{R}^3$ defined by the two equations

$$\begin{aligned} x^3 + y^3 + z^3 &= 1 \\ x + y + z &= 0 \end{aligned}$$

is a regular submanifold of $\mathbb{R}^3$.

Solution. Define $F : \mathbb{R}^3 \to \mathbb{R}^2$ by

$$(u, v) = F(x, y, z) = \left(x^3 + y^3 + z^3, x + y + z\right).$$

Then S is the level set $F^{-1}(1, 0)$. The Jacobian matrix of F is

$$J(F)=\begin{bmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \end{bmatrix}=\begin{bmatrix} 3x^2 & 3y^2 & 3z^2 \\ 1 & 1 & 1 \end{bmatrix}$$

where $u_x=\partial u/\partial x$ and so forth. The critical points of F are the points (x,y,z) where the matrix $J(F)$ has rank <2. That is precisely where all 2×2 minors of $J(F)$ are zero:

$$\begin{vmatrix} 3x^2 & 3y^2 \\ 1 & 1 \end{vmatrix}=0, \quad \begin{vmatrix} 3x^2 & 3z^2 \\ 1 & 1 \end{vmatrix}=0. \tag{9.1}$$

(The third condition

$$\begin{vmatrix} 3y^2 & 3z^2 \\ 1 & 1 \end{vmatrix}=0$$

is a consequence of these two.) Solving (9.1), we get $y=\pm x, z=\pm x$. Since $x+y+z=0$ on S, this implies that $(x,y,z)=(0,0,0)$. Since $(0,0,0)$ does not satisfy the first equation $x^3+y^3+z^3=1$, there are no critical points of F on S. Therefore, S is a regular level set. By the regular level set theorem, S is a regular submanifold of $\mathbb{R}^3$ of dimension 1 .

Example 9.13 (Special linear group). As a set, the special linear group $\mathrm{SL}(n,\mathbb{R})$ is the subset of $\mathrm{GL}(n,\mathbb{R})$ consisting of matrices of determinant 1 . Since

$$\det(AB)=(\det A)(\det B) \quad \text{and} \quad \det\left(A^{-1}\right)=\frac{1}{\det A},$$

$\mathrm{SL}(n,\mathbb{R})$ is a subgroup of $\mathrm{GL}(n,\mathbb{R})$. To show that it is a regular submanifold, we let $f:\mathrm{GL}(n,\mathbb{R})\to\mathbb{R}$ be the determinant map $f(A)=\det A$, and apply the regular level set theorem to $f^{-1}(1)=\mathrm{SL}(n,\mathbb{R})$. We need to check that 1 is a regular value of f.

Let $a_{ij}, 1\le i\le n, 1\le j\le n$, be the standard coordinates on $\mathbb{R}^{n\times n}$, and let S_{ij} denote the submatrix of $A=[a_{ij}]\in\mathbb{R}^{n\times n}$ obtained by deleting its i th row and j th column. Then $m_{ij}:=\det S_{ij}$ is the (i,j)-minor of A. From linear algebra we have a formula for computing the determinant by expanding along any row or any column: if we expand along the i th row, we obtain

$$f(A)=\det A=(-1)^{i+1}a_{i1}m_{i1}+(-1)^{i+2}a_{i2}m_{i2}+\cdots+(-1)^{i+n}a_{in}m_{in}. \tag{9.2}$$

Therefore

$$\frac{\partial f}{\partial a_{ij}}=(-1)^{i+j}m_{ij}$$

Hence, a matrix $A\in\mathrm{GL}(n,\mathbb{R})$ is a critical point of f if and only if all the $(n-1)\times(n-1)$ minors m_{ij} of A are 0 . By (9.2) such a matrix A has determinant 0 . Since every matrix in $\mathrm{SL}(n,\mathbb{R})$ has determinant 1 , all the matrices in $\mathrm{SL}(n,\mathbb{R})$ are regular points of the determinant function. By the regular level set theorem (Theorem 9.9), $\mathrm{SL}(n,\mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$ of codimension 1 ; i.e.,

$$\dim\mathrm{SL}(n,\mathbb{R})=\dim\mathrm{GL}(n,\mathbb{R})-1=n^2-1$$

8.2.5 Problems

9.1. Regular Values

Define $f:\mathbb{R}^2\to\mathbb{R}$ by

$$f(x,y)=x^3-6xy+y^2$$

Find all values $c\in\mathbb{R}$ for which the level set $f^{-1}(c)$ is a regular submanifold of $\mathbb{R}^2$.

9.2. Solution set of one Equation

Let x, y, z, w be the standard coordinates on $\mathbb{R}^4$. Is the solution set of $x^5 + y^5 + z^5 + w^5 = 1$ in $\mathbb{R}^4$ a smooth manifold? Explain why or why not. (Assume that the subset is given the subspace topology.)

9.3. Solution set of two Equations

Is the solution set of the system of equations

$$x^3 + y^3 + z^3 = 1, \quad z = xy$$

in $\mathbb{R}^3$ a smooth manifold? Prove your answer.

9.4.* Regular Submanifolds

Suppose that a subset S of $\mathbb{R}^2$ has the property that locally on S one of the coordinates is a C^∞ function of the other coordinate. Show that S is a regular submanifold of $\mathbb{R}^2$. (Note that the unit circle defined by $x^2 + y^2 = 1$ has this property. At every point of the circle, there is a neighborhood in which y is a C^∞ function of x or x is a C^∞ function of y.)

9.5. Graph of a Smooth Function

Show that the graph $\Gamma(f)$ of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$,

$$\Gamma(f) = \left\{(x, y, f(x, y)) \in \mathbb{R}^3\right\}$$

is a regular submanifold of $\mathbb{R}^3$.

9.6. Euler's Formula

A polynomial $F(x_0, \ldots, x_n) \in \mathbb{R}[x_0, \ldots, x_n]$ is homogeneous of degree k if it is a linear combination of monomials $x_0^{i_0} \cdots x_n^{i_n}$ of degree $\sum_{j=0}^n i_j = k$. Let $F(x_0, \ldots, x_n)$ be a homogeneous polynomial of degree k. Clearly, for any $t \in \mathbb{R}$,

$$F(tx_0, \ldots, tx_n) = t^k F(x_0, \ldots, x_n) \tag{9.3}$$

Show that

$$\sum_{i=0}^n x_i \frac{\partial F}{\partial x_i} = kF$$

9.7. Smooth Projective Hypersurface

On the projective space $\mathbb{R}P^n$ a homogeneous polynomial $F(x_0, \ldots, x_n)$ of degree k is not a function, since its value at a point $[a_0, \ldots, a_n]$ is not unique. However, the zero set in $\mathbb{R}P^n$ of a homogeneous polynomial $F(x_0, \ldots, x_n)$ is well defined, since $F(a_0, \ldots, a_n) = 0$ if and only if

$$F(ta_0, \ldots, ta_n) = t^k F(a_0, \ldots, a_n) = 0 \quad \text{for all } t \in \mathbb{R}^\times := \mathbb{R} - \{0\}.$$

The zero set of finitely many homogeneous polynomials in $\mathbb{R}P^n$ is called a real projective variety. A projective variety defined by a single homogeneous polynomial of degree k is called a hypersurface of degree k. Show that the hypersurface $Z(F)$ defined by $F(x_0, x_1, x_2) = 0$ is smooth if $\partial F/\partial x_0, \partial F/\partial x_1$, and $\partial F/\partial x_2$ are not simultaneously zero on $Z(F)$. (Hint: The standard coordinates on U_0, which is homeomorphic to $\mathbb{R}^2$, are $x = x_1/x_0, y = x_2/x_0$ (see

Subsection 7.7). In $U_0, F(x_0, x_1, x_2) = x_0^k F(1, x_1/x_0, x_2/x_0) = x_0^k F(1, x, y)$. Define $f(x, y) = F(1, x, y)$. Then f and F have the same zero set in U_0.)

9.8. Product of Regular Submanifolds

If S_i is a regular submanifold of the manifold M_i for $i = 1, 2$, prove that $S_1 \times S_2$ is a regular submanifold of $M_1 \times M_2$.

9.9. Complex Special Linear Group

The complex special linear group $\mathrm{SL}(n, \mathbb{C})$ is the subgroup of $\mathrm{GL}(n, \mathbb{C})$ consisting of $n \times n$ complex matrices of determinant 1 . Show that $\mathrm{SL}(n, \mathbb{C})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{C})$ and determine its dimension. (This problem requires a rudimentary knowledge of complex analysis.)

Fig. 9.5. Transversality.

9.10. The Transversality Theorem

A C^∞ map $f : N \to M$ is said to be transversal to a submanifold $S \subset M$ (Figure 9.5) if for every $p \in f^{-1}(S)$,

$$f_*(T_pN) + T_{f(p)}S = T_{f(p)}M. \tag{9.4}$$

(If A and B are subspaces of a vector space, their sum $A + B$ is the subspace consisting of all $a + b$ with $a \in A$ and $b \in B$. The sum need not be a direct sum.) The goal of this exercise is to prove the transversality theorem: if a C^∞ map $f : N \to M$ is transversal to a regular submanifold S of codimension k in M, then $f^{-1}(S)$ is a regular submanifold of codimension k in N.

When S consists of a single point c, transversality of f to S simply means that $f^{-1}(c)$ is a regular level set. Thus the transversality theorem is a generalization of the regular level set theorem. It is especially useful in giving conditions under which the intersection of two submanifolds is a submanifold.

Let $p \in f^{-1}(S)$ and $(U, x^1, \ldots, x^m)$ be an adapted chart centered at $f(p)$ for M relative to S such that $U \cap S = Z\left(x^{m-k+1}, \ldots, x^m\right)$, the zero set of the functions $x^{m-k+1}, \ldots, x^m$. Define $g : U \to \mathbb{R}^k$ to be the map

$$g = \left(x^{m-k+1}, \ldots, x^m\right).$$

(a) Show that $f^{-1}(U) \cap f^{-1}(S) = (g \circ f)^{-1}(0)$.
(b) Show that $f^{-1}(U) \cap f^{-1}(S)$ is a regular level set of the function $g \circ f : f^{-1}(U) \to \mathbb{R}^k$.
(c) Prove the transversality theorem.

8.3 §10 Categories and Functors

Many of the problems in mathematics share common features. For example, in topology one is interested in knowing whether two topological spaces are homeomorphic and in group theory one is interested in knowing whether two groups are isomorphic. This has given rise to the theory of categories and functors, which tries to clarify the structural similarities among different areas of mathematics.

A category is essentially a collection of objects and arrows between objects. These arrows, called morphisms, satisfy the abstract properties of maps and are often structure-preserving maps. Smooth manifolds and smooth maps form a category, and so do vector spaces and linear maps. A functor from one category to another preserves the identity morphism and the composition of morphisms. It provides a way to simplify problems in the first category, for the target category of a functor is usually simpler than the original category. The tangent space construction with the differential of a smooth map is a functor from the category of smooth manifolds with a distinguished point to the category of vector spaces. The existence of the tangent space functor shows that if two manifolds are diffeomorphic, then their tangent spaces at corresponding points must be isomorphic, thereby proving the smooth invariance of dimension. Invariance of dimension in the continuous category of topological spaces and continuous maps is more difficult to prove, precisely because there is no tangent space functor in the continuous category.

Much of algebraic topology is the study of functors, for example, the homology, cohomology, and homotopy functors. For a functor to be truly useful, it should be simple enough to be computable, yet complex enough to preserve essential features of the original category. For smooth manifolds, this delicate balance is achieved in the de Rham cohomology functor. In the rest of the book, we will be introducing various functors of smooth manifolds, such as the tangent bundle and differential forms, culminating in de Rham cohomology.

In this section, after defining categories and functors, we study the dual construction on vector spaces as a nontrivial example of a functor.

8.3.1 10.1 Categories

A category consists of a collection of elements, called objects, and for any two objects A and B, a set $\mathrm{Mor}(A, B)$ of elements, called morphisms from A to B, such that given any morphism $f \in \mathrm{Mor}(A, B)$ and any morphism $g \in \mathrm{Mor}(B, C)$, the composite $g \circ f \in \mathrm{Mor}(A, C)$ is defined. Furthermore, the composition of morphisms is required to satisfy two properties:
(i) the identity axiom: for each object A, there is an identity morphism $\mathbb{1}_A \in \mathrm{Mor}(A, A)$ such that for any $f \in \mathrm{Mor}(A, B)$ and $g \in \mathrm{Mor}(B, A)$,

$$f \circ \mathbb{1}_A = f \quad \text{and} \quad \mathbb{1}_A \circ g = g;$$

(ii) the associative axiom: for $f \in \mathrm{Mor}(A, B), g \in \mathrm{Mor}(B, C)$, and $h \in \mathrm{Mor}(C, D)$,

$$h \circ (g \circ f) = (h \circ g) \circ f$$

If $f \in \mathrm{Mor}(A, B)$, we often write $f : A \to B$.
Example. The collection of groups and group homomorphisms forms a category in which the objects are groups and for any two groups A and $B, \mathrm{Mor}(A, B)$ is the set of group homomorphisms from A to B.

Example. The collection of all vector spaces over $\mathbb{R}$ and $\mathbb{R}$-linear maps forms a category in which the objects are real vector spaces and for any two real vector spaces V and $W, \mathrm{Mor}(V, W)$ is the set $\mathrm{Hom}(V, W)$ of linear maps from V to W.

Example. The collection of all topological spaces together with continuous maps between them is called the continuous category.

Example. The collection of smooth manifolds together with smooth maps between them is called the smooth category.

Example. We call a pair (M, q), where M is a manifold and q a point in M, a pointed manifold. Given any two such pairs (N, p) and (M, q), let $\mathrm{Mor}((N, p), (M, q))$ be the set of all smooth maps $F : N \to M$ such that $F(p) = q$. This gives rise to the category of pointed manifolds.

Definition 10.1. Two objects A and B in a category are said to be isomorphic if there are morphisms $f : A \to B$ and $g : B \to A$ such that

$$g \circ f = \mathbb{1}_A \quad \text{and} \quad f \circ g = \mathbb{1}_B$$

In this case both f and g are called isomorphisms.
The usual notation for an isomorphism is " $\pm$ ". Thus, $A \simeq B$ can mean, for example, a group isomorphism, a vector space isomorphism, a homeomorphism, or a diffeomorphism, depending on the category and the context.

8.3.2 10.2 Functors

Definition 10.2. A (covariant) functor $\mathcal{F}$ from one category $\mathcal{C}$ to another category $\mathcal{D}$ is a map that associates to each object A in $\mathcal{C}$ an object $\mathcal{F}(A)$ in $\mathcal{D}$ and to each morphism $f : A \to B$ a morphism $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ such that
(i) $\mathcal{F}(\mathbb{1}_A) = \mathbb{1}_{\mathcal{F}(A)}$,
(ii) $\mathcal{F}(f \circ g) = \mathcal{F}(f) \circ \mathcal{F}(g)$.

Example. The tangent space construction is a functor from the category of pointed manifolds to the category of vector spaces. To each pointed manifold (N, p) we associate the tangent space T_pN and to each smooth map $f : (N, p) \to (M, f(p))$ we associate the differential $f_{*,p} : T_pN \to T_{f(p)}M$.

The functorial property (i) holds because if $1 : N \to N$ is the identity map, then its differential $\mathbb{1}_{*,p} : T_pN \to T_pN$ is also the identity map.

The functorial property (ii) holds because in this context it is the chain rule

$$(g \circ f)_{*,p} = g_{*,f(p)} \circ f_{*,p}$$

Proposition 10.3 Let $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ be a functor from a category $\mathcal{C}$ to a category $\mathcal{D}$. If $f : A \to B$ is an isomorphism in $\mathcal{C}$, then $\mathcal{F}(f) : \mathcal{F}(A) \to \mathcal{F}(B)$ is an isomorphism in $\mathcal{D}$.

Proof. Problem 10.2.
Note that we can recast Corollaries 8.6 and 8.7 in a more functorial form. Suppose $f : N \to M$ is a diffeomorphism. Then (N, p) and $(M, f(p))$ are isomorphic objects in the category of pointed manifolds. By Proposition 10.3, the tangent spaces T_pN and $T_{f(p)}M$ must be isomorphic as vector spaces and therefore have the same dimension. It follows that the dimension of a manifold is invariant under a diffeomorphism.

If in the definition of a covariant functor we reverse the direction of the arrow for the morphism $\mathcal{F}(f)$, then we obtain a contravariant functor. More precisely, the definition is as follows.

Definition 10.4. A contravariant functor $\mathcal{F}$ from one category $\mathcal{C}$ to another category $\mathcal{D}$ is a map that associates to each object A in $\mathcal{C}$ an object $\mathcal{F}(A)$ in $\mathcal{D}$ and to each morphism $f : A \to B$ a morphism $\mathcal{F}(f) : \mathcal{F}(B) \to \mathcal{F}(A)$ such that
(i) $\mathcal{F}(\mathbb{1}_A) = \mathbb{1}_{\mathcal{F}(A)}$;
(ii) $\mathcal{F}(f \circ g) = \mathcal{F}(g) \circ \mathcal{F}(f)$. (Note the reversal of order.)

Example. Smooth functions on a manifold give rise to a contravariant functor that associates to each manifold M the algebra $\mathcal{F}(M) = C^\infty(M)$ of C^∞ functions on M and to each smooth map $F : N \to M$ of manifolds the pullback map $\mathcal{F}(F) = F^* : C^\infty(M) \to C^\infty(N), F^*(h) = h \circ F$ for $h \in C^\infty(M)$. It is easy to verify that the pullback satisfies the two functorial properties:
(i) $(\mathbb{1}_M)^* = \mathbb{1}_{C^\infty(M)}$,
(ii) if $F : N \to M$ and $G : M \to P$ are C^∞ maps, then $(G \circ F)^* = F^* \circ G^* : C^\infty(P) \to C^\infty(N)$.

Another example of a contravariant functor is the dual of a vector space, which we review in the next section.

8.3.3 10.3 The Dual Functor and the Multicovector Functor

Let V be a real vector space. Recall that its dual space $V^\vee$ is the vector space of all linear functionals on V, i.e., linear functions $\alpha : V \to \mathbb{R}$. We also write

$$V^\vee = \operatorname{Hom}(V, \mathbb{R}).$$

If V is a finite-dimensional vector space with basis $\{e_1, \ldots, e_n\}$, then by Proposition 3.1 its dual space $V^\vee$ has as a basis the collection of linear functionals $\{\alpha^1, \ldots, \alpha^n\}$ defined by

$$\alpha^i(e_j) = \delta^i_j, \quad 1 \le i, j \le n.$$

Since a linear function on V is determined by what it does on a basis of V, this set of equations defines α^i uniquely.

A linear map $L : V \to W$ of vector spaces induces a linear map $L^\vee$, called the dual of L, as follows. To every linear functional $\alpha : W \to \mathbb{R}$, the dual map $L^\vee$ associates the linear functional

$$V \xrightarrow{L} W \xrightarrow{\alpha} \mathbb{R}.$$

Thus, the dual map $L^\vee : W^\vee \to V^\vee$ is given by

$$L^\vee(\alpha) = \alpha \circ L \quad \text{for } \alpha \in W^\vee.$$

Note that the dual of L reverses the direction of the arrow.

Proposition 10.5 (Functorial properties of the dual). Suppose V, W, and S are real vector spaces.
(i) If $\mathbb{1}_V : V \to V$ is the identity map on V, then $\mathbb{1}_V^\vee : V^\vee \to V^\vee$ is the identity map on $V^\vee$.
(ii) If $f : V \to W$ and $g : W \to S$ are linear maps, then $(g \circ f)^\vee = f^\vee \circ g^\vee$.

Proof. Problem 10.3. □

According to this proposition, the dual construction $\mathcal{F} : () \mapsto ()^\vee$ is a contravariant functor from the category of vector spaces to itself: for V a real vector space, $\mathcal{F}(V) = V^\vee$ and for $f \in \operatorname{Hom}(V, W), \mathcal{F}(f) = f^\vee \in \operatorname{Hom}(W^\vee, V^\vee)$. Consequently, if $f : V \to W$ is an isomorphism, then so is its dual $f^\vee : W^\vee \to V^\vee$ (cf. Proposition 10.3).

Fix a positive integer k. For any linear map $L : V \to W$ of vector spaces, define the pullback map $L^* : A_k(W) \to A_k(V)$ to be

$$(L^* f)(v_1, \ldots, v_k) = f(L(v_1), \ldots, L(v_k))$$

for $f \in A_k(W)$ and $v_1, \ldots, v_k \in V$. From the definition, it is easy to see that L^* is a linear map: $L^*(af + bg) = aL^*f + bL^*g$ for $a, b \in \mathbb{R}$ and $f, g \in A_k(W)$.

Proposition 10.6. The pullback of covectors by a linear map satisfies the two functorial properties:
(i) If $\mathbb{1}_V : V \to V$ is the identity map on V, then $\mathbb{1}_V^* = \mathbb{1}_{A_k(V)}$, the identity map on $A_k(V)$.
(ii) If $K : U \to V$ and $L : V \to W$ are linear maps of vector spaces, then

$$(L \circ K)^* = K^* \circ L^* : A_k(W) \to A_k(U)$$

Proof. Problem 10.6.
To each vector space V, we associate the vector space $A_k(V)$ of all k-covectors on V, and to each linear map $L : V \to W$ of vector spaces, we associate the pullback $A_k(L) = L^* : A_k(W) \to A_k(V)$. Then $A_k()$ is a contravariant functor from the category of vector spaces and linear maps to itself.

When $k = 1$, for any vector space V, the space $A_1(V)$ is the dual space, and for any linear map $L : V \to W$, the pullback map $A_1(L) = L^*$ is the dual map $L^\vee : W^\vee \to V^\vee$. Thus, the multicovector functor $A_k()$ generalizes the dual functor $()^\vee$.

8.3.4 Problems

10.1. Differential of the Inverse map

If $F : N \to M$ is a diffeomorphism of manifolds and $p \in N$, prove that $(F^{-1})_{*,F(p)} = (F_{*,p})^{-1}$.

10.2. Isomorphism under a Functor

Prove Proposition 10.3.

10.3. Functorial Properties of the Dual

Prove Proposition 10.5.

10.4. Matrix of the Dual map

Suppose a linear transformation $L : V \to \bar{V}$ is represented by the matrix $A = [a^i{}_j]$ relative to bases $e_1, \ldots, e_n$ for V and $\bar{e}_1, \ldots, \bar{e}_m$ for $\bar{V}$:

$$L(e_j) = \sum_i a^i_j \bar{e}_i$$

Let $\alpha^1, \ldots, \alpha^n$ and $\bar{\alpha}^1, \ldots, \bar{\alpha}^m$ be the dual bases for $V^\vee$ and $\bar{V}^\vee$, respectively. Prove that if $L^\vee(\bar{\alpha}^i) = \sum_j b^i_j \alpha^j$, then $b^i_j = a^i_j$.

10.5. Injectivity of the Dual map

(a) Suppose V and W are vector spaces of possibly infinite dimension over a field K. Show that if a linear map $L : V \to W$ is surjective, then its dual $L^\vee : W^\vee \to V^\vee$ is injective.
(b) Suppose V and W are finite-dimensional vector spaces over a field K. Prove the converse of the implication in (a).

10.6. Functorial Properties of the Pullback

Prove Proposition 10.6.

10.7. Pullback in the top Dimension

Show that if $L : V \to V$ is a linear operator on a vector space V of dimension n, then the pullback $L^* : A_n(V) \to A_n(V)$ is multiplication by the determinant of L.

8.4 §11 the Rank of a Smooth Map

In this section we analyze the local structure of a smooth map through its rank. Recall that the rank of a smooth map $f : N \to M$ at a point $p \in N$ is the rank of its differential at p. Two cases are of special interest: that in which the map f has maximal rank at a point and that in which it has constant rank in a neighborhood. Let $n = \dim N$ and $m = \dim M$. In case $f : N \to M$ has maximal rank at p, there are three not mutually exclusive possibilities:
(i) If $n = m$, then by the inverse function theorem, f is a local diffeomorphism at p.
(ii) If $n \leq m$, then the maximal rank is n and f is an immersion at p.
(iii) If $n \geq m$, then the maximal rank is m and f is a submersion at p.

Because manifolds are locally Euclidean, theorems on the rank of a smooth map between Euclidean spaces (Appendix B) translate easily to theorems about manifolds. This leads to the constant rank theorem for manifolds, which gives a simple normal form for a smooth map having constant rank on an open set (Theorem 11.1). As an immediate consequence, we obtain a criterion for a level set to be a regular submanifold, which, following [25], we call the constant-rank level set theorem. As we explain in Subsection 11.2, maximal rank at a point implies constant rank in a neighborhood, so immersions and submersions are maps of constant rank. The constant rank theorem specializes to the immersion theorem and the submersion theorem, giving simple normal forms for an immersion and a submersion. The regular level set theorem, which we encountered in Subsection 9.3, is now seen to be a consequence of the submersion theorem and a special case of the constant-rank level set theorem.

By the regular level set theorem, the preimage of a regular value of a smooth map is a manifold. The image of a smooth map, on the other hand, does not generally have a nice structure. Using the immersion theorem we derive conditions under which the image of a smooth map is a manifold.

8.4.1 11.1 Constant Rank Theorem

Suppose $f : N \to M$ is a C^∞ map of manifolds and we want to show that the level set $f^{-1}(c)$ is a manifold for some c in M. In order to apply the regular level set theorem, we need the differential f_* to have maximal rank at every point of $f^{-1}(c)$. Sometimes this is not true; even if true, it may be difficult to show. In such cases, the constant-rank level set theorem can be helpful. It has one cardinal virtue: it is not necessary to know precisely the rank of f; it suffices that the rank be constant.

The constant rank theorem for Euclidean spaces (Theorem B.4) has an immediate analogue for manifolds.

Theorem 11.1 (Constant rank theorem). Let N and M be manifolds of dimensions n and m respectively. Suppose $f : N \to M$ has constant rank k in a neighborhood of a point p in N. Then there are charts (U, ϕ) centered at p in N and (V, ψ) centered at $f(p)$ in M such that for $(r^1, \ldots, r^n)$ in $\phi(U)$,

$$\left(\psi \circ f \circ \phi^{-1}\right)\left(r^1, \ldots, r^n\right) = \left(r^1, \ldots, r^k, 0, \ldots, 0\right). \tag{11.1}$$

Proof. Choose a chart $(\bar{U}, \bar{\phi})$ about p in N and $(\bar{V}, \bar{\psi})$ about $f(p)$ in M. Then $\bar{\psi} \circ f \circ \bar{\phi}^{-1}$ is a map between open subsets of Euclidean spaces. Because $\bar{\phi}$ and $\bar{\psi}$ are diffeomorphisms, $\bar{\psi} \circ f \circ \bar{\phi}^{-1}$ has the same constant rank k as f in a neighborhood of $\bar{\phi}(p)$ in $\mathbb{R}^n$. By the constant rank theorem for Euclidean spaces (Theorem B.4) there are a diffeomorphism G of a neighborhood of $\bar{\phi}(p)$ in $\mathbb{R}^n$ and a diffeomorphism F of a neighborhood of $(\bar{\psi} \circ f)(p)$ in $\mathbb{R}^m$ such that

$$\left(F \circ \bar{\psi} \circ f \circ \bar{\phi}^{-1} \circ G^{-1}\right)\left(r^1, \ldots, r^n\right) = \left(r^1, \ldots, r^k, 0, \ldots, 0\right).$$

Set $\phi = G \circ \bar{\phi}$ and $\psi = F \circ \bar{\psi}$. $\square$

In the constant rank theorem, it is possible that the normal form (11.1) for the function f has no zeros at all: if the rank k equals m, then

$$\left(\psi \circ f \circ \phi^{-1}\right)\left(r^1, \ldots, r^n\right) = \left(r^1, \ldots, r^m\right).$$

From this theorem, the constant-rank level set theorem follows easily. By a neighborhood of a subset A of a manifold M we mean an open set containing A.

Theorem 11.2 (Constant-rank level set theorem). Let $f : N \to M$ be a C^∞ map of manifolds and $c \in M$. If f has constant rank k in a neighborhood of the level set $f^{-1}(c)$ in N, then $f^{-1}(c)$ is a regular submanifold of N of codimension k.

Proof. Let p be an arbitrary point in $f^{-1}(c)$. By the constant rank theorem there are a coordinate chart $(U, \phi) = (U, x^1, \ldots, x^n)$ centered at $p \in N$ and a coordinate chart $(V, \psi) = (V, y^1, \ldots, y^m)$ centered at $f(p) = c \in M$ such that

$$\left(\psi \circ f \circ \phi^{-1}\right)\left(r^1, \ldots, r^n\right) = \left(r^1, \ldots, r^k, 0, \ldots, 0\right) \in \mathbb{R}^m.$$

This shows that the level set $\left(\psi \circ f \circ \phi^{-1}\right)^{-1}(0)$ is defined by the vanishing of the coordinates $r^1, \ldots, r^k$.

Fig. 11.1. Constant-rank level set.

The image of the level set $f^{-1}(c)$ under ϕ is the level set $\left(\psi \circ f \circ \phi^{-1}\right)^{-1}(0)$ (Figure 11.1), since

$$\phi\left(f^{-1}(c)\right) = \phi\left(f^{-1}\left(\psi^{-1}(0)\right)\right. = \left(\psi \circ f \circ \phi^{-1}\right)^{-1}(0).$$

Thus, the level set $f^{-1}(c)$ in U is defined by the vanishing of the coordinate functions $x^1, \ldots, x^k$, where $x^i = r^i \circ \phi$. This proves that $f^{-1}(c)$ is a regular submanifold of N of codimension k.

Example 11.3 (Orthogonal group). The orthogonal group $\mathrm{O}(n)$ is defined to be the subgroup of $\mathrm{GL}(n, \mathbb{R})$ consisting of matrices A such that $A^T A = I$, the $n \times n$ identity matrix. Using the constant rank theorem, prove that $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$.

Solution. Define $f : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ by $f(A) = A^T A$. Then $\mathrm{O}(n)$ is the level set $f^{-1}(I)$. For any two matrices $A, B \in \mathrm{GL}(n, \mathbb{R})$, there is a unique matrix $C \in \mathrm{GL}(n, \mathbb{R})$ such that $B = AC$. Denote by ℓ_C and $r_C : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ the left and right multiplication by C, respectively. Since

$$f(AC) = (AC)^T AC = C^T A^T AC = C^T f(A) C$$

we have

$$\left(f \circ r_C\right)(A) = \left(\ell_{C^T} \circ r_C \circ f\right)(A)$$

Since this is true for all $A \in \mathrm{GL}(n, \mathbb{R})$,

$$f \circ r_C = \ell_{C_T} \circ r_C \circ f$$

By the chain rule,

$$f_{*,AC} \circ (r_C)_{*,A} = (\ell_{C^T})_{*,A^TAC} \circ (r_C)_{*,A^TA} \circ f_{*,A} \tag{11.2}$$

Since left and right multiplications are diffeomorphisms, their differentials are isomorphisms. Composition with an isomorphism does not change the rank of a linear map. Hence, in (11.2),

$$\operatorname{rk} f_{*,AC} = \operatorname{rk} f_{*,A}.$$

Since AC and A are two arbitrary points of $\mathrm{GL}(n,\mathbb{R})$, this proves that the differential of f has constant rank on $\mathrm{GL}(n,\mathbb{R})$. By the constant-rank level set theorem, the orthogonal group $\mathrm{O}(n) = f^{-1}(I)$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$.

Notation. If $f : N \to M$ is a map with constant rank k in a neighborhood of a point $p \in N$, its local normal form (11.1) relative to the charts $(U,\phi) = (U, x^1, \dots, x^n)$ and $(V,\Psi) = (V, y^1, \dots, y^m)$ in the constant rank theorem (Theorem 11.1) can be expressed in terms of the local coordinates $x^1, \dots, x^n$ and $y^1, \dots, y^m$ as follows.

First note that for any $q \in U$,

$$\phi(q) = \left(x^1(q), \dots, x^n(q)\right) \text{ and } \psi(f(q)) = \left(y^1(f(q)), \dots, y^n(f(q)\right).$$

Thus,

$$\begin{aligned}\left(y^1(f(q)), \dots, y^m(f(q)\right) &= \psi(f(q)) = \left(\psi \circ f \circ \phi^{-1}\right)(\phi(q)) \\ &= \left(\psi \circ f \circ \phi^{-1}\right)\left(x^1(q), \dots, x^n(q)\right) \\ &= \left(x^1(q), \dots, x^k(q)\right), 0, \dots, 0\right) \quad (\text{ by } (11.1))\end{aligned}$$

As functions on U,

$$\left(y^1 \circ f, \dots, y^m \circ f\right) = \left(x^1, \dots, x^k, 0, \dots, 0\right). \tag{11.3}$$

We can rewrite (11.3) in the following form: relative to the charts $(U, x^1, \dots, x^n)$ and $(V, y^1, \dots, y^m)$, the map f is given by

$$\left(x^1, \dots, x^n\right) \mapsto \left(x^1, \dots, x^k, 0, \dots, 0\right).$$

8.4.2 11.2 The Immersion and Submersion Theorems

In this subsection we explain why immersions and submersions have constant rank. The constant rank theorem gives local normal forms for immersions and submersions, called the immersion theorem and the submersion theorem respectively. From the submersion theorem and the constant-rank level set theorem, we get two more proofs of the regular level set theorem.

Consider a C^∞ map $f : N \to M$. Let $(U,\phi) = (U, x^1, \dots, x^n)$ be a chart about p in N and $(V,\Psi) = (V, y^1, \dots, y^m)$ a chart about $f(p)$ in M. Write $f^i = y^i \circ f$ for the i th component of f in the chart $(V, y^1, \dots, y^m)$. Relative to the charts (U,ϕ) and (V,ψ), the linear map $f_{*,p}$ is represented by the matrix $[\partial f^i/\partial x^j(p)]$ (Proposition 8.11). Hence,

$$\begin{aligned} f_{*,p} \text{ is injective } &\iff n \le m \text{ and } \operatorname{rk}\left[\partial f^i/\partial x^j(p)\right] = n, \\ f_{*,p} \text{ is surjective } &\iff n \ge m \text{ and } \operatorname{rk}\left[\partial f^i/\partial x^j(p)\right] = m. \end{aligned} \tag{11.4}$$

The rank of a matrix is the number of linearly independent rows of the matrix; it is also the number of linearly independent columns. Thus, the maximum possible rank of an $m \times n$ matrix

is the minimum of m and n. It follows from (11.4) that being an immersion or a submersion at p is equivalent to the maximality of $\operatorname{rk}[\partial f^i/\partial x^j(p)]$.

Having maximal rank at a point is an open condition in the sense that the set

$$D_{\max}(f) = \{p \in U \mid f_{*,p} \text{ has maximal rank at } p\}$$

is an open subset of U. To see this, suppose k is the maximal rank of f. Then

$$\begin{aligned} \operatorname{rk} f_{*,p} = k &\Longleftrightarrow \operatorname{rk}\left[\partial f^i/\partial x^j(p)\right] = k \\ &\Longleftrightarrow \operatorname{rk}\left[\partial f^i/\partial x^j(p)\right] \geq k \quad \text{(since } k \text{ is maximal)} \end{aligned}$$

So the complement $U - D_{\max}(f)$ is defined by

$$\operatorname{rk}\left[\partial f^i/\partial x^j(p)\right] < k$$

which is equivalent to the vanishing of all $k \times k$ minors of the matrix $[\partial f^i/\partial x^j(p)]$. As the zero set of finitely many continuous functions, $U - D_{\max}(f)$ is closed and so $D_{\max}(f)$ is open. In particular, if f has maximal rank at p, then it has maximal rank at all points in some neighborhood of p. We have proven the following proposition.

Proposition 11.4. Let N and M be manifolds of dimensions n and m respectively. If a C^∞ map $f : N \to M$ is an immersion at a point $p \in N$, then it has constant rank n in a neighborhood of p. If a C^∞ map $f : N \to M$ is a submersion at a point $p \in N$, then it has constant rank m in a neighborhood of p.

Example. While maximal rank at a point implies constant rank in a neighborhood, the converse is not true. The map $f : \mathbb{R}^2 \to \mathbb{R}^3$, $f(x, y) = (x, 0, 0)$, has constant rank 1 , but it does not have maximal rank at any point.

By Proposition 11.4, the following theorems are simply special cases of the constant rank theorem.

Theorem 11.5. Let N and M be manifolds of dimensions n and m respectively.
(i) (Immersion theorem) Suppose $f : N \to M$ is an immersion at $p \in N$. Then there are charts (U, ϕ) centered at p in N and (V, Ψ) centered at $f(p)$ in M such that in a neighborhood of $\phi(p)$,

$$\left(\psi \circ f \circ \phi^{-1}\right)\left(r^1, \ldots, r^n\right) = \left(r^1, \ldots, r^n, 0, \ldots, 0\right)$$

(ii) (Submersion theorem) Suppose $f : N \to M$ is a submersion at p in N. Then there are charts (U, ϕ) centered at p in N and (V, Ψ) centered at $f(p)$ in M such that in a neighborhood of $\phi(p)$,

$$\left(\psi \circ f \circ \phi^{-1}\right)\left(r^1, \ldots, r^m, r^{m+1}, \ldots, r^n\right) = \left(r^1, \ldots, r^m\right).$$

Corollary 11.6. A submersion $f : N \to M$ of manifolds is an open map.
Proof. Let W be an open subset of N. We need to show that its image $f(W)$ is open in M. Choose a point $f(p)$ in $f(W)$, with $p \in W$. By the submersion theorem, f is locally a projection. Since a projection is an open map (Problem A.7), there is an open neighborhood U of p in W such that $f(U)$ is open in M. Clearly,

$$f(p) \in f(U) \subset f(W).$$

Since $f(p) \in f(W)$ was arbitrary, $f(W)$ is open in M.
The regular level set theorem (Theorem 9.9) is an easy corollary of the submersion theorem. Indeed, for a C^∞ map $f : N \to M$ of manifolds, a level set $f^{-1}(c)$ is regular if and only if f is a submersion at every point $p \in f^{-1}(c)$. Fix one such point $p \in f^{-1}(c)$ and let (U, ϕ) and (V, ψ) be the charts in the submersion theorem. Then $\psi \circ f \circ \phi^{-1} = \pi : \mathbb{R}^n \supset \phi(U) \to \mathbb{R}^m$ is the projection to the first m coordinates, $\pi\left(r^1, \ldots, r^n\right) = \left(r^1, \ldots, r^m\right)$. It follows that on U,

$$\psi \circ f = \pi \circ \phi = \left(r^1, \ldots, r^m\right) \circ \phi = \left(x^1, \ldots, x^m\right).$$

Therefore,

$$f^{-1}(c) = f^{-1}\left(\psi^{-1}(\mathbf{0})\right) = (\psi \circ f)^{-1}(\mathbf{0}) = Z(\psi \circ f) = Z\left(x^1, \ldots, x^m\right)$$

showing that in the chart $(U, x^1, \ldots, x^n)$, the level set $f^{-1}(c)$ is defined by the vanishing of the m coordinate functions $x^1, \ldots, x^m$. Therefore, $(U, x^1, \ldots, x^n)$ is an adapted chart for N relative to $f^{-1}(c)$. This gives a second proof that the regular level set $f^{-1}(c)$ is a regular submanifold of N.

Since the submersion theorem is a special case of the constant rank theorem, it is not surprising that the regular level set theorem is also a special case of the constant-rank level set theorem. On a regular level set $f^{-1}(c)$, the map $f : N \to M$ has maximal rank m at every point. Since the maximality of the rank of f is an open condition, a regular level set $f^{-1}(c)$ has a neighborhood on which f has constant rank m. By the constant-rank level set theorem (Theorem 11.2), $f^{-1}(c)$ is a regular submanifold of N, giving us a third proof of the regular level set theorem.

8.4.3 11.3 Images of Smooth Maps

The following are all examples of C^∞ maps $f : N \to M$, with $N = \mathbb{R}$ and $M = \mathbb{R}^2$.
Example 11.7. $f(t) = (t^2, t^3)$.
This f is one-to-one, because $t \mapsto t^3$ is one-to-one. Since $f'(0) = (0, 0)$, the differential $f_{*,0} : T_0\mathbb{R} \to T_{(0,0)}\mathbb{R}^2$ is the zero map and hence not injective; so f is not an immersion at 0 . Its image is the cuspidal cubic $y^2 = x^3$ (Figure 11.2).

Fig. 11.2. A cuspidal cubic, not an immersion.

Example 11.8. $f(t) = (t^2 - 1, t^3 - t)$.
Since the equation $f'(t) = (2t, 3t^2 - 1) = (0, 0)$ has no solution in t, this map f is an immersion. It is not one-to-one, because it maps both $t = 1$ and $t = -1$ to the origin. To find an equation for the image $f(N)$, let $x = t^2 - 1$ and $y = t^3 - t$. Then $y = t\left(t^2 - 1\right) = tx$; so

$$y^2 = t^2x^2 = (x + 1)x^2.$$

Thus the image of f is the nodal cubic $y^2 = x^2(x + 1)$ (Figure 11.3).
Example 11.9. The map f in Figure 11.4 is a one-to-one immersion but its image, with the subspace topology induced from $\mathbb{R}^2$, is not homeomorphic to the domain $\mathbb{R}$, because there are points near $f(p)$ in the image that correspond to points in $\mathbb{R}$ far away from p. More precisely, if U is an interval about p as shown, there is no neighborhood V of $f(p)$ in $f(N)$ such that $f^{-1}(V) \subset U$; hence, f^{-1} is not continuous.

Fig. 11.3. A nodal cubic, an immersion but not one-to-one.

Fig. 11.4. A one-to-one immersion that is not an embedding.

Example 11.10. The manifold M in Figure 11.5 is the union of the graph of $y = \sin(1/x)$ on the interval $]0, 1[$, the open line segment from $y = 0$ to $y = 1$ on the y-axis, and a smooth curve joining $(0, 0)$ and $(1, \sin 1)$. The map f is a one-to-one immersion whose image with the subspace topology is not homeomorphic to $\mathbb{R}$.

Fig. 11.5. A one-to-one immersion that is not an embedding.

Notice that in these examples the image $f(N)$ is not a regular submanifold of $M = \mathbb{R}^2$. We would like conditions on the map f so that its image $f(N)$ would be a regular submanifold of M.

Definition 11.11. A C^∞ map $f : N \to M$ is called an embedding if
(i) it is a one-to-one immersion and
(ii) the image $f(N)$ with the subspace topology is homeomorphic to N under f. (The phrase "one-to-one" in this definition is redundant, since a homeomorphism is necessarily one-to-one.)

Remark. Unfortunately, there is quite a bit of terminological confusion in the literature concerning the use of the word "submanifold." Many authors give the image $f(N)$ of a one-to-one immersion $f : N \to M$ not the subspace topology, but the topology inherited from f; i.e., a subset $f(U)$ of $f(N)$ is said to be open if and only if U is open in N. With this topology, $f(N)$ is by definition homeomorphic to N. These authors define a submanifold to be the image

of any one-to-one immersion with the topology and differentiable structure inherited from f. Such a set is sometimes called an immersed submanifold of M. Figures 11.4 and 11.5 show two examples of immersed submanifolds. If the underlying set of an immersed submanifold is given the subspace topology, then the resulting space need not be a manifold at all!

For us, a submanifold without any qualifying adjective is always a regular submanifold. To recapitulate, a regular submanifold of a manifold M is a subset S of M with the subspace topology such that every point of S has a neighborhood $U \cap S$ defined by the vanishing of coordinate functions on U, where U is a chart in M.

Fig. 11.6. The figure-eight as two distinct immersed submanifolds of $\mathbb{R ^{2}$.}

Example 11.12 (The figure-eight). The figure-eight is the image of a one-to-one immersion

$$f(t) = (\cos t, \sin 2t), \quad -\pi/2 < t < 3\pi/2$$

(Figure 11.6). As such, it is an immersed submanifold of $\mathbb{R}^2$, with a topology and manifold structure induced from the open interval $]-\pi/2, 3\pi/2[$ by f. Because of the presence of a cross at the origin, it cannot be a regular submanifold of $\mathbb{R}^2$. In fact, with the subspace topology of $\mathbb{R}^2$, the figure-eight is not even a manifold.

The figure-eight is also the image of the one-to-one immersion

$$g(t) = (\cos t, -\sin 2t), \quad -\pi/2 < t < 3\pi/2$$

(Figure 11.6). The maps f and g induce distinct immersed submanifold structures on the figure-eight. For example, the open interval from A to B in Figure 11.6 is an open set in the topology induced from g, but it is not an open set in the topology induced from f, since its inverse image under f contains an isolated point $\pi/2$.

We will use the phrase "near p " to mean "in a neighborhood of p."

Theorem 11.13. If $f : N \to M$ is an embedding, then its image $f(N)$ is a regular submanifold of M.

Proof. Let $p \in N$. By the immersion theorem (Theorem 11.5), there are local coordinates $(U, x^1, \ldots, x^n)$ near p and $(V, y^1, \ldots, y^m)$ near $f(p)$ such that $f : U \to V$ has the form

$$\left(x^1, \ldots, x^n\right) \mapsto \left(x^1, \ldots, x^n, 0, \ldots, 0\right).$$

Fig. 11.7. The image of an embedding is a regular submanifold.

Thus, $f(U)$ is defined in V by the vanishing of the coordinates $y^{n+1}, \ldots, y^m$. This alone does not prove that $f(N)$ is a regular submanifold, since $V \cap f(N)$ may be larger than $f(U)$. (Think about Examples 11.9 and 11.10.) We need to show that in some neighborhood of $f(p)$ in V, the set $f(N)$ is defined by the vanishing of $m-n$ coordinates.

Since $f(N)$ with the subspace topology is homeomorphic to N, the image $f(U)$ is open in $f(N)$. By the definition of the subspace topology, there is an open set V' in M such that $V' \cap f(N) = f(U)$ (Figure 11.7). In $V \cap V'$,

$$V \cap V' \cap f(N) = V \cap f(U) = f(U),$$

and $f(U)$ is defined by the vanishing of $y^{n+1}, \ldots, y^m$. Thus, $(V \cap V', y^1, \ldots, y^m)$ is an adapted chart containing $f(p)$ for $f(N)$. Since $f(p)$ is an arbitrary point of $f(N)$, this proves that $f(N)$ is a regular submanifold of M. $\square$

Theorem 11.14. If N is a regular submanifold of M, then the inclusion $i : N \to M$, $i(p) = p$, is an embedding.

Proof. Since a regular submanifold has the subspace topology and $i(N)$ also has the subspace topology, $i : N \to i(N)$ is a homeomorphism. It remains to show that $i : N \to M$ is an immersion.

Let $p \in N$. Choose an adapted chart $(V, y^1, \ldots, y^n, y^{n+1}, \ldots, y^m)$ for M about p such that $V \cap N$ is the zero set of $y^{n+1}, \ldots, y^m$. Relative to the charts $(V \cap N, y^1, \ldots, y^n)$ for N and $(V, y^1, \ldots, y^m)$ for M, the inclusion i is given by

$$\left(y^1, \ldots, y^n\right) \mapsto \left(y^1, \ldots, y^n, 0, \ldots, 0\right)$$

which shows that i is an immersion.

In the literature the image of an embedding is often called an embedded submanifold. Theorems 11.13 and 11.14 show that an embedded submanifold and a regular submanifold are one and the same thing.

8.4.4 11.4 Smooth Maps Into a Submanifold

Suppose $f : N \to M$ is a C^∞ map whose image $f(N)$ lies in a subset $S \subset M$. If S is a manifold, is the induced map $\tilde{f} : N \to S$ also C^∞ ? This question is more subtle than it looks, because the answer depends on whether S is a regular submanifold or an immersed submanifold of M.

Example. Consider the one-to-one immersions f and $g : I \to \mathbb{R}^2$ in Example 11.12, where I is the open interval $]-\pi/2, 3\pi/2[$ in $\mathbb{R}$. Let S be the figure-eight in $\mathbb{R}^2$ with the immersed submanifold structure induced from g. Because the image of $f : I \to \mathbb{R}^2$ lies in S, the C^∞ map f induces a map $\tilde{f} : I \to S$.

The open interval from A to B in Figure 11.6 is an open neighborhood of the origin 0 in S. Its inverse image under $\tilde{f}$ contains the point $\pi/2$ as an isolated point and is therefore not open. This shows that although $f : I \to \mathbb{R}^2$ is C^∞, the induced map $\tilde{f} : I \to S$ is not continuous and therefore not C^∞.

Theorem 11.15. Suppose $f : N \to M$ is C^∞ and the image of f lies in a subset S of M. If S is a regular submanifold of M, then the induced map $\tilde{f} : N \to S$ is C^∞.

Proof. Let $p \in N$. Denote the dimensions of N, M, and S by n, m, and s, respectively. By hypothesis, $f(p) \in S \subset M$. Since S is a regular submanifold of M, there is an adapted coordinate chart $(V, \Psi) = (V, y^1, \ldots, y^m)$ for M about $f(p)$ such that $S \cap V$ is the zero set of $y^{s+1}, \ldots, y^m$, with coordinate map $\psi_S = (y^1, \ldots, y^s)$. By the continuity of f, it is possible to choose a neighborhood of p with $f(U) \subset V$. Then $f(U) \subset V \cap S$, so that for $q \in U$,

$$(\psi \circ f)(q) = \left(y^1(f(q)), \ldots, y^s(f(q)), 0, \ldots, 0\right).$$

It follows that on U,

$$\psi_S \circ \tilde{f} = \left(y^1 \circ f, \ldots, y^s \circ f\right)$$

Since $y^1 \circ f, \ldots, y^s \circ f$ are C^∞ on U, by Proposition 6.16, $\tilde{f}$ is C^∞ on U and hence at p. Since p was an arbitrary point of N, the map $\tilde{f} : N \to S$ is C^∞.

Example 11.16 (Multiplication map of $\mathrm{SL}(n, \mathbb{R})$). The multiplication map

$$\begin{aligned} \mu : \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R}) &\to \mathrm{GL}(n, \mathbb{R}), \\ (A, B) &\mapsto AB, \end{aligned}$$

is clearly C^∞ because

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$$

is a polynomial and hence a C^∞ function of the coordinates a_{ik} and b_{kj}. However, one cannot conclude in the same way that the multiplication map

$$\bar{\mu} : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$$

is C^∞. This is because $\{a_{ij}\}_{1 \le i,j \le n}$ is not a coordinate system on $\mathrm{SL}(n, \mathbb{R})$; there is one coordinate too many (See Problem 11.6).

Since $\mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$, the inclusion map

$$i : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R}) \times \mathrm{GL}(n, \mathbb{R})$$

is C^∞ by Theorem 11.14; therefore, the composition

$$\mu \circ i : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$$

is also C^∞. Because the image of $\mu \circ i$ lies in $\mathrm{SL}(n, \mathbb{R})$, and $\mathrm{SL}(n, \mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{R})$ (see Example 9.13), by Theorem 11.15 the induced map

$$\bar{\mu} : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$$

is C^∞.

8.4.5 11.5 The Tangent Plane to a Surface in $\mathbb{R}^3$

Suppose $f(x^1, x^2, x^3)$ is a real-valued function on $\mathbb{R}^3$ with no critical points on its zero set $N = f^{-1}(0)$. By the regular level set theorem, N is a regular submanifold of $\mathbb{R}^3$. By Theorem 11.14 the inclusion $i : N \to \mathbb{R}^3$ is an embedding, so at any point p in $N, i_{*,p} : T_pN \to T_p\mathbb{R}^3$ is injective. We may therefore think of the tangent plane T_pN as a plane in $T_p\mathbb{R}^3 \simeq \mathbb{R}^3$ (Figure 11.8). We would like to find the equation of this plane.

Suppose $v = \sum v^i \partial/\partial x^i|_p$ is a vector in T_pN. Under the linear isomorphism $T_p\mathbb{R}^3 \simeq \mathbb{R}^3$, we identify v with the vector $\langle v^1, v^2, v^3\rangle$ in $\mathbb{R}^3$. Let $c(t)$ be a curve lying in N with $c(0) = p$ and $c'(0) = \langle v^1, v^2, v^3\rangle$. Since $c(t)$ lies in $N, f(c(t)) = 0$ for all t. By the chain rule,

$$0 = \frac{d}{dt} f(c(t)) = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(c(t)) \left(c^i\right)'(t)$$

Fig. 11.8. Tangent plane to a surface N at p.

At $t = 0$,

$$0 = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(c(0)) \left(c^i\right)'(0) = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(p) v^i.$$

Since the vector $v = \langle v^1, v^2, v^3\rangle$ represents the arrow from the point $p = (p^1, p^2, p^3)$ to $x = (x^1, x^2, x^3)$ in the tangent plane, one usually makes the substitution $v^i = x^i - p^i$. This amounts to translating the tangent plane from the origin to p. Thus the tangent plane to N at p is defined by the equation

$$\sum_{i=1}^{3} \frac{\partial f}{\partial x^i}(p) \left(x^i - p^i\right) = 0. \tag{11.5}$$

One interpretation of this equation is that the gradient vector $\langle \partial f/\partial x^1(p), \partial f/\partial x^2(p), \partial f/\partial x^3(p)\rangle$ of f at p is normal to any vector in the tangent plane.

Example 11.17 (Tangent plane to a sphere). Let $f(x, y, z) = x^2 + y^2 + z^2 - 1$. To get the equation of the tangent plane to the unit sphere $S^2 = f^{-1}(0)$ in $\mathbb{R}^3$ at $(a, b, c) \in S^2$, we compute

$$\frac{\partial f}{\partial x} = 2x, \quad \frac{\partial f}{\partial y} = 2y, \quad \frac{\partial f}{\partial z} = 2z.$$

At $p = (a, b, c)$,

$$\frac{\partial f}{\partial x}(p) = 2a, \quad \frac{\partial f}{\partial y}(p) = 2b, \quad \frac{\partial f}{\partial z}(p) = 2c.$$

By (11.5) the equation of the tangent plane to the sphere at (a, b, c) is

$$2a(x-a)+2b(y-b)+2c(z-c)=0,$$

or

$$ax+by+cz=1,$$

since $a^2+b^2+c^2=1$.

8.4.6 Problems

11.1. Tangent Vectors to a Sphere

The unit sphere S^n in $\mathbb{R}^{n+1}$ is defined by the equation $\sum_{i=1}^{n+1}\left(x^i\right)^2=1$. For $p=\left(p^1, \ldots, p^{n+1}\right) \in S^n$, show that a necessary and sufficient condition for

$$X_p=\sum a^i \partial / \partial x^i\big|_p \in T_p\left(\mathbb{R}^{n+1}\right)$$

to be tangent to S^n at p is $\sum a^i p^i=0$.

11.2. Tangent Vectors to a Plane Curve

(a) Let $i: S^1 \hookrightarrow \mathbb{R}^2$ be the inclusion map of the unit circle. In this problem, we denote by x, y the standard coordinates on $\mathbb{R}^2$ and by $\bar{x}, \bar{y}$ their restrictions to S^1. Thus, $\bar{x}=i^* x$ and $\bar{y}=i^* y$. On the upper semicircle $U=\{(a, b) \in S^1 \mid b>0\}, \bar{x}$ is a local coordinate, so that $\partial / \partial \bar{x}$ is defined. Prove that for $p \in U$,

$$i_*\left(\left.\frac{\partial}{\partial \bar{x}}\right|_p\right)=\left.\left(\frac{\partial}{\partial x}+\frac{\partial \bar{y}}{\partial \bar{x}} \frac{\partial}{\partial y}\right)\right|_p .$$

Thus, although $i_*: T_p S^1 \rightarrow T_p \mathbb{R}^2$ is injective, $\partial / \partial \bar{x}|_p$ cannot be identified with $\partial / \partial x|_p$ (Figure 11.9).

Fig. 11.9. Tangent vector $\partial / \left.\partial \bar{x}\right|_p$ to a circle.

(b) Generalize (a) to a smooth curve C in $\mathbb{R}^2$, letting U be a chart in C on which $\bar{x}$, the restriction of x to C, is a local coordinate.

11.3.* Critical Points of a Smooth map on a Compact Manifold

Show that a smooth map f from a compact manifold N to $\mathbb{R}^m$ has a critical point. (Hint: Let $\pi : \mathbb{R}^m \to \mathbb{R}$ be the projection to the first factor. Consider the composite map $\pi \circ f : N \to \mathbb{R}$. A second proof uses Corollary 11.6 and the connectedness of $\mathbb{R}^m$.)

11.4. Differential of an Inclusion map

On the upper hemisphere of the unit sphere S^2, we have the coordinate map $\phi = (u, v)$, where

$$u(a, b, c) = a \quad \text{and} \quad v(a, b, c) = b.$$

So the derivations $\partial/\partial u|_p, \partial/\partial v|_p$ are tangent vectors of S^2 at any point $p = (a, b, c)$ on the upper hemisphere. Let $i : S^2 \to \mathbb{R}^3$ be the inclusion and x, y, z the standard coordinates on $\mathbb{R}^3$. The differential $i_* : T_pS^2 \to T_p\mathbb{R}^3$ maps $\partial/\partial u|_p, \partial/\partial v|_p$ into $T_p\mathbb{R}^3$. Thus,

$$i_*\left(\frac{\partial}{\partial u}\bigg|_p\right) = \alpha^1 \frac{\partial}{\partial x}\bigg|_p + \beta^1 \frac{\partial}{\partial y}\bigg|_p + \gamma^1 \frac{\partial}{\partial z}\bigg|_p$$
$$i_*\left(\frac{\partial}{\partial v}\bigg|_p\right) = \alpha^2 \frac{\partial}{\partial x}\bigg|_p + \beta^2 \frac{\partial}{\partial y}\bigg|_p + \gamma^2 \frac{\partial}{\partial z}\bigg|_p$$

for some constants $\alpha^i, \beta^i, \gamma^i$. Find $(\alpha^i, \beta^i, \gamma^i)$ for $i = 1, 2$.

11.5. One-to-one Immersion of a Compact Manifold

Prove that if N is a compact manifold, then a one-to-one immersion $f : N \to M$ is an embedding.

11.6. Multiplication map in $\mathrm{sl}(n, \mathbb{R})$

Let $f : \mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ be the determinant map $f(A) = \det A = \det [a_{ij}]$. For $A \in \mathrm{SL}(n, \mathbb{R})$, there is at least one (k, ℓ) such that the partial derivative $\partial f/\partial a_{k\ell}(A)$ is nonzero (Example 9.13). Use Lemma 9.10 and the implicit function theorem to prove that
(a) there is a neighborhood of A in $\mathrm{SL}(n, \mathbb{R})$ in which $a_{ij}, (i, j) \neq (k, \ell)$, form a coordinate system, and $a_{k\ell}$ is a C^∞ function of the other entries $a_{ij}, (i, j) \neq (k, \ell)$;
(b) the multiplication map

$$\bar{\mu} : \mathrm{SL}(n, \mathbb{R}) \times \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n, \mathbb{R})$$

is C^∞.

8.5 §12 the Tangent Bundle

A smooth vector bundle over a smooth manifold M is a smoothly varying family of vector spaces, parametrized by M, that locally looks like a product. Vector bundles and bundle maps form a category, and have played a fundamental role in geometry and topology since their appearance in the 1930s [39].

The collection of tangent spaces to a manifold has the structure of a vector bundle over the manifold, called the tangent bundle. A smooth map between two manifolds induces, via its differential at each point, a bundle map of the corresponding tangent bundles. Thus, the tangent bundle construction is a functor from the category of smooth manifolds to the category of vector bundles.

At first glance it might appear that the tangent bundle functor is not a simplification, since a vector bundle is a manifold plus an additional structure. However, because the tangent bundle is canonically associated to a manifold, invariants of the tangent bundle will give rise to invariants for the manifold. For example, the Chern-Weil theory of characteristic classes, which we treat in another volume, uses differential geometry to construct invariants for vector bundles. Applied to the tangent bundle, characteristic classes lead to numerical diffeomorphism invariants for a manifold called characteristic numbers. Characteristic numbers generalize, for instance, the classical Euler characteristic.

For us in this book the importance of the vector bundle point of view comes from its role in unifying concepts. A section of a vector bundle $\pi : E \to M$ is a map from M to E that maps each point of M into the fiber of the bundle over the point. As we shall see, both vector fields and differential forms on a manifold are sections of vector bundles over the manifold.

In the following pages we construct the tangent bundle of a manifold and show that it is a smooth vector bundle. We then discuss criteria for a section of a smooth vector bundle to be smooth.

8.5.1 12.1 The Topology of the Tangent Bundle

Let M be a smooth manifold. Recall that at each point $p \in M$, the tangent space T_pM is the vector space of all point-derivations of $C_p^\infty(M)$, the algebra of germs of C^∞ functions at p. The tangent bundle of M is the union of all the tangent spaces of M :

$$TM = \bigcup_{p \in M} T_pM$$

In general, if $\{A_i\}_{i \in I}$ is a collection of subsets of a set S, then their disjoint union is defined to be the set

$$\coprod_{i \in I} A_i := \bigcup_{i \in I} (\{i\} \times A_i) .$$

The subsets A_i may overlap, but in the disjoint union they are replaced by nonoverlapping copies.

In the definition of the tangent bundle, the union $\bigcup_{p \in M} T_pM$ is (up to notation) the same as the disjoint union $\amalg_{p \in M} T_pM$, since for distinct points p and q in M, the tangent spaces T_pM and T_qM are already disjoint.

Fig. 12.1. Tangent spaces to a circle.

In a pictorial representation of tangent spaces such as Figure 12.1, where M is the unit circle, it may look as though the two tangent spaces T_pM and T_qM intersect. In fact, the intersection point of the two lines in Figure 12.1 represents distinct tangent vectors in T_pM and T_qM, so that T_pM and T_qM are disjoint even in the figure.

There is a natural map $\pi : TM \to M$ given by $\pi(v) = p$ if $v \in T_pM$. (We use the word "natural" to mean that the map does not depend on any choice, for example, the choice of an atlas or of local coordinates for M.) As a matter of notation, we sometimes write a tangent vector $v \in T_pM$ as a pair (p, v), to make explicit the point $p \in M$ at which v is a tangent vector.

As defined, TM is a set, with no topology or manifold structure. We will make it into a smooth manifold and show that it is a C^∞ vector bundle over M. The first step is to give it a topology.

If $(U, \phi) = (U, x^1, \ldots, x^n)$ is a coordinate chart on M, let

$$TU = \bigcup_{p \in U} T_pU = \bigcup_{p \in U} T_pM.$$

(We saw in Remark 8.2 that $T_pU = T_pM$.) At a point $p \in U$, a basis for T_pM is the set of coordinate vectors $\partial/\,\partial x^1|_p, \ldots, \partial/\,\partial x^n|_p$, so a tangent vector $v \in T_pM$ is uniquely a linear combination

$$v = \sum_i^n c^i \frac{\partial}{\partial x^i}\bigg|_p .$$

In this expression, the coefficients $c^i = c^i(v)$ depend on v and so are functions on TU. Let $\bar{x}^i = x^i \circ \pi$ and define the map $\tilde{\phi} : TU \to \phi(U) \times \mathbb{R}^n$ by

$$v \mapsto \big(x^1(p), \ldots, x^n(p), c^1(v), \ldots, c^n(v)\big) = \big(\bar{x}^1, \ldots, \bar{x}^n, c^1, \ldots, c^n\big)\,(v). \tag{12.1}$$

Then $\tilde{\phi}$ has inverse

$$\big(\phi(p), c^1, \ldots, c^n\big) \mapsto \sum c^i \frac{\partial}{\partial x^i}\bigg|_p$$

and is therefore a bijection. This means we can use $\tilde{\phi}$ to transfer the topology of $\phi(U) \times \mathbb{R}^n$ to TU : a set A in TU is open if and only if $\tilde{\phi}(A)$ is open in $\phi(U) \times \mathbb{R}^n$, where $\phi(U) \times \mathbb{R}^n$ is given its standard topology as an open subset of $\mathbb{R}^{2n}$. By definition, TU, with the topology induced by $\tilde{\phi}$, is homeomorphic to $\phi(U) \times \mathbb{R}^n$. If V is an open subset of U, then $\phi(V) \times \mathbb{R}^n$ is an open subset of $\phi(U) \times \mathbb{R}^n$. Hence, the relative topology on TV as a subset of TU is the same as the topology induced from the bijection $\tilde{\phi}\Big|_{TV} : TV \to \phi(V) \times \mathbb{R}^n$.

Let $\phi_* : T_pU \to T_{\phi(p)}(\mathbb{R}^n)$ be the differential of the coordinate map ϕ at p. Since $\phi_*(v) = \sum c^i \partial/\, \partial r^i|_{\phi(p)} \in T_{\phi(p)}(\mathbb{R}^n) \simeq \mathbb{R}^n$ by Proposition 8.8, we may identify $\phi_*(v)$ with the column vector $\langle c^1, \ldots, c^n \rangle$ in $\mathbb{R}^n$. So another way to describe $\tilde{\phi}$ is $\tilde{\phi} = (\phi \circ \pi, \phi_*\ )$.

Let $\mathcal{B}$ be the collection of all open subsets of $T(U_\alpha)$ as U_α runs over all coordinate open sets in M :

$$\mathcal{B} = \bigcup_\alpha \{A \mid A \text{ open in } T(U_\alpha), U_\alpha \text{ a coordinate open set in } M\}.$$

Lemma 12.1. (i) For any manifold M, the set TM is the union of all $A \in \mathcal{B}$.
(ii) Let U and V be coordinate open sets in a manifold M. If A is open in TU and B is open in TV, then $A \cap B$ is open in $T(U \cap V)$.

Proof. (i) Let $\{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas for M. Then

$$TM = \bigcup_\alpha T(U_\alpha) \subset \bigcup_{A \in \mathcal{B}} A \subset TM,$$

so equality holds everywhere.
(ii) Since $T(U \cap V)$ is a subspace of TU, by the definition of relative topology, $A \cap T(U \cap V)$ is open in $T(U \cap V)$. Similarly, $B \cap T(U \cap V)$ is open in $T(U \cap V)$.
But

$$A \cap B \subset TU \cap TV = T(U \cap V).$$

Hence,

$$A \cap B = A \cap B \cap T(U \cap V) = (A \cap T(U \cap V)) \cap (B \cap T(U \cap V))$$

is open in $T(U \cap V)$. $\square$

It follows from this lemmat that the collection $\mathcal{B}$ satisfies the conditions (i) and (ii) of Proposition A. 8 for a collection of subsets to be a basis for some topology on TM. We give the tangent bundle TM the topology generated by the basis $\mathcal{B}$.

Lemma 12.2. *A* manifold M has a countable basis consisting of coordinate open sets.

Proof. Let $\{(U_\alpha, \phi_\alpha)\}$ be the maximal atlas on M and $\mathcal{B} = \{B_i\}$ a countable basis for M. For each coordinate open set U_α and point $p \in U_\alpha$, choose a basic open set $B_{p,\alpha} \in \mathcal{B}$ such that

$$p \in B_{p,\alpha} \subset U_\alpha$$

The collection $\{B_{p,\alpha}\}$, without duplicate elements, is a subcollection of $\mathcal{B}$ and is therefore countable.

For any open set U in M and a point $p \in U$, there is a coordinate open set U_α such that

$$p \in U_\alpha \subset U$$

Hence,

$$p \in B_{p,\alpha} \subset U$$

which shows that $\{B_{p,\alpha}\}$ is a basis for M.

Proposition 12.3. The tangent bundle TM of a manifold M is second countable.

Proof. Let $\{U_i\}_{i=1}^{\infty}$ be a countable basis for M consisting of coordinate open sets. Let ϕ_i be the coordinate map on U_i. Since TU_i is homeomorphic to the open subset $\phi_i(U_i) \times \mathbb{R}^n$ of $\mathbb{R}^{2n}$ and any subset of a Euclidean space is second countable (Example A. 13 and Proposition A.14), TU_i is second countable. For each i, choose a countable basis $\{B_{i,j}\}_{j=1}^{\infty}$ for TU_i. Then $\{B_{i,j}\}_{i,j=1}^{\infty}$ is a countable basis for the tangent bundle.

Proposition 12.4. The tangent bundle TM of a manifold M is Hausdorff.

Proof. Problem 12.1.

8.5.2 12.2 The Manifold Structure on the Tangent Bundle

Next we show that if $\{(U_\alpha, \phi_\alpha)\}$ is a C^∞ atlas for M, then $\left\{\left(TU_\alpha, \tilde{\phi}_\alpha\right)\right\}$ is a C^∞ atlas for the tangent bundle TM, where $\tilde{\phi}_\alpha$ is the map on TU_α induced by ϕ_α as in (12.1). It is clear that $TM = \bigcup_\alpha TU_\alpha$. It remains to check that on $(TU_\alpha) \cap (TU_\beta)$, $\tilde{\phi}_\alpha$ and $\tilde{\phi}_\beta$ are C^∞ compatible.

Recall that if $(U, x^1, \ldots, x^n), (V, y^1, \ldots, y^n)$ are two charts on M, then for any $p \in U \cap V$ there are two bases singled out for the tangent space T_pM : $\left\{\partial/\partial x^j|_p\right\}_{j=1}^n$ and $\left\{\partial/\partial y^i|_p\right\}_{i=1}^n$. So any tangent vector $v \in T_pM$ has two descriptions:

$$v = \sum_j a^j \frac{\partial}{\partial x^j}\bigg|_p = \sum_i b^i \frac{\partial}{\partial y^i}\bigg|_p. \tag{12.2}$$

It is easy to compare them. By applying both sides to x^k, we find that

$$a^k = \left(\sum_j a^j \frac{\partial}{\partial x^j}\right) x^k = \left(\sum_i b^i \frac{\partial}{\partial y^i}\right) x^k = \sum_i b^i \frac{\partial x^k}{\partial y^i}.$$

Similarly, applying both sides of (12.2) to y^k gives

$$b^k = \sum_j a^j \frac{\partial y^k}{\partial x^j}. \tag{12.3}$$

Returning to the atlas $\{(U_\alpha, \phi_\alpha)\}$, we write $U_{\alpha\beta} = U_\alpha \cap U_\beta, \phi_\alpha = (x^1, \ldots, x^n)$ and $\phi_\beta = (y^1, \ldots, y^n)$. Then

$$\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1} : \phi_\alpha(U_{\alpha\beta}) \times \mathbb{R}^n \to \phi_\beta(U_{\alpha\beta}) \times \mathbb{R}^n$$

is given by

$$\left(\phi_\alpha(p), a^1, \ldots, a^n\right) \mapsto \left(p, \sum_j a^j \frac{\partial}{\partial x^j}\bigg|_p\right) \mapsto \left(\left(\phi_\beta \circ \phi_\alpha^{-1}\right)(\phi_\alpha(p)), b^1, \ldots, b^n\right),$$

where by (12.3) and Example 6.24,

$$b^i = \sum_j a^j \frac{\partial y^i}{\partial x^j}(p) = \sum_j a^j \frac{\partial \left(\phi_\beta \circ \phi_\alpha^{-1}\right)^i}{\partial r^j}(\phi_\alpha(p)).$$

By the definition of an atlas, $\phi_\beta \circ \phi_\alpha^{-1}$ is C^∞. Therefore, $\tilde{\phi}_\beta \circ \tilde{\phi}_\alpha^{-1}$ is C^∞. This completes the proof that the tangent bundle TM is a C^∞ manifold, with $\left\{\left(TU_\alpha, \tilde{\phi}_\alpha\right)\right\}$ as a C^∞ atlas.

8.5.3 12.3 Vector Bundles

On the tangent bundle TM of a smooth manifold M, the natural projection map $\pi : TM \to M, \pi(p, v) = p$ makes TM into a C^∞ vector bundle over M, which we now define.

Given any map $\pi : E \to M$, we call the inverse image $\pi^{-1}(p) := \pi^{-1}(\{p\})$ of a point $p \in M$ the fiber at p. The fiber at p is often written E_p. For any two maps $\pi : E \to M$ and $\pi' : E' \to M$ with the same target space M, a map $\phi : E \to E'$ is said to be fiber-preserving if $\phi(E_p) \subset E'_p$ for all $p \in M$.

Exercise 12.5 (Fiber-preserving maps). Given two maps $\pi : E \to M$ and $\pi' : E' \to M$, check that a map $\phi : E \to E'$ is fiber-preserving if and only if the diagram

$$\begin{tikzcd} E \arrow[rr, "\phi"] \arrow[dr, "\pi"'] & & E' \arrow[dl, "\pi'"] \\ & M & \end{tikzcd}$$

commutes.

A surjective smooth map $\pi : E \to M$ of manifolds is said to be locally trivial of rank r if
(i) each fiber $\pi^{-1}(p)$ has the structure of a vector space of dimension r;
(ii) for each $p \in M$, there are an open neighborhood U of p and a fiber-preserving diffeomorphism $\phi : \pi^{-1}(U) \to U \times \mathbb{R}^r$ such that for every $q \in U$ the restriction

$$\phi|_{\pi^{-1}(q)} : \pi^{-1}(q) \to \{q\} \times \mathbb{R}^r$$

is a vector space isomorphism. Such an open set U is called a trivializing open set for E, and ϕ is called a trivialization of E over U.

The collection $\{(U, \phi)\}$, with $\{U\}$ an open cover of M, is called a local trivialization for E, and $\{U\}$ is called a trivializing open cover of M for E.

A C^∞ vector bundle of rank r is a triple $(\ E, M, \pi\ )$ consisting of manifolds E and M and a surjective smooth map $\pi : E \to M$ that is locally trivial of rank r. The manifold E is called the total space of the vector bundle and M the base space. By abuse of language, we say that E is a vector bundle over M. For any regular submanifold $S \subset M$, the triple $(\ \pi^{-1}S, S, \pi|_{\pi^{-1}S}\ )$ is a C^∞ vector bundle over S, called the restriction of E to S. We will often write the restriction as $E|_S$ instead of $\pi^{-1}S$.

Properly speaking, the tangent bundle of a manifold M is a triple $(\ TM, M, \pi\ )$, and TM is the total space of the tangent bundle. In common usage, TM is often referred to as the tangent bundle.

Fig. 12.2. A circular cylinder is a product bundle over a circle.

Example 12.6 (Product bundle). Given a manifold M, let $\pi : M \times \mathbb{R}^r \to M$ be the projection to the first factor. Then $M \times \mathbb{R}^r \to M$ is a vector bundle of rank r, called the product bundle of rank r over M. The vector space structure on the fiber $\pi^{-1}(p) = \{(p, v) \mid v \in \mathbb{R}^r\}$ is the obvious one:

$$(p, u) + (p, v) = (p, u + v), \quad b \cdot (p, v) = (p, bv) \text{ for } b \in \mathbb{R}.$$

A local trivialization on $M \times R$ is given by the identity map $\mathbb{1}_{M\times\mathbb{R}} : M \times \mathbb{R} \to M \times \mathbb{R}$. The infinite cylinder $S^1 \times \mathbb{R}$ is the product bundle of rank 1 over the circle (Figure 12.2).

Let $\pi : E \to M$ be a C^∞ vector bundle. Suppose $(U, \Psi) = (U, x^1, \ldots, x^n)$ is a chart on M and

$$\phi : E|_U \xrightarrow{\sim} U \times \mathbb{R}^r, \quad \phi(e) = \big(\pi(e), c^1(e), \ldots, c^r(e)\big),$$

is a trivialization of E over U. Then

$$(\psi \times \mathbb{1}) \circ \phi = \left(x^1, \ldots, x^n, c^1, \ldots, c^r\right) : E|_U \xrightarrow{\sim} U \times \mathbb{R}^n \xrightarrow{\sim} \psi(U) \times \mathbb{R}^r \subset \mathbb{R}^n \times \mathbb{R}^r$$

is a diffeomorphism of $E|_U$ onto its image and so is a chart on E. We call $x^1, \ldots, x^n$ the base coordinates and $c^1, \ldots, c^r$ the fiber coordinates of the chart $(E|_U, (\psi \times \mathbb{1}) \circ \phi)$ on E. Note that the fiber coordinates c^i depend only on the trivialization ϕ of the bundle $E|_U$ and not on the trivialization ψ of the base U.

Let $\pi_E : E \to M, \pi_F : F \to N$ be two vector bundles, possibly of different ranks. A bundle map from E to F is a pair of maps $(f, \tilde{f}), f : M \to N$ and $\tilde{f} : E \to F$, such that
(i) the diagram

$$\begin{tikzcd} E \arrow[r, "\tilde{f}"] \arrow[d, "\pi_E"'] & F \arrow[d, "\pi_F"] \\ M \arrow[r, "f"'] & N \end{tikzcd}$$

is commutative, meaning $\pi_F \circ \tilde{f} = f \circ \pi_E$;
(ii) $\tilde{f}$ is linear on each fiber; i.e., for each $p \in M$, $\tilde{f} : E_p \to F_{f(p)}$ is a linear map of vector spaces.

The collection of all vector bundles together with bundle maps between them forms a category.

Example. A smooth map $f : N \to M$ of manifolds induces a bundle map $(f, \tilde{f})$, where $\tilde{f} : TN \to TM$ is given by

$$\tilde{f}(p, v) = (f(p), f_*(v)) \in \{f(p)\} \times T_{f(p)}M \subset TM$$

for all $v \in T_pN$. This gives rise to a covariant functor T from the category of smooth manifolds and smooth maps to the category of vector bundles and bundle maps: to each manifold M, we associate its tangent bundle $T(M)$, and to each C^∞ map $f : N \to M$ of manifolds, we associate the bundle map $T(f) = (f : N \to M, \tilde{f} : T(N) \to T(M))$.

If E and F are two vector bundles over the same manifold M, then a bundle map from E to F over M is a bundle map in which the base map is the identity $\mathbb{1}_M$. For a fixed manifold M, we can also consider the category of all C^∞ vector bundles over M and C^∞ bundle maps over M. In this category it makes sense to speak of an isomorphism of vector bundles over M. Any vector bundle over M isomorphic over M to the product bundle $M \times \mathbb{R}^r$ is called a trivial bundle.

8.5.4 12.4 Smooth Sections

A section of a vector bundle $\pi : E \to M$ is a map $s : M \to E$ such that $\pi \circ s = \mathbb{1}_M$, the identity map on M. This condition means precisely that for each p in M, s maps p into the

fiber E_p above p. Pictorially we visualize a section as a cross-section of the bundle (Figure 12.3). We say that a section is smooth if it is smooth as a map from M to E.

Fig. 12.3. A section of a vector bundle.

Definition 12.7. A vector field X on a manifold M is a function that assigns a tangent vector $X_p \in T_pM$ to each point $p \in M$. In terms of the tangent bundle, a vector field on M is simply a section of the tangent bundle $\pi : TM \to M$ and the vector field is smooth if it is smooth as a map from M to TM.

Example 12.8. The formula

$$X_{(x,y)} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \begin{bmatrix} -y \\ x \end{bmatrix}$$

defines a smooth vector field on $\mathbb{R}^2$ (Figure 12.4, cf. Example 2.3).

Fig. 12.4. The vector field $(-y, x)$ in $\mathbb{R
^{2}$.}

Proposition 12.9. Let s and t be C^∞ sections of a C^∞ vector bundle $\pi : E \to M$ and let f be a C^∞ real-valued function on M. Then
(i) the sum $s + t : M \to E$ defined by

$$(s + t)(p) = s(p) + t(p) \in E_p, \quad p \in M,$$

is a C^∞ section of E.
(ii) the product fs : $M \to E$ defined by

$$(fs)(p) = f(p)s(p) \in E_p, \quad p \in M,$$

is a C^∞ section of E.

Proof.

(i) It is clear that $s + t$ is a section of E. To show that it is C^∞, fix a point $p \in M$ and let V be a trivializing open set for E containing p, with C^∞ trivialization

$$\phi : \pi^{-1}(V) \to V \times \mathbb{R}^r.$$

Suppose

$$(\phi \circ s)(q) = \big(q, a^1(q), \ldots, a^r(q)\big)$$

and

$$(\phi \circ t)(q) = \big(q, b^1(q), \ldots, b^r(q)\big)$$

for $q \in V$. Because s and t are C^∞ maps, a^i and b^i are C^∞ functions on V (Proposition 6.16). Since ϕ is linear on each fiber,

$$(\phi \circ (s + t))(q) = \big(q, a^1(q) + b^1(q), \ldots, a^r(q) + b^r(q)\big), \quad q \in V.$$

This proves that $s + t$ is a C^∞ map on V and hence at p. Since p is an arbitrary point of M, the section $s + t$ is C^∞ on M.
(ii) We omit the proof, since it is similar to that of (i). □

Denote the set of all C^∞ sections of E by $\Gamma(E)$. The proposition shows that $\Gamma(E)$ is not only a vector space over $\mathbb{R}$, but also a module over the ring $C^\infty(M)$ of C^∞ functions on M. For any open subset $U \subset M$, one can also consider the vector space $\Gamma(U, E)$ of C^∞ sections of E over U. Then $\Gamma(U, E)$ is both a vector space over $\mathbb{R}$ and a $C^\infty(U)$-module. Note that $\Gamma(M, E) = \Gamma(E)$. To contrast with sections over a proper subset U, a section over the entire manifold M is called a global section.

8.5.5 12.5 Smooth Frames

A frame for a vector bundle $\pi : E \to M$ over an open set U is a collection of sections $s_1, \ldots, s_r$ of E over U such that at each point $p \in U$, the elements $s_1(p), \ldots, s_r(p)$ form a basis for the fiber $E_p := \pi^{-1}(p)$. A frame $s_1, \ldots, s_r$ is said to be smooth or C^∞
if $s_1, \ldots, s_r$ are C^∞ as sections of E over U. A frame for the tangent bundle $TM \to M$ over an open set U is simply called a frame on U.
Example. The collection of vector fields $\partial/\partial x, \partial/\partial y, \partial/\partial z$ is a smooth frame on $\mathbb{R}^3$.

Example. Let M be a manifold and $e_1, \ldots, e_r$ the standard basis for $\mathbb{R}^n$. Define $\bar{e}_i : M \to M \times \mathbb{R}^r$ by $\bar{e}_i(p) = (p, e_i)$. Then $\bar{e}_1, \ldots, \bar{e}_r$ is a C^∞ frame for the product bundle $M \times \mathbb{R}^r \to M$.

Example 12.10 (The frame of a trivialization). Let $\pi : E \to M$ be a smooth vector bundle of rank r. If $\phi : E|_U \xrightarrow{\sim} U \times \mathbb{R}^r$ is a trivialization of E over an open set U, then ϕ^{-1} carries the C^∞ frame $\bar{e}_1, \ldots, \bar{e}_r$ of the product bundle $U \times \mathbb{R}^r$ to a C^∞ frame $t_1, \ldots, t_r$ for E over U :

$$t_i(p) = \phi^{-1}\left(\bar{e}_i(p)\right) = \phi^{-1}\left(p, e_i\right), \quad p \in U$$

We call $t_1, \ldots, t_r$ the C^∞ frame over U of the trivialization ϕ.

Lemma 12.11. Let $\phi : E|_U \to U \times \mathbb{R}^r$ be a trivialization over an open set U of a C^∞ vector bundle $E \to M$, and $t_1, \ldots, t_r$ the C^∞ frame over U of the trivialization. Then a section $s = \sum b^i t_i$ of E over U is C^∞ if and only if its coefficients b^i relative to the frame $t_1, \ldots, t_r$ are C^∞.

Proof.
($\Leftarrow$) This direction is an immediate consequence of Proposition 12.9.
($\Rightarrow$) Suppose the section $s = \sum b^i t_i$ of E over U is C^∞. Then $\phi \circ s$ is C^∞. Note that

$$(\phi \circ s)(p) = \sum b^i(p) \phi\left(t_i(p)\right) = \sum b^i(p)\left(p, e_i\right) = \left(p, \sum b^i(p) e_i\right)$$

Thus, the $b^i(p)$ are simply the fiber coordinates of $s(p)$ relative to the trivialization ϕ. Since $\phi \circ s$ is C^∞, all the b^i are C^∞.

Proposition 12.12 (Characterization of C^∞ sections). Let $\pi : E \to M$ be a C^∞ vector bundle and U an open subset of M. Suppose $s_1, \ldots, s_r$ is a C^∞ frame for E over U. Then a section $s = \sum c^j s_j$ of E over U is C^∞ if and only if the coefficients c^j are C^∞ functions on U.

Proof. If $s_1, \ldots, s_r$ is the frame of a trivialization of E over U, then the proposition is Lemma 12.11. We prove the proposition in general by reducing it to this case. One direction is quite easy. If the c^j 's are C^∞ functions on U, then $s = \sum c^j s_j$ is a C^∞ section on U by Proposition 12.9.

Conversely, suppose $s = \sum c^j s_j$ is a C^∞ section of E over U. Fix a point $p \in U$ and choose a trivializing open set $V \subset U$ for E containing p, with C^∞ trivialization $\phi : \pi^{-1}(V) \to V \times \mathbb{R}^r$. Let $t_1, \ldots, t_r$ be the C^∞ frame of the trivialization ϕ (Example 12.10). If we write s and s_j in terms of the frame $t_1, \ldots, t_r$, say $s = \sum b^i t_i$ and $s_j = \sum a_j^i t_i$, the coefficients b^i, a_j^i will all be C^∞ functions on V by Lemma 12.11. Next, express $s = \sum c^j s_j$ in terms of the t_i 's:

$$\sum b^i t_i = s = \sum c^j s_j = \sum_{i,j} c^j a_j^i t_i$$

Comparing the coefficients of t_i gives $b^i = \sum_j c^j a_j^i$. In matrix notation,

$$b = \begin{bmatrix} b^1 \\ \vdots \\ b^r \end{bmatrix} = A \begin{bmatrix} c^1 \\ \vdots \\ c^r \end{bmatrix} = Ac.$$

At each point of V, being the transition matrix between two bases, the matrix A is invertible. By Cramer's rule, A^{-1} is a matrix of C^∞ functions on V (see Example 6.21). Hence, $c = A^{-1}b$ is a column vector of C^∞ functions on V. This proves that $c^1, \ldots, c^r$ are C^∞ functions at $p \in U$. Since p is an arbitrary point of U, the coefficients c^j are C^∞ functions on U.

Remark 12.13. If one replaces "smooth" by "continuous" throughout, the discussion in this subsection remains valid in the continuous category.

8.5.6 Problems

12.1.* Hausdorff Condition on the Tangent Bundle

Prove Proposition 12.4.

12.2. Transition Functions for the Total Space of the Tangent Bundle

Let $(U, \boldsymbol{\phi}) = (U, x^1, \ldots, x^n)$ and $(V, \boldsymbol{\psi}) = (V, y^1, \ldots, y^n)$ be overlapping coordinate charts on a manifold M. They induce coordinate charts $(TU, \tilde{\phi})$ and $(TV, \tilde{\psi})$ on the total space TM of the tangent bundle (see equation (12.1)), with transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$:

$$\left(x^1, \ldots, x^n, a^1, \ldots, a^n\right) \mapsto \left(y^1, \ldots, y^n, b^1, \ldots, b^n\right).$$

(a) Compute the Jacobian matrix of the transition function $\tilde{\psi} \circ \tilde{\phi}^{-1}$ at $\phi(p)$.
(b) Show that the Jacobian determinant of the transition function $\tilde{\Psi} \circ \tilde{\phi}^{-1}$ at $\phi(p)$ is $(\det[\partial y^i / \partial x^j])^2$.

12.3. Smoothness of Scalar Multiplication

Prove Proposition 12.9(ii).

12.4. Coefficients Relative to a Smooth Frame

Let $\pi : E \to M$ be a C^∞ vector bundle and $s_1, \ldots, s_r$ a C^∞ frame for E over an open set U in M. Then every $e \in \pi^{-1}(U)$ can be written uniquely as a linear combination

$$e = \sum_{j=1}^{r} c^j(e) s_j(p), \quad p = \pi(e) \in U$$

Prove that $c^j : \pi^{-1}U \to \mathbb{R}$ is C^∞ for $j = 1, \ldots, r$. (Hint: First show that the coefficients of e relative to the frame $t_1, \ldots, t_r$ of a trivialization are C^∞.)

8.6 §13 Bump Functions and Partitions of Unity

A partition of unity on a manifold is a collection of nonnegative functions that sum to 1 . Usually one demands in addition that the partition of unity be subordinate to an open cover $\{U_\alpha\}_{\alpha \in \mathrm{A}}$. What this means is that the partition of unity $\{\rho_\alpha\}_{\alpha \in \mathrm{A}}$ is indexed by the same set as the open over $\{U_\alpha\}_{\alpha \in \mathrm{A}}$ and for each α in the index A, the support of ρ_α is contained in U_α. In particular, ρ_α vanishes outside U_α.

The existence of a C^∞ partition of unity is one of the most important technical tools in the theory of C^∞ manifolds. It is the single feature that makes the behavior of C^∞ manifolds so different from that of real-analytic or complex manifolds. In this section we construct C^∞ bump functions on any manifold and prove the existence of a C^∞ partition of unity on a compact manifold. The proof of the existence of a C^∞ partition of unity on a general manifold is more technical and is postponed to Appendix C.

A partition of unity is used in two ways: (1) to decompose a global object on a manifold into a locally finite sum of local objects on the open sets U_α of an open cover, and (2) to patch together local objects on the open sets U_α into a global object on the manifold. Thus, a partition of unity serves as a bridge between global and local analysis on a manifold. This is useful because while there are always local coordinates on a manifold, there may be no global coordinates. In subsequent sections we will see examples of both uses of a C^∞ partition of unity.

8.6.1 13.1 c^∞ Bump Functions

Recall that $\mathbb{R}^\times$denotes the set of nonzero real numbers. The support of a real-valued function f on a manifold M is defined to be the closure in M of the subset on which $f \neq 0$:

$$\operatorname{supp} f = \operatorname{cl}_M \left(f^{-1}\left(\mathbb{R}^\times\right)\right) = \text{closure of } \{q \in M \mid f(q) \neq 0\} \text{ in } M.^1$$

Let q be a point in M, and U a neighborhood of q. By a bump function at q supported in U we mean any continuous nonnegative function ρ on M that is 1 in a neighborhood of q with $\operatorname{supp} \rho \subset U$.

For example, Figure 13.1 is the graph of a bump function at 0 supported in the open interval $]-2, 2[$. The function is nonzero on the open interval $]-1, 1[$ and is zero otherwise. Its support is the closed interval $[-1, 1]$.

Example. The support of the function $f :]-1, 1[\to \mathbb{R}$, $f(x) = \tan(\pi x/2)$, is the open interval $]-1, 1[$, not the closed interval $[-1, 1]$, because the closure of $f^{-1}(\mathbb{R}^\times)$ is taken in the domain $]-1, 1[$, not in $\mathbb{R}$.

Fig. 13.1. A bump function at 0 on $\mathbb{R
$.}

The only bump functions of interest to us are C^∞ bump functions. While the continuity of a function can often be seen by inspection, the smoothness of a function always requires a formula. Our goal in this subsection is to find a formula for a C^∞ bump function as in Figure 13.1.

Example. The graph of $y = x^{5/3}$ looks perfectly smooth (Figure 13.2), but it is in fact not smooth at $x = 0$, since its second derivative $y'' = (10/9)x^{-1/3}$ is not defined there.

Fig. 13.2. The graph of $y=x^{5 / 3
$.}

In Example 1.3 we introduced the C^∞ function

[01] In this section a general point is often denoted by q, instead of p, because p resembles too much ρ, the notation for a bump function.

$$f(t) = \begin{cases} e^{-1/t} & \text{for } t > 0, \\ 0 & \text{for } t \leq 0, \end{cases}$$

with graph as in Figure 13.3.

Fig. 13.3. The graph of $f(t)$.

The main challenge in building a smooth bump function from f is to construct a smooth version of a step function, that is, a C^∞ function $g : \mathbb{R} \to \mathbb{R}$ with graph as in Figure 13.4. Once we have such a C^∞ step function g, it is then simply a matter of

$$g(t) = \begin{cases} 0 & \text{for } t \leq 0, \\ 1 & \text{for } t \geq 1. \end{cases}$$

Fig. 13.4. The graph of $g(t)$.

translating, reflecting, and scaling the function in order to make its graph look like Figure 13.1.

We seek $g(t)$ by dividing $f(t)$ by a positive function $\ell(t)$, for the quotient $f(t)/\ell(t)$ will then be zero for $t \leq 0$. The denominator $\ell(t)$ should be a positive function that agrees with $f(t)$ for $t \geq 1$, for then $f(t)/\ell(t)$ will be identically 1 for $t \geq 1$. The simplest way to construct such an $\ell(t)$ is to add to $f(t)$ a nonnegative function that vanishes for $t \geq 1$. One such nonnegative function is $f(1-t)$. This suggests that we take $\ell(t) = f(t) + f(1-t)$ and consider

$$g(t) = \frac{f(t)}{f(t) + f(1-t)} \tag{13.1}$$

Let us verify that the denominator $f(t) + f(1-t)$ is never zero. For $t > 0, f(t) > 0$ and therefore

$$f(t) + f(1-t) \geq f(t) > 0$$

For $t \leq 0, 1-t \geq 1$ and therefore

$$f(t) + f(1-t) \geq f(1-t) > 0.$$

In either case, $f(t) + f(1-t) \neq 0$. This proves that $g(t)$ is defined for all t. As the quotient of two C^∞ functions with denominator never zero, $g(t)$ is C^∞ for all t.

As noted above, for $t \leq 0$, the numerator $f(t)$ equals 0 , so $g(t)$ is identically zero for $t \leq 0$. For $t \geq 1$, we have $1 - t \leq 0$ and $f(1-t) = 0$, so $g(t) = f(t)/f(t)$ is identically 1 for $t \geq 1$. Thus, g is a C^∞ step function with the desired properties.

Given two positive real numbers $a < b$, we make a linear change of variables to map $[a^2, b^2]$ to $[0, 1]$:

$$x \mapsto \frac{x - a^2}{b^2 - a^2}.$$

Let

$$h(x) = g\left(\frac{x - a^2}{b^2 - a^2}\right)$$

Then $h : \mathbb{R} \to [0, 1]$ is a C^∞ step function such that

$$h(x) = \begin{cases} 0 & \text{for } x \leq a^2, \\ 1 & \text{for } x \geq b^2. \end{cases}$$

(See Figure 13.5.)

Fig. 13.5. The graph of $h(x)$.

Replace x by x^2 to make the function symmetric in $x : k(x) = h\left(x^2\right)$ (Figure 13.6).

Fig. 13.6. The graph of $k(x)$.

Finally, set

$$\rho(x) = 1 - k(x) = 1 - g\left(\frac{x^2 - a^2}{b^2 - a^2}\right).$$

This $\rho(x)$ is a C^∞ bump function at 0 in $\mathbb{R}$ that is identically 1 on $[-a, a]$ and has support in $[-b, b]$ (Figure 13.7). For any $q \in \mathbb{R}, \rho(x - q)$ is a C^∞ bump function at q.

Fig. 13.7. A bump function at 0 on $\mathbb{R
$.}

It is easy to extend the construction of a bump function from $\mathbb{R}$ to $\mathbb{R}^n$. To get a C^∞ bump function at $\mathbf{0}$ in $\mathbb{R}^n$ that is 1 on the closed ball $\bar{B}(\mathbf{0}, a)$ and has support in the closed ball $\bar{B}(\mathbf{0}, b)$, set

$$\sigma(x) = \rho(\|x\|) = 1 - g\left(\frac{\|x\|r^2 - a^2}{b^2 - a^2}\right). \tag{13.2}$$

As a composition of C^∞ functions, σ is C^∞. To get a C^∞ bump function at q in $\mathbb{R}^n$, take $\sigma(x - q)$.

Exercise 13.1 (Bump function supported in an open set).* Let q be a point and U any neighborhood of q in a manifold. Construct a C^∞ bump function at q supported in U.

In general, a C^∞ function on an open subset U of a manifold M cannot be extended to a C^∞ function on M; an example is the function sec (x) on the open interval $]-\pi/2, \pi/2[$ in $\mathbb{R}$. However, if we require that the global function on M agree with the given function only on some neighborhood of a point in U, then a C^∞ extension is possible.

Proposition 13.2 (C^∞ extension of a function). Suppose f is a C^∞ function defined on a neighborhood U of a point p in a manifold M. Then there is a C^∞ function $\tilde{f}$ on M that agrees with f in some possibly smaller neighborhood of p.

Fig. 13.8. Extending the domain of a function by multiplying by a bump function.

Proof. Choose a C^∞ bump function $\rho : M \to \mathbb{R}$ supported in U that is identically 1 in a neighborhood V of p (Figure 13.8). Define

$$\tilde{f}(q) = \begin{cases} \rho(q)f(q) & \text{for } q \text{ in } U, \\ 0 & \text{for } q \text{ not in } U. \end{cases}$$

As the product of two C^∞ functions on U, $\tilde{f}$ is C^∞ on U. If $q \notin U$, then $q \notin \operatorname{supp} \rho$, and so there is an open set containing q on which $\tilde{f}$ is 0 , since $\operatorname{supp} \rho$ is closed. Therefore, $\tilde{f}$ is also C^∞ at every point $q \notin U$.

Finally, since $\rho \equiv 1$ on V, the function $\tilde{f}$ agrees with f on V. $\square$

8.6.2 13.2 Partitions of Unity

If $\{U_i\}_{i \in I}$ is a finite open cover of M, a C^∞ partition of unity subordinate to $\{U_i\}_{i \in I}$ is a collection of nonnegative C^∞ functions $\{\rho_i : M \to \mathbb{R}\}_{i \in I}$ such that $\operatorname{supp} \rho_i \subset U_i$ and

$$\sum \rho_i = 1. \tag{13.3}$$

When I is an infinite set, for the sum in (13.3) to make sense, we will impose a local finiteness condition. A collection $\{A_\alpha\}$ of subsets of a topological space S is said to be locally finite if every point q in S has a neighborhood that meets only finitely many of the sets A_α. In particular, every q in S is contained in only finitely many of the A_α 's.

Example 13.3 (An open cover that is not locally finite). Let $U_{r,n}$ be the open interval $]r-\frac{1}{n}, r+\frac{1}{n}[$ on the real line $\mathbb{R}$. The open cover $\{U_{r,n} \mid r \in \mathbb{Q}, n \in \mathbb{Z}^+\}$of $\mathbb{R}$ is not locally finite.

Definition 13.4. A C^∞ partition of unity on a manifold is a collection of nonnegative C^∞ functions $\{\rho_\alpha : M \to \mathbb{R}\}_{\alpha\in \mathrm{A}}$ such that
(i) the collection of supports, $\{\operatorname{supp} \rho_\alpha\}_{\alpha\in \mathrm{A}}$, is locally finite,
(ii) $\sum \rho_\alpha = 1$.

Given an open cover $\{U_\alpha\}_{\alpha\in \mathrm{A}}$ of M, we say that a partition of unity $\{\rho_\alpha\}_{\alpha\in \mathrm{A}}$ is subordinate to the open cover $\{U_\alpha\}$ if $\operatorname{supp} \rho_\alpha \subset U_\alpha$ for every $\alpha \in \mathrm{A}$.

Since the collection of supports, $\{\operatorname{supp} \rho_\alpha\}_{\alpha\in \mathrm{A}}$, is locally finite (condition (i)), every point q lies in only finitely many of the sets $\operatorname{supp} \rho_\alpha$. Hence $\rho_\alpha(q) \neq 0$ for only finitely many α. It follows that the sum in (ii) is a finite sum at every point.

Example. Let U and V be the open intervals $]-\infty, 2[$ and $]-1, \infty[$ in $\mathbb{R}$ respectively, and let ρ_V be a C^∞ function with graph as in Figure 13.9, for example the function $g(t)$ in (13.1). Define $\rho_U = 1 - \rho_V$. Then $\operatorname{supp} \rho_V \subset V$ and $\operatorname{supp} \rho_U \subset U$. Thus, $\{\rho_U, \rho_V\}$ is a partition of unity subordinate to the open cover $\{U, V\}$.

Fig. 13.9. A partition of unity $\left\{\rho_{U}, \rho_{V}\right\}$ subordinate to an open cover $\{U, V\}$.}

Remark. Suppose $\{f_\alpha\}_{\alpha\in \mathrm{A}}$ is a collection of C^∞ functions on a manifold M such that the collection of its supports, $\{\operatorname{supp} f_\alpha\}_{\alpha\in \mathrm{A}}$, is locally finite. Then every point q in M has a neighborhood W_q that intersects $\operatorname{supp} f_\alpha$ for only finitely many α. Thus, on W_q the sum $\sum_{\alpha\in \mathrm{A}} f_\alpha$ is actually a finite sum. This shows that the function $f = \sum f_\alpha$ is well defined and C^∞ on the manifold M. We call such a sum a locally finite sum.

8.6.3 13.3 Existence of a Partition of Unity

In this subsection we begin a proof of the existence of a C^∞ partition of unity on a manifold. Because the case of a compact manifold is somewhat easier and already has some of the features of the general case, for pedagogical reasons we give a separate proof for the compact case.

Lemma 13.5. If $\rho_1, \ldots, \rho_m$ are real-valued functions on a manifold M, then

$$\operatorname{supp}\left(\sum \rho_i\right) \subset \bigcup \operatorname{supp} \rho_i$$

Proof. Problem 13.1.

Proposition 13.6. Let M be a compact manifold and $\{U_\alpha\}_{\alpha\in\mathrm{A}}$ an open cover of M. There exists a C^∞ partition of unity $\{\rho_\alpha\}_{\alpha\in\mathrm{A}}$ subordinate to $\{U_\alpha\}_{\alpha\in\mathrm{A}}$.

Proof. For each $q \in M$, find an open set U_α containing q from the given cover and let ψ_q be a C^∞ bump function at q supported in U_α (Exercise 13.1, p. 144). Because $\psi_q(q) > 0$, there is a neighborhood W_q of q on which $\psi_q > 0$. By the compactness of M, the open cover $\{W_q \mid q \in M\}$ has a finite subcover, say $\{W_{q_1}, \dots, W_{q_m}\}$. Let $\psi_{q_1}, \dots, \psi_{q_m}$ be the corresponding bump functions. Then $\psi := \sum \psi_{q_i}$ is positive at every point q in M because $q \in W_{q_i}$ for some i. Define

$$\varphi_i = \frac{\psi_{q_i}}{\psi}, \quad i = 1, \dots, m$$

Clearly, $\sum \varphi_i = 1$. Moreover, since $\psi > 0, \varphi_i(q) \neq 0$ if and only if $\psi_{q_i}(q) \neq 0$, so

$$\operatorname{supp} \varphi_i = \operatorname{supp} \psi_{q_i} \subset U_\alpha$$

for some $\alpha \in \mathrm{A}$. This shows that $\{\varphi_i\}$ is a partition of unity such that for every i, $\operatorname{supp} \varphi_i \subset U_\alpha$ for some $\alpha \in \mathrm{A}$.

The next step is to make the index set of the partition of unity the same as that of the open cover. For each $i = 1, \dots, m$, choose $\tau(i) \in \mathrm{A}$ to be an index such that

$$\operatorname{supp} \varphi_i \subset U_{\tau(i)}$$

We group the collection of function $\{\varphi_i\}$ into subcollections according to $\tau(i)$ and define for each $\alpha \in \mathrm{A}$,

$$\rho_\alpha = \sum_{\tau(i)=\alpha} \varphi_i;$$

if there is no i for which $\tau(i) = \alpha$, the sum above is empty and we define $\rho_\alpha = 0$. Then

$$\sum_{\alpha\in\mathrm{A}} \rho_\alpha = \sum_{\alpha\in\mathrm{A}} \sum_{\tau(i)=\alpha} \varphi_i = \sum_{i=1}^{m} \varphi_i = 1.$$

Moreover, by Lemma 13.5,

$$\operatorname{supp} \rho_\alpha \subset \bigcup_{\tau(i)=\alpha} \operatorname{supp} \varphi_i \subset U_\alpha$$

So $\{\rho_\alpha\}$ is a partition of unity subordinate to $\{U_\alpha\}$.

To generalize the proof of Proposition 13.6 to an arbitrary manifold, it will be necessary to find an appropriate substitute for compactness. Since the proof is rather technical and is not necessary for the rest of the book, we put it in Appendix C. The statement is as follows.

Theorem 13.7 (Existence of a C^∞ partition of unity). Let $\{U_\alpha\}_{\alpha\in\mathrm{A}}$ be an open cover of a manifold M.

(i) There is a C^∞ partition of unity $\{\varphi_k\}_{k=1}^\infty$ with every φ_k having compact support such that for each $k, \operatorname{supp} \varphi_k \subset U_\alpha$ for some $\alpha \in \mathrm{A}$.

(ii) If we do not require compact support, then there is a C^∞ partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$.

8.6.4 Problems

13.1* Support of a Finite sum

Prove Lemma 13.5.

13.2.* Locally Finite Family and Compact set

Let $\{A_\alpha\}$ be a locally finite family of subsets of a topological space S. Show that every compact set K in S has a neighborhood W that intersects only finitely many of the A_α.

13.3. Smooth Urysohn Lemma

(a) Let A and B be two disjoint closed sets in a manifold M. Find a C^∞ function f on M such that f is identically 1 on A and identically 0 on B. (Hint: Consider a C^∞ partition of unity $\{\rho_{M-A}, \rho_{M-B}\}$ subordinate to the open cover $\{M-A, M-B\}$. This lemma is needed in Subsection 29.3.)
(b) Let A be a closed subset and U an open subset of a manifold M. Show that there is a C^∞ function f on M such that f is identically 1 on A and $\operatorname{supp} f \subset U$.

13.4. Support of the Pullback of a Function

Let $F : N \to M$ be a C^∞ map of manifolds and $h : M \to \mathbb{R}$ a C^∞ real-valued function. Prove that $\operatorname{supp} F^*h \subset F^{-1}(\operatorname{supp} h)$. (Hint: First show that $(F^*h)^{-1}(\mathbb{R}^\times) \subset F^{-1}(\operatorname{supp} h)$.)

13.5* Support of the Pullback by a Projection

Let $f : M \to \mathbb{R}$ be a C^∞ function on a manifold M. If N is another manifold and $\pi : M \times N \to M$ is the projection onto the first factor, prove that

$$\operatorname{supp}(\pi^* f) = (\operatorname{supp} f) \times N$$

13.6. Pullback of a Partition of Unity

Suppose $\{\rho_\alpha\}$ is a partition of unity on a manifold M subordinate to an open cover $\{U_\alpha\}$ of M and $F : N \to M$ is a C^∞ map. Prove that
(a) the collection of supports $\{\operatorname{supp} F^*\rho_\alpha\}$ is locally finite;
(b) the collection of functions $\{F^*\rho_\alpha\}$ is a partition of unity on N subordinate to the open cover $\{F^{-1}(U_\alpha)\}$ of N.

13.7* Closure of a Locally Finite Union

If $\{A_\alpha\}$ is a locally finite collection of subsets in a topological space, then

$$\overline{\bigcup A_\alpha} = \bigcup \overline{A_\alpha} \tag{13.4}$$

where $\bar{A}$ denotes the closure of the subset A.
Remark. For any collection of subsets A_α, one always has

$$\bigcup \overline{A_\alpha} \subset \overline{\bigcup A_\alpha}$$

However, the reverse inclusion is in general not true. For example, suppose A_n is the closed interval $[0, 1-(1/n)]$ in $\mathbb{R}$. Then

$$\overline{\bigcup_{n=1}^{\infty} A_n} = \overline{[0,1)} = [0,1]$$

but

$$\bigcup_{n=1}^{\infty} \overline{A_n} = \bigcup_{n=1}^{\infty} \left[0, 1 - \frac{1}{n}\right] = [0,1)$$

If $\{A_\alpha\}$ is a finite collection, the equality (13.4) is easily shown to be true.

8.7 §14 Vector Fields

A vector field X on a manifold M is the assignment of a tangent vector $X_p \in T_pM$ to each point $p \in M$. More formally, a vector field on M is a section of the tangent bundle TM of M. It is natural to define a vector field as smooth if it is smooth as a section of the tangent bundle. In the first subsection we give two other characterizations of smooth vector fields, in terms of the coefficients relative to coordinate vector fields and in terms of smooth functions on the manifold.

Vector fields abound in nature, for example the velocity vector field of a fluid flow, the electric field of a charge, the gravitational field of a mass, and so on. The fluid flow model is in fact quite general, for as we will see shortly, every smooth vector field may be viewed locally as the velocity vector field of a fluid flow. The path traced out by a point under this flow is called an integral curve of the vector field. Integral curves are curves whose velocity vector field is the restriction of the given vector field to the curve. Finding the equation of an integral curve is equivalent to solving a system of first-order ordinary differential equations (ODE). Thus, the theory of ODE guarantees the existence of integral curves.

The set $\mathfrak{X}(M)$ of all C^∞ vector fields on a manifold M clearly has the structure of a vector space. We introduce a bracket operation $[,]thatmakesitintoaLiealgebra$. Because vector fields do not push forward under smooth maps, the Lie algebra $\mathfrak{X}(M)$ does not give rise to a functor on the category of smooth manifolds. Nonetheless, there is a notion of related vector fields that allows us to compare vector fields on two manifolds under a smooth map.

8.7.1 14.1 Smoothness of a Vector Field

In Definition 12.7 we defined a vector field X on a manifold M to be smooth if the map $X : M \to TM$ is smooth as a section of the tangent bundle $\pi : TM \to M$. In a coordinate chart $(U, \phi) = (U, x^1, \ldots, x^n)$ on M, the value of the vector field X at $p \in U$ is a linear combination

$$X_p = \sum a^i(p) \frac{\partial}{\partial x^i}\bigg|_p .$$

As p varies in U, the coefficients a^i become functions on U.
As we learned in Subsections 12.1 and 12.2, the chart $(U, \phi) = (U, x^1, \ldots, x^n)$ on the manifold M induces a chart

$$(TU, \tilde{\phi}) = \left(TU, \bar{x}^1, \ldots, \bar{x}^n, c^1, \ldots, c^n\right)$$

on the tangent bundle TM, where $\bar{x}^i = \pi^* x^i = x^i \circ \pi$ and the c^i are defined by

$$v = \sum c^i(v) \frac{\partial}{\partial x^i}\bigg|_p , \quad v \in T_pM.$$

Comparing coefficients in

$$X_p = \sum a^i(p) \frac{\partial}{\partial x^i}\bigg|_p = \sum c^i\left(X_p\right) \frac{\partial}{\partial x^i}\bigg|_p, \quad p \in U,$$

we get $a^i = c^i \circ X$ as functions on U. Being coordinates, the c^i are smooth functions on TU. Thus, if X is smooth and $(U, x^1, \ldots, x^n)$ is any chart on M, then the coefficients a^i of $X = \sum a^i \partial/\partial x^i$ relative to the frame $\partial/\partial x^i$ are smooth on U.

The converse is also true, as indicated in the following lemma.

Lemma 14.1 (Smoothness of a vector field on a chart). Let $(U, \boldsymbol{\phi}) = (U, x^1, \ldots, x^n)$ be a chart on a manifold M. A vector field $X = \sum a^i \partial/\partial x^i$ on U is smooth if and only if the coefficient functions a^i are all smooth on U.

Proof. This lemma is a special case of Proposition 12.12, with E the tangent bundle of M and s_i the coordinate vector field $\partial/\partial x^i$.

Because we have an explicit description of the manifold structure on the tangent bundle TM, a direct proof of the lemma is also possible. Since $\tilde{\phi} : TU \xrightarrow{\sim} U \times \mathbb{R}^n$ is a diffeomorphism, $X : U \to TU$ is smooth if and only if $\tilde{\phi} \circ X : U \to U \times \mathbb{R}^n$ is smooth. For $p \in U$,

$$\begin{aligned}(\tilde{\phi} \circ X)(p) = \tilde{\phi}\left(X_p\right) &= \left(x^1(p), \ldots, x^n(p), c^1\left(X_p\right), \ldots, c^n\left(X_p\right)\right) \\ &= \left(x^1(p), \ldots, x^n(p), a^1(p), \ldots, a^n(p)\right)\end{aligned}$$

As coordinate functions, $x^1, \ldots, x^n$ are C^∞ on U. Therefore, by Proposition 6.13, $\tilde{\phi} \circ X$ is smooth if and only if all the a^i are smooth on U.

This lemma leads to a characterization of the smoothness of a vector field on a manifold in terms of the coefficients of the vector field relative to coordinate frames.

Proposition 14.2 (Smoothness of a vector field in terms of coefficients). Let X be a vector field on a manifold M. The following are equivalent:

(i) The vector field X is smooth on M.

(ii) The manifold M has an atlas such that on any chart $(U, \phi) = (U, x^1, \ldots, x^n)$ of the atlas, the coefficients a^i of $X = \sum a^i \partial/\partial x^i$ relative to the frame $\partial/\partial x^i$ are all smooth.

(iii) On any chart $(U, \phi) = (U, x^1, \ldots, x^n)$ on the manifold M, the coefficients a^i of $X = \sum a^i \partial/\partial x^i$ relative to the frame $\partial/\partial x^i$ are all smooth.

Proof. (ii) $\Rightarrow$ (i): Assume (ii). By the preceding lemma, X is smooth on every chart (U, ϕ) of an atlas of M. Thus, X is smooth on M.

(i) $\Rightarrow$ (iii): A smooth vector field on M is smooth on every chart (U, ϕ) on M. The preceding lemma then implies (iii).

(iii) $\Rightarrow$ (ii): Obvious.

Just as in Subsection 2.5, a vector field X on a manifold M induces a linear map on the algebra $C^\infty(M)$ of C^∞ functions on M : for $f \in C^\infty(M)$, define Xf to be the function

$$(Xf)(p) = X_p f, \quad p \in M.$$

In terms of its action as an operator on C^∞ functions, there is still another characterization of a smooth vector field.

Proposition 14.3 (Smoothness of a vector field in terms of functions). A vector field X on M is smooth if and only if for every smooth function f on M, the function Xf is smooth on M.

Proof.

($\Rightarrow$) Suppose X is smooth and $f \in C^\infty(M)$. By Proposition 14.2, on any chart $(U, x^1, \ldots, x^n)$ on M, the coefficients a^i of the vector field $X = \sum a^i \partial/\partial x^i$ are C^∞. It follows

that $Xf = \sum a^i \partial f/\partial x^i$ is C^∞ on U. Since M can be covered by charts, Xf is C^∞ on M.
($\Leftarrow$) Let $(U, x^1, \ldots, x^n)$ be any chart on M. Suppose $X = \sum a^i \partial/\partial x^i$ on U and $p \in U$. By Proposition 13.2, for $k = 1, \ldots, n$, each x^k can be extended to a C^∞ function $\tilde{x}^k$ on M that agrees with x^k in a neighborhood V of p in U. Therefore, on V,

$$X\tilde{x}^k = \left(\sum a^i \frac{\partial}{\partial x^i}\right)\tilde{x}^k = \left(\sum a^i \frac{\partial}{\partial x^i}\right)x^k = a^k.$$

This proves that a^k is C^∞ at p. Since p is an arbitrary point in U, the function a^k is C^∞ on U. By the smoothness criterion of Proposition 14.2, X is smooth.

In this proof it is necessary to extend x^k to a C^∞ global function $\tilde{x}^k$ on M, for while it is true that $Xx^k = a^k$, the coordinate function x^k is defined only on U, not on M, and so the smoothness hypothesis on Xf does not apply to Xx^k. $\square$

By Proposition 14.3, we may view a C^∞ vector field X as a linear operator $X : C^\infty(M) \to C^\infty(M)$ on the algebra of C^∞ functions on M. As in Proposition 2.6, this linear operator $X : C^\infty(M) \to C^\infty(M)$ is a derivation: for all $f, g \in C^\infty(M)$,

$$X(fg) = (Xf)g + f(Xg).$$

In the following we think of C^∞ vector fields on M alternately as C^∞ sections of the tangent bundle TM and as derivations on the algebra $C^\infty(M)$ of C^∞ functions. In fact, it can be shown that these two descriptions of C^∞ vector fields are equivalent (Problem 19.12).

Proposition 13.2 on C^∞ extensions of functions has an analogue for vector fields.
Proposition 14.4 (C^∞ extension of a vector field). Suppose X is a C^∞ vector field defined on a neighborhood U of a point p in a manifold M. Then there is a C^∞ vector field $\tilde{X}$ on M that agrees with X on some possibly smaller neighborhood of p.

Proof. Choose a C^∞ bump function $\rho : M \to \mathbb{R}$ supported in U that is identically 1 in a neighborhood V of p (Figure 13.8). Define

$$\tilde{X}(q) = \begin{cases} \rho(q)X_q & \text{for } q \text{ in } U \\ 0 & \text{for } q \text{ not in } U \end{cases}$$

The rest of the proof is the same as in Proposition 13.2.

8.7.2 14.2 Integral Curves

In Example 12.8, it appears that through each point in the plane one can draw a circle whose velocity at any point is the given vector field at that point. Such a curve is an example of an integral curve of the vector field, which we now define.

Definition 14.5. Let X be a C^∞ vector field on a manifold M, and $p \in M$. An integral curve of X is a smooth curve $c :]a, b[\to M$ such that $c'(t) = X_{c(t)}$ for all $t \in]a, b[$. Usually we assume that the open interval $]a, b[$ contains 0 . In this case, if $c(0) = p$, then we say that c is an integral curve starting at p and call p the initial point of c. To show the dependence of such an integral curve on the initial point p, we also write $c_t(p)$ instead of $c(t)$.

Definition 14.6. An integral curve is maximal if its domain cannot be extended to a larger interval.

Example. Recall the vector field $X_{(x,y)} = \langle -y, x\rangle$ on $\mathbb{R}^2$ (Figure 12.4). We will find an integral curve $c(t)$ of X starting at the point $(1, 0) \in \mathbb{R}^2$. The condition for $c(t) = (x(t), y(t))$ to be an integral curve is $c'(t) = X_{c(t)}$, or

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} -y(t) \\ x(t) \end{bmatrix},$$

so we need to solve the system of first-order ordinary differential equations

$$\dot{x} = -y, \tag{14.1}$$
$$\dot{y} = x, \tag{14.2}$$

with initial condition $(x(0), y(0)) = (1, 0)$. From (14.1), $y = -\dot{x}$, so $\dot{y} = -\ddot{x}$. Substituting into (14.2) gives

$$\ddot{x} = -x.$$

It is well known that the general solution to this equation is

$$x = A\cos t + B\sin t. \tag{14.3}$$

Hence,

$$y = -\dot{x} = A\sin t - B\cos t. \tag{14.4}$$

The initial condition forces $A = 1, B = 0$, so the integral curve starting at (1,0) is $c(t) = (\cos t, \sin t)$, which parametrizes the unit circle.

More generally, if the initial point of the integral curve, corresponding to $t = 0$, is $p = (x_0, y_0)$, then (14.3) and (14.4) give

$$A = x_0, \quad B = -y_0,$$

and the general solution to (14.1) and (14.2) is

$$\begin{aligned} x &= x_0\cos t - y_0\sin t \\ y &= x_0\sin t + y_0\cos t, \quad t \in \mathbb{R}. \end{aligned}$$

This can be written in matrix notation as

$$c(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ \sin t & \cos t \end{bmatrix} p,$$

which shows that the integral curve of X starting at p can be obtained by rotating the point p counterclockwise about the origin through an angle t. Notice that

$$c_s\left(c_t(p)\right) = c_{s+t}(p),$$

since a rotation through an angle t followed by a rotation through an angle s is the same as a rotation through the angle $s + t$. For each $t \in \mathbb{R}, c_t : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism with inverse c_{-t}.

Let $\mathrm{Diff}(M)$ be the group of diffeomorphisms of a manifold M with itself, the group operation being composition. A homomorphism $c : \mathbb{R} \to \mathrm{Diff}(M)$ is called a one-parameter group of diffeomorphisms of M. In this example the integral curves of the vector field $X_{(x,y)} = \langle -y, x\rangle$ on $\mathbb{R}^2$ give rise to a one-parameter group of diffeomorphisms of $\mathbb{R}^2$.

Example. Let X be the vector field $x^2 d/dx$ on the real line $\mathbb{R}$. Find the maximal integral curve of X starting at $x = 2$.

Solution. Denote the integral curve by $x(t)$. Then

$$x'(t) = X_{x(t)} \quad \Longleftrightarrow \quad \dot{x}(t)\frac{d}{dx} = x^2\frac{d}{dx},$$

where $x'(t)$ is the velocity vector of the curve $x(t)$, and $\dot{x}(t)$ is the calculus derivative of the real-valued function $x(t)$. Thus, $x(t)$ satisfies the differential equation

$$\frac{dx}{dt} = x^2, \quad x(0) = 2. \tag{14.5}$$

On can solve (14.5) by separation of variables:

$$\frac{dx}{x^2} = dt. \tag{14.6}$$

Integrating both sides of (14.6) gives

$$-\frac{1}{x} = t + C, \qquad \text{or} \qquad x = -\frac{1}{t+C},$$

for some constant C. The initial condition $x(0) = 2$ forces $C = -1/2$. Hence, $x(t) = 2/(1-2t)$. The maximal interval containing 0 on which $x(t)$ is defined is $]-\infty, 1/2[$.

From this example we see that it may not be possible to extend the domain of definition of an integral curve to the entire real line.

8.7.3 14.3 Local Flows

The two examples in the preceding section illustrate the fact that locally, finding an integral curve of a vector field amounts to solving a system of first-order ordinary differential equations with initial conditions. In general, if X is a smooth vector field on a manifold M, to find an integral curve $c(t)$ of X starting at p, we first choose a coordinate chart $(U, \phi) = (U, x^1, \ldots, x^n)$ about p. In terms of the local coordinates,

$$X_{c(t)} = \sum a^i(c(t)) \frac{\partial}{\partial x^i}\bigg|_{c(t)},$$

and by Proposition 8.15,

$$c'(t) = \sum \dot{c}^i(t) \frac{\partial}{\partial x^i}\bigg|_{c(t)}$$

where $c^i(t) = x^i \circ c(t)$ is the i th component of $c(t)$ in the chart (U, ϕ). The condition $c'(t) = X_{c(t)}$ is thus equivalent to

$$\dot{c}^i(t) = a^i(c(t)) \quad \text{for } i = 1, \ldots, n \tag{14.7}$$

This is a system of ordinary differential equations (ODE); the initial condition $c(0) = p$ translates to $(c^1(0), \ldots, c^n(0)) = (p^1, \ldots, p^n)$. By an existence and uniqueness theorem from the theory of ODE, such a system always has a unique solution in the following sense.

Theorem 14.7. Let V be an open subset of $\mathbb{R}^n, p_0$ a point in V, and $f : V \to \mathbb{R}^n$ a C^∞ function. Then the differential equation

$$dy/dt = f(y), \quad y(0) = p_0$$

has a unique C^∞ solution $y :]\, a(p_0), b(p_0)\, [\to V$, where $]a(p_0), b(p_0)[$ is the maximal open interval containing 0 on which y is defined.

The uniqueness of the solution means that if $z :]\delta, \varepsilon[\to V$ satisfies the same differential equation

$$dz/dt = f(z), \quad z(0) = p_0$$

then the domain of definition $]\delta, \varepsilon[$ of z is a subset of $]a(p_0), b(p_0)[$ and $z(t) = y(t)$ on the interval $]\delta, \varepsilon[$.

For a vector field X on a chart U of a manifold and a point $p \in U$, this theorem guarantees the existence and uniqueness of a maximal integral curve starting at p.

Next we would like to study the dependence of an integral curve on its initial point. Again we study the problem locally on $\mathbb{R}^n$. The function y will now be a function of two arguments t and q, and the condition for y to be an integral curve starting at the point q is

$$\frac{\partial y}{\partial t}(t, q) = f(y(t, q)), \quad y(0, q) = q. \tag{14.8}$$

The following theorem from the theory of ODE guarantees the smooth dependence of the solution on the initial point.

Theorem 14.8. Let V be an open subset of $\mathbb{R}^n$ and $f : V \to \mathbb{R}^n$ a C^∞ function on V. For each point $p_0 \in V$, there are a neighborhood W of p_0 in V, a number $\varepsilon > 0$, and a C^∞ function

$$y :]-\varepsilon, \varepsilon[\times W \to V$$

such that

$$\frac{\partial y}{\partial t}(t, q) = f(y(t, q)), \quad y(0, q) = q$$

for all $(t, q) \in]-\varepsilon, \varepsilon[\times W$.

For a proof of these two theorems, see [7, Appendix C, pp. 359-366].

It follows from Theorem 14.8 and (14.8) that if X is any C^∞ vector field on a chart U and $p \in U$, then there are a neighborhood W of p in U, an $\varepsilon > 0$, and a C^∞ map

$$F :]-\varepsilon, \varepsilon[\times W \to U \tag{14.9}$$

such that for each $q \in W$, the function $F(t, q)$ is an integral curve of X starting at q. In particular, $F(0, q) = q$. We usually write $F_t(q)$ for $F(t, q)$.

Fig. 14.1. The flow line through q of a local flow.

Suppose s, t in the interval $]-\varepsilon, \varepsilon[$ are such that both $F_t(F_s(q))$ and $F_{t+s}(q)$ are defined. Then both $F_t(F_s(q))$ and $F_{t+s}(q)$ as functions of t are integral curves of X with initial point $F_s(q)$, which is the point corresponding to $t = 0$. By the uniqueness of the integral curve starting at a point,

$$F_t(F_s(q)) = F_{t+s}(q). \tag{14.10}$$

The map F in (14.9) is called a local flow generated by the vector field X. For each $q \in U$, the function $F_t(q)$ of t is called a flow line of the local flow. Each flow line is an integral curve of X. If a local flow F is defined on $\mathbb{R} \times M$, then it is called a global flow. Every smooth vector field has a local flow about any point, but not necessarily a global flow. A vector field having a global flow is called a complete vector field. If F is a global flow, then for every $t \in \mathbb{R}$,

$$F_t \circ F_{-t} = F_{-t} \circ F_t = F_0 = \mathbb{1}_M,$$

so $F_t : M \to M$ is a diffeomorphism. Thus, a global flow on M gives rise to a oneparameter group of diffeomorphisms of M.

This discussion suggests the following definition.

Definition 14.9. A local flow about a point p in an open set U of a manifold is a C^∞ function

$$F :]-\varepsilon, \varepsilon[\times W \to U,$$

where ε is a positive real number and W is a neighborhood of p in U, such that writing $F_t(q) = F(t, q)$, we have

(i) $F_0(q) = q$ for all $q \in W$,

(ii) $F_t(F_s(q)) = F_{t+s}(q)$ whenever both sides are defined.

If $F(t, q)$ is a local flow of the vector field X on U, then

$$F(0, q) = q \quad \text{and} \quad \frac{\partial F}{\partial t}(0, q) = X_{F(0,q)} = X_q$$

Thus, one can recover the vector field from its flow.

Example. The function $F : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2$,

$$F\left(t, \begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix},$$

is the global flow on $\mathbb{R}^2$ generated by the vector field

$$\begin{aligned} X_{(x,y)} &= \frac{\partial F}{\partial t}(t, (x, y))\bigg|_{t=0} = \begin{bmatrix} -\sin t - \cos t \\ \cos t - \sin t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}\bigg|_{t=0} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ x \end{bmatrix} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}. \end{aligned}$$

This is Example 12.8 again.

8.7.4 14.4 The Lie Bracket

Suppose X and Y are smooth vector fields on an open subset U of a manifold M. We view X and Y as derivations on $C^\infty(U)$. For a C^∞ function f on U, by Proposition 14.3 the function Yf is C^∞ on U, and the function $(XY)f := X(Yf)$ is also C^∞ on U. Moreover, because X and Y are both $\mathbb{R}$-linear maps from $C^\infty(U)$ to $C^\infty(U)$, the map $XY : C^\infty(U) \to C^\infty(U)$ is $\mathbb{R}$-linear. However, XY does not satisfy the derivation property: if $f, g \in C^\infty(U)$, then

$$\begin{aligned} XY(fg) &= X((Yf)g + fYg) \\ &= (XYf)g + (Yf)(Xg) + (Xf)(Yg) + f(XYg) \end{aligned}$$

Looking more closely at this formula, we see that the two extra terms $(Yf)(Xg)$ and $(Xf)(Yg)$ that make XY not a derivation are symmetric in X and Y. Thus, if we compute $YX(fg)$ as well and subtract it from $XY(fg)$, the extra terms will disappear, and $XY - YX$ will be a derivation of $C^\infty(U)$.

Given two smooth vector fields X and Y on U and $p \in U$, we define their Lie bracket $[X, Y]$ at p to be

$$[X, Y]_p f = (X_p Y - Y_p X) f$$

for any germ f of a C^∞ function at p. By the same calculation as above, but now evaluated at p, it is easy to check that $[X,Y]_p$ is a derivation of $C_p^\infty(U)$ and is therefore a tangent vector at p (Definition 8.1). As p varies over $U, [X,Y]$ becomes a vector field on U.

Proposition 14.10. If X and Y are smooth vector fields on M, then the vector field $[X,Y]$ is also smooth on M.

Proof. By Proposition 14.3 it suffices to check that if f is a C^∞ function on M, then so is $[X,Y]f$. But

$$[X,Y]f = (XY - YX)f$$

which is clearly C^∞ on M, since both X and Y are.
From this proposition, we see that the Lie bracket provides a product operation on the vector space $\mathfrak{X}(M)$ of all smooth vector fields on M. Clearly,

$$[Y,X] = -[X,Y]$$

Exercise 14.11 (Jacobi identity). Check the Jacobi identity:

$$\sum_{\text{cyclic}} [X,[Y,Z]] = 0.$$

This notation means that one permutes X, Y, Z cyclically and one takes the sum of the resulting terms. Written out,

$$\sum_{\text{cyclic}} [X,[Y,Z]] = [X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]]$$

Definition 14.12. Let K be a field. A Lie algebra over K is a vector space V over K together with a product $[]:, V \times V \to V$, called the bracket, satisfying the following properties: for all $a, b \in K$ and $X, Y, Z \in V$,
(i) (bilinearity) $[aX + bY, Z] = a[X,Z] + b[Y,Z]$, $[Z, aX + bY] = a[Z,X] + b[Z,Y]$,
(ii) (anticommutativity) $[Y,X] = -[X,Y]$,
(iii) (Jacobi identity) $\sum_{\text{cyclic}} [X,[Y,Z]] = 0$.

In practice, we will be concerned only with real Lie algebras, i.e., Lie algebras over $\mathbb{R}$. Unless otherwise specified, a Lie algebra in this book means a real Lie algebra.

Example. On any vector space V, define $[X,Y] = 0$ for all $X, Y \in V$. With this bracket, V becomes a Lie algebra, called an abelian Lie algebra.

Our definition of an algebra in Subsection 2.2 requires that the product be associative. An abelian Lie algebra is trivially associative, but in general the bracket of a Lie algebra need not be associative. So despite its name, a Lie algebra is in general not an algebra.

Example. If M is a manifold, then the vector space $\mathfrak{X}(M)$ of C^∞ vector fields on M is a real Lie algebra with the Lie bracket $[,]$ as the bracket.

Example. Let $K^{n\times n}$ be the vector space of all $n \times n$ matrices over a field K. Define for $X, Y \in K^{n\times n}$,

$$[X,Y] = XY - YX,$$

where XY is the matrix product of X and Y. With this bracket, $K^{n\times n}$ becomes a Lie algebra. The bilinearity and anticommutativity of $[,]$ are immediate, while the Jacobi identity follows from the same computation as in Exercise 14.11.

More generally, if A is any algebra over a field K, then the product

$$[x,y] = xy - yx, \quad x, y \in A,$$

makes A into a Lie algebra over K.

Definition 14.13. A derivation of a Lie algebra V over a field K is a K-linear map $D : V \to V$ satisfying the product rule

$$D[Y, Z] = [DY, Z] + [Y, DZ] \quad \text{for } Y, Z \in V.$$

Example. Let V be a Lie algebra over a field K. For each X in V, define $\mathrm{ad}_X : V \to V$ by

$$\mathrm{ad}_X(Y) = [X, Y].$$

We may rewrite the Jacobi identity in the form

$$[X, [Y, Z]] = [[X, Y], Z] + [Y, [X, Z]]$$

or

$$\mathrm{ad}_X[Y, Z] = [\mathrm{ad}_X Y, Z] + [Y, \mathrm{ad}_X Z],$$

which shows that $\mathrm{ad}_X : V \to V$ is a derivation of V.

8.7.5 14.5 The Pushforward of Vector Fields

Let $F : N \to M$ be a smooth map of manifolds and let $F_* : T_pN \to T_{F(p)}M$ be its differential at a point p in N. If $X_p \in T_pN$, we call $F_*(X_p)$ the pushforward of the vector X_p at p. This notion does not extend in general to vector fields, since if X is a vector field on N and $z = F(p) = F(q)$ for two distinct points $p, q \in N$, then X_p and X_q are both pushed forward to tangent vectors at $z \in M$, but there is no reason why $F_*(X_p)$ and $F_*(X_q)$ should be equal (see Figure 14.2).

Fig. 14.2. The vector field X cannot be pushed forward under the first projection $F: \mathbb{R
^{2} \rightarrow \mathbb{R}$.}

In one important special case, the pushforward F_*X of any vector field X on N always makes sense, namely, when $F : N \to M$ is a diffeomorphism. In this case, since F is injective,

there is no ambiguity about the meaning of $(F_*X)_{F(p)} = F_{*,p}(X_p)$, and since F is surjective, F_*X is defined everywhere on M.

8.7.6 14.6 Related Vector Fields

Under a C^∞ map $F : N \to M$, although in general a vector field on N cannot be pushed forward to a vector field on M, there is nonetheless a useful notion of related vector fields, which we now define.

Definition 14.14. Let $F : N \to M$ be a smooth map of manifolds. A vector field X on N is F-related to a vector field $\bar{X}$ on M if for all $p \in N$,

$$F_{*,p}(X_p) = \bar{X}_{F(p)} \tag{14.11}$$

Example 14.15 (Pushforward by a diffeomorphism). If $F : N \to M$ is a diffeomorphism and X is a vector field on N, then the pushforward F_*X is defined. By definition, the vector field X on N is F-related to the vector field F_*X on M. In Subsection 16.5, we will see examples of vector fields related by a map F that is not a diffeomorphism.

We may reformulate condition (14.11) for F-relatedness as follows.

Proposition 14.16. Let $F : N \to M$ be a smooth map of manifolds. A vector field X on N and a vector field $\bar{X}$ on M are F-related if and only if for all $g \in C^\infty(M)$,

$$X(g \circ F) = (\bar{X}g) \circ F$$

Proof.

($\Rightarrow$) Suppose X on N and $\bar{X}$ on M are F-related. By (14.11), for any $g \in C^\infty(M)$ and $p \in N$,

$$\begin{aligned} F_{*,p}(X_p)\, g &= \bar{X}_{F(p)} g && (\text{ definition of } F\text{-relatedness }), \\ X_p(g \circ F) &= (\bar{X}g)(F(p)) && \big(\text{ definitions of } F_* \text{ and } \bar{X}g\big), \\ (X(g \circ F))(p) &= (\bar{X}g)(F(p)). \end{aligned}$$

Since this is true for all $p \in N$,

$$X(g \circ F) = (\bar{X}g) \circ F$$

($\Leftarrow$) Reversing the set of equations above proves the converse.

Proposition 14.17. Let $F : N \to M$ be a smooth map of manifolds. If the C^∞ vector fields X and Y on N are F-related to the C^∞ vector fields $\bar{X}$ and $\bar{Y}$, respectively, on M, then the Lie bracket $[X, Y]$ on N is F-related to the Lie bracket $[\bar{X}, \bar{Y}]$ on M.

Proof. For any $g \in C^\infty(M)$,

$$\begin{aligned} [X, Y](g \circ F) &= XY(g \circ F) - YX(g \circ F) && (\text{ definition of } [X, Y]) \\ &= X((\bar{Y}g) \circ F) - Y((\bar{X}g) \circ F) && (\text{Proposition 14.16}) \\ &= (\bar{X}\bar{Y}g) \circ F - (\bar{Y}\bar{X}g) \circ F && (\text{Proposition 14.16}) \\ &= ((\bar{X}\bar{Y} - \bar{Y}\bar{X})g) \circ F \\ &= ([\bar{X}, \bar{Y}]g) \circ F. \end{aligned}$$

By Proposition 14.16 again, this proves that $[X, Y]$ on N and $[\bar{X}, \bar{Y}]$ on M are F related.

8.7.7 Problems

14.1.* Equality of Vector Fields

Show that two C^∞ vector fields X and Y on a manifold M are equal if and only if for every C^∞ function f on M, we have $Xf = Yf$.

14.2. Vector Field on an odd Sphere

Let $x^1, y^1, \ldots, x^n, y^n$ be the standard coordinates on $\mathbb{R}^{2n}$. The unit sphere S^{2n-1} in $\mathbb{R}^{2n}$ is defined by the equation $\sum_{i=1}^n (x^i)^2 + (y^i)^2 = 1$. Show that

$$X = \sum_{i=1}^n -y^i \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial y^i}$$

is a nowhere-vanishing smooth vector field on S^{2n-1}. Since all spheres of the same dimension are diffeomorphic, this proves that on every odd-dimensional sphere there is a nowherevanishing smooth vector field. It is a classical theorem of differential and algebraic topology that on an even-dimensional sphere every continuous vector field must vanish somewhere (see [28, Section 5, p. 31] or [16, Theorem 16.5, p. 70]). (Hint: Use Problem 11.1 to show that X is tangent to S^{2n-1}.)

14.3. Maximal Integral Curve on a Punctured Line

Let M be $\mathbb{R} - \{0\}$ and let X be the vector field d/dx on M (Figure 14.3). Find the maximal integral curve of X starting at $x = 1$.

Fig. 14.3. The vector field d/dx on $\mathbb{R -\{0\}$.}

14.4. Integral Curves in the Plane

Find the integral curves of the vector field

$$X_{(x,y)} = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} = \begin{bmatrix} x \\ -y \end{bmatrix} \quad \text{on } \mathbb{R}^2.$$

14.5. Maximal Integral Curve in the Plane

Find the maximal integral curve $c(t)$ starting at the point $(a, b) \in \mathbb{R}^2$ of the vector field $X_{(x,y)} = \partial/\partial x + x\partial/\partial y$ on $\mathbb{R}^2$.

14.6. Integral Curve Starting at a Zero of a Vector Field

(a)* Suppose the smooth vector field X on a manifold M vanishes at a point $p \in M$. Show that the integral curve of X with initial point p is the constant curve $c(t) \equiv p$.
(b) Show that if X is the zero vector field on a manifold M, and $c_t(p)$ is the maximal integral curve of X starting at p, then the one-parameter group of diffeomorphisms $c : \mathbb{R} \to \mathrm{Diff}(M)$ is the constant map $c(t) \equiv \mathbb{1}_M$.

14.7. Maximal Integral Curve

Let X be the vector field xd/dx on $\mathbb{R}$. For each p in $\mathbb{R}$, find the maximal integral curve $c(t)$ of X starting at p.

14.8. Maximal Integral Curve

Let X be the vector field $x^2 d/dx$ on the real line $\mathbb{R}$. For each $p > 0$ in $\mathbb{R}$, find the maximal integral curve of X with initial point p.

14.9. Reparametrization of an Integral Curve

Suppose $c :]a, b[\to M$ is an integral curve of the smooth vector field X on M. Show that for any real number s, the map

$$c_s :]\, a + s, b + s\, [\to M, \quad c_s(t) = c(t - s),$$

is also an integral curve of X.

14.10. Lie Bracket of Vector Fields

If f and g are C^∞ functions and X and Y are C^∞ vector fields on a manifold M, show that

$$[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X.$$

14.11. Lie Bracket of Vector Fields on $\mathbb{R}^2$

Compute the Lie bracket

$$\left[-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \frac{\partial}{\partial x}\right]$$

on $\mathbb{R}^2$.

14.12. Lie Bracket in Local Coordinates

Consider two C^∞ vector fields X, Y on $\mathbb{R}^n$:

$$X = \sum a^i \frac{\partial}{\partial x^i}, \quad Y = \sum b^j \frac{\partial}{\partial x^j},$$

where a^i, b^j are C^∞ functions on $\mathbb{R}^n$. Since $[X, Y]$ is also a C^∞ vector field on $\mathbb{R}^n$,

$$[X, Y] = \sum c^k \frac{\partial}{\partial x^k}$$

for some C^∞ functions c^k. Find the formula for c^k in terms of a^i and b^j.

14.13. Vector Field under a Diffeomorphism

Let $F : N \to M$ be a C^∞ diffeomorphism of manifolds. Prove that if g is a C^∞ function and X a C^∞ vector field on N, then

$$F_*(gX) = \left(g \circ F^{-1}\right) F_* X$$

14.14. Lie Bracket under a Diffeomorphism

Let $F : N \to M$ be a C^∞ diffeomorphism of manifolds. Prove that if X and Y are C^∞ vector fields on N, then

$$F_*[X, Y] = [F_* X, F_* Y]\,.$$

9 4 Lie Groups and Lie Algebras

A Lie group is a manifold that is also a group such that the group operations are smooth. Classical groups such as the general and special linear groups over $\mathbb{R}$ and over $\mathbb{C}$, orthogonal groups, unitary groups, and symplectic groups are all Lie groups.

A Lie group is a homogeneous space in the sense that left translation by a group element g is a diffeomorphism of the group onto itself that maps the identity element to g. Therefore, locally the group looks the same around any point. To study the local structure of a Lie group, it is enough to examine a neighborhood of the identity element. It is not surprising that the tangent space at the identity of a Lie group should play a key role.

The tangent space at the identity of a Lie group G turns out to have a canonical bracket operation $[,]thatmakesitintoaLiealgebra.ThetangentspaceT_eG$ with the bracket is called the Lie algebra of the Lie group G. The Lie algebra of a Lie group encodes within it much information about the group.

Sophus Lie
(1842-1899)

In a series of papers in the decade from 1874 to 1884, the Norwegian mathematician Sophus Lie initiated the study of Lie groups and Lie algebras. At first his work gained little notice, possibly because at the time he wrote mostly in Norwegian. In 1886, Lie became a professor in Leipzig, Germany, and his theory began to attract attention, especially after the publication of the three-volume treatise Theorie der Transformationsgruppen that he wrote in collaboration with his assistant Friedrich Engel.

Lie's original motivation was to study the group of transformations of a space as a continuous analogue of the group of permutations of a finite set. Indeed, a diffeomorphism of a manifold M can be viewed as a permutation of the points of M. The interplay of group theory, topology, and linear algebra makes the theory of Lie groups and Lie algebras

a particularly rich and vibrant branch of mathematics. In this chapter we can but scratch the surface of this vast creation. For us, Lie groups serve mainly as an important class of manifolds, and Lie algebras as examples of tangent spaces.

9.1 §15 Lie Groups

We begin with several examples of matrix groups, subgroups of the general linear group over a field. The goal is to exhibit a variety of methods for showing that a group is a Lie group and for computing the dimension of a Lie group. These examples become templates for investigating other matrix groups. A powerful tool, which we state but do not prove, is the closed subgroup theorem. According to this theorem, an abstract subgroup that is a closed subset of a Lie group is itself a Lie group. In many instances, the closed subgroup theorem is the easiest way to prove that a group is a Lie group.

The matrix exponential gives rise to curves in a matrix group with a given initial vector. It is useful in computing the differential of a map on a matrix group. As an example, we compute the differential of the determinant map on the general linear group over $\mathbb{R}$.

9.1.1 15.1 Examples of Lie Groups

We recall here the definition of a Lie group, which first appeared in Subsection 6.5.

Definition 15.1. A Lie group is a C^∞ manifold G that is also a group such that the two group operations, multiplication

$$\mu : G \times G \to G, \quad \mu(a, b) = ab,$$

and inverse

$$\imath : G \to G, \quad \imath(a) = a^{-1}$$

are C^∞.

For $a \in G$, denote by $\ell_a : G \to G, \ell_a(x) = \mu(a, x) = ax$, the operation of left multiplication by a, and by $r_a : G \to G, r_a(x) = xa$, the operation of right multiplication by a. We also call left and right multiplications left and right translations.

Exercise 15.2 (Left multiplication).* For an element a in a Lie group G, prove that the left multiplication $\ell_a : G \to G$ is a diffeomorphism.

Definition 15.3. A map $F : H \to G$ between two Lie groups H and G is a Lie group homomorphism if it is a C^∞ map and a group homomorphism.

The group homomorphism condition means that for all $h, x \in H$,

$$F(hx) = F(h)F(x). \tag{15.1}$$

This may be rewritten in functional notation as

$$F \circ \ell_h = \ell_{F(h)} \circ F \quad \text{for all } h \in H. \tag{15.2}$$

Let e_H and e_G be the identity elements of H and G, respectively. Taking h and x in (15.1) to be the identity e_H, it follows that $F(e_H) = e_G$. So a group homomorphism always maps the identity to the identity.

Notation. We use capital letters to denote matrices, but generally lowercase letters to denote their entries. Thus, the (i, j)-entry of the matrix AB is $(AB)_{ij} = \sum_k a_{ik}b_{kj}$.

Example 15.4 (General linear group). In Example 6.21, we showed that the general linear group

$$\mathrm{GL}(n,\mathbb{R}) = \left\{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\right\}$$

is a Lie group.

Example 15.5 (Special linear group). The special linear group $\mathrm{SL}(n,\mathbb{R})$ is the subgroup of $\mathrm{GL}(n,\mathbb{R})$ consisting of matrices of determinant 1. By Example 9.13, $\mathrm{SL}(n,\mathbb{R})$ is a regular submanifold of dimension n^2-1 of $\mathrm{GL}(n,\mathbb{R})$. By Example 11.16, the multiplication map

$$\bar{\mu} : \mathrm{SL}(n,\mathbb{R}) \times \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$$

is C^∞.

To see that the inverse map

$$\bar{\imath} : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$$

is C^∞, let $i : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ be the inclusion map and $i : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ the inverse map of $\mathrm{GL}(n,\mathbb{R})$. As the composite of two C^∞ maps,

$$l \circ i : \mathrm{SL}(n,\mathbb{R}) \xrightarrow{i} \mathrm{GL}(n,\mathbb{R}) \xrightarrow{l} \mathrm{GL}(n,\mathbb{R})$$

is a C^∞ map. Since its image is contained in the regular submanifold $\mathrm{SL}(n,\mathbb{R})$, the induced map $\bar{\imath} : \mathrm{SL}(n,\mathbb{R}) \to \mathrm{SL}(n,\mathbb{R})$ is C^∞ by Theorem 11.15. Thus, $\mathrm{SL}(n,\mathbb{R})$ is a Lie group.

An entirely analogous argument proves that the complex special linear group $\mathrm{SL}(n,\mathbb{C})$ is also a Lie group.

Example 15.6 (Orthogonal group). Recall that the orthogonal group $\mathrm{O}(n)$ is the subgroup of $\mathrm{GL}(n,\mathbb{R})$ consisting of all matrices A satisfying $A^TA = I$. Thus, $\mathrm{O}(n)$ is the inverse image of I under the map $f(A) = A^TA$.

In Example 11.3 we showed that $f : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ has constant rank. By the constant-rank level set theorem, $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$.

One drawback of this approach is that it does not tell us what the rank of f is, and so the dimension of $\mathrm{O}(n)$ remains unknown.

In this example we will apply the regular level set theorem to prove that $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$. This will at the same time determine the dimension of $\mathrm{O}(n)$. To accomplish this, we must first redefine the target space of f. Since A^TA is a symmetric matrix, the image of f lies in S_n, the vector space of all $n \times n$ real symmetric matrices. The space S_n is a proper subspace of $\mathbb{R}^{n\times n}$ as soon as $n \geq 2$.

Exercise 15.7 (Space of symmetric matrices).* Show that the vector space S_n of $n \times n$ real symmetric matrices has dimension $(n^2+n)/2$.

Consider the map $f : \mathrm{GL}(n,\mathbb{R}) \to S_n, f(A) = A^TA$. The tangent space of S_n at any point is canonically isomorphic to S_n itself, because S_n is a vector space. Thus, the image of the differential

$$f_{*,A} : T_A(\mathrm{GL}(n,\mathbb{R})) \to T_{f(A)}(S_n) \simeq S_n$$

lies in S_n. While it is true that f also maps $\mathrm{GL}(n,\mathbb{R})$ to $\mathrm{GL}(n,\mathbb{R})$ or $\mathbb{R}^{n\times n}$, if we had taken $\mathrm{GL}(n,\mathbb{R})$ or $\mathbb{R}^{n\times n}$ as the target space of f, the differential $f_{*,A}$ would never be surjective for any $A \in \mathrm{GL}(n,\mathbb{R})$ when $n \geq 2$, since $f_{*,A}$ factors through the proper subspace S_n of $\mathbb{R}^{n\times n}$. This illustrates a general principle: for the differential $f_{*,A}$ to be surjective, the target space of f should be as small as possible.

To show that the differential of

$$f : \mathrm{GL}(n,\mathbb{R}) \to S_n, \quad f(A) = A^TA,$$

is surjective, we compute explicitly the differential $f_{*,A}$. Since $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$, its tangent space at any $A \in \mathrm{GL}(n,\mathbb{R})$ is

$$T_A(\mathrm{GL}(n,\mathbb{R})) = T_A\left(\mathbb{R}^{n\times n}\right) = \mathbb{R}^{n\times n}$$

For any matrix $X \in \mathbb{R}^{n\times n}$, there is a curve $c(t)$ in $\mathrm{GL}(n,\mathbb{R})$ with $c(0) = A$ and $c'(0) = X$ (Proposition 8.16). By Proposition 8.18,

$$\begin{aligned} f_{*,A}(X) &= \frac{d}{dt} f(c(t))\bigg|_{t=0} \\ &= \frac{d}{dt} c(t)^T c(t)\bigg|_{t=0} \\ &= \left(c'(t)^T c(t) + c(t)^T c'(t)\right)\big|_{t=0} \quad (\text{ by Problem 15.2}) \\ &= X^T A + A^T X \end{aligned}$$

The surjectivity of $f_{*,A}$ becomes the following question: if $A \in \mathrm{O}(n)$ and B is any symmetric matrix in S_n, does there exist an $n \times n$ matrix X such that

$$X^T A + A^T X = B?$$

Note that since $\left(X^T A\right)^T = A^T X$, it is enough to solve

$$A^T X = \frac{1}{2} B, \tag{15.3}$$

for then

$$X^T A + A^T X = \frac{1}{2} B^T + \frac{1}{2} B = B$$

Equation (15.3) clearly has a solution: $X = \frac{1}{2}\left(A^T\right)^{-1} B$. So $f_{*,A} : T_A\mathrm{GL}(n,\mathbb{R}) \to S_n$ is surjective for all $A \in \mathrm{O}(n)$, and $\mathrm{O}(n)$ is a regular level set of f. By the regular level set theorem, $\mathrm{O}(n)$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$ of dimension

$$\dim \mathrm{O}(n) = n^2 - \dim S_n = n^2 - \frac{n^2+n}{2} = \frac{n^2-n}{2} \tag{15.4}$$

9.1.2 15.2 Lie Subgroups

Definition 15.8. A Lie subgroup of a Lie group G is (i) an abstract subgroup H that is (ii) an immersed submanifold via the inclusion map such that (iii) the group operations on H are C^∞.

An "abstract subgroup" simply means a subgroup in the algebraic sense, in contrast to a "Lie subgroup." The group operations on the subgroup H are the restrictions of the multiplication map μ and the inverse map ι from G to H. For an explanation of why a Lie subgroup is defined to be an immersed submanifold instead of a regular submanifold, see Remark 16.15. Because a Lie subgroup is an immersed submanifold, it need not have the relative topology. However, being an immersion, the inclusion map $i : H \hookrightarrow G$ of a Lie subgroup H is of course C^∞. It follows that the composite

$$\mu \circ (i \times i) : H \times H \to G \times G \to G$$

is C^∞. If H were defined to be a regular submanifold of G, then by Theorem 11.15, the multiplication map $H \times H \to H$ and similarly the inverse map $H \to H$ would automatically

be C^∞, and condition (iii) in the definition of a Lie subgroup would be redundant. Since a Lie subgroup is defined to be an immersed submanifold, it is necessary to impose condition (iii) on the group operations on H.

Example 15.9 (Lines with irrational slope in a torus). Let G be the torus $\mathbb{R}^2/\mathbb{Z}^2$ and L a line through the origin in $\mathbb{R}^2$. The torus can also be represented by the unit square with the opposite edges identified. The image H of L under the projection $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is a closed curve if and only if the line L goes through another lattice point, say $(m, n) \in \mathbb{Z}^2$. This is the case if and only if the slope of L is n/m, a rational number or ∞; then H is the image of finitely many line segments on the unit square. It is a closed curve diffeomorphic to a circle and is a regular submanifold of $\mathbb{R}^2/\mathbb{Z}^2$ (Figure 15.1).

If the slope of L is irrational, then its image H on the torus will never close up. In this case the restriction to L of the projection map, $f = \pi|_L : L \to \mathbb{R}^2/\mathbb{Z}^2$, is a one-toone immersion. We give H the topology and manifold structure induced from f. It

Fig. 15.1. An embedded Lie subgroup of the torus.

can be shown that H is a dense subset of the torus [3, Example III.6.15, p. 86]. Thus, H is an immersed submanifold but not a regular submanifold of the torus $\mathbb{R}^2/\mathbb{Z}^2$.

Whatever the slope of L, its image H in $\mathbb{R}^2/\mathbb{Z}^2$ is an abstract subgroup of the torus, an immersed submanifold, and a Lie group. Therefore, H is a Lie subgroup of the torus.

Exercise 15.10 (Induced topology versus subspace topology).* Suppose $H \subset \mathbb{R}^2/\mathbb{Z}^2$ is the image of a line L with irrational slope in $\mathbb{R}^2$. We call the topology on H induced from the bijection $f : L \xrightarrow{\sim} H$ the induced topology and the topology on H as a subset of $\mathbb{R}^2/\mathbb{Z}^2$ the subspace topology. Compare these two topologies: is one a subset of the other?

Proposition 15.11. If H is an abstract subgroup and a regular submanifold of a Lie group G, then it is a Lie subgroup of G.

Proof. Since a regular submanifold is the image of an embedding (Theorem 11.14), it is also an immersed submanifold.

Let $\mu : G \times G \to G$ be the multiplication map on G. Since H is an immersed submanifold of G, the inclusion map $i : H \hookrightarrow G$ is C^∞. Hence, the inclusion map $i \times i : H \times H \hookrightarrow G \times G$ is C^∞, and the composition $\mu \circ (i \times i) : H \times H \to G$ is C^∞. By Theorem 11.15, because H is a regular submanifold of G, the induced map $\bar{\mu} : H \times H \to H$ is C^∞.

The smoothness of the inverse map $\bar{\imath} : H \to H$ can be deduced from the smoothness of $\imath : G \to G$ just as in Example 15.5. $\square$

A subgroup H as in Proposition 15.11 is called an embedded Lie subgroup, because the inclusion map $i : H \to G$ of a regular submanifold is an embedding (Theorem 11.14).

Example. We showed in Examples 15.5 and 15.6 that the subgroups $\mathrm{SL}(n, \mathbb{R})$ and $\mathrm{O}(n)$

of $\mathrm{GL}(n,\mathbb{R})$ are both regular submanifolds. By Proposition 15.11 they are embedded Lie subgroups.

We state without proof an important theorem about Lie subgroups. If G is a Lie group, then an abstract subgroup that is a closed subset in the topology of G is called a closed subgroup.

Theorem 15.12 (Closed subgroup theorem). A closed subgroup of a Lie group is an embedded Lie subgroup.

For a proof of the closed subgroup theorem, see [38, Theorem 3.42, p. 110].

Examples.

(i) A line with irrational slope in the torus $\mathbb{R}^2/\mathbb{Z}^2$ is not a closed subgroup, since it is not the whole torus, but being dense, its closure is.
(ii) The special linear group $\mathrm{SL}(n,\mathbb{R})$ and the orthogonal group $\mathrm{O}(n)$ are the zero sets of polynomial equations on $\mathrm{GL}(n,\mathbb{R})$. As such, they are closed subsets of $\mathrm{GL}(n,\mathbb{R})$. By the closed subgroup theorem, $\mathrm{SL}(n,\mathbb{R})$ and $\mathrm{O}(n)$ are embedded Lie subgroups of $\mathrm{GL}(n,\mathbb{R})$.

9.1.3 15.3 The Matrix Exponential

To compute the differential of a map on a subgroup of $\mathrm{GL}(n,\mathbb{R})$, we need a curve of nonsingular matrices. Because the matrix exponential is always nonsingular, it is uniquely suited for this purpose.

A norm on a vector space V is a real-valued function $\|\cdot\| : V \to \mathbb{R}$ satisfying the following three properties: for all $r \in \mathbb{R}$ and $v, w \in V$,
(i) (positive-definiteness) $\|v\| \geq 0$ with equality if and only if $v = 0$,
(ii) (positive homogeneity) $\|rv\| = |r|\|v\|$,
(iii) (subadditivity) $\|v + w\| \leq \|v\| + \|w\|$.

A vector space V together with a norm $\|\cdot\|$ is called a normed vector space. The vector space $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$ of all $n \times n$ real matrices can be given the Euclidean norm: for $X = [x_{ij}] \in \mathbb{R}^{n\times n}$,

$$\|X\| = \left(\sum x_{ij}^2\right)^{1/2}$$

The matrix exponential e^X of a matrix $X \in \mathbb{R}^{n\times n}$ is defined by the same formula as the exponential of a real number:

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots \tag{15.5}$$

where I is the $n \times n$ identity matrix. For this formula to make sense, we need to show that the series on the right converges in the normed vector space $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$.

A normed algebra V is a normed vector space that is also an algebra over $\mathbb{R}$ satisfying the submultiplicative property: for all $v, w \in V, \|vw\| \leq \|v\|\|w\|$. Matrix multiplication makes the normed vector space $\mathbb{R}^{n\times n}$ into a normed algebra.

Proposition 15.13. For $X, Y \in \mathbb{R}^{n\times n}, \|XY\| \leq \|X\|\|Y\|$.

Proof. Write $X = [x_{ij}]$ and $Y = [y_{ij}]$ and fix a pair of subscripts (i, j). By the CauchySchwarz inequality,

$$(XY)_{ij}^2 = \left(\sum_k x_{ik}y_{kj}\right)^2 \leq \left(\sum_k x_{ik}^2\right)\left(\sum_k y_{kj}^2\right) = a_i b_j,$$

where we set $a_i = \sum_k x_{ik}^2$ and $b_j = \sum_k y_{kj}^2$. Then

$$\|XY\|^2 = \sum_{i,j}(XY)_{ij}^2 \le \sum_{i,j} a_i b_j = \left(\sum_i a_i\right)\left(\sum_j b_j\right)$$
$$= \left(\sum_{i,k} x_{ik}^2\right)\left(\sum_{j,k} y_{kj}^2\right) = \|X\|^2\|Y\|^2.$$

In a normed algebra, multiplication distributes over a finite sum. When the sum is infinite as in a convergent series, the distributivity of multiplication over the sum requires a proof.

Proposition 15.14. Let V be a normed algebra.
(i) If $a \in V$ and s_m is a sequence in V that converges to s, then as s_m converges to as.
(ii) If $a \in V$ and $\sum_{k=0}^{\infty} b_k$ is a convergent series in V, then a $\sum_k b_k = \sum_k ab_k$.

Exercise 15.15 (Distributivity over a convergent series).* Prove Proposition 15.14.

In a normed vector space V a series $\sum a_k$ is said to converge absolutely if the series $\sum \|a_k\|$ of norms converges in $\mathbb{R}$. The normed vector space V is said to be complete if every Cauchy sequence in V converges to a point in V. For example, $\mathbb{R}^{n\times n}$ is a complete normed vector space. [1] It is easy to show that in a complete normed vector space, absolute convergence implies convergence [26, Theorem 2.9.3, p. 126]. Thus, to show that a series $\sum Y_k$ of matrices converges, it is enough to show that the series $\sum \|Y_k\|$ of real numbers converges.

For any $X \in \mathbb{R}^{n\times n}$ and $k > 0$, repeated applications of Proposition 15.13 give $\left\|X^k\right\| \le \|X\|^k$. So the series $\sum_{k=0}^{\infty} \left\|X^k/k!\right\|$ is bounded term by term in absolute value by the convergent series

$$\sqrt{n} + \|X\| + \frac{1}{2!}\|X\|^2 + \frac{1}{3!}\|X\|^3 + \cdots = (\sqrt{n} - 1) + e^{\|X\|}.$$

By the comparison test for series of real numbers, the series $\sum_{k=0}^{\infty} \left\|X^k/k!\right\|$ converges. Therefore, the series (15.5) converges absolutely for any $n \times n$ matrix X.

Notation. Following standard convention we use the letter e both for the exponential map and for the identity element of a general Lie group. The context should prevent any confusion. We sometimes write $\exp(X)$ for e^X.

Unlike the exponential of real numbers, when A and B are $n \times n$ matrices with $n > 1$, it is not necessarily true that

$$e^{A+B} = e^A e^B.$$

Exercise 15.16 (Exponentials of commuting matrices). Prove that if A and B are commuting $n \times n$ matrices, then

$$e^A e^B = e^{A+B}.$$

Proposition 15.17. For $X \in \mathbb{R}^{n\times n}$,

$$\frac{d}{dt}e^{tX} = Xe^{tX} = e^{tX}X$$

Proof. Because each (i, j)-entry of the series for the exponential function e^{tX} is a power series in t, it is possible to differentiate term by term [35, Theorem 8.1, p. 173]. Hence,

[0]1 A complete normed vector space is also called a Banach space, named after the Polish mathematician Stefan Banach, who introduced the concept in 1920-1922. Correspondingly, a complete normed algebra is called a Banach algebra.

$$\begin{aligned}\frac{d}{dt}e^{tX} &= \frac{d}{dt}\left(I + tX + \frac{1}{2!}t^2X^2 + \frac{1}{3!}t^3X^3 + \cdots\right)\\ &= X + tX^2 + \frac{1}{2!}t^2X^3 + \cdots\\ &= X\left(I + tX + \frac{1}{2!}t^2X^2 + \cdots\right) = Xe^{tX}\end{aligned}$$

(Proposition 15.14(ii)).

In the second equality above, one could have factored out X as the second factor:

$$\begin{aligned}\frac{d}{dt}e^{tX} &= X + tX^2 + \frac{1}{2!}t^2X^3 + \cdots\\ &= \left(I + tX + \frac{1}{2!}t^2X^2 + \cdots\right)X = e^{tX}X\end{aligned}$$

□

The definition of the matrix exponential e^X makes sense even if X is a complex matrix. All the arguments so far carry over word for word; one merely has to replace the Euclidean norm $\|X\|^2 = \sum x_{ij}^2$ by the Hermitian norm $\|X\|^2 = \sum |x_{ij}|^2$, where $|x_{ij}|$ is the modulus of a complex number x_{ij}.

9.1.4 15.4 The Trace of a Matrix

Define the trace of an $n \times n$ matrix X to be the sum of its diagonal entries:

$$\operatorname{tr}(X) = \sum_{i=1}^{n} x_{ii}.$$

Lemma 15.18.

(i) For any two matrices $X, Y \in \mathbb{R}^{n\times n}, \operatorname{tr}(XY) = \operatorname{tr}(YX)$.
(ii) For $X \in \mathbb{R}^{n\times n}$ and $A \in \mathrm{GL}(n, \mathbb{R}), \operatorname{tr}(AXA^{-1}) = \operatorname{tr}(X)$.

Proof.

(i)

$$\begin{aligned}\operatorname{tr}(XY) &= \sum_i (XY)_{ii} = \sum_i \sum_k x_{ik}y_{ki}\\ \operatorname{tr}(YX) &= \sum_k (YX)_{kk} = \sum_k \sum_i y_{ki}x_{ik}\end{aligned}$$

(ii) Set $B = XA^{-1}$ in (i).

The eigenvalues of an $n\times n$ matrix X are the roots of the polynomial equation $\det(\lambda I - X) = 0$. Over the field of complex numbers, which is algebraically closed, such an equation necessarily has n roots, counted with multiplicity. Thus, the advantage of allowing complex numbers is that every $n \times n$ matrix, real or complex, has n complex eigenvalues, counted with multiplicity, whereas a real matrix need not have any real eigenvalue.

Example. The real matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

has no real eigenvalues. It has two complex eigenvalues, $\pm i$.
The following two facts about eigenvalues are immediate from the definitions:
(i) Two similar matrices X and AXA^{-1} have the same eigenvalues, because

$$\det\left(\lambda I - AXA^{-1}\right) = \det\left(A(\lambda I - X)A^{-1}\right) = \det(\lambda I - X).$$

(ii) The eigenvalues of a triangular matrix are its diagonal entries, because

$$\det\left(\lambda I - \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}\right) = \prod_{i=1}^{n} (\lambda - \lambda_i)\,.$$

By a theorem from algebra [19, Th. 6.4.1, p. 286], any complex square matrix X can be triangularized; more precisely, there exists a nonsingular complex square matrix A such that AXA^{-1} is upper triangular. Since the eigenvalues $\lambda_1, \ldots, \lambda_n$ of X are the same as the eigenvalues of AXA^{-1}, the triangular matrix AXA^{-1} must have the eigenvalues of X along its diagonal:

$$\begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

A real matrix X, viewed as a complex matrix, can also be triangularized, but of course the triangularizing matrix A and the triangular matrix AXA^{-1} are in general complex.

Proposition 15.19. The trace of a matrix, real or complex, is equal to the sum of its complex eigenvalues.

Proof. Suppose X has complex eigenvalues $\lambda_1, \ldots, \lambda_n$. Then there exists a nonsingular matrix $A \in \mathrm{GL}(n, \mathbb{C})$ such that

$$AXA^{-1} = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

By Lemma 15.18,

$$\mathrm{tr}(X) = \mathrm{tr}\left(AXA^{-1}\right) = \sum \lambda_i.$$

□

Proposition 15.20. For any $X \in \mathbb{R}^{n\times n}, \det\left(e^X\right) = e^{\mathrm{tr}\, X}$.
Proof.
Case 1. Assume that X is upper triangular:

$$X = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}.$$

Then

$$e^X = \sum \frac{1}{k!} X^k = \sum \frac{1}{k!} \begin{bmatrix} \lambda_1^k & & * \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & & * \\ & \ddots & \\ 0 & & e^{\lambda_n} \end{bmatrix}.$$

Hence, $\det e^X = \prod e^{\lambda_i} = e^{\sum \lambda_i} = e^{\operatorname{tr} X}$.
Case 2. Given a general matrix X, with eigenvalues $\lambda_1, \ldots, \lambda_n$, we can find a nonsingular complex matrix A such that

$$AXA^{-1} = \begin{bmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix},$$

an upper triangular matrix. Then

$$\begin{aligned} e^{AXA^{-1}} &= I + AXA^{-1} + \frac{1}{2!}\left(AXA^{-1}\right)^2 + \frac{1}{3!}\left(AXA^{-1}\right)^3 + \cdots \\ &= I + AXA^{-1} + A\left(\frac{1}{2!}X^2\right)A^{-1} + A\left(\frac{1}{3!}X^3\right)A^{-1} + \cdots \\ &= Ae^XA^{-1} \quad \text{(by Proposition 15.14(ii))}. \end{aligned}$$

Hence,

$$\begin{aligned} \det e^X &= \det\left(Ae^XA^{-1}\right) = \det\left(e^{AXA^{-1}}\right) \\ &= e^{\operatorname{tr}\left(AXA^{-1}\right)} \quad \left(\text{ by Case 1, since } AXA^{-1} \text{ is upper triangular }\right) \\ &= e^{\operatorname{tr} X} \quad \left(\text{ by Lemma 15.18 }\right) \end{aligned}$$

It follows from this proposition that the matrix exponential e^X is always nonsingular, because $\det\left(e^X\right) = e^{\operatorname{tr} X}$ is never 0 . This is one reason why the matrix exponential is so useful, for it allows us to write down explicitly a curve in $\mathrm{GL}(n, \mathbb{R})$ with a given initial point and a given initial velocity. For example, $c(t) = e^{tX} : \mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$ is a curve in $\mathrm{GL}(n, \mathbb{R})$ with initial point I and initial velocity X, since

$$c(0) = e^{0X} = e^0 = I \quad \text{and} \quad c'(0) = \frac{d}{dt}e^{tX}\bigg|_{t=0} = Xe^{tX}\big|_{t=0} = X. \tag{15.6}$$

Similarly, $c(t) = Ae^{tX} : \mathbb{R} \to \mathrm{GL}(n, \mathbb{R})$ is a curve in $\mathrm{GL}(n, \mathbb{R})$ with initial point A and initial velocity AX.

9.1.5 15.5 The Differential of det at the Identity

Let det: $\mathrm{GL}(n, \mathbb{R}) \to \mathbb{R}$ be the determinant map. The tangent space $T_I\mathrm{GL}(n, \mathbb{R})$ to $\mathrm{GL}(n, \mathbb{R})$ at the identity matrix I is the vector space $\mathbb{R}^{n\times n}$ and the tangent space $T_1\mathbb{R}$ to $\mathbb{R}$ at 1 is $\mathbb{R}$. So

$$\det_{*,I} : \mathbb{R}^{n\times n} \to \mathbb{R}$$

Proposition 15.21. For any $X \in \mathbb{R}^{n\times n}, \det_{*,I}(X) = \operatorname{tr} X$.
Proof. We use a curve at I to compute the differential (Proposition 8.18). As a curve $c(t)$ with $c(0) = I$ and $c'(0) = X$, choose the matrix exponential $c(t) = e^{tX}$. Then

$$\begin{aligned} \det_{*,I}(X) &= \frac{d}{dt}\det\left(e^{tX}\right)\bigg|_{t=0} = \frac{d}{dt}e^{t\operatorname{tr} X}\bigg|_{t=0} \\ &= (\operatorname{tr} X)e^{t\operatorname{tr} X}\big|_{t=0} = \operatorname{tr} X \end{aligned}$$

9.1.6 Problems

15.1. Matrix Exponential

For $X \in \mathbb{R}^{n\times n}$, define the partial sum $s_m = \sum_{k=0}^{m} X^k/k!$.
(a) Show that for $\ell \geq m$,

$$\|s_\ell - s_m\| \leq \sum_{k=m+1}^{\ell} \|X\|^k/k!$$

(b) Conclude that s_m is a Cauchy sequence in $\mathbb{R}^{n\times n}$ and therefore converges to a matrix, which we denote by e^X. This gives another way of showing that $\sum_{k=0}^{\infty} X^k/k!$ is convergent, without using the comparison test or the theorem that absolute convergence implies convergence in a complete normed vector space.

15.2. Product Rule for Matrix-valued Functions

Let $]a,b[$ be an open interval in $\mathbb{R}$. Suppose $A :]a,b[\to \mathbb{R}^{m\times n}$ and $B :]a,b[\to \mathbb{R}^{n\times p}$ are $m\times n$ and $n\times p$ matrices respectively whose entries are differentiable functions of $t \in]a,b[$. Prove that for $t \in]a,b[$,

$$\frac{d}{dt}A(t)B(t) = A'(t)B(t) + A(t)B'(t)$$

where $A'(t) = (dA/dt)(t)$ and $B'(t) = (dB/dt)(t)$.

15.3. Identity Component of a Lie Group

The identity component G_0 of a Lie group G is the connected component of the identity element e in G. Let μ and $\imath$ be the multiplication map and the inverse map of G.
(a) For any $x \in G_0$, show that $\mu(\{x\}\times G_0) \subset G_0$. (Hint: Apply Proposition A.43.)
(b) Show that $\imath(G_0) \subset G_0$.
(c) Show that G_0 is an open subset of G. (Hint: Apply Problem A.16.)
(d) Prove that G_0 is itself a Lie group.

15.4.* Open Subgroup of a Connected Lie Group

Prove that an open subgroup H of a connected Lie group G is equal to G.

15.5. Differential of the Multiplication map

Let G be a Lie group with multiplication map $\mu : G\times G \to G$, and let $\ell_a : G \to G$ and $r_b : G \to G$ be left and right multiplication by a and $b \in G$, respectively. Show that the differential of μ at $(a,b) \in G\times G$ is

$$\mu_{*,(a,b)}(X_a, Y_b) = (r_b)_*(X_a) + (\ell_a)_*(Y_b) \qquad \text{for } X_a \in T_a(G), \quad Y_b \in T_b(G)$$

15.6. Differential of the Inverse map

Let G be a Lie group with multiplication map $\mu : G\times G \to G$, inverse map $\imath : G \to G$, and identity element e. Show that the differential of the inverse map at $a \in G$,

$$\imath_{*,a} : T_aG \to T_{a^{-1}}G$$

is given by

$$\imath_{*,a}(Y_a) = -(r_{a^{-1}})_*(\ell_{a^{-1}})_* Y_a$$

where $(r_{a^{-1}})_* = (r_{a^{-1}})_{*,e}$ and $(\ell_{a^{-1}})_* = (\ell_{a^{-1}})_{*,a}$. (The differential of the inverse at the identity was calculated in Problem 8.8(b).)

15.7.* Differential of the Determinant map at a

Show that the differential of the determinant map det: $\mathrm{GL}(n,\mathbb{R}) \to \mathbb{R}$ at $A \in \mathrm{GL}(n,\mathbb{R})$ is given by

$$\det_{*,A}(AX) = (\det A)\operatorname{tr} X \quad \text{for } X \in \mathbb{R}^{n\times n}. \tag{15.7}$$

15.8.* Special Linear Group

Use Problem 15.7 to show that 1 is a regular value of the determinant map. This gives a quick proof that the special linear group $\mathrm{SL}(n,\mathbb{R})$ is a regular submanifold of $\mathrm{GL}(n,\mathbb{R})$.

15.9. Structure of a General Linear Group

(a) For $r \in \mathbb{R}^\times := \mathbb{R} - \{0\}$, let M_r be the $n \times n$ matrix

$$M_r = \begin{bmatrix} r & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix} = \begin{bmatrix} re_1 & e_2 & \cdots & e_n \end{bmatrix},$$

where $e_1, \ldots, e_n$ is the standard basis for $\mathbb{R}^n$. Prove that the map

$$\begin{aligned} f : \mathrm{GL}(n,\mathbb{R}) &\to \mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^\times, \\ A &\mapsto \left(AM_{1/\det A}, \det A\right), \end{aligned}$$

is a diffeomorphism.

(b) The center $Z(G)$ of a group G is the subgroup of elements $g \in G$ that commute with all elements of G :

$$Z(G) := \{g \in G \mid gx = xg \text{ for all } x \in G\}$$

Show that the center of $\mathrm{GL}(2,\mathbb{R})$ is isomorphic to $\mathbb{R}^\times$, corresponding to the subgroup of scalar matrices, and that the center of $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^\times$is isomorphic to $\{\pm 1\} \times \mathbb{R}^\times$. The group $\mathbb{R}^\times$has two elements of order 2, while the group $\{\pm 1\} \times \mathbb{R}^\times$has four elements of order 2. Since their centers are not isomorphic, $\mathrm{GL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R}) \times \mathbb{R}^\times$are not isomorphic as groups.

(c) Show that

$$\begin{aligned} h : \mathrm{GL}(3,\mathbb{R}) &\to \mathrm{SL}(3,\mathbb{R}) \times \mathbb{R}^\times \\ A &\mapsto \left((\det A)^{1/3} A, \det A\right) \end{aligned}$$

is a Lie group isomorphism.

The same arguments as in (b) and (c) prove that for n even, the two Lie groups $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{SL}(n,\mathbb{R}) \times \mathbb{R}^\times$are not isomorphic as groups, while for n odd, they are isomorphic as Lie groups.

15.10. Orthogonal Group

Show that the orthogonal group $\mathrm{O}(n)$ is compact by proving the following two statements.

(a) $\mathrm{O}(n)$ is a closed subset of $\mathbb{R}^{n\times n}$.

(b) $\mathrm{O}(n)$ is a bounded subset of $\mathbb{R}^{n\times n}$.

15.11. Special Orthogonal Group so(2)

The special orthogonal group $\mathrm{SO}(n)$ is defined to be the subgroup of $\mathrm{O}(n)$ consisting of matrices of determinant 1 . Show that every matrix $A \in \mathrm{SO}(2)$ can be written in the form

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

for some real number θ. Then prove that $\mathrm{SO}(2)$ is diffeomorphic to the circle S^1.

15.12. Unitary Group

The unitary group $\mathrm{U}(n)$ is defined to be

$$\mathrm{U}(n) = \left\{ A \in \mathrm{GL}(n, \mathbb{C}) \mid \bar{A}^T A = I \right\},$$

where $\bar{A}$ denotes the complex conjugate of A, the matrix obtained from A by conjugating every entry of A : $(\bar{A})_{ij} = \overline{a_{ij}}$. Show that $\mathrm{U}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{C})$ and that $\dim \mathrm{U}(n) = n^2$.

15.13. Special unitary group SU(2)

The special unitary group $\mathrm{SU}(n)$ is defined to be the subgroup of $\mathrm{U}(n)$ consisting of matrices of determinant 1 .

(a) Show that $\mathrm{SU}(2)$ can also be described as the set

$$\mathrm{SU}(2) = \left\{ \begin{bmatrix} a & -\bar{b} \\ b & \bar{a} \end{bmatrix} \in \mathbb{C}^{2\times 2} \,\middle|\, a\bar{a} + b\bar{b} = 1 \right\}$$

(Hint: Write out the condition $A^{-1} = \bar{A}^T$ in terms of the entries of A.)

(b) Show that $\mathrm{SU}(2)$ is diffeomorphic to the three-dimensional sphere

$$S^3 = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \right\}$$

15.14. A Matrix Exponential

Compute $\exp \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

15.15. Symplectic Group

This problem requires a knowledge of quaternions as in Appendix E. Let $\mathbb{H}$ be the skew field of quaternions. The symplectic group $\mathrm{Sp}(n)$ is defined to be

$$\mathrm{Sp}(n) = \left\{ A \in \mathrm{GL}(n, \mathbb{H}) \mid \bar{A}^T A = I \right\}$$

where $\bar{A}$ denotes the quaternionic conjugate of A. Show that $\mathrm{Sp}(n)$ is a regular submanifold of $\mathrm{GL}(n, \mathbb{H})$ and compute its dimension.

15.16. Complex Symplectic Group

Let J be the $2n \times 2n$ matrix

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

where I_n denotes the $n \times n$ identity matrix. The complex symplectic group $\mathrm{Sp}(2n, \mathbb{C})$ is defined to be

$$\mathrm{Sp}(2n, \mathbb{C}) = \left\{A \in \mathrm{GL}(2n, \mathbb{C}) \mid A^T JA = J\right\}$$

Show that $\mathrm{Sp}(2n, \mathbb{C})$ is a regular submanifold of $\mathrm{GL}(2n, \mathbb{C})$ and compute its dimension. (Hint: Mimic Example 15.6. It is crucial to choose the correct target space for the map $f(A) = A^T JA$.)

9.2 §16 Lie Algebras

In a Lie group G, because left translation by an element $g \in G$ is a diffeomorphism that maps a neighborhood of the identity to a neighborhood of g, all the local information about the group is concentrated in a neighborhood of the identity, and the tangent space at the identity assumes a special importance.

Moreover, one can give the tangent space T_eG a Lie bracket $[,]$, so that in addition to being a vector space, it becomes a Lie algebra, called the Lie algebra of the Lie group. This Lie algebra encodes in it much information about the Lie group. The goal of this section is to define the Lie algebra structure on T_eG and to identity the Lie algebras of a few classical groups.

The Lie bracket on the tangent space T_eG is defined using a canonical isomorphism between the tangent space at the identity and the vector space of left-invariant vector fields on G. With respect to this Lie bracket, the differential of a Lie group homomorphism becomes a Lie algebra homomorphism. We thus obtain a functor from the category of Lie groups and Lie group homomorphisms to the category of Lie algebras and Lie algebra homomorphisms. This is the beginning of a rewarding program, to understand the structure and representations of Lie groups through a study of their Lie algebras.

9.2.1 16.1 Tangent Space at the Identity of a Lie Group

Because of the existence of a multiplication, a Lie group is a very special kind of manifold. In Exercise 15.2, we learned that for any $g \in G$, left translation $\ell_g : G \to G$ by g is a diffeomorphism with inverse $\ell_{g^{-1}}$. The diffeomorphism ℓ_g takes the identity element e to the element g and induces an isomorphism of tangent spaces

$$\ell_{g*} = (\ell_g)_{*,e} : T_e(G) \to T_g(G).$$

Thus, if we can describe the tangent space $T_e(G)$ at the identity, then $\ell_{g*}T_e(G)$ will give a description of the tangent space $T_g(G)$ at any point $g \in G$.

Example 16.1 (The tangent space to $\mathrm{GL}(n, \mathbb{R})$ at I). In Example 8.19, we identified the tangent space $\mathrm{GL}(n, \mathbb{R})$ at any point $g \in \mathrm{GL}(n, \mathbb{R})$ as $\mathbb{R}^{n\times n}$, the vector space of all $n \times n$ real matrices. We also identified the isomorphism $\ell_{g*} : T_I(\mathrm{GL}(n, \mathbb{R})) \to T_g(\mathrm{GL}(n, \mathbb{R}))$ as left multiplication by $g : X \mapsto gX$.

Example 16.2 (The tangent space to $\mathrm{SL}(n, \mathbb{R})$ at I). We begin by finding a condition that a tangent vector X in $T_I(\mathrm{SL}(n, \mathbb{R}))$ must satisfy. By Proposition 8.16 there is a curve $c :]-\varepsilon, \varepsilon[\to \mathrm{SL}(n, \mathbb{R})$ with $c(0) = I$ and $c'(0) = X$. Being in $\mathrm{SL}(n, \mathbb{R})$, this curve satisfies

$$\det c(t) = 1$$

for all t in the domain $]-\varepsilon, \varepsilon[$. We now differentiate both sides with respect to t and evaluate at $t = 0$. On the left-hand side, we have

$$\begin{aligned}
\tfrac{d}{dt}\det(c(t))\big|_{t=0} &= (\det\circ c)_*\left(\tfrac{d}{dt}\big|_0\right)\\
&= \det{}_{*,I}\left(c_*\tfrac{d}{dt}\big|_0\right)\\
&= \det{}_{*,I}\left(c'(0)\right)\\
&= \det{}_{*,I}(X)\\
&= \operatorname{tr}(X) \qquad\qquad \text{(by the chain rule)}
\end{aligned}$$

Thus,

$$\operatorname{tr}(X) = \frac{d}{dt}1\bigg|_{t=0} = 0.$$

So the tangent space $T_I(\mathrm{SL}(n,\mathbb{R}))$ is contained in the subspace V of $\mathbb{R}^{n\times n}$ defined by

$$V = \left\{X \in \mathbb{R}^{n\times n} \mid \operatorname{tr} X = 0\right\}.$$

Since $\dim V = n^2 - 1 = \dim T_I(\mathrm{SL}(n,\mathbb{R}))$, the two spaces must be equal.

Proposition 16.3. The tangent space $T_I(\mathrm{SL}(n,\mathbb{R}))$ at the identity of the special linear group $\mathrm{SL}(n,\mathbb{R})$ is the subspace of $\mathbb{R}^{n\times n}$ consisting of all $n\times n$ matrices of trace 0 .

Example 16.4 (The tangent space to $\mathrm{O}(n)$ at I). Let X be a tangent vector to the orthogonal group $\mathrm{O}(n)$ at the identity I. Choose a curve $c(t)$ in $\mathrm{O}(n)$ defined on a small interval containing 0 such that $c(0) = I$ and $c'(0) = X$. Since $c(t)$ is in $\mathrm{O}(n)$,

$$c(t)^T c(t) = I.$$

Differentiating both sides with respect to t using the matrix product rule (Problem 15.2) gives

$$c'(t)^T c(t) + c(t)^T c'(t) = 0.$$

Evaluating at $t = 0$ gives

$$X^T + X = 0.$$

Thus, X is a skew-symmetric matrix.

Let K_n be the space of all $n\times n$ real skew-symmetric matrices. For example, for $n = 3$, these are matrices of the form

$$\begin{bmatrix} 0 & a & b\\ -a & 0 & c\\ -b & -c & 0\end{bmatrix}, \qquad \text{where } a,b,c,\in\mathbb{R}.$$

The diagonal entries of such a matrix are all 0 and the entries below the diagonal are determined by those above the diagonal. So

$$\dim K_n = \frac{n^2 - \#\text{ diagonal entries}}{2} = \frac{1}{2}\left(n^2 - n\right).$$

We have shown that

$$T_I(\mathrm{O}(n)) \subset K_n \tag{16.1}$$

By an earlier computation (see (15.4)),

$$\dim T_I(\mathrm{O}(n)) = \dim \mathrm{O}(n) = \frac{n^2 - n}{2}$$

Since the two vector spaces in (16.1) have the same dimension, equality holds.
Proposition 16.5. The tangent space $T_I(\mathrm{O}(n))$ of the orthogonal group $\mathrm{O}(n)$ at the identity is the subspace of $\mathbb{R}^{n\times n}$ consisting of all $n \times n$ skew-symmetric matrices.

9.2.2 16.2 Left-invariant Vector Fields on a Lie Group

Let X be a vector field on a Lie group G. We do not assume X to be C^∞. For any $g \in G$, because left multiplication $\ell_g : G \to G$ is a diffeomorphism, the pushforward $\ell_{g*}X$ is a well-defined vector field on G. We say that the vector field X is leftinvariant if

$$\ell_{g*}X = X$$

for every $g \in G$; this means for any $h \in G$,

$$\ell_{g*}(X_h) = X_{gh}.$$

In other words, a vector field X is left-invariant if and only if it is ℓ_g-related to itself for all $g \in G$.

Clearly, a left-invariant vector field X is completely determined by its value X_e at the identity, since

$$X_g = \ell_{g*}(X_e) \tag{16.2}$$

Conversely, given a tangent vector $A \in T_e(G)$ we can define a vector field $\tilde{A}$ on G by (16.2): $(\tilde{A})_g = \ell_{g*}A$. So defined, the vector field $\tilde{A}$ is left-invariant, since

$$\begin{aligned}
\ell_{g*}\left(\tilde{A}_h\right) &= \ell_{g*}\ell_{h*}A \\
&= (\ell_g \circ \ell_h)_* A \quad \text{(by the chain rule)} \\
&= (\ell_{gh})_*(A) \\
&= \tilde{A}_{gh}
\end{aligned}$$

We call $\tilde{A}$ the left-invariant vector field on G generated by $A \in T_eG$. Let $L(G)$ be the vector space of all left-invariant vector fields on G. Then there is a one-to-one correspondence

$$\begin{aligned}
T_e(G) &\leftrightarrow L(G), \\
X_e &\leftrightarrow X, \\
A &\mapsto \tilde{A}.
\end{aligned} \tag{16.3}$$

It is easy to show that this correspondence is in fact a vector space isomorphism.

Example 16.6 (Left-invariant vector fields on $\mathbb{R}$). On the Lie group $\mathbb{R}$, the group operation is addition and the identity element is 0 . So "left multiplication" ℓ_g is actually addition:

$$\ell_g(x) = g + x.$$

Let us compute $\ell_{g*}(d/\,dx|_0)$. Since $\ell_{g*}(d/\,dx|_0)$ is a tangent vector at g, it is a scalar multiple of $d/\,dx|_g$:

$$\ell_{g*}\left(\left.\frac{d}{dx}\right|_0\right) = a\left.\frac{d}{dx}\right|_g. \tag{16.4}$$

To evaluate a, apply both sides of (16.4) to the function $f(x) = x$:

$$a = a\frac{d}{dx}\bigg|_g f = \ell_{g*}\left(\frac{d}{dx}\bigg|_0\right) f = \frac{d}{dx}\bigg|_0 f \circ \ell_g = \frac{d}{dx}\bigg|_0 (g+x) = 1.$$

Thus,

$$\ell_{g*}\left(\frac{d}{dx}\bigg|_0\right) = \frac{d}{dx}\bigg|_g .$$

This shows that d/dx is a left-invariant vector field on $\mathbb{R}$. Therefore, the left-invariant vector fields on $\mathbb{R}$ are constant multiples of d/dx.

Example 16.7 (Left-invariant vector fields on $\mathrm{GL}(n,\mathbb{R})$). Since $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n}$, at any $g \in \mathrm{GL}(n,\mathbb{R})$ there is a canonical identification of the tangent space $T_g(\mathrm{GL}(n,\mathbb{R}))$ with $\mathbb{R}^{n\times n}$, under which a tangent vector corresponds to an $n \times n$ matrix:

$$\sum a_{ij}\frac{\partial}{\partial x_{ij}}\bigg|_g \longleftrightarrow [a_{ij}]\,. \tag{16.5}$$

We use the same letter B to denote alternately a tangent vector $B = \sum b_{ij}\partial/\,\partial x_{ij}|_I \in T_I(G(n,\mathbb{R}))$ at the identity and a matrix $B = [b_{ij}]$. Let $B = \sum b_{ij}\partial/\,\partial x_{ij}|_I \in T_I(\mathrm{GL}(n,\mathbb{R}))$ and let $\tilde{B}$ be the left-invariant vector field on $\mathrm{GL}(n,\mathbb{R})$ generated by B. By Example 8.19,

$$\tilde{B}_g = (\ell_g)_* B \longleftrightarrow gB$$

under the identification (16.5). In terms of the standard basis $\partial/\,\partial x_{ij}|_g$,

$$\tilde{B}_g = \sum_{i,j}(gB)_{ij}\frac{\partial}{\partial x_{ij}}\bigg|_g = \sum_{i,j}\left(\sum_k g_{ik}b_{kj}\right)\frac{\partial}{\partial x_{ij}}\bigg|_g .$$

Proposition 16.8. Any left-invariant vector field X on a Lie group G is C^∞.

Proof. By Proposition 14.3 it suffices to show that for any C^∞ function f on G, the function Xf is also C^∞. Choose a C^∞ curve $c : I \to G$ defined on some interval I containing 0 such that $c(0) = e$ and $c'(0) = X_e$. If $g \in G$, then $gc(t)$ is a curve starting at g with initial vector X_g, since $gc(0) = ge = g$ and

$$(gc)'(0) = \ell_{g*}c'(0) = \ell_{g*}X_e = X_g.$$

By Proposition 8.17,

$$(Xf)(g) = X_g f = \frac{d}{dt}\bigg|_{t=0} f(gc(t))$$

Now the function $f(gc(t))$ is a composition of C^∞ functions

$$\begin{aligned} G \times I &\xrightarrow{\mathbb{1}\times c} G \times G \xrightarrow{\mu} G \xrightarrow{f} \mathbb{R}, \\ (g,t) &\longmapsto (g, c(t)) \\ &\mapsto gc(t) \end{aligned}$$

as such, it is C^∞. Its derivative with respect to t,

$$F(g,t) := \frac{d}{dt}f(gc(t)),$$

is therefore also C^∞. Since $(Xf)(g)$ is a composition of C^∞ functions,

$$G \to G \times I \xrightarrow{F} \mathbb{R}$$
$$g \mapsto (g, 0) \mapsto F(g, 0) = \left.\frac{d}{dt}\right|_{t=0} f(gc(t))$$

it is a C^∞ function on G. This proves that X is a C^∞ vector field on G.
It follows from this proposition that the vector space $L(G)$ of left-invariant vector fields on G is a subspace of the vector space $\mathfrak{X}(G)$ of all C^∞ vector fields on G.

Proposition 16.9. If X and Y are left-invariant vector fields on G, then so is $[X, Y]$.
Proof. For any g in G, X is ℓ_g-related to itself, and Y is ℓ_g-related to itself. By Proposition 14.17, $[X, Y]$ is ℓ_g-related to itself.

9.2.3 16.3 The Lie Algebra of a Lie Group

Recall that a Lie algebra is a vector space $\mathfrak{g}$ together with a bracket, i.e., an anticommutative bilinear map $[] : , \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies the Jacobi identity (Definition 14.12). A Lie subalgebra of a Lie algebra $\mathfrak{g}$ is a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ that is closed under the bracket $[,].ByProposition16.9, thespaceL(G)$ of left-invariant vector fields on a Lie group G is closed under the Lie bracket $[,]andisthereforeaLie$ subalgebra of the Lie algebra $\mathfrak{X}(G)$ of all C^∞ vector fields on G.

As we will see in the next few subsections, the linear isomorphism $\varphi : T_eG \simeq L(G)$ in (16.3) is mutually beneficial to the two vector spaces, for each space has something that the other one lacks. The vector space $L(G)$ has a natural Lie algebra structure given by the Lie bracket of vector fields, while the tangent space at the identity has a natural notion of pushforward, given by the differential of a Lie group homomorphism. The linear isomorphism $\varphi : T_eG \simeq L(G)$ allows us to define a Lie bracket on T_eG and to push forward left-invariant vector fields under a Lie group homomorphism.

We begin with the Lie bracket on T_eG. Given $A, B \in T_eG$, we first map them via φ to the left-invariant vector fields $\tilde{A}, \tilde{B}$, take the Lie bracket $[\tilde{A}, \tilde{B}] = \tilde{A}\tilde{B} - \tilde{B}\tilde{A}$, and then map it back to T_eG via φ^{-1}. Thus, the definition of the Lie bracket $[A, B] \in T_eG$ should be

$$[A, B] = [\tilde{A}, \tilde{B}]_e. \tag{16.6}$$

Proposition 16.10. If $A, B \in T_eG$ and $\tilde{A}, \tilde{B}$ are the left-invariant vector fields they generate, then

$$[\tilde{A}, \tilde{B}] = [A, B]^\sim.$$

Proof. Applying $(\ )^\sim$ to both sides of (16.6) gives

$$[A, B]^\sim = \left([\tilde{A}, \tilde{B}]_e\right)^\sim = [\tilde{A}, \tilde{B}],$$

since $()^\sim$ and $()_e$ are inverse to each other. □

With the Lie bracket $[,\ ]$, the tangent space $T_e(G)$ becomes a Lie algebra, called the Lie algebra of the Lie group G. As a Lie algebra, $T_e(G)$ is usually denoted by $\mathfrak{g}$.

9.2.4 16.4 The Lie Bracket on $\mathfrak{gl}(n, \mathbb{R})$

For the general linear group $\mathrm{GL}(n, \mathbb{R})$, the tangent space at the identity I can be identified with the vector space $\mathbb{R}^{n\times n}$ of all $n \times n$ real matrices. We identified a tangent vector in $T_I(\mathrm{GL}(n, \mathbb{R}))$ with a matrix $A \in \mathbb{R}^{n\times n}$ via

$$\sum a_{ij}\frac{\partial}{\partial x_{ij}}\bigg|_I \longleftrightarrow [a_{ij}]\,. \tag{16.7}$$

The tangent space $T_I\mathrm{GL}(n,\mathbb{R})$ with its Lie algebra structure is denoted by $\mathfrak{gl}(n,\mathbb{R})$. Let $\tilde{A}$ be the left-invariant vector field on $\mathrm{GL}(n,\mathbb{R})$ generated by A. Then on the Lie algebra $\mathfrak{gl}(n,\mathbb{R})$ we have the Lie bracket $[A,B]=[\tilde{A},\tilde{B}]_I$ coming from the Lie bracket of left-invariant vector fields. In the next proposition, we identify the Lie bracket in terms of matrices.

Proposition 16.11. Let

$$A=\sum a_{ij}\frac{\partial}{\partial x_{ij}}\bigg|_I,\quad B=\sum b_{ij}\frac{\partial}{\partial x_{ij}}\bigg|_I\in T_I(\mathrm{GL}(n,\mathbb{R})).$$

If

$$[A,B]=[\tilde{A},\tilde{B}]_I=\sum c_{ij}\frac{\partial}{\partial x_{ij}}\bigg|_I, \tag{16.8}$$

then

$$c_{ij}=\sum_k a_{ik}b_{kj}-b_{ik}a_{kj}.$$

Thus, if derivations are identified with matrices via (16.7), then

$$[A,B]=AB-BA.$$

Proof. Applying both sides of (16.8) to x_{ij}, we get

$$\begin{aligned}c_{ij}&=[\tilde{A},\tilde{B}]_I x_{ij}=\tilde{A}_I\tilde{B}x_{ij}-\tilde{B}_I\tilde{A}x_{ij}\\&=A\tilde{B}x_{ij}-B\tilde{A}x_{ij}\quad\Big(\text{because }\tilde{A}_I=A,\tilde{B}_I=B\Big),\end{aligned}$$

so it is necessary to find a formula for the function $\tilde{B}x_{ij}$.
In Example 16.7 we found that the left-invariant vector field $\tilde{B}$ on $\mathrm{GL}(n,\mathbb{R})$ is given by

$$\tilde{B}_g=\sum_{i,j}(gB)_{ij}\frac{\partial}{\partial x_{ij}}\bigg|_g\quad\text{at } g\in\mathrm{GL}(n,\mathbb{R}).$$

Hence,

$$\tilde{B}_g x_{ij}=(gB)_{ij}=\sum_k g_{ik}b_{kj}=\sum_k b_{kj}x_{ik}(g).$$

Since this formula holds for all $g\in\mathrm{GL}(n,\mathbb{R})$, the function $\tilde{B}x_{ij}$ is

$$\tilde{B}x_{ij}=\sum_k b_{kj}x_{ik}$$

It follows that

$$\begin{aligned}A\tilde{B}x_{ij}&=\sum_{p,q}a_{pq}\frac{\partial}{\partial x_{pq}}\bigg|_I\left(\sum_k b_{kj}x_{ik}\right)=\sum_{p,q,k}a_{pq}b_{kj}\delta_{ip}\delta_{kq}\\&=\sum_k a_{ik}b_{kj}=(AB)_{ij}.\end{aligned}$$

Interchanging A and B gives

$$B\tilde{A}x_{ij} = \sum_k b_{ik}a_{kj} = (BA)_{ij}$$

Therefore,

$$c_{ij} = \sum_k a_{ik}b_{kj} - b_{ik}a_{kj} = (AB - BA)_{ij}$$

9.2.5 16.5 The Pushforward of Left-invariant Vector Fields

As we noted in Subsection 14.5, if $F : N \to M$ is a C^∞ map of manifolds and X is a C^∞ vector field on N, the pushforward F_*X is in general not defined except when F is a diffeomorphism. In the case of Lie groups, however, because of the correspondence between left-invariant vector fields and tangent vectors at the identity, it is possible to push forward left-invariant vector fields under a Lie group homomorphism.

Let $F : H \to G$ be a Lie group homomorphism. A left-invariant vector field X on H is generated by its value $A = X_e \in T_eH$ at the identity, so that $X = \tilde{A}$. Since a Lie group homomorphism $F : H \to G$ maps the identity of H to the identity of G, its differential $F_{*,e}$ at the identity is a linear map from T_eH to T_eG. The diagrams

$$\begin{tikzcd} T_eH \arrow[r, "F_{*,e}"] \arrow[d, "\simeq"] & T_eG \arrow[d, "\simeq"] \\ L(H) \arrow[r, dashed] & L(G), \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, mapsto] \arrow[d, mapsto] & F_{*,e}A \arrow[d, mapsto] \\ \tilde{A} \arrow[r, dashed, mapsto] & (F_{*,e}A)^\sim \end{tikzcd}$$

show clearly the existence of an induced linear map $F_* : L(H) \to L(G)$ on leftinvariant vector fields as well as a way to define it.

Definition 16.12. Let $F : H \to G$ be a Lie group homomorphism. Define $F_* : L(H) \to L(G)$ by

$$F_*(\tilde{A}) = (F_{*,e}A)^\sim$$

for all $A \in T_eH$.

Proposition 16.13. If $F : H \to G$ is a Lie group homomorphism and X is a leftinvariant vector field on H, then the left-invariant vector field F_*X on G is F-related to the left-invariant vector field X.

Proof. For each $h \in H$, we need to verify that

$$F_{*,h}(X_h) = (F_*X)_{F(h)}. \tag{16.9}$$

The left-hand side of (16.9) is

$$F_{*,h}(X_h) = F_{*,h}(\ell_{h*,e}X_e) = (F \circ \ell_h)_{*,e}(X_e),$$

while the right-hand side of (16.9) is

$$\begin{aligned} (F_*X)_{F(h)} &= (F_{*,e}X_e)_{\tilde{F}(h)} && (\text{ definition of } F_*X) \\ &= \ell_{F(h)*}F_{*,e}(X_e) && (\text{ definition of left invariance }) \\ &= \left(\ell_{F(h)} \circ F\right)_{*,e}(X_e) && (\text{ chain rule }). \end{aligned}$$

Since F is a Lie group homomorphism, we have $F \circ \ell_h = \ell_{F(h)} \circ F$, so the two sides of (16.9) are equal. □

If $F : H \to G$ is a Lie group homomorphism and X is a left-invariant vector field on H, we will call F_*X the pushforward of X under F.

9.2.6 16.6 The Differential as a Lie Algebra Homomorphism

Proposition 16.14. If $F : H \to G$ is a Lie group homomorphism, then its differential at the identity,

$$F_* = F_{*,e} : T_eH \to T_eG,$$

is a Lie algebra homomorphism, i.e., a linear map such that for all $A, B \in T_eH$,

$$F_*[A, B] = [F_*A, F_*B].$$

Proof. By Proposition 16.13, the vector field $F_*\tilde{A}$ on G is F-related to the vector field $\tilde{A}$ on H, and the vector field $F_*\tilde{B}$ is F-related to $\tilde{B}$ on H. Hence, the bracket $\left[F_*\tilde{A}, F_*\tilde{B}\right]$ on G is F-related to the bracket $[\tilde{A}, \tilde{B}]$ on H (Proposition 14.17). This means that

$$F_*\left([\tilde{A}, \tilde{B}]_e\right) = \left[F_*\tilde{A}, F_*\tilde{B}\right]_{F(e)} = \left[F_*\tilde{A}, F_*\tilde{B}\right]_e$$

The left-hand side of this equality is $F_*[A, B]$, while the right-hand side is

$$\begin{aligned} \left[F_*\tilde{A}, F_*\tilde{B}\right]_e &= [(F_*A)^\sim, (F_*B)^\sim]_e && \left(\text{ definition of } F_*\tilde{A}\right) \\ &= [F_*A, F_*B] && (\text{ definition of } [,] \text{ on } T_eG) \end{aligned}$$

Equating the two sides gives

$$F_*[A, B] = [F_*A, F_*B].$$

Suppose H is a Lie subgroup of a Lie group G, with inclusion map $i : H \to G$. Since i is an immersion, its differential

$$i_* : T_eH \to T_eG$$

is injective. To distinguish the Lie bracket on T_eH from the Lie bracket on T_eG, we temporarily attach subscripts T_eH and T_eG to the two Lie brackets respectively. By Proposition 16.14, for $X, Y \in T_eH$,

$$i_*([X, Y]_{T_eH}) = [i_*X, i_*Y]_{T_eG} \tag{16.10}$$

This shows that if T_eH is identified with a subspace of T_eG via i_*, then the bracket on T_eH is the restriction of the bracket on T_eG to T_eH. Thus, the Lie algebra of a Lie subgroup H may be identified with a Lie subalgebra of the Lie algebra of G.

In general, the Lie algebras of the classical groups are denoted by gothic letters. For example, the Lie algebras of $\mathrm{GL}(n, \mathbb{R}), \mathrm{SL}(n, \mathbb{R}), \mathrm{O}(n)$, and $\mathrm{U}(n)$ are denoted by $\mathfrak{gl}(n, \mathbb{R}), \mathfrak{sl}(n, \mathbb{R}), \mathfrak{o}(n)$, and $\mathfrak{u}(n)$, respectively. By (16.10) and Proposition 16.11, the Lie algebra structures on $\mathfrak{sl}(n, \mathbb{R}), \mathfrak{o}(n)$, and $\mathfrak{u}(n)$ are given by

$$[A, B] = AB - BA$$

as on $\mathfrak{gl}(n, \mathbb{R})$.

Remark 16.15. A fundamental theorem in Lie group theory asserts the existence of a one-to-one correspondence between the connected Lie subgroups of a Lie group G and the Lie subalgebras

of its Lie algebra $\mathfrak{g}$ [38, Theorem 3.19, Corollary (a), p. 95]. For the torus $\mathbb{R}^2/\mathbb{Z}^2$, the Lie algebra $\mathfrak{g}$ has $\mathbb{R}^2$ as the underlying vector space and the one-dimensional Lie subalgebras are all the lines through the origin. Each line through the origin in $\mathbb{R}^2$ is a subgroup of $\mathbb{R}^2$ under addition. Its image under the quotient map $\mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ is a subgroup of the torus $\mathbb{R}^2/\mathbb{Z}^2$. If a line has rational slope, then its image is a regular submanifold of the torus. If a line has irrational slope, then its image is only an immersed submanifold of the torus. According to the correspondence theorem just quoted, the one-dimensional connected Lie subgroups of the torus are the images of all the lines through the origin. Note that if a Lie subgroup had been defined as a subgroup that is also a regular submanifold, then one would have to exclude all the lines with irrational slopes as Lie subgroups of the torus, and it would not be possible to have a one-to-one correspondence between the connected subgroups of a Lie group and the Lie subalgebras of its Lie algebra. It is because of our desire for such a correspondence that a Lie subgroup of a Lie group is defined to be a subgroup that is also an immersed submanifold.

9.2.7 Problems

In the following problems the word "dimension" refers to the dimension as a real vector space or as a manifold.

16.1. Skew-hermitian Matrices

A complex matrix $X \in \mathbb{C}^{n\times n}$ is said to be skew-Hermitian if its conjugate transpose $\bar{X}^T$ is equal to $-X$. Let V be the vector space of $n \times n$ skew-Hermitian matrices. Show that $\dim V = n^2$.

16.2. Lie Algebra of a Unitary Group

Show that the tangent space at the identity I of the unitary group $\mathrm{U}(n)$ is the vector space of $n \times n$ skew-Hermitian matrices.

16.3 Lie Algebra of a Symplectic Group

Refer to Problem 15.15 for the definition and notation concerning the symplectic group $\mathrm{Sp}(n)$. Show that the tangent space at the identity I of the symplectic group $\mathrm{Sp}(n) \subset \mathrm{GL}(n, \mathbb{H})$ is the vector space of all $n \times n$ quaternionic matrices X such that $\bar{X}^T = -X$.

16.4. Lie Algebra of a Complex Symplectic Group

(a) Show that the tangent space at the identity I of $\mathrm{Sp}(2n, \mathbb{C}) \subset \mathrm{GL}(2n, \mathbb{C})$ is the vector space of all $2n \times 2n$ complex matrices X such that JX is symmetric.
(b) Calculate the dimension of $\mathrm{Sp}(2n, \mathbb{C})$.

16.5. Left-invariant Vector Fields on $\mathbb{R}^n$

Find the left-invariant vector fields on $\mathbb{R}^n$.

16.6. Left-invariant Vector Fields on a Circle

Find the left-invariant vector fields on S^1.

16.7. Integral Curves of a Left-invariant Vector Field

Let $A \in \mathfrak{gl}(n, \mathbb{R})$ and let $\tilde{A}$ be the left-invariant vector field on $\mathrm{GL}(n, \mathbb{R})$ generated by A. Show that $c(t) = e^{tA}$ is the integral curve of $\tilde{A}$ starting at the identity matrix I. Find the integral curve of $\tilde{A}$ starting at $g \in \mathrm{GL}(n, \mathbb{R})$.

16.8. Parallelizable Manifolds

A manifold whose tangent bundle is trivial is said to be parallelizable. If M is a manifold of dimension n, show that parallelizability is equivalent to the existence of a smooth frame $X_1, \ldots, X_n$ on M.

16.9. Parallelizability of a Lie Group

Show that every Lie group is parallelizable.

16.10.* the Pushforward of Left-invariant Vector Fields

Let $F : H \to G$ be a Lie group homomorphism and let X and Y be left-invariant vector fields on H. Prove that $F_*[X, Y] = [F_*X, F_*Y]$.

16.11. The Adjoint Representation

Let G be a Lie group of dimension n with Lie algebra $\mathfrak{g}$.
(a) For each $a \in G$, the differential at the identity of the conjugation map $c_a := \ell_a \circ r_{a^{-1}} : G \to G$ is a linear isomorphism $c_{a*} : \mathfrak{g} \to \mathfrak{g}$. Hence, $c_{a*} \in \mathrm{GL}(\mathfrak{g})$. Show that the map $\mathrm{Ad} : G \to \mathrm{GL}(\mathfrak{g})$ defined by $\mathrm{Ad}(a) = c_{a*}$ is a group homomorphism. It is called the adjoint representation of the Lie group G.
(b) Show that Ad: $G \to \mathrm{GL}(\mathfrak{g})$ is C^∞.

16.12. A Lie Algebra Structure on $\mathbb{R}^3$

The Lie algebra $\mathfrak{o}(n)$ of the orthogonal group $\mathrm{O}(n)$ is the Lie algebra of $n \times n$ skew-symmetric real matrices, with Lie bracket $[A, B] = AB - BA$. When $n = 3$, there is a vector space isomorphism $\varphi : \mathfrak{o}(3) \to \mathbb{R}^3$,

$$\varphi(A) = \varphi\left(\begin{bmatrix} 0 & a_1 & a_2 \\ -a_1 & 0 & a_3 \\ -a_2 & -a_3 & 0 \end{bmatrix}\right) = \begin{bmatrix} a_1 \\ -a_2 \\ a_3 \end{bmatrix} = a$$

Prove that $\varphi([A, B]) = \varphi(A) \times \varphi(B)$. Thus, $\mathbb{R}^3$ with the cross product is a Lie algebra.

10 5 Differential Forms

Differential forms are generalizations of real-valued functions on a manifold. Instead of assigning to each point of the manifold a number, a differential k-form assigns to each point a k-covector on its tangent space. For $k = 0$ and 1 , differential k-forms are functions and covector fields respectively.

Élie Cartan
(1869-1951)

Differential forms play a crucial role in manifold theory. First and foremost, they are intrinsic objects associated to any manifold, and so can be used to construct diffeomorphism invariants of a manifold. In contrast to vector fields, which are also intrinsic to a manifold, differential forms have a far richer algebraic structure. Due to the existence of the wedge product, a grading, and the exterior derivative, the set of smooth forms on a manifold is both a graded algebra and a differential complex. Such an algebraic structure is called a differential graded algebra. Moreover, the differential complex of smooth forms on a manifold can be pulled back under a smooth map, making the complex into a contravariant functor called the de Rham complex of the manifold. We will eventually construct the de Rham cohomology of a manifold from the de Rham complex.

Because integration of functions on a Euclidean space depends on a choice of coordinates and is not invariant under a change of coordinates, it is not possible to integrate functions on a manifold. The highest possible degree of a differential form is the dimension of the manifold. Among differential forms, those of top degree turn out to transform correctly under a change of coordinates and are precisely the objects that can be integrated. The theory of integration on a manifold would not be possible without differential forms.

Very loosely speaking, differential forms are whatever appears under an integral sign. In this sense, differential forms are as old as calculus, and many theorems in calculus such as Cauchy's integral theorem or Green's theorem can be interpreted as statements about differential forms. Although it is difficult to say who first gave differential forms an independent meaning, Henri Poincaré [32] and Élie Cartan [5] are generally both regarded as pioneers in this regard. In the paper [5] published in 1899, Cartan defined formally the algebra of differential forms on $\mathbb{R}^n$ as the anticommutative graded algebra over C^∞ functions generated by $dx^1, \ldots, dx^n$ in degree 1 . In the same paper one finds for the first time the exterior derivative on differential forms. The modern definition of a differential form as a section of an exterior power of the cotangent bundle appeared in the late forties [6], after the theory of fiber bundles came into being.

In this chapter we give an introduction to differential forms from the vector bundle point of view. For simplicity we start with 1-forms, which already have many of the properties of k-forms. We give various characterizations of smooth forms, and show how to multiply, differentiate, and pull back these forms. In addition to the exterior derivative, we also introduce the Lie derivative and interior multiplication, two other intrinsic operations on a manifold.

10.1 §17 Differential 1-forms

Let M be a smooth manifold and p a point in M. The cotangent space of M at p, denoted by $T_p^*(M)$ or T_p^*M, is defined to be the dual space of the tangent space T_pM :

$$T_p^*M = (T_pM)^\vee = \operatorname{Hom}(T_pM, \mathbb{R})$$

An element of the cotangent space T_p^*M is called a covector at p. Thus, a covector ω_p at p is a linear function

$$\omega_p : T_pM \to \mathbb{R}$$

A covector field, a differential 1-form, or more simply a 1 -form on M, is a function ω that assigns to each point p in M a covector ω_p at p. In this sense it is dual to a vector field on M, which assigns to each point in M a tangent vector at p. There are many reasons for the great utility of differential forms in manifold theory, among which is the fact that they can be pulled back under a map. This is in contrast to vector fields, which in general cannot be pushed forward under a map.

Covector fields arise naturally even when one is interested only in vector fields. For example, if X is a C^∞ vector field on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n, X_p = \sum a^i \partial/\partial x^i \mid p$. The coefficient a^i depends on the vector X_p. It is in fact a linear function: $T_p\mathbb{R}^n \to \mathbb{R}$, i.e., a covector at p. As p varies over $\mathbb{R}^n, a^i$ becomes a covector field on $\mathbb{R}^n$. Indeed, it is none other than the 1 -form dx^i that picks out the i th coefficient of a vector field relative to the standard frame $\partial/\partial x^1, \ldots, \partial/\partial x^n$.

10.1.1 17.1 The Differential of a Function

Definition 17.1. If f is a C^∞ real-valued function on a manifold M, its differential is defined to be the 1 -form df on M such that for any $p \in M$ and $X_p \in T_pM$,

$$(df)_p(X_p) = X_p f.$$

Instead of $(df)_p$, we also write $df|_p$ for the value of the 1 -form df at p. This is parallel to the two notations for a tangent vector: $(d/dt)_p = d/\,dt|_p$.

In Subsection 8.2 we encountered another notion of the differential, denoted by f_*, for a map f between manifolds. Let us compare the two notions of the differential.

Proposition 17.2. If $f : M \to \mathbb{R}$ is a C^∞ function, then for $p \in M$ and $X_p \in T_pM$,

$$f_*(X_p) = (df)_p(X_p) \left.\frac{d}{dt}\right|_{f(p)}.$$

Proof. Since $f_*(X_p) \in T_{f(p)}\mathbb{R}$, there is a real number a such that

$$f_*(X_p) = a \left.\frac{d}{dt}\right|_{f(p)}. \tag{17.1}$$

To evaluate a, apply both sides of (17.1) to x :

$$a = f_*(X_p)(t) = X_p(t \circ f) = X_p f = (df)_p(X_p).$$

□

This proposition shows that under the canonical identification of the tangent space $T_{f(p)}\mathbb{R}$ with $\mathbb{R}$ via

$$a \frac{d}{dt}\bigg|_{f(p)} \longleftrightarrow a,$$

f_* is the same as df. For this reason, we are justified in calling both of them the differential of f. In terms of the differential df, a C^∞ function $f : M \to \mathbb{R}$ has a critical point at $p \in M$ if and only if $(df)_p = 0$.

10.1.2 17.2 Local Expression for a Differential 1-form

Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a coordinate chart on a manifold M. Then the differentials $dx^1, \ldots, dx^n$ are 1 -forms on U.

Proposition 17.3. At each point $p \in U$, the covectors $(dx^1)_p, \ldots, (dx^n)_p$ form a basis for the cotangent space T_p^*M dual to the basis $\partial/\partial x^1|_p, \ldots, \partial/\partial x^n|_p$ for the tangent space T_pM.

Proof. The proof is just like that in the Euclidean case (Proposition 4.1):

$$(dx^i)_p\left(\frac{\partial}{\partial x^j}\bigg|_p\right) = \frac{\partial}{\partial x^j}\bigg|_p x^i = \delta_j^i$$

Thus, every 1-form ω on U can be written as a linear combination

$$\omega = \sum a_i dx^i$$

where the coefficients a_i are functions on U. In particular, if f is a C^∞ function on M, then the restriction of the 1 -form df to U must be a linear combination

$$df = \sum a_i dx^i$$

To find a_j, we apply the usual trick of evaluating both sides on $\partial/\partial x^j$:

$$(df)\left(\frac{\partial}{\partial x^j}\right) = \sum_i a_i dx^i\left(\frac{\partial}{\partial x^j}\right) \quad \Longrightarrow \quad \frac{\partial f}{\partial x^j} = \sum_i a_i \delta_j^i = a_j$$

This gives a local expression for df :

$$df = \sum \frac{\partial f}{\partial x^i} dx^i \tag{17.2}$$

10.1.3 17.3 The Cotangent Bundle

The underlying set of the cotangent bundle T^*M of a manifold M is the union of the cotangent spaces at all the points of M :

$$T^*M := \bigcup_{p \in M} T_p^*M \tag{17.3}$$

Just as in the case of the tangent bundle, the union (17.3) is a disjoint union and there is a natural map $\pi : T^*M \to M$ given by $\pi(\alpha) = p$ if $\alpha \in T_p^*M$. Mimicking the construction of the

tangent bundle, we give T^*M a topology as follows. If $(U,\phi)=(U,x^1,\ldots,x^n)$ is a chart on M and $p\in U$, then each $\alpha\in T_p^*M$ can be written uniquely as a linear combination

$$\alpha=\sum c_i(\alpha)dx^i\Big|_p$$

This gives rise to a bijection

$$\begin{aligned}\tilde{\phi}: T^*U &\to \phi(U)\times\mathbb{R}^n, \\ \alpha &\mapsto (\phi(p),c_1(\alpha),\ldots,c_n(\alpha))=(\phi\circ\pi,c_1,\ldots,c_n)(\alpha).\end{aligned} \tag{17.4}$$

Using this bijection, we can transfer the topology of $\phi(U)\times\mathbb{R}^n$ to T^*U.
Now for each domain U of a chart in the maximal atlas of M, let $\mathcal{B}_U$ be the collection of all open subsets of T^*U, and let $\mathcal{B}$ be the union of the $\mathcal{B}_U$. As in Subsection 12.1, $\mathfrak{B}$ satisfies the conditions for a collection of subsets of T^*M to be a
basis. We give T^*M the topology generated by the basis $\mathcal{B}$. As for the tangent bundle, with the maps $\tilde{\phi}=(x^1\circ\pi,\ldots,x^n\circ\pi,c_1,\ldots,c_n)$ of (17.4) as coordinate maps, T^*M becomes a C^∞ manifold, and the projection map $\pi: T^*M\to M$ becomes a vector bundle of rank n over M, justifying the "bundle" in the name "cotangent bundle." If $x^1,\ldots,x^n$ are coordinates on $U\subset M$, then $\pi^*x^1,\ldots,\pi^*x^n,c_1,\ldots,c_n$ are coordinates on $\pi^{-1}U\subset T^*M$. Properly speaking, the cotangent bundle of a manifold M is the triple (T^*M,M,π), while T^*M and M are the total space and the base space of the cotangent bundle respectively, but by abuse of language, it is customary to call T^*M the cotangent bundle of M.

In terms of the cotangent bundle, a 1 -form on M is simply a section of the cotangent bundle T^*M; i.e., it is a map $\omega: M\to T^*M$ such that $\pi\circ\omega=\mathbb{1}_M$, the identity map on M. We say that a 1 -form ω is C^∞ if it is C^∞ as a map $M\to T^*M$.

Example 17.4 (Liouville form on the cotangent bundle). If a manifold M has dimension n, then the total space T^*M of its cotangent bundle $\pi: T^*M\to M$ is a manifold of dimension $2n$. Remarkably, on T^*M there is a 1-form λ, called the Liouville form (or the Poincaré form in some books), defined independently of charts as follows. A point in T^*M is a covector $\omega_p\in T_p^*M$ at some point $p\in M$. If X_{ω_p} is a tangent vector to T^*M at ω_p, then the pushforward $\pi_*\left(X_{\omega_p}\right)$ is a tangent vector to M at p. Therefore, one can pair up ω_p and $\pi_*\left(X_{\omega_p}\right)$ to obtain a real number $\omega_p\left(\pi_*\left(X_{\omega_p}\right)\right)$. Define

$$\lambda_{\omega_p}\left(X_{\omega_p}\right)=\omega_p\left(\pi_*\left(X_{\omega_p}\right)\right)$$

The cotangent bundle and the Liouville form on it play an important role in the mathematical theory of classical mechanics [1, p. 202].

10.1.4 17.4 Characterization of c^∞ 1-Forms

We define a 1 -form ω on a manifold M to be smooth if $\omega: M\to T^*M$ is smooth as a section of the cotangent bundle $\pi: T^*M\to M$. The set of all smooth 1-forms on M has the structure of a vector space, denoted by $\Omega^1(M)$. In a coordinate chart $(U,\phi)=(U,x^1,\ldots,x^n)$ on M, the value of the 1 -form ω at $p\in U$ is a linear combination

$$\omega_p=\sum a_i(p)dx^i\Big|_p$$

As p varies in U, the coefficients a_i become functions on U. We will now derive smoothness criteria for a 1 -form in terms of the coefficient functions a_i. The development is parallel to that of smoothness criteria for a vector field in Subsection 14.1.

By Subsection 17.3, the chart (U,ϕ) on M induces a chart

$$\left(T^*U, \tilde{\phi}\right) = \left(T^*U, \bar{x}^1, \ldots, \bar{x}^n, c_1, \ldots, c_n\right)$$

on T^*M, where $\bar{x}^i = \pi^* x^i = x^i \circ \pi$ and the c_i are defined by

$$\alpha = \sum c_i(\alpha) dx^i \Big|_p, \quad \alpha \in T_p^* M$$

Comparing the coefficients in

$$\omega_p = \sum a_i(p) dx^i \Big|_p = \sum c_i\left(\omega_p\right) dx^i \Big|_p$$

we get $a_i = c_i \circ \omega$, where ω is viewed as a map from U to T^*U. Being coordinate functions, the c_i are smooth on T^*U. Thus, if ω is smooth, then the coefficients a_i of $\omega = \sum a_i dx^i$ relative to the frame dx^i are smooth on U. The converse is also true, as indicated in the following lemma.

Lemma 17.5. Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart on a manifold M. A 1-form $\omega = \sum a_i dx^i$ on U is smooth if and only if the coefficient functions a_i are all smooth.

Proof. This lemma is a special case of Proposition 12.12, with E the cotangent bundle T^*M and s_j the coordinate 1-forms dx^j. However, a direct proof is also possible (cf. Lemma 14.1).

Since $\tilde{\phi} : T^*U \to U \times \mathbb{R}^n$ is a diffeomorphism, $\omega : U \to T^*M$ is smooth if and only if $\tilde{\phi} \circ \omega : U \to U \times \mathbb{R}^n$ is smooth. For $p \in U$,

$$\begin{aligned}(\tilde{\phi} \circ \omega)(p) = \tilde{\phi}\left(\omega_p\right) &= \left(x^1(p), \ldots, x^n(p), c_1\left(\omega_p\right), \ldots, c_n\left(\omega_p\right)\right) \\ &= \left(x^1(p), \ldots, x^n(p), a_1(p), \ldots, a_n(p)\right)\end{aligned}$$

As coordinate functions, $x^1, \ldots, x^n$ are smooth on U. Therefore, by Proposition 6.13, $\tilde{\phi} \circ \omega$ is smooth on U if and only if all a_i are smooth on U.

Proposition 17.6 (Smoothness of a 1-form in terms of coefficients). Let ω be a 1 -form on a manifold M. The following are equivalent:
(i) The 1-form ω is smooth on M.
(ii) The manifold M has an atlas such that on any chart $(U, x^1, \ldots, x^n)$ of the atlas, the coefficients a_i of $\omega = \sum a_i dx^i$ relative to the frame dx^i are all smooth.
(iii) On any chart $(U, x^1, \ldots, x^n)$ on the manifold, the coefficients a_i of $\omega = \sum a_i dx^i$ relative to the frame dx^i are all smooth.

Proof. The proof is omitted, since it is virtually identical to that of Proposition 14.2.

Corollary 17.7. If f is a C^∞ function on a manifold M, then its differential df is a C^∞ 1-form on M.

Proof. On any chart $(U, x^1, \ldots, x^n)$ on M, the equality $df = \sum \left(\partial f / \partial x^i\right) dx^i$ holds. Since the coefficients $\partial f / \partial x^i$ are all C^∞, by Proposition 17.6(iii), the 1-form df is C^∞.

If ω is a 1 -form and X is a vector field on a manifold M, we define a function $\omega(X)$ on M by the formula

$$\omega(X)_p = \omega_p\left(X_p\right) \in \mathbb{R}, \quad p \in M.$$

Proposition 17.8 (Linearity of a 1-form over functions). Let ω be a 1-form on a manifold M. If f is a function and X is a vector field on M, then $\omega(fX) = f\omega(X)$.

Proof. At each point $p \in M$,

$$\omega(fX)_p = \omega_p\left(f(p) X_p\right) = f(p) \omega_p\left(X_p\right) = (f\omega(X))_p,$$

because $\omega(X)$ is defined pointwise, and at each point, ω_p is $\mathbb{R}$-linear in its argument. $\square$

Proposition 17.9 (Smoothness of a 1-form in terms of vector fields). A 1-form ω on a manifold M is C^∞ if and only if for every C^∞ vector field X on M, the function $\omega(X)$ is C^∞ on M.

Proof.

($\Rightarrow$) Suppose ω is a C^∞1-form and X is a C^∞ vector field on M. On any chart $(U, x^1, \ldots, x^n)$ on M, by Propositions 14.2 and 17.6, $\omega = \sum a_i dx^i$ and $X = \sum b^j \partial/\partial x^j$ for C^∞ functions a_i, b^j. By the linearity of 1 -forms over functions (Proposition 17.8),

$$\omega(X) = \left(\sum a_i dx^i\right)\left(\sum b^j \frac{\partial}{\partial x^j}\right) = \sum_{i,j} a_i b^j \delta_j^i = \sum a_i b^i,$$

a C^∞ function on U. Since U is an arbitrary chart on M, the function $\omega(X)$ is C^∞ on M.
($\Leftarrow$) Suppose ω is a 1 -form on M such that the function $\omega(X)$ is C^∞ for every C^∞ vector field X on M. Given $p \in M$, choose a coordinate neighborhood $(U, x^1, \ldots, x^n)$ about p. Then $\omega = \sum a_i dx^i$ on U for some functions a_i.

Fix an integer $j, 1 \leq j \leq n$. By Proposition 14.4, we can extend the C^∞ vector field $X = \partial/\partial x^j$ on U to a C^∞ vector field $\bar{X}$ on M that agrees with $\partial/\partial x^j$ in a neighborhood V_p^j of p in U. Restricted to the open set V_p^j,

$$\omega(\bar{X}) = \left(\sum a_i dx^i\right)\left(\frac{\partial}{\partial x^j}\right) = a_j.$$

This proves that a_j is C^∞ on the coordinate chart $\left(V_p^j, x^1, \ldots, x^n\right)$. On the intersection $V_p := \bigcap_j V_p^j$, all a_j are C^∞. By Lemma 17.5, the 1-form ω is C^∞ on V_p. So for each $p \in M$, we have found a coordinate neighborhood V_p on which ω is C^∞. It follows that ω is a C^∞ map from M to T^*M. $\square$

Let $\mathcal{F} = C^\infty(M)$ be the ring of all C^∞ functions on M. By Proposition 17.9, a 1form ω on M defines a map $\mathfrak{X}(M) \to \mathcal{F}, X \mapsto \omega(X)$. According to Proposition 17.8, this map is both $\mathbb{R}$-linear and $\mathcal{F}$-linear.

10.1.5 17.5 Pullback of 1-forms

If $F : N \to M$ is a C^∞ map of manifolds, then at each point $p \in N$ the differential

$$F_{*,p} : T_p N \to T_{F(p)} M$$

is a linear map that pushes forward vectors at p from N to M. The codifferential, i.e., the dual of the differential,

$$(F_{*,p})^\vee : T_{F(p)}^* M \to T_p^* N,$$

reverses the arrow and pulls back a covector at $F(p)$ from M to N. Another notation for the codifferential is $F^* = (F_{*,p})^\vee$. By the definition of the dual, if $\omega_{F(p)} \in T_{F(p)}^* M$ is a covector at $F(p)$ and $X_p \in T_p N$ is a tangent vector at p, then

$$F^*\left(\omega_{F(p)}\right)(X_p) = \left((F_{*,p})^\vee \omega_{F(p)}\right)(X_p) = \omega_{F(p)}\left(F_{*,p} X_p\right)$$

We call $F^*\left(\omega_{F(p)}\right)$ the pullback of the covector $\omega_{F(p)}$ by F. Thus, the pullback of covectors is simply the codifferential.

Unlike vector fields, which in general cannot be pushed forward under a C^∞ map, every covector field can be pulled back by a C^∞ map. If ω is a 1 -form on M, its pullback $F^*\omega$ is the 1-form on N defined pointwise by

$$(F^*\omega)_p = F^*\left(\omega_{F(p)}\right), \quad p \in N.$$

This means that

$$(F^*\omega)_p (X_p) = \omega_{F(p)} (F_* (X_p))$$

for all $X_p \in T_pN$. Recall that functions can also be pulled back: if F is a C^∞ map from N to M and $g \in C^\infty(M)$, then $F^*g = g \circ F \in C^\infty(N)$.

This difference in the behavior of vector fields and forms under a map can be traced to a basic asymmetry in the concept of a function-every point in the domain maps to only one image point in the range, but a point in the range can have several preimage points in the domain.

Now that we have defined the pullback of a 1-form under a map, a question naturally suggests itself. Is the pullback of a C^∞1-form under a C^∞ map C^∞ ? To answer this question, we first need to establish three commutation properties of the pullback: its commutation with the differential, sum, and product.

Proposition 17.10 (Commutation of the pullback with the differential). Let $F : N \to M$ be a C^∞ map of manifolds. For any $h \in C^\infty(M), F^*(dh) = d(F^*h)$.

Proof. It suffices to check that for any point $p \in N$ and any tangent vector $X_p \in T_pN$,

$$(F^*dh)_p (X_p) = (dF^*h)_p (X_p) \tag{17.5}$$

The left-hand side of (17.5) is

$$\begin{aligned}
(F^*dh)_p (X_p) &= (dh)_{F(p)} (F_* (X_p)) && (\text{ definition of the pullback of a 1-form }) \\
&= (F_* (X_p))\, h && (\text{ definition of the differential } dh) \\
&= X_p(h \circ F) && (\text{ definition of } F_*).
\end{aligned}$$

The right-hand side of (17.5) is

$$\begin{aligned}
(dF^*h)_p (X_p) &= X_p (F^*h) && (\text{ definition of } d \text{ of a function }) \\
&= X_p(h \circ F) \quad (\text{ definition of } F^* \text{ of a function }).
\end{aligned}$$

Pullback of functions and 1-forms respects addition and scalar multiplication.

Proposition 17.11 (Pullback of a sum and a product). Let $F : N \to M$ be a C^∞ map of manifolds. Suppose $\omega, \tau \in \Omega^1(M)$ and $g \in C^\infty(M)$. Then

(i) $F^*(\omega + \tau) = F^*\omega + F^*\tau$,

(ii) $F^*(g\omega) = (F^*g)(F^*\omega)$.

Proof. Problem 17.5.

Proposition 17.12 (Pullback of a C^∞ 1-form). The pullback $F^*\omega$ of a C^∞ 1-form ω on M under a C^∞ map $F : N \to M$ is C^∞1-form on N.

Proof. Given $p \in N$, choose a chart $(V, \psi) = (V, y^1, \ldots, y^n)$ in M about $F(p)$. By the continuity of F, there is a chart $(U, \phi) = (U, x^1, \ldots, x^n)$ about p in N such that $F(U) \subset V$. On $V, \omega = \sum a_i dy^i$ for some $a_i \in C^\infty(V)$. On U,

$$\begin{aligned}
F^*\omega &= \sum (F^*a_i)\, F^*\left(dy^i\right) && (\text{ Proposition 17.11}) \\
&= \sum (F^*a_i)\, dF^*y^i && (\text{Proposition 17.10}) \\
&= \sum (a_i \circ F)\, d\left(y^i \circ F\right) && (\text{definition of } F^* \text{ of a function}) \\
&= \sum_{i,j} (a_i \circ F) \frac{\partial F^i}{\partial x^j} dx^j && (\text{equation (17.2)}).
\end{aligned}$$

Since the coefficients $(a_i \circ F)\, \partial F^i / \partial x^j$ are all C^∞, by Proposition 17.5 the 1-form $F^*\omega$ is C^∞ on U and therefore at p. Since p was an arbitrary point in N, the pullback $F^*\omega$ is C^∞ on N.

Example 17.13 (Liouville form on the cotangent bundle). Let M be a manifold. In terms of the pullback, the Liouville form λ on the cotangent bundle T^*M introduced in Example 17.4 can be expressed as $\lambda_{\omega_p} = \pi^* (\omega_p)$ at any $\omega_p \in T^*M$.

10.1.6 17.6 Restriction of 1-forms to an Immersed Submanifold

Let $S \subset M$ be an immersed submanifold and $i : S \to M$ the inclusion map. At any $p \in S$, since the differential $i_* : T_pS \to T_pM$ is injective, one may view the tangent space T_pS as a subspace of T_pM. If ω is a 1-form on M, then the restriction of ω to S is the 1-form $\omega|_S$ defined by

$$(\omega|_S)_p (v) = \omega_p(v) \quad \text{for all } p \in S \text{ and } v \in T_pS$$

Thus, the restriction $\omega|_S$ is the same as ω except that its domain has been restricted from M to S and for each $p \in S$, the domain of $(\omega|_S)_p$ has been restricted from T_pM to T_pS. The following proposition shows that the restriction of 1-forms is simply the pullback of the inclusion i.

Proposition 17.14. If $i : S \hookrightarrow M$ is the inclusion map of an immersed submanifold S and ω is a 1 -form on M, then $i^*\omega = \omega|_S$.

Proof. For $p \in S$ and $v \in T_pS$,

$$\begin{aligned}
(i^*\omega)_p (v) &= \omega_{i(p)} (i_* v) && (\text{ definition of pullback })\\
&= \omega_p(v) && (\text{ both } i \text{ and } i_* \text{ are inclusions })\\
&= (\omega|_S)_p (v) && (\text{ definition of } \omega|_S)
\end{aligned}$$

To avoid too cumbersome a notation, we sometimes write ω to mean $\omega|_S$, relying on the context to make clear that it is the restriction of ω to S.

Example 17.15 (A 1-form on the circle). The velocity vector field of the unit circle $c(t) = (x, y) = (\cos t, \sin t)$ in $\mathbb{R}^2$ is

$$c'(t) = (-\sin t, \cos t) = (-y, x)$$

Thus,

$$X = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

is a C^∞ vector field on the unit circle S^1. What this notation means is that if x, y are the standard coordinates on $\mathbb{R}^2$ and $i : S^1 \hookrightarrow \mathbb{R}^2$ is the inclusion map, then at a point $p = (x, y) \in S^1$, one has $i_* X_p = -y \partial/\, \partial x|_p + x \partial/\, \partial y|_p$, where $\partial/\, \partial x|_p$ and $\partial/\, \partial y|_p$ are tangent vectors at p in $\mathbb{R}^2$. Find a 1 -form $\omega = a dx + b dy$ on S^1 such that $\omega(X) \equiv 1$.

Solution. Here ω is viewed as the restriction to S^1 of the 1 -form $a dx + b dy$ on $\mathbb{R}^2$. We calculate in $\mathbb{R}^2$, where dx, dy are dual to $\partial/\partial x, \partial/\partial y$:

$$\omega(X) = (a dx + b dy) \left(-y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} \right) = -ay + bx = 1 \tag{17.6}$$

Since $x^2 + y^2 = 1$ on $S^1, a = -y$ and $b = x$ is a solution to (17.6). So $\omega = -y dx + x dy$ is one such 1-form. Since $\omega(X) \equiv 1$, the form ω is nowhere vanishing on the circle.

Remark. In the notation of Problem 11.2, ω should be written $-\bar{y}d\bar{x} + \bar{x}d\bar{y}$, since x, y are functions on $\mathbb{R}^2$ and $\bar{x}, \bar{y}$ are their restrictions to S^1. However, one generally uses the same notation for a form on $\mathbb{R}^n$ and for its restriction to a submanifold. Since $i^*x = \bar{x}$ and $i^*dx = d\bar{x}$, there is little possibility of confusion in omitting the bar while dealing with the restriction of forms on $\mathbb{R}^n$. This is in contrast to the situation for vector fields, where $i_*\left(\partial/\,\partial\bar{x}|_p\right) \neq \partial/\,\partial x|_p$.

Example 17.16 (Pullback of a 1-form). Let $h : \mathbb{R} \to S^1 \subset \mathbb{R}^2$ be given by $h(t) = (x, y) = (\cos t, \sin t)$. If ω is the 1 -form $-ydx + xdy$ on S^1, compute the pullback $h^*\omega$.

Solution.

$$\begin{aligned} h^*(-ydx + xdy) &= -\left(h^*y\right) d\left(h^*x\right) + \left(h^*x\right) d\left(h^*y\right) \qquad \text{(by Proposition 17.11)} \\ &= -(\sin t)d(\cos t) + (\cos t)d(\sin t) \\ &= \sin^2 t dt + \cos^2 t dt = dt \end{aligned}$$

10.1.7 Problems

17.1. A 1-form on $\mathbb{R}^2 - \{(0,0)\}$

Denote the standard coordinates on $\mathbb{R}^2$ by x, y, and let

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \quad \text{and} \quad Y = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

be vector fields on $\mathbb{R}^2$. Find a 1-form ω on $\mathbb{R}^2 - \{(0,0)\}$ such that $\omega(X) = 1$ and $\omega(Y) = 0$.

17.2. Transition Formula for 1-forms

Suppose $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ are two charts on M with nonempty overlap $U \cap V$. Then a C^∞ 1-form ω on $U \cap V$ has two different local expressions:

$$\omega = \sum a_j dx^j = \sum b_i dy^i$$

Find a formula for a_j in terms of b_i.

17.3. Pullback of a 1-form on s^1

Multiplication in the unit circle S^1, viewed as a subset of the complex plane, is given by

$$e^{it} \cdot e^{iu} = e^{i(t+u)}, \quad t, u \in \mathbb{R}$$

In terms of real and imaginary parts,

$$(\cos t + i \sin t)(x + iy) = ((\cos t)x - (\sin t)y) + i((\sin t)x + (\cos t)y)$$

Hence, if $g = (\cos t, \sin t) \in S^1 \subset \mathbb{R}^2$, then the left multiplication $\ell_g : S^1 \to S^1$ is given by

$$\ell_g(x, y) = ((\cos t)x - (\sin t)y, (\sin t)x + (\cos t)y).$$

Let $\omega = -ydx + xdy$ be the 1-form found in Example 17.15. Prove that $\ell_g^*\omega = \omega$ for all $g \in S^1$.

17.4. Liouville Form on the Cotangent Bundle

(a) Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a chart on a manifold M, and let

$$\left(\pi^{-1}U, \tilde{\phi}\right) = \left(\pi^{-1}U, \bar{x}^1, \ldots, \bar{x}^n, c_1, \ldots, c_n\right)$$

be the induced chart on the cotangent bundle T^*M. Find a formula for the Liouville form λ on $\pi^{-1}U$ in terms of the coordinates $\bar{x}^1, \ldots, \bar{x}^n, c_1, \ldots, c_n$.
(b) Prove that the Liouville form λ on T^*M is C^∞. (Hint: Use (a) and Proposition 17.6.)

17.5. Pullback of a sum and a Product

Prove Proposition 17.11 by verifying both sides of each equality on a tangent vector X_p at a point p.

17.6. Construction of the Cotangent Bundle

Let M be a manifold of dimension n. Mimicking the construction of the tangent bundle in Section 12, write out a detailed proof that $\pi : T^*M \to M$ is a C^∞ vector bundle of rank n.

10.2 §18 Differential k-forms

We now generalize the construction of 1 -forms on a manifold to k-forms. After defining k-forms on a manifold, we show that locally they look no different from k forms on $\mathbb{R}^n$. In parallel to the construction of the tangent and cotangent bundles on a manifold, we construct the k th exterior power $\bigwedge^k (T^*M)$ of the cotangent bundle. A differential k-form is seen to be a section of the bundle $\bigwedge^k (T^*M)$. This gives a natural notion of smoothness of differential forms: a differential k-form is smooth if and only if it is smooth as a section of the vector bundle $\bigwedge^k (T^*M)$. The pullback and the wedge product of differential forms are defined pointwise. As examples of differential forms, we consider left-invariant forms on a Lie group.

10.2.1 18.1 Differential Forms

Recall that a k-tensor on a vector space V is a k-linear function

$$f : V \times \cdots \times V \to \mathbb{R}$$

The k-tensor f is alternating if for any permutation $\sigma \in S_k$,

$$f\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) = (\operatorname{sgn} \sigma) f\left(v_1, \ldots, v_k\right). \tag{18.1}$$

When $k = 1$, the only element of the permutation group S_1 is the identity permutation. So for 1 -tensors the condition (18.1) is vacuous and all 1 -tensors are alternating (and symmetric too). An alternating k-tensor on V is also called a k-covector on V.

For any vector space V, denote by $A_k(V)$ the vector space of alternating k-tensors on V. Another common notation for the space $A_k(V)$ is $\wedge^k (V^\vee)$. Thus,

$$\begin{aligned}
&\bigwedge^0 (V^\vee) = A_0(V) = \mathbb{R}, \\
&\bigwedge^1 (V^\vee) = A_1(V) = V^\vee, \\
&\bigwedge^2 (V^\vee) = A_2(V), \text{ and so on.}
\end{aligned}$$

In fact, there is a purely algebraic construction $\bigwedge^k(V)$, called the k th exterior power of the vector space V, with the property that $\bigwedge^k(V^\vee)$ is isomorphic to $A_k(V)$. To delve into this construction would lead us too far afield, so in this book $\bigwedge^k(V^\vee)$ will simply be an alternative notation for $A_k(V)$.

We apply the functor $A_k()$ to the tangent space T_pM of a manifold M at a point p. The vector space $A_k(T_pM)$, usually denoted by $\wedge^k\left(T_p^*M\right)$, is the space of all alternating k-tensors on the tangent space T_pM. A k-covector field on M is a function ω that assigns to each point $p \in M$ a k-covector $\omega_p \in \wedge^k\left(T_p^*M\right)$. A k-covector field is also called a differential k-form, a differential form of degree k, or simply a k-form. A top form on a manifold is a differential form whose degree is the dimension of the manifold.

If ω is a k-form on a manifold M and $X_1, \ldots, X_k$ are vector fields on M, then $\omega(X_1, \ldots, X_k)$ is the function on M defined by

$$(\omega(X_1, \ldots, X_k))(p) = \omega_p\left((X_1)_p, \ldots, (X_k)_p\right).$$

Proposition 18.1 (Multilinearity of a form over functions). Let ω be a k-form on a manifold M. For any vector fields $X_1, \ldots, X_k$ and any function h on M,

$$\omega(X_1, \ldots, hX_i, \ldots, X_k) = h\omega(X_1, \ldots, X_i, \ldots, X_k).$$

Proof. The proof is essentially the same as that of Proposition 17.8.

Example 18.2. Let $(U, x^1, \ldots, x^n)$ be a coordinate chart on a manifold. At each point $p \in U$, a basis for the tangent space T_pU is

$$\left.\frac{\partial}{\partial x^1}\right|_p, \ldots, \left.\frac{\partial}{\partial x^n}\right|_p.$$

As we saw in Proposition 17.3, the dual basis for the cotangent space T_p^*U is

$$\left(dx^1\right)_p, \ldots, \left(dx^n\right)_p$$

As p varies over points in U, we get differential 1 -forms $dx^1, \ldots, dx^n$ on U.

By Proposition 3.29 a basis for the alternating k-tensors in $\bigwedge^k\left(T_p^*U\right)$ is

$$\left(dx^{i_1}\right)_p \wedge \cdots \wedge \left(dx^{i_k}\right)_p, \quad 1 \leq i_1 < \cdots < i_k \leq n$$

If ω is a k-form on $\mathbb{R}^n$, then at each point $p \in \mathbb{R}^n, \omega_p$ is a linear combination

$$\omega_p = \sum a_{i_1 \cdots i_k}(p)\left(dx^{i_1}\right)_p \wedge \cdots \wedge \left(dx^{i_k}\right)_p$$

Omitting the point p, we write

$$\omega = \sum a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

In this expression the coefficients $a_{i_1 \cdots i_k}$ are functions on U because they vary with the point p. To simplify the notation, we let

$$\mathcal{J}_{k,n} = \{I = (i_1, \ldots, i_k) \mid 1 \leq i_1 < \cdots < i_k \leq n\}$$

be the set of all strictly ascending multi-indices between 1 and n of length k, and write

$$\omega = \sum_{I \in \mathcal{J}_{k,n}} a_I dx^I$$

where dx^I stands for $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

10.2.2 18.2 Local Expression for ak-Form

By Example 18.2, on a coordinate chart $(U, x^1, \ldots, x^n)$ of a manifold M, a k-form on U is a linear combination $\omega = \sum a_I dx^I$, where $I \in \mathcal{J}_{k,n}$ and the a_I are functions on U. As a shorthand, we write $\partial_i = \partial/\partial x^i$ for the i th coordinate vector field. Evaluating pointwise as in Lemma 3.28, we obtain the following equality on U for $I, J \in \mathcal{J}_{k,n}$:

$$dx^I\left(\partial_{j_1}, \ldots, \partial_{j_k}\right) = \delta_J^I = \begin{cases} 1 & \text{for } I = J \\ 0 & \text{for } I \neq J \end{cases} \tag{18.2}$$

Proposition 18.3 (a Wedge of Differentials in Local Coordinates).

Let $(u, x^1, \ldots, x^n)$ be a chart on a manifold and $f^1, \ldots, f^k$ smooth functions on U. Then

$$df^1 \wedge \cdots \wedge df^k = \sum_{I \in \mathcal{J}_{k,n}} \frac{\partial\left(f^1, \ldots, f^k\right)}{\partial\left(x^{i_1}, \ldots, x^{i_k}\right)} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

Proof. On U,

$$df^1 \wedge \cdots \wedge df^k = \sum_{J \in \mathcal{J}_{k,n}} c_J dx^{j_1} \wedge \cdots \wedge dx^{j_k} \tag{18.3}$$

for some functions c_J. By the definition of the differential, $df^i\left(\partial/\partial x^j\right) = \partial f^i/\partial x^j$. Applying both sides of (18.3) to the list of coordinate vectors $\partial_{i_1}, \ldots, \partial_{i_k}$, we get

$$\begin{aligned} \text{LHS } &= \left(df^1 \wedge \cdots \wedge df^k\right)\left(\partial_{i_1}, \ldots, \partial_{i_k}\right) = \det\left[\frac{\partial f^i}{\partial x^{i_j}}\right] \quad \text{by Proposition 3.27} \\ &= \frac{\partial\left(f^1, \ldots, f^k\right)}{\partial\left(x^{i_1}, \ldots, x^{i_k}\right)}, \\ \text{RHS } &= \sum_J c_J dx^J\left(\partial_{i_1}, \ldots, \partial_{i_k}\right) = \sum_J c_J \delta_I^J = c_I \qquad \text{by Lemma 18.2.} \end{aligned}$$

Hence, $c_I = \partial\left(f^1, \ldots, f^k\right)/\partial\left(x^{i_1}, \ldots, x^{i_k}\right)$.
If ($U, x^1, \ldots, x^n$) and ($V, y^1, \ldots, y^n$) are two overlapping charts on a manifold, then on the intersection $U \cap V$, Proposition 18.3 becomes the transition formula for k-forms:

$$dy^J = \sum_I \frac{\partial\left(y^{j_1}, \ldots, y^{j_k}\right)}{\partial\left(x^{i_1}, \ldots, x^{i_k}\right)} dx^I$$

Two cases of Proposition 18.3 are of special interest:
Corollary 18.4. Let $(U, x^1, \ldots, x^n)$ be a chart on a manifold, and let $f, f^1, \ldots, f^n$ be C^∞ functions on U. Then
(i) (1-forms)$df = \sum\left(\partial f/\partial x^i\right) dx^i$,
(ii) (top forms) $df^1 \wedge \cdots \wedge df^n = \det\left[\partial f^j/\partial x^i\right] dx^1 \wedge \cdots \wedge dx^n$.

Case (i) of the corollary agrees with the formula we derived in (17.2).

Exercise 18.5 (Transition formula for a 2-form).* If $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ are two overlapping coordinate charts on M, then a C^∞ 2-form ω on $U \cap V$ has two local expressions:

$$\omega = \sum_{i<j} a_{ij} dx^i \wedge dx^j = \sum_{k<\ell} b_{k\ell} dy^k \wedge dy^\ell$$

Find a formula for a_{ij} in terms of $b_{k\ell}$ and the coordinate functions $x^1, \ldots, x^n, y^1, \ldots, y^n$.

10.2.3 18.3 The Bundle Point of View

Let M be a manifold of dimension n. To better understand differential forms, we mimic the construction of the tangent and cotangent bundles and form the set

$$\bigwedge^k (T^*M) := \bigcup_{p\in M} \bigwedge^k \left(T_p^*M\right) = \bigcup_{p\in M} A_k\left(T_pM\right)$$

of all alternating k-tensors at all points of the manifold M. This set is called the k th exterior power of the cotangent bundle. There is a projection map $\pi : \wedge^k (T^*M) \to M$ given by $\pi(\alpha) = p$ if $\alpha \in \bigwedge^k \left(T_p^*M\right)$.

If $(\ U, \phi\ )$ is a coordinate chart on M, then there is a bijection

$$\bigwedge^k (T^*U) = \bigcup_{p\in U} \bigwedge^k \left(T_p^*U\right) \simeq \phi(U) \times \mathbb{R}^{\binom{n}{k}}$$

$$\alpha \in \bigwedge^k \left(T_p^*U\right) \mapsto (\phi(p), \{c_I(\alpha)\}_I)$$

where $\alpha = \sum c_I(\alpha) dx^I\big|_p \in \bigwedge^k \left(T_p^*U\right)$ and $I = (1 \le i_1 < \cdots < i_k \le n)$. In this way we can give $\bigwedge^k (T^*U)$ and hence $\bigwedge^k (T^*M)$ a topology and even a differentiable structure. The details are just like those for the construction of the tangent bundle, so we omit them. The upshot is that the projection map $\pi : \bigwedge^k (T^*M) \to M$ is a C^∞ vector bundle of rank $\binom{n}{k}$ and that a differential k-form is simply a section of this bundle. As one might expect, we define a k-form to be C^∞ if it is C^∞ as a section of the bundle $\pi : \bigwedge^k (T^*M) \to M$.

Notation. If $E \to M$ is a C^∞ vector bundle, then the vector space of C^∞ sections of E is denoted by $\Gamma(E)$ or $\Gamma(M, E)$. The vector space of all $C^\infty k$-forms on M is usually denoted by $\Omega^k(M)$. Thus,

$$\Omega^k(M) = \Gamma\left(\bigwedge^k (T^*M)\right) = \Gamma\left(M, \bigwedge^k (T^*M)\right)$$

10.2.4 18.4 Smooth k-forms

There are several equivalent characterizations of a smooth k-form. Since the proofs are similar to those for 1 -forms (Lemma 17.5 and Propositions 17.6 and 17.9), we omit them.

Lemma 18.6 (Smoothness of a k-form on a chart). Let $(\ U, x^1, \ldots, x^n\ )$ be a chart on a manifold M. A k-form $\omega = \sum a_I dx^I$ on U is smooth if and only if the coefficient functions a_I are all smooth on U.

Proposition 18.7 (Characterization of a smooth k-form). Let ω be a k-form on a manifold M. The following are equivalent:
(i) The k-form ω is C^∞ on M.
(ii) The manifold M has an atlas such that on every chart $(U, \phi) = (U, x^1, \ldots, x^n)$ in the atlas, the coefficients a_I of $\omega = \sum a_I dx^I$ relative to the coordinate frame $\left\{dx^I\right\}_{I\in\mathcal{J}_{k,n}}$ are all C^∞.
(iii) On every chart $(U, \phi) = (U, x^1, \ldots, x^n)$ on M, the coefficients a_I of $\omega = \sum a_I dx^I$ relative to the coordinate frame $\left\{dx^I\right\}_{I\in\mathcal{J}_{k,n}}$ are all C^∞.
(iv) For any k smooth vector fields $X_1, \ldots, X_k$ on M, the function $\omega\left(X_1, \ldots, X_k\right)$ is C^∞ on M.

We defined the 0 -tensors and the 0 -covectors to be the constants, that is, $L_0(V) = A_0(V) = \mathbb{R}$. Therefore, the bundle $\bigwedge^0 (T^*M)$ is simply $M \times \mathbb{R}$ and a 0 -form on M is a function on M. A C^∞ 0-form on M is thus the same as a C^∞ function on M. In our new notation,

$$\Omega^0(M) = \Gamma\left(\wedge^0\left(T^*M\right)\right) = \Gamma(M \times \mathbb{R}) = C^\infty(M).$$

Proposition 13.2 on C^∞ extensions of functions has a generalization to differential forms.

Proposition 18.8 (C^∞ extension of a form). Suppose τ is a C^∞ differential form defined on a neighborhood U of a point p in a manifold M. Then there is a C^∞ form $\tilde{\tau}$ on M that agrees with τ on a possibly smaller neighborhood of p.

The proof is identical to that of Proposition 13.2. We leave it as an exercise. Of course, the extension $\tilde{\tau}$ is not unique. In the proof it depends on p and on the choice of a bump function at p.

10.2.5 18.5 Pullback of $\boldsymbol{k}$-Forms

We have defined the pullback of 0 -forms and 1 -forms under a C^∞ map $F : N \to M$. For a C^∞0-form on M, i.e., a C^∞ function on M, the pullback F^*f is simply the composition

$$N \xrightarrow{F} M \xrightarrow{f} \mathbb{R}, \quad F^*(f) = f \circ F \in \Omega^0(N).$$

To generalize the pullback to k-forms for all $k \geq 1$, we first recall the pullback of k-covectors from Subsection 10.3. A linear map $L : V \to W$ of vector spaces induces a pullback map $L^* : A_k(W) \to A_k(V)$ by

$$\left(L^*\alpha\right)\left(v_1, \ldots, v_k\right) = \alpha\left(L\left(v_1\right), \ldots, L\left(v_k\right)\right)$$

for $\alpha \in A_k(W)$ and $v_1, \ldots, v_k \in V$.

Now suppose $F : N \to M$ is a C^∞ map of manifolds. At each point $p \in N$, the differential

$$F_{*,p} : T_pN \to T_{F(p)}M$$

is a linear map of tangent spaces, and so by the preceding paragraph there is a pullback map

$$\left(F_{*,p}\right)^* : A_k\left(T_{F(p)}M\right) \to A_k\left(T_pN\right)$$

This ugly notation is usually simplified to F^*. Thus, if $\omega_{F(p)}$ is a k-covector at $F(p)$ in M, then its pullback $F^*\left(\omega_{F(p)}\right)$ is the k-covector at p in N given by

$$F^*\left(\omega_{F(p)}\right)\left(v_1, \ldots, v_k\right) = \omega_{F(p)}\left(F_{*,p}v_1, \ldots, F_{*,p}v_k\right), \quad v_i \in T_pN$$

Finally, if ω is a k-form on M, then its pullback $F^*\omega$ is the k-form on N defined pointwise by $\left(F^*\omega\right)_p = F^*\left(\omega_{F(p)}\right)$ for all $p \in N$. Equivalently,

$$\left(F^*\omega\right)_p\left(v_1, \ldots, v_k\right) = \omega_{F(p)}\left(F_{*,p}v_1, \ldots, F_{*,p}v_k\right), \quad v_i \in T_pN \tag{18.4}$$

When $k = 1$, this formula specializes to the definition of the pullback of a 1 -form in Subsection 17.5. The pullback of a k-form (18.4) can be viewed as a composition

$$T_pN \times \cdots \times T_pN \xrightarrow{F_* \times \cdots \times F_*} T_{F(p)}M \times \cdots \times T_{F(p)}M \xrightarrow{\omega_{F(p)}} \mathbb{R}$$

Proposition 18.9 (Linearity of the pullback). Let $F : N \to M$ be a C^∞ map. If ω, τ are k-forms on M and a is a real number, then

(i) $F^*(\omega + \tau) = F^*\omega + F^*\tau$;

(ii) $F^*(a\omega) = aF^*\omega$.

Proof. Problem 18.2.

At this point, we still do not know, other than for $k = 0, 1$, whether the pullback of a $C^\infty k$-form under a C^∞ map remains C^∞. This very basic question will be answered in Subsection 19.5.

10.2.6 18.6 The Wedge Product

We learned in Section 3 that if α and β are alternating tensors of degree k and ℓ respectively on a vector space V, then their wedge product $\alpha \wedge \beta$ is the alternating ($k+\ell$)-tensor on V defined by

$$(\alpha \wedge \beta)\left(v_1, \ldots, v_{k+\ell}\right)=\sum(\operatorname{sgn} \sigma) \alpha\left(v_{\sigma(1)}, \ldots, v_{\sigma(k)}\right) \beta\left(v_{\sigma(k+1)}, \ldots, v_{\sigma(k+\ell)}\right),$$

where $v_i \in V$ and σ runs over all (k, ℓ)-shuffles of $1, \ldots, k+\ell$. For example, if α and β are 1 -covectors, then

$$(\alpha \wedge \beta)\left(v_1, v_2\right)=\alpha\left(v_1\right) \beta\left(v_2\right)-\alpha\left(v_2\right) \beta\left(v_1\right)$$

The wedge product extends pointwise to differential forms on a manifold: for a k-form ω and an ℓ-form τ on M, define their wedge product $\omega \wedge \tau$ to be the $(k+\ell)$ form on M such that

$$(\omega \wedge \tau)_p=\omega_p \wedge \tau_p$$

at all $p \in M$.

Proposition 18.10. If ω and τ are C^∞ forms on M, then $\omega \wedge \tau$ is also C^∞.

Proof. Let $(U, x^1, \ldots, x^n)$ be a chart on M. On U,

$$\omega=\sum a_I d x^I, \quad \tau=\sum b_J d x^J$$

for C^∞ function a_I, b_J on U. Their wedge product on U is

$$\omega \wedge \tau=\left(\sum a_I d x^I\right) \wedge\left(\sum b_J d x^J\right)=\sum a_I b_J d x^I \wedge d x^J$$

In this sum, $d x^I \wedge d x^J=0$ if I and J have an index in common. If I and J are disjoint, then $d x^I \wedge d x^J=\pm d x^K$, where $K=I \cup J$ but reordered as an increasing sequence. Thus,

$$\omega \wedge \tau=\sum_K\left(\sum_{\substack{I \cup J=K \\ I, J \text{ disjoint}}} \pm a_I b_J\right) d x^K$$

Since the coefficients of $d x^K$ are C^∞ on U, by Proposition 18.7, $\omega \wedge \tau$ is C^∞. $\square$

Proposition 18.11 (Pullback of a wedge product). If $F: N \rightarrow M$ is a C^∞ map of manifolds and ω and τ are differential forms on M, then

$$F^*(\omega \wedge \tau)=F^* \omega \wedge F^* \tau .$$

Proof. Problem 18.3. $\square$

Define the vector space $\Omega^*(M)$ of C^∞ differential forms on a manifold M of dimension n to be the direct sum

$$\Omega^*(M)=\bigoplus_{k=0}^n \Omega^k(M).$$

What this means is that each element of $\Omega^*(M)$ is uniquely a sum $\sum_{k=0}^n \omega_k$, where $\omega_k \in \Omega^k(M)$. With the wedge product, the vector space $\Omega^*(M)$ becomes a graded algebra, the grading being the degree of differential forms.

10.2.7 18.7 Differential Forms on a Circle

Consider the map

$$h : \mathbb{R} \to S^1, \quad h(t) = (\cos t, \sin t).$$

Since the derivative $\dot{h}(t) = (-\sin t, \cos t)$ is nonzero for all t, the map $h : \mathbb{R} \to S^1$ is a submersion. By Problem 18.8, the pullback map $h^* : \Omega^*(S^1) \to \Omega^*(\mathbb{R})$ on smooth differential forms is injective. This will allow us to identify the differential forms on S^1 with a subspace of differential forms on $\mathbb{R}$.

Let $\omega = -ydx + xdy$ be the nowhere-vanishing form on S^1 from Example 17.15. In Example 17.16, we showed that $h^*\omega = dt$. Since ω is nowhere vanishing, it is a frame for the cotangent bundle T^*S^1 over S^1, and every C^∞1-form α on S^1 can be
written as $\alpha = f\omega$ for some function f on S^1. By Proposition 12.12, the function f is C^∞. Its pullback $\bar{f} := h^*f$ is a C^∞ function on $\mathbb{R}$. Since pulling back preserves multiplication (Proposition 18.11),

$$h^*\alpha = (h^*f)(h^*\omega) = \bar{f}dt \tag{18.5}$$

We say that a function g or a 1 -form gdt on $\mathbb{R}$ is periodic of period a if $g(t+a) = g(t)$ for all $t \in \mathbb{R}$.

Proposition 18.12. For $k = 0, 1$, under the pullback map $h^* : \Omega^*(S^1) \to \Omega^*(\mathbb{R})$, smooth k-forms on S^1 are identified with smooth periodic k-forms of period 2π on $\mathbb{R}$.

Proof. If $f \in \Omega^0(S^1)$, then since $h : \mathbb{R} \to S^1$ is periodic of period 2π, the pullback $h^*f = f \circ h \in \Omega^0(\mathbb{R})$ is periodic of period 2π.

Conversely, suppose $\bar{f} \in \Omega^0(\mathbb{R})$ is periodic of period 2π. For $p \in S^1$, let s be the C^∞ inverse in a neighborhood U of p of the local diffeomorphism h and define $f = \bar{f} \circ s$ on U. To show that f is well defined, let s_1 and s_2 be two inverses of h over U. By the periodic properties of sine and cosine, $s_1 = s_2 + 2\pi n$ for some $n \in \mathbb{Z}$. Because $\bar{f}$ is periodic of period 2π, we have $\bar{f} \circ s_1 = \bar{f} \circ s_2$. This proves that f is well defined on U. Moreover,

$$\bar{f} = f \circ s^{-1} = f \circ h = h^*f \quad \text{on } h^{-1}(U).$$

As p varies over S^1, we obtain a well-defined C^∞ function f on S^1 such that $\bar{f} = h^*f$. Thus, the image of $h^* : \Omega^0(S^1) \to \Omega^0(\mathbb{R})$ consists precisely of the C^∞ periodic functions of period 2π on $\mathbb{R}$.

As for 1 -forms, note that $\Omega^1(S^1) = \Omega^0(S^1)\omega$ and $\Omega^1(\mathbb{R}) = \Omega^0(\mathbb{R})dt$. The pullback $h^* : \Omega^1(S^1) \to \Omega^1(\mathbb{R})$ is given by $h^*(f\omega) = (h^*f)\,dt$, so the image of $h^* : \Omega^1(S^1) \to \Omega^1(\mathbb{R})$ consists of C^∞ periodic 1-forms of period 2π.

10.2.8 18.8 Invariant Forms on a Lie Group

Just as there are left-invariant vector fields on a Lie group G, so also are there leftinvariant differential forms. For $g \in G$, let $\ell_g : G \to G$ be left multiplication by g. A k-form ω on G is said to be left-invariant if $\ell_g^*\omega = \omega$ for all $g \in G$. This means that for all $g, x \in G$,

$$\ell_g^*(\omega_{gx}) = \omega_x$$

Thus, a left-invariant k-form is uniquely determined by its value at the identity, since for any $g \in G$,

$$\omega_g = \ell_{g^{-1}}^*(\omega_e) \tag{18.6}$$

Example 18.13 (A left-invariant 1-form on S^1). By Problem 17.3, $\omega = -ydx + xdy$ is a left-invariant 1-form on S^1.

We have the following analogue of Proposition 16.8.

Proposition 18.14. Every left-invariant k-form ω on a Lie group G is C^∞.

Proof. By Proposition 18.7(iii), it suffices to prove that for any k smooth vector fields $X_1, \ldots, X_k$ on G, the function $\omega(X_1, \ldots, X_k)$ is C^∞ on G. Let $(Y_1)_e, \ldots, (Y_n)_e$ be a basis for the tangent space T_eG and $Y_1, \ldots, Y_n$ the left-invariant vector fields they generate. Then $Y_1, \ldots, Y_n$ is a C^∞ frame on G (Proposition 16.8). Each X_j can be written as a linear combination $X_j = \sum a_j^i Y_i$. By Proposition 12.12, the functions a_j^i are C^∞. Hence, to prove that ω is C^∞, it suffices to show that $\omega(Y_{i_1}, \ldots, Y_{i_k})$ is C^∞ for the left-invariant vector fields $Y_{i_1}, \ldots, Y_{i_k}$. But

$$\begin{aligned}(\omega(Y_{i_1}, \ldots, Y_{i_k}))(g) &= \omega_g\left((Y_{i_1})_g, \ldots, (Y_{i_k})_g\right) \\ &= \left(\ell_{g^{-1}}^*(\omega_e)\right)\left(\ell_{g*}(Y_{i_1})_e, \ldots, \ell_{g*}(Y_{i_k})_e\right) \\ &= \omega_e\left((Y_{i_1})_e, \ldots, (Y_{i_k})_e\right),\end{aligned}$$

which is a constant, independent of g. Being a constant function, $\omega(Y_{i_1}, \ldots, Y_{i_k})$ is C^∞ on G.

Similarly, a k-form ω on G is said to be right-invariant if $r_g^*\omega = \omega$ for all $g \in G$. The analogue of Proposition 18.14, that every right-invariant form on a Lie group is C^∞, is proven in the same way.

Let $\Omega^k(G)^G$ denote the vector space of left-invariant k-forms on G. The linear map

$$\Omega^k(G)^G \to \bigwedge^k (\mathfrak{g}^\vee), \quad \omega \mapsto \omega_e,$$

has an inverse defined by (18.6) and is therefore an isomorphism. It follows that $\dim \Omega^k(G)^G = \binom{n}{k}$.

10.2.9 Problems

18.1. Characterization of a Smooth k-form

Write out a proof of Proposition 18.7(i)$\Leftrightarrow$(iv).

18.2. Linearity of the Pullback

Prove Proposition 18.9.

18.3. Pullback of a Wedge Product

Prove Proposition 18.11.

18.4.* Support of a sum or Product

Generalizing the support of a function, we define the support of a k-form $\omega \in \Omega^k(M)$ to be

$$\operatorname{supp}\omega = \text{ closure of } \{p \in M \mid \omega_p \neq 0\} = \overline{Z(\omega)^c}$$

where $Z(\omega)^c$ is the complement of the zero set $Z(\omega)$ of ω in M. Let ω and τ be differential forms on a manifold M. Prove that

(a) $\operatorname{supp}(\omega + \tau) \subset \operatorname{supp}\omega \cup \operatorname{supp}\tau$,

(b) $\operatorname{supp}(\omega \wedge \tau) \subset \operatorname{supp}\omega \cap \operatorname{supp}\tau$.

18.5. Support of a Linear Combination

Prove that if the k-forms $\omega^1, \ldots, \omega^r \in \Omega^k(M)$ are linearly independent at every point of a manifold M and $a_1, \ldots, a_r$ are C^∞ functions on M, then

$$\operatorname{supp} \sum_{i=1}^{r} a_i \omega^i = \bigcup_{i=1}^{r} \operatorname{supp} a_i.$$

18.6.* Locally Finite Collection of Supports

Let $\{\rho_\alpha\}_{\alpha \in \mathrm{A}}$ be a collection of functions on M and ω a $C^\infty k$-form with compact support on M. If the collection $\{\operatorname{supp} \rho_\alpha\}_{\alpha \in \mathrm{A}}$ of supports is locally finite, prove that $\rho_\alpha \omega \equiv 0$ for all but finitely many α.

18.7. Locally Finite Sums

We say that a sum $\sum \omega_\alpha$ of differential k-forms on a manifold M is locally finite if the collection $\{\operatorname{supp} \omega_\alpha\}$ of supports is locally finite. Suppose $\sum \omega_\alpha$ and $\sum \tau_\alpha$ are locally finite sums and f is a C^∞ function on M.
(a) Show that every point $p \in M$ has a neighborhood U on which $\sum \omega_\alpha$ is a finite sum.
(b) Show that $\sum \omega_\alpha + \tau_\alpha$ is a locally finite sum and

$$\sum \omega_\alpha + \tau_\alpha = \sum \omega_\alpha + \sum \tau_\alpha$$

(c) Show that $\sum f \omega_\alpha$ is a locally finite sum and

$$\sum f \cdot \omega_\alpha = f \cdot \left(\sum \omega_\alpha \right)$$

18.8.* Pullback by a Surjective Submersion

In Subsection 19.5, we will show that the pullback of a C^∞ form is C^∞. Assuming this fact for now, prove that if $\pi : \tilde{M} \to M$ is a surjective submersion, then the pullback map $\pi^* : \Omega^*(M) \to \Omega^*(\tilde{M})$ is an injective algebra homomorphism.

18.9. Bi-invariant top Forms on a Compact, Connected Lie Group

Suppose G is a compact, connected Lie group of dimension n with Lie algebra $\mathfrak{g}$. This exercise proves that every left-invariant n-form on G is right-invariant.
(a) Let ω be a left-invariant n-form on G. For any $a \in G$, show that $r_a^* \omega$ is also left-invariant, where $r_a : G \to G$ is right multiplication by a.
(b) Since $\dim \Omega^n(G)^G = \dim \bigwedge^n (\mathfrak{g}^\vee) = 1, r_a^* \omega = f(a)\omega$ for some nonzero real number $f(a)$ depending on $a \in G$. Show that $f : G \to \mathbb{R}^\times$is a group homomorphism.
(c) Show that $f : G \to \mathbb{R}^\times$is C^∞. (Hint: Note that $f(a)\omega_e = (r_a^* \omega)_e = r_a^* (\omega_a) = r_a^* \ell_{a^{-1}}^* (\omega_e)$. Thus, $f(a)$ is the pullback of the map $\operatorname{Ad}(a^{-1}) : \mathfrak{g} \to \mathfrak{g}$. See Problem 16.11.)
(d) As the continuous image of a compact connected set G, the set $f(G) \subset \mathbb{R}^\times$is compact and connected. Prove that $f(G) = 1$. Hence, $r_a^* \omega = \omega$ for all $a \in G$.

10.3 §19 the Exterior Derivative

In contrast to undergraduate calculus, where the basic objects of study are functions, the basic objects in calculus on manifolds are differential forms. Our program now is to learn how to integrate and differentiate differential forms.

Recall that an antiderivation on a graded algebra $A = \bigoplus_{k=0}^{\infty} A^k$ is an $\mathbb{R}$-linear map $D : A \to A$ such that

$$D(\omega \cdot \tau) = (D\omega) \cdot \tau + (-1)^k \omega \cdot D\tau$$

for $\omega \in A^k$ and $\tau \in A^\ell$. In the graded algebra A, an element of A^k is called a homogeneous element of degree k. The antiderivation is of degree m if

$$\deg D\omega = \deg \omega + m$$

for all homogeneous elements $\omega \in A$.
Let M be a manifold and $\Omega^*(M)$ the graded algebra of C^∞ differential forms on M. On the graded algebra $\Omega^*(M)$ there is a uniquely and intrinsically defined antiderivation called the exterior derivative. The process of applying the exterior derivative is called exterior differentiation.

Definition 19.1. An exterior derivative on a manifold M is an $\mathbb{R}$-linear map

$$D : \Omega^*(M) \to \Omega^*(M)$$

such that
(i) D is an antiderivation of degree 1 ,
(ii) $D \circ D = 0$,
(iii) if f is a C^∞ function and X a C^∞ vector field on M, then $(Df)(X) = Xf$.

Condition (iii) says that on 0 -forms an exterior derivative agrees with the differential df of a function f. Hence, by (17.2), on a coordinate chart $(U, x^1, \ldots, x^n)$,

$$Df = df = \sum \frac{\partial f}{\partial x^i} dx^i$$

In this section we prove the existence and uniqueness of an exterior derivative on a manifold. Using its three defining properties, we then show that the exterior derivative commutes with the pullback. This will finally allow us to prove that the pullback of a C^∞ form by a C^∞ map is C^∞.

10.3.1 19.1 Exterior Derivative on a Coordinate Chart

We showed in Subsection 4.4 the existence and uniqueness of an exterior derivative on an open subset of $\mathbb{R}^n$. The same proof carries over to any coordinate chart on a manifold.

More precisely, suppose ($U, x^1, \ldots, x^n$) is a coordinate chart on a manifold M. Then any k-form ω on U is uniquely a linear combination

$$\omega = \sum a_I dx^I, \quad a_I \in C^\infty(U).$$

If D is an exterior derivative on U, then

$$\begin{aligned} D\omega &= \sum (Da_I) \wedge dx^I + \sum a_I D dx^I && \text{(by (i))} \\ &= \sum (Da_I) \wedge dx^I && \text{(by (iii) and (ii), } Dd = D^2 = 0\text{)} \\ &= \sum_I \sum_j \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I && \text{(by (iii)).} \end{aligned} \tag{19.1}$$

Hence, if an exterior derivative D exists on U, then it is uniquely defined by (19.1).
To show existence, we define D by the formula (19.1). The proof that D satisfies (i), (ii), and

(iii) is the same as in the case of $\mathbb{R}^n$ in Proposition 4.7. We will denote the unique exterior derivative on a chart (U, ϕ) by d_U.

Like the derivative of a function on $\mathbb{R}^n$, an antiderivation D on $\Omega^*(M)$ has the property that for a k-form ω, the value of $D\omega$ at a point p depends only on the values of ω in a neighborhood of p. To explain this, we make a digression on local operators.

10.3.2 19.2 Local Operators

An endomorphism of a vector space W is often called an operator on W. For example, if $W = C^\infty(\mathbb{R})$ is the vector space of C^∞ functions on $\mathbb{R}$, then the derivative d/dx is an operator on W :

$$\frac{d}{dx} f(x) = f'(x)$$

The derivative has the property that the value of $f'(x)$ at a point p depends only on the values of f in a small neighborhood of p. More precisely, if $f = g$ on an open set U in $\mathbb{R}$, then $f' = g'$ on U. We say that the derivative is a local operator on $C^\infty(\mathbb{R})$.

Definition 19.2. An operator $D : \Omega^*(M) \to \Omega^*(M)$ is said to be local if for all $k \geq 0$, whenever a k-form $\omega \in \Omega^k(M)$ restricts to 0 on an open set U in M, then $D\omega \equiv 0$ on U.

Here by restricting to 0 on U, we mean that $\omega_p = 0$ at every point p in U, and the symbol " $\equiv 0$ " means "is identically zero": $(D\omega)_p = 0$ at every point p in U. An equivalent criterion for an operator D to be local is that for all $k \geq 0$, whenever two k-forms $\omega, \tau \in \Omega^k(M)$ agree on an open set U, then $D\omega \equiv D\tau$ on U.

Example. Define the integral operator

$$I : C^\infty([a, b]) \to C^\infty([a, b])$$

by

$$I(f) = \int_a^b f(t)dt$$

Here $I(f)$ is a number, which we view as a constant function on $[a, b]$. The integral is not a local operator, since the value of $I(f)$ at any point p depends on the values of f over the entire interval $[a, b]$.

Proposition 19.3. Any antiderivation D on $\Omega^*(M)$ is a local operator.

Proof. Suppose $\omega \in \Omega^k(M)$ and $\omega \equiv 0$ on an open subset U. Let p be an arbitrary point in U. It suffices to prove that $(D\omega)_p = 0$.

Choose a C^∞ bump function f at p supported in U. In particular, $f \equiv 1$ in a neighborhood of p in U. Then $f\omega \equiv 0$ on M, since if a point q is in U, then $\omega_q = 0$, and if q is not in U, then $f(q) = 0$. Applying the antiderivation property of D to $f\omega$, we get

$$0 = D(0) = D(f\omega) = (Df) \wedge \omega + (-1)^0 f \wedge (D\omega)$$

Evaluating the right-hand side at p, noting that $\omega_p = 0$ and $f(p) = 1$, gives $0 = (D\omega)_p$.

Remark. The same proof shows that a derivation on $\Omega^*(M)$ is also a local operator.

10.3.3 19.3 Existence of an Exterior Derivative on a Manifold

To define an exterior derivative on a manifold M, let ω be a k-form on M and $p \in M$. Choose a chart $(U, x^1, \ldots, x^n)$ about p. Suppose $\omega = \sum a_I dx^I$ on U. In Subsection 19.1 we showed the existence of an exterior derivative d_U on U with the property

$$d_U\omega = \sum da_I \wedge dx^I \quad \text{on } U \tag{19.2}$$

Define $(d\omega)_p = (d_U\omega)_p$. We now show that $(d_U\omega)_p$ is independent of the chart U containing p. If $(V, y^1, \ldots, y^n)$ is another chart about p and $\omega = \sum b_J dy^J$ on V, then on $U \cap V$,

$$\sum a_I dx^I = \sum b_J dy^J$$

On $U \cap V$ there is a unique exterior derivative

$$d_{U\cap V} : \Omega^*(U \cap V) \to \Omega^*(U \cap V).$$

By the properties of the exterior derivative, on $U \cap V$

$$\begin{aligned} d_{U\cap V}\left(\sum a_I dx^I\right) &= d_{U\cap V}\left(\sum b_J dy^J\right) \\ \text{or} \quad \sum da_I \wedge dx^I &= \sum db_J \wedge dy^J \end{aligned}$$

In particular,

$$\left(\sum da_I \wedge dx^I\right)_p = \left(\sum db_J \wedge dy^J\right)_p.$$

Thus, $(d\omega)_p = (d_U\omega)_p$ is well defined, independently of the chart $(U, x^1, \ldots, x^n)$. As p varies over all points of M, this defines an operator

$$d : \Omega^*(M) \to \Omega^*(M).$$

To check properties (i), (ii), and (iii), it suffices to check them at each point $p \in M$. As in Subsection 19.1, the verification reduces to the same calculation as for the exterior derivative on $\mathbb{R}^n$ in Proposition 4.7.

10.3.4 19.4 Uniqueness of the Exterior Derivative

Suppose $D : \Omega^*(M) \to \Omega^*(M)$ is an exterior derivative. We will show that D coincides with the exterior derivative d defined in Subsection 19.3.

If f is a C^∞ function and X a C^∞ vector field on M, then by condition (iii) of Definition 19.1,

$$(Df)(X) = Xf = (df)(X).$$

Therefore, $Df = df$ on functions $f \in \Omega^0(M)$.
Next consider a wedge product of exact 1 -forms $df^1 \wedge \cdots \wedge df^k$:

$$\begin{aligned} &D\left(df^1 \wedge \cdots \wedge df^k\right) \\ &\quad = D\left(Df^1 \wedge \cdots \wedge Df^k\right) && \left(\text{ because } Df^i = df^i\right) \\ &\quad = \sum_{i=1}^{k} (-1)^{i-1} Df^1 \wedge \cdots \wedge DDf^i \wedge \cdots \wedge Df^k && (D \text{ is an antiderivation }) \\ &\quad = 0 && \left(D^2 = 0\right) \end{aligned}$$

Finally, we show that D agrees with d on any k-form $\omega \in \Omega^k(M)$. Fix $p \in M$. Choose a chart $(U, x^1, \ldots, x^n)$ about p and suppose $\omega = \sum a_I dx^I$ on U. Extend the functions $a_I, x^1, \ldots, x^n$ on U to C^∞ functions $\tilde{a}_I, \tilde{x}^1, \ldots, \tilde{x}^n$ on M that agree with $a_I, x^1, \ldots, x^n$ on a neighborhood of V of p (by Proposition 18.8). Define

$$\tilde{\omega} = \sum \tilde{a}_I d\tilde{x}^I \in \Omega^k(M).$$

Then

$$\omega \equiv \tilde{\omega} \quad \text{on } V.$$

Since D is a local operator,

$$D\omega = D\tilde{\omega} \quad \text{on } V.$$

Thus,

$$\begin{aligned}
(D\omega)_p = (D\tilde{\omega})_p &= \left(D \sum \tilde{a}_I d\tilde{x}^I\right)_p \\
&= \left(\sum D\tilde{a}_I \wedge d\tilde{x}^I + \sum \tilde{a}_I \wedge Dd\tilde{x}^I\right)_p \\
&= \left(\sum d\tilde{a}_I \wedge d\tilde{x}^I\right)_p \quad \left(\text{ because } Dd\tilde{x}^I = DD\tilde{x} = 0\right) \\
&= \left(\sum da_I \wedge dx^I\right)_p \quad (\text{ since } D \text{ is a local operator }) \\
&= (d\omega)_p
\end{aligned}$$

We have proven the following theorem.

Theorem 19.4. On any manifold M there exists an exterior derivative $d : \Omega^*(M) \to \Omega^*(M)$ characterized uniquely by the three properties of Definition 19.1.

10.3.5 19.5 Exterior Differentiation under a Pullback

The pullback of differential forms commutes with the exterior derivative. This fact, together with Proposition 18.11 that the pullback preserves the wedge product, is a cornerstone of calculations involving the pullback. Using these two properties, we will finally be in a position to prove that the pullback of a C^∞ form under a C^∞ map is C^∞.

Proposition 19.5 (Commutation of the pullback with d). Let $F : N \to M$ be a smooth map of manifolds. If $\omega \in \Omega^k(M)$, then $dF^*\omega = F^*d\omega$.

Proof. The case $k = 0$, when ω is a C^∞ function on M, is Proposition 17.10. Next consider the case $k \geq 1$. It suffices to verify $dF^*\omega = F^*d\omega$ at an arbitrary point $p \in N$. This reduces the proof to a local computation, i.e., computation in a coordinate chart. If $(V, y^1, \ldots, y^m)$ is a chart on M about $F(p)$, then on V,

$$\omega = \sum a_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}, \quad I = (i_1 < \cdots < i_k)$$

for some C^∞ functions a_I on V and

$$\begin{aligned}
F^*\omega &= \sum (F^*a_I)\, F^*dy^{i_1} \wedge \cdots \wedge F^*dy^{i_k} \quad (\text{ Proposition 18.11}) \\
&= \sum (a_I \circ F)\, dF^{i_1} \wedge \cdots \wedge dF^{i_k} \qquad \left(F^*dy^i = dF^*y^i = d\left(y^i \circ F\right) = dF^i\right)
\end{aligned}$$

So

$$dF^*\omega = \sum d\left(a_I \circ F\right) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k}$$

On the other hand,

$$\begin{aligned}
F^* d\omega &= F^*\left(\sum da_I \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_k}\right) \\
&= \sum F^* da_I \wedge F^* dy^{i_1} \wedge \cdots \wedge F^* dy^{i_k} \\
&= \sum d\left(F^* a_I\right) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k} \quad (\text{ by the case } k=0) \\
&= \sum d\left(a_I \circ F\right) \wedge dF^{i_1} \wedge \cdots \wedge dF^{i_k}.
\end{aligned}$$

Therefore,

$$dF^*\omega = F^* d\omega$$

Corollary 19.6. If U is an open subset of a manifold M and $\omega \in \Omega^k(M)$, then $(d\omega)|_U = d\left(\omega|_U\right)$.

Proof. Let $i : U \hookrightarrow M$ be the inclusion map. Then $\omega|_U = i^*\omega$, so the corollary is simply a restatement of the commutativity of d with i^*.

Example. Let U be the open set $]0, \infty[\times]0, 2\pi[$ in the (r, θ)-plane $\mathbb{R}^2$. Define $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(r, \theta) = (r\cos\theta, r\sin\theta).$$

If x, y are the standard coordinates on the target $\mathbb{R}^2$, compute the pullback $F^*(dx\wedge$ dy).

Solution. We first compute F^*dx :

$$\begin{aligned}
F^*dx &= dF^*x && \text{(Proposition 19.5)} \\
&= d(x \circ F) && \text{(definition of the pullback of a function)} \\
&= d(r\cos\theta) \\
&= (\cos\theta)dr - r\sin\theta d\theta
\end{aligned}$$

Similarly,

$$F^*dy = dF^*y = d(r\sin\theta) = (\sin\theta)dr + r\cos\theta d\theta.$$

Since the pullback commutes with the wedge product (Proposition 18.11),

$$\begin{aligned}
F^*(dx \wedge dy) &= (F^*dx) \wedge (F^*dy) \\
&= ((\cos\theta)dr - r\sin\theta d\theta) \wedge ((\sin\theta)dr + r\cos\theta d\theta) \\
&= \left(r\cos^2\theta + r\sin^2\theta\right) dr \wedge d\theta \quad (\text{ because } d\theta \wedge dr = -dr \wedge d\theta) \\
&= r dr \wedge d\theta
\end{aligned}$$

□

Proposition 19.7. If $F : N \to M$ is a C^∞ map of manifolds and ω is a $C^\infty k$-form on M, then $F^*\omega$ is a $C^\infty k$-form on N.

Proof. It is enough to show that every point in N has a neighborhood on which $F^*\omega$ is C^∞. Fix $p \in N$ and choose a chart $(V, y^1, \ldots, y^m)$ on M about $F(p)$. Let $F^i = y^i \circ F$ be the i th coordinate of the map F in this chart. By the continuity of F, there is a chart $(U, x^1, \ldots, x^n)$ on N about p such that $F(U) \subset V$. Because ω is C^∞, on V,

$$\omega = \sum_I a_I dy^{i_1} \wedge \cdots \wedge dy^{i_k}$$

for some C^∞ functions $a_I \in C^\infty(V)$ (Proposition 18.7(i) $\Rightarrow$ (ii)). By properties of the pullback,

$$\begin{aligned}
F^*\omega &= \sum (F^*a_I)\, F^*\left(dy^{i_1}\right) \wedge \cdots F^*\left(dy^{i_k}\right) && \text{(Propositions 18.9 and 18.11)}\\
&= \sum (F^*a_I)\, dF^*y^{i_1} \wedge \cdots \wedge dF^*y^{i_k} && \text{(Proposition 19.5)}\\
&= \sum (a_I \circ F)\, dF^{i_1} \wedge \cdots \wedge dF^{i_k} && \left(F^*y^i = y^i \circ F = F^i\right)\\
&= \sum_{I,J} (a_I \circ F)\, \frac{\partial\left(F^{i_1}, \ldots, F^{i_k}\right)}{\partial\left(x^{j_1}, \ldots, x^{j_k}\right)} dx^J && \text{(Proposition 18.3).}
\end{aligned}$$

Since the $a_I \circ F$ and $\partial\left(F^{i_1}, \ldots, F^{i_k}\right) / \partial\left(x^{j_1}, \ldots, x^{j_k}\right)$ are all C^∞, $F^*\omega$ is C^∞ by Proposition 18.7 (iii) $\Rightarrow$ (i).

In summary, if $F : N \to M$ is a C^∞ map of manifolds, then the pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ is a morphism of differential graded algebras, i.e., a degreepreserving algebra homomorphism that commutes with the differential.

10.3.6 19.6 Restriction of k-forms to a Submanifold

The restriction of a k-form to an immersed submanifold is just like the restriction of a 1 -form, but with k arguments. Let S be a regular submanifold of a manifold M. If ω is a k-form on M, then the restriction of ω to S is the k-form $\omega|_S$ on S defined by

$$\left(\omega|_S\right)_p\left(v_1, \ldots, v_k\right) = \omega_p\left(v_1, \ldots, v_k\right)$$

for $v_1, \ldots, v_k \in T_pS \subset T_pM$. Thus, $\left(\omega|_S\right)_p$ is obtained from ω_p by restricting the domain of ω_p to $T_pS \times \cdots \times T_pS$ (k times). As in Proposition 17.14, the restriction of k-forms is the same as the pullback under the inclusion map $i : S \hookrightarrow M$.

A nonzero form on M may restrict to the zero form on a submanifold S. For example, if S is a smooth curve in $\mathbb{R}^2$ defined by the nonconstant function $f(x, y)$, then $df = (\partial f/\partial x)dx + (\partial f/\partial y)dy$ is a nonzero 1-form on $\mathbb{R}^2$, but since f is identically zero on S, the differential df is also identically zero on S. Thus, $(df)|_S \equiv 0$. Another example is Problem 19.9.

One should distinguish between a nonzero form and a nowhere-zero or nowherevanishing form. For example, xdy is a nonzero form on $\mathbb{R}^2$, meaning that it is not identically zero. However, it is not nowhere-zero, because it vanishes on the y-axis. On the other hand, dx and dy are nowhere-zero 1-forms on $\mathbb{R}^2$.

Notation. Since pullback and exterior differentiation commute, $(df)|_S = d\left(f|_S\right)$, so one may write $df|_S$ to mean either expression.

10.3.7 19.7 A Nowhere-vanishing 1-form on the Circle

In Example 17.15 we found a nowhere-vanishing 1-form $-ydx + xdy$ on the unit circle. As an application of the exterior derivative, we will construct in a different way a nowhere-vanishing 1-form on the circle. One advantage of the new method is that it generalizes to the construction of a nowhere-vanishing top form on a smooth hypersurface in $\mathbb{R}^{n+1}$, a regular level set of a smooth function $f : \mathbb{R}^{n+1} \to \mathbb{R}$. As we will see in Section 21, the existence of a nowhere-vanishing top form is intimately related to orientations on a manifold.

Example 19.8. Let S^1 be the unit circle defined by $x^2 + y^2 = 1$ in $\mathbb{R}^2$. The 1 -form dx restricts from $\mathbb{R}^2$ to a 1 -form on S^1. At each point $p \in S^1$, the domain of $\left(dx|_{S^1}\right)_p$ is $T_p\left(S^1\right)$ instead of $T_p\left(\mathbb{R}^2\right)$:

$$\left(dx|_{S^1}\right)_p : T_p\left(S^1\right) \to \mathbb{R}$$

At $p = (1, 0)$, a basis for the tangent space $T_p\left(S^1\right)$ is $\partial/\partial y$ (Figure 19.1). Since

$$(dx)_p\left(\frac{\partial}{\partial y}\right)=0,$$

we see that although dx is a nowhere-vanishing 1-form on $\mathbb{R}^2$, it vanishes at $(1,0)$ when restricted to S^1.

Fig. 19.1. The tangent space to $S^{1 $ at $p=(1,0).}

To find a nowhere-vanishing 1 -form on S^1, we take the exterior derivative of both sides of the equation

$$x^2+y^2=1.$$

Using the antiderivation property of d, we get

$$2xdx+2ydy=0. \tag{19.3}$$

Of course, this equation is valid only at a point $(x,y)\in S^1$. Let

$$U_x=\left\{(x,y)\in S^1 \mid x\neq 0\right\} \quad \text{and} \quad U_y=\left\{(x,y)\in S^1 \mid y\neq 0\right\}.$$

By (19.3), on $U_x\cap U_y$,

$$\frac{dy}{x}=-\frac{dx}{y}.$$

Define a 1-form ω on S^1 by

$$\omega=\begin{cases}\dfrac{dy}{x} & \text{on } U_x,\\ -\dfrac{dx}{y} & \text{on } U_y.\end{cases} \tag{19.4}$$

Since these two 1-forms agree on $U_x\cap U_y, \omega$ is a well-defined 1-form on $S^1=U_x\cup U_y$. To show that ω is C^∞ and nowhere-vanishing, we need charts. Let

$$U_x^+ = \left\{(x,y) \in S^1 \mid x > 0\right\}.$$

We define similarly U_x^-, U_y^+, U_y^-(Figure 19.2). On U_x^+, y is a local coordinate, and so dy is a basis for the cotangent space $T_p^*(S^1)$ at each point $p \in U_x^+$. Since $\omega = dy/x$ on U_x^+, ω is C^∞ and nowhere zero on U_x^+. A similar argument applies to dy/x on U_x^- and $-dx/y$ on $U_y^+$and U_y^-. Hence, ω is C^∞ and nowhere vanishing on S^1.

Fig. 19.2. Two charts on the unit circle.

10.3.8 Problems

19.1. Pullback of a Differential Form

Let U be the open set $]0,\infty[\times]0,\pi[\times]0,2\pi[$ in the (ρ,ϕ,θ)-space $\mathbb{R}^3$. Define $F : U \to \mathbb{R}^3$ by

$$F(\rho,\phi,\theta) = (\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi).$$

If x, y, z are the standard coordinates on the target $\mathbb{R}^3$, show that

$$F^*(dx \wedge dy \wedge dz) = \rho^2 \sin\phi d\rho \wedge d\phi \wedge d\theta.$$

19.2. Pullback of a Differential Form

Let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$F(x,y) = \left(x^2 + y^2, xy\right).$$

If u, v are the standard coordinates on the target $\mathbb{R}^2$, compute $F^*(udu + vdv)$.

19.3. Pullback of a Differential Form by a Curve

Let τ be the 1-form $\tau = (-ydx + xdy)/(x^2 + y^2)$ on $\mathbb{R}^2 - \{\mathbf{0}\}$. Define $\gamma : \mathbb{R} \to \mathbb{R}^2 - \{\mathbf{0}\}$ by $\gamma(t) = (\cos t, \sin t)$. Compute $\gamma^*\tau$. (This problem is related to Example 17.16 in that if $i : S^1 \hookrightarrow \mathbb{R}^2 - \{\mathbf{0}\}$ is the inclusion, then $\gamma = i \circ c$ and $\omega = i^*\tau$.)

19.4. Pullback of a Restriction

Let $F : N \to M$ be a C^∞ map of manifolds, U an open subset of M, and $F|_{F^{-1}(U)} : F^{-1}(U) \to U$ the restriction of F to $F^{-1}(U)$. Prove that if $\omega \in \Omega^k(M)$, then

$$\left(F|_{F^{-1}(U)}\right)^* (\omega|_U) = (F^*\omega)|_{F^{-1}(U)}.$$

19.5. Coordinate Functions and Differential Forms

Let $f^1, \ldots, f^n$ be C^∞ functions on a neighborhood U of a point p in a manifold of dimension n. Show that there is a neighborhood W of p on which $f^1, \ldots, f^n$ form a coordinate system if and only if $(df^1 \wedge \cdots \wedge df^n)_p \neq 0$.

19.6. Local Operators

An operator $L : \Omega^*(M) \to \Omega^*(M)$ is support-decreasing if $\operatorname{supp} L(\omega) \subset \operatorname{supp} \omega$ for every k form $\omega \in \Omega^*(M)$ for all $k \geq 0$. Show that an operator on $\Omega^*(M)$ is local if and only if it is support-decreasing.

19.7. Derivations of c^∞ functions are local operators

Let M be a smooth manifold. The definition of a local operator D on $C^\infty(M)$ is similar to that of a local operator on $\Omega^*(M)$: D is local if whenever a function $f \in C^\infty(M)$ vanishes identically on an open subset U, then $Df \equiv 0$ on U. Prove that a derivation of $C^\infty(M)$ is a local operator on $C^\infty(M)$.

19.8. Nondegenerate 2-forms

A 2 -covector α on a $2n$-dimensional vector space V is said to be nondegenerate if $\alpha^n := \alpha \wedge \cdots \wedge \alpha$ (n times) is not the zero $2n$-covector. A 2 -form ω on a $2n$-dimensional manifold M is said to be nondegenerate if at every point $p \in M$, the 2 -covector ω_p is nondegenerate on the tangent space T_pM.
(a) Prove that on $\mathbb{C}^n$ with real coordinates $x^1, y^1, \ldots, x^n, y^n$, the 2-form

$$\omega = \sum_{j=1}^n dx^j \wedge dy^j$$

is nondegenerate.
(b) Prove that if λ is the Liouville form on the total space T^*M of the cotangent bundle of an n-dimensional manifold M, then $d\lambda$ is a nondegenerate 2-form on T^*M.

19.9* Vertical Planes

Let x, y, z be the standard coordinates on $\mathbb{R}^3$. A plane in $\mathbb{R}^3$ is vertical if it is defined by $ax + by = 0$ for some $(a, b) \neq (0, 0) \in \mathbb{R}^2$. Prove that restricted to a vertical plane, $dx \wedge dy = 0$.

19.10. Nowhere-vanishing Form on s^1

Prove that the nowhere-vanishing form ω on S^1 constructed in Example 19.8 is the form $-ydx + xdy$ of Example 17.15. (Hint: Consider U_x and U_y separately. On U_x, substitute $dx = -(y/x)dy$ into $-ydx + xdy$.)

19.11. A c^∞ nowhere-vanishing form on a smooth hypersurface

(a) Let $f(x,y)$ be a C^∞ function on $\mathbb{R}^2$ and assume that 0 is a regular value of f. By the regular level set theorem, the zero set M of $f(x,y)$ is a one-dimensional submanifold of $\mathbb{R}^2$. Construct a C^∞ nowhere-vanishing 1-form on M.
(b) Let $f(x,y,z)$ be a C^∞ function on $\mathbb{R}^3$ and assume that 0 is a regular value of f. By the regular level set theorem, the zero set M of $f(x,y,z)$ is a two-dimensional submanifold of $\mathbb{R}^3$. Let f_x, f_y, f_z be the partial derivatives of f with respect to x, y, z, respectively. Show that the equalities

$$\frac{dx \wedge dy}{f_z} = \frac{dy \wedge dz}{f_x} = \frac{dz \wedge dx}{f_y}$$

hold on M whenever they make sense, and therefore the three 2 -forms piece together to give a C^∞ nowhere-vanishing 2-form on M.
(c) Generalize this problem to a regular level set of $f\left(x^1, \ldots, x^{n+1}\right)$ in $\mathbb{R}^{n+1}$.

19.12. Vector Fields as Derivations of c^∞ functions

In Subsection 14.1 we showed that a C^∞ vector field X on a manifold M gives rise to a derivation of $C^\infty(M)$. We will now show that every derivation of $C^\infty(M)$ arises from one and only one vector field, as promised earlier. To distinguish the vector field from the derivation, we will temporarily denote the derivation arising from X by $\varphi(X)$. Thus, for any $f \in C^\infty(M)$,

$$(\varphi(X)f)(p) = X_p f \quad \text{for all } p \in M.$$

(a) Let $\mathcal{F} = C^\infty(M)$. Prove that $\varphi : \mathfrak{X}(M) \to \operatorname{Der}\left(C^\infty(M)\right)$ is an $\mathcal{F}$-linear map.
(b) Show that φ is injective.
(c) If D is a derivation of $C^\infty(M)$ and $p \in M$, define $D_p : C_p^\infty(M) \to C_p^\infty(M)$ by

$$D_p[f] = [D\tilde{f}] \in C_p^\infty(M),$$

where $[f]$ is the germ of f at p and $\tilde{f}$ is a global extension of f, such as those given by Proposition 18.8. Show that $D_p[f]$ is well defined. (Hint: Apply Problem 19.7.)
(d) Show that D_p is a derivation of $C_p^\infty(M)$.
(e) Prove that $\varphi : \mathfrak{X}(M) \to \operatorname{Der}\left(C^\infty(M)\right)$ is an isomorphism of $\mathcal{F}$-modules.

19.13. Twentieth-century Formulation of Maxwell's Equations

In Maxwell's theory of electricity and magnetism, developed in the late nineteenth century, the electric field $\mathbf{E} = \langle E_1, E_2, E_3 \rangle$ and the magnetic field $\mathbf{B} = \langle B_1, B_2, B_3 \rangle$ in a vacuum $\mathbb{R}^3$ with no charge or current satisfy the following equations:

$$\begin{aligned} \boldsymbol{\nabla} \times \mathbf{E} &= -\frac{\partial \mathbf{B}}{\partial t}, & \boldsymbol{\nabla} \times \mathbf{B} &= \frac{\partial \mathbf{E}}{\partial t}, \\ \operatorname{div} \mathbf{E} &= 0, & \operatorname{div} \mathbf{B} &= 0. \end{aligned}$$

By the correspondence in Subsection 4.6, the 1-form E on $\mathbb{R}^3$ corresponding to the vector field $\mathbf{E}$ is

$$E = E_1 dx + E_2 dy + E_3 dz$$

and the 2 -form B on $\mathbb{R}^3$ corresponding to the vector field $\mathbf{B}$ is

$$B = B_1 dy \wedge dz + B_2 dz \wedge dx + B_3 dx \wedge dy$$

Let $\mathbb{R}^4$ be space-time with coordinates (x, y, z, t). Then both E and B can be viewed as differential forms on $\mathbb{R}^4$. Define F to be the 2-form

$$F = E \wedge dt + B$$

on space-time. Decide which two of Maxwell's equations are equivalent to the equation

$$dF = 0$$

Prove your answer. (The other two are equivalent to $d * F = 0$ for a star-operator $*$ defined in differential geometry. See [2, Section 19.1, p. 689].)

10.4 §20 the Lie Derivative and Interior Multiplication

The only portion of this section necessary for the remainder of the book is Subsection 20.4 on interior multiplication. The rest may be omitted on first reading.

The construction of exterior differentiation in Section 19 is local and depends on a choice of coordinates: if $\omega = \sum a_I dx^I$, then

$$d\omega = \sum \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$$

It turns out, however, that this d is in fact global and intrinsic to the manifold, i.e., independent of the choice of local coordinates. Indeed, for a C^∞1-form ω and C^∞ vector fields X, Y on a manifold M, one has the formula

$$(d\omega)(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

In this section we will derive a global intrinsic formula like this for the exterior derivative of a k-form.

The proof uses the Lie derivative and interior multiplication, two other intrinsic operations on a manifold. The Lie derivative is a way of differentiating a vector field or a differential form on a manifold along another vector field. For any vector field X on a manifold, the interior multiplication l_X is an antiderivation of degree -1 on differential forms. Being intrinsic operators on a manifold, both the Lie derivative and interior multiplication are important in their own right in differential topology and geometry.

10.4.1 20.1 Families of Vector Fields and Differential Forms

A collection $\{X_t\}$ or $\{\omega_t\}$ of vector fields or differential forms on a manifold is said to be a 1 -parameter family if the parameter t runs over some subset of the real line. Let I be an open interval in $\mathbb{R}$ and let M be a manifold. Suppose $\{X_t\}$ is a 1-parameter family of vector fields on M defined for all $t \in I$ except at $t_0 \in I$. We say that the limit $\lim_{t\to t_0} X_t$ exists if every point $p \in M$ has a coordinate neighborhood $(U, x^1, \ldots, x^n)$ on which $X_t|_p = \sum a^i(t, p)\partial/\ \partial x^i|_p$ and $\lim_{t\to t_0} a^i(t, p)$ exists for all i. In this case, we set

$$\lim_{t\to t_0} X_t\Big|_p = \sum_{i=1}^n \lim_{t\to t_0} a^i(t, p) \frac{\partial}{\partial x^i}\Big|_p . \tag{20.1}$$

In Problem 20.1 we ask the reader to show that this definition of the limit of X_t as $t \to t_0$ is independent of the choice of the coordinate neighborhood $(U, x^1, \ldots, x^n)$.

A 1-parameter family $\{X_t\}_{t\in I}$ of smooth vector fields on M is said to depend smoothly on t if every point in M has a coordinate neighborhood $(U, x^1, \ldots, x^n)$ on which

$$(X_t)_p = \sum a^i(t,p)\frac{\partial}{\partial x^i}\bigg|_p, \quad (t,p) \in I \times U, \tag{20.2}$$

for some C^∞ functions a^i on $I \times U$. In this case we also say that $\{X_t\}_{t\in I}$ is a smooth family of vector fields on M.

For a smooth family of vector fields on M, one can define its derivative with respect to t at $t = t_0$ by

$$\left(\frac{d}{dt}\bigg|_{t=t_0} X_t\right)_p = \sum \frac{\partial a^i}{\partial t}(t_0,p)\frac{\partial}{\partial x^i}\bigg|_p \tag{20.3}$$

for $(t_0,p) \in I \times U$. It is easy to check that this definition is independent of the chart $(U, x^1, \ldots, x^n)$ containing p (Problem 20.3). Clearly, the derivative $d/\,dt|_{t=t_0} X_t$ is a smooth vector field on M.

Similarly, a 1 -parameter family $\{\omega_t\}_{t\in I}$ of smooth k-forms on M is said to depend smoothly on t if every point of M has a coordinate neighborhood $(U, x^1, \ldots, x^n)$ on which

$$(\omega_t)_p = \sum b_J(t,p)dx^J\bigg|_p, \quad (t,p) \in I \times U$$

for some C^∞ functions b_J on $I \times U$. We also call such a family $\{\omega_t\}_{t\in I}$ a smooth family of k-forms on M and define its derivative with respect to t to be

$$\left(\frac{d}{dt}\bigg|_{t=t_0} \omega_t\right)_p = \sum \frac{\partial b_J}{\partial t}(t_0,p)\,dx^J\bigg|_p$$

As for vector fields, this definition is independent of the chart and defines a $C^\infty k$ form $d/\,dt|_{t=t_0}\omega_t$ on M.

Notation. We write d/dt for the derivative of a smooth family of vector fields or differential forms, but $\partial/\partial t$ for the partial derivative of a function of several variables.

Proposition 20.1 (Product rule for d/dt). If $\{\omega_t\}$ and $\{\tau_t\}$ are smooth families of k-forms and ℓ-forms respectively on a manifold M, then

$$\frac{d}{dt}(\omega_t \wedge \tau_t) = \left(\frac{d}{dt}\omega_t\right) \wedge \tau_t + \omega_t \wedge \frac{d}{dt}\tau_t$$

Proof. Written out in local coordinates, this reduces to the usual product rule in calculus. We leave the details as an exercise (Problem 20.4).

Proposition 20.2 (Commutation of $d/\,dt|_{t=t_0}$ with d). If $\{\omega_t\}_{t\in I}$ is a smooth family of differential forms on a manifold M, then

$$\frac{d}{dt}\bigg|_{t=t_0} d\omega_t = d\left(\frac{d}{dt}\bigg|_{t=t_0}\omega_t\right).$$

Proof. In this proposition, there are three operations-exterior differentiation, differentiation with respect to t, and evaluation at $t = t_0$. We will first show that d and d/dt commute:

$$\frac{d}{dt}(d\omega_t) = d\left(\frac{d}{dt}\omega_t\right) \tag{20.4}$$

It is enough to check the equality at an arbitrary point $p \in M$. Let $(U, x^1, \ldots, x^n)$ be a neighborhood of p such that $\omega = \sum_J b_J dx^J$ for some C^∞ functions b_J on $I \times U$. On U,

$$\begin{aligned}\frac{d}{dt}(d\omega_t) &= \frac{d}{dt}\sum_{J,i}\frac{\partial b_J}{\partial x^i}dx^i\wedge dx^J \quad (\text{ note that there is no } dt \text{ term })\\ &= \sum_{i,J}\frac{\partial}{\partial x^i}\left(\frac{\partial b_J}{\partial t}\right)dx^i\wedge dx^J \quad (\text{ since } b_J \text{ is } C^\infty)\\ &= d\left(\sum_J \frac{\partial b_J}{\partial t}dx^J\right) = d\left(\frac{d}{dt}\omega_t\right)\end{aligned}$$

Evaluation at $t = t_0$ commutes with d, because d involves only partial derivatives with respect to the x^i variables. Explicitly,

$$\begin{aligned}\left(d\left(\frac{d}{dt}\omega_t\right)\right)\Big|_{t=t_0} &= \left(\sum_{i,J}\frac{\partial}{\partial x^i}\frac{\partial}{\partial t}b_J dx^i\wedge dx^J\right)\Big|_{t=t_0}\\ &= \sum_{i,J}\frac{\partial}{\partial x^i}\left(\frac{\partial}{\partial t}\Big|_{t=t_0} b_J\right)dx^i\wedge dx^J = d\left(\frac{\partial}{\partial t}\Big|_{t_0}\omega_t\right).\end{aligned}$$

Evaluating both sides of (20.4) at $t = t_0$ completes the proof of the proposition.

10.4.2 20.2 The Lie Derivative of a Vector Field

In a first course on calculus, one defines the derivative of a real-valued function f on $\mathbb{R}$ at a point $p \in \mathbb{R}$ as

$$f'(p) = \lim_{t\to 0}\frac{f(p+t)-f(p)}{t}.$$

The problem in generalizing this definition to the derivative of a vector field Y on a manifold M is that at two nearby points p and q in M, the tangent vectors Y_p and Y_q are in different vector spaces T_pM and T_qM and so it is not possible to compare them by subtracting one from the other. One way to get around this difficulty is to use the local flow of another vector field X to transport Y_q to the tangent space T_pM at p. This leads to the definition of the Lie derivative of a vector field.

Recall from Subsection 14.3 that for any smooth vector field X on M and point p in M, there is a neighborhood U of p on which the vector field X has a local flow; this means that there exist a real number $\varepsilon > 0$ and a map

$$\varphi :]-\varepsilon,\varepsilon[\times U \to M$$

such that if we set $\varphi_t(q) = \varphi(t,q)$, then

$$\frac{\partial}{\partial t}\varphi_t(q) = X_{\varphi_t(q)}, \quad \varphi_0(q) = q \quad \text{for } q \in U. \tag{20.5}$$

In other words, for each q in U, the curve $\varphi_t(q)$ is an integral curve of X with initial point q. By definition, $\varphi_0 : U \to U$ is the identity map. The local flow satisfies the property

$$\varphi_s \circ \varphi_t = \varphi_{s+t}$$

whenever both sides are defined (see (14.10)). Consequently, for each t the map $\varphi_t : U \to \varphi_t(U)$ is a diffeomorphism onto its image, with a C^∞ inverse φ_{-t} :

$$\varphi_{-t}\circ\varphi_t = \varphi_0 = \mathbb{1}, \quad \varphi_t\circ\varphi_{-t} = \varphi_0 = \mathbb{1}.$$

Let Y be a C^∞ vector field on M. To compare the values of Y at $\varphi_t(p)$ and at p, we use the diffeomorphism $\varphi_{-t} : \varphi_t(U) \to U$ to push $Y_{\varphi_t(p)}$ into T_pM (Figure 20.1).

Fig. 20.1. Comparing the values of Y at nearby points.

Definition 20.3. For $X, Y \in \mathfrak{X}(M)$ and $p \in M$, let $\varphi :]-\varepsilon, \varepsilon[\times U \to M$ be a local flow of X on a neighborhood U of p and define the Lie derivative $\mathcal{L}_X Y$ of Y with respect to X at p to be the vector

$$(\mathcal{L}_X Y)_p = \lim_{t\to 0} \frac{\varphi_{-t*}\left(Y_{\varphi_t(p)}\right) - Y_p}{t} = \lim_{t\to 0} \frac{(\varphi_{-t*}Y)_p - Y_p}{t} = \left.\frac{d}{dt}\right|_{t=0} (\varphi_{-t*}Y)_p .$$

In this definition the limit is taken in the finite-dimensional vector space T_pM. For the derivative to exist, it suffices that $\{\varphi_{-t*}Y\}$ be a smooth family of vector fields on M. To show the smoothness of the family $\{\varphi_{-t*}Y\}$, we write out $\varphi_{-t*}Y$ in local coordinates $x^1, \ldots, x^n$ in a chart. Let φ_t^i and φ^i be the i th components of φ_t and φ respectively. Then

$$(\varphi_t)^i (p) = \varphi^i(t,p) = \left(x^i \circ \varphi\right)(t,p).$$

By Proposition 8.11, relative to the frame $\{\partial/\partial x^j\}$, the differential φ_{t*} at p is represented by the Jacobian matrix $\left[\partial (\varphi_t)^i / \partial x^j(p)\right] = [\partial\varphi^i/\partial x^j(t,p)]$. This means that

$$\varphi_{t*}\left(\left.\frac{\partial}{\partial x^j}\right|_p\right) = \sum_i \frac{\partial \varphi^i}{\partial x^j}(t,p) \left.\frac{\partial}{\partial x^i}\right|_{\varphi_t(p)} .$$

Thus, if $Y = \sum b^j \partial/\partial x^j$, then

$$\begin{aligned}\varphi_{-t*}\left(Y_{\varphi_t(p)}\right) &= \sum_j b^j(\varphi(t,p))\varphi_{-t*}\left(\left.\frac{\partial}{\partial x^j}\right|_{\varphi_t(p)}\right) \\ &= \sum_{i,j} b^j(\varphi(t,p)) \frac{\partial \varphi^i}{\partial x^j}(-t,p) \left.\frac{\partial}{\partial x^i}\right|_p . \end{aligned} \tag{20.6}$$

When X and Y are C^∞ vector fields on M, both φ^i and b^j are C^∞ functions. The formula (20.6) then shows that $\{\varphi_{-t*}Y\}$ is a smooth family of vector fields on M. It follows that the Lie derivative $\mathcal{L}_X Y$ exists and is given in local coordinates by

$$
\begin{aligned}
(\mathcal{L}_X Y)_p &= \left.\frac{d}{dt}\right|_{t=0} \varphi_{-t*}\left(Y_{\varphi_t(p)}\right) \\
&= \sum_{i,j} \left.\frac{\partial}{\partial t}\right|_{t=0} \left(b^j(\varphi(t,p))\frac{\partial \varphi^i}{\partial x^j}(-t,p)\right) \left.\frac{\partial}{\partial x^i}\right|_p .
\end{aligned} \tag{20.7}
$$

It turns out that the Lie derivative of a vector field gives nothing new.
Theorem 20.4. If X and Y are C^∞ vector fields on a manifold M, then the Lie derivative $\mathcal{L}_X Y$ coincides with the Lie bracket $[X, Y]$.
Proof. It suffices to check the equality $\mathcal{L}_X Y = [X, Y]$ at every point. To do this, we expand both sides in local coordinates. Suppose a local flow for X is $\varphi : \,]-\varepsilon, \varepsilon[\times U \to M$, where U is a coordinate chart with coordinates $x^1, \ldots, x^n$. Let $X = \sum a^i \partial/\partial x^i$ and $Y = \sum b^j \partial/\partial x^j$ on U. The condition (20.5) that $\varphi_t(p)$ be an integral curve of X translates into the equations

$$
\frac{\partial \varphi^i}{\partial t}(t,p) = a^i(\varphi(t,p)), \quad i = 1, \ldots, n, \quad (t,p) \in \Big]-\varepsilon, \varepsilon[\times U
$$

At $t = 0, \partial\varphi^i/\partial t(0,p) = a^i(\varphi(0,p)) = a^i(p)$.
By Problem 14.12, the Lie bracket in local coordinates is

$$
[X, Y] = \sum_{i,k} \left(a^k \frac{\partial b^i}{\partial x^k} - b^k \frac{\partial a^i}{\partial x^k}\right) \frac{\partial}{\partial x^i}
$$

Expanding (20.7) by the product rule and the chain rule, we get

$$
\begin{aligned}
(\mathcal{L}_X Y)_p = &\Bigg[\sum_{i,j,k} \left(\frac{\partial b^j}{\partial x^k}(\varphi(t,p))\frac{\partial \varphi^k}{\partial t}(t,p)\frac{\partial \varphi^i}{\partial x^j}(-t,p)\right) \frac{\partial}{\partial x^i} \\
&- \sum_{i,j} \left(b^j(\varphi(t,p))\frac{\partial}{\partial x^j}\frac{\partial \varphi^i}{\partial t}(-t,p)\right) \frac{\partial}{\partial x^i}\Bigg]_{t=0} \\
= &\sum_{i,j,k} \left(\frac{\partial b^j}{\partial x^k}(p) a^k(p) \frac{\partial \varphi^i}{\partial x^j}(0,p)\right) \frac{\partial}{\partial x^i} - \sum_{i,j} \left(b^j(p)\frac{\partial a^i}{\partial x^j}(p)\right) \frac{\partial}{\partial x^i}.
\end{aligned} \tag{20.8}
$$

Since $\varphi(0,p) = p, \varphi_0$ is the identity map and hence its Jacobian matrix is the identity matrix. Thus,

$$
\frac{\partial \varphi^i}{\partial x^j}(0,p) = \delta^i_j, \qquad \text{the Kronecker delta.}
$$

So (20.8) simplifies to

$$
\mathcal{L}_X Y = \sum_{i,k} \left(a^k \frac{\partial b^i}{\partial x^k} - b^k \frac{\partial a^i}{\partial x^k}\right) \frac{\partial}{\partial x^i} = [X, Y]
$$

□

Although the Lie derivative of a vector field gives us nothing new, in conjunction with the Lie derivative of differential forms it turns out to be a tool of great utility, for example, in the proof of the global formula for the exterior derivative in Theorem 20.14.

10.4.3 20.3 The Lie Derivative of a Differential Form

Let X be a smooth vector field and ω a smooth k-form on a manifold M. Fix a point $p \in M$ and let $\varphi_t : U \to M$ be a flow of X in a neighborhood U of p. The definition of the Lie derivative of a differential form is similar to that of the Lie derivative of a vector field. However, instead of pushing a vector at $\varphi_t(p)$ to p via $(\varphi_{-t})_*$, we now pull the k-covector $\omega_{\varphi_t(p)}$ back to p via φ_t^*.

Definition 20.5. For X a smooth vector field and ω a smooth k-form on a manifold M, the Lie derivative $\mathcal{L}_X\omega$ at $p \in M$ is

$$(\mathcal{L}_X\omega)_p = \lim_{t\to 0} \frac{\varphi_t^*\left(\omega_{\varphi_t(p)}\right) - \omega_p}{t} = \lim_{t\to 0} \frac{(\varphi_t^*\omega)_p - \omega_p}{t} = \left.\frac{d}{dt}\right|_{t=0} (\varphi_t^*\omega)_p$$

By an argument similar to that for the existence of the Lie derivative $\mathcal{L}_X Y$ in Section 20.2, one shows that $\{\varphi_t^*\omega\}$ is a smooth family of k-forms on M by writing it out in local coordinates. The existence of $(\mathcal{L}_X\omega)_p$ follows.

Proposition 20.6. If f is a C^∞ function and X a C^∞ vector field on M, then $\mathcal{L}_X f = Xf$.

Proof. Fix a point p in M and let $\varphi_t : U \to M$ be a local flow of X as above. Then

$$\begin{aligned}
(\mathcal{L}_X f)_p &= \left.\frac{d}{dt}\right|_{t=0} (\varphi_t^* f)_p && (\text{ definition of } \mathcal{L}_X f) \\
&= \left.\frac{d}{dt}\right|_{t=0} (f \circ \varphi_t)(p) && (\text{ definition of } \varphi_t^* f) \\
&= X_p f && (\text{ Proposition 8.17 })
\end{aligned}$$

since $\varphi_t(p)$ is a curve through p with initial vector X_p. $\square$

10.4.4 20.4 Interior Multiplication

We first define interior multiplication on a vector space. If β is a k-covector on a vector space V and $v \in V$, for $k \geq 2$ the interior multiplication or contraction of β with v is the $(k-1)$-covector $l_v\beta$ defined by

$$(\imath_v\beta)(v_2, \ldots, v_k) = \beta(v, v_2, \ldots, v_k), \quad v_2, \ldots, v_k \in V.$$

We define $\imath_v\beta = \beta(v) \in \mathbb{R}$ for a 1 -covector β on V and $\imath_v\beta = 0$ for a 0 -covector β (a constant) on V.

Proposition 20.7. For 1 -covectors $\alpha^1, \ldots, \alpha^k$ on a vector space V and $v \in V$,

$$\imath_v\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right) = \sum_{i=1}^{k} (-1)^{i-1} \alpha^i(v) \alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k$$

where the caret $\wedge$ over α^i means that α^i is omitted from the wedge product.

Proof.

$$
\begin{aligned}
&\left(\imath_v\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right)\right)\left(v_2, \ldots, v_k\right) \\
&\quad=\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right)\left(v, v_2, \ldots, v_k\right) \\
&\quad=\det\left[\begin{array}{cccc}
\alpha^1(v) & \alpha^1\left(v_2\right) & \cdots & \alpha^1\left(v_k\right) \\
\alpha^2(v) & \alpha^2\left(v_2\right) & \cdots & \alpha^2\left(v_k\right) \\
\vdots & \vdots & \ddots & \vdots \\
\alpha^k(v) & \alpha^k\left(v_2\right) & \cdots & \alpha^k\left(v_k\right)
\end{array}\right] \qquad \text{(Proposition 3.27)} \qquad \text{(Proposition3.27)} \\
&\quad=\sum_{i=1}^k(-1)^{i+1} \alpha^i(v) \det\left[\alpha^\ell\left(v_j\right)\right]_{\substack{1 \leq \ell \leq k, \ell \neq i \\ 2 \leq j \leq k}} \qquad \text{(expansion along first column)} \\
&\quad=\sum_{i=1}^k(-1)^{i+1} \alpha^i(v)\left(\alpha^1 \wedge \cdots \wedge \widehat{\alpha^i} \wedge \cdots \wedge \alpha^k\right)\left(v_2, \ldots, v_k\right) \qquad \text{(Proposition 3)} \\
&\qquad\qquad\qquad

is a smooth function on M (Proposition 18.7(iii) $\Rightarrow$ (i)). Of course, $l_X\omega = \omega(X)$ for a 1 -form ω and $l_X f = 0$ for a function f on M. By the properties of interior multiplication at each point $p \in M$ (Proposition 20.8), the map $\imath_X : \Omega^*(M) \to \Omega^*(M)$ is an antiderivation of degree -1 such that $\boldsymbol{l}_X \circ \boldsymbol{l}_X = 0$.

Let $\mathcal{F}$ be the ring $C^\infty(M)$ of C^∞ functions on the manifold M. Because $\imath_X\omega$ is a point operator-that is, its value at p depends only on X_p and ω_p-it is $\mathcal{F}$-linear in either argument. This means that $l_X\omega$ is additive in each argument and moreover, for any $f \in \mathcal{F}$,
(i) $\imath_{fX}\omega = f\imath_X\omega$;
(ii) $l_X(f\omega) = fl_X\omega$.

Explicitly, the proof of (i) goes as follows. For any $p \in M$,

$$(\imath_{fX}\omega)_p = \imath_{f(p)X_p}\omega_p = f(p)\imath_{X_p}\omega_p = (f\imath_X\omega)_p$$

Hence, $l_{fX}\omega = fl_X\omega$. The proof of (ii) is similar. Additivity is more or less obvious.
Example 20.9 (Interior multiplication on $\mathbb{R}^2$). Let $X = x\partial/\partial x + y\partial/\partial y$ be the radial vector field and $\alpha = dx \wedge dy$ the area 2-form on the plane $\mathbb{R}^2$. Compute the contraction $\imath_X\alpha$.

Solution. We first compute $l_X dx$ and $l_X dy$:

$$\begin{aligned}\imath_X dx &= dx(X) = dx\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = x,\\ \imath_X dy &= dy(X) = dy\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right) = y.\end{aligned}$$

By the antiderivation property of l_X,

$$\imath_X\alpha = \imath_X(dx \wedge dy) = (\imath_X dx)\, dy - dx\,(\imath_X dy) = xdy - ydx,$$

which restricts to the nowhere-vanishing 1-form ω on the circle S^1 in Example 17.15.

10.4.5 20.5 Properties of the Lie Derivative

In this section we state and prove several basic properties of the Lie derivative. We also relate the Lie derivative to two other intrinsic operators on differential forms on a manifold: the exterior derivative and interior multiplication. The interplay of these three operators results in some surprising formulas.

Theorem 20.10. Assume X to be a C^∞ vector field on a manifold M.
(i) The Lie derivative $\mathcal{L}_X : \Omega^*(M) \to \Omega^*(M)$ is a derivation: it is an $\mathbb{R}$-linear map and if $\omega \in \Omega^k(M)$ and $\tau \in \Omega^\ell(M)$, then

$$\mathcal{L}_X(\omega \wedge \tau) = (\mathcal{L}_X\omega) \wedge \tau + \omega \wedge (\mathcal{L}_X\tau)\,.$$

(ii) The Lie derivative $\mathcal{L}_X$ commutes with the exterior derivative d.
(iii) (Cartan homotopy formula) $\mathcal{L}_X = dl_X + l_X d$.
(iv) ("Product" formula) For $\omega \in \Omega^k(M)$ and $Y_1, \ldots, Y_k \in \mathfrak{X}(M)$,

$$\mathcal{L}_X\left(\omega\left(Y_1, \ldots, Y_k\right)\right) = \left(\mathcal{L}_X\omega\right)\left(Y_1, \ldots, Y_k\right) + \sum_{i=1}^{k} \omega\left(Y_1, \ldots, \mathcal{L}_X Y_i, \ldots, Y_k\right).$$

Proof. In the proof let $p \in M$ and let $\varphi_t : U \to M$ be a local flow of the vector field X in a neighborhood U of p.
(i) Since the Lie derivative $\mathcal{L}_X$ is d/dt of a vector-valued function of t, the derivation property of $\mathcal{L}_X$ is really just the product rule for d/dt (Proposition 20.1). More precisely,

$$
\begin{aligned}
\left(\mathcal{L}_X(\omega \wedge \tau)\right)_p &= \left.\frac{d}{dt}\right|_{t=0} \left(\varphi_t^*(\omega \wedge \tau)\right)_p \\
&= \left.\frac{d}{dt}\right|_{t=0} \left(\varphi_t^*\omega\right)_p \wedge \left(\varphi_t^*\tau\right)_p \\
&= \left(\left.\frac{d}{dt}\right|_{t=0} \left(\varphi_t^*\omega\right)_p\right) \wedge \tau_p + \omega_p \wedge \left.\frac{d}{dt}\right|_{t=0} \left(\varphi_t^*\tau\right)_p
\end{aligned}
$$

(product rule for d/dt)

$$
= \left(\mathcal{L}_X\omega\right)_p \wedge \tau_p + \omega_p \wedge \left(\mathcal{L}_X\tau\right)_p .
$$

(ii)

$$
\begin{aligned}
\mathcal{L}_X d\omega &= \left.\tfrac{d}{dt}\right|_{t=0} \varphi_t^* d\omega && (\text{ definition of } \mathcal{L}_X) \\
&= \left.\tfrac{d}{dt}\right|_{t=0} d\varphi_t^*\omega && (d \text{ commutes with pullback }) \\
&= d\left(\left.\tfrac{d}{dt}\right|_{t=0} \varphi_t^*\omega\right) && (\text{ by Proposition 20.2}) \\
&= d\mathcal{L}_X\omega
\end{aligned}
$$

(iii) We make two observations that reduce the problem to a simple case. First, for any $\omega \in \Omega^k(M)$, to prove the equality $\mathcal{L}_X\omega = (d\imath_X + \imath_X d)\,\omega$ it suffices to check it at any point p, which is a local problem. In a coordinate neighborhood $(U, x^1, \ldots, x^n)$ about p, we may assume by linearity that ω is a wedge product $\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.

Second, on the left-hand side of the Cartan homotopy formula, by (i) and (ii), $\mathcal{L}_X$ is a derivation that commutes with d. On the right-hand side, since d and l_X are antiderivations, $dl_X + l_X d$ is a derivation by Problem 4.7. It clearly commutes with d. Thus, both sides of the Cartan homotopy formula are derivations that commute with d. Consequently, if the formula holds for two differential forms ω and τ, then it holds for the wedge product $\omega \wedge \tau$ as well as for $d\omega$. These observations reduce the verification of (iii) to checking

$$
\mathcal{L}_X f = (d\imath_X + \imath_X d)\, f \quad \text{ for } f \in C^\infty(U)
$$

This is quite easy:

$$
\begin{aligned}
(d\imath_X + \imath_X d)\, f &= \imath_X df && (\text{ because } \imath_X f = 0) \\
&= (df)(X) && (\text{ definition of } \imath_X) \\
&= Xf = \mathcal{L}_X f && (\text{ Proposition 20.6}).
\end{aligned}
$$

(iv) We call this the "product" formula, even though there is no product in $\omega\left(Y_1, \ldots, Y_k\right)$, because this formula can be best remembered as though the juxtaposition of symbols were a product. In fact, even its proof resembles that of the product formula in calculus. To illustrate this, consider the case $k = 2$. Let $\omega \in \Omega^2(M)$ and $X, Y, Z \in \mathfrak{X}(M)$. The proof looks forbidding, but the idea is quite simple. To compare the values of $\omega(Y, Z)$ at the two points $\varphi_t(p)$ and p, we subtract the value at p from the value at $\varphi_t(p)$. The trick is to add and subtract terms so that each time only one of the three variables ω, Y, and Z moves from one point to the other. By the definitions of the Lie derivative and the pullback of a function,

$$
\begin{aligned}
\left(\mathcal{L}_X(\omega(Y, Z))\right)_p &= \lim_{t \to 0} \frac{\left(\varphi_t^*(\omega(Y, Z))\right)_p - \left(\omega(Y, Z)\right)_p}{t} \\
&= \lim_{t \to 0} \frac{\omega_{\varphi_t(p)}\left(Y_{\varphi_t(p)}, Z_{\varphi_t(p)}\right) - \omega_p\left(Y_p, Z_p\right)}{t}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{t\to 0} \frac{\omega_{\varphi_t(p)}\left(Y_{\varphi_t(p)}, Z_{\varphi_t(p)}\right) - \omega_p\left(\varphi_{-t*}\left(Y_{\varphi_t(p)}\right), \varphi_{-t*}\left(Z_{\varphi_t(p)}\right)\right)}{t} && (20.9)\\
&+ \lim_{t\to 0} \frac{\omega_p\left(\varphi_{-t*}\left(Y_{\varphi_t(p)}\right), \varphi_{-t*}\left(Z_{\varphi_t(p)}\right)\right) - \omega_p\left(Y_p, \varphi_{-t*}\left(Z_{\varphi_t(p)}\right)\right)}{t} && (20.10)\\
&+ \lim_{t\to 0} \frac{\omega_p\left(Y_p, \varphi_{-t*}\left(Z_{\varphi_t(p)}\right)\right) - \omega_p\left(Y_p, Z_p\right)}{t}. && (20.11)
\end{aligned}
$$

In this sum the quotient in the first limit (20.9) is

$$
\begin{aligned}
&\frac{\left(\varphi_t^*\omega_{\varphi_t(p)}\right)\left(\varphi_{-t*}\left(Y_{\varphi_t(p)}\right), \varphi_{-t*}\left(Z_{\varphi_t(p)}\right)\right) - \omega_p\left(\varphi_{-t*}\left(Y_{\varphi_t(p)}\right), \varphi_{-t*}\left(Z_{\varphi_t(p)}\right)\right)}{t}\\
&\qquad = \frac{\varphi_t^*\left(\omega_{\varphi_t(p)}\right) - \omega_p}{t}\left(\varphi_{-t*}\left(Y_{\varphi_t(p)}\right), \varphi_{-t*}\left(Z_{\varphi_t(p)}\right)\right).
\end{aligned}
$$

On the right-hand side of this equality, the difference quotient has a limit at $t = 0$, namely the Lie derivative $(\mathcal{L}_X\omega)_p$, and by (20.6) the two arguments of the difference quotient are C^∞ functions of t. Therefore, the right-hand side is a continuous function of t and its limit as t goes to 0 is $(\mathcal{L}_X\omega)_p(Y_p, Z_p)$ (by Problem 20.2).

By the bilinearity of ω_p, the second term (20.10) is

$$
\lim_{t\to 0} \omega_p\left(\frac{\varphi_{-t*}\left(Y_{\varphi_t(p)}\right) - Y_p}{t}, \varphi_{-t*}\left(Z_{\varphi_t(p)}\right)\right) = \omega_p\left(\left(\mathcal{L}_X Y\right)_p, Z_p\right).
$$

Similarly, the third term (20.11) is $\omega_p\left(Y_p, \left(\mathcal{L}_X Z\right)_p\right)$.
Thus

$$
\mathcal{L}_X(\omega(Y, Z)) = \left(\mathcal{L}_X\omega\right)(Y, Z) + \omega\left(\mathcal{L}_X Y, Z\right) + \omega\left(Y, \mathcal{L}_X Z\right).
$$

The general case is similar. □

Remark. Unlike interior multiplication, the Lie derivative $\mathcal{L}_X\omega$ is not $\mathcal{F}$-linear in either argument. By the derivation property of the Lie derivative (Theorem 20.10(i)),

$$
\mathcal{L}_X(f\omega) = \left(\mathcal{L}_X f\right)\omega + f\mathcal{L}_X\omega = (Xf)\omega + f\mathcal{L}_X\omega.
$$

We leave the problem of expanding $\mathcal{L}_{fX}\omega$ as an exercise (Problem 20.7).
Theorem 20.10 can be used to calculate the Lie derivative of a differential form.
Example 20.11 (The Lie derivative on a circle). Let ω be the 1 -form $-ydx + xdy$ and let X be the tangent vector field $-y\partial/\partial x + x\partial/\partial y$ on the unit circle S^1 from Example 17.15. Compute the Lie derivative $\mathcal{L}_X\omega$.

Solution. By Proposition 20.6,

$$
\begin{aligned}
\mathcal{L}_X(x) &= Xx = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)x = -y,\\
\mathcal{L}_X(y) &= Xy = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right)y = x.
\end{aligned}
$$

Next we compute $\mathcal{L}_X(-ydx)$:

$$
\begin{aligned}
\mathcal{L}_X(-ydx) &= -\left(\mathcal{L}_X y\right)dx - y\mathcal{L}_X dx \quad (\mathcal{L}_X \text{ is a derivation })\\
&= -\left(\mathcal{L}_X y\right)dx - yd\mathcal{L}_X x \quad (\mathcal{L}_X \text{ commutes with } d)\\
&= -xdx + ydy
\end{aligned}
$$

Similarly, $\mathcal{L}_X(xdy) = -ydy + xdx$. Hence, $\mathcal{L}_X\omega = \mathcal{L}_X(-ydx + xdy) = 0$.

10.4.6 20.6 Global Formulas for the Lie and Exterior Derivatives

The definition of the Lie derivative $\mathcal{L}_X\omega$ is local, since it makes sense only in a neighborhood of a point. The product formula in Theorem 20.10(iv), however, gives a global formula for the Lie derivative.

Theorem 20.12 (Global formula for the Lie derivative). For a smooth k-form ω and smooth vector fields $X, Y_1, \ldots, Y_k$ on a manifold M,

$$(\mathcal{L}_X\omega)(Y_1, \ldots, Y_k) = X(\omega(Y_1, \ldots, Y_k)) - \sum_{i=1}^{k} \omega(Y_1, \ldots, [X, Y_i], \ldots, Y_k).$$

Proof. In Theorem 20.10(iv), $\mathcal{L}_X(\omega(Y_1, \ldots, Y_k)) = X(\omega(Y_1, \ldots, Y_k))$ by Proposition 20.6 and $\mathcal{L}_X Y_i = [X, Y_i]$ by Theorem 20.4.

The definition of the exterior derivative d is also local. Using the Lie derivative, we obtain a very useful global formula for the exterior derivative. We first derive the formula for the exterior derivative of a 1-form, the case most useful in differential geometry.

Proposition 20.13. If ω is a C^∞1-form and X and Y are C^∞ vector fields on a manifold M, then

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

Proof. It is enough to check the formula in a chart $(U, x^1, \ldots, x^n)$, so we may assume $\omega = \sum a_i dx^i$. Since both sides of the equation are $\mathbb{R}$-linear in ω, we may further assume that $\omega = f dg$, where $f, g \in C^\infty(U)$.

In this case, $d\omega = d(f dg) = df \wedge dg$ and

$$\begin{aligned}
d\omega(X, Y) &= df(X)dg(Y) - df(Y)dg(X) = (Xf)Yg - (Yf)Xg \\
X\omega(Y) &= X(f dg(Y)) = X(fYg) = (Xf)Yg + fXYg \\
Y\omega(X) &= Y(f dg(X)) = Y(fXg) = (Yf)Xg + fYXg \\
\omega([X, Y]) &= f dg([X, Y]) = f(XY - YX)g
\end{aligned}$$

It follows that

$$X\omega(Y) - Y\omega(X) - \omega([X, Y]) = (Xf)Yg - (Yf)Xg = d\omega(X, Y).$$

Theorem 20.14 (Global formula for the exterior derivative). Assume $k \geq 1$. For a smooth k-form ω and smooth vector fields $Y_0, Y_1, \ldots, Y_k$ on a manifold M,

$$\begin{aligned}
(d\omega)(Y_0, \ldots, Y_k) = \sum_{i=0}^{k} (-1)^i Y_i \omega\left(Y_0, \ldots, \widehat{Y}_i, \ldots, Y_k\right) \\
+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \omega\left([Y_i, Y_j], Y_0, \ldots, \widehat{Y}_i, \ldots, \widehat{Y}_j, \ldots, Y_k\right)
\end{aligned}$$

Proof. When $k = 1$, the formula is proven in Proposition 20.13.
Assuming the formula for forms of degree $k - 1$, we can prove it by induction for a form ω of degree k. By the definition of l_{Y_0} and Cartan's homotopy formula (Theorem 20.10(iii)),

$$\begin{aligned}
(d\omega)(Y_0, Y_1, \ldots, Y_k) &= (l_{Y_0} d\omega)(Y_1, \ldots, Y_k) \\
&= (\mathcal{L}_{Y_0}\omega)(Y_1, \ldots, Y_k) - (d l_{Y_0}\omega)(Y_1, \ldots, Y_k).
\end{aligned}$$

The first term of this expression can be computed using the global formula for the Lie derivative $\mathcal{L}_{Y_0}\omega$, while the second term can be computed using the global formula for d of a

form of degree $k-1$. This kind of verification is best done by readers on their own. We leave it as an exercise (Problem 20.6).

10.4.7 Problems

20.1. The Limit of a Family of Vector Fields

Let I be an open interval, M a manifold, and $\{X_t\}$ a 1-parameter family of vector fields on M defined for all $t \neq t_0 \in I$. Show that the definition of $\lim_{t\to t_0} X_t$ in (20.1), if the limit exists, is independent of coordinate charts.

20.2. Limits of Families of Vector Fields and Differential Forms

Let I be an open interval containing 0 . Suppose $\{\omega_t\}_{t\in I}$ and $\{Y_t\}_{t\in I}$ are 1-parameter families of 1 -forms and vector fields respectively on a manifold M. Prove that if $\lim_{t\to 0}\omega_t = \omega_0$ and $\lim_{t\to 0} Y_t = Y_0$, then $\lim_{t\to 0}\omega_t(Y_t) = \omega_0(Y_0)$. (Hint: Expand in local coordinates.) By the same kind of argument, one can show that there is a similar formula for a family $\{\omega_t\}$ of 2-forms: $\lim_{t\to 0}\omega_t(Y_t, Z_t) = \omega_0(Y_0, Z_0)$.

20.3.* Derivative of a Smooth Family of Vector Fields

Show that the definition (20.3) of the derivative of a smooth family of vector fields on M is independent of the chart $(U, x^1, \ldots, x^n)$ containing p.

20.4. Product Rule for d/dt

Prove that if $\{\omega_t\}$ and $\{\tau_t\}$ are smooth families of k-forms and ℓ-forms respectively on a manifold M, then

$$\frac{d}{dt}(\omega_t \wedge \tau_t) = \left(\frac{d}{dt}\omega_t\right) \wedge \tau_t + \omega_t \wedge \frac{d}{dt}\tau_t$$

20.5. Smooth Families of Forms and Vector Fields

If $\{\omega_t\}_{t\in I}$ is a smooth family of 2-forms and $\{Y_t\}_{t\in I}$ and $\{Z_t\}_{t\in I}$ are smooth families of vector fields on a manifold M, prove that $\omega_t(X_t, Y_t)$ is a C^∞ function on $I \times M$.

20.6* Global Formula for the Exterior Derivative

Complete the proof of Theorem 20.14.

20.7. F-linearity and the Lie Derivative

Let ω be a differential form, X a vector field, and f a smooth function on a manifold. The Lie derivative $\mathcal{L}_X\omega$ is not $\mathcal{F}$-linear in either variable, but prove that it satisfies the following identity:

$$\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge l_X\omega$$

(Hint: Start with Cartan's homotopy formula $\mathcal{L}_X = d\imath_X + \imath_X d$.)

20.8. Bracket of the Lie Derivative and Interior Multiplication

If X and Y are smooth vector fields on a manifold M, prove that on differential forms on M

$$\mathcal{L}_X \imath_Y - \imath_Y \mathcal{L}_X = \imath_{[X,Y]}$$

(Hint: Let $\omega \in \Omega^k(M)$ and $Y, Y_1, \ldots, Y_{k-1} \in \mathfrak{X}(M)$. Apply the global formula for $\mathcal{L}_X$ to

$$(l_Y \mathcal{L}_X \omega)(Y_1, \ldots, Y_{k-1}) = (\mathcal{L}_X \omega)(Y, Y_1, \ldots, Y_{k-1}).)$$

20.9. Interior Multiplication on $\mathbb{R}^n$

Let $\omega = dx^1 \wedge \cdots \wedge dx^n$ be the volume form and $X = \sum x^i \partial/\partial x^i$ the radial vector field on $\mathbb{R}^n$. Compute the contraction $\imath_X \omega$.

20.10. The Lie Derivative on the 2 -sphere

Let $\omega = xdy \wedge dz - ydx \wedge dy + zdx \wedge dy$ and $X = -y\partial/\partial x + x\partial/\partial y$ on the unit 2-sphere S^2 in $\mathbb{R}^3$. Compute the Lie derivative $\mathcal{L}_X \omega$.

10.5 The Change into Non-Coordinate Language and Back (????)

(???? for many year, I have this question, how to do it?)

11 6 Integration

On a manifold one integrates not functions as in calculus on $\mathbb{R}^n$ but differential forms. There are actually two theories of integration on manifolds, one in which the integration is over a submanifold and the other in which the integration is over what is called a singular chain. Singular chains allow one to integrate over an object such as a closed rectangle in $\mathbb{R}^2$:

$$[a, b] \times [c, d] := \left\{(x, y) \in \mathbb{R}^2 \mid a \leq x \leq b, c \leq y \leq d\right\},$$

which is not a submanifold of $\mathbb{R}^2$ because of its corners.
For simplicity we will discuss only integration of smooth forms over a submanifold. For integration of noncontinuous forms over more general sets, the reader may consult the many excellent references in the bibliography, for example [3, Section VI.2], [7, Section 8.2], or [25, Chapter 14].

For integration over a manifold to be well defined, the manifold needs to be oriented. We begin the chapter with a discussion of orientations on a manifold. We then enlarge the category of manifolds to include manifolds with boundary. Our treatment of integration culminates in Stokes's theorem for an n-dimensional manifold. Stokes's theorem for a surface with boundary in $\mathbb{R}^3$ was first published as a question in the Smith's Prize Exam that Stokes set at the University of Cambridge in 1854. It is not known whether any student solved the problem. According to [21, p. 150], the same theorem had appeared four years earlier in a letter of Lord Kelvin to Stokes, which only goes to confirm that the attribution of credit in mathematics is fraught with pitfalls. Stokes's theorem for a general manifold resulted from the work of many mathematicians, including Vito Volterra (1889), Henri Poincaré (1899), Edouard Goursat (1917), and Élie Cartan (1899 and 1922). First there were many special cases, then a general statement in terms of coordinates, and finally a general statement in terms of differential forms. Cartan was the master of differential forms par excellence, and it was in his work that the differential form version of Stokes's theorem found its clearest expression.

11.1 §21 Orientations

It is a familiar fact from vector calculus that line and surface integrals depend on the orientation of the curve or surface over which the integration takes place: reversing the orientation changes the sign of the integral. The goal of this section is to define orientation for n-dimensional manifolds and to investigate various equivalent characterizations of orientation.

We assume all vector spaces in this section to be finite-dimensional and real. An orientation of a finite-dimensional real vector space is simply an equivalence class of ordered bases, two ordered bases being equivalent if and only if their transition matrix has positive determinant. By its alternating nature, a multicovector of top degree turns out to represent perfectly an orientation of a vector space.

An orientation on a manifold is a choice of an orientation for each tangent space satisfying a continuity condition. Globalizing n-covectors over a manifold, we obtain differential n-forms. An orientation on an n-manifold can also be given by an equivalence class of C^∞ nowhere-vanishing n-forms, two such forms being equivalent if and only if one is a multiple of the other by a positive function. Finally, a third way to represent an orientation on a manifold is through an oriented atlas, an atlas in which any two overlapping charts are related by a transition function with everywhere positive Jacobian determinant.

11.1.1 21.1 Orientations of a Vector Space

On $\mathbb{R}^1$ an orientation is one of two directions (Figure 21.1).

Fig. 21.1. Orientations of a line.

On $\mathbb{R}^2$ an orientation is either counterclockwise or clockwise (Figure 21.2).

Fig. 21.2. Orientations of a plane.

Fig. 21.3. Right-handed orientation $\left(e_{1 , e_{2}, e_{3}\right)$ of $\mathbb{R}^3$.}

Fig. 21.4. Left-handed orientation $\left(e_{2 , e_{1}, e_{3}\right)$ of $\mathbb{R}^3$.}

On $\mathbb{R}^3$ an orientation is either right-handed (Figure 21.3) or left-handed (Figure 21.4). The right-handed orientation of $\mathbb{R}^3$ is the choice of a Cartesian coordinate system such that if you hold out your right hand with the index finger curling from the vector e_1 in the x-axis to the vector e_2 in the y-axis, then your thumb points in the direction of of the vector e_3 in the z-axis.

How should one define an orientation for $\mathbb{R}^4, \mathbb{R}^5$, and beyond? If we analyze the three examples above, we see that an orientation can be specified by an ordered basis for $\mathbb{R}^n$. Let $e_1, \ldots, e_n$ be the standard basis for $\mathbb{R}^n$. For $\mathbb{R}^1$ an orientation could be given by either e_1 or $-e_1$. For $\mathbb{R}^2$ the counterclockwise orientation is (e_1, e_2), while the clockwise orientation is (e_2, e_1). For $\mathbb{R}^3$ the right-handed orientation is (e_1, e_2, e_3), and the left-handed orientation is (e_2, e_1, e_3).

For any two ordered bases (u_1, u_2) and (v_1, v_2) for $\mathbb{R}^2$, there is a unique nonsingular 2×2 matrix $A = \left[a_j^i\right]$ such that

$$u_j = \sum_{i=1}^{2} v_i a_j^i, \quad j = 1, 2,$$

called the change-of-basis matrix from (v_1, v_2) to (u_1, u_2). In matrix notation, if we write ordered bases as row vectors, for example, $[u_1 u_2]$ for the basis (u_1, u_2), then

$$[u_1 u_2] = \begin{bmatrix} v_1 & v_2 \end{bmatrix} A.$$

We say that two ordered bases are equivalent if the change-of-basis matrix A has positive determinant. It is easy to check that this is indeed an equivalence relation on the set of all ordered bases for $\mathbb{R}^2$. It therefore partitions ordered bases into two equivalence classes. Each equivalence class is called an orientation of $\mathbb{R}^2$. The equivalence class containing the ordered basis (e_1, e_2) is the counterclockwise orientation and the equivalence class of (e_2, e_1) is the clockwise orientation.

The general case is similar. We assume all vector spaces in this section to be finite-dimensional. Two ordered bases $u = [u_1 \cdots u_n]$ and $v = [v_1 \cdots v_n]$ of a vector space V are said to be equivalent, written $u \sim v$, if $u = vA$ for an $n \times n$ matrix A with positive determinant. An orientation of V is an equivalence class of ordered bases. Any finite-dimensional vector space has two orientations. If μ is an orientation of a finite-dimensional vector space V, we denote the other orientation by $-\mu$ and call it the opposite of the orientation μ.

The zero-dimensional vector space $\{0\}$ is a special case because it does not have a basis. We define an orientation on $\{0\}$ to be one of the two signs $+$ and - .

Notation. A basis for a vector space is normally written $v_1, \ldots, v_n$, without parentheses, brackets, or braces. If it is an ordered basis, then we enclose it in parentheses: $(v_1, \ldots, v_n)$. In matrix notation, we also write an ordered basis as a row vector $[v_1 \cdots v_n]$. An orientation is an equivalence class of ordered bases, so the notation is $[(v_1, \ldots, v_n)]$, where the brackets now stand for equivalence class.

11.1.2 21.2 Orientations and n-covectors

Instead of using an ordered basis, we can also use an n-covector to specify an orientation of an n-dimensional vector space V. This approach to orientations is based on the fact that the space $\bigwedge^n (V^\vee)$ of n-covectors on V is one-dimensional.

Lemma 21.1. Let $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ be vectors in a vector space V. Suppose

$$u_j = \sum_{i=1}^{n} v_i a_j^i, \quad j = 1, \ldots, n$$

for a matrix $A = [a_j{}^i]$ of real numbers. If β is an n-covector on V, then

$$\beta(u_1, \ldots, u_n) = (\det A)\beta(v_1, \ldots, v_n).$$

Proof. By hypothesis,

$$u_j = \sum_i v_i a_j^i$$

Since β is n-linear,

$$\beta(u_1, \ldots, u_n) = \beta\left(\sum v_{i_1} a_1^{i_1}, \ldots, \sum v_{i_n} a_n^{i_n}\right) = \sum a_1^{i_1} \cdots a_n^{i_n} \beta(v_{i_1}, \ldots, v_{i_n}).$$

For $\beta(v_{i_1}, \ldots, v_{i_n})$ to be nonzero, the subscripts $i_1, \ldots, i_n$ must be all distinct. An ordered n-tuple $I = (i_1, \ldots, i_n)$ with distinct components corresponds to a permutation σ_I of $1, \ldots, n$ with $\sigma_I(j) = i_j$ for $j = 1, \ldots, n$. Since β is an alternating n-tensor,

$$\beta(v_{i_1}, \ldots, v_{i_n}) = (\operatorname{sgn} \sigma_I)\, \beta(v_1, \ldots, v_n).$$

Thus,

$$\beta\left(u_1, \ldots, u_n\right)=\sum_{\sigma_I \in S_n}\left(\operatorname{sgn} \sigma_I\right) a_1^{i_1} \cdots a_n^{i_n} \beta\left(v_1, \ldots, v_n\right)=(\det A) \beta\left(v_1, \ldots, v_n\right)$$

As a corollary, if $u_1, \ldots, u_n$ and $v_1, \ldots, v_n$ are ordered bases of a vector space V, then

$$\begin{aligned}
& \beta\left(u_1, \ldots, u_n\right) \text{ and } \beta\left(v_1, \ldots, v_n\right) \text{ have the same sign} \\
& \Longleftrightarrow \det A>0 \\
& \Longleftrightarrow u_1, \ldots, u_n \text{ and } v_1, \ldots, v_n \text{ are equivalent ordered bases.}
\end{aligned}$$

We say that the n-covector β determines or specifies the orientation $\left(v_1, \ldots, v_n\right)$ if $\beta\left(v_1, \ldots, v_n\right)>0$. By the preceding corollary, this is a well-defined notion, independent of the choice of ordered basis for the orientation. Moreover, two n-covectors β and $\beta^{\prime}$ on V determine the same orientation if and only if $\beta=a \beta^{\prime}$ for some positive real number a. We define an equivalence relation on the nonzero n-covectors on the n-dimensional vector space V by setting

$$\beta \sim \beta^{\prime} \quad \Longleftrightarrow \quad \beta=a \beta^{\prime} \text{ for some } a>0 .$$

Thus, in addition to an equivalence class of ordered bases, an orientation of V is also given by an equivalence class of nonzero n-covectors on V.

A linear isomorphism $\bigwedge^n\left(V^{\vee}\right) \simeq \mathbb{R}$ identifies the set of nonzero n-covectors on V with $\mathbb{R}-\{0\}$, which has two connected components. Two nonzero n-covectors β and $\beta^{\prime}$ on V are in the same component if and only if $\beta=a \beta^{\prime}$ for some real number $a>0$. Thus, each connected component of $\bigwedge^n\left(V^{\vee}\right)-\{0\}$ determines an orientation of V.

Example. Let e_1, e_2 be the standard basis for $\mathbb{R}^2$ and α^1, α^2 its dual basis. Then the 2-covector $\alpha^1 \wedge \alpha^2$ determines the counterclockwise orientation of $\mathbb{R}^2$, since

$$\left(\alpha^1 \wedge \alpha^2\right)\left(e_1, e_2\right)=1>0$$

Example. Let $\partial / \partial x|_p, \partial / \partial y|_p$ be the standard basis for the tangent space $T_p\left(\mathbb{R}^2\right)$, and $(d x)_p,(d y)_p$ its dual basis. Then $(d x \wedge d y)_p$ determines the counterclockwise orientation of $T_p\left(\mathbb{R}^2\right)$.

11.1.3 21.3 Orientations on a Manifold

Recall that every vector space of dimension n has two orientations, corresponding to the two equivalence classes of ordered bases or the two equivalence classes of nonzero n-covectors. To orient a manifold M, we orient the tangent space at each point in M, but of course this has to be done in a "coherent" way so that the orientation does not change abruptly anywhere.

As we learned in Subsection 12.5, a frame on an open set $U \subset M$ is an n-tuple $\left(X_1, \ldots, X_n\right)$ of possibly discontinuous vector fields on U such that at every point $p \in U$, the n-tuple ($X_{1,p}, \ldots, X_{n,p}$) of vectors is an ordered basis for the tangent space $T_p M$. A global frame is a frame defined on the entire manifold M, while a local frame about $p \in M$ is a frame defined on some neighborhood of p. We introduce an equivalence relation on frames on U :
$\left(X_1, \ldots, X_n\right) \sim\left(Y_1, \ldots, Y_n\right) \quad \Longleftrightarrow \quad\left(X_{1,p}, \ldots, X_{n,p}\right) \sim\left(Y_{1,p}, \ldots, Y_{n,p}\right)$ for all $p \in U$.
In other words, if $Y_j=\sum_i a_j^i X_i$, then two frames $\left(X_1, \ldots, X_n\right)$ and $\left(Y_1, \ldots, Y_n\right)$ are equivalent if and only if the change-of-basis matrix $A=\left[a^i{}_j\right]$ has positive determinant at every point in U.

A pointwise orientation on a manifold M assigns to each $p \in M$ an orientation μ_p of the tangent space $T_p M$. In terms of frames, a pointwise orientation on M is simply an equivalence class of possibly discontinuous frames on M. A pointwise orientation μ on M is said to be

continuous at $p \in M$ if p has a neighborhood U on which μ is represented by a continuous frame; i.e., there exist continuous vector fields $Y_1, \ldots, Y_n$ on U such that $\mu_q = [(Y_{1,q}, \ldots, Y_{n,q})]$ for all $q \in U$. The pointwise orientation μ is continuous on M if it is continuous at every point $p \in M$. Note that a continuous pointwise orientation need not be represented by a continuous global frame; it suffices that it be locally representable by a continuous local frame. A continuous pointwise orientation on M is called an orientation on M. A manifold is said to be orientable if it has an orientation. A manifold together with an orientation is said to be oriented.

Example. The Euclidean space $\mathbb{R}^n$ is orientable with orientation given by the continuous global frame $(\partial/\partial r^1, \ldots, \partial/\partial r^n)$.

Example 21.2 (The open Möbius band). Let R be the rectangle

$$R = \left\{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 1, -1 < y < 1\right\}.$$

The open Möbius band M (Figures 21.5 and 21.6) is the quotient of the rectangle R by the equivalence relation generated by

$$(0, y) \sim (1, -y). \tag{21.1}$$

The interior of R is the open rectangle

$$U = \left\{(x, y) \in \mathbb{R}^2 \mid 0 < x < 1, -1 < y < 1\right\}.$$

Fig. 21.5. Möbius band.

Fig. 21.6. Nonorientability of the Möbius band.

Suppose the Möbius band M is orientable. An orientation on M restricts to an orientation on U. To avoid confusion with an ordered pair of numbers, in this example we write an ordered basis without the parentheses. For the sake of definiteness, we first assume the orientation on

U to be given by e_1, e_2. By continuity the orientations at the points $(0,0)$ and $(1,0)$ are also given by e_1, e_2. But under the identification (21.1), the ordered basis e_1, e_2 at $(1,0)$ maps to $e_1, -e_2$ at $(0,0)$. Thus, at $(0,0)$ the orientation has to be given by both e_1, e_2 and $e_1, -e_2$, a contradiction. Assuming the orientation on U to be given by e_2, e_1 also leads to a contradiction. This proves that the Möbius band is not orientable.

Proposition 21.3. A connected orientable manifold M has exactly two orientations.
Proof. Let μ and v be two orientations on M. At any point $p \in M, \mu_p$ and v_p are orientations of T_pM. They either are the same or are opposite orientations. Define a function $f : M \to \{\pm 1\}$ by

$$f(p) = \begin{cases} 1 \text{ if } \mu_p = v_p, \\ -1 \text{ if } \mu_p = -v_p. \end{cases}$$

Fix a point $p \in M$. By continuity, there exists a connected neighborhood U of p on which $\mu = [(X_1, \ldots, X_n)]$ and $v = [(Y_1, \ldots, Y_n)]$ for some continuous vector fields X_i and Y_j on U. Then there is a matrix-valued function $A = \left[a_j^i\right] : U \to \mathrm{GL}(n, \mathbb{R})$ such that $Y_j = \sum_i a_j^i X_i$. By Proposition 12.12 and Remark 12.13, the entries a_j^i are continuous, so that the determinant $\det A : U \to \mathbb{R}^\times$is continuous also. By the intermediate value theorem, the continuous nowhere-vanishing function $\det A$ on the connected set U is everywhere positive or everywhere negative. Hence, $\mu = v$ or $\mu = -v$ on
U. This proves that the function $f : M \to \{\pm 1\}$ is locally constant. Since a locally constant function on a connected set is constant (Problem 21.1), $\mu = v$ or $\mu = -v$ on M.

11.1.4 21.4 Orientations and Differential Forms

While the definition of an orientation on a manifold as a continuous pointwise orientation is geometrically intuitive, in practice it is easier to manipulate the nowherevanishing top forms that specify a pointwise orientation. In this section we show that the continuity condition on pointwise orientations translates to a C^∞ condition on nowhere-vanishing top forms.

If f is a real-valued function on a set M, we use the notation $f > 0$ to mean that f is everywhere positive on M.

Lemma 21.4. A pointwise orientation $[(X_1, \ldots, X_n)]$ on a manifold M is continuous if and only if each point $p \in M$ has a coordinate neighborhood $(U, x^1, \ldots, x^n)$ on which the function $(dx^1 \wedge \cdots \wedge dx^n)(X_1, \ldots, X_n)$ is everywhere positive.

Proof.

($\Rightarrow$) Assume that the pointwise orientation $\mu = [(X_1, \ldots, X_n)]$ on M is continuous. This does not mean that the global frame $(X_1, \ldots, X_n)$ is continuous. What it means is that every point $p \in M$ has a neighborhood W on which μ is represented by a continuous frame $(Y_1, \ldots, Y_n)$. Choose a connected coordinate neighborhood $(U, x^1, \ldots, x^n)$ of p contained in W and let $\partial_i = \partial/\partial x^i$. Then $Y_j = \sum_i b_j^i \partial_i$ for a continuous matrix function $\left[b_j^i\right] : U \to \mathrm{GL}(n, \mathbb{R})$, the change-of-basis matrix at each point. By Lemma 21.1,

$$\left(dx^1 \wedge \cdots \wedge dx^n\right)(Y_1, \ldots, Y_n) = \left(\det\left[b_j^i\right]\right)\left(dx^1 \wedge \cdots \wedge dx^n\right)(\partial_1, \ldots, \partial_n) = \det\left[b_j^i\right]$$

which is never zero, because $[b^i{}_j]$ is nonsingular. As a continuous nowhere-vanishing real-valued function on a connected set, $(dx^1 \wedge \cdots \wedge dx^n)(Y_1, \ldots, Y_n)$ is everywhere positive or everywhere negative on U. If it is negative, then by setting $\tilde{x}^1 = -x^1$, we have on the chart $(U, \tilde{x}^1, x^2, \ldots, x^n)$ that

$$\left(d\tilde{x}^1 \wedge dx^2 \wedge \cdots \wedge dx^n\right)(Y_1, \ldots, Y_n) > 0$$

Renaming $\tilde{x}^1$ as x^1, we may assume that on the coordinate neighborhood $(U, x^1, \ldots, x^n)$ of p, the function $(dx^1 \wedge \cdots \wedge dx^n)(Y_1, \ldots, Y_n)$ is always positive.

Since $\mu = [(X_1, \ldots, X_n)] = [(Y_1, \ldots, Y_n)]$ on U, the change-of-basis matrix $C = [c^i{}_j]$ such that $X_j = \sum_i c^i_j Y_i$ has positive determinant. By Lemma 21.1 again, on U,

$$\left(dx^1 \wedge \cdots \wedge dx^n\right)(X_1, \ldots, X_n) = (\det C)\left(dx^1 \wedge \cdots \wedge dx^n\right)(Y_1, \ldots, Y_n) > 0$$

($\Leftarrow$) On the chart $(U, x^1, \ldots, x^n)$, suppose $X_j = \sum a^i_j \partial_i$. As before,

$$\left(dx^1 \wedge \cdots \wedge dx^n\right)(X_1, \ldots, X_n) = \left(\det\left[a^i_j\right]\right)\left(dx^1 \wedge \cdots \wedge dx^n\right)(\partial_1, \ldots, \partial_n) = \det\left[a^i_j\right].$$

By hypothesis, the left-hand side of the equalities above is positive. Therefore, on $U, \det\left[a^i_j\right] > 0$ and $[(X_1, \ldots, X_n)] = [(\partial_1, \ldots, \partial_n)]$, which proves that the pointwise orientation μ is continuous at p. Since p was arbitrary, μ is continuous on M.

Theorem 21.5. A manifold M of dimension n is orientable if and only if there exists a C^∞ nowhere-vanishing n-form on M.

Proof.

($\Rightarrow$) Suppose $[(X_1, \ldots, X_n)]$ is an orientation on M. By Lemma 21.4, each point p has a coordinate neighborhood $(U, x^1, \ldots, x^n)$ on which

$$\left(dx^1 \wedge \cdots \wedge dx^n\right)(X_1, \ldots, X_n) > 0 \tag{21.2}$$

Let $\{(U_\alpha, x^1_\alpha, \ldots, x^n_\alpha)\}$ be a collection of these charts that covers M, and let $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to the open cover $\{U_\alpha\}$. Being a locally finite sum, the n-form $\omega = \sum_\alpha \rho_\alpha dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha$ is well defined and C^∞ on M. Fix $p \in M$. Since $\rho_\alpha(p) \geq 0$ for all α and $\rho_\alpha(p) > 0$ for at least one α, by (21.2),

$$\omega_p(X_{1,p}, \ldots, X_{n,p}) = \sum_\alpha \rho_\alpha(p)\left(dx^1_\alpha \wedge \cdots \wedge dx^n_\alpha\right)_p(X_{1,p}, \ldots, X_{n,p}) > 0$$

Therefore, ω is a C^∞ nowhere-vanishing n-form on M.

($\Leftarrow$) Suppose ω is a C^∞ nowhere-vanishing n-form on M. At each point $p \in M$, choose an ordered basis $(X_{1,p}, \ldots, X_{n,p})$ for T_pM such that $\omega_p(X_{1,p}, \ldots, X_{n,p}) > 0$. Fix $p \in M$ and let $(U, x^1, \ldots, x^n)$ be a connected coordinate neighborhood of p. On $U, \omega = f dx^1 \wedge \cdots \wedge dx^n$ for a C^∞ nowhere-vanishing function f. Being continuous and nowhere vanishing on a connected set, f is everywhere positive or everywhere negative on U. If $f > 0$, then on the chart $(U, x^1, \ldots, x^n)$,

$$\left(dx^1 \wedge \cdots \wedge dx^n\right)(X_1, \ldots, X_n) > 0$$

If $f < 0$, then on the chart $(U, -x^1, x^2, \ldots, x^n)$,

$$\left(d\left(-x^1\right) \wedge dx^2 \wedge \cdots \wedge dx^n\right)(X_1, \ldots, X_n) > 0$$

In either case, by Lemma 21.4, $\mu = [(X_1, \ldots, X_n)]$ is a continuous pointwise orientation on M.

Example 21.6 (Orientability of a regular zero set). By the regular level set theorem, if 0 is a regular value of a C^∞ function $f(x, y, z)$ on $\mathbb{R}^3$, then the zero set $f^{-1}(0)$ is a C^∞ manifold. In Problem 19.11 we constructed a nowhere-vanishing 2-form on the regular zero set of a C^∞

function. It then follows from Theorem 21.5 that the regular zero set of a C^∞ function on $\mathbb{R}^3$ is orientable.

As an example, the unit sphere S^2 in $\mathbb{R}^3$ is orientable. As another example, since an open Möbius band is not orientable (Example 21.2), it cannot be realized as the regular zero set of a C^∞ function on $\mathbb{R}^3$.

According to a classical theorem from algebraic topology, a continuous vector field on an even-dimensional sphere must vanish somewhere [18, Theorem 2.28, p. 135]. Thus, although the sphere S^2 has a continuous pointwise orientation, any global frame (X_1, X_2) that represents the orientation is necessarily discontinuous.

If ω and ω' are two nowhere-vanishing $C^\infty n$-forms on a manifold M of dimension n, then $\omega = f\omega'$ for some nowhere-vanishing function f on M. Locally, on a chart $(U, x^1, \ldots, x^n), \omega = h dx^1 \wedge \cdots \wedge dx^n$ and $\omega' = g dx^1 \wedge \cdots \wedge dx^n$, where h and g are C^∞ nowhere-vanishing functions on U. Therefore, $f = h/g$ is also a C^∞ nowherevanishing function on U. Since U is an arbitrary chart, f is C^∞ and nowhere vanishing on M. On a connected manifold M, such a function f is either everywhere positive or everywhere negative. In this way the nowhere-vanishing $C^\infty n$-forms on a connected orientable manifold M are partitioned into two equivalence classes by the equivalence relation

$$\omega \sim \omega' \quad \Longleftrightarrow \quad \omega = f\omega' \text{ with } f > 0.$$

To each orientation $\mu = [(X_1, \ldots, X_n)]$ on a connected orientable manifold M, we associate the equivalence class of a C^∞ nowhere-vanishing n-form ω on M such that $\omega(X_1, \ldots, X_n) > 0$. (Such an ω exists by the proof of Theorem 21.5.) If $\mu \mapsto [\omega]$, then $-\mu \mapsto [-\omega]$. On a connected orientable manifold, this sets up a one-to-one correspondence

$$\{\text{ orientations on } M\} \longleftrightarrow \left\{ \begin{array}{c} \text{equivalence classes of} \\ C^\infty \text{ nowhere-vanishing} \\ n\text{-forms on } M \end{array} \right\} \tag{21.3}$$

each side being a set of two elements. By considering one connected component at a time, we see that the bijection (21.3) still holds for an arbitrary orientable manifold, each connected component having two possible orientations and two equivalence classes of C^∞ nowhere-vanishing n-forms. If ω is a C^∞ nowhere-vanishing n-form such that $\omega(X_1, \ldots, X_n) > 0$, we say that ω determines or specifies the orientation $[(X_1, \ldots, X_n)]$ and we call ω an orientation form on M. An oriented manifold can be described by a pair $(M, [\omega])$, where $[\omega]$ is the equivalence class of an orientation form on M. We sometimes write M, instead of $(M, [\omega])$, for an oriented manifold if it is clear from the context what the orientation is. For example, unless otherwise specified, $\mathbb{R}^n$ is oriented by $dx^1 \wedge \cdots \wedge dx^n$.

Remark 21.7 (Orientations on zero-dimensional manifolds). A connected manifold of dimension 0 is a point. The equivalence class of a nowhere-vanishing 0 -form on a point is either $[-1]$ or $[1]$. Hence, a connected zero-dimensional manifold is always orientable. Its two orientations are specified by the two numbers ± 1. A general zero-dimensional manifold M is a countable discrete set of points (Example 5.13), and an orientation on M is given by a function that assigns to each point of M either 1 or -1 .

A diffeomorphism $F : (N, [\omega_N]) \to (M, [\omega_M])$ of oriented manifolds is said to be orientation-preserving if $[F^*\omega_M] = [\omega_N]$; it is orientation-reversing if $[F^*\omega_M] = [-\omega_N]$.

Proposition 21.8. Let U and V be open subsets of $\mathbb{R}^n$, both with the standard orientation inherited from $\mathbb{R}^n$. A diffeomorphism $F : U \to V$ is orientation-preserving if and only if the Jacobian determinant $\det[\partial F^i/\partial x^j]$ is everywhere positive on U.

Proof. Let $x^1, \ldots, x^n$ and $y^1, \ldots, y^n$ be the standard coordinates on $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$. Then

$$\begin{aligned}
F^*\left(dy^1 \wedge \cdots \wedge dy^n\right) &= d\left(F^* y^1\right) \wedge \cdots \wedge d\left(F^* y^n\right) && (\text{ Propositions 18.11 and 19.5}) \\
&= d\left(y^1 \circ F\right) \wedge \cdots \wedge d\left(y^n \circ F\right) && (\text{ definition of pullback }) \\
&= dF^1 \wedge \cdots \wedge dF^n \\
&= \det\left[\frac{\partial F^i}{\partial x^j}\right] dx^1 \wedge \cdots \wedge dx^n && (\text{ by Corollary 18.4(ii) }).
\end{aligned}$$

Thus, F is orientation-preserving if and only if $\det\left[\partial F^i / \partial x^j\right]$ is everywhere positive on U.

11.1.5 21.5 Orientations and Atlases

Using the characterization of an orientation-preserving diffeomorphism by the sign of its Jacobian determinant, we can describe orientability of manifolds in terms of atlases.

Definition 21.9. An atlas on M is said to be oriented if for any two overlapping charts $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ of the atlas, the Jacobian determinant $\det\left[\partial y^i / \partial x^j\right]$ is everywhere positive on $U \cap V$.

Theorem 21.10. A manifold M is orientable if and only if it has an oriented atlas.

Proof.

($\Rightarrow$) Let $\mu = [(X_1, \ldots, X_n)]$ be an orientation on the manifold M. By Lemma 21.4, each point $p \in M$ has a coordinate neighborhood $(U, x^1, \ldots, x^n)$ on which

$$\left(dx^1 \wedge \cdots \wedge dx^n\right)\left(X_1, \ldots, X_n\right) > 0$$

We claim that the collection $\mathfrak{U} = \{(U, x^1, \ldots, x^n)\}$ of these charts is an oriented atlas. If $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ are two overlapping charts from $\mathfrak{U}$, then on $U \cap V$,

$$\left(dx^1 \wedge \cdots \wedge dx^n\right)\left(X_1, \ldots, X_n\right) > 0 \quad \text{and} \quad \left(dy^1 \wedge \cdots \wedge dy^n\right)\left(X_1, \ldots, X_n\right) > 0 \tag{21.4}$$

Since $dy^1 \wedge \cdots \wedge dy^n = \left(\det\left[\partial y^i / \partial x^j\right]\right) dx^1 \wedge \cdots \wedge dx^n$, it follows from (21.4) that $\det\left[\partial y^i / \partial x^j\right] > 0$ on $U \cap V$. Therefore, $\mathfrak{U}$ is an oriented atlas.

($\Rightarrow$) Suppose $\{(U, x^1, \ldots, x^n)\}$ is an oriented atlas. For each $p \in (U, x^1, \ldots, x^n)$, define μ_p to be the equivalence class of the ordered basis $\left(\partial / \partial x^1|_p, \ldots, \partial / \partial x^n|_p\right)$ for $T_p M$. If two charts $(U, x^1, \ldots, x^n)$ and $(V, y^1, \ldots, y^n)$ in the oriented atlas contain p, then by the orientability of the atlas, $\det\left[\partial y^i / \partial x^j\right] > 0$, so that $\left(\partial / \partial x^1|_p, \ldots, \partial / \partial x^n|_p\right)$ is equivalent to $\left(\partial / \partial y^1|_p, \ldots, \partial / \partial y^n|_p\right)$. This proves that μ is a well-defined pointwise orientation on M. It is continuous because every point p has a coordinate neighborhood $(U, x^1, \ldots, x^n)$ on which $\mu = [(\partial / \partial x^1, \ldots, \partial / \partial x^n)]$ is represented by a continuous frame.

Definition 21.11. Two oriented atlases $\{(U_\alpha, \boldsymbol{\phi}_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ on a manifold M are said to be equivalent if the transition functions

$$\phi_\alpha \circ \psi_\beta^{-1} : \psi_\beta\left(U_\alpha \cap V_\beta\right) \to \phi_\alpha\left(U_\alpha \cap V_\beta\right)$$

have positive Jacobian determinant for all α, β.

It is not difficult to show that this is an equivalence relation on the set of oriented atlases on a manifold M (Problem 21.3).

In the proof of Theorem 21.10, an oriented atlas $\{(U, x^1, \ldots, x^n)\}$ on a manifold M determines an orientation $U \ni p \mapsto \left[\left(\partial/\,\partial x^1|_p, \ldots, \partial/\,\partial x^n|_p\right)\right]$ on M, and conversely, an orientation $[(X_1, \ldots, X_n)]$ on M gives rise to an oriented atlas $\{(U, x^1, \ldots, x^n)\}$ on M such that $(dx^1 \wedge \cdots \wedge dx^n)(X_1, \ldots, X_n) > 0$ on U. We leave it as an exercise to show that for an orientable manifold M, the two induced maps

$$\left\{\begin{array}{l}\text{equivalence classes of}\\ \text{oriented atlases on } M\end{array}\right\} \rightleftarrows \{\text{ orientations on } M\}$$

are well defined and inverse to each other. Therefore, one can also specify an orientation on an orientable manifold by an equivalence class of oriented atlases.

For an oriented manifold M, we denote by $-M$ the same manifold but with the opposite orientation. If $\{(U, \phi)\} = \{(U, x^1, x^2, \ldots, x^n)\}$ is an oriented atlas specifying the orientation of M, then an oriented atlas specifying the orientation of $-M$ is $\{(U, \tilde{\phi})\} = \{(U, -x^1, x^2, \ldots, x^n)\}$.

11.1.6 Problems

21.1.* Locally Constant map on a Connected Space

A map $f : S \to Y$ between two topological spaces is locally constant if for every $p \in S$ there is a neighborhood U of p such that f is constant on U. Show that a locally constant map $f : S \to Y$ on a nonempty connected space S is constant. (Hint: Show that for every $y \in Y$, the inverse image $f^{-1}(y)$ is open. Then $S = \bigcup_{y \in Y} f^{-1}(y)$ exhibits S as a disjoint union of open subsets.)

21.2. Continuity of Pointwise Orientations

Prove that a pointwise orientation $[(X_1, \ldots, X_n)]$ on a manifold M is continuous if and only if every point $p \in M$ has a coordinate neighborhood $(U, \phi) = (U, x^1, \ldots, x^n)$ such that for all $q \in U$, the differential $\phi_{*,q} : T_qM \to T_{f(q)}\mathbb{R}^n \simeq \mathbb{R}^n$ carries the orientation of T_qM to the standard orientation of $\mathbb{R}^n$ in the following sense: $(\phi_* X_{1,q}, \ldots, \phi_* X_{n,q}) \sim (\partial/\partial r^1, \ldots, \partial/\partial r^n)$.

21.3. Equivalence of Oriented Atlases

Show that the relation in Definition 21.11 is an equivalence relation.

21.4. Orientation-preserving Diffeomorphisms

Let $F : (N, [\omega_N]) \to (M, [\omega_M])$ be an orientation-preserving diffeomorphism. If $\{(V_\alpha, \psi_\alpha)\} = \{(V_\alpha, y_\alpha^1, \ldots, y_\alpha^n)\}$ is an oriented atlas on M that specifies the orientation of M, show that $\{(F^{-1}V_\alpha, F^*\psi_\alpha)\} = \{(F^{-1}V_\alpha, F_\alpha^1, \ldots, F_\alpha^n)\}$ is an oriented atlas on N that specifies the orientation of N, where $F_\alpha^i = y_\alpha^i \circ F$.

21.5. Orientation-preserving or Orientation-reversing Diffeomorphisms

Let U be the open set $(0, \infty) \times (0, 2\pi)$ in the (r, θ)-plane $\mathbb{R}^2$. We define $F : U \subset \mathbb{R}^2 \to \mathbb{R}^2$ by $F(r, \theta) = (r\cos\theta, r\sin\theta)$. Decide whether F is orientation-preserving or orientation-reversing as a diffeomorphism onto its image.

21.6. Orientability of a regular level set in $\mathbb{R}^{n+1}$

Suppose $f(x^1, \ldots, x^{n+1})$ is a C^∞ function on $\mathbb{R}^{n+1}$ with 0 as a regular value. Show that the zero set of f is an orientable submanifold of $\mathbb{R}^{n+1}$. In particular, the unit n-sphere S^n in $\mathbb{R}^{n+1}$ is orientable.

21.7. Orientability of a Lie Group

Show that every Lie group G is orientable by constructing a nowhere-vanishing top form on G.

21.8. Orientability of a Parallelizable Manifold

Show that a parallelizable manifold is orientable. (In particular, this shows again that every Lie group is orientable.)

21.9. Orientability of the Total Space of the Tangent Bundle

Let M be a smooth manifold and $\pi : TM \to M$ its tangent bundle. Show that if $\{(U, \phi)\}$ is any atlas on M, then the atlas $\{(TU, \tilde{\phi})\}$ on TM, with $\tilde{\phi}$ defined in equation (12.1), is oriented. This proves that the total space TM of the tangent bundle is always orientable, regardless of whether M is orientable.

21.10. Oriented Atlas on a Circle

In Example 5.16 we found an atlas $\mathfrak{U} = \{(U_i, \phi_i)\}_{i=1}^4$ on the unit circle S^1. Is $\mathfrak{U}$ an oriented atlas? If not, alter the coordinate functions ϕ_i to make $\mathfrak{U}$ into an oriented atlas.

11.2 §22 Manifolds with Boundary

The prototype of a manifold with boundary is the closed upper half-space

$$\mathcal{H}^n = \left\{ \left(x^1, \ldots, x^n\right) \in \mathbb{R}^n \mid x^n \geq 0 \right\},$$

with the subspace topology inherited from $\mathbb{R}^n$. The points $(x^1, \ldots, x^n)$ in $\mathcal{H}^n$ with $x^n > 0$ are called the interior points of $\mathcal{H}^n$, and the points with $x^n = 0$ are called the boundary points of $\mathcal{H}^n$. These two sets are denoted by $(\mathcal{H}^n)^\circ$ and $\partial(\mathcal{H}^n)$, respectively (Figure 22.1).

Fig. 22.1. Upper half-space.

In the literature the upper half-space often means the open set

$$\left\{ \left(x^1, \ldots, x^n\right) \in \mathbb{R}^n \mid x^n > 0 \right\}.$$

We require that $\mathcal{H}^n$ include the boundary in order for it to serve as a model for manifolds with boundary.

If M is a manifold with boundary, then its boundary ∂M turns out to be a manifold of dimension one less without boundary. Moreover, an orientation on M induces an orientation on ∂M. The choice of the induced orientation on the boundary is a matter of convention, guided by the desire to make Stokes's theorem sign-free. Of the various ways to describe the boundary orientation, two stand out for their simplicity: (1) contraction of an orientation form on M with an outward-pointing vector field on ∂M and (2) "outward vector first."

11.2.1 22.1 Smooth Invariance of Domain in $\mathbb{R}^n$

To discuss C^∞ functions on a manifold with boundary, we need to extend the definition of a C^∞ function to allow nonopen domains.

Definition 22.1. Let $S \subset \mathbb{R}^n$ be an arbitrary subset. A function $f : S \to \mathbb{R}^m$ is smooth at a point p in S if there exist a neighborhood U of p in $\mathbb{R}^n$ and a C^∞ function $\tilde{f} : U \to \mathbb{R}^m$ such that $\tilde{f} = f$ on $U \cap S$. The function is smooth on S if it is smooth at each point of S.

With this definition it now makes sense to speak of an arbitrary subset $S \subset \mathbb{R}^n$ being diffeomorphic to an arbitrary subset $T \subset \mathbb{R}^m$; this will be the case if and only if there are smooth maps $f : S \to T \subset \mathbb{R}^m$ and $g : T \to S \subset \mathbb{R}^n$ that are inverse to each other.

Exercise 22.2 (Smooth functions on a nonopen set).* Using a partition of unity, show that a function $f : S \to \mathbb{R}^m$ is C^∞ on $S \subset \mathbb{R}^n$ if and only if there exist an open set U in $\mathbb{R}^n$ containing S and a C^∞ function $\tilde{f} : U \to \mathbb{R}^m$ such that $f = \tilde{f}\Big|_S$.

The following theorem is the C^∞ analogue of a classical theorem in the continuous category. We will use it to show that interior points and boundary points are invariant under diffeomorphisms of open subsets of $\mathcal{H}^n$.

Theorem 22.3 (Smooth invariance of domain). Let $U \subset \mathbb{R}^n$ be an open subset, $S \subset \mathbb{R}^n$ an arbitrary subset, and $f : U \to S$ a diffeomorphism. Then S is open in $\mathbb{R}^n$.

More succinctly, a diffeomorphism between an open subset U of $\mathbb{R}^n$ and an arbitrary subset S of $\mathbb{R}^n$ forces S to be open in $\mathbb{R}^n$. The theorem is not automatic. A diffeomorphism $f : \mathbb{R}^n \supset U \to S \subset \mathbb{R}^n$ takes an open subset of U to an open subset of S. Thus, a priori we know only that $f(U)$ is open in S, not that $f(U)$, which is S, is open in $\mathbb{R}^n$. It is crucial that the two Euclidean spaces be of the same dimension. For example, there are a diffeomorphism between the open interval $]0, 1[$ in $\mathbb{R}^1$ and the open segment $S =]0, 1[\times \{0\}$ in $\mathbb{R}^2$, but S is not open in $\mathbb{R}^2$.

Proof. Let $f(p)$ be an arbitrary point in S, with $p \in U$. Since $f : U \to S$ is a diffeomorphism, there are an open set $V \subset \mathbb{R}^n$ containing S and a C^∞ map $g : V \to \mathbb{R}^n$ such that $g|_S = f^{-1}$. Thus,

$$U \xrightarrow{f} V \xrightarrow{g} \mathbb{R}^n$$

satisfies

$$g \circ f = \mathbb{1}_U : U \to U \subset \mathbb{R}^n$$

the identity map on U. By the chain rule,

$$g_{*,f(p)} \circ f_{*,p} = \mathbb{1}_{T_pU} : T_pU \to T_pU \simeq T_p(\mathbb{R}^n)$$

the identity map on the tangent space T_pU. Hence, $f_{*,p}$ is injective. Since U and V have the same dimension, $f_{*,p} : T_pU \to T_{f(p)}V$ is invertible. By the inverse function theorem, f is locally invertible at p. This means that there are open neighborhoods U_p of p in U and $V_{f(p)}$ of $f(p)$ in V such that $f : U_p \to V_{f(p)}$ is a diffeomorphism. It follows that

$$f(p) \in V_{f(p)} = f(U_p) \subset f(U) = S$$

Since V is open in $\mathbb{R}^n$ and $V_{f(p)}$ is open in V, the set $V_{f(p)}$ is open in $\mathbb{R}^n$. By the local criterion for openness (Lemma A.2), S is open in $\mathbb{R}^n$.

Proposition 22.4. Let U and V be open subsets of the upper half-space $\mathcal{H}^n$ and $f : U \to V$ a diffeomorphism. Then f maps interior points to interior points and boundary points to boundary points.

Proof. Let $p \in U$ be an interior point. Then p is contained in an open ball B, which is actually open in $\mathbb{R}^n$ (not just in $\mathscr{H}^n$). By smooth invariance of domain, $f(B)$ is open in $\mathbb{R}^n$

(again not just in $\mathcal{H}^n$). Therefore, $f(B) \subset (\mathcal{H}^n)^\circ$. Since $f(p) \in f(B)$, $f(p)$ is an interior point of $\mathcal{H}^n$.

If p is a boundary point in $U \cap \partial\mathcal{H}^n$, then $f^{-1}(f(p)) = p$ is a boundary point. Since $f^{-1} : V \to U$ is a diffeomorphism, by what has just been proven, $f(p)$ cannot be an interior point. Thus, $f(p)$ is a boundary point.

Remark 22.5. Replacing Euclidean spaces by manifolds throughout this subsection, one can prove in exactly the same way smooth invariance of domain for manifolds: if there is a diffeomorphism between an open subset U of an n-dimensional manifold N and an arbitrary subset S of another n-dimensional manifold M, then S is open in M.

11.2.2 22.2 Manifolds with Boundary

In the upper half-space $\mathcal{H}^n$ one may distinguish two kinds of open subsets, depending on whether the set is disjoint from the boundary or intersects the boundary (Figure 22.2). Charts on a manifold are homeomorphic to only the first kind of open sets.

Fig. 22.2. Two types of open subsets of $\mathcal{H
^{n}$.}

A manifold with boundary generalizes the definition of a manifold by allowing both kinds of open sets. We say that a topological space M is locally $\mathcal{H}^n$ if every point $p \in M$ has a neighborhood U homeomorphic to an open subset of $\mathcal{H}^n$.

Definition 22.6. A topological n-manifold with boundary is a second countable, Hausdorff topological space that is locally $\mathcal{H}^n$.

Let M be a topological n-manifold with boundary. For $n \geq 2$, a chart on M is defined to be a pair (U, ϕ) consisting of an open set U in M and a homeomorphism

$$\phi : U \to \phi(U) \subset \mathcal{H}^n$$

of U with an open subset $\phi(U)$ of $\mathcal{H}^n$. As Example 22.9 (p. 254) will show, a slight modification is necessary when $n = 1$: we need to allow two local models, the right half-line $\mathcal{H}^1$ and the left half-line

$$\mathcal{L}^1 := \{x \in \mathbb{R} \mid x \leq 0\}.$$

A chart (U, ϕ) in dimension 1 consists of an open set U in M and a homeomorphism ϕ of U with an open subset of $\mathcal{H}^1$ or $\mathcal{L}^1$. With this convention, if $(U, x^1, x^2, \ldots, x^n)$ is a chart of an n-dimensional manifold with boundary, then so is $(U, -x^1, x^2, \ldots, x^n)$ for any $n \geq 1$. A manifold with boundary has dimension at least 1 , since a manifold of dimension 0 , being a discrete set of points, necessarily has empty boundary.

A collection $\{(U, \phi)\}$ of charts is a C^∞ atlas if for any two charts (U, ϕ) and (V, Ψ), the transition map

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \subset \mathcal{H}^n$$

is a diffeomorphism. A C^∞ manifold with boundary is a topological manifold with boundary together with a maximal C^∞ atlas.

A point p of M is called an interior point if in some chart (U, ϕ), the point $\phi(p)$ is an interior point of $\mathcal{H}^n$. Similarly, p is a boundary point of M if $\phi(p)$ is a boundary point of $\mathcal{H}^n$. These concepts are well defined, independent of the charts, because if (V, ψ) is another chart, then the diffeomorphism $\psi \circ \phi^{-1}$ maps $\phi(p)$ to $\psi(p)$, and so by Proposition 22.4, $\phi(p)$ and $\psi(p)$ are either both interior points or both boundary points (Figure 22.3). The set of boundary points of M is denoted by ∂M.

Fig. 22.3. Boundary charts.

Most of the concepts introduced for a manifold extend word for word to a manifold with boundary, the only difference being that now a chart can be either of two types and the local model is $\mathcal{H}^n$ (or $\mathcal{L}^1$). For example, a function $f : M \to \mathbb{R}$ is C^∞ at a boundary point $p \in \partial M$ if there is a chart (U, ϕ) about p such that $f \circ \phi^{-1}$ is C^∞ at $\phi(p) \in \mathcal{H}^n$. This in turn means that $f \circ \phi^{-1}$ has a C^∞ extension to a neighborhood of $\phi(p)$ in $\mathbb{R}^n$.

In point-set topology there are other notions of interior and boundary, defined for a subset A of a topological space S. A point $p \in S$ is said to be an interior point of A if there exists an open subset U of S such that

$$p \in U \subset S.$$

The point $p \in S$ is an exterior point of A if there exists an open subset U of S such that

$$p \in U \subset S - A.$$

Finally, $p \in S$ is a boundary point of A if every neighborhood of p contains both a point in A and a point not in A. We denote by $\operatorname{int}(A), \operatorname{ext}(A)$, and $\operatorname{bd}(A)$ the sets of interior, exterior, and boundary points respectively of A in S. Clearly, the topological space S is the disjoint union

$$S = \operatorname{int}(A) \amalg \operatorname{ext}(A) \amalg \operatorname{bd}(A).$$

In case the subset $A \subset S$ is a manifold with boundary, we call $\operatorname{int}(A)$ the topological interior and $\operatorname{bd}(A)$ the topological boundary of A, to distinguish them from the manifold interior A° and the manifold boundary ∂A. Note that the topological interior and the topological boundary of a set depend on an ambient space, while the manifold interior and the manifold boundary are intrinsic.

Example 22.7 (Topological boundary versus manifold boundary). Let A be the open unit disk in $\mathbb{R}^2$:

$$A = \left\{x \in \mathbb{R}^2 \mid \|x\| < 1\right\}.$$

Then its topological boundary $\operatorname{bd}(A)$ in $\mathbb{R}^2$ is the unit circle, while its manifold boundary ∂A is the empty set (Figure 22.4).

If B is the closed unit disk in $\mathbb{R}^2$, then its topological boundary $\operatorname{bd}(B)$ and its manifold boundary ∂B coincide; both are the unit circle.

Fig. 22.4. Interiors and boundaries.

Example 22.8 (Topological interior versus manifold interior). Let S be the upper half-plane $\mathcal{H}^2$ and let D be the subset (Figure 22.4)

$$D = \left\{(x, y) \in \mathcal{H}^2 \mid y \leq 1\right\}.$$

The topological interior of D is the set

$$\operatorname{int}(D) = \left\{(x, y) \in \mathcal{H}^2 \mid 0 \leq y < 1\right\},$$

containing the x-axis, while the manifold interior of D is the set

$$D^\circ = \left\{(x, y) \in \mathcal{H}^2 \mid 0 < y < 1\right\},$$

not containing the x-axis.
To indicate the dependence of the topological interior of a set A on its ambient space S, we might denote it by $\operatorname{int}_S(A)$ instead of $\operatorname{int}(A)$. Then in this example, the topological interior $\operatorname{int}_{\mathcal{H}^2}(D)$ of D in $\mathcal{H}^2$ is as above, but the topological interior $\operatorname{int}_{\mathbb{R}^2}(D)$ of D in $\mathbb{R}^2$ coincides with D°.

11.2.3 22.3 The Boundary of a Manifold with Boundary

Let M be a manifold of dimension n with boundary ∂M. If (U, ϕ) is a chart on M, we denote by $\phi' = \phi|_{U \cap \partial M}$ the restriction of the coordinate map ϕ to the boundary. Since ϕ maps boundary points to boundary points,

$$\phi' : U \cap \partial M \to \partial \mathcal{H}^n = \mathbb{R}^{n-1}.$$

Moreover, if $(\ U, \phi\ )$ and $(\ V, \psi\ )$ are two charts on M, then

$$\psi' \circ (\phi')^{-1} : \phi'(U \cap V \cap \partial M) \to \psi'(U \cap V \cap \partial M)$$

is C^∞. Thus, an atlas $\{(U_\alpha, \phi_\alpha)\}$ for M induces an atlas $\left\{\left(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M}\right)\right\}$ for ∂M, making ∂M into a manifold of dimension $n-1$ without boundary.

11.2.4 22.4 Tangent Vectors, Differential Forms, and Orientations

Let M be a manifold with boundary and let $p \in \partial M$. As in Subsection 2.2, two C^∞ functions $f : U \to \mathbb{R}$ and $g : V \to \mathbb{R}$ defined on neighborhoods U and V of p in M are said to be equivalent if they agree on some neighborhood W of p contained in $U \cap V$. A germ of C^∞ functions at p is an equivalence class of such functions. With the usual addition, multiplication, and scalar multiplication of germs, the set $C_p^\infty(M)$ of germs of C^∞ functions at p is an $\mathbb{R}$-algebra. The tangent space T_pM at p is then defined to be the vector space of all point-derivations on the algebra $C_p^\infty(M)$.

For example, for p in the boundary of the upper half-plane $\mathcal{H}^2, \partial/\ \partial x|_p$ and $\partial/\ \partial y|_p$ are both derivations on $C_p^\infty\left(\mathcal{H}^2\right)$. The tangent space $T_p\left(\mathcal{H}^2\right)$ is represented by a 2 -dimensional vector space with the origin at p. Since $\partial/\ \partial y|_p$ is a tangent vector to $\mathcal{H}^2$ at p, its negative $-\partial/\ \partial y|_p$

is also a tangent vector at p (Figure 22.5), although there is no curve through p in $\mathcal{H}^2$ with initial velocity $-\partial/\partial y|_p$.

Fig. 22.5. A tangent vector at the boundary.

The cotangent space T_p^*M is defined to be the dual of the tangent space:

$$T_p^*M = \mathrm{Hom}\left(T_pM, \mathbb{R}\right).$$

Differential k-forms on M are defined as before, as sections of the vector bundle $\bigwedge^k (T^*M)$. A differential k-form is C^∞ if it is C^∞ as a section of the vector bundle $\wedge^k (T^*M)$. For example, $dx \wedge dy$ is a C^∞ 2-form on $\mathcal{H}^2$. An orientation on an n manifold M with boundary is again a continuous pointwise orientation on M.

The discussion in Section 21 on orientations goes through word for word for manifolds with boundary. Thus, the orientability of a manifold with boundary is equivalent to the existence of a C^∞ nowhere-vanishing top form and to the existence of an oriented atlas. At one point in the proof of Lemma 21.4, it was necessary to replace the chart $(U, x^1, x^2, \ldots, x^n)$ by $(U, -x^1, x^2, \ldots, x^n)$. This would not have been possible for $n = 1$ if we had not allowed the left half-line $\mathcal{L}^1$ as a local model in the definition of a chart on a 1-dimensional manifold with boundary.

Example 22.9. The closed interval [0,1] is a C^∞ manifold with boundary. It has an atlas with two charts (U_1, ϕ_1) and (U_2, ϕ_2), where $U_1 = [0, 1[, \phi_1(x) = x$, and $U_2 =]0, 1], \phi_2(x) = 1-x$. With d/dx as a continuous pointwise orientation, $[0, 1]$ is an oriented manifold with boundary. However, $\{(U_1, \phi_1), (U_2, \phi_2)\}$ is not an oriented atlas, because the Jacobian determinant of the transition function $\left(\phi_2 \circ \phi_1^{-1}\right)(x) = 1 - x$ is negative. If we change the sign of ϕ_2, then $\{(U_1, \phi_1), (U_2, -\phi_2)\}$ is an oriented atlas. Note that $-\phi_2(x) = x - 1$ maps $]0, 1]$ into the left half-line $\mathcal{L}^1 \subset \mathbb{R}$. If we had allowed only $\mathcal{H}^1$ as a local model for a 1 -dimensional manifold with boundary, the closed interval $[0, 1]$ would not have an oriented atlas.

11.2.5 22.5 Outward-pointing Vector Fields

Let M be a manifold with boundary and $p \in \partial M$. We say that a tangent vector $X_p \in T_p(M)$ is inward-pointing if $X_p \notin T_p(\partial M)$ and there are a positive real number ε and a curve $c : [0, \varepsilon[\to M$ such that $c(0) = p, c((0, \varepsilon[) \subset M^\circ$, and $c'(0) = X_p$. A vector $X_p \in T_p(M)$ is outward-pointing if $-X_p$ is inward-pointing. For example, on the upper half-plane $\mathcal{H}^2$, the vector $\partial/\partial y|_p$ is inward-pointing and the vector $-\partial/\partial y|_p$ is outward-pointing at a point p on the x-axis.

A vector field along ∂M is a function X that assigns to each point p in ∂M a vector X_p in the tangent space T_pM (as opposed to $T_p(\partial M)$). In a coordinate neighborhood $(U, x^1, \ldots, x^n)$ of p in M, such a vector field X can be written as a linear combination

$$X_q = \sum_i a^i(q) \frac{\partial}{\partial x^i}\bigg|_q, \quad q \in \partial M.$$

The vector field X along ∂M is said to be smooth at $p \in M$ if there exists a coordinate neighborhood of p for which the functions a^i on ∂M are C^∞ at p; it is said to be smooth if it is smooth at every point p. In terms of local coordinates, a vector X_p is outward-pointing if and only if $a^n(p) < 0$ (see Figure 22.5 and Problem 22.3).

Proposition 22.10. On a manifold M with boundary ∂M, there is a smooth outwardpointing vector field along ∂M.

Proof. Cover ∂M with coordinate open sets $(U_\alpha, x_\alpha^1, \ldots, x_\alpha^n)$ in M. On each U_α the vector field $X_\alpha = -\partial/\partial x_\alpha^n$ along $U_\alpha \cap \partial M$ is smooth and outward-pointing. Choose a partition of unity $\{\rho_\alpha\}_{\alpha \in \mathrm{A}}$ on ∂M subordinate to the open cover $\{U_\alpha \cap \partial M\}_{\alpha \in \mathrm{A}}$. Then one can check that $X := \sum \rho_\alpha X_\alpha$ is a smooth outward-pointing vector field along ∂M (Problem 22.4).

11.2.6 22.6 Boundary Orientation

In this section we show that the boundary of an orientable manifold M with boundary is an orientable manifold (without boundary, by Subsection 22.3). We will designate one of the orientations on the boundary as the boundary orientation. It is easily described in terms of an orientation form or of a pointwise orientation on ∂M.

Proposition 22.11. Let M be an oriented n-manifold with boundary. If ω is an orientation form on M and X is a smooth outward-pointing vector field on ∂M, then $1_X\omega$ is a smooth nowhere-vanishing ($n-1$)-form on ∂M. Hence, ∂M is orientable.

Proof. Since ω and X are both smooth on ∂M, so is the contraction $\imath_X\omega$ (Subsection 20.4). We will now prove by contradiction that $t_X\omega$ is nowhere-vanishing on ∂M. Suppose $t_X\omega$ vanishes at some $p \in \partial M$. This means that $(l_X\omega)_p(v_1, \ldots, v_{n-1}) = 0$ for all $v_1, \ldots, v_{n-1} \in T_p(\partial M)$. Let $e_1, \ldots, e_{n-1}$ be a basis for $T_p(\partial M)$. Then $X_p, e_1, \ldots, e_{n-1}$ is a basis for T_pM, and

$$\omega_p(X_p, e_1, \ldots, e_{n-1}) = (\imath_X\omega)_p(e_1, \ldots, e_{n-1}) = 0.$$

By Problem 3.9, $\omega_p \equiv 0$ on T_pM, a contradiction. Therefore, $\imath_X\omega$ is nowhere vanishing on ∂M. By Theorem 21.5, ∂M is orientable.

In the notation of the preceding proposition, we define the boundary orientation on ∂M to be the orientation with orientation form $l_X\omega$. For the boundary orientation to be well defined, we need to check that it is independent of the choice of the orientation form ω and of the outward-pointing vector field X. The verification is not difficult (see Problem 22.5).

Proposition 22.12. Suppose M is an oriented n-manifold with boundary. Let p be a point of the boundary ∂M and let X_p be an outward-pointing vector in T_pM. An ordered basis $(v_1, \ldots, v_{n-1})$ for $T_p(\partial M)$ represents the boundary orientation at p if and only if the ordered basis $(X_p, v_1, \ldots, v_{n-1})$ for T_pM represents the orientation on M at p.

To make this rule easier to remember, we summarize it under the rubric "outward vector first."

Proof. For p in ∂M, let $(v_1, \ldots, v_{n-1})$ be an ordered basis for the tangent space $T_p(\partial M)$. Then

$(v_1, \ldots, v_{n-1})$ represents the boundary orientation on ∂M at p

$\Longleftrightarrow \left(\imath_{X_p}\omega_p\right)(v_1, \ldots, v_{n-1}) > 0$

$\iff \quad \omega_p\left(X_p, v_1, \ldots, v_{n-1}\right) > 0$
$\iff \quad \left(X_p, v_1, \ldots, v_{n-1}\right)$ represents the orientation on M at p. $\square$

Example 22.13 (The boundary orientation on $\partial \mathcal{H}^n$). An orientation form for the standard orientation on the upper half-space $\mathcal{H}^n$ is $\omega = dx^1 \wedge \cdots \wedge dx^n$. A smooth outward-pointing vector field on $\partial \mathcal{H}^n$ is $-\partial/\partial x^n$. By definition, an orientation form for the boundary orientation on $\partial \mathcal{H}^n$ is given by the contraction

$$\begin{aligned} \imath_{-\partial/\partial x^n}(\omega) &= -\imath_{\partial/\partial x^n}\left(dx^1 \wedge \cdots \wedge dx^{n-1} \wedge dx^n\right) \\ &= -(-1)^{n-1} dx^1 \wedge \cdots \wedge dx^{n-1} \wedge \imath_{\partial/\partial x^n}\left(dx^n\right) \\ &= (-1)^n dx^1 \wedge \cdots \wedge dx^{n-1} \end{aligned}$$

Thus, the boundary orientation on $\partial \mathcal{H}^1 = \{0\}$ is given by -1 , the boundary orientation on $\partial \mathcal{H}^2$, given by dx^1, is the usual orientation on the real line $\mathbb{R}$ (Figure 22.6(a)), and the boundary orientation on $\partial \mathcal{H}^3$, given by $-dx^1 \wedge dx^2$, is the clockwise orientation in the (x_1, x_2)-plane $\mathbb{R}^2$ (Figure 22.6(b)).

(a) Boundary orientation on $\partial \mathcal{H}^2 = \mathbb{R}$. (b) Boundary orientation on $\partial \mathcal{H}^3 = \mathbb{R}^2$.

Fig. 22.6. Boundary orientations.

Example. The closed interval $[a, b]$ in the real line with coordinate x has a standard orientation given by the vector field d/dx, with orientation form dx. At the right endpoint b, an outward vector is d/dx. Hence, the boundary orientation at b is given by $l_{d/dx}(dx) = +1$. Similarly, the boundary orientation at the left endpoint a is given by $\imath_{-d/dx}(dx) = -1$.

Example. Suppose $c : [a, b] \to M$ is a C^∞ immersion whose image is a 1-dimensional manifold C with boundary. An orientation on $[a, b]$ induces an orientation on C via the differential $c_{*,p} : T_p([a, b]) \to T_p C$ at each point $p \in [a, b]$. In a situation like this, we give C the orientation induced from the standard orientation on $[a, b]$. The boundary orientation on the boundary of C is given by $+1$ at the endpoint $c(b)$ and -1 at the initial point $c(a)$.

11.2.7 Problems

22.1. Topological Boundary Versus Manifold Boundary

Let M be the subset $[0, 1[\cup\{2\}$ of the real line. Find its topological boundary $\operatorname{bd}(M)$ and its manifold boundary ∂M.

22.2. Topological Boundary of an Intersection

Let A and B be two subsets of a topological space S. Prove that

$$\operatorname{bd}(A \cap B) \subset \operatorname{bd}(A) \cup \operatorname{bd}(B).$$

22.3.* Inward-pointing Vectors at the Boundary

Let M be a manifold with boundary and let $p \in \partial M$. Show that $X_p \in T_pM$ is inward-pointing if and only if in any coordinate chart $(U, x^1, \ldots, x^n)$ centered at p, the coefficient of $(\partial/\partial x^n)_p$ in X_p is positive.

22.4.* Smooth Outward-pointing Vector Field Along the Boundary

Show that the vector field $X = \sum \rho_\alpha X_\alpha$ defined in the proof of Proposition 22.10 is a smooth outward-pointing vector field along ∂M.

22.5. Boundary Orientation

Let M be an oriented manifold with boundary, ω an orientation form for M, and X a C^∞ outward-pointing vector field along ∂M.
(a) If τ is another orientation form on M, then $\tau = f\omega$ for a C^∞ everywhere-positive function f on M. Show that $l_X\tau = f l_X\omega$ and therefore, $l_X\tau \sim l_X\omega$ on ∂M. (Here " $\sim$ " is the equivalence relation defined in Subsection 21.4.)
(b) Prove that if Y is another C^∞ outward-pointing vector field along ∂M, then $l_X\omega \sim l_Y\omega$ on ∂M.

22.6.* Induced Atlas on the Boundary

Assume $n \geq 2$ and let (U, ϕ) and $(V, \boldsymbol{\psi})$ be two charts in an oriented atlas of an orientable n manifold M with boundary. Prove that if $U \cap V \cap \partial M \neq \varnothing$, then the restriction of the transition function $\psi \circ \phi^{-1}$ to the boundary $B := \phi(U \cap V) \cap \partial\mathcal{H}^n$,

$$\left(\psi \circ \phi^{-1}\right)\big|_B : \phi(U \cap V) \cap \partial\mathcal{H}^n \to \psi(U \cap V) \cap \partial\mathcal{H}^n$$

has positive Jacobian determinant. (Hint: Let $\phi = (x^1, \ldots, x^n)$ and $\psi = (y^1, \ldots, y^n)$. Show that the Jacobian matrix of $\psi \circ \phi^{-1}$ in local coordinates is block triangular with $J\left(\psi \circ \phi^{-1}\right)|_B$ and $\partial y^n/\partial x^n$ as the diagonal blocks, and that $\partial y^n/\partial x^n > 0$.)

Thus, if $\{(U_\alpha, \phi_\alpha)\}$ is an oriented atlas for a manifold M with boundary, then the induced atlas $\left\{\left(U_\alpha \cap \partial M, \phi_\alpha|_{U_\alpha \cap \partial M}\right)\right\}$ for ∂M is oriented.

22.7.* Boundary Orientation of the Left Half-space

Let M be the left half-space

$$\left\{\left(x^1, \ldots, x^n\right) \in \mathbb{R}^n \mid x^1 \leq 0\right\},$$

with orientation form $dx^1 \wedge \cdots \wedge dx^n$. Show that an orientation form for the boundary orientation on $\partial M = \{(0, x^2, \ldots, x^n) \in \mathbb{R}^n\}$ is $dx^2 \wedge \cdots \wedge dx^n$.

Unlike the upper half-space $\mathcal{H}^n$, whose boundary orientation takes on a sign (Example 22.13), this exercise shows that the boundary orientation for the left half-space has no sign. For this reason some authors use the left half-space as the model of a manifold with boundary, e.g., [7].

22.8. Boundary Orientation on a Cylinder

Let M be the cylinder $S^1 \times [0, 1]$ with the counterclockwise orientation when viewed from the exterior (Figure 22.7(a)). Describe the boundary orientation on $C_0 = S^1 \times \{0\}$ and $C_1 = S^1 \times \{1\}$.

(a) Oriented cylinder. (b) Radial vector field on a sphere.

Fig. 22.7. Boundary orientations.

22.9. Boundary Orientation on a Sphere

Orient the unit sphere S^n in $\mathbb{R}^{n+1}$ as the boundary of the closed unit ball. Show that an orientation form on S^n is

$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}$$

where the caret $\wedge$ over dx^i indicates that dx^i is to be omitted. (Hint: An outward-pointing vector field on S^n is the radial vector field $X = \sum x^i \partial/\partial x^i$ as in Figure 22.7(b).)

Fig. 22.8. Projection of the upper hemisphere to a disk.

22.10. Orientation on the Upper Hemisphere of a Sphere

Orient the unit sphere S^n in $\mathbb{R}^{n+1}$ as the boundary of the closed unit ball. Let U be the upper hemisphere

$$U = \left\{ x \in S^n \mid x^{n+1} > 0 \right\}.$$

It is a coordinate chart on the sphere with coordinates $x^1, \ldots, x^n$.
(a) Find an orientation form on U in terms of $dx^1, \ldots, dx^n$.
(b) Show that the projection map $\pi : U \to \mathbb{R}^n$,

$$\pi\left(x^1, \ldots, x^n, x^{n+1}\right) = \left(x^1, \ldots, x^n\right),$$

is orientation-preserving if and only if n is even (Figure 22.8).

22.11. Antipodal map on a Sphere and the Orientability of $\setminus P^n$

(a) The antipodal map $a : S^n \to S^n$ on the n-sphere is defined by

$$a\left(x^1, \ldots, x^{n+1}\right) = \left(-x^1, \ldots, -x^{n+1}\right).$$

Show that the antipodal map is orientation-preserving if and only if n is odd.
(b) Use part (a) and Problem 21.6 to prove that an odd-dimensional real projective space $\mathbb{R}P^n$ is orientable.

11.3 §23 Integration on Manifolds

In this chapter we first recall Riemann integration for a function over a closed rectangle in Euclidean space. By Lebesgue's theorem, this theory can be extended to integrals over domains of integration, bounded subsets of $\mathbb{R}^n$ whose boundary has measure zero.

The integral of an n-form with compact support in an open set of $\mathbb{R}^n$ is defined to be the Riemann integral of the corresponding function. Using a partition of unity, we define the integral of an n-form with compact support on a manifold by writing the form as a sum of forms each with compact support in a coordinate chart. We then prove the general Stokes theorem for an oriented manifold and show how it generalizes the fundamental theorem for line integrals as well as Green's theorem from calculus.

11.3.1 23.1 The Riemann Integral of a Function on $\mathbb{R}^n$

We assume that the reader is familiar with the theory of Riemann integration in $\mathbb{R}^n$, as in [26] or [35]. What follows is a brief synopsis of the Riemann integral of a bounded function over a bounded set in $\mathbb{R}^n$.

A closed rectangle in $\mathbb{R}^n$ is a Cartesian product $R = [a^1, b^1] \times \cdots \times [a^n, b^n]$ of closed intervals in $\mathbb{R}$, where $a^i, b^i \in \mathbb{R}$. Let $f : R \to \mathbb{R}$ be a bounded function defined on a closed rectangle R. The volume $\operatorname{vol}(R)$ of the closed rectangle R is defined to be

$$\operatorname{vol}(R) := \prod_{i=1}^{n} (b_i - a_i) \tag{23.1}$$

A partition of the closed interval $[a, b]$ is a set of real numbers $\{p_0, \ldots, p_n\}$ such that

$$a = p_0 < p_1 < \cdots < p_n = b.$$

A partition of the rectangle R is a collection $P = \{P_1, \ldots, P_n\}$, where each P_i is a partition of $[a^i, b^i]$. The partition P divides the rectangle R into closed subrectangles, which we denote by R_j (Figure 23.1).

We define the lower sum and the upper sum of f with respect to the partition P to be

$$L(f, P) := \sum \left(\inf_{R_j} f \right) \operatorname{vol}(R_j), \quad U(f, P) := \sum \left(\sup_{R_j} f \right) \operatorname{vol}(R_j),$$

where each sum runs over all subrectangles of the partition P. For any partition P, clearly $L(f, P) \leq U(f, P)$. In fact, more is true: for any two partitions P and P' of the rectangle R,

$$L(f, P) \leq U(f, P')$$

which we show next.

Fig. 23.1. A partition of a closed rectangle.

A partition $P' = \{P'_1, \ldots, P'_n\}$ is a refinement of the partition $P = \{P_1, \ldots, P_n\}$ if $P_i \subset P'_i$ for all $i = 1, \ldots, n$. If P' is a refinement of P, then each subrectangle R_j of P is subdivided into subrectangles R'_{jk} of P', and it is easily seen that

$$L(f, P) \leq L(f, P') \tag{23.2}$$

because if $R'_{jk} \subset R_j$, then $\inf_{R_j} f \leq \inf_{R'_{jk}} f$. Similarly, if P' is a refinement of P, then

$$U(f, P') \leq U(f, P) \tag{23.3}$$

Any two partitions P and P' of the rectangle R have a common refinement $Q = \{Q_1, \ldots, Q_n\}$ with $Q_i = P_i \cup P'_i$. By (23.2) and (23.3),

$$L(f, P) \leq L(f, Q) \leq U(f, Q) \leq U(f, P')$$

It follows that the supremum of the lower sum $L(f, P)$ over all partitions P of R is less than or equal to the infimum of the upper sum $U(f, P)$ over all partitions P of R. We define these two numbers to be the lower integral $\underline{\int}_R f$ and the upper integral $\overline{\int}_R f$, respectively:

$$\underline{\int}_R f := \sup_P L(f, P), \qquad \overline{\int}_R f := \inf_P L(f, P).$$

Definition 23.1. Let R be a closed rectangle in $\mathbb{R}^n$. A bounded function $f : R \to \mathbb{R}$ is said to be Riemann integrable if $\underline{\int}_R f = \overline{\int}_R f$; in this case, the Riemann integral of f is this common value, denoted by $\int_R f(x) dx^1 \cdots dx^n$, where $x^1, \ldots, x^n$ are the standard coordinates on $\mathbb{R}^n$.

Remark. When we speak of a rectangle $[a^1, b^1] \times \cdots \times [a^n, b^n]$ in $\mathbb{R}^n$, we have already tacitly chosen n coordinates axes, with coordinates $x^1, \ldots, x^n$. Thus, the definition of a Riemann integral depends on the coordinates $x^1, \ldots, x^n$.

If $f : A \subset \mathbb{R}^n \to \mathbb{R}$, then the extension of f by zero is the function $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ such that

$$\tilde{f}(x) = \begin{cases} f(x) & \text{for } x \in A \\ 0 & \text{for } x \notin A \end{cases}$$

Now suppose $f : A \to \mathbb{R}$ is a bounded function on a bounded set A in $\mathbb{R}^n$. Enclose A in a closed rectangle R and define the Riemann integral of f over A to be

$$\int_A f(x) dx^1 \cdots dx^n = \int_R \tilde{f}(x) dx^1 \cdots dx^n$$

if the right-hand side exists. In this way we can deal with the integral of a bounded function whose domain is an arbitrary bounded set in $\mathbb{R}^n$.

The volume $\mathrm{vol}(A)$ of a subset $A \subset \mathbb{R}^n$ is defined to be the integral $\int_A 1 dx^1 \cdots dx^n$ if the integral exists. This concept generalizes the volume of a closed rectangle defined in (23.1).

11.3.2 23.2 Integrability Conditions

In this section we describe some conditions under which a function defined on an open subset of $\mathbb{R}^n$ is Riemann integrable.

Definition 23.2. A set $A \subset \mathbb{R}^n$ is said to have measure zero if for every $\varepsilon > 0$, there is a countable cover $\{R_i\}_{i=1}^{\infty}$ of A by closed rectangles R_i such that $\sum_{i=1}^{\infty} \operatorname{vol}(R_i) < \varepsilon$.

The most useful integrability criterion is the following theorem of Lebesgue [26, Theorem 8.3.1, p. 455].

Theorem 23.3 (Lebesgue's theorem). A bounded function $f : A \to \mathbb{R}$ on a bounded subset $A \subset \mathbb{R}^n$ is Riemann integrable if and only if the set $\operatorname{Disc}(\tilde{f})$ of discontinuities of the extended function $\tilde{f}$ has measure zero.

Proposition 23.4. If a continuous function $f : U \to \mathbb{R}$ defined on an open subset U of $\mathbb{R}^n$ has compact support, then f is Riemann integrable on U.

Proof. Being continuous on a compact set, the function f is bounded. Being compact, the set $\operatorname{supp} f$ is closed and bounded in $\mathbb{R}^n$. We claim that the extension $\tilde{f}$ is continuous.

Since $\tilde{f}$ agrees with f on U, the extended function $\tilde{f}$ is continuous on U. It remains to show that $\tilde{f}$ is continuous on the complement of U in $\mathbb{R}^n$ as well. If $p \notin U$, then $p \notin \operatorname{supp} f$. Since $\operatorname{supp} f$ is a closed subset of $\mathbb{R}^n$, there is an open ball B containing p and disjoint from $\operatorname{supp} f$. On this open ball, $\tilde{f} \equiv 0$, which implies that $\tilde{f}$ is continuous at $p \notin U$. Thus, $\tilde{f}$ is continuous on $\mathbb{R}^n$. By Lebesgue's theorem, f is Riemann integrable on U.

Example 23.5. The continuous function $f :]-1, 1[\to \mathbb{R}$, $f(x) = \tan(\pi x/2)$, is defined on an open subset of finite length in $\mathbb{R}$, but is not bounded (Figure 23.2). The support of f is the open interval $]-1, 1[$, which is not compact. Thus, the function f does not satisfy the hypotheses of either Lebesgue's theorem or Proposition 23.4. Note that it is not Riemann integrable.

Fig. 23.2. The function $f(x) = \tan(\pi x/2)$ on $]-1, 1[$.

Remark. The support of a real-valued function is the closure in its domain of the subset where the function is not zero. In Example 23.5, the support of f is the open interval $]-1, 1[$, not the closed interval $[-1, 1]$, because the domain of f is $]-1, 1[$, not $\mathbb{R}$.

Definition 23.6. A subset $A \subset \mathbb{R}^n$ is called a domain of integration if it is bounded and its topological boundary $\operatorname{bd}(A)$ is a set of measure zero.

Familiar plane figures such as triangles, rectangles, and circular disks are all domains of integration in $\mathbb{R}^2$.

Proposition 23.7 Every bounded continuous function f defined on a domain of integration A in $\mathbb{R}^n$ is Riemann integrable over A.

Proof. Let $\tilde{f} : \mathbb{R}^n \to \mathbb{R}$ be the extension of f by zero. Since f is continuous on A, the extension $\tilde{f}$ is necessarily continuous at all interior points of A. Clearly, $\tilde{f}$ is continuous at all exterior points of A also, because every exterior point has a neighborhood contained entirely in $\mathbb{R}^n - A$, on which $\tilde{f}$ is identically zero. Therefore, the set $\operatorname{Disc}(\tilde{f})$ of discontinuities of $\tilde{f}$ is a subset of $\operatorname{bd}(A)$, a set of measure zero. By Lebesgue's theorem, f is Riemann integrable on A. $\square$

11.3.3 23.3 The Integral of an n-Form on $\mathbb{R}^n$

Once a set of coordinates $x^1, \ldots, x^n$ has been fixed on $\mathbb{R}^n$, n-forms on $\mathbb{R}^n$ can be identified with functions on $\mathbb{R}^n$, since every n-form on $\mathbb{R}^n$ can be written as $\omega = f(x)dx^1 \wedge \cdots \wedge dx^n$ for a unique function $f(x)$ on $\mathbb{R}^n$. In this way the theory of Riemann integration of functions on $\mathbb{R}^n$ carries over to n-forms on $\mathbb{R}^n$.

Definition 23.8. Let $\omega = f(x)dx^1 \wedge \cdots \wedge dx^n$ be a $C^\infty n$-form on an open subset $U \subset \mathbb{R}^n$, with standard coordinates $x^1, \ldots, x^n$. Its integral over a subset $A \subset U$ is defined to be the Riemann integral of $f(x)$:

$$\int_A \omega = \int_A f(x)dx^1 \wedge \cdots \wedge dx^n := \int_A f(x)dx^1 \cdots dx^n$$

if the Riemann integral exists.

In this definition the n-form must be written in the order $dx^1 \wedge \cdots \wedge dx^n$. To integrate, for example, $\tau = f(x)dx^2 \wedge dx^1$ over $A \subset \mathbb{R}^2$, one would write

$$\int_A \tau = \int_A -f(x)dx^1 \wedge dx^2 = -\int_A f(x)dx^1 dx^2$$

Example. If f is a bounded continuous function defined on a domain of integration A in $\mathbb{R}^n$, the the integral $\int_A f dx^1 \wedge \cdots \wedge dx^n$ exists by Proposition 23.7.

Let us see how the integral of an n-form $\omega = f dx^1 \wedge \cdots \wedge dx^n$ on an open subset $U \subset \mathbb{R}^n$ transforms under a change of variables. A change of variables on U is given by a diffeomorphism $T : \mathbb{R}^n \supset V \to U \subset \mathbb{R}^n$. Let $x^1, \ldots, x^n$ be the standard coordinates on U and $y^1, \ldots, y^n$ the standard coordinates on V. Then $T^i := x^i \circ T = T^*(x^i)$ is the i th component of T. We will assume that U and V are connected, and write $x = (x^1, \ldots, x^n)$ and $y = (y^1, \ldots, y^n)$. Denote by $J(T)$ the Jacobian matrix $[\partial T^i / \partial y^j]$. By Corollary 18.4(ii),

$$dT^1 \wedge \cdots \wedge dT^n = \det(J(T))dy^1 \wedge \cdots \wedge dy^n$$

Hence,

$$\begin{aligned} \int_V T^*\omega &= \int_V (T^* f)\, T^* dx^1 \wedge \cdots \wedge T^* dx^n \\ &= \int_V (f \circ T) dT^1 \wedge \cdots \wedge dT^n \\ &= \int_V (f \circ T) \det(J(T)) dy^1 \wedge \cdots \wedge dy^n \\ &= \int_V (f \circ T) \operatorname{dec}(J(T)) dy^1 \cdots dy^n \end{aligned} \tag{23.4}$$

On the other hand, the change-of-variables formula from advanced calculus gives

$$\int_U \omega = \int_U f dx^1 \cdots dx^n = \int_V (f \circ T) |\det(J(T))| dy^1 \cdots dy^n \tag{23.5}$$

with an absolute-value sign around the Jacobian determinant. Equations (23.4) and (23.5) differ by the sign of $\det(J(T))$. Hence,

$$\int_V T^*\omega = \pm \int_U \omega \tag{23.6}$$

depending on whether the Jacobian determinant $\det(J(T))$ is positive or negative.
By Proposition 21.8, a diffeomorphism $T : \mathbb{R}^n \supset V \to U \subset \mathbb{R}^n$ is orientationpreserving if and only if its Jacobian determinant $\det(J(T))$ is everywhere positive on V. Equation (23.6) shows that the integral of a differential form is not invariant under all diffeomorphisms of V with U, but only under orientation-preserving diffeomorphisms.

23.4 Integral of a Differential Form over a Manifold

Integration of an n-form on $\mathbb{R}^n$ is not so different from integration of a function. Our approach to integration over a general manifold has several distinguishing features:
(i) The manifold must be oriented (in fact, $\mathbb{R}^n$ has a standard orientation).
(ii) On a manifold of dimension n, one can integrate only n-forms, not functions.
(iii) The n-forms must have compact support.

Let M be an oriented manifold of dimension n, with an oriented atlas $\{(U_\alpha, \phi_\alpha)\}$ giving the orientation of M. Denote by $\Omega^k_{\mathrm{c}}(M)$ the vector space of $C^\infty k$-forms with compact support on M. Suppose $\{(U, \phi)\}$ is a chart in this atlas. If $\omega \in \Omega^n_{\mathrm{c}}(U)$ is an n-form with compact support on U, then because $\phi : U \to \phi(U)$ is a diffeomorphism, $\left(\phi^{-1}\right)^* \omega$ is an n-form with compact support on the open subset $\phi(U) \subset \mathbb{R}^n$. We define the integral of ω on U to be

$$\int_U \omega := \int_{\phi(U)} \left(\phi^{-1}\right)^* \omega \tag{23.7}$$

If (U, Ψ) is another chart in the oriented atlas with the same U, then $\phi \circ \psi^{-1} : \psi(U) \to \phi(U)$ is an orientation-preserving diffeomorphism, and so

$$\int_{\phi(U)} \left(\phi^{-1}\right)^* \omega = \int_{\psi(U)} \left(\phi \circ \psi^{-1}\right)^* \left(\phi^{-1}\right)^* \omega = \int_{\psi(U)} \left(\psi^{-1}\right)^* \omega$$

Thus, the integral $\int_U \omega$ on a chart U of the atlas is well defined, independent of the choice of coordinates on U. By the linearity of the integral on $\mathbb{R}^n$, if $\omega, \tau \in \Omega^n_{\mathrm{c}}(U)$, then

$$\int_U \omega + \tau = \int_U \omega + \int_U \tau$$

Now let $\omega \in \Omega^n_{\mathrm{c}}(M)$. Choose a partition of unity $\{\rho_\alpha\}$ subordinate to the open cover $\{U_\alpha\}$. Because ω has compact support and a partition of unity has locally finite supports, all except finitely many $\rho_\alpha \omega$ are identically zero by Problem 18.6. In particular,

$$\omega = \sum_\alpha \rho_\alpha \omega$$

is a finite sum. Since by Problem 18.4(b),

$$\operatorname{supp}(\rho_\alpha \omega) \subset \operatorname{supp} \rho_\alpha \cap \operatorname{supp} \omega$$

$\operatorname{supp}(\rho_\alpha \omega)$ is a closed subset of the compact set $\operatorname{supp} \omega$. Hence, $\operatorname{supp}(\rho_\alpha \omega)$ is compact. Since $\rho_\alpha \omega$ is an n-form with compact support in the chart U_α, its integral $\int_{U_\alpha} \rho_\alpha \omega$ is defined. Therefore, we can define the integral of ω over M to be the finite sum

$$\int_M \omega := \sum_\alpha \int_{U_\alpha} \rho_\alpha \omega \tag{23.8}$$

For this integral to be well defined, we must show that it is independent of the choices of oriented atlas and partition of unity. Let $\{V_\beta\}$ be another oriented atlas of M specifying the orientation of M, and $\{\chi_\beta\}$ a partition of unity subordinate to $\{V_\beta\}$. Then $\left\{\left(U_\alpha \cap V_\beta,\, \phi_\alpha|_{U_\alpha \cap U_\beta}\right)\right\}$ and $\left\{\left(U_\alpha \cap V_\beta,\, \psi_\beta|_{U_\alpha \cap U_\beta}\right)\right\}$ are two new atlases of M specifying the orientation of M, and

$$\begin{aligned}
\textstyle\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega &= \textstyle\sum_\alpha \int_{U_\alpha} \rho_\alpha \sum_\beta \chi_\beta \omega \\
&= \textstyle\sum_\alpha \sum_\beta \int_{U_\alpha} \rho_\alpha \chi_\beta \omega \qquad \left(\text{because } \textstyle\sum_\beta \chi_\beta = 1\right) \\
&= \textstyle\sum_\alpha \sum_\beta \int_{U_\alpha \cap V_\beta} \rho_\alpha \chi_\beta \omega,
\end{aligned}$$

where the last line follows from the fact that the support of $\rho_\alpha \chi_\beta$ is contained in $U_\alpha \cap V_\beta$. By symmetry, $\Sigma_\beta \int_{V_\beta} \chi_\beta \omega$ is equal to the same sum. Hence,

$$\sum_\alpha \int_{U_\alpha} \rho_\alpha \omega = \sum_\beta \int_{V_\beta} \chi_\beta \omega$$

proving that the integral (23.8) is well defined.

Proposition 23.9. Let ω be an n-form with compact support on an oriented manifold M of dimension n. If $-M$ denotes the same manifold but with the opposite orientation, then $\int_{-M} \omega = -\int_M \omega$.

Thus, reversing the orientation of M reverses the sign of an integral over M.

Proof. By the definition of an integral ((23.7) and (23.8)), it is enough to show that for every chart $(U, \phi) = (U, x^1, \ldots, x^n)$ and differential form $\tau \in \Omega^n_{\mathrm{c}}(U)$, if $(U, \bar{\phi}) = (U, -x^1, x^2, \ldots, x^n)$ is the chart with the opposite orientation, then

$$\int_{\bar{\phi}(U)} \left(\bar{\phi}^{-1}\right)^* \tau = -\int_{\phi(U)} \left(\phi^{-1}\right)^* \tau$$

Let $r^1, \ldots, r^n$ be the standard coordinates on $\mathbb{R}^n$. Then $x^i = r^i \circ \phi$ and $r^i = x^i \circ \phi^{-1}$. With $\bar{\phi}$, the only difference is that when $i = 1$,

$$-x^1 = r^1 \circ \bar{\phi} \quad \text{and} \quad r^1 = -x^1 \circ \bar{\phi}^{-1}$$

Suppose $\tau = f dx^1 \wedge \cdots \wedge dx^n$ on U. Then

$$\begin{aligned}
\left(\bar{\phi}^{-1}\right)^* \tau &= \left(f \circ \bar{\phi}^{-1}\right) d\left(x^1 \circ \bar{\phi}^{-1}\right) \wedge d\left(x^2 \circ \bar{\phi}^{-1}\right) \wedge \cdots \wedge d\left(x^n \circ \bar{\phi}^{-1}\right) \\
&= -\left(f \circ \bar{\phi}^{-1}\right) dr^1 \wedge dr^2 \wedge \cdots \wedge dr^n
\end{aligned} \tag{23.9}$$

Similarly,

$$\left(\phi^{-1}\right)^* \tau = \left(f \circ \phi^{-1}\right) dr^1 \wedge dr^2 \wedge \cdots \wedge dr^n$$

Since $\phi \circ \bar{\phi}^{-1} : \bar{\phi}(U) \to \phi(U)$ is given by

$$\left(\phi \circ \bar{\phi}^{-1}\right)\left(a^1, a^2, \ldots, a^n\right) = \left(-a^1, a^2, \ldots, a^n\right)$$

the absolute value of its Jacobian determinant is

$$\left|J\left(\phi \circ \bar{\phi}^{-1}\right)\right| = |-1| = 1. \tag{23.10}$$

Therefore,

$$\begin{aligned}
\int_{\bar{\phi}(U)} \left(\bar{\phi}^{-1}\right)^* \tau &= -\int_{\bar{\phi}(U)} \left(f \circ \bar{\phi}^{-1}\right) dr^1 \cdots dr^n \quad (\text{ by } (23.9)) \\
&= -\int_{\bar{\phi}(U)} \left(f \circ \phi^{-1}\right) \circ \left(\phi \circ \bar{\phi}^{-1}\right) \left|J\left(\phi \circ \bar{\phi}^{-1}\right)\right| dr^1 \cdots dr^n \quad (\text{ by } (23.10)) \\
&= -\int_{\phi(U)} \left(f \circ \phi^{-1}\right) dr^1 \cdots dr^n \quad (\text{ by the change-of-variables formula }) \\
&= -\int_{\phi(U)} \left(\phi^{-1}\right)^* \tau
\end{aligned}$$

The treatment of integration above can be extended almost word for word to oriented manifolds with boundary. It has the virtue of simplicity and is of great utility in proving theorems. However, it is not practical for actual computation of integrals; an n-form multiplied by a partition of unity can rarely be integrated as a closed expression. To calculate explicitly integrals over an oriented n-manifold M, it is best to consider integrals over a parametrized set.

Definition 23.10. A parametrized set in an oriented n-manifold M is a subset A together with a C^∞ map $F : D \to M$ from a compact domain of integration $D \subset \mathbb{R}^n$ to M such that $A = F(D)$ and F restricts to an orientation-preserving diffeomorphism from $\operatorname{int}(D)$ to $F(\operatorname{int}(D))$. Note that by smooth invariance of domain for manifolds (Remark 22.5), $F(\operatorname{int}(D))$ is an open subset of M. The C^∞ map $F : D \to A$ is called a parametrization of A.

If A is a parametrized set in M with parametrization $F : D \to A$ and ω is a $C^\infty n$ form on M, not necessarily with compact support, then we define $\int_A \omega$ to be $\int_D F^*\omega$. It can be shown that the definition of $\int_A \omega$ is independent of the parametrization and that in case A is a manifold, it agrees with the earlier definition of integration over a manifold. Subdividing an oriented manifold into a union of parametrized sets can be an effective method of calculating an integral over the manifold. We will not delve into this theory of integration (see [31, Theorem 25.4, p. 213] or [25, Proposition 14.7, p. 356]), but will content ourselves with an example.

Example 23.11 (Integral over a sphere). In spherical coordinates, ρ is the distance $\sqrt{x^2+y^2+z^2}$ of the point $(x, y, z) \in \mathbb{R}^3$ to the origin, φ is the angle that the vector $\langle x, y, z\rangle$ makes with the positive z-axis, and θ is the angle that the vector $\langle x, y\rangle$ in the (x, y)-plane makes with the positive x-axis (Figure 23.3(a)). Let ω be the 2 -form on the unit sphere S^2 in $\mathbb{R}^3$ given by

(a) Spherical coordinates in $\mathbb{R}^3$ (b) A parametrization by spherical coordinates

Fig. 23.3. The sphere as a parametrized set.

$$\omega = \begin{cases} \frac{dy \wedge dz}{x} & \text{for } x \neq 0, \\ \frac{dz \wedge dx}{y} & \text{for } y \neq 0, \\ \frac{dx \wedge dy}{z} & \text{for } z \neq 0. \end{cases}$$

Calculate $\int_{S^2} \omega$.

Up to a factor of 2, the form ω is the 2-form on S^2 from Problem 19.11(b). In Riemannian

geometry, it is shown that ω is the area form of the sphere S^2 with respect to the Euclidean metric. Therefore, the integral $\int_{S^2} \omega$ is the surface area of the sphere.

Solution. The sphere S^2 has a parametrization by spherical coordinates (Figure 23.3(b)):

$$F(\varphi, \theta) = (\sin\varphi\cos\theta, \sin\varphi\sin\theta, \cos\varphi)$$

on $D = \{(\varphi, \theta) \in \mathbb{R}^2 \mid 0 \leq \varphi \leq \pi, 0 \leq \theta \leq 2\pi\}$. Since

$$F^*x = \sin\varphi\cos\theta, \quad F^*y = \sin\varphi\sin\theta, \quad \text{and} \quad F^*z = \cos\varphi$$

we have

$$F^*dy = dF^*y = \cos\varphi\sin\theta d\varphi + \sin\varphi\cos\theta d\theta$$

and

$$F^*dz = -\sin\varphi d\varphi$$

so for $x \neq 0$,

$$F^*\omega = \frac{F^*dy \wedge F^*dz}{F^*x} = \sin\varphi d\varphi \wedge d\theta$$

For $y \neq 0$ and $z \neq 0$, similar calculations show that $F^*\omega$ is given by the same formula. Therefore, $F^*\omega = \sin\varphi d\varphi \wedge d\theta$ everywhere on D, and

$$\int_{S^2} \omega = \int_D F^*\omega = \int_0^{2\pi} \int_0^{\pi} \sin\varphi d\varphi d\theta = 2\pi[-\cos\varphi]_0^{\pi} = 4\pi$$

□

Integration Over a Zero-dimensional Manifold

The discussion of integration so far assumes implicitly that the manifold M has dimension $n \geq 1$. We now treat integration over a zero-dimensional manifold. A compact oriented manifold M of dimension 0 is a finite collection of points, each point oriented by +1 or -1 . We write this as $M = \sum p_i - \sum q_j$. The integral of a 0 -form $f : M \to \mathbb{R}$ is defined to be the sum

$$\int_M f = \sum f(p_i) - \sum f(q_j).$$

11.3.4 23.5 Stokes's Theorem

Let M be an oriented manifold of dimension n with boundary. We give its boundary ∂M the boundary orientation and let $i : \partial M \hookrightarrow M$ be the inclusion map. If ω is an $(n-1)$-form on M, it is customary to write $\int_{\partial M} \omega$ instead of $\int_{\partial M} i^*\omega$.

Theorem 23.12 (Stokes's theorem). For any smooth ($n-1$)-form ω with compact support on the oriented n-dimensional manifold M,

$$\int_M d\omega = \int_{\partial M} \omega$$

Proof. Choose an atlas $\{(U_\alpha, \phi_\alpha)\}$ for M in which each U_α is diffeomorphic to either $\mathbb{R}^n$ or $\mathcal{H}^n$ via an orientation-preserving diffeomorphism. This is possible since any open disk is diffeomorphic to $\mathbb{R}^n$ and any half-disk containing its boundary diameter is diffeomorphic to $\mathcal{H}^n$ (see Problem 1.5). Let $\{\rho_\alpha\}$ be a C^∞ partition of unity subordinate to $\{U_\alpha\}$. As we showed in the preceding section, the $(n-1)$-form $\rho_\alpha\omega$ has compact support in U_α.

Suppose Stokes's theorem holds for $\mathbb{R}^n$ and for $\mathscr{H}^n$. Then it holds for all the charts U_α in our atlas, which are diffeomorphic to $\mathbb{R}^n$ or $\mathcal{H}^n$. Also, note that

$$(\partial M) \cap U_\alpha = \partial U_\alpha$$

Therefore,

$$\begin{array}{lll}
\int_{\partial M} \omega & = \int_{\partial M} \sum_\alpha \rho_\alpha \omega & (\sum_\alpha \rho_\alpha = 1) \\
& = \sum_\alpha \int_{\partial M} \rho_\alpha \omega & (\sum_\alpha \rho_\alpha \omega \text{ is a finite sum by Problem 18.6 }) \\
& = \sum_\alpha \int_{\partial U_\alpha} \rho_\alpha \omega & (\operatorname{supp} \rho_\alpha \omega \subset U_\alpha) \\
& = \sum_\alpha \int_{U_\alpha} d\left(\rho_\alpha \omega\right) & (\text{ Stokes's theorem for } U_\alpha) \\
& = \sum_\alpha \int_M d\left(\rho_\alpha \omega\right) & (\operatorname{supp} d\left(\rho_\alpha \omega\right) \subset U_\alpha) \\
& = \int_M d\left(\sum \rho_\alpha \omega\right) & (\sum_\alpha \rho_\alpha \omega \text{ is a finite sum }) \\
& = \int_M d\omega &
\end{array}$$

Thus, it suffices to prove Stokes's theorem for $\mathbb{R}^n$ and for $\mathcal{H}^n$. We will give a proof only for $\mathcal{H}^2$, since the general case is similar (see Problem 23.4).

Proof of Stokes's theorem for the upper half-plane $\mathcal{H}^2$. Let x, y be the coordinates on $\mathcal{H}^2$. Then the standard orientation on $\mathcal{H}^2$ is given by $dx \wedge dy$, and the boundary orientation on $\partial \mathcal{H}^2$ is given by $\imath_{-\partial/\partial y}(dx \wedge dy) = dx$.

The form ω is a linear combination

$$\omega = f(x, y)dx + g(x, y)dy \tag{23.11}$$

for C^∞ functions f, g with compact support in $\mathcal{H}^2$. Since the supports of f and g are compact, we may choose a real number $a > 0$ large enough that the supports of f and g are contained in the interior of the square $[-a, a] \times [0, a]$. We will use the notation f_x, f_y to denote the partial derivatives of f with respect to x and y, respectively. Then

$$d\omega = \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dx \wedge dy = \left(g_x - f_y\right) dx \wedge dy$$

and

$$\begin{aligned}
\int_{\mathcal{H}^2} d\omega &= \int_{\mathcal{H}^2} g_x dxdy - \int_{\mathcal{H}^2} f_y dxdy \\
&= \int_0^\infty \int_{-\infty}^\infty g_x dxdy - \int_{-\infty}^\infty \int_0^\infty f_y dydx \\
&= \int_0^a \int_{-a}^a g_x dxdy - \int_{-a}^a \int_0^a f_y dydx
\end{aligned} \tag{23.12}$$

In this expression,

$$\int_{-a}^a g_x(x, y)dx = g(x, y)\Big]_{x=-a}^a = 0$$

because $\operatorname{supp} g$ lies in the interior of $[-a, a] \times [0, a]$. Similarly,

$$\int_0^a f_y(x, y)dy = f(x, y)\Big]_{y=0}^a = -f(x, 0)$$

because $f(x, a) = 0$. Thus, (23.12) becomes

$$\int_{\mathcal{H}^2} d\omega = \int_{-a}^a f(x, 0)dx$$

On the other hand, $\partial\mathcal{H}^2$ is the x-axis and $dy = 0$ on $\partial\mathcal{H}^2$. It follows from (23.11) that $\omega = f(x,0)dx$ when restricted to $\partial\mathcal{H}^2$ and

$$\int_{\partial\mathcal{H}^2} \omega = \int_{-a}^{a} f(x,0)dx.$$

This proves Stokes's theorem for the upper half-plane. □

11.3.5 23.6 Line Integrals and Green's Theorem

We will now show how Stokes's theorem for a manifold unifies some of the theorems of vector calculus on $\mathbb{R}^2$ and $\mathbb{R}^3$. Recall the calculus notation $\mathbf{F} \cdot d\mathbf{r} = Pdx + Qdy + Rdz$ for $\mathbf{F} = \langle P, Q, R\rangle$ and $\mathbf{r} = (x, y, z)$. As in calculus, we assume in this section that functions, vector fields, and regions of integration have sufficient smoothness or regularity properties so that all the integrals are defined.

Theorem 23.13 (Fundamental theorem for line integrals). Let C be a curve in $\mathbb{R}^3$, parametrized by $\mathbf{r}(t) = (x(t), y(t), z(t)), a \leq t \leq b$, and let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$. If $\mathbf{F} = \operatorname{grad} f$ for some scalar function f, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$$

Suppose in Stokes's theorem we take M to be a curve C with parametrization $\mathbf{r}(t), a \leq t \leq b$, and ω to be the function f on C. Then

$$\int_C d\omega = \int_C df = \int_C \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz = \int_C \operatorname{grad} f \cdot d\mathbf{r}$$

and

$$\int_{\partial C} \omega = f\Big]_{\mathbf{r}(a)}^{\mathbf{r}(b)} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

In this case Stokes's theorem specializes to the fundamental theorem for line integrals.

Theorem 23.14 (Green's theorem). If D is a plane region with boundary ∂D, and P and Q are C^∞ functions on D, then

$$\int_{\partial D} Pdx + Qdy = \int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

In this statement, dA is the usual calculus notation for $dxdy$. To obtain Green's theorem, let M be a plane region D with boundary ∂D and let ω be the 1 -form $Pdx + Qdy$ on D. Then

$$\int_{\partial D} \omega = \int_{\partial D} Pdx + Qdy$$

and

$$\begin{aligned}\int_D d\omega &= \int_D P_y dy \wedge dx + Q_x dx \wedge dy = \int_D (Q_x - P_y)\, dx \wedge dy \\ &= \int_D (Q_x - P_y)\, dxdy = \int_D (Q_x - P_y)\, dA\end{aligned}$$

In this case Stokes's theorem is Green's theorem in the plane.

11.3.6 Problems

23.1. Area of an Ellipse

Use the change-of-variables formula to compute the area enclosed by the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

in $\mathbb{R}^2$.

23.2. Characterization of Boundedness in $\mathbb{R}^n$

Prove that a subset $A \subset \mathbb{R}^n$ is bounded if and only if its closure $\bar{A}$ in $\mathbb{R}^n$ is compact.

23.3.* Integral under a Diffeomorphism

Suppose N and M are connected, oriented n-manifolds and $F : N \to M$ is a diffeomorphism. Prove that for any $\omega \in \Omega_c^k(M)$,

$$\int_N F^*\omega = \pm \int_M \omega$$

where the sign depends on whether F is orientation-preserving or orientation-reversing.

23.4.* Stokes's Theorem

Prove Stokes's theorem for $\mathbb{R}^n$ and for $\mathcal{H}^n$.

23.5. Area form on the sphere S^2

Prove that the area form ω on S^2 in Example 23.11 is equal to the orientation form

$$xdy \wedge dz - ydx \wedge dz + zdx \wedge dy$$

of S^2 in Problem 22.9.

12 7 De Rham Theory

By the fundamental theorem for line integrals (Theorem 23.13), if a smooth vector field $\mathbf{F}$ is the gradient of a scalar function f, then for any two points p and q in $\mathbb{R}^3$, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ over a curve C from p to q is independent of the curve. In this case, the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ can be computed in terms of its values at the two endpoints as $f(q) - f(p)$. Similarly, by the classical Stokes theorem for a surface, the surface integral of smooth a vector field $\mathbf{F}$ over an oriented surface S with boundary C in $\mathbb{R}^3$ can be evaluated as an integral over the curve C if $\mathbf{F}$ is the curl of another vector field. It is thus of interest to know whether a vector field $\mathbb{R}^3$ is the gradient of a function or is the curl of another vector field. By the correspondence of Section 4.6 between vector fields and differential forms, this translates into whether a differential form ω on

Henri Poincaré
(1854-1912)

$\mathbb{R}^3$ is exact.

Considerations such as these led Henri Poincaré to look for conditions under which a differential form is exact on $\mathbb{R}^n$. Of course, a necessary condition is that the form ω be closed. Poincaré proved in 1887 that for $k = 1, 2, 3$, a k-form on $\mathbb{R}^n$ is exact if and only if it is closed, a lemma that now bears his name. Vito Volterra published in 1889 the first complete proof of the Poincaré lemma for all k.

It turns out that whether every closed form on a manifold is exact depends on the topology of the manifold. For example, on $\mathbb{R}^2$ every closed k-form is exact for $k > 0$, but on the punctured plane $\mathbb{R}^2 - \{(0, 0)\}$ there are closed 1 -forms that are not exact. The extent to which closed forms are not exact is measured by the de Rham cohomology, possibly the most important diffeomorphism invariant of a manifold.

In a series of groundbreaking papers, starting

Georges de Rham
(1903-1990)

with "Analysis situs" [33] in 1895, Poincaré introduced the concept of homology and laid the foundations of modern algebraic topology. Roughly speaking, a compact submanifold with no boundary is a cycle, and a cycle is homologous to zero if it is the boundary of another manifold. The equivalence classes of cycles under the homology relation are called homology classes. In his doctoral thesis [8] in 1931, Georges de Rham showed that differential forms satisfy the same axioms as cycles and boundaries, in effect proving a duality between what are now called de Rham cohomology and singular homology with real coefficients. Although he did not define explicitly de Rham cohomology in this paper, it was implicit in his work. A formal definition of de Rham cohomology appeared in 1938 [9].

12.1 §24 De Rham Cohomology

In this section we define de Rham cohomology, prove some of its basic properties, and compute two elementary examples: the de Rham cohomology vector spaces of the real line and of the unit circle.

12.1.1 24.1 De Rham Cohomology

Suppose $\mathbf{F}(x, y) = \langle P(x, y), Q(x, y)\rangle$ is a smooth vector field representing a force on an open subset U of $\mathbb{R}^2$, and C is a parametrized curve $c(t) = (x(t), y(t))$ in U from a point p to a point q, with $a \leq t \leq b$. Then the work done by the force in moving a particle from p to q along C is given by the line integral $\int_C Pdx + Qdy$.

Such a line integral is easy to compute if the vector field $\mathbf{F}$ is the gradient of a scalar function $f(x, y)$:

$$\mathbf{F} = \operatorname{grad} f = \langle f_x, f_y\rangle ,$$

where $f_x = \partial f/\partial x$ and $f_y = \partial f/\partial y$. By Stokes's theorem, the line integral is simply

$$\int_C f_x dx + f_y dy = \int_C df = f(q) - f(p)$$

A necessary condition for the vector field $\mathbf{F} = \langle P, Q \rangle$ to be a gradient is that

$$P_y = f_{xy} = f_{yx} = Q_x.$$

The question is now the following: if $P_y - Q_x = 0$, is the vector field $\mathbf{F} = \langle P, Q \rangle$ on U the gradient of some scalar function $f(x, y)$ on U ?

In Section 4.6 we established a one-to-one correspondence between vector fields and differential 1-forms on an open subset of $\mathbb{R}^3$. There is a similar correspondence on an open subset of any $\mathbb{R}^n$. For $\mathbb{R}^2$, it is as follows:

$$\begin{aligned} \text{vector fields} &\longleftrightarrow \text{differential 1-forms,} \\ \mathbf{F} = \langle P, Q \rangle &\longleftrightarrow \omega = P dx + Q dy \\ \operatorname{grad} f = \langle f_x, f_y \rangle &\longleftrightarrow df = f_x dx + f_y dy \\ Q_x - P_y = 0 &\longleftrightarrow d\omega = (Q_x - P_y)\, dx \wedge dy = 0 \end{aligned}$$

In terms of differential forms the question above becomes the following: if the 1 form $\omega = P dx + Q dy$ is closed on U, is it exact? The answer to this question is sometimes yes and sometimes no, depending on the topology of U.

Just as for an open subset of $\mathbb{R}^n$, a differential form ω on a manifold M is said to be closed if $d\omega = 0$, and exact if $\omega = d\tau$ for some form τ of degree one less. Since $d^2 = 0$, every exact form is closed. In general, not every closed form is exact.

Let $Z^k(M)$ be the vector space of all closed k-forms and $B^k(M)$ the vector space of all exact k-forms on the manifold M. Because every exact form is closed, $B^k(M)$ is a subspace of $Z^k(M)$. The quotient vector space $H^k(M) := Z^k(M)/B^k(M)$ measures the extent to which closed k-forms fail to be exact, and is called the de Rham cohomology of M in degree k. As explained in Appendix D, the quotient vector space construction introduces an equivalence relation on $Z^k(M)$:

$$\omega' \sim \omega \quad \text{in } Z^k(M) \quad \text{iff} \quad \omega' - \omega \in B^k(M).$$

The equivalence class of a closed form ω is called its cohomology class and denoted by $[\omega]$. Two closed forms ω and ω' determine the same cohomology class if and only if they differ by an exact form:

$$\omega' = \omega + d\tau.$$

In this case we say that the two closed forms ω and ω' are cohomologous.

Proposition 24.1. If the manifold M has r connected components, then its de Rham cohomology in degree 0 is $H^0(M) = \mathbb{R}^r$. An element of $H^0(M)$ is specified by an ordered r-tuple of real numbers, each real number representing a constant function on a connected component of M.

Proof. Since there are no nonzero exact 0 -forms,

$$H^0(M) = Z^0(M) = \{ \text{ closed 0-forms } \}.$$

Supposed f is a closed 0 -form on M; i.e., f is a C^∞ function on M such that $df = 0$. On any chart $(U, x^1, \ldots, x^n)$,

$$df = \sum \frac{\partial f}{\partial x^i} dx^i.$$

Thus, $df = 0$ on U if and only if all the partial derivatives $\partial f/\partial x^i$ vanish identically on U. This in turn is equivalent to f being locally constant on U. Hence, the closed 0-forms on M are precisely the locally constant functions on M. Such a function must be constant on each connected component of M. If M has r connected components, then a locally constant function on M can be specified by an ordered set of r real numbers. Thus, $Z^0(M) = \mathbb{R}^r$.

Proposition 24.2. On a manifold M of dimension n, the de Rham cohomology $H^k(M)$ vanishes for $k > n$.

Proof. At any point $p \in M$, the tangent space T_pM is a vector space of dimension n. If ω is a k-form on M, then $\omega_p \in A_k(T_pM)$, the space of alternating k-linear functions on T_pM. By Corollary 3.31, if $k > n$, then $A_k(T_pM) = 0$. Hence, for $k > n$, the only k-form on M is the zero form.

12.1.2 24.2 Examples of de Rham Cohomology

Example 24.3 (De Rham cohomology of the real line). Since the real line $\mathbb{R}^1$ is connected, by Proposition 24.1,

$$H^0\left(\mathbb{R}^1\right) = \mathbb{R}$$

For dimensional reasons, on $\mathbb{R}^1$ there are no nonzero 2 -forms. This implies that every 1-form on $\mathbb{R}^1$ is closed. A 1-form $f(x)dx$ on $\mathbb{R}^1$ is exact if and only if there is a C^∞ function $g(x)$ on $\mathbb{R}^1$ such that

$$f(x)dx = dg = g'(x)dx$$

where $g'(x)$ is the calculus derivative of g with respect to x. Such a function $g(x)$ is simply an antiderivative of $f(x)$, for example

$$g(x) = \int_0^x f(t)dt$$

This proves that every 1 -form on $\mathbb{R}^1$ is exact. Therefore, $H^1(\mathbb{R}^1) = 0$. In combination with Proposition 24.2, we have

$$H^k\left(\mathbb{R}^1\right) = \begin{cases} \mathbb{R} & \text{for } k = 0 \\ 0 & \text{for } k \geq 1 \end{cases}$$

Example 24.4 (De Rham cohomology of a circle). Let S^1 be the unit circle in the xy-plane. By Proposition 24.1, because S^1 is connected, $H^0(S^1) = \mathbb{R}$, and because S^1 is one-dimensional, $H^k(S^1) = 0$ for all $k \geq 2$. It remains to compute $H^1(S^1)$.

Recall from Subsection 18.7 the map $h : \mathbb{R} \to S^1, h(t) = (\cos t, \sin t)$. Let $i : [0, 2\pi] \to \mathbb{R}$ be the inclusion map. Restricting the domain of h to $[\,0, 2\pi\,]$ gives a parametrization $F := h \circ i : [0, 2\pi] \to S^1$ of the circle. In Examples 17.15 and 17.16, we found a nowhere-vanishing 1 -form $\omega = -ydx + xdy$ on S^1 and showed that $F^*\omega = i^*h^*\omega = i^*dt = dt$. Thus,

$$\int_{S^1} \omega = \int_{F([0,2\pi])} \omega = \int_{[0,2\pi]} F^*\omega = \int_0^{2\pi} dt = 2\pi$$

Since the circle has dimension 1, all 1-forms on S^1 are closed, so $\Omega^1(S^1) = Z^1(S^1)$. The integration of 1 -forms on S^1 defines a linear map

$$\varphi : Z^1\left(S^1\right) = \Omega^1\left(S^1\right) \to \mathbb{R}, \quad \varphi(\alpha) = \int_{S^1} \alpha$$

Because $\varphi(\omega) = 2\pi \neq 0$, the linear map $\varphi : \Omega^1(S^1) \to \mathbb{R}$ is onto.
By Stokes's theorem, the exact 1 -forms on S^1 are in $\ker \varphi$. Conversely, we will show that all 1 -forms in $\ker \varphi$ are exact. Suppose $\alpha = f\omega$ is a smooth 1-form on S^1 such that $\varphi(\alpha) = 0$. Let $\bar{f} = h^* f = f \circ h \in \Omega^0(\mathbb{R})$. Then $\bar{f}$ is periodic of period 2π and

$$0 = \int_{S^1} \alpha = \int_{F([0,2\pi])} \alpha = \int_{[0,2\pi]} F^*\alpha = \int_{[0,2\pi]} (i^* h^* f)(t) \cdot F^*\omega = \int_0^{2\pi} \bar{f}(t)dt$$

Lemma 24.5. Suppose $\bar{f}$ is a C^∞ periodic function of period 2π on $\mathbb{R}$ and $\int_0^{2\pi} \bar{f}(u)du = 0$. Then $\bar{f}dt = d\bar{g}$ for a C^∞ periodic function $\bar{g}$ of period 2π on $\mathbb{R}$.
Proof. Define $\bar{g} \in \Omega^0(\mathbb{R})$ by

$$\bar{g}(t) = \int_0^t \bar{f}(u)du$$

Since $\int_0^{2\pi} \bar{f}(u)du = 0$ and $\bar{f}$ is periodic of period 2π,

$$\begin{aligned}\bar{g}(t+2\pi) &= \int_0^{2\pi} \bar{f}(u)du + \int_{2\pi}^{t+2\pi} \bar{f}(u)du \\ &= 0 + \int_{2\pi}^{t+2\pi} \bar{f}(u)du = \int_0^t \bar{f}(u)du = \bar{g}(t)\end{aligned}$$

Hence, $\bar{g}(t)$ is also periodic of period 2π on $\mathbb{R}$. Moreover,

$$d\bar{g} = \bar{g}'(t)dt = \bar{f}(t)dt$$

Let $\bar{g}$ be the periodic function of period 2π on $\mathbb{R}$ from Lemma 24.5. By Proposition 18.12, $\bar{g} = h^* g$ for some C^∞ function g on S^1. It follows that

$$d\bar{g} = dh^* g = h^*(dg)$$

On the other hand,

$$\bar{f}(t)dt = (h^* f)(h^*\omega) = h^*(f\omega) = h^*\alpha$$

Since $h^* : \Omega^1(S^1) \to \Omega^1(\mathbb{R})$ is injective, $\alpha = dg$. This proves that the kernel of φ consists of exact forms. Therefore, integration induces an isomorphism

$$H^1(S^1) = \frac{Z^1(S^1)}{B^1(S^1)} \xrightarrow{\sim} \mathbb{R}$$

In the next section we will develop a tool, the Mayer-Vietoris sequence, using which the computation of the cohomology of the circle becomes more or less routine.

12.1.3 24.3 Diffeomorphism Invariance

For any smooth map $F : N \to M$ of manifolds, there is a pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ of differential forms. Moreover, the pullback F^* commutes with the exterior derivative d (Proposition 19.5).

Lemma 24.6. The pullback map F^* sends closed forms to closed forms, and sends exact forms to exact forms.

Proof. Suppose ω is closed. By the commutativity of F^* with d,

$$dF^*\omega = F^* d\omega = 0$$

Hence, $F^*\omega$ is also closed.
Next suppose $\omega = d\tau$ is exact. Then

$$F^*\omega = F^*d\tau = dF^*\tau$$

Hence, $F^*\omega$ is exact.
It follows that F^* induces a linear map of quotient spaces, denoted by $F^\#$:

$$F^\# : \frac{Z^k(M)}{B^k(M)} \to \frac{Z^k(N)}{B^k(N)}, \quad F^\#([\omega]) = [F^*(\omega)]\,.$$

This is a map in cohomology,

$$F^\# : H^k(M) \to H^k(N),$$

called the pullback map in cohomology.
Remark 24.7. The functorial properties of the pullback map F^* on differential forms easily yield the same functorial properties for the induced map in cohomology:
(i) If $\mathbb{1}_M : M \to M$ is the identity map, then $\mathbb{1}_M^\# : H^k(M) \to H^k(M)$ is also the identity map.
(ii) If $F : N \to M$ and $G : M \to P$ are smooth maps, then

$$(G \circ F)^\# = F^\# \circ G^\#$$

It follows from (i) and (ii) that $\left(H^k(), F^\#\right)$ is a contravariant functor from the category of C^∞ manifolds and C^∞ maps to the category of vector spaces and linear maps. By Proposition 10.3, if $F : N \to M$ is a diffeomorphism of manifolds, then $F^\# : H^k(M) \to H^k(N)$ is an isomorphism of vector spaces.

In fact, the usual notation for the induced map in cohomology is F^*, the same as for the pullback map on differential forms. Unless there is a possibility of confusion, henceforth we will follow this convention. It is usually clear from the context whether F^* is a map in cohomology or on forms.

12.1.4 24.4 The Ring Structure on de Rham Cohomology

The wedge product of differential forms on a manifold M gives the vector space $\Omega^*(M)$ of differential forms a product structure. This product structure induces a product structure in cohomology: if $[\omega] \in H^k(M)$ and $[\tau] \in H^\ell(M)$, define

$$[\omega] \wedge [\tau] = [\omega \wedge \tau] \in H^{k+\ell}(M). \tag{24.1}$$

For the product to be well defined, we need to check three things about closed forms ω and τ :
(i) The wedge product $\omega \wedge \tau$ is a closed form.
(ii) The class $[\omega \wedge \tau]$ is independent of the choice of representative for $[\tau]$. In other words, if τ is replaced by a cohomologous form $\tau' = \tau + d\sigma$, then in the equation

$$\omega \wedge \tau' = \omega \wedge \tau + \omega \wedge d\sigma$$

we need to show that $\omega \wedge d\sigma$ is exact.
(iii) The class $[\omega \wedge \tau]$ is independent of the choice of representative for $[\omega]$.

These all follow from the antiderivation property of d. For example, in (i), since ω and τ are closed,

$$d(\omega \wedge \tau) = (d\omega) \wedge \tau + (-1)^k \omega \wedge d\tau = 0$$

In (ii),

$$d(\omega \wedge \sigma) = (d\omega) \wedge \sigma + (-1)^k \omega \wedge d\sigma = (-1)^k \omega \wedge d\sigma \quad (\text{ since } d\omega = 0)$$

which shows that $\omega \wedge d\sigma$ is exact. Item (iii) is analogous to (ii), with the roles of ω and τ reversed.

If M is a manifold of dimension n, we set

$$H^*(M) = \bigoplus_{k=0}^{n} H^k(M)$$

What this means is that an element α of $H^*(M)$ is uniquely a finite sum of cohomology classes in $H^k(M)$ for various k 's:

$$\alpha = \alpha_0 + \cdots + \alpha_n, \quad \alpha_k \in H^k(M)$$

Elements of $H^*(M)$ can be added and multiplied in the same way that one would add or multiply polynomials, except here multiplication is the wedge product. It is easy to check that under addition and multiplication, $H^*(M)$ satisfies all the properties of a ring, called the cohomology ring of M. The ring $H^*(M)$ has a natural grading by the degree of a closed form. Recall that a ring A is graded if it can be written as a direct sum $A = \bigoplus_{k=0}^{\infty} A^k$ so that the ring multiplication sends $A^k \times A^\ell$ to $A^{k\times\ell}$. A graded ring $A = \bigoplus_{k=0}^{\infty} A^k$ is said to be anticommutative if for all $a \in A^k$ and $b \in A^\ell$,

$$a \cdot b = (-1)^{k\ell} b \cdot a$$

In this terminology, $H^*(M)$ is an anticommutative graded ring. Since $H^*(M)$ is also a real vector space, it is in fact an anticommutative graded algebra over $\mathbb{R}$.

Suppose $F : N \to M$ is a C^∞ map of manifolds. Because $F^*(\omega \wedge \tau) = F^*\omega \wedge F^*\tau$ for differential forms ω and τ on M (Proposition 18.11), the linear map $F^* : H^*(M) \to H^*(N)$ is a ring homomorphism. By Remark 24.7, if $F : N \to M$ is a diffeomorphism, then the pullback $F^* : H^*(M) \to H^*(N)$ is a ring isomorphism.

To sum up, de Rham cohomology gives a contravariant functor from the category of C^∞ manifolds to the category of anticommutative graded rings. If M and N are diffeomorphic manifolds, then $H^*(M)$ and $H^*(N)$ are isomorphic as anticommutative graded rings. In this way the de Rham cohomology becomes a powerful diffeomorphism invariant of C^∞ manifolds.

12.1.5 Problems

24.1. Nowhere-vanishing 1-forms

Prove that a nowhere-vanishing 1-form on a compact manifold cannot be exact.

24.2. Cohomology in Degree Zero

Suppose a manifold M has infinitely many connected components. Compute its de Rham cohomology vector space $H^0(M)$ in degree 0 . (Hint: By second countability, the number of connected components of a manifold is countable.)

12.2 §25 the Long Exact Sequence in Cohomology

A cochain complex $\mathcal{C}$ is a collection of vector spaces $\left\{C^k\right\}_{k\in\mathbb{Z}}$ together with a sequence of linear maps $d_k : C^k \to C^{k+1}$,

$$\cdots \to C^{-1} \xrightarrow{d_{-1}} C^0 \xrightarrow{d_0} C^1 \xrightarrow{d_1} C^2 \xrightarrow{d_2} \cdots,$$

such that

$$d_k \circ d_{k-1} = 0 \tag{25.1}$$

for all k. We will call the collection of linear maps $\{d_k\}$ the differential of the cochain complex $\mathcal{C}$.

The vector space $\Omega^*(M)$ of differential forms on a manifold M together with the exterior derivative d is a cochain complex, the de Rham complex of M :

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d} \cdots, \quad d \circ d = 0.$$

It turns out that many of the results on the de Rham cohomology of a manifold depend not on the topological properties of the manifold, but on the algebraic properties of the de Rham complex. To better understand de Rham cohomology, it is useful to isolate these algebraic properties. In this section we investigate the properties of a cochain complex that constitute the beginning of a subject known as homological algebra.

12.2.1 25.1 Exact Sequences

This subsection is a compendium of a few basic properties of exactness that will be used over and over again.

Definition 25.1. A sequence of homomorphisms of vector spaces

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is said to be exact at B if $\operatorname{im} f = \ker g$. A sequence of homomorphisms

$$A^0 \xrightarrow{f_0} A^1 \xrightarrow{f_1} A^2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A^n$$

that is exact at every term except the first and the last is simply said to be an exact sequence. A five-term exact sequence of the form

$$0 \to A \to B \to C \to 0$$

is said to be short exact.

The same definition applies to homomorphisms of groups or modules, but we are mainly concerned with vector spaces.

Remark. (i) When $A = 0$, the sequence

$$0 \xrightarrow{f} B \xrightarrow{g} C$$

is exact if and only if

$$\ker g = \operatorname{im} f = 0,$$

so that g is injective.

(ii) Similarly, when $C = 0$, the sequence

$$A \xrightarrow{f} B \xrightarrow{g} 0$$

is exact if and only if

$$\operatorname{im} f = \ker g = B,$$

so that f is surjective.
The following two propositions are very useful for dealing with exact sequences.
Proposition 25.2 (A three-term exact sequence). Suppose

$$A \xrightarrow{f} B \xrightarrow{g} C$$

is an exact sequence. Then
(i) the map f is surjective if and only if g is the zero map;
(ii) the map g is injective if and only if f is the zero map.
Proof. Problem 25.1.

Proposition 25.3 (a Four-term Exact Sequence).

(i) The four-term sequence $0 \to A \xrightarrow{f} B \to 0$ of vector spaces is exact if and only if $f : A \to B$ is an isomorphism.
(ii) If

$$A \xrightarrow{f} B \to C \to 0$$

is an exact sequence of vector spaces, then there is a linear isomorphism

$$C \simeq \operatorname{coker} f := \frac{B}{\operatorname{im} f}$$

Proof. Problem 25.2.

12.2.2 25.2 Cohomology of Cochain Complexes

If $\mathcal{C}$ is a cochain complex, then by (25.1),

$$\operatorname{im} d_{k-1} \subset \ker d_k$$

We can therefore form the quotient vector space

$$H^k(\mathcal{C}) := \frac{\ker d_k}{\operatorname{im} d_{k-1}},$$

which is called the k th cohomology vector space of the cochain complex $\mathcal{C}$. It is a measure of the extent to which the cochain complex $\mathcal{C}$ fails to be exact at C^k. Elements of the vector space C^k are called cochains of degree k or k-cochains for short. A k-cochain in ker d_k is called a k-cocycle and a k-cochain in im d_{k-1} is called a k-coboundary. The equivalence class $[c] \in H^k(\mathcal{C})$ of a k-cocycle $c \in \ker d_k$ is called its cohomology class. We denote the subspaces of k-cocycles and k-coboundaries of $\mathcal{C}$ by $Z^k(\mathcal{C})$ and $B^k(\mathcal{C})$ respectively. The letter Z for cocycles comes from *Zyklen*, the German word for cycles.

To simplify the notation we will usually omit the subscript in d_k, and write $d \circ d = 0$ instead of $d_k \circ d_{k-1} = 0$.

Example. In the de Rham complex, a cocycle is a closed form and a coboundary is an exact form.

If $\mathcal{A}$ and $\mathcal{B}$ are two cochain complexes with differentials d and d' respectively, a cochain map $\varphi : \mathcal{A} \to \mathcal{B}$ is a collection of linear maps $\varphi_k : A^k \to B^k$, one for each k, that commute with d and d' :

$$d' \circ \varphi_k = \varphi_{k+1} \circ d$$

In other words, the following diagram is commutative:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{k-1} \arrow[r, "d"] \arrow[d, "\varphi_{k-1}"] & A^k \arrow[r, "d"] \arrow[d, "\varphi_k"] & A^{k+1} \arrow[r] \arrow[d, "\varphi_{k+1}"] & \cdots \\
\cdots \arrow[r] & B^{k-1} \arrow[r, "d'"'] & B^k \arrow[r, "d'"'] & B^{k+1} \arrow[r] & \cdots .
\end{tikzcd}$$

We will usually omit the subscript k in φ_k.
A cochain map $\varphi : \mathcal{A} \to \mathcal{B}$ naturally induces a linear map in cohomology

$$\varphi^* : H^k(\mathcal{A}) \to H^k(\mathcal{B})$$

by

$$\varphi^*[a] = [\varphi(a)]. \tag{25.2}$$

To show that this is well defined, we need to check that a cochain map takes cocycles to cocycles, and coboundaries to coboundaries:
(i) for $a \in Z^k(\mathcal{A}), d'(\varphi(a)) = \varphi(da) = 0$;
(ii) for $a' \in A^{k-1}, \varphi(da') = d'(\varphi(a'))$.

Example 25.4.
(i) For a smooth map $F : N \to M$ of manifolds, the pullback map $F^* : \Omega^*(M) \to \Omega^*(N)$ on differential forms is a cochain map, because F^* commutes with d (Proposition 19.5). By the discussion above, there is an induced map $F^* : H^*(M) \to H^*(N)$ in cohomology, as we saw once before, after Lemma 24.6.
(ii) If X is a C^∞ vector field on a manifold M, then the Lie derivative $\mathcal{L}_X : \Omega^*(M) \to \Omega^*(M)$ commutes with d (Theorem 20.10(ii)). By (25.2), $\mathcal{L}_X$ induces a linear map $\mathcal{L}_X^* : H^*(M) \to H^*(M)$ in cohomology.

12.2.3 25.3 The Connecting Homomorphism

A sequence of cochain complexes

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

is short exact if i and j are cochain maps and for each k,

$$0 \to A^k \xrightarrow{i_k} B^k \xrightarrow{j_k} C^k \to 0$$

is a short exact sequence of vector spaces. Since we usually omit subscripts on cochain maps, we will write i, j instead of i_k, j_k.

Given a short exact sequence as above, we can construct a linear map $d^* : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$, called the connecting homomorphism, as follows. Consider the short exact sequences in dimensions k and $k+1$:

$$\begin{tikzcd}
0 \arrow[r] & A^{k+1} \arrow[r, "i"] & B^{k+1} \arrow[r, "j"] & C^{k+1} \arrow[r] & 0 \\
0 \arrow[r] & A^k \arrow[r, "i"'] \arrow[u, "d"] & B^k \arrow[r, "j"'] \arrow[u, "d"] & C^k \arrow[r] \arrow[u, "d"] & 0.
\end{tikzcd}$$

To keep the notation simple, we use the same symbol d to denote the a priori distinct differentials d_A, d_B, d_C of the three cochain complexes. Start with $[c] \in H^k(\mathcal{C})$. Since $j : B^k \to C^k$ is onto, there is an element $b \in B^k$ such that $j(b) = c$. Then $db \in B^{k+1}$ is in $\ker j$ because

$$\begin{aligned} jdb &= djb \qquad (\text{ by the commutativity of the diagram }) \\ &= dc = 0 \quad (\text{ because } c \text{ is a cocycle }). \end{aligned}$$

By the exactness of the sequence in degree $k+1, \ker j = \operatorname{im} i$. This implies that $db = i(a)$ for some a in A^{k+1}. Once b is chosen, this a is unique because i is injective. The injectivity of i also implies that $da = 0$, since

$$i(da) = d(ia) = ddb = 0 \tag{25.3}$$

Therefore, a is a cocycle and defines a cohomology class $[a]$. We set

$$d^*[c] = [a] \in H^{k+1}(\mathcal{A})$$

In defining $d^*[c]$ we made two choices: a cocycle c to represent the cohomology class $[c] \in H^k(\mathcal{C})$ and then an element $b \in B^k$ that maps to c under j. For d^* to be well defined, one must show that the cohomology class $[a] \in H^{k+1}(\mathcal{A})$ does not depend on these choices.

Exercise 25.5 (Connecting homomorphism).* Show that the connecting homomorphism

$$d^* : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$$

is a well-defined linear map.

The recipe for defining the connecting homomorphism d^* is best remembered as a zig-zag diagram,

$$\begin{tikzcd} a \arrow[r, tail, "i"] & db & \\ & b \arrow[u, "d"'] \arrow[r, two heads, mapsto, "j"'] & c, \end{tikzcd}$$

where $a \longrightarrow db$ means that a maps to db under an injection and $b \mapsto c$ means that b maps to c under a surjection.

12.2.4 25.4 The Zig-zag Lemma

The zig-zag lemma produces a long exact sequence in cohomology from a short exact sequence of cochain complexes. It is most useful when some of the terms in the long exact sequence are known to be zero, for then by exactness, the adjacent maps will be injections, surjections, or even isomorphisms. For example, if the cohomology of one of the three cochain complexes is zero, then the cohomology vector spaces of the other two cochain complexes will be isomorphic.

Theorem 25.6 (The zig-zag lemma). A short exact sequence of cochain complexes

$$0 \to \mathcal{A} \xrightarrow{i} \mathcal{B} \xrightarrow{j} \mathcal{C} \to 0$$

gives rise to a long exact sequence in cohomology:

$$\begin{tikzcd}
 & H^{k+1}(\mathcal{A}) \arrow[r, "i^*"] & \cdots \\
 & H^k(\mathcal{A}) \arrow[r, "i^*"] & H^k(\mathcal{B}) \arrow[r, "j^*"] & H^k(\mathcal{C}) \arrow[dll, "d^*"'] \\
 & & \cdots \arrow[r, "j^*"] & H^{k-1}(\mathcal{C}) \arrow[dll, "d^*"']
\end{tikzcd}$$

where i^* and j^* are the maps in cohomology induced from the cochain maps i and j, and d^* is the connecting homomorphism.

To prove the theorem one needs to check exactness at $H^k(\mathcal{A}), H^k(\mathcal{B})$, and $H^k(\mathcal{C})$ for each k. The proof is a sequence of trivialities involving what is commonly called diagram-chasing. As an example, we prove exactness at $H^k(\mathcal{C})$.

Claim. $\operatorname{im} j^* \subset \ker d^*$.

Proof. Let $[b] \in H^k(\mathcal{B})$. Then

$$d^* j^*[b] = d^*[j(b)].$$

In the recipe above for d^*, we can choose the element in B^k that maps to $j(b)$ to be b. Then $db \in B^{k+1}$. Because b is a cocycle, $db = 0$. Following the zig-zag diagram

$$\begin{tikzcd}
0 \arrow[r, tail, "i"] & db = 0 \\
 & b \arrow[u, mapsto, "d"] \arrow[r, two heads, mapsto, "j"] & j(b),
\end{tikzcd}$$

we see that since $i(0) = 0 = db$, we must have $d^*[j(b)] = [0]$. So $j^*[b] \in \ker d^*$. $\square$

Claim. $\ker d^* \subset \operatorname{im} j^*$.

Proof. Suppose $d^*[c] = [a] = 0$, where $[c] \in H^k(\mathcal{C})$. This means that $a = da'$ for some $a' \in A^k$. The calculation of $d^*[c]$ can be represented by the zig-zag diagram

$$\begin{tikzcd}
a \arrow[r, tail, "i"] & db \\
a' \arrow[u, mapsto, "d"] & b \arrow[u, mapsto, "d"] \arrow[r, two heads, mapsto, "j"] & c,
\end{tikzcd}$$

where b is an element in B^k with $j(b) = c$ and $i(a) = db$. Then $b - i(a')$ is a cocycle in B^k that maps to c under j :

$$\begin{aligned}
d(b - i(a')) &= db - di(a') = db - id(a') = db - ia = 0, \\
j(b - i(a')) &= j(b) - ji(a') = j(b) = c.
\end{aligned}$$

Therefore,

$$j^*[b - i(a')] = [c].$$

So $[c] \in \operatorname{im} j^*$. $\square$

These two claims together imply the exactness of (25.4) at $H^k(\mathcal{C})$. As for the exactness of the cohomology sequence (25.4) at $H^k(\mathcal{A})$ and at $H^k(\mathcal{B})$, we will leave it as an exercise (Problem 25.3).

12.2.5 Problems

25.1. A Three-term Exact Sequence

Prove Proposition 25.1.

25.2. A Four-term Exact Sequence

Prove Proposition 25.2.

25.3 Long Exact Cohomology Sequence

Prove the exactness of the cohomology sequence (25.4) at $H^k(\mathcal{A})$ and $H^k(\mathcal{B})$.

25.4.* the Snake Lemma [1]

Use the zig-zag lemma to prove the following:
The snake lemma. A commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & A^1 \arrow[r] & B^1 \arrow[r] & C^1 \arrow[r] & 0 \\
0 \arrow[r] & A^0 \arrow[r] \arrow[u, "\alpha"] & B^0 \arrow[r] \arrow[u, "\beta"] & C^0 \arrow[r] \arrow[u, "\gamma"] & 0
\end{tikzcd}$$

induces a long exact sequence

$$\begin{tikzcd}
 & \operatorname{coker}\alpha \arrow[r] & \operatorname{coker}\beta \arrow[r] & \operatorname{coker}\gamma \arrow[r] & 0. \\
0 \arrow[r] & \ker\alpha \arrow[r] & \ker\beta \arrow[r] & \ker\gamma \arrow[ull, rounded corners, to path={[pos=0.5] -- ([xshift=2ex]\tikztostart.east) |- ([yshift=-2ex]\tikztotarget.south) -| ([xshift=-2ex]\tikztotarget.west) -- (\tikztotarget)}]
\end{tikzcd}$$

12.3 §26 the Mayer-vietoris Sequence

As the example of the cohomology of the real line $\mathbb{R}^1$ illustrates, calculating the de Rham cohomology of a manifold amounts to solving a canonically given system of differential equations on the manifold and, in case it is not solvable, to finding obstructions to its solvability. This is usually quite difficult to do directly. We introduce in this section one of the most useful tools in the calculation of de Rham cohomology, the Mayer-Vietoris sequence. Another tool, the homotopy axiom, will come in the next section.

12.3.1 26.1 The Mayer-vietoris Sequence

Let $\{U, V\}$ be an open cover of a manifold M, and let $i_U : U \to M, i_U(p) = p$, be the inclusion map. Then the pullback

[1] The snake lemma, also called the serpent lemma, derives its name from the shape of the long exact sequence in it, usually drawn as an S . It may be the only result from homological algebra that has made its way into popular culture. In the 1980 film It's My Turn there is a scene in which the actress Jill Clayburgh, who plays a mathematics professor, explains the proof of the snake lemma.

$$i_U^* : \Omega^k(M) \to \Omega^k(U)$$

is the restriction map that restricts the domain of a k-form on M to U : $i_U^*\omega = \omega|_U$. In fact, there are four inclusion maps that form a commutative diagram:

$$\begin{tikzcd}
 & U \arrow[dr, hook, "i_U"] & \\
U \cap V \arrow[ur, hook, "j_U"] \arrow[dr, hook, "j_V"'] & & M. \\
 & V \arrow[ur, hook, "i_V"'] &
\end{tikzcd}$$

By restricting a k-form from M to U and to V, we get a homomorphism of vector spaces

$$\begin{aligned} i : \Omega^k(M) &\to \Omega^k(U) \oplus \Omega^k(V) \\ \sigma &\mapsto (i_U^*\sigma, i_V^*\sigma) = (\sigma|_U\,, \sigma|_V) \end{aligned}$$

Define the map

$$j : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V)$$

by

$$j(\omega, \tau) = j_V^*\tau - j_U^*\omega = \tau|_{U\cap V} - \omega|_{U\cap V}\,. \tag{26.1}$$

If $U \cap V$ is empty, we define $\Omega^k(U \cap V) = 0$. In this case, j is simply the zero map. We call i the restriction map and j the difference map. Since the direct sum $\Omega^*(U) \oplus \Omega^*(V)$ is the de Rham complex $\Omega^*(U \amalg V)$ of the disjoint union $U \amalg V$, the exterior derivative d on $\Omega^*(U) \oplus \Omega^*(V)$ is given by $d(\omega, \tau) = (d\omega, d\tau)$.

Proposition 26.1. Both the restriction map i and the difference map j commute with the exterior derivative d.

Proof. This is a consequence of the commutativity of d with the pullback (Proposition 19.5). For $\sigma \in \Omega^k(M)$,

$$di\sigma = d\,(i_U^*\sigma, i_V^*\sigma) = (di_U^*\sigma, di_V^*\sigma) = (i_U^*d\sigma, i_V^*d\sigma) = id\sigma.$$

For $(\omega, \tau) \in \Omega^k(U) \oplus \Omega^k(V)$,

$$dj(\omega, \tau) = d\,(j_V^*\tau - j_U^*\omega) = j_V^*d\tau - j_U^*d\omega = jd(\omega, \tau).$$

□

Thus, i and j are cochain maps.

Proposition 26.2. For each integer $k \geq 0$, the sequence

$$0 \to \Omega^k(M) \xrightarrow{i} \Omega^k(U) \oplus \Omega^k(V) \xrightarrow{j} \Omega^k(U \cap V) \to 0 \tag{26.2}$$

is exact.

Proof. Exactness at the first two terms $\Omega^k(M)$ and $\Omega^k(U) \oplus \Omega^k(V)$ is straightforward. We leave it as an exercise (Problem 26.1). We will prove exactness at $\Omega^k(U \cap V)$.

To prove the surjectivity of the difference map

$$j : \Omega^k(U) \oplus \Omega^k(V) \to \Omega^k(U \cap V),$$

it is best to consider first the case of functions on $M = \mathbb{R}^1$. Let f be a C^∞ function on $U \cap V$ as in Figure 26.1. We have to write f as the difference of a C^∞ function on V and a C^∞ function on U.

Fig. 26.1. Writing f as the difference of a C^{∞} function on V and a C^∞ function on U.

Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$. Define $f_V : V \to \mathbb{R}$ by

$$f_V(x) = \begin{cases} \rho_U(x) f(x) & \text{for } x \in U \cap V, \\ 0 & \text{for } x \in V - (U \cap V). \end{cases}$$

Exercise 26.3 (Smooth extension of a function). Prove that f_V is a C^∞ function on V.
The function f_V is called the extension by zero of $\rho_U f$ from $U \cap V$ to V. Similarly, we define f_U to be the extension by zero of $\rho_V f$ from $U \cap V$ to U. Note that to "extend" the domain of f from $U \cap V$ to one of the two open sets, we multiply by the partition function of the other open set. Since

$$j(-f_U, f_V) = f_V|_{U\cap V} + f_U|_{U\cap V} = \rho_U f + \rho_V f = f \quad \text{on } U \cap V$$

j is surjective.
For differential k-forms on a general manifold M, the formula is similar. For $\omega \in \Omega^k(U \cap V)$, define ω_U to be the extension by zero of $\rho_V \omega$ from $U \cap V$ to U, and ω_V to be the extension by zero of $\rho_U \omega$ from $U \cap V$ to V. On $U \cap V$, $(-\omega_U, \omega_V)$ restricts to $(-\rho_V \omega, \rho_U \omega)$. Hence, j maps $(-\omega_U, \omega_V) \in \Omega^k(U) \oplus \Omega^k(V)$ to

$$\rho_V \omega - (-\rho_U \omega) = \omega \in \Omega^k(U \cap V).$$

This shows that j is surjective and the sequence (26.2) is exact at $\Omega^k(U \cap V)$. $\square$
It follows from Proposition 26.2 that the sequence of cochain complexes

$$0 \to \Omega^*(M) \xrightarrow{i} \Omega^*(U) \oplus \Omega^*(V) \xrightarrow{j} \Omega^*(U \cap V) \to 0$$

is short exact. By the zig-zag lemma (Theorem 25.6), this short exact sequence of cochain complexes gives rise to a long exact sequence in cohomology, called the Mayer-Vietoris sequence:

$$\begin{tikzcd}
\arrow[r] & H^{k+1}(M) \arrow[r, "i^*"] & \cdots . \\
\arrow[r] & H^k(M) \arrow[r, "i^*"] & H^k(U) \oplus H^k(V) \arrow[r, "j^*"] & H^k(U \cap V) \arrow[dll, "d^*"'] \\
 & \cdots \arrow[r, "j^*"] & H^{k-1}(U \cap V) \arrow[dll, "d^*"']
\end{tikzcd}$$

In this sequence i^* and j^* are induced from i and j :

$$i^*[\sigma] = [i(\sigma)] = ([\sigma|_U], [\sigma|_V]) \in H^k(U) \oplus H^k(V)$$
$$j^*([\omega], [\tau]) = [j(\omega, \tau)] = [\tau|_{U \cap V} - \omega|_{U \cap V}] \in H^k(U \cap V)$$

By the recipe of Section 25.3, the connecting homomorphism $d^* : H^k(U \cap V) \to H^{k+1}(M)$ is obtained in three steps as in the diagrams below:

$$\begin{tikzcd}
\Omega^{k+1}(M) \arrow[r, "i"] & \Omega^{k+1}(U) \oplus \Omega^{k+1}(V) \\
 & \Omega^k(U) \oplus \Omega^k(V) \arrow[u, "d"] \arrow[r, two heads, "j"] & \Omega^k(U \cap V), \\
 & \alpha \arrow[r, tail, "i", "(3)"'] & (-d\zeta_U, d\zeta_V) \arrow[r, mapsto, two heads, "j"] & 0 \\
 & & (-\zeta_U, \zeta_V) \arrow[u, mapsto, "d", "(2)"'] \arrow[r, mapsto, two heads, "j", "(1)"'] & \zeta . \arrow[u, mapsto, "d"']
\end{tikzcd}$$

(1) Starting with a closed k-form $\zeta \in \Omega^k(U \cap V)$ and using a partition of unity $\{\rho_U, \rho_V\}$ subordinate to $\{U, V\}$, one can extend $\rho_U \zeta$ by zero from $U \cap V$ to a k-form ζ_V on V and extend $\rho_V \zeta$ by zero from $U \cap V$ to a k-form ζ_U on U (see the proof of Proposition 26.2). Then

$$j(-\zeta_U, \zeta_V) = \zeta_V|_{U \cap V} + \zeta_U|_{U \cap V} = (\rho_U + \rho_V)\zeta = \zeta.$$

(2) The commutativity of the square for d and j shows that the pair $(-d\zeta_U, d\zeta_V)$ maps to 0 under j. More formally, since $jd = dj$ and since ζ is a cocycle,

$$j(-d\zeta_U, d\zeta_V) = jd(-\zeta_U, \zeta_V) = dj(-\zeta_U, \zeta_V) = d\zeta = 0.$$

It follows that the $(k+1)$-forms $-d\zeta_U$ on U and $d\zeta_V$ on V agree on $U \cap V$.
(3) Therefore, $-d\zeta_U$ on U and $d\zeta_V$ patch together to give a global ($k+1$)-form α on M. Diagram-chasing shows that α is closed (see (25.3)). By Section 25.3, $d^*[\zeta] = [\alpha] \in H^{k+1}(M)$. Because $\Omega^k(M) = 0$ for $k \leq -1$, the Mayer-Vietoris sequence starts with

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to \cdots .$$

Proposition 26.4. In the Mayer-Vietoris sequence, if U, V, and $U \cap V$ are connected and nonempty, then
(i) M is connected and

$$0 \to H^0(M) \to H^0(U) \oplus H^0(V) \to H^0(U \cap V) \to 0$$

is exact;
(ii) we may start the Mayer-Vietoris sequence with

$$0 \to H^1(M) \xrightarrow{i^*} H^1(U) \oplus H^1(V) \xrightarrow{j^*} H^1(U \cap V) \to \cdots .$$

Proof.
(i) The connectedness of M follows from a lemma in point-set topology (Proposition A.44). It is also a consequence of the Mayer-Vietoris sequence. On a nonempty, connected open set, the de Rham cohomology in dimension 0 is simply the vector space of constant functions (Proposition 24.1). By (26.1), the map

$$j^* : H^0(U) \oplus H^0(V) \to H^0(U \cap V)$$

is given by

$$(u, v) \mapsto v - u, \quad u, v \in \mathbb{R}.$$

This map is clearly surjective. The surjectivity of j^* implies that

$$\operatorname{im} j^* = H^0(U \cap V) = \ker d^*,$$

from which we conclude that $d^* : H^0(U \cap V) \to H^1(M)$ is the zero map. Thus the Mayer-Vietoris sequence starts with

$$0 \to H^0(M) \xrightarrow{i^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{j^*} \mathbb{R} \xrightarrow{d^*} 0. \tag{26.4}$$

This short exact sequence shows that

$$H^0(M) \simeq \operatorname{im} i^* = \ker j^*$$

Since

$$\ker j^* = \{(u, v) \mid v - u = 0\} = \{(u, u) \in \mathbb{R} \oplus \mathbb{R}\} \simeq \mathbb{R}$$

$H^0(M) \simeq \mathbb{R}$, which proves that M is connected.
(ii) From (i) we know that $d^* : H^0(U \cap V) \to H^1(M)$ is the zero map. Thus, in the Mayer-Vietoris sequence, the sequence of two maps

$$H^0(U \cap V) \xrightarrow{d^*} H^1(M) \xrightarrow{i^*} H^1(U) \oplus H^1(V)$$

may be replaced by

$$0 \to H^1(M) \xrightarrow{i^*} H^1(U) \oplus H^1(V)$$

without affecting exactness.

12.3.2 26.2 The Cohomology of the Circle

In Example 24.4 we showed that integration of 1-forms induces an isomorphism of $H^1(S^1)$ with $\mathbb{R}$. In this section we apply the Mayer-Vietoris sequence to give an alternative computation of the cohomology of the circle.

Cover the circle with two open arcs U and V as in Figure 26.2. The intersection $U \cap V$ is the disjoint union of two open arcs, which we call A and B. Since an open arc is diffeomorphic to an open interval and hence to the real line $\mathbb{R}^1$, the cohomology rings of U and V are isomorphic to that of $\mathbb{R}^1$, and the cohomology ring of $U \cap V$ to that of the disjoint union $\mathbb{R}^1 \amalg \mathbb{R}^1$. They fit into the Mayer-Vietoris sequence, which we arrange in tabular form:

		S^1	$U \amalg V$	$U \cap V$	
H^2	$\to$	0	$\to$	0	$\to$
H^1	$\xrightarrow{d^*} H^1(S^1)$	$\to$	0	$\to$	0
H^0	$0 \to$	$\mathbb{R}$	$\xrightarrow{i^*} \mathbb{R} \oplus \mathbb{R}$	$\xrightarrow{j^*} \mathbb{R} \oplus \mathbb{R}$	

Fig. 26.2. An open cover of the circle.

From the exact sequence

$$0 \to \mathbb{R} \xrightarrow{i^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{j^*} \mathbb{R} \oplus \mathbb{R} \xrightarrow{d^*} H^1\left(S^1\right) \to 0$$

and Problem 26.2, we conclude that $\dim H^1(S^1) = 1$. Hence, the cohomology of the circle is given by

$$H^k\left(S^1\right) = \begin{cases} \mathbb{R} & \text{for } k = 0, 1, \\ 0 & \text{otherwise .} \end{cases}$$

By analyzing the maps in the Mayer-Vietoris sequence, it is possible to write down an explicit generator for $H^1(S^1)$. First, according to Proposition 24.1, an element of $H^0(U) \oplus H^0(V)$ is an ordered pair $(u, v) \in \mathbb{R} \oplus \mathbb{R}$, representing a constant function u on U and a constant function v on V. An element of $H^0(U \cap V) = H^0(A) \oplus H^0(B)$ is an ordered pair $(a, b) \in \mathbb{R} \oplus \mathbb{R}$, representing a constant function a on A and a constant function b on B. The restriction map $j_U^* : Z^0(U) \to Z^0(U \cap V)$ is the restriction of a constant function on U to the two connected components A and B of the intersection $U \cap V$:

$$j_U^*(u) = u|_{U\cap V} = (u,u) \in Z^0(A) \oplus Z^0(B).$$

Similarly,

$$j_V^*(v) = v|_{U\cap V} = (v,v) \in Z^0(A) \oplus Z^0(B).$$

By (26.1), $j : Z^0(U) \oplus Z^0(V) \to Z^0(U \cap V)$ is given by

$$j(u,v) = v|_{U\cap V} - u|_{U\cap V} = (v,v) - (u,u) = (v-u, v-u).$$

Hence, in the Mayer-Vietoris sequence, the induced map $j^* : H^0(U) \oplus H^0(V) \to H^0(U \cap V)$ is given by

$$j^*(u,v) = (v-u, v-u).$$

The image of j^* is therefore the diagonal Δ in $\mathbb{R}^2$:

$$\Delta = \left\{(a,a) \in \mathbb{R}^2\right\}.$$

Since $H^1(S^1)$ is isomorphic to $\mathbb{R}$, a generator of $H^1(S^1)$ is simply a nonzero element. Moreover, because $d^* : H^0(U \cap V) \to H^1(S^1)$ is surjective and

$$\ker d^* = \operatorname{im} j^* = \Delta,$$

such a nonzero element in $H^1(S^1)$ is the image under d^* of an element $(a,b) \in H^0(U \cap V) \simeq \mathbb{R}^2$ with $a \neq b$.

Fig. 26.3. A generator of $H^{1} of the circle.}

So we may start with $(a,b) = (1,0) \in H^0(U \cap V)$. This corresponds to a function f with value 1 on A and 0 on B. Let $\{\rho_U, \rho_V\}$ be a partition of unity subordinate to the open cover $\{U, V\}$, and let f_U, f_V be the extensions by zero of $\rho_V f, \rho_U f$ from $U \cap V$ to U and to V, respectively. By the proof of Proposition 26.2, $j(-f_U, f_V) = f$ on $U \cap V$. From Section 25.3, $d^*(1,0)$ is represented by a 1-form on S^1 whose restriction to U is $-df_U$ and whose restriction to V is df_V. Now f_V is the function on V that is ρ_U on A and 0 on $V - A$, so df_V is a 1 -form on V whose support is contained entirely in A. A similar analysis shows that $-df_U$ restricts to the same 1 -form on A, because $\rho_U + \rho_V = 1$. The extension of either df_V or $-df_U$ by zero to a 1 -form on S^1 represents a generator of $H^1(S^1)$. It is a bump 1-form on S^1 supported in A (Figure 26.3).

The explicit description of the map j^* gives another way to compute $H^1(S^1)$, for by the exactness of the Mayer-Vietoris sequence and the first isomorphism theorem of linear algebra, there is a sequence of vector-space isomorphisms

$$H^1\left(S^1\right) = \operatorname{im} d^* \simeq \frac{\mathbb{R}\oplus\mathbb{R}}{\ker d^*} = \frac{\mathbb{R}\oplus\mathbb{R}}{\operatorname{im} j^*} \simeq \frac{\mathbb{R}\oplus\mathbb{R}}{\operatorname{im} j^*} \simeq \frac{\mathbb{R}^2}{\mathbb{R}} \simeq \mathbb{R}$$

12.3.3 26.3 The Euler Characteristic

If the cohomology vector space $H^k(M)$ of an n-manifold M is finite-dimensional for every k, we define its Euler characteristic to be the alternating sum

$$\chi(M) = \sum_{k=0}^{n} (-1)^k \dim H^k(M)$$

As a corollary of the Mayer-Vietoris sequence, the Euler characteristic of $U \cup V$ is computable from those of U, V, and $U \cap V$, as follows.

Exercise 26.5 (Euler characteristics in terms of an open cover). Suppose a manifold M has an open cover $\{U, V\}$ and the spaces M, U, V, and $U \cap V$ all have finite-dimensional cohomology. By applying Problem 26.2 to the Mayer-Vietoris sequence, prove that

$$\chi(M) - (\chi(U) + \chi(V)) + \chi(U \cap V) = 0.$$

12.3.4 Problems

26.1. Short Exact Mayer-vietoris Sequence

Prove the exactness of (26.2) at $\Omega^k(M)$ and at $\Omega^k(U) \oplus \Omega^k(V)$.

26.2. Alternating sum of Dimensions

Let

$$0 \to A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2 \xrightarrow{d_2} \cdots \to A^m \to 0$$

be an exact sequence of finite-dimensional vector spaces. Show that

$$\sum_{k=0}^{m} (-1)^k \dim A^k = 0$$

(Hint: By the rank-nullity theorem from linear algebra,

$$\dim A^k = \dim\ker d_k + \dim\operatorname{im} d_k.$$

Take the alternating sum of these equations over k and use the fact that $\dim\ker d_k = \dim\operatorname{im} d_{k-1}$ to simplify it.)

12.4 §27 Homotopy Invariance

The homotopy axiom is a powerful tool for computing de Rham cohomology. While homotopy is normally defined in the continuous category, since we are primarily interested in smooth manifolds and smooth maps, our notion of homotopy will be smooth homotopy. It differs from the usual homotopy in topology only in that all the maps are assumed to be smooth. In this section we define smooth homotopy, state the homotopy axiom for de Rham cohomology, and compute a few examples. We postpone the proof of the homotopy axiom to Section 29.

12.4.1 27.1 Smooth Homotopy

Let M and N be manifolds. Two C^∞ maps $f, g : M \to N$ are (smoothly) homotopic if there is a C^∞ map

$$F : M \times \mathbb{R} \to N$$

such that

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x)$$

for all $x \in M$; the map F is called a homotopy from f to g. A homotopy F from f to g can be viewed as a smoothly varying family of maps $\{f_t : M \to N \mid t \in \mathbb{R}\}$, where

$$f_t(x) = F(x, t), \quad x \in M,$$

such that $f_0 = f$ and $f_1 = g$. We can think of the parameter t as time and a homotopy as an evolution through time of the map $f_0 : M \to N$. If f and g are homotopic, we write

$$f \sim g.$$

Since any open interval is diffeomorphic to $\mathbb{R}$ (Problem 1.3), in the definition of homotopy we could have used any open interval containing 0 and 1 , instead of $\mathbb{R}$. The advantage of an open interval over the closed interval $[0, 1]$ is that an open interval is a manifold without boundary.

Fig. 27.1. Straight-line homotopies.

Example 27.1 (Straight-line homotopy). Let f and g be C^∞ maps from a manifold M to $\mathbb{R}^n$. Define $F : M \times \mathbb{R} \to \mathbb{R}^n$ by

$$F(x, t) = f(x) + t(g(x) - f(x)) = (1 - t)f(x) + tg(x).$$

Then F is a homotopy from f to g, called the straight-line homotopy from f to g (Figure 27.1).

Exercise 27.2 (Homotopy). Let M and N be manifolds. Prove that homotopy is an equivalence relation on the set of all C^∞ maps from M to N.

12.4.2 27.2 Homotopy Type

As usual, $\mathbb{1}_M$ denotes the identity map on a manifold M.
Definition 27.3. A map $f : M \to N$ is a homotopy equivalence if it has a homotopy inverse, i.e., a map $g : N \to M$ such that $g \circ f$ is homotopic to the identity $\mathbb{1}_M$ on M and $f \circ g$ is homotopic to the identity $\mathbb{1}_N$ on N :

$$g \circ f \sim \mathbb{1}_M \quad \text{and} \quad f \circ g \sim \mathbb{1}_N.$$

In this case we say that M is homotopy equivalent to N, or that M and N have the same homotopy type.

Example. A diffeomorphism is a homotopy equivalence.

Fig. 27.2 The punctured plane retracts to the unit circle.

Example 27.4 (Homotopy type of the punctured plane). Let $i : S^1 \to \mathbb{R}^2 - \{\mathbf{0}\}$ be the inclusion map and let $r : \mathbb{R}^2 - \{\mathbf{0}\} \to S^1$ be the map

$$r(x) = \frac{x}{\|x\|}$$

Then $r \circ i$ is the identity map on S^1.
We claim that

$$i \circ r : \mathbb{R}^2 - \{\mathbf{0}\} \to \mathbb{R}^2 - \{\mathbf{0}\}$$

is homotopic to the identity map. Note that in the definition of a smooth homotopy $F(x,t)$, the domain of t is required to be the entire real line. The straight-line homotopy

$$H(x,t) = (1-t)x + t\frac{x}{\|x\|}, \quad (x,t) \in \left(\mathbb{R}^2 - \{\mathbf{0}\}\right) \times \mathbb{R}$$

will be fine if t is restricted to the closed interval $[0,1]$. However, if t is allowed to be any real number, then $H(x,t)$ may be equal to 0 . Indeed, for $t = \|x\|/(\|x\| - 1)$, $H(x,t) = 0$, and so H does not map into $\mathbb{R}^2 - \{\mathbf{0}\}$. To correct this problem, we modify the straight-line

homotopy so that for all t the modified map $F(x,t)$ is always a positive multiple of x and hence never zero. Set

$$F(x,t) = (1-t)^2 x + t^2 \frac{x}{\|x\|} = \left((1-t)^2 + \frac{t^2}{\|x\|}\right) x.$$

Then

$$\begin{aligned} F(x,t) = 0 \iff & \quad (1-t)^2 = 0 \text{ and } \frac{t^2}{\|x\|} = 0 \\ \iff & \quad t = 1 = 0, \text{ a contradiction.} \end{aligned}$$

Therefore, $F : (\mathbb{R}^2 - \{\mathbf{0}\}) \times \mathbb{R} \to \mathbb{R}^2 - \{\mathbf{0}\}$ provides a homotopy between the identity map on $\mathbb{R}^2 - \{\mathbf{0}\}$ and $i \circ r$ (Figure 27.2). It follows that r and i are homotopy inverse to each other, and $\mathbb{R}^2 - \{\mathbf{0}\}$ and S^1 have the same homotopy type.

Definition 27.5. A manifold is contractible if it has the homotopy type of a point.
In this definition, by "the homotopy type of a point" we mean the homotopy type of a set $\{p\}$ whose single element is a point. Such a set is called a singleton set or just a singleton.

Example 27.6 (The Euclidean space $\mathbb{R}^n$ is contractible). Let p be a point in $\mathbb{R}^n$, $i : \{p\} \to \mathbb{R}^n$ the inclusion map, and $r : \mathbb{R}^n \to \{p\}$ the constant map. Then $r \circ i = \mathbb{1}_{\{p\}}$, the identity map on $\{p\}$. The straight-line homotopy provides a homotopy between the constant map $i \circ r : \mathbb{R}^n \to \mathbb{R}^n$ and the identity map on $\mathbb{R}^n$:

$$F(x,t) = (1-t)x + tr(x) = (1-t)x + tp.$$

Hence, the Euclidean space $\mathbb{R}^n$ and the set $\{p\}$ have the same homotopy type.

12.4.3 27.3 Deformation Retractions

Let S be a submanifold of a manifold M, with $i : S \to M$ the inclusion map.
Definition 27.7. A retraction from M to S is a map $r : M \to S$ that restricts to the identity map on S; in other words, $r \circ i = \mathbb{1}_S$. If there is a retraction from M to S, we say that S is a retract of M.

Definition 27.8. A deformation retraction from M to S is a map $F : M \times \mathbb{R} \to M$ such that for all $x \in M$,
(i) $F(x,0) = x$,
(ii) there is a retraction $r : M \to S$ such that $F(x,1) = r(x)$,
(iii) for all $s \in S$ and $t \in \mathbb{R}, F(s,t) = s$.

If there is a deformation retraction from M to S, we say that S is a deformation retract of M.

Setting $f_t(x) = F(x,t)$, we can think of a deformation retraction $F : M \times \mathbb{R} \to M$ as a family of maps $f_t : M \to M$ such that
(i) f_0 is the identity map on M,
(ii) $f_1(x) = r(x)$ for some retraction $r : M \to S$,
(iii) for every t the map $f_t : M \to M$ restricts to the identity on S.

We may rephrase condition (ii) in the definition as follows: there is a retraction $r : M \to S$ such that $f_1 = i \circ r$. Thus, a deformation retraction is a homotopy between the identity map $\mathbb{1}_M$ and $i \circ r$ for a retraction $r : M \to S$ such that this homotopy leaves S fixed for all time t.

Example. Any point p in a manifold M is a retract of M; simply take a retraction to be the constant map $r : M \to \{p\}$.

Example. The map F in Example 27.4 is a deformation retraction from the punctured plane $\mathbb{R}^2 - \{0\}$ to the unit circle S^1. The map F in Example 27.6 is a deformation retraction from $\mathbb{R}^n$ to a singleton $\{p\}$.

Generalizing Example 27.4, we prove the following theorem.

Proposition 27.9. If $S \subset M$ is a deformation retract of M, then S and M have the same homotopy type.

Proof. Let $F : M \times \mathbb{R} \to M$ be a deformation retraction and let $r(x) = f_1(x) = F(x, 1)$ be the retraction. Because r is a retraction, the composite

$$S \xrightarrow{i} M \xrightarrow{r} S, \quad r \circ i = \mathbb{1}_S,$$

is the identity map on S. By the definition of a deformation retraction, the composite

$$M \xrightarrow{r} S \xrightarrow{i} M$$

is f_1 and the deformation retraction provides a homotopy

$$f_1 = i \circ r \sim f_0 = \mathbb{1}_M$$

Therefore, $r : M \to S$ is a homotopy equivalence, with homotopy inverse $i : S \to M$.

12.4.4 27.4 The Homotopy Axiom for de Rham Cohomology

We state here the homotopy axiom and derive a few consequences. The proof will be given in Section 29.

Theorem 27.10 (Homotopy axiom for de Rham cohomology). Homotopic maps $f_0, f_1 : M \to N$ induce the same map $f_0^* = f_1^* : H^*(N) \to H^*(M)$ in cohomology.

Corollary 27.11. If $f : M \to N$ is a homotopy equivalence, then the induced map in cohomology

$$f^* : H^*(N) \to H^*(M)$$

is an isomorphism.

Proof (of Corollary). Let $g : N \to M$ be a homotopy inverse to f. Then

$$g \circ f \sim \mathbb{1}_M, \quad f \circ g \sim \mathbb{1}_N$$

By the homotopy axiom,

$$(g \circ f)^* = \mathbb{1}_{H^*(M)}, \quad (f \circ g)^* = \mathbb{1}_{H^*(N)}$$

By functoriality,

$$f^* \circ g^* = \mathbb{1}_{H^*(M)}, \quad g^* \circ f^* = \mathbb{1}_{H^*(N)}$$

Therefore, f^* is an isomorphism in cohomology.

Corollary 27.12. Suppose S is a submanifold of a manifold M and F is a deformation retraction from M to S. Let $r : M \to S$ be the retraction $r(x) = F(x, 1)$. Then r induces an isomorphism in cohomology

$$r^* : H^*(S) \xrightarrow{\sim} H^*(M).$$

Proof. The proof of Proposition 27.9 shows that a retraction $r : M \to S$ is a homotopy equivalence. Apply Corollary 27.11.

Corollary 27.13 (Poincaré lemma). Since $\mathbb{R}^n$ has the homotopy type of a point, the cohomology of $\mathbb{R}^n$ is

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & \text{for } k = 0, \\ 0 & \text{for } k > 0. \end{cases}$$

More generally, any contractible manifold will have the same cohomology as a point.

Example 27.14 (Cohomology of a punctured plane). For any $p \in \mathbb{R}^2$, the translation $x \mapsto x - p$ is a diffeomorphism of $\mathbb{R}^2 - \{p\}$ with $\mathbb{R}^2 - \{0\}$. Because the punctured plane $\mathbb{R}^2 - \{0\}$ and the circle S^1 have the same homotopy type (Example 27.4), they have isomorphic cohomology. Hence, $H^k(\mathbb{R}^2 - \{p\}) \simeq H^k(S^1)$ for all $k \geq 0$.

Example. The central circle of an open Möbius band M is a deformation retract of M (Figure 27.3). Thus, the open Möbius band has the homotopy type of a circle. By the homotopy axiom,

$$H^k(M) = H^k\left(S^1\right) = \begin{cases} \mathbb{R} & \text{for } k = 0, 1, \\ 0 & \text{for } k > 1. \end{cases}$$

Fig. 27.3 The Möbius band deformation retracts to its central circle.

12.4.5 Problems

27.1. Homotopy Equivalence

Let M, N, and P be manifolds. Prove that if M and N are homotopy equivalent and N and P are homotopy equivalent, then M and P are homotopy equivalent.

27.2. Contractibility and Path-connectedness

Show that a contractible manifold is path-connected.

27.3. Deformation Retraction from a Cylinder to a Circle

Show that the circle $S^1 \times \{0\}$ is a deformation retract of the cylinder $S^1 \times \mathbb{R}$.

12.5 §28 Computation of de Rham Cohomology

With the tools developed so far, we can compute the cohomology of many manifolds. This section is a compendium of some examples.

12.5.1 28.1 Cohomology Vector Space of a Torus

Cover a torus M with two open subsets U and V as shown in Figure 28.1.

Fig. 28.1. An open cover ${U, V\
$ of a torus.}

Both U and V are diffeomorphic to a cylinder and therefore have the homotopy type of a circle (Problem 27.3). Similarly, the intersection $U \cap V$ is the disjoint union of two cylinders A and B and has the homotopy type of a disjoint union of two circles. Our knowledge of the cohomology of a circle allows us to fill in many terms in the Mayer-Vietoris sequence:

$$\begin{array}{c|cccccc}
 & & M & & U \amalg V & & U \cap V \\
\hline
H^2 & \xrightarrow{d_1^*} & H^2(M) & \to & 0 & & \\
H^1 & \xrightarrow{d_0^*} & H^1(M) & \xrightarrow{i^*} & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{\beta} & \mathbb{R} \oplus \mathbb{R} \\
H^0 & 0 \to & \mathbb{R} & \to & \mathbb{R} \oplus \mathbb{R} & \xrightarrow{\alpha} & \mathbb{R} \oplus \mathbb{R}
\end{array}$$

Let $j_U : U \cap V \to U$ and $j_V : U \cap V \to V$ be the inclusion maps. Recall that H^0 of a connected manifold is the vector space of constant functions on the manifold (Proposition 24.1). If $a \in H^0(U)$ is the constant function with value a on U, then $j_U^* a = a|_{U \cap V} \in H^0(U \cap V)$ is the constant function with the value a on each component of $U \cap V$, that is,

$$j_U^* a = (a, a).$$

Therefore, for $(a, b) \in H^0(U) \oplus H^0(V)$,

$$\alpha(a, b) = b|_{U \cap V} - a|_{U \cap V} = (b, b) - (a, a) = (b - a, b - a).$$

Similarly, let us now describe the map

$$\beta : H^1(U) \oplus H^1(V) \to H^1(U \cap V) = H^1(A) \oplus H^1(B).$$

Since A is a deformation retract of U, the restriction $H^*(U) \to H^*(A)$ is an isomorphism, so if ω_U generates $H^1(U)$, then $j_U^* \omega_U$ is a generator of H^1 on A and on B. Identifying $H^1(U \cap V)$ with $\mathbb{R} \oplus \mathbb{R}$, we write $j_U^* \omega_U = (1, 1)$. Let ω_V be a generator of $H^1(V)$. The pair of real numbers

$$(a, b) \in H^1(U) \oplus H^1(V) \simeq \mathbb{R} \oplus \mathbb{R}$$

stands for $(a\omega_U, b\omega_V)$. Then

$$\beta(a, b) = j_V^* (b\omega_V) - j_U^* (a\omega_U) = (b, b) - (a, a) = (b - a, b - a).$$

By the exactness of the Mayer-Vietoris sequence,

$$\begin{aligned} H^2(M) &= \operatorname{im} d_1^* && (\text{ because } H^2(U) \oplus H^2(V) = 0) \\ &\simeq H^1(U \cap V)/\ker d_1^* && (\text{ by the first isomorphism theorem }) \\ &\simeq (\mathbb{R} \oplus \mathbb{R})/\operatorname{im} \beta \\ &\simeq (\mathbb{R} \oplus \mathbb{R})/\mathbb{R} \simeq \mathbb{R}. \end{aligned}$$

Applying Problem 26.2 to the Mayer-Vietoris sequence (28.1), we get

$$1 - 2 + 2 - \dim H^1(M) + 2 - 2 + \dim H^2(M) = 0.$$

Since $\dim H^2(M) = 1$, this gives $\dim H^1(M) = 2$.
As a check, we can also compute $H^1(M)$ from the Mayer-Vietoris sequence using our knowledge of the maps α and β :

$$\begin{aligned} H^1(M) &\simeq \ker i^* \oplus \operatorname{im} i^* \\ &\simeq \operatorname{im} d_0^* \oplus \ker \beta && (\text{ by the first isomorphism theorem }) \\ &\simeq (H^0(U \cap V)/\ker d_0^*) \oplus \ker \beta && (\text{ first isomorphism theorem for } d_0^*) \\ &\simeq ((\mathbb{R} \oplus \mathbb{R})/\operatorname{im} \alpha) \oplus \mathbb{R} \\ &\simeq \mathbb{R} \oplus \mathbb{R}. \end{aligned}$$

12.5.2 28.2 The Cohomology Ring of a Torus

A torus is the quotient of $\mathbb{R}^2$ by the integer lattice $\Lambda = \mathbb{Z}^2$. The quotient map

$$\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Lambda$$

induces a pullback map on differential forms,

$$\pi^* : \Omega^* \left(\mathbb{R}^2/\Lambda\right) \to \Omega^* \left(\mathbb{R}^2\right).$$

Since $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Lambda$ is a local diffeomorphism, its differential $\pi_* : T_q\left(\mathbb{R}^2\right) \to T_{\pi(q)}\left(\mathbb{R}^2/\Lambda\right)$ is an isomorphism at each point $q \in \mathbb{R}^2$. In particular, π is a submersion. By Problem 18.8, $\pi^* : \Omega^*\left(\mathbb{R}^2/\Lambda\right) \to \Omega^*\left(\mathbb{R}^2\right)$ is an injection.

For $\lambda \in \Lambda$, define $\ell_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ to be translation by λ,

$$\ell_\lambda(q) = q + \lambda, \quad q \in \mathbb{R}^2.$$

A differential form $\bar{\omega}$ on $\mathbb{R}^2$ is said to be invariant under translation by $\lambda \in \Lambda$ if $\ell_\lambda^* \bar{\omega} = \bar{\omega}$. The following proposition generalizes the description of differential forms on a circle given in Proposition 18.12, where Λ was the lattice $2\pi\mathbb{Z}$.

Proposition 28.1. The image of the injection $\pi^* : \Omega^*\left(\mathbb{R}^2/\Lambda\right) \to \Omega^*\left(\mathbb{R}^2\right)$ is the subspace of differential forms on $\mathbb{R}^2$ invariant under translations by elements of Λ.

Proof. For all $q \in \mathbb{R}^2$,

$$\left(\pi \circ \ell_\lambda\right)(q) = \pi(q + \lambda) = \pi(q)$$

Hence, $\pi \circ \ell_\lambda = \pi$. By the functoriality of the pullback,

$$\pi^* = \ell_\lambda^* \circ \pi^*$$

Thus, for any $\omega \in \Omega^k\left(\mathbb{R}^2/\Lambda\right), \pi^*\omega = \ell_\lambda^* \pi^* \omega$. This proves that $\pi^*\omega$ is invariant under all translations $\ell_\lambda, \lambda \in \Lambda$.

Conversely, suppose $\bar{\omega} \in \Omega^k(\mathbb{R}^2)$ is invariant under translations ℓ_λ for all $\lambda \in \Lambda$. For $p \in \mathbb{R}^2/\Lambda$ and $v_1, \ldots, v_k \in T_p(\mathbb{R}^2/\Lambda)$, define

$$\omega_p(v_1, \ldots, v_k) = \bar{\omega}_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k) \tag{28.2}$$

for any $\bar{p} \in \pi^{-1}(p)$ and $\bar{v}_1, \ldots, \bar{v}_k \in T_{\bar{p}}\mathbb{R}^2$ such that $\pi_* \bar{v}_i = v_i$. Note that once $\bar{p}$ is chosen, $\bar{v}_1, \ldots, \bar{v}_k$ are unique, since $\pi_* : T_{\bar{p}}(\mathbb{R}^2) \to T_p(\mathbb{R}^2/\Lambda)$ is an isomorphism. For ω to be well defined, we need to show that it is independent of the choice of $\bar{p}$. Now any other point in $\pi^{-1}(p)$ may be written as $\bar{p} + \lambda$ for some $\lambda \in \Lambda$. By invariance,

$$\bar{\omega}_{\bar{p}} = (\ell_\lambda^* \bar{\omega})_{\bar{p}} = \ell_\lambda^*(\bar{\omega}_{\bar{p}+\lambda})$$

So

$$\bar{\omega}_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k) = \ell_\lambda^*(\bar{\omega}_{\bar{p}+\lambda})(\bar{v}_1, \ldots, \bar{v}_k) = \bar{\omega}_{\bar{p}+\lambda}(\ell_{\lambda*}\bar{v}_1, \ldots, \ell_{\lambda*}\bar{v}_k) \tag{28.3}$$

Since $\pi \circ \ell_\lambda = \pi$, we have $\pi_*(\ell_{\lambda*}\bar{v}_i) = \pi_*\bar{v}_i = v_i$. Thus, (28.3) shows that ω_p is independent of the choice of $\bar{p}$, and $\omega \in \Omega^k(\mathbb{R}^2/\Lambda)$ is well defined. Moreover, by (28.2), for any $\bar{p} \in \mathbb{R}^2$ and $\bar{v}_1, \ldots, \bar{v}_k \in T_{\bar{p}}(\mathbb{R}^2)$,

$$\bar{\omega}_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k) = \omega_{\pi(\bar{p})}(\pi_*\bar{v}_1, \ldots, \pi_*\bar{v}_k) = (\pi^*\omega)_{\bar{p}}(\bar{v}_1, \ldots, \bar{v}_k)$$

Hence, $\bar{\omega} = \pi^*\omega$.

Let x, y be the standard coordinates on $\mathbb{R}^2$. Since for any $\lambda \in \Lambda$,

$$\ell_\lambda^*(dx) = d(\ell_\lambda^* x) = d(x + \lambda) = dx,$$

by Proposition 28.1 the 1-form dx on $\mathbb{R}^2$ is π^* of a 1-form α on the torus $\mathbb{R}^2/\Lambda$. Similarly, dy is π^* of a 1 -form β on the torus.

Note that

$$\pi^*(d\alpha) = d(\pi^*\alpha) = d(dx) = 0.$$

Since $\pi^* : \Omega^*(\mathbb{R}^2/\mathbb{Z}^2) \to \Omega^*(\mathbb{R}^2)$ is injective, $d\alpha = 0$. Similarly, $d\beta = 0$. Thus, both α and β are closed 1 -forms on the torus.

Proposition 28.2. Let M be the torus $\mathbb{R}^2/\mathbb{Z}^2$. A basis for the cohomology vector space $H^*(M)$ is represented by the forms $1, \alpha, \beta, \alpha \wedge \beta$.

Proof. Let I be the closed interval $[0,1]$, and $i : I^2 \hookrightarrow \mathbb{R}^2$ the inclusion map of the closed square I^2 into $\mathbb{R}^2$. The composite map $F = \pi \circ i : I^2 \hookrightarrow \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$ represents the torus $M = \mathbb{R}^2/\mathbb{Z}^2$ as a parametrized set. Then $F^*\alpha = i^*(\pi^*\alpha) = i^*dx$, the restriction of dx to the square I^2. Similarly, $F^*\beta = i^*dy$.

As an integral over a parametrized set,

$$\int_M \alpha \wedge \beta = \int_{F(I^2)} \alpha \wedge \beta = \int_{I^2} F^*(\alpha \wedge \beta) = \int_{I^2} dx \wedge dy = \int_0^1 \int_0^1 dxdy = 1$$

Thus, the closed 2-form $\alpha \wedge \beta$ represents a nonzero cohomology class on M. Since $H^2(M) = \mathbb{R}$ by the computation of Subsection 28.1, the cohomology class $[\alpha \wedge \beta]$ is a basis for $H^2(M)$.

Next we show that the cohomology classes of the closed 1-forms α, β on M constitute a basis for $H^1(M)$. Let $i_1, i_2 : I \to \mathbb{R}^2$ be given by $i_1(t) = (t, 0), i_2(t) = (0, t)$. Define two closed curves C_1, C_2 in $M = \mathbb{R}^2/\mathbb{Z}^2$ as the images of the maps (Figure 28.2)

$$\begin{aligned} &c_k : I \xrightarrow{i_k} \mathbb{R}^2 \xrightarrow{\pi} M = \mathbb{R}^2/\mathbb{Z}^2, \quad k = 1, 2, \\ &c_1(t) = [(t, 0)], \quad c_2(t) = [(0, t)]. \end{aligned}$$

Each curve C_i is a smooth manifold and a parametrized set with parametrization c_i.

Fig. 28.2. Two closed curves on a torus.

Moreover,

$$c_1^*\alpha = (\pi \circ i_1)^* \alpha = i_1^*\pi^*\alpha = i_1^* dx = di_1^* x = dt,$$
$$c_1^*\beta = (\pi \circ i_1)^* \beta = i_1^*\pi^*\beta = i_1^* dy = di_1^* y = 0.$$

Similarly, $c_2^*\alpha = 0$ and $c_2^*\beta = dt$. Therefore,

$$\int_{C_1} \alpha = \int_{c_1(I)} \alpha = \int_I c_1^*\alpha = \int_0^1 dt = 1$$

and

$$\int_{C_1} \beta = \int_{c_1(I)} \beta = \int_I c_1^*\beta = \int_0^1 0 = 0.$$

In the same way, $\int_{C_2} \alpha = 0$ and $\int_{C_2} \beta = 1$.
Because $\int_{C_1} \alpha \neq 0$ and $\int_{C_2} \beta \neq 0$, neither α nor β is exact on M. Furthermore, the cohomology classes $[\alpha]$ and $[\beta]$ are linearly independent, for if $[\alpha]$ were a multiple of $[\beta]$, then $\int_{C_1} \alpha$ would have to be a nonzero multiple of $\int_{C_1} \beta = 0$. By Subsection 28.1, $H^1(M)$ is two-dimensional. Hence, $[\alpha], [\beta]$ is a basis for $H^1(M)$.

In degree $0, H^0(M)$ has basis $[1]$, as is true for any connected manifold M.
The ring structure of $H^*(M)$ is clear from this proposition. Abstractly it is the algebra

$$\wedge(a, b) := \mathbb{R}[a, b]/\left(a^2, b^2, ab + ba\right), \quad \deg a = 1, \deg b = 1,$$

called the exterior algebra on two generators a and b of degree 1 .

12.5.3 28.3 The Cohomology of a Surface of Genus g

Using the Mayer-Vietoris sequence to compute the cohomology of a manifold often leads to ambiguities, because there may be several unknown terms in the sequence. We can resolve these ambiguities if we can describe explicitly the maps occurring in the sequence. Here is an example of how this might be done.

Lemma 28.3. Suppose p is a point in a compact oriented surface M without boundary, and $i : C \to M - \{p\}$ is the inclusion of a small circle around the puncture (Figure 28.3). Then the restriction map

$$i^* : H^1(M - \{p\}) \to H^1(C)$$

is the zero map.

Proof. An element $[\omega] \in H^1(M - \{p\})$ is represented by a closed 1-form ω on $M - \{p\}$. Because the linear isomorphism $H^1(C) \simeq H^1(S^1) \simeq \mathbb{R}$ is given by integration over C, to identify $i^*[\omega]$ in $H^1(C)$, it suffices to compute the integral $\int_C i^*\omega$.

Fig. 28.3. Punctured surface.

If D is the open disk in M bounded by the curve C, then $M - D$ is a compact oriented surface with boundary C. By Stokes's theorem,

$$\int_C i^*\omega = \int_{\partial(M-D)} i^*\omega = \int_{M-D} d\omega = 0,$$

because $d\omega = 0$. Hence, $i^* : H^1(M - \{p\}) \to H^1(C)$ is the zero map. $\square$

Proposition 28.4. Let M be a torus, p a point in M, and A the punctured torus $M - \{p\}$. The cohomology of A is

$$H^k(A) = \begin{cases} \mathbb{R} & \text{for } k = 0, \\ \mathbb{R}^2 & \text{for } k = 1, \\ 0 & \text{for } k > 1. \end{cases}$$

Proof. Cover M with two open sets, A and a disk U containing p. Since A, U, and $A \cap U$ are all connected, we may start the Mayer-Vietoris sequence with the $H^1(M)$ term (Proposition 26.4(ii)). With $H^*(M)$ known from Section 28.1, the Mayer-Vietoris sequence becomes

				M	$U \amalg A$	$U \cap A \sim S^1$
H^2			$\xrightarrow{d_1^*}$ $\mathbb{R}$	$\to H^2(A) \to$	0	
H^1	$0 \to \mathbb{R} \oplus \mathbb{R}$	$\xrightarrow{\beta} H^1(A)$	$\xrightarrow{\alpha}$	$H^1(S^1)$		

Because $H^1(U) = 0$, the map $\alpha : H^1(A) \to H^1(S^1)$ is simply the restriction map i^*. By Lemma 28.3, $\alpha = i^* = 0$. Hence,

$$H^1(A) = \ker \alpha = \operatorname{im} \beta \simeq H^1(M) \simeq \mathbb{R} \oplus \mathbb{R}$$

and there is an exact sequence of linear maps

$$0 \to H^1\left(S^1\right) \xrightarrow{d_1^*} \mathbb{R} \to H^2(A) \to 0.$$

Since $H^1(S^1) \simeq \mathbb{R}$, it follows that $H^2(A) = 0$. $\square$

Proposition 28.5. The cohomology of a compact orientable surface Σ_2 of genus 2 is

$$H^k(\Sigma_2) = \begin{cases} \mathbb{R} & \text{for } k = 0, 2, \\ \mathbb{R}^4 & \text{for } k = 1, \\ 0 & \text{for } k > 2. \end{cases}$$

Fig. 28.4. An open cover ${U, V\
$ of a surface of genus 2 .}

Proof. Cover Σ_2 with two open sets U and V as in Figure 28.4. Since U, V, and $U \cap V$ are all connected, the Mayer-Vietoris sequence begins with

	$M \quad U \amalg V$	$U \cap V \sim S^1$
H^2	$\to H^2(\Sigma_2) \to \quad 0$	
H^1	$0 \to H^1(\Sigma_2) \to \mathbb{R}^2 \oplus \mathbb{R}^2 \xrightarrow{\alpha} \quad \mathbb{R}$	

The map $\alpha : H^1(U) \oplus H^1(V) \to H^1(S^1)$ is the difference map

$$\alpha(\omega_U, \omega_V) = j_V^* \omega_V - j_U^* \omega_U,$$

where j_U and j_V are inclusions of an S^1 in $U \cap V$ into U and V, respectively. By Lemma 28.3, $j_U^* = j_V^* = 0$, so $\alpha = 0$. It then follows from the exactness of the Mayer-Vietoris sequence that

$$H^1(\Sigma_2) \simeq H^1(U) \oplus H^1(V) \simeq \mathbb{R}^4$$

and

$$H^2(\Sigma_2) \simeq H^1\left(S^1\right) \simeq \mathbb{R}.$$

□

A genus- 2 surface Σ_2 can be obtained as the quotient space of an octagon with its edges identified following the scheme of Figure 28.5.

To see this, first cut Σ_2 along the circle e as in Figure 28.6.
Then the two halves A and B are each a torus minus an open disk (Figure 28.7), so that each half can be represented as a pentagon, before identification (Figure 28.8). When A and B are glued together along e, we obtain the octagon in Figure 28.5.

Fig. 28.5. A surface of genus 2 as a quotient space of an octagon.

Fig. 28.6. A surface of genus 2 cut along a curve e.

Fig. 28.7. Two halves of a surface of genus 2 .

Fig. 28.8. Two halves of a surface of genus 2 .

By Lemma 28.3, if $p \in \Sigma_2$ and $i : C \to \Sigma_2 - \{p\}$ is a small circle around p in Σ_2, then the restriction map

$$i^* : H^1(\Sigma_2 - \{p\}) \to H^1(C)$$

is the zero map. This allows us to compute inductively the cohomology of a compact orientable surface Σ_g of genus g.

Exercise 28.6 (Surface of genus 3). Compute the cohomology vector space of $\Sigma_2 - \{p\}$ and then compute the cohomology vector space of a compact orientable surface Σ_3 of genus 3 .

12.5.4 Problems

28.1. Real Projective Plane

Compute the cohomology of the real projective plane $\mathbb{R}P^2$ (Figure 28.9).

Fig. 28.9. The real projective plane.

28.2. The n-sphere

Compute the cohomology of the sphere S^n.

28.3. Cohomology of a Multiply Punctured Plane

(a) Let p, q be distinct points in $\mathbb{R}^2$. Compute the de Rham cohomology of $\mathbb{R}^2 - \{p, q\}$.
(b) Let $p_1, \ldots, p_n$ be distinct points in $\mathbb{R}^2$. Compute the de Rham cohomology of $\mathbb{R}^2$ $\{p_1, \ldots, p_n\}$.

28.4. Cohomology of a Surface of Genus g

Compute the cohomology vector space of a compact orientable surface Σ_g of genus g.

28.5. Cohomology of a 3-dimensional Torus

Compute the cohomology ring of $\mathbb{R}^3/\mathbb{Z}^3$.

12.6 §29 Proof of Homotopy Invariance

In this section we prove the homotopy invariance of de Rham cohomology.
If $f : M \to N$ is a C^∞ map, the pullback maps on differential forms and on cohomology classes are normally both denoted by f^*. Since this might cause confusion in the proof of homotopy invariance, in this section we revert to our original convention of denoting the pullback of forms by

$$f^* : \Omega^k(N) \to \Omega^k(M)$$

and the induced map in cohomology by

$$f^{\#} : H^k(N) \to H^k(M).$$

The relation between these two maps is

$$f^{\#}[\omega] = [f^*\omega]$$

for $[\omega] \in H^k(N)$.

Theorem 29.1 (Homotopy axiom for de Rham cohomology). Two smoothly homotopic maps $f, g : M \to N$ of manifolds induce the same map in cohomology:

$$f^{\#} = g^{\#} : H^k(N) \to H^k(M).$$

We first reduce the problem to two special maps i_0 and $i_1 : M \to M \times \mathbb{R}$, which are the 0 -section and the 1 -section, respectively, of the product line bundle $M \times \mathbb{R} \to M$:

$$i_0(x) = (x, 0), \quad i_1(x) = (x, 1).$$

Then we introduce the all-important technique of cochain homotopy. By finding a cochain homotopy between i_0^* and i_1^*, we prove that they induce the same map in cohomology.

12.6.1 29.1 Reduction to Two Sections

Suppose f and $g : M \to N$ are smoothly homotopic maps. Let $F : M \times \mathbb{R} \to N$ be a smooth homotopy from f to g. This means that

$$F(x, 0) = f(x), \quad F(x, 1) = g(x) \tag{29.1}$$

for all $x \in M$. For each $t \in \mathbb{R}$, define $i_t : M \to M \times \mathbb{R}$ to be the section $i_t(x) = (x, t)$. We can restate (29.1) as

$$F \circ i_0 = f, \quad F \circ i_1 = g.$$

By the functoriality of the pullback (Remark 24.7),

$$f^{\#} = i_0^{\#} \circ F^{\#}, \quad g^{\#} = i_1^{\#} \circ F^{\#}.$$

This reduces proving homotopy invariance to the special case

$$i_0^{\#} = i_1^{\#}.$$

The two maps $i_0, i_1 : M \to M \times \mathbb{R}$ are obviously smoothly homotopic via the identity map

$$\mathbb{1}_{M \times \mathbb{R}} : M \times \mathbb{R} \to M \times \mathbb{R}$$

12.6.2 29.2 Cochain Homotopies

The usual method for showing that two cochain maps $\varphi, \psi : \mathcal{A} \to \mathcal{B}$ induce the same map in cohomology is to find a linear map $K : \mathcal{A} \to \mathcal{B}$ of degree -1 such that

$$\varphi - \psi = d \circ K + K \circ d$$

Such a map K is called a cochain homotopy from φ to ψ. Note that K is not assumed to be a cochain map. If a is a cocycle in $\mathcal{A}$, then

$$\varphi(a) - \psi(a) = dKa + Kda = dKa$$

is a coboundary, so that in cohomology

$$\varphi^{\#}[a] = [\varphi(a)] = [\psi(a)] = \psi^{\#}[a]$$

Thus, the existence of a cochain homotopy between φ and ψ implies that the induced maps $\varphi^{\#}$ and $\psi^{\#}$ in cohomology are equal.

Remark. Given two cochain maps $\varphi, \psi : \mathcal{A} \to \mathcal{B}$, if one could find a linear map $K : \mathcal{A} \to \mathcal{B}$ of degree -1 such that $\varphi - \psi = d \circ K$ on $\mathcal{A}$, then $\varphi^{\#}$ would be equal to $\psi^{\#}$ in cohomology. However, such a map almost never exists; it is necessary to have the term $K \circ d$ as well. The cylinder construction in homology theory [30, p. 65] shows why it is natural to consider $d \circ K + K \circ d$.

12.6.3 29.3 Differential Forms on $m \times \smallsetminus$

Recall that a sum $\sum_\alpha \omega_\alpha$ of C^∞ differential forms on a manifold M is said to be locally finite if the collection $\{\operatorname{supp} \omega_\alpha\}$ of supports is locally finite. This means that every point p in M has a neighborhood V_p such that V_p intersects only finitely many of the sets $\operatorname{supp} \omega_\alpha$. If $\operatorname{supp} \omega_\alpha$ is disjoint from V_p, then $\omega_\alpha \equiv 0$ on V_p. Thus, on V_p the locally finite sum $\sum_\alpha \omega_\alpha$ is actually a finite sum. As an example, if $\{\rho_\alpha\}$ is a partition of unity, then the sum $\sum \rho_\alpha$ is locally finite.

Let $\pi : M \times \mathbb{R} \to M$ be the projection to the first factor. In this subsection we will show that every C^∞ differential form on $M \times \mathbb{R}$ is a locally finite sum of the following two types of forms:
(I) $f(x,t)\pi^*\eta$,
(II) $f(x,t)dt \wedge \pi^*\eta$,
where $f(x,t)$ is a C^∞ function on $M \times \mathbb{R}$ and η is a C^∞ form on M.
In general, a decomposition of a differential from on $M \times \mathbb{R}$ into a locally finite sum of type-I and type-II forms is far from unique. However, once we fix an atlas $\{(U_\alpha, \phi_\alpha)\}$ on M, a C^∞ partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$, and a collection $\{g_\alpha\}$ of C^∞ functions on M such that

$$g_\alpha \equiv 1 \text{ on } \operatorname{supp} \rho_\alpha \quad \text{and} \quad \operatorname{supp} g_\alpha \subset U_\alpha$$

then there is a well-defined procedure to produce uniquely such a locally finite sum. The existence of the function g_α follows from the smooth Urysohn lemma (Problem 13.3).

In the proof of the decomposition procedure, we will need the following simple but useful lemma on the extension of a C^∞ form by zero.

Lemma 29.2. Let U be an open subset of a manifold M. If a smooth k-form $\tau \in \Omega^k(U)$ defined on U has support in a closed subset of M contained in U, then τ can be extended by zero to a smooth k-form on M.

Proof. Problem 29.1.
Fix an atlas $\{(U_\alpha, \phi_\alpha)\}$, a partition of unity $\{\rho_\alpha\}$, and a collection $\{g_\alpha\}$ of C^∞ functions as above. Then $\{\pi^{-1}U_\alpha\}$ is an open cover of $M \times \mathbb{R}$, and $\{\pi^*\rho_\alpha\}$ is a partition of unity subordinate to $\{\pi^{-1}U_\alpha\}$ (Problem 13.6).

Let ω be any $C^\infty k$-form on $M \times \mathbb{R}$ and let $\omega_\alpha = (\pi^*\rho_\alpha)\,\omega$. Since $\sum \pi^*\rho_\alpha = 1$,

$$\omega = \sum_\alpha (\pi^*\rho_\alpha)\,\omega = \sum_\alpha \omega_\alpha \tag{29.2}$$

Because $\{\operatorname{supp} \pi^*\rho_\alpha\}$ is locally finite, (29.2) is a locally finite sum. By Problem 18.4,

$$\operatorname{supp} \omega_\alpha \subset \operatorname{supp} \pi^*\rho_\alpha \cap \operatorname{supp} \omega \subset \operatorname{supp} \pi^*\rho_\alpha \subset \pi^{-1}U_\alpha$$

Let $\phi_\alpha = (x^1, \ldots, x^n)$. Then on $\pi^{-1}U_\alpha$, which is homeomorphic to $U_\alpha \times \mathbb{R}$, we have coordinates $\pi^*x^1, \ldots, \pi^*x^n, t$. For the sake of simplicity, we sometimes write x^i instead of π^*x^i. On $\pi^{-1}U_\alpha$ the k-form ω_α may be written uniquely as a linear combination

$$\omega_\alpha = \sum_I a_I dx^I + \sum_J b_J dt \wedge dx^J \tag{29.3}$$

where a_I and b_J are C^∞ functions on $\pi^{-1}U_\alpha$. This decomposition shows that ω_α is a finite sum of type-I and type-II forms on $\pi^{-1}U_\alpha$. By Problem 18.5, the supports of a_I and b_I are contained in $\operatorname{supp}\omega_\alpha$, hence in $\operatorname{supp}\pi^*\rho_\alpha$, a closed set in $M \times \mathbb{R}$. Therefore, by the lemma above, a_I and b_J can be extended by zero to C^∞ functions on $M \times \mathbb{R}$. Unfortunately, dx^I and dx^J make sense only on U_α and cannot be extended to M, at least not directly.

To extend the decomposition (29.3) to $M \times \mathbb{R}$, the trick is to multiply ω_α by π^*g_α. Since $\operatorname{supp}\omega_\alpha \subset \operatorname{supp}\pi^*\rho_\alpha$ and $\pi^*g_\alpha \equiv 1$ on $\operatorname{supp}\pi^*\rho_\alpha$, we have the equality $\omega_\alpha = (\pi^*g_\alpha)\,\omega_\alpha$. Therefore,

$$\begin{aligned}\omega_\alpha = (\pi^*g_\alpha)\,\omega_\alpha &= \sum_I a_I\,(\pi^*g_\alpha)\,dx^I + \sum_J b_J dt \wedge (\pi^*g_\alpha)\,dx^J \\ &= \sum_I a_I \pi^*\left(g_\alpha dx^I\right) + \sum_J b_J dt \wedge \pi^*\left(g_\alpha dx^J\right)\end{aligned} \tag{29.4}$$

Now $\operatorname{supp} g_\alpha$ is a closed subset of M contained in U_α, so by Lemma 29.2 again, $g_\alpha dx^I$ can be extended by zero to M. Equations (29.2) and (29.4) prove that ω is a locally finite sum of type-I and type-II forms on $M \times \mathbb{R}$. Moreover, given $\{(U_\alpha, \phi_\alpha)\}$, $\{\rho_\alpha\}$, and $\{g_\alpha\}$, the decomposition in (29.4) is unique.

12.6.4 29.4 A Cochain Homotopy between i_0^* and i_1^*

In the rest of the proof, fix an atlas $\{(U_\alpha, \phi_\alpha)\}$ for M, a C^∞ partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$, and a collection $\{g_\alpha\}$ of C^∞ functions on M as in Section 29.3. Let $\omega \in \Omega^k(M \times \mathbb{R})$. Using (29.2) and (29.4), we decompose ω into a locally finite sum

$$\omega = \sum_\alpha \omega_\alpha = \sum_{\alpha,I} a_I^\alpha \pi^*\left(g_\alpha dx_\alpha^I\right) + \sum_{\alpha,J} b_J^\alpha dt \wedge \pi^*\left(g_\alpha dx_\alpha^J\right)$$

where we now attach an index α to a_I, b_J, x^I, and x^J to indicate their dependence on α.

Define

$$K : \Omega^*(M \times \mathbb{R}) \to \Omega^{*-1}(M)$$

by the following rules:

(i) on type-I forms,

$$K\left(f\pi^*\eta\right) = 0$$

(ii) on type-II forms,

$$K\left(f dt \wedge \pi^*\eta\right) = \left(\int_0^1 f(x,t)dt\right)\eta$$

(iii) K is linear over locally finite sums.

Thus,

$$K(\omega) = K\left(\sum_\alpha \omega_\alpha\right) = \sum_{\alpha,J}\left(\int_0^1 b_J^\alpha(x,t)dt\right) g_\alpha dx_\alpha^J \tag{29.5}$$

Given the data $\{(U_\alpha, \phi_\alpha)\}, \{\rho_\alpha\}, \{g_\alpha\}$, the decomposition $\omega = \omega_\alpha$ with ω_α as in (29.4) is unique. Therefore, K is well defined. It is not difficult to show that so defined, K is the unique linear operator $\Omega^*(M \times \mathbb{R}) \to \Omega^{*-1}(M)$ satisfying (i), (ii), and (iii) (Problem 29.3), so it is in fact independent of the data $\{(U_\alpha, \phi_\alpha)\}, \{\rho_\alpha\}$, and $\{g_\alpha\}$.

12.6.5 29.5 Verification of Cochain Homotopy

We check in this subsection that

$$d \circ K + K \circ d = i_1^* - i_0^*. \tag{29.6}$$

Lemma 29.3. (i) The exterior derivative d is $\mathbb{R}$-linear over locally finite sums.
(ii) Pullback by a C^∞ map is $\mathbb{R}$-linear over locally finite sums.

Proof. (i) Suppose $\sum \omega_\alpha$ is a locally finite sum of $C^\infty k$-forms. This implies that every point p has a neighborhood on which the sum is finite. Let U be such a neighborhood. Then

$$\begin{aligned}
\left(d\sum \omega_\alpha\right)\Big|_U &= d\left(\left(\sum \omega_\alpha\right)\Big|_U\right) && \text{(Corollary 19.6)}\\
&= d\left(\sum \omega_\alpha\Big|_U\right)\\
&= \sum d\left(\omega_\alpha|_U\right) && \left(\sum \omega_\alpha\Big|_U \text{ is a finite sum}\right)\\
&= \sum (d\omega_\alpha)\Big|_U && \text{(Corollary 19.6)}\\
&= \left(\sum d\omega_\alpha\right)\Big|_U
\end{aligned}$$

Since M can be covered by such neighborhoods, $d\left(\sum \omega_\alpha\right) = \sum d\omega_\alpha$ on M. The homogeneity property $d(r\omega) = rd(\omega)$ for $r \in \mathbb{R}$ and $\omega \in \Omega^k(M)$ is trivial.
(ii) The proof is similar to (i) and is relegated to Problem 29.2.

By linearity of K, d, i_0^*, and i_1^* over locally finite sums, it suffices to check the equality (29.6) on any coordinate open set. Fix a coordinate open set ($U \times \mathbb{R}$, $\pi^* x^1, \ldots, \pi^* x^n, t$) on $M \times \mathbb{R}$. On type-I forms,

$$Kd\left(f\pi^*\eta\right) = K\left(\frac{\partial f}{\partial t}dt \wedge \pi^*\eta + \sum_i \frac{\partial f}{\partial x^i}\pi^* dx^i \wedge \pi^*\eta + f\pi^* d\eta\right)$$

In the sum on the right-hand side, the second and third terms are type-I forms; they map to 0 under K. Thus,

$$\begin{aligned}
Kd\left(f\pi^*\eta\right) &= K\left(\frac{\partial f}{\partial t}dt \wedge \pi^*\eta\right) = \left(\int_0^1 \frac{\partial f}{\partial t}dt\right)\eta\\
&= (f(x,1) - f(x,0))\eta = \left(i_1^* - i_0^*\right)\left(f(x,t)\pi^*\eta\right)
\end{aligned}$$

Since $dK\left(f\pi^*\eta\right) = d(0) = 0$, on type-I forms,

$$d \circ K + K \circ d = i_1^* - i_0^*$$

On type-II forms, by the antiderivation property of d,

$$\begin{aligned}
dK\left(fdt \wedge \pi^*\eta\right) &= d\left(\left(\int_0^1 f(x,t)dt\right)\eta\right) \\
&= \sum\left(\frac{\partial}{\partial x^i}\int_0^1 f(x,t)dt\right)dx^i \wedge \eta + \left(\int_0^1 f(x,t)dt\right)d\eta \\
&= \sum\left(\int_0^1 \frac{\partial f}{\partial x^i}(x,t)dt\right)dx^i \wedge \eta + \left(\int_0^1 f(x,t)dt\right)d\eta
\end{aligned}$$

In the last equality, differentiation under the integral sign is permissible because $f(x,t)$ is C^∞. Furthermore,

$$\begin{aligned}
Kd\left(fdt \wedge \pi^*\eta\right) &= K\left(d(fdt) \wedge \pi^*\eta - fdt \wedge d\pi^*\eta\right) \\
&= K\left(\sum_i \frac{\partial f}{\partial x^i}dx^i \wedge dt \wedge \pi^*\eta\right) - K\left(fdt \wedge \pi^* d\eta\right) \\
&= -\sum_i\left(\int_0^1 \frac{\partial f}{\partial x^i}(x,t)dt\right)dx^i \wedge \eta - \left(\int_0^1 f(x,t)dt\right)d\eta
\end{aligned}$$

Thus, on type-II forms,

$$d \circ K + K \circ d = 0$$

On the other hand,

$$i_1^*\left(f(x,t)dt \wedge \pi^*\eta\right) = 0$$

because $i_1^* dt = d i_1^* t = d(1) = 0$. Similarly, i_0^* also vanishes on type-II forms. Therefore,

$$d \circ K + K \circ d = 0 = i_1^* - i_0^*$$

on type-II forms.
This completes the proof that K is a cochain homotopy between i_0^* and i_1^*. The existence of the cochain homotopy K proves that the induced maps in cohomology $i_0^\#$ and $i_1^\#$ are equal. As we pointed out in Section 29.1,

$$f^\# = i_0^\# \circ F^\# = i_1^\# \circ F^\# = g^\#.$$

12.6.6 Problems

29.1. Extension by Zero of a Smooth k-form

Prove Lemma 29.2.

29.2. Linearity of Pullback over Locally Finite Sums

Let $h : N \to M$ be a C^∞ map, and $\sum \omega_\alpha$ a locally finite sum of $C^\infty k$-forms on M. Prove that $h^*\left(\sum \omega_\alpha\right) = \sum h^*\omega_\alpha$.

29.3. The Cochain Homotopy k

(a) Check that defined by (29.5), the linear map K satisfies the three rules in Section 29.4.
(b) Prove that a linear operator satisfying the three rules in Section 29.4 is unique if it exists.

13 Appendices

13.1 §A Point-set Topology

Point-set topology, also called "general topology," is concerned with properties that remain invariant under homeomorphisms (continuous maps having continuous inverses). The basic development in the subject took place in the late nineteenth and early twentieth centuries. This appendix is a collection of basic results from point-set topology that are used throughout the book.

13.1.1 A. 1 Topological Spaces

The prototype of a topological space is the Euclidean space $\mathbb{R}^n$. However, Euclidean space comes with many additional structures, such as a metric, coordinates, an inner product, and an orientation, that are extraneous to its topology. The idea behind the definition of a topological space is to discard all those properties of $\mathbb{R}^n$ that have nothing to do with continuous maps, thereby distilling the notion of continuity to its very essence.

In advanced calculus one learns several characterizations of a continuous map, among which is the following: a map f from an open subset of $\mathbb{R}^n$ to $\mathbb{R}^m$ is continuous if and only if the inverse image $f^{-1}(V)$ of any open set V in $\mathbb{R}^m$ is open in $\mathbb{R}^n$. This shows that continuity can be defined solely in terms of open sets.

To define open sets axiomatically, we look at properties of open sets in $\mathbb{R}^n$. Recall that in $\mathbb{R}^n$ the distance between two points p and q is given by

$$d(p,q) = \left[\sum_{i=1}^{n} \left(p^i - q^i\right)^2\right]^{1/2}$$

and the open ball $B(p,r)$ with center $p \in \mathbb{R}^n$ and radius $r > 0$ is the set

$$B(p,r) = \{x \in \mathbb{R}^n \mid d(x,p) < r\}$$

A set U in $\mathbb{R}^n$ is said to be open if for every p in U, there is an open ball $B(p,r)$ with center p and radius r such that $B(p,r) \subset U$ (Figure A.1). It is clear that the union of an arbitrary collection $\{U_\alpha\}$ of open sets is open, but the same need not be true of the intersection of infinitely many open sets.

Fig. A.1. An open set in $\mathbb{R^n}$.

Example. The intervals $]-1/n, 1/n[$, $n = 1, 2, 3, \ldots$, are all open in $\mathbb{R}^1$, but their intersection $\bigcap_{n=1}^{\infty}]-1/n, 1/n[$ is the singleton set $\{0\}$, which is not open.

What is true is that the intersection of a finite collection of open sets in $\mathbb{R}^n$ is open. This leads to the definition of a topology on a set.

Definition A.1. A topology on a set S is a collection $\mathcal{T}$ of subsets containing both the empty set $\varnothing$ and the set S such that $\mathcal{T}$ is closed under arbitrary unions and finite intersections; i.e., if $U_\alpha \in \mathcal{T}$ for all α in an index set A , then $\bigcup_{\alpha \in \mathrm{A}} U_\alpha \in \mathcal{T}$ and if $U_1, \ldots, U_n \in \mathcal{T}$, then $\bigcap_{i=1}^n U_i \in \mathcal{T}$.

The elements of $\mathcal{T}$ are called open sets and the pair ($S, \mathcal{T}$) is called a topological space. To simplify the notation, we sometimes simply refer to a pair ($S, \mathcal{T}$) as "the topological space S " when there is no chance of confusion. A neighborhood of a point p in S is an open set U containing p. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are two topologies on set S and $\mathcal{T}_1 \subset \mathcal{T}_2$, then we say that $\mathcal{T}_1$ is coarser than $\mathcal{T}_1$, or that $\mathcal{T}_2$ is finer than $\mathcal{T}_1$. A coarser topology has fewer open sets; conversely, a finer topology has more open sets.

Example. The open subsets of $\mathbb{R}^n$ as we understand them in advanced calculus form a topology on $\mathbb{R}^n$, the standard topology of $\mathbb{R}^n$. In this topology a set U is open in $\mathbb{R}^n$ if and only if for every $p \in U$, there is an open ball $B(p, \varepsilon)$ with center p and radius ε contained in U. Unless stated otherwise, $\mathbb{R}^n$ will always have its standard topology.

The criterion for openness in $\mathbb{R}^n$ has a useful generalization to a topological space.

Lemma A. 2 (Local criterion for openness). Let S be a topological space. A subset A is open in S if and only iffor every $p \in A$, there is an open set V such that $p \in V \subset A$.

Proof.

($\Rightarrow$) If A is open, we can take $V = A$.
($\Leftarrow$) Suppose for every $p \in A$ there is an open set V_p such that $p \in V_p \subset A$. Then

$$A \subset \bigcup_{p \in A} V_p \subset A,$$

so that equality $A = \bigcup_{p \in A} V_p$ holds. As a union of open sets, A is open. $\square$

Example. For any set S, the collection $\mathcal{T} = \{\varnothing, S\}$ consisting of the empty set $\varnothing$ and the entire set S is a topology on S, sometimes called the trivial or indiscrete topology. It is the coarsest topology on a set.

Example. For any set S, let $\mathcal{T}$ be the collection of all subsets of S. Then $\mathcal{T}$ is a topology on S, called the discrete topology. A singleton set is a set with a single element. The discrete topology can also be characterized as the topology in which every singleton subset $\{p\}$ is open. A topological space having the discrete topology is called a discrete space. The discrete topology is the finest topology on a set.

The complement of an open set is called a closed set. By de Morgan's laws from set theory, arbitrary intersections and finite unions of closed sets are closed (Problem A.3). One may also specify a topology by describing all the closed sets.

Remark. When we say that a topology is closed under arbitrary union and finite intersection, the word "closed" has a different meaning from that of a "closed subset."

Example A. 3 (Finite-complement topology on $\mathbb{R}^1$). Let $\mathcal{T}$ be the collection of subsets of $\mathbb{R}^1$ consisting of the empty set $\varnothing$, the line $\mathbb{R}^1$ itself, and the complements of finite sets. Suppose F_α and F_i are finite subsets of $\mathbb{R}^1$ for $\alpha \in$ some index set A and $i = 1, \ldots, n$. By de Morgan's laws,

$$\bigcup_\alpha \left(\mathbb{R}^1 - F_\alpha\right) = \mathbb{R}^1 - \bigcap_\alpha F_\alpha \quad \text{and} \quad \bigcap_{i=1}^n \left(\mathbb{R}^1 - F_i\right) = \mathbb{R}^1 - \bigcup_{i=1}^n F_i.$$

Since the arbitrary intersection $\bigcap_{\alpha \in \mathrm{A}} F_\alpha$ and the finite union $\bigcup_{i=1}^n F_i$ are both finite, $\mathcal{T}$ is closed under arbitrary unions and finite intersections. Thus, $\mathcal{T}$ defines a topology on $\mathbb{R}^1$, called the finite-complement topology.

For the sake of definiteness, we have defined the finite-complement topology on $\mathbb{R}^1$, but of course, there is nothing specific about $\mathbb{R}^1$ here. One can define in exactly the same way the finite-complement topology on any set.

Example A. 4 (Zariski topology). One well-known topology is the Zariski topology from algebraic geometry. Let K be a field and let S be the vector space K^n. Define a subset of K^n to be Zariski closed if it is the zero set $Z(f_1, \ldots, f_r)$ of finitely many polynomials $f_1, \ldots, f_r$ on K^n. To show that these are indeed the closed subsets of
a topology, we need to check that they are closed under arbitrary intersections and finite unions.

Let $I = (f_1, \ldots, f_r)$ be the ideal generated by $f_1, \ldots, f_r$ in the polynomial ring $K[x_1, \ldots, x_n]$. Then $Z(f_1, \ldots, f_r) = Z(I)$, the zero set of all the polynomials in the ideal I. Conversely, by the Hilbert basis theorem [11, §9.6, Th. 21], any ideal in $K[x_1, \ldots, x_n]$ has a finite set of generators. Hence, the zero set of finitely many polynomials is the same as the zero set of an ideal in $K[x_1, \ldots, x_n]$. If $I = (f_1, \ldots, f_r)$ and $J = (g_1, \ldots, g_s)$ are two ideals, then the product ideal IJ is the ideal in $K[x_1, \ldots, x_n]$ generated by all products $f_i g_j, 1 \leq i \leq r, 1 \leq j \leq s$. If $\{I_\alpha\}_{\alpha \in \mathrm{A}}$ is a family of ideals in $K[x_1, \ldots, x_n]$, then their sum $\sum_\alpha I_\alpha$ is the smallest ideal in $K[x_1, \ldots, x_n]$ containing all the ideals I_α.

Exercise A. 5 (Intersection and union of zero sets). Let I_α, I, and J be ideals in the polynomial ring $K[x_1, \ldots, x_n]$. Show that

$$\bigcap_\alpha Z(I_\alpha) = Z\left(\sum_\alpha I_\alpha\right) \tag{i}$$

and

$$Z(I) \cup Z(J) = Z(IJ) \tag{ii}$$

The complement of a Zariski-closed subset of K^n is said to be Zariski open. If $I = (0)$ is the zero ideal, then $Z(I) = K^n$, and if $I = (1) = K[x_1, \ldots, x_n]$ is the entire ring, then $Z(I)$ is the empty set $\varnothing$. Hence, both the empty set and K^n are Zariski open. It now follows from Exercise A. 5 that the Zariski-open subsets of K^n form a topology on K^n, called the Zariski topology on K^n. Since the zero set of a polynomial on $\mathbb{R}^1$ is a finite set, the Zariski topology on $\mathbb{R}^1$ is precisely the finite-complement topology of Example A.3.

13.1.2 A. 2 Subspace Topology

Let $(S, \mathcal{T})$ be a topological space and A a subset of S. Define $\mathcal{T}_A$ to be the collection of subsets

$$\mathcal{T}_A = \{U \cap A \mid U \in \mathcal{T}\}.$$

By the distributive property of union and intersection,

$$\bigcup_\alpha (U_\alpha \cap A) = \left(\bigcup_\alpha U_\alpha\right) \cap A$$

and

$$\bigcap_i (U_i \cap A) = \left(\bigcap_i U_i\right) \cap A,$$

which shows that $\mathcal{T}_A$ is closed under arbitrary unions and finite intersections. Moreover, $\varnothing, A \in \mathcal{T}_A$. So $\mathcal{T}_A$ is a topology on A, called the subspace topology or the
relative topology of A in S, and elements of $\mathcal{T}_A$ are said to be open in A. To emphasize the

fact that an open set U in A need not be open in S, we also say that U is open relative to A or relatively open in A. The subset A of S with the subspace topology $\mathcal{T}_A$ is called a subspace of S.

If A is an open subset of a topological space S, then a subset of A is relatively open in A if and only if it is open in S.

Example. Consider the subset $A = [0, 1]$ of $\mathbb{R}^1$. In the subspace topology, the halfopen interval $[\ 0, 1/2\ [$ is open relative to A, because

$$\left[0, \frac{1}{2}[=\right] - \frac{1}{2}, \frac{1}{2}[\cap A.$$

(See Figure A.2.)

Fig. A.2. A relatively open subset of $[0, 1]$.

13.1.3 A. 3 Bases

It is generally difficult to describe directly all the open sets in a topology $\mathcal{T}$. What one can usually do is to describe a subcollection $\mathfrak{B}$ of $\mathcal{T}$ such that any open set is expressible as a union of open sets in $\mathcal{B}$.

Definition A.6. A subcollection $\mathcal{B}$ of a topology $\mathcal{T}$ on a topological space S is a basis for the topology $\mathcal{T}$ if given an open set U and point p in U, there is an open set $B \in \mathcal{B}$ such that $p \in B \subset U$. We also say that $\mathcal{B}$ generates the topology $\mathcal{T}$ or that $\mathcal{B}$ is a basis for the topological space S.

Example. The collection of all open balls $B(p, r)$ in $\mathbb{R}^n$, with $p \in \mathbb{R}^n$ and r a positive real number, is a basis for the standard topology of $\mathbb{R}^n$.

Proposition A.7. A collection $\mathfrak{B}$ of open sets of S is a basis if and only if every open set in S is a union of sets in $\mathcal{B}$.

Proof.

($\Rightarrow$) Suppose $\mathcal{B}$ is a basis and U is an open set in S. For every $p \in U$, there is a basic open set $B_p \in \mathcal{B}$ such that $p \in B_p \subset U$. Therefore, $U = \bigcup_{p \in U} B_p$.

($\Leftarrow$) Suppose every open set in S is a union of open sets in $\mathcal{B}$. Given an open set U and a point p in U, since $U = \bigcup_{B_\alpha \in \mathcal{B}} B_\alpha$, there is a $B_\alpha \in \mathcal{B}$ such that $p \in B_\alpha \subset U$. Hence, $\mathcal{B}$ is a basis. $\square$

The following proposition gives a useful criterion for deciding whether a collection $\mathcal{B}$ of subsets is a basis for some topology.

Proposition A.8. A collection $\mathfrak{B}$ of subsets of a set S is a basis for some topology $\mathcal{T}$ on S if and only if

(i) S is the union of all the sets in $\mathcal{B}$, and

(ii) given any two sets B_1 and $B_2 \in \mathcal{B}$ and a point $p \in B_1 \cap B_2$, there is a set $B \in \mathcal{B}$ such that $p \in B \subset B_1 \cap B_2$ (Figure A.3).

Fig. A.3. Criterion for a basis.

Proof.

($\Rightarrow$) (i) follows from Proposition A.7.
(ii) If $\mathcal{B}$ is a basis, then B_1 and B_2 are open sets and hence so is $B_1 \cap B_2$. By the definition of a basis, there is a $B \in \mathcal{B}$ such that $p \in B \subset B_1 \cap B_2$.
($\Leftarrow$) Define $\mathcal{T}$ to be the collection consisting of all sets that are unions of sets in $\mathcal{B}$. Then the empty set $\varnothing$ and the set S are in $\mathcal{T}$ and $\mathcal{T}$ is clearly closed under arbitrary union. To show that $\mathcal{T}$ is closed under finite intersection, let $U = \bigcup_\mu B_\mu$ and $V = \bigcup_v B_v$ be in $\mathcal{T}$, where $B_\mu, B_v \in \mathcal{B}$. Then

$$U \cap V = \left(\bigcup_\mu B_\mu\right) \cap \left(\bigcup_v B_v\right) = \bigcup_{\mu,v} (B_\mu \cap B_v)\,.$$

Thus, any p in $U \cap V$ is in $B_\mu \cap B_v$ for some μ, v. By (ii) there is a set B_p in $\mathcal{B}$ such that $p \in B_p \subset B_\mu \cap B_v$. Therefore,

$$U \cap V = \bigcup_{p \in U \cap V} B_p \in \mathcal{T}.$$

□

Proposition A.9. Let $\mathcal{B} = \{B_\alpha\}$ be a basis for a topological space S, and A a subspace of S. Then $\{B_\alpha \cap A\}$ is a basis for A.
Proof. Let U' be any open set in A and $p \in U'$. By the definition of subspace topology, $U' = U \cap A$ for some open set U in S. Since $p \in U \cap A \subset U$, there is a basic open set B_α such that $p \in B_\alpha \subset U$. Then

$$p \in B_\alpha \cap A \subset U \cap A = U',$$

which proves that the collection $\{B_\alpha \cap A \mid B_\alpha \in \mathcal{B}\}$ is a basis for A. □

13.1.4 A. 4 First and Second Countability

First and second countability of a topological space have to do with the countability of a basis. Before taking up these notions, we begin with an example. We say that a point in $\mathbb{R}^n$ is rational if all of its coordinates are rational numbers. Let $\mathbb{Q}$ be the set of rational numbers and $\mathbb{Q}^+$the set of positive rational numbers. From real analysis, it is well known that every open interval in $\mathbb{R}$ contains a rational number.

Lemma A.10. Every open set in $\mathbb{R}^n$ contains a rational point.
Proof. An open set U in $\mathbb{R}^n$ contains an open ball $B(p, r)$, which in turn contains an open cube $\prod_{i=1}^n I_i$, where I_i is the open interval $]p^i - (r/\sqrt{n}), p^i + (r/\sqrt{n})[$ (see Problem A.4). For each i,

let q^i be a rational number in I_i. Then $(q^1, \ldots, q^n)$ is a rational point in $\prod_{i=1}^n I_i \subset B(p,r) \subset U$. $\square$

Proposition A.11. The collection $\mathcal{B}_{\text{rat}}$ of all open balls in $\mathbb{R}^n$ with rational centers and rational radii is a basis for $\mathbb{R}^n$.

Fig. A.4. A ball with rational center q and rational radius $r/2$.

Proof. Given an open set U in $\mathbb{R}^n$ and point p in U, there is an open ball $B(p, r')$ with positive real radius r' such that $p \in B(p, r') \subset U$. Take a rational number r in $]0, r'[$. Then $p \in B(p,r) \subset U$. By Lemma A.10, there is a rational point q in the smaller ball $B(p, r/2)$. We claim that

$$p \in B\left(q, \frac{r}{2}\right) \subset B(p,r). \tag{A.1}$$

(See Figure A.4.) Since $d(p,q) < r/2$, we have $p \in B(q, r/2)$. Next, if $x \in B(q, r/2)$, then by the triangle inequality,

$$d(x,p) \leq d(x,q) + d(q,p) < \frac{r}{2} + \frac{r}{2} = r.$$

So $x \in B(p,r)$. This proves the claim (A.1). Because $p \in B(q, r/2) \subset U$, the collection $\mathcal{B}_{\text{rat}}$ of open balls with rational centers and rational radii is a basis for $\mathbb{R}^n$. $\square$

Both of the sets $\mathbb{Q}$ and $\mathbb{Q}^+$are countable. Since the centers of the balls in $\mathcal{B}_{\text{rat}}$ are indexed by $\mathbb{Q}^n$, a countable set, and the radii are indexed by $\mathbb{Q}^+$, also a countable set, the collection $\mathcal{B}_{\text{rat}}$ is countable.

Definition A.12. A topological space is said to be second countable if it has a countable basis.

Example A.13. Proposition A. 11 shows that $\mathbb{R}^n$ with its standard topology is second countable. With the discrete topology, $\mathbb{R}^n$ would not be second countable. More generally, any uncountable set with the discrete topology is not second countable.

Proposition A.14. A subspace A of a second-countable space S is second countable.

Proof. By Proposition A.9, if $\mathcal{B} = \{B_i\}$ is a countable basis for S, then $\mathcal{B}_A := \{B_i \cap A\}$ is a countable basis for A.

Definition A.15. Let S be a topological space and p a point in S. A basis of neighborhoods at p or a neighborhood basis at p is a collection $\mathcal{B} = \{B_\alpha\}$ of neighborhoods of p such that for any neighborhood U of p, there is a $B_\alpha \in \mathcal{B}$ such that $p \in B_\alpha \subset U$. A topological space S is first countable if it has a countable basis of neighborhoods at every point $p \in S$.

Example. For $p \in \mathbb{R}^n$, let $B(p, 1/n)$ be the open ball of center p and radius $1/n$ in $\mathbb{R}^n$. Then $\{B(p, 1/n)\}_{n=1}^\infty$ is a neighborhood basis at p. Thus, $\mathbb{R}^n$ is first countable.

Example. An uncountable discrete space is first countable but not second countable. Every second-countable space is first countable (the proof is left to Problem A.18).

Suppose p is a point in a first-countable topological space and $\{V_i\}_{i=1}^\infty$ is a countable neighborhood basis at p. By taking $U_i = V_1 \cap \cdots \cap V_i$, we obtain a countable descending sequence

$$U_1 \supset U_2 \supset U_3 \supset \cdots$$

that is also a neighborhood basis at p. Thus, in the definition of first countability, we may assume that at every point the countable neighborhood basis at the point is a descending sequence of open sets.

13.1.5 A. 5 Separation Axioms

There are various separation axioms for a topological space. The only ones we will need are the Hausdorff condition and normality.

Definition A.16. A topological space S is Hausdorff if given any two distinct points x, y in S, there exist disjoint open sets U, V such that $x \in U$ and $y \in V$. A Hausdorff space is normal if given any two disjoint closed sets F, G in S, there exist disjoint open sets U, V such that $F \subset U$ and $G \subset V$ (Figure A.5).

Fig. A.5. The Hausdorff condition and normality.

Proposition A.17. Every singleton set (a one-point set) in a Hausdorff space S is closed.

Proof. Let $x \in S$. For any $y \in S - \{x\}$, by the Hausdorff condition there exist an open set $U \ni x$ and an open set $V \ni y$ such that U and V are disjoint. In particular,

$$y \in V \subset S - U \subset S - \{x\}.$$

By the local criterion for openness (Lemma A.2), $S - \{x\}$ is open. Therefore, $\{x\}$ is closed. □

Example. The Euclidean space $\mathbb{R}^n$ is Hausdorff, for given distinct points x, y in $\mathbb{R}^n$, if $\varepsilon = \frac{1}{2}d(x, y)$, then the open balls $B(x, \varepsilon)$ and $B(y, \varepsilon)$ will be disjoint (Figure A.6).

Fig. A.6. Two disjoint neighborhoods in $\mathbb{R ^{n}$.}

Example A. 18 (Zariski topology). Let $S = K^n$ be a vector space of dimension n over a field K, endowed with the Zariski topology. Every open set U in S is of the form $S - Z(I)$, where I is an ideal in $K[x_1, \ldots, x_n]$. The open set U is nonempty if and only if I is not the zero ideal. In the Zariski topology any two nonempty open sets intersect: if $U = S - Z(I)$ and $V = S - Z(J)$ are nonempty, then I and J are nonzero ideals and

$$\begin{aligned} U \cap V &= (S - Z(I)) \cap (S - Z(J)) \\ &= S - (Z(I) \cup Z(J)) && (\text{ de Morgan's law }) \\ &= S - Z(IJ), && (\text{ Exercise A.5 }) \end{aligned}$$

which is nonempty because IJ is not the zero ideal. Therefore, K^n with the Zariski topology is not Hausdorff.

Proposition A.19. Any subspace A of a Hausdorff space S is Hausdorff.

Proof. Let x and y be distinct points in A. Since S is Hausdorff, there exist disjoint neighborhoods U and V of x and y respectively in S. Then $U \cap A$ and $V \cap A$ are disjoint neighborhoods of x and y respectively in A. $\square$

13.1.6 A. 6 Product Topology

The Cartesian product of two sets A and B is the set $A \times B$ of all ordered pairs (a, b) with $a \in A$ and $b \in B$. Given two topological spaces X and Y, consider the collection B of subsets of $X \times Y$ of the form $U \times V$, with U open in X and V open in Y. We will call elements of $\mathcal{B}$ basic open sets in $X \times Y$. If $U_1 \times V_1$ and $U_2 \times V_2$ are in $\mathcal{B}$, then

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2),$$

which is also in $\mathcal{B}$ (Figure A.7). From this, it follows easily that $\mathcal{B}$ satisfies the conditions of Proposition A. 8 for a basis and generates a topology on $X \times Y$, called the product topology. Unless noted otherwise, this will always be the topology we assign to the product of two topological spaces.

Fig. A.7. Intersection of two basic open subsets in $X \times Y$.

Proposition A.20. Let $\{U_i\}$ and $\{V_j\}$ be bases for the topological spaces X and Y, respectively. Then $\{U_i \times V_j\}$ is a basis for $X \times Y$.

Proof. Given an open set W in $X \times Y$ and point $(x, y) \in W$, we can find a basic open set $U \times V$ in $X \times Y$ such that $(x, y) \in U \times V \subset W$. Since U is open in X and $\{U_i\}$ is a basis for X,

$$x \in U_i \subset U$$

for some U_i. Similarly,

$$y \in V_j \subset V$$

for some V_j. Therefore,

$$(x, y) \in U_i \times V_j \subset U \times V \subset W.$$

By the definition of a basis, $\{U_i \times V_j\}$ is a basis for $X \times Y$. □

Corollary A.21. The product of two second-countable spaces is second countable.

Proposition A.22. The product of two Hausdorff spaces X and Y is Hausdorff.

Proof. Given two distinct points $(x_1, y_1), (x_2, y_2)$ in $X \times Y$, without loss of generality we may assume that $x_1 \neq x_2$. Since X is Hausdorff, there exist disjoint open sets U_1, U_2 in X such that $x_1 \in U_1$ and $x_2 \in U_2$. Then $U_1 \times Y$ and $U_2 \times Y$ are disjoint neighborhoods of (x_1, y_1) and (x_2, y_2) (Figure A.8), so $X \times Y$ is Hausdorff. □

Fig. A.8. Two disjoint neighborhoods in $X \times Y$.

The product topology can be generalized to the product of an arbitrary collection $\{X_\alpha\}_{\alpha\in\mathrm{A}}$ of topological spaces. Whatever the definition of the product topology, the projection maps $\pi_{\alpha_i} : \prod_\alpha X_\alpha \to X_{\alpha_i}, \pi_{\alpha_i}\left(\prod x_\alpha\right) = x_{\alpha_i}$ should all be continuous. Thus, for each open set U_{α_i} in X_{α_i}, the inverse image $\pi_{\alpha_i}^{-1}\left(U_{\alpha_i}\right)$ should be open in $\Pi_\alpha X_\alpha$. By the properties of open sets, a finite intersection $\bigcap_{i=1}^r \pi_{\alpha_i}^{-1}\left(U_{\alpha_i}\right)$ should also be open. Such a finite intersection is a set of the form $\prod_{\alpha\in A} U_\alpha$, where U_α is open in X_α and $U_\alpha = X_\alpha$ for all but finitely many $\alpha \in A$. We define the product topology on the Cartesian product $\prod_{\alpha\in\mathrm{A}} X_\alpha$ to be the topology with basis consisting of sets of this form. The product topology is the coarsest topology on $\prod_\alpha X_\alpha$ such that all the projection maps $\pi_{\alpha_i} : \prod_\alpha X_\alpha \to X_{\alpha_i}$ are continuous.

13.1.7 A. 7 Continuity

Let $f : X \to Y$ be a function of topological spaces. Mimicking the definition from advanced calculus, we say that f is continuous at a point p in X if for every neighborhood V of $f(p)$ in Y, there is a neighborhood U of p in X such that $f(U) \subset V$. We say that f is continuous on X if it is continuous at every point of X.

Proposition A. 23 (Continuity in terms of open sets). A function $f : X \to Y$ is continuous if and only if the inverse image of any open set is open.

Proof.

($\Rightarrow$) Suppose V is open in Y. To show that $f^{-1}(V)$ is open in X, let $p \in f^{-1}(V)$. Then $f(p) \in V$. Since f is assumed to be continuous at p, there is a neighborhood U of p such that $f(U) \subset V$. Therefore, $p \in U \subset f^{-1}(V)$. By the local criterion for openness (Lemma A.2), $f^{-1}(V)$ is open in X.
($\Leftarrow$) Let p be a point in X, and V a neighborhood of $f(p)$ in Y. By hypothesis, $f^{-1}(V)$ is open in X. Since $f(p) \in V, p \in f^{-1}(V)$. Then $U = f^{-1}(V)$ is a neighborhood of p such that $f(U) = f\left(f^{-1}(V)\right) \subset V$, so f is continuous at p.

Example A. 24 (Continuity of an inclusion map). If A is a subspace of X, then the inclusion map $i : A \to X, i(a) = a$ is continuous.

Proof. If U is open in X, then $i^{-1}(U) = U \cap A$, which is open in the subspace topology of A.

Example A. 25 (Continuity of a projection map). The projection $\pi : X \times Y \to X$, $\pi(x, y) = x$, is continuous.

Proof. Let U be open in X. Then $\pi^{-1}(U) = U \times Y$, which is open in the product topology on $X \times Y$.

Proposition A.26. The composition of continuous maps is continuous: if $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

Proof. Let V be an open subset of Z. Then

$$(g \circ f)^{-1}(V) = f^{-1}\left(g^{-1}(V)\right)$$

because for any $x \in X$,

$$x \in (g \circ f)^{-1}(V) \text{ iff } g(f(x)) \in V \text{ iff } f(x) \in g^{-1}(V) \text{ iff } x \in f^{-1}\left(g^{-1}(V)\right).$$

By Proposition A.23, since g is continuous, $g^{-1}(V)$ is open in Y. Similarly, since f is continuous, $f^{-1}\left(g^{-1}(V)\right)$ is open in X. By Proposition A. 23 again, $g \circ f : X \to Z$ is continuous.

If A is a subspace of X and $f : X \to Y$ is a function, the restriction of f to A,

$$f|_A : A \to Y,$$

is defined by

$$(f|_A)(a) = f(a)$$

With $i : A \to X$ being the inclusion map, the restriction $f|_A$ is the composite $f \circ i$. Since both f and i are continuous (Example A.24) and the composition of continuous functions is continuous (Proposition A.26), we have the following corollary.

Corollary A.27. The restriction $f|_A$ of a continuous function $f : X \to Y$ to a subspace A is continuous.

Continuity may also be phrased in terms of closed sets.

Proposition A. 28 (Continuity in terms of closed sets). A function $f : X \to Y$ is continuous if and only if the inverse image of any closed set is closed.

Proof. Problem A.9.

A map $f : X \to Y$ is said to be open if the image of every open set in X is open in Y; similarly, $f : X \to Y$ is said to be closed if the image of every closed set in X is closed in Y.

If $f : X \to Y$ is a bijection, then its inverse map $f^{-1} : Y \to X$ is defined. In this context, for any subset $V \subset Y$, the notation $f^{-1}(V)$ a priori has two meanings. It can mean either the inverse image of V under the map f,

$$f^{-1}(V) = \{x \in X \mid f(x) \in V\}$$

or the image of V under the map f^{-1},

$$f^{-1}(V) = \left\{f^{-1}(y) \in X \mid y \in V\right\}.$$

Fortunately, because $y = f(x)$ if and only if $x = f^{-1}(y)$, these two meanings coincide.

13.1.8 A. 8 Compactness

While its definition may not be intuitive, the notion of compactness is of central importance in topology. Let S be a topological space. A collection $\{U_\alpha\}$ of open subsets of S is said to cover S or to be an open cover of S if $S \subset \bigcup_\alpha U_\alpha$. Of course, because S is the ambient space, this condition is equivalent to $S = \bigcup_\alpha U_\alpha$. A subcover of an open cover is a subcollection whose union still contains S. The topological space S is said to be compact if every open cover of S has a finite subcover.

With the subspace topology, a subset A of a topological space S is a topological space in its own right. The subspace A can be covered by open sets in A or by open sets in S. An open cover of A in S is a collection $\{U_\alpha\}$ of open sets in S that covers A. In this terminology, A is compact if and only if every open cover of A in A has a finite subcover.

Proposition A.29. A subspace A of a topological space S is compact if and only if every open cover of A in S has a finite subcover.

Proof.

($\Rightarrow$) Assume A compact and let $\{U_\alpha\}$ be an open cover of A in S. This means that $A \subset \cup_\alpha U_\alpha$. Hence,

$$A \subset \left(\bigcup_\alpha U_\alpha\right) \cap A = \bigcup_\alpha (U_\alpha \cap A)$$

Fig. A.9. An open cover of A in S.

Since A is compact, the open cover $\{U_\alpha \cap A\}$ has a finite subcover $\{U_{\alpha_i} \cap A\}_{i=1}^r$. Thus,

$$A \subset \bigcup_{i=1}^r (U_{\alpha_i} \cap A) \subset \bigcup_{i=1}^r U_{\alpha_i},$$

which means that $\{U_{\alpha_i}\}_{i=1}^r$ is a finite subcover of $\{U_\alpha\}$.
($\Leftarrow$) Suppose every open cover of A in S has a finite subcover, and let $\{V_\alpha\}$ be an open cover of A in A. Then each V_α is equal to $U_\alpha \cap A$ for some open set U_α in S. Since

$$A \subset \bigcup_\alpha V_\alpha \subset \bigcup_\alpha U_\alpha,$$

by hypothesis there are finitely many sets U_{α_i} such that $A \subset \bigcup_i U_{\alpha_i}$. Hence,

$$A \subset \left(\bigcup_i U_{\alpha_i}\right) \cap A = \bigcup_i (U_{\alpha_i} \cap A) = \bigcup_i V_{\alpha_i}.$$

So $\{V_{\alpha_i}\}$ is a finite subcover of $\{V_\alpha\}$ that covers A. Therefore, A is compact. □

Proposition A.30. A closed subset F of a compact topological space S is compact.
Proof. Let $\{U_\alpha\}$ be an open cover of F in S. The collection $\{U_\alpha, S - F\}$ is then an open cover of S. By the compactness of S, there is a finite subcover $\{U_{\alpha_i}, S - F\}$ that covers S, so $F \subset \bigcup_i U_{\alpha_i}$. This proves that F is compact. □

Proposition A.31. In a Hausdorff space S, it is possible to separate a compact subset K and a point p not in K by disjoint open sets; i.e., there exist an open set $U \supset K$ and an open set $V \ni p$ such that $U \cap V = \varnothing$.

Proof. By the Hausdorff property, for every $x \in K$, there are disjoint open sets $U_x \ni x$ and $V_x \ni p$. The collection $\{U_x\}_{x \in K}$ is a cover of K by open subsets of S. Since K is compact, it has a finite subcover $\{U_{x_i}\}$.

Let $U = \bigcup_i U_{x_i}$ and $V = \bigcap_i V_{x_i}$. Then U is an open set of S containing K. Being the intersection of finitely many open sets containing p, V is an open set containing p. Moreover, the set

$$U \cap V = \bigcup_i (U_{x_i} \cap V)$$

is empty, since each $U_{x_i} \cap V$ is contained in $U_{x_i} \cap V_{x_i}$, which is empty. □

Proposition A.32. Every compact subset K of a Hausdorff space S is closed
Proof. By the preceding proposition, for every point p in $S - K$, there is an open set V such that $p \in V \subset S - K$. This proves that $S - K$ is open. Hence, K is closed.

Exercise A. 33 (Compact Hausdorff space).* Prove that a compact Hausdorff space is normal. (Normality was defined in Definition A.16.)

Proposition A.34. The image of a compact set under a continuous map is compact.
Proof. Let $f : X \to Y$ be a continuous map and K a compact subset of X. Suppose $\{U_\alpha\}$ is a cover of $f(K)$ by open subsets of Y. Since f is continuous, the inverse images $f^{-1}(U_\alpha)$ are all open. Moreover,

$$K \subset f^{-1}(f(K)) \subset f^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha f^{-1}(U_\alpha)$$

So $\{f^{-1}(U_\alpha)\}$ is an open cover of K in X. By the compactness of K, there is a finite subcollection $\{f^{-1}(U_{\alpha_i})\}$ such that

$$K \subset \bigcup_i f^{-1}(U_{\alpha_i}) = f^{-1}\left(\bigcup_i U_{\alpha_i}\right)$$

Then $f(K) \subset \bigcup_i U_{\alpha_i}$. Thus, $f(K)$ is compact.
Proposition A.35. A continuous map $f : X \to Y$ from a compact space X to a Hausdorff space Y is a closed map.

Proof. Let F be a closed subset of the compact space X. By Proposition A.30, F is compact. As the image of a compact set under a continuous map, $f(F)$ is compact in Y (Proposition A.34). As a compact subset of the Hausdorff space $Y, f(F)$ is closed (Proposition A.32).

A continuous bijection $f : X \to Y$ whose inverse is also continuous is called a homeomorphism.

Corollary A.36. A continuous bijection $f : X \to Y$ from a compact space X to a Hausdorff space Y is a homeomorphism.

Proof. By Proposition A.28, to show that $f^{-1} : Y \to X$ is continuous, it suffices to prove that for every closed set F in X, the set $(f^{-1})^{-1}(F) = f(F)$ is closed in Y, i.e., that f is a closed map. The corollary then follows from Proposition A.35.

Exercise A. 37 (Finite union of compact sets). Prove that a finite union of compact subsets of a topological space is compact.

We mention without proof an important result. For a proof, see [29, Theorem 26.7, p. 167, and Theorem 37.3, p. 234].

Theorem A. 38 (The Tychonoff theorem). The product of any collection of compact spaces is compact in the product topology.

13.1.9 A. 9 Boundedness in $\mathbb{R}^n$

A subset A of $\mathbb{R}^n$ is said to be bounded if it is contained in some open ball $B(p, r)$; otherwise, it is unbounded.

Proposition A.39. A compact subset of $\mathbb{R}^n$ is bounded.
Proof. If A were an unbounded subset of $\mathbb{R}^n$, then the collection $\{B(0, i)\}_{i=1}^\infty$ of open balls with radius increasing to infinity would be an open cover of A in $\mathbb{R}^n$ that does not have a finite subcover. □

By Propositions A. 39 and A.32, a compact subset of $\mathbb{R}^n$ is closed and bounded. The converse is also true.

Theorem A. 40 (The Heine-Borel theorem). A subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

For a proof, see for example [29].

13.1.10 A. 10 Connectedness

Definition A.41. A topological space S is disconnected if it is the union $S = U \cup V$ of two disjoint nonempty open subsets U and V (Figure A.10). It is connected if it is not disconnected. A subset A of S is disconnected if it is disconnected in the subspace topology.

Fig. A.10. A disconnected space.

Proposition A.42. A subset A of a topological space S is disconnected if and only if there are open sets U and V in S such that
(i) $U \cap A \neq \varnothing, V \cap A \neq \varnothing$,
(ii) $U \cap V \cap A = \varnothing$,
(iii) $A \subset U \cup V$.

A pair of open sets in S with these properties is called a separation of A (Figure A.11).

Proof. Problem A.15. □

Fig. A.11. A separation of A.

Proposition A.43. The image of a connected space X under a continuous map $f : X \to Y$ is connected.

Proof. Suppose $f(X)$ is not connected. Then there is a separation $\{U, V\}$ of $f(X)$ in Y. By the continuity of f, both $f^{-1}(U)$ and $f^{-1}(V)$ are open in X. We claim that $\{f^{-1}(U), f^{-1}(V)\}$ is a separation of X.
(i) Since $U \cap f(X) \neq \varnothing$, the open set $f^{-1}(U)$ is nonempty.
(ii) If $x \in f^{-1}(U) \cap f^{-1}(V)$, then $f(x) \in U \cap V \cap f(X) = \varnothing$, a contradiction. Hence, $f^{-1}(U) \cap f^{-1}(V)$ is empty.
(iii) Since $f(X) \subset U \cup V$, we have $X \subset f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.

The existence of a separation of X contradicts the connectedness of X. This contradiction proves that $f(X)$ is connected. □

Proposition A.44. In a topological space S, the union of a collection of connected subsets A_α having a point p in common is connected.

Proof. Suppose $\cup_\alpha A_\alpha = U \cup V$, where U and V are disjoint open subsets of $\cup_\alpha A_\alpha$. The point $p \in \bigcup_\alpha A_\alpha$ belongs to U or V. Assume without loss of generality that $p \in U$.

For each α,

$$A_\alpha = A_\alpha \cap (U \cup V) = (A_\alpha \cap U) \cup (A_\alpha \cap V).$$

The two open sets $A_\alpha \cap U$ and $A_\alpha \cap V$ of A_α are clearly disjoint. Since $p \in A_\alpha \cap U$, $A_\alpha \cap U$ is nonempty. By the connectedness of A_α, $A_\alpha \cap V$ must be empty for all α. Hence,

$$V = \left(\bigcup_\alpha A_\alpha\right) \cap V = \bigcup_\alpha (A_\alpha \cap V)$$

is empty. So $\bigcup_\alpha A_\alpha$ must be connected. □

13.1.11 A. 11 Connected Components

Let x be a point in a topological space S. By Proposition A.44, the union C_x of all connected subsets of S containing x is connected. It is called the connected component of S containing x.

Proposition A.45. Let C_x be a connected component of a topological space S. Then a connected subset A of S is either disjoint from C_x or is contained entirely in C_x.

Proof. If A and C_x have a point in common, then by Proposition A.44, $A \cup C_x$ is a connected set containing x. Hence, $A \cup C_x \subset C_x$, which implies that $A \subset C_x$. □

Accordingly, the connected component C_x is the largest connected subset of S containing x in the sense that it contains every connected subset of S containing x.

Corollary A.46. For any two points x, y in a topological space S, the connected components C_x and C_y either are disjoint or coincide.

Proof. If C_x and C_y are not disjoint, then by Proposition A.45, they are contained in each other. In this case, $C_x = C_y$. □

As a consequence of Corollary A.46, the connected components of S partition S into disjoint subsets.

13.1.12 A. 12 Closure

Let S be a topological space and A a subset of S.

Definition A.47. The closure of A in S, denoted by $\bar{A}, \mathrm{cl}(A)$, or $\mathrm{cl}_S(A)$, is defined to be the intersection of all the closed sets containing A.

The advantage of the bar notation $\bar{A}$ is its simplicity, while the advantage of the $\mathrm{cl}_S(A)$ notation is its indication of the ambient space S. If $A \subset B \subset S$, then the closure of A in B and the closure of A in S need not be the same. In this case, it is useful to have the notations $\mathrm{cl}_B(A)$ and $\mathrm{cl}_M(A)$ for the two closures.

As an intersection of closed sets, $\bar{A}$ is a closed set. It is the smallest closed set containing A in the sense that any closed set containing A contains $\bar{A}$.

Proposition A. 48 (Local characterization of closure). Let A be a subset of a topological space S. A point $p \in S$ is in the closure $\mathrm{cl}(A)$ if and only if every neighborhood of p contains a point of A (Figure A.12).

Here by "local," we mean a property satisfied by a basis of neighborhoods at a point.

Proof. We will prove the proposition in the form of its contrapositive:

$$p \notin \mathrm{cl}(A) \quad \Longleftrightarrow \quad \text{there is a neighborhood of } p \text{ disjoint from } A.$$

($\Rightarrow$) Suppose

$$p \notin \mathrm{cl}(A) = \bigcap \{F \text{ closed in } S \mid F \supset A\}.$$

Then $p \notin$ some closed set F containing A. It follows that $p \in S - F$, an open set disjoint from A.
($\Longleftarrow$) Suppose $p \in$ an open set U disjoint from A. Then the complement $F := S - U$ is a closed set containing A and not containing p. Therefore, $p \notin \mathrm{cl}(A)$. $\square$

Fig. A.12. Every neighborhood of p contains a point of A.

Example. The closure of the open disk $B(\mathbf{0}, r)$ in $\mathbb{R}^2$ is the closed disk

$$\bar{B}(\mathbf{0}, r) = \left\{p \in \mathbb{R}^2 \mid d(p, \mathbf{0}) \leq r\right\}.$$

Definition A.49. A point p in S is an accumulation point of A if every neighborhood of p in S contains a point of A other than p. The set of all accumulation points of A is denoted by $\mathrm{ac}(A)$.

If U is a neighborhood of p in S, we call $U - \{p\}$ a deleted neighborhood of p. An equivalent condition for p to be an accumulation point of A is to require that every deleted neighborhood of p in S contain a point of A. In some books an accumulation point is called a limit point.
Example. If $A = [0, 1[\cup\{2\}$ in $\mathbb{R}^1$, then the closure of A is $[0, 1] \cup \{2\}$, but the set of accumulation points of A is only the closed interval [0,1].

Proposition A.50. Let A be a subset of a topological space S. Then

$$\mathrm{cl}(A) = A \cup \mathrm{ac}(A).$$

Proof.

($\supset$) By definition, $A \subset \mathrm{cl}(A)$. By the local characterization of closure (Proposition A.48), $\mathrm{ac}(A) \subset \mathrm{cl}(A)$. Hence, $A \cup \mathrm{ac}(A) \subset \mathrm{cl}(A)$.
(C) Suppose $p \in \mathrm{cl}(A)$. Either $p \in A$ or $p \notin A$. If $p \in A$, then $p \in A \cup \mathrm{ac}(A)$. Suppose $p \notin A$. By Proposition A.48, every neighborhood of p contains a point of A, which cannot be p, since $p \notin A$. Therefore, every deleted neighborhood of p contains a point of A. In this case,

$$p \in \mathrm{ac}(A) \subset A \cup \mathrm{ac}(A).$$

So $\mathrm{cl}(A) \subset A \cup \mathrm{ac}(A)$. $\square$

Proposition A.51. A set A is closed if and only if $A = \bar{A}$.

Proof.

($\Leftarrow$) If $A = \bar{A}$, then A is closed because $\bar{A}$ is closed.
($\Rightarrow$) Suppose A is closed. Then A is a closed set containing A, so that $\bar{A} \subset A$. Because $A \subset \bar{A}$, equality holds. $\square$

Proposition A.52. If $A \subset B$ in a topological space S, then $\bar{A} \subset \bar{B}$.
Proof. Since $\bar{B}$ contains B, it also contains A. As a closed subset of S containing A, it contains $\bar{A}$ by definition.

Exercise A. 53 (Closure of a finite union or finite intersection). Let A and B be subsets of a topological space S. Prove the following:
(a) $\overline{A \cup B} = \bar{A} \cup \bar{B}$,
(b) $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$.

The example of $A =]a, 0[$ and $B =]0, b[$ in the real line shows that in general, $\overline{A \cap B} \neq \bar{A} \cap \bar{B}$.

13.1.13 A. 13 Convergence

Let S be a topological space. A sequence in S is a map from the set $\mathbb{Z}^+$of positive integers to S. We write a sequence as $\langle x_i \rangle$ or $x_1, x_2, x_3, \ldots$.
Definition A.54. The sequence $\langle x_i \rangle$ converges to p if for every neighborhood U of p, there is a positive integer N such that for all $i \geq N, x_i \in U$. In this case we say that p is a limit of the sequence $\langle x_i \rangle$ and write $x_i \to p$ or $\lim_{i \to \infty} x_i = p$.

Proposition A.55 (Uniqueness of the limit). In a Hausdorff space S, if a sequence $\langle x_i \rangle$ converges to p and to q, then $p = q$.

Proof. Problem A.19.
Thus, in a Hausdorff space we may speak of the limit of a convergent sequence.
Proposition A. 56 (The sequence lemma). Let S be a topological space and A a subset of S. If there is a sequence $\langle a_i \rangle$ in A that converges to p, then $p \in \mathrm{cl}(A)$. The converse is true if S is first countable.

Proof.

($\Rightarrow$) Suppose $a_i \to p$, where $a_i \in A$ for all i. By the definition of convergence, every neighborhood U of p contains all but finitely many of the points a_i. In particular, U contains a point in A. By the local characterization of closure (Proposition A.48), $p \in \mathrm{cl}(A)$.
($\Leftarrow$) Suppose $p \in \mathrm{cl}(A)$. Since S is first countable, we can find a countable basis of neighborhoods $\{U_n\}$ at p such that

$$U_1 \supset U_2 \supset \cdots$$

By the local characterization of closure, in each U_i there is a point $a_i \in A$. We claim that the sequence $\langle a_i \rangle$ converges to p. If U is any neighborhood of p, then by the definition of a basis of neighborhoods at p, there is a U_N such that $p \in U_N \subset U$. For all $i \geq N$, we then have

$$U_i \subset U_N \subset U$$

Therefore, for all $i \geq N$,

$$a_i \in U_i \subset U$$

This proves that $\langle a_i \rangle$ converges to p. $\square$

Problems

A.1. Set Theory

If U_1 and U_2 are subsets of a set X, and V_1 and V_2 are subsets of set Y, prove that

$$(U_1 \times V_1) \cap (U_2 \times V_2) = (U_1 \cap U_2) \times (V_1 \cap V_2)\,.$$

A.2. Union and Intersection

Suppose $U_1 \cap V_1 = U_2 \cap V_2 = \varnothing$ in a topological space S. Show that the intersection $U_1 \cap U_2$ is disjoint from the union $V_1 \cup V_2$. (Hint: Use the distributive property of an intersection over a union.)

A.3. Closed Sets

Let S be a topological space. Prove the following two statements.
(a) If $\{F_i\}_{i=1}^n$ is a finite collection of closed sets in S, then $\bigcup_{i=1}^n F_i$ is closed.
(b) If $\{F_\alpha\}_{\alpha\in \mathrm{A}}$ is an arbitrary collection of closed sets in S, then $\bigcap_\alpha F_\alpha$ is closed.

A.4. Cubes Versus Balls

Prove that the open cube $]-a, a[^n$ is contained in the open ball $B(\mathbf{0}, \sqrt{n}a)$, which in turn is contained in the open cube $]-\sqrt{n}a, \sqrt{n}a[^n$. Therefore, open cubes with arbitrary centers in $\mathbb{R}^n$ form a basis for the standard topology on $\mathbb{R}^n$.

A.5. Product of Closed Sets

Prove that if A is closed in X and B is closed in Y, then $A \times B$ is closed in $X \times Y$.

A.6. Characterization of a Hausdorff Space by its Diagonal

Let S be a topological space. The diagonal Δ in $S \times S$ is the set

$$\Delta = \{(x, x) \in S \times S\}$$

Prove that S is Hausdorff if and only if the diagonal Δ is closed in $S \times S$. (Hint: Prove that S is Hausdorff if and only if $S \times S - \Delta$ is open in $S \times S$.)

A.7. Projection

Prove that if X and Y are topological spaces, then the projection $\pi : X \times Y \to X, \pi(x, y) = x$, is an open map.

A.8. The $\varepsilon - \delta$ Criterion for Continuity

Prove that a function $f : A \to \mathbb{R}^m$ is continuous at $p \in A$ if and only if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x \in A$ satisfying $d(x, p) < \delta$, one has $d(f(x), f(p)) < \varepsilon$.

A.9. Continuity in Terms of Closed Sets

Prove Proposition A.28.

A.10. Continuity of a map Into a Product

Let X, Y_1, and Y_2 be topological spaces. Prove that a map $f = (f_1, f_2) : X \to Y_1 \times Y_2$ is continuous if and only if both components $f_i : X \to Y_i$ are continuous.

A.11. Continuity of the Product map

Given two maps $f : X \to X'$ and $g : Y \to Y'$ of topological spaces, we define their product to be

$$f \times g : X \times Y \to X' \times Y', \quad (f \times g)(x, y) = (f(x), g(y)).$$

Note that if $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ are the two projections, then $f \times g = (f \circ \pi_1, f \circ \pi_2)$. Prove that $f \times g$ is continuous if and only if both f and g are continuous.

A.12. Homeomorphism

Prove that if a continuous bijection $f : X \to Y$ is a closed map, then it is a homeomorphism (cf. Corollary A.36).

A.13.* the Lindelöf Condition

Show that if a topological space is second countable, then it is Lindelöf; i.e., every open cover has a countable subcover.

A.14. Compactness

Prove that a finite union of compact sets in a topological space S is compact.

A.15.* Disconnected Subset in Terms of a Separation

Prove Proposition A.42.

A.16. Local Connectedness

A topological space S is said to be locally connected at $p \in S$ if for every neighborhood U of p, there is a connected neighborhood V of p such that $V \subset U$. The space S is locally connected if it is locally connected at every point. Prove that if S is locally connected, then the connected components of S are open.

A.17. Closure

Let U be an open subset and A an arbitrary subset of a topological space S. Prove that $U \cap \bar{A} \neq \varnothing$ if and only if $U \cap A \neq \varnothing$.

A.18. Countability

Prove that every second-countable space is first countable.

A.19.* Uniqueness of the Limit

Prove Proposition A.55.

A.20.* Closure in a Product

Let S and Y be topological spaces and $A \subset S$. Prove that

$$\mathrm{cl}_{S\times Y}(A \times Y) = \mathrm{cl}_S(A) \times Y$$

in the product space $S \times Y$.

A.21. Dense Subsets

A subset A of a topological space S is said to be dense in S if its closure$_{\mathrm{cl}_S(A) \text{ equals } S}$.
(a) Prove that A is dense in S if and only if for every $p \in S$, every neighborhood U of p contains a point of A.
(b) Let K be a field. Prove that a Zariski-open subset U of K^n is dense in K^n. (Hint: Example A.18.)

13.2 B. The Inverse Function Theorem on $\mathbb{R}^n$ and Related Results

This appendix reviews three logically equivalent theorems from real analysis: the inverse function theorem, the implicit function theorem, and the constant rank theorem, which describe the local behavior of a C^∞ map from $\mathbb{R}^n$ to $\mathbb{R}^m$. We will assume the inverse function theorem and from it deduce the other two in the simplest cases. In Section 11 these theorems are applied to manifolds in order to clarify the local behavior of a C^∞ map when the map has maximal rank at a point or constant rank in a neighborhood.

13.2.1 B. 1 the Inverse Function Theorem

A C^∞ map $f : U \to \mathbb{R}^n$ defined on an open subset U of $\mathbb{R}^n$ is locally invertible or a local diffeomorphism at a point p in U if f has a C^∞ inverse in some neighborhood of p. The inverse function theorem gives a criterion for a map to be locally invertible. We call the matrix $Jf = [\partial f^i/\partial x^j]$ of partial derivatives of f the Jacobian matrix of f and its determinant $\det[\partial f^i/\partial x^j]$ the Jacobian determinant of f.

Theorem B. 1 (Inverse function theorem). Let $f : U \to \mathbb{R}^n$ be a C^∞ map defined on an open subset U of $\mathbb{R}^n$. At any point p in U, the map f is invertible in some neighborhood of p if and only if the Jacobian determinant $\det[\partial f^i/\partial x^j(p)]$ is not zero.

For a proof, see for example [35, Theorem 9.24, p. 221]. Although the inverse function theorem apparently reduces the invertibility of f on an open set to a single number at p, because the Jacobian determinant is a continuous function, the nonvanishing of the Jacobian determinant at p is equivalent to its nonvanishing in a neighborhood of p.

Since the linear map represented by the Jacobian matrix $Jf(p)$ is the best linear approximation to f at p, it is plausible that f is invertible in a neighborhood of p if and only if $Jf(p)$ is also, i.e., if and only if $\det(Jf(p)) \neq 0$.

13.2.2 B. 2 the Implicit Function Theorem

In an equation such as $f(x, y) = 0$, it is often impossible to solve explicitly for one of the variables in terms of the other. If we can show the existence of a function $y = h(x)$, which we may or may not be able to write down explicitly, such that $f(x, h(x)) = 0$, then we say that $f(x, y) = 0$ can be solved implicitly for y in terms of x. The implicit function theorem provides a sufficient condition on a system of equations $f^i(x^1, \ldots, x^n) = 0, i = 1, \ldots, m$, under which locally a set of variables can be solved implicitly as C^∞ functions of the other variables.

Example. Consider the equation

$$f(x,y) = x^2 + y^2 - 1 = 0.$$

The solution set is the unit circle in the xy-plane.

Fig. B.1. The unit circle.

From the picture we see that in a neighborhood of any point other than $(\pm 1, 0)$, y is a function of x. Indeed,

$$y = \pm\sqrt{1 - x^2}$$

and either function is C^∞ as long as $x \neq \pm 1$. At $(\pm 1, 0)$, there is no neighborhood on which y is a function of x.

On a smooth curve $f(x,y) = 0$ in $\mathbb{R}^2$,

$$\begin{aligned}
&y \text{ can be expressed as a function of } x \text{ in a neighborhood of a point } (a,b)\\
&\iff \text{the tangent line to } f(x,y) = 0 \text{ at } (a,b) \text{ is not vertical}\\
&\iff \text{the normal vector } \operatorname{grad} f := \langle f_x, f_y \rangle \text{ to } f(x,y) = 0 \text{ at } (a,b)\\
&\text{is not horizontal}\\
&\iff f_y(a,b) \neq 0.
\end{aligned}$$

The implicit function theorem generalizes this condition to higher dimensions. We will deduce the implicit function theorem from the inverse function theorem.

Theorem B. 2 (Implicit function theorem). Let U be an open subset in $\mathbb{R}^n \times \mathbb{R}^m$ and $f : U \to \mathbb{R}^m$ a C^∞ map. Write $(x,y) = (x^1, \ldots, x^n, y^1, \ldots, y^m)$ for a point in U. At a point $(a,b) \in U$ where $f(a,b) = 0$ and the determinant $\det[\partial f^i / \partial y^j(a,b)]$ is nonzero, there exist a neighborhood $A \times B$ of (a,b) in U and a unique function $h : A \to B$ such that in $A \times B \subset U \subset \mathbb{R}^n \times \mathbb{R}^m$,

$$f(x,y) = 0 \iff y = h(x).$$

Moreover, h is C^∞.

Proof. To solve $f(x,y) = 0$ for y in terms of x using the inverse function theorem, we first turn it into an inverse problem. For this, we need a map between two open

Fig. B.2. $F^{-1}
$ maps the u-axis to the zero set of f.}

sets of the same dimension. Since $f(x,y)$ is a map from an open set U in $\mathbb{R}^{n+m}$ to $\mathbb{R}^m$, it is natural to extend f to a map $F : U \to \mathbb{R}^{n+m}$ by adjoining x to it as the first n components:

$$F(x,y) = (u,v) = (x, f(x,y)).$$

To simplify the exposition, we will assume in the rest of the proof that $n = m = 1$. Then the Jacobian matrix of F is

$$JF = \begin{bmatrix} 1 & 0 \\ \partial f/\partial x & \partial f/\partial y \end{bmatrix}.$$

At the point (a,b),

$$\det JF(a,b) = \frac{\partial f}{\partial y}(a,b) \neq 0.$$

By the inverse function theorem, there are neighborhoods U_1 of $(\ a,b\ )$ and V_1 of $F(a,b) = (a,0)$ in $\mathbb{R}^2$ such that $F : U_1 \to V_1$ is a diffeomorphism with C^∞ inverse F^{-1} (Figure B.2). Since $F : U_1 \to V_1$ is defined by

$$\begin{aligned} u &= x \\ v &= f(x,y) \end{aligned}$$

the inverse map $F^{-1} : V_1 \to U_1$ must be of the form

$$\begin{aligned} x &= u \\ y &= g(u,v) \end{aligned}$$

for some C^∞ function $g : V_1 \to \mathbb{R}$. Thus, $F^{-1}(u,v) = (u, g(u,v))$.
The two compositions $F^{-1} \circ F$ and $F \circ F^{-1}$ give

$$\begin{aligned} (x,y) &= \left(F^{-1} \circ F\right)(x,y) = F^{-1}(x, f(x,y)) = (x, g(x, f(x,y))), \\ (u,v) &= \left(F \circ F^{-1}\right)(u,v) = F(u, g(u,v)) = (u, f(u, g(u,v))). \end{aligned}$$

Hence,

$$\begin{aligned} y &= g(x, f(x,y)) \quad \text{for all } (x,y) \in U_1, \\ v &= f(u, g(u,v)) \quad \text{for all } (u,v) \in V_1. \end{aligned} \tag{B.2}$$

If $f(x,y) = 0$, then (B.1) gives $y = g(x,0)$. This suggests that we define $h(x) = g(x,0)$ for all $x \in \mathbb{R}^1$ for which $(x,0) \in V_1$. The set of all such x is homeomorphic to $V_1 \cap (\mathbb{R}^1 \times \{0\})$ and is an open subset of $\mathbb{R}^1$. Since g is C^∞ by the inverse function theorem, h is also C^∞.

Claim. For $(x,y) \in U_1$ such that $(x,0) \in V_1$,

$$f(x,y) = 0 \quad \Longleftrightarrow \quad y = h(x).$$

Proof (of Claim).
($\Rightarrow$) As we saw already, from (B.1), if $f(x, y) = 0$, then

$$y = g(x, f(x, y)) = g(x, 0) = h(x). \tag{B.3}$$

($\Leftarrow$) If $y = h(x)$ and in (B.2) we set $(u, v) = (x, 0)$, then

$$0 = f(x, g(x, 0)) = f(x, h(x)) = f(x, y).$$

By the claim, in some neighborhood of $(a, b) \in U_1$, the zero set of $f(x, y)$ is precisely the graph of h. To find a product neighborhood of (a, b) as in the statement of the theorem, let $A_1 \times B$ be a neighborhood of $(\ a, b\ )$ contained in U_1 and let $A = h^{-1}(B) \cap A_1$. Since h is continuous, A is open in the domain of h and hence in $\mathbb{R}^1$. Then $h(A) \subset B$,

$$A \times B \subset A_1 \times B \subset U_1, \quad \text{and} \quad A \times \{0\} \subset V_1.$$

By the claim, for $(x, y) \in A \times B$,

$$f(x, y) = 0 \quad \Longleftrightarrow \quad y = h(x).$$

Equation (B.3) proves the uniqueness of h.
Replacing a partial derivative such as $\partial f/\partial y$ with a Jacobian matrix $[\partial f^i/\partial y^j]$, we can prove the general case of the implicit function theorem in exactly the same way. Of course, in the theorem $y^1, \ldots, y^m$ need not be the last m coordinates in $\mathbb{R}^{n+m}$; they can be any set of m coordinates in $\mathbb{R}^{n+m}$.

Theorem B.3. The implicit function theorem is equivalent to the inverse function theorem.

Proof. We have already shown, at least for one typical case, that the inverse function theorem implies the implicit function theorem. We now prove the reverse implication.

So assume the implicit function theorem, and let $f : U \to \mathbb{R}^n$ be a C^∞ map defined on an open subset U of $\mathbb{R}^n$ such that at some point $p \in U$, the Jacobian determinant
$\det [\partial f^i/\partial x^j(p)]$ is nonzero. Finding a local inverse for $y = f(x)$ near p amounts to solving the equation

$$g(x, y) = f(x) - y = 0$$

for x in terms of y near $(p, f(p))$. Note that $\partial g^i/\partial x^j = \partial f^i/\partial x^j$. Hence,

$$\det \left[\frac{\partial g^i}{\partial x^j}(p, f(p))\right] = \det \left[\frac{\partial f^i}{\partial x^j}(p)\right] \neq 0.$$

By the implicit function theorem, x can be expressed in terms of y locally near $(p, f(p))$; i.e., there is a C^∞ function $x = h(y)$ defined in a neighborhood of $f(p)$ in $\mathbb{R}^n$ such that

$$g(x, y) = f(x) - y = f(h(y)) - y = 0.$$

Thus, $y = f(h(y))$. Since $y = f(x)$,

$$x = h(y) = h(f(x)).$$

Therefore, f and h are inverse functions defined near p and $f(p)$ respectively.

13.2.3 B. 3 Constant Rank Theorem

Every C^∞ map $f : U \to \mathbb{R}^m$ on an open set U of $\mathbb{R}^n$ has a rank at each point p in U, namely the rank of its Jacobian matrix $[\partial f^i/\partial x^j(p)]$.

Theorem B. 4 (Constant rank theorem). If $f : \mathbb{R}^n \supset U \to \mathbb{R}^m$ has constant rank k in a neighborhood of a point $p \in U$, then after a suitable change of coordinates near p in U and $f(p)$ in $\mathbb{R}^m$, the map f assumes the form

$$\left(x^1, \ldots, x^n\right) \mapsto \left(x^1, \ldots, x^k, 0, \ldots, 0\right).$$

More precisely, there are a diffeomorphism G of a neighborhood of p in U sending p to the origin in $\mathbb{R}^n$ and a diffeomorphism F of a neighborhood of $f(p)$ in $\mathbb{R}^m$ sending $f(p)$ to the origin in $\mathbb{R}^m$ such that

$$(F \circ f \circ G)^{-1}\left(x^1, \ldots, x^n\right) = \left(x^1, \ldots, x^k, 0, \ldots, 0\right).$$

Proof (for $n = m = 2, k = 1$). Suppose $f = (f^1, f^2) : \mathbb{R}^2 \supset U \to \mathbb{R}^2$ has constant rank 1 in a neighborhood of $p \in U$. By reordering the functions f^1, f^2 or the variables x, y, we may assume that $\partial f^1/\partial x(p) \neq 0$. (Here we are using the fact that f has rank ≥ 1 at p.) Define $G : U \to \mathbb{R}^2$ by

$$G(x, y) = (u, v) = \left(f^1(x, y), y\right).$$

The Jacobian matrix of G is

$$JG = \begin{bmatrix} \partial f^1/\partial x & \partial f^1/\partial y \\ 0 & 1 \end{bmatrix}$$

Since $\det JG(p) = \partial f^1/\partial x(p) \neq 0$, by the inverse function theorem there are neighborhoods U_1 of $p \in \mathbb{R}^2$ and V_1 of $G(p) \in \mathbb{R}^2$ such that $G : U_1 \to V_1$ is a diffeomorphism. By making U_1 a sufficiently small neighborhood of p, we may assume that f has constant rank 1 on U_1.

On V_1,

$$(u, v) = \left(G \circ G^{-1}\right)(u, v) = \left(f^1 \circ G^{-1}, y \circ G^{-1}\right)(u, v).$$

Comparing the first components gives $u = \left(f^1 \circ G^{-1}\right)(u, v)$. Hence,

$$\begin{aligned} \left(f \circ G^{-1}\right)(u, v) &= \left(f^1 \circ G^{-1}, f^2 \circ G^{-1}\right)(u, v) \\ &= \left(u, f^2 \circ G^{-1}(u, v)\right) \\ &= (u, h(u, v)), \end{aligned}$$

where we set $h = f^2 \circ G^{-1}$.

Because $G^{-1} : V_1 \to U_1$ is a diffeomorphism and f has constant rank 1 on U_1, the composite $f \circ G^{-1}$ has constant rank 1 on V_1. Its Jacobian matrix is

$$J\left(f \circ G^{-1}\right) = \begin{bmatrix} 1 & 0 \\ \partial h/\partial u & \partial h/\partial v \end{bmatrix}.$$

For this matrix to have constant rank $1, \partial h/\partial \nu$ must be identically zero on V_1. (Here we are using the fact that f has rank ≤ 1 in a neighborhood of p.) Thus, h is a function of u alone and we may write

$$\left(f \circ G^{-1}\right)(u, v) = (u, h(u))$$

Finally, let $F : \mathbb{R}^2 \to \mathbb{R}^2$ be the change of coordinates $F(x, y) = (x, y - h(x))$. Then

$$\left(F \circ f \circ G^{-1}\right)(u, v) = F(u, h(u)) = (u, h(u) - h(u)) = (u, 0).$$

Example B.5. If a C^∞ map $f : \mathbb{R}^n \supset U \to \mathbb{R}^n$ defined on an open subset U of $\mathbb{R}^n$ has nonzero Jacobian determinant $\det(Jf(p))$ at a point $p \in U$, then by continuity it has nonzero Jacobian determinant in a neighborhood of p. Therefore, it has constant rank n in a neighborhood of p.

13.2.4 Problems

B.1.* the Rank of a Matrix

The rank of a matrix A, denoted by $\operatorname{rk} A$, is defined to be the number of linearly independent columns of A. By a theorem in linear algebra, it is also the number of linearly independent rows of A. Prove the following lemma.
Lemma. Let A be an $m \times n$ matrix (not necessarily square), and k a positive integer. Then $\operatorname{rk} A \geq k$ if and only if A has a nonsingular $k \times k$ submatrix. Equivalently, $\operatorname{rk} A \leq k - 1$ if and only if all $k \times k$ minors of A vanish. ($Ak \times k$ minor of a matrix A is the determinant of a $k \times k$ submatrix of A.)

B.2.* Matrices of Rank at Most r

For an integer $r \geq 0$, define D_r to be the subset of $\mathbb{R}^{m \times n}$ consisting of all $m \times n$ real matrices of rank at most r. Show that D_r is a closed subset of $\mathbb{R}^{m \times n}$. (Hint: Use Problem B.1.)

B.3.* Maximal Rank

We say that the rank of an $m \times n$ matrix A is maximal if $\operatorname{rk} A = \min(m, n)$. Define $D_{\max}$ to be the subset of $\mathbb{R}^{m \times n}$ consisting of all $m \times n$ matrices of maximal rank r. Show that $D_{\max}$ is an open subset of $\mathbb{R}^{m \times n}$. (Hint: Suppose $n \leq m$. Then $D_{\max} = \mathbb{R}^{m \times n} - D_{n-1}$. Apply Problem B.2.)

B.4.* Degeneracy Loci and Maximal-rank Locus of a map

Let $F : S \to \mathbb{R}^{m \times n}$ be a continuous map from a topological space S to the space $\mathbb{R}^{m \times n}$. The degeneracy locus of rank r of F is defined to be

$$D_r(F) := \{x \in S \mid \operatorname{rk} F(x) \leq r\}.$$

(a) Show that the degeneracy locus $D_r(F)$ is a closed subset of S. (Hint: $D_r(F) = F^{-1}(D_r)$, where D_r was defined in Problem B.2.)
(b) Show that the maximal-rank locus of F,

$$D_{\max}(F) := \{x \in S \mid \operatorname{rk} F(x) \text{ is maximal }\},$$

is an open subset of S.

B.5. Rank of a Composition of Linear Maps

Suppose V, W, V', W' are finite-dimensional vector spaces.
(a) Prove that if the linear map $L : V \to W$ is surjective, then for any linear map $f : W \to W'$, $\operatorname{rk}(f \circ L) = \operatorname{rk} f$.
(b) Prove that if the linear map $L : V \to W$ is injective, then for any linear map $g : V' \to V$, $\operatorname{rk}(L \circ g) = \operatorname{rk} g$.

B.6. Constant Rank Theorem

Generalize the proof of the constant rank theorem (Theorem B.4) in the text to arbitrary n, m, and k.

B.7. Equivalence of the Constant Rank Theorem and the Inverse Function Theorem

Use the constant rank theorem (Theorem B.4) to prove the inverse function theorem (Theorem B.1). Hence, the two theorems are equivalent.

13.3 §C Existence of a Partition of Unity in General

This appendix contains a proof of Theorem 13.7 on the existence of a C^∞ partition of unity on a general manifold.

Lemma C.1. Every manifold M has a countable basis all of whose elements have compact closure.

Recall that if A is a subset of a topological space X, the notation $\bar{A}$ denotes the closure of A in X.

Proof (of Lemma C.1). Start with a countable basis $\mathcal{B}$ for M and consider the subcollection $\mathcal{S}$ of elements in $\mathcal{B}$ that have compact closure. We claim that $\mathcal{S}$ is again a basis. Given an open subset $U \subset M$ and point $p \in U$, choose a neighborhood V of p such that $V \subset U$ and V has compact closure. This is always possible since M is locally Euclidean.

Since $\mathcal{B}$ is a basis, there is an open set $B \in \mathcal{B}$ such that

$$p \in B \subset V \subset U.$$

Then $\bar{B} \subset \bar{V}$. Because $\bar{V}$ is compact, so is the closed subset $\bar{B}$. Hence, $B \in \mathcal{S}$. Since for any open set U and any $p \in U$, we have found a set $B \in \mathcal{S}$ such that $p \in B \subset U$, the collection $\mathcal{S}$ of open sets is a basis.

Proposition C.2. Every manifold M has a countable increasing sequence of subsets

$$V_1 \subset \overline{V_1} \subset V_2 \subset \overline{V_2} \subset \cdots,$$

with each V_i open and $\bar{V}_i$ compact, such that M is the union of the V_i 's (Figure C.1).

Proof. By Lemma C.1, M has a countable basis $\{B_i\}_{i=1}^\infty$ with each $\overline{B_i}$ compact. Any basis of M of course covers M. Set $V_1 = B_1$. By compactness, $\overline{V_1}$ is covered by finitely many of the B_i 's. Define i_1 to be the smallest integer ≥ 2 such that

$$\overline{V_1} \subset B_1 \cup B_2 \cup \cdots \cup B_{i_1}.$$

Suppose open sets $V_1, \ldots, V_m$ have been defined, each with compact closure. As before, by compactness, $\overline{V_m}$ is covered by finitely many of the B_i 's. If i_m is the smallest integer $\geq m+1$ and $\geq i_{m-1}$ such that

$$\overline{V_m} \subset B_1 \cup B_2 \cup \cdots \cup B_{i_m}$$

then we set

$$V_{m+1} = B_1 \cup B_2 \cup \cdots \cup B_{i_m}.$$

Since a finite union of compact sets is compact and

$$\overline{V_{m+1}} \subset \overline{B_1} \cup \overline{B_2} \cup \cdots \cup \overline{B_{i_m}}$$

is a closed subset of a compact set, $\overline{V_{m+1}}$ is compact. Since $i_m \geq m+1, B_{m+1} \subset V_{m+1}$. Thus,

$$M = \bigcup B_i \subset \bigcup V_i \subset M.$$

This proves that $M = \bigcup_{i=1}^{\infty} V_i$. $\square$

Fig. C.1. A nested open cover.

Define V_0 to be the empty set. For each $i \geq 1$, because $\overline{V_{i+1}} - V_i$ is a closed subset of the compact $\overline{V_{i+1}}$, it is compact. Moreover, it is contained in the open set $V_{i+2} - \overline{V_{i-1}}$.

Theorem 13.7 (Existence of a C^∞ partition of unity). Let $\{U_\alpha\}_{\alpha \in \mathrm{A}}$ be an open cover of a manifold M.
(i) There is a C^∞ partition of unity $\{\varphi_k\}_{k=1}^{\infty}$ with every φ_k having compact support such that for each $k, \operatorname{supp} \varphi_k \subset U_\alpha$ for some $\alpha \in \mathrm{A}$.
(ii) If we do not require compact support, then there is a C^∞ partition of unity $\{\rho_\alpha\}$ subordinate to $\{U_\alpha\}$.

Proof.

(i) Let $\{V_i\}_{i=0}^{\infty}$ be an open cover of M as in Proposition C.2, with V_0 the empty set. The idea of the proof is quite simple. For each i, we find finitely many smooth bump functions ψ_j^i on M, each with compact support in the open set $V_{i+2} - \overline{V_{i-1}}$ as well as in some U_α, such that their sum $\sum_j \psi_j^i$ is positive on the compact set $\overline{V_{i+1}} - V_i$. The collection $\left\{\operatorname{supp} \psi_j^i\right\}$ of supports over all i, j will be locally finite. Since the compact sets $\overline{V_{i+1}} - V_i$ cover M, the locally finite sum $\psi = \sum_{i,j} \psi_j^i$ will be positive on M. Then $\left\{\psi_j^i / \psi\right\}$ is a C^∞ partition of unity satisfying the conditions in (i).

We now fill in the details. Fix an integer $i \geq 1$. For each p in the compact set $\overline{V_{i+1}} - V_i$, choose an open set U_α containing p from the open cover $\{U_\alpha\}$. Then p is in the open set $U_\alpha \cap \left(V_{i+2} - \overline{V_{i-1}}\right)$. Let ψ_p be a C^∞ bump function on M that is positive on a neighborhood W_p of p and has support in $U_\alpha \cap \left(V_{i+2} - \overline{V_{i-1}}\right)$. Since $\operatorname{supp} \psi_p$ is a closed set contained in the compact set $\overline{V_{i+2}}$, it is compact.

The collection $\left\{W_p \mid p \in \overline{V_{i+1}} - V_i\right\}$ is an open cover of the compact set $\overline{V_{i+1}} - V_i$, and so there is a finite subcover $\{W_{p_1}, \ldots, W_{p_m}\}$, with associated bump functions $\psi_{p_1}, \ldots, \psi_{p_m}$. Since m, W_{p_j}, and ψ_{p_j} all depend on i, we relabel them as $m(i)$, $W_1^i, \ldots, W_{m(i)}^i$, and $\psi_1^i, \ldots, \psi_{m(i)}^i$.

In summary, for each $i \geq 1$, we have found finitely many open sets $W_1^i, \ldots, W_{m(i)}^i$ and finitely many C^∞ bump functions $\psi_1^i, \ldots, \psi_{m(i)}^i$ such that
(1) $\psi_j^i > 0$ on W_j^i for $j = 1, \ldots, m(i)$;
(2) $W_1^i, \ldots, W_{m(i)}^i$ cover the compact set $\overline{V_{i+1}} - V_i$;
(3) $\operatorname{supp} \psi_j^i \subset U_{\alpha_{ij}} \cap \left(V_{i+2} - \overline{V_{i-1}}\right)$ for some $\alpha_{ij} \in \mathrm{A}$;
(4) $\operatorname{supp} \psi_j^i$ is compact.

As i runs from 1 to ∞, we obtain countably many bump functions $\{\psi_j^i\}$. The collection of their supports, $\{\operatorname{supp}\psi_j^i\}$, is locally finite, since only finitely many of these sets intersect any V_i. Indeed, since

$$\operatorname{supp}\psi_j^\ell \subset V_{\ell+2} - \overline{V_{\ell-1}}$$

for all ℓ, as soon as $\ell \geq i+1$,

$$\left(\operatorname{supp}\psi_j^\ell\right) \cap V_i = \text{ the empty set } \varnothing$$

Any point $p \in M$ is contained in the compact set $\overline{V_{i+1}} - V_i$ for some i, and therefore $p \in W_j^i$ for some (i,j). For this $(i,j), \psi_j^i(p) > 0$. Hence, the sum $\psi := \sum_{i,j}\psi_j^i$ is locally finite and everywhere positive on M. To simplify the notation, we now relabel the countable set $\{\psi_j^i\}$ as $\{\psi_1, \psi_2, \psi_3, \ldots\}$. Define

$$\varphi_k = \frac{\psi_k}{\psi}$$

Then $\sum \varphi_k = 1$ and

$$\operatorname{supp}\varphi_k = \operatorname{supp}\psi_k \subset U_\alpha$$

for some $\alpha \in \mathrm{A}$. So $\{\varphi_k\}$ is a partition of unity with compact support such that for each k, $\operatorname{supp}\varphi_k \subset U_\alpha$ for some $\alpha \in \mathrm{A}$.
(ii) For each $k = 1, 2, \ldots$, let $\tau(k)$ be an index in A such that

$$\operatorname{supp}\varphi_k \subset U_{\tau(k)}$$

as in the preceding paragraph. Group the collection $\{\varphi_k\}$ according to $\tau(k)$ and define

$$\rho_\alpha = \sum_{\tau(k)=\alpha} \varphi_k$$

if there is a k with $\tau(k) = \alpha$; otherwise, set $\rho_\alpha = 0$. Then

$$\sum_{\alpha\in\mathrm{A}} \rho_\alpha = \sum_{\alpha\in\mathrm{A}} \sum_{\tau(k)=\alpha} \varphi_k = \sum_{k=1}^{\infty} \varphi_k = 1$$

By Problem 13.7,

$$\operatorname{supp}\rho_\alpha \subset \bigcup_{\tau(k)=\alpha} \operatorname{supp}\varphi_k \subset U_\alpha$$

Hence, $\{\rho_\alpha\}$ is a C^∞ partition of unity subordinate to $\{U_\alpha\}$.

13.4 §D Linear Algebra

This appendix gathers together a few facts from linear algebra used throughout the book, especially in Sections 24 and 25.

The quotient vector space is a construction in which one reduces a vector space to a smaller space by identifying a subspace to zero. It represents a simplification, much like the formation of a quotient group or of a quotient ring. For a linear map $f : V \to W$ of vector spaces, the first isomorphism theorem of linear algebra gives an isomorphism between the quotient space $V/\ker f$ and the image of f. It is one of the most useful results in linear algebra.

We also discuss the direct sum and the direct product of a family of vector spaces, as well as the distinction between an internal and an external direct sum.

13.4.1 D. 1 Quotient Vector Spaces

If V is a vector space and W is a subspace of V, a coset of W in V is a subset of the form

$$v + W = \{v + w \mid w \in W\}$$

for some $v \in V$.
Two cosets $v+W$ and $v'+W$ are equal if and only if $v' = v+w$ for some $w \in W$, or equivalently, if and only if $v' - v \in W$. This introduces an equivalence relation on V :

$$v \sim v' \quad \Longleftrightarrow \quad v' - v \in W \quad \Longleftrightarrow \quad v + W = v' + W.$$

A coset of W in V is simply an equivalence class under this equivalence relation. Any element of $v + W$ is called a representative of the coset $v + W$.

The set V/W of all cosets of W in V is again a vector space, with addition and scalar multiplication defined by

$$\begin{aligned}(u + W) + (v + W) &= (u + v) + W,\\ r(v + W) &= rv + W\end{aligned}$$

for $u, v \in V$ and $r \in \mathbb{R}$. We call V/W the quotient vector space or the quotient space of V by W.

Example D.1. For $V = \mathbb{R}^2$ and W a line through the origin in $\mathbb{R}^2$, a coset of W in $\mathbb{R}^2$ is a line in $\mathbb{R}^2$ parallel to W. (For the purpose of this discussion, two lines in $\mathbb{R}^2$ are parallel if and only if they coincide or fail to intersect. This definition differs from the usual one in plane geometry in allowing a line to be parallel to itself.) The quotient space $\mathbb{R}^2/W$ is the collection of lines in $\mathbb{R}^2$ parallel to W (Figure D.1).

Fig. D.1. Quotient vector space of $\mathbb{R
^{2}$ by W.}

13.4.2 D. 2 Linear Transformations

Let V and W be vector spaces over $\mathbb{R}$. A map $f : V \to W$ is called a linear transformation, a vector space homomorphism, a linear operator, or a linear map over $\mathbb{R}$ if for all $u, v \in V$ and $r \in \mathbb{R}$,

$$\begin{aligned}f(u + v) &= f(u) + f(v),\\ f(ru) &= rf(u).\end{aligned}$$

Example D.2. Let $V = \mathbb{R}^2$ and W a line through the origin in $\mathbb{R}^2$ as in Example D.1. If L is a line through the origin not parallel to W, then L will intersect each line in $\mathbb{R}^2$ parallel to W in one and only one point. This one-to-one correspondence

$$\begin{aligned} L &\to \mathbb{R}^2/W, \\ v &\mapsto v + W, \end{aligned}$$

preserves addition and scalar multiplication, and so is an isomorphism of vector spaces. Thus, in this example the quotient space $\mathbb{R}^2/W$ can be identified with the line L.

If $f : V \to W$ is a linear transformation, the kernel of f is the set

$$\ker f = \{v \in V \mid f(v) = 0\}$$

and the image of f is the set

$$\operatorname{im} f = \{f(v) \in W \mid v \in V\}.$$

The kernel of f is a subspace of V and the image of f is a subspace of W. Hence, one can form the quotient spaces $V/\ker f$ and $W/\operatorname{im} f$. This latter space, $W/\operatorname{im} f$, denoted by coker f, is called the cokernel of the linear map $f : V \to W$.

For now, denote by K the kernel of f. The linear map $f : V \to W$ induces a linear map $\bar{f} : V/K \to \operatorname{im} f$, by

$$\bar{f}(v + K) = f(v).$$

It is easy to check that $\bar{f}$ is linear and bijective. This gives the following fundamental result of linear algebra.

Theorem D. 3 (The first isomorphism theorem). Let $f : V \to W$ be a homomorphism of vector spaces. Then f induces an isomorphism

$$\bar{f} : \frac{V}{\ker f} \xrightarrow{\sim} \operatorname{im} f.$$

13.4.3 D. 3 Direct Product and Direct Sum

Let $\{V_\alpha\}_{\alpha \in I}$ be a family of real vector spaces. The direct product $\prod_\alpha V_\alpha$ is the set of all sequences (v_α) with $v_\alpha \in V_\alpha$ for all $\alpha \in I$, and the direct sum $\bigoplus_\alpha V_\alpha$ is the subset of the direct product $\prod_\alpha V_\alpha$ consisting of sequences (v_α) such that $v_\alpha = 0$ for all but finitely many $\alpha \in I$. Under componentwise addition and scalar multiplication,

$$\begin{aligned} (v_\alpha) + (w_\alpha) &= (v_\alpha + w_\alpha), \\ r(v_\alpha) &= (rv_\alpha), \quad r \in \mathbb{R}, \end{aligned}$$

both the direct product $\prod_\alpha V_\alpha$ and the direct sum $\oplus_\alpha V_\alpha$ are real vector spaces. When the index set I is finite, the direct sum coincides with the direct product. In particular, for two vector spaces A and B,

$$A \oplus B = A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

The sum of two subspaces A and B of a vector space V is the subspace

$$A + B = \{a + b \in V \mid a \in A, b \in B\}.$$

If $A \cap B = \{0\}$, this sum is called an internal direct sum and written $A \oplus_i B$. In an internal direct sum $A \oplus_i B$, every element has a representation as $a + b$ for a unique $a \in A$ and a unique $b \in B$. Indeed, if $a + b = a' + b' \in A \oplus_i B$, then

$$a - a' = b' - b \in A \cap B = \{0\}.$$

Hence, $a = a'$ and $b = b'$.
In contrast to the internal direct sum $A \oplus_i B$, the direct sum $A \oplus B$ is called the external direct sum. In fact, the two notions are isomorphic: the natural map

$$\begin{aligned} \varphi : A \oplus B &\to A \oplus_i B, \\ (a, b) &\mapsto a + b \end{aligned}$$

is easily seen to be a linear isomorphism. For this reason, in the literature the internal direct sum is normally denoted by $A \oplus B$, just like the external direct sum.

If $V = A \oplus_i B$, then A is called a complementary subspace to B in V. In Example D.2, the line L is a complementary subspace to W, and we may identify the quotient vector space $\mathbb{R}^2/W$ with any complementary subspace to W.

In general, if W is a subspace of a vector space V and W' is a complementary subspace to W, then there is a linear map

$$\begin{aligned} \varphi : W' &\to V/W, \\ w' &\mapsto w' + W. \end{aligned}$$

Exercise D.4. Show that $\varphi : W' \to V/W$ is an isomorphism of vector spaces.
Thus, the quotient space V/W may be identified with any complementary subspace to W in V. This identification is not canonical, for there are many complementary subspaces to a given subspace W and there is no reason to single out any one of them. However, when V has an inner product $\langle , \rangle$, *onecansingleoutacanonical* complementary subspace, the orthogonal complement of W :

$$W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}.$$

Exercise D.5. Check that $W^\perp$ is a complementary subspace to W.
In this case, there is a canonical identification $W^\perp \xrightarrow{\sim} V/W$.
Let $f : V \to W$ be a linear map of finite-dimensional vector spaces. It follows from the first isomorphism theorem and Problem D. 1 that

$$\dim V - \dim(\ker f) = \dim(\operatorname{im} f).$$

Since the dimension is the only isomorphism invariant of a vector space, we therefore have the following corollary of the first isomorphism theorem.

Corollary D.6. If $f : V \to W$ is a linear map of finite-dimensional vector spaces, then there is a vector space isomorphism

$$V \simeq \ker f \oplus \operatorname{im} f$$

(The right-hand side is an external direct sum because $\ker f$ and $\operatorname{im} f$ are not subspaces of the same vector space.)

13.4.4 Problems

D.1. Dimension of a Quotient Vector Space

Prove that if $w_1, \ldots, w_m$ is a basis for W that extends to a basis $w_1, \ldots, w_m, v_1, \ldots, v_n$ for V, then $v_1 + W, \ldots, v_n + W$ is a basis for V/W. Therefore,

$$\dim V/W = \dim V - \dim W.$$

D.2. Dimension of a Direct sum

Prove that if $a_1, \ldots, a_m$ is a basis for a vector space A and $b_1, \ldots, b_n$ is a basis for a vector space B, then $(a_i, 0), (0, b_j), i = 1, \ldots, m, j = 1, \ldots, n$, is a basis for the direct sum $A \oplus B$. Therefore,

$$\dim A \oplus B = \dim A + \dim B.$$

13.5 §E Quaternions and the Symplectic Group

First described by William Rowan Hamilton in 1843, quaternions are elements of the form

$$q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d, \quad a, b, c, d \in \mathbb{R},$$

that add componentwise and multiply according to the distributive property and the rules

$$\begin{aligned} &\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1 \\ &\mathbf{ij} = \mathbf{k}, \quad \mathbf{jk} = \mathbf{i}, \quad \mathbf{ki} = \mathbf{j} \\ &\mathbf{ij} = -\mathbf{ji}, \quad \mathbf{jk} = -\mathbf{kj}, \quad \mathbf{ki} = -\mathbf{ik} \end{aligned}$$

A mnemonic for the three rules $\mathbf{ij} = \mathbf{k}, \mathbf{jk} = \mathbf{i}, \mathbf{ki} = \mathbf{j}$ is that in going clockwise around the circle

the product of two successive elements is the next one. Under addition and multiplication, the quaternions satisfy all the properties of a field except the commutative property for multiplication. Such an algebraic structure is called a skew field or a division ring. In honor of Hamilton, the usual notation for the skew field of quaternions is $\mathbb{H}$.

A division ring that is also an algebra over a field K is called a division algebra over K. The real and complex fields $\mathbb{R}$ and $\mathbb{C}$ are commutative division algebras over $\mathbb{R}$. By a theorem of

Ferdinand Georg Frobenius [13] from 1878, the skew field $\mathbb{H}$ of quaternions has the distinction of being the only (associative) division algebra over $\mathbb{R}$ other than $\mathbb{R}$ and $\mathbb{C}$.[1]

In this appendix we will derive the basic properties of quaternions and define the symplectic group in terms of quaternions. Because of the familiarity of complex matrices, quaternions are often represented by 2×2 complex matrices. Correspondingly, the symplectic group also has a description as a group of complex matrices.

One can define vector spaces and formulate linear algebra over a skew field, just as one would for vector spaces over a field. The only difference is that over a skew field it is essential to keep careful track of the order of multiplication. A vector space over $\mathbb{H}$ is called a quaternionic vector space. We denote by $\mathbb{H}^n$ the quaternionic vector space of n-tuples of quaternions. There are many potential pitfalls stemming from a choice of left and right, for example:

(1) Should $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be written on the left or on the right of a scalar?
(2) Should scalars multiply on the left or on the right of $\mathbb{H}^n$?
(3) Should elements of $\mathbb{H}^n$ be represented as column vectors or as row vectors?
(4) Should a linear transformation be represented by multiplication by a matrix on the left or on the right?
(5) In the definition of the quaternion inner product, should one conjugate the first or the second argument?
(6) Should a sesquilinear form on $\mathbb{H}^n$ be conjugate-linear in the first or the second argument?

The answers to these questions are not arbitrary, since the choice for one question may determine the correct choices for all the others. A wrong choice will lead to inconsistencies.

13.5.1 E. 1 Representation of Linear Maps by Matrices

Relative to given bases, a linear map of vector spaces over a skew field will also be represented by a matrix. Since maps are written on the left of their arguments as in $f(x)$, we will choose our convention so that a linear map f corresponds to left multiplication by a matrix. In order for a vector in $\mathbb{H}^n$ to be multiplied on the left by a matrix, the elements of $\mathbb{H}^n$ must be column vectors, and for left multiplication by a matrix to be a linear map, scalar multiplication on $\mathbb{H}^n$ should be on the right. In this way, we have answered (1), (2), (3), and (4) above.

Let K be a skew field and let V and W be vector spaces over K, with scalar multiplication on the right. A map $f : V \to W$ is linear over K or K-linear if for all $x, y \in V$ and $q \in K$,

$$\begin{aligned} f(x+y) &= f(x) + f(y), \\ f(xq) &= f(x)q. \end{aligned}$$

An endomorphism or a linear transformation of a vector space V over K is a K linear map from V to itself. The endomorphisms of V over K form an algebra over K, denoted by $\mathrm{End}_K(V)$. An endomorphism $f : V \to V$ is invertible if it has a twosided inverse, i.e., a linear map $g : V \to V$ such that $f \circ g = g \circ f = \mathbb{1}_V$. An invertible endomorphism of V is also called an automorphism of V. The general linear group $\mathrm{GL}(V)$ is by definition the group of all automorphisms of the vector space V. When $V = K^n$, we also write $\mathrm{GL}(n, K)$ for $\mathrm{GL}(V)$.

To simplify the presentation, we will discuss matrix representation only for endomorphisms of the vector space K^n. Let e_i be the column vector with 1 in the i th row and 0 everywhere else. The set $e_1, \ldots, e_n$ is called the standard basis for K^n. If $f : K^n \to K^n$ is K-linear, then

$$f(e_j) = \sum_i e_i a_j^i$$

[01] If one allows an algebra to be nonassociative, then there are other division algebras over $\mathbb{R}$, for example Cayley's octonians.

for some matrix $A = \left[a_j^i\right] \in K^{n\times n}$, called the matrix of f (relative to the standard basis). Here a_j^i is the entry in the i th row and j th column of the matrix A. For $x = \sum_j e_j x^j \in K^n$,

$$f(x) = \sum_j f\left(e_j\right) x^j = \sum_{i,j} e_i a_j^i x^j$$

Hence, the i th component of the column vector $f(x)$ is

$$(f(x))^i = \sum_j a_j^i x^j$$

In matrix notation,

$$f(x) = Ax$$

If $g : K^n \to K^n$ is another linear map and $g\left(e_j\right) = \sum_i e_i b_j^i$, then

$$\left(f \circ g\right)\left(e_j\right) = f\left(\sum_k e_k b_j^k\right) = \sum_k f\left(e_k\right) b_j^k = \sum_{i,k} e_i a_k^i b_j^k$$

Thus, if $A = \left[a_j^i\right]$ and $B = \left[b_j^i\right]$ are the matrices representing f and g respectively, then the matrix product AB is the matrix representing the composite $f \circ g$. Therefore, there is an algebra isomorphism

$$\operatorname{End}_K\left(K^n\right) \xrightarrow{\sim} K^{n\times n}$$

between endomorphisms of K^n and $n \times n$ matrices over K. Under this isomorphism, the group $\mathrm{GL}(n, K)$ corresponds to the group of all invertible $n \times n$ matrices over K.

13.5.2 E. 2 Quaternionic Conjugation

The conjugate of a quaternion $q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ is defined to be

$$\bar{q} = a - \mathbf{i}b - \mathbf{j}c - \mathbf{k}d$$

It is easily shown that conjugation is an antihomomorphism from the ring $\mathbb{H}$ to itself: it preserves addition, but under multiplication,

$$\overline{pq} = \bar{q}\bar{p} \quad \text{for } p, q \in \mathbb{H}$$

The conjugate of a matrix $A = \left[a_j^i\right] \in \mathbb{H}^{m\times n}$ is $\bar{A} = \left[\overline{a_j^i}\right]$, obtained by conjugating each entry of A. The transpose A^T of the matrix A is the matrix whose (i, j)-entry is the (j, i)-entry of A. In contrast to the case for complex matrices, when A and B are quaternion matrices, in general

$$\overline{AB} \neq \bar{A}\bar{B}, \quad \overline{AB} \neq \bar{B}\bar{A}, \quad \text{and } (AB)^T \neq B^T A^T.$$

However, it is true that

$$\overline{AB}^T = \bar{B}^T \bar{A}^T$$

as one sees by a direct computation.

13.5.3 E. 3 Quaternionic Inner Product

The quaternionic inner product on $\mathbb{H}^n$ is defined to be

$$\langle x, y\rangle = \sum_i \overline{x^i} y^i = \bar{x}^T y, \quad x, y \in \mathbb{H}^n,$$

with conjugation on the first argument $x = \langle x^1, \ldots, x^n\rangle$. For any $q \in \mathbb{H}$,

$$\langle xq, y\rangle = \bar{q}\langle x, y\rangle \quad \text{and} \quad \langle x, yq\rangle = \langle x, y\rangle q.$$

If conjugation were on the second argument, then the inner product would not have the correct linearity property with respect to scalar multiplication on the right.

For quaternion vector spaces V and W, we say that a map $f : V \times W \to \mathbb{H}$ is sesquilinear over $\mathbb{H}$ if it is conjugate-linear on the left in the first argument and linear on the right in the second argument: for all $v \in V, w \in W$, and $q \in \mathbb{H}$,

$$\begin{aligned} f(vq, w) &= \bar{q} f(v, w), \\ f(v, wq) &= f(v, w) q. \end{aligned}$$

In this terminology, the quaternionic inner product is sesquilinear over $\mathbb{H}$.

13.5.4 E. 4 Representations of Quaternions by Complex Numbers

A quaternion can be identified with a pair of complex numbers:

$$q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d = (a + \mathbf{i}b) + \mathbf{j}(c - \mathbf{i}d) = u + \mathbf{j}v \longleftrightarrow (u, v).$$

Thus, $\mathbb{H}$ is a vector space over $\mathbb{C}$ with basis $1, \mathbf{j}$, and $\mathbb{H}^n$ is a vector space over $\mathbb{C}$ with basis $e_1, \ldots, e_n, \mathbf{j}e_1, \ldots, \mathbf{j}e_n$.

Proposition E.1. Let q be a quaternion and let u, v be complex numbers.
(i) If $q = u + \mathbf{j}v$, then $\bar{q} = \bar{u} - \mathbf{j}v$.
(ii) $\mathbf{j}u\mathbf{j}^{-1} = \bar{u}$.

Proof. Problem E.1.

By Proposition E.1(ii), for any complex vector $v \in \mathbb{C}^n$, one has $\mathbf{j}v = \bar{v}\mathbf{j}$. Although $\mathbf{j}e_i = e_i\mathbf{j}$, elements of $\mathbb{H}^n$ should be written as $u + \mathbf{j}v$, not as $u + v\mathbf{j}$, so that the map $\mathbb{H}^n \to \mathbb{C}^{2n}, u + \mathbf{j}v \mapsto (u, v)$, will be a complex vector space isomorphism.

For any quaternion $q = u + \mathbf{j}v$, left multiplication $\ell_q : \mathbb{H} \to \mathbb{H}$ by q is $\mathbb{H}$-linear and a fortiori $\mathbb{C}$-linear. Since

$$\begin{aligned} \ell_q(1) &= u + \mathbf{j}v \\ \ell_q(\mathbf{j}) &= (u + \mathbf{j}v)\mathbf{j} = -\bar{v} + \mathbf{j}\bar{u}, \end{aligned}$$

the matrix of ℓ_q as a $\mathbb{C}$-linear map relative to the basis $1, \mathbf{j}$ for $\mathbb{H}$ over $\mathbb{C}$ is the 2×2 complex matrix $\begin{bmatrix} u & -\bar{v} \\ v & \bar{u} \end{bmatrix}$. The map $\mathbb{H} \to \operatorname{End}_{\mathbb{C}}(\mathbb{C}^2), q \mapsto \ell_q$ is an injective algebra homomorphism over $\mathbb{R}$, giving rise to a representation of the quaternions by 2×2 complex matrices.

13.5.5 E. 5 Quaternionic Inner Product in Terms of Complex Components

Let $x = x_1 + \mathbf{j}x_2$ and $y = y_1 + \mathbf{j}y_2$ be in $\mathbb{H}^n$, with $x_1, x_2, y_1, y_2 \in \mathbb{C}^n$. We will express the quaternionic inner product $\langle x, y\rangle$ in terms of the complex vectors $x_1, x_2, y_1, y_2 \in \mathbb{C}^n$.

By Proposition E.1,

$$\begin{aligned}\langle x, y\rangle &= \bar{x}^T y = \left(\bar{x}_1^T - \mathbf{j}x_2^T\right)\left(y_1 + \mathbf{j}y_2\right) && (\text{ since } \bar{x} = \bar{x}_1 - \mathbf{j}x_2) \\ &= \left(\bar{x}_1^T y_1 + \bar{x}_2^T y_2\right) + \mathbf{j}\left(x_1^T y_2 - x_2^T y_1\right) \quad (\text{ since } x_2^T\mathbf{j} = \mathbf{j}\bar{x}_2^T \text{ and } \bar{x}_1^T\mathbf{j} = \mathbf{j}x_1^T)\end{aligned}$$

Let

$$\langle x, y\rangle_1 = \bar{x}_1^T y_1 + \bar{x}_2^T y_2 = \sum_{i=1}^n \bar{x}_1^i y_1^i + \bar{x}_2^i y_2^i$$

and

$$\langle x, y\rangle_2 = x_1^T y_2 - x_2^T y_1 = \sum_{i=1}^n x_1^i y_2^i - x_2^i y_1^i$$

So the quaternionic inner product $\langle,\rangle isthesumofaHermitianinnerproductand\mathbf{j}$ times a skew-symmetric bilinear form on $\mathbb{C}^{2n}$:

$$\langle,\rangle = \langle,\rangle_1 + \mathbf{j}\langle,\rangle_2.$$

Let $x = x_1 + \mathbf{j}x_2 \in \mathbb{H}^n$. By skew-symmetry, $\langle x, x\rangle_2 = 0$, so that

$$\langle x, x\rangle = \langle x, x\rangle_1 = \|x_1\|^2 + \|x_2\|^2 \geq 0.$$

The norm of a quaternionic vector $x = x_1 + \mathbf{j}x_2$ is defined to be

$$\|x\| = \sqrt{\langle x, x\rangle} = \sqrt{\|x_1\|^2 + \|x_2\|^2}$$

In particular, the norm of a quaternion $q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ is

$$\|q\| = \sqrt{a^2 + b^2 + c^2 + d^2}$$

13.5.6 E. 6 H-linearity in Terms of Complex Numbers

Recall that an $\mathbb{H}$-linear map of quaternionic vector spaces is a map that is additive and commutes with right multiplication r_q for any quaternion q.

Proposition E.2. Let V be a quaternionic vector space. A map $f : V \to V$ is $\mathbb{H}$-linear if and only if it is $\mathbb{C}$-linear and $f \circ r_{\mathbf{j}} = r_{\mathbf{j}} \circ f$.

Proof. ($\Rightarrow$) Clear.

($\Longleftarrow$) Suppose f is $\mathbb{C}$-linear and f commutes with $r_{\mathbf{j}}$. By $\mathbb{C}$-linearity, f is additive and commutes with r_u for any complex number u. Any $q \in \mathbb{H}$ can be written as $q = u + \mathbf{j}v$ for some $u, v \in \mathbb{C}$; moreover, $r_q = r_{u+\mathbf{j}v} = r_u + r_v \circ r_{\mathbf{j}}$ (note the order reversal in $r_{\mathbf{j}v} = r_v \circ r_{\mathbf{j}}$). Since f is additive and commutes with r_u, r_v, and $r_{\mathbf{j}}$, it commutes with r_q for any $q \in \mathbb{H}$. Therefore, f is $\mathbb{H}$-linear.

Because the map $r_{\mathbf{j}} : \mathbb{H}^n \to \mathbb{H}^n$ is neither $\mathbb{H}$-linear nor $\mathbb{C}$-linear, it cannot be represented by left multiplication by a complex matrix. If $q = u + \mathbf{j}v \in \mathbb{H}^n$, where $u, v \in \mathbb{C}^n$, then

$$r_{\mathbf{j}}(q) = q\mathbf{j} = (u + \mathbf{j}v)\mathbf{j} = -\bar{v} + \mathbf{j}\bar{u}$$

In matrix notation,

$$r_{\mathbf{j}}\left(\begin{bmatrix} u \\ v \end{bmatrix}\right) = \begin{bmatrix} -\bar{v} \\ \bar{u} \end{bmatrix} = c\left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}\right) = -c\left(J\begin{bmatrix} u \\ v \end{bmatrix}\right), \tag{E.1}$$

where c denotes complex conjugation and J is the 2×2 matrix $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

13.5.7 E. 7 Symplectic Group

Let V be a vector space over a skew field K with conjugation, and let $B : V \times V \to K$ be a bilinear or sesquilinear function over K. Such a function is often called a bilinear or sesquilinear form over K. A K-linear automorphism $f : V \to V$ is said to preserve the form B if

$$B(f(x), f(y)) = B(x, y) \quad \text{for all } x, y \in V$$

The set of these automorphisms is a subgroup of the general linear group $\mathrm{GL}(V)$.
When K is the skew field $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, and B is the Euclidean, Hermitian, or quaternionic inner product respectively on K^n, the subgroup of $\mathrm{GL}(n, K)$ consisting of automorphisms of K^n preserving each of these inner products is called the orthogonal, unitary, or symplectic group and denoted by $\mathrm{O}(n), \mathrm{U}(n)$, or $\mathrm{Sp}(n)$ respectively. Naturally, the automorphisms in these three groups are called orthogonal, unitary, or symplectic automorphisms.

In particular, the symplectic group is the group of automorphisms f of $\mathbb{H}^n$ such that

$$\langle f(x), f(y)\rangle = \langle x, y\rangle \quad \text{for all } x, y \in \mathbb{H}^n$$

In terms of matrices, if A is the quaternionic matrix of such an f, then

$$\langle f(x), f(y)\rangle = \overline{Ax}^T Ay = \bar{x}^T \bar{A}^T Ay = \bar{x}^T y \quad \text{for all } x, y \in \mathbb{H}^n$$

Therefore, $f \in \mathrm{Sp}(n)$ if and only if its matrix A satisfies $\bar{A}^T A = I$. Because $\mathbb{H}^n = \mathbb{C}^n \oplus \mathbf{j}\mathbb{C}^n$ is isomorphic to $\mathbb{C}^{2n}$ as a complex vector space and an $\mathbb{H}$-linear map is necessarily $\mathbb{C}$-linear, the group $\mathrm{GL}(n, \mathbb{H})$ is isomorphic to a subgroup of $\mathrm{GL}(2n, \mathbb{C})$ (see Problem E.2).

Example. Under the algebra isomorphisms $\mathrm{End}_{\mathbb{H}}(\mathbb{H}) \simeq \mathbb{H}$, elements of $\mathrm{Sp}(1)$ correspond to quaternions $q = a + \mathbf{i}b + \mathbf{j}c + \mathbf{k}d$ such that

$$\bar{q}q = a^2 + b^2 + c^2 + d^2 = 1.$$

These are precisely quaternions of norm 1. Therefore, under the chain of real vector space isomorphisms $\mathrm{End}_{\mathbb{H}}(\mathbb{H}) \simeq \mathbb{H} \simeq \mathbb{R}^4$, the group $\mathrm{Sp}(1)$ maps to S^3, the unit 3sphere in $\mathbb{R}^4$.

The complex symplectic group $\mathrm{Sp}(2n, \mathbb{C})$ is the subgroup of $\mathrm{GL}(2n, \mathbb{C})$ consisting of automorphisms of $\mathbb{C}^{2n}$ preserving the skew-symmetric bilinear form $B : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$,

$$B(x, y) = \sum_{i=1}^{n} x^i y^{n+i} - x^{n+i} y^i = x^T Jy, \quad J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix},$$

where I_n is the $n \times n$ identity matrix. If $f : \mathbb{C}^{2n} \times \mathbb{C}^{2n} \to \mathbb{C}$ is given by $f(x) = Ax$, then

$$\begin{aligned} f \in \mathrm{Sp}(2n, \mathbb{C}) &\iff B(f(x), f(y)) = B(x, y) \text{ for all } x, y \in \mathbb{C}^{2n} \\ &\iff (Ax)^T JAy = x^T \left(A^T JA\right) y = x^T Jy \text{ for all } x, y \in \mathbb{C}^{2n} \\ &\iff A^T JA = J \end{aligned}$$

Theorem E.3. Under the injection $\mathrm{GL}(n, \mathbb{H}) \hookrightarrow \mathrm{GL}(2n, \mathbb{C})$, the symplectic group $\mathrm{Sp}(n)$ maps isomorphically to the intersection $\mathrm{U}(2n) \cap \mathrm{Sp}(2n, \mathbb{C})$.

Proof.

$f \in \mathrm{Sp}(n)$
$\iff f : \mathbb{H}^n \to \mathbb{H}^n$ is $\mathbb{H}$-linear and preserves the quaternionic inner product
$\iff f : \mathbb{C}^{2n} \to \mathbb{C}^{2n}$ is $\mathbb{C}$-linear, $f \circ r_{\mathbf{j}} = r_{\mathbf{j}} \circ f$, and f preserves the Hermitian inner product and the standard skew-symmetric bilinear form on $\mathbb{C}^{2n}$ (by Proposition E. 2 and Section E.5)
$\iff f \circ r_{\mathbf{j}} = r_{\mathbf{j}} \circ f$ and $f \in \mathrm{U}(2n) \cap \mathrm{Sp}(2n, \mathbb{C})$.

We will now show that if $f \in \mathrm{U}(2n)$, then the condition $f \circ r_{\mathbf{j}} = r_{\mathbf{j}} \circ f$ is equivalent to $f \in \mathrm{Sp}(2n, \mathbb{C})$. Let $f \in \mathrm{U}(2n)$ and let A be the matrix of f relative to the standard basis in $\mathbb{C}^{2n}$. Then

$$\begin{aligned}
(f \circ r_{\mathbf{j}})(x) = (r_{\mathbf{j}} \circ f)(x) &\text{ for all } x \in \mathbb{C}^{2n} \\
&\Longleftrightarrow -Ac(Jx) = -c(JAx) \text{ for all } x \in \mathbb{C}^{2n} \quad (\text{ by (E.1)}) \\
&\Longleftrightarrow c(\bar{A}Jx) = c(JAx) \text{ for all } x \in \mathbb{C}^{2n} \\
&\Longleftrightarrow \bar{A}Jx = JAx \text{ for all } x \in \mathbb{C}^{2n} \\
&\Longleftrightarrow J = \bar{A}^{-1}JA \\
&\Longleftrightarrow J = A^T JA \\
&\Longleftrightarrow f \in \mathrm{Sp}(2n, \mathbb{C})
\end{aligned}$$

Therefore, the condition $f \circ r_{\mathbf{j}} = r_{\mathbf{j}} \circ f$ is redundant if $f \in \mathrm{U}(2n) \cap \mathrm{Sp}(2n, \mathbb{C})$. By the first paragraph of this proof, there is a group isomorphism $\mathrm{Sp}(n) \simeq \mathrm{U}(2n) \cap \mathrm{Sp}(2n, \mathbb{C})$.

13.5.8 Problems

E.1. Quaternionic Conjugation

Prove Proposition E.1.

E.2. Complex Representation of an $\overline{\sim}$-linear map

Suppose an $\mathbb{H}$-linear map $f : \mathbb{H}^n \to \mathbb{H}^n$ is represented relative to the standard basis $e_1, \ldots, e_n$ by the matrix $A = u + \mathbf{j}v \in \mathbb{H}^{n \times n}$, where $u, v \in \mathbb{C}^{n \times n}$. Show that as a $\mathbb{C}$-linear map, $f : \mathbb{H}^n \to \mathbb{H}^n$ is represented relative to the basis $e_1, \ldots, e_n, \mathbf{j}e_1, \ldots, \mathbf{j}e_n$ by the matrix $\begin{bmatrix} u & -\bar{v} \\ v & \bar{u} \end{bmatrix}$.

E.3. Symplectic and Unitary Groups of Small Dimension

For a field K, the special linear group $\mathrm{SL}(n, K)$ is the subgroup of $\mathrm{GL}(n, K)$ consisting of all automorphisms of K^n of determinant 1 , and the special unitary group $\mathrm{SU}(n)$ is the subgroup of $\mathrm{U}(n)$ consisting of unitary automorphisms of $\mathbb{C}^n$ of determinant 1 . Prove the following identifications or group isomorphisms.
(a) $\mathrm{Sp}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})$.
(b) $\mathrm{Sp}(1) \simeq \mathrm{SU}(2)$. (Hint: Use Theorem E. 3 and part (a).)
(c)

$$\mathrm{SU}(2) \simeq \left\{ \begin{bmatrix} u & -\bar{v} \\ v & \bar{u} \end{bmatrix} \in \mathbb{C}^{2 \times 2} \middle| \, u\bar{u} + v\bar{v} = 1 \right\}.$$

(Hint: Use part (b) and the representation of quaternions by 2×2 complex matrices in Subsection E.4.)

13.6 Solutions to Selected Exercises Within the Text

3.6 Inversions

As a matrix, $\tau = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 4 & 5 & 1 \end{bmatrix}$. Scanning the second row, we see that τ has four inversions: $(2,1), (3,1), (4,1), (5,1)$.

3.13 Symmetrizing Operator

A k-linear function $h : V \to \mathbb{R}$ is symmetric if and only if $\tau h = h$ for all $\tau \in S_k$. Now

$$\tau(Sf) = \tau \sum_{\sigma \in S_k} \sigma f = \sum_{\sigma \in S_k} (\tau\sigma) f.$$

As σ runs over all elements of the permutation groups S_k, so does $\tau\sigma$. Hence,

$$\sum_{\sigma \in S_k} (\tau\sigma) f = \sum_{\tau\sigma \in S_k} (\tau\sigma) f = Sf.$$

This proves that $\tau(Sf) = Sf$.

3.15 Alternating Operator

$$f(v_1, v_2, v_3) - f(v_1, v_3, v_2) + f(v_2, v_3, v_1) - f(v_2, v_1, v_3) + f(v_3, v_1, v_2) - f(v_3, v_2, v_1).$$

3.20 Wedge Product of two 2-covectors

$$\begin{aligned}
&(f \wedge g)(v_1, v_2, v_3, v_4) \\
&\quad = f(v_1, v_2) g(v_3, v_4) - f(v_1, v_3) g(v_2, v_4) + f(v_1, v_4) g(v_2, v_3) \\
&\qquad + f(v_2, v_3) g(v_1, v_4) - f(v_2, v_4) g(v_1, v_3) + f(v_3, v_4) g(v_1, v_2)
\end{aligned}$$

3.22 Sign of a Permutation

We can achieve the permutation τ from the initial configuration $1, 2, \ldots, k + \ell$ in k steps.
(1) First, move the element k to the very end across the ℓ elements $k + 1, \ldots, k + \ell$. This requires ℓ transpositions.
(2) Next, move the element $k - 1$ across the ℓ elements $k + 1, \ldots, k + \ell$.
(3) Then move the element $k - 2$ across the same ℓ elements, and so on.

Each of the k steps requires ℓ transpositions. In the end we achieve τ from the identity using ℓk transpositions.

Alternatively, one can count the number of inversions in the permutation τ. There are k inversions starting with $k + 1$, namely, $(k + 1, 1), \ldots, (k + 1, k)$. Indeed, for each $i = 1, \ldots, \ell$, there are k inversions starting with $k + i$. Hence, the total number of inversions in τ is $k\ell$. By Proposition 3.8, $\operatorname{sgn}(\tau) = (-1)^{k\ell}$.

4.3 A Basis for 3-covectors

By Proposition 3.29, a basis for $A_3(T_p(\mathbb{R}^4))$ is $(dx^1 \wedge dx^2 \wedge dx^3)_p, (dx^1 \wedge dx^2 \wedge dx^4)_p$, $(dx^1 \wedge dx^3 \wedge dx^4)_p, (dx^2 \wedge dx^3 \wedge dx^4)_p$.

4.4 Wedge Product of a 2 -form with a 1 -form

The $(2, 1)$-shuffles are $(1 < 2, 3), (1 < 3, 2), (2 < 3, 1)$, with respective signs,,+-+ . By equation (3.6),

$$(\omega \wedge \tau)(X, Y, Z) = \omega(X, Y)\tau(Z) - \omega(X, Z)\tau(Y) + \omega(Y, Z)\tau(X)$$

6.14 Smoothness of a map to a Circle

Without further justification, the fact that both $\cos t$ and $\sin t$ are C^∞ proves only the smoothness of $(\cos t, \sin t)$ as a map from $\mathbb{R}$ to $\mathbb{R}^2$. To show that $F : \mathbb{R} \to S^1$ is C^∞, we need to cover S^1 with charts (U_i, ϕ_i) and examine in turn each $\phi_i \circ F : F^{-1}(U_i) \to \mathbb{R}$. Let $\{(U_i, \phi_i) \mid i = 1, \ldots, 4\}$ be the atlas of Example 5.16. On $F^{-1}(U_1)$, $\phi_1 \circ F(t) = (x \circ F)(t) = \cos t$ is C^∞. On $F^{-1}(U_3)$, $\phi_3 \circ F(t) = \sin t$ is C^∞. Similar computations on $F^{-1}(U_2)$ and $F^{-1}(U_4)$ prove the smoothness of F.

6.18 Smoothness of a map to a Cartesian Product

Fix $p \in N$, let (U, ϕ) be a chart about p, and let $(V_1 \times V_2, \psi_1 \times \psi_2)$ be a chart about $(f_1(p), f_2(p))$. We will be assuming either (f_1, f_2) smooth or both f_i smooth. In either case, (f_1, f_2) is continuous. Hence, by choosing U sufficiently small, we may assume $(f_1, f_2)(U) \subset V_1 \times V_2$. Then

$$(\psi_1 \times \psi_2) \circ (f_1, f_2) \circ \phi^{-1} = \left(\psi_1 \circ f_1 \circ \phi^{-1}, \psi_2 \circ f_2 \circ \phi^{-1}\right)$$

maps an open subset of $\mathbb{R}^n$ to an open subset of $\mathbb{R}^{m_1+m_2}$. It follows that (f_1, f_2) is C^∞ at p if and only if both f_1 and f_2 are C^∞ at p.

7.11 Real Projective Space as a Quotient of a Sphere

Define $\bar{f} : \mathbb{R}P^n \to S^n/\sim$ by $\bar{f}([x]) = \left[\frac{x}{\|x\|}\right] \in S^n/\sim$. This map is well defined because $\bar{f}([tx]) = \left[\frac{tx}{|tx|}\right] = \left[\pm\frac{x}{\|x\|}\right] = \left[\frac{x}{\|x\|}\right]$. Note that if $\pi_1 : \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$ and $\pi_2 : S^n \to S^n/\sim$ are the projection maps, then there is a commutative diagram

$$\begin{tikzcd}
\mathbb{R}^n - \{0\} \arrow[r, "f"] \arrow[d, "\pi_1"'] & S^n \arrow[d, "\pi_2"] \\
\mathbb{R}P^n \arrow[r, "\bar{f}"'] & S^n/\sim .
\end{tikzcd}$$

By Proposition 7.1, $\bar{f}$ is continuous because $\pi_2 \circ f$ is continuous.
Next define $g : S^n \to \mathbb{R}^{n+1} - \{0\}$ by $g(x) = x$. This map induces a map $\bar{g} : S^n/\sim \to \mathbb{R}P^n$, $\bar{g}([x]) = [x]$. By the same argument as above, $\bar{g}$ is well defined and continuous. Moreover,

$$\bar{g} \circ \bar{f}([x]) = \left[\frac{x}{\|x\|}\right] = [x]$$
$$\bar{f} \circ \bar{g}([x]) = [x]$$

so $\bar{f}$ and $\bar{g}$ are inverses to each other.

8.14 Velocity Vector Versus the Calculus Derivative

As a vector at the point $c(t)$ in the real line, $c'(t)$ equals $a d/dx|_{c(t)}$ for some scalar a. Applying both sides of the equality to x, we get $c'(t)x = a dx/dx|_{c(t)} = a$. By the definition of $c'(t)$,

$$a = c'(t)x = c_*\left(\left.\frac{d}{dt}\right|_{c(t)}\right)x = \left.\frac{d}{dt}\right|_{c(t)} x \circ c = \left.\frac{d}{dt}\right|_{c(t)} c = \dot{c}(t).$$

Hence, $c'(t) = \dot{c}(t)d/\,dx|_{c(t)}$.

13.1 Bump Function Supported in an Open set

Let (V, ϕ) be a chart centered at q such that V is diffeomorphic to an open ball $B(0, r)$. Choose real numbers a and b such that

$$\bar{B}(0, a) \subset B(0, b) \subset \bar{B}(0, b) \subset B(0, r)$$

With the σ given in (13.2), the function $\sigma \circ \phi$, extended by zero to M, gives the desired bump function.

15.2 Left Multiplication

Let $i_a : G \to G \times G$ be the inclusion map $i_a(x) = (a, x)$. It is clearly C^∞. Then $\ell_a(x) = ax = (\mu \circ i_a)(x)$. Since $\ell_a = \mu \circ i_a$ is the composition of two C^∞ maps, it is C^∞. Moreover, because it has a two-sided C^∞ inverse $\ell_{a^{-1}}$, it is a diffeomorphism.

15.7 Space of Symmetric Matrices

Let

$$A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ * & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & a_{nn} \end{bmatrix}$$

be a symmetric matrix. The symmetry condition $a_{ji} = a_{ij}$ implies that the entries below the diagonal are determined by the entries above the diagonal, and that there are no further conditions on the the entries above or on the diagonal. Thus, the dimension of S_n is equal to the number of entries above or on the diagonal. Since there are n such entries in the first row, $n - 1$ in the second row, and so on,

$$\dim S_n = n + (n-1) + (n-2) + \cdots + 1 = \frac{n(n+1)}{2}$$

15.10 Induced Topology Versus the Subspace Topology

A basic open set in the induced topology on H is the image under f of an open interval in L. Such a set is not open in the subspace topology. A basic open set in the subspace topology on H is the intersection of H with the image of an open ball in $\mathbb{R}^2$ under the projection $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2$; it is a union of infinitely many open intervals. Thus, the subspace topology is a subset of the induced topology, but not vice versa.

15.15 Distributivity over a Convergent Series

(i) We may assume $a \neq 0$, for otherwise there is nothing to prove. Let $\varepsilon > 0$. Since $s_m \to s$, there exists an integer N such that for all $m \geq N$, $\|s - s_m\| < \varepsilon/\|a\|$. Then for $m \geq N$,

$$\|as - as_m\| \leq \|a\|\,\|s - s_m\| < \|a\|\left(\frac{\varepsilon}{\|a\|}\right) = \varepsilon.$$

Hence, $as_m \to as$.
(ii) Set $s_m = \sum_{k=0}^{m} b_k$ and $s = \sum_{k=0}^{\infty} b_k$. The convergence of the series $\sum_{k=0}^{\infty} b_k$ means that $s_m \to s$. By (i), $as_m \to as$, which means that the sequence $as_m = \sum_{k=0}^{m} ab_k$ converges to $a\sum_{k=0}^{\infty} b_k$. Hence, $\sum_{k=0}^{\infty} ab_k = a\sum_{k=0}^{\infty} b_k$.

18.5 Transition Formula for a 2-form

$$\begin{aligned}
a_{ij} &= \omega\left(\partial/\partial x^i, \partial/\partial x^j\right) = \sum_{k<\ell} b_{k\ell} dy^k \wedge dy^\ell \left(\partial/\partial x^i, \partial/\partial x^j\right) \\
&= \sum_{k<\ell} b_{k\ell} \left(dy^k\left(\partial/\partial x^i\right) dy^\ell\left(\partial/\partial x^j\right) - dy^k\left(\partial/\partial x^j\right) dy^\ell\left(\partial/\partial x^i\right)\right) \\
&= \sum_{k<\ell} b_{k\ell}\left(\frac{\partial y^k}{\partial x^i}\frac{\partial y^\ell}{\partial x^j} - \frac{\partial y^k}{\partial x^j}\frac{\partial y^\ell}{\partial x^i}\right) = \sum_{k<\ell} b_{k\ell}\frac{\partial\left(y^k, y^\ell\right)}{\partial\left(x^i, x^j\right)}
\end{aligned}$$

Alternative solution: by Proposition 18.3,

$$dy^k \wedge dy^\ell = \sum_{i<j} \frac{\partial\left(y^k, y^\ell\right)}{\partial\left(x^i, x^j\right)} dx^i \wedge dx^j$$

Hence,

$$\sum_{i<j} a_{ij} dx^i \wedge dx^j = \sum_{k<\ell} b_{k\ell} dy^k \wedge dy^\ell = \sum_{i<j}\sum_{k<\ell} b_{k\ell}\frac{\partial\left(y^k, y^\ell\right)}{\partial\left(x^i, x^j\right)} dx^i \wedge dx^j$$

Comparing the coefficients of $dx^i \wedge dx^j$ gives

$$a_{ij} = \sum_{k<\ell} b_{k\ell}\frac{\partial\left(y^k, y^\ell\right)}{\partial\left(x^i, x^j\right)}$$

22.2 Smooth Functions on a Nonopen set

By definition, for each p in S there are an open set U_p in $\mathbb{R}^n$ and a C^∞ function $\tilde{f}_p : U_p \to \mathbb{R}^m$ such that $f = \tilde{f}_p$ on $U_p \cap S$. Extend the domain of $\tilde{f}_p$ to $\mathbb{R}^n$ by defining it to be zero on $\mathbb{R}^n - U_p$. Let $U = \bigcup_{p\in S} U_p$. Choose a partition of unity $\{\sigma_p\}_{p\in S}$ on U subordinate to the open cover $\{U_p\}_{p\in S}$ of U and define the function $\tilde{f} : U \to \mathbb{R}^m$ by

$$\tilde{f} = \sum_{p\in S} \sigma_p \tilde{f}_p \tag{*}$$

Because $\operatorname{supp}\sigma_p \subset U_p$, the product $\sigma_p\tilde{f}_p$ is zero and hence smooth outside U_p; as a product of two C^∞ functions on $U_p, \sigma_p\tilde{f}_p$ is C^∞ on U_p. Therefore, $\sigma_p\tilde{f}_p$ is C^∞ on $\mathbb{R}^n$. Since the sum $(*)$ is locally finite, $\tilde{f}$ is well defined and C^∞ on $\mathbb{R}^n$ for the usual reason. (Every point $q \in U$ has a neighborhood W_q that intersects only finitely many of of the sets $\operatorname{supp}\sigma_p, p \in S$. Hence, the sum$(*)$ is a finite sum on W_q.)

Let $q \in S$. If $q \in U_p$, then $\tilde{f}_p(q) = f(q)$, and if $q \notin U_p$, then $\sigma_p(q) = 0$. Thus, for $q \in S$, one has

$$\tilde{f}(q) = \sum_{p\in S} \sigma_p(q)\tilde{f}_p(q) = \sum_{p\in S} \sigma_p(q) f(q) = f(q)$$

25.5 Connecting Homomorphism

The proof that the cohomology class of a is independent of the choice of b as preimage of c can be summarized in the commutative diagram

$$\begin{tikzcd}
da'' = a - a' \arrow[r, tail, "i"] & db - db' & \\
a'' \arrow[u, tail, "d"] \arrow[r, tail, "i"'] & b - b' \arrow[u, tail, "d"'] \arrow[r, two heads, mapsto, "j"'] & 0.
\end{tikzcd}$$

Suppose $b, b' \in B^k$ both map to c under j. Then $j(b - b') = jb - jb' = c - c = 0$. By the exactness at $B^k, b - b' = i(a'')$ for some $a'' \in A^k$.

With the choice of b as preimage, the element $d^*[c]$ is represented by a cocycle $a \in A^{k+1}$ such that $i(a) = db$. Similarly, with the choice of b' as preimage, the element $d^*[c]$ is represented by a cocycle $a' \in A^{k+1}$ such that $i(a') = db'$. Then $i(a - a') = d(b - b') = di(a'') = id(a'')$. Since i is injective, $a - a' = da''$, and thus $[a] = [a']$. This proves that $d^*[c]$ is independent of the choice of b.

The proof that the cohomology class of a is independent of the choice of c in the cohomology class $[c]$ can be summarized by the commutative diagram

$$\begin{tikzcd}
a - a' \arrow[r, tail, "i"] & db - db' = 0 & \\
 & b - b' \arrow[u, tail, "d"] \arrow[r, two heads, mapsto, "j"] & c - c' \\
 & b'' \arrow[u, tail, "d"] \arrow[r, two heads, mapsto, "j"'] & c''. \arrow[u, tail, "d"']
\end{tikzcd}$$

Suppose $[c] = [c'] \in H^k(\mathcal{C})$. Then $c - c' = dc''$ for some $c'' \in C^{k-1}$. By the surjectivity of $j : B^{k-1} \to C^{k-1}$, there is a $b'' \in B^{k-1}$ that maps to c'' under j. Choose $b \in B^k$ such that $j(b) = c$ and let $b' = b - db'' \in B^k$. Then $j(b') = j(b) - jdb'' = c - dj(b'') = c - dc'' = c'$. With the choice of b as preimage, $d^*[c]$ is represented by a cocycle $a \in A^{k+1}$ such that $i(a) = db$. With the choice of b' as preimage, $d^*[c]$ is represented by a cocycle $a' \in A^{k+1}$ such that $i(a') = db'$. Then

$$i(a - a') = d(b - b') = ddb'' = 0.$$

By the injectivity of $i, a = a'$, so $[a] = [a']$. This shows that $d^*[c]$ is independent of the choice of c in the cohomology class $[c]$. $\square$

A. 33 Compact Hausdorff Space

Let S be a compact Hausdorff space, and A, B two closed subsets of S. By Proposition A.30, A and B are compact. By Proposition A.31, for any $a \in A$ there are disjoint open sets $U_a \ni a$ and $V_a \supset B$. Since A is compact, the open cover $\{U_a\}_{a \in A}$ for A has a finite subcover $\{U_{a_i}\}_{i=1}^n$. Let $U = \bigcup_{i=1}^n U_{a_i}$ and $V = \bigcap_{i=1}^n V_{a_i}$. Then $A \subset U$ and $B \subset V$. The open sets U and V are disjoint because if $x \in U \cap V$, then $x \in U_{a_i}$ for some i and $x \in V_{a_i}$ for the same i, contradicting the fact that $U_{a_i} \cap V_{a_i} = \varnothing$. $\square$

13.7 Hints and Solutions to Selected End-of-section Problems

Problems with complete solutions are starred (*). Equations are numbered consecutively within each problem.

1.2* a c^∞ function very flat at 0

(a) Assume $x > 0$. For $k = 1, f'(x) = (1/x^2)\, e^{-1/x}$. With $p_2(y) = y^2$, this verifies the claim. Now suppose $f^{(k)}(x) = p_{2k}(1/x)e^{-1/x}$. By the product rule and the chain rule,

$$\begin{aligned} f^{(k+1)}(x) &= p_{2k-1}\left(\frac{1}{x}\right) \cdot \left(-\frac{1}{x^2}\right) e^{-1/x} + p_{2k}\left(\frac{1}{x}\right) \cdot \frac{1}{x^2} e^{-1/x} \\ &= \left(q_{2k+1}\left(\frac{1}{x}\right) + q_{2k+2}\left(\frac{1}{x}\right)\right) e^{-1/x} \\ &= p_{2k+2}\left(\frac{1}{x}\right) e^{-1/x}, \end{aligned}$$

where $q_n(y)$ and $p_n(y)$ are polynomials of degree n in y. By induction, the claim is true for all $k \geq 1$. It is trivially true for $k = 0$ also.
(b) For $x > 0$, the formula in (a) shows that $f(x)$ is C^∞. For $x < 0, f(x) \equiv 0$, which is trivially C^∞. It remains to show that $f^{(k)}(x)$ is defined and continuous at $x = 0$ for all k.

Suppose $f^{(k)}(0) = 0$. By the definition of the derivative,

$$f^{(k+1)}(0) = \lim_{x \to 0} \frac{f^{(k)}(x) - f^{(k)}(0)}{x} = \lim_{x \to 0} \frac{f^{(k)}(x)}{x}.$$

The limit from the left is clearly 0 . So it suffices to compute the limit from the right:

$$\begin{aligned} \lim_{x \to 0^+} \frac{f^{(k)}(x)}{x} &= \lim_{x \to 0^+} \frac{p_{2k}\left(\frac{1}{x}\right) e^{-1/x}}{x} = \lim_{x \to 0^+} p_{2k+1}\left(\frac{1}{x}\right) e^{-1/x} \\ &= \lim_{y \to \infty} \frac{p_{2k+1}(y)}{e^y} \quad \left(\text{replacing } \frac{1}{x} \text{ by } y\right). \end{aligned} \tag{1.2.1}$$

Applying l'Hôpital's rule $2k + 1$ times, we reduce this limit to 0 . Hence, $f^{(k+1)}(0) = 0$. By induction, $f^{(k)}(0) = 0$ for all $k \geq 0$.

A similar computation as (1.2.1) for $\lim_{x \to 0} f^{(k)}(x) = 0$ proves that $f^{(k)}(x)$ is continuous at $x = 0$.
1.3 (b) $h(t) = (\pi/(b - a))(t - a) - (\pi/2)$.

1.5

(a) The line passing through $(0, 0, 1)$ and (a, b, c) has a parametrization

$$x = at, \quad y = bt, \quad z = (c - 1)t + 1.$$

This line intersects the xy-plane when

$$z = 0 \Leftrightarrow t = \frac{1}{1 - c} \Leftrightarrow (x, y) = \left(\frac{a}{1 - c}, \frac{b}{1 - c}\right).$$

To find the inverse of g, write down a parametrization of the line through $(u, v, 0)$ and $(0, 0, 1)$ and solve for the intersection of this line with S.

1.6* Taylor's Theorem with Remainder to Order 2

To simplify the notation, we write $\mathbf{0}$ for $(0,0)$. By Taylor's theorem with remainder, there exist C^∞ functions g_1, g_2 such that

$$f(x,y) = f(\mathbf{0}) + xg_1(x,y) + yg_2(x,y) \tag{1.6.1}$$

Applying the theorem again, but to g_1 and g_2, we obtain

$$g_1(x,y) = g_1(\mathbf{0}) + xg_{11}(x,y) + yg_{12}(x,y) \tag{1.6.2}$$
$$g_2(x,y) = g_2(\mathbf{0}) + xg_{21}(x,y) + yg_{22}(x,y) \tag{1.6.3}$$

Since $g_1(\mathbf{0}) = \partial f/\partial x(\mathbf{0})$ and $g_2(\mathbf{0}) = \partial f/\partial y(\mathbf{0})$, substituting (1.6.2) and (1.6.3) into (1.6.1) gives the result.

1.7* a Function with a Removable Singularity

In Problem 1.6, set $x = t$ and $y = tu$. We obtain

$$f(t,tu) = f(\mathbf{0}) + t\frac{\partial f}{\partial x}(\mathbf{0}) + tu\frac{\partial f}{\partial y}(\mathbf{0}) + t^2(\cdots)$$

where

$$(\cdots) = g_{11}(t,tu) + ug_{12}(t,tu) + u^2 g_{22}(t,tu)$$

is a C^∞ function of t and u. Since $f(\mathbf{0}) = \partial f/\partial x(\mathbf{0}) = \partial f/\partial y(\mathbf{0}) = 0$,

$$\frac{f(t,tu)}{t} = t(\cdots)$$

which is clearly C^∞ in t, u and agrees with g when $t = 0$.

1.8 See Example 1.2(ii).

3.1 $f = \sum g_{ij}\alpha^i \otimes \alpha^j$.

3.2

(a) Use the formula $\dim\ker f + \dim\operatorname{im} f = \dim V$.

(b) Choose a basis $e_1, \ldots, e_{n-1}$ for $\ker f$, and extend it to a basis $e_1, \ldots, e_{n-1}, e_n$ for V. Let $\alpha^1, \ldots, \alpha^n$ be the dual basis for $V^\vee$. Write both f and g in terms of this dual basis.

3.3 We write temporarily α^I for $\alpha^{i_1} \otimes \cdots \otimes \alpha^{i_k}$ and e_J for $(e_{j_1}, \ldots, e_{j_k})$.

(a) Prove that $f = \sum f(e_I)\,\alpha^I$ by showing that both sides agree on all (e_J). This proves that the set $\{\alpha^I\}$ spans.

(b) Suppose $\sum c_I\alpha^I = 0$. Applying both sides to e_J gives $c_J = \sum c_I\alpha^I(e_J) = 0$. This proves that the set $\{\alpha^I\}$ is linearly independent.

3.9 To compute $\omega(v_1, \ldots, v_n)$ for any $v_1, \ldots, v_n \in V$, write $v_j = \sum_i e_i a^i_j$ and use the fact that ω is multilinear and alternating.

3.10* Linear Independence of Covectors

($\Rightarrow$) If $\alpha^1, \ldots, \alpha^k$ are linearly dependent, then one of them is a linear combination of the others. Without loss of generality, we may assume that

$$\alpha^k = \sum_{i=1}^{k-1} c_i\alpha^i$$

In the wedge product $\alpha^1 \wedge \cdots \wedge \alpha^{k-1} \wedge \left(\sum_{i=1}^{k-1} c_i \alpha^i\right)$, every term has a repeated α^i. Hence, $\alpha^1 \wedge \cdots \wedge \alpha^k = 0$.
($\Leftarrow$) Suppose $\alpha^1, \ldots, \alpha^k$ are linearly independent. Then they can be extended to a basis $\alpha^1, \ldots, \alpha^k, \ldots, \alpha^n$ for $V^\vee$. Let $v_1, \ldots, v_n$ be the dual basis for V. By Proposition 3.27,

$$\left(\alpha^1 \wedge \cdots \wedge \alpha^k\right)(v_1, \ldots, v_k) = \det\left[\alpha^i(v_j)\right] = \det\left[\delta^i_j\right] = 1$$

Hence, $\alpha^1 \wedge \cdots \wedge \alpha^k \neq 0$.

3.11* Exterior Multiplication

($\Leftarrow$) Clear because $\alpha \wedge \alpha = 0$.
($\Rightarrow$) Suppose $\alpha \wedge \gamma = 0$. Extend α to a basis $\alpha^1, \ldots, \alpha^n$ for $V^\vee$, with $\alpha^1 = \alpha$. Write $\gamma = \sum c_J \alpha^J$, where J runs over all strictly ascending multi-indices $1 \leq j_1 < \cdots < j_k \leq n$. In the sum $\alpha \wedge \gamma = \sum c_J \alpha \wedge \alpha^J$, all the terms $\alpha \wedge \alpha^J$ with $j_1 = 1$ vanish, since $\alpha = \alpha^1$. Hence,

$$0 = \alpha \wedge \gamma = \sum_{j_1 \neq 1} c_J \alpha \wedge \alpha^J$$

Since $\left\{\alpha \wedge \alpha^J\right\}_{j_1 \neq 1}$ is a subset of a basis for $A_{k+1}(V)$, it is linearly independent, and so all c_J are 0 if $j_1 \neq 1$. Thus,

$$\gamma = \sum_{j_1 = 1} c_J \alpha^J = \alpha \wedge \left(\sum_{j_1 = 1} c_J \alpha^{j_2} \wedge \cdots \wedge \alpha^{j_k}\right).$$

4.1$\omega(X) = yz, d\omega = -dx \wedge dz$.
4.2 Write $\omega = \sum_{i<j} c_{ij} dx^i \wedge dx^j$. Then $c_{ij}(p) = \omega_p(e_i, e_j)$, where $e_i = \partial/\partial x^i$. Calculate $c_{12}(p), c_{13}(p)$, and $c_{23}(p)$. The answer is $\omega_p = p^3 dx^1 \wedge dx^2$.
4.3$dx = \cos\theta dr - r\sin\theta d\theta, dy = \sin\theta, dr + r\cos\theta d\theta, dx \wedge dy = r dr \wedge d\theta$.
4.4$dx \wedge dy \wedge dz = \rho^2 \sin\phi d\rho \wedge d\phi \wedge d\theta$.
4.5$\alpha \wedge \beta = (a_1 b_1 + a_2 b_2 + a_3 b_3)\, dx^1 \wedge dx^2 \wedge dx^3$.
5.3 The image $\phi_4(U_{14}) = \left\{(x, z) \mid -1 < z < 1, 0 < x < \sqrt{1 - z^2}\right\}$.

The transition function $\left(\phi_1 \circ \phi_4^{-1}\right)(x, z) = \phi_1(x, y, z) = (y, z) = \left(-\sqrt{1 - x^2 - z^2}, z\right)$ is a C^∞ function of x, z.

5.4* Existence of a Coordinate Neighborhood

Let U_β be any coordinate neighborhood of p in the maximal atlas. Any open subset of U_β is again in the maximal atlas, because it is C^∞ compatible with all the open sets in the maximal atlas. Thus $U_\alpha := U_\beta \cap U$ is a coordinate neighborhood such that $p \in U_\alpha \subset U$.

6.3* Group of Automorphisms of a Vector Space

The manifold structure $\mathrm{GL}(V)_e$ is the maximal atlas on $\mathrm{GL}(V)$ containing the coordinate chart $(\mathrm{GL}(V), \phi_e)$. The manifold structure $\mathrm{GL}(V)_u$ is the maximal atlas on $\mathrm{GL}(V)$ containing the coordinate chart $(\mathrm{GL}(V), \phi_u)$. The two maps $\phi_e : \mathrm{GL}(V) \to \mathbb{R}^{n \times n}$ and $\phi_u : \mathrm{GL}(V) \to \mathbb{R}^{n \times n}$ are C^∞ compatible, because $\phi_e \circ \phi_u^{-1} : \mathrm{GL}(n, \mathbb{R}) \to \mathrm{GL}(n, \mathbb{R})$ is conjugation by the change-ofbasis matrix from u to e. Therefore, the two maximal atlases are in fact the same.

7.4* Quotient Space of a Sphere with Antipodal Points Identified

(a) Let U be an open subset of S^n. Then $\pi^{-1}(\pi(U)) = U \cup a(U)$, where $a : S^n \to S^n, a(x) = -x$, is the antipodal map. Since the antipodal map is a homeomorphism, $a(U)$ is open, and hence $\pi^{-1}(\pi(U))$ is open. By the definition of quotient topology, $\pi(U)$ is open. This proves that π is an open map.
(b) The graph R of the equivalence relation $\sim$ is

$$R = \{(x, x) \in S^n \times S^n\} \cup \{(x, -x) \in S^n \times S^n\} = \Delta \cup (\mathbb{1} \times a)(\Delta)$$

By Corollary 7.8, because S^n is Hausdorff, the diagonal Δ in $S^n \times S^n$ is closed. Since $\mathbb{1} \times a : S^n \times S^n \to S^n \times S^n, (x, y) \mapsto (x, -y)$ is a homeomorphism, $(\mathbb{1} \times a)(\Delta)$ is also closed. As a union of the two closed sets Δ and $(\mathbb{1} \times a)(\Delta), R$ is closed in $S^n \times S^n$. By Theorem 7.7, $S^n/\sim$ is Hausdorff.

7.5* Orbit Space of a Continuous Group Action

Let U be an open subset of S. For each $g \in G$, since right multiplication by g is a homeomorphism $S \to S$, the set Ug is open. But

$$\pi^{-1}(\pi(U)) = \cup_{g \in G} Ug$$

which is a union of open sets, hence is open. By the definition of the quotient topology, $\pi(U)$ is open. $\square$

7.9* Compactness of Real Projective Space

By Exercise 7.11 there is a continuous surjective map $\pi : S^n \to \mathbb{R}P^n$. Since the sphere S^n is compact, and the continuous image of a compact set is compact (Proposition A.34), $\mathbb{R}P^n$ is compact. $\square$

8.1* Differential of a map

To determine the coefficient a in $F_*(\partial/\partial x) = a\partial/\partial u + b\partial/\partial v + c\partial/\partial w$, we apply both sides to u to get

$$F_*\left(\frac{\partial}{\partial x}\right) u = \left(a\frac{\partial}{\partial u} + b\frac{\partial}{\partial v} + c\frac{\partial}{\partial w}\right) u = a.$$

Hence,

$$a = F_*\left(\frac{\partial}{\partial x}\right) u = \frac{\partial}{\partial x}(u \circ F) = \frac{\partial}{\partial x}(x) = 1.$$

Similarly,

$$b = F_*\left(\frac{\partial}{\partial x}\right) v = \frac{\partial}{\partial x}(v \circ F) = \frac{\partial}{\partial x}(y) = 0$$

and

$$c = F_*\left(\frac{\partial}{\partial x}\right) w = \frac{\partial}{\partial x}(w \circ F) = \frac{\partial}{\partial x}(xy) = y.$$

So $F_*(\partial/\partial x) = \partial/\partial u + y\partial/\partial w$. $\square$
8.3 One can directly calculate $a = F_*(X)u$ and $b = F_*(X)v$ or more simply, one can apply Problem 8.2. The answer is $a = -(\sin\alpha)x - (\cos\alpha)y, b = (\cos\alpha)x - (\sin\alpha)y$.

8.5* Velocity of a Curve in Local Coordinates

We know that $c'(t) = \sum a^j \partial/\partial x^j$. To compute a^i, evaluate both sides on x^i :

$$a^i = \left(\sum a^j \frac{\partial}{\partial x^j}\right) x^i = c'(t)x^i = c_*\left(\frac{d}{dt}\right) x^i = \frac{d}{dt}\left(x^i \circ c\right) = \frac{d}{dt}c^i = \dot{c}^i(t).$$

8.6$c'(0) = -2y\partial/\partial x + 2x\partial/\partial y$.

8.7* Tangent Space to a Product

If $(U, \phi) = (U, x^1, \ldots, x^m)$ and $(V, \psi) = (V, y^1, \ldots, y^n)$ are charts about p in M and q in N respectively, then by Proposition 5.18, a chart about $(\ p, q\ )$ in $M \times N$ is

$$(U \times V, \phi \times \psi) = (U \times V, (\pi_1^*\phi, \pi_2^*\psi)) = \left(U \times V, \bar{x}^1, \ldots, \bar{x}^n, \bar{y}^1, \ldots, \bar{y}^n\right),$$

where $\bar{x}^i = \pi_1^* x^i$ and $\bar{y}^i = \pi_2^* y^i$. Let $\pi_{1*}\left(\partial/\partial \bar{x}^j\right) = \sum a_j^i \partial/\partial x^i$. Then

$$a_j^i = \pi_{1*}\left(\frac{\partial}{\partial \bar{x}^j}\right) x^i = \frac{\partial}{\partial \bar{x}^j}\left(x^i \circ \pi_1\right) = \frac{\partial \bar{x}^i}{\partial \bar{x}^j} = \delta_j^i.$$

Hence,

$$\pi_{1*}\left(\frac{\partial}{\partial \bar{x}^j}\right) = \sum_i \delta_j^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^j}.$$

This really means that

$$\pi_{1*}\left(\left.\frac{\partial}{\partial \bar{x}^j}\right|_{(p,q)}\right) = \left.\frac{\partial}{\partial x^j}\right|_p. \tag{8.7.1}$$

Similarly,

$$\pi_{1*}\left(\frac{\partial}{\partial \bar{y}^j}\right) = 0, \quad \pi_{2*}\left(\frac{\partial}{\partial \bar{x}^j}\right) = 0, \quad \pi_{2*}\left(\frac{\partial}{\partial \bar{y}^j}\right) = \frac{\partial}{\partial y^j}. \tag{8.7.2}$$

A basis for $T_{(p,q)}(M \times N)$ is

$$\left.\frac{\partial}{\partial \bar{x}^1}\right|_{(p,q)}, \ldots, \left.\frac{\partial}{\partial \bar{x}^m}\right|_{(p,q)}, \left.\frac{\partial}{\partial \bar{y}^1}\right|_{(p,q)}, \ldots, \left.\frac{\partial}{\partial \bar{y}^n}\right|_{(p,q)}.$$

A basis for $T_pM \times T_qN$ is

$$\left(\left.\frac{\partial}{\partial x^1}\right|_p, 0\right), \ldots, \left(\left.\frac{\partial}{\partial x^m}\right|_p, 0\right), \left(0, \left.\frac{\partial}{\partial y^1}\right|_q\right), \ldots, \left(0, \left.\frac{\partial}{\partial y^n}\right|_q\right).$$

By (8.7.1) and (8.7.2), the linear map $(\ \pi_{1*}, \pi_{2*}\ )$ maps a basis of $T_{(p,q)}(M \times N)$ to a basis of $T_pM \times T_qN$. It is therefore an isomorphism. $\square$

8.8 (a) Let $c(t)$ be a curve starting at e in G with $c'(0) = X_e$. Then $\alpha(t) = (c(t), e)$ is a curve starting at (e, e) in $G \times G$ with $\alpha'(0) = (X_e, 0)$. Compute $\mu_{*,(e,e)}$ using $\alpha(t)$.

8.9* Transforming Vectors to Coordinate Vectors

Let $(V, y^1, \ldots, y^n)$ be a chart about p. Suppose $(X_j)_p = \sum_i a_j^i \partial/\partial y^i|_p$. Since $(X_1)_p, \ldots, (X_n)_p$ are linearly independent, the matrix $A = \left[a_j^i\right]$ is nonsingular.

Define a new coordinate system $x^1, \ldots, x^n$ by

$$y^i = \sum_{j=1}^{n} a_j^i x^j \quad \text{for } i = 1, \ldots, n \tag{8.9.1}$$

By the chain rule,

$$\frac{\partial}{\partial x^j} = \sum_i \frac{\partial y^i}{\partial x^j} \frac{\partial}{\partial y^i} = \sum a_j^i \frac{\partial}{\partial y^i}$$

At the point p,

$$\left.\frac{\partial}{\partial x^j}\right|_p = \sum a_j^i \left.\frac{\partial}{\partial y^i}\right|_p = (X_j)_p .$$

In matrix notation,

$$\begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix} = A \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix}, \quad \text{so} \quad \begin{bmatrix} x^1 \\ \vdots \\ x^n \end{bmatrix} = A^{-1} \begin{bmatrix} y^1 \\ \vdots \\ y^n \end{bmatrix}.$$

This means that (8.9.1) is equivalent to $x^j = \sum_{i=1}^n (A^{-1})_i^j \, y^i$.

8.10 (a) For all $x \le p, f(x) \le f(p)$. Hence,

$$f'(p) = \lim_{x \to p^-} \frac{f(x) - f(p)}{x - p} \ge 0. \tag{8.10.1}$$

Similarly, for all $x \ge p, f(x) \le f(p)$, so that

$$f'(p) = \lim_{x \to p^+} \frac{f(x) - f(p)}{x - p} \le 0 \tag{8.10.2}$$

The two inequalities (8.10.1) and (8.10.2) together imply that $f'(p) = 0$.

9.1$c \in \mathbb{R} - \{0, -108\}$.

9.2 Yes, because it is a regular level set of the function $f(x, y, z, w) = x^5 + y^5 + z^5 + w^5$.

9.3 Yes; see Example 9.12.

9.4* Regular Submanifolds

Let $p \in S$. By hypothesis there is an open set U in $\mathbb{R}^2$ such that on $U \cap S$ one of the coordinates is a C^∞ function of the other. Without loss of generality, we assume that $y = f(x)$ for some C^∞ function $f : A \subset \mathbb{R} \to B \subset \mathbb{R}$, where A and B are open sets in $\mathbb{R}$ and $V := A \times B \subset U$. Let $F : V \to \mathbb{R}^2$ be given by $F(x, y) = (x, y - f(x))$. Since F is a diffeomorphism onto its image, it can be used as a coordinate map. In the chart $(V, x, y - f(x)), V \cap S$ is defined by the vanishing of the coordinate $y - f(x)$. This proves that S is a regular submanifold of $\mathbb{R}^2$.

9.5 $(\mathbb{R}^3, x, y, z - f(x, y))$ is an adapted chart for $\mathbb{R}^3$ relative to $\Gamma(f)$.

9.6 Differentiate (9.3) with respect to t.

9.10* the Transversality Theorem

(a) $f^{-1}(U) \cap f^{-1}(S) = f^{-1}(U \cap S) = f^{-1}\left(g^{-1}(0)\right) = (g \circ f)^{-1}(0)$.
(b) Let $p \in f^{-1}(U) \cap f^{-1}(S) = f^{-1}(U \cap S)$. Then $f(p) \in U \cap S$. Because S is a fiber of g, the pushforward $g_*\left(T_{f(p)}S\right)$ equals 0 . Because $g : U \to \mathbb{R}^k$ is a projection, $g_*\left(T_{f(p)}M\right) = T_0\left(\mathbb{R}^k\right)$. Applying g_* to the transversality equation (9.4), we get

$$g_* f_*\left(T_p N\right) = g_*\left(T_{f(p)}M\right) = T_0\left(\mathbb{R}^k\right)$$

Hence, $g \circ f : f^{-1}(U) \to \mathbb{R}^k$ is a submersion at p. Since p is an arbitrary point of $f^{-1}(U) \cap f^{-1}(S) = (g \circ f)^{-1}(0)$, this set is a regular level set of $g \circ f$.
(c) By the regular level set theorem, $f^{-1}(U) \cap f^{-1}(S)$ is a regular submanifold of $f^{-1}(U) \subset N$. Thus every point $p \in f^{-1}(S)$ has an adapted chart relative to $f^{-1}(S)$ in N.
10.7 Let $e_1, \ldots, e_n$ be a basis for V and $\alpha^1, \ldots, \alpha^n$ the dual basis for $V^\vee$. Then a basis for $A_n(V)$ is $\alpha^1 \wedge \cdots \wedge \alpha^n$ and $L^*\left(\alpha^1 \wedge \cdots \wedge \alpha^n\right) = c\alpha^1 \wedge \cdots \wedge \alpha^n$ for some constant c. Suppose $L\left(e_j\right) = \sum_i a_j^i e_i$. Compute c in terms of a_j^i.
11.1 Let $c(t) = \left(x^1(t), \ldots, x^{n+1}(t)\right)$ be a curve in S^n with $c(0) = p$ and $c'(0) = X_p$. Differentiate $\sum_i \left(x^i\right)^2(t) = 1$ with respect to t. Let H be the plane $\{(a^1, a^2, a^3) \in \mathbb{R}^3 \mid \sum a^i p^i = 0\}$. Show that $T_p\left(S^2\right) \subset H$. Because both sets are linear spaces and have the same dimension, equality holds.

11.3* Critical Points of a Smooth map on a Compact Manifold

First Proof. Suppose $f : N \to \mathbb{R}^m$ has no critical point. Then it is a submersion. The projection to the first factor, $\pi : \mathbb{R}^m \to \mathbb{R}$, is also a submersion. It follows that the composite $\pi \circ f : N \to \mathbb{R}$ is a submersion. This contradicts the fact that as a continuous function from a compact manifold to $\mathbb{R}$, the function $\pi \circ f$ has a maximum and hence a critical point (see Problem 8.10). Second Proof. Suppose $f : N \to \mathbb{R}^m$ has no critical point. Then it is a submersion. Since a submersion is an open map (Corollary 11.6), the image $f(N)$ is open in $\mathbb{R}^m$. But the continuous image of a compact set is compact and a compact subset of $\mathbb{R}^m$ is closed and bounded. Hence, $f(N)$ is a nonempty proper closed subset of $\mathbb{R}^m$. This is a contradiction, because being connected, $\mathbb{R}^m$ cannot have a nonempty proper subset that is both open and closed.
11.4 At $p = (a, b, c), i_*\left(\partial/\left.\partial u\right|_p\right) = \partial/\partial x - (a/c)\partial/\partial z$, and $i_*\left(\partial/\left.\partial v\right|_p\right) = \partial/\partial y - (b/c)\partial/\partial z$.
11.5 Use Proposition A. 35 to show that f is a closed map. Then apply Problem A.12.

12.1* the Hausdorff Condition on the Tangent Bundle

Let (p, v) and (q, w) be distinct points of the tangent bundle TM.
Case 1: $p \neq q$. Because M is Hausdorff, p and q can be separated by disjoint neighborhoods U and V. Then TU and TV are disjoint open subsets of TM containing (p, v) and (q, w), respectively.

Case 2: $p = q$. Let (U, ϕ) be a coordinate neighborhood of p. Then (p, v) and (p, w) are distinct points in the open set TU. Since TU is homeomorphic to the open subset $\phi(U) \times \mathbb{R}^n$ of $\mathbb{R}^{2n}$, and any subspace of a Hausdorff space is Hausdorff, TU is Hausdorff. Therefore, (p, v) and (p, w) can be separated by disjoint open sets in TU.

13.1* Support of a Finite sum

Let A be the set where $\sum \rho_i$ is not zero and A_i the set where ρ_i is not zero:

$$A = \left\{x \in M \mid \sum \rho_i(x) \neq 0\right\}, \quad A_i = \{x \in M \mid \rho_i(x) \neq 0\}$$

If $\sum \rho_i(x) \neq 0$, then at least one $\rho_i(x)$ must be nonzero. This implies that $A \subset \cup A_i$. Taking the closure of both sides gives $\mathrm{cl}(A) \subset \overline{\bigcup A_i}$. For a finite union, $\overline{\bigcup A_i} = \bigcup \overline{A_i}$ (Exercise A.53). Hence,

$$\operatorname{supp}\left(\sum \rho_i\right) = \mathrm{cl}(A) \subset \overline{\bigcup A_i} = \bigcup \overline{A_i} = \bigcup \operatorname{supp} \rho_i$$

13.2* Locally Finite Family and Compact set

For each $p \in K$, let W_p be a neighborhood of p that intersects only finitely many of the sets A_α. The collection $\{W_p\}_{p \in K}$ is an open cover of K. By compactness, K has a finite subcover $\{W_{p_i}\}_{i=1}^r$. Since each W_{p_i} intersects only finitely many of the A_α, the finite union $W := \bigcup_{i=1}^r W_{p_i}$ intersects only finitely many of the A_α.
13.3 (a) Take $f = \rho_{M-B}$.

13.5* Support of the Pullback by a Projection

Let $A = \{p \in M \mid f(p) \neq 0\}$. Then $\operatorname{supp} f = \mathrm{cl}_M(A)$. Observe that

$$(\pi^* f)(p, q) = f(p) \neq 0 \quad \text{iff } p \in A.$$

Hence,

$$\{(p, q) \in M \times N \mid (\pi^* f)(p, q) \neq 0\} = A \times N$$

So

$$\operatorname{supp}(\pi^* f) = \mathrm{cl}_{M \times N}(A \times N) = \mathrm{cl}_M(A) \times N = (\operatorname{supp} f) \times N$$

by Problem A.20.

13.7* Closure of a Locally Finite Union

Since $A_\alpha \subset \bigcup A_\alpha$, taking the closure of both sides gives

$$\overline{A_\alpha} \subset \overline{\bigcup A_\alpha}$$

Hence, $\bigcup \overline{A_\alpha} \subset \overline{\bigcup A_\alpha}$.
($\subset$) Instead of proving $\overline{\bigcup A_\alpha} \subset \bigcup \overline{A_\alpha}$, we will prove the contrapositive: if $p \notin \bigcup \overline{A_\alpha}$, then $p \notin \overline{\bigcup A_\alpha}$. Suppose $p \notin \bigcup \overline{A_\alpha}$. By local finiteness, p has a neighborhood W that meets only finitely many of the A_α 's, say $A_{\alpha_1}, \ldots, A_{\alpha_m}$ (see the figure below).

Since $p \notin \overline{A_\alpha}$ for any $\alpha, p \notin \bigcup_{i=1}^m \overline{A_{\alpha_i}}$. Note that W is disjoint from A_α for all $\alpha \neq \alpha_i$, so $W - \bigcup_{i=1}^m \overline{A_{\alpha_i}}$ is disjoint from A_α for all α. Because $\bigcup_{i=1}^m \overline{A_{\alpha_i}}$ is closed, $W - \bigcup_{i=1}^m \overline{A_{\alpha_i}}$ is an open set containing p disjoint from $\cup A_\alpha$. By the local characterization of closure (Proposition A.48), $p \notin \overline{\bigcup A_\alpha}$. Hence, $\overline{\bigcup A_\alpha} \subset \bigcup \overline{A_\alpha}$.

14.1* Equality of Vector Fields

The implication in the direction ($\Rightarrow$) is obvious. For the converse, let $p \in M$. To show that $X_p = Y_p$, it suffices to show that $X_p[H] = Y_p[H]$ for any germ $[h]$ of C^∞ functions in $C_p^\infty(M)$. Suppose $h : U \to \mathbb{R}$ is a C^∞ function that represents the germ $[h]$. We can extend it to a C^∞ function $\tilde{h} : M \to \mathbb{R}$ by multiplying it by a C^∞ bump function supported in U that is identically 1 in a neighborhood of p. By hypothesis, $X\tilde{h} = Y\tilde{h}$. Hence,

$$X_p\tilde{h} = (X\tilde{h})_p = (Y\tilde{h})_p = Y_p\tilde{h} \tag{14.1.1}$$

Because $\tilde{h} = h$ in a neighborhood of p, we have $X_p h = X_p\tilde{h}$ and $Y_p h = Y_p\tilde{h}$. It follows from (14.1.1) that $X_p h = Y_p h$. Thus, $X_p = Y_p$. Since p is an arbitrary point of M, the two vector fields X and Y are equal.

14.6 Integral Curve Starting at a Zero of a Vector Field

(a)* Suppose $c(t) = p$ for all $t \in \mathbb{R}$. Then

$$c'(t) = 0 = X_p = X_{c(t)}$$

for all $t \in \mathbb{R}$. Thus, the constant curve $c(t) = p$ is an integral curve of X with initial point p. By the uniqueness of an integral curve with a given initial point, this is the maximal integral curve of X starting at p.
14.8$c(t) = 1/((1/p) - t)$ on $(-\infty, 1/p)$.
14.10 Show that both sides applied to a C^∞ function h on M are equal. Then use Problem 14.1.
14.11 $-\partial/\partial y$.
14.12$c^k = \sum_i \left(a^i \frac{\partial b^k}{\partial x^i} - b^i \frac{\partial a^k}{\partial x^i}\right)$.
14.14 Use Example 14.15 and Proposition 14.17.
15.3
(a) Apply Proposition A.43.
(b) Apply Proposition A.43.
(c) Apply Problem A.16.
(d) By (a) and (b), the subset G_0 is a subgroup of G. By (c), it is an open submanifold.

15.4* Open Subgroup of a Connected Lie Group

For any $g \in G$, left multiplication $\ell_g : G \to G$ by g maps the subgroup H to the left coset gH. Since H is open and ℓ_g is a homeomorphism, the coset gH is open. Thus, the set of cosets gH, $g \in G$, partitions G into a disjoint union of open subsets. Since G is connected, there can be only one coset. Therefore, $H = G$.
15.5 Let $c(t)$ be a curve in G with $c(0) = a, c'(0) = X_a$. Then $(c(t), b)$ is a curve through (a, b) with initial velocity $(X_a, 0)$. Compute $\mu_{*,(a,b)}(X_a, 0)$ using this curve (Proposition 8.18). Compute similarly $\mu_{*,(a,b)}(0, Y_b)$.

15.7* Differential of the Determinant map

Let $c(t) = Ae^{tX}$. Then $c(0) = A$ and $c'(0) = AX$. Using the curve $c(t)$ to calculate the differential yields

$$\begin{aligned}\det_{A,*}(AX) &= \frac{d}{dt}\bigg|_{t=0} \det(c(t)) = \frac{d}{dt}\bigg|_{t=0} (\det A)\det e^{tX} \\ &= (\det A)\frac{d}{dt}\bigg|_{t=0} e^{t\operatorname{tr} X} = (\det A)\operatorname{tr} X\end{aligned}$$

15.8* Special Linear Group

If $\det A = 1$, then Exercise 15.7 gives

$$\det_{*,A}(AX) = \operatorname{tr} X$$

Since $\operatorname{tr} X$ can assume any real value, $\det_{*,A} : T_A GL(n,\mathbb{R}) \to \mathbb{R}$ is surjective for all $A \in \det^{-1}(1)$. Hence, 1 is a regular value of det.

15.10

(a) $\mathrm{O}(n)$ is defined by polynomial equations.
(b) If $A \in \mathrm{O}(n)$, then each column of A has length 1 .
15.11 Write out the conditions $A^T A = I, \det A = 1$. If $a^2 + b^2 = 1$, then $(\ a, b\ )$ is a point on the unit circle, and so $a = \cos\theta, b = \sin\theta$ for some $\theta \in [0, 2\pi]$.

15.14

$$\begin{bmatrix} \cosh 1 & \sinh 1 \\ \sinh 1 & \cosh 1 \end{bmatrix},$$

where $\cosh t = \left(e^t + e^{-t}\right)/2$ and $\cosh t = \left(e^t - e^{-t}\right)/2$ are hyperbolic cosine and sine, respectively.
15.16 The correct target space for f is the vector space $K_{2n}(\mathbb{C})$ of $2n \times 2n$ skew-symmetric complex matrices.
16.4 Let $c(t)$ be a curve in $\mathrm{Sp}(n)$ with $c(0) = I$ and $c'(0) = X$. Differentiate $c(t)^T J c(t) = J$ with respect to t.
16.5 Mimic Example 16.6. The left-invariant vector fields on $\mathbb{R}^n$ are the constant vector fields $\sum_{i=1}^n a^i \partial/\partial x^i$, where $a^i \in \mathbb{R}$.
16.9 A basis $X_{1,e}, \ldots, X_{n,e}$ for the tangent space $T_e(G)$ at the identity gives rise to a frame consisting of left-invariant vector fields $X_1, \ldots, X_n$.

16.10* the Pushforward of Left-invariant Vector Fields

Under the isomorphisms $\varphi_H : T_e H \xrightarrow{\sim} L(H)$ and $\varphi_G : T_e G \xrightarrow{\sim} L(G)$, the Lie brackets correspond and the pushforward maps correspond. Thus, this problem follows from Proposition 16.14 by the correspondence.

A more formal proof goes as follows. Since X and Y are left-invariant vector fields, $X = \tilde{A}$ and $Y = \tilde{B}$ for $A = X_e$ and $B = Y_e \in T_e H$. Then

$$\begin{aligned}
F_*[X,Y] &= F_*[\tilde{A},\tilde{B}] = F_*\left([A,B]^{\sim}\right) && (\text{ Proposition 16.10}) \\
&= \left(F_*[A,B]\right)^{\sim} && (\text{ definition of } F_* \text{ on } L(H)) \\
&= [F_*A, F_*B]^{\sim} && (\text{ Proposition 16.14}) \\
&= \left[(F_*A)^{\sim}, (F_*B)^{\sim}\right] && (\text{ Proposition 16.10}) \\
&= \left[F_*\tilde{A}, F_*\tilde{B}\right] && (\text{ definition of } F_* \text{ on } L(H)) \\
&= [F_*X, F_*Y]\,.
\end{aligned}$$

16.11 (b) Let $(U, x^1, \ldots, x^n)$ be a chart about e in G. Relative to this chart, the differential c_{a*} at e is represented by the Jacobian matrix $[\partial\left(x^i \circ c_a\right) / \partial x^j|_e]$. Since $c_a(x) = axa^{-1}$ is a C^∞ function of x and a, all the partial derivatives $\partial\left(x^i \circ c_a\right) / \partial x^j|_e$ are C^∞ and therefore $\mathrm{Ad}(a)$ is a C^∞ function of a.
17.1$\omega = (xdx + ydy)/\left(x^2 + y^2\right)$.
17.2$a_j = \sum_i b_i \partial y^i / \partial x^j$.
17.4 (a) Suppose $\omega_p = \sum c_i dx^i|_p$. Then

$$\lambda_{\omega_p} = \pi^*\left(\omega_p\right) = \sum c_i \pi^*\left(dx^i|_p\right) = \sum c_i\left(\pi^* dx^i\right)_{\omega_p} = \sum c_i\left(d\pi^* x^i\right)_{\omega_p} = \sum c_i\left(d\bar{x}^i\right)_{\omega_p}$$

Hence, $\lambda = \sum c_i d\bar{x}^i$.

18.4* Support of a sum or Product

(a) If $(\omega + \tau)(p) \neq 0$, then $\omega(p) \neq 0$ or $\tau(p) \neq 0$. Hence,

$$Z(\omega + \tau)^c \subset Z(\omega)^c \cup Z(\tau)^c$$

Taking the closure of both sides and using the fact that $\overline{A \cup B} = \bar{A} \cup \bar{B}$, we get

$$\operatorname{supp}(\omega + \tau) \subset \operatorname{supp}\omega \cup \operatorname{supp}\tau$$

(b) Suppose $(\omega \wedge \tau)_p \neq 0$. Then $\omega_p \neq 0$ and $\tau_p \neq 0$. Hence,

$$Z(\omega \wedge \tau)^c \subset Z(\omega)^c \cap Z(\tau)^c$$

Taking the closure of both sides and using the fact that $\overline{A \cap B} \subset \bar{A} \cap \bar{B}$, we get

$$\operatorname{supp}(\omega \wedge \tau) \subset \overline{Z(\omega)^c \cap Z(\tau)^c} \subset \operatorname{supp}\omega \cap \operatorname{supp}\tau$$

18.6* Locally Finite Supports

Let $p \in \operatorname{supp}\omega$. Since $\{\operatorname{supp}\rho_\alpha\}$ is locally finite, there is a neighborhood W_p of p in M that intersects only finitely many of the sets $\operatorname{supp}\rho_\alpha$. The collection $\{W_p \mid p \in \operatorname{supp}\omega\}$ covers $\operatorname{supp}\omega$. By the compactness of $\operatorname{supp}\omega$, there is a finite subcover $\{W_{p_1}, \ldots, W_{p_m}\}$. Since each W_{p_i} intersects only finitely many $\operatorname{supp}\rho_\alpha, \operatorname{supp}\omega$ intersects only finitely many $\operatorname{supp}\rho_\alpha$.

By Problem 18.4,

$$\operatorname{supp}\left(\rho_\alpha\omega\right) \subset \operatorname{supp}\rho_\alpha \cap \operatorname{supp}\omega$$

Thus, for all but finitely many $\alpha, \operatorname{supp}\left(\rho_\alpha\omega\right)$ is empty; i.e., $\rho_\alpha\omega \equiv 0$.

18.8* Pullback by a Surjective Submersion

The fact that $\pi^* : \Omega^*(M) \to \Omega^*(\tilde{M})$ is an algebra homomorphism follows from Propositions 18.9 and 18.11.

Suppose $\omega \in \Omega^k(M)$ is a k-form on M for which $\pi^*\omega = 0$ in $\Omega^k(\tilde{M})$. To show that $\omega = 0$, pick an arbitrary point $p \in M$, and arbitrary vectors $v_1, \ldots, v_k \in T_pM$. Since π is surjective, there is a point $\tilde{p} \in \tilde{M}$ that maps to p. Since π is a submersion at $\tilde{p}$, there exist $\tilde{v}_1, \ldots, \tilde{v}_k \in T_{\tilde{p}}\tilde{M}$ such that $\pi_{*,\tilde{p}}\tilde{v}_i = v_i$. Then

$$\begin{aligned} 0 &= (\pi^*\omega)_{\tilde{p}}(\tilde{v}_1, \ldots, \tilde{v}_k) \\ &= \omega_{\pi(\tilde{p})}(\pi_*\tilde{v}_1, \ldots, \pi_*\tilde{v}_k) \quad (\text{ definition of } \pi^*\omega) \\ &= \omega_p(v_1, \ldots, v_k) \end{aligned}$$

Since $p \in M$ and $v_1, \ldots, v_k \in T_pM$ are arbitrary, this proves that $\omega = 0$. Therefore, $\pi^* : \Omega^*(M) \to \Omega^*(\tilde{M})$ is injective.

18.9 (c) Because $f(a)$ is the pullback by $\mathrm{Ad}(a^{-1})$, we have $f(a) = \det(\mathrm{Ad}(a^{-1}))$ by Problem 10.7. According to Problem 16.11, $\mathrm{Ad}(a^{-1})$ is a C^∞ function of a.

19.1$F^*(dx \wedge dy \wedge dz) = d(x \circ F) \wedge d(y \circ F) \wedge d(z \circ F)$. Apply Corollary 18.4(ii).

19.2$F^*(udu + vdv) = (2x^3 + 3xy^2)\,dx + (3x^2y + 2y^3)\,dy$.

19.3$c^*\omega = dt$.

19.5* Coordinates and Differential Forms

Let $(V, x^1, \ldots, x^n)$ be a chart about p. By Corollary 18.4(ii),

$$df^1 \wedge \cdots \wedge df^n = \det\left[\frac{\partial f^i}{\partial x^j}\right] dx^1 \wedge \cdots \wedge dx^n$$

So $(df^1 \wedge \cdots \wedge df^n)_p \neq 0$ if and only if $\det[\partial f^i/\partial x^j(p)] \neq 0$. By the inverse function theorem, this condition is equivalent to the existence of a neighborhood W on which the map $F := (f^1, \ldots, f^n) : W \to \mathbb{R}^n$ is a C^∞ diffeomorphism onto its image. In other words, $(W, f^1, \ldots, f^n)$ is a chart.

19.7 Mimic the proof of Proposition 19.3.

19.9* Vertical Plane

Since $ax + by$ is the zero function on the vertical plane, its differential is identically zero:

$$adx + bdy = 0$$

Thus, at each point of the plane, dx is a multiple of dy or vice versa. In either case, $dx \wedge dy = 0$.

19.11

(a) Mimic Example 19.8. Define

$$U_x = \{(x, y) \in M \mid f_x \neq 0\} \quad \text{and} \quad U_y = \{(x, y) \in M \mid f_y \neq 0\},$$

where f_x, f_y are the partial derivatives $\partial f/\partial x, \partial f/\partial y$ respectively. Because 0 is a regular value of f, every point in M satisfies $f_x \neq 0$ or $f_y \neq 0$. Hence, $\{U_x, U_y\}$ is an open cover of M. Define $\omega = dy/f_x$ on U_x and $-dx/f_y$ on U_y. Show that ω is globally defined on M. By the implicit function theorem, in a neighborhood of a point $(a, b) \in U_x, x$ is a C^∞ function of y. It follows that y can be used as a local coordinate and the 1-form dy/f_x is C^∞ at (a, b). Thus, ω is C^∞ on U_x. A similar argument shows that ω is C^∞ on U_y.

(b) On $M, df = f_x dx + f_y dy + f_z dz \equiv 0$.
(c) Define $U_i = \{p \in \mathbb{R}^{n+1} \mid \partial f/\partial x^i(p) \neq 0\}$ and

$$\omega = (-1)^{i-1} \frac{dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^{n+1}}{\partial f/\partial x^i} \qquad \text{on } U_i$$

19.13 $\boldsymbol{\nabla} \times \mathbf{E} = -\partial \mathbf{B}/\partial t$ and $\operatorname{div} \mathbf{B} = 0$.

20.3* Derivative of a Smooth Family of Vector Fields

Let $(V, y^1, \ldots, y^n)$ be another coordinate neighborhood of p such that

$$X_t = \sum_j b^j(t,q) \frac{\partial}{\partial y^j} \qquad \text{on } V$$

On $U \cap V$,

$$\frac{\partial}{\partial x^i} = \sum_j \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j}$$

Substituting this into (20.2) in the text and comparing coefficients with the expression for X_t above, we get

$$b^j(t,q) = \sum_i a^i(t,q) \frac{\partial y^j}{\partial x^i}$$

Since $\partial y^j/\partial x^i$ does not depend on t, differentiating both sides of this equation with respect to t gives

$$\frac{\partial b^j}{\partial t} = \sum_i \frac{\partial a^i}{\partial t} \frac{\partial y^j}{\partial x^i}$$

Hence,

$$\sum_j \frac{\partial b^j}{\partial t} \frac{\partial}{\partial y^j} = \sum_{i,j} \frac{\partial a^i}{\partial t} \frac{\partial y^j}{\partial x^i} \frac{\partial}{\partial y^j} = \sum_i \frac{\partial a^i}{\partial t} \frac{\partial}{\partial x^i}$$

20.6* Global Formula for the Exterior Derivative

By Theorem 20.12,

$$\begin{aligned}(\mathcal{L}_{Y_0}\omega)(Y_1, \ldots, Y_k) &= Y_0(\omega(Y_1, \ldots, Y_k)) - \sum_{j=1}^k \omega(Y_1, \ldots, [Y_0, Y_j], \ldots, Y_k) \\ &= Y_0(\omega(Y_1, \ldots, Y_k)) + \sum_{j=1}^k (-1)^j \omega\left([Y_0, Y_j], Y_1, \ldots, \widehat{Y_j}, \ldots, Y_k\right) \qquad (20.6.1)\end{aligned}$$

By the induction hypothesis, Theorem 20.14 is true for $(k-1)$-forms. Hence,

$$
\begin{aligned}
-\left(dl_{Y_0}\omega\right)\left(Y_1,\ldots,Y_k\right) &= -\sum_{i=1}^{k}(-1)^{i-1}Y_i\left(\left(l_{Y_0}\omega\right)\left(Y_1,\ldots,\widehat{Y}_i,\ldots,Y_k\right)\right)\\
&\quad -\sum_{1\le i<j\le k}(-1)^{i+j}\left(l_{Y_0}\omega\right)\left(\left[Y_i,Y_j\right],Y_1,\ldots,\widehat{Y}_i,\ldots,\widehat{Y}_j,\ldots,Y_k\right)\\
&= \sum_{i=1}^{k}(-1)^{i}Y_i\left(\omega\left(Y_0,Y_1,\ldots,\widehat{Y}_i,\ldots,Y_k\right)\right)\\
&\quad +\sum_{1\le i<j\le k}(-1)^{i+j}\omega\left(\left[Y_i,Y_j\right],Y_0,Y_1,\ldots,\widehat{Y}_i,\ldots,\widehat{Y}_j,\ldots,Y_k\right) \qquad (20.6.2)
\end{aligned}
$$

Adding (20.6.1) and (20.6.2) gives

$$
\begin{aligned}
\sum_{i=0}^{k}(-1)^{i}Y_i\left(\omega\left(Y_0,\ldots,\widehat{Y}_i,\ldots,Y_k\right)\right) &+ \sum_{j=1}^{k}(-1)^{j}\omega\left(\left[Y_0,Y_j\right],\widehat{Y}_0,Y_1,\ldots,\widehat{Y}_j,\ldots,Y_k\right)\\
&+ \sum_{1\le i<j\le k}(-1)^{i+j}\omega\left(\left[Y_i,Y_j\right],Y_0,\ldots,\widehat{Y}_i,\ldots,\widehat{Y}_j,\ldots,Y_k\right)
\end{aligned}
$$

which simplifies to the right-hand side of Theorem 20.14.

21.1* Locally Constant map on a Connected Space

We first show that for every $y \in Y$, the inverse $f^{-1}(y)$ is an open set. Suppose $p \in f^{-1}(y)$. Then $f(p) = y$. Since f is locally constant, there is a neighborhood U of p such that $f(U) = \{y\}$. Thus, $U \subset f^{-1}(y)$. This proves that $f^{-1}(y)$ is open.

The equality $S = \bigcup_{y\in Y} f^{-1}(y)$ exhibits S as a disjoint union of open sets. Since S is connected, this is possible only if there is just one such open set $S = f^{-1}(y_0)$. Hence, f assumes the constant value y_0 on S.

21.5 The map F is orientation-preserving.

21.6 Use Problem 19.11(c) and Theorem 21.5.

21.9 See Problem 12.2.

22.1 The topological boundary $\mathrm{bd}(M)$ is $\{0,1,2\}$; the manifold boundary ∂M is $\{0\}$.

22.3* Inward-pointing Vectors at the Boundary

($\Longleftarrow$) Suppose $(U,\phi) = (U,x^1,\ldots,x^n)$ is a chart for M centered at p such that $X_p = \sum a^i\partial/\,\partial x^i|_p$ with $a^n > 0$. Then the curve $c(t) = \phi^{-1}\left(a^1t,\ldots,a^nt\right)$ in M satisfies

$$
c(0) = p, \quad c(]0,\varepsilon[) \subset M^\circ, \quad \text{and } c'(0) = X_p. \qquad (22.3.1)
$$

So X_p is inward-pointing.

($\Rightarrow$) Suppose X_p is inward-pointing. Then $X_p \notin T_p(\partial M)$ and there is a curve $c : [0,\varepsilon[\to M$ such that (22.3.1) holds. Let $(U,\phi) = (U,x^1,\ldots,x^n)$ be a chart centered at p. On $U \cap M$, we have $x^n \ge 0$. If $(\phi\circ c)(t) = (c^1(t),\ldots,c^n(t))$, then $c^n(0) = 0$ and $c^n(t) > 0$ for $t > 0$. Therefore, the derivative of c^n at $t = 0$ is

$$
\dot{c}^n(0) = \lim_{t\to 0^+}\frac{c^n(t)-c^n(0)}{t} = \lim_{t\to 0^+}\frac{c^n(t)}{t} \ge 0
$$

Since $X_p = \sum_{i=1}^n \dot{c}^i(0)\partial/\,\partial x^i|_p$, the coefficient of $\partial/\,\partial x^n|_p$ in X_p is $\dot{c}^n(0)$. In fact, $\dot{c}^n(0) > 0$ because if $\dot{c}^n(0)$ were 0 , then X_p would be tangent to ∂M at p.

22.4* Smooth Outward-pointing Vector Field Along the Boundary

Let $p \in \partial M$ and let $(U, x^1, \ldots, x^n)$ be a coordinate neighborhood of p. Write

$$X_{\alpha,p} = \sum_{i=1}^{n} a^i\left(X_{\alpha,p}\right) \frac{\partial}{\partial x^i}\bigg|_p$$

Then

$$X_p = \sum_\alpha \rho_\alpha(p) X_{\alpha,p} = \sum_{i=1}^{n} \sum_\alpha \rho_\alpha(p) a^i\left(X_{\alpha,p}\right) \frac{\partial}{\partial x^i}\bigg|_p .$$

Since $X_{\alpha,p}$ is outward-pointing, the coefficient $a^n\left(X_{\alpha,p}\right)$ is negative by Problem 22.3. Because $\rho_\alpha(p) \geq 0$ for all α with $\rho_\alpha(p)$ positive for at least one α, the coefficient $\sum_\alpha \rho_\alpha(p) a^i\left(X_{\alpha,p}\right)$ of $\partial/\,\partial x^n|_p$ in X_p is negative. Again by Problem 22.3, this proves that X_p is outward-pointing.

The smoothness of the vector field X follows from the smoothness of the partition of unity ρ_α and of the coefficient functions $a^i\left(X_{\alpha,p}\right)$ as functions of p.

22.6* Induced Atlas on the Boundary

Let $r^1, \ldots, r^n$ be the standard coordinates on the upper half-space $\mathcal{H}^n$. As a shorthand, we write $a = (a^1, \ldots, a^{n-1})$ for the first $n-1$ coordinates of a point in $\mathcal{H}^n$. Since the transition function

$$\psi \circ \phi^{-1} : \phi(U \cap V) \to \psi(U \cap V) \subset \mathcal{H}^n$$

takes boundary points to boundary points and interior points to interior points,
(i) $\left(r^n \circ \psi \circ \phi^{-1}\right)(a, 0) = 0$, and
(ii) $\left(r^n \circ \psi \circ \phi^{-1}\right)(a, t) > 0$ for $t > 0$,
where $(a, 0)$ and (a, t) are points in $\phi(U \cap V) \subset \mathcal{H}^n$.
Let $x^j = r^j \circ \phi$ and $y^i = r^i \circ \psi$ be the local coordinates on the charts (U, ϕ) and (V, ψ) respectively. In particular, $y^n \circ \phi^{-1} = r^n \circ \boldsymbol{\psi} \circ \boldsymbol{\phi}^{-1}$. Differentiating (i) with respect to r^j gives

$$\frac{\partial y^n}{\partial x^j}\bigg|_{\phi^{-1}(a,0)} = \frac{\partial\left(y^n \circ \phi^{-1}\right)}{\partial r^j}\bigg|_{(a,0)} = \frac{\partial\left(r^n \circ \psi \circ \phi^{-1}\right)}{\partial r^j}\bigg|_{(a,0)} = 0 \quad \text{for } j = 1, \ldots, n-1.$$

From (i) and (ii),

$$\begin{aligned}
\frac{\partial y^n}{\partial x^n}\bigg|_{\phi^{-1}(a,0)} &= \frac{\partial\left(y^n \circ \phi^{-1}\right)}{\partial r^n}\bigg|_{(a,0)} \\
&= \lim_{t \to 0^+} \frac{\left(y^n \circ \phi^{-1}\right)(a, t) - \left(y^n \circ \phi^{-1}\right)(a, 0)}{t} \\
&= \lim_{t \to 0^+} \frac{\left(y^n \circ \phi^{-1}\right)(a, t)}{t} \geq 0,
\end{aligned}$$

since both t and $\left(y^n \circ \phi^{-1}\right)(a, t)$ are positive.
The Jacobian matrix of $J = [\partial y^i / \partial x^j]$ of the overlapping charts U and V at a point $p = \phi^{-1}(a, 0)$ in $U \cap V \cap \partial M$ therefore has the form

$$J = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^{n-1}} & \frac{\partial y^1}{\partial x^n} \\ \vdots & \ddots & \vdots & \vdots \\ \frac{\partial y^{n-1}}{\partial x^1} & \cdots & \frac{\partial y^{n-1}}{\partial x^{n-1}} & \frac{\partial y^{n-1}}{\partial x^n} \\ 0 & \cdots & 0 & \frac{\partial y^n}{\partial x^n} \end{pmatrix} = \begin{pmatrix} A & * \\ 0 & \frac{\partial y^n}{\partial x^n} \end{pmatrix},$$

where the upper left $(n-1)\times(n-1)$ block $A=\left[\partial y^i/\partial x^j\right]_{1\le i,j\le n-1}$ is the Jacobian matrix of the induced charts $U\cap\partial M$ and $V\cap\partial M$ on the boundary. Since $\det J(p)>0$ and $\partial y^n/\partial x^n(p)>0$, we have $\det A(p)>0$. $\square$

22.7* Boundary Orientation of the Left Half-space

Because a smooth outward-pointing vector field along ∂M is $\partial/\partial x^1$, by definition an orientation form of the boundary orientation on ∂M is the contraction

$$l_{\partial/\partial x^1}\left(dx^1\wedge dx^2\wedge\cdots\wedge dx^n\right)=dx^2\wedge\cdots\wedge dx^n$$

22.8 Viewed from the top, C_1 is clockwise and C_0 is counterclockwise.
22.9 Compute $l_X\left(dx^1\wedge\cdots\wedge dx^{n+1}\right)$.
22.10 (a) An orientation form on the closed unit ball is $dx^1\wedge\cdots\wedge dx^{n+1}$ and a smooth outward-pointing vector field on U is $\partial/\partial x^{n+1}$. By definition, an orientation form on U is the contraction

$$\imath_{\partial/\partial x^{n+1}}\left(dx^1\wedge\cdots\wedge dx^{n+1}\right)=(-1)^n dx^1\wedge\cdots\wedge dx^n$$

22.11 (a) Let ω be the orientation form on the sphere in Problem 22.9. Show that $a^*\omega=(-1)^{n+1}\omega$.
23.1 Let $x=au$ and $y=bv$.
23.2 Use the Heine-Borel theorem (Theorem A.40).

23.3* Integral under a Diffeomorphism

Let $\{(U_\alpha,\phi_\alpha)\}$ be an oriented atlas for M that specifies the orientation of M, and $\{\rho_\alpha\}$ a partition of unity on M subordinate to the open cover $\{U_\alpha\}$. Assume that $F:N\to M$ is orientation-preserving. By Problem 21.4, $\{(F^{-1}(U_\alpha),\phi_\alpha\circ F)\}$ is an oriented atlas for N that specifies the orientation of N. By Problem 13.6, $\{F^*\rho_\alpha\}$ is a partition of unity on N subordinate to the open cover $\{F^{-1}(U_\alpha)\}$.

By the definition of the integral,

$$\begin{aligned}\int_N F^*\omega&=\sum_\alpha\int_{F^{-1}(U_\alpha)}(F^*\rho_\alpha)(F^*\omega)\\&=\sum_\alpha\int_{F^{-1}(U_\alpha)}F^*(\rho_\alpha\omega)\\&=\sum_\alpha\int_{(\phi_\alpha\circ F)(F^{-1}(U_\alpha))}(\phi_\alpha\circ F)^{-1*}F^*(\rho_\alpha\omega)\\&=\sum_\alpha\int_{\phi_\alpha(U_\alpha)}\left(\phi_\alpha^{-1}\right)^*(\rho_\alpha\omega)\\&=\sum_\alpha\int_{U_\alpha}\rho_\alpha\omega=\int_M\omega\end{aligned}$$

If $F:N\to M$ is orientation-reversing, then $\{(F^{-1}(U_\alpha),\phi_\alpha\circ F)\}$ is an oriented atlas for N that gives the opposite orientation of N. Using this atlas to calculate the integral as above gives $-\int_N F^*\omega$. Hence in this case $\int_M\omega=-\int_N F^*\omega$.

23.4* Stokes's Theorem for $\mathbb{R}^n$ and for $\mathcal{H}^n$

An ($n-1$)-form ω with compact support on $\mathbb{R}^n$ or $\mathcal{H}^n$ is a linear combination

$$\omega = \sum_{i=1}^{n} f_i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \tag{23.4.1}$$

Since both sides of Stokes's theorem are $\mathbb{R}$-linear in ω, it suffices to check the theorem for just one term of the sum (23.4.1). So we may assume

$$\omega = f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n$$

where f is a C^∞ function with compact support in $\mathbb{R}^n$ or $\mathcal{H}^n$. Then

$$\begin{aligned} d\omega &= \frac{\partial f}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge dx^{i-1} \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \\ &= (-1)^{i-1} \frac{\partial f}{\partial x^i} dx^1 \wedge \cdots \wedge dx^i \wedge \cdots \wedge dx^n \end{aligned}$$

Since f has compact support in $\mathbb{R}^n$ or $\mathcal{H}^n$, we may choose $a > 0$ large enough that $\operatorname{supp} f$ lies in the interior of the cube $[-a, a]^n$.

Stokes's theorem for $\mathbb{R}^n$

By Fubini's theorem, one can first integrate with respect to x^i :

$$\begin{aligned} \int_{\mathbb{R}^n} d\omega &= \int_{\mathbb{R}^n} (-1)^{i-1} \frac{\partial f}{\partial x^i} dx^1 \cdots dx^n \\ &= (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x^i} dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= (-1)^{i-1} \int_{\mathbb{R}^{n-1}} \left(\int_{-a}^{a} \frac{\partial f}{\partial x^i} dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n \end{aligned}$$

But

$$\begin{aligned} \int_{-a}^{a} \frac{\partial f}{\partial x^i} dx^i &= f\left(\ldots, x^{i-1}, a, x^{i+1}, \ldots\right) - f\left(\ldots, x^{i-1}, -a, x^{i+1}, \ldots\right) \\ &= 0 - 0 = 0 \end{aligned}$$

because the support of f lies in the interior of $[-a, a]^n$. Hence, $\int_{\mathbb{R}^n} d\omega = 0$.

The right-hand side of Stokes's theorem is $\int_{\partial \mathbb{R}^n} \omega = \int_{\varnothing} \omega = 0$, because $\mathbb{R}^n$ has empty boundary. This verifies Stokes's theorem for $\mathbb{R}^n$.

Stokes's Theorem for $\langle^n$

Case 1: $i \neq n$.

$$\begin{aligned} \int_{\mathcal{H}^n} d\omega &= (-1)^{i-1} \int_{\mathcal{H}^n} \frac{\partial f}{\partial x^i} dx^1 \cdots dx^n \\ &= (-1)^{i-1} \int_{\mathcal{H}^{n-1}} \left(\int_{-\infty}^{\infty} \frac{\partial f}{\partial x^i} dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n \\ &= (-1)^{i-1} \int_{\mathcal{H}^{n-1}} \left(\int_{-a}^{a} \frac{\partial f}{\partial x^i} dx^i \right) dx^1 \cdots \widehat{dx^i} \cdots dx^n \end{aligned}$$

As for $\int_{\partial \mathcal{H}^n} \omega$, note that $\partial \mathcal{H}^n$ is defined by the equation $x^n = 0$. Hence, on $\partial \mathcal{H}^n$, the 1 form dx^n is identically zero. Since $i \neq n, \omega = f dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n \equiv 0$ on $\partial \mathcal{H}^n$, so $\int_{\partial \mathcal{H}^n} \omega = 0$. Thus, Stokes's theorem holds in this case.

Case 2: $i = n$.

$$\begin{aligned}\int_{\mathcal{H}^n} d\omega &= (-1)^{n-1} \int_{\mathcal{H}^n} \frac{\partial f}{\partial x^n} dx^1 \cdots dx^n \\ &= (-1)^{n-1} \int_{\mathbb{R}^{n-1}} \left(\int_0^\infty \frac{\partial f}{\partial x^n} dx^n \right) dx^1 \cdots dx^{n-1}\end{aligned}$$

In this integral

$$\begin{aligned}\int_0^\infty \frac{\partial f}{\partial x^n} dx^n &= \int_0^a \frac{\partial f}{\partial x^n} dx^n \\ &= f\left(x^1, \ldots, x^{n-1}, a\right) - f\left(x^1, \ldots, x^{n-1}, 0\right) \\ &= -f\left(x^1, \ldots, x^{n-1}, 0\right)\end{aligned}$$

Hence,

$$\int_{\mathcal{H}^n} d\omega = (-1)^n \int_{\mathbb{R}^{n-1}} f\left(x^1, \ldots, x^{n-1}, 0\right) dx^1 \cdots dx^{n-1} = \int_{\partial \mathcal{H}^n} \omega$$

because $(-1)^n \mathbb{R}^{n-1}$ is precisely $\partial \mathcal{H}^n$ with its boundary orientation. So Stokes's theorem also holds in this case.

23.5 Take the exterior derivative of $x^2 + y^2 + z^2 = 1$ to obtain a relation among the 1 -forms dx, dy, and dz on S^2. Then show for example that for $x \neq 0$, one has $dx \wedge dy = (z/x) dy \wedge dz$.

24.1 Assume $\omega = df$. Derive a contradiction using Problem 8.10(b) and Proposition 17.2.

25.4* the Snake Lemma

If we view each column of the given commutative diagram as a cochain complex, then the diagram is a short exact sequence of cochain complexes

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

By the zig-zag lemma, it gives rise to a long exact sequence in cohomology. In the long exact sequence, $H^0(\mathcal{A}) = \ker \alpha, H^1(\mathcal{A}) = A^1 / \operatorname{im} \alpha = \operatorname{coker} \alpha$, and similarly for $\mathcal{B}$ and $\mathcal{C}$.

26.2 Define $d_{-1} = 0$. Then the given exact sequence is equivalent to a collection of short exact sequences

$$0 \to \operatorname{im} d_{k-1} \to A^k \xrightarrow{d_k} \operatorname{im} d_k \to 0, \quad k = 0, \ldots, m-1.$$

By the rank-nullity theorem,

$$\dim A^k = \dim\left(\operatorname{im} d_{k-1}\right) + \dim\left(\operatorname{im} d_k\right).$$

When we compute the alternating sum of the left-hand side, the right-hand side will cancel to 0 .

28.1 Let U be the punctured projective plane $\mathbb{R}P^2 - \{p\}$ and V a small disk containing p. Because U can be deformation retracted to the boundary circle, which after identification is in fact $\mathbb{R}P^1, U$ has the homotopy type of $\mathbb{R}P^1$. Since $\mathbb{R}P^1$ is homeomorphic to $S^1, H^*(U) \simeq H^*\left(S^1\right)$. Apply the Mayer-Vietoris sequence. The answer is $H^0\left(\mathbb{R}P^2\right) = \mathbb{R}, H^k\left(\mathbb{R}P^2\right) = 0$ for $k > 0$.

28.2$H^k\left(S^n\right) = \mathbb{R}$ for $k = 0, n$, and $H^k\left(S^n\right) = 0$ otherwise.

28.3 One way is to apply the Mayer-Vietoris sequence to $U = \mathbb{R}^2 - \{p\}, V = \mathbb{R}^2 - \{q\}$.

A.13* the Lindelöf Condition

Let $\{B_i\}_{i\in I}$ be a countable basis and $\{U_\alpha\}_{\alpha\in \mathrm{A}}$ an open cover of the topological space S. For every $p \in U_\alpha$, there exists a B_i such that

$$p \in B_i \subset U_\alpha$$

Since this B_i depends on p and α, we write $i = i(p, \alpha)$. Thus,

$$p \in B_{i(p,\alpha)} \subset U_\alpha$$

Now let J be the set of all indices $j \in I$ such that $j = i(p, \alpha)$ for some p and some α. Then $\bigcup_{j\in J} B_j = S$ because every p in S is contained in some $B_{i(p,\alpha)} = B_j$.

For each $j \in J$, choose an $\alpha(j)$ such that $B_j \subset U_{\alpha(j)}$. Then $S = \bigcup_j B_j \subset \bigcup_j U_{\alpha(j)}$. So $\left\{U_{\alpha(j)}\right\}_{j\in J}$ is a countable subcover of $\{U_\alpha\}_{\alpha\in \mathrm{A}}$.

A.15* Disconnected Subset in Terms of a Separation

($\Rightarrow$) By (iii),

$$A = (U \cap V) \cap A = (U \cap A) \cup (V \cap A).$$

By (i) and (ii), $U \cap A$ and $V \cap A$ are disjoint nonempty open subsets of A. Hence, A is disconnected.

($\Leftarrow$) Suppose A is disconnected in the subspace topology. Then $A = U' \cup V'$, where U' and V' are two disjoint nonempty open subsets of A. By the definition of the subspace topology, $U' = U \cap A$ and $V' = V \cap A$ for some open sets U, V in S.

(i) holds because U' and V' are nonempty.

(ii) holds because U' and V' are disjoint.

(iii) holds because $A = U' \cup V' \subset U \cup V$.

A.19* Uniqueness of the Limit

Suppose $p \neq q$. Since S is Hausdorff, there exist disjoint open sets U_p and U_q such that $p \in U_p$ and $q \in U_q$. By the definition of convergence, there are integers N_p and N_q such that for all $i \geq N_p, \quad x_i \in U_p$ and for all $i \geq N_q, \quad x_i \in U_q$. This is a contradiction, since $U_p \cap U_q$ is the empty set.

A.20* Closure in a Product

(C) By Problem A.5, $\mathrm{cl}(A) \times Y$ is a closed set containing $A \times Y$. By the definition of closure, $\mathrm{cl}(A \times Y) \subset \mathrm{cl}(A) \times Y$.

($\supset$) Conversely, suppose $(p, y) \in \mathrm{cl}(A) \times Y$. If $p \in A$, then $(p, y) \in A \times Y \subset \mathrm{cl}(A \times Y)$. Suppose $p \notin A$. By Proposition A.50, p is an accumulation point of A. Let $U \times V$ be any basis open set in $S \times Y$ containing (p, y). Because $p \in \mathrm{ac}(A)$, the open set U contains a point $a \in A$ with $a \neq p$. So $U \times V$ contains the point $(a, y) \in A \times Y$ with $(a, y) \neq (p, y)$. This proves that (p, y) is an accumulation point of $A \times Y$. By Proposition A. 50 again, $(p, y) \in \mathrm{ac}(A \times Y) \subset \mathrm{cl}(A \times Y)$. This proves that $\mathrm{cl}(A) \times Y \subset \mathrm{cl}(A \times Y)$.

B.1* the Rank of a Matrix

($\Rightarrow$) Suppose $\mathrm{rk}\, A \geq k$. Then one can find k linearly independent columns, which we call $a_1, \ldots, a_k$. Since the $m \times k$ matrix $[a_1 \cdots a_k]$ has rank k, it has k linearly independent rows $b^1, \ldots, b^k$. The matrix B whose rows are $b^1, \ldots, b^k$ is a $k \times k$ submatrix of A, and $\mathrm{rk}\, B = k$. In

other words, B is a nonsingular $k \times k$ submatrix of A.
($\Leftarrow$) Suppose A has a nonsingular $k \times k$ submatrix B. Let $a_1, \ldots, a_k$ be the columns of A such that the submatrix $[a_1 \cdots a_k]$ contains B. Since $[a_1 \cdots a_k]$ has k linearly independent rows, it also has k linearly independent columns. Thus, $\operatorname{rk} A \geq k$.

B.2* Matrices of Rank at Most r

Let A be an $m \times n$ matrix. By Problem B.1, $\operatorname{rk} A \leq r$ if and only if all $(r+1) \times (r+1)$ minors $m_1(A), \ldots, m_s(A)$ of A vanish. As the common zero set of a collection of continuous functions, D_r is closed in $\mathbb{R}^{m \times n}$.

B.3* Maximal Rank

For definiteness, suppose $n \leq m$. Then the maximal rank is n and every matrix $A \in \mathbb{R}^{m \times n}$ has rank $\leq n$. Thus,

$$D_{\max} = \left\{A \in \mathbb{R}^{m \times n} \mid \operatorname{rk} A = n\right\} = \mathbb{R}^{m \times n} - D_{n-1}$$

Since D_{n-1} is a closed subset of $\mathbb{R}^{m \times n}$ (Problem B.2), $D_{\max}$ is open in $\mathbb{R}^{m \times n}$.

B.4* Degeneracy Loci and Maximal-rank Locus of a map

(a) Let D_r be the subset of $\mathbb{R}^{m \times n}$ consisting of matrices of rank at most r. The degeneracy locus of rank r of the map $F : S \to \mathbb{R}^{m \times n}$ may be described as

$$D_r(F) = \{x \in S \mid F(x) \in D_r\} = F^{-1}(D_r).$$

Since D_r is a closed subset of $\mathbb{R}^{m \times n}$ (Problem B.2) and F is continuous, $F^{-1}(D_r)$ is a closed subset of S.
(b) Let $D_{\max}$ be the subset of $\mathbb{R}^{m \times n}$ consisting of all matrices of maximal rank. Then $D_{\max}(F) = F^{-1}(D_{\max})$. Since $D_{\max}$ is open in $\mathbb{R}^{m \times n}$ (Problem B.3) and F is continuous, $F^{-1}(D_{\max})$ is open in S.
B. 7 Use Example B.5.

List of Notations

$\mathbb{R}^n$	Euclidean space of dimension n (p. 4)
$p = (p^1, \ldots, p^n)$	point in $\mathbb{R}^n$ (p. 4)
C^∞	smooth or infinitely differentiable (p. 4)
$\partial f / \partial x^i$	partial derivative with respect to x^i(pp.4, 67)
$f^{(k)}(x)$	the k th derivative of $f(x)$(p.5)
$B(p, r)$	open ball in $\mathbb{R}^n$ with center p and radius r (pp. 7, 317)
$]\, a, b[$	open interval in $\mathbb{R}^1$ (p. 8)
$T_p\left(\mathbb{R}^n\right)$ or $T_p R^n$	tangent space to $\mathbb{R}^n$ at p (p. 10)
$v = \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \langle v^1, \ldots, v^n \rangle$	column vector (p. 11)
$\{e_1, \ldots, e_n\}$	standard basis for $\mathbb{R}^n$ (p. 11)
$D_v f$	directional derivative of f in the direction of v at p (p.11)
$x \sim y$	equivalence relation (p. 11)
C_p^∞	algebra of germs of C^∞ functions at p in $\mathbb{R}^n$ (p. 12)
$\mathcal{D}_p\left(\mathbb{R}^n\right)$	vector space of derivations at p in $\mathbb{R}^n$ (p. 13)
$\mathfrak{X}(U)$	vector space of C^∞ vector fields on U (p. 15)
$\operatorname{Der}(A)$	vector space of derivations of an algebra A (p. 17)
δ_j^i	Kronecker delta (p. 13)
$\operatorname{Hom}(V, W)$	vector space of linear maps $f : V \to W$ (p. 19)
$V^\vee = \operatorname{Hom}(V, \mathbb{R})$	dual of a vector space (p. 19)
V^k	Cartesian product $V \times \cdots \times V$ of k copies of V (p. 22)
$L_k(V)$	vector space of k-linear functions on V (p. 22)

$(a_1 a_2 \cdots a_r)$	cyclic permutation, r-cycle (p. 20)
$(\ ab\ )$	transposition (p. 20)
S_k	group of permutations of k objects (p. 20)
$\operatorname{sgn}(\sigma)$ or $\operatorname{sgn}\sigma$	sign of a permutation (p. 20)
$A_k(V)$	vector space of alternating k-linear functions on V (p. 23)
σf	a function f acted on by a permutation σ (p. 23)
e	identity element of a group (p. 24)
$\sigma \cdot x$	left action of σ on x (p. 24)
$x \cdot \sigma$	right action of σ on x (p. 24)
Sf	symmetrizing operator applied to f (p.24)
Af	alternating operator applied to f (p. 24)
$f \otimes g$	tensor product of multilinear functions f and g (p. 25)
$f \wedge g$	wedge product of multicovectors f and g (p. 26)
$B = \left[b^i_j\right]$ or $[b_{ij}]$	matrix whose (i,j)-entry is b^i_j or b_{ij} (p. 30)
$\det\left[b^i_j\right]$ or $\det[b_{ij}]$	determinant of the matrix $\left[b^i_j\right]$ or $[b_{ij}]$ (p. 30)
$\wedge(V)$	exterior algebra of a vector space (p. 30)
$I = (i_1, \ldots, i_k)$	multi-index (p. 31)
e_I	k-tuple $(e_{i_1}, \ldots, e_{i_k})$ (p. 31)
α^I	k-covector $\alpha^{i_1} \wedge \cdots \wedge \alpha^{i_k}$ (p. 31)
$T^*_p(\mathbb{R}^n)$ or $T^*_p\mathbb{R}^n$	cotangent space to $\mathbb{R}^n$ (p. 34)
df	differential of a function (pp. 34, 191)
dx^I	$dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ (p. 36)
$\Omega^k(U)$	vector space of $C^\infty k$-forms on U (pp. 36, 203)
$\Omega^*(U)$	direct sum $\bigoplus_{k=0}^n \Omega^k(U)$ (p. 37, 206)
$\omega(X)$	the function $p \mapsto \omega_p(X_p)$ (p. 37)
$\mathcal{F}(U)$ or $C^\infty(U)$	ring of C^∞ functions on U (p. 38)
$d\omega$	exterior derivative of ω (p. 38)
f_x	$\partial f/\partial x$, partial derivative of f with respect to x (p. 38)
$\bigoplus_{k=0}^\infty A^k$	direct sum of $A^0, A^1, \ldots$ (p. 30)
$\operatorname{grad} f$	gradient of a function f (p. 41)
$\operatorname{curl}\mathbf{F}$	curl of a vector field $\mathbf{F}$ (p. 41)
$\operatorname{div}\mathbf{F}$	divergence of a vector field $\mathbf{F}$ (p. 41)
$H^k(U)$	k th de Rham cohomology of U (p. 43)

$\{U_\alpha\}_{\alpha\in \mathrm{A}}$	open cover (p. 48)
$(U,\phi),(U,\phi:U\to\mathbb{R}^n)$	chart or coordinate open set (p. 48)
$\mathbb{1}_U$	identity map on U (p. 49)
$U_{\alpha\beta}$	$U_\alpha\cap U_\beta$ (p. 50)
$U_{\alpha\beta\gamma}$	$U_\alpha\cap U_\beta\cap U_\gamma$ (p. 50)
$\mathfrak{U}=\{(U_\alpha,\phi_\alpha)\}$	atlas (p. 50)
$\mathbb{C}$	complex plane (p. 50)
?	disjoint union (pp. 51, 129)
$\phi_\alpha\vert_{U_\alpha\cap V}$	restriction of ϕ_α to $U_\alpha\cap V$ (p. 54)
$\Gamma(f)$	graph of f (p. 54)
$K^{m\times n}$	vector space of $m\times n$ matrices with entries in K (p. 54)
$\mathrm{GL}\ (n,K)$	general linear group over a field K (p. 54)
$M\times N$	product manifold (p. 55)
$f\times g$	Cartesian product of two maps (p. 55)
S^n	unit sphere in $\mathbb{R}^{n+1}$ (p. 58)
F^*h	pullback of a function h by a map F (p. 60)
$J(f)=[\partial F^i/\partial x^j]$	Jacobian matrix (p. 68)
$\det[\partial F^i/\partial x^j]$	Jacobian determinant (p. 68)
$\frac{\partial\left(F^1,\ldots,F^n\right)}{\partial(x^1,\ldots,x^n)}$	Jacobian determinant (p. 68)
$\mu:G\times G\to G$	multiplication on a Lie group (p. 66)
$\imath:G\to G$	inverse map of a Lie group (p. 66)
$K^\times$	nonzero elements of a field K (p. 66)
S^1	unit circle in $\mathbb{C}^\times$(p. 66)
$A=[a_{ij}]\,,[a^i{}_j]$	matrix whose (i,j)-entry is a_{ij} or a^i_j (p. 67)
$S/\sim$	quotient (p. 71)
$[\ x\ ]$	equivalence class of x (p.71)
$\pi^{-1}(U)$	inverse image of U under π (p. 71)
$\mathbb{R}P^n$	real projective space of dimension n (p. 76)
$\Vert x\Vert$	modulus of x (p. 77)
$a^1\wedge\cdots\wedge\widehat{a^i}\wedge\cdots\wedge a^n$	the caret ^ means to omit a^i (p. 80)
$G(k,n)$	Grassmannian of k-planes in $\mathbb{R}^n$ (p. 82)
$\mathrm{rk}\,A$	rank of a matrix A (p. 82 (p. 344)
$C^\infty_p(M)$	germs of C^∞ functions at p in M (p. 87)

$T_p(M)$ or T_pM	tangent space to M at p (p. 87)
$\partial/\ \partial x^i\big\vert_p$	coordinate tangent vector at p (p. 87)
$d/\ dt\big\vert_p$	coordinate tangent vector of a 1-dimensional manifold (p. 87)
$F_{*,p}$ or F_*	differential of F at p (p. 87)
$c(t)$	curve in a manifold (p. 92)
$c'(t) := c_*\left(\frac{d}{dt}\big\vert_{t_0}\right)$	derivative of a real-valued function (p. 92)
ϕ_S	coordinate map on a submanifold S (p. 100)
$f^{-1}(\{c\})$ or $f^{-1}(c)$	level set (p. 103)
$Z(f) = f^{-1}(0)$	zero set (p. 103)
$\mathrm{SL}\ (n, K)$	special linear group over a field K (pp. 107, 109)
m_{ij} or $m_{ij}(A)$	(i, j)-minor of a matrix A(p.107)
$\mathrm{Mor}(A, B)$	the set of morphisms from A to B (p. 110)
$\mathbb{1}_A$	identity map on A (p. 110)
(M, q)	pointed manifold (p. 111)
$\simeq$	isomorphism (p. 111)
$\mathcal{F}, \mathcal{G}$	functors (p. 111)
$\mathcal{C}, \mathcal{D}$	categories (p. 111)
$\{e_1, \ldots, e_n\}$	basis for a vector space V (p. 113)
$\{\alpha^1, \ldots, \alpha^n\}$	dual basis for $V^\vee$ (p. 113)
$L^\vee$	dual of linear map L (p. 113)
$\mathrm{O}(n)$	orthogonal group (p. 117)
A^T	transpose of a matrix A (p. 117)
ℓ_g	left multiplication by g (p.117)
r_g	right multiplication by g (p.117)
$D_{\max}\ (F)$	maximal rank locus of $F : S \to \mathbb{R}^{m\times n}$ (pp. 118, 345)
$i : N \to M$	inclusion map (p. 123)
TM	tangent bundle (p. 129)
$\tilde{\phi}$	coordinate map on the tangent bundle (p. 130)
$E_p := \pi^{-1}(p)$	fiber at p of a vector bundle (p. 133)
X	vector field (p. 136)
X_p	tangent vector at p (p. 136)
$\Gamma(U, E)$	vector space of C^∞ sections of E over U (p. 137)

$\Gamma(E) := \Gamma(M, E)$	vector space of C^∞ sections of E over M (p. 137)
$\operatorname{supp} f$	support of a function f (p. 140)
$\bar{B}(p, r)$	closed ball in $\mathbb{R}^n$ with center p and radius r (p. 143)
$\bar{A}, \operatorname{cl}(A), \operatorname{orcl}_S(A)$	closure of a set A in S (pp. 148, 334)
$c_t(p)$	integral curve through p (p. 152)
$\operatorname{Diff}(M)$	group of diffeomorphisms of M (p. 153)
$F_t(q) = F(t, q)$	local flow (p. 156)
$[X, Y]$	Lie bracket of vector fields, bracket in a Lie algebra (pp. 157, 158)
$\mathfrak{X}(M)$	Lie algebra of C^∞ vector fields on M (p. 158)
S_n	vector space of $n \times n$ real symmetric matrices (p. 166)
$\mathbb{R}^2/\mathbb{Z}^2$	torus (p. 167)
$\|X\|$	norm of a matrix (p. 169
$\exp(X)$ or e^X	exponential of a matrix X (p. 170)
$\operatorname{tr}(X)$	trace (p. 171)
$Z(G)$	center of a group G (p. 176)
SO(n)	special orthogonal group (p. 176)
$\mathrm{U}(n)$	unitary group (p. 176)
$\mathrm{SU}(n)$	special unitary group (p. 177)
$\mathrm{Sp}(n)$	compact symplectic group (p. 177)
J	the matrix $\begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}$ (p. 177)
I_n	$n \times n$ identity matrix (p. 177)
$\mathrm{Sp}(2n, \mathbb{C})$	complex symplectic group (p. 177)
K_n	space of $n \times n$ real skew-symmetric matrices (p. 179)
$\tilde{A}$	left-invariant vector field generated by $A \in T_eG$ (p. 180)
$L(G)$	Lie algebra of left-invariant vector fields on G (p. 180)
$\mathfrak{g}$	Lie algebra (p. 182)
$\mathfrak{h} \subset \mathfrak{g}$	Lie subalgebra (p. 182)
$\mathfrak{gl}(n, \mathbb{R})$	Lie algebra of $\mathrm{GL}(n, \mathbb{R})$ (p. 183)
$\mathfrak{sl}(n, \mathbb{R})$	Lie algebra of $\mathrm{SL}(\ n, \mathbb{R}\ )$ (p. 186)
$\mathfrak{o}(n)$	Lie algebra of $\mathrm{O}(n)$ (p. 186)
$\mathfrak{u}(n)$	Lie algebra of $\mathrm{U}(n)$ (p. 186)
$(df)_p$, $df\|_p$	value of a 1 -form at p (p. 191)

$T_p^*(M)$ or T_p^*M	cotangent space at p (p. 190)
T^*M	cotangent bundle (p. 192)
$F^* : T_{F(p)}^*M \to T_p^*N$	codifferential (p. 196)
$F^*\omega$	pullback of a differential form ω by F (pp. 196, 205)
$\bigwedge^k (V^\vee) = A_k(V)$	k-covectors on a vector space V (p. 200)
ω_p	value of a differential form ω at p (p. 200)
$\mathcal{J}_{k,n}$	the set of strictly ascending multi-indices
	$1 \le i_1 < \cdots < i_k \le n$ (p.201)
$\wedge^k (T^*M)$	k th exterior power of the cotangent bundle (p. 203)
$\Omega^k(G)^G$	left-invariant k-forms on a Lie group G (p. 208)
$\operatorname{supp}\omega$	support of a k-form (p. 208)
$d\omega$	exterior derivative of a differential form ω (p.213)
$\omega\vert_S$	restriction of a differential from ω to a submanifold S (p. 216)
$\hookrightarrow$	inclusion map (p. 216)
$\mathcal{L}_X Y$	the Lie derivative of a vector field Y along X (p. 224)
$\mathcal{L}_X \omega$	the Lie derivative of a differential form ω along X (p. 226)
$t_v\omega$	interior multiplication of ω by v (p. 227)
$(v_1, \ldots, v_n)$	ordered basis (p. 237)
$[v_1, \ldots, v_n]$	ordered basis as a matrix (p. 238)
$(M, [\omega])$	oriented manifold with orientation $[\omega]$ (p. 244)
$-M$	the oriented manifold having the opposite orientation as M (p. 246)
$\mathcal{H}^n$	closed upper half-space (p. 248)
M°	interior of a manifold with boundary (pp. 248, 252)
∂M	boundary of a manifold with boundary (pp. 248, 251)
$\mathcal{L}^1$	left half-line (p. 251)
$\operatorname{int}(A)$	topological interior of a subset A (p. 252)
$\operatorname{ext}(A)$	exterior of a subset A (p. 252)
$\operatorname{bd}(A)$	topological boundary of a subset A (p. 252)
$\{p_0, \ldots, p_n\}$	partition of a closed interval (p. 260)
$P = \{P_1, \ldots, P_n\}$	partition of a closed rectangle (p. 260)
$L(f, P)$	lower sum of f with respect to a partition P (p. 260)
$U(f, P)$	upper sum of f with respect to a partition P (p. 260)

$\int_R f$	upper integral of f over a closed rectangle R (p. 261)
$\underline{\int}_R f$	lower integral of f over a closed rectangle R (p. 261)
$\int_R f(x)\,\lvert dx^1 \cdots dx^n\rvert$	Riemann integral of f over a closed rectangle R (p. 261)
$\int_U \omega$	Riemann integral of a differential form ω over U (p. 263)
$\mathrm{vol}(A)$	volume of a subset A of $\mathbb{R}^n$ (p. 262)
$\mathrm{Disc}(f)$	set of discontinuities of a function f (p. 262)
$\Omega_c^k(M)$	vector space of $C^\infty k$-forms with compact support on M (p. 265)
$Z^k(M)$	vector space of closed k-forms on M (p. 275)
$B^k(M)$	vector space of exact k-forms on M (p. 275)
$H^k(M)$	de Rham cohomology of M in degree k (p. 275)
$[\omega]$	cohomology class of ω (p. 275)
$F^\#$ or F^*	induced map in cohomology (p. 278)
$H^*(M)$	the cohomology ring $\oplus_{k=0}^n H^k(M)$ (p. 279)
$\mathcal{C} = \left(\left\{C^k\right\}_{k\in\mathbb{Z}}, d\right)$	cochain complex (p. 281)
$(\Omega^*(M), d)$	de Rham complex (p. 281)
$H^k(\mathcal{C})$	k th cohomology of C (p. 283)
$Z^k(\mathcal{C})$	subspace of k-cocycles (p. 283)
$B^k(\mathcal{C})$	subspace of k-coboundaries (p. 283
$d^* : H^k(\mathcal{C}) \to H^{k+1}(\mathcal{A})$	connecting homomorphism (p. 284)
$\to$	injection or maps to under an injection (p. 285)
$\mapsto$	maps to under a surjection (p. 285)
$i_U : U \to M$	inclusion map of U in M (p. 288)
$j_U : U \cap V \to U$	inclusion map of $U \cap V$ in U (p. 288)
$\to$	surjection (p. 291)
$\chi(M)$	Euler characteristic of M (p. 295)
$f \sim g$	f is homotopic to g (p. 296)
Σ_g	compact orientable surface of genus g (p. 310)
$d(p, q)$	distance between p and q (p. 317)
$(\ a, b\ )$	open interval (p. 318)
$(S, \mathcal{T})$	a set S with a topology $\mathcal{T}$ (p. 318)
$Z\,(f_1, \ldots, f_r)$	zero set of $f_1, \ldots, f_r$ (p. 319)
$Z(I)$	zero set of all the polynomials in an ideal I (p. 320)

IJ	the product ideal (p. 320)
$\sum_\alpha I_\alpha$	sum of ideals (p. 320)
$\mathcal{T}_A$	subspace topology or relative topology of A (p. 320)
$\mathbb{Q}$	the set of rational numbers (p. 323)
$\mathbb{Q}^+$	the set of positive rational numbers (p. 323)
$A \times B$	Cartesian product of two sets A and B (p. 326)
C_x	connected component of a point x (p. 333)
$\mathrm{ac}(A)$	the set of accumulation points of A (p. 335)
$\mathbb{Z}^+$	the set of positive integers (p. 336)
D_r	the set of matrices of rank $\leq r$ in $\mathbb{R}^{m\times n}$ (p. 345)
$D_{\max}$	the set of matrices of maximal rank in $\mathbb{R}^{m\times n}$ (p. 345)
$D_r(F)$	degeneracy locus of rank r of a map $F : S \to \mathbb{R}^{m\times n}$ (p. 345)
$\ker f$	kernel of a homomorphism f (p. 350)
$\mathrm{im}\, f$	image of a map f (p. 350)
$\mathrm{coker}\ f$	cokernel of a homomorphism f (p. 350)
$v + W$	coset of a subspace W (p. 349)
V/W	quotient vector space of V by W (p. 349)
$\Pi_\alpha V_\alpha, A \times B$	direct product (p. 351)
$\oplus_\alpha V_\alpha, A \oplus B$	direct sum (p. 351)
$A + B$	sum of two vector subspaces (p. 351)
$A \oplus_i B$	internal direct sum (p. 351)
$W^\perp$	W "perp," orthogonal complement of W (p. 352)
$\mathbb{H}$	skew field of quaternions (p. 353)
$\mathrm{End}_K(V)$	algebra of endomorphisms of V over K (p. 354)

14 Lectures on Classical Differential Geometry by Ivanov, A.A. Tuzhilin

14.1 1 Curves in Euclidean space

Классическая дифференциальная геометрия изучает объекты, расположенные в конечномерном евклидовом пространстве. Мы начнем с напоминания необходимых определений и обозначений из математического анализа и линейной алгебры.

14.1.1 1.1 Symbols and Reminders

Через $\mathbb{R}^n$ мы будем обозначать n-мерное арифметическое евклидово линейное пространство, т.е. множество всех упорядоченных наборов $x = (x^1, \ldots, x^n)$ вещественных чисел x^i, на котором введена естественная структура линейного пространства, а также стандартное скалярное произведение

$$\left\langle \left(v^1, \ldots, v^n\right), \left(w^1, \ldots, w^n\right) \right\rangle = \sum_{i=1}^n v^i w^i.$$

Элемент из $\mathbb{R}^n$, состоящий из $(n-1)$-го нуля и одной единицы, расположенной на i-ом месте, обозначим через e_i. Векторы $e_1, \ldots, e_n$ образуют ортонормированный базис пространства $\mathbb{R}^n$, который мы будем называть стандартньм базисом в $\mathbb{R}^n$, а координаты x^i вектора $x = \sum_{i=1}^n x^i e_i$ относительно этого базиса - стандартными координатами. Векторы из $\mathbb{R}^n$ будем также называть точками.

Скалярное произведение порождает норму $\|v\| = \sqrt{\langle v, v\rangle}$ и функцию расстояния $|vw| = \|v - w\|$. С помощью функции расстояния определяются замкнутый шар $B_r^n(x)$, открытый шар $U_r^n(x)$ и сфера $S_r^{n-1}(x)$ радиуса r с центром в точке x :

$$B_r^n(x) = \{y \in \mathbb{R}^n : |xy| \leq r\},$$
$$U_r^n(x) = \{y \in \mathbb{R}^n : |xy| < r\},$$
$$S_r^{n-1}(x) = \{y \in \mathbb{R}^n : |xy| = r\}.$$

Пустое множество, а также произвольные объединения открытых шаров называются открытыми множествами в $\mathbb{R}^n$. Эквивалентная формулировка: подмножество

евклидова пространства открыто, если и только если оно вместе с каждой точкой содержит и некоторый открытый шар с центром в этой точке. Легко видеть, что произвольные объединения, а также конечные пересечения открытых множеств являются открытыми множествами (проверьте).

Перейдем теперь к определению кривых. Мы рассмотрим несколько подходов, которые приводят, вообще говоря, к разным теориям. Однако, если рассматривать "достаточно малые" фрагменты возникающих объектов, эти теории оказываются эквивалентными. Мы начнем с параметрических кривых, затем определим кривые-графики для векторнозначных функций, и, наконец, введем в рассмотрение неявные кривые, которые задаются в виде решения систем уравнений, удовлетворяющих условиям теоремы о неявной функции.

14.1.2 1.2 Parametric curves

Обозначим через $I \subset \mathbb{R}$ конечный или бесконечный промежуток одного из следующих видов: отрезок $[a, b]$, интервал $(\ a, b\ )$, полуинтервал $(\ a, b\ ]$ или $[a, b\ )$. Для каждой точки $t \in I$ ее окрестностью будем называть каждое множество вида $I \cap (t - \varepsilon_1, t + \varepsilon_2)$, где ε_1 и ε_2 - произвольные положительные числа [01].

Пример 1.1. Каждая окрестность точки a отрезка $[a, b]$ является или полуинтервалом вида $[a, a + \varepsilon)$, или совпадает со всем отрезком $[a, b]$.

Рассмотрим отображение $\gamma : I \to \mathbb{R}^n$. Для каждого $t \in I$ точка $\gamma(t) \in \mathbb{R}^n$ имеет вид $(x^1(t), \ldots, x^n(t))$, поэтому отображение γ порождает n функций $x^i(t)$, которые называются координатньми функциями отображения γ. Напомним, что отображение γ является непрерывным (векторнозначным отображением), если и только если все его координатные функции непрерывны.

Определение 1.2. Непрерывное отображение $\gamma : I \to \mathbb{R}^n$ называется параметрической кривой, а переменная $t \in I$ - ее параметром.

Замечание 1.3. Параметрическую кривую можно представить себе как математическую формализацию движения материальной точки в пространстве $\mathbb{R}^n$, при этом t соответствует времени, а $\gamma(t)$ - положению материальной точки в момент времени t; образ кривой γ можно воспринимать как траекторию движения материальной точки.

Параметрические кривые в $\mathbb{R}^3$ называются пространственньмии, а на плоскости $\mathbb{R}^2$ - плоскими. Кривые, встречавшиеся в курсе аналитической геометрии, - прямые, окружности, эллипсы, ветви гипербол, параболы, - можно представить как образы плоских параметрических кривых.

Определение 1.4. Если при некоторых $t \neq t'$ точки $\gamma(t)$ и $\gamma(t')$ совпадают, то $P = \gamma(t) = \gamma(t')$ называется точкой самопересечения параметрической кривой γ. Если γ не имеет точек самопересечения, то ее называют вложенной.

[01] В общей топологии окрестность точки понимается в более общем смысле, см. лекцию П. В рамках данной лекции нам достаточно будет рассматривать именно такие "интервальные" окрестности.

Пример 1.5. Параметрические кривые $\gamma(t) = (A\cos(at), B\sin(bt+c)), t \in [0, 2\pi]$, называются кривыми Лиссажу, см. рис. 1.1. Эти кривые хорошо известны в технике: именно они возникают на экране осциллографа при сравнении частот двух электрических колебаний (на экран выводится светящаяся точка, координаты которой задаются как $\gamma(t)$, в результате точка заметает кривую Лиссажу). Здесь A, B - амплитуды колебаний, a, b - частоты, c - сдвиг фаз. Замкнутая устойчивая картинка на экране соответствует соизмеримости частот, т.е., в наших обозначениях, рациональности отношения a/b.

Рис. 1.1: Кривая Лиссажу для $A = B = a = 1, b = 3$ и $c = 0$. Точки самопересечения лежат на одной оси, значит одна частота кратна другой.

Замечание 1.6. Интуитивное представление о параметрической кривой как о траектории движения материальной точки иногда оказывается несостоятельным. Так, например, известные из курса математического анализа кривые Пеано отображают отрезок на весь треугольник или весь квадрат, см. Дополнительный материал 1.10.1. Таких "неприятностей" не возникает, если ввести дополнительные требования на рассматриваемые кривые.

Следуя традиции, мы будем называть бесконечно дифференцируемые функции гладкими. [Д]

Определение 1.7. Параметрическая кривая $\gamma : I \to \mathbb{R}^n$ называется гладкой, если все ее координатные функции $x^i(t)$ - гладкие.

Для каждого значения параметра t_0 гладкой параметрической кривой $\gamma(t)$ определен вектор $\dot{\gamma}(t_0) = (\dot{x}^1(t_0), \ldots, \dot{x}^n(t_0))$, составленный из производных координатных функций, вычисленных в точке t_0.

Определение 1.8. Вектор $\dot{\gamma}(t_0)$ называется вектором скорости параметрической кривой γ в точке $\gamma(t_0)$, соответствующим значению параметра $t = t_0$.

Замечание 1.9. В точках самопересечения гладкой параметрической кривой определено несколько векторов скорости. Чтобы выделить один, нужно указать значение параметра.

Точка гладкой параметрической кривой, в которой вектор скорости обращается в ноль, называется особой или сингулярной, а остальные точки - неособыми или регулярными.

[02] Иногда в дифференциальной геометрии гладкой называют функцию непрерывно дифференцируемую столько раз, сколько требуется для решения конкретной задачи.

Определение 1.10. Гладкая параметрическая кривая называется регулярной, если ее вектор скорости всюду отличен от нуля.

Замечание 1.11. Образ гладкой параметрической кривой, в отличие, скажем, от кривой Пеано, не может содержать никакого открытого шара. Это вытекает из теоремы Сарда, известной из курса математического анализа, см. также лекцию 14.

Пример 1.12. Интуитивное представление о гладкости кривой как об отсутствии изломов оказывается не всегда верным в рамках данного определения. Рассмотрим полукубическую параболу - плоскую гладкую параметрическую кривую $\gamma(t) = (t^3, t^2)$, $t \in [-1, 1]$, изображенную на рис. 1.2. Вектор скорости $\dot{\gamma}(t)$ в начале координат равен нулю, а его направление, т.е. нормированный вектор $\dot{\gamma}(t)/\|\dot{\gamma}(t)\|$, имеет разрыв, т.е. направление скорости меняется скачком. В этом случае говорят, что кривая имеет излом. Ясно, что регулярная кривая не может иметь изломов.

Рис. 1.2: Полукубическая парабола имеет излом.

Пусть $t(s)$ - некоторая строго монотонная непрерывная функция, отображающая промежуток I' на промежуток I, а $\gamma(t), t \in I$, - некоторая параметрическая кривая. Тогда $\delta(s) = \gamma(t(s))$ - тоже некоторая параметрическая кривая.

Определение 1.13. Отображение $t : I' \to I$ называется заменой параметра, а про кривую $\delta(s) = \gamma(t(s))$ говорят, что она получена из кривой γ заменой параметра $t = t(s)$.

Если кривая δ получена из γ заменой параметра, то, очевидно, γ и δ имеют одинаковые образы. Однако, замена параметра гладкой кривой может привести к потере гладкости.

Пример 1.14. Если в гладкой параметрической кривой $\gamma(t) = (t, 0)$ сделать замену $t = t(s)$, для которой функция $t(s)$ равна s при $s \leq 0$ и $2s$ при $s \geq 0$, то полученная кривая $\gamma(t(s))$ не будет дифференцируемой в $s = 0$.

Замечание 1.15. Если замена параметра $t = t(s)$ и обратная замена параметра $s = s(t)$ - гладкие, то регулярность кривой $\gamma(t)$ равносильна регулярности кривой $\delta(s)$, полученной из $\gamma(t)$ такой заменой параметра.

Замечание 1.16. Замена параметризации может превратить нерегулярную параметрическую кривую в регулярную. Например, параметрическая кривая γ с координатными функциями $x(t) = t^3, y(t) = t^6$ не регулярна. Положив $\tau = t^3$, мы получим регулярную параметрическую кривую $x(\tau) = \tau, y(\tau) = \tau^2$. При этом, отображение $\tau(t) = t^3$ не обладает гладким обратным, так как производная $\tau'(t)$ обращается в нуль.

Замечание 1.17. Представление подмножества в виде образа параметрической кривой неоднозначно, так как одна и та же траектория-подмножество может быть поразному пройдена. При этом неоднозначность возникает не только "при выборе скоростного режима" (монотонные замены параметра), но и при определении порядка прохождении

перекрестков (точек самопересечения), а также из-за возможности пройти траекторию несколько раз или развернуться.

Предположим, что две регулярные параметрические кривые γ_1 и γ_2 пересекаются, т.е. для некоторых значений параметров выполнено равенство $\gamma_1(t_1) = \gamma_2(t_2) = P$.

Определение 1.18. Углом между γ_1 и γ_2 в их точке пересечения P, соответствующим значениям параметров t_1 и t_2, называется угол между векторами скоростей $\dot{\gamma}_1(t_1)$ и $\dot{\gamma}_2(t_2)$.

Упражнение 1.19. Напишите в явном виде выражение для угла между пересекающимися регулярными параметрическими кривыми.

14.1.3 1.3 Graph Curves

Пусть $f : I \to \mathbb{R}^{n-1}$ - непрерывная векторнозначная функция.
Определение 1.20. Множество $\mathrm{Gr}(f) = \{(t, f(t)) \mid t \in I\} \subset \mathbb{R}^n$ называется кривойграфиком в пространстве $\mathbb{R}^n$.

По определению, множество $\mathrm{Gr}(f)$ - образ непрерывного отображения $\gamma_f(t) = (t, f(t))$, представляющего собой параметрическую кривую, про которую будем говорить, что она соответствует кривой-графику $\mathrm{Gr}(f)$. Если отображение f гладкое, то $\mathrm{Gr}(f)$ назовем гладкой кривой-графиком.

Замечание 1.21. Параметрическая кривая γ_f, соответствующая гладкой кривойграфику, всегда регулярна, так как $\dot{\gamma}_f(t) = (1, \dot{f}(t)) \neq 0$.

Замечание 1.22. Образ параметрической кривой не всегда является кривой-графиком. Действительно, кривая-график однозначно проецируется на координатную прямую, а, например, дуга окружности угловой величины больше π - нет.

Лемма 1.23. Пусть $\gamma(t) = (x^1(t), \ldots, x^n(t))$ - регулярная параметрическая кривая. Для каждого $t_0 \in I$ существует такая его окрестность $U \subset I$, что множество $\gamma(U)$ является кривой-графиком $\mathrm{Gr}(f)$ для некоторой гладкой векторнозначной функции f (возможно, после перенумерации координат).

Доказательство. Так как $\dot{\gamma}(t_0) \neq 0$, для некоторого i имеем $\dot{x}^i(t_0) \neq 0$. Без ограничения общности будем считать, что $i = 1$ (иначе перенумеруем координаты). Тогда при всех t из некоторой окрестности U точки t_0 функция $x^1(t)$ строго монотонна и имеет ненулевую производную, поэтому на промежутке $I' = x^1(U)$ существует гладкая обратная функция $t = t(x^1)$. Но тогда функция

$$f\left(x^1\right) = \left(x^2\left(t\left(x^1\right)\right), \ldots, x^n\left(t\left(x^1\right)\right)\right)$$

является гладкой и порождает искомую кривую-график $\mathrm{Gr}(f)$. $\square$

14.1.4 1.4 Implicit curves

Пусть $F^i : \Omega \to \mathbb{R}, i = 2, \ldots, n$, - гладкие функции, заданные на области $\Omega \subset \mathbb{R}^n$, и $c = (c^2, \ldots, c^n)$ - некоторый постоянный вектор. Предположим, что семейство функций F^i удовлетворяет условиям теоремы о неявной функции, т.е. множество M_c всех решений системы уравнений

$$F^i\left(x^1, \ldots, x^n\right) = c^i, \quad i = 2, \ldots, n,$$

не пусто, и ранг матрицы Якоби $J = (\partial F^i / \partial x^j)$ в каждой точке из M_c равен $n - 1$.
Определение 1.24. Множество M_c называется неявной кривой.
Замечание 1.25. Каждая гладкая кривая-график $\mathrm{Gr}(f)$ векторнозначной функции $f = (f^2, \ldots, f^n)$, определенной на некотором интервале (a, b), является также и неявной кривой, заданной системой уравнений $x^i - f^i(x^1) = 0, i = 2, \ldots, n$, на области $(a, b) \times \mathbb{R}^{n-1}$.

Обратный результат снова не имеет места: например окружность $(x^1)^2 + (x^2)^2 = 1$ не представляется в виде кривой-графика.

Лемма 1.26. Пусть P - произвольная точка неявной кривой M_c. Тогда в некоторой окрестности точки P в $\mathbb{R}^n$ множество M_c можно представить как гладкую кривую-график для некоторой векторнозначной функц 44 (возможно, после перенумерации координат).

Доказательство. Без ограничения общности, предположим, что минор, порожденный столбцами матрицы Якоби J в точке P, начиная со второго, отличен от нуля. Тогда, по теореме о неявной функции, в некоторой окрестности точки P система уравнений, задающая M_c, разрешима в том смысле, что переменные $x^2, \ldots, x^n$ выражаются через x^1. Это означает, что часть множества M_c, попавшая в эту окрестность, представляется в виде $(x^1, x^2(x^1), \ldots, x^n(x^1))$, т.е. является гладкой кривой-графиком. □

14.1.5 1.5 Local equivalence

Подведем итог. В предыдущих разделах мы описали три разных подхода к определению понятия кривой. Как мы уже видели выше, эти три определения задают "в целом" разные объекты. Однако, в силу лемм 2.23 и 1.26 имеет место следующий результат.

Theorem 1.27.

С локальной точки зрения три способа задания регулярной кривой эквивалентны в следующем смысле:

- если $\gamma : (a, b) \to \mathbb{R}^n$ - регулярная параметрическая кривая, то для каждого $t \in (a, b)$ существует такая окрестность U точки t в интервале (a, b), что ее образ $\gamma(U)$ является и гладкой кривой-графиком, и неявной кривой;
- для каждой точки P кривой-графика $\mathrm{Gr}(f)$ произвольной гладкой функции f, заданной на некотором интервале (a, b), существует такая окрестность $U \subset \mathbb{R}^n$, что $U \cap \mathrm{Gr}(f)$ есть и образ регулярной параметрической кривой, и неявная кривая;
- для каждой точки P неявной кривой M_c существует такая окрестность $U \subset \mathbb{R}^n$, что $U \cap M_c$ есть и образ регулярной параметрической кривой, и криваяграфик в $\mathbb{R}^n$.

Приведем одно очевидное, но полезное следствие локальной эквивалентности параметрического задания кривой и представления ее в виде кривой-графика.

Следствие 1.28. Пусть $\gamma : I \to \mathbb{R}^n$ - регулярная параметрическая кривая. Для каждого $t \in I$ существует такая окрестность U точки t в промежутке I, что ограничение отображения γ на U взаимно однозначно с образом.

Ниже мы будем заниматься локальной теорией регулярных кривых, поэтому, опираясь на теорему 1.27, будем использовать следующее соглашение.

Соглашение 1.29. Там, где это не вызовет недоразумений, мы будем говорить просто о регулярной кривой, подразумевая что соответствующий объект задан удобным для нас способом (или как кривая-график, или как неявная кривая, или как регулярная параметрическая кривая). Причем в последнем случае, если не оговорено противное, параметризующее отображение предполагается взаимно однозначными с образом, а замены параметризации - гладкими и с гладкими обратными, т.е. как в замечании 1.15.

Замечание 1.30. В рамках принятого только что соглашения 1.29 можно рассматривать регулярную кривую как подмножество в объемлющем пространстве.

14.1.6 1.6 Curve length, natural parameter

Общее определение длины кривой обычно использует приближение кривой вписанными ломаными. Для случая гладкой параметрической кривой $\gamma : [a, b] \to \mathbb{R}^n$ мы сразу дадим определение через интеграл. Связь двух этих подходов обсуждается в курсе математического анализа, см. также добавление 1.10.3.

Определение 1.31. Длиной $\ell(\gamma)$ гладкой параметрической кривой $\gamma : [a, b] \to \mathbb{R}^n$ называется величина интеграла $\int_a^b \|\dot{\gamma}(t)\| dt$.

Предложение 1.32. Длина гладкой параметрической кривой не меняется при замене параметризации.

Доказательство. Действительно, пусть $\gamma(t)$ - гладкая кривая, $t = t(s)$ - замена параметра, где $t \in [t_0, t_1]$, а $s \in [s_0, s_1]$. Положим $\tilde{\gamma}(s) = \gamma(t(s))$. По теореме о замене параметра в определенном интеграле имеем:

$$\ell(\tilde{\gamma}) = \int_{s_0}^{s_1} \left\| \frac{d\tilde{\gamma}}{ds} \right\| ds = \int_{t(s_0)}^{t(s_1)} \left\| \frac{d\gamma}{dt} \right\| \cdot \left| \frac{dt}{ds} \right| \frac{ds}{dt} dt$$

Осталось избавиться от модуля в правой части. Так как $t(s)$ - строго монотонная функция, ее производная по s знакопостоянна. Если $t(s)$ монотонно возрастает, то производная dt/ds неотрицательна и модуль можно опустить. При этом, $t(s_0) = t_0$ и $t(s_1) = t_1$, поэтому правая часть равна длине $\ell(\gamma)$ параметрической кривой γ, т.е. утверждение доказано. Если же функция $t(s)$ монотонно убывает, то $dt/ds < 0$, и модуль раскрывается с минусом. Но $t(s_0) = t_1$ и $t(s_1) = t_0$, что и дает недостающий "минус" в правой части. Доказательство закончено.

Из утверждения 1.32 следует возможность корректно определить длину кривойграфика и длину неявной кривой как длины соответствующих (взаимно однозначных с образом) параметрических кривых.

Понятие длины позволяет выбрать на кривой некоторое естественное семейство параметризаций, соответствующих движению по траектории с единичной по величине скоростью.

Определение 1.33. Параметр $s, s \in [a, b]$, гладкой параметрической кривой $\gamma(s)$ называется натуральным, если величина $s - a$ равна длине части кривой γ между точками $\gamma(a)$ и $\gamma(s)$ для любого $s \in [a, b]$.

Statement

1.34. Параметр s на гладкой кривой $\gamma(s)$ является натуральным, если и только если $\|\dot{\gamma}(s)\| = 1$.

Доказательство. Пусть s - натуральный параметр, т.е.

$$s = a + \int_a^s \|\dot{\gamma}(\sigma)\| d\sigma$$

Дифференцируя левую и правую части равенства по s, получаем требуемое. Обратно, если $\|\dot{\gamma}(s)\| = 1$, то

$$s - a = \int_a^s 1 d\sigma = \int_a^s \|\dot{\gamma}(\sigma)\| d\sigma$$

поэтому $s - a$ равно длине части кривой γ между точками $\gamma(a)$ и $\gamma(s)$, т.е. s - натуральный параметр, что и требовалось доказать.

Замечание 1.35. Если t - произвольный параметр гладкой кривой $\gamma : [a, b] \to \mathbb{R}^n$, то отображение $\varphi(t) = \int_a^t \|\dot{\gamma}(\tau)\| d\tau$ задает замену параметра, приводящую к натуральному

параметру $s = \varphi(t)$. Однако, после такой замены кривая может перестать быть гладкой. Дело в том, что обратная функция $\varphi^{-1}(s)$ не обязана быть гладкой. Как видно из следующего утверждения, гладкость полученной в результате кривой равносильна регулярности исходной.

Statement

1.36. Переход к натуральному параметру сохраняет гладкость параметрической кривой, если и только если эта кривая регулярна.

Доказательство. Если кривая $\gamma(t)$ регулярна, то отображение

$$\varphi : t \mapsto \int_a^t \|\dot{\gamma}(\tau)\| d\tau$$

имеет гладкое обратное (так как $\dot{\varphi}(t) = \|\dot{\gamma}(t)\| \neq 0$), поэтому результирующая кривая $\gamma\left(\varphi^{-1}(s)\right)$ - это гладкая кривая с натуральным параметром $s = \varphi(t)$. Обратное утверждение следует из предложения [.34. □

Упражнение 1.37. Если s_1 и s_2 - два натуральных параметра на кривой γ, то $s_1 = \pm s_2 + c$ для некоторого числа $c \in \mathbb{R}$.

Определение 1.38. Вектором ускорения гладкой параметрической кривой γ в точке $\gamma(t)$ называется вектор $\ddot{\gamma}(t) = (\ddot{x}^1(t), \ldots, \ddot{x}^n(t))$.

Statement

1.39. Пусть s - натуральный параметр на регулярной кривой γ. Тогда в каждой точке кривой γ вектор ускорения $\ddot{\gamma}$ перпендикулярен вектору скорости $\dot{\gamma}$.

Доказательство. Действительно, $\langle \dot{\gamma}(s), \dot{\gamma}(s) \rangle = 1$ в силу предложения 1.34. Дифференцируя это равенство по s, получаем

$$2\langle \dot{\gamma}(s), \ddot{\gamma}(s) \rangle = 0$$

что и требовалось доказать. □

14.1.7 1.7 Regular Curve Curvature

Чтобы охарактеризовать степень искривления отдельных участков кривой, обычно используют величину ускорения при равномерном движении по рассматриваемой кривой.

Определение 1.40. Пусть γ - произвольная регулярная кривая, параметризованная натуральным параметром s. Величина $k(s) = \|\ddot{\gamma}(s)\|$ называется кривизной кривой γ в точке $\gamma(s)$, а величина $R(s) = 1/k(s)$ - радиусом кривизны в точке $\gamma(s)$ (если $k(s) = 0$, то полагают $R(s) = \infty$).

Из задачи 1.37 следует, что $k(s)$ и $R(s)$ не зависят от выбора натурального параметра.

Пример 1.41. Рассмотрим окружность радиуса r, заданную в параметрическом виде так: $\gamma(s) = r(\cos(s/r), \sin(s/r)), s \in [0, 2\pi r]$. Тогда $\gamma'(s) = (-\sin(s/r), \cos(s/r))$, и параметр s - натуральный в соответствии с предложением 1.34. Далее, $\gamma''(s) = -1/r(\cos(s/r), \sin(s/r))$, откуда $k(s) = 1/r$, а $R(s) = r$.

Утверждение 1.42. Если t - произвольный параметр на регулярной кривой γ, то кривизна в точке $\gamma(t)$ может быть вычислена по формуле

$$k(t) = \frac{\|\dot{\gamma}(t) \times \ddot{\gamma}(t)\|}{\|\dot{\gamma}(t)\|^3},$$

где через $\|v \times w\|$ обозначена площадь параллелограмма, натянутого на векторы v и w из $\mathbb{R}^n$.

В случае кривой на плоскости $\mathbb{R}^2$ формула для кривизны в явном виде имеет вид

$$k(t) = \frac{|\dot{x}(t)\ddot{y}(t) - \dot{y}(t)\ddot{x}(t)|}{(\dot{x}(t)^2 + \dot{y}(t)^2)^{3/2}}$$

где (x, y) - стандартнье координаты на плоскости, а в случае пространства $\mathbb{R}^3$ в числителе стоит модуль векторного произведения векторов $\dot{\gamma}(t)$ и $\ddot{\gamma}(t)$.

Доказательство. Частные случаи получаются из общей формулы, геометрической интерпретации определителя и определения векторного произведения, поэтому достаточно доказать только общую формулу. Пусть сначала $t = s$ - натуральный параметр. Тогда

$$\dot{\gamma} = \tau, \quad \|\dot{\gamma}\| = 1, \quad \ddot{\gamma} = k\nu, \quad \|[\dot{\gamma}, \ddot{\gamma}]\| = k,$$

где последнее равенство имеет место так как векторы $\dot{\gamma}$ и $\ddot{\gamma}$ перпендикулярны, поэтому в этом случае утверждение имеет место.

Теперь сделаем замену параметра. Оказывается, что общая формула при этом не изменятся. Действительно, обозначая исходный параметр через s, новый параметр через t, а дифференцирования по параметрам - соответствующими нижними индексами, получаем

$$\gamma_s = \gamma_t t_s, \quad \gamma_{ss} = \gamma_{tt} t_s^2 + \gamma_t t_{ss}$$

откуда

$$[\gamma_s, \gamma_{ss}] = [\gamma_t, \gamma_{tt}]\, t_s^3$$

и, наконец,

$$\frac{\|[\gamma_s, \gamma_{ss}]\|}{\|\gamma_s\|^3} = \frac{\|[\gamma_t, \gamma_{tt}]\|}{\|\gamma_t\|^3}$$

Доказательство закончено.

Определение 1.43. Регулярная параметрическая кривая $\gamma(t)$ называется бирегулярной, если ее кривизна всюду отлична от нуля.

Из утверждения 1.42 получается следующий результат.

Следствие 1.44. Регулярная кривая γ бирегулярна, если и только если ее векторы скорости $\dot{\gamma}(t)$ и ускорения $\ddot{\gamma}(t)$ линейно независимы в каждой точке $\gamma(t)$ в произвольной параметризации.

Конструкция 1.45. Пусть $\gamma(s)$ - натурально параметризованная бирегулярная кривая в $\mathbb{R}^n$. Обозначим через τ единичный вектор скорости $\dot{\gamma}(s)$ в точке $\gamma(s)$, а через ν - нормированный вектор ускорения в точке $\gamma(s)$, т.е. $\nu = \ddot{\gamma}(s)/\|\ddot{\gamma}(s)\|$. Из предложения 1.39 следует, что векторы τ и ν ортогональны.

Определение 1.46. Вектор ν называется главной нормалью бирегулярной кривой γ в точке $\gamma(s)$.

14.1.8 1.8 Flat curves

Рассмотрим случай плоских кривых более подробно. Воспользуемся конструкцией 1.45. Случай $n = 2$ отличается от общего тем, что пара векторов (τ, ν) образует ортонормированный базис.

Определение 1.47. Пара векторов (τ, ν) называется репером Френе плоской бирегулярной кривой в соответствующей её точке.

Theorem 1.48 (Формулы Френе).

Пусть γ - натурально параметризованная плоская бирегулярная кривая, us - натуральный параметр. Тогда справедливы следующие соотношения:

$$\begin{cases} \dot{\tau} = k\nu, \\ \dot{\nu} = -k\tau, \end{cases}$$

где $k = k(s)$ - кривизна кривой γ в точке $\gamma(s)$.

Доказательство. Первое равенство мгновенно следует из определения векторов τ и ν.

Докажем второе равенство. Дифференцируя тождество $\langle \nu, \nu \rangle = 1$, находим, что $\langle \dot{\nu}, \nu \rangle = 0$, т.е. вектор $\dot{\nu}$ перпендикулярен ν и поэтому коллинеарен τ (здесь мы пользуемся тем, что кривая - плоская). Следовательно,

$$\dot{\nu} = \langle \dot{\nu}, \tau \rangle \tau.$$

Покажем, что $\langle \dot{\nu}, \tau \rangle = -k$. Для этого продифференцируем тождество $\langle \nu, \tau \rangle = 0$. Получим

$$0 = \langle \dot{\nu}, \tau \rangle + \langle \nu, \dot{\tau} \rangle = \langle \dot{\nu}, \tau \rangle + \langle \nu, k\nu \rangle = \langle \dot{\nu}, \tau \rangle + k$$

что и требовалось доказать.

Для плоских кривых можно определить понятие ориентированной кривизны, которое в некоторых случаях оказывается более удобным.

Конструкция 1.49. Фиксируем стандартную ориентацию плоскости, т.е. объявим положительно ориентированными те базисы, матрицы перехода от которых к стандартному базису имеют положительный определитель, а все остальные - отрицательно

Покажем теперь, что построенная кривая γ - искомая. Действительно, вычислив скорость и ускорение кривой γ, заметим, что она регулярна, натурально параметризована, и ее ориентированная кривизна равна $f(s)$. Доказательство закончено.

Замечание 1.54. Мы нашли явные формулы для восстановления регулярной кривой γ по ее ориентированной кривизне.

Замечание 1.55. Любая гладкая функция может быть реализована как ориентированная кривизна некоторой плоской регулярной кривой.

Замечание 1.56. Чтобы задать на плоскости регулярную кривую (с точностью до движения), достаточно задать одну гладкую функцию - ориентированную кривизну этой кривой.

Замечание 1.57. Теорему 1.53 можно переформулировать и для обычной кривизны кривой. Чтобы воспользоваться разработанной выше техникой, основанной на формулах Френе, необходимо требовать бирегулярность кривой и, следовательно, положительность кривизны. Следующий результат доказывается аналогично теореме 1.53.

Theorem

1.58. Пусть $f(s)$ - произвольная положительная гладкая функция, заданная на промежутке I. Тогда существует плоская бирегулярная кривая γ, натурально параметризованная параметром s, такая, что ее кривизна $k(s)$ в каждой точке $\gamma(s)$ равна $f(s)$, причем такая кривая единственна с точностью до движения плоскости.

Замечание 1.59. Если попытаться распространить теорему 1.58 на случай произвольной регулярной кривой, то придется учесть, что кривизна такой кривой может

обращаться в нуль. Однако, это приведет к дополнительной неоднозначности. Действительно, рассмотрим неотрицательную функцию $f(s)$ положительную на всем промежутке I, за исключением единственной его внутренней точки s_0. Тогда теорема 1.58 применима к ограничению функции f на каждый из двух промежутков I_1 и I_2, на которые распадается I после выбрасывания точки s_0. Обозначим через $\gamma_i, i = 1, 2$, бирегулярную кривую восстановленную по функции $f|_{I_i}$. Обе кривые γ_i можно по непрерывности продолжить в точку s_0. В результате получим регулярные кривые, ускорения которых обращаются в нуль в точке s_0. Если совместить точки $\gamma_1(s_0)$ и $\gamma_2(s_0)$, а также вектора скоростей $\dot{\gamma}_1(s_0)$ и $\dot{\gamma}_2(s_0)$ с помощью некоторого движения Φ, то результате получится регулярная натурально параметризованная кривая, определенная на всем промежутке I и имеющая кривизну $f(s)$. Но движение Φ не определено однозначно: если отразить одну из кривых γ_i относительно прямой, проходящей через точку $\gamma_i(s_0)$ в направлении $\dot{\gamma}_i(s_0)$, то полученная в результате кривая также удовлетворяет всем нашим требованиям.

Определение 1.60. Соотношение $k_o(s) = f(s)$ называется натуральным уравнением плоской кривой. Решить натуральное уравнение означает найти натурально параметризованную регулярную кривую $\gamma(s)$, ориентированная кривизна которой равна $f(s)$.

Замечание 1.61. Часто натуральным уравнением называют соотношение $k(s) = f(s)$ на обычную кривизну плоской кривой. Однако из теоремы 1.53 и замечания 1.59 следует, что уравнение на ориентированную кривизну разрешимо для более широкого класса

функций, а множество его решений устроено проще. Поэтому мы назвали натуральным уравнением условие именно на ориентированную кривизну регулярной плоской кривой.

14.1.9 1.9 Problems

Задача 1.62. Круг радиуса a катится по прямой без скольжения. Составить уравнение траектории точки, жестко связанной с кругом и находящейся на расстоянии d от его центра (при $d = a$ циклоида, при $d < a$ - укороченная циклоида, при $d > a$ удлиненная ииклоида).

Задача 1.63. Ввести натуральную параметризацию на следующих кривых:

(1) отрезке прямой $ax + by + c = 0$;

(2) дуге окружности $x^2 + y^2 = R^2$;

(3) графике функции $y = x^2$.

Задача 1.64. Записать явные формулы для длины кривой-графика гладкой функции $f : I \to \mathbb{R}^{n-1}$ и для неявной кривой, заданной системой уравнений $F_i(x) = c_i$.

Задача 1.65. Вычислить кривизну следующих кривых:

(1) окружности радиуса R;

(2) цепной линии, заданной графиком функции $y = a\,\mathrm{ch}(x/a), a > 0$;

(3) эллипса с полуосями a и b в его вершинах;

(4) циклоиды, укороченной циклоиды, удлиненной циклоиды.

Задача 1.66. Решить натуральное уравнение $k = 1/s$.

Задача 1.67. Описать все плоские кривые постоянной кривизны.

Задача 1.68. Докажите, что объединение двух отрезков, стыкующихся под произвольным углом, можно представить как образ некоторой гладкой параметрической кривой (см. дополнительный материал 1.10.2).

Задача 1.69. Сколько, с точностью до движения, существует плоских регулярных кривых с кривизной $k(s) = \cos(s) + 1, s \in [0, 4\pi]$.

Задача 1.70. Не используя теорему Сарда докажите, что образ регулярной

параметрической кривой не может содержать никакого открытого шара.
Задача 1.71. Приведите пример непрерывной кривой $\gamma : [0,1] \to \mathbb{R}^n$, для которой ограничение $\gamma|_I$ на любой невырожденный промежуток $I \subset [0,1]$ не представляется в виде кривой-графика.
Задача 1.72. Пусть $\gamma(s)$ - натурально параметризованная бирегулярная кривая. Центром кривизны кривой γ в точке $\gamma(s)$ называется точка $\gamma(s) + R(s)\nu(s)$, где $R(s) = 1/k(s)$ - радиус кривизны, а $\nu(s)$ - главная нормаль. Множество центров кривизны называется каустикой, или эволютой кривой γ. Построить каустику для эллипса и циклоиды.

Задача 1.73. Овалом называется простая замкнутая кривая, кривизна которой отлична от нуля во всех ее точках (овал ограничивает строго выпуклую область). Вершиной овала называется точка, в которой кривизна имеет локальные минимум или максимум. Доказать, что каждый овал имеет по меньшей мере четыре вершины.

14.2 1.10 Curves in Euclidean space: Supplemental material

14.2.1 1.10.1 Peano curve in a triangle

Кривая Пеано - непрерывная кривая, заполняющая квадрат или треугольник, может быть построена, например, следующим образом.

Конструкция 1.74. Возьмем равнобедренный прямоугольный треугольник T, разобьем его на два равнобедренных прямоугольных треугольника T_1 и T_2, проведя высоту h из вершины прямого угла. Обозначим через r_1 отрезок, соединяющий центры треугольников $T_j, j = 1, 2$.

Чтобы построить ломаную r_2, в каждом из треугольников $T_j, j = 1, 2$, построим такую же ломаную, как во всем треугольнике T на первом шаге. Другими словами, возьмем два экземпляра треугольника T с уже построенной ломаной r_1, уменьшим их до размера треугольников T_j и отождествим каждый из экземпляров с соответствующим треугольником $T_j, j = 1, 2$. Полученное объединение отрезков достроим до связной ломаной, соединив две концевые вершины, ближайшие к высоте h.

Наконец, на $(i+1)$-м шаге построим ломаную r_{i+1} из ломаной r_i, полученной на i-м шаге. Для этого вновь разобьем треугольник T на два треугольника T_1 и T_2, проведя в T высоту h из вершины прямого угла. Возьмем два экземпляра треугольника T с уже построенной ломаной r_i, уменьшим их до размера треугольников T_j и отождествим каждый из них с соответствующим треугольником $T_j, j = 1, 2$. Полученное объединение двух ломаных, лежащих в треугольнике T, достроим до связной ломаной, соединив две концевые вершины этих ломаных, ближайшие к высоте h.

Продолжим этот процесс до бесконечности (рис. П.3). Предельная кривая γ называется кривой Пеано.

Несложно показать, что кривая Пеано проходит через каждую точку треугольника. Поэтому ее образ - это треугольник T.

Рис. 1.3: Кривая Пеано в треугольнике.

Отметим, что можно построить много- и даже бесконечномерные аналоги кривой Пеано (теорема Мазуркевича).

Упражнение 1.75. Показать, что кривая Пеано непрерывна и проходит через каждую точку треугольника T. Найти число прообразов точек кривой Пеано.

14.2.2 1.10.2 Polyline as a smooth parametric curve

Рассмотрим отображение L отрезка $[-1,1]$ в плоскость $\mathbb{R}^2$, заданное координатными функциями $x^i(t)$:

$$x^1(t) = \begin{cases} 0 & \text{при } t \in [-1,0], \\ e^{-1/t^2} & \text{при } t \in (0,1]; \end{cases} \qquad x^2(t) = \begin{cases} e^{-1/t^2} & \text{при } t \in [-1,0), \\ 0 & \text{при } t \in [0,1]. \end{cases}$$

Легко проверить, что функции $x^1(t)$ и $x^2(t)$ являются гладкими. Графики координатных функций $x^i(t), i = 1,2$, и сама гладкая кривая L изображены на рис. 1.4. Отметим, что построенная гладкая кривая не регулярна. Переход к натуральому параметру на ней возможен только с потерей гладкости.

Рис. 1.4: Координатные функции и образ гладкой кривой L.

14.2.3 1.10.3 The length of the curve as the limit of the lengths of inscribed polylines

В разделе 1.6 мы определили длину гладкой параметрической кривой через интеграл. Более геометрический и универсальный подход состоит в определении длины кривой с помощью вписанных в нее ломаных.

Конструкция 1.76. Пусть ξ - некоторое разбиение отрезка $[a, b]$ точками $a = t_0 < t_1 < \cdots < t_n = b$. Напомним, что диаметром δ_ξ разбиения ξ называется максимальное из чисел $t_i - t_{i-1}, i = 1, \ldots, n$. Разбиение ξ и кривая γ порождают ломаную L_ξ, последовательные вершины которой - суть точки $\gamma(t_i)$. Ломаную L_ξ будем называть вписанной в γ ломаной, соответствующей разбиению ξ. Напомним, что длина $\ell(L)$ произвольной ломаной L - это сумма длин всех ее ребер. Таким образом,

$$\ell(L_\xi) = \sum_{i=1}^{n} \|\gamma(t_i) - \gamma(t_{i-1})\|.$$

Определение 1.77. Непрерывная параметрическая кривая γ называется измеримой, если существует предел

$$\ell(\gamma) = \lim_{\delta_\xi \to 0} \ell(L_\xi).$$

Предел $\ell(\gamma)$ называется длиной измеримой кривой γ.

Упражнение 1.78. Доказать, что гладкая параметрическая кривая γ измерима и ее длина, как измеримой кривой, равна

$$\int_a^b \|\dot{\gamma}(t)\| dt$$

Другими словами, два определения длины гладкой кривой эквивалентны.

Упражнение 1.79. Доказать аналог утверждения 1.32 об инвариантности длины относительно замены параметризации для произвольной измеримой кривой.

14.2.4 1.10.4 Osculating Circle

Определение 1.80. Центром кривизны бирегулярной кривой γ в точке $\gamma(t)$ называется точка $\gamma(t) + R(t)\nu(t)$, где через $R(t)$ и $\nu(t)$ обозначены, соответственно, радиус кривизны и главная нормаль кривой γ в этой точке.

Выясним геометрический смысл радиуса и центра кривизны. Для этого определим порядок касания кривых и покажем, что окружность, касающаяся кривой в данной точке P с максимальным порядком, имеет радиус, равный радиусу кривизны кривой в точке P, а центр ее находится в центре кривизны кривой в P.

Рассмотрим на плоскости пару регулярных кривых γ_1 и γ_2 и предположим, что они пересекаются в точке P. Отнесем обе кривых к натуральному параметру s, выбрав в качестве начальной точку пересечения P. Отметим, что выбор таких параметризаций неоднозначен, так как мы можем двигаться по кривым в разных направлениях. Для каждой такой параметризации $\gamma_1(0) - \gamma_2(0) = 0$ и для некоторого $m \geq 0$ выполняется равенство $\gamma_1(s) - \gamma_2(s) = o\left(s^m\right)$ при $s \to 0$ (напомним, что векторнозначная функция $\gamma(s)$ называется $o\left(s^m\right)$ при $s \to 0$, если $\|\gamma(s)\|/s^m \to 0$ при $s \to 0$).

Определение 1.81. Максимальное m, для которого справедливо соотношение $\gamma_1(s) - \gamma_2(s) = o\left(s^m\right)$ при $s \to 0$, где максимум берется также и по всевозможным натуральным параметризациям кривых γ_i, называется порядком касания кривых $\gamma_1 u \gamma_2$.

Пример 1.82. Пусть $\gamma(s)$ - произвольная регулярная кривая, параметризованная натуральным параметром, и $P = \gamma(0)$ - точка на ней. Рассмотрим семейство прямых на плоскости, проходящих через эту точку и тоже параметризованных натуральным параметром. Каждая такая прямая l имеет вид $l_E(s) = Es + \gamma(0)$, где E - единичный вектор. Найдем порядок касания прямой l_E и кривой γ в точке $s = 0$. Имеем:

$$\gamma(s) - l_E(s) = \gamma(0) + \dot{\gamma}(0)s + o(s) - \gamma(0) - Es = (\dot{\gamma}(0) - E)s + o(s), \quad s \to 0.$$

Поэтому все прямые, кроме $l_{\pm\dot{\gamma}(0)}$, имеют порядок касания 0 , а порядок касания прямых $l_{\pm\dot{\gamma}(0)}$ не меньше 1. Таким образом, максимальный порядок касания имеет именно прямая, направленная вдоль вектора скорости, ее называют касательной прямой.

Перейдем теперь к окружностям, касающимся плоской регулярной кривой $\gamma(s)$, параметризованной натуральным параметром, в точке $P = \gamma(0)$. Предположим сначала, что кривизна кривой $\gamma(s)$ отлична от нуля в точке $s = 0$. Пусть $c(s)$ - натурально параметризованная окружность, касающаяся $\gamma(s)$ в точке $P = c(0)$ с порядком касания не меньше 2. Тогда вектора скоростей и вектора ускорений кривых $\gamma(s)$ и $c(s)$ в точке P совпадают. Следовательно, у этих кривых совпадают главные нормали, кривизны и центры кривизны. Поэтому центр окружности $c(s)$ находится в центре кривизны кривой γ в точке $\gamma(s)$, а радиус равен радиусу кривизны $R(s)$ кривой γ.

Пусть теперь кривизна кривой $\gamma(s)$ в точке $s = 0$ равна нулю, т.е. равно нулю ее ускорение. Поскольку у натурально параметризованной прямой ускорение также равно нулю, прямая, касающаяся кривой γ в точке $\gamma(0)$, имеет порядок касания не меньше, чем 2 . Если прямые рассматривать как окружности бесконечного радиуса, то, опять же, заключаем, что и в этом случае существует единственная окружность (бесконечного радиуса), касающаяся кривой $\gamma(s)$ с порядком не меньше 2 (у других окружностей в натуральной параметризации ускорение отлично от нуля).

Итак, мы доказали следующее утверждение.

Statement

1.83. Среди всех окружностей, проходящих через некоторую фиксированную точку P регулярной кривой, существует единственная окружность, для которой ее порядок касания с кривой больше или равен 2. Радиус этой окружности равен радиусу кривизны R кривой в точке P, а и,ентр находится в центре кривизны.

Определение 1.84. Окружность из утверждения 1.83 называется соприкасающейся окружностью кривой γ в точке P.

Упражнение 1.85. Рассмотрим не только окружности, а, вообще, все кривые второго порядка, проходящие через точку регулярной кривой. Показать, что среди таких кривых найдется одна, порядок касания которой не меньше 4 . Определить тип и параметры кривой второго порядка из предыдущего упражнения в терминах кривизны исходной кривой.

14.2.5 1.10.5 On explicit solutions of the natural equation

Натуральное уравнение в явном виде часто не решается - интегралы, как правило, не берутся. Скажем, если мы рассмотрим натуральное уравнение $k_o(s) = as + b$, то даже в этом, на первый взгляд простом случае, уже появляются специальные функции - интегралы Френеля. Однако численно найти решение не составляет труда (рис. 1.5).

Рис. 1.5: Кривая с ориентированной кривизной $k_{o (s)=3 s+1, s \in[-5,5]$.}

14.2.6 1.10.6 Evolutes & Evolvents

Конструкция 1.86. Пусть γ - регулярная кривая на плоскости и $\varepsilon \in \mathbb{R}$ - некоторое число. Построим новую кривую γ_ε, заменив каждую точку $\gamma(s)$ точкой, полученной смещением точки $\gamma(s)$ на вектор $\varepsilon\nu_o(s)$, т.е. $\gamma_\varepsilon(s) = \gamma(s) + \varepsilon\nu_o(s)$. Эта кривая называется (ориентированной) ε-эквидистантой кривой γ.

При малых ε эквидистанта γ_ε по-прежнему является регулярной кривой. Однако, когда ε возрастает, у эквидистант могут появляться особые точки. Опишем их.

Пусть кривая γ параметризована натуральным параметром s. Тогда вектор скорости ε эквидистанты в силу формул Френе имеет вид

$$\dot{\gamma}_\varepsilon(s) = \dot{\gamma}(s) + \varepsilon\dot{\nu}_o(s) = (1 - \varepsilon k_o(s))\,\tau(s),$$

откуда следует, что точка $\gamma_\varepsilon(s)$ является особой, если и только если $k_o(s) = 1/\varepsilon$. Если последнее условие выполнено, то, очевидно, соответствующая точка эквидистанты имеет вид $\gamma_\varepsilon(s) = \gamma(s) + \frac{1}{k_o(s)}\nu_o(s)$ и, в силу утверждения 1.83, совпадает с центром кривизны кривой γ в точке $\gamma(s)$. Итак, имеет место следующее утверждение.

Statement

1.87. Вектор скорости ε-эквидистанты обращается в нуль, если и только если в соответствующей точке ориентированная кривизна исходной кривой равна $1/\varepsilon$. Множество особых точек эквидистант - это в точности множество центров кривизны исходной кривой.

Определение 1.88. Множество центров кривизны регулярной кривой γ называется каустикой, или эволютой кривой γ.

Следствие 1.89. Эволюта - это множество всех особых точек всех эквидистант.

Пример 1.90. Найдем, например, эволюту ичклоиды, т.е. плоской кривой $\gamma(\varphi)$, заданной в виде

$$x(\varphi) = R\varphi - R\sin\varphi, \quad y(\varphi) = R - R\cos\varphi.$$

Параметр φ не является натуральным, поэтому для вычисления кривизны воспользуемся утверждением 1.52. Имеем

$$\dot\gamma(\varphi) = (R - R\cos\varphi, R\sin\varphi), \quad \ddot\gamma(\varphi) = (R\sin\varphi, R\cos\varphi),$$
$$\|\dot\gamma(\varphi)\| = R\sqrt{2(1-\cos\varphi)} = 2R\left|\sin\frac{\varphi}{2}\right|,$$
$$\dot\gamma(\varphi)\times\ddot\gamma(\varphi) = R^2(\cos\varphi - 1) = -2R^2\sin^2\frac{\varphi}{2},$$

откуда

$$k_o(\varphi) = \frac{-2R\sin^2(\varphi/2)}{8R^3|\sin(\varphi/2)|^3} = -\frac{1}{4R|\sin(\varphi/2)|}$$

Рис. 1.6: Циклоида и ее эволюта.

Напомним, что вектор ориентированной главной нормали ν_o получается поворотом на $\pi/2$ нормированного вектора скорости, поэтому

$$\nu_o(\varphi) = \frac{1}{2R\left|\sin\frac{\varphi}{2}\right|}(-R\sin\varphi, R - R\cos\varphi),$$

откуда эволюта циклоиды может быть записана в следующем виде:
$(R\varphi - R\sin\varphi, R - R\cos\varphi) - 4R\left|\sin\frac{\varphi}{2}\right| \frac{1}{2R\left|\sin\frac{\varphi}{2}\right|}(-R\sin\varphi, R - R\cos\varphi) =$

$$= (R\varphi - R\sin\varphi, R - R\cos\varphi) - 2(-R\sin\varphi, R - R\cos\varphi) =$$
$$=(R\varphi + R\sin\varphi, R\cos\varphi - R).$$

Заметим, что полученная эволюта - это снова циклоида, только смещенная (рис. 1.6). Итак, доказано следующее утверждение.

Statement

1.91. Эволюта циклоиды - это снова циклоида.
Упражнение 1.92. Описать эквидистанты для циклоиды.
Упражнение 1.93. Построить эквидистанты и эволюту эллипса (рис. L.7).

Рис. 1.7: Построение эволюты эллипса: жирная точка - центр кривизны, тонкий прямолинейный отрезок - радиус соответствующей соприкасающейся окружности.

Рассмотрим теперь следующую обратную задачу. Пусть на плоскости задана регулярная кривая δ. Существует ли кривая γ, для которой δ является эволютой?

Предположим, что $\gamma(s)$ - это искомая кривая, параметризованная натуральным параметром. Тогда, по определению эволюты, имеем:

$$\delta(s) = \gamma(s) + \frac{\nu_o(s)}{k_o(s)}.$$

Продифференцировав это равенство по s, получаем в силу формул Френе

$$\frac{d\delta}{ds} = \dot{\gamma} + \frac{\dot{\nu}_o}{k_o} - \frac{\nu_o \dot{k}_o}{k_o^2} = -\frac{\nu_o \dot{k}_o}{k_o^2}.$$

Таким образом, доказана следующая лемма.

Лемма 1.94. Нормаль κ регулярной кривой, проведенная в точке P, является касательной к эволюте этой кривой в соответствующей точке $P + \frac{1}{k_o}\nu_o$. Другими словами, регулярная кривая перпендикулярна каждой касательной своей эволюты.

Конструкция 1.95. Пусть $\gamma(s)$ - регулярная кривая, параметризованная натуральным параметром. Построим в каждой ее точки касательную прямую и проведем из некоторой начальной точки кривую δ, перпендикулярную всем построенным касательным. Эта кривая δ может быть записана в виде $\delta(s) = \gamma(s) + \tau(s)f(s)$, где $f(s)$ - некоторая гладкая функция. Условие перпендикулярности дает следующее уравнение на $f(s)$:

$$\langle \dot{\delta}, \tau \rangle = \langle \dot{\gamma}(s) + \dot{\tau}(s)f(s) + \tau(s)\dot{f}(s), \tau(s) \rangle = 0, \quad \text{т.е.} \quad \dot{f}(s) = -1.$$

Поэтому $f(s) = C - s$, и окончательно, $\delta(s) = \gamma(s) + \tau(s)(C - s)$.
Определение 1.96. Кривая δ, перпендикулярная ко всем касательным данной кривой, называется эвольвентой этой кривой.

Statement

1.97. Каждая эвольвента натурально параметризованной кривой $\gamma(s)$ может быть записана в виде $\delta(s) = \gamma(s) + \tau(s)(C - s)$.

Замечание 1.98. Эвольвенте соответствует следующая механическая модель. Представим себе, что на кривую $\gamma(s)$ наклеена нерастяжимая нить, начальный отрезок длины C которой свисает в точке $\gamma(0)$. Натянем нить вдоль касательной к кривой в точке $\gamma(0)$ и будем сматывать ее с кривой так, чтобы она в каждый момент времени оставалась натянутой по соответствующей касательной. Конец нити в этом случае опишет эвольвенту с параметром C.

Следствие 1.99. Каждая регулярная кривая является некоторой эвольвентой своей эволюты.

Упражнение 1.100. Найти эвольвенты окружности и циклоиды.

14.2.7 1.10.7 Huygens Pendulum

В 1657 г. Гюйгенс (Huygens) изобрел и изготовил особо точные (для того времени) часы, за что получил огромную денежную премию, а принципиальная схема этих часов стала эмблемой XVII века. Такое внимание к часам в эпоху географических открытий, бурного развития колоний, мореплавания и т.п. объяснить легко - точные часы необходимы для вычисления географической долготы. Периодические свойства маятников были уже известны во времена Гюйгенса, однако проблема состояла в том, что период колебаний, скажем, кругового маятника (точка движется по дуге окружности) меняется при затухании, что крайне затрудняет измерение времени с помощью часов, в основе которых лежит такой маятник [3]. Чтобы преодолеть эту трудность, нужно чтобы маятник двигался не по окружности, а по какой-нибудь другой кривой. Подходящая кривая - циклоида - была найдена Гюйгенсом. Он показал, что период колебания циклоидального маятника не зависит от начального положения точки на дуге циклоиды. Удивительно, что, как выяснилось через 150 лет, кривая, обладающая таким свойством, единственна. Этот факт был установлен в 1823 г. Абелем (Abel), который для этого решил одно из первых в истории интегральных уравнений. Однако изучение теоремы Абеля выходит за рамками данного курса. Мы только убедимся, следуя Гюйгенсу, что циклоида действительно обладает искомым свойством.

Рассмотрим дугу циклоиды, выпуклую вниз и заданную параметрически в следующем виде:

$$x(t) = R(t + \pi + \sin t), \quad y(t) = R - R\cos t, \quad t \in [-\pi, \pi].$$

Легко сосчитать, что если выбрать в качестве начальной точки самую нижнюю точку O циклоиды, т.е. точку $t = 0$, то натуральный параметр s будет иметь следующий простой вид: $s = \sqrt{8Ry}$. Чтобы найти уравнение движения маятника, запишем закон сохранения энергии:

$$\frac{mv^2}{2} + mgy = \text{const}$$

Тогда, если h_0 - это начальная высота точки, а начальная скорость равна нулю, то $v^2 = 2g\,(h_0 - y)$. Учитывая, что при движении от начальной точки расстояние s до точки O убывает, получаем следующее дифференциальное уравнение: $ds/dt = -\sqrt{2g\,(h_0 - y)}$. Но $y = s^2/(8R)$, поэтому имеем окончательно:

[03] Зависимость периода T от угла φ размаха колебаний кругового маятника имеет вид $T \approx 2\pi\sqrt{l/g}\,\big(1 + \varphi^2/16\big)$, где l - длина маятника и g - ускорение свободного падения.

$$\frac{ds}{dt} = -\sqrt{\frac{g}{4R}\,(s_0^2 - s^2)}.$$

Интегрируя это дифференциальное уравнение с разделяющимися переменными, находим, что $s = s_0 \cos\Big(t\sqrt{g/(4R)}\Big)$, поэтому период колебаний циклоидального маятника равен $2\pi\sqrt{4R/g}$, что не зависит от начального положения точки на циклоиде. Итак, мы доказали следующее важное свойство циклоиды.

Statement

1.101. Время движения материальной точки по ииклоиде к положению равновесия без начальной скорости не зависит от начального положения точки на и,иклоиде.

Это свойство циклоиды называется таутохронностью.
Однако, мало было открыть это замечательное свойство циклоидального маятника. Нужно было еще решить нетривиальную инженерную задачу: заставить маятник колебаться вдоль циклоиды. Для этого Гюйгенс воспользовался тем уже известным нам фактом, что среди эвольвент циклоиды есть ее эволюта, которая, как мы знаем, конгруэнтна самой циклоиде. В итоге была получена конструкция, изображенная на рис. 1.8. Движение математического маятника ограничено двумя пластинами, имеющими форму дуг циклоид. Поэтому маятник в свою очередь сам описывает циклоиду и, значит, совершает таутохронные колебания.

Рис. 1.8: Маятник Гюйгенса.

14.3 2 Curves in three-dimensional space

Рассмотрим теперь регулярные кривые в трехмерном евклидовом пространстве $\mathbb{R}^3$. В предыдущем разделе мы построили репер Френе для плоской бирегулярной кривой и изучили его зависимость от натурального параметра, что, в конечном счете, позволило нам решить натуральное уравнение. В трехмерном пространстве двух векторов - скорости и главной нормали, - не хватает для репера, поэтому нужны дополнительные построения.

14.3.1 2.1 Frenet's Formulas

Конструкция 2.1. Пусть $\gamma(s)$ - натурально параметризованная бирегулярная кривая в $\mathbb{R}^3$. Как и выше, пусть τ - вектор скорости $\dot{\gamma}(s)$, и ν - главная нормаль к γ в точке $\gamma(s)$. Обозначим через β - векторное произведение $[\tau, \nu]$. Тройка (τ, ν, β) образует положительно ориентированный ортонормальный базис в $\mathbb{R}^3$.

Определение 2.2. Вектор β называется бинормалью к бирегулярной кривой γ в точке $\gamma(s)$, а базис (τ, ν, β) - репером Френе кривой γ в точке $\gamma(s)$.

Лемма 2.3. Производная $\dot{\beta}(s)$ вектора бинормали по натуральному параметру коллинеарна вектору главной нормали ν.

Доказательство. По формуле Лейбница имеем

$$\dot{\beta} = [\tau, \nu]^{\cdot} = [\dot{\tau}, \nu] + [\tau, \dot{\nu}] = [\tau, \dot{\nu}],$$

где последнее равенство справедливо, так как вектор $\dot{\tau}$, равный $\ddot{\gamma}$, коллинеарен ν и, значит, $[\dot{\tau}, \nu] = 0$. Отсюда, в частности, следует, что вектор $\dot{\beta}$ ортогонален τ. Далее, так как вектор $\beta(s)$ - единичный при каждом s, то $\dot{\beta}$ перпендикулярен β. Следовательно, $\dot{\beta}$ коллинеарен вектору ν, что и требовалось. $\square$

Определение 2.4. Величина $\varkappa = -\langle \dot{\beta}, \nu \rangle$ называется кручением бирегулярной кривой γ в точке $\gamma(s)$.

Замечание 2.5. Из определения и леммы 2.3 вытекает, что $\dot{\beta} = -\varkappa\nu$. Ясно, что кручение $\varkappa$ является гладкой функцией натурального параметра s.

Theorem

2.6 (Формулы Френе). В сделанных предположениях справедливы следующие соотношения:

$$\begin{cases} \dot{\tau} = k\nu, \\ \dot{\nu} = -k\tau + \varkappa\beta, \\ \dot{\beta} = -\varkappa\nu. \end{cases}$$

где $ku\varkappa$ - кривизна и кручение бирегулярной кривой γ в точке $\gamma(s)$, соответственно.

Доказательство. Первое и третье уравнения - это фактически определения кривизны и кручения. Осталось доказать второе равенство.

Так как вектор $\dot{\nu}$ перпендикулярен ν, то его можно разложить по векторам τ и β. Положим $\dot{\nu} = a\tau + b\beta$. В силу ортонормальности репера Френе имеем $a = \langle \dot{\nu}, \tau \rangle$ и $b = \langle \dot{\nu}, \beta \rangle$. Мы должны показать, что $a = -k$, а $b = \varkappa$.

Так как $\langle \nu, \tau \rangle = \langle \nu, \beta \rangle = 0$, то, дифференцируя эти равенства, заключаем, что

$$0 = \langle \dot{\nu}, \tau \rangle + \langle \nu, \dot{\tau} \rangle = a + k, \quad 0 = \langle \dot{\nu}, \beta \rangle + \langle \nu, \dot{\beta} \rangle = b - \varkappa,$$

что и требовалось доказать.

Приведем теперь явные формулы для вычисления кривизны и кручения кривой в произвольной параметризации.

Theorem

2.7. Пусть $\gamma(t)$ - бирегулярная кривая. Тогда

$$k = \frac{\|[\dot{\gamma}, \ddot{\gamma}]\|}{\|\dot{\gamma}\|^3}, \quad \varkappa = \frac{(\dot{\gamma}, \ddot{\gamma}, \dddot{\gamma})}{\|[\dot{\gamma}, \ddot{\gamma}]\|^2},$$

где через (u, v, w) обозначено смешанное произведение векторов u, v и w из $\mathbb{R}^3$.

Доказательство. Пусть сначала $t = s$ - натуральный параметр. Тогда

$$\dot{\gamma} = \tau, \quad \|\dot{\gamma}\| = 1, \quad \ddot{\gamma} = k\nu, \quad \|[\dot{\gamma}, \ddot{\gamma}]\| = k$$

и

$$\dddot{\gamma} = \dot{k}\nu + k(-k\tau + \varkappa\beta), \quad (\dot{\gamma}, \ddot{\gamma}, \dddot{\gamma}) = k^2\varkappa,$$

поэтому в этом случае теорема имеет место.
Сделаем замену параметра. Оказывается, что приведенные формулы при этом не изменятся. Действительно, обозначая начальный параметр через s, новый параметр через t, а дифференцирования по параметрам - соответствующими нижними индексами, получаем

$$\gamma_s = \gamma_t t_s, \quad \gamma_{ss} = \gamma_{tt} t_s^2 + \gamma_t t_{ss}, \quad \gamma_{sss} = \gamma_{ttt} t_s^3 + 3\gamma_{tt} t_s t_{ss} + \gamma_t t_{sss},$$

откуда

$$[\gamma_s, \gamma_{ss}] = [\gamma_t, \gamma_{tt}]\, t_s^3, \quad (\gamma_s, \gamma_{ss}, \gamma_{sss}) = (\gamma_t, \gamma_{tt}, \gamma_{ttt})\, t_s^6$$

и, наконец,

$$\frac{\|[\gamma_s, \gamma_{ss}]\|}{\|\gamma_s\|^3} = \frac{\|[\gamma_t, \gamma_{tt}]\|}{\|\gamma_t\|^3} \quad \text{и} \quad \frac{(\gamma_s, \gamma_{ss}, \gamma_{sss})}{\|[\gamma_s, \gamma_{ss},]\|^2} = \frac{(\gamma_t, \gamma_{tt}, \gamma_{ttt})}{\|[\gamma_t, \gamma_{tt}]\|^2}.$$

Доказательство закончено.
Упражнение 2.8. Пусть t - произвольный параметр пространственной бирегулярной кривой γ. Проверить справедливость следующих формул для вычисления векторов бинормали и главной нормали:

$$\beta(t) = \frac{[\dot\gamma, \ddot\gamma]}{\|[\dot\gamma, \ddot\gamma]\|}, \quad \nu(t) = [\beta, \tau].$$

14.3.2 2.2 Natural equations

Оказывается, кривизна и кручение полностью определяют форму бирегулярной кривой в трехмерном пространстве.

Theorem

2.9. Пусть $f(s)$ и $g(s)$ - две гладкие функции на отрезке $[a, b]$, причем функция $f(s)$ положительна. Тогда существует единственная, с точностью до сохраняющего ориентацию движения пространства, натурально параметризованная кривая $\gamma(s)$, такая, что ее кривизна $k(s)$ и кручение $\varkappa(s)$ равны соответственно $f(s)ug(s)$.

Доказательство. Формулы Френе, в которые вместо кривизны подставлена функция f, а вместо кручения - функция g, можно рассматривать как систему линейных дифференциальных уравнений первого порядка с девятью неизвестными - компонентами векторов τ, ν и β. Эту систему будем обозначать символом ($*$). Нам будем удобно записать ее в матричном виде. Пусть $\tau = (\tau^1, \tau^2, \tau^3), \nu = (\nu^1, \nu^2, \nu^3), \beta = (\beta^1, \beta^2, \beta^3)$, тогда система (*) имеет вид

$$\begin{pmatrix} \dot\tau^1 & \dot\tau^2 & \dot\tau^3 \\ \dot\nu^1 & \dot\nu^2 & \dot\nu^3 \\ \dot\beta^1 & \dot\beta^2 & \dot\beta^3 \end{pmatrix} = \begin{pmatrix} 0 & f & 0 \\ -f & 0 & g \\ 0 & -g & 0 \end{pmatrix} \begin{pmatrix} \tau^1 & \tau^2 & \tau^3 \\ \nu^1 & \nu^2 & \nu^3 \\ \beta^1 & \beta^2 & \beta^3 \end{pmatrix}.$$

Матрицу из системы (*), строками которой являются координаты векторов τ, ν и β, обозначим через η, а матрицу, содержащую f и g, - через Ω. Таким образом, система $(*)$ имеет вид $\dot\eta = \Omega\eta$. Отметим, что Ω - кососимметричная матрица.

Выберем в $\mathbb{R}^3$ некоторый ортонормальный положительно ориентированный репер (τ_0, ν_0, β_0). По теореме существования и единственности решения задачи Коши для системы дифференциальных уравнений существуют и единственны вектор-функции

$\tau(s), \nu(s), \beta(s)$, удовлетворяющие системе (*) и начальным условиям $\tau(a) = \tau_0, \nu(a) = \nu_0, \beta(a) = \beta_0$. Положим

$$\gamma(s) = \gamma_0 + \int_a^s \tau(t)dt$$

где γ_0 - произвольный вектор, и покажем, что γ - искомая кривая, т.е. γ - бирегулярная кривая, параметризованная натуральным параметром s, её кривизна $k(s)$ равна $f(s)$, а кручение $\varkappa(s)$ равно $g(s)$.

Лемма 2.10. При каждом s тройка $(\tau(s), \nu(s), \beta(s))$ образует ортонормальный положительно ориентированный репер.

Доказательство. Ортонормированность репера $(\tau(s), \nu(s), \beta(s))$ равносильна ортогональности матрицы $\eta(s)$ при каждом s, т.е. $\eta^T(s)\eta(s) = E$, где E - единичная матрица. Так как начальное условие (τ_0, ν_0, β_0) представляет собой ортонормальный репер, то при $s = a$ матрица $\eta(a)$ ортогональна. Поэтому достаточно проверить, что матрица $\eta^T(s)\eta(s)$ не зависит от s. Для этого продифференцируем выражение $\eta^T(s)\eta(s)$ по s и воспользуемся тем, что $\dot{\eta}(s) = \Omega\eta$. Имеем:

$$\left(\eta^T\eta\right)^{\cdot} = \dot{\eta}^T\eta + \eta^T\dot{\eta} = \eta^T\Omega^T\eta + \eta^T\Omega\eta = \eta^T\left(\Omega^T + \Omega\right)\eta = 0,$$

где последнее равенство следует из кососимметричности матрицы Ω, что и требовалось.
Положительность ориентации репера $(\tau(s), \nu(s), \beta(s))$ при каждом s следует из того, что репер (τ_0, ν_0, β_0) ориентирован положительно по предположению, а определитель матрицы $\eta(s)$ не обращается в нуль ни при каком s, значит ориентация не меняется. Лемма доказана.

Проверим, что γ - искомая кривая. Так как $\dot{\gamma}(s) = \tau(s)$, то из леммы 2.10 следует, что γ - регулярна, и s - натуральный параметр. Продифференцируем еще раз и воспользуемся тем, что γ - решение системы $(*)$: $\ddot{\gamma}(s) = \dot{\tau}(s) = f(s)\nu(s)$. По предположению функция $f(s)$ положительна, а $\nu(s)$ - единичный вектор в силу леммы 2.10, поэтому $f(s)$ совпадает с кривизной, а $\nu(s)$ - с главной нормалью кривой γ в точке $\gamma(s)$. Так как $f(s) > 0$ при любом s, то кривая γ - бирегулярна. Далее, поскольку каждый из ортонормальных реперов $(\tau(s), \nu(s), \beta(s))$ положительно ориентирован, то $\beta(s)$ совпадает с бинормалью к γ. Наконец, по условию $\dot{\beta}(s) = -g(s)\nu(s)$, откуда $g(s)$ совпадает с кручением кривой γ. Таким образом, доказано существование кривой γ.

Теперь докажем теперь единственность кривой γ. Пусть имеются две натурально параметризованные параметром s кривые γ_1 и γ_2 с одинаковыми кривизной k и кручением $\varkappa$. С помощью движения совместим их начальные точки и начальные реперы Френе. Полученные кривые вновь обозначим теми же буквами. Пусть $(\tau_i(s), \nu_i(s), \beta_i(s))$ - семейство реперов Френе для кривой γ_i. Заметим, что эти реперы удовлетворяют одной и той же системе обыкновенных дифференциальных уравнений Френе и имеют одинаковые начальные условия. Отсюда по теореме единственности решения задачи Коши для обыкновенного дифференциального уравнения реперы Френе кривых γ_i совпадают в соответствующих точках. В частности, совпадают векторы $\tau_i(s)$. Наконец, так как $\gamma_i(s)$ является решением дифференциального уравнения $\dot{\gamma}_i(s) = \tau_i(s)$ с одинаковыми начальными условиями, кривые γ_i совпадают. Теорема доказана.

Определение 2.11. Если $f(s)$ и $g(s)$ - две гладкие функции, такие, что $f(s) > 0$ при всех s, то условия $k(s) = f(s), \varkappa(s) = g(s)$ на кривизну k и кручение $\varkappa$ кривой γ называются натуральными уравнениями пространственной кривой. Решить натуральные уравнения

означает найти натурально параметризованную бирегулярную кривую $\gamma(s)$ с кривизной $f(s)$ и кручением $g(s)$.

Следствие 2.12. Натуральное уравнение пространственной кривой всегда имеет решение. Более того, это решение единственно с точностью до сохраняющего ориентацию движения пространства $\mathbb{R}^3$.

Замечание 2.13. Напомним, что в плоском случае по одной гладкой функции - ориентированной кривизне - однозначно восстанавливается регулярная (не обязательно бирегулярная) кривая. Дело в том, что в двумерном случае семейство ориентированных главных нормалей регулярной кривой продолжается по непрерывности в те точки кривой, в которых ускорение обращается в нуль, т.е. нарушается бирегулярность. В трехмерном пространстве такое продолжение, вообще говоря, невозможно, так как в окрестности точки, в которой нарушается бирегулярность, прямая, проведенная вдоль главной нормали, может меняться скачком.

Упражнение 2.14. Привести пример регулярной кривой $\gamma(s)$, для которой выполнено равенство $(\dot\gamma, \ddot\gamma, \dddot\gamma) = 0$, но кривая γ не лежит ни в одной двумерной плоскости. Может ли такая кривая быть бирегулярной? Исследовать поведение главной нормали построенной кривой в окрестности точек, в которых нарушается бирегулярность.

Задача 2.21. Описать класс бирегулярных кривых в $\mathbb{R}^3$ с постоянным кручением.

Задача 2.22. Доказать, что кривизна бирегулярной кривой в $\mathbb{R}^3$ пропорциональна кручению, если и только если найдется постоянный ненулевой вектор u, такой, что $\langle u, \tau\rangle = \mathrm{const}$.

Задача 2.23. Доказать, что бирегулярная кривая в $\mathbb{R}^3$ с ненулевым кручением $\varkappa$ лежит на сфере радиуса R тогда, и только тогда, когда справедливо соотношение

$$R^2 = \frac{1}{k^2}\left(1 + \frac{(k')^2}{(\varkappa k)^2}\right),$$

где k - кривизна кривой.

14.3.3 2.4 Slope lines

Бирегулярная кривая называется линией откоса, если вдоль нее отношение кручения к кривизне не меняется, т.е. $k(s)/\varkappa(s) = \mathrm{const}$. Опишем класс таких кривых.

Положим для удобства $k(s)/\varkappa(s) = \operatorname{tg}\alpha$, где α - некоторый фиксированный угол. Тогда указанное соотношение можно переписать в виде

$$k(s)\cos\alpha - \varkappa(s)\sin\alpha = 0$$

Умножая первое из уравнений Френе на $\cos\alpha$, а последнее - на $\sin\alpha$ и вычитая из первого уравнения последнее, получаем

$$\tau'(s)\cos\alpha - \beta'(s)\sin\alpha = \nu(s)(k(s)\cos\alpha - \varkappa(s)\sin\alpha) = 0.$$

Поэтому единичный вектор $m = \tau(s)\cos\alpha - \beta(s)\sin(a)$ не меняется вдоль кривой. В то же время, $\langle m, \tau(s)\rangle = \cos\alpha$, поэтому угол между вектором m и касательным вектором к кривой γ постоянен и равен α. Проведем через каждую точку кривой прямую в направлении вектора m. Получим цилиндрическую поверхность. Очевидно, кривая γ лежит на этой цилиндрической поверхности, причем при разворачивании поверхности в плоскость, кривая превращается в прямую, пересекающуюся с образующей прямой под углом α. Оказывается, что верно и обратное утверждение.

Statement

2.24. Всякая линия откоса есть такая линия на подходящей ицилиндрической поверхности, которая переходит в прямую после разворачивания поверхности в плоскость. Обратно: всякая кривая, обладающая указанным свойством, есть линия откоса.

Доказательство. Нам осталось доказать лишь обратное утверждение. Пусть m - направляющий вектор прямых, образующих цилиндрическую поверхность. Тогда по условию $\langle m, \tau(s)\rangle = \cos\alpha = \text{const}$. Продифференцировав это равенство по s, получим $\langle m, k(s)\nu(s)\rangle = 0$. Поскольку кривая бирегулярна, ее кривизна не обращается в нуль. Поэтому $\langle m, \nu(s)\rangle = 0$ для любого s, т.е. существуют такие функции $x(s)$ и $y(s)$, что $m = x(s)\tau(s) + y(s)\beta(s)$. Но, напомним, что $\langle m, \tau(s)\rangle = \cos\alpha$, откуда $x(s) = \text{const} = \cos\alpha$, а $y(s) = \text{const} = \sin\alpha$, т.е. $m = \tau(s)\cos\alpha + \beta(s)\sin\alpha$. Продифференцировав последнее равенство, из формул Френе получим $k(s)\cos\alpha + \varkappa(s)\sin\alpha = 0$, т.е. $\varkappa(s)/k(s) = \text{const}$.

Итак, γ - это линия откоса. Утверждение доказано.

14.4 3. Surfaces and First Fundamental Form

Понятие кривых естественным образом обобщается на объекты большей размерности, которые называются поверхностями. Как и в случае кривых, мы рассмотрим несколько подходов к определению понятия поверхности, которые приводят к, вообще говоря, разным, но, тем не менее, локально эквивалентным объектам.

14.4.1 3.1 Parametric Surfaces

Обозначим через Ω область в евклидовом пространстве $\mathbb{R}^k$, т.е., напомним, его открытое связное подмножество.

Определение 3.1. Непрерывной параметрической поверхностью размерности k в n мерном евклидовом пространстве $\mathbb{R}^n, n \geq k$, называется произвольное непрерывное отображение $r : \Omega \to \mathbb{R}^n$ из некоторой области $\Omega \subset \mathbb{R}^k$ в $\mathbb{R}^n$.

Такое отображение r задается набором из n координатных функций $x^i\left(u^1, \ldots, u^k\right)$, где $u^1 \ldots, u^k$ - стандартные координаты в $\mathbb{R}^k \supset \Omega$, а $x^1, \ldots, x^n$ - стандартные координаты в $\mathbb{R}^n$. При этом непрерывность отображения r равносильна непрерывности всех функций $x^i\left(u^1, \ldots, u^k\right)$. Координаты u^i называются параметрами для r, или координатами на параметрической поверхности r.

Непрерывная параметрическая поверхность r может, вообще говоря, иметь самопересечения, т.е. могут существовать такие точки P и P' из Ω, что $r(P) = r\left(P'\right)$.

Определение 3.2. Если отображение r взаимно-однозначно с образом, то параметрическая поверхность r называется вложенной.

Определение 3.3. Будем называть непрерывную параметрическую поверхность $r : \Omega \to \mathbb{R}^n$ гладкой, если все ее координатные функции $x^i\left(u^1, \ldots, u^k\right)$ - гладкие, т.е., напомним, непрерывно дифференцируемые бесконечное число раз.

Для каждой точки ($u^1, \ldots, u^k$) области Ω гладкой параметрической поверхности определены векторы $r_{u^i} = \left(x^1_{u^i}, \ldots, x^n_{u^i}\right)$, где через $x^j_{u^i}$ обозначена частная производная функции x^j по параметру u^i в этой точке.

Определение 3.4. Векторы $r_{u^j}, j = 1, \ldots, k$, называются базисными касательными векторами поверхности r в точке $\left(u^1, \ldots, u^k\right)$.

Замечание 3.5. Одна и та же точка образа параметрической поверхности может соответствовать разным точкам параметризующей области (именно так дело обстоит в точках самопересечения). Поэтому, чтобы выделить один набор базисных касательных векторов, нужно указывать значения параметров.

Отметим, что интуитивное представление о гладкости поверхности не соответствует данному определению, что иллюстрируется следующим упражнением.

Упражнение 3.6. Показать, что конус и двугранный угол можно представить как образы гладких параметрических поверхностей.

Для поверхностей "изломы" могут возникать лишь в тех точках, в которых базисные касательные векторы линейно зависимы. Такие точки называются особыми, или сингулярными, а все остальные - регулярными, или неособыми.

Определение 3.7. Гладкая параметрическая поверхность называется регулярной, если ее базисные касательные векторы линейно независимы в каждой точке параметризующей области.

Замечание 3.8. Условие регулярности, очевидно, равносильно тому, что матрица Якоби ($x^i_{u^j}$) параметризующего отображения r имеет во всех точках максимальный ранг (равный, очевидно, k).

Пусть $r : \Omega \to \mathbb{R}^n$ - непрерывная параметрическая поверхность. Рассмотрим взаимнооднозначное непрерывное вместе со своим обратным отображение $\varphi : \Omega' \to \Omega$ области $\Omega' \subset \mathbb{R}^k$ на область $\Omega \subset \mathbb{R}^k$. Каждое такое отображение порождает новую непрерывную параметрическую поверхность $r \circ \varphi : \Omega' \to \mathbb{R}^n$.

Определение 3.9. Отображение φ называется заменой параметризации, а про поверхность r о φ говорят, что она получена из поверхности r заменой параметризации.

Ясно, что если φ - замена параметризации, то φ^{-1} также является заменой параметризацией. Параметрические поверхности r и $r \circ \varphi$, полученные друг из друга заменой параметризации, имеют совпадающие образы в $\mathbb{R}^n$.

Замечание 3.10. Как и в случае параметрических кривых, гладкая замена параметризации может нарушить регулярность параметрической поверхности. Однако, если оба отображения φ и φ^{-1}, задающие замену параметризации, - гладкие, то отличающиеся на такую замену параметризации параметрические поверхности регулярны одновременно.

14.4.2 3.2 Graphic Surfaces

Пусть $f : \Omega \to \mathbb{R}^{n-k}$ - векторнозначная функция, заданная на области $\Omega \subset \mathbb{R}^k$, и $u^1, \ldots, u^k$ - стандартные координаты на $\mathbb{R}^k$.

Определение 3.11. Множество

$$\mathrm{Gr}(f) = \left\{ \left(u^1, \ldots, u^k, f\left(u^1, \ldots, u^k\right)\right) \mid \left(u^1, \ldots, u^k\right) \in \Omega \right\} \subset \mathbb{R}^n$$

называется поверхностью-графиком в пространстве $\mathbb{R}^n$.

По определению, поверхность-график $\mathrm{Gr}(f)$ представляет собой образ k-мерной параметрической поверхности $r_f : \Omega \to \mathbb{R}^n$, где

$$r_f : \left(u^1, \ldots, u^k\right) \mapsto \left(u^1, \ldots, u^k, f\left(u^1, \ldots, u^k\right)\right)$$

про которую будем говорить, что она соответствует поверхности-графику $\mathrm{Gr}(f)$. Если функция f - гладкая, то $\mathrm{Gr}(f)$ будем называть гладкой поверхностью-графиком. В этом случае, очевидно, r_f - гладкая параметрическая поверхность.

Замечание 3.12. Гладкая параметрическая поверхность, соответствующая гладкой поверхности-графику является регулярной параметрической поверхностью, поскольку матрица Якоби отображения r_f содержит единичную подматрицу размером $k \times k$.

Обратное, как и в случае кривых, разумеется, не верно (кривая график взаимно однозначно проецируется на некоторую k-мерную плоскость).

Пример 3.13. Образ вложенной регулярной поверхности

$$r : (u, v) \mapsto (\cos u, \sin u, v), \quad u \in (0, 2\pi), \quad v \in \mathbb{R},$$

(это прямой круговой цилиндр с выброшенной прямой) не представим в виде поверхности-графика, поскольку его проекция на любую плоскость не взаимно-однозначна с образом.

Лемма 3.14. Для любой регулярной параметрической поверхности $r : \Omega \to \mathbb{R}^n u$ любой точки $P \in \Omega$ существует такая окрестность $U \subset \Omega$ этой точки, что образ отображения $r|_U$ является графиком некоторой гладкой функции (возможно, после перенумерации координат в $\mathbb{R}^n$).

Доказательство. Пусть x^i - стандартные координаты в $\mathbb{R}^n$, а u^i - координаты на поверхности r. В силу регулярности поверхности r ранг матрицы ($x^i_{u^j}(P)$) равен k. Без ограничения общности будем считать, что квадратная матрица $\left(x^i_{u^j}(P)\right)^k_{i,j=1}$ невырождена (иначе - перенумеруем координаты x^j). Тогда по теореме об обратной функции отображение

$$\psi : \left(u^1, \ldots, u^k\right) \mapsto \left(x^1\left(u^1, \ldots, u^k\right), \ldots, x^k\left(u^1, \ldots, u^k\right)\right),$$

ограниченное на некоторую окрестность U точки P, имеет гладкое обратное отображение $\varphi : \psi(U) \to U$, где

$$\varphi : \left(x^1 \ldots, x^k\right) \mapsto \left(u^1\left(x^1, \ldots, x^k\right), \ldots, u^k\left(x^1, \ldots, x^k\right)\right)$$

являющееся, по определению, заменой параметризации для поверхности $r : U \to \mathbb{R}^n$. Согласно определению отображения φ имеем

$$x^i\left(u^1\left(x^1, \ldots, x^k\right), \ldots, u^k\left(x^1, \ldots, x^k\right)\right) = x^i, \quad i = 1, \ldots, k,$$

Положим $\psi(U) = \Omega'$. Тогда параметрическая поверхность $r \circ \varphi : \Omega' \to \mathbb{R}^n$ имеет вид

$$\begin{aligned} r \circ \varphi : & \left(x^1, \ldots, x^k\right) \mapsto \\ & \mapsto \left(x^1, \ldots, x^k, x^{k+1}\left(u^1\left(x^1, \ldots, x^k\right), \ldots, u^k\left(x^1, \ldots, x^k\right)\right), \ldots,\right. \\ & \left. x^n\left(u^1\left(x^1, \ldots, x^k\right), \ldots, u^k\left(x^1, \ldots, x^k\right)\right)\right), \end{aligned}$$

т.е. ее образ $r \circ \varphi\left(\Omega'\right) = r(U)$ является графиком гладкой векторнозначной функции $f : \Omega' \to \mathbb{R}^{n-k}$, где

$$\begin{aligned} f\left(x^1, \ldots, x^k\right) = & \left(x^{k+1}\left(u^1\left(x^1, \ldots, x^k\right), \ldots, u^k\left(x^1, \ldots, x^k\right)\right), \ldots,\right. \\ & \left. x^n\left(u^1\left(x^1, \ldots, x^k\right), \ldots, u^k\left(x^1, \ldots, x^k\right)\right)\right). \end{aligned}$$

Лемма доказана.

14.4.3 3.3 Implicit surfaces

Пусть $F : U \to \mathbb{R}^{n-k}$ - гладкая векторнозначная функция, заданная на некоторой области $U \subset \mathbb{R}^n$, и пусть c - некоторый постоянный вектор из $\mathbb{R}^{n-k}$. Предположим, что система уравнений $F\left(x^1, \ldots, x^n\right) = c$ удовлетворяет условиям теоремы о неявной функции, а именно, множество $M_c = \left\{\left(x^1, \ldots, x^n\right) \mid F\left(x^1, \ldots, x^n\right) = c\right\}$ всех решений этой системы не пусто, и ранг матрицы Якоби $\left(\partial F^i / \partial x^j\right)$ в каждой точке из M_c максимален и равен $n-k$.

Определение 3.15. Множество M_c называется неявной поверхностью.
Замечание 3.16. Поверхность-график произвольной гладкой векторнозначной функции $f : \Omega \to \mathbb{R}^{n-k}, \Omega \subset \mathbb{R}^k$, является неявной поверхностью, поскольку может быть задана как множество решений системы уравнений $x^i - f^{i-k}\left(x^1, \ldots, x^k\right) = 0, i = k+1, \ldots, n$, где через f^i обозначены компоненты вектора f. Обратное неверно.

Упражнение 3.17. Проверьте невырожденность матрицы Якоби системы уравнений из замечания 3.16.

Пример 3.18. Неявная поверхность - двумерная сфера, заданная уравнением $x^2 + y^2 + z^2 = R^2$, не является графиком функции, так как не проецируется однозначно ни на какую двумерную плоскость.

Лемма 3.19. Пусть P - произвольная точка неявной поверхности M_c. Тогда в некоторой окрестности точки $P \in M_c$ в пространстве $\mathbb{R}^n$ множество M_c можно представить как гладкую поверхность-график некоторой векторнозначной функции f (возможно, после перенумерации координат).

Доказательство. По условию ранг матрицы Якоби ($\partial F^i/\partial x^j$) в точке P равен $n-k$. Предположим, что невырожденной оказалась подматрица ($\partial F^i/\partial x^j$), $i = 1, \ldots, n-k$ и $j = k+1, \ldots, n$ (иначе - перенумеруем координаты). Обозначим через P' проекцию точки P на координатную плоскость $x^{k+1} = \cdots = x^n = 0$. Тогда по теореме о неявной функции в некоторой окрестности U' точки P' (на этой плоскости) определена гладкая векторнозначная функция $f : U' \to \mathbb{R}^{n-k}$, такая, что

$$F\left(x^1, \ldots, x^k, f\left(x^1, \ldots, x^k\right)\right) = c \quad \text{для всех } \left(x^1, \ldots, x^k\right) \in U',$$

т.е. точки $\left(x^1, \ldots, x^k, f\left(x^1, \ldots, x^k\right)\right)$, где $\left(x^1, \ldots, x^k\right) \in U'$, принадлежат M_c. Таким образом, положив $U = U' \times \mathbb{R}^{n-k}$, заключаем, что множество $U \cap M_c$ является графиком гладкой функции f. Лемма доказана.

14.4.4 3.4 Local equivalence

Подведем итог. Мы описали три разных подхода к определению понятия поверхности. При этом, каждая поверхность-график представляет собой также и неявную поверхность, и образ регулярной параметрической поверхности, однако, неявная поверхность, вообще говоря, не обязана быть поверхностью-графиком, а образ регулярной параметрической поверхности не обязан быть неявной поверхностью. Таким образом, в целом, все три определения поверхности задают разные объекты. С другой стороны, как было показано выше, см. Леммы 3.14 и 3.19, с локальной точки зрения эти три подхода к определению регулярной поверхности эквивалентны в следующем смысле.

Theorem 3.20.

С локальной точки зрения три способа задания регулярной поверхности эквивалентны в следующем смысле:

- если $r : \Omega \to \mathbb{R}^n$ - регулярная параметрическая поверхность, то для каждого $u \in \Omega$ существует такая окрестность U точки и в области Ω, что ее образ $r(U)$ является поверхностью-графиком для некоторой гладкой векторнозначной функции;
- для каждой точки P неявной поверхности M_c существует такая окрестность $U \subset \mathbb{R}^n$, что $U \cap M_c$ есть поверхность-график.

Так как каждая поверхность-график может быть представлена как образ регулярной параметрической поверхности, у которой параметризующее отображение взаимно однозначно с образом, то имеет место следующий результат.

Следствие 3.21. Пусть $r : \Omega \to \mathbb{R}^n$ - регулярная параметрическая поверхность. Для каждого $u \in \Omega$ существует такая окрестность U точки и в области Ω, что ограничение отображения r на U взаимно однозначно с образом.

Сначала мы будем заниматься локальной теорией регулярных поверхностей, поэтому, опираясь на теорему 3.20, будем использовать следующее соглашение.

Соглашение 3.22. Там, где это не вызовет недоразумений, мы будем говорить просто о регулярной поверхности, подразумевая, что соответствующий объект задан удобным

для нас способом (или как поверхность-график, или как неявная поверхность, или как регулярная параметрическая поверхность или её образ). Причем в последнем случае, если не оговорено противное, параметризующее отображение предполагается взаимно однозначными с образом, а замены параметризации гладкими и с гладкими обратными.

Замечание 3.23. Глобальные характеристики поверхностей изучаются в теории многообразий, см. ниже.

14.4.5 3.5 Surfaces and Mappings

В рамках принятого соглашения 3.22, регулярную поверхность можно рассматривать как некоторое подмножество в объемлющем пространстве, поэтому имеет смысл говорить об отображениях поверхности и в поверхность. Однако, чтобы определить гладкость таких отображений, необходимы дополнительные построения.

Конструкция 3.24. Пусть $M \subset \mathbb{R}^n$ - регулярная поверхность, и $f : M \to \mathbb{R}^m$ - (векторнозначная) функция на поверхности. Пусть $r_M : \Omega_M \to \mathbb{R}^n$ - (взаимно-однозначное с образом) параметризующее отображение поверхности M. Положим $\tilde{f} = f \circ r_M$. Обратно, так как отображение r_M взаимно-однозначно с образом, то для каждой функции $\tilde{f} : \Omega_M \to \mathbb{R}^m$ однозначно определена функция $f : M \to \mathbb{R}^m$, такая, что $\tilde{f} = f \circ r_M$, а именно $f = \tilde{f} \circ r_M^{-1}$, где $r_M^{-1} : M \to \Omega_M$.

Аналогично, пусть задано отображение $\gamma : I \to M$ промежутка I в регулярную поверхность M. Тогда определено отображение $\tilde{\gamma} = r_M^{-1} \circ \gamma : I \to \Omega_M$, и наоборот, для каждого отображения $\tilde{\gamma} : I \to \Omega_M$ из промежутка I в параметризующую область Ω_M однозначно определено отображение $\gamma = r_M \circ \tilde{\gamma} : I \to M$, причем $\tilde{\gamma} = r_M^{-1} \circ \gamma$.

Наконец, пусть $N \subset \mathbb{R}^n$ - ещё одна регулярная поверхность, и $r_N : \Omega_N \to \mathbb{R}^n$ - ее (взаимно-однозначное с образом) параметризующее отображение. Пусть $F : M \to N$ отображение поверхности M в поверхность N. Тогда в силу взаимной однозначности параметризующих отображений определено отображение $\tilde{F} : \Omega_M \to \Omega_N$ следующего вида: $\tilde{F} = r_N^{-1} \circ F \circ r_M$. Обратно, для каждого отображения $\tilde{F} : \Omega_M \to \Omega_N$ однозначно определено отображение поверхностей $F : M \to N$, такое, что $\tilde{F} = r_N^{-1} \circ F \circ r_M$.

Определение 3.25. Функция $\tilde{f}$ и отображения $\tilde{F}$ и $\tilde{\gamma}$ называются координатными представлениями функции f и отображений F и γ соответственно.

Поскольку соответствие между отображениями и их координатными представлениями взаимно-однозначно, мы, как правило, не будем их различать.

Замечание 3.26. Координатное представление представляет собой обычное отображение из области в $\mathbb{R}^n$ в область пространство $\mathbb{R}^k$. Именно такие отображения рассматриваются в математическом анализе, и поэтому можно говорить об их непрерывности, дифференцируемости и т.д. без каких-либо оговорок.

Определение 3.27. Гладкое отображение F одной регулярной поверхности в другую называется регулярным, если регулярным является его координатное представление, т.е. если ранг матрицы Якоби отображения $\tilde{F}$ максимален в каждой точке.

14.4.6 3.6 Surface curves

Пусть $M \subset \mathbb{R}^n$ - регулярная поверхность и $\gamma : I \to \mathbb{R}^n$ - параметрическая кривая.

Определение 3.28. Говорят, что кривая γ лежит на M, если $\gamma(I) \subset M$. Лежащая на поверхности параметрическая кривая называется гладкой, если она гладкая как отображение промежутка в поверхность, т.е., если её координатное представление в смысле определения 3.25 - гладкое.

Замечание 3.29. Если γ - гладкая параметрическая кривая, лежащая на регулярной поверхности, то соответствующая параметрическая кривая в $\mathbb{R}^n$ тоже гладкая. В этом случае координатное представление кривой γ задает регулярную кривую в $\mathbb{R}^k$, если и только если кривая γ регулярна.

Замечание 3.30. Пусть $r : \Omega \to \mathbb{R}^n$ - параметризация регулярной поверхности M, $u^1, \ldots, u^k$ - координаты в пространстве $\mathbb{R}^k$, в котором лежит области Ω, и $x^1, \ldots, x^n$ координаты в объемлющем пространстве. Координатное представление кривой γ задается набором функций $u^i(t), t \in I, i = 1, \ldots, k$, которые называются координатными функциями кривой. Сама параметрическая кривая γ записывается через координатные функции так: $\gamma(t) = r\left(u^1(t), \ldots, u^k(t)\right)$. Координатные функции кривой γ имеют вид $x^i(t) = x^i\left(u^1(t), \ldots, u^k(t)\right)$, где $x^i\left(u^1, \ldots, u^k\right)$ - координатные функции поверхности r.

Пример 3.31. Пусть $u^1, \ldots, u^k$ - координаты на регулярной поверхности M. Через каждую точку на M проходит k так называемых координатных линий, вдоль каждой из которых меняется лишь одна координата u^i, а остальные - постоянны. А именно, если $P \in M$ - точка на поверхности с координатами ($u_0^1, \ldots, u_0^k$), то i-й координатной линией, проходящей через точку P, называется кривая, координатное представление которой имеет вид:

$$u^j(t) = u_0^j, \quad j \neq i; \qquad u^i(t) = t, \quad t \in \left(u_0^i - \varepsilon, u_0^i + \varepsilon\right),$$

где ε выбрано настолько малым, чтобы интервал с центром в точке $\left(u_0^1, \ldots, u_0^k\right)$, имеющий длину 2ε и параллельный i-ой координатной оси не выходил за пределы параметризующей области.

14.4.7 3.7 Tangent Space

Пусть $P \in M$ - произвольная точка регулярной поверхности M, и $\gamma : I \to \mathbb{R}^n$ гладкая кривая, лежащая на M. Пусть $\gamma\left(t_0\right) = P$, т.е. γ проходит через P.

Определение 3.32. Вектор скорости $\dot{\gamma}\left(t_0\right) \in \mathbb{R}^n$ кривой γ в точке P назовем касательным вектором к поверхности M в точке P. Совокупность касательных векторов в точке P, построенных по всем гладким кривым на поверхности M, проходящим через эту точку, называется касательным пространством к поверхности M в точке P и обозначается через $T_P M$.

Утверждение 3.33. Пусть $P \in M$ - произвольная точка регулярной k-мерной поверхности $M \subset \mathbb{R}^n$. Тогда касательное пространство $T_P M$ является линейным пространством размерности k.

Доказательство. Пусть, как и выше, $x^i = x^i\left(u^1, \ldots, u^k\right), i = 1, \ldots, n$, - координатные функции поверхности M. Возьмем произвольную кривую $\gamma(t)$ на поверхности, проходящую через точку P, и пусть $u^j = u^j(t)$ - координатные функции этой кривой, а $P = \gamma\left(t_0\right)$. Тогда i-я координата соответствующего касательного вектора имеет вид

$$\left.\frac{dx^i}{dt}\right|_{t=t_0} = \sum_{j=1}^{k} \left.\frac{\partial x^i}{\partial u^j}\right|_P \left.\frac{du^j}{dt}\right|_{t=t_0}.$$

Таким образом, вектор скорости кривой γ - произвольный элемент касательного пространства $T_P M$, - это линейная комбинация вида

$$\begin{pmatrix} \frac{\partial x^1}{\partial u^1} \\ \vdots \\ \frac{\partial x^n}{\partial u^1} \end{pmatrix} \frac{du^1}{dt} + \cdots + \begin{pmatrix} \frac{\partial x^1}{\partial u^k} \\ \vdots \\ \frac{\partial x^n}{\partial u^k} \end{pmatrix} \frac{du^k}{dt},$$

где векторы-столбцы $\left(\partial x^1/\partial u^i, \ldots, \partial x^n/\partial u^i\right)^T, i = 1, \ldots, k$, не зависят от выбора кривой γ и однозначно определяются координатными функциями $x^i\left(u^1, \ldots, u^k\right)$ поверхности M и точкой P. В то же время, в силу произвольности гладкой кривой γ числа $du^i/\,dt|_{t=t_0}$ произвольны. Поэтому касательное пространство $T_P M$ - это линейное пространство, натянутое на векторы $\partial_{u^1}, \ldots, \partial_{u^k}$, где через ∂_{u^i} обозначен вектор-столбец

$$\left(\partial x^1/\partial u^i, \ldots, \partial x^n/\partial u^i\right)^T.$$

Матрица, составленная из векторов-столбцов ∂_{u^i}, - это в точности матрица Якоби отображения $x^i = x^i\left(u^1, \ldots, u^k\right)$, задающего поверхность M. Ранг этой матрицы, в силу регулярности поверхности M, равен k, поэтому векторы $\partial_{u^1}, \ldots, \partial_{u^k}$ линейно независимы и образуют базис в линейном пространстве $T_P M$.

Замечание 3.34. Теперь понятно, почему в определении 3.4 векторы ∂_{u^j} называются базисными касательными векторами,

Определение 3.35. Базис в касательном пространстве $T_P M$, составленный из векторов ∂_{u^j}, называется каноническим базисом в точке P, соответствующим координатам $u^1, \ldots, u^k$ на поверхности. Коэффициенты разложения касательного вектора к поверхности по элементам канонического базиса (∂_{u^i}) называются компонентами или координатами этого вектора в этом каноническом базисе или его координатами (компонентами) относительно системь координат ($u^1, \ldots, u^k$).

Замечание 3.36. Векторы, образующие канонический базис, суть ни что иное, как вектора скоростей соответствующих координатных линий. Действительно, вектор скорости кривой с координатным представлением $u^j(t)$ равен линейной комбинации векторов ∂_{u^p} с коэффициентами du^p/dt. Но, по определению i-й координатной линии, см. пример 3.31, $du^p/dt = \delta_i^p$, поэтому касательный вектор к i-й координатной линии совпадает с базисным вектором ∂_{u^i}.

Утверждение 3.37. При замене координат на регулярной поверхности, векторы канонического базиса в каждой точке поверхности преобразуются с помощью матрицы Якоби этой замены координат, которая играет роль матрицы перехода от одного базиса к другому. Компоненты одного и того же касательного вектора в разных системах координат связаны между собой с помощью этой матрицы перехода.

Доказательство. Пусть $v^i = v^i\left(u^1, \ldots, u^k\right)$ - замена координат на регулярной поверхности M. Тогда в каждой точке $P \in M$ определены сразу два канонических базиса: (∂_{u^i}) и (∂_{v^i}). Из определения векторов канонических базисов и теоремы о дифференцировании сложной функции получаем:

$$\partial_{v^i} = \begin{pmatrix} \frac{\partial x^1}{\partial v^i} \\ \vdots \\ \frac{\partial x^n}{\partial v^i} \end{pmatrix} = \begin{pmatrix} \sum_j \frac{\partial x^1}{\partial u^j} \frac{\partial u^j}{\partial v^i} \\ \vdots \\ \sum_j \frac{\partial x^n}{\partial u^j} \frac{\partial u^j}{\partial v^i} \end{pmatrix} = \sum_j \begin{pmatrix} \frac{\partial x^1}{\partial u^j} \\ \vdots \\ \frac{\partial x^n}{\partial u^j} \end{pmatrix} \frac{\partial u^j}{\partial v^i},$$

т.е. окончательно имеем:

$$\partial_{v^i} = \sum_j \partial_{u^j} \frac{\partial u^j}{\partial v^i},$$

или, в матричном виде,

$$(\partial_{v^1}, \ldots, \partial_{v^k}) = (\partial_{u^1}, \ldots, \partial_{u^k})\, J, \quad J = \left(\frac{\partial u^j}{\partial v^i}\right)$$

Далее, пусть $a \in T_P M$ произвольный касательный вектор, и пусть

$$a = \sum_{j=1}^{k} a^j \partial_{u^j} = \sum_{i=1}^{k} \tilde{a}^i \partial_{v^i},$$

т.е. $\left(a^1, \ldots, a^k\right)$ и $\left(\tilde{a}^1, \ldots, \tilde{a}^k\right)$ - компоненты вектора a в базисах (∂_{u^i}) и (∂_{v^i}) соответственно. Тогда из теорем линейной алгебры о преобразовании компонент вектора при замене базиса вытекает, что

$$\left(\tilde{a}^1, \ldots, \tilde{a}^k\right)^T = J^{-1}\left(a^1, \ldots, a^k\right)^T, \quad J = \left(\frac{\partial u^j}{\partial v^i}\right)$$

или, поскольку матрицы $\left(\frac{\partial u^j}{\partial v^i}\right)$ и $\left(\frac{\partial v^i}{\partial u^j}\right)$ взаимно обратны, в координатах имеем:

$$\tilde{a}^i = \sum_{\alpha=1}^{k} \frac{\partial v^i}{\partial u^\alpha} a^\alpha, \quad i = 1, \ldots, k$$

Утверждение доказано.

Упражнение 3.38. Получите формулу преобразования компонент касательного вектора как следствие теоремы о производной сложной функции: запишите координатные функции кривой в системе координат $v^1, \ldots, v^k$ в виде $v^i(t) = v^i\left(u^1(t), \ldots, u^k(t)\right)$ и вычислите производную по t.

14.4.8 3.8 Differential

Напомним, что в математическом анализе каждое дифференцируемое отображение порождает семейство дифференциалов - линейных отображений, приближающих исходное в соответствующих точках. На самом деле, эти линейные отображения действуют на касательных пространствах, и могут быть построены также для отображений поверхностей.

Конструкция 3.39. Пусть $F : M \to N$ - гладкое отображение регулярных поверхностей, $P \in M$ - произвольная точка поверхности M и $v \in T_P M$ - произвольный касательный вектор. Напомним, что каждый такой вектор является по определению вектором скорости некоторой кривой γ, проходящей через точку P. Пусть $v = \dot{\gamma}(t_0)$, где γ - гладкая кривая на M и $\gamma(t_0) = P$. Построим отображение $dF|_P : T_P M \to T_{F(P)} N$, положив

$$dF|_P(v) = \left.\frac{d}{dt}\right|_{t=t_0} F(\gamma(t))$$

Заметим, что $F \circ \gamma$ - гладкая кривая на поверхности N, проходящая через точку $F(P)$, а именно, $F(\gamma(t_0)) = F(P)$, поэтому отображение $dF|_P$ определено корректно.

Определение 3.40. Отображение $dF|_P : T_P M \to T_{F(P)} N$ называется дифференциалом отображения F в точке P.

Замечание 3.41. В частности, конструкция 3.39 позволяет определить дифференциал гладкой функции $f : M \to \mathbb{R}$ на поверхности M в точке $P \in M$ как отображение $df|_P : T_P M \to T_{f(P)}\mathbb{R} = \mathbb{R}$, а также дифференциал гладкой кривой $\gamma : I \to M$ в точке $t \in I$ как $d\gamma|_t : T_t I = \mathbb{R} \to T_{\gamma(t)} M$.

Утверждение 3.42. Дифференциал гладкого отображения регулярных поверхностей - линейное отображение соответстьующих касательных пространств. В канонических базисах систем координат он задается матрицей Якоби координатного представления отображения, вычисленной в соответствующей точке.

Доказательство. Пусть $u^1, \ldots, u^m$ - координаты на поверхности M, а $v^1, \ldots, v^n$ - координаты на поверхности N, точка P имеет координаты $u_0^1, \ldots, u_0^m$, и $v^i = v^i\left(u^1, \ldots, u^m\right)$ - координатные функции отображения F. Рассмотрим произвольный вектор $v \in T_P M$, и пусть $a = \dot{\gamma}\left(t_0\right)$ для некоторой проходящей через P гладкой кривой γ на $M, \gamma\left(t_0\right) = P$, координатные функции которой имеют вид $u^i(t)$. Тогда компоненты a^i вектора a в каноническом базисе (∂_{u^j}) имеют вид $a^i = \dot{u}^i\left(t_0\right)$.

Вычислим вектор $dF|_P(a)$. Координатные функции кривой $F \circ \gamma$ имеют вид $v^j(t) = v^j\left(u^1(t), \ldots, u^n(t)\right)$, поэтому

$$\dot{v}^j\left(t_0\right) = \sum_{\alpha=1}^m \left.\frac{\partial v^j}{\partial u^\alpha}\right|_P \dot{u}^\alpha\left(t_0\right) = \sum_{\alpha=1}^m \left.\frac{\partial v^j}{\partial u^\alpha}\right|_P a^\alpha.$$

Таким образом, компоненты $\dot{v}^j\left(t_0\right)$ вектора $dF|_P(a)$ в базисе (∂_{v^j}) получаются из столбца компонент вектора a в базисе (∂_{u^i}) умножением на матрицу Якоби координатного представления отображения F в точке P. Это доказывает линейность дифференциала и требуемую формулу.

14.4.9 3.9 First fundamental form

Пусть, как и выше, P - произвольная точка регулярной k-мерной поверхности $M \subset \mathbb{R}^n$. Скалярное произведение векторов в $\mathbb{R}^n$ порождает скалярное произведение касательных векторов из $T_P M$.

Конструкция 3.43. Пусть ξ и η - касательные векторы к поверхности M в точке P. Определим функцию $\mathfrak{G}(P) : T_P M \times T_P M \to \mathbb{R}$ так: $\mathfrak{G}(\xi, \eta) = \langle \xi, \eta \rangle$, где справа стоит стандартное скалярное произведение в $\mathbb{R}^n$. Другими словами, функция $\mathfrak{G}(P)$ есть ограничение стандартного скалярного произведения на подпространство $T_P M$.

Утверждение 3.44. *В* произвольной точке P регулярной поверхности M функция $\mathfrak{G}(P)$ - это положительно определенная симметричная билинейная форма на касательном пространстве $T_P M$.

Доказательство. Линейность по каждому аргументу, положительная определенность и симметричность функции $\mathfrak{G}(P)$ вытекают из соответствующих свойств евклидова скалярного произведения.

Определение 3.45. Билинейная форма $\mathfrak{G}(P)$ называется первой фундаментальной формой поверхности в точке P или индуцированной метрикой.

Замечание 3.46. Иногда первой фундаментальной формой поверхности называют соответствующую квадратичную форму. Как известно из линейной алгебры, имеется взаимно однозначное соответствие между симметричными билинейными формами и квадратичными формами, поэтому мы будем называть первой фундаментальной формой поверхности и то и другое.

Выясним, как выглядит матрица (g_{ij}) билинейной формы $\mathfrak{G}(P)$ в каноническом базисе.

Утверждение 3.47. Пусть $x^i\left(u^1, \ldots, u^k\right)$ - координатные функции регулярной поверхности. Тогда компоненты g_{ij} матрицы первой фундаментальной формы имеют вид

$$g_{ij}\left(u^1,\ldots,u^k\right)=\sum_{p=1}^{n}\frac{\partial x^p}{\partial u^i}\frac{\partial x^p}{\partial u^j}$$

В частности, g_{ij} - гладкие функции координат u^i, т.е. гладко зависят от точки P.

Доказательство. Пусть ξ и η - произвольные касательные векторы к поверхности M в точке P, а $\xi=\left(\xi^1,\ldots,\xi^k\right)$ и $\eta=\left(\eta^1,\ldots,\eta^k\right)$ - их компоненты в этих координатах. Тогда $\xi=\sum_i\xi^i\partial_{u^i}$ и $\eta=\sum_j\eta^j\partial_{u^j}$, где $\{\partial_{u^i}\}$ - канонический базис в точке P. Поэтому, с одной стороны,

$$\mathfrak{G}(P)(\xi,\eta)=\langle\xi,\eta\rangle=\left\langle\sum_{i=1}^{k}\xi^i\partial_{u^i},\sum_{j=1}^{k}\eta^j\partial_{u^j}\right\rangle=\sum_{i,j=1}^{k}\langle\partial_{u^i},\partial_{u^j}\rangle\,\xi^i\eta^j,$$

а с другой

$$\mathfrak{G}(P)(\xi,\eta)=\sum_{i,j=1}^{k}g_{ij}(P)\xi^i\eta^j$$

откуда $g_{ij}=\langle\partial_{u^i},\partial_{u^j}\rangle$, и нам остается воспользоваться видом векторов канонического базиса, см. утверждение 5.33. $\square$

Определение 3.48. Числа $g_{ij}(P)$ называются компонентами индучированной метрики или компонентами первой фундаментальной формы поверхности M в координатах $u^1,\ldots,u^k$, а матрица $G(P)=(g_{ij}(P))$ - матрицей первой фундаментальной формы или матрицей индучированной метрики.

Теперь из стандартных теорем линейной алгебры об изменении матрицы билинейной формы при замене базиса получаем следующий результат.

Следствие 3.49. При замене координат $v^i=v^i\left(u^1,\ldots,u^k\right)$ матрица первой фундаментальной формы поверхности меняется по закону изменения матрии, билинейньх форм: $G(u)=J^T\tilde{G}(v(u))J$, где G - матрица первой фундаментальной формью в координатах $u^1,\ldots,u^k,\tilde{G}$ - матрица той же формъы в координатах $v^1,\ldots,v^k$, а $J=\left(\frac{\partial v^i}{\partial u^j}\right)$ - матрица Якоби замень координат.

Замечание 3.50. Этот же результат можно легко получить и непосредственно, воспользовавшись теоремой о дифференцировании сложной функции. В покомпонентной записи он выглядит так:

$$g_{ij}\left(u^1,\ldots,u^k\right)=\sum_{pq}\tilde{g}_{pq}\left(v^1\left(u^1,\ldots,u^k\right),\ldots,v^k\left(u^1,\ldots,u^k\right)\right)\frac{\partial v^p}{\partial u^i}\frac{\partial v^q}{\partial u^j},$$

где частные производные вычисляются в точке ($u^1,\ldots,u^k$).

Замечание 3.51. Если задана первая фундаментальная форма, то для вычисления скалярного произведения касательных векторов, их длин и углов между ними нет необходимости вычислять их координаты в объемлющем пространстве. Достаточно знать их координаты на поверхности. В частности, нам не важен конкретный вид отображения r, задающего поверхность. Иногда, чтобы подчеркнуть этот факт, говорят, что скалярное произведение вычисляется во внутренних терминах, а индуцированную метрику называют внутренней метрикой поверхности, или внутренним скалярным произведением. Здесь слово "внутренний" подчеркивает независимость вычисления скалярных произведений от объемлющего пространства.

14.4.10 3.10 Lengths and angles

В терминах первой фундаментальной формы легко записываются длины касательных векторов и углы между ними и, как следствие, - длины кривых на поверхности и углы между кривыми.

Утверждение 3.52. Пусть $u^1, \ldots, u^k$ - координаты на регулярной поверхности, $g_{ij}(u)$ - компоненты её первой фундаментальной формы в этих координатах, $u\left(u^1(t), \ldots, u^k(t)\right), t \in [a, b]$, - координатные функции гладкой кривой γ, лежащей на этой поверхности. Тогда

$$\ell(\gamma) = \int_a^b \|\dot{\gamma}(t)\| dt = \int_a^b \sqrt{\sum_{i,j} g_{ij} \dot{u}^i(t) \dot{u}^j(t)} dt$$

Определение 3.53. Пусть γ_1 и γ_2 - две кривые на поверхности, пересекающиеся в точке P. Тогда углом между кривыми $\gamma_1 u \gamma_2$ в точке P называется меньший из двух углов между касательными векторами к этим кривым в точке P.

Напомним, что угол между векторами определяется через скалярное произведение. А именно, если a и b - произвольные векторы, то косинус угла между ними, по определению, равен

$$\frac{\langle a, b \rangle}{\|a\| \|b\|}$$

Утверждение 3.54. Пусть P - точка регулярной поверхности, u кривые $\gamma_p, p = 1{,}2$, лежат на ней u пересекаются в точке P. Пусть $u^1, \ldots, u^k$ - координаты на этой поверхности, $g_{ij}(u)$ - компоненты её первой фундаментальной формы в этих координатах, а $u_p^i = u_p^i(t), p = 1, 2$, - координатные функции кривых γ_p. Тогда косинус угла между кривыми γ_p может быть вычислен по формуле

$$\frac{\sum g_{\alpha\beta}(P) \dot{u}_1^\alpha \dot{u}_2^\beta}{\sqrt{\sum g_{\alpha\beta}(P) \dot{u}_1^\alpha \dot{u}_1^\beta} \sqrt{\sum g_{\alpha\beta}(P) \dot{u}_2^\alpha \dot{u}_2^\beta}}.$$

Замечание 3.55. В дифференциальной геометрии первую квадратичную форму часто называют "квадратом элемента длины" поверхности и используют так называемую "запись в дифференциалах". При этом саму форму $\mathfrak{G}$ обозначают через ds^2 - квадрат элемента длины, - и, в координатах $u^1, \ldots, u^k$ записывают в виде

$$ds^2 = \sum_{\alpha\beta} g_{\alpha\beta} du^\alpha du^\beta$$

Можно относиться к этой форме записи просто как к удобному формализму. Например, если записать квадрат элемента длины в объемлющем пространстве $\mathbb{R}^n$ со стандартными координатами $x^1, \ldots, x^n$ в виде

$$ds^2 = \left(dx^1\right)^2 + \cdots + \left(dx^n\right)^2$$

подставить в это выражение явный вид задающих поверхность координатных функций $x^i\left(u^1, \ldots, u^k\right)$, вычислить дифференциалы $dx^i = \sum_j \partial x^i / \partial u^j du^j$ и записать полученную формулу как полином от du^j, то получится выражение для первой фундаментальной формы поверхности (проверьте это).

С другой стороны, эта форма записи приобретает смысл, если договориться обозначать через $\{du^i\}$ линейные функционалы на касательном пространстве $T_P M$, образующие в двойственном пространстве $T_P^* M$ двойственный базис по отношению к

каноническому базису $\{\partial_{u^i}\}$. Напомним, что двойственный базис в T_P^*M однозначно определяется из соотношений $du^i(\partial_{u^j}) = \delta_j^i$ (здесь, как обычно, через δ_j^i обозначены символы Кронекера). При этом если $a = \left(a^1, \ldots, a^k\right)$ - касательный вектор, заданный своими координатами в каноническом базисе $\{\partial_{u^i}\}$, то значение функционала du^j на векторе a равно a^j. Теперь выражение для ds^2 можно воспринимать как запись билинейной формы в виде комбинации произведений линейных форм (т.е. как элемент пространства $T_P^*M \otimes T_P^*M$). Значение формы ds^2 на паре векторов a и b вычисляется следующим образом:

$$ds^2(a,b) = \sum_{\alpha\beta} g_{\alpha\beta} du^\alpha(a) du^\beta(b) = \sum_{\alpha\beta} g_{\alpha\beta} a^\alpha b^\beta$$

где $a = (a^i)$ и $b = (b^i)$, что полностью согласуется с данными выше определениями. В Дополнительном материале приведены примеры вычисления первой фундаментальной формы и угла между кривыми.

14.4.11 3.11 Surface isometries

Общее понятие изометрии как отображения, сохраняющего расстояния относится скорее к метрической геометрии, т.е. к геометрии метрических пространств общего вида. Здесь мы дадим определение и приведем ряд результатов, основанных на дифференциально-геометрическом подходе.

Определение 3.56. Пусть заданы две k-мерные регулярные поверхности $M_i, i = 1, 2$. Будем говорить, что поверхности M_1 и M_2 изометричны, если существует гладкое, взаимно-однозначное, регулярное отображение $\psi : M_1 \to M_2$, сохраняющее длины всех переходящих друг в друга гладких кривых.

Замечание 3.57. Условие сохранения длин означает следующее: если $\gamma_1 : I \to M_1-$ произвольная гладкая кривая на поверхности M_1 и $\gamma_2 = \psi \circ \gamma_1$ - соответствующая кривая на поверхности M_2, то длина кривой γ_1, вычисленная относительно первой фундаментальной формы $\mathfrak{G}_1$ поверхности M_1, совпадает с длиной кривой γ_2, вычисленной относительно первой фундаментальной формы $\mathfrak{G}_2$ поверхности M_2.

Утверждение 3.58. Гладкое взаимно-однозначное регулярное отображение $\psi : M_1 \to M_2$ задает изометрию регулярных поверхностей $M_i, i = 1, 2$, если и только если отображение ψ "сохраняет первую фундаментальную форму" в следующем смысле:

$$G_1 = J^T G_2 J,$$

где G_i - матрица первой фундаментальной формы поверхности M_i, aJ - матрица Якоби отображения ψ в соответствующих координатах.

Доказательство. Проверим достаточность. Рассмотрим произвольную гладкую кривую γ на поверхности M_1. Зафиксируем на M_1 некоторые координаты $u^1, \ldots, u^k$, а на M_2 - координаты $v^1, \ldots, v^k$. Пусть $u^i(t)$ - координатные функции кривой γ. Если $v^i\left(u^1, \ldots, u^k\right)$ - координатные функции, задающие отображение ψ, то образ кривой γ

при отображении ψ имеет вид $v^i(t) = v^i\left(u^1(t) \ldots, u^k(t)\right)$. Обозначим через g_{ij} элементы матрицы $G_1(u)$ первой фундаментальной формы поверхности M_1, а через h_{ij} - элементы матрицы $G_2(v)$ первой фундаментальной формы поверхности M_2 в выбранных координатах. Тогда длина кривой γ на M_1 имеет вид

$$\ell(\gamma) = \int \sqrt{\sum_{ij} g_{ij} \dot{u}^i \dot{u}^j} dt$$

В то же время длина кривой $\psi \circ \gamma$ на M_2 вычисляется как

$$\ell(\psi \circ \gamma) = \int \sqrt{\sum_{ij} h_{ij} \dot{v}^i \dot{v}^j} dt = \int \sqrt{\sum_{ij} h_{ij} \sum_{\alpha} \frac{\partial v^i}{\partial u^\alpha} \dot{u}^\alpha \sum_{\beta} \frac{\partial v^j}{\partial u^\beta} \dot{u}^\beta} dt =$$

$$= \int \sqrt{\sum_{\alpha\beta} \left(\sum_{ij} h_{ij} \frac{\partial v^i}{\partial u^\alpha} \frac{\partial v^j}{\partial u^\beta} \right) \dot{u}^\alpha \dot{u}^\beta} dt = \int \sqrt{\sum_{\alpha\beta} g_{\alpha\beta} \dot{u}^\alpha \dot{u}^\beta} dt = \ell(\gamma),$$

поскольку

$$\sum_{ij} h_{ij} \frac{\partial v^i}{\partial u^\alpha} \frac{\partial v^j}{\partial u^\beta} = g_{\alpha\beta}$$

по предположению, что и требовалось.

Докажем справедливость обратного утверждения. Пусть ψ - некоторое регулярное взаимно-однозначное отображение из M_1 в M_2, сохраняющее длины всех кривых. Это означает, что для произвольной кривой γ при сделанных выше обозначениях справедливо приведенное выше интегральное равенство. Из произвольности кривой γ следует совпадение подынтегральных выражений для каждого вектора. Действительно, для вектора $a \in T_P(M)$ достаточно рассмотреть семейство кривых $\gamma_s : [0, s] \to M$, координатные функции которых имеют вид $u^i(t) = u_0^i + a^i t$, где $\left(u_0^1, \ldots, u_0^k\right)$ – координаты точки P, и a^i - компонент вектора a. Тогда интегральное равенство имеет вид

$$\ell\left(\gamma_s\right) = \int_0^s \sqrt{\sum_{ij} g_{ij} \dot{u}^i \dot{u}^j} dt = \ell\left(\psi \circ \gamma_s\right) = \int_0^s \sqrt{\sum_{ij} h_{ij} \dot{v}^i \dot{v}^j} dt$$

откуда, дифференцируя его по верхнему пределу в $s = 0$, получаем равенство подынтегральных выражений:

$$\sum_{ij} g_{ij} a^i a^j = \sum_{\alpha\beta} \left(\sum_{ij} h_{ij} \frac{\partial v^i}{\partial u^\alpha} \frac{\partial v^j}{\partial u^\beta} \right) a^\alpha a^\beta.$$

Далее, из произвольности вектора a вытекает, что $g_{\alpha\beta} = \sum_{ij} h_{ij} \left(\partial v^i / \partial u^\alpha\right) \left(\partial v^j / \partial u^\beta\right)$ в каждой точке поверхности M_1. Последнее, по определению, означает, что отображение ψ сохраняет первую квадратичную форму. Доказательство закончено. □

Замечание 3.59. Условие утверждения [3.58 означает, что дифференциал отображения ψ, рассматриваемый как отображение евклидовых пространств ($T_P M_1, g_1(P)$) и $\left(T_{\psi(P)} M, g_2\right)$, где g_i - скалярные произведения, заданные первыми фундаментальными формами, сохраняет скалярные произведения.

14.4.12 3.12 Problems

Задача 3.63. Составить параметрическое уравнение поверхности, образованной касательными к данной кривой $\gamma = \gamma(u)$. Такая поверхность называется развертывающейся поверхностью. Исследовать развертывающуюся поверхность на регулярность.

Задача 3.64. Вокруг оси Oz вращается плоская кривая $x = f(v), z = g(v)$. Составить параметрические уравнения поверхности вращения. Рассмотреть частный случай, когда меридиан задан уравнением $x = f(z)$.

Задача 3.65. Прямая движется поступательно с постоянной скоростью, пересекая другую прямую под прямым углом, и одновременно равномерно вращается вокруг этой

прямой. Составить уравнение поверхности, которую описывает движущаяся прямая (прямой геликоид).

Задача 3.66. Составить уравнение поверхности, образованной вращением трактрисы

$$\gamma(v) = \left\{ a \ln \operatorname{tg}\left(\frac{\pi}{4} + \frac{v}{2}\right) - a \sin v, a \cos v \right\}$$

вокруг ее асимптоты (псевдосфера).

Задача 3.67. Вычислить первую квадратичную форму следующих поверхностей:
(1) $r(u,v) = (a\cos u \cos v, a \sin u \cos v, a \sin v)$(сфера);
(2) $r(u,v) = (a\cos u \cos v, b \sin u \cos v, c \sin v)$(эллипсоид);
(3) $r(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$ (поверхность вращения).

Задача 3.68. Первая квадратичная форма поверхности имеет вид

$$ds^2 = du^2 + \left(u^2 + a^2\right) dv^2$$

Требуется вычислить:

- периметр криволинейного треугольника, образованного пересечением кривых

$$u = \pm\frac{1}{2}av^2, \quad v = 1$$

- углы этого криволинейного треугольника;
- площадь треугольника, образованного пересечением кривых $u = \pm av, v = 1$.

Задача 3.69. Найти на поверхности

$$x = u\cos v, \quad y = u \sin v, \quad z = a \ln\left(u + \sqrt{u^2 - a^2}\right)$$

кривые, пересекающие каждую кривую $v = \text{const}$ под постоянным углом θ (локсодромы).

14.4.13 3.13.1 Examples of calculation of induced metric

Рассмотрим двумерную плоскость Π в $\mathbb{R}^3$, заданную параметрически в виде $r\left(u^1, u^2\right) = r_0 + a_1 u^1 + a_2 u^2$, где r_0, a_1 и a_2 - фиксированные векторы, причем a_1 и a_2 - линейно независимы. Поскольку $\partial_u i = a_i, i = 1, 2$, поверхность невырождена в каждой своей точке. Первая квадратичная форма $\mathfrak{G}_\Pi$ плоскости постоянна и имеет вид

$$ds^2 = \langle a_1, a_1\rangle \left(du^1\right)^2 + 2\langle a_1, a_2\rangle du^1 du^2 + \langle a_2, a_2\rangle \left(du^2\right)^2$$

или

$$G(u) = \begin{pmatrix} \langle a_1, a_1\rangle & \langle a_1, a_2\rangle \\ \langle a_2, a_1\rangle & \langle a_2, a_2\rangle \end{pmatrix} = \begin{pmatrix} a_1^1 & a_1^2 \\ a_2^1 & a_2^2 \end{pmatrix} \begin{pmatrix} a_1^1 & a_2^1 \\ a_1^2 & a_2^2 \end{pmatrix} = A^T A,$$

где A - матрица, столбцы которой суть векторы a_1 и a_2 соответственно.
В качестве следующего примера возьмем стандартную сферу радиуса R в $\mathbb{R}^3$, заданную параметрически так:

$$\begin{aligned} x^1 &= R\cos\vartheta\cos\varphi \\ x^2 &= R\cos\vartheta\sin\varphi \\ x^3 &= R\sin\vartheta \end{aligned}$$

В координатах (ϑ, φ) первая квадратичная форма сферы имеет вид

$$ds^2 = R^2(d\vartheta)^2 + R^2\cos^2\vartheta(d\varphi)^2.$$

Длина четверти меридиана $\varphi = \varphi_0, \vartheta \in [0, \pi/2]$, например, вычисляется следующим образом:

$$\int_0^{\pi/2} R\sqrt{1+\cos^2\vartheta\cdot 0} = R\pi/2$$

Рассмотрим теперь поверхность-график функции $z = f(x, y)$. Тогда соответствующее параметрическое представление имеет вид

$$r = (x, y, f(x, y)), \quad r_x = (1, 0, f_x), \quad r_y = (0, 1, f_y),$$

а значит, матрица метрики -

$$\begin{pmatrix} 1+f_x^2 & f_xf_y \\ f_xf_y & 1+f_y^2 \end{pmatrix}$$

т.е. $ds^2 = (1+f_x^2)\,dx^2 + 2f_xf_ydxdy + \left(1+f_y^2\right)dy^2$.

Упражнение 3.70. Вычислить первую квадратичную форму поверхности, заданной как график гладкой функции $x^n = f\left(x^1, \ldots, x^{n-1}\right)$.

Упражнение 3.71. Вычислить первую квадратичную форму поверхности, заданной как множество решений уравнения $F\left(x^1, \ldots, x^n\right) = 0, dF \neq 0$.

Вычислим угол между меридианом и параллелью на стандартной сфере S^2 радиуса R. Как и выше, параметризуем сферу координатами (ϑ, φ). Пусть γ_1 - это параллель, координатные функции которой имеют вид $\vartheta(t) = \vartheta_0, \varphi(t) = t$, и γ_2 - это меридиан с координатными функциями $\vartheta(t) = t, \varphi(t) = \varphi_0$. Они пересекаются в точке $P = (\vartheta_0, \varphi_0)$. Касательные векторы $\dot\gamma_1 = (0, 1)$ и $\dot\gamma_2 = (1, 0)$. Далее, первая квадратичная форма сферы, как уже известно, в этих координатах записывается в виде

$$ds^2 = R^2(d\vartheta)^2 + R^2\cos^2\vartheta(d\varphi)^2$$

поэтому $\langle\dot\gamma_1, \dot\gamma_2\rangle = 0$, т.е. косинус угла между ними тоже равен нулю, а угол $-\pi/2$.

14.4.14 3.13.2 Rhumb line

Найдем на стандартной двумерной сфере кривую, пересекающую все меридианы под одним и тем же углом α (обладающая этим свойством кривая называется локсодромой). Пусть снова на сфере выбраны координаты (ϑ, φ) и меридианы имеют вид $\vartheta = t, \varphi = \varphi_0$. Далее, пусть локсодрома параметризована следующим образом: $\vartheta = \vartheta(\varphi), \varphi = t$. Тогда касательные векторы к ней и меридиану имеют соответственно вид $(\dot\vartheta, 1)$ и $(1, 0)$, поэтому по определению локсодромы имеем

$$\frac{R^2\dot\vartheta}{R\sqrt{R^2\dot\vartheta^2 + R^2\cos^2\vartheta}} = \cos\alpha.$$

Получаем следующее дифференциальное уравнение на $\vartheta(t)$:

$$\dot\vartheta = \cos\vartheta\operatorname{ctg}\alpha$$

(мы воспользовались тем, что $\vartheta \in [-\pi/2, \pi/2]$, и предположили, что $\operatorname{ctg}\alpha > 0$). Решая это уравнение, находим

$$\varphi = \frac{\operatorname{tg}\alpha}{2}\log\left(\frac{1+\sin\vartheta}{1-\sin\vartheta}\right) + C.$$

На рис. 3.1 изображена локсодрома для $\alpha = 0,45\pi$ (т.е. близкого к $\pi/2$).

Рис. 3.1: Локсодрома; $\alpha = 0,45\pi$.

14.4.15 3.13.3 First quadratic form and volume

В терминах первой квадратичной формы также естественно записывается k-мерный объем поверхности. Пусть заданы регулярная параметрическая поверхность $r : \Omega \to \mathbb{R}^n, \Omega \subset \mathbb{R}^k, \left(u^1, \ldots, u^k\right)$ – координаты на r и $P \in \Omega$ - произвольная точка. Обозначим через $\Pi(P)$ параллелепипед, натянутый на векторы $\{\partial_u{}^i\}$ канонического репера, а через $|\Pi(P)|$ - его k-мерный объем. Тогда k-мерным объемом, или просто объемом параметрической поверхности r, называется величина

$$\operatorname{vol}_k(M) = \int_\Omega |\Pi(P)| du^1 \cdots du^k$$

Напомним, как определяется и вычисляется k-мерный объем параллелепипеда. Хорошо известно, что двумерный объем (площадь) параллелограмма может быть вычислен как произведение одномерного объема (длины) его основания на высоту. Это подводит к следующему индуктивному определению площади параллелепипеда: k-мерный объем параллелепипеда, натянутого на векторы $a_1, \ldots, a_k, a_i \in \mathbb{R}^n$, равен произведению ($k-1$)-мерного объема параллелепипеда, натянутого на векторы $a_1, \ldots, a_{k-1}$, (площадь основания) на высоту, т.е. длину ортогональной проекции вектора a_k на ортогональное дополнение к $a_1, \ldots, a_{k-1}$. Напомним следующий результат из линейной алгебры.

Statement

3.72. Если определить объем k-мерного параллелепипеда по индукции как произведение объема его основания - ($k-1$)-мерного параллелепипеда, на высоту, то полученное число может быть вычислено по формуле $\sqrt{\det G}$, где $G = A^T A$ - матрица Грамма системь векторов $a_1, \ldots, a_k$, на которые натянут параллелепипед.

Доказательство. Пусть сначала $k = 2$. Тогда

$$\begin{aligned}\det(G) &= \|a_1\|^2 \|a_2\|^2 - \langle a_1, a_2\rangle^2 = \\ &= \|a_1\|^2 \|a_2\|^2 \left(1 - \cos^2\varphi\right) = \|a_1\|^2 \|a_2\|^2 \sin^2\varphi\end{aligned}$$

где φ - угол между векторами a_1 и a_2. В итоге получаем

$$\sqrt{\det(G)} = \|a_1\| \left(\|a_2\| \sin\varphi\right),$$

т.е. определенная нами площадь равна длине вектора a_1 - длине основания, умноженной на длину проекции вектора a_2 на нормаль к a_1, проведенную в плоскости (a_1, a_2). Таким образом, в случае $k = 2$ данное утверждение имеет место.

Рассмотрим теперь общий случай. Представим вектор a_k в виде суммы его проекции T на плоскость Π_{k-1}, натянутую на векторы $a_1, \ldots, a_{k-1}$, и нормальной составляющей N. Воспользовавшись равенством $\langle a_k, a_i\rangle = \langle T, a_i\rangle + \langle N, a_i\rangle$ и полилинейностью определителя как функции строк и столбцов, найдем, что $\det\left(A^T A\right) = \det\left(\tilde{A}^T \tilde{A}\right)$, где матрица $\tilde{A}$ получена из A заменой столбца a_k на столбец N. Осталось заметить, что последняя строка и столбец матрицы $\tilde{A}^T \tilde{A}$ содержат ровно один ненулевой элемент - последний, который равен $\|N\|^2$, поэтому имеет место равенство

$$\det\left(\tilde{A}^T \tilde{A}\right) = \det\left(A_{k-1}^T A_{k-1}\right) \|N\|^2,$$

где A_{k-1} - матрица, составленная из первых $k-1$ вектора $a_1, \ldots, a_{k-1}$. Но по предположению индукции этот определитель равен квадрату площади параллелепипеда, натянутого на эти векторы, т.е. квадрату площади основания, а длина нормальной составляющей N - это и есть длина высоты. Утверждение доказано.

Таким образом, возвращаясь к определению k-мерного объема поверхности, можно сказать, что k мерный объем $|\Pi(P)|$ параллелепипеда $\Pi(P)$, натянутого на векторы канонического репера, может быть вычислен как корень из определителя матрицы Грамма канонического репера, т.е. матрицы первой квадратичной формы:

$$\mathrm{vol}_k(M) = \int_\Omega |\Pi(P)| du^1 \cdots du^k = \int_\Omega \sqrt{\det(G)} du^1 \cdots du^k$$

Следующее утверждение следует из теоремы о замене переменных в кратном интеграле и из следствия 3.49.

Statement

3.73. Определенный нами k-мерный объем поверхности не меняется при замене параметризации.

Таким образом, имеет смысл говорить о k-мерном объеме поверхности, понимая под ним объем произвольного представителя этой поверхности.

Упражнение 3.74. Выписать формулы для вычисления площади гиперповерхности, заданной в виде графика функции, а также неявно.

14.4.16 3.13.4 Euclidean Metric Isometries

Найдем все изометрии евклидовой метрики $ds^2 = \sum_i (dx^i)^2$ пространства $\mathbb{R}^n$. Отметим, что в линейной алгебре было получено описание всех линейньх отображений пространства $\mathbb{R}^n$ в себя, сохраняющих евклидово скалярное произведение (т.е. евклидову метрику). Ими оказались так называемые ортогональные преобразования. Напомним, что линейное преобразование называется ортогональным, если его матрица A в ортонормированном базисе обладает свойством $AA^T = E$. Однако из результатов линейной алгебры не следует, что не существует какого-нибудь нелинейного регулярного отображения $\mathbb{R}^n$ на себя, сохраняющего метрику.

Пусть $\psi : \mathbb{R}^n \to \mathbb{R}^n$ - изометрия $\mathbb{R}^n$. Выберем в каждом из экземпляров $\mathbb{R}^n$ декартовы координаты, которые обозначим через $(x^1, \ldots, x^n)$ и $(\ y^1, \ldots, y^n\ )$ соответственно. Тогда отображение ψ может быть записано в виде $y^i = y^i\,(x^1, \ldots, x^n)$.

В силу утверждения [3.58 в каждой точке $\mathbb{R}^n$ выполнено равенство

$$\sum_{ij} \delta_{ij} \frac{\partial y^i}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\beta} = \delta_{\alpha\beta}$$

где $\delta_{\alpha\beta}$ - символы Кронекера. Выполним суммирование по i в левой части предыдущего равенства:

$$\sum_j \frac{\partial y^j}{\partial x^\alpha} \frac{\partial y^j}{\partial x^\beta} = \delta_{\alpha\beta}$$

Продифференцировав последнее равенство по x^γ, получим для произвольной тройки α, β, γ :

$$\sum_j \left(\frac{\partial^2 y^j}{\partial x^\alpha \partial x^\gamma} \frac{\partial y^j}{\partial x^\beta} + \frac{\partial y^j}{\partial x^\alpha} \frac{\partial^2 y^j}{\partial x^\beta \partial x^\gamma} \right) = 0.$$

Чтобы выразить отсюда вторые производные функций y^j, вычтем из этого уравнения точно такое же для тройки β, γ, α, а затем, прибавим еще одно для тройки γ, α, β. В результате будем иметь:

$$\sum_j \left(\frac{\partial^2 y^j}{\partial x^\alpha \partial x^\gamma} \frac{\partial y^j}{\partial x^\beta} + \frac{\partial y^j}{\partial x^\alpha} \frac{\partial^2 y^j}{\partial x^\beta \partial x^\gamma} - \frac{\partial^2 y^j}{\partial x^\beta \partial x^\alpha} \frac{\partial y^j}{\partial x^\gamma} - \right.$$
$$\left. - \frac{\partial y^j}{\partial x^\beta} \frac{\partial^2 y^j}{\partial x^\gamma \partial x^\alpha} + \frac{\partial^2 y^j}{\partial x^\gamma \partial x^\beta} \frac{\partial y^j}{\partial x^\alpha} + \frac{\partial y^j}{\partial x^\gamma} \frac{\partial^2 y^j}{\partial x^\alpha \partial x^\beta} \right) = 0$$

Приводя подобные члены, заключаем, что

$$2 \sum_j \frac{\partial^2 y^j}{\partial x^\beta \partial x^\gamma} \frac{\partial y^j}{\partial x^\alpha} = 0$$

Итак, получили однородную линейную систему уравнений на неизвестные величины $\partial^2 y^j / \partial x^\beta \partial x^\gamma, j = 1, \ldots, n$, где β, γ - произвольная фиксированная пара индексов. Матрица этого уравнения по построению невырождена, откуда следует, что существует единственное решение

$$\frac{\partial^2 y^j}{\partial x^\beta \partial x^\gamma} = 0$$

Поэтому все функции $y^j(x^1,\ldots,x^n)$ - линейны, а следовательно, все изометрии пространства $\mathbb{R}^n$ с евклидовой метрикой - суть линейные ортогональные преобразования и сдвиги.

14.5 4 Surfaces. The second fundamental form

В данной лекции ограничимся рассмотрением гиперповерхностей в $\mathbb{R}^n$. Общий случай будет кратко разобран в разделе "Дополнительный материал".

14.5.1 4.1 Definition of the second form

Пусть M - регулярная гиперповерхность в евклидовом пространстве $\mathbb{R}^n$.

Конструкция 4.1. Обозначим через ($x^1,\ldots,x^n$) стандартные координаты $\mathbb{R}^n$, а через $(u^1,\ldots,u^{n-1})$ - координаты на поверхности. В каждой точке P на M определен канонический базис $\{\partial_{u^i}\}$ касательного пространства T_PM. Поэтому в каждой точке P поверхности однозначно определен единичный вектор $N(P)$, ортогональный всем векторам ∂_{u^i} и дополняющий систему векторов $\{\partial_{u^i}\}$ до положительно ориентированного базиса в $\mathbb{R}^n$. Другими словами, вектор $N(P)$ однозначно определяется из соотношений $\langle\partial_{u^i},N(P)\rangle=0,\langle N(P),N(P)\rangle=1$ и, наконец,

$$\det\left(\partial_{u^1},\ldots,\partial_{u^{n-1}},N(P)\right)>0.$$

Так как векторы ∂_{u^i} гладко зависят от точки поверхности, то и вектор $N(P)$ тоже гладко зависит от точки.

Определение 4.2. Вектор $N(P)$ называется (ориентированной) поле нормалью к поверхности M, соответствующей координатам (u^i). Также будем говорить, что N это поле ориентированных нормалей, имея ввиду, что вектор $N(P)$ задан в каждой точке $P\in M$.

Утверждение 4.3. Предположим, что на поверхности M заданы две системы координат: $(u^1,\ldots,u^{n-1}),(v^1,\ldots,v^{n-1}),uv^i = v^i(u^1,\ldots,u^{n-1})$. Тогда векторы ориентированных нормалей в точке P, соответствующие координатам (u^i) $u(v^i)$, отличаются на знак определителя матрицы Якоби ($\partial v^i/\partial u^j$) замены координат.

Доказательство. Прямая, проходящая через вектор $N(P)$ в точке P, однозначно определяется касательным пространством T_PM и не зависит от выбора базиса в нем. Для завершения доказательства осталось воспользоваться утверждением [3.37. □

Следствие 4.4. На регулярной гиперповерхности существуют в точности два ориентированных поля нормалей, которые в каждой точке поверхности противонаправлены.

Замечание 4.5. Пусть ($u^1,\ldots,u^{n-1}$) - координаты на поверхности M. Ясно, что все замены $v^i=v^i(u^1,\ldots,u^{n-1})$ координат на M можно разделить на два непересекающихся класса, в зависимости от знака определителя матрицы Якоби. Замена координат внутри каждого из классов имеет, очевидно, положительный якобиан. Из сказанного выше следует, что выбор одного из двух полей нормалей равносилен выбору одного из классов замен координат. Такой выбор часто называют выбором ориентации поверхности M. Отметим, что в дальнейшем аналогично будет введено понятие ориентации гладкого многообразия.

Конструкция 4.6. Пусть M - регулярная гиперповерхность в $\mathbb{R}^n$ и $P\in M$ - некоторая ее точка. Фиксируем одно из двух полей нормалей к поверхности M и обозначим его через

N. Пусть $\gamma(t)$ - произвольная кривая, лежащая на M и проходящая через P, $\gamma(t_0) = P$, и $\xi \in T_P M$ - вектор скорости $\dot{\gamma}(t_0)$ кривой γ в точке P. Построим функцию $\mathfrak{q} : T_P M \to \mathbb{R}$, положив $\mathfrak{q}(\xi) = \langle \ddot{\gamma}(t_0), N \rangle$.

Утверждение 4.7. Пусть на регулярной гиперповерхности M фиксировано поле нормалей N. Функция $\mathfrak{q}$ является квадратичной формой на линейном пространстве $T_P M$. Если $(u^1, \ldots, u^{n-1})$ - координаты на M, то матрица квадратичной формы $\mathfrak{q}$ в соответствующем каноническом базисе (∂_{u^i}) имеет вид $q_{ij} = \langle r_{u^i u^j}, N \rangle$.
Доказательство. Пусть $(u^1, \ldots, u^{n-1})$ - координаты на поверхности M и $u^i(t)$ - координатные функции кривой γ. Тогда $\xi = (\dot{u}^1, \ldots, \dot{u}^{n-1})$ и

$$\ddot{\gamma} = \left(\sum_i r_{u^i} \dot{u}^i \right)^{\cdot} = \sum_{ij} r_{u^i u^j} \dot{u}^i \dot{u}^j + \sum_i r_{u^i} \ddot{u}^i,$$

поэтому

$$\mathfrak{q}(\xi) = \langle \ddot{\gamma}, N \rangle = \sum_{ij} \langle r_{u^i u^j}, N \rangle \dot{u}^i \dot{u}^j,$$

поскольку $\langle r_{u^i}, N \rangle = 0$. Таким образом, $\mathfrak{q}(\xi)$ зависит лишь от компонент вектора ξ и величин $\langle r_{u^i u^j}(P), N(P) \rangle$, которые не зависят от вектора ξ, а зависят только от точки P. Более того, при фиксированной P величина $\mathfrak{q}(\xi)$ равна значению квадратичной формы на векторе ξ, где форма эта задана в базисе (∂_{u^i}) матрицей $Q(u) = (\langle r_{u^i u^j}(P), N(P) \rangle)$. $\square$

Следствие 4.8. Пусть $u^1, \ldots, u^{n-1} u v^1, \ldots, v^{n-1}$ - две системы координат на регулярной поверхности M. Тогда в каждой точке P матрицы $Q(u) = (\langle r_{u^i u^j}(P), N(P) \rangle) uQ(v) = (\langle r_{v^i v^j}(P), N(P) \rangle)$ связаным между собой по правилу $Q(u) = J^t Q(v) J$, где $J = \left(\frac{\partial v^i}{\partial u^j} \right)$ - матрица Якоби заменьк координат, вычисленная в точке P.

Доказательство. Величина $\mathfrak{q}(\xi) = \langle \ddot{\gamma}, N \rangle$ не зависит от выбора координат (напомним, что мы зафиксировали поле нормалей N, т.е. рассматриваем только замены координат с положительным якобианом). С другой стороны,

$$\mathfrak{q}(\xi) = \sum_{ij} \langle r_{v^i v^j}, N \rangle \dot{v}^i \dot{v}^j = \sum_{ij} \langle r_{u^i u^j}, N \rangle \dot{u}^i \dot{u}^j,$$

где $\dot{u}^i$ и $\dot{v}^i$ - компоненты одного и того же касательного вектора ξ в координатах u^i и v^i соответственно. Поэтому матрицы $Q(u)$ и $Q(v)$ - суть матрицы одной и той же квадратичной формы в разных базисах. Осталось применить теорему из линейной алгебры об изменении матрицы квадратичной формы. $\square$

Замечание 4.9. Закон преобразования величин $\langle r_{u^i u^j}, N \rangle$ при замене координат легко проверить и прямым подсчетом (см. Дополнительный материал).

Определение 4.10. Квадратичная форма $\mathfrak{q}$ на касательном пространстве $T_P M$ называется второй фундаментальной формой, или второй квадратичной формой, гиперповерхности M в точке P (по отношению к нормали N).

Замечание 4.11. Если заменить поле N на $-N$, то вторая фундаментальная форма изменит знак.

Замечание 4.12. В отличие от первой фундаментальной формы поверхности, вторая фундаментальная форма не обязана быть ни невырожденной, ни положительно определенной.

Замечание 4.13. Вторую фундаментальную форму также часто записывают в дифференциальном виде:

$$dq^2 = \sum_{ij} q_{ij} du^i du^j$$

Пример 4.14. Пусть двумерная гиперповерхность задана в виде графика $z = f(x, y)$. Тогда соответствующее параметрическое представление имеет вид

$$r = (x, y, f(x, y)), \quad r_x = (1, 0, f_x), \quad r_y = (0, 1, f_y),$$
$$r_{xx} = (0, 0, f_{xx}), \quad r_{xy} = (0, 0, f_{xy}), r_{yy} = (0, 0, f_{yy}),$$
$$N = \frac{[r_x, r_y]}{\|[r_x, r_y]\|} = \frac{(-f_x, -f_y, 1)}{\sqrt{1 + f_x^2 + f_y^2}}$$

и, значит, матрица второй фундаментальной формы в координатах (x, y) выглядит следующим образом:

$$\frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{pmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{pmatrix}.$$

Упражнение 4.15. Вычислить вторую форму гиперповерхности, заданной в виде графика гладкой функции $x^n = f(x^1, \ldots, x^{n-1})$.

Упражнение 4.16. Вычислить вторую фундаментальную форму гиперповерхности, заданной неявно в виде $F(x^1, \ldots, x^n) = 0$, где $dF \neq 0$, в любой ее точки.

14.5.2 4.2 The geometric meaning of the second form is the curvatures of flat sections

Пусть M - регулярная гиперповерхность, ориентированная с помощью поля нормалей N, P - произвольная ее точка, и $\mathfrak{q}$ - вторая фундаментальная форма поверхности. Мы начнем со следующего общего результата.

Theorem

4.17 (Об отношении форм). Пусть $\gamma-$ произвольная регулярная кривая на M, проходящая через точку P. Обозначим через $\xi \in T_P M$ её вектор скорости в точке P, а через k - ее кривизну. Тогда или $ku\mathfrak{q}(\xi)$ одновременно равны нулю, или кривая γ бирегулярна в точке Pu

$$k \cos\theta = \frac{\mathfrak{q}(\xi)}{\mathfrak{G}(\xi)}$$

где θ - угол между главной нормалью ν кривой γ и нормалью N к поверхности M в точке P (напомним, что вторая фундаментальная форма вычисляется по отношению κN).

Доказательство. Рассмотрим первую возможность. Условие $k = 0$ равносильно линейной зависимости векторов $\dot{\gamma}$ и $\ddot{\gamma}$, что, в свою очередь, возможно, если и только если $\ddot{\gamma}$ лежит в касательной плоскости $T_P M$. Значит, условие $k = 0$ равносильно перпендикулярности векторов $\ddot{\gamma}$ и N, что равносильно условию $\mathfrak{q}(\xi) = 0$. В первом случае теорема доказана.

Рассмотрим теперь случай $k \neq 0$. Покажем сначала, что равенство, которое мы доказываем, не меняется при регулярной замене параметра на кривой γ. Действительно, левая часть вообще не зависит от параметризации, а вектор скорости кривой при замене растягивается в некоторое число раз. С другой стороны, каждая квадратичная форма -

однородная функция степени однородности 2, поэтому $\mathfrak{q}(\lambda\xi) = \lambda^2\mathfrak{q}(\xi)$ и $\mathfrak{G}(\lambda\xi) = \lambda^2\mathfrak{G}(\xi)$ для любых вектора $\xi \in T_PM$ и числа $\lambda \neq 0$. Поэтому

$$\frac{\mathfrak{q}(\xi)}{\mathfrak{G}(\xi)} = \frac{\mathfrak{q}(\lambda\xi)}{\mathfrak{G}(\lambda\xi)}.$$

Таким образом, теорему достаточно доказать для натурально параметризованной кривой γ.

Выберем на кривой γ натуральный параметр s, такой, что $\gamma(s_0) = P, \dot{\gamma}(s_0) = \xi$, тогда $\ddot{\gamma}(s_0) = k\nu, \mathfrak{G}(\xi) = 1$ и

$$\frac{\mathfrak{q}(\xi)}{\mathfrak{G}(\xi)} = \langle \ddot{\gamma}(s_0), N\rangle = \langle k\nu, N\rangle = k\langle \nu, N\rangle = k\cos\theta,$$

что и требовалось доказать.

Чтобы выяснить геометрический смысл второй фундаментальной формы, рассмотрим на поверхности кривые специального вида.

Конструкция 4.18. Пусть $\subset \mathbb{R}^n$ - произвольная, проходящая через $P \in M$ двумерная аффинная плоскость, не лежащая в T_PM. Несложно показать, что она пересекает поверхность M около точки P по некоторой регулярной кривой γ.

Определение 4.19. Кривая γ называется плоским сечением, проходящим через P. Если плоскость Π проходит через нормаль N к поверхности M в точке P, то плоское сечение γ называется нормальным. Нам также будет удобно говорить, что сечение γ проведено в направлении вектора $\dot{\gamma}(t_0)$, где t - произвольный параметр на γ, такой, что $\gamma(t_0) = P$.

Следствие 4.20. Пусть $\xi \in T_PM$ - произвольный ненулевой вектор, γ - плоское сечение, проведенное через P в направлении $\xi, u\Pi-$ плоскость сечения. Фиксируем в плоскости Π некоторый единичны вектор m перпендикулярны ξ и фиксируем ориентацию этой плоскости с помощью базиса (ξ, m). Обозначим через k_o ориентированную кривизну плоского сечения γ в точке P. Тогда

$$k_o\cos\theta = \frac{\mathfrak{q}(\xi)}{\mathfrak{G}(\xi)}$$

где θ - угол между m и нормалью N к поверхности M в точке P.

Доказательство. Заметим, что знак выражения $k_o\cos\theta$ не зависит от выбора вектора m, поскольку при замене m на $-m$ меняется знак и у ориентированной кривизны, и у $\cos\theta$. Поэтому, если кривая γ бирегулярна в точке P, то достаточно выбрать $m = \nu$ и применить теорему 4.17. Если же кривая не является бирегулярной, то ориентированная кривизна равна нулю, а величина $\mathfrak{q}(\xi)$ равна нулю снова по теореме 4.17. Следствие доказано.

Следствие 4.21. Значение $\mathfrak{q}(\xi)$ второй фундаментальной формы поверхности M на единичном векторе $\xi \in T_PM$ равно ориентированной кривизне k_o плоского нормального сечения γ, проведенного через P в направлении ξ (плоскость сечения ориентирована с помощью базиса (ξ, N)). В частности, для обычной кривизны плоского нормального сечения выполнено равенство $k = |\mathfrak{q}(\xi)|$.

Определение 4.22. Ориентированной кривизной нормального сечения, проведенного через точку P в направлении $\xi \in T_PM$, (без явного указания ориентации плоскости сечения) будем называть его ориентированную кривизну, соответствующую ориентации плоскости сечения базисом (ξ, N).

Замечание 4.23. Для ориентаций плоскости сечения базисами ($\pm\xi, N$) ориентированные кривизны нормального сечения совпадают.

Конструкция 4.24. Фиксируем касательный вектор $\xi \in T_PM$ и проведем через него два плоских сечения, одно - нормальное, которое мы обозначим через γ_n, и еще одно, не

нормальное, которое обозначим через γ. На вопрос о том, как связаны кривизны этих сечений отвечает Теорема Менье.

Следствие 4.25 (Теорема Менье).

Пусть $\xi \in T_P M$ - ненулевой вектор, $\gamma u \gamma_n$ некоторое плоское и плоское нормальное сечения, проведенные через P в одном и том же направлении ξ, а kuk_n - их кривизны. Тогда или одновременно $k = 0uk_n = 0$, или $k_n = k\cos\alpha$, где α - угол между главными нормалями κ этим сечениям.

Доказательство. По следствию 4.21 из равенства нулю одной из кривизн равносильно равенству нулю величины $\mathfrak{q}(\xi)$, что, в свою очередь, равносильно равенству нулю второй кривизны снова по следствию 4.21.

Пусть теперь $k \neq 0$. Выберем в качестве нормали N к поверхности в точке P главную нормаль к плоскому нормальному сечению γ_n. По теореме 4.17

$$k\cos\alpha = \frac{\mathfrak{q}(\xi)}{\mathfrak{G}(\xi)} = k_n \cos 0 = k_n$$

что и требовалось доказать. □

Замечание 4.26. В дифференциальной геометрии поверхностей сложилась следующая терминология. Объекты и величины, определяемые первой фундаментальной формой, принято называть элементами внутренней геометрии. Напомним, что саму первую форму называют иногда внутренним скалярным произведением. К внутренней геометрии, например, принадлежат такие понятия, как длина кривых, углы между кривыми, площади областей. Однако, как отмечалось выше, первая фундаментальная форма определяет не всё. Например, в терминах первой фундаментальной формы нельзя вычислить кривизну кривой на поверхности. Объекты и величины, для определения которых недостаточно первой фундаментальной формы, называют элементами внешней геометрии поверхности, или геометрии погружения.

14.5.3 4.3 Principal curvatures and principal directions

Пусть, как и выше, P - произвольная точка регулярной гиперповерхности M в $\mathbb{R}^n$. Тогда, как уже известно, в точке P определены две квадратичные формы: $\mathfrak{G}$ и $\mathfrak{q}$. Следующая теорема, известная в линейной алгебре как теорема о паре квадратичных форм, позволяет выбрать удобный базис, в котором обе формы имеют простой вид.

Statement

4.27. Пусть в линейном пространстве $\mathbb{R}^n$, в котором фиксирован произвольный базис, задано скалярное произведение $\langle\cdot,\cdot\rangle$ с помощью симметричной невырожденной, положительно определенной матрицы G. Пусть, кроме того, задана квадратичная форма $\mathfrak{q}$. Тогда в $\mathbb{R}^n$ существует такой базис (e_i), ортонормальный относительно указанного скалярного произведения, что форма $\mathfrak{q}$ имеет в нем диагональный вид. При этом если Q - это матрица формы $\mathfrak{q}$ в исходном базисе, то базис (e_i) и соответствующие собственные числа формы Q относительно формы G могут быть найдены из следующего характеристического уравнения:

$$\det(Q - \lambda G) = 0.$$

Доказательство. Как известно из линейной алгебры (теорема о собственном базисе квадратичной формы), в пространстве $\mathbb{R}^n$ существует базис (f_i), ортонормированный относительно невырожденной, положительно определенной квадратичной формы,

задающей скалярное произведение. Пусть A - матрица перехода к этому базису, другими словами, компоненты X произвольного вектора в исходном базисе связаны с его компонентами X_f в базисе (f_i) так: $X_f = AX$. Тогда матрица G_f квадратичной формы $\mathfrak{G}$, задающей скалярное произведение, связана с матрицей G этой квадратичной формы

в исходном базисе: $A^T G_f A = G$. Аналогично, $A^T Q_f A = Q$. Отметим, что, так как базис (f_i) выбран ортонормированным относительно нашего скалярного произведения, матрица G_f - это единичная матрица, т.е. $A^T A = G$.

Теперь воспользуемся теоремой о приведении квадратичной формы $\mathfrak{q}$ в евклидовом пространстве $\mathbb{R}^n$ к главным осям. В силу этой теоремы форма $\mathfrak{q}$ имеет ортонормированный собственный базис (e_i). Отметим, что, так как базис (e_i) ортонормированный, матрица формы $\mathfrak{G}$ в нем по-прежнему единичная. Собственные числа и собственные векторы формы q могут быть найдены, как известно, при решении характеристического уравнения $\det(Q_f - \lambda E) = 0$. Умножив это уравнение на не равный нулю квадрат определителя матрицы перехода A и внеся матрицу A под знак определителя, получим

$$\det A^T \det(Q_f - \lambda E) \det A = \det\left(A^T Q_f A - \lambda A E A^T\right) = \det(Q - \lambda G).$$

Таким образом, характеристическое уравнение эквивалентно уравнению $\det(Q - \lambda G) = 0$, что и требовалось доказать.

Определение 4.28. Пусть G и Q - матрицы первой и второй фундаментальных форм регулярной гиперповерхности M в некоторой ее точке P. Корни уравнения $\det(Q - \lambda G) = 0$ называются главными кривизнами поверхности M в точке P. Если λ_0 - главная кривизна, то векторы из $T_P M$, соответствующие нетривиальным решениям линейного уравнения $(Q - \lambda_0 G)\, X = 0$, называются главными направлениями поверхности M в точке P.

Замечание 4.29. Если λ_0 - главная кривизна, то $\det(Q - \lambda_0 G) = 0$, поэтому уравнение $(Q - \lambda_0 G)\, X = 0$ имеет нетривиальные решения.

Замечание 4.30. Из утверждения 4.27 следует, что главные кривизны и главные направления поверхности не зависят от выбора координат на поверхности, и что из векторов главных направлений можно составить базис пространства $T_P M$.

Следствие 4.31. Пусть P - произвольная точка регулярной гиперповерхности M. Тогда в базисе из векторов главных направлений в касательном пространстве $T_P M$ первая и вторая фундаментальные формы поверхности будут иметь вид

$$ds^2 = \sum_i \left(du^i\right)^2, \quad dq^2 = \sum_i \lambda_i \left(du^i\right)^2$$

соответственно. При этом числа λ_i являются главными кривизнами поверхности M в точке P.

Следующий результат, получающийся из следствия 4.21, проясняет геометрический смысл главных кривизн.

Следствие 4.32. Главная кривизна λ_i поверхности M в точке P равна ориентированной кривизне нормального сечения поверхности в точке P вдоль соответствующего главного направления.

С помощью главных кривизн и главных направлений можно легко вычислить кривизну любого нормального сечения.

Конструкция 4.33. Фиксируем в касательном пространстве $T_P M$ к поверхности M в точке P произвольный касательный вектор V единичной длины. Выберем в касательном пространстве $T_P M$ ортонормированный базис, состоящий из векторов главных направлений. Тогда координаты вектора V в этом базисе могут быть записаны в виде

$$V = (\cos\varphi_1, \ldots, \cos\varphi_{n-1})$$

где φ_i - угол между вектором V и i-м базисным вектором. В силу следствий 4.21 и 4.31 ориентированная кривизна $k_n(V)$ нормального сечения поверхности в точке P в направлении V может быть вычислена следующим образом:

$$k_n(V) = dq^2(V, V) = \sum_i \lambda_i \left(\cos\varphi_i\right)^2$$

Особенно полезна эта формула в случае двумерной поверхности в $\mathbb{R}^3$.
Следствие 4.34 (Формула Эйлера). Пусть M^2 - двумерная регулярная поверхность в $\mathbb{R}^3$ и P - точка из M. Тогда ориентированная кривизна $k_n(V)$ нормального сечения в точке P в направлении $V \in T_P M$ может быть вычислена следующим образом:

$$k_n(V) = \lambda_1(\cos\varphi)^2 + \lambda_2(\sin\varphi)^2$$

где λ_1, λ_2 - главные кривизны поверхности в точке $M, a\varphi$ - угол между вектором V и главным направлением, соответствующим главной кривизне λ_1.

Следствие 4.35. Главные кривизны и главные направления двумерной регулярной поверхности в точке P соответствуют наибольшему и наименьшему значениям функции $\lambda_1(\cos\varphi)^2 + \lambda_2(\sin\varphi)^2$, значение которой равно ориентированной кривизне нормального сечения, проведенного в направлении вектора $(\cos\varphi, \sin\varphi)$.

Доказательство. Производная функции $k_n(\varphi) = \lambda_1(\cos\varphi)^2 + \lambda_2(\sin\varphi)^2$ по φ имеет вид

$$k_n'(\varphi) = 2\left(\lambda_2 - \lambda_1\right)\cos\varphi\sin\varphi = \left(\lambda_2 - \lambda_1\right)\sin 2\varphi$$

поэтому, если $\lambda_1 \neq \lambda_2$, то точки экстремума функции $k(\varphi)$ - это точки вида $\varphi = k\pi$ и $\varphi = \pi/2 + k\pi$, т.е. как раз главные направления. Если же $\lambda_1 = \lambda_2$, то все ориентированные кривизны нормальных сечений одинаковы. Доказательство закончено.

14.5.4 4.4 Mean and Gaussian curvatures

Как и в линейной алгебре, иногда бывает полезно исследовать инвариантные функции главных кривизн, а не сами главные кривизны.

Определение 4.36. Сумма главных кривизн поверхности M в точке P называется средней кривизной поверхности M в точке P и обозначается $H(P)$. Произведение главных кривизн называется гауссовой кривизной и обозначается $K(P)$.

Утверждение 4.37. Средняя и гауссова кривизны поверхности не зависят от выбора координат на поверхности. При этом, если G и Q - матрицы первой u второй квадратичных форм поверхности в точке P в некотором базисе, то средняя и гауссова кривизны могут быть вычислены следующим образом:

$$H(P) = \operatorname{trace}\left(G^{-1}Q\right), \quad K(P) = \det\left(G^{-1}Q\right) = \frac{\det Q}{\det G}$$

Доказательство. Первое утверждение очевидно, так как сами главные кривизны не зависят от выбора параметризации. Второе утверждение тоже очевидно, так как приведенные формулы справедливы в базисе, составленном из главных направлений, а в силу первого утверждения и инвариантности этих формул - и в любом другом базисе. □

Знак гауссовой кривизны двумерной поверхности имеет простой геометрический смысл.

Утверждение 4.38. Пусть P - произвольная точка двумерной регулярной поверхности $M \subset \mathbb{R}^3$. Если в точке P гауссова кривизна $K(P)$ поверхности M положительна, то достаточно малая окрестность точки P поверхности расположена целиком в одном из двух полупространств, определяемых касательной плоскостью $T_P M$, рассматриваемой как

аффинная плоскость, проходящая через точку P пространства $\mathbb{R}^3$. Если же $K(P) < 0$, то это не так, а именно, любая окрестность точки P пересекается с внутренностью обоих полупространств.

Доказательство. Действительно, пусть гауссова кривизна $K(P)$ положительна. В двумерном случае это равносильно тому, что обе главных кривизны поверхности M в точке P имеют один и тот же знак, т.е. их главные нормали лежат в одном и том же полупространстве по отношению к касательной плоскости T_PM. Но тогда из формулы Эйлера вытекает, что главная нормаль любого нормального сечения поверхности в точке P определена и расположена в том же полупространстве. Отсюда следует, что в том же полупространстве находятся и сами сечения (в некоторой достаточно малой окрестности). Если же $K(P) < 0$, то из сказанного выше очевидно, что нормальные сечения вдоль главных направлений лежат в разных полупространствах по отношению к T_PM. Доказательство закончено. □

Замечание 4.39. Утверждение 4.38 можно обобщить на многомерный случай, потребовав одинаковости знаков всех главных кривизн поверхности в точке P.

Определение 4.40. Точка двумерной поверхности, в которой $K > 0$, называется точкой выпуклости, или сферической точкой. Точка, в которой $K < 0$, называется седловой.

Геометрический смысл средней кривизны менее очевиден. Определенное представление о нем дает теорема Лапласа-Пуассона (см. Дополнительный материал).

Следующее важное свойство средней кривизны, объясняет отчасти ее название.
Утверждение 4.41. Пусть $P-$ произвольная точка гиперповерхности $M \subset \mathbb{R}^n uN$ - нормаль M в точке P. Пусть $v_1, \ldots, v_{n-1}$ - произвольный набор попарно

ортогональных единичных векторов из T_PM. Обозначим через k_i ориентированную кривизну нормального сечения поверхности M вдоль направления v_i. Тогда сумма чисел k_i равна средней кривизне поверхности M в точке P.

Доказательство. Это очевидно из следствия 4.31. Действительно, пусть $A = \left(a_i^j\right) -$ матрица, столбцы которой - суть координаты векторов v_i в ортонормированном базисе из главных направлений поверхности M в точке P. Тогда, по определению, матрица A ортогональна. Величина k_i равна

$$\lambda_1 \left(a_i^1\right)^2 + \cdots + \lambda_{n-1} \left(a_i^{n-1}\right)^2$$

поэтому, суммируя все k_i, меняя порядок суммирования и пользуясь тем свойством ортогональной матрицы, что скалярный квадрат любой ее строки равен единице, получаем

$$\sum_i k_i = \sum_i \left(\sum_\alpha \lambda_\alpha \left(a_i^\alpha\right)^2 \right) = \sum_\alpha \left(\lambda_\alpha \sum_i \left(a_i^\alpha\right)^2 \right) = \sum_\alpha \lambda_\alpha 1 = H(P),$$

что и требовалось доказать.
Пример 4.42. Пусть поверхность задана графиком функции $z = f(x, y)$. Тогда (см. выше вид матриц первой и второй фундаментальных форм) средняя и гауссова кривизны в соответствии с утверждением 4.37 имеют вид

$$H = \frac{\left(1 + f_y^2\right) f_{xx} - 2 f_x f_y f_{xy} + \left(1 + f_x^2\right) f_{yy}}{\left(1 + f_x^2 + f_y^2\right)^{3/2}}, \quad K = \frac{f_{xx} f_{yy} - f_{xy}^2}{\left(1 + f_x^2 + f_y^2\right)^2}$$

Пример 4.43. Пусть $M = S^2$ - стандартная сфера радиуса R. В качестве N выберем внутренние нормали к S^2, т.е. нормали, направленные к центру сферы. Так как каждое плоское нормальное сечение является окружностью радиуса R, кривизна каждого такого сечения равна $1/R$. Поэтому кривизны главных нормальных сечений также равны $1/R$, и, значит, средняя кривизна равна $2/R$, а гауссова $-1/R^2$.

Пример 4.44. Пусть $M = T^2$ - тор в $\mathbb{R}^3$, полученный вращением окружности радиуса r вокруг оси, находящейся в плоскости этой окружности на расстоянии $R > r$ от её центра. Пусть $P \in T^2$ - точка, расположенная ближе всего к оси вращения. Вычислим среднюю кривизну $H(P)$ тора в этой точке. Воспользуемся утверждением 4.41. Через точку P проходит два ортогональных плоских сечения - окружность радиуса r (образующая) и окружность радиуса $R - r$ (направляющая). Пусть нормаль N направлена наружу тора. Тогда $H(P) = 1/(R - r) - 1/r$

14.5.5 4.5 Bonn Theorem

Возникает вопрос: а всё ли "знают" про поверхность ее первая и вторая квадратичные формы? Может быть, следует определить еще несколько квадратичных форм, или достаточно уже определенных двух? Сформулируем задачу более формально.

Пусть Ω - область, в точках которой заданы две квадратичные формы g и q, гладко зависящие от точки. Если фиксированы координаты, то можно считать, что просто

заданы матрицы G и Q этих форм, гладко зависящие от координат. Предположим, что форма g невырождена и положительно определена. Можно ли задать на Ω параметрическую поверхность r так, чтобы ее первая квадратичная форма совпадала с g, а вторая - с q ?

Оказывается, в общем случае ответ - отрицательный. Дело в том, что первая и вторая формы не являются независимыми. Связь между ними выражается в так называемых условиях Петерсона-Кодацич [П. Мы поговорим о них в следующей лекции.] Бонне удалось показать, что этих условий достаточно для восстановления поверхности.

Theorem

4.45 (Бонне, 1867). Пусть на Ω заданы положительно определенная квадратичная форма g и произвольная квадратичная форма q, удовлетворяющие условиям Петерсона-Кодац,ии. Тогда существует единственная (с точностью до движения объемлющего пространства) регулярная поверхность на Ω, такая, что ее первая квадратичная форма совпадает cg, а вторая квадратичная форма $-cq$.

14.5.6 4.6 Problems

Задача 4.46. Вычислить вторую фундаментальную форму для поверхности, заданной неявной функцией $F(x, y, z)$. Вычислить среднюю и гауссову кривизны этой поверхности.

Задача 4.47. Вычислить среднюю и гауссову кривизны для гиперповерхности, заданной графиком $x^n = f\left(x^1, \ldots, x^{n-1}\right)$. Вычислить среднюю и гауссову кривизны для поверхности, заданной неявной функцией $F\left(x^1, \ldots, x^n\right)$.

Задача 4.48. Вычислить вторую фундаментальную форму, а также среднюю и гауссову кривизны следующих поверхностей вращения:

- $r = (R\cos u\cos v, R\cos u\sin v, R\sin u)$(с дера);
- $r = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)(mop)$;
- $r = \left(a\,\mathrm{ch}\,\frac{u}{a}\cos v, a\,\mathrm{ch}\,\frac{u}{a}\sin v, u\right)$ (катеноид).

[01] Впервые часть этих условий была найдена Гауссом. Полная система уравнений была независимо найдена Петерсоном (1853), Майнарди (1856) и Кодацци (1867).

Задача 4.49. Вычислить среднюю и гауссову кривизны поверхности, полученной вращением графика функции $x = f(z) > 0$ вокруг оси z.

Задача 4.50. Пусть S - некоторая данная поверхность. Отложим на нормалях к поверхности S в одном направлении отрезки постоянной длины. Концы отложенных отрезков описывают поверхность S^*, параллельную поверхности S. Если уравнение поверхности S есть $r = r(u, v)$, то уравнение S^* имеет вид

$$\rho = r(u, v) + an(u, v)$$

где $n(u, v)$ - единичный вектор нормали к S.
Выразить первую и вторую фундаментальные формы поверхности S^* через первую и вторую фундаментальные формы поверхности S.

Выразить гауссову кривизну K^* и среднюю кривизну H^* поверхности S^*, параллельной поверхности S, через гауссову и среднюю кривизны поверхности S.

14.5.7 4.7 Supplemental material

14.5.8 4.7.1 Other proof of claim 4.7

Итак, пусть (u^i) и (v^i) - две системы координат на регулярной поверхности и N - фиксированное поле нормалей на ней. Определим, как связаны наборы чисел $q_{ij} = \langle r_{u^i u^j}, N\rangle$ и $\tilde{q}_{\alpha\beta} = \langle r_{v^\alpha u^\beta}, N\rangle$. Изучим сначала, как связаны векторы вторых частных производных от r. Так как

$$\frac{\partial r}{\partial u^i} = \sum_\alpha \frac{\partial r}{\partial v^\alpha}\frac{\partial v^\alpha}{\partial u^i},$$

то

$$\frac{\partial^2 r}{\partial u^i \partial u^j} = \sum_\alpha \frac{\partial}{\partial u^j}\left(\frac{\partial r}{\partial v^\alpha}\frac{\partial v^\alpha}{\partial u^i}\right) = \sum_\alpha \left(\sum_\beta \frac{\partial^2 r}{\partial v^\alpha \partial v^\beta}\frac{\partial v^\beta}{\partial u^j}\frac{\partial v^\alpha}{\partial u^i} + \frac{\partial r}{\partial v^\alpha}\frac{\partial^2 v^\alpha}{\partial u^i \partial u^j}\right).$$

Умножив теперь левую и правую части предыдущего равенства скалярно на вектор нормали N и заметив, что вектор $\partial r/\partial v^\alpha$ - это просто вектор $r_{v^\alpha} = \partial_{v^\alpha}$ из канонического репера и поэтому перпендикулярен нормали N, получим:

$$\left\langle \frac{\partial^2 r}{\partial u^i \partial u^j}, N\right\rangle = \sum_{\alpha\beta}\left\langle \frac{\partial^2 r}{\partial v^\alpha \partial v^\beta}, N\right\rangle \frac{\partial v^\beta}{\partial u^j}\frac{\partial v^\alpha}{\partial u^i}, \qquad \text{т.е.} \qquad q_{ij} = \sum_{\alpha\beta}\tilde{q}_{\alpha\beta}\frac{\partial v^\beta}{\partial u^j}\frac{\partial v^\alpha}{\partial u^i}$$

Обозначим матрицу (q_{ij}) через $Q(u)$, матрицу ($\tilde{q}_{\alpha\beta}$) через $\tilde{Q}(v)$, а матрицу Якоби ($\partial v^\alpha/\partial u^i$) через J. Тогда последнее соотношение принимает вид

$$Q(u) = J^T \tilde{Q}(v(u)) J$$

14.5.9 4.7.2 Second cylinder shape and Meunier's theorem

Рассмотрим прямой круговой цилиндр в трехмерном евклидовом пространстве $\mathbb{R}^3$, заданный параметрически в виде

$$x^1\left(u^1, u^2\right) = R\cos u^1, \quad x^2\left(u^1, u^2\right) = R\sin u^1, \quad x^3\left(u^1, u^2\right) = u^2.$$

Обозначим через N поле нормалей к цилиндру, направленное наружу. Рассмотрим плоскость Π, пересекающую вертикальную координатную ось Ox^3 под углом α. Эта

плоскость пересекает цилиндр по некоторому эллипсу. Найдем кривизну этого эллипса в его вершине P, наиболее удаленной от центра эллипса. Пусть V - касательный вектор к эллипсу в точке P и Π_N - двумерная плоскость, проходящая через векторы $N(P)$ и V. Эта плоскость, очевидно, пересекает цилиндр по окружности радиуса R (этим и хороша выбранная нами точка эллипса). Из следствия 4.25 получаем, что кривизна $k(P)$ эллипса в точке P может быть вычислена по формуле $k(P)\cos(\pi/2-\alpha) = 1/R$, т.е. $k(P) = 1/(R\sin\alpha)$. В частности, если $\alpha = \pi/2$ (плоскость Π совпадает с Π_N), то $k(P) = 1/R$, а если $\alpha = 0$ (плоскость Π пересекает цилиндр по прямой, параллельной оси Ox^3), то $k(P) = \infty$.

Тот же результат, впрочем, можно получить и непосредственно из теоремы 4.17. Для этого нам придется вычислить обе квадратичные формы цилиндра, что и сделаем. Векторы канонического базиса цилиндра имеют вид

$$\partial_{u^1} = \left(-R\sin u^1, R\cos u^1, 0\right), \quad \partial_{u^2} = (0,0,1).$$

Представим вектор нормали N в виде нормированного векторного произведения:

$$N = [\partial_{u^1}, \partial_{u^2}] \,/\, \|[\partial_{u^1}, \partial_{u^2}]\| \,.$$

Получим: $N = (\cos u^1, \sin u^1, 0)$. Далее, первая квадратичная форма цилиндра в координатах (u^1, u^2) имеет вид

$$ds^2 = R^2\left(du^1\right)^2 + \left(du^2\right)^2$$

Чтобы найти вторую квадратичную форму цилиндра, вычислим вторые производные параметризующего отображения r. Получим:

$$\frac{\partial^2 r}{\partial u^1 \partial u^1} = \left(-R\cos u^1, -R\sin u^1, 0\right), \quad \frac{\partial^2 r}{\partial u^1 \partial u^2} = 0, \quad \frac{\partial^2 r}{\partial u^2 \partial u^2} = 0$$

Вторая квадратичная форма цилиндра будет иметь вид

$$dq^2 = -R\left(du^1\right)^2$$

В координатах (u^1, u^2) вектор V принимает вид $V = (1/R, 0)$. Вектор главной нормали $n(P)$ к эллипсу направлен внутрь цилиндра, вектор $N(P)$ нормали к поверхности - наружу цилиндра, поэтому угол между векторами $n(P)$ и $N(P)$ равен $\pi/2+\alpha$. Итак, из теоремы 4.17 получаем:

$$k(P)\cos(\pi/2+\alpha) = \frac{-R(1/R)^2}{1} = -1/R$$

поэтому $k(P) = 1/(R\sin\alpha)$, как и выше.

14.5.10 4.7.3 Main Cylinder Curvatures

Рассмотрим снова прямой круговой цилиндр

$$x^1\left(u^1, u^2\right) = R\cos u^1, \quad x^2\left(u^1, u^2\right) = R\sin u^1, \quad x^3\left(u^1, u^2\right) = u^2.$$

Выше были вычислены его первая и вторая квадратичные формы в этой параметризации. В матричном виде ответ можно записать так:

$$G = \begin{pmatrix} R^2 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} -R & 0 \\ 0 & 0 \end{pmatrix}.$$

Характеристическое уравнение имеет вид

$$\det\begin{pmatrix} -R-\lambda R^2 & 0 \\ 0 & -\lambda \end{pmatrix} = R\lambda(1+R\lambda) = 0,$$

и главные кривизны цилиндра равны 0 и $-1/R$. Главные направления находятся из линейных уравнений

$$\begin{pmatrix} -R & 0 \\ 0 & 0 \end{pmatrix} X = 0, \quad \begin{pmatrix} 0 & 0 \\ 0 & 1/R \end{pmatrix} X = 0,$$

и соответственно равны (0,1) и (1,0). Другими словами, главные направления кругового цилиндра - это направления его параллелей и меридианов.

В этом случае формула Эйлера для кривизны нормального сечения имеет вид

$$k(\varphi) = \frac{\cos^2\varphi}{R}.$$

График этой функции на интересующем нас отрезке $[0, 2\pi]$ изображен на рис. 4.1.

Рис. 4.1: Кривизны нормальных сечений кругового цилиндра.

Отметим, что, зная формулу Эйлера и принимая во внимание взаимную ортогональность главных направлений, можно определить главные кривизны и главные направления цилиндра, не вычисляя его фундаментальные формы. Действительно, рассмотрим всевозможные нормальные сечения цилиндра, проходящие через фиксированную точку P. Все эти сечения, кроме одного особого, представляют собой эллипсы, особое же сечение - это прямолинейная образующая цилиндра. Фиксируем внутреннюю нормаль к цилиндру. Векторы ускорений всех сечений-эллипсов направлены внутрь цилиндра, поэтому в формуле Эйлера следует выбрать знак "плюс" для произвольного направления $V = (\cos\varphi, \sin\varphi) : k_n(\varphi) = \lambda_1(\cos\varphi)^2 + \lambda_2(\sin\varphi)^2$. Далее, кривизна прямой равна нулю, поэтому на этом сечении достигается наименьшее возможное значение функции $k_n(\varphi)$. Следовательно, направление меридиана - главное, и соответствующая главная кривизна равна 0 . Второе главное направление ортогонально первому, значит, это направление - направление параллели-окружности. Поэтому вторая главная кривизна равна кривизне окружности, т.е. $1/R$.

14.5.11 4.7.4 Laplace-Poisson theorem

Понятие средней кривизны двумерной поверхности было впервые определено Томасом Юнгом (1805), а затем независимо Пьером Симоном Лапласом (1806). Лаплас

изучал форму поверхности раздела двух физических сред, находящихся в равновесии, и выяснил, что средняя кривизна поверхности раздела пропорциональна разности давлений в разделяемых средах. Этот результат в механике сплошной среды называется теоремой Лапласа. Коэффициент пропорциональности называется коэффициентом поверхностного натяжения.

Приведем здесь схему рассуждений Лапласа. Рассмотрим малый участок поверхности раздела двух физических сред, в одной из которых давление равно p_1, а в другой $-p_2$ (в окрестности поверхности). Представим себе малое возмущение поверхности, при котором каждая её точка движется вдоль нормального поля N, направленного, скажем, в сторону второй среды. При этом, если поверхность смещается на δN, то объем пространства между исходной и возмущенной поверхностями равен $\delta N dS$, где dS - элемент площади исходной поверхности. Полная работа δA при таком возмущении складывается из работы по изменению объема, которая есть $(\ p_2 - p_1\ )$ $\delta N dS$, и работы поверхностных сил, которая, в свою очередь, есть $\sigma \delta S$. Здесь через δS обозначено изменение площади поверхности, а через σ - коэффициент поверхностного натяжения. Условие термодинамического равновесия записывается так:

$$\delta A = (p_2 - p_1)\, \delta N dS + \sigma \delta S = 0$$

Вычислим площадь возмущенной поверхности. Рассмотрим пару главных нормальных сечений, проведенных через фиксированную точку P, и пусть $r_i\,(t_i)$ - их натуральные параметризации. Элемент dS площади невозмущенной поверхности можно записать в виде $dt_1 dt_2$. Рассмотрим возмущения главных сечений в их плоскостях вдоль вектора $N(P)$ нормали к поверхности в этой точке. Эти возмущения имеют вид $r_i + \varepsilon N$. Однако, по определению, вектор нормали N и вектор главной нормали n_i к кривой r_i коллинеарны, поэтому возмущения имеют вид $r_i + \varepsilon s_i n_i$, где $s_i = \langle n_i, N \rangle$. Из плоских формул Френе следует, что элемент длины возмущенной кривой r_i имеет вид $(1 + \varepsilon s_i k_i)\, dt_i$, что в силу следствий 4.25 и 4.32 можно переписать в виде $(\ 1 + \varepsilon \lambda_i\ )\ dt_i$. Здесь через λ_i обозначена главная кривизна, соответствующая i-му главному направлению. Наконец, элемент площади возмущенной поверхности равен произведению элементов длин возмущенных кривых γ_i (эти кривые ортогональны), поэтому приращение δS площади равно

$$(1 + \varepsilon \lambda_1)\,(1 + \varepsilon \lambda_2)\, dt_1 dt_2 - dS = ((\lambda_1 + \lambda_2)\, \varepsilon + o(\varepsilon))\, dS$$

Подставляя полученный результат в условие термодинамического равновесия, заменяя в нем ε на δN и пренебрегая членами порядка $o(\varepsilon)$, получаем

$$((p_2 - p_1) + \sigma\,(\lambda_1 + \lambda_2))\, dS \delta N = 0$$

Отсюда окончательно получаем уравнение Лапласа: $\sigma(P) H(P) = p_1 - p_2$.
Итак, доказан следующий результат.

Theorem

4.51 (Лаплас-Пуассон). Средняя кривизна поверхности раздела двух равновесньх физических сред пропорциональна разности давлений в этих средах вблизи поверхности раздела.

14.5.12 4.7.5 Mean and Gaussian curvatures of a two-dimensional surface

Рассмотрим двумерную поверхность, заданную параметрически. Пусть (u, v) - координаты на этой поверхности, и метрика ds^2 и вторая форма $d\sigma^2$ имеют вид

$$ds^2 = Edu^2 + 2Fdudv + Gdv^2, \quad d\sigma^2 = Ldu^2 + 2Mdudv + Ndv^2$$

Тогда

$$H = \frac{GL - 2FM + EN}{EG - F^2}, \quad K = \frac{LN - M^2}{EG - F^2}.$$

Отметим, что такие буквенные обозначения для компонент первой и второй фундаментальных форм являются традиционными в классической теории поверхностей.

14.5.13 4.7.6 On the choice of coordinates on the surface

Часто бывает полезно выбрать такое представление поверхности (т.е. такие координаты на поверхности), для которого первая и/или вторая фундаментальная формы имеют наиболее простой вид. Как известно, при замене параметризации матрицы первой и второй форм меняются, как матрицы квадратичных форм, причем в качестве матрицы замены надо взять соответствующую матрицу Якоби.

Пусть M - поверхность, заданная параметрически с помощью отображения $r : \Omega \to \mathbb{R}^n$, и $(u^1, \ldots, u^{n-1})$ - соответствующие координаты на этой поверхности. Пусть $P \in M$. Выберем произвольную невырожденную

Рассмотрим двумерную поверхность M в $\mathbb{R}^3$, заданную параметрически в виде $x = r(u, v)$, где $x = (\ x^1, x^2, x^3\ )$ - стандартные координаты в $\mathbb{R}^3$. Предположим, что координаты $(\ u, v\ )$ являются конформными (по теореме 4.54 такие координаты всегда можно выбрать), т.е. индуцированная метрика ds^2 в координатах (u, v) имеет вид $ds^2 = \lambda(u, v)\,(du^2 + dv^2)$, где $\lambda(u, v)$ - некоторая положительная функция.

Theorem

4.56. Пусть двумерная поверхность M в $\mathbb{R}^3$ задана параметрически в виде $r(u, v)$, причем координаты (u, v) - конформны. Пусть $N = [r_u, r_v]\,/\,\|[r_u, r_v]\|$ - единичная нормаль к поверхности M. Тогда

$$\Delta r = \frac{\partial^2 r}{\partial u^2} + \frac{\partial^2 r}{\partial v^2} = r_{uu} + r_{vv} = \lambda H N$$

где H - средняя кривизна поверхности M, вычисленная по отношению к нормали N. В частности, лапласиан Δr радиус-вектора r перпендикулярен поверхности M.

Доказательство. Покажем сначала, что $\Delta r \perp M$. Действительно,

$$\langle \Delta r, r_u \rangle = \langle r_{uu} + r_{vv}, r_u \rangle = \langle r_{uu}, r_u \rangle + \langle r_{vv}, r_u \rangle =$$
$$= \frac{1}{2}\langle r_u, r_u \rangle_u + \langle r_v, r_u \rangle_v - \langle r_v, r_{uv} \rangle = \frac{1}{2}\lambda_u + 0 - \frac{1}{2}\langle r_v, r_v \rangle_u = \frac{1}{2}\lambda_u - \frac{1}{2}\lambda_u = 0$$

Аналогично показывается, что $\langle \Delta r, r_v \rangle = 0$.
Вычислим теперь $\langle \Delta r, N \rangle$:

$$\langle \Delta r, N \rangle = \langle r_{uu}, N \rangle + \langle r_{vv}, N \rangle = \operatorname{tr} Q.$$

Так как $G^{-1}Q = \frac{1}{\lambda}Q$, то $H = \operatorname{tr}(G^{-1}Q) = \operatorname{tr} Q/\lambda$, откуда

$$\langle \Delta r, N\rangle = \lambda H \quad \Rightarrow \quad \Delta r = \lambda H N$$

что и требовалось доказать.
Радиус-вектор $r(u,v)$ назовем гармоническим, если

$$\Delta r = \frac{\partial^2 r}{\partial u^2} + \frac{\partial^2 r}{\partial v^2} = 0$$

Приведем важное следствие предыдущей теоремы.
Следствие 4.57. Двумерная поверхность M в $\mathbb{R}^3$, заданная параметрически в виде $r(u,v)$, где (u,v) конформные координаты, является минимальной, если u только если радиус-вектор $r(u,v)$ - гармонический.

Если поверхность задана графиком функции $z = f(x,y)$, то условие ее минимальности, в силу проделанных ранее явных вычислений средней кривизны, записывается в виде

$$\left(1+f_y^2\right) f_{xx} - 2 f_x f_y f_{xy} + \left(1+f_x^2\right) f_{yy} = 0$$

Это уравнение, найденное Лагранжем, называется уравнением минимальных поверхностей. Интересно отметить, что, хотя для малых областей плоскости XY существует много разных решений этого уравнения, решений, определенных на всей плоскости, не так уж много. Одним из очевидных решений является произвольная линейная функция. Оказывается, что других таких решений нет. Приведем без доказательства следующую теорему, принадлежащую Бернштейну.

Theorem

4.58 (Бернштейн). Если $f(x,y)$ - решение уравнения минимальных поверхностей, определенное на всей плоскости XY, то функция f - линейна.

Приведем еще некоторые примеры минимальных поверхностей.
Пример 4.59. Опишем все минимальные поверхности вращения, т.е. поверхности вращения нулевой средней кривизны. Поверхность вращения, напомним, получается при вращении графика положительной гладкой функции f вокруг оси изменения параметра.

Зададим поверхность вращения M как отображение $r : U \to \mathbb{R}^3$:

$$x^1(\varphi, z) = f(z)\cos\varphi, \quad x^2(\varphi, z) = f(z)\sin\varphi, \quad x^3(\varphi, z) = z.$$

Векторы канонического базиса в точке (φ, z) имеют вид

$$\partial_\varphi = (-f(z)\sin\varphi, f(z)\cos\varphi, 0), \quad \partial_z = (f'(z)\cos\varphi, f'(z)\sin\varphi, 1)$$

поэтому первая квадратичная форма поверхности вращения равна

$$(f(z))^2 (d\varphi)^2 + \left(1 + (f'(z))^2\right)(dz)^2.$$

Зададим поле нормалей на поверхности M как нормированное векторное произведение элементов канонического базиса:

$$N(\varphi, z) = \frac{[\partial_\varphi, \partial_z]}{\|[\partial_\varphi, \partial_z]\|} = \frac{1}{\sqrt{1 + (f'(z))^2}} \left(\cos\varphi, \sin\varphi, -f'(z)\right).$$

Отметим, что это внешняя нормаль. Вычисляя вторую квадратичную форму поверхности M, получим:

$$dq^2 = \frac{-f(z)(d\varphi)^2 + f''(z)(dz)^2}{\sqrt{1 + (f'(z))^2}}.$$

Рис. 4.2: Катеноид (слева) и геликоид (справа) - поверхности нулевой средней кривизны.

Найдем теперь среднюю кривизну поверхности вращения, воспользовавшись для этого утверждением 4.37:

$$H(P) = \operatorname{trace}\left(G^{-1}Q\right) = \frac{f''(z)(f(z))^2 - f(z)\left(1 + (f'(z))^2\right)}{\sqrt{1 + (f'(z))^2}}.$$

Таким образом, условие равенства нулю средней кривизны поверхности вращения можно записать в виде

$$f''(z)f(z) - (f'(z))^2 - 1 = 0.$$

Решая полученное дифференциальное уравнение, находим: $f(z) = a\,\mathrm{ch}(z/a) + b$. Это так называемые *ка* теноидь - поверхности, полученные при вращении цепной линии. Их можно наблюдать, опустив в мыльную воду проволочный контур, состоящий из двух окружностей. Один из катеноидов ($a = 1, b = 0$) показан на рис. 4.2 .

Еще одним примером минимальных поверхностей является геликоид, представляющий собой поверхность, образованную прямой линией, которая равномерно движется вдоль некоторой оси и равномерно вращается вокруг нее (см. рис. 4.2). При специальном выборе координат в $\mathbb{R}^3$ геликоид можно записать в виде

$$(x = u\cos v, y = u\sin v, z = av),$$

где a - некоторая ненулевая постоянная (проверьте, что геликоид является минимальной поверхностью).
Напомним, что линейчатой поверхностью называется поверхность, порождаемая прямой линией при произвольном ее движении. Оказывается, что имеет место следующий результат (докажите).

Theorem

4.60 (Каталан). Линейчатая поверхность минимальна, если и только если она является или плоскостью, или геликоидом (см. рис. 4.2).

14.5.14 4.7.9 Rotation surfaces of constant Gaussian curvature

Некоторые поверхности постоянной гауссовой кривизны нам уже известны - сфера ($K = 1/R^2$) и плоскость ($K = 0$). В данном пункте построим ряд более сложных поверхностей постоянной гауссовой кривизны, которые можно найти среди поверхностей вращения.

Пользуясь приведенными выше выражениями для первой и второй квадратичных форм поверхности вращения графика функции f, заданной параметрически, найдем ее гауссову кривизну как отношение определителей форм:

$$K = \frac{-f''(z)}{f(z)\left(1 + f'(z)^2\right)^2}.$$

Решения этого дифференциального уравнения порождают поверхности вращения постоянной гауссовой кривизны, примеры которых приведены на рис. 4.3 и 4.4 .

Рис. 4.3: Поверхности вращения постоянной отрицательной кривизны.

Рис. 4.4: Поверхности вращения постоянной положительной кривизны.

Существуют и гораздо более сложные поверхности постоянной гауссовой кривизны. На рис. 4.5 слева представлена поверхность Дини, ее кривизна отрицательна, а справа - поверхность постоянной положительной гауссовой кривизны.

Рис. 4.5: Поверхности постоянной гауссовой кривизны.

Рис. 4.5: Поверхности постоянной гауссовой кривизны.

14.5.15 4.7.10 Case of arbitrary codimensionality

Выше в данной лекции мы ограничивались рассмотрением гиперповерхностей. Здесь вкратце обсудим общий случай. Итак, пусть M - произвольная k-мерная поверхность в $\mathbb{R}^n$. Рассмотрим произвольную точку $P \in M$. Пусть T_PM - касательное пространство к поверхности в этой точке, а $(T_PM)^\perp$ - ортогональное дополнение к T_PM. Для произвольного вектора $X \in \mathbb{R}^n$ обозначим через X^T и X^N его ортогональные проекции на подпространства T_PM и $(T_PM)^\perp$ соответственно.

Пусть $\gamma(t)$ - произвольная кривая, лежащая на M, такая, что $\gamma(0) = P$, и ξ - вектор скорости $\dot\gamma(0)$ кривой γ в точке P. Обозначим через $\mathfrak{q}(\xi)$ вектор $(\ddot\gamma)^N$.

Пусть $r\left(u^1, \ldots, u^k\right)$ - параметрическое задание поверхности M, и кривая γ задается в координатах $\left(u^1, \ldots, u^k\right) : u^i = u^i(t), i = 1, \ldots, k$. Тогда $\xi = \left(\dot u^1, \ldots, \dot u^k\right)$ и

$$\ddot\gamma = \left(\sum_i r_{u^i}\dot u^i\right)^{\cdot} = \sum_{ij} r_{u^iu^j}\dot u^i\dot u^j + \sum_i r_{u^i}\ddot u^i, \quad \text{откуда} \quad \mathfrak{q}(\xi) = \sum_{ij}\left(r_{u^iu^j}\right)^N \dot u^i\dot u^j,$$

поскольку $\left(r_u{}^i\right)^N = 0$.

Таким образом, величина $\mathfrak{q}(\xi)$ зависит не от конкретного вида кривой γ, а лишь от компонентов вектора скорости ξ и векторов $q_{ij}(P) = \left(r_{u^iu_j}(P)\right)^N(P)$.

Пусть $\partial_{u^i} = \partial_{u^i}(P)$ - векторы канонического базиса. Фиксируем в ортогональном дополнении $(T_PM)^\perp$ к касательному пространству T_PM ортонормированный базис $N_1, \ldots, N_{n-k}$, такой, что векторы

$$\partial_u 1, \ldots, \partial_{uk}, N_1, \ldots, N_{n-k}$$

образуют положительно ориентированный базис в $\mathbb{R}^n$. Легко проверить, что для каждой достаточно близкой к P точки Q векторы $N_i(Q)$ можно выбрать так, чтобы их

координаты гладко зависели от этой точки. Тогда $q_{ij}(P) = \sum_\alpha \langle r_{u^i u^j}, N_\alpha \rangle N_\alpha$.

Векторы $q_{ij}(P)$ задают билинейную форму на линейном пространстве $T_P M$ со значениями в $(T_P M)^N$ (т.е. билинейную функцию, значения которой - суть векторы из $(T_P M)^N$). Она называется второй $\oint yrda$ ментальной формой поверхности. Многие конструкции, обсуждавшиеся выше, могут быть обобщены на общий случай. В частности, след этой билинейной формы называется вектором средней кривизны поверхности. Поверхности нулевой средней кривизны (называемые, как и в двумерном случае, минимальными) также обладают рядом важных и интересных свойств, что делает их предметом интенсивного изучения.

14.6 5 Elements of differential calculus on surfaces

Материал данной лекции обычно не входит в стандартный курс классической дифференциальной геометрии, а излагается во втором семестре (в существенно более общем случае - случае римановых многообразий). Однако, по нашему мнению, знакомство с ним не только углубит понимание классической теории поверхностей, но, что более важно, облегчит восприятие более сложных конструкций тензорного анализа.

14.6.1 5.1 Weingarten-Gauss derivational formulas

В практических задачах часто приходится рассматривать семейства векторов, зависящих от точки поверхности. Например, это могут быть векторы скоростей частиц жидкости, текущей по поверхности, векторы направления магнитного поля, и т.п.

Определение 5.1. Гладкое отображение $X : M \to \mathbb{R}^n$ регулярной поверхности M в объемлющее пространство $\mathbb{R}^n$ называется (гладким) векторным полем на M. Если при этом $X(P) \in T_P M$ для всех $P \in M$, то векторное поле X называется касательным векторным полем.

Пример 5.2. Поле нормалей на гиперповерхности является гладким векторным полем в смысле предыдущего определения.

Ниже в этом разделе мы ограничимся случаем гиперповерхностей.

Конструкция 5.3. Пусть на регулярной гиперповерхности M с полем нормалей N задано векторное поле X. Если на M фиксированы координаты $u^1, \dots, u^{n-1}$, то в каждой точке P вектор $X(P)$ можно разложить по базису, полученному из канонического базиса (∂_{u^i}) добавлением вектора нормали:

$$X = X^1 \partial_{u^1} + \dots + X^{n-1} \partial_{u^{n-1}} + X^n N.$$

скалярно на N, найдем, что $q_{ij} = \langle r_{ij}, N \rangle$, т.е. матрица q_{ij} - это не что иное, как матрица второй квадратичной формы поверхности. Точно так же, умножая равенство

$$N_j = \sum_\alpha b_j^\alpha r_\alpha + c_j N, \qquad (**)$$

скалярно на N, находим, что $\langle N, N_j \rangle = c_j$. С другой стороны, векторное поле N по модулю равно единице, поэтому, $\langle N, N \rangle = 1$ и, дифференцируя это последнее равенство по u^j заключаем, что $\langle N, N_j \rangle = 0$. Таким образом, $c_j = 0$.

Найдем теперь коэффициенты Γ_{ij}^α. Прежде всего, заметим, что так как r_{ij} и q_{ij} симметричны по i и j, то из равенства ($*$) следует, что то же самое верно и для любых Γ_{ij}^k, т.е. $\Gamma_{ij}^k = \Gamma_{ji}^k$. Далее, умножим равенство (*) скалярно на вектор r_k. Получим

$$\langle r_{ij}, r_k \rangle = \sum_\alpha \Gamma_{ij}^\alpha g_{\alpha k}$$

где $g_{\alpha k} = \langle r_\alpha, r_k \rangle$ - матрица первой квадратичной формы поверхности. В то же время

$$\frac{\partial g_{ij}}{\partial u^k} = \frac{\partial \langle r_i, r_j \rangle}{\partial u^k} = \langle r_{ik}, r_j \rangle + \langle r_i, r_{jk} \rangle = \sum_\alpha \Gamma^\alpha_{ik} g_{\alpha j} + \sum_\alpha \Gamma^\alpha_{jk} g_{\alpha i}.$$

Последнее равенство справедливо для любого набора индексов i, j и k. Поэтому, для фиксированной тройки индексов i, j и k можно записать систему из трех таких уравнений, добавив еще два, получающихся циклической перестановкой индексов. В итоге получим систему следующего вида:

$$\begin{aligned}
\frac{\partial g_{ij}}{\partial u^k} &= \sum_\alpha \Gamma^\alpha_{ik} g_{\alpha j} + \sum_\alpha \Gamma^\alpha_{jk} g_{\alpha i}, \\
\frac{\partial g_{jk}}{\partial u^i} &= \sum_\alpha \Gamma^\alpha_{ji} g_{\alpha k} + \sum_\alpha \Gamma^\alpha_{ki} g_{\alpha j}, \\
\frac{\partial g_{ki}}{\partial u^j} &= \sum_\alpha \Gamma^\alpha_{kj} g_{\alpha i} + \sum_\alpha \Gamma^\alpha_{ij} g_{\alpha k}.
\end{aligned}$$

Теперь вычтем первое уравнение из суммы двух последних. Воспользовавшись симметрией Γ^k_{ij}, получим

$$\frac{\partial g_{jk}}{\partial u^i} + \frac{\partial g_{ki}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^k} = 2 \sum_\alpha \Gamma^\alpha_{ij} g_{\alpha k}$$

Фиксируем произвольные i и j и выпишем такие уравнения для всех $k = 1, \ldots, n-1$. Мы получим систему линейных уравнений на неизвестные величины Γ^k_{ij}, матрица которой - это матрица $g_{\alpha k}$ первой квадратичной формы. Так как эта матрица невырождена, полученная система всегда имеет единственное решение, которое легко получить, умножая нашу систему на матрицу, обратную к матрице (g_{ij}). Эту матрицу в дифференциальной геометрии обычно обозначают через g^{ij}. Тогда ответ записывается в виде

$$\Gamma^k_{ij} = \frac{1}{2} \sum_\alpha \left(g^{k\alpha} \left(\frac{\partial g_{j\alpha}}{\partial u^i} + \frac{\partial g_{\alpha i}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\alpha} \right) \right).$$

Нам осталось найти коэффициенты b^α_j. Умножив равенство (**) скалярно на r_k, получим

$$\langle N_i, r_k \rangle = \sum_j b^j_i \langle r_j, r_k \rangle = \sum_j b^j_i g_{jk},$$

Чтобы вычислить скалярное произведение в левой части последнего равенства, заметим, что $\langle N, r_k \rangle = 0$, поэтому

$$0 = \frac{\partial \langle N, r_k \rangle}{\partial u^i} = \langle N_i, r_k \rangle + \langle N, r_{ki} \rangle.$$

Однако $\langle N, r_{ki} \rangle = q_{ki}$, где q_{ki} - элемент второй квадратичной формы поверхности. Поэтому получаем следующие линейные уравнения на неизвестные b^j_i (индекс i фиксирован):

$$-q_{ki} = \sum_j b^j_i g_{jk}, \quad k = 1, \ldots, n-1.$$

Матрица этой линейной системы совпадает с (невырожденной) матрицей первой квадратичной формы поверхности, поэтому система имеет единственное решение, которое записывается через матрицу (g^{ij}), обратную матрице (g_{ij}) так:

$$b_i^j = -\sum_k q_{ik} g^{kj}$$

Доказательство утверждения закончено.

Соглашение 5.9. Матрица, обратная к матрице первой фундаментальной формы поверхности в некоторых фиксированных координатах, обозначается той же буквой, но с верхними индексами. Например, если g_{ij} - матрица первой фундаментальной формы, то g^{ij} - обратная к ней.

Упражнение 5.10. Пусть g_{ij} и h_{ij} - матрицы первой фундаментальной формы регулярной поверхности в разных системах координат. Вывести формулы, связывающие компоненты матриц g^{ij} и h^{ij}.

Определение 5.11. Набор величин

$$\Gamma_{ij}^k = \frac{1}{2} \sum_\alpha \left(g^{k\alpha} \left(\frac{\partial g_{j\alpha}}{\partial u^i} + \frac{\partial g_{\alpha i}}{\partial u^j} - \frac{\partial g_{ij}}{\partial u^\alpha} \right) \right)$$

называется символами Кристоффеля поверхности M в точке P в системе координат ($u^1, \ldots, u^{n-1}$).

Собирая вместе формулы, полученные в ходе доказательства утверждения 5.7, получаем следующий результат.

Утверждение 5.12. Для вторых производных радиус-вектора неособой поверхности M имеют место следующие деривационные формулы Гаусса-Вейнгартена:

$$\frac{\partial}{\partial u^j} \partial_{u^i} = \sum_\alpha \Gamma_{ij}^\alpha \partial_{u^\alpha} + q_{ij} N,$$

где q_{ij} - матрица второй квадратичной формы поверхности, а Γ_{ij}^k - символы Кристоффеля поверхности M.

Производные нормального векторного поля N на поверхности имеют вид

$$\frac{\partial N}{\partial u^i} = \sum_\alpha b_i^\alpha \partial_{u^\alpha}$$

где коэффициенты b_i^α выражаются через первую u вторую квадратичные формы поверхности так:

$$b_i^\alpha = -\sum_k q_{ik} g^{k\alpha}$$

Упражнение 5.13. Выяснить, как преобразуются символы Кристоффеля при замене параметризации поверхности.

Упражнение 5.14. Вычислить символы Кристоффеля для плоскости, цилиндра, сферы, двумерной поверхности вращения.

Замечание 5.15. Коэффициенты деривационных формул связаны соотношениями, полученными при вычислении двумя разными способами третьих смешанных производных радиус-вектора и вторых смешанных производных вектора нормали. Часть из них была впервые записана Гауссом, а остальные - Петерсоном, Майнарди и Кодацци. Эти дифференциальные уравнения в частных производных на компоненты квадратичных форм поверхности принято называть уравнениями Петерсона-Кодацци. Именно они образуют набор необходимых и достаточных условий разрешимости системы, к которой сводится задача о восстановлении поверхности по первой и второй квадратичным формам. Однако эта тема выходит за рамки нашего курса. Мы выведем лишь уравнения Гаусса для двумерной поверхности в $\mathbb{R}^3$. Из этих уравнений Гаусс получил свою знаменитую Theorema Egregium - "Блистательную теорему".

14.6.2 5.2 Gaussian Theorem

Theorem

5.16 (C. F. Gauss). Гауссова кривизна двумерной поверхности не меняется при изометрии. Другими словами, двумерные изометричные поверхности имеют в соответствующих точках одинаковую гауссову кривизну.

Доказательство. Идея доказательства состоит в том, чтобы получить из деривационных формул уравнения Гаусса, приравняв друг к другу соответствующие коэффициенты в выражениях для третьих производных радиус-вектора поверхности. Уравнения Гаусса позволяют выразить гауссову кривизну поверхности через элементы ее первой квадратичной формы и их производные, откуда и следует утверждение теоремы. Действительно, напомним, что на изометричных поверхностях можно выбрать такие координаты, что их первые квадратичные формы совпадут, т.е. совпадут элементы соответствующих матриц. Поэтому в таких координатах равны друг другу и соответствующие производные компонент первых квадратичных форм. Таким образом, в этих координатах совпадают все функции, входящие в выражения для гауссовой кривизны, а значит и сами гауссовы кривизны.

Итак, пусть M - регулярная гиперповерхность, заданная параметрически в виде $r : \Omega \to \mathbb{R}^n$, и пусть $u^1, \ldots, u^{n-1}$ - координаты на M. Как и выше, будем обозначать нижним индексом i производную вектора по координате u^i. В этих обозначениях, напомним, вектор канонического репера ∂_{u^i} - это вектор r_i. Тогда, сохраняя обозначения предыдущего пункта и пользуясь утверждением 5.12, найдем, что

$$\begin{aligned} r_{iij} = (r_{ii})_j &= \left(\sum_k \Gamma_{ii}^k r_k + q_{ii} N\right)_j = \\ &= \sum_k \frac{\partial \Gamma_{ii}^k}{\partial u^j} r_k + \sum_k \Gamma_{ii}^k \left(\sum_\alpha \Gamma_{kj}^\alpha r_\alpha + q_{kj} N\right) + \frac{\partial q_{ii}}{\partial u^j} N + q_{ii} \sum_\alpha b_j^\alpha r_\alpha = \\ &= \sum_\beta \left(\frac{\partial \Gamma_{ii}^\beta}{\partial u^j} + \sum_k \Gamma_{ii}^k \Gamma_{kj}^\beta + q_{ii} b_j^\beta\right) r_\beta + \left(\sum_k \Gamma_{ii}^k q_{kj} + \frac{\partial q_{ii}}{\partial u^j}\right) N. \end{aligned}$$

В то же время аналогично

$$\begin{aligned} r_{iji} = (r_{ij})_i &= \\ &= \sum_\beta \left(\frac{\partial \Gamma_{ij}^\beta}{\partial u^i} + \sum_k \Gamma_{ij}^k \Gamma_{ki}^\beta + q_{ij} b_i^\beta\right) r_\beta + \left(\sum_k \Gamma_{ij}^k q_{ki} + \frac{\partial q_{ij}}{\partial u^i}\right) N. \end{aligned}$$

Очевидно, что $r_{iij} = r_{iji}$. Приравняв коэффициенты при r_β и воспользовавшись выражением для b_i^β, получим для любого набора индексов i, j и β следующее равенство:

$$\begin{aligned} \frac{\partial \Gamma_{ii}^\beta}{\partial u^j} - \frac{\partial \Gamma_{ij}^\beta}{\partial u^i} + \sum_k \left(\Gamma_{ii}^k \Gamma_{kj}^\beta - \Gamma_{ij}^k \Gamma_{ki}^\beta\right) &= q_{ij} b_i^\beta - q_{ii} b_j^\beta = \\ &= \sum_\alpha \left(q_{ii} q_{j\alpha} - q_{ij} q_{i\alpha}\right) g^{\alpha\beta} \end{aligned}$$

Умножим последнее равенство на $g_{\beta j}$ и просуммируем по всем β. Поскольку $\sum_\beta g^{\alpha\beta} g_{\beta j} = \delta_\alpha^j$ - символ Кронекера, получим в итоге уравнения Гаусса:

$$\sum_\beta g_{\beta j} \left\{\frac{\partial \Gamma_{ii}^\beta}{\partial u^j} - \frac{\partial \Gamma_{ij}^\beta}{\partial u^i} + \sum_k \left(\Gamma_{ii}^k \Gamma_{kj}^\beta - \Gamma_{ij}^k \Gamma_{ki}^\beta\right)\right\} = q_{ii} q_{jj} - q_{ij} q_{ij}.$$

Заметим, что до этого момента мы нигде не пользовались тем, что поверхность M двумерна. Теперь запишем уравнение Гаусса для двумерной поверхности при $i = 1$ и $j = 2$:

$$\sum_{\beta} g_{\beta 2} \left\{ \frac{\partial \Gamma_{11}^{\beta}}{\partial u^2} - \frac{\partial \Gamma_{12}^{\beta}}{\partial u^1} + \sum_{k} \left(\Gamma_{11}^{k} \Gamma_{k2}^{\beta} - \Gamma_{12}^{k} \Gamma_{k1}^{\beta} \right) \right\} = \det Q.$$

Здесь Q - матрица второй квадратичной формы. Однако в силу утверждения 4.37 гауссова кривизна K равна отношению определителей первой и второй квадратичных форм. Из этого факта и уравнения Гаусса следует, что гауссова кривизна выражается только через элементы первой квадратичной формы. Доказательство закончено.

Приведем пример того, как используется теорема 5.16.

Следствие 5.17. Никакую окрестность точки на стандартной двумерной сфере невозможно изометрично отобразить на плоскость.

Доказательство. Действительно, в противном случае их гауссовы кривизны совпадали бы. Однако гауссова кривизна сферы радиуса R равна $1/R^2$, а гауссова кривизна плоскости равна нулю. Доказательство закончено.

Упражнение 5.18. Выведите все уравнения Петерсона-Кодацци.

14.6.3 5.3 Covariant derivative of tangent vector Field

Воспользуемся деривационными формулами, чтобы вычислить производную произвольного касательного векторного поля.

Конструкция 5.19. Пусть X - касательное векторное поле на регулярной поверхности M, на которой фиксированы координаты u^i. Как и выше, представим векторное поле X в виде линейной комбинации координатных векторных полей: $X = \sum_{\alpha} X^{\alpha} \partial_{u^{\alpha}}$. Тогда, дифференцируя каждое слагаемое по формуле Лейбница и применяя утверждение 5.12, получим

$$\begin{aligned} \frac{\partial X}{\partial u^i} &= \sum_{\alpha} \frac{\partial X^{\alpha}}{\partial u^i} \partial_{u^{\alpha}} + \sum_{\beta} X^{\beta} \frac{\partial}{\partial u^i} \partial_{u^{\beta}} = \\ &= \sum_{\alpha} \frac{\partial X^{\alpha}}{\partial u^i} \partial_{u^{\alpha}} + \sum_{\beta} \left(X^{\beta} \left(\sum_{\alpha} \Gamma_{i\beta}^{\alpha} \partial_{u^{\alpha}} + q_{i\beta} N \right) \right) = \\ &= \sum_{\alpha} \left(\frac{\partial X^{\alpha}}{\partial u^i} + \sum_{\beta} \Gamma_{i\beta}^{\alpha} X^{\beta} \right) \partial_{u^{\alpha}} + \left(\sum_{\beta} q_{i\beta} X^{\beta} \right) N. \end{aligned}$$

Замечание 5.20. Производная касательного векторного поля X по координате u^i имеет, вообще говоря, нормальную составляющую и поэтому не является касательным векторным полем. Полезно определить на касательных векторных полях другое дифференцирование, такое, чтобы его результат был бы касательным векторным полем.

Определение 5.21. Ковариантной производной $\nabla_i X$ касательного векторного поля X по координате u^i называется касательное векторное поле, полученное в каждой точке из вектора $\partial X / \partial u^i$ ортогональным проецированием на касательную плоскость:

$$\nabla_i X = \left(\frac{\partial X}{\partial u^i} \right)^T,$$

где через $(\cdot)^T$ обозначена операция ортогонального проецирования на касательную плоскость.

Непосредственно из определения получаем следующее выражение для координат векторного поля $\nabla_i X$.

Утверждение 5.22. Пусть в окрестности точки P регулярной гиперповерхности M задано касательное векторное поле X, имеющее в координатах ($u^1, \ldots, u^{n-1}$) вид $X = \sum_\alpha X^\alpha \partial_{u^\alpha}$. Тогда координаты касательного векторного поля $\nabla_i X$ относительно ($u^1, \ldots, u^{n-1}$) могут быть вычислены следующим образом:

$$(\nabla_i X)^j = \frac{\partial X^j}{\partial u^i} + \sum_\alpha \Gamma^j_{i\alpha} X^\alpha,$$

где Γ^k_{ij} - символьк Кристоффеля поверхности M.

Замечание 5.23. Операция ковариантного дифференцирования полностью определяется первой фундаментальной формой поверхности.

Упражнение 5.24. Проверить следующие свойства операции ковариантного дифференцирования:

- Линейность: $\nabla_i(\alpha X + \beta Y) = \alpha \nabla_i X + \beta \nabla_i Y$, где X и Y - произвольные касательные векторные поля, α и β - произвольные числа.
- Правило Лейбница: $\nabla_i(fX) = (\partial f/\partial u^i)\, X + f \nabla_i X$, где X - произвольное касательное векторное поле, f - произвольная гладкая функция на поверхности.

Замечание 5.25. Символы Кристоффеля Γ^k_{ij} имеют следующий прозрачный геометрический смысл: это коэффициенты разложения по векторам канонического базиса ковариантной производной j-го координатного поля по i-й координате:

$$\nabla_i \partial_{u^j} = \sum_\alpha \Gamma^\alpha_{ij} \partial_{u^\alpha}$$

Замечание 5.26. Удобно распространить операцию ковариантного дифференцирования на функции на поверхности, положив, по определению, $\nabla_i f = \partial f/\partial u^i$ для произвольной функции i.

Утверждение 5.27. Если X и Y - произвольные касательные векторные поля на поверхности, то

$$\nabla_i \langle X, Y \rangle = \langle \nabla_i X, Y \rangle + \langle X, \nabla_i Y \rangle .$$

Замечание 5.28. Иногда этот факт формулируют так: ковариантное дифференцирование сохраняет первую фундаментальную форму поверхности, или метрика поверхности ковариантно постоянна. Точный смысл этих выражений станет ясен в следующем семестре.

Доказательство. Воспользуемся определением ковариантной производной:

$$\frac{\partial X}{\partial u^i} = \left(\frac{\partial X}{\partial u^i}\right)^T + \left(\frac{\partial X}{\partial u^i}\right)^N = \nabla_i X + \left(\frac{\partial X}{\partial u^i}\right)^N,$$

где через $(\cdot)^N$ обозначена ортогональная проекция на нормаль к поверхности. Для любого касательного поля Y

$$\left\langle \frac{\partial X}{\partial u^i}, Y \right\rangle = \langle \nabla_i X, Y \rangle ,$$

так как нормальный вектор $(Z)^N$ перпендикулярен касательному вектору Y для любого исходного вектора Z. Итак,

$$\nabla_i \langle X, Y \rangle = \frac{\partial \langle X, Y \rangle}{\partial u^i} = \left\langle \frac{\partial X}{\partial u^i}, Y \right\rangle + \left\langle X, \frac{\partial Y}{\partial u^i} \right\rangle = \langle \nabla_i X, Y \rangle + \langle X, \nabla_i Y \rangle ,$$

что и требовалось доказать. □

Замечание 5.29. Утверждение 5.27 может быть доказано также прямым подсчетом. Эта довольно громоздкая выкладка приведена в Дополнительном материале.

Как и в случае обычного дифференцирования, определим ковариантную производную по направлению произвольного касательного векторного поля.

Определение 5.30. Ковариантной производной поля Y вдоль поля X называется касательное векторное поле

$$\nabla_X Y = \sum_\alpha X^\alpha \nabla_\alpha Y.$$

Утверждение 5.31. Пусть X и Y - касательные векторные поля на регулярной гиперповерхности M с координатами $u^1, \ldots, u^{n-1}$. Тогда $\nabla_X Y$ - касательное векторное поле, компоненты которого имеют вид

$$(\nabla_X Y)^i = \sum_\alpha \frac{\partial Y^i}{\partial u^\alpha} X^\alpha + \sum_{k\alpha} \Gamma^i_{\alpha k} X^\alpha Y^k$$

Доказательство. В силу определений и утверждения 5.22 имеем:

$$\nabla_X Y = \sum_\alpha X^\alpha \nabla_\alpha Y = \sum_\alpha \left(X^\alpha \sum_\beta (\nabla_\alpha Y)^\beta r_\beta \right) =$$
$$= \sum_{\alpha,\beta} X^\alpha \left(\frac{\partial Y^\beta}{\partial u^\alpha} + \sum_k \Gamma^\beta_{\alpha k} Y^k \right) r_\beta =$$
$$= \sum_\beta \left(\sum_\alpha \frac{\partial Y^\beta}{\partial u^\alpha} X^\alpha + \sum_{k\alpha} \Gamma^\beta_{\alpha k} X^\alpha Y^k \right) r_\beta,$$

т.е. мы записали $\nabla_X Y$ в виде линейной комбинации векторов канонического базиса, что и требовалось. □

Из утверждений 5.27 и 5.31 немедленно получается следующий результат.

Следствие 5.32. Пусть на поверхности M задано три касательных векторных поля X_1, X_2 и Y. Тогда

$$\nabla_Y \langle X_1, X_2 \rangle = \langle \nabla_Y X_1, X_2 \rangle + \langle X_1, \nabla_Y X_2 \rangle$$

Конструкция 5.33. Пусть X - векторное поле на поверхности M и γ - регулярная кривая на M, параметризованная параметром t. Рассмотрим ограничение $X(t)$ векторного поля X на кривую γ. Обозначим через $\nabla_{\dot\gamma} X$ ковариантную производную поля X вдоль поля скоростей $\dot\gamma$ кривой γ. Получим векторное поле, определенное в точках кривой $\gamma(t)$.

Определение 5.34. Векторное поле $\nabla_{\dot\gamma} X$, заданное в точках кривой $\gamma(t)$ называется производной векторного поля X вдоль кривой γ.

Замечание 5.35. Геометрический смысл ковариантного дифференцирования вдоль кривой ясен: мы дифференцируем поле $X(t)$ по t в точке $\gamma(t)$ и результат дифференцирования ортогонально проецируем на $T_{\gamma(t)}M$.

Следующее утверждение проверяется прямым подсчетом.

Утверждение 5.36. Если $u^1, \ldots, u^{n-1}$ - регулярные координаты на M и $u^i(t)$ координатные функции кривой γ, то компоненты поля $\nabla_{\dot\gamma} X$ имеют вид

$$(\nabla_{\dot\gamma} X)^k = \frac{d}{dt} X^k + \sum_{ip} \Gamma^k_{ip} \dot u^i X^p$$

В частности, поле $\nabla_{\dot\gamma} X$ полностью определяется значениями поля X в точках кривой γ, т.е. ограничением $X|_\gamma$ поля X на эту кривую.

Замечание 5.37. Из утверждения 5.36 вытекает, что можно корректно определить производную вдоль кривой γ от векторного поля, заданного только в точках этой кривой, например, от поля скоростей кривой.

14.6.4 5.4 Supplemental material

14.6.5 5.4.1 On multidimensional generalizations of the Gaussian theorem

Theorem

Гаусса имеет ряд нетривиальных обобщений на случаи поверхностей произвольной размерности. Например, оказывается, аналогично можно доказать, что все четные элементарные симметричные многочлены от главных кривизн, т.е. выражения вида

$$\sum_{i_1 < \cdots < i_{2p}} k_{i_1} \cdots k_{2p}$$

где $2 \le 2p \le n$, определяются первой квадратичной формой гиперповерхности в $\mathbb{R}^n$.

14.6.6 5.4.2 Other proof of claim 5.27

Пусть $(u^1, \ldots, u^{n-1})$ - координаты на поверхности. В силу утверждения 5.22 имеем

$$\langle \nabla_i X, Y \rangle = \sum_{\alpha\beta} g_{\alpha\beta} \left(\frac{\partial X^\alpha}{\partial u^i} Y^\beta + \sum_\gamma \Gamma^\alpha_{i\gamma} X^\gamma Y^\beta \right)$$

Точно так же

$$\langle X, \nabla_i Y \rangle = \sum_{\alpha\beta} g_{\alpha\beta} \left(X^\alpha \frac{\partial Y^\beta}{\partial u^i} + X^\alpha \sum_\gamma \Gamma^\beta_{i\gamma} Y^\gamma \right).$$

Складывая эти два равенства и пользуясь определением символов Кристоффеля, получаем

$$\begin{aligned}
\langle \nabla_i X, Y \rangle + \langle X, \nabla_i Y \rangle = \sum_{\alpha\beta} g_{\alpha\beta} \left(\frac{\partial X^\alpha}{\partial u^i} Y^\beta + X^\alpha \frac{\partial Y^\beta}{\partial u^i} \right) + \\
+ \frac{1}{2} \sum_{\alpha\beta} \left\{ g_{\alpha\beta} \sum_\gamma \left(\sum_k g^{\alpha k} \left(\frac{\partial g_{ik}}{\partial u^\gamma} + \frac{\partial g_{\gamma k}}{\partial u^i} - \frac{\partial g_{i\gamma}}{\partial u^k} \right) \right) X^\gamma Y^\beta + \right. \\
\left. + \sum_\gamma \left(\sum_k g^{\beta k} \left(\frac{\partial g_{ik}}{\partial u^\gamma} + \frac{\partial g_{\gamma k}}{\partial u^i} - \frac{\partial g_{i\gamma}}{\partial u^k} \right) \right) X^\alpha Y^\gamma \right\}
\end{aligned}$$

Теперь заметим, что после преобразований типа $\sum_\alpha g_{k\alpha} g^{\alpha l} = \delta^l_k$ в выражениях для символов Кристоффеля в последней формуле четыре из шести слагаемых сокращаются, а два оставшихся сворачиваются. В итоге имеем:

$$\langle \nabla_i X, Y\rangle + \langle X, \nabla_i Y\rangle = \sum_{\alpha\beta} g_{\alpha\beta}\left(\frac{\partial X^\alpha}{\partial u^i}Y^\beta + X^\alpha\frac{\partial Y^\beta}{\partial u^i}\right) + \sum_{\alpha\beta}\frac{\partial g_{\alpha\beta}}{\partial u^i}X^\alpha Y^\beta$$

Последнее выражение совпадает с ковариантной производной скалярного произведения векторных полей X и Y :

$$\nabla_i\langle X, Y\rangle = \frac{\partial}{\partial u^i}\left(\sum_{\alpha\beta} g_{\alpha\beta}X^\alpha Y^\beta\right) =$$
$$= \sum_{\alpha\beta}\left(\frac{\partial g_{\alpha\beta}}{\partial u^i}X^\alpha Y^\beta + g_{\alpha\beta}\frac{\partial X^\alpha}{\partial u^i}Y^\beta + g_{\alpha\beta}X^\alpha\frac{\partial Y^\beta}{\partial u^i}\right)$$

Доказательство закончено.

14.7 6 Geodetic on surfaces

В данной лекции разработанная выше техника используется для определения важного понятия геодезической линии на поверхности. Обычно в курсах дифференциальной геометрии понятие геодезической вводится позднее - сразу для общего случая римановых многообразий. Нам, однако, кажется, что предварительное знакомство с геодезическими на обычных поверхностях облегчает понимание сути дела.

14.7.1 6.1 Definition and simplest properties of geodetic

Воспользуемся теперь разработанной выше техникой для вычисления вектора ускорения произвольной гладкой кривой на поверхности.

Конструкция 6.1. Пусть M - регулярная гиперповерхность, заданная параметрически в виде $r : \Omega \to \mathbb{R}^n$, и пусть γ - гладкая кривая на M с параметром s. Фиксируем на M какие-нибудь координаты ($u^1, \ldots, u^{n-1}$), и пусть $u^i(s), i = 1, \ldots, n$, - координатные функции кривой γ. Обозначим через $x^k\left(u^1, \ldots, u^{n-1}\right)$ координатные функции поверхности M. Как и выше, будем обозначать дифференцирование по u^i нижним индексом i, а по параметру s - точкой. Тогда вектор скорости $\dot\gamma$ кривой γ как элемент касательного пространства к поверхности имеет вид

$$\dot\gamma = \sum_\alpha r_\alpha \dot u^\alpha.$$

Продифференцируем еще раз. Получим следующее выражение для вектора ускорения $\ddot\gamma$:

$$\frac{d}{ds}\dot\gamma = \sum_\alpha \frac{d}{ds}\left(r_\alpha \dot u^\alpha\right) = \sum_\alpha\left(\sum_\beta r_{\alpha\beta}\dot u^\alpha \dot u^\beta + r_\alpha \ddot u^\alpha\right).$$

Перепишем последнее выражение, воспользовавшись деривационными формулами (см. утверждение 5.12), и затем переменим порядок суммирования. Получим:

$$\ddot\gamma = \sum_\alpha\left(\sum_\beta\left(\sum_k \Gamma^k_{\alpha\beta} r_k + q_{\alpha\beta}N\right)\dot u^\alpha \dot u^\beta + r_\alpha \ddot u^\alpha\right) =$$
$$= \sum_k\left(\sum_{\alpha\beta}\Gamma^k_{\alpha\beta}\dot u^\alpha \dot u^\beta + \ddot u^k\right) r_k + \left(\sum_{\alpha\beta} q_{\alpha\beta}\dot u^\alpha \dot u^\beta\right) N.$$

Выполним ряд формальных преобразований. Представив производную по параметру s в виде

$$\frac{d}{ds} = \sum_{\alpha} \dot{u}^{\alpha} \frac{\partial}{\partial u^{\alpha}}$$

можно переписать вторую производную $\ddot{u}^k$ в виде

$$\ddot{u}^k = \sum_{\alpha} \dot{u}^{\alpha} \frac{\partial \dot{u}^k}{\partial u^{\alpha}}.$$

Тогда в силу утверждения 5.31 выражение для вектора ускорения можно представить как

$$\ddot{\gamma} = \nabla_{\dot{\gamma}} \dot{\gamma} + \mathfrak{q}(\dot{\gamma}, \dot{\gamma}) N,$$

где через $\mathfrak{q}$ обозначена вторая квадратичная форма поверхности.

Замечание 6.2. Выражение $\nabla_{\dot{\gamma}} \dot{\gamma}$ в данном контексте имеет лишь следующий формальный смысл: это касательный вектор к поверхности, определенный в точках кривой, компоненты которого в каноническом базисе $\{r_k\}$ имеют вид

$$\sum_{\alpha\beta} \Gamma^k_{\alpha\beta} \dot{u}^{\alpha} \dot{u}^{\beta} + \ddot{u}^k$$

Отметим, что так как векторное поле $\dot{\gamma}$ задано лишь в точках кривой γ, операция $\nabla_i \dot{\gamma}$ не определена. На самом деле нетрудно показать, что поле $\dot{\gamma}$ можно продолжить на некоторую окрестность кривой γ в M и что результат ковариантного дифференцирования вдоль $\dot{\gamma}$ в точках кривой не зависит от такого продолжения. (В дальнейшем мы проделаем это при существенно более общих предположениях.)

Итак, доказано следующее утверждение.

Утверждение 6.3. Вектор ускорения кривой γ на гиперповерхности в каждой точке раскладывается в сумму касательного вектора $\nabla_{\dot{\gamma}} \dot{\gamma}$ и нормального вектора $\mathfrak{q}(\dot{\gamma}, \dot{\gamma}) N$.

Замечание 6.4. Предположим, что кривая γ соответствует движению некоторой материальной точки по поверхности M. Тогда её ускорение $\ddot{\gamma}$ пропорционально сумме сил, действующих на точку (второй закон Ньютона). Причем одна из этих сил присутствует всегда - это так называемая сила реакции поверхности, направленная по нормали к ней. Если предположить, что никакие другие силы на точку не действуют, т.е. точка движется по поверхности свободно, то ее вектор ускорения должен быть коллинеарен вектору нормали. Кривые, для которых это так, обладают многими замечательными свойствами и выделяются в отдельный класс.

Утверждение 6.21. Если $F : M_1 \to M_2$ - изометрия регулярных поверхностей, то F переводит геодезические в геодезические. Более формально, образ геодезической $\gamma_1 \subset M_1$ при изометрии F - это геодезическая на M_2, и, обратно, кривая γ_2 - геодезическая на M_2, если и только если она является F-образом некоторой геодезической на M_1.

Доказательство. Коэффициенты в уравнениях геодезических - это символы Кристоффеля, которые, напомним, вычисляются в терминах первой квадратичной формы поверхности, поэтому сохраняются при изометриях. Утверждение доказано.

14.7.2 6.2 Examples, Claireaux's theorem

В данном разделе приводится несколько простейших примеров геодезических, а также обсуждаются свойства геодезических на поверхностях вращения.

Утверждение 6.22. Геодезические на стандартной евклидовой плоскости - это всевозможные прямые линии.

Доказательство. Пусть $\mathbb{R}^2$ - стандартная евклидова плоскость с декартовыми координатами (x, y). Напомним, что метрика на $\mathbb{R}^2$ имеет вид $ds^2 = dx^2 + dy^2$. Поэтому все символы Кристоффеля тождественно равны нулю, и уравнения геодезических принимают вид

$$\ddot{x} = 0, \quad \ddot{y} = 0.$$

Их решения - это всевозможные прямые на плоскости, и только они.
Упражнение 6.23. Описать геодезические на следующих поверхностях:

- прямом цилиндре в $\mathbb{R}^3$ над плоской кривой;
- круговом конусе, заданном в $\mathbb{R}^3$ уравнением $x^2 + y^2 - z^2 = 0$.

Конструкция 6.24. Пусть в евклидовом пространстве $\mathbb{R}^3$ задана поверхности вращения, параметризация которой имеет выглядит так:

$$x(\varphi, z) = f(z)\cos\varphi, \quad y(\varphi, z) = f(z)\sin\varphi, \quad z(\varphi, z) = z.$$

Здесь $f(z)$ - строго положительная гладкая функция. Поскольку канонический базис имеет вид

$$\partial_\varphi = (-f(z)\sin\varphi, f(z)\cos\varphi, 0), \quad \partial_z = (f'(z)\cos\varphi, f'(z)\sin\varphi, 1)\,,$$

первая квадратичная форма поверхности вращения записывается как

$$ds^2 = (f(z))^2 d\varphi^2 + \left(1 + (f'(z))^2\right) dz^2.$$

Вектор нормали к поверхности вращения найдем как нормированное векторное произведение векторов канонического репера. Получим:

$$N = \frac{1}{1 + f'(z)^2}\left(\cos\varphi, \sin\varphi, -f'(z)\right).$$

Утверждение 6.25. Пусть $M \subset \mathbb{R}^3$ - поверхность вращения графика положительной гладкой функц $f(z)$. Тогда каждый меридиан на M является геодезической. Среди параллелей геодезическими являются те и только те, которые соответствуют критическим точкам функции f.

Доказательство. Начнем с параллелей (координатных линий $z = z_0$). Каждая из них представляет собой окружность, лежащую в плоскости, перпендикулярной оси Oz. Вектор ускорения натурально параметризованной плоской окружности в точке $P = (\varphi, z_0)$ коллинеарен вектору ($\cos\varphi, \sin\varphi, 0$). Этот вектор коллинеарен вектору $N(P)$ нормали к поверхности, если и только если $f'(z_0) = 0$. Итак, параллель $z = z_0$ является геодезической, если и только если функция $f(z)$ имеет в z_0 критическую точку.

Рассмотрим теперь меридиан (координатную линию $\varphi = \varphi_0$). Он тоже является плоской кривой, лежащей в плоскости, содержащей ось Oz и получающейся при повороте координатной плоскости Oxz вокруг Oz на угол φ_0. Пусть сначала $\varphi_0 = 0$. Тогда меридиан лежит в плоскости Oxz и задается в этой плоскости в виде $x - f(z) = 0$. Поэтому вектор нормали к меридиану в этой плоскости может быть записан, как $(1, -f'(z))$. Вернувшись обратно в $\mathbb{R}^3$, получим вектор $(1, 0, -f'(z))$. Меридиан с произвольным φ_0 получается из рассмотренного поворотом на угол φ_0 вокруг оси Oz.

Таким образом, вектор нормали к меридиану $\varphi = \varphi_0$ в точке $P = (\varphi_0, z)$ имеет вид $(\cos\varphi, \sin\varphi, -f'(z))$, т.е. коллинеарен вектору $N(P)$, поэтому любой меридиан является геодезической на поверхности вращения. □

Следствие 6.26. Геодезические на стандартной двумерной сфере - это большие круги, и только они.

Доказательство. То, что большие круги являются геодезическими, следует из утверждения 6.25. То, что других геодезических нет, вытекает из следствия 6.13. Доказательство закончено. □

Конструкция 6.27. Запишем уравнения геодезических на поверхности вращения. Для этого удобно перепараметризовать ее, выбрав на графике функции f натуральный параметр. Будем считать, что эта кривая задана, как плоская натурально параметризованная кривая в плоскости Oxz в виде $x = x(t), z = z(t)$, где $x(t) \neq 0$. Тогда поверхность вращения можно параметризовать как

$$x(\varphi, t) = x(t)\cos\varphi, \quad y(\varphi, t) = x(t)\sin\varphi, \quad z(\varphi, t) = z(t).$$

В этой параметризации канонический базис имеет вид

$$\partial_\varphi = (-x(t)\sin\varphi, x(t)\cos\varphi, 0), \quad \partial_t = (x'(t)\cos\varphi, x'(t)\sin\varphi, z'(t)),$$

поэтому первая квадратичная форма выглядит следующим образом:

$$ds^2 = x(t)^2 d\varphi^2 + \left(x'(t)^2 + z'(t)^2\right) dt^2 = x(t)^2 \left(d\varphi^2 + \frac{dt^2}{x(t)^2}\right)$$

Перейдем к новому параметру $\tau = \tau(t)$, такому, что $d\tau = dt/x(t)$. Окончательно первая квадратичная форма поверхности вращения в параметрах (φ, τ) имеет вид

$$ds^2 = \rho(\tau)^2 \left(d\varphi^2 + d\tau^2\right),$$

где $\rho(\tau) = x(t(\tau))$. Отметим, что геометрический смысл функции $\rho(\tau)$ по-прежнему очень прост - это расстояние от точки (φ, τ) поверхности до оси вращения.

Чтобы записать уравнения геодезических на поверхности вращения, нам нужна матрица G первой квадратичной формы и обратная к ней матрица G^{-1}, которые могут быть записаны, как

$$G = \begin{pmatrix} \rho(\tau)^2 & 0 \\ 0 & \rho(\tau)^2 \end{pmatrix}, \quad G^{-1} = \begin{pmatrix} 1/\rho(\tau)^2 & 0 \\ 0 & 1/\rho(\tau)^2 \end{pmatrix}.$$

Выпишем символы Кристоффеля для поверхности вращения. Поскольку матрица G^{-1} диагональна, в выражении для каждого символа Кристоффеля будет лишь одно слагаемое. Положив $u^1 = \varphi, u^2 = \tau$ и обратив внимание на то, что первая квадратичная форма не зависит от u^1, получаем:

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial u^1} + \frac{\partial g_{11}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^1}\right) = 0 \\
\Gamma^1_{12} &= \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^1}\right) = \frac{2\rho(\tau)\rho'(\tau)}{2\rho(\tau)^2} = \frac{\rho'(\tau)}{\rho(\tau)} = \Gamma^1_{21}, \\
\Gamma^1_{22} &= \frac{1}{2} g^{11} \left(\frac{\partial g_{21}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^1}\right) = 0.
\end{aligned}$$

Точно так же получаем

$$\Gamma_{11}^2 = \frac{1}{2} g^{22} \left(\frac{\partial g_{12}}{\partial u^1} + \frac{\partial g_{21}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2} \right) = \frac{-2\rho(\tau)\rho'(\tau)}{2\rho(\tau)^2} = \frac{-\rho'(\tau)}{\rho(\tau)},$$
$$\Gamma_{12}^2 = \frac{1}{2} g^{22} \left(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^1} \right) = 0 = \Gamma_{21}^2,$$
$$\Gamma_{22}^2 = \frac{1}{2} g^{22} \left(\frac{\partial g_{22}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^2} \right) = \frac{2\rho(\tau)\rho'(\tau)}{2\rho(\tau)^2} = \frac{\rho'(\tau)}{\rho(\tau)}.$$

В итоге уравнения геодезических имеют вид

$$\begin{cases} 0 = \ddot{\varphi} + 2\frac{\rho'(\tau)}{\rho(\tau)}\dot{\varphi}\dot{\tau} \\ 0 = \ddot{\tau} + \frac{\rho'(\tau)}{\rho(\tau)}\left(\dot{\tau}^2 - \dot{\varphi}^2\right) \end{cases}$$

Мы используем приведенные выше вычисления для доказательства теоремы Клеро о первом интеграле уравнении геодезической на поверхности вращения.

Theorem

6.28 (Клеро). Пусть $\gamma(s)$ - произвольная геодезическая на поверхности, полученной вращением плоской регулярной кривой вокруг прямой, лежащей в той же плоскости, причем ось вращения и кривая не пересекаются. Тогда произведение $c(s)$ расстояния $\rho(\tau)$ от оси вращения до точки $\gamma(s)$ на синус угла α между $\gamma(s)$ и соответствующим меридианом есть величина постоянная вдоль γ :

$$c(s) = \rho(\tau(s)) \sin \alpha(s) = \text{ const.}$$

Доказательство. Пусть $\gamma(s) = (\varphi(s), \tau(s))$ - решение уравнения геодезических. Вычислим угол α между геодезической и соответствующим меридианом (φ_0, τ), с которым геодезическая γ пересекается в точке $\varphi(s_0) = \varphi_0, \tau_0 = \tau(s_0)$. Касательный вектор к геодезической имеет вид $\dot{\gamma} = (\dot{\varphi}, \dot{\tau})$, а касательный вектор к меридиану - $(0, 1)$ (здесь точкой обозначена производная по параметру s, а штрихом - по параметру τ). Поэтому

$$\cos \alpha = \frac{\dot{\tau}\rho(\tau)^2}{\sqrt{\rho(\tau)^2\left(\dot{\varphi}^2 + \dot{\tau}^2\right)}\sqrt{\rho(\tau)^2}} = \frac{\dot{\tau}}{\sqrt{\dot{\varphi}^2 + \dot{\tau}^2}}$$

и соответственно

$$\sin \alpha = \frac{\dot{\varphi}}{\sqrt{\dot{\varphi}^2 + \dot{\tau}^2}}$$

(при этом мы учли, что $\rho(\tau) > 0$ по предположению).
Продифференцируем функцию $c(s) = \rho(\tau) \sin \alpha$ по параметру s. Получим:

$$\dot{c}(s) = \frac{d}{ds}\left(\frac{\rho(\tau)\dot{\varphi}}{\sqrt{\dot{\varphi}^2 + \dot{\tau}^2}}\right) = \frac{\left(\rho'\dot{\tau}\dot{\varphi} + \rho\ddot{\varphi}\right)\left(\dot{\varphi}^2 + \dot{\tau}^2\right) - \rho\dot{\varphi}(\dot{\varphi}\ddot{\varphi} + \dot{\tau}\ddot{\tau})}{\left(\dot{\varphi}^2 + \dot{\tau}^2\right)^{3/2}}.$$

Перепишем числитель последнего выражения в виде

$$\rho\dot{\tau}^2\left(\ddot{\varphi} + 2\frac{\rho'}{\rho}\dot{\varphi}\dot{\tau}\right) - \rho\dot{\varphi}\dot{\tau}\left(\ddot{\tau} + \frac{\rho'}{\rho}\left(\dot{\tau}^2 - \dot{\varphi}^2\right)\right).$$

Так как выражения в больших скобках равны нулю в силу уравнений геодезических, заключаем, что $c(s) = \text{const}$ вдоль произвольной геодезической, что и требовалось.

С помощью теоремы Клеро можно решить, например, следующую задачу.
Упражнение 6.29. Описать все поверхности вращения, на которых имеются замкнутые геодезические.

Каждая вариация Φ задает семейство кривых $\gamma_\tau(t) = \Phi(\tau, t)$, причем $\gamma_0(t) = \gamma(t)$. Кроме того, в каждой точке $\gamma(t)$ вариация Φ определяет векторное поле

$$E(t) = \frac{\partial \Phi(0,t)}{\partial \tau},$$

которое называется полем вариации. Поле вариации - это касательное векторное поле к другому семейству кривых, задаваемых отображением Φ, а именно, к кривым $\nu_t(\tau) = \Phi(\tau, t)$. Отметим, что в концевых точках поле вариации вовсе не совпадает с касательными векторами к концевым кривым. Более точно, имеет место следующая лемма (здесь и ниже точкой обозначается дифференцирование по t, а штрихом дифференцирование по τ).

Лемма 6.38. Векторы скоростей концевых кривых в начальный момент $\tau = 0$ удовлетворяют следующим соотношениям:

$$p'(0) = t_1'(0)\dot{\gamma}\left(t_1\right) + E\left(t_1\right), \qquad q'(0) = t_2'(0)\dot{\gamma}\left(t_2\right) + E\left(t_2\right).$$

Доказательство. Докажем, например, первое соотношение. По определению, имеем:

$$p'(\tau) = \frac{d\Phi\left(\tau, t_1(\tau)\right)}{d\tau} = \frac{\partial \Phi\left(\tau, t_1(\tau)\right)}{\partial \tau} + \frac{\partial \Phi\left(\tau, t_1(\tau)\right)}{\partial t}\frac{dt_1(\tau)}{d\tau}.$$

Подставив значение $\tau = 0$ и вспомнив, что $t_1(0) = t_1$, получим

$$p'(0) = \frac{\partial \Phi\left(0, t_1(0)\right)}{\partial \tau} + \frac{\partial \Phi\left(0, t_1(0)\right)}{\partial t} t_1'(0) = E\left(t_1\right) + \dot{\gamma}\left(t_1\right) t_1'(0).$$

Доказательство закончено.
Рассмотрим функцию $\ell(\tau)$, определенную на отрезке $[-\tau_0, \tau_0]$ и равную длине кривой γ_τ. Вычислим производную этой функции в $\tau = 0$. Дополнительно предположим для простоты, что кривая $\gamma(t)$ натурально параметризована. Запишем функцию $\ell(\tau)$ в виде

$$\ell(\tau) = \int_{t_1(\tau)}^{t_2(\tau)} \|\dot{\gamma}_\tau\| \, dt$$

и воспользуемся теоремой о дифференцировании интеграла, зависящего от параметра. Получим:

$$\frac{d\ell(\tau)}{d\tau} = t_2'(\tau)\left\|\dot{\gamma}_\tau\left(t_2(\tau)\right)\right\| - t_1'(\tau)\left\|\dot{\gamma}_\tau\left(t_1(\tau)\right)\right\| + \int_{t_1(\tau)}^{t_2(\tau)} \|\dot{\gamma}_\tau(t)\|' \, dt$$

Вычислим производную $\|\dot{\gamma}_\tau(t)\|'$. Для этого понадобится следующая лемма.
Лемма 6.39. При сделанных выше обозначениях имеет место следующее равенство:

$$\nabla_\tau \dot{\gamma}_\tau(t) = \nabla_t \nu_t'(\tau).$$

Доказательство. Пусть на поверхности фиксированы координаты ($u^1, \ldots, u^{n-1}$). Обозначим через $u^i(\tau, t)$ координатное представление вариации Φ. Тогда k-я компонента поля $\nabla_\tau \dot{\gamma}_\tau(t)$ имеет вид

$$\sum_\alpha \frac{\partial u^\alpha}{\partial \tau} \nabla_\alpha \dot{u}^k = \frac{\partial^2 u^k}{\partial t \partial \tau} + \sum_{\alpha\beta} \Gamma_{\alpha\beta}^k \dot{u}^\alpha \left(u^\beta\right)' = \sum_\alpha \frac{\partial u^\alpha}{\partial t} \nabla_\alpha \left(u^k\right)',$$

т.е. совпадает с k-й компонентой поля $\nabla_t \nu_t'(\tau)$. Доказательство закончено.

Теперь с помощью леммы 6.39 найдем:

$$\|\dot{\gamma}_\tau(t)\|' = \nabla_\tau \sqrt{\langle \dot{\gamma}_\tau, \dot{\gamma}_\tau \rangle} = \frac{1}{2} \frac{\nabla_\tau \langle \dot{\gamma}_\tau, \dot{\gamma}_\tau \rangle}{\sqrt{\langle \dot{\gamma}_\tau, \dot{\gamma}_\tau \rangle}} = \frac{\langle \nabla_t \nu'_t, \dot{\gamma}_\tau \rangle}{\sqrt{\langle \dot{\gamma}_\tau, \dot{\gamma}_\tau \rangle}}.$$

Подставим полученное выражение в формулу для производной функции длины $\ell(\tau)$, положим $\tau = 0$, воспользуемся предположением о том, что кривая $\gamma(t)$ натурально параметризована, и тем, что $E(t) = \nu'_t(0)$ по определению. Получим:

$$\frac{d\ell(0)}{d\tau} = t'_2(0) - t'_1(0) + \int_{t_1}^{t_2} \langle \nabla_t E(t), \dot{\gamma} \rangle \, dt$$

Далее,

$$\nabla_t \langle E(t), \dot{\gamma} \rangle = \langle \nabla_t E(t), \dot{\gamma} \rangle + \langle E(t), \nabla_t \dot{\gamma} \rangle \,,$$

поэтому выражение для производной имеет вид

$$\begin{aligned} \frac{d\ell(0)}{d\tau} &= t'_2(0) - t'_1(0) + \int_{t_1}^{t_2} \left(\nabla_t \langle E(t), \dot{\gamma} \rangle - \langle E(t), \nabla_t \dot{\gamma} \rangle \right) dt = \\ &= t'_2(0) - t'_1(0) + \langle E\left(t_2\right), \dot{\gamma}\left(t_2\right) \rangle - \langle E\left(t_1\right), \dot{\gamma}\left(t_1\right) \rangle - \int_{t_1}^{t_2} \langle E(t), \nabla_t \dot{\gamma} \rangle \, dt \end{aligned}$$

Теперь, представив $t'_i(0)$ в виде $t'_i(0) \left\langle \dot{\gamma}\left(t_i\right), \dot{\gamma}\left(t_i\right) \right\rangle, i = 1, 2$, заметим, что

$$t'_i(0) + \langle E\left(t_i\right), \dot{\gamma}\left(t_i\right) \rangle = \langle t'_i(0) \dot{\gamma}\left(t_i\right) + E\left(t_i\right), \dot{\gamma}\left(t_i\right) \rangle \,.$$

Наконец, воспользовавшись леммой 6.38 и определением производной вдоль кривой, перепишем выражение для производной в виде

$$\frac{d\ell(0)}{d\tau} = \langle q'(0), \dot{\gamma}\left(t_2\right) \rangle - \langle p'(0), \dot{\gamma}\left(t_1\right) \rangle - \int_{t_1}^{t_2} \langle E(t), \nabla_{\dot{\gamma}} \dot{\gamma} \rangle \, dt$$

Итак, доказано следующее важное утверждение.

Statement

6.40 (Первая вариация длины). Пусть $\Phi(\tau, t)$ - произвольная вариация кривой $\gamma : [t_1, t_2] \to \mathrm{M}$, а $p(\tau)$ и $q(\tau)$ - концевые кривые вариации Φ. Тогда первая производная длины $\ell(\tau)$ кривой $\gamma_\tau(t) = \Phi(\tau, t)$ в начальный момент времени $\tau = 0$ имеет вид

$$\frac{d\ell(0)}{d\tau} = \langle q'(0), \dot{\gamma}\left(t_2\right) \rangle - \langle p'(0), \dot{\gamma}\left(t_1\right) \rangle - \int_{t_1}^{t_2} \langle E(t), \nabla_{\dot{\gamma}} \dot{\gamma} \rangle \, dt$$

Из утверждения 6.40 вытекает несколько важных следствий.

Следствие 6.41 (Стационарность геодезических). Любая геодезическая $\gamma(t)$ на поверхности M является кривой стационарной длины по отношению κ произвольной вариации кривой γ с закрепленными концами, т.е. производная длины геодезической при таких вариациях в начальный момент времени равна нулю.

Следствие 6.42. Если кривая γ - геодезическая, то для произвольной ее вариации $\Phi(\tau, t)$ производная функции длины $\ell(\tau)$ в начальный момент времени $\tau = 0$ имеет вид

$$\frac{d\ell(0)}{d\tau} = \langle q'(0), \dot{\gamma}\left(t_2\right) \rangle - \langle p'(0), \dot{\gamma}\left(t_1\right) \rangle$$

где $p(\tau)$ и $q(\tau)$ - концевые кривые вариации Φ.

14.8 7 Curvilinear coordinates in the area and on the surface

Выше мы уже видели, как выбор подходящих координат на поверхности позволяет проще решать те или иные задачи. В этой лекции излагается общая теория координат, обобщающих линейные замены, рассматриваемые в линейной алгебре и аналитической геометрии. Если при линейных заменах координатные линии всегда оставались прямолинейными, то в обобщенных координатах они могут, вообще говоря, отличаться от прямых. Чтобы подчеркнуть это отличие, обобщенные координаты часто называют криволинейными.

Общая идея, лежащая в основе понятия координат, состоит в "оцифровке" точек пространства. Иными словами, каждой точке из рассматриваемой области пространства $\mathbb{R}^n$ приписывается n чисел, причем разным точкам приписываются разные наборы чисел.

Пример 7.1. Каждую точку P плоскости $\mathbb{R}^2$ можно задать ее стандартными евклидовыми координатами (x, y), а можно поставить в соответствие точке P пару чисел (r, φ), где r - расстояние от P до начала координат O, а φ - угол от направления оси OX до направления радиус-вектора точки P. Напомним, что пара (r, φ) называется полярными координатами точки P.

Отметим возникающие здесь сложности. Во-первых, если P совпадает с O, то направление радиус-вектора OP не определено. Во-вторых, для точек P, отличных от O, угол от направления оси OX до направления вектора OP определен неоднозначно (с точностью до 2π). Таким образом, нам не удается корректно определить полярные координаты на всей плоскости $\mathbb{R}^2$. Однако, если ограничиться областью Ω, полученной из $\mathbb{R}^2$ выбрасыванием, скажем, неотрицательной полуоси OX_+, и положить, что φ меняется в пределах от 0 до 2π, получим корректно определенную "оцифровку" области Ω, являющуюся примером криволинейной системы координат. Именно такими объектами будем здесь заниматься.

14.8.1 7.1 Determination of curvilinear coordinates

Пусть Ω - некоторая область в $\mathbb{R}^n$ и $(x^1, \ldots, x^n)$ - евклидовы координаты в $\mathbb{R}^n$. Рассмотрим еще один экземпляр $\mathbb{R}^n$ (обозначим его через $\mathbb{R}_1^n$) со стандартными координатами $(y^1, \ldots, y^n)$.

Определение 7.2. Система из n функций $y^i = y^i\left(x^1, \ldots, x^n\right), i = 1, \ldots, n$, заданных на Ω, называется непрерывной системой координат, если отображение $\psi : \Omega \to \mathbb{R}_1^n$, заданное в виде

$$\begin{cases} y^1 = y^1\left(x^1, \ldots, x^n\right), \\ \ldots \\ y^n = y^n\left(x^1, \ldots, x^n\right), \end{cases}$$

является непрерывным и взаимно-однозначным с образом, а обратное к нему отображение ψ^{-1}, заданное системой функций

$$\begin{cases} x^1 = x^1\left(y^1, \ldots, y^n\right), \\ \ldots \\ x^n = x^n\left(y^1, \ldots, y^n\right), \end{cases}$$

также непрерывно. Набор чисел ($y^1(P), \ldots, y^n(P)$) для произвольной точки $P \in \Omega$ называется криволинейными координатами точки P (по отношению к системе координат, заданной функциями y^i, или, что то же самое, заданной отображением ψ).

Определение 7.3. Если все функции $x^i\left(y^1, \ldots, y^n\right)$ - гладкие, то координаты y^i называются гладкими. Для гладкой системы координат определена матрица Якоби J,

составленная из частных производных ($\partial x^i/\partial y^j$), и якобиан $|J|$, равный определителю матрицы Якоби.

Замечание 7.4. Отметим, что из гладкости системы координат не следует гладкость отображения ψ (достаточно рассмотреть на прямой $\mathbb{R}^1$ пример замены координат, заданной в виде $x = y^3$; в точке $y = 0$ обратное отображение, т.е. отображение ψ гладким не является).

Определение 7.5. Гладкая система координат называется регулярной, если ее якобиан всюду отличен от нуля, иными словами, матрица Якоби всюду невырождена.

Из теоремы об обратной функции вытекает следующее утверждение.

Утверждение 7.6. В сделанных обозначениях, гладкая система координат является регулярной, если и только если отображение ψ - гладкое. В этом случае матрица Якоби отображения ψ равна J^{-1}.

Замечание 7.7. Отметим, что при решении конкретных задач намного проще бывает проверять, что якобиан отличен от нуля, чем искать обратное отображение и проверять его гладкость.

Конструкция 7.8. Пусть $(y_0^1, \ldots, y_0^n)$ - криволинейные координаты некоторой точки $P \in \Omega$. Для каждого i рассмотрим кривую $\delta_i(t)$, определенную для t, близких к y_0^i, так:

$$\begin{cases} x^1(t) = x^1\left(y_0^1, \ldots, y_0^{i-1}, t, y_0^{i+1}, \ldots, y_0^n\right), \\ \ldots \\ x^n(t) = x^n\left(y_0^1, \ldots, y_0^{i-1}, t, y_0^{i+1}, \ldots, y_0^n\right). \end{cases}$$

Аналогично, для фиксированного набора индексов $(i_1, \ldots, i_k)$, $k < n$, и для t^{i_p}, близких к $y_0^{i_p}$, зададим параметрически k-мерную поверхность:

$$\begin{cases} x^1\left(t^{i_1}, \ldots, t^{i_k}\right) = x^1\left(y_0^1, \ldots, t^{i_1}, \ldots, t^{i_k}, \ldots, y^n\right) \\ \ldots \\ x^n\left(t^{i_1}, \ldots, t^{i_k}\right) = x^n\left(y_0^1, \ldots, t^{i_1}, \ldots, t^{i_k}, \ldots, y^n\right) \end{cases}$$

Определение 7.9. Построенная кривая $\delta_i(t)$ называется i-й координатной кривой, проходящей через точку P, а построенные поверхности - k-мерными координатными поверхностями. Если $k = n - 1$, то k-мерные координатные поверхности называются координатными гиперповерхностями.

Легко проверить справедливость следующего утверждения.

Утверждение 7.10. Координатные кривые и поверхности регулярной системы координат являются регулярными кривыми и регулярными поверхностями соответственно.

Конструкция 7.11. Пусть в области Ω заданы две регулярные системы координат, скажем коринаты ($y^1, \ldots, y^n$), заданные с помощью отображения ψ, и координаты ($z^1, \ldots, z^n$), заданные с помощью отображения ξ. Поскольку отображения, задающие эти координаты, взаимно-однозначны, на их образах определены "сквозные" отображения $\psi \circ \xi^{-1}$ и $\xi \circ \psi^{-1}$, которые задаются наборами функций $y^i\left(z^1, \ldots, z^n\right)$ и $z^i\left(y^1, \ldots, y^n\right)$ соответственно. В силу регулярности координат, определены матрицы Якоби $J(y, z) = (\partial y^i/\partial z^j)$ и $J(z, y) = (\partial z^i/\partial y^j)$ этих систем функций.

Определение 7.12. Отображения $\psi \circ \xi^{-1}$ и $\xi \circ \psi^{-1}$ называются заменами координат, а матрицы $J(y, z)$ и $J(z, y)$ - матрицами Якоби этих замен.

Замечание 7.13. Регулярные криволинейные координаты $\psi : \Omega \to \mathbb{R}_1^n$ на области $\Omega \subset \mathbb{R}^n$ - это фактически регулярная вложенная параметрическая поверхность размерности n в $\mathbb{R}_1^n$. Поэтому многие дальнейшие конструкции и вычисления аналогичны встречавшимся выше в теории поверхностей. Однако мы приводим формулировки и краткие доказательства, на которые теперь можно взглянуть с несколько иной точки зрения.

14.8.2 7.2 Examples of regular coordinate systems

Перечисленные ниже криволинейные системы координат являются наиболее распространенными.

Пример 7.14 (Евклидовы координаты). Если отображение ψ определено на всем $\mathbb{R}^n$ и является тождественным, то криволинейная система координат, порожденная ψ, совпадает с самой евклидовой системой координат. При этом матрица Якоби в каждой точке $\mathbb{R}^n$ постоянна и равна единичной матрице, а якобиан равен 1 , поэтому это - регулярная система координат; координатные кривые - это параллельные координатным осям прямые, проходящие через данную точку; координатные k-мерные поверхности - это k-мерные плоскости, проходящие через данную точку параллельно некоторой k-мерной координатной плоскости.

Пример 7.15 (Линейная система координат). Линейное отображение ψ, заданное на $\mathbb{R}^n$, порождает на $\mathbb{R}^n$ криволинейную систему координат, если и только если матрица A_ψ отображения ψ невырождена. Матрица Якоби линейной системы постоянна и совпадает с матрицей $(\ A_\psi)^{-1}$; якобиан равен $1/\det(A_\psi)$. Координатные кривые и k-мерные координатные поверхности - это соответствующие прямые и k-мерные плоскости.

Пример 7.16 (Полярная система координат). Если $(\ x, y\ )$ - стандартные координаты на плоскости, то положим

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$$

где $r > 0$, а $0 < \varphi < 2\pi$. Описанное только что отображение задает регулярную систему координат (r, φ) на области $\Omega = \mathbb{R}^2 \backslash \{(x, 0) \mid x \geq 0\}$. Эта система координат называется полярной. Матрица Якоби равна

$$\begin{pmatrix} \cos\varphi & -r\sin\varphi \\ \sin\varphi & r\cos\varphi \end{pmatrix},$$

а якобиан равен r. Координатные линии - открытые лучи, выходящие из начала координат (за исключением положительной полуоси OX_+), а также окружности с центром в O (проколотые в точке пересечения с лучом OX_+).

Отметим, что полярной системой координат называется любая регулярная система, основанная на функциях $(\ r, \varphi\ )$ (при разном корректном выборе области Ω и области изменения функций r и φ).

Пример 7.17 (Цилиндрические координаты). Если $(\ x, y, z\ )$ - стандартные координаты в $\mathbb{R}^3$, то положим

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \\ z = z \end{cases}$$

где $r > 0$, а $0 < \varphi < 2\pi, -\infty < z < \infty$. Описанное только что отображение задает регулярную систему координат (r, φ, z) в подходящей области. Эта система координат называется и,илиндрической.

Упражнение 7.18. Опишите какую-нибудь максимальную область, в которой цилиндрическая система координат корректно определена, найдите матрицу Якоби, якобиан, координатные кривые, координатные поверхности и докажите регулярность этой системы.

Пример 7.19 (Сферические координаты). Если $(\ x, y, z\ )$ - стандартные евклидовы координаты, то для каждой точки $P = (x, y, z)$ определим три числа (r, θ, φ), выбрав в качестве r расстояние между P и началом координат O, в качестве θ - угол между

вектором OP и плоскостью XY, а в качестве φ - угол между радиус-вектором проекции точки P на плоскость XY и положительным направлением оси X. Чтобы избежать неприятностей, аналогичных тем, которые возникли при рассмотрении полярных координат, функции (r, θ, φ) зададим в области $\Omega = \mathbb{R}^3 \backslash \Pi$, где Π - замкнутая полуплоскость в координатной плоскости XZ, ограниченная осью OZ и содержащая положительную полуось OX_+. Кроме того, будем считать, что $r > 0, -\pi/2 < \theta < \pi/2$ и $0 < \varphi < 2\pi$. Связь между координатами (x, y, z) и (r, θ, φ) в явном виде выглядит так:

$$\begin{cases} x = r\cos\theta\cos\varphi \\ y = r\cos\theta\sin\varphi \\ z = r\sin\theta \end{cases}$$

При описанных ограничениях функции (r, θ, φ) задают регулярную криволинейную систему координат, называемую сферической.

Упражнение 7.20. Для сферической системы координат найдите матрицу Якоби, якобиан, координатные кривые, координатные поверхности и докажите регулярность этой системы.

Упражнение 7.21. Часто в качестве сферической координаты θ точки $P \in \mathbb{R}^3$ выбирают угол между направлением оси OZ и вектором OP. В этом случае угол θ меняется в пределах $0 < \theta < \pi$. Запишите соответствующее отображение ψ, опишите область, в которой корректно определены соответствующие криволинейные координаты, найдите матрицу Якоби, якобиан, координатные кривые, координатные поверхности и докажите регулярность этой системы.

14.8.3 7.3 Tangent Space to Area

Как мы уже отмечали, область $\Omega \subset \mathbb{R}^n$ можно рассматривать как частный случай регулярной поверхности. Поэтому для нее определены касательные векторы и касательное пространство в каждой её точке. В данном пункте мы приведем независимые определения.

Определение 7.22. Касательным вектором κ области Ω в точке $T_P\Omega$ называется вектор скорости произвольной гладкой кривой, проходящей через P, вычисленный в этой точке. Касательным пространством $T_P\Omega$ в точке P области $\Omega \subset \mathbb{R}^n$ называется множество вычисленных в точке P векторов скоростей всевозможных гладких кривых, проходящих через P, т.е. множество всех касательных векторов к области в этой точке.

Конструкция 7.23. Пусть ($x^1, \ldots, x^n$) - регулярные координаты в $\Omega \subset \mathbb{R}^n$ и $v \in T_P\Omega$ - вектор скорости кривой $(x^1(t), \ldots, x^n(t))$, причем $P = (x^1(t_0), \ldots, x^n(t_0))$. Тогда вектор v в координатах $(x^1, \ldots, x^n)$ имеет вид $(\dot{x}^1(t_0), \ldots, \dot{x}^n(t_0))$.

Определение 7.24. Величины ($\dot{x}^1(t_0), \ldots, \dot{x}^n(t_0)$) называются компонентами касательного вектора v или координатами касательного вектора v в системе координат $x^1, \ldots, x^n$.

Конструкция 7.25. Пусть снова ($x^1, \ldots, x^n$) - регулярные координаты в $\Omega \subset \mathbb{R}^n$. Рассмотрим i-ую координатную кривую, проходящую через точку P. Тогда компоненты ее вектора скорости, т.е. компоненты соответствующего касательного вектора из $T_P\Omega$, имеют вид: $(0, \ldots, 0, 1, 0, \ldots, 0)$, где единица стоит на i-ом месте.

Определение 7.26. Векторы скоростей координатных линий регулярной системы координат $x^1, \ldots, x^n$ обозначаются через ∂_{x^i} и называются векторами канонического базиса, соответствующего системе координат $(x^1, \ldots, x^n)$.

Утверждение 7.27. Касательное пространство $T_P\Omega$ представляет собой n-мерное линейное пространство. Если $x^1, \ldots, x^n$ - регулярная система координат в области, то векторы $\partial_{x^i}, i = 1, \ldots, n$, образуют базис в $T_P\Omega$, и каждый касательный вектор $v \in T_P\Omega$

представляется в виде $v = \sum_i v^i \partial_{x^i}$, где v^i- компоненты вектора v в системе координат $x^1, \ldots, x^n$.

Доказательство. Пусть фиксирована система координат. Тогда каждый касательный вектор задает набор из n чисел - своих компонент, который является элементом арифметического пространства $\mathbb{R}^n$. С другой стороны, для любого вектора $a = (a^1, \ldots, a^n)$ из арифметического пространства легко найти гладкую кривую в области, проходящую через точку P с вектором скорости a. Действительно, если P имеет координаты $(x_0^1, \ldots, x_0^n)$, то достаточно рассмотреть кривую с координатными функциями $x^i(t) = x_0^i + a^i t$, где t меняется в достаточно малой окрестности нуля. Таким образом, множество $T_P\Omega$ отождествляется с арифметическим пространством $\mathbb{R}^n$, что и превращает его в линейное пространство.

Линейная независимость векторов ∂_{x^i} и формула $v = \sum_i v^i \partial_{x^i}$ тривиально следует из вида векторов ∂_{x^i}, описанного в конструкции 7.25. Утверждение доказано.

Утверждение 7.28. Пусть $v \in T_P\Omega$ - касательный вектор к области Ω, в которой задано две регулярных системь координат $x^1, \ldots, x^n$ и $y^1, \ldots, y^n$. Тогда канонические базисы $(\partial_{x^i})\, u\, \left(\partial_{y^j}\right)$ связаны между собой с помощью матрицы Якоби $J(y, x)$ заменью координат $y(x)$, вычисленной в точке P, а именно,

$$\partial_{x^i} = \sum_{\alpha=1}^{n} \frac{\partial y^\alpha}{\partial x^i} \partial_{y^\alpha}$$

или, в матричном виде,

$$(\partial_{x^1}, \ldots, \partial_{x^n}) = (\partial_{y^1}, \ldots, \partial_{y^n})\, J(y, x).$$

Если $v^1, \ldots, v^n u \tilde{v}^1, \ldots, \tilde{v}^n$ обозначают компоненты вектора v в координатах $x^1, \ldots, x^n$ и $y^1, \ldots, y^n$ соответственно, то

$$\tilde{v}^i = \sum_{\alpha=1}^{n} \frac{\partial y^i}{\partial x^\alpha} v^\alpha$$

Доказательство. Начнем с формулы пересчета компонент вектора. Пусть $\gamma(t)$ - такая гладкая кривая в области что $\gamma(t_0) = P$ и $\dot{\gamma}(0) = v$. Если $x^i(t)$ и $y^j(t)$ - координатные функции кривой γ в системах координат $x^1, \ldots, x^n$ и $y^1, \ldots, y^n$, соответственно, ТО

$$y^j(t) = y^j\left(x^1(t), \ldots, x^n(t)\right)$$

где $y^j(x^1, \ldots, x^n)$ - функции замены координат. Тогда, по теореме о дифференцировании сложной функции имеем

$$\tilde{v}^j = \dot{y}^j(t_0) = \sum_{\alpha=1}^{n} \frac{\partial y^j}{\partial x^\alpha} \dot{x}^\alpha(t_0) = \sum_{\alpha=1}^{n} \frac{\partial y^j}{\partial x^\alpha} v^\alpha,$$

что и требовалось.

Применив полученную формулу к вектору ∂_{x^k}, найдем, что его i-ая компонента в координатах $y^1, \ldots, y^n$ имеет вид

$$(\partial_{x^k})^i = \frac{\partial y^i}{\partial x^k},$$

откуда

$$\partial_{x^k} = \sum_{\alpha} \frac{\partial y^\alpha}{\partial x^k} \partial_{y^\alpha}$$

что и требовалось.

Замечание 7.29. Из утверждения 7.28 вытекает, что структура линейного пространства на $T_P\Omega$, построенная при доказательстве утверждения Z.27, не зависит от выбора системы координат, участвовавшей в доказательстве.

14.8.4 7.4 Euclidean metric in curvilinear coordinates

Как и в общем случае регулярной поверхности, на касательных пространствах к области определено скалярное произведение.

Конструкция 7.30. Для любых двух касательных векторов v и w из $T_P\Omega$ определено их евклидово скалярное произведение $\langle v, w\rangle$ как векторов в $\mathbb{R}^n$. Тем самым, для каждой точки $P \in \Omega$ определена билинейная форма $g(P) : T_P\Omega \times T_P\Omega \to \mathbb{R}$.

Определение 7.31. Семейство билинейных форм $g(P)$ называется евклидовой метрикой в области Ω.

Утверждение 7.32. Пусть $x^1, \dots, x^n$ - декартовы координаты в $\mathbb{R}^n$, а $y^1, \dots, y^n$ - регулярные координаты в области Ω. Матрица $G = (g_{ij})$) евклидовой метрики в точке P в базисе (∂_{y^i}) имеет вид

$$g_{ij} = \sum_{\alpha=1}^{n} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\alpha}{\partial y^j}.$$

Если $z^1, \dots, z^n$ еще одна регулярная система координат в Ω, а $H = (h_{pq})$ - матрица евклидовой метрики в точке P в базисе (∂_{z^j}), то матриц, G и H связаны между собой с помощью матрицы Якоби замены координат так:

$$h_{ij} = \sum_{k,l} \frac{\partial y^k}{\partial z^i} \frac{\partial y^l}{\partial z^j} g_{kl}.$$

Доказательство. Пусть $(\tilde{v}^1, \dots, \tilde{v}^n)$ и $(\tilde{w}^1, \dots, \tilde{w}^n)$ - компоненты векторов v и w по отношению к y^i. Выразим скалярное произведение векторов v и w через эти компоненты. Имеем

$$v = \sum_i \tilde{v}^i \partial_{y^i}, \quad w = \sum_i \tilde{w}^i \partial_{y^i}, \quad \langle v, w\rangle = \sum_{i,j} \left\langle \partial_{y^i}, \partial_{y^j} \right\rangle \tilde{v}^i \tilde{w}^j$$

Таким образом, матрица g_{ij} евклидовой метрики в базисе (∂_{y^i}) совпадает с матрицей Грама $= \left\langle \partial_{y^i}, \partial_{y^j} \right\rangle$ базиса $(\partial_{y^1}, \dots, \partial_{y^n})$. Записав векторы ∂_{y^i} в декартовых координатах, получаем

$$g_{ij} = \sum_k \frac{\partial x^k}{\partial y^i} \frac{\partial x^k}{\partial y^j}$$

Теперь найдем связь между матрицами H и G. Так как $h_{ij} = \langle \partial_{z^i}, \partial_{z^j}\rangle$, имеем:

$$h_{ij} = \langle \partial_{z^i}, \partial_{z^j}\rangle = \left\langle \sum_k \frac{\partial y^k}{\partial z^i} \partial_{y^k}, \sum_l \frac{\partial y^l}{\partial z^j} \partial_{y^l} \right\rangle =$$

$$= \sum_{k,l} \frac{\partial y^k}{\partial z^i} \frac{\partial y^l}{\partial z^j} \left\langle \partial_{y^k}, \partial_{y^l} \right\rangle = \sum_{k,l} \frac{\partial y^k}{\partial z^i} \frac{\partial y^l}{\partial z^j} g_{kl}.$$

Иными словами, $H = J(y, z)^T G J(y, z)$, где $(\cdot)^T$ обозначает транспонирование матрицы.

Замечание 7.33. Компоненты метрики удобно вычислять на языке дифференциалов:

$$ds^2 = \sum_k \left(dx^k\right)^2 = \sum_k \left(\sum_i \frac{\partial x^k}{\partial y^i} dy^i\right)^2 =$$

$$= \sum_{i,j} \left(\sum_k \frac{\partial x^k}{\partial y^i} \frac{\partial x^k}{\partial y^j}\right) dy^i dy^j = \sum_{i,j} g_{ij} dy^i dy^j$$

Упражнение 7.34. Найдите вид евклидовой метрики в полярных, цилиндрических и сферических координатах. А именно, проверьте, что она имеет следующий вид:

- в полярной системе координат (r, φ) -

$$ds^2 = dr^2 + r^2 d\varphi^2$$

- в цилиндрической системе координат $(\ r, \varphi, z\ )$ -

$$ds^2 = dr^2 + r^2 d\varphi^2 + dz^2$$

- в сферической системе координат (r, θ, φ) -

$$ds^2 = dr^2 + r^2 d\theta^2 + r^2 \cos^2\theta d\varphi^2$$

14.8.5 7.5 Regular coordinates on the surface

Понятие регулярных (криволинейных) координат естественным образом переносится на случай регулярных поверхностей.

Определение 7.35. Пусть M - регулярная k-мерная поверхность в $\mathbb{R}^n$. Регулярное взаимно-однозначное с образом отображение $\psi : M \to \mathbb{R}^k$ называется регулярными координатами на поверхности M.

Замечание 7.36. Пусть регулярная поверхность M в $\mathbb{R}^n$, задана параметрически, $r : \Omega \to \mathbb{R}^n$ - параметризующее отображение, и пусть $\psi : M \to \mathbb{R}^k$ - регулярные координаты на M. Если $u^1, \ldots, u^k$ - евклидовы координаты в $\mathbb{R}^k \supset \Omega$, а $y^1, \ldots, y^k$ - евклидовы координаты в $\mathbb{R}^k \supset \psi(M)$, то возникает набор гладких функций $y^i\left(u^1, \ldots, u^k\right)$, $i = 1, \ldots, k$, - координатное представление отображения ψ, - причем матрица Якоби $\frac{\partial y^i}{\partial u^j}$ невырождена. Поэтому отображение $\psi \circ r$ задаете регулярные координаты на области Ω. Положим $\Omega' = \psi(r(\Omega))$. Ясно, что Ω' - область в $\mathbb{R}^k$. Отображение $\psi^{-1} : \Omega' \to M \subset \mathbb{R}^n$ задает в $\mathbb{R}^n$ регулярную поверхность $r' : \Omega' \to \mathbb{R}^n$, которая отличается от исходной заменой параметризации. Если $x^1, \ldots, x^n$ - евклидовы координаты в $\mathbb{R}^n$, координатные функции параметризующего отображения r имеют вид $x^i\left(u^1, \ldots, u^k\right)$, а координатные функции отображения ψ имеют вид $y^i\left(u^1, \ldots, u^k\right)$, то координатные функции отображения r' имеют вид

$$x^j\left(y^1, \ldots, y^k\right) = x^i\left(u^1\left(y^1, \ldots, y^k\right), \ldots, u^k\left(y^1, \ldots, y^k\right)\right).$$

Замечание 7.37. Выше описано, как ведет себя первая фундаментальная форма поверхности при замене координат. Из сделанного замечания вытекает, что эти же формулы работают и в регулярных координатах на поверхности.

14.8.6 7.6 Stereographic coordinates of the sphere

В качестве примера мы построим удобную систему координат на сфере.
Конструкция 7.38. Рассмотрим двумерную сферу радиуса R. Пусть (x, y, z) - стандартные координаты в $\mathbb{R}^3$, тогда сфера S^2 задается уравнением $x^2+y^2+z^2=R^2$. Обозначим через N северный полюс сферы S^2, т.е. точку с координатами $(\ 0,0,R\ )$, а через S - южный полюс, т.е. точку с координатами $(\ 0,0,-R\ )$. Пусть - координатная плоскость $z=0$. Зададим отображение

$$\nu : S^2 \backslash \{N\} \to \Pi$$

следующим образом. Для каждой точки $P \in S^2 \backslash \{N\}$ обозначим через P' единственную точку пересечения прямой NP с плоскостью Π. Положим $\nu(P)=P'$. Отображение $\nu : S^2 \backslash \{N\} \to \mathbb{R}^2$ задает на сфере систему координат (ниже мы покажем что это регулярные координаты).

Определение 7.39. Отображение ν называется стереографической проекицей из северного полюса N. Соответствующие координаты на сфере S^2 без северного полюса называются стереографическими.

Следующий результат получается прямым подсчетом.

Theorem

7.43. Индуцированная метрика $d\sigma^2$ на сфере S^2 радиуса R в стереографических координатах имеет вид

$$d\sigma^2 = \frac{4R^4}{\left(u^2+v^2+R^2\right)^2}\left(du^2+dv^2\right) = \frac{4R^4}{\left(\rho^2+R^2\right)^2}\left(d\rho^2+\rho^2 d\varphi^2\right)$$

где (u, v) - декартовы, а (ρ, φ) - полярные координаты в плоскости проеки,ии.
Упражнение 7.44. Обобщить только что приведенные выкладки на случай гиперсферы в $\mathbb{R}^n$.

14.8.7 7.7 Problems

Определение 7.45. Криволинейная система координат $(\ q^1, \ldots, q^n\ )$, заданная в области пространства $\mathbb{R}^n$, называется ортогональной, если в ней евклидова метрика имеет вид $H_1^2\left[dq^1\right]^2+\cdots+H_n^2\left[dq^n\right]^2$, где $H_i, i=1, \ldots, n$, - положительные функции координат $(\ q^1, \ldots, q^n\ )$, называемые коэффициентами Ламе.

Задача 7.46. Вычислить якобиан перехода $J=\left|\frac{\partial x^s}{\partial q^k}\right|$ от декартовых координат (x^i) к ортогональным криволинейным координатам $(\ q^i\ )$ в пространстве $\mathbb{R}^n$.

Задача 7.47. Эллиптические координатьс в $\mathbb{R}^3$

$$q^1=\lambda, q^2=\mu, q^3=z$$

определяются с помощью формул

$$x=c\lambda\mu, \quad y=c\sqrt{\left(\lambda^2-1\right)\left(1-\mu^2\right)}, \quad z=z$$

где c - масштабный множитель.

- Найти координатные поверхности эллиптических координат.
- Вычислить коэффициенты Ламе.

14.8.8 7.8 Supplemental material

14.8.9 7.8.1 Stereographic Projection Properties

Отметим следующее важное свойство стереографической проекции.

Theorem

7.48. Стереографическая проекция $\nu : S^2 \to \Pi$ переводит каждую (невырожденную) окружностъ на сфере S^2, не проходящую через северный полюс N, в некоторую окружность на плоскости Π, а окружность на S^2, проходящую через N, - в некоторую прямую на Π.

Доказательство. Второе утверждение теоремы очевидно, так как каждая окружность на S^2, проходящая через N, переходит, по определению, в прямую пересечения плоскости, задающей эту окружность, с плоскостью .

Докажем первое утверждение теоремы. Для простоты положим $R = 1$. Пусть окружность $\gamma \subset S^2$ получается, как пересечение плоскости $'$ и сферы S^2. Пусть плоскость $'$ задается уравнением $ax + by + cz = d$. Подставляя в это уравнение выражение для x, y и z через стереографические координаты u и v, получаем

$$ax + by + cz = a\frac{2u}{u^2+v^2+1} + b\frac{2v}{u^2+v^2+1} + c\frac{u^2+v^2-1}{u^2+v^2+1} = \frac{2au + 2bv + c\left(u^2+v^2-1\right)}{u^2+v^2+1} = d,$$

откуда

$$2au + 2bv + c\left(u^2+v^2-1\right) - d\left(u^2+v^2+1\right) = (c-d)\left(u^2+v^2\right) + 2au + 2bv - (d+c) = 0.$$

Так как при u^2 и v^2 стоят одинаковые коэффициенты, то это уравнение второго порядка задает или окружность (если $c \neq d$ и есть хотя бы два разных решения), или прямую (если $c = d$ и $a^2+b^2 \neq 0$), или точку, или пустое множество. Но, по предположению, окружность γ невырождена (состоит более чем из одной точки) и не проходит через N, поэтому ее образ состоит более чем из одной точки и ограничен, откуда немедленно заключаем, что $\nu(\gamma)$ - окружность. Утверждение доказано. $\square$

14.8.10 7.8.2 Multidimensional stereographic projection

Понятие стереографической проекции легко обобщается на случай стандартной сферы $S^n \subset \mathbb{R}^{n+1}$ радиуса R. Пусть в $\mathbb{R}^{n+1}$ заданы стандартные евклидовы координаты ($x^1, \ldots, x^{n+1}$). В качестве параметризующей области выберем координатную плоскость , заданную уравнением $x^{n+1} = 0$. Стандартные декартовы координаты в этой плоскости обозначим через ($u^1, \ldots, u^n$).

Пусть сфера S^n задана как множество решений уравнения $\sum \left(x^i\right)^2 = R^2$. Обозначим через N точку сферы с координатами $(0, \ldots, 0, 1)$ и назовем N северным полюсом сферы. Пусть P - произвольная точка из $S^n \backslash \{N\}$. Выпустим из точки N луч, проходящий через P. Очевидно, построенный луч пересекает плоскость Π в единственной точке, которую обозначим через $\nu(P)$. Таким образом, возникает взаимно-однозначное соответствие $\nu : S^n \backslash \{N\} \to \Pi$ между точками сферы S^n, из которой удален северный полюс, и точками плоскости Π. Отображение ν называется стереографической проеки,ией.

Рис. 7.1: Стереографическая проекция.

Запишем отображение ν в координатах. Ясно, что координаты ($u^1, \ldots, u^n$) точки $\nu(P)$ и первые n координат ($x^1, \ldots, x^n$) точки $P \in S^n \backslash \{N\}$ пропорциональны, причем коэффициент пропорциональности равен

коэффициенту подобия k между треугольниками $NO\nu(P)$ и $NO'P$, где через O обозначено начало координат в $\mathbb{R}^{n+1}$, а через O' - проекция точки P на ось Ox^{n+1} (рис. 7.1). Более точно, $x^i = u^i|PN|/|\nu(P)N|$, $i = 1, \ldots, n$. В свою очередь, величину $|PN|$ вычислим из подобия треугольников $\nu(P)NO$ и SNP, где S - точка сферы, противоположная северному полюсу (ее, кстати, называют южным полюсом): $|PN| = |NS||NO|/|\nu(P)N|$. Поэтому, учитывая, что $|NS| = 2|NO| = 2R$, окончательно получаем:

$$x^i = \frac{2R^2 u^i}{|\nu(P)N|^2} = \frac{2R^2 u^i}{\sum_j (u^j)^2 + R^2}, \quad i = 1, \ldots, n.$$

Воспользовавшись задающим сферу уравнением, находим, что

$$\left(x^{n+1}\right)^2 = R^2 \frac{\left(R^2 - \sum_j (u^j)^2\right)^2}{\left(R^2 + \sum_j (u^j)^2\right)^2}.$$

Осталось заметить, что $(n+1)$-я координата x^{n+1} точки P положительна тогда и только тогда, когда точка $\nu(P)$ лежит вне круга радиуса R. Это дает возможность избавиться от модулей после извлечения корня в последнем равенстве. Окончательно имеем:

$$x^{n+1} = R \frac{\sum_j (u^j)^2 - R^2}{R^2 + \sum_j (u^j)^2}.$$

Рис. 7.2: Стереографическая проекция из северного полюса.

Первая квадратичная форма сферы в координатах ($u^1, \ldots, u^n$) выглядит, как

$$ds^2 = 4R^4 \frac{\sum_j (du^j)^2}{\left(R^2 + \sum_j (u^j)^2\right)^2}.$$

14.8.11 7.8.3 Normal coordinates

С помощью геодезических определим на поверхности так называемые нормальные координаты, которые оказываются удобными для многих вычислений. Итак, пусть $M = \{r : U \to \mathbb{R}^{n+1}\}$ - регулярная поверхность, заданная в параметрическом виде, P - некоторая точка на M и $T_P M$ - касательное пространство к поверхности M в точке P. Пусть V - произвольный касательный вектор из $T_P M$. Тогда в силу следствия 6. 3 существует единственная геодезическая $\xi_V(s)$, удовлетворяющая начальным условиям $\xi_V(0) = P$ и $\dot{\xi}_V(0) = V$ (здесь точкой обозначено дифференцирование по s). Пусть a - произвольное вещественное число. Тогда для вектора aV существует своя такая геодезическая $\xi_{aV}(s)$. Оказывается, геодезические $\xi_V(s)$ и $\xi_{aV}(s)$ отличаются на перепараметризацию.

Лемма 7.49. При сделанных выше обозначениях

$$\xi_{aV}(s) = \xi_V(as)$$

Доказательство. Действительно, касательный вектор геодезической $\xi_V(as)$ в точке $P = \xi_V(0)$ имеет вид

$$\left.\frac{d\xi_V(as)}{ds}\right|_{s=0} = \dot{\xi}_V(0)a = aV$$

поэтому геодезическая $\xi_V(as)$ удовлетворяет тем же начальным условиям, что и геодезическая $\xi_{aV}(s)$. Теперь утверждение леммы получается из следствия 6. 3.

Определим экспоненичальное отображение $\exp_P$ некоторой окрестности нуля в касательном пространстве $T_P M$ в поверхность M :

$$\exp_P : V \mapsto \xi_V(1)$$

Отображение $\exp_P$ - гладкое в области определения (в силу теоремы о гладкой зависимости решения задачи Коши от начальных условий). Оказывается, что отображение $\exp_P$ является также регулярным в некоторой окрестности нуля.

Statement

7.50. Пусть P - произвольная точка регулярной поверхности M. Тогда отображение $\exp_P$ задает регулярные координаты в некоторой окрестности точки P. Другими словами, отображение $\exp_P$ гладко u взаимно-однозначно отображает некоторую окрестность нуля из T_PM на окрестностъ точки P в M, причем матрица Якоби этого отображения невырождена.

Доказательство. Для доказательства утверждения достаточно показать, что матрица Якоби отображения $\exp_P$ невырождена в точке $0 \in T_PM$. Действительно, если это так, то матрица Якоби невырождена и в некоторой окрестности нуля, и тогда утверждение следует из теоремы о неявной функции.

Итак, по определению, $\exp_P(V) = \xi_V(1)$. Фиксируем на M какие-нибудь регулярные координаты $(u^1, \ldots, u^n)$, и пусть $u^i(s, V)$ - координатное представление геодезической $\xi_V(s)$. Если мы фиксируем какой-нибудь базис в T_PM и в этом базисе $V = (v^1, \ldots, v^n)$, то координатное представление отображения $\exp_P$ в координатах $(v^1, \ldots, v^n)$ на T_PM и $(u^1, \ldots, u^n)$ на M имеет вид

$$u^i = u^i(1, V) = u^i\left(1, v^1, \ldots, v^n\right)$$

Пусть t - произвольное число. Поскольку в силу леммы 7.49$\xi_{tV}(s) = \xi_V(st)$, в координатах имеем

$$u^i(s, tV) = u^i(st, V). \qquad (*)$$

Продифференцируем соотношение (*) по t и положим $t = 0$. Дифференцируя левую часть, получим

$$\left.\frac{\partial u^i(s, tV)}{\partial t}\right|_{t=0} = \sum_j \left.\frac{\partial u^i(s, tV)}{\partial v^j}\right|_{t=0} v^j = \sum_j \frac{\partial u^i(s, 0)}{\partial v^j} v^j$$

Дифференцируя правую часть соотношения (*), имеем

$$\left.\frac{\partial u^i(st, V)}{\partial t}\right|_{t=0} = \left.\frac{\partial u^i(st, V)}{\partial s}\right|_{t=0} s = \frac{\partial u^i(0, V)}{\partial s} s = \left(\dot{\xi}_V(0)\right)^i s = v^i s$$

Приравняв полученные выражения и положив в них $s = 1$, получим окончательно

$$\sum_j \frac{\partial u^i(1, 0)}{\partial v^j} v^j = v^i$$

для произвольного вектора V. Поэтому матрица Якоби ($\frac{\partial u^i(1,0)}{\partial v^j}$) отображения $\exp_P$ в точке $V = 0$ равна единичной матрице и, в частности, невырождена. Утверждение доказано.

Определение 7.51. Пусть в касательном пространстве T_PM фиксирован произвольный ортонормированный базис (e_i) и ($v^1, \ldots, v^n$) - декартовы координаты в T_PM, порожденные этим базисом. Регулярные координаты, порожденные отображением

$$V = \left(v^1, \ldots, v^n\right) \mapsto \exp_P(V)$$

переводящим некоторую окрестность точки $0 \in T_PM$ на некоторую окрестность точки P в M, называются нормальными координатами с центром в точке P, порожденными базисом (e_i). Сама окрестность точки P, в которой заданы нормальные координаты, а также соответствующая окрестность точки 0 в T_PM называются нормальными окрестностями.

Пусть в некоторой нормальной окрестности точки P фиксированы нормальные координаты. Рассмотрим в касательном пространстве T_PM сферу $S^n(\varepsilon)$ радиуса ε, целиком лежащую в нормальной окрестности точки $0 \in T_PM$. Образ $\exp_P(S^n(\varepsilon))$ сферы $S^n(\varepsilon)$ при экспоненциальном отображении называется геодезической сферой с центром в точке P и радиусом ε. Имеет место следующее интересное утверждение.

Statement

7.52 (Лемма Гаусса). Пусть Q - произвольная точка геодезической сферы Σ с центром в точке P. Тогда геодезическая, соединяющая точки P и Q и целиком лежащая в нормальной окрестности точки P, - единственна. Более того, эта геодезическая приходит на геодезическую сферу Σ под прямым углом. Последнее означает, что геодезическая перпендикулярна любой регулярной кривой, проходящей через Q и лежащей в Σ.

Доказательство. Первое утверждение следует из взаимной однозначности отображения $\exp_P$ в нормальной окрестности. Докажем второе утверждение.

Пусть $\gamma(t)$ - геодезическая, соединяющая точки P и Q и лежащая в нормальной окрестности точки P, и пусть $\sigma(\tau), \tau \in [-\tau_0, \tau_0]$ - произвольная регулярная кривая на геодезической сфере Σ, такая, что $\sigma(0) = Q$. Рассмотрим семейство геодезических $\gamma_\tau(t)$, целиком лежащих в нормальной окрестности точки P и соединяющих P с точкой $\sigma(\tau) \in \Sigma$. В силу первого утверждения каждая геодезическая $\gamma_\tau(t)$ однозначно определена. Ясно, что семейство геодезических $\gamma_\tau(t)$ порождает вариацию $\Phi(\tau, t) = \gamma_\tau(t)$ геодезической $\gamma(t) = \gamma_0(t)$. При этом, по определению, одна концевая кривая $p(\tau)$ вариации Φ состоит из одной точки P, а другая $-q(\tau)$ совпадает с кривой $\sigma(\tau)$. Воспользуемся формулой первой вариации длины (см. следствие 6.42). Поскольку длины всех геодезических $\gamma_\tau(t)$ равны между собой и равны радиусу сферы Σ, имеем

$$0 = \frac{d\ell(0)}{d\tau} = \langle \sigma'(0), \dot{\gamma} \rangle$$

где вектор скорости $\dot{\gamma}$ вычисляется в точке Q, т.е. геодезическая γ перпендикулярна кривой σ, лежащей на сфере Σ. Доказательство закончено.

Упражнение 7.53. Пусть ($u^1, \ldots, u^n$) - нормальные координаты с центром в точке P. Показать, что в этих координатах матрица первой квадратичной формы в точке P равна единичной, а символы Кристоффеля в точке P равны нулю.

14.8.12 7.8.4 Semi-geodetic coordinates

Определенные в предыдущем разделе нормальные координаты являются частным случаем так называемых координат Ферми. Координаты Ферми будут определены в следующем семестре для подмногообразия в римановом многообразии, а сейчас ограничимся другим частным случаем - так называемыми полугеодезическими координатами на двумерной поверхности.

Пусть $M = \{r : U \to \mathbb{R}^3\}$ - регулярная двумерная поверхность и $\gamma : I \to \mathbb{R}^3$ - регулярная кривая на M. Пусть $E(t)$ - гладкое касательное векторное поле единичной длины, заданное в точках кривой γ и перпендикулярное вектору скорости $\dot{\gamma}(t)$ в точке $\gamma(t)$. Построим отображение некоторой окрестности отрезка $I \times \{0\}$ в прямом произведении $I \times \mathbb{R}^1$ в поверхность M. Из каждой точки $\gamma(t)$ кривой γ выпустим геодезическую $\nu_t(s)$, определенную на некотором интервале $[-s_0(t), s_0(t)]$ и удовлетворяющую начальным условиям $\nu_t(0) = \gamma(t), \nu_t'(0) = E(t)$. Точке $(t, s) \in I \times \mathbb{R}^1$ поставим в соответствие точку $\nu_t(s)$ поверхности M. Полученное отображение обозначим через $\exp_\gamma$ и назовем экспоненичальным отображением нормалей вдоль γ.

Замечание 7.54. Так как $\|E(t)\| = 1$, все геодезические $\nu_t(s)$ параметризованы натурально.

Лемма 7.55. Экспоненицальное отображение $\exp_\gamma$ нормалей вдоль γ является гладким взаимно-однозначным и регулярным в некоторой окрестности отрезка $I \times 0$ в полосе $I \times \mathbb{R}^1$.

Доказательство. Действительно, ограничение отображения $\exp_\gamma$ на $I \times 0$ совпадает с отображением γ, а ограничение отображения $\exp_\gamma$ на $t \times \mathbb{R}^1$ совпадает с ограничением отображения $\exp_{\gamma(t)}$ на прямую в касательной плоскости $T_{\gamma(t)}M$, проходящую через 0 в направлении вектора $E(t)$. Поэтому матрица Якоби отображения $\exp_\gamma$ невырождена в каждой точке отрезка $I \times 0$ и, следовательно, так как отрезок компактен, в некоторой окрестности отрезка $I \times 0$. Лемма доказана.

В силу леммы 7.55 корректно следующее определение.

Определение 7.56. Регулярные координаты, порожденные в некоторой окрестности кривой γ отображением $\exp_\gamma$, называются полугеодезическими координатами, ассоциированными с γ.

Имеет место следующий важный аналог леммы Гаусса (см. утверждение 7.52).

Statement

7.57 (Обобщенная лемма Гаусса). На произвольной двумерной регулярной поверхности полугеодезические координаты, заданные в окрестности регулярной натурально параметризованной кривой, ортогональны.

Доказательство. Пусть $\exp_\gamma : (t, s) \to M$ - полугеодезические координаты в окрестности натурально параметризованной кривой $\gamma(t)$. Обозначим через $\nu_t(s)$ координатные линии $t = \text{const}$, а через $\gamma_s(t)$ - координатные линии $s = \text{const}$. Нужно показать, что для любого t координатная линия $\nu_t(s)$ пересекает все координатные линии $\gamma_s(t)$ под прямым углом. Фиксируем некоторое s_0. Семейство геодезических $\nu_t(s)$, приходящих на кривую $\gamma_{s_0}(t)$, порождают гладкую вариацию геодезической $\nu_{t_0}(s)$. При этом концевые кривые этой гладкой вариации совпадают с $\gamma_0(t)$ и $\gamma_{s_0}(t)$, а сама вариация - это просто ограничение отображения $\exp_\gamma$ на прямоугольник $[t_0 - \varepsilon, t_0 + \varepsilon] \times [0, s_0]$. Длины всех геодезических $\nu_t(s)$ при этом одинаковы и равны s_0. Воспользуемся формулой первой вариации длины геодезической (см. следствие 6.42). Будем обозначать дифференцирование по s штрихом, а по t - точкой. Имеем:

$$0 = \frac{d\ell(0)}{dt} = \left\langle \dot{\gamma}_{s_0}(t_0), \nu'_{t_0}(s_0) \right\rangle - \left\langle \dot{\gamma}(t_0), \nu'_{t_0}(0) \right\rangle =$$
$$= \left\langle \dot{\gamma}_{s_0}(t_0), \nu'_{t_0}(s_0) \right\rangle - \left\langle \dot{\gamma}(t_0), E(t_0) \right\rangle$$

Второе слагаемое равно нулю по определению, поэтому и первое слагаемое равно нулю. В силу произвольности выбора s_0 и t_0 доказательство закончено.

14.9 8 Riemann and pseudo-Riemann metrics

В данной лекции мы обобщим понятия евклидовой и индуцированной метрик. Это позволит нам расширить класс примеров метрик, не заботясь о построении конкретной поверхности и индуцированной метрики на ней.

Замечание 8.1. Индуцированная на поверхности метрика в общем случае не является евклидовой (например, метрика на стандартной сфере). Однако у той и другой много общего: обе позволяют вычислять скалярное произведение касательных векторов и, как следствие, длины кривых, углы между кривыми и т.д.; обе меняются по одинаковому закону при замене (регулярных) координат. Определим общий объект, называемый

билинейной формой, частными случаями которой являются евклидова и индуцированная метрики.

14.9.1 8.1 Bilinear shape in the area

Пусть M - регулярная k-мерная поверхность в $\mathbb{R}^n, k \leq n$, (напомним, что при $k = n$ мы имеем дело просто с областью в $\mathbb{R}^n$).

Определение 8.2. Будем говорить, что на поверхности (области) M задана билинейная форма b, если в каждой точке $P \in M$ определена билинейная форма $b(P)$ на касательном пространстве $T_P M$, т.е., напомним, линейное по каждому аргументу отображение $b(P) : T_P M \times T_P M \to \mathbb{R}$.

Конструкция 8.3. Пусть $u^1, \ldots, u^k$ - регулярные координаты на поверхности M, на которой задана билинейная форма b. Тогда в каждой точке $P \in M$ определен канонический базис (∂_{u^i}) в касательном пространстве $T_P M$, поэтому определена матрица $b_{ij}(P)$ билинейной формы $b(P)$, а именно, $b_{ij}(P) = b(P)\left(\partial_{u^i}, \partial_{u^j}\right)$. При этом величины $b_{ij}(P) = b_{ij}\left(u^1, \ldots, u^k\right)$ естественно рассматривать как функции на параметризующей области.

Замечание 8.18. Рассмотрим следующую обратную задачу. Пусть в некоторой области $\Omega \subset \mathbb{R}^k$ в координатах ($u^1, \ldots, u^k$) задана некоторая риманова метрика своими компонентами $g_{\alpha\beta}\left(u^1, \ldots, u^k\right)$. Является ли эта метрика индуцированной? Другими словами, существует ли такая регулярная параметрическая поверхность $r : \Omega \to \mathbb{R}^n$, что соответствующая индуцированная метрика на Ω совпадает с заданной? Эта задача весьма сложна и выходит за рамки данного курса. В двумерном случае положительный ответ дает теорема Бонне.

14.9.2 8.2 Pseudo-Euclidean spaces

Рассмотрим следующий важный пример.

Конструкция 8.19. Пусть $\Omega \subset \mathbb{R}^n$ - область в $\mathbb{R}^n$, и $x^1, \ldots, x^n$ - линейные координаты в $\mathbb{R}^n$, как в арифметическом линейном пространстве. Рассмотрим в каждой точке области Ω билинейную форму, матрица которой в каноническом базисе (∂_{x^i}) диагональна, причем на диагонали стоит s штук -1 и $n - s$ штук $+1$. Другими словами, в дифференциальной записи эта форма имеет вид

$$-\sum_{i=1}^{s}\left(dx^i\right)^2 + \sum_{i=s+1}^{n}\left(dx^i\right)^2 .$$

Полученная билинейная форма является псевдоевклидовой метрикой.

Определение 8.20. Пространство $\mathbb{R}^n$ на котором задана псевдоевклидова метрика из 8.19, называется псевдоевклидовым пространством индекса s и обозначается через $\mathbb{R}^n_s$, а координаты $x^1, \ldots, x^n$ - псевдоевклидовыми. Соответствующая билинейная форма называется псевдоскалярным произведением. Пространство $\mathbb{R}^n_1$ называется пространством Минковского.

Определение 8.21. Псевдосферой $S^{n-1}_s(R)$ индекса s и радиуса R называется подмножество пространства $\mathbb{R}^n_s$, заданное уравнением

$$-\sum_{i=1}^{s}\left(x^i\right)^2 + \sum_{j=s+1}^{n}\left(x^j\right)^2 = R^2.$$

Замечание 8.22. В отличие от обычной сферы в евклидовом пространстве, псевдосфера может иметь не только положительный радиус, но и нулевой, а также чисто мнимый

(комплексный) радиус, т.е. радиус $R = i\rho$ (при этом $R^2 = -\rho^2$). Ясно, что псевдосфера нулевого радиуса является конусом, который называется изотропным конусом.

Пример 8.23. В качестве примера (рис. 8.1) рассмотрим плоскость $\mathbb{R}^2_1$ с фиксированными стандартными координатами (x^1, x^2), в которых псевдоевклидова метрика записывается в виде $-(dx^1)^2 + (dx^2)^2$. Тогда изотропный конус - это пара прямых $x^1 = \pm x^2$. Псевдосферы вещественного радиуса расположены в области, ограниченной изотропным конусом и определяемой неравенством $|x^2| > |x^1|$. Они представляют

собой гиперболы $-(x^1)^2 + (x^2)^2 = R^2$. Аналогично, в ограниченной изотропным конусом дополнительной области $|x^1| > |x^2|$ расположены псевдосферы мнимого радиуса. Они тоже суть гиперболы $-(x^1)^2 + (x^2)^2 = -R^2$.

Рис. 8.1: Плоскость Минковского и псевдосферы.

Упражнение 8.24. Показать, что линейные преобразования плоскости $\mathbb{R}^2_1$, сохраняющие псевдоскалярное произведение, образуют группу гиперболических поворотов, которая состоит из всех матриц вида

$$\begin{pmatrix} \pm \operatorname{ch}\varphi & \pm \operatorname{sh}\varphi \\ \pm \operatorname{sh}\varphi & \pm \operatorname{ch}\varphi \end{pmatrix},$$

где знаки в каждом из столбцов - одинаковы.

14.9.3 8.3 Spacetime

Рассмотрим важный для физики частный случай псевдоевклидова пространства пространство Минковского $\mathbb{R}^4_1$, называемое в физике пространственно-временным континуумом. Это пространство фигурирует в специальной теории относительности.

Конструкция 8.25. Каждой точке $x = (x^1, x^2, x^3)$ пространства $\mathbb{R}^3$ и каждому моменту времени t поставим в соответствие точку (t, x^1, x^2, x^3) четырехмерного пространства, называемую событием. Превратим четырехмерное пространство в псевдоевклидово, выбрав качестве псевдоевклидовых координат координаты ($x^0 = ct, x^1, x^2, x^3$), где c - постоянная, интерпретируемая как скорость света в вакууме. Таким образом, псевдоевклидово скалярное произведение векторов

$$P = \left(P^0 = ct_P, P^1, P^2, P^3\right) \quad \text{и} \quad Q = \left(Q^0 = ct_Q, Q^1, Q^2, Q^3\right)$$

имеет вид

$$\langle P, Q\rangle_1 = -P^0Q^0 + P^1Q^1 + P^2Q^2 + P^3Q^3 = -c^2 t_P t_Q + P^1Q^1 + P^2Q^2 + P^3Q^3.$$

Псевдоевклидово расстояние между точками P и Q, т.е. величина $\sqrt{\langle P-Q, P-Q\rangle_1}$, называется пространственно-временным интервалом между событиями P и Q; векторы из изотропного конуса, т.е. векторы V, такие, что $\langle V, V\rangle_1 = 0$, называются световыми (поэтому изотропный конус называется световым); векторы P, для которых
$\langle P, P\rangle_1 < 0$, называются времениподобными, а векторы P, для которых $\langle P, P\rangle_1 > 0$, - пространственно-подобными.

Основной постулат специальной теории относительности гласит, что скорость любого объекта не может превосходить скорости света c. Переформулируем его на геометрическом языке. Каждой траектории $x(t) = (x^1(t), x^2(t), x^3(t))$ движения материальной точки соответствует кривая $m(t) = (ct, x(t))$ в пространстве-времени, называемая мировой линией. Ясно, что $\langle \dot{m}, \dot{m}\rangle_1 = -c^2 + \langle v, v\rangle$, где $v = \dot{x}$ - вектор скорости материальной точки. Постулат $\langle v, v\rangle < c^2$ равносилен условию $\langle \dot{m}, \dot{m}\rangle_1 < 0$. Другими словами, вектор скорости мировой линии должен быть времениподобен.

14.9.4 8.4 Pseudosphere geometry

Изучим более подробно геометрию псевдосферы $S_1^n(iR)$.
Конструкция 8.26. Пусть M - регулярная поверхность в пространстве $\mathbb{R}_s^{n+1}$, заданная параметрически в виде $r : \Omega \to \mathbb{R}_s^{n+1}$. Точно так же, как и в случае евклидова пространства, заданное в $\mathbb{R}_s^{n+1}$ псевдоевклидово скалярное произведение индуцирует в каждой касательной плоскости T_PM поверхности M симметричную билинейную форму (которая может быть вырожденной).

Определение 8.27. Если билинейная форма из 8.26 является римановой метрикой, то поверхность M называется пространственноподобной.

Конструкция 8.28. Рассмотрим в пространстве $\mathbb{R}_1^{n+1}$ псевдосферу S_1^n чисто мнимого радиуса iR. По определению, она задается уравнением

$$-\left(x^1\right)^2 + \left(x^2\right)^2 + \cdots + \left(x^{n+1}\right)^2 = -R^2.$$

Ясно, что псевдосфера S_1^n состоит из двух компонент, разделенных координатной гиперплоскостью $x^1 = 0$ (при $n = 2$ это обычный двуполостный гиперболоид). Рассмотрим половинку псевдосферы, заданную неравенством $x^1 > 0$, и произвольную точку P из этой половинки соединим с точкой $S = (-R, 0, \ldots, 0)$ отрезком SP, см. рис. 8.2. Этот отрезок пересекает плоскость $x^1 = 0$ в некоторой точке, которую обозначим через $\sigma(P)$. Определим отображение σ, положив $\sigma : P \mapsto \sigma(P)$. Ясно, что отображение σ взаимно-однозначно отображает полусферу $S_1^n\,(x^1 > 0)$ на открытый шар радиуса R в плоскости $x^1 = 0$. Обозначим через ($u^1, \ldots, u^n$) стандартные координаты в плоскости $x^1 = 0$ (для точек плоскости $x^1 = 0$, по определению, положим $x^i = u^{i-1}$, $i = 2, \ldots, n+1$).

Рис. 8.2: Стереографическая проекция псевдосферы.

Определение 8.29. Точка $N = (R, 0, \ldots, 0)$ называется северным полюсом псевдосферы, точка $S = (-R, 0, \ldots, 0)$ - южным полюсом, а отображение $\sigma : S_1^n\,(x^1 > 0) \to \mathbb{R}^n\,(x^1 = 0)$ - стереографической проекцией из южного полюса.

Лемма 8.30. Стереографическая проекиця взаимно-однозначно отображает половину псевдосферы $S_1^n\,(x^1 > 0)$ на открытый шар радиуса R в плоскости $x^1 = 0$. Во введенных выше обозначениях, если ($u^1, \ldots, u^n$) - координаты точки $\sigma(P)$, то координаты $(x^1, \ldots, x^{n+1})$ точки P могут быть вычислены следующим образом:

$$x^1 = R\frac{R^2 + \sum_i (u^i)^2}{R^2 - \sum_i (u^i)^2}, \quad x^{i+1} = \frac{2R^2 u^i}{R^2 - \sum_i (u^i)^2}, \quad i = 1, \ldots, n.$$

Доказательство. Действительно, поскольку треугольники $S\sigma(P)O$ и SPO' подобны (рис. 8.2), имеют место следующие равенства:

$$\frac{u^i}{x^{i+1}} = \frac{R}{R + x^1}, \quad i = 1, \ldots, n.$$

Подставляя их в уравнение псевдосферы, получаем квадратное уравнение на x^1. Решая его и отбрасывая решение $x^1 = -R$, соответствующее южному полюсу, находим

$$x^1 = R\frac{R^2 + \sum_i (u^i)^2}{R^2 - \sum_i (u^i)^2}.$$

Подставляя это соотношение в предыдущие, получаем

$$x^{i+1} = \frac{2R^2 u^i}{R^2 - \sum_i (u^i)^2}, \quad i = 1, \ldots, n,$$

что и требовалось доказать. □

Определение 8.31. Регулярные координаты ($u^1, \ldots, u^n$), заданные стереографической проекцией на полусфере $S_1^n\,(x^1 > 0)$ называются координатами стереографической проекиии.

Из леммы 8.30 с помощью непосредственных, хотя и довольно громоздких, вычислений получаем следующее утверждение (проверьте).

Утверждение 8.32. Псевдосфера мнимого радиуса iR с центром в начале координат в пространстве Минковского $\mathbb{R}_1^{n+1}$ является пространственноподобной поверхностью.

Индуцированная на ней риманова метрика в координатах стереографической проекции (соответствующая риманова метрика в шаре радиуса R) имеет вид

$$ds^2 = \frac{4R^4}{\left(R^2 - \sum_i (u^i)^2\right)^2} \sum_i \left(du^i\right)^2$$

Определение 8.33. Открытый шар радиуса R в $\mathbb{R}^n$ с метрикой из утверждения 8.32 называется шаром Пуанкаре (при $n = 2$ - кругом Пуанкаре).

Замечание 8.34. Ниже псевдосфера используется для построения примера метрики постоянной отрицательной кривизны и для изучения замечательной геометрии таких метрик, обнаруженной Лобачевским.

Замечание 8.35. Отметим, что метрика на (псевдо-)сфере, записанная в стереографических координатах, отличается от евклидовой на функциональный множитель, т.е. имеет вид $\lambda^2(u) \sum_i (du_i)^2$.

Определение 8.36. Метрики, отличающиеся от евклидовой на функциональный множитель, называются конформно-евклидовыми.

Утверждение 8.37. Предположим, что в области $\Omega \subset \mathbb{R}^n$ заданы две римановых метрики ds^2 и $d\sigma^2$, отличающиеся на положительную функцию λ^2, т.е. $d\sigma^2 = \lambda^2 ds^2$. Тогда величины угла между парой пересекающихся кривых, вычисленные по отношению к обеим этим метрикам, одинаковы.

Доказательство. Пусть γ и δ - две кривые в Ω, пересекающиеся в точке P под углом α по отношению к ds^2 и под углом β по отношению к $d\sigma^2$. Обозначим через ξ и η векторы скоростей этих кривых. Тогда

$$\cos\alpha = \frac{ds^2(\xi,\eta)}{ds(\xi)ds(\eta)} = \frac{\lambda^2 ds^2(\xi,\eta)}{\lambda ds(\xi)\lambda ds(\eta)} = \frac{d\sigma^2(\xi,\eta)}{d\sigma(\xi)d\sigma(\eta)} = \cos\beta$$

что и требовалось доказать. Здесь через $ds^2(\xi,\eta)$ и $d\sigma^2(\xi,\eta)$ обозначены скалярные произведения векторов ξ и η по отношению к метрикам ds^2 и $d\sigma^2$ соответственно, а через $ds(\xi)$ и $d\sigma(\xi)$ - длина вектора ξ, вычисленная по отношению к этим метрикам.

Следствие 8.38. Стереографическая проекция сохраняет углы между кривыми на сфере S^n.

Следствие 8.39. Углы между кривыми на псевдосфере S_1^n, вычисленные в индучированной метрике, совпадают с евклидовыми углами.

14.9.5 8.5 Problems

Задача 8.40. Доказать, что меньший из двух отрезков большого круга, на которые его делят две точки A и B, является кратчайшей кривой на сфере среди всех регулярных кривых, соединяющих A и B.

Задача 8.41. Вычислить длину окружности и площадь круга радиуса ρ на сфере (радиус понимается в смысле метрики на сфере).

Задача 8.42. Показать, что сумма углов сферического треугольника равна $\pi + S$, где S - площадь этого треугольника.

Задача 8.43. Пусть ABC - сферический треугольник на сфере $S^2; \alpha, \beta, \gamma$ - величины углов при вершинах $A, B, C; a, b, c$ - длины сторон BC, AC, AB. Доказать теорему косинусов:

$$\cos a = \cos b \cos c + \sin b \sin c \cos\alpha.$$

Задача 8.44. Пусть L^2 - круг Пуанкаре радиуса R и A, B - точки на одном из его диаметров. Доказать, что отрезок AB является кратчайшей кривой среди всех регулярных кривых, соединяющих A и B в круге Пуанкаре.

Задача 8.45. Вычислить длину окружности и площадь круга радиуса ρ с центром в начале координат в круге Пуанкаре (радиус понимается в смысле соответствующей римановой метрики).

14.9.6 8.6.1 Two-dimensional surfaces of constant Gaussian curvature

В этом разделе разработанная выше техника применяется для изучения поверхностей постоянной кривизны.

Знаменитая теорема Гаусса, доказанная выше, утверждает, что изометричные двумерные поверхности имеют одинаковую гауссову кривизну. Покажем, что если гауссова кривизна поверхности постоянна, то верно и обратное, т.е. две поверхности одинаковой постоянной гауссовой кривизны локально изометричны.

Для доказательства этого факта введем на поверхности координаты специального вида - полугеодезические координаты, ассоциированные с натурально параметризованной геодезической (см. Дополнительные материалы к лекции 7). А именно, пусть M, как и выше, - регулярная двумерная поверхность и пусть $\gamma(t)$ - натурально параметризованная геодезическая на M. Введем в окрестности геодезической $\gamma(t)$ ассоциированные с ней полугеодезические координаты $\exp_\gamma : (t,s) \to M$. В этих координатах первая квадратичная форма поверхности M записывается особенно просто.

Лемма 8.46. Пусть $\exp_\gamma : (t,s) \to M$ - полугеодезические координаты на регулярной поверхности M, ассоциированные с натурально параметризованной геодезической $\gamma(t)$. Первая квадратичная форма поверхности в этих координатах имеет вид

$$ds^2 + G(t,s)dt^2$$

причем функция G удовлетворяет следующим соотношениям:

$$G(t,0) = 1, \quad \frac{\partial G(t,0)}{\partial s} = 0$$

Доказательство. В силу утверждения 7.57 координаты (t,s) являются ортогональными, поэтому первая квадратичная форма имеет вид

$$E(t,s)ds^2 + G(t,s)dt^2$$

Функция $E(t,s)$ равна, по определению, скалярному квадрату вектора ∂_s, т.е. вектора скорости соответствующей геодезической $\nu_t(s)$, поэтому $E(t,s) = 1$. Далее, функция $G(t,0)$ равна скалярному квадрату вектора скорости натурально параметризованной геодезической $\gamma_0(t) = \gamma(t)$, поэтому $G(t,0) = 1$. Осталось вычислить производную от $G(t,s)$ по s при $s = 0$. Имеем:

$$\frac{\partial G(t,s)}{\partial s} = \frac{\partial \langle \partial_t, \partial_t \rangle}{\partial s} = \nabla_s \langle \dot\gamma_s, \dot\gamma_s \rangle = 2 \langle \nabla_s \dot\gamma_s, \dot\gamma_s \rangle$$

Отображение $\exp_\gamma$ задает гладкую вариацию геодезической $\nu_t(s)$ (для каждого значения t), поэтому применима лемма 6.39. В результате получаем:

$$\frac{\partial G(t,s)}{\partial s} = 2 \langle \nabla_s \dot\gamma_s, \dot\gamma_s \rangle = 2 \langle \nabla_t \nu_t', \dot\gamma_s \rangle$$

Наконец, так как векторы ν_t' и $\dot\gamma_s$ ортогональны,

$$\langle \nabla_t \nu_t', \dot{\gamma}_s \rangle = - \langle \nu_t', \nabla_t \dot{\gamma}_s \rangle$$

поэтому окончательно имеем

$$\frac{\partial G(t,s)}{\partial s} = 2 \langle \nabla_t \nu_t', \dot{\gamma}_s \rangle = -2 \langle \nu_t', \nabla_t \dot{\gamma}_s \rangle .$$

Положив в последнем равенстве $s = 0$ и воспользовавшись тем, что γ_0 - геодезическая и поэтому $\nabla_t \dot{\gamma} = 0$, получаем, наконец, что

$$\frac{\partial G(t,0)}{\partial s} = 0$$

Лемма доказана.

Вычислим теперь гауссову кривизну поверхности в координатах (t, s). Для этого воспользуемся формулой Гаусса:

$$\sum_\beta g_{\beta 2} \left\{ \frac{\partial \Gamma_{11}^\beta}{\partial u^2} - \frac{\partial \Gamma_{12}^\beta}{\partial u^1} + \sum_k \left(\Gamma_{11}^k \Gamma_{k2}^\beta - \Gamma_{12}^k \Gamma_{k1}^\beta \right) \right\} = \det Q$$

где Q - матрица второй квадратичной формы поверхности; g_{ij} - матрица первой квадратичной формы и Γ_{ij}^k - символы Кристоффеля. В силу диагональности матрицы первой квадратичной формы получаем для гауссовой кривизны K следующее выражение:

$$K = \frac{g_{22}}{\det(g_{ij})} \left\{ \frac{\partial \Gamma_{11}^2}{\partial u^2} - \frac{\partial \Gamma_{12}^2}{\partial u^1} + \sum_k \left(\Gamma_{11}^k \Gamma_{k2}^2 - \Gamma_{12}^k \Gamma_{k1}^2 \right) \right\}.$$

Вычислим символы Кристоффеля. Положим $u^1 = s, u^2 = t$ и заметим, что матрица первой квадратичной формы и обратная ей матрица диагональны; более того, $g_{11} = g^{11} = 1$. Получим:

$$\begin{aligned}
\Gamma_{11}^1 &= \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial u^1} + \frac{\partial g_{11}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^1} \right) = 0, \\
\Gamma_{12}^1 &= \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^1} \right) = 0 = \Gamma_{21}^1, \\
\Gamma_{22}^1 &= \frac{1}{2} g^{11} \left(\frac{\partial g_{21}}{\partial u^2} + \frac{\partial g_{21}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^1} \right) = \frac{-G_s}{2},
\end{aligned}$$

где дифференцирования по t и s обозначены нижними индексами. Точно так же

$$\begin{aligned}
\Gamma_{11}^2 &= \frac{1}{2} g^{22} \left(\frac{\partial g_{12}}{\partial u^1} + \frac{\partial g_{21}}{\partial u^1} - \frac{\partial g_{11}}{\partial u^2} \right) = 0, \\
\Gamma_{12}^2 &= \frac{1}{2} g^{22} \left(\frac{\partial g_{12}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^1} - \frac{\partial g_{12}}{\partial u^1} \right) = \frac{G_s}{2G} = \Gamma_{21}^2, \\
\Gamma_{22}^2 &= \frac{1}{2} g^{22} \left(\frac{\partial g_{22}}{\partial u^2} + \frac{\partial g_{22}}{\partial u^2} - \frac{\partial g_{22}}{\partial u^2} \right) = \frac{G_t}{2G}.
\end{aligned}$$

Подставим выражения для символов Кристоффеля в формулу Гаусса для кривизны. Тогда

$$K = \frac{G}{G}\left\{0 - \frac{\partial}{\partial s}\left(\frac{G_s}{2G}\right) + 0 + 0 - 0 - \left(\frac{G_s}{2G}\right)^2\right\} =$$
$$= -\frac{1}{2}\left(\frac{G_s}{G}\right)_s - \frac{1}{4}\left(\frac{G_s}{G}\right)^2 = \frac{1}{4}\left(\frac{G_s}{G}\right)^2 - \frac{1}{2}\frac{G_{ss}}{G} = -\frac{(\sqrt{G})_{ss}}{\sqrt{G}}. \qquad (*)$$

Предположим теперь, что гауссова кривизна K поверхности M постоянна. Тогда получаем следующее дифференциальное уравнение на функцию $\sqrt{G}$:

$$(\sqrt{G})_{ss} + K\sqrt{G} = 0$$

Это уравнение легко решается:

$$\sqrt{G} = \begin{cases} A\cos(a(s+B)), & \text{если } K = a^2 > 0, \\ As + B, & \text{если } K = 0, \\ A\,\mathrm{ch}(a(s+B)), & \text{если } K = -a^2 < 0, \end{cases}$$

где a - некоторая постоянная, A и B - произвольные функции от t. Однако, в силу леммы 8.46, для любого t должны иметь место следующие соотношения:

$$1 = \sqrt{G(t,0)} = \begin{cases} A\cos(aB), & \text{при } K > 0, \\ B, & \text{при } K = 0, \\ A\,\mathrm{ch}(aB), & \text{при } K < 0. \end{cases} \qquad (*)$$

Так как

$$(\sqrt{G})_s = \frac{1}{2}\frac{G_s}{\sqrt{G}} = \begin{cases} -Aa\sin(a(s+B)), & \text{при } K > 0 \\ A, & \text{при } K = 0 \\ Aa\,\mathrm{sh}(a(s+B)), & \text{при } K < 0 \end{cases}$$

для любого t имеют также место равенства

$$0 = (\sqrt{G(t,0)})_s = \begin{cases} -Aa\sin(aB), & \text{если } K = a^2 > 0, \\ A, & \text{если } K = 0, \\ Aa\,\mathrm{sh}(aB), & \text{если } K = -a^2 < 0. \end{cases} \qquad (**)$$

Осталось заметить, что из тождеств $(*)$ и $(**)$ однозначно находятся функции A и B в каждом из трех рассматриваемых случаев. Окончательно получаем:

$$\sqrt{G} = \begin{cases} \cos(as), & \text{если } K = a^2 > 0, \\ 1, & \text{если } K = 0, \\ \mathrm{ch}(as), & \text{если } K = -a^2 < 0. \end{cases}$$

Таким образом, задание постоянной гауссовой кривизны однозначно определяет первую квадратичную форму поверхности в некоторой окрестности отрезка любой натурально параметризованной геодезической на ней. Итак, доказана следующая теорема.

Theorem

8.47. Пусть $M_1 u M_2$ - двумерные регулярные поверхности постоянной ауссовой кривизны K и пусть P_i - произвольная точка на $M_i, i = 1, 2$. Тогда у точек P_i на $M_i, i = 1, 2$, существуют изометричные окрестности. Другими словами, поверхности $M_1 u M_2$ локально изометричны.

Следствие 8.48. Двумерная поверхность нулевой ауссовой кривизны локально изометрична плоскости. Двумерная поверхность положительной аауссовой кривизны a^2 локально изометрична стандартной сфере радиуса $1/a$.

Осталось выяснить, как выглядят поверхности постоянной отрицательной кривизны. Пока про них известно только, что в полугеодезических координатах их метрика имеет вид $ds^2 = du^2 + (\mathrm{ch}(au))^2 dv^2$, где a - некоторая константа (квадрат которой равен кривизне поверхности с точностью до знака).

Упражнение 8.49. Выяснить, существуют ли поверхности вращения постоянной отрицательной гауссовой кривизны. Если да, то описать их.

Рассмотрим псевдосферу в трехмерном пространстве Минковского $\mathbb{R}^3_1$. Выше метрика псевдосферы была записана в стереографических координатах. Здесь, однако, будет удобно переписать эту метрику в других координатах. Для этого сначала введем на пространстве Минковского так называемые псевдосферические координаты, положив:

$$x^1(\rho, \chi, \varphi) = \rho \,\mathrm{ch}\, \chi, \quad x^2(\rho, \chi, \varphi) = \rho \,\mathrm{sh}\, \chi \cos \varphi, \quad x^3(\rho, \chi, \varphi) = \rho \,\mathrm{sh}\, \chi \sin \varphi$$

Легко проверить, что это - действительно регулярные координаты в $\mathbb{R}^3_1$ при $\rho > 0$ и $\varphi \in (0, 2\pi)$. В то же время координатная поверхность $\rho = R$ совпадает с псевдосферой чисто мнимого радиуса iR. Наконец, псевдоевклидово скалярное произведение в этих координатах имеет вид

$$-(d\rho)^2 + \rho^2 (d\chi)^2 + \rho^2 (\mathrm{sh}\, \chi)^2 (d\varphi)^2$$

поэтому на псевдосфере радиуса iR получаем метрику следующего вида

$$ds^2 = R^2 \left((d\chi)^2 + (\mathrm{sh}\, \chi)^2 (d\varphi)^2 \right)$$

Введем новые параметры $\xi = R\chi$ и $\psi = R\varphi$. Тогда метрика псевдосферы примет вид

$$ds^2 = (d\xi)^2 + \left(\mathrm{sh}\, \frac{\xi}{R} \right)^2 (d\psi)^2$$

Для метрик такого вида выше уже вычислялась гауссова кривизна K по формуле Гаусса. Для псевдосферы радиуса iR по формуле (*) получаем $K = -1/R^2$. Итак, доказано следующее утверждение.

Statement

8.50. Гауссова кривизна псевдосферы S^2_1 радиуса iR в $\mathbb{R}^3_1$ постоянна и равна $-1/R^2$.
Из теоремы 8.47 немедленно получаем следствие.
Следствие 8.51. Двумерная поверхность отрицательной гауссовой кривизньс $-a^2$ локально изометрична псевдосфере радиуса i/a.

Упражнение 8.52. Показать, что геодезическими на круге радиуса R с метрикой Лобачевского являются дуги окружностей, пересекающие ограничивающую этот круг окружность под прямым углом, и только они.

Упражнение 8.53. Пусть на верхней полуплоскости $(x, y) \in \mathbb{R}^2 \mid y > 0$ задана риманова метрика

$$ds^2 = R^2 \frac{dx^2 + dy^2}{y^2}$$

Эта метрика называется метрикой Лобачевского на верхней полуплоскости. Вычислить гауссову кривизну верхней полуплоскости в метрике Лобачевского. Как устроены геодезические в этой метрике?

14.10 9 Lobachevsky geometry

Эта лекция посвящена геометрии метрик постоянной отрицательной кривизны - геометрии Лобачевского. Отметим, однако, что Лобачевский открыл свою знаменитую геометрию совсем из других соображений, с краткого неформального обсуждения которых мы и начнем.

14.10.1 9.1 Non-Euclidean geometries

Принято считать, что "школьная" геометрия, или евклидова геометрия, была аксиоматизирована в знаменитых "Началах" Евклида 3 в. до н.э. Свои построения Евклид базировал на так называемых постулатах, в числе которых был и знаменитый пятый постулат, который, в одной из своих многочисленных формулировок, выглядит так:

через данную точку вне прямой можно провести единственную параллельную ей (т.е. не пересекающуюся с ней) прямую.

Пятый постулат похож на теорему, поэтому многие комментаторы "Начал" Евклида пытались вывести его из остальных постулатов евклидовой геометрии, однако их попытки не увенчались успехом.

Существенной переработкой аксиом евклидовой геометрии занялся Гильберт. В его аксиоматике (1899) выделяются три основных неопределяемых понятия - точка, прямая и плоскость, и основные неопределяемые отношения: "принадлежит", "между", "конгруэнтность". Аксиомы делятся на пять групп: аксиомы принадлежности, порядка, движения, непрерывности и параллельности.

Оказывается, пятый постулат Евклида (аксиома параллельности) является независимой аксиомой: существуют геометрии, в которых выполняются все аксиомы, кроме аксиомы параллельности.

14.10.2 9.1.1 Elliptical geometry

Опишем сначала геометрию, в которой пятый постулат не выполнен, а из остальных аксиом выполняются почти все.

Конструкция 9.1. Рассмотрим стандартную сферу $S^2 \subset \mathbb{R}^3$ единичного радиуса с центром в нуле. Точками будут точки из S^2, прямыми - большие круги (пересечение сферы S^2 с плоскостями, проходящими через ее центр), а движениями - преобразования сферы, индуцированные ортогональными преобразованиями пространства $\mathbb{R}^3$.

Замечание 9.2. Ясно, что в такой геометрии вообще нет параллельных прямых. Недостаток - через диаметрально противоположные точки сферы проходит больше одной (фактически бесконечно много) прямых. От этого неприятного обстоятельства можно легко избавиться, отождествив противоположные точки x и $-x$. В результате получим проективную плоскость $\mathbb{R}P^2$ и естественную проекцию $\pi : S^2 \to \mathbb{R}P^2$. В качестве прямых на $\mathbb{RP}P^2$ рассмотрим образы больших кругов при проекции π. Теперь через две разные точки проходит в точности одна прямая. Однако имеется еще одно

неприятное обстоятельство: как на сфере, так и на $\mathbb{R}P^2$ не определено корректно понятие "между". Можно показать, что на $\mathbb{RP}^2$ выполняются все аксиомы евклидовой геометрии, за исключением аксиом параллельности и порядка. Построенная только что геометрия на $\mathbb{RP}^2$ называется эллиптической, или проективной.

14.10.3 9.1.2 Lobachevsky geometry (hyperbolic geometry)

Можно ли построить пример геометрии, где выполняются все постулаты, кроме пятого? Ответ на этот вопрос положительный. Геометрия, в которой это так, называется геометрией Лобачевского. Оказывается, чтобы ее получить, надо вместо сферы взять псевдосферу.

Конструкция 9.3. Рассмотрим пространство $\mathbb{R}^3_1$ с псевдоевклидовыми координатами (x, y, z). Псевдосфера мнимого радиуса iR (двуполостный гиперболоид) задается уравнением

$$-x^2 + y^2 + z^2 = -R^2.$$

Обозначим половинку этой псевдосферы, выделенную условием $x > 0$, через $L^2(R)$ и рассмотрим на $L^2(R)$ геометрию, в которой прямые - это пересечения $L^2(R)$ с плоскостями, проходящими через начало координат в $\mathbb{R}^3_1$, а движения - преобразования $L^2(R)$, индуцированные преобразованиями пространства $\mathbb{R}^3_1$, сохраняющими псевдоскалярное произведение и переводящими $L^2(R)$ в себя.

Определение 9.4. Поверхность $L^2(R)$ называется плоскостью Лобачевского, а определенная на ней геометрия - геометрией Лобачевского, или гиперболической геометрией.

Оказывается, построенная геометрия удовлетворяет всем аксиомам евклидовой геометрии, кроме аксиомы параллельности. Мы докажем только последнее.

Утверждение 9.5. Пусть ℓ - произвольная прямая на плоскости Лобачевского $L^2(R)$, и P - произвольная точка из $L^2(R)$, не лежащая на ℓ. Тогда на $L^2(R)$ существует бесконечно много прямых, проходящих через P и не пересекающих ℓ.

Доказательство. Действительно, пусть ℓ - прямая в $L^2(R)$, порожденная плоскостью $\mathcal{L}$, и $P \in L^2(R)$ - произвольная точка, не лежащая на ℓ. Обозначим через C множество

всех векторов из $\mathcal{L}$ неотрицательной (псевдоевклидовой) длины. Легко видеть, что это множество не пусто и ограничено пересечением плоскости $\mathcal{L}$ с изотропным конусом, поэтому само является невырожденным конусом с центром в начале координат (другими словами, C - это подмножество плоскости, ограниченное парой пересекающихся в начале координат прямых).

Рассмотрим произвольную прямую m', лежащую в C, и пусть $\mathcal{L}'$ - плоскость, проходящая через P и m'. Так как $\mathcal{L} \cap \mathcal{L}' = m'$ и прямая m' не пересекает $L^2(R)$, то соответствующая плоскости $\mathcal{L}'$ прямая ℓ' на $L^2(R)$, проходящая через P, не пересекает прямой ℓ. Из построения видно, что существует бесконечно много таких прямых ℓ' (так как имеется бесконечно много прямых m' из конуса C), что и требовалось доказать. $\square$

14.10.4 9.2 Poincaré model in a circle

Для того чтобы представить себе геометрию Лобачевского более наглядно, удобно воспользоваться другими координатами на $L^2(R)$, а именно координатами стереографической проекции (см. выше).

Конструкция 9.6. Обозначим через (x, y, z) - координаты в $\mathbb{R}^3_1$, причем будем предполагать, что псевдоскалярное произведение в $\mathbb{R}^3_1$ имеет вид $ds^2 = dx^2 + dy^2 - dz^2$. Рассмотрим стереографическую проекцию плоскости Лобачевского $L^2(R)$ из южного

полюса $(0, 0, -R)$ на координатную плоскость $Oxy = \Pi$, стандартные координаты на которой обозначим через (u, v). Стереографическая проекция описана в лемме 8.30. Она переводит $L^2(R)$ во внутренность круга $D^2(R)$ радиуса R с центром в начале координат. Метрика на плоскости Лобачевского в координатах стереографической проекции, как было показано в предыдущей лекции, имеет вид

$$ds^2 = \frac{4R^4}{\left(R^2 - u^2 - v^2\right)^2} \left((du)^2 + (dv)^2\right).$$

Определение 9.7. Круг $D^2(R)$ на плоскости с координатами (u, v), на котором задана метрика

$$ds^2 = \frac{4R^4}{\left(R^2 - u^2 - v^2\right)^2} \left((du)^2 + (dv)^2\right),$$

называется моделью Пуанкаре плоскости Лобачевского в круге или, кратко, кругом Пуанкаре. Окружность, ограничивающая круг $D^2(R)$, называется абсолютом.

Следствие 9.8. Угль между кривыми на плоскости Лобачевского в модели Пуанкаре совпадают с евклидовыми углами, т.е. углами, измеренными в евклидовой метрике.

Доказательство. Это следует из конформно-евклидовой формы метрики на круге Пуанкаре. □

Утверждение 9.9. Стереографическая проекиия $\sigma : L^2(R) \to \Pi$ переводит каждую прямую Лобачевского в дугу окружности, перпендикулярную абсолюту, или в диаметр диска $D^2(R)$.

Доказательство. Доказательство получается прямым подсчетом (см. также дополнительный материал). □

Теперь невыполнимость аксиомы параллельности становится еще более наглядной (рис. 9.1).

Замечание 9.10. Модель Пуанкаре в круге, в сущности, - это просто удобная система координат на псевдосфере. Как уже не раз отмечалось выше, в разных задачах удобны разные координаты. Поэтому естественно ожидать, что, кроме модели Пуанкаре в круге, существуют и другие стандартные модели. Для понимания некоторых из них нам понадобится понятие дробнолинейного преобразования плоскости.

Рис. 9.1. Пятый постулат в модели Пуанкаре в круге: через точку P проходит бесконечно много прямых, параллельных прямой l.

Замечание 9.11. Геометрию Лобачевского часто называют геометрией постоянной отрицательной кривизны. Как мы уже знаем, гауссова кривизна двумерной поверхности полностью задается ее первой фундаментальной формой, т.е. метрикой, поэтому гауссова кривизна естественно определяется и для области с римановой метрикой. Оказывается, что гауссова кривизна плоскости Лобачевского, как нетрудно убедиться, проведя вычисления, например в модели Пуанкаре, постоянна и равна $-1/R^2$. Кроме того, геодезические на круге Пуанкаре - суть прямые плоскости Лобачевского (см. Дополнительный материал).

14.10.5 9.3 Fractional-linear transformations

При изучении геометрии Лобачевского оказывается очень полезным комплексный язык. Напомним необходимые конструкции.

Конструкция 9.12. Каждую точку плоскости $\mathbb{R}^2$, на которой введены стандартные декартовы координаты (u, v), можно рассматривать как комплексное число $z = u + iv$, где i - мнимая единица. Также полезна так называемая тригонометрическая форма записи комплексного числа, а именно, если $(\ r, \varphi\ )$ - полярные координаты на $\mathbb{R}^2$, то комплексное число z равно $re^{i\varphi}$, где $e^{i\varphi} = \cos\varphi + i\sin\varphi$.

Утверждение 9.13. Арифметические операции над комплексными числами и комплексное сопряжение задают преобразования плоскости, являющиеся композициями сдвига, растяжения, осевой симметрии и инверсии относительно единичной окружности с иентром в нуле.

Доказательство. Пусть $c = c_1 + ic_2$ - некоторое фиксированное комплексное число. Преобразование плоскости, ставящее в соответствие каждой точке z точку $z + c$, т.е. в явном виде

$$z = u + iv \mapsto z + c = (u + c_1) + i\,(v + c_2)\,,$$

представляет собой сдвиг плоскости (т.е. параллельный перенос) на вектор c.
Далее, напомним, что комплексное число $\bar{z} = u - iv$ называется сопряженным с $z =$

$u + iv$. Преобразование, ставящее в соответствие каждому числу z число $\bar{z}$, является осевой симметрией относительно координатной оси Ou.

Пусть теперь $a = r_a e^{i\varphi_a}$ - некоторое ненулевое комплексное число, записанное в тригонометрической форме. Преобразование, ставящее в соответствие каждому числу $z = re^{i\varphi}$ число az, имеет вид

$$z = re^{i\varphi} \mapsto az = rr_a e^{i(\varphi+\varphi_a)},$$

иными словами, полярный радиус точки z увеличивается в r_a раз (растяжение плоскости в r_a раз), а полярный угол φ изменяется на φ_a (поворот плоскости на угол φ_a вокруг начала координат). Таким образом, рассмотренное преобразование есть композиция растяжения и поворота.

Наконец, рассмотрим теперь преобразование, ставящее в соответствие каждой точке z точку $1/z$. В явном виде

$$z = re^{i\varphi} \mapsto \frac{1}{r}e^{-i\varphi},$$

поэтому полярный радиус r меняется на $1/r$ (при сохранении полярного угла φ это - инверсия относительно стандартной единичной окружности с центром в нуле), а полярный угол меняется с φ на $-\varphi$ (осевая симметрия относительно оси u).

Конструкция 9.14. Пополним обычную (комплексную) плоскость $\mathbb{C} = \mathbb{R}^2$, добавив к ней бесконечность ∞. Для этого рассмотрим стереографическую проекцию ν сферы S^2 из северного полюса N на плоскость $\mathbb{C}$ и отождествим точки из $S^2 \backslash N$ с их образами на плоскости, а точку N - с бесконечностью. Тем самым, расширенная комплексная плоскость $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ - это вся сфера S^2.

Определение 9.15. Расширенной комплексной плоскостью $\overline{\mathbb{C}}$ называется сфера S^2 вместе с описанным только что отождествлением ее точек с комплексными числами.

Конструкция 9.16. Пусть a, b, c, d - комплексные числа. Рассмотрим отображение, заданное формулой

$$\Phi : z \mapsto \frac{az+b}{cz+d}$$

Отметим, что Φ определено на всей плоскости $\mathbb{C}$, за исключение точки $-d/c$, в которой знаменатель дроби, стоящей в правой части, обращается в ноль. Кроме того, отображение Φ есть отображение в точку, если и только если $ad-bc = 0$ (в этом случае числитель дроби пропорционален знаменателю), поэтому, так как нас будут интересовать преобразования плоскости, мы будем предполагать, что $ad - bc \neq 0$. Доопределим Φ до отображения расширенной комплексной плоскости на себя, положив $\Phi(-d/c) = \infty$, а $\Phi(\infty) = a/c$.

Определение 9.17. Построенное отображение $\Phi : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ называется дробно-линейньм преобразование плоскости с матрицей $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, которая, в свою очередь, называется матрицей дробно-линейного преобразования Φ.

Утверждение 9.18. Каждое дробно-линейное преобразование представимо в одном из двух видов:

$$z \mapsto Az + B \quad \text{или} \quad z \mapsto \frac{A}{z+B} + C,$$

где A, B, C - некоторые комплексные числа. Поэтому каждое дробно-линейное преобразование есть или композиция поворота, растяжения и сдвига (в первом случае), или композиция сдвига, инверсии, осевой симметрии, поворота, растяжения и сдвига (во втором случае).

Доказательство. Возможны следующие два случая: $c = 0$ и $c \neq 0$. В первом случае $d \neq 0$ (матрица преобразования невырождена), поэтому

$$\frac{az+b}{cz+d} = \frac{a}{d}z + \frac{b}{d} = Az + B$$

т.е. дробно-линейное преобразование есть композиция поворота, растяжения и сдвига. Во втором случае

$$\frac{az+b}{cz+d} = \frac{a(z+d/c)+b-ad/c}{c(z+d/c)} = \frac{A}{z+B} + C$$

поэтому рассматриваемое преобразование есть композиция сдвига, инверсии, осевой симметрии, поворота, растяжения и сдвига. Утверждение доказано. □

Следствие 9.19. Каждое дробно-линейное преобразование переводит прямые и окружности в прямые и окружности, а также сохраняет угль между кривыми.

Доказательство. Действительно, этими свойствами обладают все преобразования, перечисленные в утверждении 9.18, а значит и их композиции. □

Утверждение 9.20. Множество всех дробно-линейных преобразований расширенной комплексной плоскости образует группу относительно операции композиции. В частности, дробно-линейные преобразования - действительно преобразования, т.е. взаимно однозначные отображения расширенной плоскости на себя.

Доказательство. Рассмотрим композицию $\mu \circ \nu$ двух дробно-линейных преобразований:

$$\mu : z \mapsto \frac{az+b}{cz+d} \quad \text{и} \quad \nu : z \mapsto \frac{a'z+b'}{c'z+d'}.$$

Имеем:

$$\mu \circ \nu(z) = \frac{a\frac{a'z+b'}{c'z+d'} + b}{c\frac{a'z+b'}{c'z+d'} + d} = \frac{(aa'+bc')\,z + ab' + bd'}{(ca'+dc')\,z + cb' + dd'},$$

т.е. эта композиция также является дробно-линейным преобразованием, причем его матрица $M_{\mu\circ\nu}$ совпадает с произведением матриц $M_\mu = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ и $M_\nu = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ преобразований μ и ν, $M_{\mu\circ\nu} = M_\mu M_\nu$, поэтому невырождена. Ассоциативность композиции имеет место для произвольных отображений и хорошо известна. Тождественное

преобразование $z \;:\!\mapsto\; z$, которое, очевидно, является дробно линейным (ему соответствует единичная матрица), представляет собой нейтральный элемент. Наконец, дробно-линейное преобразование с матрицей $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}$ будет обратным для дробнолинейного преобразования с матрицей $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, поскольку матрица композиции есть произведение матриц. Таким образом, множество дробно линейных преобразований - группа. Из обратимости дробно-линейного преобразования вытекает его взаимно однозначность. Утверждение доказано.

Конструкция 9.21. Обозначим через $\mathrm{GL}(2,\mathbb{C})$ группу невырожденных комплексных (2×2)-матриц. Рассмотрим отображение ψ из группы $\mathrm{GL}(2,\mathbb{C})$ в группу дробнолинейных преобразований:

$$\psi : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az+b}{cz+d} \right)$$

Утверждение 9.22. Отображение ψ является эпиморфизмом (т.е. сюрьективным гомоморфизмом групп). Группа дробно-линейных преобразований изоморфна группе $\mathrm{GL}(2,\mathbb{C})/\mathbb{C}^*$.

Доказательство. Действительно, $\psi(AB) = \psi(A) \circ \psi(B)$ для любых $A, B \in \mathrm{GL}(2,\mathbb{C})$, что мы, фактически проверили при доказательстве утверждения 9.20. Далее, если M_μ - матрица дробно линейного преобразования μ, то $\psi(M_\mu) = \mu$, поэтому ψ - сюръекция. Вычислим теперь ядро эпиморфизма ψ. Если $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ задает тождественное дробно-линейное отображение $z \mapsto z$, то

$$z = \frac{az+b}{cz+d}, \quad \text{при всех } z, \text{ откуда} \quad cz^2 + dz = az + b, \Rightarrow \quad c = b = 0, a = d.$$

Итак, ядро отображения ψ состоит из всех диагональных невырожденных матриц вида $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ и изоморфно группе $\mathbb{C}^*$ всех ненулевых комплексных чисел (с операцией умножения). Остается применить теорему о гомоморфизме. Утверждение доказано.

Конструкция 9.23. Множество комплексных (2×2)-матриц с единичным определителем образует группу, которая, напомним, называется комплексной специальной линейной группой, обозначается через $\mathrm{SL}(2,\mathbb{C})$ и является подгруппой в группе $\mathrm{GL}(2,\mathbb{C})$. Обозначим через ψ' ограничение отображения ψ, описанного в конструкции 9.21, на $\mathrm{SL}(2,\mathbb{C})$.

Имеет место следующий аналог утверждения 9.22, доказательство которого мы оставляем читателю в качестве упражнения.

Утверждение 9.24. Отображение ψ' - эпиморфизм. Группа дробно-линейных преобразований изоморфна группе $\mathrm{SL}(2,\mathbb{C})/\{\pm E\}$, где через $E \in \mathrm{SL}(2,\mathbb{C})$ обозначена единичная матрица, т.е. изоморфна группе $\mathrm{SL}(2,\mathbb{C})/\mathbb{Z}_2$.

14.10.6 9.4 Recording the metric in a comprehensive form

Работая с дробно-линейными (комплексными) преобразованиями плоскости, удобно уметь записывать метрику на ней в комплексной форме. Особенно удобным это оказывается в случае конформно-евклидовых метрик.

Конструкция 9.25. Пусть на плоскости со стандартными координатами u, v задана конформно-евклидова метрика $ds^2 = \lambda(u,v)\,(du^2 + dv^2)$. Поскольку $z = u + iv$ и $\bar{z} = u - iv$, введем в рассмотрение следующие комплексные дифференциалы:

$$dz = du + idv, \quad d\bar{z} = du - idv$$

Легко сосчитать, что $du^2 + dv^2 = dzd\bar{z}$, поэтому конформно-евклидова метрика $ds^2 = \lambda(u,v)\,(du^2 + dv^2)$ в комплексном виде выглядит так:

$$ds^2 = \mu(z,\bar{z})dzd\bar{z}$$

где функция $\mu(z,\bar{z})$ получается из $\lambda(u,v)$ подстановкой $u = (z+\bar{z})/2, v = (z - bz)/(2i)$.
Замечание 9.26. Поясним конструкцию 9.25 более формально. Пусть $\mathbb{R}^2$ - стандартная плоскость с евклидовыми координатами u, v. Соответствующий базис в линейном пространстве $\mathbb{R}^2$ - это базис $\{\partial_u, \partial_v\}$. Рассмотрим комплексификацию $\mathbb{C}^2$ пространства $\mathbb{R}^2$, т.е. $\mathbb{C}^2$ - это множество всевозможных линейных комбинаций векторов ∂_u и ∂_v, но уже с комплексными коэффициентами. Ясно, что исходное $\mathbb{R}^2$ - подпространство в $\mathbb{C}^2$ и, кроме того, $\{\partial_u, \partial_v\}$ - базис в $\mathbb{C}^2$. Выберем теперь в $\mathbb{C}^2$ новый базис $\{\partial_z, \partial_{\bar{z}}\}$, т.е. сделаем линейную замену координат, положив

$$\partial_z = \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}\right), \quad \partial_{\bar z} = \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial u} + i\frac{\partial}{\partial v}\right).$$

Если (u, v) - координаты некоторого вектора V из $\mathbb{C}^2$ в базисе $\{\partial_u, \partial_v\}$, то координаты $(z, \bar z)$ этого вектора в базисе $\{\partial_z, \partial_{\bar z}\}$ имеют вид $z = u + iv$ и $\bar z = u - iv$. Соответствующие линейные формы dz и $d\bar z$ связаны с линейными формами du и dv так: $dz = du + idv$ и $d\bar z = du - idv$.

Далее, любая риманова метрика на плоской области, как мы знаем, - это билинейная форма, заданная в каждой точке области (на соответствующем касательном пространстве, которое отождествляется с $\mathbb{R}^2$). Она задает билинейную форму (комплекснозначную) на комплексификации $\mathbb{C}^2$ (с той же матрицей в том же исходном базисе). Именно эта форма в базисе $\{\partial_z, \partial_{\bar z}\}$ имеет указанный в конструкции 9.25 вид. Интересующая нас метрика - это ограничение этой формы на подпространство $\mathbb{R}^2 \subset \mathbb{C}^2$.

Пример 9.27. Приведем примеры комплексной записи известных метрик.
(1) $ds^2 = dzd\bar z$ (метрика плоскости), $z = u+iv, (u, v)$ - стандартные евклидовы координаты;
(2) $ds^2 = \frac{4R^4 dzd\bar z}{(R^2+|z|^2)^2}$ (метрика сферы радиуса R), $z = u + iv, (u, v)$ - стереографические координаты;
(3) $ds^2 = \frac{4R^4 dzd\bar z}{(R^2-|z|^2)^2}$ (метрика плоскости Лобачевского $L^2(R)$ в модели Пуанкаре в круге), $z = u + iv, (u, v)$ - стереографические координаты.

Утверждение 9.28. Дробно-линейное преобразование сохраняет конформно-евклидов вид метрики. В частности, для конформно-евклидовых метрик дробно-линейные преобразования сохраняют углы между кривыми.
Доказательство. Пусть $z = \frac{aw+b}{cw+d}$ - дробно-линейное преобразование. Тогда

$$dz = \frac{a(cw + d) - c(aw + b)}{(cw + d)^2}dw = \frac{ad - cb}{(cw + d)^2}dw, \quad d\bar z = \frac{\overline{ad - cb}}{\overline{(cw + d)^2}}d\bar w$$

поэтому

$$ds^2 = \lambda dzd\bar z = \lambda\frac{|ad - cb|^2}{|cw + d|^4}dwd\bar w$$

т.е. метрика по-прежнему имеет конформно-евклидов вид, что и требовалось. □

14.10.7 9.5 Top half-plane model

Мы будем искать изометрии плоскости Лобачевского среди дробно-линейных преобразований. Для этого мы построим еще одну модель плоскости Лобачевского.

Конструкция 9.29. Выберем на плоскости Лобачевского в модели Пуанкаре в круге новую систему координат. Для этого рассмотрим дробно-линейное преобразование плоскости, заданное в виде

$$w = -iR\frac{z + iR}{z - iR}$$

Это дробно-линейное преобразование отображает круг Пуанкаре на верхнюю (открытую) полуплоскость. Чтобы проверить это, выясним сначала, куда перейдет окружность $S^1 = \{|z| = R\}$, т.е. абсолют. Так как дробно-линейные преобразования переводят прямые и окружности в прямые и окружности, то образ каждой окружности однозначно задается образами любых трех ее точек. В качестве таких точек возьмем $-R, -iR$ и R. Имеем:

$$-R \mapsto -iR\frac{-R+iR}{-R-iR} = -iR\frac{-1+i}{-1-i} = -iR\frac{(-1+i)(-1+i)}{2} = -R,$$
$$-iR \mapsto 0,$$
$$R \mapsto -iR\frac{R+iR}{R-iR} = -iR\frac{1+i}{1-i} = -iR\frac{(1+i)(1+i)}{2} = -iR\frac{2i}{2} = R,$$

поэтому абсолют переходит в вещественную координатную прямую. Отсюда следует, что внутренность круга $D^2(R)$ при рассматриваемом преобразовании переходит или в верхнюю, или в нижнюю полуплоскость. Но поскольку точка $0 \in D^2(R)$ переходит в точку iR, внутренность $D^2(R)$ переходит именно в верхнюю полуплоскость. Из следствия 9.19 вытекает, что прямые модели Пуанкаре переходят в окружности и прямые, перпендикулярные вещественной оси.

Утверждение 9.30. В модели верхней полуплоскости метрика Лобачевского имеет вид

$$ds^2 = R^2\frac{du^2+dv^2}{v^2} = -\frac{4R^2 dw d\bar{w}}{(w-\bar{w})^2}$$

где $w = u + iv-$ соответствующая комплексная координата.

Доказательство. Доказательство получается прямым подсчетом, аналогичным приведенному в доказательстве утверждения 9.28, см. также дополнительный материал.

Определение 9.31. Верхняя полуплоскость $v > 0$ с метрикой $ds^2 = (du^2 + dv^2)/v^2$, где u, v - стандартные координаты на плоскости, называетсямоделью Пуанкаре плоскости Лобачевского в верхней полуплоскости.

14.10.8 9.6 Lobachevsky plane isometries

Опишем дробно-линейные преобразования плоскости, являющиеся изометриями плоскости Лобачевского в модели Пуанкаре в верней полуплоскости. Для простоты в дальнейших вычислениях положим $R = 1$.

Определение 9.32. Дробно-линейное преобразование называется вещественным, если все коэффициенты a, b, c и d его матрицы вещественны.

Утверждение 9.33. Вещественное дробно-линейное преобразование сохраняет верхнюю полуплоскость, если и только если матрица этого преобразования имеет положительный определитель.

Доказательство. Действительно, вещественная ось переходит, очевидно, в вещественную ось, поэтому достаточно вычислить мнимую часть образа такого преобразования, чтобы понять, на какую именно полуплоскость отображается верхняя полуплоскость. Имеем

$$\operatorname{Im} z = \operatorname{Im}\left(\frac{aw+b}{cw+d}\right) = \operatorname{Im}\left(\frac{ac|w|^2+bd+adw+bc\bar{w}}{|cw+d|^2}\right) = \frac{(ad-bc)\operatorname{Im} w}{|cw+d|^2}$$

поэтому, для любого w, такого, что $\operatorname{Im} w > 0$, условие $\operatorname{Im} z > 0$ выполнено, если и только если $ad - bc > 0$, что и требовалось.

Замечание 9.34. На самом деле, не сложно показать, что любое дробно-линейное преобразование, сохраняющее верхнюю полуплоскость, может быть представлено в виде вещественного с положительным определителем. Обычно это утверждение доказывают в курсе теории функций комплексного переменного.

Утверждение 9.35. Множество всех вещественных дробно-линейных преобразований с положительным определителем образует группу, изоморфную $\mathrm{SL}(2,\mathbb{R})/\{\pm E\}$, где E - единичная матрица.

Доказательство. Действительно, группа $\mathrm{SL}(2,\mathbb{R})$ является подгруппой в $\mathrm{SL}(2,\mathbb{C})$, образ ограничения ψ'' на нее гомоморфизма ψ' из утверждения 9.24 состоит из вещественных дробно-линейных преобразований с определителем 1. С другой стороны, если $z = \frac{aw+b}{cw+d}$ - вещественное преобразование с положительным определителем Δ, то поделив числитель и знаменатель на вещественное число $\sqrt{\Delta}$, мы не изменим наше преобразование, однако превратим матрицу $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ в вещественную матрицу с единичным определителем, поэтому образ ψ'' - это множество всех вещественных дробно-линейных преобразований с положительным определителем. Поэтому последнее образует группу, изоморфную фактор-группе по ядру $\{\pm E\}$ гомоморфизма ψ'', что и требовалось.

Утверждение 9.36. Вещественные дробно-линейные преобразования, сохраняющие верхнюю полуплоскость, являются движениями плоскости Лобачевского.
Доказательство. Рассмотрим такое дробно-линейное преобразование $z = \frac{aw+b}{cw+d}$, т.е., в силу утверждения 9.35, преобразование с матрицей из $\mathrm{SL}(2,\mathbb{R})$. Так как

$$ds^2 = -\frac{4dzd\bar{z}}{(z-\bar{z})^2}$$

$$ds^2 = -\frac{4\frac{a(cw+d)-c(aw+b)}{(cw+d)^2}dw\frac{a(c\bar{w}+d)-c(a\bar{w}+b)}{(c\bar{w}+d)^2}d\bar{w}}{\left(\frac{aw+b}{cw+d}-\frac{a\bar{w}+b}{c\bar{w}+d}\right)^2}$$

то

Последнее выражение преобразуем к виду

$$-\frac{4\frac{ad-cb}{(cw+d)^2}\frac{ad-cb}{(c\bar{w}+d)^2}dwd\bar{w}}{\left(\frac{(aw+b)(c\bar{w}+d)-(cw+d)(a\bar{w}+b)}{(cw+d)(c\bar{w}+d)}\right)^2} = -\frac{4\frac{(ad-cb)^2}{|cw+d|^4}}{\left(\frac{(ad-bc)(w-\bar{w})}{|cw+d|^2}\right)^2}$$

и получим

$$ds^2 = -\frac{4dwd\bar{w}}{(w-\bar{w})^2}$$

что и требовалось доказать.
Итак, доказана следующая теорема.

Theorem

9.37. Группа всех движений плоскости Лобачевского в модели Пуанкаре в верхней полуплоскости содержит подгруппу, состоящую из вещественных дробнолинейных преобразований, сохраняющих верхнюю полуплоскость. Эта подгруппа изоморфна группе $\mathrm{SL}(2,\mathbb{R})/\{E,-E\}$.
Замечание 9.38. Чтобы получить полную группу движений плоскости Лобачевского в модели верхней полуплоскости, надо добавить еще композиции только что описанных вещественных дробно-линейных преобразований с осевой симметрией $z \mapsto -\bar{z}$.

14.10.9 9.7 Problems

Задача 9.39. Доказать, что если $w = f(z)$ - дробно-линейное отображение, то для любых четырех точек z, z_1, z_2, z_3 и их образов w, w_1, w_2, w_3 выполняется следующее равенство:

$$\frac{z-z_1}{z-z_3}:\frac{z_2-z_1}{z_2-z_3}=\frac{w-w_1}{w-w_3}:\frac{w_2-w_1}{w_2-w_3}.$$

Иными словами, двойное отношение

$$\frac{z-z_1}{z-z_3}:\frac{z_2-z_1}{z_2-z_3}$$

сохраняется при дробно-линейных преобразованиях. Вывести отсюда, что любые три различные точки можно перевести в любые другие три различные точки с помощью некоторого дробно-линейного преобразования, причем такое преобразование единственно.

Задача 9.40. Задайте дробно-линейное преобразование $z \mapsto -z$ матрицей из $\mathrm{SL}(2,\mathbb{C})$.
Задача 9.41. Сколько неподвижных точек может иметь в вещественное дробно-линейное преобразование с положительным определителем? Как они расположены?

Задача 9.42. Доказать, что отрезок прямой на плоскости Лобачевского между точками A и B есть кратчайшая кривая, соединяющая эти точки.

Задача 9.43. Расстояние между точками на плоскости Лобачевского - это длина отрезка прямой Лобачевского между этими точками. Найти расстояние между точками на плоскости Лобачевского в модели Пуанкаре и в модели верхней полуплоскости.

Задача 9.44. Как выглядят окружности на плоскости Лобачевского? Вычислить длину окружности и площадь круга радиуса ρ на плоскости Лобачевского (радиус понимается в смысле метрики на плоскости Лобачевского).

Задача 9.45. Пусть ABC - треугольник на плоскости Лобачевского; α, β, γ - величины углов при вершинах $A, B, C; a, b, c$ - длины сторон BC, AC, AB. Доказать две теоремы косинусов:

$$\operatorname{ch} c = \operatorname{ch} a \operatorname{ch} b - \operatorname{sh} a \operatorname{sh} b \cos\gamma$$
$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta\operatorname{ch} c$$

Доказать теорему синусов на плоскости Лобачевского:

$$\frac{\operatorname{sh} a}{\sin\alpha}=\frac{\operatorname{sh} b}{\sin\beta}=\frac{\operatorname{sh} c}{\sin\gamma}$$

Задача 9.46. Во всякий ли треугольник на плоскости Лобачевского можно вписать окружность? Всякий ли треугольник на плоскости Лобачевского можно описать окружностью?

14.11 9.8 Lobachevsky geometry: Supplemental material

14.11.1 9.8.1 Stereographic Projection Properties

Проверим, что стереографическая проекция $\sigma : L^2(R) \to \Pi$ переводит каждую прямую Лобачевского в дугу окружности, перпендикулярную абсолюту, или в диаметр диска $D^2(R)$, т.е. докажем утверждение 9.9 .

Пусть ℓ - прямая на плоскости Лобачевского, полученная пересечением плоскости Лобачевского $L^2(R)$ с плоскостью $\mathcal{L}$, проходящей через начало координат. Если $\mathcal{L}$ проходит через северный полюс $N = (0,0,R)$, то южный полюс S и все прямые, соединяющие S и точки из ℓ, лежат в плоскости $\mathcal{L}$, поэтому стереографический образ прямой ℓ есть диаметр диска $D^2(R)$, полученный пересечением плоскости $\mathcal{L}$ и $D^2(R)$, проходящей через центр круга $D^2(R)$.

Пусть теперь ℓ не проходит через северный полюс N. Пусть плоскость $\mathcal{L}$ задается уравнением $ax + by + cz = 0$. Из сделанных выше предположений следует, что c отлично

от нуля. Подставляя в это уравнение выражение для x, y и z через стереографические координаты u и v, получаем

$$ax + by + cz = a\frac{2R^2u}{R^2 - u^2 - v^2} + b\frac{2R^2v}{R^2 - u^2 - v^2} + cR\frac{R^2 + u^2 + v^2}{R^2 - u^2 - v^2} = \\ = \frac{2aR^2u + 2bR^2v + cR\left(R^2 + u^2 + v^2\right)}{R^2 - u^2 - v^2}$$

откуда, разделив на $cR \neq 0$, имеем

$$2\frac{aR}{c}u + 2\frac{bR}{c}v + \left(R^2 + u^2 + v^2\right) = 0 \Rightarrow \\ \Rightarrow \left(u + \frac{aR}{c}\right)^2 + \left(v + \frac{bR}{c}\right)^2 = R^2\left(\frac{a^2 + b^2}{c^2} - 1\right).$$

Последнее уравнение задает некоторую окружность $\bar{S}^1$ в плоскости П. Радиус R абсолюта S^1, радиус $R\sqrt{\left(a^2 + b^2\right)/c^2 - 1}$ окружности $\bar{S}^1$ и расстояние между центрами окружностей S^1 и $\bar{S}^1$, т.е. $R\sqrt{\left(a^2 + b^2\right)/c^2}$, образуют пифагорову тройку: сумма квадратов радиусов равна квадрату расстояния между центрами. Поэтому рассматриваемые окружности ортогональны, что и требовалось доказать.

14.11.2 9.8.2 Metric in the upper half-plane model

Докажем утверждение 9.30. Напомним, что дробно-линейное преобразование, переводящее круг Пуанкаре в верхнюю полуплоскость, можно записать, как $w = -iR\frac{z+iR}{z-iR}$, а метрика Лобачевского в модели Пуанкаре имеет (в комплексной форме) вид $ds^2 = \frac{4R^4 dz d\bar{z}}{\left(R^2 - |z|^2\right)^2}$.

Найдем обратное дробно-линейное преобразование, переводящее модель верхней полуплоскости в модель Пуанкаре. В силу утверждения 9.22, обратное преобразование задается обратной матрицей и поэтому имеет вид:

$$z = iR\frac{-R - iw}{R - iw} = iR\frac{w - iR}{w + iR}$$

Отсюда получаем:

$$dz = -\frac{2R^2}{(w + iR)^2}dw, \quad d\bar{z} = -\frac{2R^2}{(w + iR)^2}d\bar{w}, \quad dzd\bar{z} = \frac{4R^4 dw d\bar{w}}{|w + iR|^4}$$

Далее,

$$\left(R^2 - |z|^2\right)^2 = R^4\left(1 - \frac{w - iR}{w + iR}\frac{\overline{w - iR}}{\overline{w + iR}}\right)^2 = -\frac{4R^6(w - \bar{w})^2}{|w + iR|^4}$$

поэтому

$$ds^2 = \frac{4R^4 dz d\bar{z}}{\left(R^2 - |z|^2\right)^2} = -\frac{\frac{16R^8 dw d\bar{w}}{|w+iR|^4}}{\frac{4R^6(w-\bar{w})^2}{|w+iR|^4}} = -\frac{4R^2 dw d\bar{w}}{(w - \bar{w})^2} = R^2\frac{du^2 + dv^2}{v^2}$$

14.11.3 9.8.3 Geodetic on the plane of Lobachevsky

В этом разделе опишем геодезические на плоскости Лобачевского. Так как отражение относительно вертикальной прямой является изометрией плоскости Лобачевского в модели верхней полуплоскости и соответствующий вертикальный луч состоит из неподвижных точек этой изометрии, то, в силу утверждения б.36, каждый вертикальный луч является геодезической на плоскости Лобачевского в модели верхней полуплоскости. Далее, так как каждая прямая на плоскости Лобачевского может быть переведена изометрией в вертикальный луч, то каждая прямая на плоскости Лобачевского является геодезической. Сославшись на теорему существования и единственности геодезических, получаем следующий результат.

Statement

9.47. Геодезические на плоскости Лобачевского в модели верхней полуплоскости суть прямые Лобачевского в этой модели, и только они.

Производя подходящую замену координат, получаем следствие.

Следствие 9.48. Геодезические на плоскости Лобачевского в модели Пуанкаре суть прямые Лобачевского в этой модели, и только они. Геодезические на псевдосфере чисто мнимого радиуса в метрике, индучированной из объемлющего пространства Минковского, суть сечения двумерными плоскостями, проходящими через начало координат.

Упражнение 9.49. Для каждой прямой Лобачевского в модели верхней полуплоскости найдите изометрию, оставляющую на месте все точки этой прямой.

14.11.4 9.8.4 Lobachevsky plane motions in the Poincaré model in a circle

Чтобы описать движения плоскости Лобачевского в модели Пуанкаре, достаточно сделать замену координат. Приведем соответствующий результат.

Дробно-линейные преобразования, сохраняющие единичный круг, являются движениями метрики Лобачевского в модели Пуанкаре. Матрицы $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ таких преобразований удовлетворяют условиям

$$|a|^2 - |c|^2 = 1, \quad |b|^2 - |d|^2 = -1, \quad a\bar{b} - c\bar{d} = 0$$

и поэтому записываются в виде

$$\begin{pmatrix} a & b \\ \bar{b} & \bar{a} \end{pmatrix}, \text{ где } |a|^2 - |b|^2 = 1.$$

Группа таких матриц обозначается через $\mathrm{SU}(1,1)$.

Другое удобное представление дробно-линейных преобразований, сохраняющих единичный круг, -

$$z \mapsto e^{i\varphi}\frac{z - z_0}{1 - \bar{z}_0 z}, \quad \varphi \in \mathbb{R}, \quad |z_0| < 1$$

Чтобы получить полную группу движений плоскости Лобачевского, необходимо добавить сопряжение $z \mapsto \bar{z}$.

Упражнение 9.50. Для каждой прямой Лобачевского в модели Пуанкаре найдите изометрию, оставляющую на месте все точки этой прямой.

14.11.5 9.8.5 Distance between points on the Lobachevsky plane

Приведем ряд фактов, иллюстрирующих метрические соотношения на плоскости Лобачевского.

Упражнение 9.51. Рассмотрим модель Пуанкаре $D^2(R)$. Пусть O центр круга $D^2(R)$. Диаметры круга $D^2(R)$, по определению, являются прямыми в геометрии Лобачевского. Показать, что прямолинейный отрезок OA в метрике Лобачевского имеет длину

$$l_L = 2R \operatorname{arcth} \frac{l_E}{R}$$

где $l_E = |OA|$ - евклидова длина отрезка OA.

Упражнение 9.52. Показать, что длина окружности радиуса ρ с центром в точке O на плоскости Лобачевского $L^2(R)$ равна $2\pi R \operatorname{sh}(\rho/R)$.

Упражнение 9.53. Рассмотрим модель верхней полуплоскости. Пусть P_1 и P_2 - точки на верхней полуплоскости, имеющие полярные координаты $(\ r_0, \varphi_i\ )$, $i = 1, 2$. Через P_1 и P_2 проходит единственная прямая на плоскости Лобачевского, которая совпадает с координатной линией $r = r_0$ полярных координат. Показать, что длина отрезка P_1P_2 прямой на плоскости Лобачевского может быть вычислена по формуле

$$|P_1P_2|_L = \log\left(\frac{\operatorname{th}(\varphi_2/2)}{\operatorname{th}(\varphi_1/2)}\right)$$

Упражнение 9.54. Пусть P_1 и P_2 - точки на верхней полуплоскости, лежащие на одной вертикальной прямой, и пусть $(\ x_i, y_i\ )$ - их декартовы координаты. Показать, что длина отрезка P_1P_2 прямой на плоскости Лобачевского может быть вычислена по формуле

$$|P_1P_2|_L = \log\left(\frac{y_2}{y_1}\right)$$

Упражнение 9.55. Вычислить длину отрезка прямой на плоскости Лобачевского, соединяющего любые две точки P_1 и P_2 верхней полуплоскости.

14.11.6 9.8.6 Triangles on the Lobachevsky plane

Соотношения между элементами треугольника на плоскости Лобачевского в целом аналогичны имеющимся на евклидовой плоскости, хотя вид конкретных формул довольно сильно отличается. Докажем здесь несколько характерных утверждений и приведем ряд задач, позволяющих получить более полное представление о геометрии Лобачевского. Всюду ниже длину единственного отрезка прямой на плоскости Лобачевского, соединяющего две фиксированных точки A и B, вычисленную в метрике Лобачевского, будем обозначать через $|AB|_L$. Отметим также, что при доказательстве разных утверждений удобно пользоваться разными моделями геометрии Лобачевского.

Statement

9.56 (Неравенство треугольника). Пусть ABC - произволъный треуголъник на плоскости Лобачевского. Тогда имеет место следующее неравенство:

$$|AB|_L + |BC|_L \geq |AC|_L$$

Доказательство. Проведем доказательство непосредственно на псевдосфере S_1^2 радиуса iR в $\mathbb{R}_1^3$. Прямым на плоскости Лобачевского, напомним, соответствуют плоские сечения половинки псевдосферы $S_1^2\,(x^1 > 0)$ плоскостями, проходящими через нуль. Пусть A и B - произвольные точки на этой половине псевдосферы. Обозначим через e_A

и e_B соответствующие векторы в $\mathbb{R}^3_1$. Рассмотрим плоскость Π_{AB}, проходящую через нуль и точки A и B.

Поскольку псевдоскалярный квадрат векторов e_A и e_B равен $-R^2$, псевдоскалярное произведение, индуцированное на Π_{AB} из $\mathbb{R}^3_1$, - индефинитно и невырождено, поэтому Π_{AB} - это $\mathbb{R}^2_1$. Ясно, что на Π_{AB}, как и на любой плоскости Минковского, можно сделать такую линейную замену координат, сохраняющую псевдоскалярное произведение, что в новых координатах, которые обозначим через $(\ x, y\ )$, псевдоскалярное произведение запишется в виде $-dx^2 + dy^2$. Пересечение плоскости Π_{AB} с псевдосферой чисто мнимого радиуса iR - это псевдо-окружность того же радиуса iR, которая в координатах $(\ x, y\ )$ имеет вид $-x^2 + y^2 = -R^2$.

Пусть вектор e_A имеет координаты (x_A, y_A), а вектор $e_B - (x_B, y_B)$. Вычислим длину дуги гиперболы между точками A и B. Для этого параметризуем гиперболу в виде $x(\chi) = R\operatorname{ch}\chi, y(\chi) = R\operatorname{sh}\chi$, положим $x_A = R\operatorname{ch}\chi_A, y_A = R\operatorname{sh}\chi_A, x_B = R\operatorname{ch}\chi_B, y_B = R\operatorname{sh}\chi_B$ и $\chi_B > \chi_A$. Тогда

$$|AB|_L = \int_{\chi_A}^{\chi_B} \sqrt{-R^2(\operatorname{sh}\chi)^2 + R^2(\operatorname{ch}\chi)^2}d\chi =$$
$$= R\left(\chi_B - \chi_A\right) = R\left(\operatorname{arcch}\left(x_B/R\right) - \operatorname{arcch}\left(x_A/R\right)\right)$$

Теперь, воспользовавшись тем, что $\operatorname{ch}(\alpha - \beta) = \operatorname{ch}\alpha\operatorname{ch}\beta - \operatorname{sh}\alpha\operatorname{sh}\beta$, а также основным гиперболическим тождеством $\operatorname{sh}\alpha = \sqrt{(\operatorname{ch}\alpha)^2 - 1}$, получим:

$$\operatorname{ch}\left(\frac{|AB|_L}{R}\right) = \frac{x_A x_B}{R^2} - \frac{\sqrt{x_A^2 - R^2}\sqrt{x_B^2 - R^2}}{R^2} = \frac{x_A x_B - y_A y_B}{R^2} = \frac{-<e_A, e_B>}{R^2},$$

где через $\langle e_A, e_B\rangle$ обозначено псевдоскалярное произведение векторов e_A и e_B. Поэтому

$$|AB|_L = R\operatorname{arcch}\left(\frac{-\langle e_A, e_B\rangle}{R^2}\right).$$

Полученный нами результат имеет и самостоятельный интерес. Напомним, что поскольку $\cos(x/i) = \operatorname{ch}x$, то предыдущее соотношение можно переписать в виде

$$\cos\left(\frac{|AB|_L}{iR}\right) = \frac{\langle e_A, e_B\rangle}{(iR)^2}.$$

Statement

9.57. Косинус отношения длины дуги AB псевдо-окружности к ее радиусу равен псевдоскалярному произведению радиус-векторов точек A и B, деленному на квадрат радиуса окружности. B этом смъсле отношение длины дуги окружности κ радиусу может играть роль меры угла между радиус-векторами концевых точек этой дуги.

Вернемся к доказательству утверждения. Пусть A, B и C - произвольные точки на половинке $S^2_1\,(x^1 > 0\ )$ псевдосферы и e_A, e_B и e_C - соответствующие векторы из $\mathbb{R}^3_1$. Из только что полученной формулы следует, что неравенство треугольника эквивалентно соотношению

$$\operatorname{arcch}\left(\frac{-\langle e_A, e_B\rangle}{R^2}\right) + \operatorname{arcch}\left(\frac{-\langle e_B, e_C\rangle}{R^2}\right) \geq \operatorname{arcch}\left(\frac{-\langle e_A, e_C\rangle}{R^2}\right).$$

Отметим, что все попарные скалярные произведения времениподобных векторов e_A, e_B и e_C отрицательны, поэтому, применяя к последнему неравенству функцию ch и

воспользовавшись формулой гиперболического косинуса суммы, получим следующее эквивалентное неравенство:

$$\langle e_A, e_B\rangle \langle e_B, e_C\rangle + \sqrt{\langle e_A, e_B\rangle^2 - R^4}\sqrt{\langle e_B, e_C\rangle^2 - R^4} \geq -R^2 \langle e_A, e_C\rangle ,$$

которое можно переписать в виде

$$\sqrt{\langle e_A, e_B\rangle^2 - R^4}\sqrt{\langle e_B, e_C\rangle^2 - R^4} \geq -R^2 \langle e_A, e_C\rangle - \langle e_A, e_B\rangle \langle e_B, e_C\rangle$$

Возводя обе части в квадрат, имеем

$$\begin{aligned}\langle e_A, e_B\rangle^2 \langle e_B, e_C\rangle^2 - R^4 \langle e_A, e_B\rangle^2 - R^4 \langle e_B, e_C\rangle^2 + R^8 \\ \geq R^4 \langle e_A, e_C\rangle^2 + \langle e_A, e_B\rangle^2 \langle e_B, e_C\rangle^2 + 2R^2 \langle e_A, e_C\rangle \langle e_A, e_B\rangle \langle e_B, e_C\rangle\end{aligned}$$

или, приводя подобные, перенося все в одну сторону и сокращая на R^2,

$$0 \geq -R^6 + 2 \langle e_A, e_C\rangle \langle e_A, e_B\rangle \langle e_B, e_C\rangle + R^2 \langle e_A, e_C\rangle^2 + R^2 \langle e_A, e_B\rangle^2 + R^2 \langle e_B, e_C\rangle^2$$

Осталось заметить, что последнее выражение совпадает с определителем матрицы Грамма системы векторов (e_A, e_B, e_C). Знак этого определителя не меняется при замене координат, поэтому он отрицателен. Тем самым неравенство треугольника доказано.

Упражнение 9.58. Показать, что длина прямолинейного отрезка, соединяющего две любые фиксированные точки на плоскости Лобачевского, меньше длины любой другой кривой, соединяющей эти точки.

Следующее утверждение, как несложно проверить, эквивалентно наличию бесконечного числа прямых, проходящих через фиксированную точку и параллельных данной прямой.

Statement

9.59. Сумма углов произвольного треугольника ABC на плоскости Лобачевского строго меньше π.

Доказательство. Проще всего провести доказательство в модели Пуанкаре на круге. Действительно, сначала совместим с помощью движения плоскости Лобачевского одну из вершин треугольника с центром круга. Ясно, что сумма углов при этом не изменится. Теперь очевидно (рис. 9.2), что сумма углов треугольника на плоскости Лобачевского меньше, чем сумма углов евклидова треугольника с теми же вершинами (мы воспользовались тем, что углы в метрике Лобачевского совпадают с евклидовыми углами). Доказательство закончено.

Оказывается, на плоскости Лобачевского существует связь между суммой углов треугольника и его площадью.

Рис. 9.2. Сумма углов треугольника на плоскости Лобачевского меньше π.

Statement

9.60. Пусть ABC - произвольный треугольник на плоскости Лобачевского $L^2(R)$ и α, $\beta, \gamma-$ его угль в вершинах A, B, C. Тогда, если S - площадь треугольника ABC, то $S = R^2(\pi - \alpha - \beta - \gamma)$.

Доказательство. Воспользуемся теперь моделью на верхней полуплоскости. Напомним, что площадь области в римановой метрике - это интеграл по этой области от корня из определителя матрицы метрики. На верхней полуплоскости определитель метрики равен R^4/y^4, поэтому следует интегрировать функцию R^2/y^2.

Рис. 9.3: Вычисление площади треугольников: с углами $\alpha, \pi/2, 0(a)$, с углами α, β и $0()$, с произвольными углами ( ).

Докажем сначала требуемое равенство для треугольника с углами $\alpha, \pi/2$ и 0 . Для этого движением плоскости Лобачевского переведем параллельные стороны рассматриваемого треугольника в параллельные оси ординат лучи, причем будем предполагать, что один из этих лучей лежит на самой оси ординат (рис. 9.3). Обозначив через r евклидов радиус окружности и заметив, что угол между радиусом этой окружности, приходящим в вершину A, и осью ординат равен α, получим требуемое:

$$S = \int_0^{r\cos\alpha} dx \int_{\sqrt{r^2-x^2}}^{\infty} \frac{R^2 dy}{y^2} = R^2 \int_0^{r\cos\alpha} \frac{dx}{\sqrt{r^2-x^2}} = R^2\left(\frac{\pi}{2} - \alpha\right)$$

Пусть теперь углы треугольника ABC - суть α, β и 0 . Без ограничения общности можно предполагать, что стороны этого треугольника параллельны оси ординат. Пусть для определенности параллельные стороны AC и BC треугольника ABC лежат по разные стороны от вертикального луча, перпендикулярного соответствующей стороне AB окружности (см. рис. 9.3). Площадь этого треугольника складывается из площадей двух треугольников, углы которых равны соответственно $\alpha, \pi/2, 0$ и $\beta, \pi/2, 0$. Поэтому для площади такого треугольника имеем:

$$S = R^2(\pi - \pi/2 - \alpha) + R^2(\pi - \pi/2 - \beta) = R^2(\pi - \alpha - \beta),$$

что и требовалось доказать.
Отметим, что вершина треугольника с нулевым углом может лежать и на вещественной оси. Действительно, треугольник с вершиной на вещественной оси в точке a переводится в треугольник с вершиной на бесконечности изометрией плоскости Лобачевского вида $w(z) = -1/(z - a)$ (проверьте, что это изометрия).

Рассмотрим, наконец, треугольник общего вида. Продолжим одну из сторон треугольника, скажем AB, за точку B до пересечения с абсолютом в точке D и проведем луч CD (см. рис. 9.3). Тогда треугольники ACD и CBD имеют нулевой угол в точке D. Обозначим угол CBD через δ, а угол BCD - через ε. Искомая площадь S треугольника ABC может быть найдена, как разность площадей треугольников ACD и CBD :

$$S = R^2(\pi - \alpha - (\gamma + \varepsilon)) - R^2(\pi - \varepsilon - \delta) = R^2(\pi - \alpha - (\pi - \delta) - \gamma) = R^2(\pi - \alpha - \beta - \gamma).$$

Таким образом, утверждение полностью доказано. □

14.12 10 Topological Spaces and Continuous Mappings

В это лекции мы начнем знакомство с общей топологии - части математики, которая изучает наиболее общие свойства геометрических объектов, не меняющиеся при непрерывных преобразованиях. Понятие непрерывности знакомо нам из математического анализа. В основе его лежит, на самом деле, некое представление об "окрестности" элемента рассматриваемого множеств. В математическом анализе окрестность элемента (точки) определяется с помощью функции расстояния. В общей топология понятие окрестности определяется наиболее общим образом с помощью специальной дополнительной структуры - так называемой топологии.

14.12.1 10.1 Topological spaces

В данном разделе мы даем определения основных объектов, изучаемых в общей топологии, - топологических пространств, а также непрерывных отображений таких пространств.

14.12.2 10.1.1 Topology and topological space

Определение 10.1. Пусть X - произвольное множество и $\tau = \{U_\alpha\}$ - некоторое семейство подмножеств множества X. Семейство τ называется топологией, если оно удовлетворяет следующим свойствам:
(1) само пространство X и пустое множество принадлежат τ;
(2) объединение любого семейства множеств из τ принадлежит τ;
(3) пересечение любого конечного семейства множеств из τ также принадлежит τ.

Определение 10.2. Множество X с фиксированной топологией τ называется топологическим пространством и обозначается через (X, τ). Элементы множества X

называются точками. Множества из τ называются открытыми в (X, τ). Подмножество F топологического пространства называется замкнутым, если его дополнение $X \backslash F$ открыто.

Определение 10.3. Окрестностью точки x топологического пространства (X, τ) будем называть произвольное открытое подмножество U, т.е. $U \in \tau$, содержащее x.

Замечание 10.4. Ниже, там, где из контекста понятно, о какой именно топологии идет речь, мы часто будем обозначать топологическое пространство (X, τ) просто через X.

Пример 10.5. Пусть X - произвольное множество. Семейство τ_d, состоящее из всех подмножеств множества X, является, очевидно, топологией. Эта топология называется дискретной, а само пространство - дискретным топологическим пространством. Каждое подмножество дискретного топологического пространства является одновременно и открытым и замкнутым. Любое подмножество дискретного топологического пространства является окрестностью каждой своей точки. В частности, каждая точка является своей окрестностью.

Пример 10.6. Семейство $\tau_0 = \{X, \emptyset\}$ является топологией на X. Эта топология называется тривиальной. Каждая точка топологического пространства (X, τ_0) обладает единственной окрестностью - множеством X. Каждое собственное (т.е. не совпадающее со всем X и не пустое) подмножество в X не является ни открытым, ни замкнутым.

14.12.3 10.1.2 Metric Spaces

Метрические пространства являются важным частным случаем топологических пространств. Напомним соответствующие определения.

Определение 10.7. Пусть X - произвольное множество. Метрикой, или функцией расстояния на X, называется неотрицательная вещественная функция ρ, определенная на декартовом произведении $X \times X$ и обладающая следующими свойствами:
(1) $\rho(x, y) = 0$, если и только если $x = y$ (невырожденность);
(2) $\rho(x, y) = \rho(y, x)$ для любых x и y из X (симметричность);
(3) $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ для любых точек x, y, z из X (неравенство треугольника).

Определение 10.8. Множество X вместе с введенной на нем метрикой ρ называется метрическим пространством и обозначается чрез (X, ρ). Элементы метрического пространства называются точками. Значение $\rho(x, y)$ метрики ρ на паре точек x и y называется расстоянием между точками.

Замечание 10.9. Часто для краткости, там где вид функции расстояния понятен из контекста, мы будем обозначать метрическое пространство (X, ρ) просто через X.

Пример 10.10. Пусть $X = \mathbb{R}^n$. Положим

$$\rho_p(x, y) = \sqrt[p]{\sum_{i=1}^{n} |x^i - y^i|^p},$$

где $x = (x^1, \ldots, x^n)$ и $y = (y^1, \ldots, y^n)$ - точки из $\mathbb{R}^n$, а $p \in (0, +\infty)$. Функция ρ_p удовлетворяет аксиомам метрики. При $p = 2$ она называется евклидовой метрикой на $\mathbb{R}^n$.

Пример 10.11. Пусть $X = C[a, b]$ - множество непрерывных функций на отрезке $[a, b]$. Положим

$$\rho^0(f, g) = \max_{t \in [a,b]} |f(t) - g(t)|$$

где f и g - произвольные элементы из $C[a,b]$. Несложно проверить, что ρ^0 удовлетворяет аксиомам метрики. Поэтому пара ($C[a,b],\rho^0$) - метрическое пространство. Метрика ρ^0 называется метрикой равномерной сходимости.

Определение 10.12. Пусть x - произвольная точка метрического пространства (X,ρ). Открытым шаром с центром в x радиуса ε называется множество всех точек y из X, таких, что $\rho(x,y)<\varepsilon$. Точка a множества $A\subset X$ называется внутренней точкой для A, если в A содержится некоторый открытый шар $O_\varepsilon(a)$. Множество всех внутренних точек множества A называется внутренностью множества A и обозначается через $\operatorname{int} A$. Множество A называется открытым в X (как в метрическом пространстве), если его внутренность совпадает с ним самим: $A=\operatorname{int} A$.

Утверждение 10.13. Пусть (X,ρ) - произвольное метрическое пространство. Объединение любого семейства множеств, открытых в X, открыто в X, *и* пересечение конечного числа множеств, открытых в X, также открыто в X.

Доказательство. Действительно, пусть $\{A_\alpha\}$ - произвольное семейство открытых подмножеств пространства X. Обозначим через A объединение всех множеств A_α, и пусть a - произвольная точка множества A. Тогда существует такое α_0, что $a\in A_{\alpha_0}$, а множество A_{α_0} - открыто по условию. Поэтому существует открытый шар $O_\varepsilon(a)$, целиком лежащий в A_α. Но тогда $O_\varepsilon(a)\subset A$, и, значит, каждая точка множества A внутренняя, т.е., по определению, A - открыто.

Пусть теперь α пробегает конечное число значений: $\alpha=1,\dots,N$. Обозначим через A' пересечение всех множеств A_α, и пусть a - произвольная точка из A'. Тогда $a\in A_\alpha$ для каждого $\alpha=1,\dots,n$. Поэтому для каждого α существует такое ε_α, что шар $O_{\varepsilon_\alpha}(a)$ целиком лежит в A_α. Положим $\varepsilon=\min_\alpha\varepsilon_\alpha$ и заметим, что $\varepsilon>0$, так как набор множеств A_α конечен. Тогда $O_\varepsilon(a)\subset O_{\varepsilon_\alpha}(a)$ для любого α, и поэтому $O_\varepsilon(a)\subset A'$, т.е. каждая точка a из A' является внутренней для A'. Доказательство закончено.

Следствие 10.14. Семейство всех открытых подмножеств метрического пространства образуют топологию.

Определение 10.15. Пусть (X,ρ) - метрическое пространство. Топология из следствия 110.14, т.е. топология, состоящая из всех открытых подмножеств множества X (открытых как в метрическом пространстве), называется метрической или порожденной метрикой ρ и обозначается через τ_ρ.

Пример 10.16. На произвольном не пустом множестве X рассмотрим функцию расстояния, равную 1 на любой паре различных элементов из X. Несложно проверить, что соответствующая метрическая топология совпадает с дискретной топологией на множестве X. Поэтому дискретная топология является метрической.

Упражнение 10.17. Сколько различных топологий можно ввести на множестве X, состоящем из двух элементов? Каждая ли из них является метрической?

Замечание 10.18. В метрическом пространстве понятие замкнутого множества можно естественно определить с помощью метрики. Напомним соответствующую конструкцию и покажем, что она эквивалентна топологической.

Конструкция 10.19. Пусть (X,ρ) - метрическое пространство и $A\subset X$ - произвольное его подмножество. Точка $x\in X$ называется точкой прикосновения множества A, если любой открытый шар $O_\varepsilon(x)$ пересекается с A. В частности, каждая точка $a\in A$ является точкой прикосновения множества A, хотя обратное неверно.

Замыканием $\bar A$ подмножества A называется множество всех точек прикосновения множества A. Ясно, что $A\subset\bar A$; обратное, вообще говоря, неверно. Если замыкание $\bar A$ множества A совпадает с самим A, т.е. $\bar A=A$, то множество A называется замкнутьмм в X.

Пример 10.20. Пусть $(a,b)\subset\mathbb{R}^1$. Множество всех точек прикосновения интервала (a,b) - это отрезок $[a,b]$. В частности, a и b - точки прикосновения множества (a,b). Поэтому

замыкание $\overline{(a,b)}$ совпадает с отрезком $[a,b]$.

Утверждение 10.21. Пусть (X,ρ) - некоторое метрическое пространство. Подмножество $A \subset X$ является открытым, если и только если его дополнение, т.е. множество $X \backslash A$, замкнуто.

Доказательство. Пусть сначала A - открыто. Рассмотрим произвольную точку $a \in A$. По определению, существует открытый шар $O_\varepsilon(a)$, целиком лежащий в A. Но тогда этот шар не пересекается с $X \backslash A$, поэтому a не является точкой прикосновения множества $X \backslash A$. Но тогда $\overline{X \backslash A} = X \backslash A$, т.е. дополнение до A замкнуто в X.

Обратно: пусть теперь множество $X \backslash A$ замкнуто в X. Это означает, что никакая точка из A не является точкой прикосновения для $X \backslash A$. Пусть a - произвольная точка из A. Тогда существует открытый шар $O_\varepsilon(a)$ не пересекающийся с $X \backslash A$ и, значит, этот шар целиком лежит в A. Последнее означает, что каждая точка из A является внутренней для A, т.е. A открыто. Доказательство закончено.

14.12.4 10.1.3 Topological interior and topological closure

Использую понятие окрестности точки, можно перенести понятие внутренности и замыкания со случая метрического пространства на общий случай топологического пространства.

Определение 10.22. Пусть (X,τ) - топологическое пространство. Точка a множества $A \subset X$ называется внутренней точкой для A, если в A содержится некоторая окрестность U точки a. Множество всех внутренних точек множества A называется внутренностью множества A и обозначается через $\operatorname{int} A$.

Точка $x \in X$ называется точкой прикосновения множества $A \subset X$, если любая окрестность точки x пересекается с A. Замыканием $\bar{A}$ множества A называется множество всех его точек.

Утверждение 10.23. Пусть (X,τ) - топологическое пространство. Множество $A \subset X$ является открытым, если и только если $\operatorname{int} A = A$. Множество $A \subset X$ является замкнутым, если и только если $\bar{A} = A$.

Доказательство. Каждое открытое множество само является окрестностью каждой своей точки, поэтому совпадает со своей внутренностью. Обратно, если каждая точка множества A является внутренней, то для каждой точки $a \in A$ существует открытое множество $U(a) \subset A$, содержащее a. Возьмем объединение $U = \cup_{a \in A} U(a)$. По определению, U - открыто. Кроме того, так как каждое $U(a) \subset A$, то само множество U содержится в A. В то же время $A \subset U$, так как каждая точка $a \in U(a)$. Поэтому $U = A$ и, в частности, A - открыто.

Далее, множество A - замкнуто, если и только если $X \backslash A$ открыто. Последнее, как мы только что показали, равносильно тому, что каждая точка из $X \backslash A$ обладает окрестностью, не пересекающейся с A, что, в свою очередь, равносильно включению $\bar{A} \subset A$, т.е. $\bar{A} = A$. Доказательство закончено.

Следствие 10.24. Пусть A - произвольное подмножество топологического пространства $(\ X,\rho\ )$. Тогда внутренность $\operatorname{int} A$ множества A открыта, а замыкание $\bar{A}$ множества A - замкнуто.

14.12.5 10.2 Continuous mappings

Определение 10.25. Отображение $f : X \to Y$ топологического пространства (X,τ_X) в топологическое пространство $(\ Y,\tau_Y\ )$ называется непрерывным в точке $x_0 \in X$, если для любой окрестности $V\left(f\left(x_0\right)\right) \in \tau_Y$ точки $f\left(x_0\right)$ в пространстве Y существует такая

окрестность $U(x_0) \in \tau_X$ точки x_0 в пространстве X, что $f(U(x_0)) \subset V(f(x_0))$. Отображение f называется непрерывным, если оно непрерывно в каждой точке $x \in X$.

Пример 10.26. Пусть (X, ρ_X) и (Y, ρ_Y) - метрические пространства. Превратим их в топологические пространства, наделив стандартными метрическими топологиями. Тогда определение непрерывности принимает следующий стандартный для метрических пространств вид. Отображение $f : X \to Y$ метрических пространств называется непрерывным в точке x_0, если для любого $\varepsilon > 0$ существует такое $\delta(\varepsilon) > 0$, что $f(U_\delta(x_0)) \subset V_\varepsilon(f(x_0))$, т.е. из $\rho_X(x_0, x) < \delta(\varepsilon)$ следует, что $\rho_Y(f(x_0), f(x)) < \varepsilon$.

Пример 10.27. Еще один важный частный случай - это отображения топологического пространства X в числа. Такие отображения называют функциями. Функция $f : X \to \mathbb{R}$ называется непрерывной в точке $x_0 \in X$, если для любого $\varepsilon > 0$ существует такая зависящая от ε окрестность $U(x_0)$ этой точки в пространстве X, что для любой точки x из $U(x_0)$ имеет место неравенство $|f(x_0) - f(x)| < \varepsilon$.

Theorem

10.28. Отображение $f : X \to Y$ непрерывно, если и только если выполнено одно из следующих эквивалентных условий:

- прообраз любого открытого множества $A \subset Y$ открыт в X;
- прообраз любого замкнутого множества замкнут.

Доказательство. Пусть сначала f непрерывно. Рассмотрим произвольное открытое множество $A \subset Y$ и обозначим через $f^{-1}(A)$ его прообраз при отображении f. Пусть $x \in f^{-1}(A)$ - произвольная точка из прообраза; обозначим через $y = f(x)$ образ точки x при отображении f. Тогда y принадлежит открытому множеству A. Возьмем A в качестве окрестности точки $y : V(y) = A$. Так как отображение f непрерывно в точке x, то существует окрестность $U(x)$ точки x, такая, что $f(U(x)) \subset V(y) = A$, но тогда $U(x) \subset A$, и множество $f^{-1}(A)$ открыто по утверждению (10.23).

Обратно, пусть прообраз произвольного открытого подмножества из Y открыт в X. Фиксируем произвольную точку $x \in X$, и пусть, как и выше, $y = f(x)$. Рассмотрим произвольную окрестность $V(y)$ точки Y. По определению, $V(y)$ - открытое множество, значит, $f^{-1}(V(y))$ - открытое подмножество в X, причем $x \in f^{-1}(V(y))$. Выберем в качестве окрестности $U(x)$ точки x множество $f^{-1}(V(y))$. Тогда $f(U(x)) \subset V(y)$, поэтому отображение f непрерывно в произвольной точке x, что и требовалось доказать.

Для завершения доказательства утверждения осталось показать, что сформулированные в нем условия непрерывности эквивалентны. Это немедленно следует из соотношения $f^{-1}(Y \backslash A) = X \backslash f^{-1}(A)$, справедливого для произвольного множества $A \subset Y$ и отображения f. Доказательство закончено. $\square$

Замечание 10.29. Вообще говоря, образ открытого (замкнутого) множества при непрерывном отображении не обязан быть открытым (замкнутым). Приведите соответствующие примеры.

Следствие 10.30. Композиция непрерывных отображений непрерывна.

Упражнение 10.31. Описать все непрерывные отображения из произвольного топологического пространства X в пространство с тривиальной топологией. Описать все непрерывные отображения из пространства с дискретной топологией в произвольное топологическое пространство.

Определение 10.32. Пусть (X_1, τ_1) и (X_2, τ_2) - топологические пространства. Взаимно-однозначное непрерывное отображение $\varphi : X_1 \to X_2$ называется гомеоморфизмом пространств X_1 и X_2, если обратное отображение также непрерывно. Если существует

гомеоморфизм пространств X_1 и X_2, то сами пространства называются гомеоморфными. Отображение $\varphi : X_1 \to X_2$ называется вложением X_1 в X_2, если оно задает гомеоморфизм между X_1 и $\varphi(X_1) \subset X_2$. Свойство пространства называется топологическим, если оно сохраняется при гомеоморфизме.

Упражнение 10.33. Показать, что следующие подпространства $\mathbb{R}^1$ попарно не гомеоморфны: интервал (a, b), полуинтервал $(a, b]$, отрезок $[a, b]$.

Упражнение 10.34. Привести пример топологических пространств X и Y, таких, что существуют непрерывные взаимно-однозначные отображения f и g, где $f : X \to Y$, и $g : Y \to X$, однако пространства X и Y не гомеоморфны.

14.12.6 10.3 Standard Structures

В даном разделе приводятся примеры стандартных конструкций, позволяющих строить новые примеры топологических пространств из уже имеющихся.

Конструкция 10.35. Пусть (X, τ) - топологическое пространство, и Y - произвольное его подмножество. Положим $\tau_Y = \{U \cap Y \mid U \in \tau\}$. Легко проверить, что τ_Y является топологией на множестве Y.

Определение 10.36. Топологическое пространство (Y, τ_Y) называется подпространством в (X, τ), а топология τ_Y - индучированной (из пространства X) топологией.

Пример 10.37. Рассмотрим плоскость $\mathbb{R}^2$ как топологическое пространство с топологией τ, порожденной евклидовой метрикой (именно такая топология рассматривается по умолчанию на евклидовых пространствах), фиксируем на плоскости стандартные декартовы координаты (x, y). Подмножество S^1, заданное уравнением $x^2 + y^2 = 1$, с индуцированной топологией называется окружностью.

Упражнение 10.38. Рассмотрим плоскость $\mathbb{R}^2$ и на ней - треугольник T (замкнутую ломаную из трех звеньев) с индуцированной топологией. Проверьте, что треугольник T гомеоморфен окружности S^1.

Пример 10.39. Рассмотрим плоскость $\mathbb{R}^2$, фиксируем на плоскости стандартные декартовы координаты, и пусть Y - объединение координатных осей. Тогда каждая окрестность начала координат в индуцированной топологии τ_Y содержит объединение двух открытых интервалов, каждый из которых - окрестность нуля на своей координатной оси.

Упражнение 10.40. Пусть (X, τ) - топологическое пространство, и $i : Y \to X$ тождественное вложение подмножества $Y \subset X$ в множество X. Покажите, что i - непрерывное отображение из (Y, τ_Y) в (X, τ).

Конструкция 10.41. Пусть (X, τ) - топологическое пространство и $\varphi : X \to M-$ некоторое отображение X на множество M. Положим $\tau_\varphi = \{U \subset M \mid \varphi^{-1}(U) \in \tau\}$. Несложно показать, что τ_φ задает на M топологию.

Определение 10.42. Говорят, что топология τ_φ порождена отображением φ или *не* ренесена с X на M с помощью отображения f.

Упражнение 10.43. Показать, что отображение $\varphi : X \to M$ топологического пространства (X, τ) в пространство (M, τ_φ) непрерывно.

Определение 10.44. Пусть (X, τ_X) и (Y, τ_Y) - топологические пространства, причем X и Y не имеют общих точек. Множество $X \cup Y$ с топологией $\tau = \tau_X \cup \tau_Y$ называется несвязной суммой топологических пространств X и Y.

Конструкция 10.45. Рассмотрим два топологических пространства - (X, τ_X) и (Y, τ_Y). Напомним, что декартовым произведением множеств X и Y называется множество всевозможных упорядоченных пар (x, y), где $x \in X$ и $y \in Y$. Превратим множество $X \times Y$ в топологическое пространство. Множество $U \subset X \times Y$ назовем открытым, если и

только если оно представимо в виде объединения некоторого семейства множеств вида $U \times V$, где $U \in \tau_X$, а $V \in \tau_Y$. Легко проверить, что определенное нами семейство открытых множеств удовлетворяет аксиомам топологии.

Определение 10.46. Построенная только что топология обозначается через $\tau_X \times \tau_Y$ и называется топологией прямого произведения.

Замечание 10.47. Это определение легко обобщается на случай декартова произведения любого конечного семейства топологических пространств.

Упражнение 10.48. Пусть $\pi : X \times Y \to X$ - стандартная проекция декартова произведения на сомножитель, т.е. $\pi : (x, y) \mapsto x$. Показать, что отображение X - непрерывно.

Упражнение 10.49. Рассмотрим два экземпляра X и Y вещественной прямой со стандартной метрической топологией. Поверьте, что в этом случае топология прямого произведения совпадает с обычной метрической топологией на плоскости.

Конструкция 10.50. Пусть X - топологическое пространство с топологией τ, на котором задано некоторое отношение эквивалентности $\sim$. Обозначим через $X/\sim$ множество классов эквивалентности, и пусть $\pi : X \to X/\sim$ - стандартная проекция, переводящая элемент $x \in X$ в содержащий его класс эквивалентности. Тогда проекция π порождает на $X/\sim$ некоторую топологию τ_π (см. конструкцию 11.41).

Определение 10.51. Топология τ_π называется фактор-топологией, а пространство ($X/\sim, \tau_\pi$) - фактор-пространством.

Упражнение 10.52. Показать, что стандартная проекция π пространства на факторпространство непрерывна.

Пример 10.53. Пусть $X = [0, 2\pi] \subset \mathbb{R}$ - отрезок с топологией индуцированной топологией вещественной прямой. Зададим на X отношение эквивалентности σ, объявив эквивалентными точки 0 и 2π (все остальные точки отрезка эквивалентны только сами себе). Тогда топологическое пространство $[0, 2\pi]/\sigma$ естественным образом отождествляется с окружностью S^1 из примера 11.37.

14.12.7 10.4 Problems

Задача 10.54. Перечислите все топологии на множестве из двух элементов. Приведите пример топологии на множестве из трех элементов, отличной от дискретной и антидискретной. Приведите пример такого семейства подмножеств множества X из трех элементов, которое содержит пустое множество, все X, но не является топологией.

Задача 10.55. Пусть топологическое пространство X представлено в виде объединения двух своих замкнутых подмножеств F_1 и F_2. Показать, что отображение f пространства X в произвольное топологическое пространство Y непрерывно, если и только если ограничение f на каждое из $F_i, i = 1, 2$, непрерывно.

Задача 10.56. Докажите, что если M и N - два подмножества топологического пространства, то:
(1) $M \subset N$ влечет $\bar{M} \subset \bar{N}$;
(2) $\overline{\bar{M}} = \bar{M}$;
(3) $\overline{M \cup N} = \bar{M} \cup \bar{N}$;
(4) приведите пример, когда $\overline{M \cap N} \neq \bar{M} \cap \bar{N}$;
(5) $M \subset N$ влечет $\operatorname{int} M \subset \operatorname{int} N$;
(6) $\operatorname{int}(\operatorname{int} M) = \operatorname{int} M$;
(7) $\operatorname{int}(M \cap N) = \operatorname{int} M \cap \operatorname{int} N$;
(8) приведите пример, когда $\operatorname{int}(M \cup N) \neq \operatorname{int} M \cup \operatorname{int} N$;
(9) докажите, что

$$\partial M = (\bar{M} \backslash M) \cup (M \backslash \operatorname{int} M) = (\bar{M} \backslash M) \cup (M \cap \overline{CM})$$

где через CM обозначено дополнение множества M.
Задача 10.57. Привести пример непрерывного взаимно-однозначного соответствия, не являющегося гомеоморфизмом со своим образом.

Задача 10.58. Положим $\partial X = \bar{X} \backslash \operatorname{int} X$. Показать, что $\partial(X \times Y) = (\partial X \times Y) \cup (X \times \partial Y)$.
Задача 10.59. Доказать, что отрезок и буква "T" (как подмножество плоскости, на которой она нарисована) не гомеоморфны.

Задача 10.60. Гомеоморфны ли буквы "T" и "X" (как подмножества плоскости)?

14.12.8 10.5.1 Examples of metric spaces

Приведем ряд стандартных примеров метрических пространств.
Пример 10.61. Пусть $X = \mathbb{R}^n$. Расстояние между точками

$$x = (x_1, \ldots, x_n) \quad \text{и} \quad y = (y_1, \ldots, y_n)$$

из $\mathbb{R}^n$ может быть определено многими способами. Приведем два из них, отличающиеся от стандартного евклидового расстояния:

$$\rho_\infty(x, y) = \max_i \{|x_i - y_i|\}$$

$$\rho_1(x, y) = \sum_{i=1}^{n} |x_i - y_i| \quad - \text{ манхеттенское расстояние.}$$

Нетрудно показать, что функции ρ_1 и ρ_∞ являются метриками на $\mathbb{R}^n$, поэтому и ($\mathbb{R}^n, \rho_1$), и ($\mathbb{R}^n, \rho_\infty$) метрические пространства.

Пример 10.62. Пусть $X = C^k[a, b], 0 \leq k \leq \infty$, - множество k раз непрерывно дифференцируемых функций на отрезке $[a, b]$. Положим

$$\rho^k(f, g) = \max_{0 \leq n \leq k} \max_{t \in [a,b]} \left| f^{(n)}(t) - g^{(n)}(t) \right|$$

где f и g - произвольные элементы из $C^k[a, b]$ и $f^{(n)}$ - производная n-го порядка. Определенная так функция ρ^k удовлетворяет аксиомам метрики. Поэтому пара ($C^k[a, b], \rho^k$) - метрическое пространство.

Пример 10.63. Пусть A - произвольное непустое подмножество метрического пространства X с метрикой ρ. Обозначим через ρ_A ограничение функции $\rho : X \times X \to \mathbb{R}$ на множество $A \times A$. Ясно, что ρ_A удовлетворяет аксиомам метрики. Метрическое пространство (A, ρ_A) называется подпространством пространства X, а метрика ρ_A - индучированной метрикой.

14.12.9 10.5.2 Properties of metric spaces

Докажем следующий результат.

Statement

10.64. Пусть (X, ρ) - метрическое пространство. Тогда шаровая окрестность $O_\varepsilon(x)$ произвольной точки $x \in X$ является открытым множеством.

Доказательство. Пусть y - произвольная точка из $O_\varepsilon(x)$. Тогда, по определению, $\rho(x, y) < \varepsilon$, поэтому существует такое число $\delta > 0$, что $\rho(x, y) + \delta < \varepsilon$. Для доказательства

утверждения достаточно проверить, что шаровая окрестность $O_\delta(y)$ точки y целиком содержится в $O_\varepsilon(x)$. Действительно, пусть z - произвольная точка из $O_\delta(y)$. Тогда в силу неравенства треугольника

$$\rho(z,x) \le \rho(z,y) + \rho(y,x) < \delta + \rho(x,y) < \varepsilon$$

т.е. $z \in O_\varepsilon(x)$, что и требовалось доказать.

Наконец, оказывается, что операции замыкания и взятия внутренности, определенные выше на подмножествах метрического пространства, естественным образом согласованы с понятиями открытых и замкнутых множеств.

Statement

10.65. Пусть (X,ρ) - метрическое пространство и A - его произвольное подмножество. Тогда множество $\bar{A}$ замкнуто, а множество $\operatorname{int} A$ - открыто.

Доказательство. Покажем сначала, что $\operatorname{int} A$ - открыто. Каждая точка a из $\operatorname{int} A$ обладает окрестностью $O_\varepsilon(a)$, целиком лежащей в A. Но тогда $\operatorname{int}(O_\varepsilon(a)) \subset \operatorname{int} A$, а в силу утверждения 10.64 $\operatorname{int}(O_\varepsilon(a)) = O_\varepsilon(a)$, поэтому множество $\operatorname{int}(A)$ содержит шаровую окрестность своей точки a и, следовательно, открыто.

Далее, точка x из X не принадлежит замыканию $\bar{A}$ множества A, если и только если $\rho(a,A) = \varepsilon > 0$, поэтому $X \backslash \bar{A} = \operatorname{int}(X \backslash A)$, а последнее множество открыто. Тогда, в силу утверждения 10.21, множество $\bar{A}$ - замкнуто. Доказательство закончено.

14.12.10 10.5.3 Examples of topological spaces

Приведем еще несколько стандартных конструкций топологических пространств.

Пример 10.66. Пусть $X \cup Y$ - несвязная сумма топологических пространств X и Y. Предположим, что фиксированы точки x_0 в X и y_0 в Y. Зададим на множестве $X \cup Y$ отношение эквивалентности, положив $x_0 \sim y_0$. Тогда фактор-пространство $X/\sim$ называется связной суммой, или букетом пространств X и Y по точкам $x_0 \in X$ и $y_0 \in Y$. Это пространство обозначают через $(X,x_0) \vee (Y,y_0)$, или просто $X \vee Y$.

Пример 10.67. Пусть Y - подпространство топологического пространства X. Зададим на X отношение эквивалентности, положив $x_1 \sim x_2$, если и только если $x_i \in Y$. Говорят, что фактор-пространство $X/\sim$ получено из X факторизацией по множеству Y. Это пространство часто обозначают через X/Y. Например, если D^n - стандартный замкнутый диск в $\mathbb{R}^n$, а S^{n-1} - его граничная сфера, то $D^n/\partial D^n$ - стандартная n-мерная сфера S^n.

Пример 10.68. Пусть $K = [-1,1] \times [-1,1]$ - квадрат. Отождествляя граничные точки разными способами, получим хорошо известные пространства, а именно отождествляя точки вида:

- $(\ -1,t\ )$ и $(\ 1,t\ )$, получаем цилиндр;
- $(\ -1,t\ )$ и $(\ 1,-t\ )$, получаем лист Мебиуса;
- $(\ -1,t\ )$ и $(\ 1,t\ )$, а также $(\ s,-1\ )$ и $(\ s,1\ )$, получаем тор;
- $(\ -1,t\ )$ и $(\ 1,t\ )$, а также $(\ s,-1\ )$ и $(\ -s,1\ )$, получаем бутылку Клейна;
- $(\ -1,t\ )$ и $(\ 1,-t\ )$, а также $(\ s,-1\ )$ и $(\ -s,1\ )$, получаем проективную плоскость.

Напомним, что проективная плоскость определяется как множество всех прямых в $\mathbb{R}^3$, проходящих через 0 . Так как каждая такая прямая пересекает стандартную сферу S^2 в точности по двум точкам вида x и $-x$, проективную плоскость можно

рассматривать как фактор-пространство, полученное из сферы S^2 отождествлением точек x и $-x$. Последнее эквивалентно фактор-пространству, полученному из замкнутой полусферы отождествлением диаметрально противоположных точек экватора. Осталось применить естественный гомеоморфизм квадрата K на замкнутую полусферу, переводящий симметричные относительно начала координат точки в симметричные.

14.13 11 Classes of topological spaces

Среди огромного разнообразия топологических пространств естественно выделить некоторые классы, обладающие специальными топологически инвариантными свойствами. В данной лекции рассматриваются некоторые из таких классов.

14.13.1 11.1 connectedness

Определение 11.1. Топологическое пространство X называется связным, если его нельзя представить в виде объединения двух непересекающихся непустых подмножеств A_1 и A_2, каждое из которых является открытым в X. Если такое представление возможно, пространство X называется несвязным.

Замечание 11.2. Если $X = A_1 \cup A_2$, где A_i - открытые подмножества в X, и $A_1 \cap A_2 = \emptyset$, то каждое A_i является дополнением открытого множества $A_j, j \neq i$, и поэтому замкнуто в X. Такие одновременно открытые и замкнутые подмножества топологического пространства иногда называют открыто-замкнутыми.

Определение 11.3. Подмножество A топологического пространства X называется связным, если связным является топологическое пространство A с топологией, индуцированной из X.

Утверждение 11.4. Отрезок $[a, b] \subset \mathbb{R}^1$ - связен.

Доказательство. Предположим противное, и пусть $[a, b] = A \cup B$, где множества A и B открыты, непусты и не пересекаются. Предположим для определенности, что $a \in A$. Рассмотрим множество всех таких ε, что полуинтервал $[a, a + \varepsilon\ )$ содержится в A, и пусть ε_0 - точная верхняя грань этого множества. Тогда, так как A - открыто, $\varepsilon_0 > 0$. Поскольку множество A замкнуто, точка $a + \varepsilon_0$ принадлежит A. Но тогда, если только $a + \varepsilon_0 \neq b$, существует открытый интервал $(\ a + \varepsilon_0 - \delta, a + \varepsilon_0 + \delta\ )$, целиком лежащий в A, и поэтому полуинтервал $[a, a + (\varepsilon_0 + \delta)\ )$ содержится в A, что противоречит выбору ε_0.

Поэтому $a+\varepsilon_0 = b$. Но тогда $A = [a, b]$ и множество B пусто. Полученное противоречие завершает доказательство. $\square$

Упражнение 11.5. Показать, что интервал (a, b) связен.

Упражнение 11.6. Пусть $X \cup Y$ - несвязная сумма произвольных топологических пространств. Показать, что пространство $X \cup Y$ несвязно.

Упражнение 11.7. Показать, что букет (см. пример 10.66) связных топологических пространств по любой паре точек является связным.

Приведем несколько утверждений, позволяющих проверять связность топологического пространства.

Утверждение 11.8. Пусть некоторое топологическое пространство X представлено в виде объединения своих подмножеств X_α, каждое из которых связно. Предположим, что пересечение $\cap_\alpha X_\alpha$ непусто. Тогда пространство X связно.

Доказательство. Предположим противное, т.е. $X = A \cup B$, где A и B - открыты, непусты, причем $A \cap B = \emptyset$. Тогда каждое X_α можно представить в виде $X_\alpha = (X_\alpha \cap A) \cup (X_\alpha \cap B)$. Однако, так как по условию X_α связно, то или $X_\alpha \cap A = \emptyset$, или $X_\alpha \cap B = \emptyset$, т.е. каждое из X_α целиком лежит или в A, или в B. В то же время, так как A и B не

пусты, существуют $X_{\alpha_0} \subset A$ и $X_{\alpha_1} \subset B$. Однако $A \cap B = \emptyset$, поэтому и $X_{\alpha_0} \cap X_{\alpha_1} = \emptyset$, что противоречит условию. Утверждение доказано. $\square$

Определение 11.9. Максимальное по включению связное подпространство пространства X называется его связной компонентой или компонентой связности.

Упражнение 11.10. Покажите, что связные компоненты пространства X попарно не пересекаются и являются замкнутыми подмножествами пространства X.

Замечание 11.11. Связные компоненты не обязаны быть открытыми подмножествами. В качестве примера достаточно рассмотреть подмножество X вещественной прямой, полученное из отрезка $[0,1]$ выбрасыванием последовательности $\{1/n\}$. Тогда $\{0\} \subset X$ - связная компонента пространства X.

Упражнение 11.12. Показать, что пространство X связно, если и только если каждое его открыто-замкнутое подмножество или пусто, или совпадает со всем X.

Упражнение 11.13. Сколько компонент связности имеет пространство X, наделенное дискретной топологией?

Утверждение 11.14. Предположим, что в топологическом пространстве X для каждой пары его различных точек (x, y) найдется связное подмножество C_{xy}, содержащее x и y. Тогда пространство X связно.

Доказательство. Предположим противное, и пусть $X = A \cup B$, где A и B - открыты, непусты, причем $A \cap B = \emptyset$. Выберем точки $a \in A, b \in B$ и рассмотрим связное множество C_{ab}, содержащее a и b. Однако множество C_{ab} разлагается в объединение двух своих непустых, открытых и непересекающихся подмножеств $C_{ab} \cap A$ и $C_{ab} \cap B$, что противоречит связности C_{ab}. Доказательство закончено. $\square$

Утверждение 11.15. Образ связного подмножества топологического пространства при непрерывном отображении связен.

Доказательство. В самом деле, если образ $Y = f(X)$ связного множества X при непрерывном отображении f несвязен, то, по определению, $Y = A \cup B$, где A и B - открыты, непусты и не пересекаются. Но тогда $X = f^{-1}(A) \cup f^{-1}(B)$, причем множества $f^{-1}(A)$ и $f^{-1}(B)$ непусты, открыты и не пересекаются, т.е. X несвязно. Полученное противоречие и завершает доказательство.

Следствие 11.16. Связность топологического пространства - топологический инвариант, т.е. сохраняется при гомеоморфизмах.

Следствие 11.17 (Теорема Вейерштрасса). Непрерывная функция, заданная на связном топологическом пространстве X, принимает все промежуточные значения.

Доказательство. Действительно, если, скажем, значение y_0 не принимается непрерывной функцией $f : X \to \mathbb{R}$, но принимаются некоторые значения больше и меньше y_0, то образ $f(X)$ распадается в объединение двух открытых, непустых, непересекающихся подмножеств $f(X) \cap (y_0, +\infty)$ и $f(X) \cap (-\infty, y_0)$. Последнее противоречит утверждению Т.5. Следствие доказано.

Часто удобно пользоваться другим, более сильным понятием связности.

Определение 11.18. Топологическое пространство X называется линейно связным, если любые две точки из X можно соединить непрерывной кривой, т.е. для любых x и y из X существует непрерывное отображение $\gamma : [0,1] \to X$, такое, что $\gamma(0) = x$ и $\gamma(1) = y$.

В силу утверждения Т.Т4 каждое линейно связное пространство является связным. Обратное, вообще говоря, неверно.

Упражнение 11.19. Рассмотрим замыкание X в плоскости $\mathbb{R}^2(x, y)$ графика функции $y = \sin(1/x)$. Ясно, что X состоит из графика функции $y = \sin(1/x)$ и вертикального отрезка, соединяющего точки ($0, -1$) и (0,1). Показать, что пространство X (топология индуцирована из $\mathbb{R}^2$) связно, но не линейно связно.

14.13.2 11.2 Separability axioms

Определение 11.20. Две различные точки топологического пространства X называются отделимьмми, если у них существуют непересекающиеся окрестности. Более общо, два произвольных подмножества A_1 и A_2 топологического пространства называются отделимыми, если у них существуют непересекающиеся окрестности, т.е. непересекающиеся открытые множества U_1 и U_2, такие, что $A_i \subset U_i, i = 1, 2$.

Определение 11.21. Топологическое пространство X называется хаусдорфовым, если любые его две различные точки отделимы.

Пример 11.22. Рассмотрим множество X, состоящее из двух элементов x и y, и зададим на X топологию, положив $\tau = \{\emptyset, X, \{x\}\}$. Очевидно, точки x и y из X неотделимы, так как точка y обладает единственной окрестностью X.

Конструкция 11.23. Пусть X - произвольное бесконечное множество. Зададим на X топологию, объявив замкнутыми все возможные конечные подмножества из X и само множество X. Полученная топология называется топологией Зарисского.

Упражнение 11.24. Проверьте, что топология Зарисского - действительно топология. Докажите, что полученное топологическое пространство не является хаусдорфовым.

Пример 11.25. Топологическое пространство X, наделенное дискретной топологией, хаусдорфово.

Утверждение 11.26. Каждая точка хаусдорфова пространства является его замкнутым подмножеством.

Доказательство. Пусть X - хаусдорфово топологическое пространство и x - произвольная точка из X. Так как любая точка $y \in X \backslash \{x\}$ отделима от x, т.е. существует открытое множество U_y, содержащее y и не содержащее x, то множество $X \backslash \{x\}$ представимо в виде $\cup_{y \neq x} U_y$ и поэтому открыто в X. Следовательно, $\{x\}$ - замкнуто. Доказательство закончено.

Упражнение 11.27. Пусть X и Y - хаусдорфовы топологические пространства. Показать, что их декартово произведение, несвязная сумма и букет также являются хаусдорфовыми.

Упражнение 11.28. Является ли хаусдорфовость топологическим инвариантом?

Определение 11.29. Топологическое пространство называется нормальным, если оно хаусдорфово, и, более того, любые его два непересекающихся замкнутых подмножества отделимы.

Класс нормальных пространств достаточно широк.

Утверждение 11.30. Каждое метрическое пространство нормально.

Доказательство. Пусть (X, ρ) - метрическое пространство. Покажем сначала, что топологическое пространство (X, τ_ρ), где τ_ρ - соответствующая метрическая топология, хаусдорфово. Действительно, для произвольных различных точек x и y пространства X рассмотрим открытые шары $O_\varepsilon(x)$ и $O_\varepsilon(y)$, где $\varepsilon = \rho(x, y)/3$. Из неравенства треугольника следует, что эти шары не пересекаются, что и доказывает хаусдорфовость пространства X.

Пусть теперь A_1, A_2 - два произвольных непересекающихся замкнутых подмножества в X. Для каждой точки x из множества A_1 выберем число $\varepsilon(x) > 0$, положив $\varepsilon(x) = (1/3)\rho(x, A_2)$. Аналогично для каждой точки y из A_2 определим число $\varepsilon'(y) = (1/3)\rho(y, A_1)$. Построим открытые множества U_1 и U_2, положив

$$U_1 = \cup_{x \in A_1} O_{\varepsilon(x)}(x), \quad U_2 = \cup_{y \in A_2} O_{\varepsilon'(y)}(y).$$

Очевидно, $U_i \supset A_i, i = 1, 2$. Покажем, что U_1 и U_2 не пересекаются. Действительно, пусть существует $z \in U_1 \cap U_2$. Тогда имеются такие $x \in A_1$ и $y \in A_2$, что $z \in O_{\varepsilon(x)}(x)$ и $z \in O_{\varepsilon'(y)}(y)$. Но тогда

$$\rho(z,x) < (1/3)\rho\left(x, A_2\right) \leq (1/3)\rho(x,y)$$

$$\rho(z,y) < (1/3)\rho\left(y, A_1\right) \leq (1/3)\rho(y,x)$$

Складывая последние два неравенства, получаем

$$\rho(y,z) + \rho(z,x) < (2/3)\rho(y,x)$$

что противоречит неравенству треугольника. Утверждение доказано.
Упражнение 11.31. Является ли нормальность топологическим инвариантом?

14.13.3 11.3 Compact

Определение 11.32. Система открытых множеств $\{U_\alpha\}$ топологического пространства X называется открытым покрытием пространства X, если $\cup_\alpha U_\alpha = X$.

Определение 11.33. Топологическое пространство называется компактным или компактом, если всякое его открытое покрытие $\{U_\alpha\}$ содержит конечное покрытие $\{U_{\alpha_i}\}_{i=1}^N$ этого пространства. Подмножество Y топологического пространства называется компактным, если подпространство Y - компактное пространство.

Пример 11.34. Отрезок $[a,b]$ в топологии, индуцированной из $\mathbb{R}^1$, компактен, а интервал (a,b) - нет.

Пример 11.35. Каждое замкнутое ограниченное подмножество метрического пространства $\mathbb{R}^n$ компактно (теорема Бореля).

Приведем несколько полезных свойств компактных пространств.
Утверждение 11.36. Замкнутое подмножество компактного топологического пространства компактно.

Доказательство. Действительно, пусть $\{V_\alpha\}$ - произвольное открытое покрытие замкнутого подмножества A компактного топологического пространства X. Тогда каждое V_α - это пересечение некоторого открытого в X множества U_α со множеством A. Набор открытых множеств $\{U_\alpha\} \cup \{X \backslash A\}$ образует открытое покрытие пространства X, из которого можно выбрать конечное подпокрытие $\{U_{\alpha_k}\} \cup \{X \backslash A\}$. Но тогда конечная система множеств $\{V_{\alpha_k}\}$ образует конечное подпокрытие множества A. Утверждение доказано.

Утверждение 11.37. Каждое хаусдорфово компактное топологическое пространство нормально.

Доказательство. Пусть X - хаусдорфово компактное пространство, A - произвольное замкнутое подмножество в X и x - точка из X, не лежащая в A. Покажем, что $\{x\}$ и A отделимы в X. Действительно, в силу хаусдорфовости пространства X, для каждой точки $a \in A$ существуют окрестности $U(a)$ точки a и $V(x,a)$ точки x (вообще говоря, зависящая от a), такие, что $U(a) \cap V(x,a) = \emptyset$. Семейство $U(a)$ образует

открытое покрытие множества A, в котором, в силу утверждения Т.36, можно выбрать конечное подпокрытие $\{U\left(a_k\right)\}_{k=1}^N$. Поэтому окрестность $\cap_{k=1}^N V\left(x, a_k\right)$ точки x определена и не пересекается с содержащим A открытым множеством $\cup_{k=1}^N U\left(a_k\right)$.

Рассмотрим теперь два произвольных непересекающихся замкнутых множества A_1 и A_2 в компактном пространстве X. Как мы только что доказали, для любой точки $a \in A_1$ существуют открытая окрестность $U(a)$ точки a и содержащее множество A_2 открытое множество V_a, такие, что $U(a) \cap V_a = \emptyset$. Семейство $\{U(a)\}$ образует открытое покрытие замкнутого множества A_1, в котором, в силу утверждения [Т.36, можно выделить конечное подпокрытие $\{U\left(a_k\right)\}_{k=1}^N$. Поэтому определено открытое множество $\cap_{k=1}^N V_{a_k}$, содержащее

множество A_2 и не пересекающееся с содержащим множество A_1 открытым множеством $\cup_{k=1}^{N} U(a_k)$. Утверждение доказано. □

Вообще говоря, компактное подпространство топологического пространства не является замкнутым (сравните с утверждением II.36). Однако имеет место следующий результат.

Утверждение 11.38. Пусть A - компактное подпространство в хаусдорфовом пространстве X. Тогда A - замкнуто.

Доказательство. Рассмотрим произвольную точку x из $X \backslash A$. Достаточно показать, что x не является точкой прикосновения для A. В силу хаусдорфовости пространства X, для каждой точки a из A существуют окрестности $U(a)$ точки a и $V(x,a)$ точки x, такие, что $U(a) \cap V(x,a) = \emptyset$. Семейство $\{U(a)\}$ образует открытое покрытие компакта A, поэтому в нем можно выделить конечное подпокрытие $\{U(a_k)\}_{k=1}^{N}$. Следовательно, определена открытая окрестность $\cap_{k=1}^{N} V(x, a_k)$ точки x, которая не пересекается со множеством A, т.е. $x \notin \bar{A}$. Утверждение доказано. □

Компактность пространства является топологическим инвариантом. А именно, имеет место следующее утверждение.

Утверждение 11.39. Пусть $f : X \to Y$ - непрерывное отображение компактного топологического пространства X в произвольное топологическое пространство Y. Тогда образ $f(X)$ компактен в Y.

Доказательство. Действительно, рассмотрим произвольное открытое покрытие $\{U_\alpha\}$ множества $f(X)$. Тогда система множеств $\{f^{-1}(U_\alpha)\}$ образует открытое покрытие пространства X. В силу компактности X в ней можно выделить конечное подпокрытие, образ которого и даст искомое конечное покрытие для $f(X)$. Доказательство закончено. □

Непрерывные функции на компактных пространствах обладают теми же свойствами, что и непрерывные функции на отрезке.

Следствие 11.40 (Теорема Вейерштрасса-Коши). Функияя, непрерывная на компактном топологическом пространстве X, ограничена и принимает наибольшее u наименьшее значения.

Доказательство. В силу утверждения Т.39 образ $f(X)$ функции f компактен, а в силу утверждения Ш. 38 - замкнут. Если бы образ $f(X)$ не был бы ограничен, то система открытых интервалов $U_n = (-n, n)$ покрывала бы $f(X)$, но не содержала бы конечного подпокрытия, поэтому множество $f(X)$ ограничено. Далее, очевидно, что точки $\sup_{x \in X} f(x)$ и $\inf_{x \in X} f(x)$ являются точками прикосновения замкнутого множества $f(X)$ и поэтому принадлежат $f(X)$. Утверждение доказано. □

Следующий результат полезен при доказательстве гомеоморфности топологических пространств.

Следствие 11.41. Непрерывное взаимно-однозначное отображение f компактного пространства X в хаусдорфово пространство Y является гомеоморфизмом.

Доказательство. Достаточно показать непрерывность обратного отображения, что, в силу теоремы 10.28, равносильно замкнутости прообраза произвольного замкнутого подмножества $F \subset X$ при отображении f^{-1}, т.е. замкнутости $f(F) \subset Y$. Но замкнутое подмножество компакта - компакт (см. утверждение 11.36). Далее, образ компакта F при непрерывном отображении f - тоже компакт (см. утверждение 11.39). Наконец, компактное подмножество $f(F)$ хаусдорфова пространства Y замкнуто в силу утверждения 1.38. Доказательство закончено. □

Упражнение 11.42. Проверить следующие свойства компактов:

- если пространство X представимо в виде конечного объединения своих компактных подмножеств, то X компактно;

- декартово произведение компактных пространств компактно.

В заключение сформулируем несколько специфических свойств компактных метрических пространств (отметим, что в общем случае эти свойства не имеют места).

Упражнение 11.43. Пусть X - метрическое пространство. Показать, что следующие свойства эквивалентны:

- пространство X компактно;
- всякая последовательность $\{x_n\}$ элементов из X содержит сходящуюся подпоследовательность;
- всякая последовательность непустых замкнутых подмножеств A_n, таких, что $A_{n+1} \subset A_n$, имеет непустое пересечение.

Упражнение 11.44. Привести пример метрического пространства X и замкнутого ограниченного некомпактного подмножества в нем (другими словами, в общем случае теорема Бореля неверна).

В следующей лекции с помощью топологических пространств определяются основные объекты дифференциальной геометрии - гладкие многообразия.

14.13.4 11.4 Problems

Задача 11.45. Доказать, что замыкание связного множества связно.
Задача 11.46. Доказать, что:
(1) при непрерывном отображении образ связного (линейно связного, компактного) пространства связен (линейно связен, компактен);
(2) прообраз хаусдорфова пространства при непрерывном отображении, взаимнооднозначном с образом, также является хаусдорфовым пространством.

Задача 11.47. Привести пример непрерывного отображения, для которого:
(1) прообраз связного (линейно связного, компактного) пространства не является связным (линейно связным, компактным);
(2) образ хаусдорфова пространства не является хаусдорфовым пространством.

Задача 11.48. Доказать, что подмножество плоскости, состоящее из графика функции $y = \sin 1/x, x \neq 0$, и отрезка $[-1, 1]$ оси ординат, наделенное индуцированной топологией, является связным, но не линейно связным.

Задача 11.49. Топологией Зарисского на множестве X называется система множеств, состоящая из пустого множества, а также из всех подмножеств множества X, дополнения до которых конечны.
(1) Проверьте, что эта система удовлетворяет аксиомам топологии.
(2) Докажите, что если X бесконечно, то с топологией Зарисского оно не является хаусдорфовым.
(3) Опишите все непрерывные отображения из пространства с топологией Зарисского в прямую со стандартной топологией, заданной функцией расстояния между точками.

Задача 11.50. Выяснить, какие из следующих матричных групп связны, какие компактны: $\mathrm{O}(n), \mathrm{SO}(n), \mathrm{GL}(n), \mathrm{SL}(n)$.

Задача 11.51. Говорят, что последовательность точек x_n топологического пространства x сходится к точке $x \in X$, называемой пределом последовательности x_n, если для любой окрестности U точки x существует такое N, что при каждом $n > N$ выполняется $x_n \in U$. Докажите, что:
(1) в хаусдорфовом пространстве предел сходящейся последовательности определен однозначно;

(2) в антидискретном пространстве X (пространстве с топологией, состоящей лишь из пустого множества и всего X) любая последовательность точек сходится к любой точке.

Опишите все сходящиеся последовательности в пространстве с топологией Зарисского.

14.13.5 11.5.1 Functional separability

Рассматриваемая в данном пункте версия аксиомы отделимости очень полезна с технической точки зрения.
Нам понадобится следующая терминология. Открытое покрытие $\{U_\alpha\}$ пространства X называется эффективным, если из него нельзя выбросить ни одного элемента U_α так, чтобы в результате снова получилось открытое покрытие. Пусть заданы два открытых покрытия пространства $X : U = \{U_\alpha\}$ и $V = \{V_\alpha\}$. Говорят, что покрытие V измельчает покрытие U, или вписано в покрытие U, если каждое множество V_β лежит в некотором множестве U_α.

Начнем со следующего важного свойства нормальных пространств.

Statement

11.52. Пусть X - нормальное топологическое пространство $uU = \{U_i\}, i = 1, \ldots, N$, - некоторое его конечное открытое покрытие. Тогда существует вписанное в U покрытие $V = \{V_i\}$, $i = 1, \ldots, N$, причем $\bar{V}_i \subset U_i$ для каждого i.

Доказательство. Если $N = 1$, то можно взять $V = U$.
Пусть теперь $N \geq 2$. Предположим сначала, что покрытие U эффективно. Рассмотрим замкнутые множества $X \backslash U_1$ и $X \backslash \bigcup_{i=2}^N U_i$. В силу эффективности покрытия U оба эти множества непусты. Кроме того, они замкнуты и не пересекаются. Поэтому, в силу нормальности пространства X, существует открытое множество V_1, такое, что

$$X \backslash \cup_{i=2}^N U_i \subset V_1 \subset \overline{V_1} \subset U_1$$

Система множеств $\{V_1, U_2, \ldots, U_N\}$ снова образует эффективное покрытие пространства X, поэтому к ней можно применить аналогичную процедуру, построив множество $V_2 \subset \bar{V}_2 \subset U_2$, такое, что $\{V_1, V_2, \ldots, U_N\}$ - эффективное покрытие. Продолжим эту процедуру до тех пор, пока не построим требуемое измельчение V покрытия U. Для эффективных покрытий утверждение доказано.

Если же покрытие U неэффективно, то его можно представить в виде объединения конечного числа эффективных покрытий, для каждого из которых можно построить покрытие V. Объединение соответствующих покрытий V и даст искомое покрытие. Утверждение доказано.

Два подмножества A_1 и A_2 топологического пространства X называются функцчонально отделимъu, если существует непрерывная функция $f : X \to [a, b], a < b$, такая, что $f|_{A_1} = a$, и $f|_{A_2} = b$. Если два замкнутых множества A_1 и A_2 функционально отделимы в X, то они отделимы и в обычном смысле. Действительно, достаточно взять число $c \in (a, b)$ и рассмотреть два открытых множества $f^{-1}([a, c))$ и $f^{-1}((c, b])$. Эти множества не пересекаются, причем первое из них содержит A_1, а второе $-A_2$. Оказывается, для "хороших" пространств имеет место и обратное утверждение.

Statement

11.53 (Лемма Урысона). Пусть X - нормальное топологическое пространство и A_0, A_1 - два его замкнутых непересекающихся подмножества. Тогда существует непрерывная

функция $f : X \to [0,1]$, такая, что $f|_{A_0} \equiv 0uf\big|_{A_1} \equiv 1$, т.е. множества A_0 и A_1 функичонально отделимы.

Доказательство. Пусть построена такая система открытых множеств U_r, занумерованных всеми двоичнорациональными (т.е. вида $p/2^n$) числами $r \in [0,1]$, что:

множество A_0 содержится в U_0;

(2) замыкание $\bar{U}_1$ множества U_1 содержится в $X \backslash A_1$;

(3) множества U_r последовательно вложены, т.е. если $r < r'$, то $\bar{U}_r \subset U_{r'}$.

Тогда систему открытых множеств $\{U_r\}$ можно расширить, добавив множества $U_t = \cup_{r<t} U_r$, где t произвольное вещественное число из $[0,1]$. Система множеств $\{U_t\}$ удовлетворяет свойству 3 . Действительно, для любых чисел t и t' из $[0,1]$, где $t < t'$, найдутся рациональные числа r и r', такие, что $t < r < r' < t'$. Тогда $U_t \subset U_r$ и $U_{r'} \subset U_{t'}$. Но при этом $\bar{U}_t \subset \bar{U}_r$, а $\bar{U}_r \subset U_{r'}$, откуда получаем, что $\bar{U}_t \subset U_{t'}$.

Определим функцию $f : X \to [0,1]$, положив $f(x) = \sup\{t \mid x \notin U_t\}$. Покажем, что функция f непрерывна на X. Фиксируем произвольную точку x_0 и число $\varepsilon > 0$ и положим $t_0 = f(x_0)$. По определению функции f имеем: $x_0 \notin \bar{U}_{t_0-\varepsilon/2}$ и $x_0 \in U_{t_0+\varepsilon/2}$. Рассмотрим окрестность $V = U_{t_0+\varepsilon/2} \backslash \bar{U}_{t_0-\varepsilon/2}$ точки x_0. Тогда, так как любая точка x из V принадлежит $U_{t_0+\varepsilon/2}$ и не принадлежит $\bar{U}_{t_0-\varepsilon/2}$, имеет место неравенство $t_0 - \varepsilon/2 \le f(x) \le t_0 + \varepsilon/2$, т.е. $|f(x_0) - f(x)| < \varepsilon$, что и доказывает непрерывность функции f. Наконец, функция f равна тождественно нулю на множестве A_0 и единице на A_1.

Для доказательства утверждения осталось построить систему открытых множеств U_r, удовлетворяющую требованиям $1-3$. В силу нормальности пространства X для любого замкнутого множества A и его окрестности U найдется такая другая окрестность V множества A, что $A \subset V \subset \bar{V} \subset U$. Для краткости будем обозначать через $V \Subset U$ тот факт, что $V \subset \bar{V} \subset U$.

Итак, $A_0 = \bar{A}_0 \subset X \backslash A_1$, поэтому существует открытое множество U_0, такое, что $A_0 \subset U_0 \Subset X \backslash A_1$. Точно так же $\bar{U}_0$ - замкнутое множество, содержащееся в открытом $X \backslash A_1$, поэтому найдется открытое множество U_1, такое, что $U_0 \Subset U_1$. Предположим теперь, что для всех двоично-рациональных чисел вида $r = p/2^n$,

где $0 \le p \le 2^n$, уже построены открытые множества U_r, для которых выполнено включение $U_{p/2^n} \Subset U_{(p+1)/2^n}$. Тогда, так как $\bar{U}_{p/2^n}$ - замкнутое множество, содержащееся в открытом множестве $U_{(p+1)/2^n}$, то найдется открытое множество $U_{(2p+1)/2^{n+1}}$, такое, что $U_{p/2^n} \Subset U_{(2p+1)/2^{n+1}} \Subset U_{(p+1)/2^n}$. Таким образом, по индукции мы построили требуемое семейство множеств $\{U_r\}$ для всех двоично-рациональных чисел r из отрезка $[0,1]$. Утверждение доказано.

Из леммы Урысона следует важное утверждение о продолжении непрерывной функции, заданной на замкнутом подмножестве нормального топологического пространства.

Statement

11.54. Пусть X - некоторое нормальное топологическое пространство, F - его произвольное замкнутое подмножество $uf : F \to \mathbb{R}^1$ - непрерывная функция на F. Тогда функция f продолжается до непрерывной функици $g : X \to \mathbb{R}^1$, где $g \mid F \equiv f$. Если при этом функция f ограничена некоторой константой c, т.е. $|f(x)| < c$, то u функц $_{\text{ил}}$ g может быть выбрана ограниченной той же константой c.

Доказательство. Предположим сначала, что функция f ограничена: существует константа c, такая, что $|f(x)| < c$. Положим $\varphi_0(x) = f(x)$ и рассмотрим два замкнутых подмножества A_0 и B_0 множества F, положив

$$A_0 = \{x \mid \varphi_0(x) \le -c/3\} \quad \text{и} \quad B_0 = \{x \mid \varphi_0(x) \ge c/3\}.$$

Множества A_0 и B_0 не пересекаются и замкнуты в пространстве X (так как F замкнуто), поэтому, в силу утверждения 11.53, существует непрерывная функция $f_0 : X \to [-c/3, c/3]$, такая, что $f_0|_{A_0} \equiv -c/3$, а $f_0|_{A_1} \equiv c/3$. В частности, $|f_0(x)| \le c/3$ для всех $x \in X$ и $|\varphi_0(x) - f_0(x)| \le 2c/3$.

Определим на F новую функцию $\varphi_1(x) = \varphi_0(x) - f_0(x)$. Эта непрерывная функция ограничена на F константой $2c/3$. Поэтому, повторяя конструкцию предыдущего абзаца, можно построить два непересекающихся замкнутых множества

$$A_1 = \{x \mid \varphi_1(x) \le -2c/9\} \quad \text{и} \quad B_1 = \{x \mid \varphi_1(x) \ge 2c/9\}$$

и непрерывную функцию $f_1 : X \to [-2c/9, 2c/9]$, такую, что

$$f_1|_{A_1} \equiv -2c/9 \quad \text{и} \quad f_1|_{B_1} \equiv 2c/9$$

В частности, $|f_1(x)| \le 2c/9$ и $|\varphi_1(x) - f_1(x)| \le 4c/9$.
Повторяя этот процесс бесконечное число раз, построим две последовательности функций $f_n : X \to \mathbb{R}^1$ и $\varphi_n : F \to \mathbb{R}^1$, такие, что $\varphi_{n+1}(x) = \varphi_n(x) - f_n(x)$ и выполнены оценки

$$|f_n(x)| \le (2/3)^n (c/3), \quad |\varphi_{n+1}(x)| \le (2/3)^{n+1} c$$

Далее, при всех x из F имеем

$$f(x) = \varphi_0(x) = f_0(x) + \varphi_1(x) = f_0(x) + f_1(x) + \varphi_2(x) \cdots = \cdots = \\ = f_0(x) + f_1(x) + \cdots + f_n(x) + \varphi_{n+1}(x)$$

Так как $|f_n(x)| \le (2/3)^n (c/3)$, то ряд $\sum_{n=1}^{\infty} f_n(x)$ сходится равномерно на всем пространстве X и, поэтому, функция $g(x) = \sum_{n=1}^{\infty} f_n(x)$ непрерывна на всем X. Так как $|\varphi_{n+1}(x)| \le (2/3)^{n+1} c$, функция $g(x)$ совпадает с $f(x)$ на всем множестве F. Наконец,

$$|g(x)| \le \sum_{n=1}^{\infty} |f_n(x)| \le (c/3) \sum_{n=1}^{\infty} (2/3)^n = c$$

что и завершает доказательство утверждения для случая ограниченной функции.
Пусть теперь функция f неограничена. Рассмотрим произвольный гомеоморфизм $h : \mathbb{R}^1 \to (-1, 1)$. Тогда композиция $h \circ f$ - это непрерывная ограниченная функция на пространстве X, и для нее, как мы уже доказали, существует продолжение $g : X \to [-1, 1]$. Значения ± 1 функция g принимает на некотором замкнутом множестве F_1, причем F_1 не пересекается с F. Согласно утверждению Т.5.3 существует непрерывная функция $\xi : X \to [0, 1]$, равная нулю на множестве F_1 и единице на F. Тогда функция $u(x) = g(x)\xi(x)$ совпадает с функцией $h \circ f$ на множестве F и переводит все пространство X в открытый интервал $(-1, 1)$. Положим, наконец, $w(x) = h^{-1} \circ u(x)$. На множестве F функции w и f совпадают, поэтому w - это искомое продолжение. Утверждение доказано.

14.13.6 11.5.2 Unit breakdown

Разовьем полезную для дальнейшего технику, позволяющую работать с непрерывными функциями на топологических пространствах в целом.

Пусть f - непрерывная функция на топологическом пространстве X. Носителем функции f называется замыкание множества всех точек x из X, в которых функция f отлична от нуля. Будем обозначать носитель функции f через $\operatorname{supp} f$. По определению, вне своего носителя функция тождественно равна нулю. Полезным техническим

приемом в топологии является разложение непрерывной функции в сумму непрерывных функций, носители которых в определенном смысле малы.

Пусть $U = \{U_\alpha\}$ - некоторое открытое покрытие пространства X. Покрытие U называется локально конечным, если каждая точка x пространства X имеет такую окрестность $O(x)$, которая пересекается лишь с конечным набором элементов покрытия U.

Пусть $U = \{U_\alpha\}$ - локально конечное покрытие пространства X. Система функций $\varphi_\alpha : X \to \mathbb{R}^1$, удовлетворяющая условиям:

- для каждого α и $x \in X$ имеет место неравенство $0 \le \varphi_\alpha(x) \le 1$;
- для каждого α носитель функции φ_α содержится в соответствующем элементе покрытия: $\operatorname{supp} \varphi_a \subset U_\alpha$;
- $\sum_\alpha \varphi_\alpha(x) \equiv 1$,

называется разбиением единицы, подчиненным покрытию U. Отметим, что сумма из определения разбиения единицы имеет смысл, так как в каждой точке x лишь конечное число слагаемых отлично от нуля.

Оказывается, что для любого локально конечного покрытия нормального пространства X существует подчиненное ему разбиение единицы. Докажем здесь этот факт только для конечных покрытий.

Statement

11.55. Пусть X - нормальное пространство и $\{U_\alpha\}$ - его конечное покрытие. Тогда существует подчиненное этому покрытию разбиение единицы.

Доказательство. Пусть $\{U_\alpha\}$ - конечное покрытие. В силу утверждения Т.5.2 существует измельчение $\{V_\alpha\}$ покрытия $\{U_\alpha\}$, причем $V_\alpha \Subset U_\alpha$. Воспользовавшись утверждением 11.53, построим на пространстве X непрерывную функцию ψ_α, такую, что $\psi_\alpha|_{\bar{V}} \equiv 1$, $\psi_\alpha|_{X \setminus U_\alpha} \equiv 0$ и $0 \le \psi_\alpha(x) \le 1$. Это означает, что носитель $\operatorname{supp} \psi_\alpha$ содержится в U_α и $\psi_\alpha(x) > 0$ при $x \in V_\alpha$. Положим $\psi(x) = \sum_\alpha \psi_\alpha(x)$. Эта функция непрерывна. Покажем теперь, что $\psi(x) > 0$ на всем пространстве X. Действительно, для любой точки x из X существует окрестность V_{α_0}, такая, что $x \in V_{\alpha_0}$, и, в частности, $\psi_{\alpha_0}(x) > 0$. Но тогда и $\psi(x) \ge \psi_{\alpha_0} > 0$. Определим, наконец, непрерывные функции $\varphi_\alpha(x)$, положив $\varphi_\alpha(x) = \psi_\alpha(x)/\psi(x)$. Очевидно, $\operatorname{supp} \varphi_\alpha = \operatorname{supp} \psi_\alpha \subset U_\alpha$ и, кроме того, $0 \le \varphi_\alpha \le 1$, а

$$\sum_\alpha \varphi_\alpha(x) = \sum_\alpha \psi_\alpha(x)/\psi(x) = \psi(x)/\psi(x) = 1$$

Итак, мы построили разбиение единицы, подчиненное покрытию $\{U_\alpha\}$. Утверждение доказано.

Замечание 11.56. Для перенесения доказательства утверждения 11.55 на локально конечное покрытие нужно получить обобщение утверждения 11.52 на случай такого покрытия.

Упражнение 11.57. Пусть X - евклидово пространство $\mathbb{R}^n$. Показать, что в этом случае можно построить разбиение единицы с гладкими функциями φ_a.

14.14 12 Diversity

Идея, лежащая в основе понятия многообразия, впервые, по-видимому, возникла при построении географических карт земной поверхности. Такие построения сводятся к отображению сферической поверхности Земли на плоскость. Естественно требовать,

чтобы используемые отображения были взаимно-однозначны и непрерывны, т.е. являлись гомеоморфизмами. Однако можно показать, что сфера не гомеоморфна никакой части плоскости, иными словами, нельзя построить карту всей Земли. Кроме того, при попытке построения карт достаточно больших кусков земной поверхности неизбежно возникают искажения (напомним, что сфера и плоскость локально не изометричны, так как имеют разные гауссовы кривизны). Поэтому картографы разбивают поверхность Земли, т.е. сферу, на несколько достаточно маленьких перекрывающихся кусочков и отдельно проецируют каждый из них на плоскость, получая тем самым атлас карт. Исходная сфера - Земля - получается при обратной склейке этих кусочков. Неформально говоря, многообразие размерности n - это топологическое пространство, склеенное из областей пространства $\mathbb{R}^n$, как поверхность Земли из карт атласа.

Отметим, что с многообразиями мы уже не раз сталкивались в нашем курсе, а именно, регулярные кривые и поверхности - все это многообразия. Перейдем теперь к формальным определениям.

14.14.1 12.1 Topological manifolds

Определение 12.1. *A* тласом размерности n на топологическом пространстве X называется семейство $\mathcal{A} = \{(\mathcal{U}_\alpha, \varphi_\alpha)\}$, где $\{U_\alpha\}$ - покрытие пространства X его открытыми связными подмножествами, каждое из которых гомеоморфно некоторой области V_α в $\mathbb{R}^n$, а $\varphi_\alpha : U_\alpha \to V_\alpha$ - соответствующие гомеоморфизмы. Пара $(U_\alpha, \varphi_\alpha)$ называется картой, гомеоморфизм φ_α - координатным гомеоморфизмом. Если в $\mathbb{R}^n$ фиксированы некоторые координаты, то говорят, что карта $(U_\alpha, \varphi_\alpha)$ задает в окрестности U_α координаты, сопоставляя каждой точке $P \in U_\alpha$ координаты вектора $\varphi_\alpha(P) \in \mathbb{R}^n$. Если $U_\alpha \cap U_\beta$ непусто, то возникает гомеоморфизм

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha \left(U_\alpha \cap U_\beta\right) \subset \mathbb{R}^n \to \varphi_\beta \left(U_\alpha \cap U_\beta\right) \subset \mathbb{R}^n,$$

который называется отображением склейки, или функцией перехода (от одной карты к другой или от одних координат к другим).

Замечание 12.2. Выбор карты $(U_\alpha, \varphi_\alpha)$ в окрестности данной точки P на многообразии задает в окрестности этой точки координаты (считаем, что в открытом подмножестве V_α пространства $\mathbb{R}^n$ заданы некоторые криволинейные координаты $(x^1, \ldots, x^n)$), т.е. каждой точке $P \in U_\alpha$ ставится в соответствие набор чисел $\varphi_\alpha(P) = (x_0^1, \ldots, x_0^n)$.

Замечание 12.3. Пусть точка P принадлежит двум разным картам U_α, U_β и пусть $(x^1, \ldots, x^n)$ - координаты в V_α, а $(y^1, \ldots, y^n)$ - координаты в V_β. Тогда в окрестности $U_\alpha \cap U_\beta$ заданы две системы координат, определенные картами $(U_\alpha, \varphi_\alpha)$ и (U_β, φ_β) : каждой точке $P \in U_\alpha \cap U_\beta$ ставятся в соответствие два набора чисел $\varphi_\alpha(P) = (x_0^1, \ldots, x_0^n)$ и $\varphi_\beta(P) = (y_0^1, \ldots, y_0^n)$. При этом отображение склейки может быть записано в виде $x^i = x^i\left(y^1, \ldots, y^n\right)$ или $y^i = y^i\left(x^1, \ldots, x^n\right)$, т.е. отображение склейки задает формулы пересчета одних координат через другие. Поэтому отображение склейки иногда называется просто заменой координат.

Определение 12.4. Хаусдорфово топологическое пространство называется топологическим многообразием размерности n, если на нем существует не более чем счетный атлас размерности n.

Пример 12.5. Простейший пример многообразия - это открытое связное подмножество $\Omega \subset \mathbb{R}^n$. На нем можно выбрать атлас, состоящий из одной карты $(\Omega, \mathrm{id} : \Omega \to \Omega)$, где id - тождественное отображение.

Пример 12.6. Следующий тривиальный пример многообразия - это график непрерывной функции $f : \mathbb{R}^n \to \mathbb{R}$, т.е. подмножество $M \subset \mathbb{R}^n \times \mathbb{R}$ следующего вида

$$M = \{(x, f(x)) \mid x \in \mathbb{R}^n\}.$$

Атлас на M состоит из одной карты

$$\left(M, \varphi : \left(x^1, \ldots, x^n, f\left(x^1, \ldots, x^n\right)\right) \mapsto \left(x^1, \ldots, x^n\right)\right).$$

Пример 12.7. Стандартная сфера $S^n \subset \mathbb{R}^{n+1}$, заданная в евклидовых координатах $(x^1, \ldots, x^{n+1})$ как множество вида $\left\{\sum_i (x^i)^2 = 1\right\}$, с топологией, индуцированной из объемлющего пространства, превращается в многообразие, если в качестве атласа выбрать $2n+2$ карты вида

$$\left(U_i^+ = \left\{x^i > 0\right\}, \varphi_i^+ : \left(x^1, \ldots, x^{n+1}\right) \mapsto \left(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}\right)\right),$$
$$\left(U_i^- = \left\{x^i < 0\right\}, \varphi_i^- : \left(x^1, \ldots, x^{n+1}\right) \mapsto \left(x^1, \ldots, \widehat{x^i}, \ldots, x^{n+1}\right)\right),$$

где $\widehat{x^i}$ означает, что координата x^i пропущена, а $i = 1, \ldots, n+1$.
Пример 12.8. Другой пример атласа на сфере можно получить с помощью стереографических проекций. Пусть $N = (0, 0, \ldots, 1) \in S^n$ - северный полюс, а $S = (0, 0, \ldots, -1)$ - южный полюс сферы S^n. Обозначим через φ_N и φ_S стереографические проекции из северного и южного полюсов соответственно на координатную гиперплоскость
$\Pi = \{x^{n+1} = 0\}$. Так как стереографические проекции являются гомеоморфизмами, $(S^n \backslash \{N\}, \varphi_N)$ и $(S^n \backslash \{S\}, \varphi_S)$ задают на S^n структуру n-мерного многообразия. Таким образом, на сфере можно выбрать атлас, состоящий из двух карт.

Упражнение 12.9. Выпишите функции перехода для атласов из примеров [12.7 и [2.8]

Упражнение 12.10. Показать, что если X - топологическое многообразие размерности n, то на X существует не более чем счетный атлас $\{(U_\alpha, \varphi_\alpha)\}$, где каждый координатный гомеоморфизм имеет вид $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$, т.е. без ограничения общности можно предполагать, что координатные гомеоморфизмы отображают карты на все $\mathbb{R}^n$.

Упражнение 12.11. Показать, что если X - топологическое многообразие размерности n, то на X существует не более чем счетный атлас $\{(U_\alpha, \varphi_\alpha)\}$, где каждый координатный гомеоморфизм имеет вид $\varphi_\alpha : U_\alpha \to D^n$, где $D^n \subset \mathbb{R}^n$ - открытый шар в пространстве $\mathbb{R}^n$.

14.14.2 12.2 Functions and displays

Конструкция 12.12. Пусть M^n - некоторое топологическое многообразие, $f : M \to \mathbb{R}^1$ - непрерывная функция на M. Рассмотрим произвольную карту (U, φ) многообразия M, т.е. гомеоморфизм $\varphi : U \to \mathbb{R}^n$. Тогда определена функция $f = f \circ \varphi^{-1}$, отображающая $\mathbb{R}^n$ в $\mathbb{R}^1$. Обозначим через $(x^1, \ldots, x^n)$ координаты в $\mathbb{R}^n$, которые, напомним, предполагаются фиксированными. Тогда $\tilde{f}$ может быть записана в этих координатах в виде $\tilde{f}(x^1, \ldots, x^n)$.
Определение 12.13. Функцию $\tilde{f}(x^1, \ldots, x^n)$ будем называть координатным представлением функции f в карте (U, φ), или координатным представлением функции f в координатах ($x^1, \ldots, x^n$). При этом часто, допуская некоторую вольность речи, будем отождествлять разнье функции f и $\tilde{f}$ и записывать координатное представление $\tilde{f}$ функции f в виде $f(x^1, \ldots, x^n)$.

Конструкция 12.14. Пусть имеются два многообразия M^m, N^n и пусть задано непрерывное отображение $F : M \to N$ из многообразия M в многообразие N. Фиксируем в многообразии M точку P, и пусть $Q = F(P)$. Рассмотрим карту (W, ξ) многообразия N, такую, что $Q \in W$. Тогда, в силу непрерывности отображения F, существует открытое в M связное множество $U \subset M$, содержащее точку P, такое, что $F(U) \subset W$.

При этом множество U можно выбрать целиком лежащим в некоторой карте многообразия M. Ограничение отображения φ на U также является картой, которую без ограничения общности можно считать входящей в наш атлас.

Итак, существует карта (U, φ) многообразия M, такая, что $P \in U$ и $F(U) \subset W$. Отображение F порождает отображение $\tilde{F} = \xi \circ F \circ \varphi^{-1}$, заданное на области $\varphi(U) \subset \mathbb{R}^m$ и переводящее ее в область $\xi(W) \subset \mathbb{R}^n$.

Обозначим через ($x^1, \ldots, x^m$) координаты на M, порожденные картой (U, φ), а через $(y^1, \ldots, y^n)$ - координаты на N, порожденные картой (W, ξ). Отображение $\tilde{F}$ в этих координатах может быть записано в виде

$$y^i = \tilde{F}^i \left(x^1, \ldots, x^m\right), \quad i = 1, \ldots, n.$$

Определение 12.15. Отображение называется координатным представлением отображения F в картах $(U, \varphi) u (W, \xi)$, или координатным представлением отображения F в координатах $(x^1, \ldots, x^m)$ и $(y^1, \ldots, y^n)$. При этом часто, допуская некоторую вольность речи, будем отождествлять разные отображения F и $\tilde{F}$ и записывать координатное представление $\tilde{F}$ отображения F как $F^i\left(x^1, \ldots, x^m\right)$.

14.14.3 12.3 Smooth manifolds

Пусть M^n - некоторое топологическое многообразие, $f : M^n \to \mathbb{R}$ - непрерывная функция на M^n и P - произвольная точка из M^n. Как определить дифференцируемость f в точке P ? Естественно было бы воспользоваться координатным представлением функции f в каких-нибудь координатах в окрестности точки P, которое, напомним, является обычной функцией на области в $\mathbb{R}^n$, и определить дифференцируемость f в $P \in M^n$ как обычную дифференцируемость ее координатного представления в точке $\varphi^{-1}(P)$. Собственно, для этого и нужна карта, т.е. система координат. Однако, как мы уже отмечали, точка P может принадлежать нескольким картам, и, тем самым, в окрестности точки P может быть определено несколько систем координат. Тогда естественно требовать, чтобы определение дифференцируемости функции в точке P не зависело от выбора карты, или, другими словами, чтобы все системы координат на многообразии были бы равноправны. Ясно, что это "равноправие" есть в точности условие на функции перехода. Оно приводит к следующему определению.

Определение 12.16. Топологическое многообразие с атласом $\mathcal{A} = \{(\mathcal{U}_\alpha, \varphi_\alpha)\}$ называется гладким, если все отображения склейки атласа $\mathcal{A}$ - гладкие. Атлас $\mathcal{A}$ в этом случае тоже называется гладким.

Замечание 12.17. Напомним, что в дифференциальной геометрии под словом "гладкий" понимают непрерывную дифференцируемость столько раз, сколько нужно, хотя, конечно, можно определить многообразие класса $C^r, 0 \leq r \leq \infty$ (случай $r = 0$ соответствует топологическим многообразиям), а также вещественно-аналитические и, для четных n, комплексно-аналитические многообразия.

Ниже, как правило, мы будем иметь дело с бесконечно гладкими многообразиями (если противное не оговорено специально).

Замечание 12.18. Если структура топологического многообразия задана атласом, состоящим из одной карты, то оно автоматически является гладким. (Функций перехода нет вообще, гладкость проверять не у чего.)

Замечание 12.19. Существуют примеры топологических многообразий, на которых нельзя ввести структуру гладкого многообразия.

Определение 12.20. Пусть на топологическом многообразии M заданы два гладких атласа: $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ и $\mathcal{A}' = \{(U'_\alpha, \varphi'_\alpha)\}$. Атласы $\mathcal{A}$ и $\mathcal{A}'$ называются эквивалентны4,

если их объединение $\mathcal{A} \cup \mathcal{A}'$ также является гладким атласом (т.е. все функции перехода между всеми картами атласов $\mathcal{A}$ и $\mathcal{A}'$ - гладкие).

Упражнение 12.21. Эквивалентны ли следующие два атласа на вещественной прямой: $\mathcal{A} = \{(U = R^1(x), \varphi(x) = x)\}$ и $\mathcal{A}' = \{(U' = \mathbb{R}^1(x), \varphi'(x) = x^3)\}$?

Конструкция 12.22. Пусть $\mathcal{A}$ - некоторый гладкий атлас на M. Обозначим через $\mathcal{A}_{\max}$ максимальный по включению гладкий атлас на M, эквивалентный $\mathcal{A}$. Другими словами, если $\mathcal{A}'$ - произвольный эквивалентный $\mathcal{A}$ атлас на M, то $\mathcal{A}'$, по определению, содержится в $\mathcal{A}_{\max}$. Ясно, что два атласа на M эквивалентны, если и только если они содержатся в одном и том же максимальном атласе.

Определение 12.23. Максимальные гладкие атласы на многообразии M называют гладкими структурами многообразия M. В дальнейшем, говоря о гладком многообразии, будем всегда предполагать фиксированной некоторую его гладкую структуру.

14.14.4 12.4 Examples of smooth manifolds

Пример 12.24. Рассмотренные выше многообразия, т.е. область Ω, график непрерывной функции и сфера S^n, являются гладкими. (В первых двух случаях это очевидно, в третьем - следует из явного вида функций перехода.)

Упражнение 12.25. Выше на сфере S^n было построено два атласа (атлас из $2n+2$ "координатных" полусфер, которые проецируются на соответствующие координатные плоскости, и атлас из двух проколотых сфер со стереографическими координатами). Проверьте, являются ли эти атласы эквивалентными.

Пример 12.26. Важным примером гладкого многообразия является проективное пространство $\mathbb{RP}P^n$. Напомним, что проективным пространством $\mathbb{RP}P^n$ размерности n называется множество всех прямых в $\mathbb{R}^{n+1}$, проходящих через начало координат. Введем на $\mathbb{RP}^n$ функцию расстояния $\rho(P, Q)$, равную углу между прямыми P и Q (проверьте, что функция ρ удовлетворяет всем аксиомам расстояния). Функция расстояния ρ превращает $\mathbb{RP}^n$ в метрическое, а значит и в топологическое, пространство. Зададим теперь на $\mathbb{RP}^n$ карты следующим образом.

Обозначим через U_i множество всех проходящих через начало координат прямых в $\mathbb{R}^{n+1}$, не перпендикулярных i-му координатному вектору. Иными словами, прямая ℓ с направляющим вектором $e = (x^1, \ldots, x^{n+1})$ входит в U_i, если и только если $x^i \neq 0$. Отображение $\varphi_i : U_i \to \mathbb{R}^n$ определим по формуле

$$\varphi_i : \ell \mapsto \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i} \right).$$

(Проверьте, что φ_i - гомеоморфизмы.)
Выпишем в явном виде отображения склейки. Рассмотрим две карты (U_i, φ_i) и (U_j, φ_j). Пусть $\varphi_{ij} : \varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ - отображение склейки. Если $e = (x^1, \ldots, x^{n+1})$ - направляющий вектор прямой $\ell \in U_i \cap U_j$, то

$$\varphi_i(\ell) = \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i} \right) = \left(\alpha^1, \ldots, \alpha^{i-1}, \alpha^{i+1}, \ldots, \alpha^{n+1} \right),$$
$$\varphi_j(\ell) = \left(\frac{x^1}{x^j}, \ldots, \frac{x^{j-1}}{x^j}, \frac{x^{j+1}}{x^j}, \ldots, \frac{x^{n+1}}{x^j} \right) = \left(\beta^1, \ldots, \beta^{j-1}, \beta^{j+1}, \ldots, \beta^{n+1} \right),$$

поэтому

$$\beta^k = \frac{x^i}{x^j} \alpha^k = \frac{\alpha^k}{\alpha^j} \quad \text{при} \quad k \neq i, \quad \text{и} \quad \beta^i = \frac{x^i}{x^j} = \frac{1}{\alpha^j}.$$

Таким образом, отображения склейки являются гладкими, и, значит, $\mathbb{R}\mathrm{P}^n$ с построенным атласом - гладкое n-мерное многообразие.

Пример 12.27. Пусть M_1 и M_2 - два гладких многообразия. Введем на прямом произведении $M_1 \times M_2$ топологических пространств M_1, M_2 структуру гладкого многообразия. А именно, пусть $\{(U_\alpha, \varphi_\alpha)\}$ и $\{(V_\alpha, \psi_\alpha)\}$ - атласы на многообразиях M_1 и M_2 соответственно. Тогда система карт $\{(U_\alpha \times V_\beta, \varphi_\alpha \times \psi_\beta)\}$ образует на $M_1 \times M_2$ гладкий атлас (проверьте!).

14.14.5 12.5 Smooth functions, smooth mappings, diffeomorphisms

Пусть M - гладкое многообразие (напомним, на M, по определению, фиксирован некоторый гладкий атлас $\mathcal{A}$) и $f : M \to \mathbb{R}^1$ - непрерывная функция на M.

Определение 12.28. Функция f называется гладкой в точке $P \in M$, если для некоторой карты $(U, \varphi) \in \mathcal{A}$, такой, что $P \in U$, координатное представление функции f является гладким в точке $\varphi(P)$. Функция f называется гладкой на многообразии M, если она гладкая во всех точках многообразия M.

Далее, пусть M и N - два гладких многообразия и $F : M \to N$ - непрерывное отображение из M в N.

Определение 12.29. Отображение F называется гладким в точке $P \in M$, если для некоторых карт (U, φ) на M и (W, ξ) на N, таких, что $P \in U$ и $F(U) \subset W$ (напомним, что такие карты существуют для любой точки $P \in M$), соответствующее координатное представление отображения F является гладким в точке $\varphi(P)$. Отображение F называется гладким, если оно является гладким в каждой точке $P \in M$.

Замечание 12.30. Гладкая функция - это просто частный случай гладкого отображения многообразий: многообразие M отображается в многообразие $\mathbb{R}^1$ со стандартной гладкой структурой.

Утверждение 12.31. Определения гладкой функции и гладкого отображения зависят только от гладких структур рассматриваемых гладких многообразий и не зависят от выбора координат (в фиксированных гладких структурах).

Доказательство. Действительно, если (U, φ) и (U', φ') - две карты на M, содержащие точку P, и f - непрерывная функция на M, то координатные представления $f \circ \varphi^{-1}$ и $f \circ (\varphi')^{-1}$ функции f в этих разных картах одновременно являются гладкими в точках $\varphi(P)$ и $\varphi'(P)$ соответственно, поскольку отличаются друг от друга на гладкую функцию перехода $\varphi' \circ \varphi^{-1}$. Аналогично, если $F : M \to N-$ гладкое отображение гладких многообразий, то при заменах координат в окрестности точки $P \in M$ и точки $F(P) \in N$ координатные представления функции F отличаются на гладкие функции перехода. $\square$

Замечание 12.32. Аналогично можно определить функции и отображения класса C^r, а также вещественно-аналитические и комплексно-аналитические функции (отображения). Отметим, что класс гладкости многообразия должен быть не ниже определяемого класса гладкости отображения.

Конструкция 12.33. Пусть $F : M^m \to N^n$ - гладкое отображение гладких многообразий, P - произвольная точка из M и $Q = F(P)$. Рассмотрим координатное представление функции F в координатах $(x^1, \ldots, x^m)$ в окрестности $P \in M$ и $(y^1, \ldots, y^n)$ в окрестности Q, которое запишем в виде

$$y^i = f^i\left(x^1, \ldots, x^m\right), \quad i = 1, \ldots, n.$$

Тогда определена матрица Якоби этого координатного представления, т.е. матрица

$$J(P) = \left(\frac{\partial y^i}{\partial x^j}(P)\right)$$

Определение 12.34. Матрица $J(P)$, построенная выше, называется матрицей Якоби отображения F в точке P в координатах $(x^1, \ldots, x^m)$ и $(y^1, \ldots, y^n)$.

Упражнение 12.35. Проверить, что при замене координат матрица Якоби изменяется, как матрица линейного отображения, где в качестве матриц перехода выступают матрицы Якоби соответствующих функций перехода.

Пример 12.36. Пусть M - гладкое многообразие. Гладкой кривой γ на многообразии M называется гладкое отображение интервала I (со стандартной гладкой структурой) в многообразие M. Если точка P из M лежит в образе отображения γ, т.е. $P = \gamma(t_0)$, $t_0 \in I$, то будем говорить, что кривая γ проходит через точку P. Матрица Якоби отображения γ в точке t_0 состоит из одного столбца и называется вектором скорости кривой γ в точке P (в соответствующих координатах). Далее, если $\gamma : I \to M-$ гладкая кривая на M и $\tau : I' \to I$ - взаимно-однозначное гладкое в обе стороны отображение интервала I' в интервал I, то отображение $\tilde{\gamma} = \gamma \circ \tau : I' \to M$ - снова гладкая кривая. Будем говорить, что кривая $\tilde{\gamma}$ получена из кривой γ заменой параметризации. С точностью до параметризации всегда можно считать, что кривая параметризована, скажем, интервалом $(-1, 1)$.

Напомним, что на классе топологических пространств мы определили отношение эквивалентности с помощью понятия гомеоморфизма. Однако гомеоморфизмы не сохраняют структуру гладкого многообразия (хотя карта по-прежнему переходит в карту, гладкость функций перехода разрушается). Поэтому требуется усилить определение гомеоморфизма так, чтобы гладкая структура сохранялась.

Определение 12.37. Пусть F - гомеоморфизм гладкого многообразия M в гладкое многообразие N. Если отображение F является гладким и, более того, обратное отображение F^{-1} также является гладким, то отображение F называется диффеоморфизмом. Если существует диффеоморфизм M в N, то говорят, что многообразия M и N диффеоморфны.

Упражнение 12.38. Выше, см. задачу [2.2], мы уже видели пример двух неэквивалентных гладких атласов на прямой $\mathbb{R}^1$. Обозначим через M_1 и M_2 соответствующие гладкие многообразия, а именно, M_1 - это прямая $\mathbb{R}^1$ с картой ($U = \mathbb{R}^1, \varphi_1 : x \mapsto x$), а M_2 - это прямая с картой ($U = \mathbb{R}^1, \varphi_2 : x \mapsto x^3$). Показать, что M_1 и M_2 диффеоморфны.

Замечание 12.39. Две гладкие структуры на одном и том же топологическом многообразии называются эквивалентными, если соответствующие гладкие многообразия диффеоморфны. Например, гладкие структуры на прямой, описанные в задаче [2.2], эквивалентны. Оказывается, в размерности $n = 4$, и только в ней, на $\mathbb{R}^n$ существуют гладкие структуры, не эквивалентные стандартной гладкой структуре на $\mathbb{R}^n$.

Замечание 12.40. Отношение диффеоморфности является отношением эквивалентности на классе гладких многообразий, и мы, как правило, не будем различать диффеоморфные многообразия.

Свойства многообразий, сохраняющиеся при диффеоморфизмах, называются дифференицальными инвариантами. Такие свойства являются важными характеристиками многообразий. Одним из таких инвариантов является размерность многообразия.

Theorem

12.41. Если гладкие многообразия M и N диффеоморфны, то их размерности совпадают.

Доказательство. Пусть $F : M \to N$ - диффеоморфизм. Обозначим через m и n размерности многообразий M и N соответственно. Фиксируем произвольную точку P на многообразии M, обозначим через Q ее образ при отображении F и рассмотрим координатные представления отображений F и F^{-1} в координатах ($x^1, \ldots, x^m$) в

окрестности точки P на M и ($y^1, \ldots, y^n$) в окрестности точки Q на N. Эти координатные представления - суть векторнозначные функции

$$\left(y^1, \ldots, y^n\right) = F\left(x^1, \ldots, x^m\right), \quad \left(x^1, \ldots, x^m\right) = F^{-1}\left(y^1, \ldots, y^n\right),$$

причем эти функции взаимно-обратны:

$$F\left(F^{-1}\left(y^1, \ldots, y^n\right)\right) = \left(y^1, \ldots, y^n\right), \quad F^{-1}\left(F\left(x^1, \ldots, x^m\right)\right) = \left(x^1, \ldots, x^m\right).$$

Обозначим через $dF(P)$ и $dF^{-1}(Q)$ матрицы Якоби координатных представлений отображений F и F^{-1} в точках P и Q соответственно . Напомним стандартную теорему о дифференцировании сложной функции.

Лемма 12.42. Пусть $U_i, i = 1, 2, 3,$ - область в евклидовом пространстве $\mathbb{R}^{n_i}$. Пусть заданы гладкие отображения $F : U_1 \to U_2, G : U_2 \to U_3$ и $H = G \circ F$ - композиция отображений F и G. Тогда отображение H также является гладким, и если $P-$ точка из U_1, то для матрии, Якоби выполнено следующее соотношение:

$$dH(P) = dG(F(P)) \cdot dF(P)$$

Применим этот результат к композициям $F \circ F^{-1}$ и $F^{-1} \circ F$. Получим:

$$E_m = dF^{-1}(Q) \cdot dF(P), \quad E_n = dF(P) \cdot dF^{-1}(Q)$$

где E_k - единичная матрица размера $k \times k$, так как матрица Якоби тождественного отображения - единичная матрица.

Поскольку матрицы $dF(P)$ и $dF^{-1}(Q)$ - прямоугольные, $(n \times m)$ и $(m \times n)$ соответственно, имеем

$$\operatorname{rank} dF(P) \leq \min(m, n) \quad \text{и} \quad \operatorname{rank} dF^{-1}(Q) \leq \min(m, n).$$

Вспомним, что ранг произведения матриц не превосходит ранга каждого из сомножителей. Получаем: $m \leq \min(m, n)$ и $n \leq \min(m, n)$, поэтому $\max(m, n) \leq \min(m, n)$, т.е. $m = n$, что и требовалось доказать.

Замечание 12.43. Отметим, что аналогичный результат для гомеоморфных топологических многообразий - весьма нетривиален и находится поэтому за рамками настоящего курса.

Конструкция 12.44. Пусть X - топологическое пространство, M - гладкое многообразие и F - гомеоморфизм из X в M. Тогда на X можно ввести структуру гладкого многообразия так, чтобы отображение F стало диффеоморфизмом. Для этого достаточно построить на X следующий атлас: для каждой карты (U, φ) на M рассмотрим карту ($F^{-1}(U), \varphi \circ F$). Очевидно, что полученный в результате атлас задает на X гладкую структуру, которая называется перенесенной на X с M с помощью отображения F.

14.14.6 12.6 Setting manifolds by equations - the geometric meaning of the implicit function theorem

Один из наиболее часто встречающихся в приложениях объектов - это подмножество евклидового пространства $\mathbb{R}^n$, заданное системой уравнений.

Конструкция 12.45. Пусть задана система уравнений

[01] Здесь мы, избегая громоздких формул, допускаем определенную вольность, отождествляя отображение с его координатным представлением, а точку многообразия - с ее образом при координатном гомеоморфизме. Формально следует писать $d\tilde{F}(\varphi(P))$.

$$f^i\left(x^1, \ldots, x^n\right)=c^i, \quad i=1, \ldots, k.$$

Множество решений этой системы уравнений удобно представлять как множество уровня векторнозначной функции $F=\left(f^1, \ldots, f^k\right)$:

$$M_c=\left\{\left(x^1, \ldots, x^n\right) \mid F\left(x^1, \ldots, x^n\right)=c\right\},$$

где $c=\left(c^1, \ldots, c^k\right)$ - постоянный вектор.

Theorem

12.46. Если множество M_c не пусто и матрица Якоби отображения F имеет в каждой точке из M_c максимальный ранг, равный k, то M_c является гладким многообразием размерности $n-k$. При этом, в подходящей окрестности U произвольной точки из M_c в качестве координат можно выбрать некоторые $n-k$ декартовых координат объемлющего пространства $\mathbb{R}^n$. Другими словами, в качестве координатного гомеоморфизма можно выбрать проекцию окрестности U на подходящую $(n-k)$-мерную координатную плоскость.

Доказательство. Пусть P - произвольная точка из M_c. По условию, ранг матрицы Якоби $dF(P)$ отображения F в точке P равен k. Это означает, что имеется набор из k столбцов матрицы Якоби такой, что определитель соответствующего ($k \times k$)минора матрицы $dF(P)$ отличен от нуля. Без ограничения общности предположим, что это столбцы с номерами $1, \ldots, k$. Обозначим через $\tilde{P}$ образ точки P при стандартной проекции на координатную плоскость $\mathbb{R}^{n-k}$, определенную соотношениями $x^1=\cdots=x^k=0$. Тогда по теореме о неявной функции, известной из математического анализа, в некоторой окрестности W точки $\tilde{P}$ в координатной плоскости $\mathbb{R}^{n-k}$ существуют гладкие функции

$$x^i\left(x^{k+1}, \ldots, x^n\right), \quad i=1, \ldots, k,$$

разрешающие нашу систему уравнений $F\left(x^1, \ldots, x^n\right)=c$, т.е.

$$F\left(x^1\left(x^{k+1}, \ldots, x^n\right), \ldots, x^k\left(x^{k+1}, \ldots, x^n\right), x^{k+1}, \ldots, x^n\right)=c$$

для всех $\left(x^{k+1}, \ldots, x^n\right)$ из W.

Обозначим через U окрестность точки P в M_c, проектирующуюся на W. В качестве карты в окрестности точки P на M_c возьмем пару (U, π), где π - стандартная проекция на координатную плоскость $\mathbb{R}^{n-k}$. Обратное отображение π^{-1} задается в виде

$$\begin{aligned} \pi^{-1}\left(x^{k+1}, \ldots, x^n\right)= \\ =\left(x^1\left(x^{k+1}, \ldots, x^n\right), \ldots, x^k\left(x^{k+1}, \ldots, x^n\right), x^{k+1}, \ldots, x^n\right). \end{aligned}$$

По построению, для каждой точки ($x^{k+1}, \ldots, x^n$) из $W \subset \mathbb{R}^{n-k}$ точка $\pi^{-1}\left(x^{k+1}, \ldots, x^n\right)$ принадлежит множеству уровня M_c. Отображения π и π^{-1} непрерывны и взаимнообратны. Итак, для каждой точки $P \in M_c$ мы построили карту. Тем самым доказано, что M_c - топологическое многообразие размерности $n-k$.

Осталось проверить гладкость многообразия M_c. Предположим, что точка P содержится как в построенной выше карте (U, π), так и в другой карте (U', π'). Напомним, что во всех картах в качестве координат выступают некоторые $n-k$ декартовых координат объемлющего пространства. В карте (U, π) координаты - это ($x^{k+1}, \ldots, x^n$), а в карте $(U', \pi')-\left(x^{i_1}, \ldots, x^{i_{n-k}}\right)$. Тогда функции перехода имеют вид

$$x^{i_p} = \begin{cases} x^{i_p}\left(x^{k+1}, \ldots, x^n\right), & \text{если } i_p \leq k \\ x^{i_p}, & \text{если } i_p > k. \end{cases}$$

Но, по построению, функции $x^{i_p}, p = 1, \ldots, k$, являются гладкими, что и доказывает гладкость многообразия M_c. $\square$

Упражнение 12.47. Доказательство теоремы 12.46, приведенное выше, содержит неточность. Найдите и устраните ее.

Следующие два утверждения являются важными частными случаями теоремы 22.46 .
Следствие 12.48. Пусть f - гладкая функция на $\mathbb{R}^n$. Рассмотрим множество уровня $M_c = \{(x^1, \ldots, x^n) \mid f(x^1, \ldots, x^n) = c\}$ функции f. Если дифференицал df отличен от нуля во всех точках множества M_c, то M_c является многообразием размерности $n-1$.

Следствие 12.49. Регулярные поверхности в евклидовом пространстве, изученные нами в классическом случае, являются гладкими многообразиями.

Пример 12.50. Сфера S^n, заданная в $\mathbb{R}^{n+1}$ уравнением $\sum_k \left(x^k\right)^2 = 1$, является гладким многообразием, так как дифференциал уравнения обращается в нуль только в начале координат, которое не принадлежит сфере.

Пример 12.51. Рассмотрим специальную линейную группу $SL(n, \mathbb{R})$ вещественных матриц размера $n \times n$. Напомним, что по определению $SL(n, \mathbb{R})$ - множество всех квадратных вещественных матриц с определителем 1. Множество всех вещественных матриц размера $n \times n$ естественным образом отождествляется с линейным пространством $\mathbb{R}^{n^2}$ (для этого достаточно записать квадратную матрицу X в виде одной длинной строки). Покажем, что $SL(n, \mathbb{R})$ является гладким многообразием размерности $n^2 - 1$. Действительно, $SL(n, \mathbb{R})$ может быть задано как множество уровня функции $\det X$, определенной на пространстве $\mathbb{R}^{n^2}$ всех квадратных матриц:

$$SL(n, \mathbb{R}) = \left\{X \in \mathbb{R}^{n^2} \mid \det X = 1\right\}.$$

Поэтому, в силу следствия 12.48, достаточно проверить, что $d(\det)$ не обращается в нуль в точках $X \in SL(n, \mathbb{R})$. Вычислим дифференциал функции $\det(X)$ в декартовых координатах $(x_{ij}), 1 \leq i, j \leq n$, пространства $\mathbb{R}^{n^2}$. Для этого заметим, что

$$\frac{\partial \det(X)}{\partial x_{ij}} = (-1)^{i+j} X_{ij},$$

где через X_{ij} обозначен минор матрицы X, соответствующий ее элементу x_{ij} (это следует из формулы разложения определителя по строке или столбцу). Осталось заметить, что одновременно все миноры X_{ij} не могут равняться нулю (иначе $\det X = 0$), поэтому $d(\det(X))$ не равен нулю, если $X \in SL(n, \mathbb{R})$. Таким образом, мы находимся в условиях следствия [12.48, из которого вытекает, что $SL(n, \mathbb{R})$ - гладкое многообразие размерности $n^2 - 1$.

14.14.7 12.7 Problems

Задача 12.52. Показать, что объединение двух координатных осей не является многообразием.

Задача 12.53. Задать на двумерном торе $T^2 = S^1 \times S^1$ структуру двумерного топологического многообразия.

Задача 12.54. Привести пример различных гладких структур на прямой $\mathbb{R}^1$.
Задача 12.55. Проверить эквивалентность двух описанных выше атласов на стандартной сфере.

Задача 12.56. Рассмотрим на прямой $\mathbb{R}$ атлас, состоящий из двух карт ($\mathbb{R}, \varphi : x \mapsto x$), $(\mathbb{R}, \psi : x \mapsto x^3)$ и задающий на $\mathbb{R}$ структуру топологического многообразия. Показать, что оно не является гладким. Модифицировать предыдущий пример так, чтобы получилось многообразие класса C^2, но не C^3.

Задача 12.57. Показать, что комплексное проективное пространство $\mathbb{CP}^n$ является комплексно-аналитическим многообразием комплексной размерности n.

Задача 12.58. Показать, что гладкие многообразия $\mathbb{R}^1(x)$ с картой ($\mathbb{R}^1, \varphi(x) = x$) и $\mathbb{R}^1(y)$ с картой $(\mathbb{R}^1, \varphi(y) = y^3)$ диффеоморфны.

Задача 12.59. Показать, что любое некомпактное связное одномерное многообразие диффеоморфно $\mathbb{R}^1$, а любое компактное связное одномерное многообразие диффеоморфно окружности S^1.

Задача 12.60. Рассмотрим в $\mathbb{R}^4$ поверхность, заданную системой уравнений

$$\begin{cases} (x^1)^2 + (x^2)^2 = 1 \\ (x^3)^2 + (x^4)^2 = 1 \end{cases}$$

Показать, что эта поверхность в $\mathbb{R}^4$ является гладким многообразием размерности 2 и это многообразие диффеоморфно двумерному тору $T^2 = S^1 \times S^1$.

Задача 12.61. Показать, что следующие матричные группы являются гладкими многообразиями, и определить их размерность:

$$\begin{aligned} GL(n, \mathbb{R}) &= \left\{ X \in \mathbb{R}^{n^2} \mid \det X \neq 0 \right\}, \\ O(n, \mathbb{R}) &= \left\{ X \in \mathbb{R}^{n^2} \mid XX^T = E \right\}, \\ SO(n, \mathbb{R}) &= \left\{ X \in \mathbb{R}^{n^2} \mid XX^T = E, \quad \det X = 1 \right\}, \\ U(n, \mathbb{R}) &= \left\{ X \in \mathbb{C}^{n^2} = \mathbb{R}^{2n^2} \mid X\bar{X}^T = E \right\}, \\ SU(n, \mathbb{R}) &= \left\{ X \in \mathbb{C}^{n^2} = \mathbb{R}^{2n^2} \mid X\bar{X}^T = E, \quad \det X = 1 \right\}. \end{aligned}$$

Исследовать эти матричные группы на компактность и линейную связность.

Задача 12.62. Показать, что многообразие $O(2)$ гомеоморфно объединению двух окружностей. Чему равны радиусы этих окружностей? Как расположены эти окружности в $\mathbb{R}^4$ (зацеплены ли они)?

14.14.8 12.8 Supplemental material

14.14.9 12.8.1 Pathological Examples

Приводимые здесь примеры иллюстрируют важность всех требований из определения многообразия.

Пример 12.67. Приведем пример топологического пространства, для которого выполняются все условия из определения многообразия, кроме хаусдорфовости. Для этого возьмем два экземпляра прямой $\mathbb{R}$ и отождествим все точки с одинаковыми координатами, за исключением точек 0 . Полученное топологическое пространство, наделенное фактор-топологией, называется прямой Александрова. Докажите, что это пространство не является хаусдорфовым (точки 0 не отделимы), хотя все остальные условия из определения многообразия выполняются.

Пример 12.68. Приведем пример топологического пространства, для которого выполняются все условия из определения многообразия, кроме существования счетной системы карт. Представим двумерную плоскость $\mathbb{R}^2$ как дизъюнктивное объединение

параллельных прямых (с топологией дизъюнктивного объединения). Каждая прямая гомеоморфна $\mathbb{R}^1$, однако в построенном пространстве X нельзя выбрать никакого счетного атласа.

14.14.10 12.8.2 Setting the structure of a smooth manifold on a set

По определению структура гладкого многообразия задается на некотором хаусдорфовом топологическом пространстве. Предположим теперь, что у нас есть произвольное множество X (без какой бы то ни было топологии) и фиксирована не более чем счетная система его подмножеств $\{U_\alpha\}$, такая, что:

- подмножества U_α покрывают X;
- для каждого U_α фиксировано взаимно-однозначное отображение $\varphi_\alpha : U_\alpha \to \mathbb{R}^n$;
- если функции перехода, т.е. композиции вида $\varphi_\alpha \circ \varphi_\beta^{-1}$, определена, то она является гладкой.

Тогда на множестве X можно ввести структуру хаусдорфова топологического пространства, и относительно этой структуры X будет гладким многообразием.

Действительно, достаточно объявить открытыми в X те и только те подмножества, образы пересечений которых с любым из U_α открыты в $\mathbb{R}^n$.

Упражнение 12.69. Проверить, что в результате действительно получится некоторая топология на X. Показать, что X - хаусдорфово. Показать, что семейство $\{(U_\alpha, \varphi_a)\}$ задает на X структуру гладкого многообразия.

Замечание 12.70. Отметим, что если проделать описанную только что процедуру для гладкого многообразия M, то в результате получим на M топологию, эквивалентную исходной. В этом смысле гладкая структура многообразия определяет его топологию (как, скажем, метрика определяет топологию метрического пространства).

14.14.11 12.8.3 Homeomorphism $\mathbb{RP}(3)$ и $\mathrm{SO}(3)$

Многие из рассмотренных нами многообразий "устроены одинаково" с точки зрения топологии, т.е. гомеоморфны друг другу. Приведем пример.

Theorem

12.71. Трехмерное проективное пространство $\mathbb{RP}(3)$ гомеоморфно группе $\mathrm{SO}(3)$ матрии, специальных (т.е. с определителем 1) ортогональных преобразований трехмерного евклидова пространства $\mathbb{R}^3$.

Доказательство. Напомним, что каждое преобразование $A \in \mathrm{SO}(3), A \neq E$ представляет собой вращение вокруг некоторой оси ℓ на некоторый угол $\varphi, 0 < \varphi \leq \pi$. Рассмотрим замкнутый шар D^3 радиуса π с центром в начале координат O и определим отображение $\nu : D^3 \to \mathrm{SO}(3)$. Для этого поставим в соответствие точке $O \in D^3$ тождественное преобразование E; отличной от O точке $P \in D^3$, не лежащей на границе шара D^3, вращение A вокруг прямой OP на угол $\varphi = |P|$, причем из двух возможных преобразований этого типа выберем то, которое удовлетворяет следующему условию: если - плоскость, ортогональная к ℓ, и $v \in \Pi$ произвольный ненулевой вектор, то потребуем, чтобы репер (v, Av, OP) был положительно ориентирован. Наконец, каждой граничной точке P шара D^3 поставим в соответствие вращение вокруг прямой OP на угол π (такое вращение определено однозначно). Образ отображения ν совпадает с $\mathrm{SO}(3)$; отображение ν взаимнооднозначно на внутренности шара D^3; каждой паре

диаметрально противоположных точек из границы шара D^3 ставится в соответствие одно и тоже вращение, причем разным парам таких точек сопоставляются разные вращения, отличные от тех, которые соответствуют точкам внутренности шара D^3. Так как ось вращения и

угол поворота непрерывно зависят от коэффициентов матрицы поворота, то SO(3) гомеоморфно диску D^3, у которого отождествлены диаметрально противоположные точки границы.

В то же время каждая точка проективного пространства $\mathbb{R}P^3$ - это некоторая проходящая через начало координат прямая ℓ в $\mathbb{R}^4$, где фиксированы координаты ($x^1, \ldots, x^4$). Рассмотрим стандартную сферу S^3, и пусть $S^3_+ = S^3 \cap \{x^4 \geq 0\}$ - замкнутая полусфера. Ясно, что S^3_+ гомеоморфна замкнутому трехмерному диску (гомеоморфизм задается проекцией на трехмерное пространство $\{x^4 = 0\}$). Если ℓ не лежит в $\{x^4 = 0\}$, то она пересекает полусферу S^3_+ в единственной точке. Если же ℓ лежит в $\{x^4 = 0\}$, то она пересекает полусферу в двух диаметрально противоположных точках. Таким образом, $\mathbb{R}P^3$ гомеоморфно диску S^3_+, у которого отождествляются противоположные точки границы. Доказательство закончено.

14.15 13 Tangent space to diversity, differential

Напомним, что касательными векторами к поверхности M в данной точке $P \in M$ называются вычисленные в точке P векторы скоростей гладких кривых, лежащих на поверхности M и проходящих через точку P. В этом определении использовано то обстоятельство, что поверхность рассматривалась как подмножество векторного пространства.

В случае абстрактных многообразий дело обстоит несколько сложнее (абстрактное гладкое многообразие нигде, вообще говоря, не лежит). Тем не менее, имеется естественное определение касательного вектора в точке к гладкому многообразию M, использующее возможность ввести в окрестности каждой точки многообразия локальные координаты. Мы приведем три эквивалентных определения касательного вектора (в разных задачах оказывается удобным использовать разные определения).

14.15.1 13.1 Tangent vector definitions

Итак, пусть M - гладкое многообразие размерности n и $P \in M$ - некоторая точка. Мы начнем с так называемого "алгебраического" или тензорного определения.

Определение 13.1. Касательный вектор V в точке P - это соответствие, сопоставляющее каждой системе координат ($x^1, \ldots, x^n$) в P набор из n чисел $(v^1, \ldots, v^n)$, которые называются компонентами касательного вектора V в координатах $(x^1, \ldots, x^n)$, причем, если ($x^1, \ldots, x^n$) и ($y^1, \ldots, y^n$) - две системы координат в окрестности точки P и

$$V : \left(x^1, \ldots, x^n\right) \mapsto \left(v^1, \ldots, v^n\right), \quad \text{и} \quad V : \left(y^1, \ldots, y^n\right) \mapsto \left(w^1, \ldots, w^n\right),$$

то $w^i = \sum_j \frac{\partial y^i}{\partial x^j} v^j$, где $J = \left(\frac{\partial y^i}{\partial x^j}\right)$ - матрица Якоби функции перехода из координат x в координаты y, вычисленная в точке P (сравните с законом изменения касательного к поверхности вектора при замене криволинейных координат).

координат. Определив сложение и умножение касательных векторов на числа как соответствующие покомпонентные операции на этих наборах, мы введем на множестве $T_P M$ всех касательных векторов структуру n-мерного векторного пространства. Осталось заметить, что эти операции не зависят от выбора системы координат $(x^1, \ldots, x^n)$. Утверждение доказано.

Конструкция 13.5. Пусть $(x^1, \ldots, x^n)$ - фиксированная система координат в окрестности точки P многообразия M. Из доказательства утверждения ТЗ.4 вытекает, что следующий набор касательных векторов, заданных в соответствии с леммой ТЗ.3, образует базис в линейном пространстве $T_P M$:

$$E_i : \left(x^1, \ldots, x^n\right) \mapsto (0, \ldots, 0, 1, 0, \ldots, 0), \quad i = 1, \ldots, n,$$

где единица расположена на i-ом месте. Если $V \in T_P M$ - произвольный касательный вектор и $V : (x^1, \ldots, x^n) \mapsto (v^1, \ldots, v^n)$, то $V = \sum_i v_i E_i$.

Определение 13.6. Базис $\{E_1, \ldots, E_n\}$ называется каноническим базисом в $T_P M$, соответствующим координатам $x^1, \ldots, x^n$.

Замечание 13.7. Чтобы подчеркнуть, каким именно координатам соответствует канонический базис, для векторов E_i часто используют обозначения ∂_{x^i}. Смысл этих обозначений будет объяснен ниже, см. замечание 13.25.

Конструкция 13.8. Пусть $\gamma : I \to M$ - гладкая кривая на многообразии M, проходящая через точку $P = \gamma(t_0)$. Сопоставим локальным координатам ($x^1, \ldots, x^n$) набор компонент вектора скорости кривой γ в точке P в этих координатах. Более точно, если $x^i(t), i = 1, \ldots, n$, - координатные функции кривой γ, то

$$\dot{\gamma} : \left(x^1, \ldots, x^n\right) \mapsto \left(\dot{x}^1, \ldots, \dot{x}^n\right).$$

Лемма 13.9. Построенное выше соответствие $\dot{\gamma}$ является касательным вектором κ многообразию M в точке P.

Доказательство. Действительно, пусть как и выше $x^i(t)$ - координатное представление кривой γ в координатах ($x^1, \ldots, x^n$). Рассмотрим в окрестности точки P еще одну систему координат $(y^1, \ldots, y^n)$, и пусть $y^i(t)$ - координатное представление кривой γ в них. Тогда, по теореме о производной сложной функции,

$$\dot{y}^i(t_0) = \sum_{k=1}^{n} \frac{\partial y^i}{\partial x^k}(P) \dot{x}^k(t_0),$$

т.е. наборы чисел

$$\left(\dot{x}^1(t_0), \ldots, \dot{x}^n(t_0)\right), \quad \left(\dot{y}^1(t_0), \ldots, \dot{y}^n(t_0)\right)$$

действительно связаны между собой как компоненты касательного вектора.
Определение 13.10. Касательный вектор $\dot{\gamma}$ называется вектором скорости кривой γ в точке P.

Дадим теперь "геометрическое" определение касательного вектора.

Конструкция 13.11. Пусть γ_1 и γ_2 - две гладкие кривые на гладком многообразии M, такие, что $\gamma_1(0) = \gamma_2(0) = P$. Фиксируем произвольную систему координат в окрестности P. Заметим, что если векторы скоростей кривых γ_i в точке P в этой системе координат, т.е. векторы $\dot{\gamma}_1(0)$ и $\dot{\gamma}_2(0)$, равны, то они равны и в любой другой системе координат (как мы уже видели в доказательстве леммы 13.9, при замене координат компоненты вектора скорости пересчитываются с помощью матрицы Якоби). Таким образом, корректно определено следующее отношение эквивалентности: проходящие через точку P кривые γ_1 и γ_2 эквивалентны в P, если их векторы скоростей в P равны в некоторой (а, значит, и в любой) системе координат.

Определение 13.12. Класс эквивалентности кривых, проходящих через точку P, называется касательным вектором в точке P.

Утверждение 13.13. "Алгебраическое" и "геометрическое" определения касательного вектора эквивалентны. Более формально, по каждому касательному вектору в смысле

одного определения, можно взаимно однозначно построить касательный вектор в смысле второго определения.

Доказательство. В соответствие с конструкцией [3.8, каждой гладкой кривой, проходящей через точку P, соответствует касательный вектор в смысле "алгебраического" определения. При этом, из конструкции [3.1] вытекает, что этот вектор одинаков для эквивалентных кривых, т.е. он строится по классу эквивалентности или, другими словами, по касательному вектору в смысле "геометрического" определения. Более того, для разных классов эквивалентности получаются, очевидно, разные касательные векторы.

Остается заметить, что так может быть получен произвольный касательный вектор $V : (x^1, \ldots, x^n) \mapsto (v^1, \ldots, v^n)$. Действительно, пусть $P = (P^1, \ldots, P^n)$ в координатах $(x^1, \ldots, x^n)$, и γ - гладкая кривая, заданная в этих координатах так: $(P^1 + v^1 t, \ldots, P^n + v^n t$). Тогда V совпадает с вектором скорости кривой γ в точке P в смысле определения 13.10. Утверждение доказано.

Замечание 13.14. Соответствие, построенное в доказательстве утверждения [13.13, переносит структуру n-мерного векторного пространства $T_P M$ на множество классов эквивалентности кривых, проходящих через точку P.

Замечание 13.15. Пусть $(x^1, \ldots, x^n)$ - некоторые координаты в окрестности точки $P = (x_0^1, \ldots, x_0^n)$. Тогда проходящая через P кривая γ_i, координатные функции которой имеют вид

$$x^j(t) = x_0^j, \quad j \neq i, \quad x^i(t) = x_0^i + t$$

называется i-й координатной линией. Вектор скорости координатной линии γ_i в точке P совпадает с вектором E_i канонического базиса касательного пространства $T_P M$, соответствующего ($x^1, \ldots, x^n$). Поэтому, с точки зрения "геометрического" определения, векторы канонического базиса - это классы $[\gamma_i]$ эквивалентности координатных линий, проходящих через точку P

Перейдем, наконец, к третьему, так называемому "дифференциальному", определению касательного вектора.

Определение 13.16. Операция ψ, сопоставляющая каждой гладкой функции, определенной в окрестности точки P, число $\psi(f)$, такая, что

$$\psi(\alpha f + \beta g) = \alpha \psi(f) + \beta \psi(g), \quad \psi(fg) = \psi(f) g(P) + f(P) \psi(g),$$

называется дифференирированием в точке P. Последнее соотношение называется *пра* вилом Лейбница.

Пример 13.17. Пусть f - некоторая гладкая функция, определенная в окрестности точки P, и γ - произвольная гладкая кривая, проходящая через точку $P, \gamma(t_0) = P$. Тогда естественным образом определяется производная функции f вдоль кривой γ (в точке P):

$$\dot{\gamma}(f) = \left. \frac{df(\gamma(t))}{dt} \right|_{t=t_0}$$

Эта операция, как легко проверить, линейна и удовлетворяет правилу Лейбница и поэтому является дифференцированием в точке P.

Пример 13.18. Пусть в окрестности точки P фиксированы координаты ($x^1, \ldots, x^n$), и γ_i - это i-ая координатная линия. Тогда

$$\dot{\gamma}_i(f) = \left. \frac{\partial f}{\partial x^i} \right|_{t=t_0}$$

для произвольной гладкой функции f. Таким образом, операция взятия частной производной в точке - это дифференцирование в точке в смысле определения [3.16.

Следующее утверждение непосредственно следует из определений.

Утверждение 13.19. Если φ и $\psi-$ два дифференировования в точке P, то их линейная комбинация с вещественными коэффициентами α и β, определенная формулой

$$(\alpha\varphi + \beta\psi)(f) = \alpha\varphi(f) + \beta\psi(f)$$

где f - произвольная гладкая функция, тоже является дифференицрованием в точке P. Множество всех дифферениций в точке P образует векторное пространство.

Определение 13.20. Касательным вектором в точке P называется произвольное дифференцирование в этой точке, имеющее в некоторых координатах x^i вид

$$f \mapsto \sum_i v^i \frac{\partial f}{\partial x_i}(P)$$

где v^i - некоторый набор чисел.

Замечание 13.21. Операция из определения 13.20 действительно является дифференцированием в силу утверждения 13.19 и примера 13.18, так как представляет собой линейную комбинацию дифференцирований (частных производных) с вещественными коэффициентами.

Замечание 13.22. На самом деле можно показать, что любое дифференцирование представляет собой линейную комбинацию частных производных с вещественными коэффициентами (см. дополнительный материал).

Утверждение 13.23. Пусть в координатах ($x^1, \ldots, x^n$) в окрестности точки $P \in$ M операция дифференцирования ψ имеет вид

$$\psi : f \mapsto \sum_i v^i \frac{\partial f}{\partial x_i}(P),$$

где v^i - некоторый набор чисел. Тогда, если $(y^1, \ldots, y^n)$ - другие координаты в окрестности точки P, то

$$\psi : f \mapsto \sum_j w^j \frac{\partial f}{\partial y_j}(P), \quad \text{где} \quad w^j = \sum_i v^i \frac{\partial y^j}{\partial x^i}(P)$$

Доказательство. Действительно, по теореме о производной сложной функции, имеем:

$$\sum_i v^i \frac{\partial f}{\partial x^i} = \sum_i v^i \left[\sum_j \frac{\partial y^j}{\partial x^i}\frac{\partial f}{\partial y^j}\right] = \sum_j \left[\sum_i v^i \frac{\partial y^j}{\partial x^i}\right]\frac{\partial f}{\partial y^j} = \sum_j w^j \frac{\partial f}{\partial y^j},$$

что и требовалось.

Theorem

13.24. Три приведенные выше определения касательного вектора эквивалентны.

Доказательство. Эквивалентность первых двух определений уже доказана в утверждении [3.13. Эквивалентность первого и третьего определений немедленно вытекает из утверждения [3.23. Действительно, $(x^1, \ldots, x^n)$ - координаты в окрестности точки P, и V - касательный вектор из $T_P M$ в смысле первого определения,

$$V : \left(x^1, \ldots, x^n\right) \mapsto \left(v^1, \ldots, v^n\right),$$

то ему соответствует дифференцирование

$$\partial_V : f \mapsto \sum_i v^i \frac{\partial f}{\partial x_i}(P)$$

и наоборот. Построенное соответствие определено корректно, так как в силу утверждения [3.23 компоненты вектора V меняются при замене координат по тому же правилу, что и коэффициенты, стоящие при частных производных в формуле для дифференцирования ∂_V. Теорема доказана.

Замечание 13.25. Пусть ($x^1, \ldots, x^n$) - координаты в окрестности точки P. Тогда из замечания 13.15 и примера 13.18 вытекает, что с точки зрения "дифференциального" определения векторы канонического базиса пространства $T_P M$, соответствующего координатам ($x^1, \ldots, x^n$), - это дифференцирования, представляющие собой частные производные вдоль соответствующих координат, $E_i = \frac{\partial}{\partial x^i}$. Поэтому векторы канонического базиса часто обозначаются через $\frac{\partial}{\partial x^i}$ или, кратко, через ∂_{x^i}.

14.15.2 13.2 Tangential delamination

В этом разделе мы построим еще один важный пример многообразия - касательное расслоение.

Конструкция 13.26. Пусть M - произвольное гладкое многообразие и $\{(U_\alpha, \varphi_\alpha)\}$ его атлас. Рассмотрим множество $T(M)$ всех касательных пространств к многообразию M

$$T(M) = \cup_{P \in M} T_P M$$

и превратим данное множество в гладкое многообразие. Отметим, что элементы множества $T(M)$ - это все касательные векторы во всех точках многообразия M.

Определим отображение $\pi : T(M) \to M$, сопоставив каждому вектору V из $T(M)$ ту единственную точку $P \in M$, для которой выполнено $V \in T_P M$. Пусть теперь $(x^1, \ldots, x^n)$ - координаты, определенные на карте U многообразия M. Рассмотрим множество $TU = \pi^{-1}(U)$ и определим отображение $\Phi_U : TU \to \mathbb{R}^{2n}$ так:

$$\Phi_U : V \mapsto \left(x_0^1, \ldots, x_0^n, v^1, \ldots, v^n\right) \in \mathbb{R}^{2n}$$

где V - элемент из $T_P M, (x_0^1, \ldots, x_0^n)$ и $(v^1, \ldots, v^n)$ - координаты точки P и компоненты касательного вектора V в координатах $(x^1, \ldots, x^n)$ соответственно. Поскольку касательный вектор однозначно определяется своими компонентами, отображение Φ_U взаимно-однозначно с образом.

Введем на $T(M)$ топологию так, чтобы для каждой карты U на M соответствующее отображение Φ_U было бы гомеоморфизмом. Подмножество $X \subset T(M)$ назовем открытым в $T(M)$, если и только если Φ_U-образ пересечения X с произвольным множеством вида TU открыт в $\mathbb{R}^{2n}$ для каждой карты U на M. Тогда прообраз любого открытого подмножества из $\mathbb{R}^{2n}$ при отображении Φ_U открыт в $T(M)$, поэтому Φ_U непрерывно, и Φ_U-образ каждого открытого подмножества из $T(M)$ открыт в $\mathbb{R}^{2n}$, поэтому Φ_U^{-1} тоже непрерывно.

Итак, мы ввели на $T(M)$ топологию и построили атлас $\{(TU_\alpha, \Phi_\alpha)\}$, где $\Phi_\alpha = \Phi_{U_\alpha}$, т.е. $T(M)$ - топологическое многообразие размерности $2n$.

Утверждение 13.27. Построенный атлас $\{(TU_\alpha, \Phi_\alpha)\}$ - гладкий, тем самьм, $T(M)$ - гладкое $2n$-мерное многообразие.

Доказательство. Возьмем на $T(M)$ любые две карты $\mathcal{U}_\alpha = (TU_\alpha, \Phi_\alpha)$ и $\mathcal{U}_\beta = (TU_\beta, \Phi_\beta)$ такие, что $U_\alpha \cap U_\beta$ не пусто (только такие карты из построенного атласа пересекаются). Пусть $(x_\alpha^1, \ldots, x_\alpha^n, v_\alpha^1, \ldots, v_\alpha^n)$ и $\left(x_\beta^1, \ldots, x_\beta^n, v_\beta^1, \ldots, v_\beta^n\right)$ - соответствующие координаты. Тогда функция перехода между этими двумя картами имеет вид

$$x_\beta^k = x_\beta^k\left(x_\alpha^1, \ldots, x_\alpha^n\right), \quad k = 1, \ldots, n$$
$$v_\beta^k = \sum_{p=1}^{n} \frac{\partial x_\beta^k}{\partial x_\alpha^p}\left(x_\alpha^1, \ldots, x_\alpha^n\right) v_\alpha^p, \quad k = 1, \ldots, n.$$

Эти функции гладкие, так как первые - это функции перехода между картами атласа многообразия M, а вторые - многочлены от компонент матрицы Якоби замены координат на многообразии M (которые, конечно, гладко зависят от точки) и координат v_α^p на $T(M)$. Утверждение доказано.

Определение 13.28. Гладкое многообразие $T(M)$, построенное выше, называется *ка* сательным расслоением многообразия M.

Замечание 13.29. Термин "расслоение" не является случайным. Многообразие $T(M)$ представляет собой объединение одинаковых подмножеств - слоев. В данном случае это прообразы точек $x \in M$ при проекции π, построенной выше. Каждый такой прообраз $\pi^{-1}(x)$ совпадает с касательным пространством T_xM, т.е. изоморфен линейному пространству $\mathbb{R}^n$.

Замечание 13.30. Касательные расслоения возникают при описании механических систем. Напомним, что множество всевозможных положений элементов механической системы в механике называют конфигурационным пространством. Выше отмечалось, что в реальных задачах конфигурационное пространство обычно представляет собой гладкое многообразие. В механике часто бывает необходимо учитывать не только расположение элементов системы, но и их скорости, т.е. рассматривать большее пространство - фазовое пространство, элементами которого являются пары (положение, скорость). Фазовое пространство механической системы естественно отождествляется с касательным расслоением к ее конфигурационному пространству.

Замечание 13.31. Аналогично определяется кокасательное расслоение T^*M к многообразию M. Слоями этого расслоения являются кокасательные пространства T_P^*M к точкам многообразия, т.е. пространства линейных функций на соответствующих касательных пространствах T_PM. Кокасательные расслоения возникают в гамильтоновой модели классической механики (при инвариантной интерпретации импульсы - суть линейные функции на векторах).

14.15.3 13.3 Definition of differential

Перейдем к построению основ дифференциального исчисления гладких отображений многообразий. Это даст возможность изучать локальные свойства таких отображений, пользуясь техническими средствами математического анализа. Фактически, мы просто "пересадим" соответствующие понятия с областей в $\mathbb{R}^n$ на гладкие многообразия, воспользовавшись имеющейся на них гладкой структурой.

Конструкция 13.32. Пусть $F : M \to N$ - гладкое отображение m-мерного гладкого многообразия M в n-мерное гладкое многообразие N и $v \in T_PM$ - произвольный касательный вектор. Пусть γ - кривая, такая, что $\gamma(0) = P$ и $\dot\gamma(0) = v$. Рассмотрим образ кривой γ при отображении F. Если кривая γ задается гладким отображением $\gamma : (-1, 1) \to M$, то ее образ - это гладкая кривая в N, заданная отображением $F \circ \gamma$. Кривая F о γ проходит через образ $F(P)$ точки P. Обозначим через w вектор скорости кривой $F \circ \gamma$ в точке $F(P)$. По определению, $w \in T_{F(P)}N$.

Определение 13.33. Дифферениал dF_P отображения F в точке P - это отображение $dF_P : T_PM \to T_{F(P)}N$ касательных пространств, заданное так: $dF_P(v) = w$.

Утверждение 13.34. Дифферениал отображения $F-$ это линейное отображение касательных пространств. Если $F(P) = Q$, и в окрестностях точек P и Q фиксированы

системы координат, то, в соответствующих канонических базисах, матрица дифференциала dF_P - это матрица Якоби отображения F в этих координатах.

Доказательство. Пусть $(x^1, \ldots, x^m)$ - координаты на M в окрестности точки P и $(y^1, \ldots, y^n)$ - координаты на N в окрестности точки Q, а $y^i = y^i(x^1, \ldots, x^m)$ - координатное представление отображения F. Пусть $x^i = x^i(t)$ - координатное представление кривой γ. Тогда координатное представление кривой $F \circ \gamma$ имеет вид $y^i = y^i(x^1(t), \ldots, x^n(t))$. В результате получаем

$$dF_P(v) = \frac{d(F \circ \gamma)}{dt}(0) = \left(\sum_j \frac{\partial y^1}{\partial x^j}(P) v^j, \ldots, \sum_j \frac{\partial y^n}{\partial x^j}(P) v^j \right) = Jv,$$

где J - матрица Якоби отображения F в точке P в локальных координатах ($x^1, \ldots, x^m$) и $(y^1, \ldots, y^n)$. Утверждение доказано. $\square$

Упражнение 13.35. Пусть $F : M \to N$ - гладкое отображение гладких многообразий. Проверить, что для произвольной гладкой функции h на N и произвольного вектора $v \in T_P M$ имеет место равенство

$$dF_P(v)(h) = v(h \circ F).$$

Пример 13.36. В частном случае гладкой функции $f : M \to \mathbb{R}$, заданной на гладком многообразии M, в каждой точке $P \in M$ имеем линейный функционал $df_P : T_P M \equiv \mathbb{R}^m \to T_{f(P)}\mathbb{R}^1 \equiv \mathbb{R}^1$. Матрица этого функционала в локальных координатах ($x^1, \ldots, x^m$) на M - это строка, состоящая из частных производных $\partial f / \partial x^i$. Этот функционал называют дифферениц алом функц u f в точке P и обозначают $df|_P$. Подчеркнем, что строка, состоящая из частных производных функции, задает линейный функционал, а не вектор, как иногда учат в математическом анализе.

Упражнение 13.37. Пусть $f : M_1 \to M_2$ и $g : M_2 \to M_3$ - гладкие отображения гладких многообразий, $g \circ f$ - их композиция. Показать, что $d(g \circ f) = dg \circ df$.

14.15.4 13.4 Local Properties of Mappings

Рассмотрим локальные свойства гладких отображений, дифференциалы которых в некотором смысле невырождены, или так называемые свойства "общего положения". Начнем с аналога теоремы о неявной функции.

Определение 13.38. Пусть $F : M \to N$ - гладкое отображение гладких многообразий. Точка $P \in M$ называется регулярной точкой отображения F, если дифференциал dF_P отображения F в точке P является эпиморфизмом, т.е. образ дифференциала dF_P совпадает со всем пространством $T_{F(P)}N$. Точка $Q \in N$ называется регулярной точкой отображения F, или регулярным значением отображения F, если всякая точка из прообраза $F^{-1}(Q)$ этой точки является регулярной для отображения F.

Замечание 13.39. Если прообраз точки Q при отображении F пуст, то, по определению, точка Q является регулярным значением отображения F.

Замечание 13.40. Отметим, что регулярность отображения в точке $P \in M$ влечет следующее соотношение на размерности многообразий:

$$m = \dim M \geq \dim N = n = \operatorname{rank} dF_P.$$

Определение 13.41. Точка P многообразия M называется критической точкой отображения M, если дифференциал dF_P не является эпиморфизмом, т.е. если P не является регулярной точкой отображения F. Точка Q называется критическим значением отображения F, если существует критическая точка $P \in M$ отображения F, такая, что $F(P) = Q$.

Замечание 13.42. Если $\dim M < \dim N$, то любая точка многообразия M является критической точкой отображения F. Поэтому множество критических значений отображения F в этом случае совпадает с $F(M)$.

Определение 13.43. Если все точки многообразия M являются регулярными для отображения F, то отображение F называется субмерсией.

Theorem

13.44. Пусть $F : M \to N-$ гладкое отображение гладких многообразий, $Q \in N$ - регулярное значение отображения F, и прообраз точки Q не пуст. Тогда прообраз $W = F^{-1}(Q)$ точки Q является гладким многообразием размерности $\dim M - \dim N$. При этом в качестве координат в окрестности каждой точки из W можно взять некоторые координаты объемлющего многообразия.

Доказательство. Переформулируем условие регулярности точки Q в координатах. Обозначим через P какую-нибудь точку из $F^{-1}(Q)$. Пусть $(x^1, \ldots, x^m)$ и $(y^1, \ldots, y^n)$ - координаты в окрестности точек P и Q соответственно и пусть $y^i = f^i(x^1, \ldots, x^m)$, $i = 1, \ldots, n$, - координатное представление отображения F. Рассмотрим множество $W = F^{-1}(Q)$ и с помощью координатного гомеоморфизма отождествим некоторую окрестность точки P в W с подмножеством $\tilde{W}$ пространства $\mathbb{R}^m$. Тогда $\tilde{W}$ задается в $\mathbb{R}^m$ набором уравнений

$$f^i\left(x^1, \ldots, x^m\right) = y^i(Q), \quad i = 1, \ldots, n.$$

Дифференциал dF_P отображения F в точке P матрицей Якоби отображения F в точке P, поэтому условие эпиморфности дифференциала - это условие равенства ранга этой матрицы Якоби размерности многообразия N, т.е. $\operatorname{rank} dF_P = n$. Но тогда применима теорема 12.46, и множество $\tilde{W}$ является гладким многообразием размерности $m - n$, причем в качестве координат в нем могут быть выбраны некоторые координаты из ($x^1, \ldots, x^m$). Поэтому и W является гладким многообразием, так как для получения атласа на W достаточно рассмотреть композицию координатных гомеоморфизмов, построенных нами для $\tilde{W}$, и гомеоморфизма, отождествляющего W и $\tilde{W}$. Доказательство закончено.

Таким образом, эпиморфность дифференциала отвечает за то, что прообраз соответствующей точки "устроен хорошо". Оказывается, мономорфность дифференциала отвечает за "хорошие" свойства образа.

Определение 13.45. Пусть $F : M \to N$ - гладкое отображение гладких многообразий. Отображение F назовем погружением, или иммерсией, если в каждой точке P из M дифференциал dF_P отображения F в точке P является мономорфизмом, т.е. отображение dF_P взаимно-однозначно с образом.

Замечание 13.46. Если дифференциал отображения $F : M \to N$ - мономорфизм, то выполнено следующее соотношение на размерности многообразий:

$$\operatorname{rank} dF_P = m = \dim M \leq \dim N = n.$$

Определение 13.47. Погружение F называется вложением, если оно задает гомеоморфизм многообразия M с образом $F(M)$ (на $F(M)$ рассматривается топология, индуцированная из N).

Замечание 13.48. Если $F : M \to N$ - произвольное погружение, то, по теореме о неявной функции, у каждой точки $P \in M$ существует такая окрестность U, что ограничение отображения F на U - гомеоморфизм с образом. Поэтому каждое погружение локально является вложением.

Пример 13.49. Рассмотрим семейство F_a отображений прямой $\mathbb{R}^1(x)$ в плоскость $\mathbb{R}^2(y^1, y^2)$, заданное в виде $F_a : x \mapsto (x^2, x^3 + ax)$. Матрица Якоби этого отображения

имеет вид ($2x, 3x^2 + a$). Поэтому, если $a \neq 0$, отображение F_a является погружением. Если же $a = 0$, то в точке $x = 0$ ранг матрицы Якоби равен нулю и наше отображение - не погружение. Далее, как легко проверить, отображение F_a является вложением, если и только если $a > 0$. Если же $a < 0$, то кривая $F_a(x)$ имеет самопересечение. Вид кривой $F_a(x)$ при разных значениях параметра a приведен на рис. ГЗ. 1.

Рис. 13.1: Семейство кривых $F_{a (x)$.}

Образ многообразия при погружении может и не быть многообразием: например, кривая $F_a(x)$ при $a < 0$ не является многообразием (в топологии, индуцированной из $\mathbb{R}^2$).

Определение 13.50. Подмножество M гладкого многообразия N размерности n называется гладким подмногообразием размерности m, если для каждой точки $P \in M$ существует такая карта (U, φ) многообразия $N, P \in U$, что множество $\varphi(M \cap U)$ является m-мерной регулярной поверхностью в $\mathbb{R}^n$.

Легко проверить, что справедливо следующее утверждение.

Statement

13.51. Произвольное m-мерное гладкое подмногообразие $M \subset N$ является m-мерным гладким многообразием. Отображение включения $M \hookrightarrow N$ является вложением. Обратно, образ $F(M)$ произвольного вложения $F : M \to N$ является подмногообразием.

Замечание 13.52. Образ погружения называется иногда погруженным подмногообразием, хотя подмногообразием он, вообще говоря, не является.

Пример 13.53. Пусть $f : \mathbb{R}^1 \to \mathbb{R}^2$ - взаимно-однозначное погружение, заданное так, как показано на рис. [3.2. Каждая окрестность точки $f(a)$ в индуцированной из $\mathbb{R}^2$ топологии содержит, кроме интервала вида ($f(b), f(c)$), еще и некоторый интервал, прообраз которого является лучом. Поэтому отображение f^{-1} не является непрерывным, так как, например, прообраз интервала (b, c), равный интервалу ($f(b), f(c)$), не есть открытое множество. Значит, отображение f не является вложением. Отметим, что подмножество $f(\mathbb{R}^1)$ плоскости $\mathbb{R}^2$ не является подмногообразием.

Рис. 13.2: Взаимно-однозначное погружение прямой в плоскость.

Упражнение 13.54. Показать, что в предположениях теоремы ПЗ.44 многообразие $W = F^{-1}(Q)$ является подмногообразием в M.

Упражнение 13.55. Показать, что взаимно-однозначное с образом погружение компактного многообразия в произвольное многообразие является вложением.

В дальнейшем нам понадобятся следующие полезные примеры подмногообразий.

Пример 13.56. Пусть M - произвольное гладкое многообразие и $T(M)$ - касательное расслоение к M. Тогда определено подмножество M_0 касательного расслоения, состоящее из всех нулевых касательных векторов к многообразию M. Легко проверить, что M_0 - гладкое подмногообразие в $T(M)$, причем M_0 диффеоморфно M. Для этого достаточно рассмотреть отображение $i : M \to T_P M$, ставящее в соответствие точке $P \in M$ нулевой касательный вектор из $T_P M$, и убедиться, что это отображение - вложение. Многообразие M_0 обычно называют нулевым сечением касательного расслоения. Поскольку M_0 - замкнутое подмножество в $T(M)$, то непустое дополнение $T(M)\backslash M_0$ является открытым $2n$-мерным гладким многообразием.

Пример 13.57. Пусть M - гладкое многообразие и $M \times M$ - прямое произведение M на себя. Рассмотрим в этом $2n$-мерном многообразии подмножество $\Delta = \{(x, x) \mid x \in M\}$, называемое диагональю. Можно проверить, что Δ - гладкое подмногообразие в $M \times M$ размерности n, причем Δ диффеоморфно M. Для этого достаточно рассмотреть отображение $i : M \to M \times M$, переводящее точку $P \in M$ в точку ($P \times P$), и убедиться, что i - вложение. Дополнение $M \times M\backslash\Delta$ является открытым $2n$-мерным гладким многообразием.

Большинство известных нам примеров многообразий - это подмногообразия некоторого евклидового пространства $\mathbb{R}^N$ достаточно большой размерности. Оказывается, это неслучайно. В следующей лекции будет показано, что произвольное многообразие может быть вложено в евклидово пространство, причем размерность этого пространства можно оценить.

14.15.5 13.5 Problems

Задача 13.58. Вычислить образ горизонтального единичного вектора, касательного к двумерной сфере, при действии дифференциала стереографической проекции из северного полюса.

Задача 13.59. Доказать, что дифференциалы dx^i координатных функций x^i, вычисленные в точке P гладкого многообразия M, образуют базис векторного пространства $T_P^* M$, двойственного $T_P M$. Пространство $T_P^* M$ называется кокасательным. Проверить, что базисы $dx^1, \ldots, dx^n$ и $\partial_{x^1}, \ldots, \partial_{x^n}$ двойственны друг другу, т.е. $dx^i\left(\partial_{x^i}\right) = \delta_j^i$.

Задача 13.60. Перенести на случай гладкой функции на гладком многообразии понятие локального экстремума. Доказать, что в точках локального экстремума дифференциал гладкой функции равен нулю.

Задача 13.61. Построить пример функции на плоскости, у которой дифференциал нигде не равен нулю. Показать, что дифференциал произвольной гладкой функции на сфере S^2 обращается в нуль по крайней мере 2 раза. Что будет в случае тора $T^2 = S^1 \times S^1$, а также в случае проективной плоскости $\mathbb{RP}^2$?

Задача 13.62. Построить вложение проективной плоскости $\mathbb{RP}^2$ в $\mathbb{R}^5$ и погружение в $\mathbb{R}^3$ (написать формулы).

Задача 13.63. Построить вложение тора T^n, т.е. прямого произведения n экземпляров окружности, в пространство $\mathbb{R}^{n+1}$.

Задача 13.64. Построить вложение $S^2 \times S^2$ в $\mathbb{R}^5$.

Задача 13.65. Какому многообразию гомеоморфно касательное расслоение к окружности?

Задача 13.66. Гомеоморфно ли касательное расслоение к двумерной сфере прямому произведению сферы на плоскость?

Отметим, что

$$h_k(P) = \int_0^1 \frac{\partial f}{\partial x^k}(P)dt = \frac{\partial f}{\partial x^k}(P)$$

Применим теперь полученное только что разложение к функциям h_k. Тогда

$$h_k\left(x^1, \ldots, x^n\right) = h_k(P) + \sum_{p=1}^{n} \left(x^p - x_0^p\right) h_{kp}\left(x^1, \ldots, x^n\right)$$

где $h_{kp}\left(x^1, \ldots, x^n\right)$ - снова некоторые гладкие функции. Окончательно для функции f можно записать:

$$\begin{aligned} f\left(x^1, \ldots, x^n\right) =& f(P) + \sum_{k=1}^{n} \left(x^k - x_0^k\right) h_k\left(x^1, \ldots, x^n\right) = \\ =& f(P) + \sum_{k=1}^{n} \left(x^k - x_0^k\right) h_k(P) + \\ &+ \sum_{k,p=1}^{n} \left(x^k - x_0^k\right)\left(x^p - x_0^p\right) h_{kp}\left(x^1, \ldots, x^n\right) = \\ =& f(P) + \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(P)\left(x^k - x_0^k\right) + \\ &+ \sum_{k,p=1}^{n} \left(x^k - x_0^k\right)\left(x^p - x_0^p\right) h_{kp}\left(x^1, \ldots, x^n\right) \end{aligned}$$

Лемма доказана.

Вернемся к доказательству утверждения. Воспользуемся леммой 13.69 для вычисления $\mathcal{D}(f)$. В силу свойств операции дифференцирования имеем

$$\begin{aligned} \mathcal{D}(f) =& \mathcal{D}(f(P)) + \sum_{k=1}^{n} \frac{\partial f}{\partial x^k}(P)\mathcal{D}\left(\left(x^k - x_0^k\right)\right) + \\ &+ \sum_{k,p=1}^{n} \mathcal{D}\left(\left(x^k - x_0^k\right)\left(x^p - x_0^p\right) h_{kp}\left(x^1, \ldots, x^n\right)\right). \end{aligned}$$

Однако первое и последнее слагаемые в этой сумме равны нулю согласно утверждению 13.67. Поэтому

$$\mathcal{D}(f)=\sum_{k=1}^{n}\frac{\partial f}{\partial x^k}(P)\mathcal{D}\left(\left(x^k-x_0^k\right)\right)$$

Определим касательный вектор V, положив его компоненты ($v^1,\ldots,v^n$) в координатах ($x^1,\ldots,x^n$) равными $v^k=\mathcal{D}\left(\left(x^k-x_0^k\right)\right)$. Тогда

$$\mathcal{D}(f)=\sum_{k=1}^{n}\frac{\partial f}{\partial x^k}(P)v^k=V(f)$$

Таким образом, мы установили существование искомого касательного вектора V. Для доказательства единственности вектора V достаточно заметить, что если таких векторов два, то их разность Δ - тоже касательный вектор, причем $\Delta(f)=0$ для любой функции f. Но отсюда следует, что $\Delta=0$, т.е. векторы одинаковы. Утверждение доказано.

14.15.6 13.6.2 Local device of submersions and immersions

Оказывается, локально все субмерсии и все иммерсии устроены одинаково.

Statement

13.70. Пусть $P\in M$ - регулярная точка гладкого отображения $F:M\to N$. Тогда существуют локальные координаты $(x^1,\ldots,x^m)$ в окрестности $U\subset M$ точки P и ($y^1,\ldots,y^n$) в окрестности $V\subset N$ точки $F(P)$, такие, что $F(U)\subset V$, и отображение F записывается в этих координатах в виде

$$y^i\left(x^1,\ldots,x^m\right)=x^i,\quad 1\le i\le n$$

Доказательство. Действительно, пусть $(\tilde{x}^1,\ldots,\tilde{x}^m)$ и $(\tilde{y}^1,\ldots,\tilde{y}^n)$ - произвольные координаты в U и V соответственно и $\tilde{y}^i=f^i\left(\tilde{x}^1,\ldots,\tilde{x}^m\right)$ - координатные функции отображения F. По определению ранг матрицы Якоби отображения F в точке P равен n, поэтому после перенумерации координат можно считать, что первые n столбцов матрицы Якоби линейно независимы. Тогда отображение

$$x^i=\begin{cases}f^i\left(\tilde{x}^1,\ldots,\tilde{x}^m\right), & i=1,\ldots,n\\ \tilde{x}^i, & i=n+1,\ldots,m\end{cases}$$

задает замену координат в окрестности точки P. В этих координатах отображение F имеет вид

$$\tilde{y}^i=x^i,\quad i=1,\ldots,n,$$

что и требовалось доказать. □

Из утверждения [13.70] следует, что локально все субмерсии устроены одинаково, а именно - это проекция $\mathbb{R}^m$ на $\mathbb{R}^n$, переводящая точку ($x^1,\ldots,x^m$) из $\mathbb{R}^m$ в точку ($x^1,\ldots,x^n$) из $\mathbb{R}^n$.

Все иммерсии локально устроены одинаково в следующем смысле.

Statement

13.71. Пусть $F : M \to N$ - произвольное погружение и P - некоторая точка из M. Тогда существуют локальные координаты $(x^1, \ldots, x^m)$ в окрестности $U \subset M$ точки $Pu\,(y^1, \ldots, y^n)$ в окрестности $V \subset N$ точки $F(P)$, такие, что $F(U) \subset V$, и отображение F записывается в этих координатах в виде

$$y^i\left(x^1, \ldots, x^m\right) = x^i, \quad 1 \le i \le m, \quad y^i\left(x^1, \ldots, x^m\right) = 0, \quad m+1 \le i \le n.$$

Доказательство. Действительно, пусть $(\tilde{x}^1, \ldots, \tilde{x}^m)$ и $(\tilde{y}^1, \ldots, \tilde{y}^n)$ - произвольные координаты в окрестностях $U \ni P$ и $V \ni F(P)$ и пусть $\tilde{y}^i = f^i\left(\tilde{x}^1, \ldots, \tilde{x}^m\right), i = 1, \ldots, n, -$ координатное представление погружения F. По определению ранг матрицы Якоби отображения F в точке P равен m, поэтому после перенумерации координат на V можно предположить, что первые m строк этой матрицы линейно независимы. По теореме о неявной функции существуют гладкие функции $\tilde{x}^j = \tilde{x}^j\left(\tilde{y}^1, \ldots, \tilde{y}^m\right), j = 1, \ldots, m$, такие, что множество $F(U)$ задается в координатах $(\ \tilde{y}^1, \ldots, \tilde{y}^n\ )$ уравнениями

$$\tilde{y}^i = f^i\left(\tilde{x}^1\left(\tilde{y}^1, \ldots, \tilde{y}^m\right), \ldots, \tilde{x}^m\left(\tilde{y}^1, \ldots, \tilde{y}^m\right)\right), \quad m+1 \le i \le n.$$

Для завершения доказательства осталось сделать замены координат:

$$y^j = \begin{cases} \tilde{y}^j, & j = 1, \ldots, m, \\ \tilde{y}^j - f^j\left(\tilde{x}^1\left(\tilde{y}^1, \ldots, \tilde{y}^m\right), \ldots, \tilde{x}^m\left(\tilde{y}^1, \ldots, \tilde{y}^m\right)\right), & j = m+1, \ldots, n, \end{cases}$$

и

$$x^j = \tilde{y}^j\left(\tilde{x}^1, \ldots, \tilde{x}^m\right), \quad j = 1, \ldots, m.$$

Утверждение доказано. □

Утверждение 13.77 означает, что локально каждое погружение устроено как стандартное вложение пространства $\mathbb{R}^m$ в пространство $\mathbb{R}^n$ в виде координатного подпространства, т.е. как вложение, переводящее точку $(\ x^1, \ldots, x^m\ )$ пространства $\mathbb{R}^m$ в точку $(\ x^1, \ldots, x^m, 0, \ldots, 0\ )$ из $\mathbb{R}^n$.

14.16 14 Injections of manifolds into Euclidean space

В данной лекции будет показано, что произвольное компактное многообразие может быть вложено в евклидово пространство подходящей размерности. Сначала мы построим вложение в пространство очень большой, но конечной размерности, а затем докажем так называемую слабую теорему Уитни, позволяющую понизить размерность объемлющего пространства до удвоенной размерности многообразия плюс один.

Замечание 14.1. Теорема остается верной и для некомпактных многообразий, но доказательство для этого случая более громоздко.

Замечание 14.2. Результаты данной лекции означают, что любое многообразие можно рассматривать как подмногообразие в подходящем пространстве $\mathbb{R}^N$. Однако во многих задачах, например при описании конфигурационных или фазовых пространств физических систем, структура многообразия возникает естественно, в то время как вложение в евклидово пространство не имеет явного физического смысла. Кроме того, если нас, скажем, интересует топологическое устройство рассматриваемого многообразия, то для его описания часто достаточно знать его внутренние свойства, т.е. свойства, не зависящие от вложения в объемлющее пространство. Тем не менее, возможность реализации многообразий в виде подмногообразий в $\mathbb{R}^N$ позволяет достаточно легко доказать многие важные факты о самих многообразиях.

14.16.1 14.1 Existence of an attachment

Напомним, что носителем $\operatorname{supp} f$ функции $f : M \to \mathbb{R}^1$ называется замыкание множества всех точек x из M, в которых функция f отлична от нуля. Нам понадобится следующая техническая лемма, известная из математического анализа.

Лемма 14.3. Пусть D_r^n - открытый шар в $\mathbb{R}^n$ с центром в начале координат O и радиуса r. Для каждого $\varepsilon > 0$ существует гладкая функция $f : \mathbb{R}^n \to \mathbb{R}^1$, такая, что носитель $\operatorname{supp} f$ функции f лежит в D_r^n, функия f тождественно равна 1 на шаре $D_{r-\varepsilon}^n u 0 \leq f \leq 1$.

Замечание 14.4. Чтобы построить такую функцию, достаточно, например, воспользоваться хорошо известной гладкой функцией f, равной 0 на отрицательной полуоси, монотонно возрастающей от 0 до 1 на отрезке $[0,1]$, и равной 1 на луче $[1,\infty\ )$. Эта функция может быть задана, например, в виде

$$f(x) = \begin{cases} 0, & x \leq 0; \\ \exp\left[-\frac{1}{x^2}\exp\left(-\frac{1}{1-x^2}\right)\right], & 0 < x < 1; \\ 1, & x \geq 1. \end{cases}$$

График функции f приведен на рис. 14.1 .

Рис. 14.1: Одномерная гладкая "ступенька".

Theorem

14.5. Пусть M - произвольное компактное гладкое многообразие. Тогда существует вложение $\Phi : M \to \mathbb{R}^N$ многообразия M в евклидово пространство подходящей размерности.

Доказательство. В силу компактности многообразия M на нем можно выбрать конечный атлас $\{(U_\alpha, \varphi_\alpha)\}_{\alpha=1}^K$, причем можно считать без ограничения общности, что каждый координатный гомеоморфизм φ_α переводит U_α в открытый шар D_α.

Конструкция 14.6. Итак, пусть $\{(U_\alpha, \varphi_\alpha)\}_{\alpha=1}^K$ - атлас, карты которого гомеоморфны шарам D_α. Рассмотрим систему меньших открытых шаров $D'_\alpha \subset D_\alpha$ с теми же центрами, таких, что система открытых множеств $U'_\alpha = \varphi_\alpha^{-1}(D'_\alpha)$ по-прежнему покрывает M. Обозначим через h_α гладкую функцию на $\mathbb{R}^n$ с носителем внутри D_α, тождественно равную 1 на D'_α, и строго меньшую 1 на $D_\alpha \backslash D'_\alpha$ (такая функция существует в силу леммы 14.3). Построим гладкое отображение $\psi_\alpha : M \to \mathbb{R}^n$, положив

$$\psi_\alpha(P) = \begin{cases} h_\alpha(\varphi_\alpha(P))\,\varphi_\alpha(P), & P \in U_\alpha, \\ 0, & P \in M \backslash U_\alpha. \end{cases}$$

Очевидно, $\psi_\alpha(P) = \varphi_\alpha(P)$, если $P \in U'_\alpha$. Мы построили K гладких отображений $\{\psi_\alpha\}_{\alpha=1}^K$ из M в $\mathbb{R}^n$. Зададим теперь гладкое отображение $\bar{\Phi} : M \to \mathbb{R}^{nK}$, положив

$$\bar{\Phi}(P) = (\psi_1(P), \ldots, \psi_K(P)) \in \mathbb{R}^{nK}.$$

Лемма 14.7. Отображение $\bar{\Phi}$ является погружением.
Доказательство. Пусть P - произвольная точка из M. Тогда P содержится в некотором множестве $U'_\alpha \subset U_\alpha$. Пусть $(x_\alpha^1, \ldots, x_\alpha^n)$ локальные координаты в карте U_α, порожденные φ_α, а координаты в $\mathbb{R}^{nK}$ обозначим через

$$\left(y_1^1, \ldots, y_1^n, \ldots, y_\alpha^1, \ldots, y_\alpha^n, \ldots, y_K^1, \ldots, y_K^n\right).$$

Матрица Якоби отображения $\bar{\Phi}$ в точке P в локальных координатах (x_α^j) и (y_β^i) - это прямоугольная матрица вида $\left(\frac{\partial y_\beta^i}{\partial x_\alpha^j}\right)$ (здесь индекс α фиксирован, а остальные пробегают значения от 1 до n). В частности, матрица Якоби отображения $\bar{\Phi}$ содержит блок $\left(\frac{\partial y_\alpha^i}{\partial x_\alpha^j}\right)$. Но так как на множестве U'_α отображение ψ_α совпадает с φ_α, координатное представление отображения ψ_α в координатах $(x_\alpha^1, \ldots, x_\alpha^n)$ и $(y_\alpha^1, \ldots, y_\alpha^n)$ имеет вид

$$y_\alpha^i\left(x_\alpha^1, \ldots, x_\alpha^n\right) = x_\alpha^i, \quad i = 1, \ldots, n.$$

Поэтому $\left(\frac{\partial y_\alpha^i}{\partial x_\alpha^j}\right)$ - единичная матрица размера $n \times n$. Таким образом, ранг матрицы Якоби отображения $\bar{\Phi}$ равен n во всех точках многообразия M, следовательно $\bar{\Phi}-$ погружение. Лемма доказана.

Теперь подправим отображение $\bar{\Phi}$ так, чтобы получить вложение.
Конструкция 14.8. Определим функции $\hat{h}_\alpha : M \to \mathbb{R}^1$, положив

$$\hat{h}_\alpha(P) = \begin{cases} h_\alpha\left(\varphi_\alpha(P)\right), & P \in U_\alpha \\ 0, & P \in M \backslash U_\alpha \end{cases}$$

Построим новое отображение $\Phi : M \to \mathbb{R}^N$, где $N = Kn + K$, положив

$$\Phi(P) = \left(\bar{\Phi}(P), \hat{h}_1(P), \ldots, \hat{h}_K(P)\right),$$

т.е. мы просто приписали к $\bar{\Phi}(P)$ вектор длины K, составленный из чисел $\hat{h}_\alpha(P)$.

Лемма 14.9. Отображение Φ является вложением.
Доказательство. Очевидно, что Φ - снова погружение, так как ранг матрицы Якоби отображения Φ не меньше, чем у матрицы Якоби для $\bar{\Phi}$.

Покажем теперь, что Φ взаимно-однозначно. Пусть P и Q - произвольные различные точки из M. Поскольку система открытых множеств $\{U'_\alpha\}$ покрывает многообразие M, то найдется такое U'_α, которое содержит P, поэтому $\hat{h}_\alpha(P) = 1$. Если $\hat{h}_\alpha(Q) = 1$, то это означает, что Q также лежит в U'_α и, следовательно, $\psi_\alpha(P) \neq \psi_\alpha(Q)$ (отображение ψ_α на U'_α совпадает с гомеоморфизмом φ_α и не может склеивать различные точки). Поэтому в рассмотренном случае $\Phi(P) \neq \Phi(Q)$. Если же $\hat{h}_\alpha(Q) \neq 1$, то и подавно $\Phi(P) \neq \Phi(Q)$, так как они различаются координатой, соответствующей $\hat{h}_\alpha$. Таким образом, Φ взаимно-однозначно с образом.

Наконец, многообразие M компактно, отображение Φ - взаимно-однозначное погружение, значит, Φ - вложение в силу следствия 11.41.

Теорема доказана. $\square$

Замечание 14.10. На самом деле размерность объемлющего многообразия может быть оценена сверху существенно более эффективно. Соответствующие результаты называются

теоремами Уитни. В следующем разделе мы докажем так называемый слабый вариант теоремы Уитни, для чего используем теорему Сарда из математического анализа.

14.16.2 14.2 Sard's theorem

Говоря неформально, теорема Сарда утверждает, что у любого гладкого отображения регулярных значений "гораздо больше" чем сингулярных. В этом раделе мы напомним необходимые определения и результаты из математического анализа и переформулируем их для интересующего нас случая гладких многообразий.

Определение 14.11. Подмножество A евклидового пространства $\mathbb{R}^n$ называется множеством меры нуль, если для любого $\varepsilon > 0$ существует такое конечное или счетное покрытие множества A открытыми шарами в $\mathbb{R}^n$, что сумма их n-мерных объемов меньше ε.

Замечание 14.12. Шары в этом определении можно заменить на кубы или параллелепипеды со сторонами, параллельными осям координат. В результате, как легко проверить, получится эквивалентное определение.

Statement

14.13. Множества меры нуль обладают следующими свойствами:
(1) объединение конечного, или счетного, семейства множеств $\{A_i\}$ меры нуль является множеством меры нуль;
(2) для любого гладкого отображения $F : U \to V$, где U и V - открытые подмножества в $\mathbb{R}^n, u$ любого множества $A \subset U$ меры нуль множество $F(A)$ тоже является множеством меры нуль;
(3) множество меры нуль не имеет внутренних точек.

Доказательство. Первое свойство очевидно: покрыв каждое из множеств A_i набором шаров, сумма объемов которых меньше $\varepsilon/2^i$, получим покрытие множества $\cup_i A_i$ набором шаров, суммарный объем которых меньше ε.

В силу первого свойства, второе свойство достаточно доказать в предположении, что множество A содержится в некотором замкнутом кубе $Q \subset U$. Тогда, согласно теореме Лагранжа, существует некоторая постоянная M, такая, что

$$\|F(x) - F(y)\| \leq M\|x - y\|.$$

Поэтому образ шара радиуса r, содержащегося в кубе Q, отображается во множество, содержащееся в шаре радиуса $r \cdot M$. Следовательно, если покрыть множество A шарами, суммарный объем которых меньше ε/M^n, то множество $F(A)$ может быть покрыто шарами суммарного объема меньше ε.

Для доказательства свойства 3 достаточно установить, что если конечное семейство кубов $\{Q_i\}$ в $\mathbb{R}^n$ со сторонами, параллельными осям координат, покрывает куб Q, то сумма объемов кубов Q_i не меньше объема куба Q (и, значит, ограничена снизу положительной константой, не зависящей от покрытия, откуда Q не может иметь меру нуль). Доказательство этого утверждения оставим в качестве упражнения. □

Определение 14.14. Подмножество A гладкого многообразия M называется множеством меры нуль, если существует такое конечное или счетное семейство карт $\{(U_i, \varphi_i)\}$ на M, покрывающих множество A, что каждое множество $\varphi_i(A \cap U_i)$ имеет меру нуль в пространстве $\mathbb{R}^n$.

Упражнение 14.15. Покажите, что утверждение 14.13 остается справедливым для случая многообразий.

Определение 14.16. Подмножество A топологического пространства называется нигде не плотным, если его замыкание $\bar{A}$ не имеет внутренних точек. Подмножество, являющееся объединением конечного, или счетного, семейства нигде не плотных множеств называется тощим.

Упражнение 14.17. Приведите пример топологического пространства, которое само является тощим множеством.

Следующее утверждение непосредственно следует из определений.

Утверждение 14.18. Любое замкнутое множество меры нуль нигде не плотно. Если множество меры нуль представимо в виде конечного или счетного объединения замкнутых множеств, то оно является тощим.

Определение 14.19. Множество меры нуль, представимое в виде конечного или счетного объединения замкнутых множеств будем называть ® нуль-тощими.

Теперь сформулируем теорему Сарда [r .

Theorem

14.20 (Сард). Пусть $F : M \to N$ - гладкое отображение гладких многообразий. Тогда множество $C(F)$ его критических значений является нуль-тощим множеством, т.е. представимо в виде не более чем счетного объединения замкнутых множеств меры нуль. Если многообразие M - компактно, то множество $C(F)$ замкнуто и нигде не плотно.

Теорема Сарда - это, фактически, результат математического анализа, поэтому не будем здесь ее доказывать полностью. В Дополнительном материале к этой лекции выполнена лишь вся "геометрическая" работа, а здесь разобран единственный нужный нам для доказательства теоремы Уитни случай, когда $\dim M < \dim N$.

Следствие 14.21. Если $\dim M < \dim N$, то для любого гладкого отображения $F : M \to N$ множество $N \backslash F(M)$ не пусто.

Доказательство следствия. Это очевидно, так как в условиях следствия множество $F(M)$ представляет собой множество критических значений отображения F и, следовательно, по теореме Сарда, является нуль-тощим. В частности, $F(M)$ имеет меру нуль, а потому даже не содержит внутренних точек. Доказательство закончено. □

14.16.3 14.3 Whitney's "weak" theorem

Напомним, что размерность пространства $\mathbb{R}^N$, в которое нам удалось вложить многообразие M при доказательстве теоремы 14.5, была очень велика (она равнялась количеству карт в атласе многообразия, умноженному на $\dim M + 1$).

Theorem

14.22 (Уитни). Пусть M - гладкое компактное многообразие размерности n. Тогда существует вложение $F : M \to \mathbb{R}^{2n+1}$.

Доказательство. Воспользуемся теоремой 14.5 и построим вложение Φ многообразия M в евклидово пространство $\mathbb{R}^N$. Рассмотрим произвольную, проходящую через начало координат прямую ℓ в пространстве $\mathbb{R}^N$ и обозначим через π_ℓ ортогональную проекцию пространства $\mathbb{R}^N$ вдоль прямой ℓ на ортогональное подпространство размерности на единицу меньше: $\pi_\ell : \mathbb{R}^N \to \mathbb{R}^{N-1}$. Попробуем понизить размерность, для чего

[01] Эта терминология предложена М. М. Постниковым в его замечательной книге "Гладкие многообразия".

[2] Заметим, что до Сарда эта теорема была доказана Брауном. Кроме того, независимо от Сарда её доказал Дубовицкий.

рассмотрим композицию $\pi_\ell \circ \Phi : M \to \mathbb{R}^{N-1}$. Нужно выбрать прямую ℓ так, чтобы композиция $\pi_\ell \circ \Phi$ по-прежнему являлась бы вложением (этот способ называется методом проекций). Для этого достаточно проверить выполнение двух условий:
(1) мономорфность дифференциала композиции $\pi_\ell \circ \Phi$ (тогда эта композиция является погружением);
(2) биективность композиции $\pi_\ell \circ \Phi$ (взаимно однозначное погружение $\pi_\ell \circ \Phi$ компактного многообразия M является вложением).

Наша цель - показать, что прямую ℓ можно выбрать так, чтобы оба условия были выполнены.

Рассмотрим первое условие. Пусть P - произвольная точка из M. Обозначим через Π_P образ $d\Phi\left(T_P M\right)$ касательного пространства $T_P M$ в точке P к многообразию M под действием дифференциала $d\Phi$ отображения Φ в точке P. Так как Φ - вложение, то Π_P - это n-мерное подпространство в $\mathbb{R}^N \equiv T_{\Phi(P)}\mathbb{R}^N$. Для того чтобы дифференциал отображения $\pi_\ell \circ \Phi$ был мономорфизмом в точке P, необходимо и достаточно, чтобы прямая ℓ не принадлежала бы подпространству Π_P (напомним, что и прямая, и подпространство проходят через нуль).

Конструкция 14.23. Обозначим через W_1 многообразие размерности $2n$, полученное из касательного расслоения $T(M)$ к многообразию M выбрасыванием нулевого сечения M_0. Построим гладкое отображение h_1 многообразия W_1 в проективное пространство $\mathbb{R}\mathrm{P}^{N-1}$, ставя в соответствие ненулевому касательному вектору $V \in T_P M$ проходящую через нуль прямую в $\mathbb{R}^N$ с направляющим вектором $d\Phi(V) \in \mathbb{R}^N$. Ясно, что прямая ℓ не содержится в образе никакой плоскости Π_P и, значит, удовлетворяет первому условию, если и только если она не лежит в образе отображения h_1.
Лемма 14.24. Отображение $h_1 : W_1 \to \mathbb{R}P^{N-1}$ является гладким.

Доказательство. Запишем отображение h_1 в локальных координатах. Пусть $(x^1, \ldots, x^n)$ - локальные координаты на M в окрестности $P, (x^1, \ldots, x^n, v^1, \ldots, v^n)$ - соответствующие координаты на $W_1 \subset T(M)$ в окрестности V и $\left(y^1, \ldots, y^N\right)$ - стандартные координаты в $\mathbb{R}^N$, а ($y^1 : \ldots : y^N$) - соответствующие однородные координаты в $\mathbb{RP}^{N-1}$. Если $y^i = y^i\left(x^1, \ldots, x^n\right), i = 1, \ldots, n, -$ координатное представление вложения Φ, то отображение h_1 имеет вид

$$h_1 : \left(x^1, \ldots, x^n, v^1, \ldots, v^n\right) \mapsto \left(\sum_{k=1}^n \frac{\partial y^1}{\partial x^k} v^k : \ldots : \sum_{k=1}^n \frac{\partial y^N}{\partial x^k} v^k\right) \in \mathbb{R}\mathrm{P}^{N-1},$$

где все производные вычисляются в точке P. Это отображение очевидно является гладким. $\square$

Рассмотрим теперь второе условие. Отсутствие взаимной однозначности композиции $\pi_\ell \circ \Phi$ эквивалентно существованию в пространстве $\mathbb{R}^N$ прямой, параллельной ℓ и проходящей через две различные точки: $\Phi(P)$ и $\Phi(Q)$. Отметим, что так как Φ вложение, условие $\Phi(P) \neq \Phi(Q)$ равносильно $P \neq Q$.

Конструкция 14.25. Построим гладкое отображение h_2 из $2n$-мерного многообразия $W_2 = (M \times M) \backslash \Delta$, где через Δ, как обычно, обозначена диагональ, в проективное пространство $\mathbb{R}\mathrm{P}^{N-1}$, сопоставив каждой паре различных точек (P, Q) многообразия M прямую в $\mathbb{R}\mathrm{P}^{N-1}$, параллельную прямой $\Phi(P)\Phi(Q)$ в $\mathbb{R}^N$. Ясно, что для взаимной однозначности проекции вдоль прямой ℓ, необходимо и достаточно, чтобы ℓ не лежала в образе отображения h_2.
Лемма 14.26. Отображение $h_2 : W_2 \to \mathbb{R}\mathrm{P}^{N-1}$ является гладким.
Доказательство. Запишем отображение h_2 в локальных координатах. Обозначим через $(x^1, \ldots, x^n), (\bar{x}^1, \ldots, \bar{x}^n)$ локальные координаты в окрестностях точек P и Q на многообразии M, порождающие локальные координаты в окрестности точки $(P, Q) \in \tilde{W}_2$. Пусть, далее, $\left(y^1, \ldots, y^N\right)$ - стандартные координаты в пространстве $\mathbb{R}^N$, а (

$y^1 : \ldots : y^N\big)$ - соответствующие однородные координаты в $\mathbb{R}\mathrm{P}^{N-1}$. Тогда, если $y^i = y^i\left(x^1, \ldots, x^n\right), i = 1, \ldots, N, -$ координатное представление отображения Φ, отображение h_2 имеет вид

$$h_2 : (P, Q) = \left(x^1(P), \ldots, x^n(P), \bar{x}^1(Q), \ldots, \bar{x}^n(Q)\right) \mapsto$$
$$\mapsto \left(y^1(x(P)) - y^1(\bar{x}(Q)) : \ldots : y^N(x(P)) - y^N(\bar{x}(Q))\right).$$

Поэтому отображение h_2 - гладкое. $\square$

Конструкция 14.27. Рассмотрим многообразие $X = W_1 \sqcup W_2$, являющееся дизъюнктивным объединением многообразий W_1 и W_2. Заметим, что X - гладкое многообразие размерности $2n$. Определим отображение H из X в $\mathbb{R}P^{N-1}$, положив

$$H(x) = \begin{cases} h_1(x), & x \in W_1, \\ h_2(x), & x \in W_2. \end{cases}$$

14.16.4 14.4 Problems

Задача 14.31. Проделать предложенную в доказательстве теоремы о существовании вложения конструкцию для случая стандартной окружности (атлас из двух карт). Применить к полученному вложению метод проекций.

Задача 14.32. Докажите утверждение 14.18.

Задача 14.33. Показать с помощью метода проекций, что каждое компактное гладкое многообразие размерности n можно погрузить в евклидово пространство размерности $2n$.

Задача 14.34. Построить вложение проективного пространства $\mathbb{RP}^n$ в подходящее евклидово пространство.

Задача 14.35. Показать, что двумерное компактное многообразие нельзя вложить в $\mathbb{R}^3$ так, чтобы в каждой точке полученной поверхности гауссова кривизна была: (1) отрицательной; (2) неположительной.

14.16.5 14.5 Supplemental material

14.16.6 14.5.1 To prove Sard's theorem

Теорема Сарда может быть выведена из следующего утверждения, которое в математическом анализе тоже часто называют теоремой Сарда.

Statement

14.36. Пусть $F : U \to \mathbb{R}^n$ - гладкое отображение открытого множества $U \subset \mathbb{R}^m$ в пространство $\mathbb{R}^n$ и пусть $K \subset U$ - произвольное компактное подмножество множества критических точек отображения F. Тогда множество $F(K)$ имеет меру нуль.

Покажем, как теорема Сарда получается из утверждения 14.36.

Доказательство. Рассмотрим произвольное открытое подмножество O многообразия M. Пересечение множества критических точек отображения F со множеством O совпадает со множеством критических точек ограничения $F|_O$ отображения F на множество O. Поэтому если $\{(U_\alpha, \varphi_\alpha)\}$ - произвольный атлас многообразия M, то множество $C(F)$ критических значений отображения F может быть представлено в виде

$$C(F) = \cup_\alpha C\left(F|_{U_\alpha}\right)$$

Пусть (V_α, ψ_α) - такая карта на многообразии N, что $F(U_\alpha) \subset V_\alpha$. Тогда множество $C\left(F|_{U_\alpha}\right)$ - это образ при гомеоморфизме ψ_α^{-1} множества $C(g_\alpha)$ критических значений сквозного отображения

$$g_\alpha = \psi_\alpha \circ F|_{U_\alpha} \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha) \subset \mathbb{R}^m \to \mathbb{R}^n$$

В то же время множество критических точек отображения g_α, как подмножество в $\varphi_\alpha(U_\alpha) \subset \mathbb{R}^m$, задается набором уравнений (равенство нулю соответствующих миноров) и, следовательно, является замкнутым. Как и всякое замкнутое подмножество в $\mathbb{R}^m$, множество критических точек отображения g_α представимо в виде не более чем счетного объединения компактных множеств. Поэтому каждое множество

$$C\left(F|_{U_\alpha}\right) = \psi_\alpha^{-1}(C(g_\alpha))$$

является, в силу утверждения 14.36, утверждения 14.13 и хаусдорфовости многообразия N, объединением не более чем счетного семейства замкнутых множеств меры нуль. Значит, таким будет и все множество $C(F)$. Итак, мы показали, что $C(F)$ - нуль-тощее множество.

Если же многообразие M компактно, то на нем можно выбрать конечный атлас, что позволяет представить множество всех критических точек отображения F в виде конечного объединения компактных множеств K_i (по числу карт). Все множества $F(K_i)$ замкнуты и нуль-тощие, поэтому они нигде не плотны, а значит, конечное объединение $C(F)$ этих множеств тоже нигде не плотно. Доказательство закончено.

Итак, для завершения доказательства теоремы Сарда осталось доказать утверждение 14.36. Мы этого делать не будем, сославшись на любой стандартный курс математического анализа. Однако для целостности изложения разберем тривиальный случай $m < n$.

Доказательство утверждения 14.30*при* $\mathbf{m} < \mathbf{n}$. Представим пространство $\mathbb{R}^n$ в виде прямого произведения $\mathbb{R}^n = \mathbb{R}^m \times \mathbb{R}^{n-m}$. Построим открытое подмножество $U \times \mathbb{R}^{n-m} \subset \mathbb{R}^n$ и новое отображение $\Phi = F \circ \pi_1 : U \times \mathbb{R}^{n-m} \to \mathbb{R}^n$, где π_1 - проекция на первый сомножитель. Тогда для каждого компакта $K \subset U$ его образ $F(K)$ при отображении F совпадает с образом $\Phi(K \times 0)$ множества $K \times 0 \subset U \times \mathbb{R}^{n-m}$ при отображении Φ. Но множество $K \times 0$ имеет меру нуль в пространстве $\mathbb{R}^n$ (его можно покрыть конечным множеством кубиков сколь угодно малой высоты). Поэтому, в силу утверждения 14.13, множество $F(K) = \Phi(K \times 0)$ тоже имеет меру нуль, что и требовалось доказать.

Отметим, что следствие 14.21 нами полностью доказано. Именно им мы воспользовались при доказательстве теоремы Уитни.

14.16.7 14.5.2 Examples of dives

Приведем некоторые известные примеры погружений. На рис. 14.2 изображено погружение бутылки Клейна в трехмерное пространство, а на рис. 14.3 - погружение проективной плоскости, известное как "скрещенный колпак". Оба эти многообразия компактны и неориентируемы, поэтому построить их вложение в трехмерное пространство нельзя (проверьте).

Рис. 14.2: Погружение бутылки Клейна в $\mathbb{R ^{3}$.}

Рис. 14.3: Погружение проективной плоскости в $\mathbb{R ^{3}$.}

14.17 15 Additional structures: Riemannian metric, orientability

Лекция посвящена дополнительным структурам, которые вводятся на гладких многообразиях для расширения возможностей при работе с ними. Например, на абстрактном топологическом пространстве понятие расстояния между его точками не определено. В случае гладких многообразий имеется естественный способ задать расстояние, используя в качестве дополнительной структуры аналог первой квадратичной формы поверхности - так называемую риманову метрику. Другие примеры дополнительных структур - это ориентация, симплектическая структура, аффинная связность.

14.17.1 15.1 Riemannian metric, Riemannian manifolds

Как мы уже отмечали, риманова метрика является аналогом первой фундаментальной формы поверхности или римановой метрики в области евклидова пространства.

Конструкция 15.1. Пусть M - произвольное гладкое многообразие размерности m. Предположим, что в каждой точке $P \in M$ в касательном пространстве $T_P M$ фиксирована некоторая билинейная форма $b(P)$. Рассмотрим произвольные координаты $(x^1, \ldots, x^m)$, определенные на M в некоторой окрестности U. Тогда для каждой точки $P \in U$ в касательном пространстве $T_P M$ имеется канонический базис ∂_i. В этом базисе форма $b(P)$ задается матрицей $B(P) = (b_{ij}(P))$, где $b_{ij}(P) = b(P)\,(\partial_i, \partial_j)$. Компоненты $b_{ij}(P)$ являются функциями от координат точки $P \in U$, т.е. функциями от $(x^1, \ldots, x^m)$.

Определение 15.2. Если в каждой точке P многообразия M задана билинейная форма $b(P) : T_PM \times T_PM \to \mathbb{R}$, то говорят, что на многообразии задана билинейная форма b. Величины $b_{ij}(P) = b(P)(\partial_i, \partial_j)$ называются компонентами формы b в координатах $(x^1, \ldots, x^m)$.

Утверждение 15.3. Пусть $(x^1, \ldots, x^m)$ u $(y^1, \ldots, y^m)$ - две системь координат, а $b_{ij} u c_{ij}$ - компоненты билинейной формы b в первой u второй системах координат соответственно. Тогда во всех точкам многообразия, где эти координаты определены одновременно, функции $b_{ij} u c_{ij}$ связаны так:

$$b_{ij} = \sum_{\alpha,\beta=1}^{m} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} c_{\alpha\beta}.$$

Доказательство. Пусть $P \in M$ - точка, в которой определены обе системы координат, ∂_{x^i} - векторы канонического базиса первой из этих систем, а ∂_{y^i} - второй. Тогда $\partial_{x^i} = \sum_\alpha \frac{\partial y^\alpha}{\partial x^i} \partial_{y^\alpha}$, поэтому

$$b_{ij} = b(\partial_{x^i}, \partial_{x^j}) = b\left(\sum_\alpha \frac{\partial y^\alpha}{\partial x^i} \partial_{y^\alpha}, \sum_\beta \frac{\partial y^\beta}{\partial x^j} \partial_{y^\beta}\right) =$$

$$= \sum_{\alpha,\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} b\left(\partial_{y^\alpha}, \partial_{y^\beta}\right) = \sum_{\alpha,\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} c_{\alpha\beta}.$$

Следствие 15.4. Если компоненты билинейной формы b в координатах ($x^1, \ldots, x^m$) гладко зависят от $(x^1, \ldots, x^m)$, то для любой другой системы координат ($y^1, \ldots, y^m$), имеющей $c(x^1, \ldots, x^m)$ общую область определения, компоненты b в $(y^1, \ldots, y^m)$ также будут гладко зависеть от ($y^1, \ldots, y^m$).

Определение 15.5. Будем говорить, что на многообразии M задана риманова метрика, если на M задано скалярное произведение (т.е. невырожденная положительно определенная симметричная билинейная форма), которое гладко зависит от точки в том смысле, что в каждой системе координат ($x^1, \ldots, x^m$) на M компоненты этого скалярного произведения являются гладкими функциями от ($x^1, \ldots, x^m$).

Многообразие, на котором задана риманова метрика называется римановым многообразием.

Замечание 15.6. Из следствия 15.4 вытекает, что достаточно проверить гладкость компонент римановой метрики в каждой точке в одной системе координат.

Замечание 15.7. Пусть (U, φ) - карта на римановом многообразии M. Отождествляя U и $\varphi(U) \subset \mathbb{R}^m$ с помощью координатного гомоморфизма, мы получаем на области $\varphi(U)$ риманову метрику в смысле лекции 8. С этой точки зрения, риманова метрика на многообразии "сшита" из метрик, заданных в координатных областях.

Имеется альтернативный способ определения римановой метрики, не использующий напрямую касательные векторы и билинейные формы.

Определение 15.8. Говорят, что на гладком многообразии M задана риманова метрика, если в каждой системе координат ($x^1, \ldots, x^m$) на M в области определения этих координат задан набор величин $g_{ij}(x^1, \ldots, x^m), i, j = 1, \ldots, m$, которые гладко зависят от $(x^1, \ldots, x^m)$ и называются компонентами римановой метрики в этих координатах, и выполнены следующие два условия:
(1) матрицы (g_{ij}) являются симметричными и положительно определенными в каждой точке области определения;
(2) компоненты римановой метрики в разных системах координат, связаны друг с

другом по формуле из утверждения 15.3 во всех точкам многообразия, где эти координаты определены одновременно.

Утверждение 15.9. Два определения римановой метрики эквивалентны.

Доказательство. Если дана риманова метрика в смысле определения 15.5, то наборы величин $g_{ij}\left(x^1,\ldots,x^m\right)$ из определения 15.8 - это компоненты скалярного произведения в координатах ($x^1,\ldots,x^m$). Требуемые свойства вытекают из определения и утверждения 15.3.

Обратно, пусть задана риманова метрика в смысле определения 15.8. Определим на M скалярное произведение так. Пусть $(x^1,\ldots,x^m)$ - некоторые координаты на M, и P - произвольная точка из соответствующей карты U. Сопоставим паре векторов v и w из T_PM число $\langle v,w\rangle_P=\sum_{i,j=1}^m g_{ij}v^iw^j$, где g_{ij} компоненты римановой метрики, а v^i и w^j - компоненты касательных векторов v и w в координатах $(x^1,\ldots,x^m)$ соответственно. Заметим, что мы определили билинейную форму на T_PM, которая, в силу свойств матрицы (g_{ij}), является скалярным произведением, причем $\langle\partial_{x^i},\partial_{x^j}\rangle_P=g_{ij}(P)$, поэтому это скалярное произведение гладко зависит от точки (в силу гладкости g_{ij}). Таким образом, мы построили в U риманову метрику в смысле первого определения, причем g_{ij} представляют собой компоненты этой римановой метрики в координатах $(x^1,\ldots,x^m)$.

Остается проверить, что построенная нами риманова метрика не зависит от выбора координат ($x^1,\ldots,x^m$) и, значит, корректно определена на всем многообразии M. Пусть в окрестности точки $P\in M$ заданы две системы координат ($x^1,\ldots,x^m$) и ($y^1,\ldots,y^m$), и пусть g_{ij} и h_{ij} компоненты римановой метрики из определения [5.8] в этих координатах соответственно. Тогда для произвольных касательных векторов $v,w\in T_PM$, имеющих в $(x^1,\ldots,x^m)$ компоненты v^i и w^i, а в $(y^1,\ldots,y^m)$ - компоненты ξ^i и η^i соответственно, числа $\sum_{i,j=1}^m g_{ij}v^iw^j$ и $\sum_{i,j=1}^m h_{ij}\xi^i\eta^j$ равны. Действительно, так как

$$v^i=\sum_{i=1}^m\frac{\partial x^i}{\partial y^p}\xi^p\quad\text{ и }\quad w^i=\sum_{i=1}^m\frac{\partial x^i}{\partial y^p}\eta^p,$$

имеем

$$\sum_{i,j}g_{ij}v^iw^j=\sum_{i,j,\alpha,\beta,p,q}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j}h_{\alpha\beta}\frac{\partial x^i}{\partial y^p}\xi^p\frac{\partial x^j}{\partial y^q}\eta^q=\sum_{\alpha,\beta,p,q}h_{\alpha\beta}\delta_p^\alpha\delta_q^\beta\xi^p\eta^q=\sum_{\alpha,\beta}h_{\alpha\beta}\xi^\alpha\eta^\beta.$$

□

Пример 15.10. Рассмотрим многообразие $\mathbb{R}^m$, на котором фиксированы стандартные декартовы координаты ($x^1,\ldots,x^m$). Зададим в каждой точке $P\in\mathbb{R}^m$ риманову метрику, положив в координатах ($x^1,\ldots,x^m$) компоненты g_{ij} равными символам Кронекера δ_{ij}, а в любых других координатах $(y^1,\ldots,y^m)$ - выражениям

$$\sum_{p,q}\delta_{pq}\frac{\partial x^p}{\partial y^i}\frac{\partial x^q}{\partial y^j}=\sum_p\frac{\partial x^p}{\partial y^i}\frac{\partial x^p}{\partial y^j}.$$

поэтому

$$\langle v,w\rangle_P=\sum_j\left(\sum_p\frac{\partial y^j}{\partial x^p}v^p\sum_q\frac{\partial y^j}{\partial x^q}w^q\right)=\sum_{j,p,q}\frac{\partial y^j}{\partial x^p}\frac{\partial y^j}{\partial x^q}v^pw^q.$$

Если ∂_p обозначают векторы канонического базиса системы координат $(x^1,\ldots,x^m)$, то α-ая координата такого вектора в этом базисе равна δ_p^α, поэтому

$$g_{pq}=\langle\partial_p,\partial_q\rangle_P=\sum_{j,\alpha,\beta}\frac{\partial y^j}{\partial x^\alpha}\frac{\partial y^j}{\partial x^\beta}\delta_p^\alpha\delta_q^\beta=\sum_j\frac{\partial y^j}{\partial x^p}\frac{\partial y^j}{\partial x^q}$$

так что функции g_{pq} гладкие, поэтому $\langle\cdot,\cdot\rangle_P$ является римановой метрикой, а g_{pq} - ее компонентами.

Определение 15.15. Метрика $\langle\cdot,\cdot\rangle_P$ на подмногообразии M называется индучированной из $\mathbb{R}^n$.

Пример 15.16. Первая квадратичная форма регулярной поверхности в евклидовом пространстве - это частный случай индуцированной римановой метрики.

Имеет место важное утверждение, следующее из теоремы Уитни.

Theorem

15.17. На произвольном (компактном) гладком многообразии существует, по крайней мере, одна риманова метрика.

Доказательство. Действительно, в силу теоремы Уитни каждое такое многообразие можно вложить в $\mathbb{R}^N$, т.е. реализовать в виде подмногообразия $\mathbb{R}^N$. Но тогда индуцированная из $\mathbb{R}^N$ метрика и является искомой. Доказательство закончено.

Как и первая квадратичная форма поверхности, риманова метрика может быть использована для вычисления длин, углов и т.д.

Определение 15.18. Пусть M - риманово многообразие. Длиной касательного вектора $v \in T_PM$ называется число $\|v\| = \sqrt{\langle v,v\rangle_P}$, где через $\langle\cdot,\cdot\rangle_P$ обозначается скалярное произведение на T_PM в римановой метрике.

Далее, если $\gamma : [a,b] \to M$ - произвольная гладкая кривая на римановом многообразии M, то в каждой точке $\gamma(t)$ кривой γ определен ее вектор скорости $\gamma'(t) \in T_{\gamma(t)}M$. Длина $\|\gamma'(t)\|$ этого вектора в рассматриваемой римановой метрике является непрерывной функцией параметра t, поэтому следующее определение корректно.

Определение 15.19. Величина

$$\ell(\gamma) = \int_a^b \|\gamma'(t)\|\,dt$$

называется длиной гладкой кривой γ на римановом многообразии M.

Замечание 15.20. Это определение легко обобщается на случай кусочно гладкой кривой γ на римановом многообразии. Для этого достаточно разбить γ на участки гладкости и определить длину кусочно гладкой кривой как сумму длин всех ее гладких участков.

Определение 15.21. Пусть $\gamma_1(t)$ и $\gamma_2(s)$ - гладкие кривые на римановом многообразии M, пересекающиеся в некоторой точке P из M. Пусть $\gamma_1(t_0) = \gamma_2(s_0) = P$, тогда угол α между $\gamma_1 u \gamma_2$ в точка P определяется из соотношений

$$\cos\alpha = \frac{\langle\gamma_1'(t_0), \gamma_2'(s_0)\rangle_P}{\|\gamma_1'(t_0)\|\,\|\gamma_2'(s_0)\|}, \quad 0 \le \alpha \le \pi$$

Понятие индуцированной метрики (конструкция 15.13) обобщается на случай произвольного погружения в риманово многообразие.

Конструкция 15.22. Пусть $f : M \to W$ - погружение гладкого многообразия M в риманово многообразие W, и v, w - произвольные касательные векторы из T_PM. Положим

$$\langle v,w\rangle_P = \langle df(v), df(w)\rangle_{f(P)}$$

Следующая лемма доказывается точно также как лемма 15.14.

Лемма 15.23. Пусть $f : M \to W$ - погружение гладкого многообразия M в риманово многообразие W. Семейство функций $\langle\cdot,\cdot\rangle_P : T_PM \times T_PM \to \mathbb{R}, P \in M$, задает риманову метрику на многообразии M. Если $(x^1,\ldots,x^m)$ - некоторые координаты на M в окрестности $U, a\,(y^1,\ldots,y^n)$ - координаты на W в окрестности $f(P)$ и h_{ij} - компоненты

римановой метрики на W в координатах $(y^1, \ldots, y^n)$, то компоненты g_{pq} римановой метрики $\langle \cdot, \cdot \rangle_P$ на M в координатах $(x^1, \ldots, x^m)$ имеют вид

$$g_{pq} = \sum_{i,j} \frac{\partial y^j}{\partial x^p} \frac{\partial y^j}{\partial x^q} h_{ij},$$

где $y^j\left(x^1, \ldots, x^m\right)$ – координатное представление погружения f.

Определение 15.24. Метрика $\langle \cdot, \cdot \rangle_P$ на многообразии M называется индучированной из W погружением f.

Упражнение 15.25. Докажите лемму 15.23.

Конструкция 15.26. Если риманово многообразие M связное, то риманова метрика определяет функцию расстояния на M следующим образом. Пусть P и Q - произвольные точки из M. Обозначим через $\Omega(P, Q)$ множество всевозможных кусочно гладких кривых на M, соединяющих P и Q, и для каждой $\gamma \in \Omega(P, Q)$ пусть $\ell(\gamma)$ обозначает длину кривой γ. Тогда положим

$$\rho(P, Q) = \inf\{\ell(\gamma) \mid \gamma \in \Omega(P, Q)\}.$$

Полученная метрика называется внутренней метрикой.

Упражнение 15.27. Докажите, что ρ является метрикой на M.

14.17.2 15.2 Isometries

На классе римановых многообразий определяется более сильное отношение эквивалентности, чем просто диффеоморфность. Это отношение эквивалентности "уважает" риманову метрику.

Определение 15.28. Пусть M и N - произвольные римановы многообразия. Гладкое отображение $f : M \to N$ называется изометричным, если в каждой точке $P \in M$ дифференциал $df : T_P M \to T_{f(P)} N$ сохраняет скалярные произведения, т.е. для любых $v, w \in T_P M$ выполняется

$$\langle v, w \rangle_P = \langle df(v), df(w) \rangle_{f(P)}.$$

Изометричный диффеоморфизм называется изометрией. Два римановых многообразия называются изометричными, если существует изометрия, переводящее одно из них в другое.

Утверждение 15.29. Изометрии сохраняют длины кривых и углы между ними.

Компоненты римановых метрик изометричных многообразий отличаются друг от друга на замену координат в следующем смъисле. Пусть $f : M \to N$– изометрия, $(x^1, \ldots, x^n)$ - координаты в окрестности точки P, а $(y^1, \ldots, y^n)$ - координаты в окрестности точки $f(P)$, и $y^i\left(x^1, \ldots, x^n\right)$ - координатное представление изометрии f. Кроме того, пусть g_{ij} - компоненты римановой метрики на M в координатах $(x^1, \ldots, x^n)$, а h_{pq} - компоненты римановой метрики на N в координатах ($y^1, \ldots, y^n$). Тогда

$$g_{ij} = \sum_{\alpha,\beta} h_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j}.$$

Отображение, обратное κ изометрии также является изометрией.

Доказательство. Первая часть утверждения немедленно вытекает из определений изометрии, длины и угла.

Проверим формулу из второй части. Если ∂_{x^i} - векторы канонического базиса в $T_P M$ для координат x^i, то координаты вектора ∂_{x^i} в этом базисе равны δ_i^α, поэтому $df\left(\partial_{x^i}\right)^\alpha = \frac{\partial y^\alpha}{\partial x^i}$. Тогда

$$g_{ij} = \langle \partial_i, \partial_j \rangle_P = \langle df(\partial_i), df(\partial_j) \rangle_{f(P)} = \sum_{\alpha,\beta} h_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j},$$

что и требовалось.
Наконец, обратное отображение к изометрии существует и является диффеоморфизмом, так как изометрия - диффеоморфизм. Остается заметить, что формулы, доказанные только что, обратимы, а именно, в сделанных обозначениях,

$$h_{pq} = \sum_{\alpha,\beta} g_{\alpha\beta} \frac{\partial x^\alpha}{\partial y^p} \frac{\partial x^\beta}{\partial y^q},$$

касательных пространствах $T_P M$, где P - произвольная точка из области определения координат, имеют одну и ту же ориентацию. Если ориентация существует, то многообразие M называется ориентируемым, иначе - неориентируемьм. Если ориентация фиксирована, то многообразие называется ориентированным.

Пример 15.36. Пусть Ω - произвольное открытое подмножество $\mathbb{R}^n$. Рассмотрим в $\mathbb{R}^n$ стандартные евклидовы координаты и соответствующие канонические базисы $(\partial_1, \ldots, \partial_n)$ касательных пространств $T_P\Omega, P \in \Omega$. Ориентируем все $T_P M$ с помощью базиса $(\partial_1, \ldots, \partial_n)$. Тогда, по определению, семейство таких ориентаций будет непрерывно зависеть от точки из Ω, так что Ω - ориентируемое многообразие. В частности, само $\mathbb{R}^n$ является ориентируемым многообразием.

Утверждение 15.37. Пусть $M \subset \mathbb{R}^n$ - гладкое k-мерное многообразие, $1 \leq k < n$, являющееся решением системь

$$\left\{ F_1\left(x^1, \ldots, x^n\right) = 0, \ldots, F_{n-k}\left(x^1, \ldots, x^n\right) = 0 \right\},$$

удовлетворяющей условию теорем о неявной функции: функции F_i- гладкие, и в каждой точке из M матрица Якоби ($\partial F_i / \partial x^j$) имеет максимально возможный ранг $n-k$. Тогда многообразие M ориентируемо.

Доказательство. Напомним, что в каждой точке $P \in M$ градиенты

$$\nabla F_i = \left(\partial F_i / \partial x^1, \ldots, \partial F_i / \partial x^n\right)$$

функций F_i линейно независимы и ортогональны касательному пространству $T_P M$. Зададим в каждом $T_P M$ ориентацию, назвав базис ($f_1, \ldots, f_k$) пространства $T_P M$ положительно ориентированным, если и только если построенный с помощью него базис $(f_1, \ldots, f_k, \nabla F_1, \ldots, \nabla F_{n-k})$ пространства $\mathbb{R}^n$ положительно ориентирован по отношению к стандартной ориентации пространства $\mathbb{R}^n$. Покажем, что так заданное семейство ориентаций касательных пространств $T_P M$ многообразия M непрерывно зависит от точки. Выберем произвольную точку $P \in M$ и в некоторой ее связной окрестности $U \subset M$ рассмотрим локальные координаты ($y^1, \ldots, y^k$) и соответствующие канонические базисы ($\partial_1, \ldots, \partial_k$) касательных пространств $T_P M, P \in U$, тогда векторы базисов

$$B = (\partial_1, \ldots, \partial_k, \nabla F_1, \ldots, \nabla F_{n-k})$$

пространства $\mathbb{R}^n$ непрерывно меняются при изменении точки $P \in U$, так что если $C(P)$ обозначает матрицу перехода от стандартного базиса ($e_1, \ldots, e_n$) пространства $\mathbb{R}^n$ к базису B в точке P, то $\det C(P)$ является нигде не обращающейся в ноль непрерывной функцией, определенной на связном множестве, поэтому эта функция имеет везде один и тот же знак. Но тогда, по определению заданной выше ориентации касательных пространств $T_P M$, все канонические базисы ($\partial_1, \ldots, \partial_k$) имеют одну и ту же

ориентацию. Итак, заданное семейство ориентаций непрерывно зависит от точки, поэтому задает ориентацию многообразия M, так что M - ориентируемо.

Следствие 15.38. Сфера $S^{n-1} \subset \mathbb{R}^n$, ортогональная группа $\mathrm{O}(n) \subset \mathbb{R}^{n^2}$, группа $\mathrm{SL}(n) \subset \mathbb{R}^{n^2}$ матрии, с определителем 1 являются ориентируемьми гладкими многообразиями.

Иногда бывает проще доказать ориентируемость гладкого многообразия, выбрав на нем некоторый специальный атлас. Мы приведем еще одно определение ориентируемости и докажем его эквивалентность предыдущему.

Напомним, что для каждого гладкого n-мерного многообразия M пара (U, φ), состоящая из связного открытого множества $U \subset M$ и гомеоморфизма φ между U и некоторой областью из $\mathbb{R}^n$, называется гладкой картой, если функции перехода между ней и картами атласа многообразия M являются гладкими. Максимальный атлас на M это семейство всех гладких карт на M. Как мы уже уславливались, все рассматриваемые карты на гладком многообразии M являются элементами, а все рассматриваемые атласы - подмножествами из максимального атласа. Более того, говоря про карты и атласы на гладком многообразии, мы будем опускать прилагательное "гладкий".

Определение 15.39. Атлас на гладком многообразии M называется ориентированным, если якобианы всех его функции перехода положительны. Гладкое многообразие, на котором задан ориентированный атлас, называется ориентированным. Если на гладком многообразии существует ориентированный атлас, то многообразие называется ориентируемьм.

В дальнейшем нам понадобятся некоторые технические результаты, к формулировке и доказательству которых мы и переходим.

Лемма 15.40. Пусть (U, φ) - произвольная гладкая карта на ориентированном в смысле определения 15.35 гладком многообразии M. Обозначим через $x = (x^1, \ldots, x^n)$ порожденные картой (U, φ) координаты на $U \subset M$, а через $\partial = (\partial_1, \ldots, \partial_n)$ - канонические базисы, соответствующие этим координатам. Тогда для любых точек PuQ из U ориентации канонических базисов ∂ в $T_P M$ и $T_Q M$ одинаковы.

Доказательство. Так как U - связное открытое подмножество многообразия, оно также является линейно связным, так что точки P и Q соединяются некоторой непрерывной кривой $\gamma(t)$, лежащей в U. Определим функцию $n(t)$ параметра t, положим ее равной 1 , если базис ∂ в точке $\gamma(t)$ имеет положительную ориентацию, и -1 в противном случае. Так как выбранное семейство ориентаций непрерывно зависит от точки, для каждого t_0 существует связная окрестность $V \subset U$ точки $\gamma(t_0)$ и такие локальные координаты $y = (y^1, \ldots, y^n)$ в V, что ориентации всех канонических базисов $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, соответствующих координатам y, одинаковы. Обозначим через C матрицу перехода от базиса ε к базису ∂, тогда $\det C$ всюду в V отличен от нуля. В силу связности V, функция $\det C$ имеет везде в V один и тот же знак, так что и все базисы ∂ имеют одну и ту же ориентация в V, поэтому для всех t таких, что $\gamma(t) \in V$, функция $n(t)$ постоянна. Так как кривая γ непрерывна, последнее имеет место в некоторой окрестности точки t_0. Итак, функция $n(t)$ является локально постоянной, поэтому $n(t)$ - постоянна на всей области определения и, значит, ориентации канонических базисов ∂ в P и Q одинаковы. $\square$

Определение 15.41. Пусть M - ориентированное в смысле определения 15.35 гладкое многообразие и (U, φ) - некоторая карта на нем. Карту (U, φ) назовем положительной, если определенные ей канонические базисы во всех точках из U положительно ориентированы. Если же все такие базисы отрицательно ориентированы, то карту (U, φ) назовем отрицательной.

Из леммы 15.40 вытекает следующий результат.

Следствие 15.42. У ориентированного в смысле определения 15.35 гладкого многообразия каждая карта из максимального атласа или положительная, или отрицательная.

Лемма 15.43. Пусть (U, φ) и (U', φ') - две карты на гладком многообразии M, для которых $U \cap U' \neq \emptyset$, а $x = (x^1, \ldots, x^n)$ и $y = (y^1, \ldots, y^n)$ - координаты, порожденные этими картами. Выберем произвольную точку $P \in U \cap U'$ и обозначим через $\partial_x = (\partial_{x^1}, \ldots, \partial_{x^n})$ и $\partial_y = (\partial_{y^1}, \ldots, \partial_{y^n})$ канонические базисы в $T_P M$, соответствующие этим локальным координатам. Тогда якобиан функции перехода $\varphi' \circ \varphi^{-1}$ в точке $\varphi(P)$ положителен тогда и только тогда, когда базисы $\partial_x u \partial_y$ одинаково ориентированы.

Доказательство. Компоненты касательного вектора ∂_{x^i} в базисе $\left\{\partial_{y^j}\right\}$ имеют вид $(\partial y^1/\partial x^i, \ldots, \partial y^n/\partial x^i)$, так что матрица Якоби функции перехода от координат x к координатам y представляет собой матрицу перехода от базиса ∂_x к базису ∂_y. □

Конструкция 15.44. Пусть $x = (x^1, \ldots, x^n)$ - координаты, заданные картой $(\ U, \varphi\ )$. Рассмотрим новые координаты $y = (y^1, \ldots, y^n)$ в U, положив $y^1 = -x^1$ и $y^i = x^i$ при всех $i > 1$. Тем самым, мы получим новую карту $(\ U, \varphi'\ )$.

Лемма 15.45. В сделанных обозначениях, канонические базисы, соответствующие координатам x и y имеют противоположные ориентации.

Будем говорить, что карта $(\ U, \varphi'\ )$ получается из $(\ U, \varphi\ )$ заменой ориентации. Из леммы [5.4] вытекает следующий результат.

Следствие 15.46. Пусть $\mathcal{A}$ - произвольный атлас на гладком ориентируемом в смысле определения 15.35 многообразии M, тогда, изменив ориентации всех его отрицательных карт, мы получим ориентированный атлас $\mathcal{A}'$ в смысле определения 15.39.

Доказательство. Так как в атласе $\mathcal{A}'$ все карты положительны, то, по лемме 15.43, все якобианы функций перехода также положительны, поэтому $\mathcal{A}'$ - ориентированный атлас. □

Утверждение 15.47. Определения 15.35 и 15.39 эквивалентны, т.е. гладкое многообразие, ориентируемое в смысле одного из этих определений, является также ориентируемым и в смысле другого.

Доказательство. Предположим сначала, что гладкое многообразие M является ориентируемым в смысле определения 15.35, т.е. M допускает такой выбор ориентаций всех его касательных пространств, при котором эти ориентации непрерывно зависят от точки многообразия. Тогда, по следствию 15.46, на M существует ориентированный атлас, так что M ориентируемо и в смысле определения 5.39.

Обратно, пусть M ориентируемо в смысле определения 15.39, т.е. на нем существует ориентированный атлас $\mathcal{A}$. Выберем произвольную точку $P \in M$, тогда, по лемме 15.43, канонические базисы локальных координат, порожденных всеми содержащими P картами из атласа $\mathcal{A}$, имеют одну и ту же ориентацию. Эту ориентацию пространства $T_P M$ выберем в качестве положительной. Тогда все канонические базисы,

порожденные каждой картой из $\mathcal{A}$, будут иметь положительную ориентацию, в частности, выбранное семейство ориентаций касательных пространств $T_P M$ непрерывно зависит от точки, так что многообразие M ориентируемо в смысле определения 15.35. Утверждение доказано. □

Следующий результат оказывается полезным при доказательстве неориентируемости.

Следствие 15.48. Пусть M - произвольное гладкое многообразие $u\mathcal{A}$ - произвольный его атлас. Тогда M ориентируемо, если и только если атлас $\mathcal{A}$ можно ориентировать заменами ориентации в некоторых его картах.

Доказательство. Пусть многообразие M ориентируемо, тогда возможность ориентировать атлас $\mathcal{A}$ заменами ориентаций некоторых его карт фактически составляет результат следствия [5.46. Обратно, если нам удалось ориентировать атлас $\mathcal{A}$, то многообразие M ориентируемо по определению 15.39. □

Пример 15.49. Покажем, как работаем следствие 15.48. Рассмотрим лист Мёбиуса M, склеенный из квадрата $\{(x, y) : |x| \leq 1, |y| < 1\}$ посредством отождествления каждой пары точек $(-1, -y)$ и $(1, y)$. Введем на M атлас из двух карт $(U_i, \varphi_i)\,, i = 1, 2$, положив

$U_1 = \{(x,y) \in M : |x| < 2/3\}, \varphi_1 = \mathrm{id}, U_2 = \{(x,y) \in M : |x| > 1/3\}$, $\varphi_2 = \mathrm{id}$ при $x > 0$ и $\varphi_2(x,y) = (x+2,-y)$ при $x < 0$. Тогда $U_1 \cap U_2$ состоит из двух полосок $V_1 = \{(x,y) \in M : -2/3 < x < -1/3\}$ и $V_2 = \{(x,y) \in M : 1/3 < x < 2/3\}$. Рассмотрим функции перехода $\varphi_{21} = \varphi_2 \circ \varphi_1^{-1}$. Тогда $\varphi_{21}(x,y) = (x,y)$ на $\varphi_1(V_2)$ и $\varphi_{21}(x,y) = (x+2,-y)$ на $\varphi_1(V_1)$. Но тогда на $\varphi_1(V_2)$ якобианы этой замены положительны, а на $\varphi_1(V_1)$ - отрицательны, так что построенный атлас ориентированным не является. Ясно, что делая замены ориентации в этих двух картах, мы получим еще три атласа. Отметим, что замены ориентации, выполненные одновременно во всех картах, не меняют свойства атласа быть ориентированным или нет. Таким образом, достаточно рассмотреть еще ровно один атлас, получающийся заменой ориентации в одной из построенных карт. Заменим ориентацию в карте (U_2, φ_2), в результате получим карту (U_2, φ_2'), для которой $\varphi_2'(x,y) = (-x,y)$ при $x > 0$ и $\varphi_2'(x,y) = (-x-2,-y)$ при $x > 0$. Тогда функция перехода $\varphi_{21}' = \varphi_2' \circ \varphi_1^{-1}$ действует так: $\varphi_{21}'(x,y) = (-x,y)$ на $\varphi_1(V_2)$ и $\varphi_{21}(x,y) = (-x-2,-y)$ на $\varphi_1(V_1)$. Поэтому теперь на $\varphi_1(V_2)$ якобиан функции перехода отрицателен, а на $\varphi_1(V_1)$ - положителен, так что построенный атлас снова не является ориентированным. Применяя следствие 15.48, заключаем, что лист Мёбиуса неориентируем.

Наконец, чтобы дать третье определение ориентируемости, нам понадобится сначала определить непрерывное семейство базисов вдоль непрерывной кривой на гладком многообразии.

Определение 15.50. Пусть $\gamma(t), t \in [a,b]$, - непрерывная кривая на гладком многообразии M. При каждом t выберем в $T_{\gamma(t)}M$ некоторый базис $E(t) = (e_1(t), \ldots, e_n(t))$. Будем говорить, что семейство базисов $E(t)$ непрерывно вдоль γ, если при каждом t_0 существуют некоторые координаты в окрестности точки $\gamma(t_0)$, в которых компоненты $e_i(t)^\alpha$ векторов $e_i(t)$ являются непрерывными функциями параметра t в некоторой окрестности точки t_0.

Лемма 15.51. Для каждой непрерывной кривой $\gamma(t), t \in [a,b]$, на гладком многообразии M существует семейство базисов $E(t)$ касательных пространств $T_{\gamma(t)}M$, непрерывно зависящее от t.

Доказательство. Пусть $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}$ - произвольный атлас на M, тогда каждая точка $\gamma(t)$ кривой γ лежит в некотором U_α. В силу непрерывности кривой, для каждого $t_0 \in [a,b]$ существует такая его связная окрестность $V(t_0) \subset [a,b]$, что точки $\gamma(t)$ при всех $t \in V(t_0)$ содержатся в одном и том же U_α. Семейство $\{V(t_0)\}_{t_0 \in [a,b]}$ является открытым покрытием отрезка $[a,b]$, поэтому в нем существует конечное подпокрытие $\mathcal{V} = \{V_1, \ldots, V_k\}$ (в силу компактности отрезка $[a,b]$). Выкинем из этого конечного подпокрытия все V_p, содержащиеся в других V_q этого же подпокрытия. Без ограничения общности, будем считать, что мы сразу выбрали такое подпокрытие $\mathcal{V}$. Раз так, то у любых двух различных интервалов V_p и V_q не могут совпадать как левые концы, так и правые; более того, если левый конец интервала V_p меньше левого конца интервала V_q, то и правый конец V_p меньше правого конца V_q. Упорядочим элементы покрытия $\mathcal{V}$ по возрастанию их левых концов. Без ограничения общности, сразу будем считать, что элементы V_p расположены именно в таком порядке в списке $\{V_1, \ldots, V_k\}$. Тогда каждое множество $W_p = V_p \cap V_{p+1}$ - непустой интервал (иначе $\mathcal{V}$ не покрывает отрезок). Отметим, что левый конец w_p интервала W_p совпадает с левым концом интервала V_{p+1}, поэтому w_p также монотонно возрастают с ростом p. Легко видеть, что для достаточно малого $\varepsilon > 0$ каждая точка $t_p = w_p + \varepsilon$ лежит в W_p. Кроме того, t_p также монотонно возрастают с ростом p.

Будем теперь строить семейство $E(t)$. Для этого обозначим через $\partial_p = \left(\partial_{x_p^1}, \ldots, \partial_{x_p^n}\right)$ канонические базисы, порожденные картой (V_p, φ_p). В каждой точке $t \in [a, t_1]$ положим $E(t)$ равным базису ∂_1 касательного пространства $T_{\gamma(t)}M$. Далее, в касательном пространстве $T_{\gamma(t_1)}M$ имеется два базиса: ∂_1 и ∂_2. Пусть C_1 - матрица перехода от первого из этих базисов ко второму, т.е. $\partial_1 = \partial_2 C_1$. В каждой точке $t \in [t_1, t_2]$ положим

$E(t)$ равным "подправленному" базису ∂_2 касательного пространства $T_{\gamma(t)}M$, а именно, базису $\partial_2 C_1$. В силу выбора матрицы C_1, в точке t_1 выбранный базис совпадает с уже построенным на предыдущем шаге. В касательном пространстве $T_{\gamma(t_1)}M$ имеется два базиса: ∂_2 и ∂_3. Пусть C_2 - матрица перехода от первого из этих базисов ко второму, т.е. $\partial_2 = \partial_3 C_2$. В каждой точке $t \in [t_2, t_3]$ положим $E(t)$ равным "подправленному" базису ∂_3 касательного пространства $T_{\gamma(t)}M$, а именно, базису $\partial_3 C_2 C_1$. В силу выбора матриц C_1 и C_2, в точке t_2 выбранный базис совпадает с уже построенным на предыдущем шаге. Продолжая этот процесс, мы за конечное число шагов построим семейство $E(t)$, компоненты каждого вектора которого постоянны в каждой карте V_i. $\square$

Конструкция 15.52. Пусть P и Q - две произвольные точки гладкого многообразия M, соединенные непрерывной кривой $\gamma(t), t \in [0,1], P = \gamma(a), Q = \gamma(b)$. Ориентируем касательное пространство $T_P M$. Рассмотрим непрерывное вдоль γ семейство $E(t)$ базисов касательных пространств $T_{\gamma(t)}M$. Ориентируем пространство $T_Q M$ так, чтобы ориентации базисов $E(a)$ и $E(b)$ были одинаковы.

Лемма 15.53. Полученная в $T_Q M$ ориентация не зависит от выбора семейства $E(t)$.
Доказательство. Пусть $E'(t)$ - другое непрерывное вдоль γ семейство базисов касательных пространств $T_{\gamma(t)}M$. Тогда при каждом t определена матрица $C(t)$ перехода

от $E(t)$ к $E'(t)$, поэтому функция $\det C(t)$ всюду отлична от нуля. С другой стороны, в силу непрерывности семейств $E(t)$ и $E'(t)$, эта функция также непрерывна и, значит, она или всюду положительна, или всюду отрицательна, так что ориентация базисов $E(a)$ и $E'(a)$ одинакова тогда и только тогда, когда одинакова ориентация базисов $E(b)$ и $E'(b)$. Последнее доказывает, что ориентация пространства $T_Q M$, полученная с помощью семейства $E'(t)$, такая же, как и полученная с помощью $E(t)$. $\square$

Замечание 15.54. Вообще говоря, полученная в $T_Q M$ ориентация может зависеть от выбора кривой γ.

Лемма 15.53 обосновывает корректность следующего определения.
Определение 15.55. Во введенных выше обозначениях, будем говорить, что полученная на $T_Q M$ ориентация получена в результате переноса ориентации пространства $T_P M$ вдоль кривой γ.

Определение 15.56. Гладкое многообразие M назовем ориентируемым, если перенос ориентации вдоль непрерывных кривых, соединяющих одну и ту же пару точек из M, не зависит от выбора таких кривых.

Утверждение 15.57. Определение 15.56 эквивалентно двум предыдущим определениям ориентируемости.

Доказательство. Пусть гладкое многообразие M является ориентированным в смысле определения 15.56. Тогда выберем в каждой компоненте связности M_i многообразия M по одной точке P_i, зафиксируем некоторую ориентацию для каждого пространства $T_{P_i}M$ и соединим точку P_i непрерывными кривыми со всеми остальными точками компоненты M_i. По лемме 15.5], вдоль каждой из построенных кривых мы можем перенести выбранную ориентацию пространства $T_{P_i}M$ в концевую точку этой кривой, что мы и сделаем. Тем самым, мы ориентировали все касательные пространства к многообразию M. Покажем, что построенное семейство ориентаций непрерывно зависит от точки.

Пусть $P \in M_i$ - произвольная точка и (U, φ) - некоторая карта, содержащая P. Обозначим через $\partial = (\partial_1, \ldots, \partial_n)$ заданное картой (U, φ) семейство канонических базисов касательных пространств в точках области U. Для каждой точки $Q \in U$ обозначим через γ_Q некоторую непрерывную кривую в U, соединяющую P и Q, а через E_Q семейство базисов пространств $T_{\gamma_Q(t)}$, представляющее собой ограничение семейства ∂ на кривую γ_Q. Тогда E_Q - непрерывное вдоль γ_Q семейство базисов, причем ориентации, полученные перенесением из P во все точки Q, непрерывны в P, так как все базисы ∂ имеют одну и ту же ориентацию. В силу независимости переноса ориентации от

выбранной кривой, ориентацию пространства $T_{P_i}M$ можно перенести в точки Q так: сначала перенести эту ориентацию из P_i в P вдоль некоторой непрерывной кривой γ, а затем перенести полученную в T_PM ориентацию в точки Q с помощью кривых γ_Q. В результате получится тот же самый перенос ориентации из P_i, что и перенос ориентации из P, полученный после переноса в P ориентации из P_i вдоль кривой γ. Таким образом, мы доказали непрерывность построенного семейства ориентаций в точке P и, в силу произвольности точки P, - на всем M.

Обратно, пусть многообразие M ориентируемо в смысле первых двух определений. Выберем на M семейство ориентаций, непрерывно зависящих от точки. Пусть $\gamma(t)$ произвольная непрерывная кривая на M, соединяющая точки $P = \gamma(a)$ и $Q = \gamma(b)$. Выберем в T_PM произвольную ориентацию и перенесем ее в T_QM с помощью непрерывного вдоль γ семейства базисов $E(t)$. Из непрерывности семейств $E(t)$ и непрерывной зависимости ориентации от точки вытекает, что у каждого $t_0 \in [a, b]$ существует такая окрестность, что для любого t из этой окрестности все $E(t)$ имеют одну и ту же ориентацию. Отсюда вытекает, что базис $E(b)$ имеет ту же ориентацию, что и базис $E(a)$ (доказательство такое же, как и в лемме 15.40). Таким образом, при переносе ориентации из T_PM в T_QM вдоль любой непрерывной кривой эта ориентация сохраняется, так что перенос ориентации между любыми точками из одной компоненты связности не зависит от выбора кривых, соединяющих эти точки. □

Упражнение 15.58. Докажите, что третье определение ориентируемости гладкого многообразия M эквивалентно следующему: для любой точки $P \in M$ и любой непрерывной петли γ, выходящей из P, т.е. непрерывной кривой $\gamma(t), t \in [a, b]$, для которой $\gamma(a) = \gamma(b) = P$, ориентация касательного пространства T_PM остается неизменной при переносе вдоль γ.

Приведем несколько примеров.

Упражнение 15.59. Показать, что атлас стереографической проекции на сфере является ориентированным. Тем самым, атлас стереографической проекции является ориентируемым, и ориентируемость стандартной сферы доказана с помощью определения 15.39.

Упражнение 15.60. Показать, что если атлас многообразия состоит из одной карты, то многообразие ориентируемо. Если атлас многообразия состоит из двух карт, пересечение которых связно, то многообразие ориентируемо.

Рис. 15.1: Склейка листа Мебиуса из прямоугольника.

Пример 15.61. Простейший пример неориентируемого многообразия - это лист Мебиуса. Напомним, что лист Мебиуса получается склейкой двух противоположных сторон $a \times (0, 1)$ и $b \times (0, 1)$ прямоугольника $[a, b] \times (0, 1)$ по эквивалентности

$(a, y) \sim (b, 1-y)$,
$y \in (0,1)$. На рис. 15.1 эта склейка условно изображена стрелками. Стороны, помеченные стрелками, следует склеивать так, чтобы направления стрелок совмещались при склеивании.

Покажем, что лист Мебиуса неориентируем с помощью определения 15.56. Для этого рассмотрим центральную окружность $[a, b] \times 1/2$, зададим в ее точке $P = (x, 1/2)$ базис (e_1, e_2), где e_1 направлен по касательной к окружности, а e_2 - ортогонален e_1 (рис. 15.2). При переносе ориентации из точки P по окружности в ту же точку P, мы получим базис ($e_1, -e_2$), что и требовалось доказать.

Рис. 15.2: Перенос базиса по центральной окружности листа Мебиуса.

14.17.3 15.4 Problems

Задача 15.62. Пусть M - связное риманово многообразие. Зададим на нем функцию расстояния ρ, положив $\rho(P, Q) = \inf\{\ell(\gamma)\}$, где точная нижняя грань берется по всевозможным кусочно-гладким кривым, соединяющим P и Q.

Показать, что пара (M, ρ) - метрическое пространство. Проверить, что метрическая топология, порожденная M, эквивалентна топологии многообразия.

Задача 15.63. Пусть $i : M \to W$ - вложение многообразия M в риманово многообразие W и $\gamma : [a, b] \to M$ - гладкая кривая на M. Показать, что длина кривой γ, вычисленная в индуцированной вложением i метрике на M, совпадает с длиной кривой $i \circ \gamma$ на W (в римановой метрике на W).

Задача 15.64. Докажите, что катеноид ($\operatorname{ch} u \cos v, \operatorname{ch} u \sin v, u$) локально изометричен геликоиду ($\operatorname{sh} u \sin v, -\operatorname{sh} u \cos v, v$).

Задача 15.65. Докажите, что: 1) декартово произведение ориентируемых многообразий ориентируемо; 2) многообразие размерности n, содержащее неориентируемое подмногообразие размерности n, неориентируемо; 3) регулярная поверхность в $\mathbb{R}^n$, заданная системой неявных функций, - ориентируемое многообразие; 4) сфера S^n, тор $T^n = S^1 \times \cdots \times S^1$, группы $\mathrm{GL}(n), \mathrm{O}(n), \mathrm{SO}(n), \mathrm{SL}(n)$ ориентируемы, а бутылка Клейна K^2 и проективная плоскость $\mathbb{RP}^2$ - нет; 5) проективное пространство $\mathbb{RP}P^n$ ориентируемо тогда и только тогда, когда n - нечетно.

Задача 15.66. Показать, что касательное расслоение к произвольному многообразию ориентируемо.

14.17.4 15.5 Supplemental material

14.17.5 15.5.1 Examples: Klein bottle

Бутылкой Клейна, напомним, называется многообразие, полученное склейкой противоположных сторон квадрата, изображенной схематически на рис. 15.3. Две противоположных стороны, помеченные буквой a, склеиваются без переворачивания, а две другие, помеченные буквой b, - с переворачиванием.

Рис. 15.3: Склейка бутылки Клейна из квадрата (слева вверху) и из двух листов Мебиуса.

Заметим, что бутылка Клейна может быть получена склейкой двух листов Мебиуса. Действительно, разрезав квадрат по двум параллельным отрезкам, соединяющим середины сторон b с соответствующими вершинами квадрата (см. рис. 15.3) и склеив затем по стороне a, получим два листа Мебиуса. В частности, отсюда следует, что бутылка Клейна - это неориентируемое многообразие.

14.17.6 15.5.2 Examples: projective space $\mathbb{RP}^3$

Проверим ориентируемость проективного пространства $\mathbb{R}\mathrm{P}^3$. Напомним, что на этом многообразии имеется стандартный атлас из четырех карт, который в терминах однородных координат ($x^1 : x^2 : x^3 : x^4$) задается в виде

$$\begin{aligned}
&U_1\left(x^1 \neq 0\right), \quad \left(x^1 : x^2 : x^3 : x^4\right) \mapsto \left(\tfrac{x^2}{x^1}, \tfrac{x^3}{x^1}, \tfrac{x^4}{x^1}\right), \\
&U_2\left(x^2 \neq 0\right), \quad \left(x^1 : x^2 : x^3 : x^4\right) \mapsto \left(\tfrac{x^1}{x^2}, \tfrac{x^3}{x^2}, \tfrac{x^4}{x^2}\right), \\
&U_3\left(x^3 \neq 0\right), \quad \left(x^1 : x^2 : x^3 : x^4\right) \mapsto \left(\tfrac{x^1}{x^3}, \tfrac{x^2}{x^3}, \tfrac{x^4}{x^3}\right), \\
&U_4\left(x^4 \neq 0\right), \quad \left(x^1 : x^2 : x^3 : x^4\right) \mapsto \left(\tfrac{x^1}{x^4}, \tfrac{x^2}{x^4}, \tfrac{x^3}{x^4}\right).
\end{aligned}$$

Однако, как легко проверить, этот атлас не является ориентированным (скажем, определитель матрицы перехода от U_1 к U_4 всюду отрицателен). Поэтому, чтобы превратить атлас в ориентированный, следует переменить порядок координат в некоторых картах. Например, это можно сделать так:

$$\begin{aligned}
&U_1\left(x^1 \neq 0\right), \quad \left(x^1 : x^2 : x^3 : x^4\right) \mapsto \left(\tfrac{x^2}{x^1}, \tfrac{x^3}{x^1}, \tfrac{x^4}{x^1}\right), \\
&U_2\left(x^2 \neq 0\right), \quad \left(x^1 : x^2 : x^3 : x^4\right) \mapsto \left(\tfrac{x^4}{x^2}, \tfrac{x^3}{x^2}, \tfrac{x^1}{x^2}\right), \\
&U_3\left(x^3 \neq 0\right), \quad \left(x^1 : x^2 : x^3 : x^4\right) \mapsto \left(\tfrac{x^1}{x^3}, \tfrac{x^2}{x^3}, \tfrac{x^4}{x^3}\right), \\
&U_4\left(x^4 \neq 0\right), \quad \left(x^1 : x^2 : x^3 : x^4\right) \mapsto \left(\tfrac{x^2}{x^4}, \tfrac{x^1}{x^4}, \tfrac{x^3}{x^4}\right)
\end{aligned}$$

(мы переставили пару координат во второй и четвертой картах). Теперь все матрицы перехода имеют положительный якобиан. Например, обозначив координаты в карте U_1 через (a^1, a^2, a^3), запишем функции перехода к карте U_4 в виде

$$\left(a^1, a^2, a^3\right) \mapsto \left(\frac{a^1}{a^3}, \frac{1}{a^3}, \frac{a^2}{a^3}\right)$$

Тогда определитель матрицы Якоби равен $1/\left(a^3\right)^2 > 0$.

14.18 16 Classification of connected compact two-dimensional manifolds

Полное доказательство теоремы классификации (с точностью до гомеоморфизма) связных компактных двумерных многообразий [®] мы приводить не будем и ограничимся демонстрацией основных идей. [2] Отметим, что одномерных компактных многообразий совсем мало: все они гомеоморфны окружности. В двумерном случае ситуация существенно богаче и поддается классификации. Для более высоких размерностей полное эффективное описание неизвестно.

14.18.1 16.1 Polygon gluing

Начнем с уже известных нам примеров двумерных многообразий.
Пример 16.1. Напомним, что выше мы получили бутылку Клейна из квадрата, отождествив его стороны по определенному правилу. Аналогично можно получить три других многообразия, а именно, сферу, тор и проективную плоскость, см. рис. 16.1).

Рис. 16.1: Склейки квадрата.

Попробуем взглянуть на эти четыре примера с единой точки зрения.
Конструкция 16.2. Пусть P - произвольный плоский многоугольник с $2k$ сторонами. Разобьем множество его сторон на k непересекающихся пар и пометим стороны, попавшие в одну и ту же пару, одинаковыми буквами. В итоге, у нас будет k различных букв $a_1, \ldots, a_k$, расставленных на $2k$ сторонах. Превратим многоугольник P в двумерное (топологическое) многообразие, отождествив стороны, помеченные одинаковыми буквами. Напомним, однако, что существуют два различных гомеоморфизма отрезка на

[01] Обычно в формулировку этой теоремы включают условие замкнутости, т.е. отсутствия края. Однако мы пока не рассматривали многообразий с краем, поэтому все компактные многообразия замкнуты автоматически.
[2] Наш подход отличается от "метода приклейки ленточек," использованного в курсе наглядной геометрии и топологии, см. [10]

себя. Чтобы различать эти два разных способа отождествления сторон, поступим так. Фиксируем направление обхода многоугольника по периметру, например по часовой стрелке. Начиная с произвольной стороны, будем последовательно перебирать стороны многоугольника, расставляя на них стрелки (или, если угодно, ± 1) по следующему правилу. Если сторона, помеченная буквой a_i, встречается первый раз, то ставим на ней стрелку, сонаправленную с выбранным направлением обхода многоугольника (припишем $+1$). Если же сторона встречается второй раз, то имеются две возможности:

- стороны склеиваются "без переворота", т.е. концевая, в смысле направления обхода, точка первой встретившейся стороны склеивается с первой точкой второй встретившейся стороны (как противоположные стороны квадрата при склейке его в тор);
- стороны склеиваются "с переворотом", т.е. концевая точка первой встретившейся стороны склеивается с концевой точкой второй встретившейся стороны (как противоположные стороны квадрата при склейке проективной плоскости).

В первом случае пометим сторону стрелкой, направленной против движения вокруг многоугольника (припишем -1), а во втором - стрелкой, сонаправленной с движением вокруг многоугольника (припишем $+1$). Отметим, что после такой разметки при склеивании стрелок их направления совмещаются. В результате получаем разметку сторон нашего многоугольника буквами и стрелками (или ± 1). Ясно, что эта разметка однозначно задает склейку двумерного многообразия.

Каждой такой разметке мы сопоставим некоторое слово, выписав последовательно буквы, стоящие на сторонах нашего многоугольника, и приписав каждой букве степень -1 , если соответствующая сторона помечена -1 .

Определение 16.3. Построенное слово W будем называть словом склейки, а о полученном в результате склейки многоугольника P в соответствии со словом W компактном двумерном многообразии M будем говорить, что оно склеено из P по слову W. Будем обозначать это многообразие $P(W)$.

Упражнение 16.4. Докажите, что полученное в результате описанной склейки многоугольника (т.е. факторизации по соответствующему отношению эквивалентности) топологическое пространство (с фактор топологией) действительно является топологическим многообразием. Для этого укажите гомеоморфные кругу окрестности всех его точек.

Конструкция 16.5. Если задано произвольное слово W, составленное из k пар различных букв, каждая из которых может встречаться в степени ± 1, то по слову W можно восстановить двумерное компактное многообразие, склеенное из некоторого $2k$-угольника P. А именно, возьмем произвольный $2k$-угольник P и, начав с произвольной его стороны, припишем последовательным сторонам многоугольника P последовательные буквы слова W в соответствующих степенях ± 1. Склеим стороны, помеченные одинаковыми буквами. При этом если обе стороны помечены буквой в одинаковой степени, то выполним склейку "с переворотом", а если степени разные то "без переворота".

Следующее утверждение непосредственно вытекает из построений.

Statement

16.6. Пусть фиксировано произвольное разбиение множества сторон $2k$-угольника P на непересекающиеся пары u задано некоторое отождествление этих пар сторон. Тогда эта склейка многоугольника однозначно определяется некоторым словом W.

Обратно: каждое слово W, состоящее из k пар различных букв, каждая из которых может встречаться в степени ± 1, задает некоторую склейку многоугольника P в двумерное компактное многообразие.

Пример 16.7. Пусть Q - это уже знакомый нам квадрат. Тогда, как видно из рис. 16.1, $Q\left(abb^{-1}a^{-1}\right)$ - это сфера, $Q\left(bab^{-1}a^{-1}\right)$ - тор, $Q\left(baba^{-1}\right)$ - бутылка Клейна, $Q\left(ab^{-1}ab^{-1}\right)$

- проективная плоскость.

Замечание 16.8. Заметим, что разным словам могут отвечать одни и те же многообразия. Например, $Q(abab)$ - тоже проективная плоскость, а $P\left(aa^{-1}\right)$ - сфера.

14.18.2 16.2 Sphere sealing

Рассмотрим еще один способ получения двумерных компактных многообразий.
Конструкция 16.9. Возьмем стандартную двумерную сферу и вырежем из нее два открытых непересекающихся диска. Полученное топологическое пространство - сфера с двумя дырками - гомеоморфно цилиндру $S^1 \times [0,1]$, а его граница - объединению

двух окружностей. Рассмотрим еще один экземпляр цилиндра $[0,1] \times S^1$. Склеим цилиндр и сферу с двумя дырками, отождествив их граничные окружности.

Вся ручка снаружи

Половина ручки внутри, половина ручки снаружи

Рис. 16.2: Разные способы приклейки цилиндра-ручки к сфере с дырками.

Заметим, что имеется, вообще говоря, четыре способа приклеить цилиндр к сфере с дырками (рис. 16.2). При этом если цилиндр целиком лежит снаружи или целиком внутри исходной сферы, то получается компактное двумерное многообразие, гомеоморфное тору (фактически мы склеили два цилиндра), а в двух других случаях гомеоморфное бутылке Клейна.

Определение 16.10. Операцию приклейки цилиндра по двум граничным окружностям назовем приклейкой ручки. Если цилиндр-ручка при этом целиком лежит по одну сторону от поверхности, то будем называть такую ручку ориентированной. В противном случае будем говорить о неориентируемой ручке.

Следующая лемма вытекает непосредственно из определений.

Лемма 16.11. Двумерный тор гомеоморфен сфере с одной ориентированной ручкой. Бутылка Клейна гомеоморфна сфере с одной неориентируемой ручкой.

Конструкция 16.12. Рассмотрим проективную плоскость, склеенную из квадрата: $\mathbb{R}\mathrm{P}^2 = Q(abab)$, и вырежем из нее диск, центр которого, для удобства, поместим на сторону квадрата (рис. 16.3). То, что осталось, гомеоморфно листу Мебиуса. Таким образом, проективная плоскость может быть получена так: нужно вырезать из сферы диск и заклеить образовавшуюся дырку листом Мебиуса, отождествив граничные окружности.

Определение 16.13. Операция приклейки листа Мебиуса по граничной окружности называется приклейкой пленки Мебиуса.

Лемма 16.14. Проективная плоскость гомеоморфна сфере с одной приклеенной пленкой Мебиуса.

Рис. 16.3: Проективная плоскость - склейка диска и листа Мебиуса.

Упражнение 16.15. С помощью рис. 15.3 показать, что бутылка Клейна гомеоморфна сфере, заклеенной двумя пленками Мебиуса. Сравнить с доказательством леммы 16.11, нарисовать приклеиваемую в этой лемме ручку на квадрате. Таким образом, приклеивание неориентируемой ручки эквивалентно приклеиванию двух пленок Мебиуса.

Конструкция 16.16. Мы определили две операции: приклейка ориентированной ручки и приклейка пленки Мебиуса, и у нас естественно возникают две серии двумерных компактных многообразий:

- многообразие $M_g^2, g \geq 0$, получающееся из сферы с $2g$ дырками приклеиванием g штук ориентированных ручек и называемое сферой с g ручками;
- многообразие $N_\mu^2, \mu \geq 1$, получающееся из сферы с μ дырками приклейкой μ пленок Мебиуса.

Замечание 16.17. Очевидно, что любое двумерное многообразие, в которое вклеена пленка Мебиуса, неориентируемо, так как, перенеся репер вдоль центральной окружности листа Мебиуса, мы получим репер противоположной ориентации. Напротив, многообразие, полученное приклейкой к сфере любого числа ориентированных ручек, разбивает объемлющее пространство на две части и поэтому ориентируемо.

Упражнение 16.18. Показать, что сфера, к которой приклеено k ручек и m пленок Мебиуса, гомеоморфна сфере с $2k + m$ пленками Мебиуса.

Упражнение 16.19. Докажите, что для любых g и μ многообразие M_g^2 может быть вложено в $\mathbb{R}^3$, а многообразие M_μ^2 - погружено в $\mathbb{R}^3$.

14.18.3 16.3 Classification theorem

Основным результатом данного раздела является следующая классификационная теорема.

Theorem

16.20. Любое гладкое связное компактное (замкнутое) двумерное многообразие гомеоморфно или сфере с g ручками, $g \geq 0$, или сфере с μ пленками Мебиуса, $\mu \geq 1$. Сферы с ручками (при разных g) и сферы с пленками Мебиуса (при разных μ) попарно негомеоморфны.

Мы не будем приводить полное доказательство этой теоремы, а лишь поясним основные идеи.

14.18.4 16.3.1 Triangulations

Обсуждаемое в данном разделе понятие триангуляции является мощным инструментом современной топологии, позволяющим сводить многие топологические задачи к комбинаторным. Фактически идея состоит в том, что многообразие можно заменить гомеоморфным многогранником с треугольными гранями.

Определение 16.21. Пусть M - связное компактное двумерное многообразие. Будем говорить, что на M задана некоторая триангуляция, если на M задан конечный набор отрезков гладких вложенных кривых $\gamma_i : [0,1] \to M$ - ребер триангуляции, пересекающихся между собой только по концевым точкам $\gamma_i(0), \gamma_i(1)$ - вершинам триангуляции, и при этом выполнены следующие условия:

- концы каждого ребра триангуляции различны, т.е. $\gamma_i(0) \neq \gamma_i(1)$ (отсутствие петель);
- множество $M \setminus (\cup \gamma_i([0,1]))$, полученное в результате выбрасывания из многообразия M всех ребер триангуляции, представляет собой несвязное объединение открытых дисков (замыкания которых называются треугольниками триангуляu, uu), причем граница каждого такого диска состоит из трех ребер триангуляции;
- любые два треугольника триангуляции или не пересекаются, или их пересечение совпадает с некоторой их общей вершиной или с некоторым общим ребром триангуляции.

Упражнение 16.22. Постройте триангуляции сферы, тора, бутылки Клейна и проективной плоскости. Из какого минимального числа треугольников могут состоять триангуляции этих поверхностей?

Имеет место следующий результат, который приводится без доказательства.

Statement

16.23. Любое двумерное гладкое компактное связное (замкнутое) многообразие допускает триангуляцию.

Конструкция 16.24. Пусть M - гладкое связное компактное (замкнутое) двумерное многообразие и пусть задана некоторая его триангуляция, существующая в силу утверждения [6.23. Ориентируем все ребра триангуляции, т.е. расставим на них стрелки, и пометим ребра триангуляции буквами, приписывая разным ребрам различные буквы. Разрежем многообразие M по всем ребрам триангуляции. Получим конечный набор треугольников - замкнутых топологических дисков, все стороны которых помечены буквами и стрелками, причем каждая буква встречается ровно два раза (при разрезании вдоль стороны триангуляции мы сохраняем стрелку и букву на обоих берегах разреза).

Будем теперь склеивать треугольники триангуляции, на каждом следующем шаге приклеивая в точности один треугольник в точности по одной стороне. Из построения, определения триангуляции и связности многообразия M следует, что последовательно будут приклеены все треугольники и в результате получится множество P, гомеоморфное двумерному диску. Граница этого множества разбита на четное количество $2k$ ребер триангуляции. Эти ребра некоторым образом разбиты на пары ребер, помеченных одинаковыми буквами, и, кроме того, каждое ребро снабжено стрелкой.

Лемма 16.25. Каждое гладкое связное компактное (замкнутое) двумерное многообразие может быть получено из некоторого $2k$-угольника некоторой склейкой пар его сторон.

Доказательство. Из конструкции 16.24 ясно, что множество P гомеоморфно плоскому $2k$-угольнику, на границе которого определена некоторое слово склейка. Склеивая многоугольник P по этому слову, получим исходное многообразие. Лемма доказана.

14.18.5 16.3.2 Canonical polygon glues

Напомним, что каждая склейка многоугольника задается некоторым словом, хотя разные слова могут приводить к одному и тому же многообразию.

Утверждение 16.26. Имеют место следующие гомеоморфизмы:
(1) многообразие $P\left(aa^{-1}\right)$ гомеоморфно сфере;
(2) многообразие $P\left(a_1b_1a_1^{-1}b_1^{-1}\cdots a_nb_na_n^{-1}b_n^{-1}\right)$ гомеоморфно сфере c п ручками;
(3) многообразие $P\left(a_1a_1\cdots a_na_n\right)$, гомеоморфно сфере с n пленками Мебиуса.

Определение 16.27. Склейки, описанные в утверждении 16.26, называются каноническими.

Лемма 16.28. Произвольное многообразие, полученное склейкой из $2k$ угольника, гомеоморфно или сфере с g ручками, или сфере $c\mu$ пленками Мебиуса.

Доказательство леммы 16.28 можно провести с помощью набора нескольких гомеоморфизмов, которые перестраивают многоугольник P и слово склейки. Эти гомеоморфизмы описываются в терминах разрезания многоугольника P по некоторой диагонали, помеченной буквой и стрелкой, и склейки двух полученных многоугольников с размеченными сторонами по паре сторон. При этом меняется слово склейки, но не многообразие. Можно показать, что с помощью таких операций произвольное слово склейки можно преобразовать к одному из трех канонических видов, приведенных в утверждении 16.26. Мы не будем здесь приводить эту конструкцию, отсылая заинтересованных к книге Мищенко А. С. и Фоменко А. Т. "Курс дифференциальной геометрии и топологии", где соответствующий алгоритм подробно изложен и снабжен замечательными рисунками.

14.18.6 16.3.3 Last step, Euler characteristic

Чтобы завершить доказательство классификационной теоремы, следует установить, что многообразия $M_g^2, g \geq 0$, и N_μ^2 попарно негомеоморфны. Для этого нам понадобится понятие эйлеровой характеристики.

Определение 16.29. Пусть τ - произвольная триангуляция двумерного многообразия M. Назовем эйлеровой характеристикой $\chi(\tau)$ триангуляции τ следующее целое число:

$$\chi(\tau) = F - E + V,$$

где F, E и V - количества треугольников (граней), ребер и вершин триангуляции соответственно.

Имеет место следующее утверждение, доказательство которого полностью аналогично доказательству для частного случая $M = S^2$, которое приводится в курсе наглядной геометрии и топологии.

Утверждение 16.30. Эйлерова характеристика триангуляции многообразия M не зависит от триангуляции и определяется только многообразием M.

Определение 16.31. Назовем эйлеровой характеристикой двумерного многообразия M эйлерову характеристику любой его триангуляции.

Statement

16.32. Эйлерова характеристика многообразия является его топологическим инвариантом, т.е. если два многообразия гомеоморфны, их эйлеровы характеристики совпадают.

Замечание 16.33. Можно теперь вычислить эйлеровы характеристики всех двумерных компактных связных многообразий, построив подходящие триангуляции. Если получатся разные числа, то тем самым будет доказана негомеоморфность соответствующих многообразий. Однако построение триангуляций - дело сложное и трудоемкое. Мы поступим иначе.

Конструкция 16.34. Пусть M_1 и M_2 - два двумерных многообразия и P_i - точка на многообразии M_i. Пусть U_i - малая круговая окрестность точки P_i. Выбросим из многообразия M_i диск U_i. В результате получится топологическое пространство, граница которого гомеоморфна окружности. Склеим два этих пространства по граничной окружности.

Определение 16.35. Построенное компактное многообразие назовем связной суммой многообразий M_1 и M_2 и обозначим через $M_1 \# M_2$.

Утверждение 16.36. Пусть $M_1 u M_2$ - двумерные компактные связные многообразия. Тогда

$$\chi\left(M_1 \# M_2\right)=\chi\left(M_1\right)+\chi\left(M_2\right)-2.$$

Доказательство. Рассмотрим достаточно мелкие триангуляции многообразий M_i. Тогда при построении их связной суммы в качестве дисков можно взять треугольники триангуляций. Отождествляя эти треугольники, склеим их по сторонам. В итоге получим некоторую триангуляцию τ связной суммы.

Пусть V_i, E_i и F_i - соответственно количества вершин, ребер и треугольников триангуляций многообразий M_i, а V, E и F - соответствующие числа для триангуляции τ. Тогда

$$V=V_1+V_2-3, \quad E=E_1+E_2-3, \quad F=F_1+F_2-2$$

откуда и получаем требуемое соотношение. □

Лемма 16.37. Эйлерова характеристика сферы равна 2 .

Доказательство. Ясно, что $S^2 \# S^2=S^2$, откуда $\chi\left(S^2\right)=2 \chi\left(S^2\right)-2$, поэтому $\chi\left(S^2\right)=2$. □

Лемма 16.38. Эйлерова характеристика тора равна 0 .

Доказательство. Тор можно получить из сферы выбрасыванием двух дисков и последующей склейкой друг с другом двух граничных окружностей. Если сфера уже была триангулирована, а вместо дисков мы возьмем треугольники триангуляции, то получим некоторую триангуляцию τ тора. Рассуждая так же, как при доказательстве утверждения 16.36, получим

$$V=V_1-3, \quad E=E_1-3, \quad F=F_1-2,$$

где V, E и F - количества вершин, ребер и треугольников триангуляции τ, а V_1, E_1 и F_1 - соответствующие числа для исходной триангуляции сферы. Поэтому

$$\chi\left(T^2\right)=F-E+V=\left(F_1-2\right)-\left(E_1-3\right)+\left(V_1-3\right)=\chi\left(S^2\right)-2=0.$$

□

Лемма 16.39. Эйлерова характеристика сферы с g ручками равна $2-2g$. Доказательство. Если M_g^2 - это сфера с g ручками, то

$$M_g^2 = \underbrace{T^2 \# \cdots \# T^2}_{g},$$

и

$$\chi\left(M_g^2\right) = \underbrace{\chi\left(T^2\right) + \cdots + \chi\left(T^2\right)}_{g} - 2(g-1) = 2 - 2g.$$

□

Лемма 16.40. Эйлерова характеристика сферы с μ пленкам Мебиуса равна $2-\mu$. Доказательство. Заметим сначала, что если M_μ^2 - сфера с μ пленками Мебиуса, то

$$M_\mu^2 = \underbrace{\mathbb{RP}^{\not\simeq} \# \cdots \# \mathbb{R}P^2}_{\mu}.$$

Но так как $\mathbb{RP}^2 \# T^2 = \mathbb{R}P^2 \# \mathbb{R}P^2 \# \mathbb{R}P^2$, то

$$\chi\left(\mathbb{R}P^2\right) + \chi\left(T^2\right) - 2 = 3\chi\left(\mathbb{R}P^2\right) - 4,$$

т.е. $\chi\left(\mathbb{RP}^2\right) = 1$. Поэтому

$$\chi\left(M_\mu^2\right) = \underbrace{\chi\left(\mathbb{R}P^2\right) + \cdots + \chi\left(\mathbb{R}P^2\right)}_{\mu} - 2(\mu-1) = 2 - \mu.$$

Следствие 16.41. Сферы cg ручками и сферы с μ пленками Мебиуса попарно недиффеоморфны.

Доказательство. Осталось заметить, что диффеоморфизм сохраняет свойство ориентируемости, поэтому никакая сфера с $g \geq 0$ ручками недиффеоморфна сфере с $\mu \geq 1$ пленками Мебиуса.

Следствием 16.41 завершается доказательство теоремы классификации.

15 Ivanov 2

Эта книга является продолжением наших «Лекций по классической дифференциальной геометрии» и представляет собой подробные конспекты лекций, читанных авторами на механико-математическом факультете МГУ имени М. В. Ломоносова для студентов второго и третьего курсов на протяжении более чем десяти лет. Напомним, что традиционно годовой курс дифференциальной геометрии делится на два семестра. Представленные 15 лекций относятся ко второму из пих, который называется «Дифференциальная геометрия и топология».

Книга предназначена главишм образом для студентов и аспираптов математических, физических и других естественнонаучных специальностей, как систематически изучающих геометрию, так и желающих продолжить беглое знакомство с предметом. Отметим, что эта книга, в комплекте с предыдущей, может быть использована при постановке полноценного курса геометрии для студентов университетов.

План курса. Фактически, этот второй семестр курса мог бы быть озаглавлен «Тензорный анализ на многообразиях». Тензоры и тензорные поля, простейшими примерами которых являются векторные поля и дифференциалы функций, - основные объекты изучения многих математических, механических и физических дисциплин. Мы начнем со стандартных определений тензора и тензорного поля на многообразии. Затем будет построена теория внешних дифференциальных форм (кососимметрических

тензорных полей). Оказывается, именно дифференциальные формы представляют собой естественный объект интегрирования по гладкому многообразию. Мы докажем основную теорему интегрального исчисления (формулу Стокса), обобщающую известную формулу Ньютона-Лейбница, а в качестве приложения изучим когомологии де Рама - алгебраический объект, сопоставляемый многообразию, инвариантный при диффеоморфизмах и гомотопиях, а также отвечающий за разрешимость на многообразии дифференциальных уравнений специального вида. Затем мы вернемся к произвольным тензорным полям и построим для них общую теорию тензорного («ковариантного») дифференцирования. Несмотря на то, что таких дифференцирований оказывается много, все они, в некотором смысле, аналогичны ковариантному дифференцированию из теории поверхностей (теория Вейнгартена-Гаусса). В этих терминах можно определить аналог геодезических на поверхностях и параллельный перенос тензоров вдоль кривых, позволяющий сравнивать значения тензорного поля в разных точках многообразия. Будет изучен «маленький монстр» дифференциальной геометрии - тензор кривизны, являющийся аналогом гауссовой кривизны из теории поверхностей и характеризующий локальную «непохожесть» многообразия на евклидово пространство. Мы также познакомимся с понятием степени отображения - мощным топологическим ин-

вариантом, позволяющим решать некоторые глобальные задачи дифференциальной геометрии. В заключении, мы рассмотрим общий подход к вариационным задачам, записав общие уравнения Эйлера-Лагранжа для таких геометрических вариационных функционалов, как длина, энергия и площадь.

Структура книги. Как и в первом семестре, каждая лекция представляет собой подробный конспект реальной лекции, рассчитанной на два академических часа. Затем приводится список задач, которые могут быть использованы при проведении семинарских занятий по теме этой лекции. Наконец, каждая лекция снабжена Дополнительными материалами, которые содержат не вошедшие в основной тест лекции примеры, конструкции и результаты, полезные для более глубокого знакомства с предметом, а также часть громоздких доказательств, не поместившихся в основной текст (обычно на лекциях студентам сообщается лишь идея такого доказательства и предлагается восстановить подробности самостоятельно).

Авторы пользуются случаем поблагодарить всех, кто учил и учит нас понимать и любить Геометрию. Прежде всего, нашего учителя, академика Анатолия Тимофеевича Фоменко, чьи замечательные лекции мы сами слушали на мехмате. Мы очень признательны профессорам А. С. Мищенко, В. В. Трофимову, Ю. П. Соловьеву, чьи лекции и семинары нам посчастливилось посещать, а также нашим коллегамгеометрам Л. А. Алании, А. В. Болсинову, В. Л. Голо, Д. П. Ильютко, Е. А. Кудрявцевой, В. М. Мануйлову, Н. Ю. Нецветаеву, И. М. Никонову, А. А. Ошемкову, Ф. Ю. Попеленскому, И. Х. Сабитову, А. Б. Скопенкову, Е. В. Троицкому, А. И. Шафаревичу за многочисленные интересные обсуждения, обмен опытом и полезные советы. Разумеется, эта книга никогда не была бы написана без нашего постоянного общения со студентами и аспирантами механико-математического факультета, которые пытались учиться по этим лекциям, за что мы им также очень признательны.

Мы пытались сделать изложение материала независимым, но не от первого семестра наших лекций. Все понятия, введенные в наших Лекциях по классической дифференциальной геометрии, предполагаются известными и используются без пояснений.

В заключении отметим, что в настоящий момент существует много замечательных учебников по дифференциальной геометрии. Их далеко не полный список приведен в конце книги.

15.1 1: Tensors and tensor fields

В любой конкретной задаче, в конечном итоге, требуется определить поведение каких-нибудь числовых характеристик изучаемой системы. В простейшем случае эти числовые характеристики представляют собой функции, в общем случае они организуются в более сложные образования. Например, рассмотрим вектор в линейном пространстве $\mathbb{R}^n$. Чтобы производить реальные вычисления, нужно сначала фиксировать какие-нибудь координаты на конфигурационном пространстве рассматриваемой системы, скажем фиксировать базис в $\mathbb{R}^n$. Но, как только появляются координаты, так сразу требуется выяснить как преобразуются наши числовые данные при замене координат. В простейшем случае случае функции, значение функции в точке конфигурационного пространства не зависит от системы координат. Такие величины часто называют скалярньми. Однако в случае вектора в линейном пространстве $\mathbb{R}^n$ это уже не так: компоненты вектора при линейной замене координат преобразуются с помощью матрицы перехода этой замены. В этом семестре мы изучим естественное обобщение понятия «вектор» понятие «тензор». Тензоры, говоря неформально, это числовые данные, заданные в линейном пространстве и меняющиеся при замене координат в линейном пространстве наиболее простым образом - полилинейным. Возможно, понятие тензора уже появлялось в курсе линейной алгебры и/или в курсе механики сплошной среды. Замечания о различии подходов дифференциальной геометрии и механики сплошной среды см. ниже.

В дифференциальной геометрии в качестве конфигурационных пространств выступают, конечно же, гладкие многообразия. В каждой точке P гладкого многообразия M имеется естественно определенное линейное пространство T_PM - касательное пространство к многообразию в данной точке. Поэтому в каждой точке многообразия можно задать тензор, а именно, тензор в этом линейном пространстве. Обычно в дифференциальной геометрии рассматривают не просто тензоры в касательном пространстве в какой-то одной точке, а тензорные поля, т.е. тензоры в касательных пространствах, гладко зависящие от точки многообразия. Однако, имея это в виду, мы начнем с простейших примеров тензоров, которые нам уже встречались, и с напоминания того, что такое тензоры в одном линейном пространстве.

Замечание. Будем использовать обозначения удобные в дифференциальной геометрии. Так, например, базис в линейном пространстве $\mathbb{R}^n = T_PM$ всегда соответствует некоторым локальным координатам x^i в окрестности точки P на гладком многообразии M, поэтому обозначается через $\{\partial_{x^i}\}$, а матрица перехода от базиса к базису в нашем случае - это матрица Якоби соответствующей замены координат в окрестности точки P, поэтому записывается в виде частных производных. В остальном наши определения ничем не отлича-

ются от стандартных определений из курса линейной алгебры [1].

15.1.1 1.1. The simplest examples of tensors

Мы будем пользоваться следующим соглашением, которое используют в тензорном исчислении для упрощения записи выражений в «новых» координатах. Пусть x^i - «старые» координаты. Вместо того, чтобы каждые следующие «новые» координаты обозначать новыми буквами, скажем y^j, где i и j - независимые индексы, мы будем добавлять штрихи к старым буквам: $x^{i'}$, где i и i' - снова независимые индексы. При этом, поскольку последняя запись является громоздкой и неудобной, мы будем сокращенно писать $x^{i'}$, опуская штрих у x, и подразумевая, что «штрих у индекса относится и к x и к i ». Если же необходимо вместо i' подставить его значение, например, записать координату $x^{i'}$ при $i' = 1$, т.е. первую «новую» координату, то будем

писать так: $x^{1'}$.

Касательный вектор. Пусть M - гладкое n-мерное многообразие, P - его произвольная точка, и T_PM - касательное пространство в этой точке. Предположим, что фиксированы локальные координаты $(x^1, \ldots, x^n)$. Тогда каждый касательный вектор $V \in T_PM$ в этих координатах однозначно определяется своими компонентами ($v^1, \ldots, v^n$) (относительно канонического базиса ∂_{x^i}). Если в окрестности точки P заданы еще одни локальные координаты ($x^{1'}, \ldots, x^{n'}$), то компоненты $\left(v^{1'}, \ldots, v^{n'}\right)$ вектора V в этих «новых» «штрихованных» координатах, по определению, пересчитываются так:

$$v^{i'} = \sum_{i=1}^{n} \frac{\partial x^{i'}}{\partial x^i} v^i$$

где $\left(\frac{\partial x^{i'}}{\partial x^i}\right)$ - матрица Якоби замены координат в точке P. Отметим, что эта формула в точности совпадает с формулой пересчета компонент вектора при замене базиса, известной из линейной алгебры. Действительно, канонические базисы ∂_{x^i} и $\partial_{x^{i'}}$ в касательном пространстве T_PM связаны, как известно, так:

$$\partial_{x^i} = \sum_{i'=1}^{n} \frac{\partial x^{i'}}{\partial x^i} \partial_{x^{i'}}$$

т.е. матрица Якоби $J = \left(\frac{\partial x^{i'}}{\partial x^i}\right)$ замены координат - это матрица перехода от базиса $\{\partial_{x^{i'}}\}$ к базису $\{\partial_{x^i}\}$. В матричном виде формула для

пересчета компонент вектора записывается так: $V' = JV$ (здесь V' и V - столбцы компонент рассматриваемого вектора в «новых» и «старых» координатах соответственно). Этот закон преобразования называется тензорньмм законом преобразования компонент вектора.

Дифференциал функции - ковектор. Пусть снова M - гладкое n-мерное многообразие, P - произвольная точка из M, и f - гладкая функция на M. Тогда, как мы уже знаем, определено линейное отображение $df|_P$ линейного пространства $T_PM = \mathbb{R}^n$ в $\mathbb{R}^1$, т.е. линейная функция на T_PM, называемая дифференциалом функции f. Папомним, что в линейной алгебре линейные функции называются линейньлми функчионалами. Если в окрестности точки P фиксированы локальные координаты ($x^1, \ldots, x^n$), то, напомним, дифференциал функции f однозначно опрсдслястся матрицсй, состоящсй из одной строки, которая в координатах $(x^1, \ldots, x^n)$ (т.е. в каноническом базисе ∂_{x^i}) имеет вид

$$df = \left(\frac{\partial f}{\partial x^1}, \ldots, \frac{\partial f}{\partial x^n}\right)$$

где все частные производные вычисляются в точке P. Если в окрестности точки P заданы еще одни локальные координаты ($x^{1'}, \ldots, x^{n'}$), то компоненты матриц дифференциала в разных системах координат частные производные, связаны между собой так:

$$\frac{\partial f}{\partial x^{i'}} = \sum_{i=1}^{n} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial f}{\partial x^i}$$

Отметим, что эта формула отличается от формулы преобразования компонент вектора. Однако, эта формула тоже известна из линейной алгебры.

[01] В данной книге приведены все необходимые определения. В качестве учебника по линейной алгебре можно воспользоваться, например, замечательными книгами И. М. Гельфанда или А. И. Кострикина и Ю. И. Манина

Действительно, рассмотрим двойственное пространство T_P^*M к касательному пространству T_PM, т.е. пространство всех линейных функций на пространстве T_PM. Если фиксированы локальные координаты $(x^1, \ldots, x^n)$ в окрестности точки P, и $\{\partial_{x^i}\}$ - канонический базис в линейном пространстве T_PM, то каждая линейная функция ξ, т.е. элемент пространства T_P^*M, однозначно определяется своими значениями на базисных векторах: если V - произвольный вектор из T_PM с компонентами $(v^1, \ldots, v^n)$, то

$$\xi(V) = \xi\left(\sum_{k=1}^{n} v^i \partial_{x^i}\right) = \sum_{k=1}^{n} v^i \xi\left(\partial_{x^i}\right) = \sum_{k=1}^{n} v^i \xi_i,$$

где числа $\xi_i = \xi\left(\partial_{x^i}\right)$ называются компонентами линейной функции ξ в координатах $(x^1, \ldots, x^n)$. На самом деле, напомним, в линейном пространстве T_P^*M каноническим образом определен двойственный базис
$\{dx^i \in T_P^*M\}$, где линейный функционал dx^i однозначно определяется из соотношений

$$dx^i\left(\partial_{x^j}\right) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

(Отметим, что функционал dx^i - это дифференциал гладкой функции, заданной в рассматриваемой карте, а именно функции, сопоставляющей точке P ее i-ую координату $x^i(P)$.) Линейная функция ξ однозначно представима в виде

$$\xi = \sum_{k=1}^{n} \xi_k dx^k$$

Так вот, пусть теперь в окрестности точки P заданы еще одни локальные координаты $(\ x^{1'}, \ldots, x^{n'}\ )$. Тогда, как легко проверить, компоненты линейного функционала ξ относительно разных систем координат связаны по уже знакомой нам формуле для компонент дифференциала:

$$\xi_{i'} = \sum_{i=1}^{n} \frac{\partial x^i}{\partial x^{i'}} \xi_i$$

В матричном виде эта формула записывается в виде $\xi' = \left(J^{-1}\right)^T \xi$, где J - матрица Якоби $(\ \frac{\partial x^{i'}}{\partial x^i}\ )$ замены координат, а ξ и ξ' - столбцы из соответствующих компонент функционала. Отметим, что иногда эту же формулу записывают в виде $\xi' = \xi J^{-1}$, обозначая через ξ' и ξ строки из компонент функционала.

В тензорном анализе линейные функционалы называются ковекторами, а закон преобразования компонент линейного функционала, т.е. ковектора, называется тензорным законом преобразования компонент ковектора. Таким образом, дифференциал функции является ковектором, и его компоненты меняются по ковекторному закону.

Билинейная форма. Приведем еще один пример. Пусть снова M - гладкое n-мерное многообразие, и T_PM - касательное пространство к M в некоторой точке $P \in M$. Рассмотрим на T_PM билинейную форму B, т.е. отображение $B : T_PM \times T_PM \to \mathbb{R}^1$, линейное по каждому аргументу. Из линейной алгебры хорошо известно, что если в линейном пространстве фиксирован базис, то билинейная форма однозначно задается своими значениями на всевозможных парах базисных векторов. Эти значения образуют матрицу билинейной формы.

Действительно, если $(\ x^1, \ldots, x^n\ )$ - локальные координаты в окрестности точки P, а $V = (v^1, \ldots, v^n)$ и $W = (w^1, \ldots, w^n)$ - произвольные векторы из T_PM, заданные своими компонентами, то

$$B(V,W) = B\left(\sum_{\alpha=1}^{n} v^\alpha \partial_{x^\alpha}, \sum_{\beta=1}^{n} w^\beta \partial_{x^\beta}\right) = \sum_{\alpha,\beta=1}^{n} B\left(\partial_{x^\alpha}, \partial_{x^\beta}\right) v^\alpha v^\beta$$

т.е, положив $b_{\alpha\beta} = B\left(\partial_{x^\alpha}, \partial_{x^\beta}\right)$, получаем:

$$B(V,W) = \sum_{\alpha,\beta=1}^{n} b_{\alpha\beta} v^\alpha v^\beta.$$

Числа $b_{\alpha\beta}$ называются компонентами билинейной формы B в координатах $(x^1, \ldots, x^n)$.

Если теперь в окрестности точки P заданы другие локальные координаты ($x^{1'}, \ldots, x^{n'}$), то компоненты $b_{\alpha'\beta'} = B\left(\partial_{x^{\alpha'}}, \partial_{x^{\beta'}}\right)$ билинейной формы B в этих координатах могут быть вычислены через компоненты b_{pq} формы B в координатах $(x^1, \ldots, x^n)$ так:

$$\begin{aligned} b_{\alpha'\beta'} &= B\left(\partial_{x^{\alpha'}}, \partial_{x^{\beta'}}\right) = B\left(\sum_{p=1}^{n} \frac{\partial x^{\alpha'}}{\partial x^p} \partial_{x^p}, \sum_{q=1}^{n} \frac{\partial x^{\beta'}}{\partial x^q} \partial_{x^q}\right) \\ &= \sum_{p,q=1}^{n} \frac{\partial x^{\alpha'}}{\partial x^p} \frac{\partial x^{\beta'}}{\partial x^q} B\left(\partial_{x^p}, \partial_{x^q}\right) \end{aligned}$$

т.е. окончательно:

$$b_{\alpha'\beta'} = \sum_{p,q=1}^{n} \frac{\partial x^{\alpha'}}{\partial x^p} \frac{\partial x^{\beta'}}{\partial x^q} b_{pq}.$$

Этот закон преобразования называется тензорным законом преобразования компонент билинейной формы.

Отметим, что нам уже встречался пример билинейных форм, заданных на гладких многообразиях: это риманова метрика.

Упражнение 1.1. Рассмотреть линейный оператор $dF(P) : T_P M \to T_P M$, являющийся дифференциалом гладкого отображения $F : M \to M$, где $F(P) = P$. Выяснить, как преобразуются компоненты матрицы оператора $dF(P)$ при замене координат. Сравнить полученный закон с тензорным законом преобразования компонент билинейной формы.

15.1.2 1.2. General tensor definition

В предыдущем разделе мы напомнили определения хорошо известных из линейной алгебры и геометрии объектов: векторов, ковекторов, линейных операторов и билинейных форм. Все эти объекты, на самом деле, инвариантны, т.е. не зависят от выбора локальных координат (скажем, если оператор $\mathcal{A}$ переводит вектор V в вектор W, то он это делает в не зависимости от выбора локальных координат; оператор «объективен», а координаты - «субъективны»; оператор «не знает» в каких координатах мы записываем его матрицу). Координатное представление (компоненты) каждого из указанных объектов при замене координат меняется, но меняется по строго определенному правилу - по тензорному закону.

Тензоры на линейном пространстве $T_P M$, которые мы сейчас определим, являются естественным обобщением всех этих понятий. Мы дадим здесь два определения тензора, одно из которых координатное, а другое инвариантное.

Координатное определение тензора и тензорного поля. Данное в этом пункте определение особенно полезно при конкретных вычислениях.

Определение. Тензором типа (p, q) ранга $p+q$ в касательном пространстве $T_P M$ называется соответствие T, сопоставляющее каждой локальной системе координат (

$x^1, \ldots, x^n$) набор из n^{p+q} чисел $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$, где все верхние и нижние индексы независимо принимают значения от 1 до n, а n - размерность линейного пространства $T_P M$, т.е. размерность многообразия M. Числа $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ называются координатами или компонентами тензора T в локальных координатах $(x^1, \ldots, x^n)$. При этом, если ($x^{1'}, \ldots, x^{n'}$) - другая локальная система координат в окрестности точки P, то компоненты $T^{i'_1 \ldots i'_p}_{j'_1 \ldots j'_q}$ тензора T в этой новой системе координат связаны с его компонентами в старой системе координат по следующему правилу, называемому тензорньлм законом:

$$T^{i'_1 \ldots i'_p}_{j'_1 \ldots j'_q} = \sum_{i_1=1}^{n} \cdots \sum_{i_p=1}^{n} \sum_{j_1=1}^{n} \cdots \sum_{j_q=1}^{n} \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_q}}{\partial x^{j'_q}} T^{i_1 \ldots i_p}_{j_1 \ldots j_q}.$$

Наконец, если на гладком многообразии M задано отображение $\mathbb{T}$, ставящее в соответствие каждой точке P многообразия некоторый тензор $T = \mathbb{T}(P)$ типа (p, q), причем, в каждой локальной системе координат $(x^1, \ldots, x^n)$, компоненты тензора T гладко зависят от $(x^1, \ldots, x^n)$, то говорят, что на многообразии M задано (гладкое) тензорное поле $\mathbb{T}$ muna (p, q).

Удобно считать произвольную константу тензором типа (0,0), а произвольную гладкую функцию - тензорным полем типа $(0, 0)$. Тензор (тензорное поле) типа $(0, 0)$ принято называть скаляром (соответственно, полем скаляров).

Очевидно, вектор является тензором типа $(1, 0)$ ранга 1 , ковектор тензором типа $(0, 1)$ ранга 1 , линейный оператор - тензором типа $(1, 1)$ ранга 2 , а билинейная форма - тензором типа $(0, 2)$ ранга 2 . Несколько позднее мы построим еще несколько важных примеров тензоров, а сейчас, прежде чем давать второе, инвариантное определение тензора, мы примем важное соглашение.

Формула, выражающая тензорный закон, очень громоздка. Она содержит $p+q$ знаков суммирования. Чтобы немного сократить записи тензорных вычислений, в тензорном анализе принято опускать знак

суммирования, если оно ведется по паре повторяющихся переменных индексов, один из которьх верхний, а другой - нижний. Отметим, что все суммы, входящие в запись тензорного закона, именно такие: например, индекс суммирования i_1 в правой части первый раз появляется в выражении $\frac{\partial x^{i'_\perp}}{\partial x^{i_1}}$, в котором является нижним (стоит в знаменателе), а второй раз - в выражении $T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$, где является верхним. Таким образом, в силу принятого соглашения, тензорный закон записывается в более компактной форме так:

$$T^{i'_1 \ldots i'_p}_{j'_1 \ldots j'_q} = \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_q}}{\partial x^{j'_q}} T^{i_1 \ldots i_p}_{j_1 \ldots j_q}.$$

Замечание. Так как все системы координат равноправны, тензорный закон может быть записан и в следующем, эквивалентном виде:

$$T^{i_1 \ldots i_p}_{j_1 \ldots j_q} = \frac{\partial x^{i_1}}{\partial x^{i'_1}} \cdots \frac{\partial x^{i_p}}{\partial x^{i'_p}} \frac{\partial x^{j'_1}}{\partial x^{j_1}} \cdots \frac{\partial x^{j'_q}}{\partial x^{j_q}} T^{i'_1 \ldots i'_p}_{j'_1 \ldots j'_q}. \qquad (*)$$

Мы определили тензор в $T_P M$ как некоторое соответствие, сопоставляющее каждым локальным координатам в окрестности точки P наборы чисел, связанные между собой по тензорному закону. Возникает естественный вопрос: как задавать такие соответствия, т.е. тензоры? Неужели, нужно задавать наборы чисел для всех систем координат? Ответ на этот вопрос дает следующая простая лемма.

Лемма 1.1. Пусть $(x^1, \ldots, x^n)$ - произвольные локальльге координатью в окрестности точки P многообразия M размерности n. Сопоставим им произвольный набор $\left(T^{i_1 \ldots i_p}_{j_1 \ldots j_q}\right)$

из n^{p+q} чисел. Любой другой системе координат $(\ x^{1'},\dots,x^{n'}\ )$ поставим в соответствие набор чисел $\left(T^{i'_1\dots i'_p}_{j'_1\dots j'_q}\right)$, вычисленный по числам $T^{i_1\dots i_p}_{j_1\dots j_q}$ и матрице Якоби заменью координат по тензорному закону, т.е.

$$T^{i'_1\dots i'_p}_{j'_1\dots j'_q}=\frac{\partial x^{i'_1}}{\partial x^{i_1}}\cdots\frac{\partial x^{i'_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x^{j'_1}}\cdots\frac{\partial x^{j_q}}{\partial x^{j'_q}}T^{i_1\dots i_p}_{j_1\dots j_q}.$$

Построенное соответствие задает тензор в T_PM. Таким образом, чтобы задать тензор типа $(\ p,q\ )$ в пространстве T_PM достаточно задать его компоненты в какой-нибудь одной системе координат, а во всех остальных системах координат пересчитать компоненты по тензорному закону.

Доказательство. Для доказательствалеммы достаточно проверить, что если

$$T^{i'_1\dots i'_p}_{j'_1\dots j'_q}=\frac{\partial x^{i'_1}}{\partial x^{i_1}}\cdots\frac{\partial x^{i'_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x^{j'_1}}\cdots\frac{\partial x^{j_q}}{\partial x^{j'_q}}T^{i_1\dots i_p}_{j_1\dots j_q}$$

Доказательство. Действительно, пусть $\xi^1,\dots,\xi^p$ - произвольные ковекторы из T^*_PM, а $V_1,\dots,V_q$ - произвольные векторы из T_PM. Обозначим компоненты вектора V_i в каноническом базисе $\{\partial_{x^i}\}$ через $(v^1_i,\dots,v^n_i)$, а компоненты ковектора ξ^i в каноническом базисе $\{dx^j\}$ через $(\xi^i_1,\dots,\xi^i_n)$. Тогда, в силу полилинейности отображения T имеем:

$$\begin{aligned}T\left(\xi^1,\dots,\xi^p;V_1,\dots,V_q\right)&=T\left(\xi^1_{i_1}dx^{i_1},\dots,\xi^p_{i_p}dx^{i_p};v^{j_1}_1\partial_{x^{j_1}},\dots,v^{j_q}_q\partial_{x^{j_q}}\right)=\\&=\xi^1_{i_1}\cdots\xi^p_{i_p}v^{j_1}_1\cdots v^{j_q}_qT\left(dx^{i_1},\dots,dx^{i_p};\partial_{x^{j_1}},\dots,\partial_{x^{j_q}}\right).\end{aligned}$$

Поэтому, зная значения полилинейного отображения T на базисных векторах и ковекторах, мы однозначно восстановим его на любых других, и наоборот, чтобы задать полилинейное отображение, достаточно задать его значения на комбинациях элементов канонических базисов. Лемма доказана.

Итак, обозначив значение $T\left(dx^{i_1},\dots,dx^{i_p};\partial_{x^{j_1}},\dots,\partial_{x^{j_q}}\right)$ полилинейного отображения через $T^{i_1\dots i_p}_{j_1\dots j_q}$, мы однозначно определим по полилинейному отображению T соответствие $\tilde T$, сопоставляющее локальным координатам $(x^1,\dots,x^n)$ набор чисел $T^{i_1\dots i_p}_{j_1\dots j_q}$. Проверим, что соответствие $\tilde T$ действительно определяет тензор в смысле координатного определения. Для этого нужно убедиться, что наборы чисел, сопоставляемые соответствием $\tilde T$ разным локальным координатам, связаны по тензорному закону.

Проверим это. Пусть $(x^1,\dots,x^n)$ и $(\ x^{1'},\dots,x^{n'}\ )$ - две системы локальных координат. Наше соответствие $\tilde T$, которое мы построили по полилинейному отображению, устроено так:

$$\begin{aligned}\tilde T:&\quad\left(x^1,\dots,x^n\right)\mapsto\quad T^{i_1\dots i_p}_{j_1\dots j_q}=T\left(dx^{i_1},\dots,dx^{i_p};\partial_{x^{j_1}},\dots,\partial_{x^{j_q}}\right),\\\tilde T:&\quad\left(x^{1'},\dots,x^{n'}\right)\mapsto\quad T^{i'_1\dots i'_p}_{j'_1\dots j'_q}=T\left(dx^{i'_1},\dots,dx^{i'_p};\partial_{x^{j'_1}},\dots,\partial_{x^{j'_q}}\right).\end{aligned}$$

Чтобы найти связь между числами $T^{i_1\dots i_p}_{j_1\dots j_q}$ и $T^{i'_1\dots i'_p}_{j'_1\dots j'_q}$, воспользуемся связью между элементами канонических базисов в T_PM и T^*_PM :

$$\partial_{x^{i'}}=\frac{\partial x^i}{\partial x^{i'}}\partial_{x^i},\quad dx^{i'}=\frac{\partial x^{i'}}{\partial x^i}dx^i$$

Получим:

$$
\begin{aligned}
T^{i'_1\cdots i'_p}_{j'_1\cdots j'_q} &= T\left(dx^{i'_1},\ldots,dx^{i'_p};\partial_{x^{j'_1}},\ldots,\partial_{x^{j'_q}}\right) = \\
&= T\left(\frac{\partial x^{i'_1}}{\partial x^{i_1}}dx^{i_1},\ldots,\frac{\partial x^{i'_p}}{\partial x^{i_p}}dx^{i_p};\frac{\partial x^{j_1}}{\partial x^{j'_1}}\partial_{x^{j_1}},\ldots,\frac{\partial x^{j_q}}{\partial x^{j'_q}}\partial_{x^{j_q}}\right) = \\
&= \frac{\partial x^{i'_1}}{\partial x^{i_1}}\cdots\frac{\partial x^{i'_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x^{j'_1}}\cdots\frac{\partial x^{j_q}}{\partial x^{j'_q}}T\left(dx^{i_1},\ldots,dx^{i_p};\partial_{x^{j_1}},\ldots,\partial_{x^{j_q}}\right) = \\
&= \frac{\partial x^{i'_1}}{\partial x^{i_1}}\cdots\frac{\partial x^{i'_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x^{j'_1}}\cdots\frac{\partial x^{j_q}}{\partial x^{j'_q}}T^{i_1\ldots i_p}_{j_1\ldots j_q}.
\end{aligned}
$$

Это и есть тензорный закон. Таким образом, наше соответствие $\tilde{T}$ действительно задает тензор, и мы показали, что по каждому полилинейному отображению T однозначно строится тензор в смысле координатного определения.

Обратно, пусть задан тензор T в смысле координатного определения, и пусть снова фиксированы локальные координаты ($x^1,\ldots,x^n$). Тогда, в силу леммы 1.2, по набору компонент $T^{i_1\ldots i_p}_{j_1\ldots j_q}$ тензора T, можно однозначно определить полилинейное отображение $\tilde{T}$, положив

$$
\tilde{T}\left(dx^{i_1},\ldots,dx^{i_p};\partial_{x^{j_1}},\ldots,\partial_{x^{j_q}}\right) = T^{i_1\ldots i_p}_{j_1\ldots j_q}
$$

Для завершения доказательства остается убедиться, что определенное нами полилинейное отображение $\tilde{T}$ определено корректно, т.е. не зависит от локальной системы координат ($x^1,\ldots,x^n$), фигурирующей в определении $\tilde{T}$.

Проверим это. Пусть ($x^{1'},\ldots,x^{n'}$) - другая система локальных координат. Тогда соотношения

$$
S\left(dx^{i'_1},\ldots,dx^{i'_p};\partial_{x^{j'_1}},\ldots,\partial_{x^{j'_q}}\right) = T^{i'_1\cdots i'_p}_{j'_1\cdots j'_q}
$$

определяют, вообще говоря, другое полилинейное отображение S. Вычислим значение отображения $\tilde{T}$ на канонических базисных векторах штрихованной системы координат. Получим:

$$
\begin{aligned}
&\tilde{T}\left(dx^{i'_1},\ldots,dx^{i'_p};\partial_{x^{j'_1}},\ldots,\partial_{x^{j'_q}}\right) = \\
&= \tilde{T}\left(\frac{\partial x^{i'_1}}{\partial x^{i_1}}dx^{i_1},\ldots,\frac{\partial x^{i'_p}}{\partial x^{i_p}}dx^{i_p};\frac{\partial x^{j_1}}{\partial x^{j'_1}}\partial_{x^{j_1}},\ldots,\frac{\partial x^{j_q}}{\partial x^{j'_q}}\partial_{x^{j_q}}\right) = \\
&= \frac{\partial x^{i'_1}}{\partial x^{i_1}}\cdots\frac{\partial x^{i'_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x^{j'_1}}\cdots\frac{\partial x^{j_q}}{\partial x^{j'_q}}\tilde{T}\left(dx^{i_1},\ldots,dx^{i_p};\partial_{x^{j_1}},\ldots,\partial_{x^{j_q}}\right) = \\
&= \frac{\partial x^{i'_1}}{\partial x^{i_1}}\cdots\frac{\partial x^{i'_p}}{\partial x^{i_p}}\frac{\partial x^{j_1}}{\partial x^{j'_1}}\cdots\frac{\partial x^{j_q}}{\partial x^{j'_q}}T^{i_1\ldots i_p}_{j_1\ldots j_q} = \\
&= T_{j'_1\ldots j'_q}{}^{i'_1\ldots i'_p} = S\left(dx^{i'_1},\ldots,dx^{i'_p};\partial_{x^{j'_1}},\ldots,\partial_{x^{j'_q}}\right)
\end{aligned}
$$

что и означает совпадение полилинейных отображений $\tilde{T}$ и S. Таким образом, по каждому тензору в смысле координатного определения, мы построили корректно определенное полилинейное отображение. Утверждение доказано.

Замечание. Тензорное поле $\mathbb{T}$ можно определить как гладко зависящее от точки многообразия семейство полилинейных отображений описанного выше типа. Это означает, что если фиксированы локальные координаты ($x^1,\ldots,x^n$) в карте U, то для каждой точки $P\in U$ значение

$$
\mathbb{T}\left(P,dx^{i_1},\ldots,dx^{i_p};\partial_{x^{j_1}},\ldots,\partial_{x^{j_q}}\right)
$$

поля $\mathbb{T}$ на любом фиксированном наборе базисных ковекторов $dx^{i_1}, \ldots, dx^{i_p}$ и векторов $\partial_{x^{j_1}}, \ldots, \partial_{x^{j_q}}$ гладко зависит от $P \in U$. Другими словами, если определить аналог касательного расслоения - расслоение $T^{(p,q)}M$ со слоем

$$T^{(p,q)} = \underbrace{T_P^\star M \times \cdots \times T_P^\star M}_{p \text{ раз}} \times \underbrace{T_P M \times \cdots \times T_P M}_{q \text{ раз}},$$

то тензорное поле $\mathbb{T}$ - это гладкое отображение $\mathbb{T} : T^{(p,q)}M \to \mathbb{R}^1$ такое, что для любой фиксированной точки P соответиствующее отображение слоя полилинейно.

Замечание. Каждое полилинейное отображение $A : (T_P M)^q \to T_P M$ со значениями в векторном пространстве $T_P M$ однозначно определяет тензор типа $(1, q)$. Действительно, определим отображение $T_A : T_P^* M \times (T_P M)^q \to \mathbb{R}^1$, положив для любых векторов $V_1, \ldots, V_n$ и произвольного ковектора ξ

$$T_A : (\xi, V_1, \ldots, V_q) \mapsto \xi\left(A\left(V_1, \ldots, V_q\right)\right).$$

Очевидно, T_A - полилинейное отображение, т.е. тензор типа $(1, q)$. Обратно, если T - тензор типа $(1, q)$, то определено отображение $A_T : (T_P M)^q \to T_P M$, устроенное так:

$$\xi\left(A_T\left(V_1, \ldots, V_q\right)\right) = T\left(\xi, V_1, \ldots, V_q\right)$$

для произвольного ковектора ξ и векторов $V_1, \ldots, V_n$.

15.1.3 1.3. Linear space of tensors

Рассмотрим множество $\mathbb{T}_q^p$ всех тензоров фиксированного типа (p, q) в касательном пространстве $T_P M$ в точке P к многообразию M. Поскольку линейная комбинация полилинейных отображений тоже является полилинейным отображением, множество $\mathbb{T}_q^p$ образует линейное пространство. Чтобы определить размерность этого пространства, предъявим естественный базис в нем. А именно, пусть фиксированы локальные координаты ($x^1, \ldots, x^n$). Определим полилинейные отображения

$$\Delta_{i_1 \ldots i_p}^{j_1 \ldots j_q} : (T_P^* M)^p \times (T_P M)^q \to \mathbb{R}^1$$

положив для любых ковекторов $\xi^1, \ldots, \xi^p$ и векторов $V_1, \ldots, V_q$

$$\Delta_{i_1 \ldots i_p}^{j_1 \ldots j_q}\left(\xi^1, \ldots, \xi^p; V_1, \ldots, V_q\right) = \partial_{x^{i_1}}\left(\xi^1\right) \cdots \partial_{x^{i_p}}\left(\xi^p\right) dx^{j_1}\left(V_1\right) \cdots dx^{j_q}\left(V_q\right).$$

Другими словами,

$$\Delta_{i_1 \ldots i_p}^{j_1 \ldots j_q}\left(dx^{\alpha_1}, \ldots, dx^{\alpha_p}; \partial_{x^{\beta_1}}, \ldots, \partial_{x^{\beta_q}}\right) = \\ = \begin{cases} 1, & i_k = \alpha_k, \quad k = 1, \ldots, p; \quad j_l = \beta_l, \quad l = 1, \ldots, q; \\ 0, & \text{во всех остальных случаях.} \end{cases}$$

В силу леммы 1.2, каждое полилинейное отображение T можно представить в виде

$$T = T_{j_1 \ldots j_q}^{i_1 \ldots i_p} \Delta_{i_1 \ldots i_p}^{j_1 \ldots j_q},$$

где

$$T_{j_1 \ldots j_q}^{i_1 \ldots i_p} = T\left(dx^{i_1}, \ldots, dx^{i_p}; \partial_{x^{j_1}}, \ldots, \partial_{x^{j_q}}\right).$$

При этом, линейная комбинация

$$\Lambda = \lambda^{i_1 \dots i_p}_{j_1 \dots j_q} \Delta^{j_1 \dots j_q}_{i_1 \dots i_p}$$

не равна нулю, если хотя бы один коэффициент, скажем $\lambda^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q}$, отличен от нуля. Действительно, достаточно вычислить значение полилинейного отображения Λ на наборе векторов

$$dx^{\alpha_1}, \dots, dx^{\alpha_p}; \partial_{x^{\beta_1}}, \dots, \partial_{x^{\beta_q}}$$

которое будет, очевидно, равно $\lambda^{\alpha_1 \dots \alpha_p}_{\beta_1 \dots \beta_q}$ и, поэтому, отлично от нуля. Итак, мы показали, что полилинейные отображения

$$\Delta^{j_1 \dots j_q}_{i_1 \dots i_p},$$

где все индексы i_k и j_l пробегают независимо всевозможные значения от 1 до n, образуют базис в пространстве $\mathbb{T}^p_q$. Таким образом, мы доказали следующее полезное утверждение.

Утверждение 1.2. Множество $\mathbb{T}^p_q$ всех тензоров типа (p, q) образует линейное пространство размерности n^{p+q}. Если фиксированы локальные координаты $(x^1, \dots, x^n)$, то в качестве базиса в пространстве $\mathbb{T}^p_q$ можно выбрать полилинейные отображения

$$\Delta^{j_1 \dots j_q}_{i_1 \dots i_p},$$

где все индексы $i_k u j_l$ пробегают независимо всевозможные значения от 1 до п. Этот базис называется каноническим.

Замечание. Часто полилинейные отображения $\Delta^{j_1 \dots j_q}_{i_1 \dots i_p}$ обозначают так:

$$\Delta^{j_1 \dots j_q}_{i_1 \dots i_p} = \partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q}.$$

Естественность этих более громоздких обозначений будет понятна из следующей лекции.

Отметим, что если $(x^1, \dots, x^n)$ и $\left(x^{1'}, \dots, x^{n'}\right)$ - две системы координат в окрестности точки P, то в пространстве $\mathbb{T}^p_q$ тензоров типа $(\ p, q\ )$ в точке P возникает два канонических базиса

$$\Delta^{j_1 \dots j_q}_{i_1 \dots i_p} \quad \text{и} \quad \Delta^{j'_1 \dots j'_q}_{i'_1 \dots i'_p}$$

которые связаны между собой, как несложно проверить, следующим образом:

$$\Delta^{j'_1 \dots j'_q}_{i'_1 \dots i'_p} = \frac{\partial x^{i_1}}{\partial x^{i'_1}} \dots \frac{\partial x^{i_p}}{\partial x^{i'_p}} \frac{\partial x^{j'_1}}{\partial x^{j_1}} \dots \frac{\partial x^{j'_q}}{\partial x^{j_q}} \Delta^{j_1 \dots j_q}_{i_1 \dots i_p}.$$

15.1.4 Problems

Задача 1.1. Пусть на двойственном пространстве $T^*_P M$ к касательному пространству, т.е. на пространстве линейных функционалов на $T_P M$, задана билинейная форма $B : T^*_P M \times T^*_P M \to \mathbb{R}^1$. Если фиксированы локальные координаты $(\ x^1, \dots, x^n\ )$, то форма B однозначно определяется своими значениями на всевозможных парах ковекторов из канонического базиса $(dx^i)^n_{i=1}$. Эти значения называются компонентами формы B в координатах $(x^1, \dots, x^n)$. Вывести закон преобразования компонент формы B при замене координат. Совпадает ли он с законом преобразования компонент билинейной формы на $T_P M$?

Задача 1.2. Назовем тензор инвариантным, если его компоненты не меняются ни при каких заменах координат. Доказать, что набор δ^j_i символов Кронекера задает

инвариантный тензор типа $(1,1)$. Проверить, что тот же самый набор δ_{ij} задает уже не инвариантный тензор типа $(0,2)$.

Задача 1.3. Описать все инвариантные тензоры типа $(1,1),(1,2),(2,1)$ и $(2,2)$. Существуют ли инвариантные тензоры типа $(\ p,q\ )$, если $p \neq q$?

Задача 1.4. Каким полилинейным отображениям соответствуют вектор, ковектор и линейный оператор? Выпишите явные формулы в локальных координатах.

Задача 1.5. Показать, что операция взятия смешанного произведения в $\mathbb{R}^3$ является тензором типа $(0,3)$ и найти его компоненты в стандартных декартовых координатах.

Задача 1.6. Показать, что операция взятия векторного произведения в $\mathbb{R}^3$ задает тензор типа $(1,2)$ и найти его компоненты в стандартных декартовых координатах.

Задача 1.7. На плоскости $\mathbb{R}^2$ фиксированы декартова $(\ x,y\ )$ и полярная $(\ r,\varphi\ )$ системы координат. В декартовых координатах задана точка $P=(\sqrt{3},1)$ и вектор $V \in T_P\mathbb{R}^2, V=(3,4)$. В полярных координатах задан ковектор $\xi \in T_P^*\mathbb{R}^2, \xi=(5,6)$. Вычислить $\xi(V)$.

15.1.5 Extra Topics

1.1. Линейный оператор - тензор типа $(1,1)$. Пусть снова M - гладкое n мерное многообразие, P - некоторая точка из M, и пусть $F: M \to M$ - гладкое отображение многообразия M в себя, причем $F(P)=P$, т.е. P - неподвижная точка отображения F. Тогда, как мы уже знаем, определено линейное отображение $dF|_P : T_PM \to T_PM$ касательного пространства T_PM в себя - дифференциал гладкого отображения F в точке P. Если на многообразии в окрестности точки P заданы локальные координаты $(\ x^1,\ldots,x^n\ )$, то, напомним, линейное отображение $dF|_P$ задается матрицей

$$\left(\frac{\partial f^i}{\partial x^j}\right) \tag{*}$$

где $x^i = f^i\left(x^1,\ldots,x^n\right), i=1,\ldots,n,$ - координатное представление отображения F в координатах $(\ x^1,\ldots,x^n\ )$. Пусть теперь на многообразии M в окрестности точки P заданы еще одни локальные координаты $(\ x^{1'},\ldots,x^{n'}\ )$. Тогда, естественно, линейное отображение $dF|_P$ задается в этих новых координатах уже другой матрицей, элементы которой связаны с элементами матрицы (*) так:

$$\frac{\partial f^{i'}}{\partial x^{j'}} = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial f^i}{\partial x^j}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}},$$

где $x^{i'} = f^{i'}\left(x^{1'},\ldots,x^{n'}\right)$ - координатное представление отображения F в координатах $(\ x^{1'},\ldots,x^{n'}\ )$. Действительно, координатное представление отображения F в координатах $(\ x^{1'},\ldots,x^{n'}\ )$ можно записать в виде

$$\begin{aligned} x^{i'} = x^{i'}\Big(f^1\Big(x^1\Big(x^{1'},\ldots,x^{n'}\Big),\ldots,x^n\Big(x^{1'},\ldots,x^{n'}\Big)\Big),\ldots \\ \ldots,f^n\Big(x^1\Big(x^{1'},\ldots,x^{n'}\Big),\ldots,x^n\Big(x^{1'},\ldots,x^{n'}\Big)\Big)\Big), \end{aligned}$$

поэтому остается воспользоваться теоремой о дифференцировании сложной функции. Отметим, что закон преобразования элементов матрицы дифференциала также известен нам из линейной алгебры и совпадает с законом изменения матрицы линейного оператора при замене координат в линейном пространстве. Действительно, если A - произвольный линейный оператор, заданный на T_PM, и $(\ x^1,\ldots,x^n\ )$ локальные координаты в окрестности точки P, то оператор A однозначно задается компонентами образов векторов канонического базиса $(\partial_{x^i})_{i=1}^n$ в этом же самом базисе. А именно, если

$V = (v^1, \ldots, v^n)$ - произвольный касательный вектор из $T_P M$, заданный своими координатами, то

$$A(V) = A\left(\sum_{\alpha=1}^{n} v^\alpha \partial_{x^\alpha}\right) = \sum_{\alpha=1}^{n} v^\alpha A\left(\partial_{x^\alpha}\right),$$

и, положив $A\left(\partial_{x^\alpha}\right) = a_\alpha^1 \partial_{x^1} + \cdots + a_\alpha^n \partial_{x^n}$, получим

$$A(V) = \sum_{\alpha=1}^{n} v^\alpha \left(\sum_{\beta=1}^{n} a_\alpha^\beta \partial_{x^\beta}\right) = \sum_{\beta=1}^{n} \left(\sum_{\alpha=1}^{n} a_\alpha^\beta v^\alpha\right) \partial_{x^\beta}.$$

Числа a_α^β называются компонентами линейного оператора A в координатах x^i. Пусть теперь ($x^{1'}, \ldots, x^{n'}$) - другие локальные координаты в окрестности точки P. Тогда компоненты линейного оператора A в этих новых координатах определяются из соотношения

$$A\left(\partial_{x^{\alpha'}}\right) = \sum_{\beta'=1}^{n} a_{\alpha'}^{\beta'} \partial_{x^{\beta'}}$$

Воспользовавшись связью между векторами из канонических базисов, перепишем последнее соотношение в виде:

$$A\left(\sum_{p=1}^{n} \frac{\partial x^p}{\partial x^{\alpha'}} \partial_{x^p}\right) = \sum_{\beta'=1}^{n} a_{\alpha'}^{\beta'} \left(\sum_{q=1}^{n} \frac{\partial x^q}{\partial x^{\beta'}} \partial_{x^q}\right),$$

откуда, по определению компонент оператора A в координатах ($x^1, \ldots, x^n$),

$$\sum_{p=1}^{n} \frac{\partial x^p}{\partial x^{\alpha'}} \left(\sum_{q=1}^{n} a_p^q \partial_{x^q}\right) = \sum_{\beta'=1}^{n} a_{\alpha'}^{\beta'} \left(\sum_{q=1}^{n} \frac{\partial x^q}{\partial x^{\beta'}} \partial_{x^q}\right).$$

Приравняв коэффициенты при базисных векторах $\partial_x q$, найдем:

$$\sum_{p=1}^{n} \frac{\partial x^p}{\partial x^{\alpha'}} a_p^q = \sum_{\beta'=1}^{n} a_{\alpha'}^{\beta'} \frac{\partial x^q}{\partial x^{\beta'}},$$

откуда, умножив на обратную матрицу, получаем окончательно:

$$a_{\alpha'}^{\beta'} = \sum_{p,q=1}^{n} \frac{\partial x^p}{\partial x^{\alpha'}} \frac{\partial x^{\beta'}}{\partial x^q} a_p^q$$

Эти формулы называются тензорным законом преобразования компонент линейного оператора.

Таким образом, линейный оператор - это тензор типа $(1,1)$. С точки зрения инвариантного определения, тензор A типа $(1,1)$ представляет собой два-линейное отображение $A : T_P^* M \times T_P M \to \mathbb{R}$. Построим по каждому такому отображению A соответствующий линейный оператор $L_A : T_P M \to T_P M$. Положим

$$\xi\left(L_A(V)\right) = A(\xi, V)$$

для произвольных $\xi \in T_P^* M$ и $V \in T_P M$. Ясно, что значение оператора на каждом векторе $V \in T_P M$ определено однозначно. Кроме того, в локальных координатах ($x^1, \ldots, x^n$) компоненты вектора $L_A(V)$ могу' быть вычислены так:

$$L_A(V)^i = dx^i\left(L_A(V)\right) = A\left(dx^i, V\right) = a^i_j V^j$$

где a^i_j - компоненты тензора A в координатах $((^1, \ldots, (^n)\, x)$. Поэтому L_A - действительно линейный оператор. Обратно, та же формула однозначно определяет тензор A типа $(1,1)$ по линейному оператору L_A.
1.2. Одна очень простая задача. Пусть задан тензор T типа (1,2). Найти его компоненту $T^{1'}_{2'3'}$ в локальных координатах ($x^{1'}, \ldots, x^{n'}$), если даны все его компоненты T^i_{jk} в локальных координатах $(x^1, \ldots, x^n)$.

На всякий случай, приведем ответ к этой очень популярной на экзаменах задаче:

$$T^{1'}_{2'3'} = \frac{\partial x^{1'}}{\partial x^i}\frac{\partial x^j}{\partial x^{2'}}\frac{\partial x^k}{\partial x^{3'}} T^i_{jk}$$

по индексам i, j и k подразумевается суммирование.
1.3. Как строить примеры тснзорных полей. Мы дали общес определение тепзорного поля, но как построить пример такого поля на гладком многообразии? Конечно, можно рассмотреть тензорное поле любого типа, равное нулю в каждой точке многообразия. А как построить нетривиальный, т.е. не нулевой, пример? В прошлом

семестре мы уже видели, как теорема Уитни о существовании вложения многообразия в евклидово пространство подходящей размерности позволяет построить пример римановой метрики, т.е. тензорного поля типа $(0,2)$. Эта же теорема приходит на помощь и в общем случае. А именно, ограничение гладкой функции, заданной на $\mathbb{R}^n$, на подмногообразие $M \subset \mathbb{R}^n$ является гладкой функцией на многообразии M. Так можно построить пример не постоянной функции на многообразии. Взяв дифференциал, мы получаем не нулевое ковекторное поле, т.е. тензорное поле типа $(0,1)$. Используя алгебраические операции, описанные в следующей лекции, можно построить примеры тензорных полей любого типа.

15.2 2: Algebraic operations on tensors and tensor fields

В данном лекции мы определим несколько операций, позволяющих конструировать новые тензоры (тензорные поля) из уже имеющихся. Такие операции обычно называются тензорньлми. Для простоты, мы будем вести речь о тензорах в фиксированной точке многообразия, но все определяемые ниже операции естественно переносятся и на тензорные поля. Там где это необходимо, мы приводим соответствующие комментарии.

15.2.1 2.1. Linear Combination

Пусть имеется два тензора P и T одинакового типа (p,q). Тогда для любых двух чисел λ и μ определен новый тензор $S = \lambda P + \mu T$ того же типа, называемый линейной комбинацией тензоров P и T с коэффициентами $\lambda u \mu$. Здесь сложение и умножение на числа происходит в линейном пространстве $\mathbb{T}^p_q$ тензоров типа (p, q), поэтому S - действительно тензор типа (p, q). Если фиксированы локальные координаты ($x^1, \ldots, x^n$), то, очевидно, компоненты тензора S выражаются через компонент тензоров P и T так:

$$S^{i_1 \ldots i_p}_{j_1 \ldots j_q} = \lambda P^{i_1 \ldots i_p}_{j_1 \ldots j_q} + \mu T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$$

Замечание. Отметим, что если мы работаем с тензорными полями, то в качестве коэффициентов в линейной комбинации выступают гладкие функции на многообразии.

15.2.2 2.2. Permutation of indices of the same type

Пусть задан тензор T типа (p,q), и предположим, что задана некоторая перестановка $\sigma \in S_q$ множества из q элементов. Построим новый тензор σT, который определяется так: для любых ковекторов $\xi^1,\ldots,\xi^p$ и векторов $V_1,\ldots,V_q$ положим

$$_\sigma T\left(\xi^1,\ldots,\xi^p;V_1,\ldots,V_q\right)=T\left(\xi^1,\ldots,\xi^p;V_{\sigma(1)},\ldots,V_{\sigma(q)}\right)$$

т.е. мы переставили векторные аргументы полилинейного отображения T с помощью перестановки σ. Очевидно, $_\sigma T$ - полилинейное отображение, т.е. мы действительно определили некоторый новый тензор $_\sigma T$. Говорят, что тензор σT получен из тензора T перестановкой σ нижних индексов.

Нам будет полезна следующая лемма.

Лемма 2.1. Пусть σ и τ - перестановки из S_q. Тогда $_\tau\left(_\sigma T\right)=\,_{\tau\sigma}T$.

Доказательство. Действительно, для произвольных векторов $V_1,\ldots,V_q$ имеем:

$$\begin{aligned}{}_\tau\left(_\sigma T\right)\left(V_1,\ldots,V_q\right)&={}_\sigma T\left(V_{\tau(1)},\ldots,V_{\tau(q)}\right)=T\left(V_{\tau(\sigma(1))},\ldots,V_{\tau(\sigma(q))}\right)=\\&={}_{\tau\sigma}T\left(V_1,\ldots,V_q\right).\end{aligned}$$

Лемма доказана.

Пусть $(x^1,\ldots,x^n)$ - локальная система координат, и $T^{i_1\ldots i_p}_{j_1\ldots j_q}$ - компоненты тензора T относительно ($x^1,\ldots,x^n$). Найдем компоненты тензора $_\sigma T$, для чего вычислим его значение на произвольной комбинации базисных векторов. Получим:

$$\begin{aligned}\sigma^{i_1\ldots i_p}_{j_1\ldots j_q}&={}_\sigma T\left(dx^{i_1},\ldots,dx^{i_p};\partial_{x^{j_1}},\ldots,\partial_{x^{j_q}}\right)\\&=T\left(dx^{i_1},\ldots,dx^{i_p};\partial_{x^{j_{\sigma(1)}}},\ldots,\partial_{x^{j_{\sigma(q)}}}\right)=T^{i_1\ldots i_p}_{j_{\sigma(1)}\ldots j_{\sigma(q)}}\end{aligned}$$

Итак, доказана следующая лемма.

Лемма 2.2. Если тензор $_\sigma T$ получен из тензора T перестановкой σ нижних индексов, то компоненты тензора $_\sigma T$ в локальной системе координат $(x^1,\ldots,x^n)$ выражаются так:

$$_\sigma T^{i_1\ldots i_p}_{j_1\ldots j_q}=T^{i_1\ldots i_p}_{j_{\sigma(1)}\ldots\ldots j_{\sigma(q)}}$$

где $T^{i_1\ldots i_p}_{j_1\ldots j_q}$– компоненты тензора T в $(x^1,\ldots,x^n)$.

Замечание. Отметим, что перестановка меняет местами расположение индексов а не их значения. Например, если $\sigma=(1,2,3)$ - циклическая перестановка, т.е. $(\sigma(1),\sigma(2),\sigma(3))=(2,3,1)$ то, скажем,

$$_\sigma T_{789}=T_{897},$$

и вообще,

$$_\sigma T_{j_1j_2j_3}=T_{j_2j_3j_1}$$

Аналогично определяется перестановка верхних индексов: если T тензор типа (p,q), и π - произвольная перестановка из S_p, то тензор $^\pi T$ определяется так: для любых ковекторов $\xi^1,\ldots,\xi^p$ и векторов $V_1,\ldots,V_q$ положим

$$^\pi T\left(\xi^1,\ldots,\xi^p;V_1,\ldots,V_q\right)=T\left(\xi^{\pi(1)},\ldots,\xi^{\pi(p)};V_1,\ldots,V_q\right).$$

Говорят, что тензор $^\pi T$ получен из тензора T перестановкой π верхних индексов. Имеет место следующее утверждение.

Лемма 2.3. Если тензор $^\pi T$ получен из тензора T перестановкой π верхних индексов, то компоненты тензора $^\pi T$ в локальной системе координат $(x^1,\ldots,x^n)$ выражаются так:

$$^{\pi}T^{i_1\dots i_p}_{j_1\dots j_q} = T^{i_{\pi(1)}\dots i_{\pi(p)}}_{j_1\dots j_q},$$

где $T^{i_1\dots i_p}_{j_1\dots j_q}$ – компоненты тензора T в $(x^1,\dots,x^n)$.
Замечание. Менять местами верхние и нижние индексы, вообще говоря, нельзя, см. задачу 2.2.

15.2.3 2.3. свертка

Если предыдущие операции было удобно определять в инвариантной форме, то операцию свертки нам будет проще определить в локальных координатах. Разберем сначала простейший случай тензора типа $(1,1)$ - линейного оператора.

Итак, пусть A - тензор типа $(1,1)$ в T_PM, и ($x^1,\dots,x^n$) - произвольные локальные координаты в окрестности точки P. Обозначим, как обычно, через A^i_j компоненты тензора A в координатах $(x^1,\dots,x^n)$. Определим скаляр $C^1_1(A)$, положив $C^1_1(A) = A^i_i$. Эта величина называется сверткой линейного оператора или его следом.

Покажем, что величина $C^1_1(A)$ не зависит от выбора локальных координат, фигурирующих в определении. Для этого рассмотрим в окрестности точки P другую систему локальных координат ($x^{1'},\dots,x^{n'}$), и пусть $A^{i'}_{j'}$ - компоненты тензора A в этих новых координатах. Нам нужно проверить, что $A^{i'}_{i'} = A^i_i$. Действительно, воспользовавшись тензорным законом, получаем:

$$A^{i'}_{i'} = \frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{i'}}A^i_j = \delta^j_i A^i_j = A^i_i,$$

где δ^j_i - символы Кронекера, что и требовалось. Итак, для каждого тепзора типа $(1,1)$, мы построили скаляр, т.е. величину, пе зависящую от локальных координат, называемую сверткой этого линейного оператора. Отметим, что след тензора A типа $(1,1)$ можно, также, записать в виде $C^1_1(A) = A\left(dx^i;\partial_{x^i}\right)$.

Рассмотрим теперь общий случай. Пусть задан тензор T типа (p,q), где $p\geq 1$ и $q\geq 1$, и пусть $T^{i_1\dots i_p}_{j_1\dots j_q}$ - его компоненты в локальных координатах $(x^1,\dots,x^n)$. Фиксируем два целых числа α и β, такие что $1\leq\alpha\leq p$ и $1\leq\beta\leq q$. Построим новый тензор $C^\alpha_\beta(T)$ типа $(p-1,q-1)$, компоненты которого в координатах ($x^1,\dots,x^n$) имеют вид:

$$\left(C^\alpha_\beta(T)\right)^{i_1\dots i_{p-1}}_{j_1\dots j_{q-1}} = T^{i_1\dots i_{\alpha-1}k i_\alpha\dots i_{p-1}}_{j_1\dots j_{\beta-1}k j_\beta\dots j_{q-1}}$$

Тензор $C^\alpha_\beta(T)$ называется сверткой тензора T по верхнему индексу с номером α и нижнему индексу с номером β. Чтобы проверить, что $C^\alpha_\beta(T)$ - корректно определенный тензор, следует рассмотреть еще одни локальные координаты ($x^{1'},\dots,x^{n'}$), в которых определен, вообще говоря, другой тензор $\tilde{C}^\alpha_\beta(T)$, компоненты которого в координатах ($x^{1'},\dots,x^{n'}$) имеют вид

$$\left(\tilde{C}^\alpha_\beta(T)\right)^{i'_1\dots i'_{p-1}}_{j'_1\dots j'_{q-1}} = T^{i'_1\dots i'_{\alpha-1}k' i'_\alpha\dots i'_{p-1}}_{j'_1\dots j'_{\beta-1}k' j'_\beta\dots j'_{q-1}},$$

и, воспользовавшись тензорным законом, убедиться что $\tilde{C}^\alpha_\beta(T)$ совпадает с $C^\alpha_\beta(T)$.

Упражнение 2.1. Проверить совпадение тензоров $C^\alpha_\beta(T)$ и $\tilde{C}^\alpha_\beta(T)$ с помощью тензорного закона.

Особый интерес представляет случай тензоров типа (p,p). В этом случае с помощью последовательного применения p штук последовательных операций свертки можно построить по тензору скаляр, который называется полной сверткой.

Упражнение 2.2. Зависит ли полная свертка тензора от того, по каким парам индексов производится сворачивание?

15.2.4 2.4. Tensor product

Пусть в T_PM заданы два тензора: тензор T типа (p,q) и тензор S типа (u,v). Определим новый тензор $T\otimes S$ типа $(\ p+u, q+v\ )$, положив для произвольных ковекторов $\xi^1,\dots,\xi^{p+u}$, и векторов $V_1,\dots,V_{q+v}$

$$\begin{aligned}&(T\otimes S)\left(\xi^1,\dots,\xi^{p+u};V_1,\dots,V_{q+v}\right)=\\&\quad=T\left(\xi^1,\dots,\xi^p;V_1,\dots,V_q\right)S\left(\xi^{p+1},\dots,\xi^{p+u};V_{q+1},\dots,V_{q+v}\right)\end{aligned}$$

Тривиальным образом проверяется, что $T\otimes S$ действительно тензор, т.е. что отображение $T\otimes S$ - полилинейно. Действительно, например

$$\begin{aligned}&(T\otimes S)(\dots,a\xi+b\psi,\dots)=T(\dots,a\xi+b\psi,\dots)S(\dots)=\\&\quad=aT(\dots,\xi,\dots)S(\dots)+bT(\dots,\psi,\dots)S(\dots)=\\&\quad=a(T\otimes S)(\dots,\xi,\dots)+b(T\otimes S)(\dots,\psi,\dots)\end{aligned}$$

где ковектор $a\xi+b\psi$ стоит на месте с номером, не превосходящим p. Тензор $T\otimes S$ называется тензорным произведением тензоров T и S.

Нам будут полезны следующие свойства тензорного произведения.

Лемма 2.4. Тензорное произведение обладает следующими свойствами.

- Ассоичативность: для любых тензоров T,S и R, имеем:

$$T\otimes(S\otimes R)=(T\otimes S)\otimes R$$

- Дистрибутивность: для любых тензоров T_1,T_2 из $\mathbb{T}^p_q$, и S_1,S_2 из $\mathbb{T}^u_v$, и для любых чисел $\alpha_1 u\alpha_2$,

$$\begin{aligned}(\alpha_1T_1+\alpha_2T_2)\otimes S_1&=\alpha_1\left(T_1\otimes S_1\right)+\alpha_2\left(T_2\otimes S_1\right),\\T_1\otimes(\alpha_1S_1+\alpha_2S_2)&=\alpha_1\left(T_1\otimes S_1\right)+\alpha_2\left(T_1\otimes S_2\right)\end{aligned}$$

Замечание. Тензорное произведение тензорных полей дистрибутивно относительно умножения на гладкие функции α_i.

Замечание. Тензорное произведение, вообще говоря, не коммутативно. Например, если (x,y) - координаты на плоскости, то $dx\otimes dy\neq dy\otimes dx$. В общем случае тензоры $T\otimes S$ и $S\otimes T$ отличаются перестановкой индексов.

Пусть $(x^1,\dots,x^n)$ - локальная система координат в окрестности точки P. Вычислим компоненты тензора $T\otimes S$ в этих координатах. Получим:

$$\begin{aligned}(T\otimes S)^{i_1\dots i_{p+u}}_{j_1\dots j_{q+v}}=&(T\otimes S)\left(dx^{i_1},\dots,dx^{i_{p+u}};\partial_{x^{j_1}},\dots,\partial_{x^{j_{q+v}}}\right)=\\=&T\left(dx^{i_1},\dots,dx^{i_p};\partial_{x^{j_1}},\dots,\partial_{x^{j_q}}\right)\cdot\\&\cdot S\left(dx^{i_{p+1}},\dots,dx^{i_{p+u}};\partial_{x^{j_{q+1}}},\dots,\partial_{x^{j_{q+v}}}\right)=\\=&T^{i_1\dots i_p}_{j_1\dots j_q}S^{i_{p+1}\dots i_{p+u}}_{j_{q+1}\dots j_{q+v}}\end{aligned}$$

Таким образом, имеется следующий результат.

Лемма 2.5. Компоненты тензорного произведения $T\otimes S$ тензоров TuS вычисляются по формуле

$$(T\otimes S)^{i_1\dots i_{p+u}}_{j_1\dots j_{q+v}}=T^{i_1\dots i_p}_{j_1\dots j_q}S^{i_{p+1}\dots i_{p+u}}_{j_{q+1}\dots j_{q+v}}.$$

Значимость операции тензорного произведения определяется следующим утверждением.

Утверждение 2.1. Пространство $\mathbb{T}^p_q$ всех тензоров типа (p,q) можно представить в виде тензорного произведения p экземпляров пространства $T_PM = \mathbb{T}^1_0$ и q экземпляров $T^*_PM = \mathbb{T}^0_1$.

Доказательство. Действительно, в утверждении 1.2 мы построили базис пространства $\mathbb{T}^p_q$, состоящий из отображений вида

$$\partial_{x^{i_1}} \otimes \cdots \otimes \partial_{x^{i_p}} \otimes dx^{j_1} \otimes \cdots \otimes dx^{j_q}.$$

Теперь ясно, что обозначения для этих полилинейных отображений были выбраны не случайно: каждое из них является тензорным произведением p соответствующих векторов $\partial_{x^i k}$ канонического базиса пространства T_PM и q соответствующих ковекторов dx^{j_l} канонического базиса пространства T^*_PM. Утверждение доказано.

Пример. Пусть фиксированы локальные координаты $(\ x^1, \ldots, x^n\ )$, и пусть $dx^{i_1}, \ldots, dx^{i_q}$ - произвольные q ковекторов соответствующего канонического базиса. Построим тензор типа $(\ 0, q\ )$, положив

$$T = dx^{i_1} \otimes \cdots \otimes dx^{i_q}$$

Нам будет полезна следующая лемма, доказательство которой получается прямым подсчетом.

Лемма 2.6. В сделанных выше обозначениях, для произвольной перестановки $\sigma \in S_q$, имеем:

$${}_\sigma T = dx^{i\sigma_{\sigma^{-1}(1)}} \otimes \cdots \otimes dx^{i_{\sigma^{-1}(q)}}$$

Замечание. Отметим, что по определению тензорного произведения имеет место следующее тождество для произведения базисных элементов:

$$\left(\partial_{x^{i_1}} \otimes \cdots \partial_{x^{i_p}} \otimes dx^{j_1} \otimes \cdots dx^{j_q}\right) \otimes \left(\partial_{x^{\alpha_1}} \otimes \cdots \partial_{x^{\alpha_p}} \otimes dx^{\beta_1} \otimes \cdots dx^{b_q}\right) =$$
$$= \partial_{x^{i_1}} \otimes \cdots \partial_{x^{i_p}} \otimes \partial_{x^{\alpha_1}} \otimes \cdots \partial_{x^{\alpha_p}} \otimes dx^{j_1} \otimes \cdots dx^{j_q} \otimes dx^{\beta_1} \otimes \cdots dx^{b_q}$$

В частности, $dx^i \otimes \partial_{x^j} = \partial_{x^j} \otimes dx^i$.

15.2.5 2.5. Lowering and raising the index

Предположим, что фиксирован некоторый тензор A типа $(\ 0{,}2\ )$. Тензор A называется невырожденньлм, если в какой-нибудь, а значит и в любой локальной системе координат $(\ x^1, \ldots, x^n\ )$ определитель матрицы $(\ A_{ij}\ )$ компонент тензора A отличен от нуля. В этом случае существует обратная матрица $(A_{ij})^{-1}$, элементы которой мы обозначим через A^{ij}.

Лемма 2.7. В сделанных выше обозначениях, соответствие

$$\left(x^1, \ldots, x^n\right) \mapsto \left(A^{ij}\right)$$

задает тензор типа $(2,0)$.

Доказательство. Действительно, пусть $\left(x^{1'}, \ldots, x^{n'}\right)$ - другая система локальных координат, $A_{i'j'}$ - компоненты тензора A в этих новых координатах, и $(\ A^{i'}j'\ )$ - элементы матрицы, обратной к невырожденной матрице $(\ A_{i'j'}\ )$. Последнее означает, что $A_{i'\alpha'}A^{\alpha'j'} = \delta^{j'}_{i'}$. Применив тензорный закон к тензорам A и δ, получим

$$\frac{\partial x^{j'}}{\partial x^p}\frac{\partial x^q}{\partial x^{i'}}\delta^p_q = \frac{\partial x^i}{\partial x^{i'}}\frac{\partial x^\alpha}{\partial x^{\alpha'}}A_{i\alpha}A^{\alpha'j'}$$

поэтому

$$\delta_q^p = \frac{\partial x^p}{\partial x^{j'}} \left(\frac{\partial x^{i'}}{\partial x^q} \frac{\partial x^i}{\partial x^{i'}} \right) \frac{\partial x^\alpha}{\partial x^{\alpha'}} A_{i\alpha} A^{\alpha' j'} = \frac{\partial x^p}{\partial x^{j'}} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \left(\delta_q^i A_{i\alpha} \right) A^{\alpha' j'} =$$
$$= A_{q\alpha} \left(\frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^p}{\partial x^{j'}} A^{\alpha' j'} \right)$$

Отсюда, в силу единственности обратной матрицы, заключаем:

$$\left(\frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^p}{\partial x^{j'}} A^{\alpha' j'} \right) = A^{\alpha p},$$

т.е. для элементов матрицы, обратной к матрице тензора A, выполняется тензорный закон, что и требовалось.

Таким образом, если фиксирован невырожденный тензор A типа $(0,2)$, то определен однозначно и тензор типа $(2,0)$, компоненты которого в каждых локальных координатах образуют обратную матрицу к матрице компонент тензора A. Это тензор будем обозначать через A^{-1} и называть обратньbм по отношению к A.

Пример. Важным примером невырожденного тензора типа $(0,2)$ является риманова метрика. Именно она обычно используется для подъема и опускания индексов тензоров на римановых многообразиях.

Итак, пусть в точке P фиксирован невырожденный тензор A типа $(0,2)$. Пусть, далее, T - тензор типа $(\ p, q\ )$, причем $p \geq 1$. Определим новый тензор L типа $(\ p-1, q+1\ )$, положив $L = C_2^1(A \otimes T)$. Таким образом, тензор L получается в результате применения двух последовательных операций - тензорного произведения и свертки. Говорят, что тензор L получен из тензора T опусканием первого индекса с помощью тензора A. Аналогично можно определить опускание k-го индекса: $C_2^k(A \otimes T)$, где $1 \leq k \leq p$.

Далее, $q \geq 1$, и $1 \leq k \leq q$, то можно построить новый тензор U типа $(\ p+1, q-1\ )$, положив $U = C_k^2\left(A^{-1} \otimes T\right)$. Говорят, что тензор U получен из тензора T поднятием k-го индекса с помощью тензора A.

Выясним, как выглядит операция опускания и поднятия индекса в локальных координатах. Следующая лемма очевидна.

Лемма 2.8. Если тензоры L и U получень из тензора T типа $(\ p, q\ )$ операциями опускания и поднятия первого индекса с помощью тензора A типа $(0,2)$, то в произвольных локальных координатах $(x^1, \ldots, x^n)$ их компоненты вычисляются так:

$$L_{j_1 \ldots j_{q+1}}^{i_1 \ldots i_{p-1}} = A_{j_1 \alpha} T_{j_2 \ldots j_{q+1}}^{\alpha i_1 \ldots i_{p-1}}, \quad U_{j_1 \ldots j_{q-1}}^{i_1 \ldots i_{p+1}} = A^{i_1 \alpha} T_{\alpha j_1 \ldots j_{q-1}}^{i_2 \ldots i_{p+1}}.$$

Непосредственно из леммы 2.8 вытекает следующий полезный факт.

Лемма 2.9. Операции опускания и поднятия индекса взаимно обратны в следующем смысле: если у тензора T сначала, скажем, опустить а затем поднять индекс с помощью одного и того же тензора A, то в результате получится исходный тензор T с точностью до перестановки индексов.

Доказательство. Действительно, если $T_{j_1 \ldots j_q}^{i_1 \ldots i_p}$ - компоненты тензора T в координатах $(\ x^1, \ldots, x^n\ )$, и мы опустим у T верхний k-ый индекс с помощью тензора $A = (A_{ij})$, то получим тензор L с компонентами

$$L_{j_1 \ldots j_{q+1}}^{i_1 \ldots i_{p-1}} = A_{j_1, \alpha} T_{j_2 \ldots j_{q+1}}^{i_1 \ldots i_{k-1} \alpha i_k \ldots i_{p-1}}$$

Если мы теперь поднимем у L первый индекс, то получим тензор U с компонентами

$$U_{j_1 \ldots j_q}^{i_1 \ldots i_p} = A^{i_1 \beta} L_{\beta j_1 \ldots j_q}^{i_2 \ldots i_p} = A^{i_1 \beta} A_{\beta, \alpha} T_{j_1 \ldots j_q}^{i_2 \ldots i_k \alpha i_{k+1} \ldots i_p} = T_{j_1 \ldots j_q}^{i_2 \ldots i_k i_1 i_{k+1} \ldots i_p}.$$

Тензор U, очевидно, отличается от исходного тензора T на перестановку

$$\begin{pmatrix} 1 & 2 & \dots & k-1 & k & k+1 & \dots & n \\ 2 & 3 & \dots & k & 1 & k+1 & \dots & n \end{pmatrix},$$

что и требовалось.

Замечание. Если фиксирован невырожденный тензор типа $(0,2)$, то операции опускания и поднятия индексов задают каноническое, т.е. не зависящее от локальных координат, отождествление двойственных пространств T_PM и T_P^*M. Действительно, опустив индекс у вектора, мы получим ковектор, и наоборот, подняв индекс у ковектора, мы получим вектор.

Замечание. Если невырожденный тензор A типа $(0,2)$, с помощью которого мы опускаем или поднимаем индекс, в некоторой системе координат имеет вид δ_{ij} - символы Кронекера, то в этих координатах операция опускания и поднятия индекса тривиальна: соответствующие компоненты тензоров просто совпадают.

15.2.6 2.6. Symmetry and alternation

Пусть T - произвольный тензор типа $(\ 0,q\ )$. Построим по нему новые тензоры $S(T)$ и $A(T)$ того же типа, положив

$$S(T) = \frac{1}{q!}\sum_{\sigma\in S_q} {}_\sigma T, \quad A(T) = \frac{1}{q!}\sum_{\sigma\in S_q} (-1)^\sigma {}_\sigma T,$$

где суммирование берется по всем перестановкам σ множества из q элементов. Говорят, что тензор $S(T)$ получен из тензора T симметрированием или симметризацией, а тензор $A(T)$ - альтернированием. Ясно, что симметризация и альтернирование - это результат последовательного применения перестановок индексов и взятия линейной комбинации тензоров, поэтому $S(T)$ и $A(T)$ - действительно тензоры.

Лемма 2.10. Компоненты тензоров $S(T)$ и $A(T)$ имеют в локальных координатах $(x^1,\dots,x^n)$ следующий вид:

$$S(T)_{j_1\dots j_q} = \frac{1}{q!}\sum_{\sigma\in S_q} T_{j_{\sigma(1)}\dots j_{\sigma(q)}}, \quad A(T)_{j_1\dots j_q} = \frac{1}{q!}\sum_{\sigma\in S_q} (-1)^\sigma T_{j_{\sigma(1)}\dots j_{\sigma(q)}}.$$

Нам будут полезна следующая очевидная лемма.

Лемма 2.11. Операции альтернирования и симметрирования линейны:
$A\left(\alpha_1 T_1 + \alpha_2 T_2\right) = \alpha_1 A\left(T_1\right) + \alpha_2 A\left(T_2\right), \quad S\left(\alpha_1 T_1 + \alpha_2 T_2\right) = \alpha_1 S\left(T_1\right) + \alpha_2 S\left(T_2\right).$

Тензор T типа $(0,q)$ называется симметричным, если для любой перестановки $\sigma \in S_q$ имеет место равенство $T = {}_\sigma T$. Тензор T типа $(0,q)$ называется кососимметричным, если для любой перестановки $\sigma \in S_q$ имеет место соотношение $T = (-1)^\sigma {}_\sigma T$. Имеет место следующий результат.

Лемма 2.12. Для произвольного тензора T, Тогда тензор $S(T)$ - симметричен, а тензор $A(T)$ - антисимметричен. Симметрирование не меняет симметричных тензоров, и обращает в нуль любой кососимметричный тензор. Аналогично, альтернирование не меняет кососимметричных тензоров, и обращает любой симметричный тензор в нуль.

Доказательство. Докажем, например, последнее утверждение. Если T - симметричный тензор, то $T = {}_\sigma T$, поэтому

$$A(T) = \frac{1}{q!}\sum_{\sigma\in S_q} (-1)^\sigma {}_\sigma T = \frac{T}{q!}\left(\sum_{\sigma \text{ четная}} 1 + \sum_{\sigma \text{ нечетная}} -1\right) = 0,$$

так как четных перестановок столько же, сколько нечетных, что и требовалось. Остальные утверждения доказываются аналогично.

Ясно, что линейная комбинация симметричных (кососимметричных) тензоров представляет собой симметричный (соответственно, кососимметричный) тензор. Поэтому множество всех симметричных (кососимметричных) тензоров представляет собой линейное подпространство в пространстве всех тензоров данного типа.

Замечание. Мы определили операциях симметрирования и альтернирования для тензоров типа ($0, q$), хотя их можно аналогично определить и для тензоров типа ($p, 0$), а также и для тензоров произвольного типа, проводя симметрирование или альтернирование по индексам одного типа.

15.2.7 2.7. Partial alternation

В дальнейшем нам также понадобится следующее обобщение понятия альтернирования. Пусть T, как и выше, тензор типа ($0, q$), и предположим, что фиксировано некоторое подмножество U группы перестановок S_q (подчеркнем, что U вовсе не обязано быть подгруппой). Тогда частичной альтернацией тензора T по U называется тензор $A_U(T)$ следующего вида

$$A_U(T) = \frac{1}{|U|} \sum_{\sigma \in U} (-1)^{\sigma} {}_{\sigma}T,$$

где через $|U|$ обозначено количество элементов множества U.
Имеет место следующий полезный для нас факт.
Лемма 2.13. Пусть T - тензор типа $(0, q), uU$ - подмножество группы перестановок S_q. Тогда

$$A\left(A_U(T)\right) = A(T)$$

т.е. альтернация "съедает" любую частичную альтернацию.

Доказательство. Действительно,

$$A\left(A_U(T)\right) = A\left(\frac{1}{|U|} \sum_{\sigma \in U} (-1)^{\sigma} {}_{\sigma}T\right)$$

В силу линейности альтернации, перепишем правую часть в виде

$$\frac{1}{|U|} \sum_{\sigma \in U} (-1)^{\sigma} A\left({}_{\sigma}T\right) = \frac{1}{|U|} \sum_{\sigma \in U} \sum_{\tau \in S_q} \frac{1}{q!} (-1)^{\sigma} (-1)^{\tau} {}_{\tau}\left({}_{\sigma}T\right).$$

Обозначим композицию $\sigma\tau$ через π. Отметим, что для каждой фиксированной перестановки σ, если τ пробегает всю группу S_q, то и произведение $\sigma\tau$ тоже пробегает S_q. В силу леммы 2.1, получим:

$$\frac{1}{|U|} \sum_{\sigma \in U} \frac{1}{q!} \left(\sum_{\pi \in S_q} (-1)^{\pi} {}_{\pi}T\right) = \frac{1}{|U|} \sum_{\sigma \in U} A(T) = A(T).$$

Лемма доказана.
Сформулируем полезное следствие.
Следствие 2.1. Пусть T и Q - произвольные тензоры типа $(0, q)$ и $(0, u)$ соответственно. Тогда

$$A(A(T) \otimes Q) = A(T \otimes Q) = A(T \otimes A(Q))$$

15.2.8 Problems

Задача 2.1. Пусть в $\mathbb{R}^3$ фиксированы стандартные координаты (x, y, z). Рассмотрим тензорное поле T типа $(1,2)$, заданное так:

$$T = (x+y)\partial_x \otimes dx \otimes dx + (y-z)\partial_z \otimes dy \otimes dz$$

Фиксируем точку $P = (1,2,3)$, ковектор $\xi = (2,3,4)$ из $T_P^*\mathbb{R}^3$ и векторы $V_1 = (3,4,5)$ и $V_2 = (4,5,6)$ из $T_P\mathbb{R}^3$. Вычислить $T(P)\,(\xi, V_1, V_2)$. Та же задача, но (другое) тензорное поле T задано в цилиндрических координатах (r, φ, z) так:

$$T = (r+z)\partial_\varphi \otimes d\varphi \otimes dz + (r-z)\partial_z \otimes dr \otimes d\varphi$$

Задача 2.2. Рассмотреть пример перестановки верхнего и нижнего индексов у линейного оператора, т.е. тензора типа $(1,1)$. Убедиться, что эта операция не тензорная, т.е. полученный в результате объект не является тензором (а именно, матрицы, полученные такой перестановкой в разных системах координат, задают разные линейные операторы).

Задача 2.3. Сколько существует способов получить полную свертку тензора типа (p,p) ? Все ли полные свертки различны?

Задача 2.4. Пусть T - произвольный тензор типа (p,q), а $\xi^1, \ldots, \xi^p$ и $V_1, \ldots, V_q$ - произвольные p ковекторов и q векторов. Рассмотрим тензор

$$S = T \otimes \xi^1 \otimes \cdots \otimes \xi^p \otimes V_1 \otimes \cdots \otimes V_q$$

типа ($p|q, q|p$). Показать, что полная свертка тснзора S

$$C^1_{q+1}\left(\ldots\left(C^p_{q+p}\left(C^{p+1}_1\left(\ldots\left(C^{p+q}_q(S)\ldots\right)\right.\right.\right.$$

равна значению тензора T на наборе $\xi^1, \ldots, \xi^p; V_1, \ldots, V_q$.

Задача 2.5. Выписать формулы для вычисления дифференциала функции (ковектора) в цилиндрических и сферических координатах в $\mathbb{R}^3$. Поднять у полученных ковекторов индекс с помощью евклидовой метрики δ (записанной в тех же координатах). Сравнить ответ с формулами для градиента из стандартного учебника по математическому анализу.

Задача 2.6. Найти размерности пространств симметричных и кососимметричных тензоров типа ($0, q$). Верно ли, что любой тензор типа ($0, q$) может быть представлен в виде суммы симметричного и кососимметричного тепзоров типа ($0, q$)?

15.2.9 Extra Topics

2.1. Проверка корректности определения свертки. Вместо того, чтобы проверять корректность определения свертки с помощью тензорного закона, можно проделать это на языке инвариантного определения, используя только свертку линейного оператора.

Фиксируем произвольные ($p-1$) ковекторов $\xi^1, \ldots, \xi^{p-1}$ и произвольные ($q-1$) векторов $V_1, \ldots, V_{q-1}$. Тогда, для произвольных чисел $1 \le \alpha \le p$ и $1 \le \beta \le q$, по тензору T типа (p,q) можно построить тензор A типа $(1,1)$ следующим образом:

$$A(\psi, W) = T\left(\xi^1, \ldots, \xi^{\alpha-1}, \psi, \xi^\alpha, \ldots, \xi^{p-1}; V_1, \ldots, V_{\beta-1}, W, V_\beta, \ldots, V_{q-1}\right).$$

Тензор A зависит от ξ^i и V_j полилинейно. Определим теперь функцию $C^\alpha_\beta(T)$ на $(T_P^*M)^{p-1} \times (T_PM)^{q-1}$, положив ее значение на наборе $(\xi^1, \ldots, \xi^{p-1}; V_1, \ldots, V_{q-1})$ равным следу соответствующего линейного оператора A. Поскольку взятие следа линейного оператора - тензорная операция, функция $C^\alpha_\beta(T)$ очевидно является полилинейной,

поэтому определяет тензор типа ($p-1, q-1$), который называется сверткой тензора T по верхнему индексу с номером α и нижнему индексу с номером β. Чтобы убедиться, что это определение свертки совпадает с данным выше координатным определением, вычислим компоненты только что определенного тензора в локальных координатах $(x^1, \ldots, x^n)$. Получим:

$$\begin{aligned} C^\alpha_\beta(T)^{i_1\ldots i_{p-1}}_{j_1\ldots j_{q-1}} &= C^\alpha_\beta(T)\left(dx^{i_1}, \ldots, dx^{i_{p-1}}; \partial_{x^{j_1}}, \ldots, \partial_{x^{j_{q-1}}}\right) = \\ &= A\left(dx^k; \partial_{x^k}\right) = T\left(dx^{i_1}, \ldots, dx^{i_{\alpha-1}}, dx^k, dx^\alpha, \ldots, dx^{i_{p-1}}\right. \\ &\quad \left.\partial_{x^{j_1}}, \ldots, \partial_{x^{j_{\beta-1}}}, \partial_{x^k}, \partial_{x^\beta}, \ldots \partial_{x^{j_{q-1}}}\right) = \\ &= T^{i_1,\ldots,i_{\alpha-1},k,i_\alpha,\ldots,i_{p-1}}_{j_1,\ldots,j_{\beta-1},k,j_\beta,\ldots,j_{q-1}} \end{aligned}$$

что и требовалось. Таким образом, мы показали, что два определения свертки совпадают, и, тем самым, установили корректность первого, координатного определения.
2.2. Некоммутативность тензорного произведения. Пусть на плоскости фиксированы координаты (x^1, x^2). Рассмотрим тензорные произведения $dx^1 \otimes dx^2$ и $dx^2 \otimes dx^1$. Тогда для любых векторов $V = (v^1, v^2)$ и $W = (w^1, w^2)$ имеем:

$$dx^1 \otimes dx^2(V, W) = v^1 w^2, \quad \text{а} \quad dx^2 \otimes dx^1(V, W) = v^2 w^1$$

Поэтому, если векторы V и W линейно независимы, то

$$dx^1 \otimes dx^2(V, W) \neq dx^2 \otimes dx^1(V, W)$$

Последнее означает, что отображения $dx^1 \otimes dx^2$ и $dx^2 \otimes dx^1$ различны.
2.3. Тензорное произведение и перестановка индексов. Приведем тут доказательство формулы из леммы 2.6 , а именно, если

$$T = dx^{i_1} \otimes \cdots \otimes dx^{i_q}$$

где dx^i - базисные ковекторы, соответствующие локальным координатам ($x^1, \ldots, x^n$), то для произвольной перестановки $\sigma \in S_q$ выполнено:

$${}_\sigma T = dx^{i\sigma_\sigma{}^{-1}(1)} \otimes \cdots \otimes dx^{i_\sigma{}^{-1}(q)}$$

По определению операции перестановки, для любых векторов канонического базиса $\partial_{x^{\alpha}1}, \ldots, \partial_{x^{\alpha}q}$ имеем:

$$\begin{aligned} \sigma T\left(\partial_x \alpha_1, \ldots, \partial_x \alpha_q\right) &= T\left(\partial_x \alpha_{\sigma(1)}, \ldots, \partial_x{}^\alpha \sigma(q)\right) = \\ &= \left(dx^{i_1} \otimes \cdots \otimes dx^{i_q}\right)\left(\partial_x{}^\alpha \sigma(1), \ldots, \partial_x{}^\alpha \sigma(q)\right) = \\ &= dx^{i_1}\left(\partial_x \alpha_{\sigma(1)}\right) \cdots dx^{i_q}\left(\partial_x{}^\alpha \sigma(q)\right) = \delta^{i_1}_{\alpha_{\sigma(1)}} \cdots \delta^{i_q}_{\alpha_{\sigma(q)}}. \end{aligned}$$

Полученное число отлично от нуля (и равно единице), если и только если $\alpha_{\sigma(k)} = i_k$ для всех $k = 1, \ldots, q$. Это условие переписывается в виде $\alpha_k = i_{\sigma-1(k)}$. Таким образом,

$$\sigma T\left(\partial_x \alpha_1, \ldots, \partial_x \alpha_q\right) = 1$$

если и только если $\alpha_k = i_{\sigma-1(k)}$, а на всех остальных наборах базисных ковекторов значение тензора σT равно нулю, что и завершает доказательство.
2.4. Опускание и поднятие индексов в евклидовых координатах. Если тензор A, с помощью которого мы опускаем или поднимаем индекс, в некоторой системе координат имеет вид δ_{ij} - символы Кронекера, то в этих координатах операция опускания и поднятия индекса тривиальна: соответствующие компоненты тензоров просто совпадают. Поэтому, в этом случае, все ипдсксы можно ститать, скажем, нижними. Эта

ситуация имеет место, например, если работать с тензорами в евклидовом пространстве, т.е. линейном пространстве со скалярным произведением, поднимая и опуская индексы с помощью невырожденной билинейной формы, задающей это скалярное произведение, при условии, что фиксированы евклидовы координаты (координаты, в которых матрица скалярного произведения равна единичной матрице).

Например в математическом анализе, работая в евклидовом пространстве $\mathbb{R}^n$ с евклидовыми координатами ($x^1, \ldots, x^n$), не различают векторы и ковекторы, молчаливо предполагая, что индексы поднимаются и опускаются с помощью евклидового метрического тензора δ, имеющего, по определению, в евклидовых координатах компоненты δ_{ij}. Отметим, однако, что тензор, соответствующий набору чисел δ_{ij} в координатах ($x^1, \ldots, x^n$), не является, к сожалению, постоянным: при замене координат его компоненты меняются по тензорному закону:

$$\delta_{i'j'} = \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \delta_{ij} = \sum_i \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^i}{\partial x^{j'}}.$$

Таким образом, равенство компонент $\delta_{i'j'}$ символам Кронекера есть в точности условие ортогональности матрицы Якоби замены координат. Поэтому, пока мы в математическом анализе делаем ортогональные замены координат, никаких проблем не возникает. Однако, как только замена координат перестает быть ортогональной, возникают, например, «странные», с точки зрения математического анализа, формулы для вычисления компонент «вектора градиента функции». В математическом анализе иногда градиент определяют как вектор, компоненты которого равны компонентам дифференциала. Так можно поступать, как мы теперь понимаем, только в евклидовых координатах. На самом деле, кажущаяся странность формул для градиента объясняется тем, что дифференциал является ковектором, и чтобы получить из него вектор градиента, чего обычно и хотят в математическом анализе, нужно поднять индекс с помощью непостоянного тензора δ.

2.5. Симметричные линейные операторы. В линейной алгебре часто идет речь о симметричных или кососимметричных линейных операторах. Однако, линейный оператор - этот тензор типа $(1,1)$, поэтому, чтобы определить для него понятие симметричности или кососимметричности, следует фиксировать невырожденный тензор A типа (0,2). Тогда, если L - линейный оператор, то опустив сначала его индекс с помощью тензора A, можно говорить о его симметричности или кососимметричности в тензорном смысле. В линейной алгебре в качестве тензора A берут обычно евклидов метрический тензор. Вообще, если фиксирована произвольная риманова метрика g_{ij}, то симметричность линейного оператора L относительно этого невырожденного тензора типа $(0,2)$ эквивалентна, как легко проверить, следующему соотношению, выполненному для произвольных векторов V и W :

$$\langle L(V), W \rangle = \langle V, L(W) \rangle,$$

где $\langle \cdot, \cdot \rangle$ - скалярное произведение, соответствующее римановой метрике g_{ij}.

2.6. Тензоры в механике сплошной среды. В механикс сплошной среды [2] также активно используется тензорный анализ. При этом, особенность рассматриваемой модели сплошной среды состоит в том, что всегда предполагается заданным некоторый метрический тензор g_{ij}. Поэтому всегда можно поднимать и опускать индексы. В механике сплошной среды принято говорить, что в результате операции опускания или поднятия индекса получается «тот же тензор, но записанный в другой форме». Так, скажем, вектор V в одной и той же системе координат может быть задан как своими обычными «контравариантными» компонентами v^i, т.е. компонентами с верхними индексами, так и «ковариантными» компонентами, т.е. компонентами $v_i = g_{i\alpha} v^\alpha$.

При этом, чтобы избежать путаницы с перестановками индексов при их поднятииопускании, порядок всех индексов считается фиксированным, и он, по определению, сохраняется при поднятии-опускании индексов. Скажем, тензор ранга 3 может быть записан с первым верхним, вторым нижним и третьим снова верхним индексами. Вообще, для тензора ранга r имеется 2^r разных «способов записи». При записи соответствующих формул используют точки на месте пропущенных индексов, например $T^{i\cdot k}_{\cdot j}$. Связь между разными формами одного и того же тензора выражается формулами типа $T^{i\cdot k}_{\cdot j\cdot} = g^{k\alpha} g_{j\beta} T^{i\beta}_{\cdot\cdot\alpha}$.

Таким образом, с точки зрения дифференциальной геометрии, в механике сплошной среды молчаливо предполагаются фиксированными изоморфизмы между пространствами тензоров одного ранга (но разного типа). Эти изоморфизмы порождаются, как и положено, невырожденным тензором типа $(0,2)$ и перестановками индексов. Это надо иметь ввиду, переходя с одного языка на другой.

15.3 3: Algebra of external differential forms

В нескольких следующих разделах мы более подробно изучим кососимметрические тензорные поля типа ($0,q$). Оказывается, для таких тензорных полей можно построить содержательную теорию дифференцирования и интегрирования. Сам термин «дифференциальные формы» объясняется естественной записью этих тензоров в локальных координатах в дифференциальном виде (напомним, что каждым локальным координатам соответствует канонический базис $(dx^i)_{i=1}^n$ пространства $T^*_P M$).

15.3.1 3.1. The space of skew-symmetric tensors

Напомним, что тензор T типа $(0,q)$ называется кососимметрическим, если для произвольной перестановки $\sigma \in S_q$ выполнено соотношение ${}_\sigma T = (-1)^\sigma \cdot T$.

Пример. Произвольный ковектор $\xi \in T^*_P M$ представляет собой кососимметрический тензор (нетривиальных перестановок множества из одного элемента нет).

Пример. Тензор T типа (0,2) является кососимметрическим, если в любой системе локальных координат соответствующая матрица кососимметрична: $T_{ij} = -T_{ji}$.

Напомним, что множество всех кососимметрических тензоров типа $(0,q)$ образует линейное пространство. Найдем размерность этого пространства. Пусть фиксированы локальные координаты ($x^1, \ldots, x^n$). Для каждого набора натуральных чисел $(i_1, \ldots, i_q)$, где $1 \le i_1 < \ldots < i_q \le n$ определим кососимметричное полилинейное отображение, которое обозначим через $dx^{i_1} \wedge \cdots \wedge dx^{i_q}$, положив

$$dx^{i_1} \wedge \cdots \wedge dx^{i_q}\,(V_1, \ldots, V_q) = \det \begin{pmatrix} v_1^{i_1} & v_1^{i_2} & \ldots & v_1^{i_q} \\ v_2^{i_1} & v_2^{i_2} & \ldots & v_2^{i_q} \\ \vdots & \vdots & \ddots & \vdots \\ v_q^{i_1} & v_q^{i_2} & \ldots & v_q^{i_q} \end{pmatrix},$$

где $V_i = (v_i^1, \ldots, v_i^n)$. В правой части стоит определитель матрицы, k-ая строка которой состоит из компонент вектора V_k с номерами $i_1, \ldots, i_q$. Так как определитель - кососимметричная полилинейная функция строк, определенное нами отображение действительно полилинейно и кососимметрично.

Замечание. Отметим, что определение тензора $dx^{i_1} \wedge \cdots \wedge dx^{i_q}$ можно переписать в виде

[02] В качестве классического учебника по механике сплошной среды следует назвать замечательную книгу Л. И. Седова.

$$dx^{i_1} \wedge \cdots \wedge dx^{i_q}\left(V_1, \ldots, V_q\right) = \det \begin{pmatrix} dx^{i_1}\left(V_1\right) & dx^{i_2}\left(V_1\right) & \ldots & dx^{i_q}\left(V_1\right) \\ \vdots & \vdots & \ddots & \vdots \\ dx^{i_1}\left(V_q\right) & dx^{i_2}\left(V_q\right) & \ldots & dx^{i_q}\left(V_q\right) \end{pmatrix}$$

В такой форме это определение легко обобщается на случай произвольного набора ковекторов, не обязательно базисных. В частности, очевидно, что если $q > n$, то функции $dx^{i_1} \wedge \cdots \wedge dx^{i_q}$ всегда равны нулю (в определителе будут одинаковые столбцы).

Утверждение 3.1. Система $\left\{dx^{i_1} \wedge \cdots \wedge dx^{i_q}\right\}$ кососимметрических тензоров, $1 \leq i_1 < \cdots < i_q \leq n$, образует базис в пространстве кососимметричных тензоров типа $(\ 0, q\ )$.

Доказательство. Пусть T - произвольный кососимметрический тензор типа $(\ 0, q\ )$. Покажем, что он может быть представлен в виде

$$T = \sum_{1 \leq i_1 < \cdots < i_q \leq n} T_{i_1 \ldots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q} \tag{*}$$

Для этого вычислим значение правой и левой части на произвольном наборе базисных векторов $(\partial_{x^\alpha}, \ldots, \partial_{x^{\alpha_q}})$. В левой части, очевидно, получим:

$$T\left(\partial_{x^{\alpha_1}}, \ldots, \partial_{x^{\alpha_q}}\right) = T_{\alpha_1 \ldots \alpha_q}.$$

Чтобы вычислить правую часть, докажем следующую лемму.

Лемма 3.1. Справедливо следующее равенство:

$$dx^{i_1} \wedge \cdots \wedge dx^{i_q} = \sum_{\sigma \in S_q} (-1)^\sigma dx^{i_{\sigma(1)}} \otimes \cdots \otimes dx^{i_{\sigma(q)}}$$

Доказательство. Действительно, по определению,

$$dx^{i_1} \wedge \cdots \wedge dx^{i_q}\left(V_1, \ldots, V_q\right) = \det \begin{pmatrix} dx^{i_1}\left(V_1\right) & \ldots & dx^{i_q}\left(V_1\right) \\ \vdots & \ddots & \vdots \\ dx^{i_1}\left(V_q\right) & \ldots & dx^{i_q}\left(V_q\right) \end{pmatrix} =$$

$$= \sum_{\sigma \in S_q} (-1)^\sigma dx^{i_{\sigma(1)}}\left(V_1\right) \cdots dx^{i_{\sigma(q)}}\left(V_q\right) =$$

$$= \sum_{\sigma \in S_q} (-1)^\sigma dx^{i_{\sigma(1)}} \otimes \cdots \otimes dx^{i_{\sigma(q)}}\left(V_1, \ldots, V_q\right)$$

что и требовалось.

В силу леммы 3.1, вычисляя значение правой части выражения (*) на наборе векторов $(\partial_{x^{\alpha_1}}, \ldots, \partial_{x^{\alpha_q}})$, получим:

$$\sum_{1 \leq i_1 < \cdots < i_q \leq n} T_{i_1 \ldots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}\left(\partial_{x^{\alpha_1}}, \ldots, \partial_{x^{\alpha_q}}\right) =$$

$$= \sum_{1 \leq i_1 < \cdots < i_q \leq n} T_{i_1 \ldots i_q} \sum_{\sigma \in S_q} (-1)^\sigma dx^{i_{\sigma(1)}} \otimes \cdots \otimes dx^{i_{\sigma(q)}}\left(\partial_{x^\alpha}, \ldots, \partial_{x^{\alpha_q}}\right) =$$

$$= \sum_{1 \leq i_1 < \cdots < i_q \leq n} T_{i_1 \ldots i_q} \sum_{\sigma \in S_q} (-1)^\sigma \delta_{\alpha_1}^{i_{\sigma(1)}} \cdots \delta_{\alpha_q}^{i_{\sigma(q)}}.$$

Выражение

$$\delta_{\alpha_1}^{i_{\sigma(1)}} \cdots \delta_{\alpha_q}^{i_{\sigma(q)}}$$

отлично от нуля (и равно единице) в том и только том случае, если $i_{\sigma(k)} = \alpha_k$ для всех $k = 1, \ldots, q$. Чтобы это было выполнено, необходимо чтобы наборы $(i_1, \ldots, i_q)$ и $(\alpha_1, \ldots, \alpha_q)$ совпадали как множества. Последние возможно, только если все индексы α_k различны. Если все α_k различны, то среди всех упорядоченных наборов $(i_1, \ldots, i_q)$ имеется ровно один, совпадающий как множество с набором $(\alpha_1, \ldots, \alpha_q)$. Поэтому, в этом случае, имеем:

$$\sum_{1 \le i_1 < \cdots < i_q \le n} T_{i_1 \ldots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q} \left(\partial_{x^{\alpha_1}}, \ldots, \partial_{x^{\alpha_q}}\right) = T_{i_1 \ldots i_q} (-1)^{\sigma}$$

где $\sigma \in S_q$ - та самая единственная перестановка, для которой $i_{\sigma(k)} = \alpha_k$, а наборы $(i_1, \ldots, i_q), 1 \le i_1 < \cdots < i_q \le n$, и $(\alpha_1, \ldots, \alpha_q)$ совпадают как множества. Наконец, в силу косой симметрии тензора T и равенств $i_{\sigma(k)} = \alpha_k$, можно записать следующее

$$(-1)^{\sigma} T_{i_1 \ldots i_q} = T_{i_{\sigma(1)} \ldots i_{\sigma(q)}} = T_{\alpha_1 \ldots \alpha_q}$$

Поэтому окончательно:

$$\sum_{1 \le i_1 < \cdots < i_q \le n} T_{i_1 \ldots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q} \left(\partial_{x^{\alpha_1}}, \ldots, \partial_{x^{\alpha_q}}\right) = T_{\alpha_1 \ldots \alpha_q}$$

Осталось заметить, что если среди индексов α_k есть совпадающие, то, в силу косой симметрии тензора T, снова получаем

$$\sum_{1 \le i_1 < \cdots < i_q \le n} T_{i_1 \ldots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q} \left(\partial_{x^{\alpha}1}, \ldots, \partial_{x^{\alpha}q}\right) = 0 = T_{\alpha_1 \ldots \alpha_q}.$$

Формула (*) доказана.
Для завершения доказательства утверждения осталось проверить, что тензоры $dx^{i_1} \wedge \cdots \wedge dx^{i_q}$ линейно независимы. Однако из определения

этих полилинейных отображений вытекает, что на каждом из наборов $\left(\partial_{x^{\alpha^1}}, \ldots, \partial_{x^{\alpha^q}}\right)$, где $1 \le \alpha_1 < \cdots < \alpha_q \le n$, ровно одно такое отображение отлично от нуля (а именно, $dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_q}$), откуда и следует нетривиальность произвольной их линейной комбинации. Доказательство закончено.

Пространство кососимметрических тензоров типа ($0, q$) на пространстве $T_P M$ обычно обозначают через $\Lambda^q\left(T_P M\right)$.

Следствие 3.1. Если $q \le n$, то размерность пространства $\Lambda^q\left(T_P M\right)$ кососимметричных тензоров типа $(0, q)$ на касательном пространстве κn-мерному многообразию M равна биномиальному коэффициенту C_n^q. Если $q > n$, то размерность $\Lambda^q\left(T_P M\right)$ равна нулю.

Пример. Рассмотрим пространство $\Lambda^n\left(T_P M\right)$ кососимметрических тензоров типа ($0, n$), где n - размерность многообразия M. В силу следствия 3.1, это пространство одномерно. Пусть T - произвольный тензор из $\Lambda^n\left(T_P M\right)$. Из утверждения 3.1 вытекает, что в произвольных локальных координатах $(x^1, \ldots, x^n)$ он представим в виде $T = T_{1 \ldots n} dx^1 \wedge \cdots \wedge dx^n$. Изучим, как меняется компонента $T_{1 \ldots n}$ при замене координат. Пусть ($x^{1'}, \ldots, x^{n'}$) - еще одна система локальных координат в окрестности точки P. Тогда, применяя тензорный закон, получаем

$$T_{1' \ldots n'} = \frac{\partial x^{i_1}}{\partial x^{1'}} \cdots \frac{\partial x^{i_n}}{\partial x^{n'}} T_{i_1 \ldots i_n} = \sum_{\text{все } i_k \text{ различны}} + \sum_{\text{не все } i_k \text{ различны}},$$

где вторая сумма равна нулю в силу косой симметрии тензора T. Далее, если набор индексов $(i_1, \ldots, i_n)$ состоит из попарно различных чисел, то его можно воспринимать как результат применения некоторой однозначно определенной перестановки $\sigma \in S_n$. Поэтому, продолжая равенство, запишем

$$T_{1'\ldots n'} = \sum_{\sigma\in S_n} \frac{\partial x^{\sigma(1)}}{\partial x^{1'}} \cdots \frac{\partial x^{\sigma(n)}}{\partial x^{n'}} T_{\sigma(1)\ldots\sigma(n)} =$$

$$= \sum_{\sigma\in S_n} (-1)^\sigma \frac{\partial x^{\sigma(1)}}{\partial x^{1'}} \cdots \frac{\partial x^{\sigma(n)}}{\partial x^{n'}} T_{1\ldots n} = T_{1\ldots n} \det\left(\frac{\partial x^i}{\partial x^{j'}}\right).$$

Таким образом, нами доказана следующая полезная лемма.

Лемма 3.2. Компонента $T_{1\ldots n}$ кососимметричного тензора типа $(0,n)$ на n-мерном многообразии при замене координат умножается на определитель матрицы Якоби этой заменьт:

$$T_{1'\ldots n'} = T_{1\ldots n} \det\left(\frac{\partial x^i}{\partial x^{j'}}\right).$$

Компонента $T_{1\ldots n}$ называется существенной компонентой тензора T из $\Lambda^n(T_P M)$.

15.3.2 3.2. External differential shapes

Пусть M - гладкое многообразие.

Определение. Внешней дифференциальной формой степени q на M называется гладкое тензорное поле кососимметрических тензоров типа $(0,q)$.

Пусть ω^q - внешняя дифференциальная форма степени q на многообразии M. Если на M фиксированы некоторые локальные координаты $(x^1,\ldots,x^n)$, то в каждой точке P из области определения $U\subset M$ этих локальных координат, кососимметрический тензор ω^q типа $(0,q)$ может быть записан, в силу утверждения 3.1, в виде

$$\omega^q(P) = \sum_{1\le i_1<\cdots<i_q\le n} \omega_{i_1\ldots i_q}(P) dx^{i_1}\wedge\cdots\wedge dx^{i_q}$$

где $\omega_{i_1\ldots i_q}$ - гладкие функции, определенные на области U многообразия M. Если точка P принадлежит пересечению $U\cap U'$ областей определения U и U' локальных координат $(x^1,\ldots,x^n)$ и $\left(x^{1'},\ldots,x^{n'}\right)$, то форму ω^q можно записать в этих координатах в виде

$$\omega^q(P) = \sum_{1\le i'_1<\cdots<i'_q\le n} \omega_{i'_1\ldots i'_q}(P) dx^{i'_1}\wedge\cdots\wedge dx^{i'_q}$$

Функции $\omega_{i_1\ldots i_q}(P)$ и $\omega_{i'_1\ldots i'_q}(P)$ - это компоненты одного и того же тензорного поля ω^q, поэтому они на пересечении $U\cap U'$ связаны по тензорному закону.

Если ω^q - внешняя форма на многообразии M, то в каждой точке P многообразия M задано, тем самым, кососимметричное полилинейное отображение из $(T_P M)^q$ в числа, которое мы обозначим через $\omega^q|_P$.

Пример. Пусть f - гладкая функция на многообразии M. Тогда ее дифференциал df является 1 -формой на многообразии M. Если $(x^1,\ldots,x^n)$ локальные координаты, то форма df записывается в виде $df=\frac{\partial f}{\partial x^i}dx^i$. Если $V\in T_P M$ - касательный вектор, заданный своим компонентами ($v^1,\ldots,v^n$) в координатах $(x^1,\ldots,x^n)$, то

$$df|_P(V) = \left.\frac{\partial f}{\partial x^i}\right|_P v^i$$

Линейная комбинация внешних дифференциальных форм степени q (с функциональными коэффициентами) является дифференциальной формой степени q. Поэтому дифференциальные формы степени q на M образуют линейное пространство,

которое мы обозначим через $\Omega^q(M)$. Удобно рассматривать гладкие функции на M как внешние формы степени 0 , тогда $\Omega^0(M)$ - это пространство всех гладких функций на M.

Объединение $\cup_{q=0}^n \Omega^q(M)$ обозначим через $\Omega^*(M)$ (напомним, что при $q > n$ все дифференциальные формы степени q на n-мерном многообразии тривиальны, т.е. равны нулю).

15.3.3 3.3. External Multiplication

Превратим пространство $\Omega^*(M)$ в алгебру, введя на нем операцию умножения. Обычное тензорное произведение не подходит, так как тензорное произведение кососимметрических тензоров не обязано быть кососимметричным. Пусть $P \in \Omega^p(M)$ и $Q \in \Omega^q(M)$ - дифференциальные формы на M. Обозначим через $P \wedge Q$ дифференциальную форму степени $p+q$, определенную так:

$$P \wedge Q = \frac{(p+q)!}{p!q!} A(P \otimes Q)$$

Так как результат альтернирования - это кососимметрический тензор, $P \wedge Q$ действительно представляет собой элемент пространства $\Omega^{p+q}(M)$. Форма $P \wedge Q$ называется внешним произведением форм P и Q, а сама операция $\wedge : \Omega^p(M) \times \Omega^q(M) \to \Omega^{p+q}(M)$ - внешним умножением.

Докажем простейшие свойства внешнего произведения.

Утверждение 3.2. Внешнее произведение билинейно, ассоциативно и кососкоммутативно в следующем смысле: $P \wedge Q = (-1)^{pq} Q \wedge P$.

Доказательство. Билинейность немедленно вытекает из дистрибутивности тензорного произведения и линейности альтернирования.

Проверим ассоциативность. Пусть $P \in \Omega^p, Q \in \Omega^q$, и $R \in \Omega^r$. Тогда $(P \wedge Q) \wedge R = \left(\frac{(p+q)!}{p!q!} A(P \otimes Q)\right) \wedge R = \frac{(p+q)!}{p!q!} \frac{(p+q+r)!}{(p+q)!r!} A(A(P \otimes Q) \otimes R)$.

Осталось воспользоваться следствием 2.1 и ассоциативностью тензорного произведения. В итоге, получаем

$$(P \wedge Q) \wedge R = \frac{(p+q+r)!}{p!q!r!} A(P \otimes Q \otimes R),$$

что и доказывает ассоциативность внешнего умножения.

Для доказательства косой симметрии вычислим сначала значение полилинейного отображения $A(P \otimes Q)$, где как и выше $P \in \Omega^p$, и $Q \in \Omega^q$, на произвольном наборе векторов $V_1, \ldots, V_{p+q}$. Получим:

$$\begin{aligned} A(P \otimes Q)\,(V_1, \ldots, V_{p+q}) = \\ = \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q}} (-1)^\sigma (P \otimes Q)\left(V_{\sigma(1)}, \ldots, V_{\sigma(p+q)}\right) = \\ = \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q}} (-1)^\sigma P\left(V_{\sigma(1)}, \ldots, V_{\sigma(p)}\right) Q\left(V_{\sigma(p+1)}, \ldots, V_{\sigma(p+q)}\right) \end{aligned}$$

Обозначим через τ перестановку из S_{p+q}, переводящую набор чисел $(1, \ldots, p+q)$ в набор $(q+1, \ldots, q+p, 1, \ldots, q)$. Очевидно, $(-1)^\tau$ равна $(-1)^{pq}$. Вычислим значение тензора ${}_\tau A(P \otimes Q)$ на том же наборе векторов $V_1, \ldots, V_{p+q}$. Получим:

$$\begin{aligned} {}_\tau A(P \otimes Q)\,(V_1, \ldots, V_{p+q}) = A(P \otimes Q)\,(V_{q+1}, \ldots, V_{q+p}, V_1, \ldots, V_q) = \\ = \frac{1}{(p+q)!} \sum_{\sigma \in S_{p+q}} (-1)^\sigma P\left(V_{\sigma(q+1)}, \ldots, V_{\sigma(q+p)}\right) Q\left(V_{\sigma(1)}, \ldots, V_{\sigma(q)}\right) \end{aligned}$$

Сравнивая полученное выражение с предыдущим, заключаем, что

$$_\tau A(P \otimes Q) = A(Q \otimes P)$$

С другой стороны, тензор $A(P \otimes Q)$ кососимметричен, поэтому

$$_\tau A(P \otimes Q) = (-1)^\tau A(P \otimes Q)$$

Окончательно получаем:

$$A(Q \otimes P) = {}_\tau A(P \otimes Q) = (-1)^{pq} A(P \otimes Q)$$

что и требовалось. Утверждение доказано.
Выясним, как устроены внешние произведения базисных 1 -форм dx^i.
Пример. Пусть $(x^1, \ldots, x^n)$ - система локальных координат на многообразии, и dx^1 и dx^2 - базисные ковекторы. Найдем, чему равно внешнее произведение $dx^1 \wedge dx^2$. Воспользовавшись определением и леммой 2.6, получим:

$$\begin{aligned} dx^1 \wedge dx^2 &= \frac{2!}{1!1!} A\left(dx^1 \otimes dx^2\right) = 2\frac{1}{2!}\sum_{\sigma \in S_2} (-1)^\sigma {}_\sigma \left(dx^1 \otimes dx^2\right) \\ &= \sum_{\sigma \in S_2} (-1)^\sigma dx^{\sigma^{-1}(1)} \otimes dx^{\sigma^{-1}(2)} = \sum_{\tau \in S_2} (-1)^\tau dx^{\tau(1)} \otimes dx^{\tau(2)} \end{aligned}$$

(Тут мы воспользовались тем, что четность перестановок σ и σ^{-1} одинакова.) Но тогда, в силу леммы 3.1 , мы получаем, что внешнее произведение ковекторов dx^1 и dx^2 совпадает в каждой точке с элементом $dx^1 \wedge dx^2$ базиса пространства Λ^2. В частности,

$$dx^1 \wedge dx^2 \left(V_1, V_2\right) = \det \begin{pmatrix} v_1^1 & v_1^2 \\ v_2^1 & v_2^2 \end{pmatrix}$$

где $V_1 = (v_1^1, v_1^2)$, и $V_2 = (v_2^1, v_2^2)$. Оказывается, и вообще все базисные элементы ($dx^{i_1} \wedge \cdots \wedge dx^{i_q}$) пространства Λ^q на самом деле - внешние произведения соответствующих базисных ковекторов.

Следующая лемма доказывается прямым подсчетом.

Лемма 3.3. Для произвольного набора цельх чисел $(\alpha_1, \ldots, \alpha_k)$, таких что $1 \leq \alpha_i \leq n$, и произвольного набора векторов $V_1, \ldots, V_k$ имеет место равенство

$$dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}\left(V_1, \ldots, V_k\right) = \det \begin{pmatrix} dx^{\alpha_1}\left(V_1\right) & dx^{\alpha_2}\left(V_1\right) & \ldots & dx^{\alpha_k}\left(V_1\right) \\ \vdots & \vdots & \ddots & \vdots \\ dx^{\alpha_1}\left(V_k\right) & dx^{\alpha_2}\left(V_k\right) & \ldots & dx^{\alpha_k}\left(V_k\right) \end{pmatrix}.$$

В частности, внешнее произведение ковекторов $dx^{\alpha_1}, \ldots, dx^{\alpha_k}$, аде $1 \leq \alpha_1 < \cdots \alpha_k \leq n$, совпадает с базисньлм элементом $dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$ пространства Λ^k, определенным в предыдущем разделе.

Следствие 3.2. Пусть $(\alpha_1, \ldots, \alpha_k)$ - произвольный набор цельх чисел, таких что $1 \leq \alpha_i \leq n$. Рассмотрим внешнее произведение $\omega^k = dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$. Тогда

- форма ω^k равна нулю если и только если среди чисел α_i встречаются одинаковые;
- для любой перестановки $\sigma \in S_k$ имеет место равенство

$$dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k} = (-1)^\sigma dx^{\alpha_{\sigma(1)}} \wedge \cdots \wedge dx^{\alpha_{\sigma(k)}}$$

Замечание. Выше мы дали инвариантное определение внешнего умножения форм. Можно поступить иначе. А именно, определив внешнее умножение базисных ковекторов

по формуле из леммы 3.3, а внешнее произведение базисных форм по формуле $(dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p}) \wedge \left(dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_q}\right) = dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_p} \wedge dx^{\beta_1} \wedge \cdots \wedge dx^{\beta_q}$, можно продолжить эту операцию на все формы по дистрибутивности. После этого останется проверить, что определение не зависит от выбора координат.

В дальнейшем нам также будет полезно следующий технический результат.

Следствие 3.3. Пусть T - произвольная внешняя форма степени q. Тогда

$$\sum_{1 \le i_1 < \cdots < i_q \le n} T_{i_1 \ldots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q} = \frac{1}{q!} T_{i_1 \ldots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q}$$

Итак, мы определили на множестве всех внешних форм на многообразии M операцию внешнего умножения, превратив тем самым линейное пространство $\Omega^*(M) = \cup_{p=0}^m \Omega^p(M)$ в алгебру. Эта алгебра называется алгеброй внешних дифферениальных форм на многообразии или, коротко, внешней алгеброй.

Выясним, наконец, как записывается операция внешнего умножения в локальных координатах. Пусть $P \in \Lambda^p$ и $Q \in \Lambda^q$ - дифференциальные формы, и ($x^1, \ldots, x^n$) - локальная система координат. Тогда, в силу следствия 3.3 , формы P и Q можно записать в виде

$$P = \frac{1}{p!} P_{i_1 \ldots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}, \quad \text{и} \quad Q = \frac{1}{q!} Q_{j_1 \ldots j_q} dx^{j_1} \wedge \cdots \wedge dx^{j_q}.$$

Пользуясь свойствами внешнего умножения и следствием 3.3, имеем:

$$\begin{aligned}
P \wedge Q &= \frac{1}{p!q!} P_{i_1 \ldots i_p} Q_{j_1 \ldots j_q} dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_q} = \\
&= \frac{1}{p!q!} \sum_{1 \le \alpha_1 < \cdots < \alpha_{p+q} \le n} \sum_{\sigma \in S_{p+q}} P_{\alpha_{\sigma(1)} \ldots \alpha_{\sigma(p)}} Q_{\alpha_{\sigma(p+1)} \ldots \alpha_{\sigma(p+q)}} \\
&= \frac{1}{p!q!} \sum_{1 \le \alpha_1 < \cdots < \alpha_{p+q} \le n} \left(\sum_{\sigma \in S_{p+q}} (-1)^\sigma P_{\alpha_{\sigma(1)} \ldots \alpha_{\sigma(p)}} Q_{\alpha_{\sigma(p+1)} \ldots \alpha_{\sigma(p+q)}} \wedge \cdots \wedge dx^{\alpha_{\sigma(p+q)}} = \right. \\
&\cdot dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_{p+q}}
\end{aligned}$$

Итак, доказано следующее утверждение.

Утверждение 3.3. Пусть $P \in \Lambda^p u Q \in \Lambda^q$ - внешние формьс. Тогда компоненты формь $P \wedge Q$ в локальных координатах выражаются через компоненты форм P и Q так:

$$P \wedge Q_{\alpha_1 \ldots \alpha_{p+q}} = \frac{1}{p!q!} \sum_{\sigma \in S_{p+q}} (-1)^\sigma P_{\alpha_{\sigma(1)} \ldots \alpha_{\sigma(p)}} Q_{\alpha_{\sigma(p+1)} \ldots \alpha_{\sigma(p+q)}}.$$

Замечание. Пользоваться формулой из утверждения 3.3, перемножая конкретные формы, не очень удобно. Часто проще раскрыть скобки и привести подобные члены, преобразуя все возникающие мономы $dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ так, чтобы индексы i_p были упорядочены.

Упражнение 3.1. Пусть M^n - n-мерное ориентированное риманово многообразие, и $(x^1, \ldots, x^n)$ - локальные координаты на M. Обозначим через $g(x)$ определитель матрицы метрики g_{ij} в координатах $(x^1, \ldots, x^n)$. Определим на M внешнюю форму vol_n степени n, положив ее в координатах ($x^1, \ldots, x^n$) равной

$$\mathrm{vol}_n = \sqrt{g} dx^1 \wedge \cdots \wedge dx^n$$

если эти координаты согласованы с ориентацией многообразия M, и

$$\mathrm{vol}_n = -\sqrt{g}dx^1 \wedge \cdots \wedge dx^n$$

в противном случае. Проверить, что vol_n - корректно определенная внешняя n форма на M.

Упражнение 3.2. Пусть M - ориентированное риманово многообразие размерности n. Определим отображение $* : \Omega^k(M) \to \Omega^{n-k}(M)$, называемое звездочка Ходжа. Пусть $\sqrt{g}dx^1 \wedge \ldots \wedge dx^n$ - форма объема на M, и обозначим через $\varepsilon_{i_1 \ldots i_n}$ ее компоненты в координатах $(x^1, \ldots, x^n)$. Пусть T - произвольная дифференциальная форма из $\Omega^k(M)$, и $T_{i_1 \ldots i_k}$ - ее компоненты в координатах ($x^1, \ldots, x^n$). Определим форму $*T \in \Omega^{n-k}$, положив

$$(*T)_{i_{k+1} \ldots i_n} = \frac{1}{k!} \varepsilon_{i_1 \ldots i_n} T^{i_1 \ldots i_k},$$

где $T^{i_1 \ldots i_k}$ - тензор, полученный из T поднятием индексов с помощью римановой метрики g_{ij}.

Проверить, что $*T$ - корректно определенная форма из $\Omega^{n-k}(M)$.

15.3.4 3.4. Shapes & Displays

Пусть $F : M \to N$ - гладкое отображение гладких многообразий M и N. Предположим, что на многообразии N задана внешняя форма ω^q стспсни q. Тогда, оказывастся, на многообразии M сстественно определяется внешняя форма той же степени q, которую мы обозначим через $F^*(\omega^q)$. Действительно, если P - произвольная точка на M, и $V_1, \ldots, V_q$ - произвольные векторы из $T_P M$, то положим

$$F^*(\omega^q)|_P (V_1, \ldots, V_q) = \omega^q|_{F(P)} \left(dF|_P (V_1), \ldots, dF|_P (V_q) \right).$$

Кососимметричность и полилинейность построенного отображения очевидна. Говорят, что форма $F^*(\omega^q)$ получена из формы ω^q перенесением помощью отображения F.

Замечание. Дифференциал гладкого отображения $F : M \to N$, напомним, отображает каждое касательное пространство $T_P M$ в касательное пространство $T_{F(P)} N$, перенося тем самым касательные векторы к многообразию M в касательные векторы к многообразию N, т.е. «вперед». Дифференциальные формы переносятся отображением F в противоположную сторону, т.е. «назад». Отметим, что векторные поля, вообще говоря, «вперед» не переносятся. Дело в том, что отображение F может не быть взаимно однозначным. Тогда в одной точке образа мы можем получить несколько (даже бесконечно много) векторов. Переносить векторные поля можно только взаимно однозначными отображениями. С дифференциальными формами, как мы видим, дело обстоит куда проще: они переносятся назад любым гладким отображением, причем в результате получается форма, заданная на всем многообразии.

Таким образом, каждое отображение $F : M \to N$ порождает отображение алгебр внешних дифференциальных форм $F^* : \Omega^*(N) \to \Omega^*(M)$. Оказывается, это отображение уважает структуру внешней алгебры. А именно, имеет место следующее утверждение.

Утверждение 3.4. Отображение F^* линейно. Более того, для любых форм $P \in \Omega^p(N)$ и $Q \in \Omega^q(N)$ на многообразии N имеет место следующее соотношение:

$$F^*(P \wedge Q) = F^*(P) \wedge F^*(Q).$$

Доказательство. Линейность отображения F^* очевидна из определения. Далее, в силу линейности отображения F^*, второе свойство достаточно проверить для мономов, т.е. для

форм $P = f dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ и $Q = g dx^{j_1} \wedge \cdots \wedge dx^{j_q}$. Для случая мономов требуемое свойство проверяется прямым подсчетом.

Упражнение 3.3. Показать, что если задано два отображения $F_1 : M_1 \to M_2$ и $F_2 : M_2 \to M_3$, то $(F_2 \circ F_1)^* = F_1^* \circ F_2^*$.

Выясним, как вычисляются компоненты формы $F^*(\omega^q)$ через компоненты формы ω^q в локальных координатах. Пусть $(x^1, \ldots, x^m)$ и $(y^1, \ldots, y^n)$ - локальнyс координаты в окрестности точек P и $F(P)$ соответственно. Тогда для произвольного набора ($\partial_{x^{i_1}}, \ldots, \partial_{x^{i_q}}$) векторов канонического базиса пространства $T_P M$ имеем:

$$\begin{aligned} F^*(\omega^q)|_P (\partial_{x^{i_1}}, \ldots, \partial_{x^{i_q}}) &= \omega^q|_{F(P)} (F_*|_P (\partial_{x^{i_1}}), \ldots, F_*|_P (\partial_{x^{i_q}})) = \\ &= \omega^q|_{F(P)} \left(\frac{\partial y^{\alpha_1}}{\partial x^{i_1}} \partial_{y^{\alpha_1}}, \ldots, \frac{\partial y^{\alpha_q}}{\partial x^{i_q}} \partial_{y^{\alpha_q}} \right) = \\ &= \frac{\partial y^{\alpha_1}}{\partial x^{i_1}} \cdots \frac{\partial y^{\alpha_q}}{\partial x^{i_q}} \omega^q \Big|_{F(P)} (\partial_{y^{\alpha_1}}, \ldots, \partial_{y^{\alpha_q}}) \end{aligned}$$

где $y^i(x^1, \ldots, x^m)$ - координатное представление отображения F.
Таким образом, имеет место следующее утверждение.
Утверждение 3.5. Комполепты формы $F^*(\omega)$, полученной из формы ω перенесением с помощью отображения $F : M \to N$, выражаются через компоненты формы ω по следующей формуле

$$(F^*(\omega))_{i_1 \ldots i_q}(P) = \frac{\partial y^{\alpha_1}}{\partial x^{i_1}}(P) \cdots \frac{\partial y^{\alpha_q}}{\partial x^{i_q}}(P) \omega_{\alpha_1 \ldots \alpha_q}(F(P)).$$

15.3.5 Problems

Задача 3.1. Пусть $\omega \in \Omega^2(\mathbb{R}^3)$ - внешняя 2-форма на многообразии $\mathbb{R}^3$, с декартовыми координатами (x, y, z), заданная так: $\omega = (x + y^2)\, dx \wedge dz + (y + z) dy \wedge dz$. Фиксируем точку $P = (1, 2, 3)$ и пару векторов $a = (4, 5, 6)$ и $b = (7, 8, 9)$ из $T_P(\mathbb{R}^3)$. Вычислить $\omega|_P(a, b)$.

Задача 3.2. Построить пример не нулевой гладкой 2 -формы на трехмерной сфере S^3.

Задача 3.3. Показать, что $*1$ - это форма объема на M, где 1 - это гладкая функция на ориентируемом римановом многообразии M, тождественно равная единице.

Задача 3.4. Пусть F - гладкая фупкция па ориептируемом римаповом мпогообразии M. Найти $*F$.

Задача 3.5. Пусть T - тензорное поле типа $(0, 1)$ на евклидовом пространстве $\mathbb{R}^3$. Показать, что в евклидовых координатах

$$(*T)_{23} = T_1, \quad (*T)_{31} = T_2, \quad (*T)_{12} = T_3$$

Задача 3.6. Пусть (x, y, z) - евклидовы координаты в евклидовом пространстве $\mathbb{R}^3$. Вычислить

$$*dx, \quad *dy, \quad *dz, \quad *(dx \wedge dy), \quad *(dx \wedge dz), \quad *(dy \wedge dz), \quad *(dx \wedge dy \wedge dz)$$

Задача 3.7. Вычислить $*(*1)$, и более общо, $*(*T)$ на ориентированном римановом многообразии M.

Задача 3.8. Пусть $f : S^1 \to \mathbb{R}^2$ стандартное вложение окружности в плоскость:

$$x = R\cos\varphi, \quad y = R\sin\varphi$$

а $\omega_1 = x dy + y dx$ и $\omega_2 = (x + y) dx \wedge dy$ - внешние формы на плоскости. Вычислить $f^*(\omega_1)$ и $f^*(\omega_2)$.

15.3.6 Extra Topics

3.1. Примеры внешних форм. Приведем несколько примеров.

1. Пусть на многообразии $M = \mathbb{R}^2$ фиксированы стандартные декартовы координаты (x, y). Зададим на $\mathbb{R}^2$ внешнюю форму ω^2 степени 2 , положив

$$\omega^2 = \left(x^2 + y\right) dx \wedge dy$$

Вычислим значение формы ω^2 на касательных векторах $V = (1, 2)$ и $W = (3, 4)$ из касательного пространства $T_P\mathbb{R}^2$ к плоскости в точке $P = (5, 6)$. Форма $\omega^2|_P$ имеет вид $\omega^2|_P = (5^2 + 6)\, dx \wedge dy = 31 dx \wedge dy$. Поэтому

$$\omega^2\big|_P (V, W) = 31 dx \wedge dy(V, W) = 31 \det \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} = -62$$

2. Пусть M^n - n-мерное ориентированное риманово многообразие. Обозначим через g гладкую функцию на M, равную определителю матрицы метрики в точках многообразия. Определим на M внешнюю форму vol_n степени n, положив ее равной

$$\mathrm{vol}_n = \sqrt{g} dx^1 \wedge \cdots \wedge dx^n$$

в локальных координатах, согласованных с ориентацией многообразия M, и

$$\mathrm{vol}_n = -\sqrt{g} dx^1 \wedge \cdots \wedge dx^n$$

в остальных локальных координатах. Легко проверить, что приведенные формулы действительно задают на M дифференциальную форму. В самом деле, если g_{ij} — компоненты римановой метрики в координатах $(x^1, \ldots, x^n)$, и $\left(x^{1'}, \ldots, x^{n'}\right)$ - другая система локальных координат на M, то компоненты $g_{i'j'}$ римановой метрики в новых координатах выглядят так:

$$g_{i'j'} = g_{ij} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}}$$

Поэтому

$$\det\left(g_{i'j'}\right) = \left(\det \frac{\partial x^i}{\partial x^{i'}}\right)^2 \det\left(g_{ij}\right),$$

откуда, пользуясь леммой 3.2 , заключаем что

$$(\mathrm{vol}_n)_{1'\ldots n'} = (\mathrm{vol}_n)_{1\ldots n} \det\left(\frac{\partial x^i}{\partial x^{i'}}\right) = \sqrt{\det\left(g_{ij}\right)} \det\left(\frac{\partial x^i}{\partial x^{i'}}\right) = \\ = \operatorname{sign} \det\left(\frac{\partial x^i}{\partial x^{i'}}\right) \sqrt{\det\left(g_{i'j'}\right)}$$

т.е. выражение $\sqrt{g}$ меняется при замене координат как существенная компонента дифференциальной формы степени n, что и требовалось. Форма vol_n называется формой объема риманова многообразия M (Подумайте, почему?).

3.2. Внешнее умножение базисных ковекторов. Пусть $(x^1, \ldots, x^n)$ - система координат на многообразии. Оказывается, базисные элементы $dx^{i_1} \wedge \cdots \wedge dx^{i_q}$ пространства Λ^q на

самом деле - внешние произведения соответствующих базисных ковекторов, а именно, для произвольного набора целых чисел ($\alpha_1, \ldots, \alpha_k$), таких что $1 \leq \alpha_i \leq n$, и произвольного набора векторов $V_1, \ldots, V_k$ имеет место равенство

$$dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}(V_1, \ldots, V_k) = \det \begin{pmatrix} dx^{\alpha_1}(V_1) & dx^{\alpha_2}(V_1) & \ldots & dx^{\alpha_k}(V_1) \\ \vdots & \vdots & \ddots & \vdots \\ dx^{\alpha_1}(V_k) & dx^{\alpha_2}(V_k) & \ldots & dx^{\alpha_k}(V_k) \end{pmatrix}$$

В частности, внешнее произведение форм $dx^{\alpha_1}, \ldots, dx^{\alpha}_k$, где $1 \leq \alpha_1 < \cdots \alpha_k \leq n$, совпадает с базисным элементом $dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$ пространства Λ^k. Этот результат уже был сформулирован выше в виде леммы 3.3

Действительно, воспользовавшись ассоциативностью внешнего произведения, и следствием 2.1, запишем форму $dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}$ так:

$$\begin{aligned} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k} &= dx^{\alpha_1} \wedge (dx^{\alpha_2} \wedge (\cdots (dx^{\alpha_{k-1}} \wedge dx^{\alpha_k}) \cdots)) = \\ &= dx^{\alpha_1} \wedge \left(dx^{\alpha_2} \wedge \cdots \left(dx^{\alpha_{k-2}} \wedge \left[\frac{2!}{1!1!} A\left(dx^{\alpha_{k-1}} \otimes dx^{\alpha_k}\right) \right] \right) \cdots \right) = \\ &= dx^{\alpha_1} \wedge \left(dx^{\alpha_2} \wedge \cdots \left[\frac{3!}{2!1!} \frac{2!}{1!1!} A\left(dx^{\alpha_{k-2}} \otimes A\left(dx^{\alpha_{k-1}} \otimes dx^{\alpha_k}\right)\right) \right] \cdots \right) = \\ &= \cdots = k! A\left(dx^{\alpha_1} \otimes \cdots \otimes dx^{\alpha_k}\right) \end{aligned}$$

Но тогда, в силу определения альтернирования и леммы 2.6 можно переписать последнее выражение в виде

$$\begin{aligned} dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k} &= k! A\left(dx^{\alpha_1} \otimes \cdots \otimes dx^{\alpha_k}\right) = \\ &= k! \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^{\sigma}{}_{\sigma}\left(dx^{\alpha_1} \otimes \cdots \otimes dx^{\alpha_k}\right) = \\ &= \sum_{\sigma \in S_k} (-1)^{\sigma} dx^{\alpha_{\sigma^{-1}(1)}} \otimes \cdots \otimes dx^{\alpha_{\sigma^{-1}(k)}} = \\ &= \sum_{\pi \in S_k} (-1)^{\pi} dx^{\alpha_{\pi(1)}} \otimes \cdots \otimes dx^{\alpha_{\pi(k)}} \end{aligned}$$

где через π мы переобозначили перестановку σ^{-1}. Поэтому

$$dx^{\alpha_1} \wedge \cdots \wedge dx^{\alpha_k}(V_1, \ldots, V_k) = \sum_{\pi \in S_k} (-1)^{\pi} dx^{\alpha_{\pi(1)}}(V_1) \cdots dx^{\alpha_{\pi(k)}}(V_k)$$

что и требовалось.

3.3. Доказательство следствия 3.3. Действительно,

$$T_{i_1 \ldots i_q} dx^{i_1} \wedge \cdots \wedge dx^{i_q} = \sum_{\text{все } i_k \text{ различны}} + \sum_{\text{не все } i_k \text{ различны}}$$

причем вторая сумма равна нулю. Представим теперь первую сумму в виде

$$\sum_{1 \leq i_1 < \cdots < i_q \leq n} \left(\sum_{\sigma \in S_q} T_{i_{\sigma(1)} \cdots i_{\sigma(q)}} dx^{i_{\sigma(1)}} \wedge \cdots \wedge dx^{i_{\sigma(q)}} \right)$$

Но, для каждого упорядоченного набора $1 \leq i_1 < \cdots < i_q \leq n$ соответствующее слагаемое в скобках в силу кососимметричности тензора T и следствия 3.2 переписывается в виде

$$\sum_{\sigma\in S_q} T_{i_{\sigma(1)}\dots i_{\sigma(q)}} dx^{i_{\sigma(1)}}\wedge\dots\wedge dx^{i_{\sigma(q)}} = T_{i_1\dots i_q} dx^{i_1}\wedge\dots\wedge dx^{i_q} \sum_{\sigma\in S_q}((-1)^\sigma(-1)^\sigma)$$

и, таким образом, это слагаемое равно

$$q! T_{i_1\dots i_q} dx^{i_1}\wedge\dots\wedge dx^{i_q}$$

Поэтому окончательно

$$T_{i_1\dots i_q} dx^{i_1}\wedge\dots\wedge dx^{i_q} = q! \sum_{1\le i_1<\dots<i_q\le n} T_{i_1\dots i_q} dx^{i_1}\wedge\dots\wedge dx^{i_q}$$

что и требовалось.

3.4. Пример вычисления звездочки Ходжа. Пусть в пространстве $\mathbb{R}^3$ в координатах (x, y, z) задана риманова метрика (g_{ij}), где

$$(g_{ij}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 9 \end{pmatrix}, \quad \left(g^{ij}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1/9 \end{pmatrix}, \quad g = \sqrt{\det(g_{ij})} = 6$$

а также дифференциальные формы dz и $dx\wedge dy$. Вычислим 2-форму $\omega = *(dz)$. Вычисления проведем в координатах (x, y, z). Заметим, что компоненты формы dz имеют вид $(0,0,1)$. После поднятия индекса, получаем векторное поле с компонентами $T^i = g^{i\alpha}(dz)_\alpha$, равными ($0, 0, 1/9$). По определению,

$$\omega_{ij} = \frac{1}{k!}\varepsilon_{ij\beta}T^\beta,$$

где $\varepsilon_{ij\beta}$ - компоненты формы объема, а $k = 1$ как степень формы dz. В свою очередь, форма объема имеет вид $g dx\wedge dy\wedge dz$, а ее компонента $\varepsilon_{ijk} = g\,\mathrm{sign}\{i,j,k\}$, где $\mathrm{sign}\{i,j,k\}$ равен нулю, если среди $\{i,j,k\}$ есть совпадающие, и равен знаку перестановки (i, j, k) в противном случае. Итак,

$$\omega_{12} = \varepsilon_{12\beta}T^\beta = \varepsilon_{123}T^3 = \frac{6}{9} = \frac{2}{3}, \quad ,\omega_{13} = \varepsilon_{132}T^2 = 0, \quad \omega_{23} = \varepsilon_{231}T^1 = 0$$

поэтому $*(dz) = \frac{2}{3}dx\wedge dy$.

Теперь вычислим $\xi = *(dx\wedge dy)$. Компоненты T^{ij} тензорного поля, полученного из $dx\wedge dy$ поднятием индексов, можно записать в виде следующей матрицы:

$$\left(T^{ij}\right) = \begin{pmatrix} 0 & 1/4 & 0 \\ -1/4 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Снова по определению,

$$\xi_i = \frac{1}{2!}\varepsilon_{i\alpha\beta}T^{\alpha\beta}$$

Поэтому

$$\xi_1 = \frac{1}{2}\varepsilon_{1\alpha\beta}T^{\alpha\beta} = \frac{1}{2}\left(\varepsilon_{123}T^{23} + \varepsilon_{132}T^{32}\right) = \varepsilon_{123}T^{23} = 0$$

где мы сначала выписали все потенциально ненулевые слагаемые а затем воспользовались косой симметрией. Аналогично,

$$\xi_2 = \varepsilon_{231}T^{31} = 0, \quad \xi_3 = \varepsilon_{312}T^{12} = 6\frac{1}{1} = \frac{3}{2}$$

откуда $*(dx\wedge dy) = \frac{3}{2}dz$. Поэтому $*(*(dz)) = dz$ и $*(*(dx\wedge dy)) = dx\wedge dy$.

15.4 4: Differentiating and Integrating Forms

В прошлой лекции мы определили алгебру внешних дифференциальных форм. Теперь введем на дифференциальных формах операции интегрирования и дифференцирования. Нам также понадобится расширить понятие многообразия, определив многообразия с краем.

15.4.1 4.1. External differentiation

Внешняя дифференциальная форма степени нуль - это просто гладкая функция на многообразии. Тогда, как мы уже знаем, каждой функции можно поставить в соответствие кососимметричное (автоматически) тензорное поле типа $(0,1)$ - дифференциал этой функции. Таким образом, определено отображение $d:\Omega^0(M)\to\Omega^1(M)$, ставящее в соответствие каждой 0 -форме $f\in\Omega^0(M)$ ковектор, т.е. 1 -форму, df. Если $(x^1,\ldots,x^n)$ - координаты на многообразии M, и $f\left(x^1,\ldots,x^n\right)$ - координатное представление функции f, то, в этих координатах, $df=\frac{\partial f}{\partial x^i}dx^i$.

В общем случае, для произвольного натурального p, построим отображение $d:\Omega^p(M)\to\Omega^{p+1}(M)$. Мы определим его в локальных координатах. Пусть ($x^1,\ldots,x^n$) - локальные координаты на многообразии M, и P - произвольная форма из $\Omega^p(M)$, которая в этих координатах записывается в виде

$$P=\sum_{1\le i_1<\cdots<i_p\le n}P_{i_1\ldots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

Коэффициенты $P_{i_1\ldots i_p}$ - это гладкие функции в карте $U\left(x^1,\ldots,x^n\right)$ на многообразии M. Определим внешний дифференциал (внешнюю производную) dP формы P, положив

$$dP=\sum_{1\le i_1<\cdots<i_p\le n}d\left(P_{i_1\ldots i_p}\right)\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

Отметим, что если записать форму P в виде

$$P=\frac{1}{p!}P_{i_1\ldots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

то определенная только что ее внешняя производная запишется в виде

$$dP=\frac{1}{p!}d\left(P_{i_1\ldots i_p}\right)\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}$$

Лемма 4.1. Операция внешнего дифференцирования определена корректно, т.е. не зависит от выбора локальной системы координат, участвующей в ее определении.

Доказательство. Пусть в некоторой окрестности $U\subset M$ заданы две локальных системы координат $(x^1,\ldots,x^n)$ и $\left(x^{1'},\ldots,x^{n'}\right)$. Тогда дифференциальная форма $P\in\Omega^p(M)$ в U может быть записана двумя способами:

$$P=\frac{1}{p!}P_{i_1\ldots i_p}dx^{i_1}\wedge\cdots\wedge dx^{i_p}=\frac{1}{p!}P_{i'_1\ldots i'_p}dx^{i'_1}\wedge\cdots\wedge dx^{i'_p}$$

где компоненты $P_{i_1\ldots i_p}$ и $P_{i'_1\ldots i'_p}$ связаны по тензорному закону. Тогда в координатах ($x^{1'},\ldots,x^{n'}$) внешняя производная dP, в соответствие с определением, вычисляется так:

$$dP=\frac{1}{p!}d\left(P_{i'_1\ldots i'_p}\right)\wedge dx^{i'_1}\wedge\cdots\wedge dx^{i'_p}=\frac{1}{p!}\frac{\partial P_{i'_1\ldots i'_p}}{\partial x^{\alpha'}}dx^{\alpha'}\wedge dx^{i'_1}\wedge\cdots\wedge dx^{i'_p}$$

Воспользовавшись тензорным законом, получим:

$$p!dP = \frac{\partial}{\partial x^{\alpha'}}\left(\frac{\partial x^{i_1}}{\partial x^{i'_1}}\cdots\frac{\partial x^{i_p}}{\partial x^{i'_p}}P_{i_1\ldots i_p}\right)dx^{\alpha'}\wedge dx^{i'_1}\wedge\cdots\wedge dx^{i'_p}$$

Так как вторые производные $\frac{\partial^2 x^{i_k}}{\partial x^{i'_l}\partial x^{\alpha'}}$ симметричны по i'_l и α', а внешнее произведение $dx^{\alpha'}\wedge dx^{i'_l}$ — кососимметрично, все слагаемые, содержащие эти вторые производные сокращаются. В результате получаем:

$$\begin{aligned}
p!dP &= \frac{\partial P_{i_1\ldots i_p}}{\partial x^{\alpha'}}\frac{\partial x^{i_1}}{\partial x^{i'_1}}\cdots\frac{\partial x^{i_p}}{\partial x^{i'_p}}dx^{\alpha'}\wedge dx^{i'_1}\wedge\cdots\wedge dx^{i'_p} = \\
&= \frac{\partial P_{i_1\ldots i_p}}{\partial x^{\alpha}}\frac{\partial x^{\alpha}}{\partial x^{\alpha'}}dx^{\alpha'}\wedge\frac{\partial x^{i_1}}{\partial x^{i'_1}}dx^{i'_1}\wedge\cdots\wedge\frac{\partial x^{i_p}}{\partial x^{i'_p}}dx^{i'_p} = \\
&= \frac{\partial P_{i_1\ldots i_p}}{\partial x^{\alpha}}dx^{\alpha}\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_p}
\end{aligned}$$

Но выражение справа - это, с точностью до деления на $p!$, внешняя производная dP формы P, записанная в координатах ($x^1,\ldots,x^n$). Лемма доказана.

Установим простейшие свойства внешнего дифференциала.

Утверждение 4.1. Пусть $P\in\Omega^p(M)$ и $Q\in\Omega^q(M)$ - дифферениальные формь на M.

- Для произвольных постоянных $c_1 uc_2$ имеет место равенство: $d\left(c_1P+c_2Q\right)=c_1d(P)+c_2d(Q)$ (линейность).
- Имеет место следующий аналог правила Лейбница: $d(P\wedge Q)=d(P)\wedge Q+(-1)^pP\wedge d(Q)$.
- Идемпотентность: $d(dP)=0$.
- Для произвольного гладкого отображения $F:X\to M$ имеет место равенство $F^*(dP)=d\left(F^*P\right)$.

Доказательство. Первое свойство очевидно. [3]

В силу линейности, второе свойство достаточно проверить для случая мономов, т.е. для $P=fdx^{i_1}\wedge\cdots\wedge dx^{i_p}$ и $Q=gdx^{j_1}\wedge\cdots\wedge dx^{j_q}$. Имеем:

$$\begin{aligned}
P\wedge Q = \left(fdx^{i_1}\wedge\cdots\wedge dx^{i_p}\right)\wedge\left(gdx^{j_1}\wedge\cdots\wedge dx^{j_q}\right) = \\
= fg\left(dx^{i_1}\wedge\cdots\wedge dx^{i_p}\right)\wedge\left(dx^{j_1}\wedge\cdots\wedge dx^{j_q}\right)
\end{aligned}$$

поэтому

$$\begin{aligned}
&d(P\wedge Q) = d(fg)\wedge\left(dx^{i_1}\wedge\cdots\wedge dx^{i_p}\right)\wedge\left(dx^{j_1}\wedge\cdots\wedge dx^{j_q}\right) = \\
&= \frac{\partial fg}{\partial x^{\alpha}}dx^{\alpha}\wedge\left(dx^{i_1}\wedge\cdots\wedge dx^{i_p}\right)\wedge\left(dx^{j_1}\wedge\cdots\wedge dx^{j_q}\right) = \\
&= \left(\frac{\partial f}{\partial x^{\alpha}}g+f\frac{\partial g}{\partial x^{\alpha}}\right)dx^{\alpha}\wedge\left(dx^{i_1}\wedge\cdots\wedge dx^{i_p}\right)\wedge\left(dx^{j_1}\wedge\cdots\wedge dx^{j_q}\right) = \\
&= \left(\frac{\partial f}{\partial x^{\alpha}}dx^{\alpha}\wedge\left(dx^{i_1}\wedge\cdots\wedge dx^{i_p}\right)\right)\wedge\left(gdx^{j_1}\wedge\cdots\wedge dx^{j_q}\right)+ \\
&+(-1)^p\left(fdx^{i_1}\wedge\cdots\wedge dx^{i_p}\right)\wedge\left(\frac{\partial g}{\partial x^{\alpha}}dx^{\alpha}\wedge\left(dx^{j_1}\wedge\cdots\wedge dx^{j_q}\right)\right) = \\
&= dP\wedge Q+(-1)^pP\wedge dQ
\end{aligned}$$

что и требовалось. Отметим, что здесь $(-1)^p$ - четность перестановки, «меняющей местами» dx^{α} и моном $dx^{i_1}\wedge\cdots\wedge dx^{i_p}$.

Докажем третье свойство. Для моіюмов вида $f dx^{i_1} \wedge \cdots \wedge dx^{i_p}$ оно немедленно вытекает из симметричности вторых частных производных $\frac{\partial^2 f}{\partial x^i \partial x^j}$ и косой симметрии внешнего произведения $dx^i \wedge dx^j$.

Наконец, в силу линейности внешнего дифференцирования и операции перенесения, последнее свойство тоже достаточно проверить для монома $P = f dx^{i_1} \wedge \cdots \wedge dx^{i_p}$. Обозначим через $(t^1, \ldots, t^m)$ локальные координаты на X, и пусть $x^i = x^i (t^1, \ldots, t^m)$ - координатное предста-

где $f = \omega_{1 \ldots n}$. Положим

$$\int_W \omega^n = \int_{\varphi(W)} \cdots \int f \left(x^1, \ldots, x^n\right) dx^1 \cdots dx^n$$

где справа стоит обычный кратный интеграл по области $\varphi(W) \subset \mathbb{R}^n$. Это число назовем интегралом от формы ω^n по области W.

Величина, стоящая в правой части, не зависит от выбора локальных координат. Действительно, пусть (U', φ') - еще одна карта на M, и $(x^{1'}, \ldots, x^{n'})$ - соответствующие локальные координаты. Предположим, что $W \subset U \cap U'$. Тогда форма ω^n записывается в этих координатах в виде

$$g \left(x^{1'}, \ldots, x^{n'}\right) dx^{1'} \wedge \cdots \wedge dx^{n'}$$

где, как мы вычислили в лемме 3.2 , функции f и g связаны так:

$$g \left(x^{1'}, \ldots, x^{n'}\right) = f \left(x^1 \left(x^{1'}, \ldots, x^{n'}\right), \ldots, x^n \left(x^{1'}, \ldots, x^{n'}\right)\right) \det \left(\frac{\partial x^i}{\partial x^{j'}}\right).$$

Но тогда, воспользовавшись формулой замены переменных в кратном интеграле и тем фактом, что якобиан функции перехода $\varphi \circ (\varphi')^{-1}$ положителен, можно переписать интеграл от формы ω^n по W в «новых» координатах $\left(x^{1'}, \ldots, x^{n'}\right)$ так:

$$\begin{aligned}
&\int_W \omega^n = \int_{\varphi'(W)} \cdots \int g \left(x^{1'}, \ldots, x^{n'}\right) dx^{1'} \cdots dx^{n'} = \\
&= \int_{\varphi'(W)} \cdots \int_{\varphi'(W)} f \left(x^1 \left(x^{1'}, \ldots, x^{n'}\right), \ldots, x^n \left(x^{1'}, \ldots, x^{n'}\right)\right) \det \frac{\partial x^i}{\partial x^{j'}} dx^{1'} \cdots dx^{n'} = \\
&= \int_{\left(\varphi \circ (\varphi')^{-1}\right)(\varphi'(W))} f \left(x^1 \left(x^{1'}, \ldots, x^{n'}\right), \ldots, x^n \left(x^{1'}, \ldots, x^{n'}\right)\right) \left|\det \frac{\partial x^i}{\partial x^{j'}}\right| dx^{1'} \cdots dx^{n'} = \\
&= \int_{\varphi(W)} \cdots \int^n f \left(x^1, \ldots, x^n\right) dx^1 \cdots dx^n
\end{aligned}$$

Осталось заметить, что в правой части мы получили в точности интеграл от формы ω^n, записанный в координатах $(x^1, \ldots, x^n)$. Итак, мы доказали следующую лемму.

Лемма 4.3. Интеграл от формы ω^n по области $W \subset M$, целиком лежащей в одной карте, не зависит от выбора ориентированных локальных координат.

Чтобы определить интеграл от формы ω^n в общем случае, нам придется вспомнить кое-что из прошлого семестра. А именно, тогда мы уже пользовались существованием разбиения единицьb, подчиненного (локально) конечному покрытию $\{U_\alpha\}$ топологического пространства M. Последнее означает, что существует набор функций $\{f_\alpha\}$ на M, таких что

[03] На самом деле, мы определили только дифференциал формы фиксированной степени p, а не произвольного элемента алгебры Ω^*, каким является $c_1 P + c_2 Q$. Поэтому для $p \neq q$ формула $d(P + Q) = dP + dQ$ имеет место по определению.

- множество значений каждой функции f_α лежит в отрезке $[0,1]$;
- носитель функции f_α содержится в соответствующем элементе покрытия, т.е. $\operatorname{supp} f_\alpha \subset U_\alpha$;
- для каждой точки x из M имеет место равенство $\sum_\alpha f_\alpha(x) = 1$.

Более того, было отмечено, что если U_α - открытые множества в $\mathbb{R}^n$, то все функции f_α могут быть выбраны гладкими. Поскольку координатный гомеоморфизм φ_α отождествляет каждую карту U_α с областью в $\mathbb{R}^n$, отсюда вытекает следующий результат.

Утверждение 4.2. Пусть M - гладкое многообразие, $u\,\{(U_\alpha, \varphi_\alpha)\}$ локально конечный атлас на M. Тогда существует разбиение единиц, подиненное покрытию $\{U_\alpha\}$, состоящее из гладких функиии на M.

Итак, пусть снова $\omega^n \in \Omega^n(M)$ - форма с компактным носителем, и $\{(U_\alpha, \varphi_\alpha)\}$ - ориентированный локально конечный атлас на ориентированном многообразии M. Пусть $\{f_\alpha\}$ - разбиение единицы, подчиненное атласу $\{U_\alpha\}$. Так как носитель формы ω^n компактен, он содержится в объединении конечного числа карт $U_1, \ldots, U_N$. Определим интеграл от формы ω^n по многообразию M, положив

$$\int_M \omega^n = \sum_{i=1}^N \int_{U_i} f_i \omega^n$$

Отметим, что все интегралы, стоящие в сумме справа берутся по областям, целиком лежащим в карте, поэтому определены.

Лемма 4.4. Определение интеграла от формы ω^n по многообразию M не зависит ни от выбора разбиения единиц, ни от выбора ориентированного атласа на ориентированном многообразии M.

Доказательство. Без ограничения общности можно считать, что атлас карт $\{(U_\alpha, \varphi_\alpha)\}$ конечен. Действительно, для этого достаточно вместо многообразия M рассмотреть его подмножество $\cup_{i=1}^N U_i$, которое тоже, очевидно, является многообразием.

Итак, пусть на M имеется два конечных атласа $\{(U_\alpha, \varphi_\alpha)\}_{\alpha=1}^N$ и $\{(W_\beta, \psi_\beta)\}_{\beta=1}^M$, и $\{f_\alpha\}$ и $\{g_\beta\}$ - подчиненные им разбиения единицы.

Рассмотрим набор функций $\{h_{\alpha\beta} = f_\alpha g_\beta\}$. Легко видеть, что этот набор функций представляет собой разбиение единицы для системы множеств $\{V_{\alpha\beta} = U_\alpha \cap W_\beta\}$. Действительно, первые два свойства из определения разбиения единицы очевидны, а

$$\sum_{\alpha,\beta} h_{\alpha\beta} = \sum_\alpha \left(f_\alpha \sum_\beta g_\beta \right) = \sum_\alpha f_\alpha = 1,$$

что и требовалось.

Проверим теперь совпадение интегралов от формы ω, определенных с помощью разных атласов. Имеем:

$$\sum_\alpha \int_{U_\alpha} f_\alpha \omega = \sum_\alpha \int_{U_\alpha} \left(\sum_\beta g_\beta \right) f_\alpha \omega = \sum_{\alpha,\beta} \int_{U_\alpha} (f_\alpha g_\beta)\, \omega =$$
$$= \sum_{\alpha,\beta} \int_{U_\alpha \cap W_\beta} h_{\alpha\beta} \omega = \sum_{\alpha\beta} \int_{V_{\alpha\beta}} h_{\alpha\beta} \omega,$$

т.е. интеграл, определенный с помощью покрытия $\{U_\alpha\}$ совпадает с интегралом, определенным с помощью покрытия $\{V_{\alpha\beta}\}$. Аналогично проверяется и что интеграл,

определенный с помощью $\{W_\beta\}$ совпадает с интегралом, определенным с помощью $\{V_{\alpha\beta}\}$. Лемма доказана.

Перечислим простейшие свойства интеграла.

Утверждение 4.3. Пусть M - гладкое ориентированное многообразие.

- Интегрирование внешних форм по M линейно:

$$\int_M (\lambda_1\omega_1 + \lambda_2\omega_2) = \lambda_1 \int_M \omega_1 + \lambda_2 \int_M \omega_2$$

для любых форм $\omega_1 u \omega_2$ с компактным носителем и любых чисел $\lambda_1 u \lambda_2$.

- Обозначим через $\tilde{M}$ многообразие M с противоположной ориентацией. Тогда

$$\int_{\tilde{M}} \omega = -\int_M \omega$$

для любой формы ω с компактным носителем.

Доказательство. Первое свойство очевидно.

В силу аддитивности интеграла, второе свойство достаточно проверять для формы, носитель которой содержится в одной карте. Пусть $\{(U_\alpha, \varphi_\alpha)\}$ - ориентированный атлас на M. Тогда атлас с противоположной ориентацией, т.е. атлас на $\tilde{M}$ можно получить заменив знак

первой координатной функции в каждой системе локальных координат. Другими словами, если ($x^1, \ldots, x^n$) - локальные координаты на M в карте (U, φ), то в качестве локальных координат на $\tilde{M}$ можно взять $(y^1, \ldots, y^n) = (-x^1, x^2, \ldots, x^n)$. Соответствующую карту на $\tilde{M}$ обозначим через $(U, \tilde{\varphi})$.

Пусть в карте $(U, \tilde{\varphi})$ форма ω имеет вид $f\left(y^1, \ldots, y^n\right) dy^1 \wedge \cdots \wedge dy^n$. Тогда в силу леммы 3.2

$$\begin{aligned}\int_{U\subset\tilde{M}} \omega &= \int \cdots \int_{\tilde{\varphi}(U)} f\left(y^1, \ldots, y^n\right) dy^1 \cdots dy^n = \\ &= \int_{\tilde{\varphi}(U)} \cdots \int g\left(x^1\left(y^1, \ldots, y^n\right), \ldots, x^n\left(y^1, \ldots, y^n\right)\right) \det\left(\frac{\partial x^i}{\partial y^j}\right) dy^1 \cdots dy^n\end{aligned}$$

где $g\left(x^1, \ldots, x^n\right)$ - существенная компонента формы ω в локальных координатах $(x^1, \ldots, x^n)$. Воспользуемся теоремой о замене переменной в кратном интеграле, учитывая что $\det\left(\frac{\partial x^i}{\partial y^j}\right) < 0$. Получим:

$$\begin{aligned}&\int_{U\subset\tilde{M}} \omega = \\ &= \int_{\tilde{\varphi}(U)} \cdots \int_r g\left(x^1\left(y^1, \ldots, y^n\right), \ldots, x^n\left(y^1, \ldots, y^n\right)\right) \det\left(\frac{\partial x^i}{\partial y^j}\right) dy^1 \cdots dy^n = \\ &= -\int_{\varphi(U)} \cdots \int_r g\left(x^1, \ldots, x^n\right) dx^1 \cdots dx^n = -\int_{U\subset M} \omega\end{aligned}$$

Утверждение доказано.

Итак, мы определили интегрирование дифференциальных форм по многообразию и изучили простейшие свойства этой операции. Теперь естественно перенести на случай многообразий известные из курса математического анализа формулы Грина, Стокса и

Остроградского-Гаусса (соответствующий результат имеет место и называется общей теоремой Стокса). Однако в этих формулах интегрирование производиться не по многообразиям, а по несколько более общим объектам: к соответствующим многообразиям (области на плоскости или в пространстве, поверхности в пространстве) добавляются граничные (в топологическом смысле) точки. Поэтому, чтобы построить соответствующее обобщение, мы начнем с определения аналога области с границей, который называется многообразисм с красм.
Замечание. Отрезок $[a, b] \subset \mathbb{R}^1$ не является многообразием, так как его концевые точки a и b не имеют окрестностей, гомеоморфных открытым дискам. Поэтому, пока, величина $\int_a^b f(x)dx$, где x - координата на прямой, а $f(x)dx$ - гладкая 1-форма на ней, не определена.

15.4.2 4.3. Edge Manifolds

Пусть N - гладкое многообразие. Замыкание M открытого подмножества из N называется многообразием с краем, если существует такая гладкая функция $f : N \to \mathbb{R}$, что $M = \{x \in N \mid f(x) \geq 0\}$, и дифференциал функции f во всех точках x, где $f(x) = 0$, отличен от нуля. В этом случае множество точек, где $f(x) = 0$, называется краем многообразия M и обозначается через ∂M. Размерностью многообразия M с краем ∂M называстся размсрность многообразия N. Компактныс многообразия без края называют замкнутьлми.

Отметим, что если край ∂M пуст, то M является обычным гладким многообразием (без края). Из прошлого семестра мы знаем, что край многообразия M, как множество решений регулярного уравнения, обладает структурой гладкого многообразия (без края), причем в качестве карт можно рассмотреть пересечения карт многообразия N с краем ∂M. Если размерность многообразия M равняется n, то размерность его края ∂M равна $n - 1$.

Эквивалентное определение многообразия с краем получается, если немного модифицировать определение многообразия. А именно, хаусдорфово топологическое пространство M называется многообразием *с* краем, если его можно покрыть не более чем счетным семейством открытых множеств, каждое из которых гомеоморфно некоторому открытому подмножеству в полупространстве $\mathbb{R}^n_+ = \{(x^1, \ldots, x^n) \mid x^1 \geq 0\}$. Последнее означает, что точки пространства M разбиваются на два класса: точки, обладающие окрестностью, гомеоморфной пространству $\mathbb{R}^n$ (такие точки называют внутренними); и точки, обладающие лишь окрестностью, гомсоморфной полупространству $\mathbb{R}^n_+$ (такис точки называют краевьлми). Край ∂M - это множество всех краевых точек из M.

В качестве примера многообразия с краем можно взять любую область в $\mathbb{R}^n$, (топологическая) граница которой задана неявно гладкой функцией в виде $f(x) = 0$, причем $df \neq 0$ на этой границе.

В каждом конкретном случае удобно пользоваться то одним, то другим из приведенных определений. Отметим, что все понятия, введенные нами выше для случая многообразий без края дословно переносятся и на случай многообразий с краем. Для них также легко можно переформулировать и доказанные нами теоремы. В качестве примера, рассмотрим аналог теоремы о задании многообразия системой уравнений.

Пусть $F : \mathbb{R}^n \to \mathbb{R}^k, k \leq n$, - вектор-функция, и предположим, что во всех точках множества уровня $M = \{P \in \mathbb{R}^n \mid F(P) = 0\}$ дифференциал dF отображения F имеет максимальный ранг, равный k. Обозначим через $(x^1, \ldots, x^n)$ и $\left(y^1, \ldots, y^k\right)$ декартовы координаты в $\mathbb{R}^n$ и $\mathbb{R}^k$ соответственно, и пусть $y^i = f^i\left(x^1, \ldots, x^n\right)$ - координатное представление функции F. Рассмотрим систему из $k - 1$ уравнения и одного неравен-

Определение. Ориентация края ∂M ориентированного многообразия M, противоположная построенной в доказательстве теоремы 4.2, называется согласованной с ориентацией многообразия M или канонической.

Замечание. Гсомстрически, каноническая ориснтация края ∂M ориснтированного многообразия M^n устроена так. Рассмотрим в точке P края ∂M базис $e_1, \ldots, e_n$ касательного пространства $T_P M$, где векторы $e_2, \ldots, e_n$ касаются края, а вектор e_1 направлен от края ∂M «наружу» многообразия M (формально последнее означает, что, если ($x^1, \ldots, x^n$) - локальные координаты в окрестности краевой точки P, в которых край ∂M задается условием $x_1 = 0$, то вектор $e_1 \in T_P M$ имеет отрицательную составляющую по координате x^1). Ориентация базиса $e_2, \ldots, e_n$ согласована с ориентацией края, порожденной ориентацией многообразия M, если и только если ориентация базиса $e_1, \ldots, e_n$ согласована с ориентацией многообразия M.

Пример. На рис. 1 изображена ориентация края цилиндра - двух окружностей, согласованная с ориентацией цилиндра, заданной базисом (e_1, e_2), где e_1 - вектор, параллельный меридиану, а e_2 - параллели, см. рис. 1.

Рис. 1. Ориентация цилиндра и его края

Замечание. Край неориентируемого многообразия может быть ориентируемым многообразием. Например, край неориентируемого листа Мебиуса ориентируемая окружность. Поэтому продолжить ориентацию с края на многообразие можно не всегда.

15.4.3 Problems

Задача 4.1. Пусть на многообразии $\mathbb{R}^3$ со стандартными координатами (x, y, z) задана 1 -форма $\omega = xdy + ydx$. Вычислить $d\omega$.

Задача 4.2. Пусть $f : S^1 \to \mathbb{R}^2$ стандартное вложение окружности в плоскость:

$$x = R\cos\varphi, \quad y = R\sin\varphi$$

а $\omega_1 = xdy + ydx$ - внешняя форма на плоскости. Вычислить $df^*(\omega_1)$.

Задача 4.3. Пусть M - риманово многообразие размерности n. Вычислить $d\,\mathrm{vol}_n$.

Задача 4.4. Iусть S^2 - стандартно вложенная в $\mathbb{R}^3$ сфера радиуса R, и (g_{ij}) индуцированная метрика на S^2. Вычислить

$$\int_{S^2} \mathrm{vol}_2$$

Задача 4.5. Покажите, что полноторий, ограниченный тором, полученным при вращении вокруг оси z окружности радиуса r в плоскости xz с центром в точке ($R, 0, 0$), $0 < r < R$, является многообразием с краем.

Задача 4.6. [I] Iто произойдет с канонической ориентацией края многообразия, если поменять ориентацию многообразия с краем на противоположную?

15.4.4 Extra Topics

4.1. Доказательство леммы 4.2. Пусть

$$P - \sum_{1 \le j_1 < \cdots < j_p \le n} P_{j_1 \ldots j_p} dx^{j_1} \wedge \cdots \wedge dx^{j_p}$$

- внешняя форма степени p на многообразии M, записанная в локальных координатах $(x^1, \ldots, x^n)$, и $R = dP \in \Omega^{p+1}(M)$. Тогда, по определению,

$$R = \sum_{1 \le j_1 < \cdots < j_p \le n} d\left(P_{j_1 \ldots j_p}\right) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_p}.$$

Поэтому

$$R = \sum_{1 \le j_1 < \cdots < j_p \le n} \left(\sum_{\alpha=1}^{n} \frac{\partial T_{j_1 \ldots j_p}}{\partial x^\alpha} dx^\alpha \right) \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_p} =$$

$$= \sum_{1 \le j_1 < \cdots < j_p \le n} \left(\sum_{\alpha \notin \{j_1, \ldots, j_p\}} \frac{\partial T_{j_1 \ldots j_p}}{\partial x^\alpha} dx^\alpha \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_p} \right).$$

Отметим, что индексы j_k упорядочены по возрастанию. Переставив dx^α так, чтобы полученный набор из ($p+1$)-го индекса стал упорядоченным по возрастанию и переобозначив индексы, получаем

$$R = \sum_{1 \le i_1 < \cdots < i_{p+1} \le n} \left(\sum_{\alpha=1}^{p+1} \frac{\partial T_{i_1 \ldots i_{\alpha-1} i_{\alpha+1} \ldots i_{p+1}}}{\partial x^{i_\alpha}} (-1)^{\alpha+1} \right) \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_{p+1}},$$

что и требовалось.

4.2. Об интегрировании форм. Предположим, что атлас гладкого компактного многообразия M содержит карту (U, φ), такую что $M \backslash U$ представляет собой множество меры нуль. Такое часто случается в примерах, скажем стандартная карта стереографической проекции - это сфера без точки. В сделанных предположениях имеет место следующее утверждение.

Утверждение 4.4. Для любой внешней формы ω старшей степени на M имеет место равенство

$$\int_U \omega = \int_M \omega$$

Доказательство проводится стандартными средствами математического анализа и оставляется в качестве упражнения.

4.3. Два определения многообразия с краем. Рассмотрим $M \subset N$ - многообразие с краем, край которого ∂M задан невырожденной на ∂M функцией $f : N \to \mathbb{R}$, и пусть $n = \dim N$. Построим на M атлас обобщенных карт. Во внутренних точках множества M, т.е. в точках из множества $M \backslash \partial M$, в качестве карт искомого атласа можно взять соответствующие карты многообразия N, пересекая их, если нужно с открытым множеством $M \backslash \partial M$.

Рассмотрим теперь точку P из ∂M. Функция f является гладкой функцией на N, причем дифференциал этой функции в точках из ∂M по определению отличен от нуля, т.е. функция f является субмерсией в точках из M. Поэтому множество ∂M является многообразием размерности ($n-1$). Более того, ∂M является подмногообразием в N.

Воспользовавшись утверждением о нормальной форме субмерсии, см. прошлый семестр, выберем в окрестности точки P такие координаты $(\ t^1, \ldots, t^n\ )$ на N, что функция f запишется в них так: $f\left(t^1, \ldots, t^n\right) = t^1$. В частности, множество ∂M в этом случае совпадет с множеством $\{X \in N \mid t^1(X) = 0\}$, а множество $N \cap M$ - с множеством $\{X \in N \mid t^1(X) \geq 0\}$. Но последнее в точности означает, что ограничение соответствующего координатного гомеоморфизма на $N \cap M$ задает гомеоморфизм на $\mathbb{R}^n_+$. Таким образом, мы определим карты во всех точках края ∂M. Остается проверить, что все функции перехода в объединенном атласе для внутренних и краевых точек многообразия M - гладкие. Это, фактически, вытекает из гладкости многообразия N.

Обратно, если многообразие с краем задано как хаусдорфово топологическое пространство с не более чем счетным атласом обобщенных карт, то следует вложить это многообразие в $\mathbb{R}^N$ (существование такого вложения для достаточно больших N доказывается дословным повторением рассуждений для многообразий без края). Затем, в малой окрестности края, следует расширить многообразие с краем, продолжив отображения вложения в малой окрестности границ параметризующих полупространств. Полученное в результате вложенное многообразие N будет содержать многообразие с краем M, которое будет задаваться в нем неравенством $f \geq 0$, где функция f в каждой (краевой) карте задается равенством вида $f(x) = x^1$.

15.5 5: The Stokes Theorem

В данном разделе мы докажем основную теорему интегрального исчисления, обобщающую многочисленные формулы интегрирования, известные из математического анализа.

15.5.1 5.1. The submanifold integral and the Stokes formula

Пусть M - гладкое многообразие, и $i : V^k \to M^n$ - гладкое k-мерное подмногообразие в M, заданное вложением. Пусть на M задана внешняя форма ω степени k, и пусть носитель формы $i^*(\omega)$ - компактное подмножество в V. Тогда интегралом от формь ω по подмногообразию V называется величина

$$\int_V \omega = \int_V i^*(\omega)$$

Отметим, что $i^*(\omega)$ - это форма степени k с компактным носителем на k-мерном многообразии V, поэтому интеграл в левой части определен. Форму $i^*(V)$ иногда обозначают чсрсз $\omega|_V$ и называют ограничеиисм формы ω на подмногообразие V.

Теперь все готово для формулировки основной теоремы интегрального исчисления.

Теорема 5.1 (Формула Стокса). Пусть M - гладкое п-мерное ориентированное многообразие с краем $\partial M, и \omega \in \Omega^{n-1}(M)$ - внешняя дифференциальная форма степени $(n-1)$ с компактным носителем. Тогда

$$\int_M d\omega = \int_{\partial M} \omega,$$

где ориентачия края ЭМ согласована с ориентачией многообразия M.

Доказательство. В силу аддитивности интеграла, теорему достаточно доказать для формы, носитель которой лежит внутри одной карты (U, φ) многообразия M. Более того, из тех же соображений можно предполагать, что если $(\ x^1, \ldots, x^n\ )$ - локальные координаты карты $(\ U, \varphi\ )$, то форма ω представляет собой моном вида

$$\omega = f\left(x^1, \ldots, x^n\right) dx^1 \wedge \cdots \wedge dx^{k-1} \wedge dx^{k+1} \wedge \cdots \wedge dx^n$$

Дифференциал формы ω в этом случае имеет вид

$$d\omega = (-1)^{k-1} \frac{\partial f}{\partial x^k} dx^1 \wedge \cdots \wedge dx^n$$

Пусть сначала (U, φ) - «внутренняя» карта на M, т.е. $\varphi : U \to \mathbb{R}^n$. Покажем, что в этом случае интеграл от $d\omega$ по U равен нулю. Действительно, перейдя от кратного интеграла к повторному, получаем:

$$\int_U d\omega = \int \cdots \int (-1)^{k-1} \frac{\partial f}{\partial x^k} dx^1 \cdots dx^n =$$
$$= (-1)^{k-1} \int_{\mathbb{R}^{n-1}} \cdots \int dx^1 \cdots dx^{k-1} dx^{k+1} \cdots dx^n \int_{-\infty}^{+\infty} \frac{\partial f}{\partial x^k} dx^k$$

Но внутрснний интсграл равсн нулю в силу формулы Ньютона-Лсйбница, так как форма ω имеет компактный носитель:

$$\int_{-\infty}^{+\infty} \frac{\partial f}{\partial x^k} dx^k = f\left(x^1, \ldots, x^n\right)\Big|_{-\infty}^{+\infty} = 0$$

что и требовалось.

Пусть теперь (U, φ) - «краевая» карта на M, т.е. $\varphi : U \to \mathbb{R}^n_+ (x^1 \geq 0)$. Здесь следует рассмотреть два случая: $k = 1$ и $k > 1$. Пусть сначала $k > 1$. Так как на крае ∂M координата x^1 постоянна и равна нулю, ограничение $\omega|_{\partial M}$ формы ω на край ∂M в этом случае равно нулю. С другой стороны, интеграл от формы $d\omega$ по карте U в этом случае равен нулю по тем же соображениям, что и выше. Случай $k > 1$ разобран.

Пусть теперь $k = 1$. Форма ω в этом случае имеет вид

$$f\left(x^1, \ldots, x^n\right) dx^2 \wedge \cdots \wedge dx^n$$

и ее дифференциал dw вычисляется так:

$$dw = \frac{\partial f}{\partial x^1} dx^1 \wedge \cdots \wedge dx^n$$

Вычислим интеграл от $d\omega$ по карте U. Имеем:

$$\int_U d\omega = \int_{\mathbb{R}^n_+ (x^1 \geq 0)} \cdots \int_0 \frac{\partial f}{\partial x^1} dx^1 \cdots dx^n = \int_{\mathbb{R}^{n-1}(x^1=0)} \cdots \int_{\mathbb{R}^{n-1}(x^1=0)} dx^2 \cdots dx^n \int_0^{+\infty} \frac{\partial f}{\partial x^1} dx^1 =$$
$$= -\int_{\ldots} \cdots \int^n \left(0, x^2, \ldots, x^n\right) dx^2 \cdots dx^n$$

С другой стороны, интеграл от формы ω по множеству $U \cap \partial M$ записывается так:

$$\int_{U \cap \partial M} \omega = \int_{\mathbb{R}^{n-1}(x^1=0)} \cdots \int f\left(0, x^2, \ldots, x^n\right) dx^2 \cdots dx^n$$

В математическом анализе определенные только что интегралы от функций называют интегралами первого рода. Эти интегралы зависят от метрики, так как при их вычислении используется форма объема подмногообразия. В отличие от таких интегралов, интегралы от обычных дифференциальных форм называют интегралами втюрого рода.

В качестве следствий из теоремы Стокса, получим классические формулы Грина, Стокса и Гаусса-Остроградского.

15.5.2 5.3. Green's formula

Пусть на евклидовой плоскости $\mathbb{R}^2$ задана регулярная замкнутая кривая $\gamma : [a, b] \to \mathbb{R}^2$, ограничивающая замкнутую область U, и пусть $\gamma(t) = (x(t), y(t))$ - координатное представление для γ (здесь (x, y) - стандартные декартовы координаты на плоскости). Более того, предположим, что при движении по кривой γ область U остается слева, т.е. ориентация кривой γ такова, что базис ($N, \dot{\gamma}$) имеет положительную ориентацию (по отношению к стандартной ориентации плоскости). Здесь N - поле внешних нормалей к области U вдоль ее границы, т.е. вдоль кривой γ, а $\dot{\gamma}$ - вектор скорости кривой γ.

Рис. 2. Формула Грина

Пусть $P(x, y)$ и $Q(x, y)$ - гладкие функции, определенные в некоторой области $\hat{U}$, содержащей область U. Обозначим через $X = (P, Q)$ соответствующее векторное поле. Тогда определен интеграл

$$\int_\gamma P(x,y)dx + Q(x,y)dy =$$
$$= \int_a^b [P(x(t), y(t))\dot{x}(t) + Q(x(t), y(t))\dot{y}(t)]dt = \int_a^b \langle X, \dot{\gamma}\rangle dt$$

называемый циркуляцией векторного поля X вдоль кривой γ. Классическая формула Грина утверждает, что

$$\int_a^b \langle X, \dot{\gamma}\rangle dt = \int_U \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy$$

Покажем, что эта формула является следствием теоремы Стокса. Действительно, будем рассматривать область U как многообразие с краем, ориентированное с помощью стандартной ориентации плоскости: в каждом касательном пространстве ориентация задана стандартным репером (e_1, e_2). Тогда γ - это край ∂U многообразия U. Каноническая ориентация края γ - это выбор того из двух направлений касательных к γ векторов v, при котором (N, v) - положительно ориентированный репер на плоскости, где N - внешняя нормаль к U. Эта ориентация соответствует ориентации кривой γ векторами скоростей $\dot{\gamma}$, так как при обходе кривой γ (т.е. при изменении параметра t от a до b) область U остается слева.

Пусть $\omega = P(x, y)dx + Q(x, y)dy$ - дифференциальная 1-форма. Тогда $d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$. По теореме Стокса, имеем

$$\int_{\gamma=\partial U} \omega = \int_U d\omega$$

а это и есть формула Грина.
Отметим, что все наши рассуждения были проведены в декартовых координатах (x, y) на евклидовой плоскости. Именно в этих координатах, в которых евклидова метрика имеет вид (δ_{ij}), векторному полю $X = P\partial_x + Q\partial_y$ соответствует дифференциальная форма $\omega = Pdx + Qdy$. Напомним, что соответствие между векторами и ковекторами зависит, вообще говоря, от выбора римановой метрики.

Пусть теперь на плоскости заданы произвольные регулярные координаты (x^1, x^2), и $X = X^1\partial_{x^1} + X^2\partial_{x^2}$ - векторное поле, заданное в этих координатах. Как записать формулу Грина не переходя в декартовы координаты?

Как мы уже знаем, чтобы построить по векторному полю дифференциальную 1 - форму, нужно воспользоваться операцией опускания индекса. Пусть (g_{ij}) - компоненты метрики в координатах (x^1, x^2). Тогда соответствующая векторному полю X форма ω имеет вид $\omega = \omega_k dx^k$, где $\omega_k = g_{k\alpha}X^\alpha$. Если $x^i = x^i(t)$ - координатное представление кривой γ, то ограничение формы ω на γ имеет вид:

$$\omega|_\gamma = \omega_k dx^k\big|_\gamma = g_{k\alpha}X^\alpha \dot{x}^k dt = \langle X, \dot{\gamma}\rangle dt,$$

где в последнем выражении стоит скалярное произведение поля X на вектор скорости кривой γ в метрике (g_{ij}). Таким образом, выражение для циркуляции не зависит от выбора координат. Теорема Стокса утверждает, что

$$\int_\gamma \omega = \int_U d\omega$$

Однако, дифференциал от формы ω теперь имеет следующий вид:

$$d\omega = \frac{\partial g_{2\alpha}X^\alpha}{\partial x^1} - \frac{\partial g_{1\alpha}X^\alpha}{\partial x^2},$$

т.е. окончательно имеем:

$$\int_a^b \langle X, \dot{\gamma}\rangle dt = \int_U \left(\frac{\partial g_{2\alpha}X^\alpha}{\partial x^1} - \frac{\partial g_{1\alpha}X^\alpha}{\partial x^2}\right) dx^1 dx^2$$

15.5.3 5.4. Gauss-Ostrogradsky formula

Напомним необходимые определения из курса математического анализа.
Пусть x^i - стандартные декартовы координаты в $\mathbb{R}^3$, X - векторное поле на области $\Omega' \subset \mathbb{R}^3$, содержащей замкнутую область Ω, причем граница $\partial\Omega$ области Ω представляет собой регулярную поверхность, которую мы обозначим через M. Пусть X^i - компоненты поля X, тогда функция

$$\operatorname{div} X = \sum_{i=1}^{3} \frac{\partial X^i}{\partial x^i}$$

называется дивергениией поля X. Отметим, что дивергенция - это функция на Ω', поэтому ее значения не зависят от координат (в то время как определение дано в евклидовых координатах).

Далее, пусть N - поле внешних единичных нормалей к границе M. В каждой точке $P \in M$ ориентируем касательную плоскость T_PM базисом e_1, e_2 так, чтобы тройка (N, e_1, e_2) была положительно ориентирована (мы предполагаем, что в $\mathbb{R}^3$ фиксирован стандартный базис, задающий положительную ориентацию пространства $\mathbb{R}^3$). В каждой точке $P \in M$ вычислим скалярное произведение векторов $X(P)$ и $N(P)$. Получаем гладкую функцию $\langle X, N\rangle$ на M. На поверхности M определена индуцированная метрика, превращающая

M в риманово многообразие. Поэтому опредслен интсграл от любой гладкой функции на M (напомним, что интеграл от функции по риманову многообразию - это интеграл от дифференциальной формы, полученной умножением формы объема на эту функцию).

Определение. Интеграл от функции $\langle X, N\rangle$ по поверхности M называется потоком векторного поля X через поверхность M.

Теорема 5.2 (Формула Гаусса-Остроградского). Поток векторного поля X через границу $M = \partial\Omega$ области Ω равен интегралу по области Ω от дивергенции поля X, т.е.

$$\int_M \langle X, N\rangle * 1 = \int_\Omega \operatorname{div} X * 1$$

Замечание. В формуле Гаусса-Остроградского слева и справа стоят интегралы от функций по римановым многообразиям. Формы $*1$ - это формы объема на этих (разных) многообразиях.

Покажем, что теорема Остроградского-Гаусса является частным случаем общей формулы Стокса. Для этого достаточно построить такую дифференциальную 2 -форму ω на Ω', что ее ограничение на M совпадает с формой $\langle X, N\rangle * 1$, а ее дифференциал $d\omega$ имеет вид $\operatorname{div} X * 1$. Рассмотрим 1 -форму T на $\mathbb{R}^3$, полученную из поля X опусканием индексов. Тогда в качестве формы ω можно взять $*T$. Проверим, что ω искомая 2-форма.

Проделаем это в координатах. В евклидовых координатах (x^1, x^2, x^3) имеем: $\operatorname{div} X * 1 = \operatorname{div} X dx^1 \wedge dx^2 \wedge dx^3$. С другой стороны, если X^i- координаты поля X, то $T_i = X^i$ (так как метрика у нас евклидова, и координаты - тоже евклидовы), и

$$\omega = \sum_{i<j} \omega_{ij} dx^i \wedge dx^j = X^1 dx^2 \wedge dx^3 + X^2 dx^3 \wedge dx^1 + X^3 dx^1 \wedge dx^2 \qquad (*)$$

поэтому

$$d\omega = \left(\sum_i \frac{\partial X^i}{\partial x^i}\right) dx^1 \wedge dx^2 \wedge dx^3 = \operatorname{div} X dx^1 \wedge dx^2 \wedge dx^3 = \operatorname{div} X * 1$$

Нам осталось доказать следующее утверждение.

Утверждение 5.1. Пусть T - это 1-форма на Ω, получающаяся из поля X опусканием индексов, $a\omega = *T$. Тогда ограничение $\omega|_M$ формь ω на поверхность M задается так:

$$\omega|_M = \langle X, N\rangle * 1.$$

Доказательство. Пусть (u^1, u^2) - локальные координаты на M, в которых M задается в виде параметрической регулярной поверхности $r(u^1, u^2)$, причем

$$N = \frac{r_{u^1} \times r_{u^2}}{\|r_{u^1} \times r_{u^2}\|},$$

(g_{ij}) - матрица индуцированной метрики, и $g = \det(g_{ij})$.

Воспользусмся только что получснной формулой (*) для формы ω и вычислим $\omega|_M$. Так как

$$dx^i = \frac{\partial x^i}{\partial u^k} du^k$$

ТО

$$dx^i \wedge dx^j = \frac{\partial x^i}{\partial u^k}\frac{\partial x^j}{\partial u^l} du^k \wedge du^l = \left(\frac{\partial x^i}{\partial u^1}\frac{\partial x^j}{\partial u^2} - \frac{\partial x^i}{\partial u^2}\frac{\partial x^j}{\partial u^1}\right) du^1 \wedge du^2$$

Положим

$$A^{ij} = \frac{\partial x^i}{\partial u^1}\frac{\partial x^j}{\partial u^2} - \frac{\partial x^i}{\partial u^2}\frac{\partial x^j}{\partial u^1}.$$

Таким образом,

$$\omega|_M = \left(X^1 A^{23} + X^2 A^{31} + X^3 A^{12}\right) du^1 \wedge du^2$$

Далее,

$$r_{u^1} \times r_{u^2} = \begin{vmatrix} \partial_{x^1} & \partial_{x^2} & \partial_{x^3} \\ \frac{\partial x^1}{\partial u^1} & \frac{\partial x^2}{\partial u^1} & \frac{\partial x^3}{\partial u^1} \\ \frac{\partial x^1}{\partial u^2} & \frac{\partial x^2}{\partial u^2} & \frac{\partial x^3}{\partial u^2} \end{vmatrix} = \left(A^{23}, A^{31}, A^{12}\right)$$

Напомним, что (g_{ij}) - это матрица Грамма системы векторов r_{u^1}, r_{u^2}.
Лемма 5.1. Пусть a $ub-$ дв вектора в $\mathbb{R}^n, G-$ их матрица Грамма, $ug = \det G$. Тогда площадь S параллелограмма П, натянутого на векторы a ub, равна $\sqrt{g}$.

Доказательство. Действительно, пусть φ - угол между a и b. Тогда

$$\det G = \|a\|^2 \mid b\,\|^2 - \langle a, b\rangle^2 = \|\,a\,\|^2 \|\,b\,\|^2 - \|\,a\,\|^2 \|\,b\|^2 \cos^2\varphi = \\ = \|a\|^2 \|b\|^2 \sin^2\varphi = S^2$$

что и требовалось.
Из леммы 5.1 вытекает, что

$$\|r_{u^1} \times r_{u^2}\| = \sqrt{g}$$

поэтому $r_{u^1} \times r_{u^2} = \sqrt{g}N$. В итоге получаем

$$\omega|_M = \langle X, r_{u^1} \times r_{u^2}\rangle\, du^1 \wedge du^2 = \langle X, N\rangle \sqrt{g} du^1 \wedge du^2 = \langle X, N\rangle * 1$$

что и требовалось.

15.5.4 5.5. Stokes formula for surfaces

Пусть $\Omega' \subset \mathbb{R}^2$ - некоторая область, и $\Omega \subset \Omega'$ - замкнутая компактная область с гладкой границей $\partial\Omega$, состоящей из конечного числа регулярных кривых. Пусть (u^1, u^2) - декартовы координаты в $\mathbb{R}^2 \supset \Omega' \supset \Omega$, и $r : \Omega' \to \mathbb{R}^3$ - регулярная поверхность. Ограничение отображения r на Ω называется регулярной поверхностью с краем $r : \partial\Omega \to \mathbb{R}^3$, а область Ω - параметрической областью этой поверхности. Если r является вложением, то можно отождествить такую поверхность с ее образом и рассматривать поверхность как подмногообразие с краем, лежащее в $\mathbb{R}^3$. В дальнейшем, для простоты, мы будем предполагать, что r - вложение.

Итак, пусть M - регулярная поверхность в $\mathbb{R}^3$ с краем ∂M, заданная параметрически в виде

$$r\left(u^1, u^2\right) = \left(x^1\left(u^1, u^2\right), x^2\left(u^1, u^2\right), x^3\left(u^1, u^2\right)\right),$$

где (x^1, x^2, x^3) - стандартные декартовы координаты в $\mathbb{R}^3$, а (u^1, u^2) координаты на поверхности M.

Обозначим через N каноническое поле единичных нормалей κ поверхности M, полученное так:

$$N = \frac{r_{u^1} \times r_{u^2}}{\|r_{u^1} \times r_{u^2}\|}$$

Пусть $g_{ij} = \langle r_{u^i}, r_{u^j} \rangle$ - компоненты индуцированной на M метрики, а g - определитель матрицы (g_{ij}).

Рассмотрим произвольное векторное поле X в $\mathbb{R}^3$. Напомним, что ротором $\operatorname{rot} X$ поля X называется векторное поле на $\mathbb{R}^3$, имеющее в декартовых координатах (x^1, x^2, x^3) следующий вид

$$\operatorname{rot} X = \| \begin{matrix} \partial_{x^1} & \partial_{x^2} & \partial_{x^3} \\ \partial/\partial x^1 & \partial/\partial x^2 & \partial/\partial x^3 \\ X^1 & X^2 & X^3 \end{matrix} \| =$$
$$= \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}, \frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}, \frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2} \right)$$

Так же, как и выше, поток векторного поля $\operatorname{rot} X$ через поверхность M равен интегралу от функции $\langle \operatorname{rot} X, N \rangle$ по M :

$$\int_M \langle \operatorname{rot} X, N \rangle * 1 = \int_M \langle \operatorname{rot} X, N \rangle \sqrt{g} du^1 \wedge du^2$$

Далее, обозначим через γ_i регулярные замкнутые кривые в $\mathbb{R}^3$ (одномерные подмногообразия), являющиеся связными компонентами края ∂M. В касательной плоскости $T_P M$ в каждой точке $P \in \gamma_i$ выберем единичный вектор n, перпендикулярный γ_i и направленный наружу поверхности M. Выберем на γ_i параметр $t \in [a_i, b_i]$ (и, тем самым, ориентируем γ_i) так, чтобы пара ($n, \dot{\gamma}_i$) была положительно ориентирована относительно ориентации, заданной координатами (u^1, u^2). Отметим, что именно такая ориентация края ∂M является канонической.

Поле X и выбранная параметризация $\gamma_i(t)$ каждой кривой γ_i порождают на параметризующем кривую γ_i отрезке $[a_i, b_i]$ функцию $\langle X, \dot{\gamma}_i \rangle$. Легко видеть, что интеграл

$$\int_{a_i}^{b_i} \langle X, \dot{\gamma}_i \rangle \, dt$$

не меняется при монотонно возрастающих заменах параметра t. Этот интеграл называется ирркуляцией векторного поля X вдоль кривой γ_i. Отметим, что если в качестве параметра t выбрать натуральный параметр s, то длины всех векторов $\dot{\gamma}_i$ будут равны 1 , и, фактически, вектор $e = \dot{\gamma}_i$ в точке $P = \gamma_i(s)$ будет тем из двух единичных векторов касательной прямой $T_P \partial M$, для которого репер (n, e) положительно ориентирован. Более того, рассматривая ∂M как риманово многообразие с индуцированной метрикой, мы получаем, что ds - это форма одномерного объема (т.е. длины), соответствующая выбранной (канонической) ориентации края ∂M. Таким образом, циркуляция векторного поля X вдоль γ_i это интеграл по канонически ориентированной компоненте γ_i края ∂M от функции $\langle X, e \rangle$. Сумму циркуляций поля X по всем компонентам γ_i края ∂M назовем иирркуляцией поля X вдоль края ∂M поверхности M и обозначим через

$$\int_{\partial M} \langle X, e \rangle * 1 = \sum_i \int_{\gamma_i} \langle X, \dot{\gamma}_i \rangle \, dt$$

Теорема 5.3 (Формула Стокса для поверхностей). Поток ротора $\operatorname{rot} X$ векторного поля X через компактную регулярную поверхность $M\mathbb{R}^3$ с гладким краем ∂M равен ирркуляции поля X вдоль края ∂M :

$$\int_M \langle \operatorname{rot} X, N\rangle * 1 = \int_{\partial M} \langle X, e\rangle * 1$$

В координатах:

$$\int_\Omega \langle \operatorname{rot} X, N\rangle du^1 \wedge du^2 = \int_{\partial M} \langle X, \partial \dot{M}\rangle dt = \sum_i \int_{a_i}^{b^i} \langle X, \dot{\gamma}_i\rangle \, dt$$

Покажем, что формула Стокса для поверхностей является частным случаем общей формулы Стокса. Для этого достаточно построить такую дифференциальную 1 -форму ω на M, что ее ограничение на ∂M совпадает с $\langle X, e\rangle ds$, а ее дифференциал $d\omega$ имеет вид

$$\langle \operatorname{rot} X, N\rangle \sqrt{g} du^1 \wedge du^2$$

Пусть T - это 1 -форма в $\mathbb{R}^3$, полученная из X опусканием индекса. В качестве формы ω возьмем ограничение формы T на поверхность M. Проверим, что такая форма удовлетворяет требуемым условиям.

Іусть s - натуральный параметр на кривой γ_i, выбранный так, как описано выше, и $\dot{\gamma}_i = (\dot{\gamma}_i^1, \dot{\gamma}_i^2, \dot{\gamma}_i^3)$. Тогда

$$\omega|_{\gamma_i} = T|_{\gamma_i} = \sum_k X^k \dot{\gamma}_i^k ds = \langle X, \dot{\gamma}_i\rangle \, ds = \langle X, e\rangle ds$$

Определим 2-форму μ, положив $\mu = dT$; 1-форму η, положив $\eta = *\mu$; и пусть Y - векторное поле, полученное из 1 -формы η поднятием индекса.

Утверждение 5.2. Поле Y является ротором векторного поля X.
Доказательство. Действительно, форма $\mu = dT$ имеет вид
$\left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}\right) dx^2 \wedge dx^3 + \left(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}\right) dx^3 \wedge dx^1 + \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\right) dx^1 \wedge dx^2$,
и
$\eta = *\mu = \left(\frac{\partial X^3}{\partial x^2} - \frac{\partial X^2}{\partial x^3}\right) dx^1 + \left(\frac{\partial X^1}{\partial x^3} - \frac{\partial X^3}{\partial x^1}\right) dx^2 + \left(\frac{\partial X^2}{\partial x^1} - \frac{\partial X^1}{\partial x^2}\right) dx^3$,
откуда немедленно заключаем, что $Y = \operatorname{rot} X$, что и требовалось.
Заметим теперь, что $*\eta = *^2\mu = \mu = dT$, поэтому, по утверждению 5.1,

$$(dT)|_M = \mu|_M = *\eta|_M = \langle Y, N\rangle \sqrt{g} du^1 \wedge du^2.$$

Окончательно, $(dT)|_M = d\left(T|_M\right) = d\omega$ в силу перестановочности дифференциала и операции перенесения форм, поэтому

$$d\omega = \langle Y, N\rangle \sqrt{g} du^1 \wedge du^2$$

и, таким образом, форма ω удовлетворяет всем требуемым свойствам.

15.5.5 Problems

Задача 5.1. Проверить, что определение потока векторного поля через поверхность не зависит от выбора координат.

Задача 5.2. Показать, что в криволинейных координатах (x^1, x^2, x^3) дивергенция векторного поля X выглядит так:

$$\operatorname{div} X = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^k} \left(\sqrt{g} X^k\right)$$

Задача 5.3. Записать выражение для ротора векторного поля в криволинейных координатах.

Задача 5.4. Пусть замкнутая поверхность $M \subset \mathbb{R}^3$ ограничивает замкнутую область $W \subset \mathbb{R}^3$, и n - поле нормалей к поверхности. Пусть φ и ψ - гладкие функции на области $U \supset W$. Проверить, что

$$\int_W (\varphi \Delta \psi + \langle \operatorname{grad} \varphi, \operatorname{grad} \psi \rangle) * 1 = \int_M \varphi \frac{\partial \psi}{\partial n} * 1$$

где Δ - лапласиан, а $\partial/\partial n$ - производная по нормали.

15.5.6 Extra Topics

5.1. Теорема о вычетах и ряды Лорана. Пусть $f(x,y) = u(x,y) + iv(x,y)$ комплекснозначная функция комплексного переменного $z = x + iy$, определенная в некоторой области $\Omega \subset \mathbb{R}^2 \equiv \mathbb{C}$. Напомним, что $\bar{z} = x - iy$. Будем рассматривать z и $\bar{z}$ как независимые переменные. Выразим x и y через z и $\bar{z}$,

$$x = \frac{z + \bar{z}}{2}, \quad y = \frac{z - \bar{z}}{2i}$$

и подставим в выражения для u и v. Получим

$$f(z, \bar{z}) = u\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + iv\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right)$$

Определение. Функция $f(z, \bar{z})$ называется комплексно-аналитической или голоморфной, если она не зависит от $\bar{z}$. Иными словами,

$$\frac{\partial f(z, \bar{z})}{\partial \bar{z}} = 0$$

Продифференцируем функцию $f(z, \bar{z})$ по $\bar{z}$ по правилу из теоремы о дифференцировании сложной функции. Получим

$$\begin{aligned}
\frac{\partial f(z, \bar{z})}{\partial \bar{z}} &= \frac{\partial}{\partial \bar{z}} u\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) + i \frac{\partial}{\partial \bar{z}} v\left(\frac{z + \bar{z}}{2}, \frac{z - \bar{z}}{2i}\right) = \\
&= \frac{\partial u}{\partial x}\frac{1}{2} - \frac{\partial u}{\partial y}\frac{1}{2i} + i\left(\frac{\partial v}{\partial x}\frac{1}{2} - \frac{\partial v}{\partial y}\frac{1}{2i}\right) = \\
&= \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u + iv) = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) f = \\
&= \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)
\end{aligned}$$

Таким образом,

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$$

и условие голоморфности функции f равносильно выполнению следующей системы уравнений в частных производных, которая называется условиями Коши-Римана:

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \end{cases}$$

Пусть теперь

$$\begin{aligned}
f(z)dz &= (u(x,y) + iv(x,y))(dx + idy) = \\
&= u(x,y)dx - v(x,y)dy + i(u(x,y)dy + v(x,y)dx) = \omega_1 + i\omega_2
\end{aligned}$$

- комплекснозначная 1 -форма, где

$$\omega_1 = u(x,y)dx - v(x,y)dy, \quad \omega_2 = u(x,y)dy + v(x,y)dx$$

- вещественнозначные 1 -формы.

Утверждение 5.3. Функияя f - голоморфна, если и только если дифферениал 1− форм $\omega_1 u \omega_2$ равен нулю.

Доказательство. Действительно,

$$d\omega_1 = -\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dx \wedge dy$$
$$d\omega_2 = \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx \wedge dy$$

откуда непосредственно видно, что равенство нулю дифференциалов форм ω_i равносильно выполнению условий Коши-Римана. Доказательство закончено.

Определим интеграл от формы $f(z)dz$ вдоль гладкой кривой $\gamma : [a,b] \to \mathbb{R}^2$ с координатным представлением $\gamma(t) = (x(t), y(t))$, положив

$$\int_\gamma f(z)dz = \int_\gamma [u(x,y)dx - v(x,y)dy] + i\int_\gamma [u(x,y)dy + v(x,y)dx] =$$
$$= \int_\gamma [u(x(t),y(t))\dot{x}(t) - v(x(t),y(t))\dot{y}(t)]dt +$$
$$+ i\int_\gamma [u(x(t),y(t))\dot{y}(t) + v(x(t),y(t))\dot{x}(t)]dt$$

Пусть $\Omega \subset \mathbb{R}^2$ - произвольная компактная (вообще говоря, многосвязная, т.е., возможно, с дырками) область с гладкой границей, состоящей из конечного числа регулярных замкнутых кривых γ_i. Выберем на каждой кривой γ_i направление обхода, при котором ограниченная γ_i область плоскости (это, как правило, не Ω) обходится против часовой стрелки. Для краткости, будем говорить, что такая кривая γ_i*ори* ентирована против часовой стрелки. Кривую γ_i с противоположной ориентацией обозначим через $-\gamma_i$.

Ясно, что Ω - двумерное многообразие с краем. Введем на крае $\partial\Omega = \cup_i \gamma_i$ каноническую ориентацию. Пусть γ_1 - та из кривых γ_i, которая ограничивает область, содержащую все остальные γ_i. Легко видеть, что при канонической ориентации $\partial\Omega$ каждая кривая γ_i проходится в таком направлении, что область Ω все время остается слева. Последнее означает, что (в терминах введенных выше ориентаций кривых γ_i) канонически ориентированный край $\partial\Omega$ записывается так: $\partial\Omega = \gamma_1 \cup (-\gamma_2) \cup \ldots$.

Воспользуемся общей формулой Стокса (в действительности, естественным обобщением формулы Грина на многосвязные области). Имеем

$$\int_{\partial\Omega} f(z)dz = -\int_\Omega \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) dx \wedge dy + i\int_\Omega \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx \wedge dy$$

поэтому, если функция $f(z)$ голоморфна, то

$$\int_{\partial\Omega} f(z)dz = \int_{\gamma_1} f(z)dz + \sum_{i\geq 2} \int_{-\gamma_i} f(z)dz = 0$$

Т.е.

$$\int_{\gamma_1} f(z)dz = \sum_{i\geq 2} \int_{\gamma_i} f(z)dz$$

Таким образом, мы доказали следующее утверждение.
Утверждение 5.4. Пусть $f(z)$ - голоморфная функция в компактной области Ω с гладким краем $\partial\Omega$, состоящим из конечного числа замкнутых регулярных кривых γ_i. Пусть γ_1 - та из кривых γ_i, которая ограничивает область, содержащую

все остальные γ_i. Ориентируем каждую кривую $\gamma_i, i = 1, 2, \ldots$, против часовой стрелки. Тогда

$$\int_{\gamma_1} f(z)dz = \sum_{i\geq 2} \int_{\gamma_i} f(z)dz$$

В частности, если область Ω односвязна, т.е. $\partial\Omega$ состоит из одной кривой γ_1, то

$$\int_{\gamma_1} f(z)dz = 0$$

Рассмотрим функцию $f(z) = z^n, n \in \mathbb{Z}$. Если $n \geq 0$, то эта функция определена и голоморфна на всей комплексной плоскости $\mathbb{C}$, поэтому, в силу утверждения 5.4, для вложенной регулярной замкнутой кривой γ на комплексной плоскости $\mathbb{C}$ (такие кривые будем для краткости называть контурами) интеграл $\int_\gamma z^n dz$ равен нулю. Если $n < 0$, то для любого контура γ, не охватывающего 0 (ограниченная контуром γ область не содержит 0), также выполняется $\int_\gamma z^n dz = 0$. Если же контур γ охватывает 0 , то для $n < 0$ интеграл $\int_\gamma z^n dz$ не зависит от выбора контура, если только фиксировано направление обхода, скажем, против часовой стрелки. Действительно, для любых двух так ориентированных контуров γ_1 и γ_2, охватывающих 0 , выберем третий коптур γ, также охватьвающий 0 и пе пересекающийся с γ_i. Ориептируем γ так же, как и контуры γ_i. Тогда при каждом $i = 1, 2$ контуры γ_i и γ ограничивают двусвязную компактную область Ω_i, в которой функция $f(z)$ голоморфна. Из утверждения 5.4 вытекает, что

$$\int_{\gamma_i} z^n dz = \int_\gamma z^n dz$$

откуда и получаем требуемое.
Пусть S^1 - стандартная единичная окружность с центром в 0 , а γ - произвольный контур, охватывающий 0 . Ориентируем γ и S^1 против часовой стрелки. Тогда

$$\int_\gamma z^n dz = \int_{S^1} z^n dz$$

Явные вычисления дают

$$\begin{aligned}
&\int_{S^1} z^n dz = \int_0^{2\pi} e^{in\varphi} de^{i\varphi} = \int_0^{2\pi} e^{in\varphi} d(\cos\varphi + i\sin\varphi) = \\
&= \int_0^{2\pi} e^{in\varphi}(-\sin\varphi + i\cos\varphi)d\varphi = \int_0^{2\pi} e^{in\varphi} ie^{i\varphi} d\varphi = \int_0^{2\pi} ie^{i(1+n)\varphi} d\varphi = \\
&= i\int_0^{2\pi} \cos(1+n)\varphi d\varphi - \int_0^{2\pi} \sin(1+n)\varphi d\varphi
\end{aligned}$$

поэтому

$$\int_\gamma z^n dz = \begin{cases} 2\pi i, & \text{при } n = -1, \text{ и} \\ 0 & \text{в противном случае.} \end{cases}$$

Делая замену координат $z \mapsto z - a, a \in \mathbb{C}$, для контура γ, охватывающего точку a, получаем

$$\int_\gamma (z-a)^n dz = \begin{cases} 2\pi i, & \text{при } n = -1, \text{ и} \\ 0 & \text{в противном случае.} \end{cases}$$

Имеет место следующая теорема, которая доказывается в курсе теории функций комплексного переменного.

Теорема 5.4 (О разложении функции в ряд Лорана). Пусть $\Omega \subset \mathbb{C}$ - некоторая область, a - внутренняя точка области $\Omega, uf(z)$ - голоморфная функция в $\Omega \backslash \{a\}$. Тогда существуют такие $0 < r < R$, что в кольце $K = \{z | r \leq |z-a| \leq R\}$ функция $f(z)$ может быть представлена в виде равномерно сходящегося ряда:

$$f(z) = \sum_{n=-\infty}^{+\infty} c_n (z-a)^n$$

который называется рядом Лорана. В силу равномерной сходимости, ряд Лорана можно почленно интегрировать и дифференцировать.

Определение. Коэффициент c_{-1} ряда Лорана называется вычетом ϕ ункиии f в точке a и обозначается через $\operatorname{res}_a f$.

Из приведенных выше результатов вытекает следующий результат.

Следствие 5.1. В предположениях теоремы 5.4, для любого контура γ, лежащего в K, охватывающего a и ориентированного против часовой стрелки, имеем:

$$\int_\gamma f(z) dz = \sum_{n=-\infty}^{\infty} \int_\gamma c_n (z-a)^n = 2\pi i \operatorname{res}_a f(z)$$

Пусть $\Omega \subset \mathbb{C}$ - некоторая компактная односвязная область с гладкой границей, состоящей из одного контура $\partial\Omega = \gamma$. Пусть $a_1, \ldots, a_m$ - некоторые внутренние (не лежащие на γ) точки из Ω, и $f(z)$ - голоморфная функция на $\Omega \backslash \{a_1, \ldots, a_m\}$. Как всегда, ориентируем γ против часовой стрелки. Следующая теорема сводит вычисление интеграла от формы $f(z)dz$ по контуру γ к вычислению вычетов функции f в точках a_k.

Теорема 5.5 (О вычетах). В сделанных выше предположениях,

$$\frac{1}{2\pi i} \int_\gamma f(z) dz = \sum_{k=1}^{m} \operatorname{res}_{a_k} f$$

Доказательство. Окружим особые точки a_k малыми окружностями S_k, лежащими внутри Ω, и пусть D_k - открытый диск, ограниченный S_k. Обозначим через Ω' область, полученную из Ω выкидыванием всех дисков D_k. Тогда Ω' - компактная область с гладкой границей, и $f(z)$ - голоморфная в Ω' функция. Ориентировав все S_k против часовой стрелки и воспользовавшись утверждением 5.4 и следствием 5.1, получаем:

$$\frac{1}{2\pi i} \int_\gamma f(z) dz = \frac{1}{2\pi i} \sum_{k=1}^{m} \int_{S_k} f(z) dz = \sum_{k=1}^{m} \operatorname{res}_{a_k} f$$

Теорема доказана.

15.6 6: Cogomology de Rama

Одна из самых сложных и часто встречающихся топологических задач дифференциальной геометрии состоит в проверке гомеоморфности двух данных

многообразий. Теория когомологий де Рама, которая обсуждается в этом разделе, позволяет построить по каждому многообразию M некоторый алгебраический объект - группы когомологий [4], который оказывается топологическим (на самом деле, даже гомотопическим, см. ниже) инвариантом. А именно, группы когомологий гомеоморфных многообразий изоморфны. Поэтому эти группы могут использоваться для доказательства негомеоморфности многообразий: если группы различны, то многообразия не гомеоморфны. С другой стороны, (что не менее важно) группы когомологий отвечают за разрешимость на многообразиях дифференциальных уравнений специального вида. Тем самым, когомологии позволяют понять природу топологических препятствий такой разрешимости.

15.6.1 6.1. Definition of de Rham cohomology groups

Пусть M - гладкое многообразие, и ω - внешняя дифференциальная форма на M.

Форма ω называется замкнутой, если ее внешний дифференциал $d\omega$ равен нулю. Форма ω называется точной, если она является «полным дифференциалом», т.е. если существует форма $\tilde{\omega}$ на M, такая что $\omega = d\tilde{\omega}$. Ясно, что множество $Z^k(M)$ всех замкнутых форм степени k на M и множество $B^k(M)$ всех точных форм степени k на M образуют (бесконечномерные) линейные подпространства в (бесконечномерном) линейном пространстве $\Omega^k(M)$ всех внешних форм степени k на M. Напомним, см. утверждение 4.1, что $d(d\omega) = 0$ для любой формы ω, поэтому каждая точная форма является замкнутой: $B^k(M) \subset Z^k(M)$. Отсюда вытекает, что корректно определено фактор-пространство $Z^k(M)/B^k(M)$ пространства всех замкнутых форм степени k по пространству всех точных форм степени k. Неожиданно оказывается, что это пространство конечномерно.

Определение. Пространство $Z^k(M)/B^k(M)$ называется k-ой группой когомологий де Рама многообразия M и обозначается через $H^k(M)$.

Замечание. Краткое напоминание понятия фактор-группы и фактор-пространства содержится в дополнительных материалах к этой лекции.

Нам будет удобно записать это определение в других обозначениях. Линейный оператор внешнего дифференцирования d действует, как мы

знаем, из пространства $\Omega^k(M)$ внешних форм степени k в пространство $\Omega^{k+1}(M)$ внешних форм степени $k+1$. Для удобства, будем обозначать оператор d через d_k, указывая тем самым явно то пространство, на котором этот оператор в данный момент рассматривается. Тогда мы имеем следующую цепочку «сквозного действия» линейного оператора внешнего дифференцирования:

$$\cdots \xrightarrow{d_{k-2}} \Omega^{k-1}(M) \xrightarrow{d_{k-1}} \Omega^k(M) \xrightarrow{d_k} \Omega^{k+1}(M) \xrightarrow{d_{k+1}} \cdots .$$

В этих обозначениях, k-ая группа когомологий может быть определена следующим образом: $H^k(M) = \operatorname{Ker}(d_k) / \operatorname{Im}(d_{k-1})$, поскольку, по определению, $\operatorname{Ker}(d_k) = Z^k(M)$, а $\operatorname{Im}(d_{k-1}) = B^k(M)$.

Замечание. Так как $\Omega^k(M) = 0$ при $k > n$, где n - размерность многообразия, можно считать, что описанная выше цепочка на самом деле конечна, а именно, она заканчивается так:

$$\cdots \xrightarrow{d_{n-2}} \Omega^{n-1}(M) \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_n} 0.$$

[4] Отметим, что сопоставление топологическому объекту алгебраического - это центральная идея алгебраической топологии.

Поэтому $\operatorname{Ker}(d_n) = \Omega^n(M)$ и $H^n(M) = \Omega^n(M)/\operatorname{Im}(d_{n-1})$.
Замечание. Мы определили группы когомологий $H^k(M)$ для $k \geq 1$. Удобно определить так же 0 -ую группу $H^0(M)$, продлив цепочку «сквозного действия» операторов внешнего дифференцирования влево так:

$$0 \xrightarrow{d_{-1}} \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots .$$

Тогда группа $H^0(M)$ по определению полагается равной $\operatorname{Ker}(d_0)/\operatorname{Im}(d_{-1})$, т.е. $H^0(M) = \operatorname{Ker}(d_0)$.

Замечание. На этой последней конструкции основан алгебраический подход к определению когомологий. А именно, если Ω^k - произвольные абелевы группы, и $d_k : \Omega^k \to \Omega^{k+1}$ - гомоморфизм, такой что $d_k \circ d_{k-1} = 0$, то можно определить так называемые когомологии цепного комплекса $\left\{\Omega^k, d_k\right\}_{k=0}^{\infty}$, положив $H^k = \operatorname{Ker}(d_k)/\operatorname{Im}(d_{k-1})$. Однако, эта общая конструкция лежит за пределами нашего курса.

Элемент $\omega + B^k(M)$ группы $H^k(M)$ будем обозначать через $[\omega]$ и называть k-мерным классом когомологий. Две замкнутых внешних формы ω_1 и ω_2, принадлежащие одному и тому же классу $[\omega]$, называются когомологичными. Ясно, что замкнутые формы ω_1 и ω_2 когомологичны если и только если их разность $\omega_1 - \omega_2$ точна.

Пример. Пусть M - связное многообразие. Вычислим группу $H^0(M)$. Для этого, по определению, нужно найти ядро $\operatorname{Ker}(d_0)$ оператора внешнего дифференцирования d_0 на пространстве $\Omega^0(M)$ всех внешних форм степени 0 на многообразии M, т.е. на пространстве гладких функций на M. Очевидно, $\operatorname{Ker}(d_0)$ - это пространство всех постоянных функций на M, поэтому $H^0(M) = \mathbb{R}^1$.

Вычислим теперь группу $H^1(S^1)$. Так как $\dim(S^1) = 1$, снова заключаем, что $\operatorname{Ker}(d_1) = \Omega^1(S^1)$. Осталось выяснить, как устроен образ $\operatorname{Im}(d_0)$ оператора d_0. Пусть $\omega \in \Omega^1(S^1)$ - внешняя 1 -форма на S^1, принадлежащая образу $\operatorname{Im}(d_0)$. Это означает, что существует гладкая функция $g \in \Omega^0(S^1)$, такая что $dg = \omega$. Но тогда, так как окружность - замкнутое многообразие, по теореме Стокса имеем:

$$\int_{S^1} \omega = \int_{S^1} dg = \int_{\partial S^1} g = 0$$

То есть имеет место следующее включение

$$\operatorname{Im}(d_0) \subset \left\{\omega \in \Omega^1\left(S^1\right) \mid \int_{S^1} \omega = 0\right\}$$

Оказывается, верно и обратное. Действительно, пусть $\omega \in \Omega^1(S^1)$ - такая внешняя 1 -форма, что $\int_{S^1} \omega = 0$. Пусть окружность S^1 параметризована стандартным угловым параметром $\varphi \in [0, 2\pi)$. Тогда каждая 1 -форма на окружности может быть записана в виде $f(\varphi)d\varphi$, где f - гладкая функция на окружности. Последнее означает, что f порождает гладкую периодическую с периодом 2π функцию на прямой, которую мы будем обозначать той же буквой f. В частности, $f(\varphi + 2\pi) = f(\varphi)$ для любого $\varphi \in \mathbb{R}^1$.

Занишем нашу форму ω в виде $\omega = \int(\varphi)d\varphi$. Рассмотрим функцию $g(\varphi)$ на $\mathbb{R}^1$, определенную так:

$$g(\varphi) = \int_0^{\varphi} f(t)dt, \quad \varphi \in \mathbb{R}^1$$

Функция g - гладкая периодическая функция на прямой с периодом 2π, и, поэтому, задает функцию на окружности. Действительно,

$$g(\varphi+2\pi)=\int_0^{\varphi+2\pi} f(t)dt=\int_0^{\varphi} f(t)dt+\int_{\varphi}^{\varphi+2\pi} f(t)dt=g(\varphi)$$

так как последний интеграл совпадает с интегралом от формы ω по окружности S^1, который, напомним, равен нулю в силу выбора формы ω. Остается заметить, что $dg=\omega$. Итак, доказана следующая лемма.

Лемма 6.1. Форма $\omega\in\Omega^1\left(S^1\right)$ является точной если и только если интеграл от нее по S^1 равен нулю.

Рассмотрим теперь отображение $i:\Omega^1\left(S^1\right)\to\mathbb{R}^1$, переводящее форму ω из $\Omega^1\left(S^1\right)$ в число $\int_{S^1}\omega$. Отображение i является гомоморфизмом линейных пространств, причем, в силу леммы 6.1, ядро $\operatorname{Ker} i$ этого гомоморфизма совпадает с $\operatorname{Im}\left(d_0\right)$. Кроме того, легко видеть, что образ гомоморфизма i совпадает со всем пространством $\mathbb{R}^1$ (проверьте это!). Осталось воспользоваться теоремой о гомоморфизме:

$$\mathbb{R}^1=\operatorname{Im} i=\Omega^1\left(S^1\right)/\operatorname{Ker} i=\operatorname{Ker}\left(d_1\right)/\operatorname{Im}\left(d_0\right)=H^1\left(S^1\right)$$

Итак, нами доказано следующее утверждение.

Утверждение 6.2. Группы когомологий окружности S^1 выглядят так:

$$H^k\left(S^1\right)=\begin{cases}\mathbb{R}^1 & \text{при } k=0 \text{ и } k=1\\ 0 & \text{при } k\geq 2\end{cases}$$

15.6.2 6.2. Cohomology and mapping

Пусть M и N - гладкие многообразия, и $f:M\to N$ - гладкое отображение. Как мы уже знаем, отображение f порождает линейное отображение f^* на пространствах внешних форм: $f^*:\Omega^k(N)\to\Omega^k(M)$. Более того, так как отображение f^* перестановочно с операцией внешнего дифференцирования, см. утверждение 4.1, оно, очевидно, переводит замкнутые формы в замкнутые, а точные - в точные. Поэтому отображение f^* корректно определено на группах когомологий. А именно, если $[\omega]\in H^k(N)$ - произвольный класс когомологий многообразия N, и ω - некоторый его представитель, т.е. (замкнутая) форма из $[\omega]$, то определим $f^*([\omega])$ равным $[f^*(\omega)]$. Ясно, что $d\left(f^*(\omega)\right)=f^*(d\omega)=f^*(0)=0$, поэтому $f^*(\omega)$ - замкнутая форма, и $[f^*(\omega)]$ определен. Кроме того, если $\omega_i, i=1,2,$ - два разных представителя класса $[\omega]$, то, по определению, $\omega_1-\omega_2=d\varphi$ для некоторой формы φ на N. Поэтому

$$f^*\left(\omega_1\right)-f^*\left(\omega_2\right)=f^*\left(\omega_1-\omega_2\right)=f^*(d\varphi)=d\left(f^*(\varphi)\right)$$

т.е. формы $f^*\left(\omega_1\right)$ и $f^*\left(\omega_2\right)$ когомологичны или, другими словами, $\left[f^*\left(\omega_1\right)\right]=\left[f^*\left(\omega_2\right)\right]$. Итак, доказано следующее утверждение.

Утверждение 6.3. Пусть $f:M\to N$ - гладкое отображение гладких многообразий. Отображение $f^*:\Omega^k(N)\to\Omega^k(M)$ индучирует гомоморфизм групп когомологий $f^*:H^k(N)\to H^k(M)$. Этот гомоморфизм определяется так: $f^*([\omega])=[f^*(\omega)]$, где ω - произвольная замкнутая форма степени k на N.

Упражнение 6.1. Показать, что если $1_M:M\to M$ - тождественное отображение, то 1_M^* - тождественное отображение когомологий.

Упражнение 6.2. Показать, что если $f:M_1\to M_2$, и $g:M_2\to M_3$ - гладкие отображения гладких многообразий, то $(g\circ f)^*=f^*\circ g^*$.

Из двух предыдущих упражнений вытекает следующий результат.

Утверждение 6.4. Если многообразия M и N диффеоморфнья, то *их* группы когомологий $H^k(M)$ и $H^k(N)$ совпадают.

Замечание. Ниже мы увидим, что группы когомологий сохраняются не только при диффеоморфизмах, но и при существенно более общих преобразованиях - гомотопических эквивалентностях.

15.6.3 6.3. Homotopies and cohomology

Пусть X и Y - топологические пространства, и $f_i : X \to Y, i = 0, 1$, непрерывные отображения пространства X в пространство Y. Отображения f_0 и f_1 называются гомотопными, если существует такое непрерывное отображение $F : X \times [0, 1] \to Y$, что $F(x, 0) = f_0(x)$, и $F(x, 1) = f_1(x)$ для каждого $x \in X$. Отображение F в этом случае называется голотопией отображения f_0 в отображение f_1. Если X и Y - гладкие многообразия, f_i - гладкие отображения, то мы будем говорить, что отображения f_0 и f_1 гладко гомотопны или, для краткости, гомотопны, если существует гомотопия F отображения f_0 в f_1, являющаяся гладким отображением из $X \times [0, 1]$ в Y.

Говорят, что топологические пространства X и Y гомотопически эквивалентны, если существуют непрерывные отображения $f : X \to Y$ и $g : Y \to X$, такие что отображение $f \circ g$ гомотопно 1_Y, а отображение $g \circ f$ гомотопно 1_X, где, напомним, через 1_X обозначается тождественное отображение пространства X на себя. Два многообразия называются гомотопически эквивалентньлми, если существуют гладкие отображения $f : X \to Y$ и $g : Y \to X$, такие что $f \circ g$ гладко гомотопно 1_Y, а $g \circ f$ гладко гомотопно 1_X.

Упражнение 6.3. Проверьте, что гомотопическая эквивалентность - отношение эквивалентности на множестве гладких многообразий.

Пример. Пусть X - одноточечное пространство, а Y - пространство $\mathbb{R}^n$. Возьмем в качестве f отображение, переводящее X в начало координат пространства $\mathbb{R}^n$, а в качестве g - отображение всего $\mathbb{R}^n$ в точку X. Тогда, очевидно, $g \circ f = 1_X$, а $f \circ g$ - отображение всего $\mathbb{R}^n$ в начало координат. Покажем, что $f \circ g$ гомотопно $1_{\mathbb{R}^n}$. Для этого достаточно рассмотреть отображение $F : \mathbb{R}^n \times [0, 1] \to \mathbb{R}^n$, устроенное так: $F : (y, t) \mapsto ty$. Очевидно, $F(y, 0) = 0 \in \mathbb{R}^n$ для любого y, и $F(y, 1) = y$ для любого y, т.е. $F(y, 0) = f \circ g$, а $F(y, 1) = 1_{\mathbb{R}^n}$. Более того, гомотопия F гладкая. Итак, нульмерное многообразие $X = \{*\}$ и n-мерное многообразие $Y = \mathbb{R}^n$ гомотопически эквивалентны. Поэтому R^n и R^k гомотопически эквивалентны для любых n и k.

Теорема 6.1. Гомотопные отображения $\int_0 u \int_1, \int_i : X \to Y, i = 0, 1$, гладких многообразий X и Y индучируют один и тот же гомоморфизм групп когомологий, т.е. $f_0^* = f_1^* : II^k(Y) \to II^k(X)$.

Доказательство. Для доказательства теоремы достаточно построить такое линейное отображение $K : \Omega^k(Y) \to \Omega^{k-1}(X)$, что имеет место тождество

$$(f_0^* - f_1^*)(\omega) = \pm(dK - Kd)(\omega), \quad \omega \in \Omega^k(Y) \tag{*}$$

Действительно, если отображение K построено, и ω произвольная замкнутая форма, представляющая класс $[\omega] \in H^k(Y)$, то замкнутые формы
$f_0^*(\omega)$ и $f_1^*(\omega)$ отличаются на точную форму

$$(f_0^* - f_1^*)(\omega) = \pm(dK - Kd)(\omega) = \pm d(K(\omega)) \mp K(d\omega) = \pm d(K(\omega))$$

где последнее равенство имеет место в силу замкнутости формы ω. Поэтому классы когомологий $[f_0^*(\omega)]$ и $[f_1^*(\omega)]$ совпадают, что и требовалось.

Перейдем к построению отображения K (его часто называют цепной гомотопией). Гомотопность отображений f_0 и f_1 означает, что существует такое гладкое отображение $F : X \times [0, 1] \to Y$, что $F(x, 0) = f_0(x)$ и $F(x, 1) = f_1(x)$. Отображение F порождает гомоморфизм пространств всех внешних форм $F^* : \Omega^k(Y) \to \Omega^k(X \times [0, 1])$.

Будем говорить, что форма $\Omega \in \Omega^k(X \times [0,1])$ не зависит от dt, если она равна нулю на любом наборе векторов $(\xi_1, \ldots, \xi_{k-1}, \partial/\partial t)$, где ξ_i векторы, касательные к X. В локальной системе координат ($x^1, \ldots, x^n, t$) на прямом произведении $X \times [0,1]$ последнее означает, что в разложении

$$\Omega(x,t) = \sum_{1 \le i_1 \cdots i_k \le n} \Omega_{i_1 \cdots i_k}\left(x^1, \ldots, x^n, t\right) dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

отсутствую мономы, содержащие сомножитель dt. Такую форму Ω можно рассматривать как однопараметрическое семейство форм $\Omega_t(x) = \Omega(x,t)$ на многообразии X. Формально, если $\varphi_t : X \to X \times [0,1]$ вложение вида $\varphi_t : x \mapsto (x,t)$, то $\Omega_t = \varphi_t^*(\Omega(x,t))$, т.е. ограничение формы $\Omega(x,t)$ на многообразие X, вложенное в прямое произведение $X \times [0,1]$ как $X \times \{t\}$.

Следующая лемма легко проверяется в локальных координатах.

Лемма 6.2. Каждая внешняя форма $\Omega \in \Omega^k(X \times [0,1])$ однозначным образом представима в виде $\Omega = \Omega' + \Omega'' \wedge dt$, где формы $\Omega' u \Omega''$ не зависят от dt.

Отметим, что если форма $\Omega \in \Omega^k(X \times [0,1])$ не зависит от dt, т.е. $\Omega = \Omega_t$ - однопараметрическое семейство форм на X, то внешний дифференциал $d\Omega$ имеет вид

$$d\Omega = d_x \Omega_t \pm \frac{\partial}{\partial t}\left(\Omega_t\right) \wedge dt$$

где d_x - внешнее дифференцирование на X, а знак перед вторым слагаемым зависит от четности степени формы Ω_t. В частности, последняя формула дает разложение из леммы 6.2 для дифференциала формы $d\Omega$.

Определим теперь отображение $K : \Omega^k(Y) \to \Omega^{k-1}(X)$ так. Для каждой формы $\omega \in \Omega^k(Y)$ построим форму $\Omega = F^*(\omega)$, разложим ее по формуле из леммы 6.2 , получив $\Omega = \Omega' + \Omega'' \wedge dt = \Omega'_t + \Omega''_t \wedge dt$, и положим

$$K(\omega) = \int_0^1 \Omega''_t dt$$

Следствие 6.2 (Лемма Пуанкаре). Всякая замкнутая форма на $\mathbb{R}^n$, $n \ge 1$, точна. Другими словами, группы когомологий пространства $\mathbb{R}^n$ имеют вид:

$$H^k\left(\mathbb{R}^n\right) = \begin{cases} \mathbb{R}^1 & \text{при } k = 0 \\ 0 & \text{при } k \ge 1 \end{cases}$$

15.6.4 6.4. Cohomology and Stokes' general formula

Общая формула Стокса во многих случаях помогает вычислять когомологии многообразий.

Утверждение 6.5. Если M - гладкое n-мерное замкнутое ориентируемое многообразие, то $H^n(M) \neq 0$.

Доказательство. Действительно, зададим на M некоторую риманову метрику (например, вложив M в $\mathbb{R}^N$ и рассмотрев индуцированную метрику). Так как M - ориентируемо, на нем корректно определена форма объема $\omega = *1$. Интеграл от ω по M равен объему многообразия M, и поэтому не равен нулю. Следовательно, ω не является точной формой. Действительно, если $\omega = d\eta$, то по теореме Стокса имеем

$$\int_M \omega = \int_M d\eta = \int_{\partial M} \eta = 0$$

так как $\partial M = \emptyset$, противоречие. С другой стороны, форма ω, будучи формой максимальной степени, является замкнутой. Итак, на M существует замкнутая не точная n-форма. Доказательство закончено.

Замечание. Построение групп когомологий многообразия с помощью дифференциальных форм - это далеко не единственный подход. Более общие топологические конструкции (так называемые симплициальные и сингулярные гомологии и когомологии) можно найти, например, в замечательной книге А. Т. Фоменко, Д. Б. Фукс, "Курс гомотопической топологии", а общий алгебраический подход изложен, например, в классической книге H. Стинрод, С. Эйленберг, "Основания алгебраической топологии".

15.6.5 Problems

Задача 6.1. Пусть M - многообразие, состоящее из k компонент связности. Вычислить $H^0(M)$.

Задача 6.2. Показать, что если два топологических пространства гомеоморфны, то они гомотопически эквивалентны.

Задача 6.3. [т] Iему гомотопически эквивалентно $\mathbb{R}^3$ с выброшенной прямой (с двумя выброшенными точками, с выброшенной окружностью)?

Задача 6.4. Показать, что $\mathbb{R}^n \backslash \{0\}$ гомотопически эквивалентно S^{n-1}. Чему гомотопически эквивалентно $\mathbb{R}^n$ с выброшенным R^k, с выброшенной сферой $S^k, k < n$?

Задача 6.5. Вычислить когомологии двумерной сферы S^2 и тора T^2.

15.6.6 Extra Topics

6.1. Фактор-группы и фактор-пространства. Подгруппа H группы G называется нормальной, если для каждого элемента $g \in G$ имеет место равенство левых и правых смежных классов $gH = Hg$ подгруппы H. В этом случае, на множестве, скажем, левых смежных классов можно корректно определить групповую операцию так:

$$gH \cdot g'H = (g_1 g_2)\, H, \qquad \text{где } g_1 \in gH \text{ и } g_2 \in g'H.$$

Таким образом, левые смежные классы нормальной подгруппы образуют группу, которая называется $\oint$ *актор*-группой и обозначается через G/H.

Пример. Если группа - коммутативна, то любая ее подгруппа - нормальна, поэтому для любой такой подгруппы определена фактор-группа. В частности, любое подпространство произвольного линейного пространства является в нем нормальной подгруппой (относительно операции сложения векторов). Соответствующая факторгруппа называется фактор-пространством.

Пример. Если $\varphi : A \to B$ - гомоморфизм групп, то его ядро

$$\operatorname{Ker} \varphi = \{a \in A \mid \varphi(a) = e_B\}$$

являетсгя пормальпой подгруппой в Λ. Имест место так пазывасмая теорема о гомоморфизме: группа $\varphi(A)$ изоморфна фактор-группе $A/\operatorname{Ker}\varphi$. Для случая линейных пространств это означает, что размерность образа линейного оператора равен разности размерностей пространства-прообраза и ядра этого оператора.

Более подробную информацию можно найти в любом учебнике по алгебре, например, в замечательных книжках А. И. Кострикина «Введение в алгебру» и Э. Б. Винберга «Курс высшей алгебры».

6.2. Когомологии и векторные поля на плоскости. Когомологии плоскости $\mathbb{R}^2$ можно вычислить непосредственно. Как мы уже знаем, $H^0(\mathbb{R}^2) = \mathbb{R}$, и $H^k(\mathbb{R}^2) = 0$ при $k \geq 3$.

Вычислим сначала $H^1(\mathbb{R}^2)$. Для этого заметим, что если (x, y) - декартовы координаты на плоскости, то

$$\operatorname{Ker} d_1 = \left\{ \omega = Pdx + Qdy \,\middle|\, \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x} \right\}, \quad \operatorname{Im} d_0 = \left\{ dF \mid F \in \Omega^0\left(\mathbb{R}^2\right) \right\}$$

Покажем, что $\operatorname{Ker} d_1 = \operatorname{Im} d_0$. Для этого по форме $\omega \in \operatorname{Ker} d_1$ построим функцию

$$F(x, y) = \int_\gamma \omega$$

где γ - произвольная кусочно-регулярная кривая, соединяющая начало координат и точку (x, y). В силу формулы Грина, условие замкнутости формы ω равносильно независимости значения интеграла в правой части от выбора кривой γ (проверьте). Покажем, что $dF = \omega$. Для того, чтобы вычислить частные производные функции F, удобно в качестве γ рассмотреть двухзвенные ломаные, ребра которых параллельны координатным осям. Отметим, что на отрезке, параллельном оси OX, форма Qdy равна нулю, а на отрезке, параллельном оси OY, равна нулю форма Pdx. Имеем:

$$F(x, y) = \int_0^x P(t, 0)dt + \int_0^y Q(x, t)dt = \int_0^y Q(0, t)dt + \int_0^x P(t, y)dt$$

откуда

$$\frac{\partial F}{\partial x} = P(x, y), \quad \frac{\partial F}{\partial y} = Q(x, y),$$

что и требовалось. Таким образом, $H^1(\mathbb{R}^2) = 0$.

На физическом языке этот результат формулируется так. Снова, пусть (x, y) декартовы координаты на плоскости. Векторное поле X называется потенииальным, если существует такая функция F, что $X = \operatorname{grad} F = (\partial F/\partial x, \partial F/\partial y)$. Поле $X = (P, Q)$ называется безвихревьлм, если

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Ясно, что каждое потенциальное поле является безвихревым. Действительно, если $P = \partial F/\partial x$, а $Q = \partial F/\partial y$, то

$$\frac{\partial P}{\partial y} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial Q}{\partial x}$$

Равенство нулю первых когомологий плоскости означает в точности, что верно и обратное: каждое безвихревое поле на плоскости является потенциальным. Отметим, что если рассмотреть плоскость с выколотыми точками, то это уже не так.

Далее, докажем, что $H^2(\mathbb{R}^2) = 0$. Ясно, что $Z^2(\mathbb{R}^2) = \Omega^2(\mathbb{R}^2)$, поэтому достаточно показать, что для любой 2 -формы $\Omega = F(x, y)dx \wedge dy$ существует такая форма $\omega = P(x, y)dx + Q(x, y)dy$, что $\Omega = d\omega$. Иными словами, мы должны для каждой гладкой функции F найти такие функции P и Q, что

$$F = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

Легко видеть, что этому равенству удовлетворяют функции

$$Q(x, y) = \frac{1}{2} \int_0^x F(t, y)dt, \quad \text{и} \quad P(x, y) = -\frac{1}{2} \int_0^y F(x, t)dt$$

Итак, мы доказали следующее утверждение.
Утверждение 6.6. Когомологии стандартной плоскости $\mathbb{R}^2$ выглядят так:

$$H^0\left(\mathbb{R}^2\right)=\mathbb{R}, \quad H^1\left(\mathbb{R}^2\right)=H^2\left(\mathbb{R}^2\right)=0$$

6.3. Когомологии и векторные поля в пространстве. Рассмотрим пространство $\mathbb{R}^3$. Векторное поле X называется потенииальньлм, если оно является градиентом некоторой функции F, т.е. $X=\operatorname{grad} F$. Поле X называется называется безвихревьля, если $\operatorname{rot} X=0$. Поле X, являющееся ротором другого поля Y, т.е. $X=\operatorname{rot} Y$, называется соленоидальным. Поле X называется бездивергентным или полем без источников, если $\operatorname{div} X=0$.

Пусть (x, y, z) - декартовы координаты в $\mathbb{R}^3$. Отождествим теперь векторные поля $X=(P, Q, R)$ с 1 -формами $\omega=P dx+Q dy+R dz$. Эту операцию отождествления (операцию опускания индексов с помощью евклидовой метрики δ_{ij}) обозначим через I. Таким образом, $I(X)=\omega$. Пусть F - гладкая функция. Напомним некоторые полезные нам тождества:

$$I(\operatorname{grad} F)=d F, \quad I(\operatorname{rot} X)=*(d \omega), \quad \text{и} \quad \operatorname{div} X=*d(*\omega)$$

Таким образом,

1. поле X потенциально, если и только если форма $\omega=I(X)$ точна;
2. поле X безвихревое, т.е. $\operatorname{rot} X=0$, если и только если форма $\omega=I(X)$ замкнута;
3. поле X бездивергентно, т.е. $\operatorname{div} X=0$, если и только если 2 -форма $*\omega$ замкнута.

Лемма Пуанкаре, в применении к случаю $\mathbb{R}^3$, дает следующий результат:

Утверждение 6.7. Поле в $\mathbb{R}^3$ потенчиально, если и только если оно безвихревое (это эквивалентно $H^1\left(\mathbb{R}^3\right)=0$). Поле X бездивергентно, если и только если оно соленоидально (это эквивалентно $H^2\left(\mathbb{R}^3\right)=0$).

Доказательство. Потенциальность поля X равносильна условию $\omega=d F$ для некоторой функции F (потенциала поля X), но последнее условие равносильно (по лемме Пуанкаре) условию $d \omega=0$, что равносильно условию rot $X=0$ (операции I и * устанавливают изоморфизмы соответствующих векторных пространств).

Далее, бездивергентность поля X равносильна условию $d(*\omega)=0$, т.е. условию замкнутости формы $*\omega$. Последнее условие равносильно (в силу леммы Пуанкаре) точности этой формы, т.е. условию $*\omega=d \eta$ для некоторой 1 -формы η. Применяя операцию $*$ к обеим частям последнего равенства, получаем $I(X)=\omega=*d \eta=I(\operatorname{rot} Y)$, где Y - векторное поле, соответствующее 1 -форме η. Поэтому $X=\operatorname{rot} Y$, а это и есть условие соленоидальности поля X. Доказательство закончено.

6.4. Симплектические многообразия. Симплектические многообразия играют важную роль в приложениях, особенно в механике и вариационном исчислении.

Определение. Многообразие M называется симплектическим, если на нем задана дифференциальная 2 -форма ω, обладающая следующими двумя свойствами.

1. Форма ω замкнута, т.е. $d \omega=0$.
2. В каждой точке P из M внешняя форма $\omega(P)$ невырождена, т.е. определитель матрицы (ω_{ij}) компонент формы ω в локальных координатах в окрестности точки P отличен от нуля. Другими словами, для каждого вектора $v \in T_P M$ существует вектор $w \in T_P M$, такой что $\omega(v, w) \neq 0$.
   Такая форма ω называется симплектической структурой на многообразии M.

Пример. Пространство $\mathbb{R}^{2n}$ с координатами ($x^1, \ldots, x^n, y^1, \ldots, y^n$) и внешней формой $\omega = dx^1 \wedge dy^1 + \cdots + dx^n \wedge dy^n$ является симплектическим многообразием (докажите).

Пример. Пусть M - гладкое многообразие, и T^*M - кокасательное расслоение к многообразию M. Обозначим через $\pi : T^*M \to M$ естественную проекцию, сопоставляющую каждому ковектору $\xi \in T_P^*M$ точку $P \in M$. Определим на T^*M следующую 1-форму α. Пусть $W = (P, \xi) \in T_P^*M$ - произвольная точка из T^*M (напомним, что W - это ковектор ξ, заданный в точке $P \in M$), и $v \in T_W(T^*M)$ - касательный вектор. Тогда $\pi_*(v) \in T_PM$. Положим $\alpha(v) = \xi(\pi_*(v))$. Ясно, что α задает в каждой точке $W \in T^*M$ линейную функцию на $T_W(T^*M)$, т.е. α - это 1 -форма на кокасательном расслоении. Определим замкнутую 2 -форму ω так: $\omega = d\alpha$. Эта форма оказывается невырожденной. Чтобы убедиться в этом, достаточно записать эту форму в локальных координатах. Пусть ($x^1, \ldots, x^n$) - некоторая карта на M, и ($x^1 \ldots, x^n, p_1, \ldots, p_n$) - соответствующая карта на T^*M. Тогда легко видеть, что в этих координатах форма α имеет вид $\alpha = p_i dx^i$, поэтому $\omega = d\alpha = dp_i \wedge dx^i-$ невырожденная замкнутая форма.

Имеет место следующее простое утверждение.

Утверждение 6.8. Размерность произвольного симплектического многообразия - четное число.

Доказательство. Действительно, если $(x^1, \ldots, x^n)$ - произвольные координаты в окрестности произвольной точки P симплектического n-мерного многообразия M, и ω_{ij} - компоненты симплектической структуры ω в этих координатах, то, по определению, $\Omega = (\omega_{ij})$ - кососимметрическая невырожденная матрица, откуда

$$0 \neq \det \Omega = \det \Omega^T = \det(-\Omega) = (-1)^n \det \Omega$$

поэтому $(-1)^n = 1$, т.е. n четно, что и требовалось.

Симплектическая структура ω устанавливает изоморфизм между ковекторами и векторами (как и любое другое невырожденное тензорное поле типа $(0,2)$ с помощью опускания-поднятия индекса). Если V - касательный вектор, то ему соответствует ковектор ξ, такой что для любого другого вектора W имеет место равенство $\xi(W) = \omega(V, W)$. Обозначим через $I : T_P^*M \to T_PM$ соответствующий изоморфизм (проверьте, что этот изоморфизм - поднятие индекса ковектора с помощью невырожденного тензора ω типа $(0,2)$).

Если H - гладкая функция на M, то векторное поле $I(dH)$ называется ее косьлм градиентом и обозначается через sgrad H. Другими словами, для любого вектора W выполнено: $W(H) = dH(W) = \omega(\operatorname{sgrad} H, W)$. Если $M = \mathbb{R}^{2n}$ со стандартной симплектической структурой $\omega = \sum dx^i \wedge dy^i$ в координатах ($x^1, \ldots, x^n, y^1, \ldots, y^n$), то

$$\operatorname{sgrad} H = \left(\frac{\partial H}{\partial y^1}, \ldots, \frac{\partial H}{\partial y^n}, -\frac{\partial H}{\partial x^1}, \ldots, -\frac{\partial H}{\partial x^n} \right).$$

Действительно, если $\operatorname{sgrad}(H) = A = (A^1, \ldots, A^{2n})$, и $W = (W^1, \ldots, W^{2n})$, то

$$dH(W) = \frac{\partial H}{\partial x^i} W^i + \frac{\partial H}{\partial y^i} W^{i+n}$$

И

$$\omega(A, W) = A^1 W^{n+1} - A^{n+1} W^1 + A^2 W^{n+2} - A^{n+2} W^2 + \cdots + A^n W^{2n} - A^{2n} W^n,$$

откуда, приравнивая коэффициенты при одинаковых W^j, и получаем требуемое.

Известная из механики система уравнений Гамильтона на этом языке записывается так: $(\dot{x}, \dot{y}) = I(dH(x)) = \operatorname{sgrad} H$.

Существования симплектической структуры на многообразии M накладывает ряд жестких ограничений на устройство M. Мы уже видели, что размерность M в этом случае обязана быть четной. Приведем еще один результат такого типа.

Утверждение 6.9. Пусть M - замкнутое (т.е. компактное без края) симплектическое многообразие размерности $2n$. Тогда $H^2(M) \neq 0$.

Доказательство. Пусть ω - симплектическая структура на M. Покажем, что форма ω не точна (это и будет означать, что $H^2(M) \neq 0$). Действительно, если $\omega = d\alpha$, то форма

$$\Omega_k = \underbrace{\omega \wedge \cdots \wedge \omega}_{k \text{ раз}}$$

также будет точной, так как $\Omega_k = d(\alpha \wedge \omega \wedge \cdots \wedge \omega)$. Но тогда, по теореме Стокса, интеграл от формы Ω_n по замкнутому многообразию M будет равен нулю, что невозможно, так как существенная компонсента формы Ω_n нигдс не обращастся в нуль. Доказательство закончено.

Упражнение 6.4. Покажите, что каждое симплектическое многообразие ориентируемо.

Упражнение 6.5. При каких n на сфере S^{2n} можно задать симплектическую структуру?

Отметим в заключение, что симплектические многообразия обладают следующим важным свойством, позволяющим распространять любые рассуждения "из одной точки" на целую окрестность.

Теорема 6.2 (Дарбу). В достаточно малой окрестности каждой точки симплектического многообразия можно выбрать такие локальные координаты (x, y), что симплектическая структура в них принимает вид $\sum dx^i \wedge dy^i$.

15.7 7: Covariant differentiation

В предыдущем разделе мы построили теорию внешнего дифференцирования для дифференциальных форм - кососимметричных тензорных полей. Цель настоящего раздела - построить общую теорию дифференцирования для произвольных тензорных полей.

Прежде всего отметим, что обычная операция взятия частных производных от компонент тензорного поля не является тензорной. А именно, пусть T - произвольный тензорное поле типа (p, q) на гладком многообразии M. Рассмотрим соответствие, сопоставляющее локальным координатам $(x^1, \ldots, x^n)$ набор величин $\frac{\partial T^{i_1 \ldots i_p}_{j_1 \ldots j_q}}{\partial x^k}$. Пусть $\left(x^{1'}, \ldots, x^{n'}\right)$ – другая система локальных координат на M. Ей, по определению, это соответствие сопоставляет набор величин $\frac{\partial T^{i'_1 \ldots i'}_{j'_1 \ldots j'_q}}{\partial x^{k'}}$. Воспользовавшись тензорным законом для компонент тензора T, получим:

$$\begin{aligned}
\frac{\partial T^{i'_1 \ldots i'_p}_{j'_1}}{\partial x^{k'}} &= \frac{\partial}{\partial x^{k'}} \left(T^{i_1 \ldots i_p}_{j_1 \ldots j_q} \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_q}}{\partial x^{j'_q}} \right) = \\
&= \frac{\partial T^{i_1 \ldots i_p}_{j_1 \ldots j_q}}{\partial x^k} \frac{\partial x^k}{\partial x^{k'}} \frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_q}}{\partial x^{j'_q}} + \\
&\qquad + T^{i_1 \ldots i_p}_{j_1 \ldots j_q} \frac{\partial}{\partial x^{k'}} \left(\frac{\partial x^{i'_1}}{\partial x^{i_1}} \cdots \frac{\partial x^{i'_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial x^{j'_1}} \cdots \frac{\partial x^{j_q}}{\partial x^{j'_q}} \right).
\end{aligned}$$

Первое слагаемое - это в точности тензорный закон для рассматриваемого соответствия, а второе слагаемое, вообще говоря, не равно нулю.

Замечание. Отметим, что если замена координат линейная, то второе слагаемое - нуль.

Однако, оказывается, операцию частного дифференцирования можно слегка подправить так, что она станет тензорной.

15.7.1 7.1. Euclidean connectivity

Мы начнсм со случая свклидового пространства $\mathbb{R}^n$. Зафиксирусм в нем глобальные евклидовы координаты ($z^1, \ldots, z^n$). Отметим, что возможность фиксировать глобальные координаты в $\mathbb{R}^n$ лежит в основе нашей конструкции. При построении операции дифференцирования тензорных полей на $\mathbb{R}^n$, которую мы обозначим через ∇, будем исходить из следующих соображений.

- Операция ∇ должна быть тензорной. Другими словами, если T тензорное поле типа (p, q), то результат дифференцирования ∇T должен быть тензорным полем типа ($p, q+1$).
- В евклидовых координатах $(z^1, \ldots, z^n)$ компоненты $(\nabla T)^{i_1 \ldots i_p}_{j_1 \ldots j_{q+1}}$ тензорного поля ∇T должны совпадать с частными производными $\frac{\partial T^{i_1 \ldots j_p}_{j_1 \ldots j_q}}{\partial z^{j_{q+1}}}$ компонент тензорного поля T. Результат применения операции ∇, т.е. тензорное поле ∇T, будем называть ковариантной производной тензорного поля T (в евклидовой связности).

По определению, чтобы найти компоненты тензорного поля ∇T в произвольных координатах $(x^1, \ldots, x^n)$, следует перейти в евклидовы координаты $(z^1, \ldots, z^n)$, там выполнить обычное частное дифференцирование, а затем вернуться обратно. Проделаем это. Пусть $T^{a_1 \ldots a_p}_{b_1 \ldots b_q}$ - компоненты тензорного поля T в (криволинейных) координатах ($x^1, \ldots, x^n$). Тогда его компоненты $\tilde{T}^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}$ в евклидовых координатах имеют вид

$$\tilde{T}^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q} = \frac{\partial z^{\alpha_1}}{\partial x^{a_1}} \cdots \frac{\partial z^{\alpha_p}}{\partial x^{a_p}} \frac{\partial x^{b_1}}{\partial z^{\beta_1}} \cdots \frac{\partial x^{b_q}}{\partial z^{\beta_q}} T^{a_1 \ldots a_p}_{b_1 \ldots b_q}.$$

Компоненты тензорного поля ∇T в евклидовых координатах по определению равны обычным частным производным:

$$(\widetilde{\nabla T})^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q \gamma} = \frac{\partial}{\partial z^\gamma} \left(\frac{\partial z^{\alpha_1}}{\partial x^{a_1}} \cdots \frac{\partial z^{\alpha_p}}{\partial x^{a_p}} \frac{\partial x^{b_1}}{\partial z^{\beta_1}} \cdots \frac{\partial x^{b_q}}{\partial z^{\beta_q}} T^{a_1 \ldots a_p}_{b_1 \ldots b_q} \right)$$

Последнее выражение удобно переписать в виде

$$\begin{aligned}(\widetilde{\nabla T})^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q \gamma} = \left(\frac{\partial T^{a_1 \ldots a_p}_{b_1 \ldots b_q}}{\partial x^c} \frac{\partial x^c}{\partial z^\gamma} \right) \frac{\partial z^{\alpha_1}}{\partial x^{a_1}} \cdots \frac{\partial z^{\alpha_p}}{\partial x^{a_p}} \frac{\partial x^{b_1}}{\partial z^{\beta_1}} \cdots \frac{\partial x^{b_q}}{\partial z^{\beta_q}} + \\ + T^{a_1 \ldots a_p}_{b_1 \ldots b_q} \left(\frac{\partial^2 z^{\alpha_1}}{\partial x^{a_1} \partial x^c} \frac{\partial x^c}{\partial z^\gamma} \right) \frac{\partial z^{\alpha_2}}{\partial x^{a_2}} \cdots \frac{\partial z^{\alpha_p}}{\partial x^{a_p}} \frac{\partial x^{b_1}}{\partial z^{\beta_1}} \cdots \frac{\partial x^{b_q}}{\partial z^{\beta_q}} + \cdots + \\ + T^{a_1 \ldots a_p}_{b_1 \ldots b_q} \left(\frac{\partial^2 x^{b_1}}{\partial z^{\beta_1} \partial z^\gamma} \right) \frac{\partial z^{\alpha_1}}{\partial x^{a_1}} \cdots \frac{\partial z^{\alpha_p}}{\partial x^{a_p}} \frac{\partial x^{b_2}}{\partial z^{\beta_2}} \cdots \frac{\partial x^{b_q}}{\partial z^{\beta_q}} + \cdots\end{aligned}$$

где многоточия обозначают слагаемые, отвечающие дифференцированию других сомножителей вида $\frac{\partial z^\alpha}{\partial x^a}$ и $\frac{\partial x^b}{\partial z^\beta}$ по правилу Лейбница.

Чтобы найти компоненты тензорного поля ∇T в координатах ($x^1, \ldots, x^n$), вновь воспользуемся тензорным законом. Получим:

$$\begin{aligned}(\nabla T)^{i_1 \ldots i_p}_{j_1 \ldots j_q k} = \frac{\partial x^{i_1}}{\partial z^{\alpha_1}} \cdots \frac{\partial x^{i_p}}{\partial z^{\alpha_p}} \frac{\partial z^{\beta_1}}{\partial x^{j_1}} \cdots \frac{\partial z^{\beta_q}}{\partial x^{j_q}} \frac{\partial z^\gamma}{\partial x^k} (\widetilde{\nabla T})^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q \gamma} = \\ = \frac{\partial T^{i_1 \ldots j_p}_{j_1 \ldots j_q}}{\partial x^k} + T^{a_1 i_2 \ldots i_p}_{j_1 \ldots j_q} \underbrace{\frac{\partial^2 z^{\alpha_1}}{\partial x^{a_1} \partial x^k} \frac{\partial x^{i_1}}{\partial z^{\alpha_1}}} + \cdots + T^{i_1 \ldots i_p}_{b_1 j_2 \ldots j_q} \underbrace{\frac{\partial^2 x^{b_1}}{\partial z^{\beta_1} \partial z^\gamma} \frac{\partial z^{\beta_1}}{\partial x^{j_1}} \frac{\partial z^\gamma}{\partial x^k}} + \cdots .\end{aligned}$$

Отметим, что в последнее выражение входят функции двух типов, выделенные снизу фигурными скобками. Оказывается, впрочем, что на самом деле это одна и та же функция.

Лемма 7.1. Имеет место равенство

$$\frac{\partial^2 x^i}{\partial z^\alpha \partial z^\beta}\frac{\partial z^\alpha}{\partial x^j}\frac{\partial z^\beta}{\partial x^k} = -\frac{\partial^2 z^\alpha}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial z^\alpha}.$$

Доказательство. Дсйствитсльно, продиффсрснцировав очевиднос равенство $\frac{\partial x^i}{\partial z^\alpha}\frac{\partial z^\alpha}{\partial x^j} = \delta^i_j$ по x^k получим:

$$0 = \frac{\partial^2 x^i}{\partial z^\alpha \partial z^\beta}\frac{\partial z^\beta}{\partial x^k}\frac{\partial z^\alpha}{\partial x^j} + \frac{\partial x^i}{\partial z^\alpha}\frac{\partial^2 z^\alpha}{\partial x^j \partial x^k}$$

что и требовалось.

Положим

$$\Gamma^i_{jk} = \frac{\partial^2 z^\alpha}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial z^\alpha} \qquad (*)$$

Тогда компоненты тензорного поля ∇T в координатах ($x^1, \ldots, x^n$) перепишутся в более компактном виде так:

$$(\nabla T)^{i_1 \ldots i_p}_{j_1 \ldots j_q k} = \frac{\partial T^{i_1 \ldots i_p}_{j_1 \ldots j_q}}{\partial x^k} + T^{a_1 i_2 \ldots i_p}_{j_1 \ldots j_q}\Gamma^{i_1}_{a_1 k} + \cdots + T^{i_1 \ldots i_{p-1} a_p}_{j_1 \ldots j_q}\Gamma^{i_p}_{a_p k} -$$
$$- T^{i_1 \ldots i_p}_{b_1 j_2 \ldots j_q}\Gamma^{b_1}_{j_1 k} - \cdots - T^{i_1 \ldots i_p}_{j_1 \ldots j_{q-1} b_q}\Gamma^{q_q}_{j_q k}.$$

Отметим, что величины Γ^i_{jk}, определенные соотношением ($*$), не являются компонентами тензорного поля.

Лемма 7.2. Пусть $(x^1, \ldots, x^n)$ u $\left(x^{1'}, \ldots, x^{n'}\right)$ – две криволинейных системь координат в $\mathbb{R}^n$. Пусть

$$\Gamma^i_{jk} = \frac{\partial^2 z^\alpha}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial z^\alpha}, \quad u \quad \Gamma^{i'}_{j'k'} = \frac{\partial^2 z^\alpha}{\partial x^{j'} \partial x^{k'}}\frac{\partial x^{i'}}{\partial z^\alpha}$$

где $(z^1, \ldots, z^n)$ евклидовы координаты в $\mathbb{R}^n$. Тогда

$$\Gamma^{i'}_{j'k'} = \Gamma^i_{jk}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}} + \frac{\partial^2 x^\beta}{\partial x^{j'} \partial x^{k'}}\frac{\partial x^{i'}}{\partial x^\beta}.$$

Доказательство. Действительно,

$$\Gamma^{i'}_{j'k'} = \frac{\partial}{\partial x^{k'}}\left(\frac{\partial z^\alpha}{\partial x^{j'}}\right)\frac{\partial x^{i'}}{\partial z^\alpha} = \frac{\partial}{\partial x^{k'}}\left(\frac{\partial z^\alpha}{\partial x^j}\frac{\partial x^j}{\partial x^{j'}}\right)\frac{\partial x^{i'}}{\partial z^\alpha} =$$
$$= \left(\frac{\partial^2 z^\alpha}{\partial x^j \partial x^k}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^j}{\partial x^{j'}} + \frac{\partial z^\alpha}{\partial x^j}\frac{\partial^2 x^j}{\partial x^{j'} \partial x^{k'}}\right)\frac{\partial x^{i'}}{\partial z^\alpha} =$$
$$= \left(\frac{\partial^2 z^\alpha}{\partial x^j \partial x^k}\frac{\partial x^i}{\partial z^\alpha}\right)\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}} + \frac{\partial z^\alpha}{\partial x^j}\frac{\partial^2 x^j}{\partial x^{j'} \partial x^{k'}}\frac{\partial x^{i'}}{\partial z^\alpha} =$$
$$= \Gamma^i_{jk}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}} + \frac{\partial^2 x^j}{\partial x^{j'} \partial x^{k'}}\frac{\partial x^{i'}}{\partial x^j},$$

что и требовалось.

Итак, нами получен следующий важный результат.

Теорема 7.1. Па тензорных полях на $\mathbb{R}^n$ существует тензорная операция ∇, ставящая в соответствие каждому тензорному полю T типа (p, q) тензорное поле ∇T типа $(p, q+1)$ и обладающая следующими свойствами.

- В евклидовых координатах $(z^1, \ldots, z^n)$ операция ∇ совпадает *с* объчным частным дифференцированием, т.е. если $\tilde{T}^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ $-$ $-$ поненты поля T в евклидовых координатах, то в евклидовых координатах компоненть $\widetilde{\nabla T}^{i_1 \ldots i_p}_{j_1 \ldots j_{q+1}}$ тензора ∇T имеют вид

$$\widetilde{\nabla T}^{i_1 \ldots i_p}_{j_1 \ldots j_q k} = \frac{\partial}{\partial z^k} \left(\tilde{T}^{i_1 \ldots i_p}_{j_1 \ldots j_q} \right)$$

- В произвольных криволинейных координатах $(x^1, \ldots, x^n)$ компоненты тензоров T и ∇T связаны так:

$$(\nabla T)^{i_1 \ldots i_p}_{j_1 \ldots j_q k} = \frac{\partial T^{i_1 \ldots i_p}_{j_1 \ldots j_q}}{\partial x^k} + T^{a_1 i_2 \ldots i_p}_{j_1 \ldots j_q} \Gamma^{i_1}_{a_1 k} + \cdots + T^{i_1 \ldots i_{p-1} a_p}_{j_1 \ldots j_q} \Gamma^{i_p}_{a_p k} - \\ - T^{i_1 \ldots i_p}_{b_1 j_2 \ldots j_q} \Gamma^{b_1}_{j_1 k} - \cdots - T^{i_1 \ldots i_p}_{j_1 \ldots j_{q-1} b_q} \Gamma^{b_q}_{j_q k}$$

где Γ^i_{jk} - гладкие функции на $\mathbb{R}^n$, соответствующие системе локальных координат $(x^1, \ldots, x^n)$, и определенные соотношениями

$$\Gamma^i_{jk} = \frac{\partial^2 z^\alpha}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial z^\alpha}$$

- Соответствие

$$\Gamma : \left(x^1, \ldots, x^n \right) \mapsto \left\{ \Gamma^i_{jk} \right\},$$

сопоставляющее координатам $(x^1, \ldots, x^n)$ набор функций Γ^i_{jk} не является тензорным полем. Функиии Γ^i_{jk} при замене координат преобразуются по правилу

$$\Gamma^{i'}_{j'k'} = \Gamma^i_{jk} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} + \frac{\partial^2 x^j}{\partial x^{j'} \partial x^{k'}} \frac{\partial x^{i'}}{\partial x^j},$$

где $\Gamma^{i'}_{j'k'}$ — набор функиий, соответствующих криволинейньлм координатам ($x^{1'}, \ldots, x^{n'}$).

Итак, нами доказана теорема существования тензорной операции дифференцирования в $\mathbb{R}^n$. Отметим, что мы существенно использовали наличие в $\mathbb{R}^n$ выделенной - евклидовой системы координат, в которых

операция дифференцирования по нашему определению должна была выглядеть особенно просто.

Прежде чем переходить к общему случаю, условимся о терминологии и обозначениях. Компоненту $(\nabla T)^{i_1 \ldots i_p}_{j_1 \ldots j_q k}$ тензора ∇T принято обозначать или через $\nabla_k T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$, или, особенно в физической литературе, через $T^{i_1 \ldots i_p}_{j_1 \ldots j_q ; k}$. Функции Γ^i_{jk} называются символами Кристоффеля евклидовой связности в координатах ($x^1, \ldots, x^n$), а их закон преобразования, установленный в лемме 7.2, - законом преобразования символов Кристоффеля.

15.7.2 7.2. Affine Connectivity

Пусть теперь M - произвольное гладкое многообразие.
Определение. Говорят, что на многообразии M задана аффинная связность Γ, если задано соответствие, сопоставляющее каждым локальным координатам $(\ x^1, \ldots, x^n\ )$ в окрестности произвольной точки P из M набор гладких функций $\Gamma^i_{jk}(x^1, \ldots, x^n)$. При этом, если $\left(x^{1'}, \ldots, x^{n'}\right)$ – другие локальные координаты в окрестности точки P, то соответствующие им функции $\Gamma^{i'}_{j'k'}$ связаны с функциями Γ^i_{jk} по следующему закону

$$\Gamma^{i'}_{j'k'} = \Gamma^i_{jk} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} + \frac{\partial^2 x^j}{\partial x^{j'} \partial x^{k'}} \frac{\partial x^{i'}}{\partial x^j}.$$

Функции Γ^i_{jk} называются символами Кристоффеля аффинной связности Γ, или ее компонентами в координатах $(x^1, \ldots, x^n)$.

Если на многообразии M задана аффинная связность Γ, то на тензорных полях на многообразии M определена операция ∇ ковариантного дифференцирования относительно аффинной связности Γ. А именно, если T - произвольное тензорное поле типа (p, q), то его ковариантной производной в связности Γ называется тензорное поле ∇T, компоненты которого в произвольных координатах $(x^1, \ldots, x^n)$ вычисляются так:

$$(\nabla T)^{i_1 \ldots i_p}_{j_1 \ldots j_q k} = \frac{\partial T^{i_1 \ldots i_p}_{j_1 \ldots j_q}}{\partial x^k} + T^{a_1 i_2 \ldots i_p}_{j_1 \ldots j_q} \Gamma^{i_1}_{a_1 k} + \cdots + T^{i_1 \ldots i_{p-1} a_p}_{j_1 \ldots j_q} \Gamma^{i_p}_{a_p k} - \\ - T^{i_1 \ldots i_p}_{b_1 j_2 \ldots j_q} \Gamma^{b_1}_{j_1 k} - \cdots - T^{i_1 \ldots i_p}_{j_1 \ldots j_{q-1} b_q} \Gamma^{b_q}_{j_q k}.$$

Замечание. Пока, вообще говоря, не откуда не вытекает, что на произвольном гладком многообразии M можно ввести хотя бы одну аффинную связность.

Пример. Пусть $M = \mathbb{R}^n$. Тогда построенное нами в предыдущем пункте соответствие, сопоставляющее произвольным координатам в $\mathbb{R}^n$ соответствующие символы Кристоффеля евклидовой связности, является, как следует из леммы 7.2, аффинной связностью на многообразии $\mathbb{R}^n$.

Пример. Пусть M - регулярная гиперповерхность в $\mathbb{R}^{n|1}$. В прошлом семестре мы вывели деривационные формулы Гаусса-Вейнгартена и определили ковариантное дифференцирование векторных полей на M. При этом на поверхности M возникли символы Кристоффеля, которые, напомним, в координатах $(\ u^1, \ldots, u^n\ )$ явно записывались через первую фундаментальную форму (g_{ij}) так:

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\alpha} \left(\frac{\partial g_{j\alpha}}{\partial u^k} + \frac{\partial g_{k\alpha}}{\partial u^j} - \frac{\partial g_{jk}}{\partial u^\alpha} \right)$$

Несложные, хотя и громоздкие, вычисления показывают, что соответствие, сопоставляющее координатам $(\ u^1, \ldots, u^n\ )$ набор функций Γ^i_{jk}, является аффинной связностью на многообразии M.

Пусть M - произвольное гладкое многообразие с аффинной связностью Γ. Рассмотрим соответствие Ω, сопоставляющее каждой локальной системе координат $(x^1, \ldots, x^n)$ набор чисел $\Omega^i_{jk} = \Gamma^i_{jk} - \Gamma^i_{kj}$.

Лемма 7.3. В сделанных выше обозначениях, соответствие Ω задает на многообразии тензорное поле типа $(1, 2)$.

Доказательство. Действительно, это немедленно вытекает из того, что нетензорная добавка в законе преобразований символов Кристоффеля симметрична по нижним индексам. Лемма доказана.

Тензорное поле Ω называется тензором кручения аффинной связности Γ. Связность Γ называется симметричной, если ее тензор кручения равен нулю.

Пример. Евклидова связность на пространстве $\mathbb{R}^n$ является симметричной, так как ее символы Кристоффеля могут быть записаны в виде

$$\Gamma^i_{jk} = \frac{\partial^2 z^\alpha}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial z^\alpha}$$

симметричном по нижним индексам j и k. Произвольная связность, отметим, вовсе не обязана быть симметричной.

Упражнение 7.1. Проверить симметричность аффинной связности построенной выше на гиперповерхности $M \subset \mathbb{R}^{n+1}$.

15.7.3 7.3. Covariant directional derivative

Пусть T - тензорное поле типа (p, q), и X - векторное поле, т.е. тензорное поле типа (1,0) на многообразии M.

Определение. Ковариантной производной $\nabla_X T$ поля T вдоль векторного поля X называется следующее тензорное поле типа (p, q):

$$\nabla_X T = C^1_{q+1}(X \otimes \nabla T)$$

Пусть $(x^1, \ldots, x^n)$ - локальные координаты на многообразии M. Тогда, по определению, компоненты тензорного поля $\nabla_X T$ могут быть записаны в виде

$$(\nabla_X T)^{i_1 \ldots i_p}_{j_1 \ldots j_q} = X^\alpha \nabla_\alpha T^{i_1 \ldots i_p}_{j_1 \ldots j_q},$$

где $(X^1, \ldots, X^n)$ - компоненты поля X в координатах $(x^1, \ldots, x^n)$. В частности, если поле X совпадает с вектором ∂_{x^k} канонического базиса, то

$$\left(\nabla_{\partial_{x^k}} T\right)^{i_1 \ldots i_p}_{j_1 \ldots j_q} = \nabla_k T^{i_1 \ldots i_p}_{j_1 \ldots j_q}.$$

Операция дифференцирования $\nabla_{\partial_{x^k}}$ вдоль векторного поля ∂_{x^k} называется ковариантным дифференцированием по k-ой координате и обозначается обычїю для краткости просто через ∇_k. В этих обозначениях ковариантное дифференцирование вдоль векторного поля X можно представить в локальных координатах ($x^1, \ldots, x^n$) в виде

$$\nabla_X = X^k \nabla_k.$$

Определение ковариантной производной вдоль векторного поля позволяет дать естественную интерпретацию символов Кристоффеля аффинной связности.

Лемма 7.4. Пусть $(x^1, \ldots, x^n)$ - произвольные локальные координаты на гладком многообразии M с аффинной связностью Γ. Тогда

$$\nabla_{\partial_{x^i}} \partial_{x^j} = \Gamma^\alpha_{ji} \partial_{x^\alpha}$$

Доказательство. Действительно, запишем ковариантную производную векторного поля ∂_{x^j} вдоль поля ∂_{x^i}. По определению получим:

$$\left(\nabla_{\partial_{x^i}} \partial_{x^j}\right)^k = (\partial_{x^i})^\alpha \nabla_\alpha (\partial_{x^j})^k = (\partial_{x^i})^\alpha \left(\frac{\partial}{\partial x^\alpha} (\partial_{x^j})^k + \Gamma^k_{\beta\alpha} (\partial_{x^j})^\beta \right).$$

Поскольку координаты $(\partial_{x^j})^k$ базисного вектора постоянны и равны δ^k_j, получаем

$$\left(\nabla_{\partial_{x^i}} \partial_{x^j}\right)^k = \delta^\alpha_i \Gamma^k_{\beta\alpha} \delta^\beta_j = \Gamma^k_{ji}$$

откуда и вытекает требуемое. Лемма доказана.
Пусть γ - произвольная гладкая кривая на многообразии M. Тогда ковариантной производной $\nabla_{\gamma'}T$ тензорного поля T вдоль γ называется ковариантная производная поля T вдоль вектора скорости γ' кривой γ. Если $(x^1, \ldots, x^n)$ - произвольная локальная система координат, и $x^i = x^i(t)$ - координатное представление кривой γ, то, по определению,

компоненты тензорного поля $\nabla_{\gamma'}T$ вычисляются так:

$$\begin{aligned}(\nabla_{\gamma'}T)^{i_1\ldots i_p}_{j_1\ldots j_q} &= \frac{dx^\alpha}{dt}\nabla_\alpha T^{i_1\ldots i_p}_{j_1\ldots j_q} = \\ &= \frac{dx^\alpha}{dt}\left(\frac{\partial}{\partial x^\alpha}T^{i_1\ldots i_p}_{j_1\ldots j_q} + \text{ слагаемые с символами Кристоффеля }\right) = \\ &= \frac{d}{dt}T^{i_1\ldots i_p}_{j_1\ldots j_q} + \frac{dx^\alpha}{dt}(\text{ слагаемые с символами Кристоффеля })\end{aligned}$$

Из полученной формулы вытекает следующий факт.
Лемма 7.5. Ковариантная производная тензорного поля T вдоль кривой γ зависит только от значения поля T на кривой γ.

Замечание. Чтобы вычислить обычную ковариантную производную в некоторой точке следует знать значение тензорного поля в некоторой открытой окрестности этой точки. Поэтому, с формальной точки зрения, если тензорное поле T задано только в точках кривой γ, то производная $\nabla_{\gamma'}T$ не определена, так как не определена производная ∇T, входящая в определение производной по направлению. Чтобы выйти из этого затруднительного положения, поступают следующим образом. Пусть $P = \gamma(t_0)$ произвольная точка на кривой γ. Продолжим поле T, заданное вдоль кривой, на малую окрестность U точки P в многообразии, т.е. построим поле $\tilde{T}$, определенное в окрестности U и совпадающее с T в точках кривой. Ясно, что такое продолжение существует (докажите!). Определим ковариантную производную $\nabla_{\gamma'}T(P)$ поля T вдоль γ, положив

$$\nabla_{\gamma'}T = \nabla_{\gamma'}\tilde{T}$$

Стоящее справа выражение корректно определено, а из леммы 7.5 вытекает, что оно не зависит от продолжения T поля T за пределы кривой γ. В дальнейшем мы будем ковариантно дифференцировать вдоль кривых тензорные поля определенные только вдоль этих кривых без специальных оговорок.

Замечание. Значение ковариантной производной тензорного поля вдоль кривой γ, вообще говоря, зависит от параметризации кривой. Действительно, пусть $\gamma : [a, b] \to M$ - гладкая кривая на многообразии M, и $\tau : [c, d] \to [a, b]$ - замена параметризации (т.е. некоторый диффеоморфизм отрезков). Обозначим через $t \in [a, b]$ параметр кривой γ, через $\tilde{\gamma} = \gamma \circ \tau$ - кривую полученную из γ заменой параметра, и через $s \in [c, d]$ - параметр кривой $\tilde{\gamma}$. Пусть $(x^1, \ldots, x^n)$ - локальные координаты, и $x^i(t)$ - координатное представление кривой γ. Тогда координатное представление $\tilde{x}^i(s)$ имеет вид $\tilde{x}^i(s) = x^i(\tau(s))$. Поэтому для произвольного тензорного поля T имеем:

$$\nabla_{d\tilde{\gamma}/ds}T = \frac{d\tilde{x}^\alpha}{ds}\nabla_\alpha T = \frac{dx^\alpha}{dt}\frac{dt}{ds}\nabla_\alpha T = \frac{dt}{ds}\nabla_{d\gamma/dt}T$$

Итак, доказана следующая лемма.
Лемма 7.6. При замене параметра $t = \tau(s)$ кривой $\gamma(t)$, ковариантная производная вдоль кривой умножается на производную $d\tau/ds$.

15.7.4 7.4. Euclidean coordinates for connectivity

При определении евклидовой связности мы пользовались наличием на пространстве $\mathbb{R}^n$ «хороших» координат.

Определение. Пусть M - гладкое многообразие с аффинной связностью Γ. Локальные координаты ($x^1, \ldots, x^n$) на M называются евклидовыли для связности Γ, если символы Кристоффеля связности Γ в координатах $(x^1, \ldots, x^n)$ тождественно равны нулю в области определения этих координат.

Например, если $M = \mathbb{R}^n$, и Γ - евклидова связность, то стандартные евклидовы координаты ($z^1, \ldots, z^n$) в $\mathbb{R}^n$ являются, по определению евклидовой связности, евклидовыми для связности Γ. Отметим, что евклидовы координаты для связности Γ могут и не существовать.

Замечание. Пусть M - гладкое многообразие и Γ - несимметричная связность на M. Тогда для связности Γ не существует евклидовых координат. Действительно, если ($x^1, \ldots, x^n$) - евклидовы координаты связности Γ, то в этих координатах тензор кручения связности Γ тождественно равен нулю. Но тогда тензор кручения равен нулю в любых координатах, что означает симметричность связности Γ. Противоречие. Таким образом, необходимым условием существования евклидовых координат для аффинной связности является ее симметричность.

15.7.5 Problems

Задача 7.1. Чему равны символы Кристоффеля евклидовой связности в евклидовой системе координат?

Задача 7.2. Выписать формулы для компонент тензора ∇T в случае, когда T является вектором, ковектором, линейным оператором, билинейной формой.

Задача 7.3. Пусть M - гладкое многообразие с аффинной связностью Γ, и δ тензорное поле типа $(1,1)$, компоненты которого в координатах ($x^1, \ldots, x^n$) имеют вид δ^i_j (символы Кронекера). Вычислить $\nabla_k \delta^i_j$.

Задача 7.4. Пусть $M = \mathbb{R}^n$. Рассмотрим в $\mathbb{R}^n$ произвольные криволинейные координаты $(x^1, \ldots, x^n)$ и сопоставим им набор гладких функций $\left\{\Gamma^i_{jk}\right\}$. Каждой другой системе координат $\left(x^{1'}, \ldots, x^{n'}\right)$ на $\mathbb{R}^n$ сопоставим набор функций $\left\{\Gamma^{i'}_{j'k'}\right\}$, вычисленных в соответствии с законом изменения символов Кристоффеля при замене координат:

$$\Gamma^{i'}_{j'k'} = \Gamma^i_{jk} \frac{\partial x^{i'}}{\partial x^i} \frac{\partial x^j}{\partial x^{j'}} \frac{\partial x^k}{\partial x^{k'}} + \frac{\partial^2 x^j}{\partial x^{j'} \partial x^{k'}} \frac{\partial x^{i'}}{\partial x^j}.$$

Проверить, что построенное соответствие задает аффинную связность на $\mathbb{R}^n$, т.е. если $\{\Gamma_{j''k''}{}''\}$ - набор функций, сопоставленный системе координат ($x^{1''}, \ldots, x^{n''}$), то справедливо равенство

$$\Gamma^{i''}_{j''k''} = \Gamma^{i'}_{j'k'} \frac{\partial x^{i''}}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^{j''}} \frac{\partial x^{k'}}{\partial x^{k''}} + \frac{\partial^2 x^{j'}}{\partial x^{j''} \partial x^{k''}} \frac{\partial x^{i''}}{\partial x^{j'}}?$$

Другими словами, закон преобразования символов Кристоффеля при замене координат является транзитивным?

Задача 7.5. Пусть M - гладкое многообразие с аффинной связностью I', и T' произвольное тензорное поле на M типа $(1,2)$. Сопоставим каждой системе координат $(x^1, \ldots, x^n)$ набор функций $\left\{\Gamma^i_{jk} + T^i_{jk}\right\}$. Проверить, что в результате получится некоторая аффинная связность на M.

Задача 7.6. Привести пример несимметричной связности на $\mathbb{R}^n$.

Задача 7.7. Пусть $\omega \in \Omega^k(M)$ - внешняя форма степени k на многообразии M, а Γ - симметричная аффинная связность на M. Как связаны тензорные поля $A(\nabla\omega)$ и $d\omega$?

15.7.6 Extra Topics

7.1. Транзитивность закона преобразования символов Кристоффеля. Рассмотрим в $M = \mathbb{R}^n$ произвольные криволинейные координаты ($x^1, \ldots, x^n$) и сопоставим им набор гладких функций $\left\{\Gamma^i_{jk}\right\}$. Каждой другой системе координат $\left(x^{1'}, \ldots, x^{n'}\right)$ на $\mathbb{R}^n$ сопоставим набор функций $\left\{\Gamma^{i'}_{j'k'}\right\}$, вычисленных в соответствии с законом изменения символов Кристоффеля при замене координат:

$$\Gamma^{i'}_{j'k'} = \Gamma^i_{jk}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}} + \frac{\partial^2 x^j}{\partial x^{j'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^j}.$$

Проверим, что построенное нами соответствие задает аффинную связность на $\mathbb{R}^n$, т.е. если $\left\{\Gamma^{i''}_{j''k''}\right\}$ - набор функций, сопоставленный системе координат ($x^{1''}, \ldots, x^{n''}$), а именно,

$$\Gamma^{i''}_{j''k''} = \Gamma^i_{jk}\frac{\partial x^{i''}}{\partial x^i}\frac{\partial x^j}{\partial x^{j''}}\frac{\partial x^k}{\partial x^{k''}} + \frac{\partial^2 x^j}{\partial x^{j''}\partial x^{k''}}\frac{\partial x^{i''}}{\partial x^j},$$

то справедливо ли равенство

$$\Gamma^{i''}_{j''k''} = \Gamma^{i'}_{j'k'}\frac{\partial x^{i''}}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^{j''}}\frac{\partial x^{k'}}{\partial x^{k''}} + \frac{\partial^2 x^{j'}}{\partial x^{j''}\partial x^{k''}}\frac{\partial x^{i''}}{\partial x^{j'}}.$$

Действительно,

$$\begin{aligned}
\Gamma^{i''}_{j''k''} = \Gamma^i_{jk}\frac{\partial x^{i''}}{\partial x^i}\frac{\partial x^j}{\partial x^{j''}}\frac{\partial x^k}{\partial x^{k''}} + \frac{\partial^2 x^j}{\partial x^{j''}\partial x^{k''}}\frac{\partial x^{i''}}{\partial x^j} = \Gamma^i_{jk}\frac{\partial x^{i''}}{\partial x^{i'}}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{j'}}{\partial x^{j''}}\frac{\partial x^k}{\partial x^{k'}}\frac{\partial x^{k'}}{\partial x^{k''}} + \\
+\frac{\partial}{\partial x^{k''}}\left(\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^{j'}}{\partial x^{j''}}\right)\frac{\partial x^{i''}}{\partial x^{i'}}\frac{\partial x^{i'}}{\partial x^j} = \left(\Gamma^i_{jk}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}}\right)\frac{\partial x^{i''}}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^{j''}}\frac{\partial x^{k'}}{\partial x^{k''}} + \\
+\left(\frac{\partial^2 x^j}{\partial x^{j'}\partial x^{k'}}\frac{\partial x^{k'}}{\partial x^{k''}}\frac{\partial x^{j'}}{\partial x^{j''}} + \frac{\partial x^j}{\partial x^{j'}}\frac{\partial^2 x^{j'}}{\partial x^{j''}\partial x^{k''}}\right)\frac{\partial x^{i''}}{\partial x^{i'}}\frac{\partial x^{i'}}{\partial x^j} = \\
= \left(\Gamma^i_{jk}\frac{\partial x^{i'}}{\partial x^i}\frac{\partial x^j}{\partial x^{j'}}\frac{\partial x^k}{\partial x^{k'}} + \frac{\partial^2 x^j}{\partial x^{j'}\partial x^{k'}}\frac{\partial x^{i'}}{\partial x^j}\right)\frac{\partial x^{i''}}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^{j''}}\frac{\partial x^{k'}}{\partial x^{k''}} + \\
+\frac{\partial x^j}{\partial x^{j'}}\frac{\partial^2 x^{j'}}{\partial x^{j''}\partial x^{k''}}\frac{\partial x^{i''}}{\partial x^{i'}}\frac{\partial x^{i'}}{\partial x^j} = \Gamma^{i'}_{j'k'}\frac{\partial x^{i''}}{\partial x^{i'}}\frac{\partial x^{j'}}{\partial x^{j''}}\frac{\partial x^{k'}}{\partial x^{k''}} + \frac{\partial^2 x^{j'}}{\partial x^{j''}\partial x^{k''}}\frac{\partial x^{i''}}{\partial x^{j'}}
\end{aligned}$$

Пример. Пусть на многообразии M имеются глобальные координаты ($z^1, \ldots, z^n$). Сопоставим им набор символов Кристоффеля Γ^i_{jk} тождественно равный нулю. Для построенной связности выделенные координаты ($z^1, \ldots, z^n$) будут евклидовыми. Сравните описанную конструкцию со случаем евклидового пространства.

Упражнение 7.2. Какие еще бывают транзитивные законы преобразования, кроме тензорного и закона символов Кристоффеля?

15.8 8: Properties of covariant differentiation

В данной лекции мы продолжим изучение ковариантного дифференцирования, а также построим важный пример аффинной связности - риманову связность.

15.8.1 8.1. Algebraic properties of covariant differentiation

Перечислим простейшие свойства операции ковариантного дифференцирования.

Утверждение 8.1. Пусть M - гладкое многообразие, Γ - аффинная связность на нем, $P \in M$ - некоторая его точка, и X - произвольное векторное поле, заданное в окрестности точки P. Тогда операция ковариантного дифференцирования в аффинной связности Γ обладает следующими свойствами.

1. Результат ∇T применения операции ковариантного дифферениирования к тензорному полю T типа (p, q) является тензорным полем типа $(p, q+1)$. Результат $\nabla_X T$ применения операции ковариантного дифференцирования по направлению векторного поля $X\kappa$ тензорному полю T типа (p, q) является тензорным полем того же типа (p, q).

2. Операция ∇ линейна: для любых постоянных α и β и тензорных полей T и S типа (p, q) имеют место равенства $\nabla(\alpha T+\beta S)=\alpha \nabla T+\beta \nabla S, u \nabla_X(\alpha T+\beta S)=\alpha \nabla_X T+\beta \nabla_X S$.
   3) Результат применения операции $\nabla\kappa$ функции f совпадает с дифференциалом этой функции: $\nabla f=d f$; в локальных координатах $(x^1, \ldots, x^n)$ имеем: $\nabla f=\frac{\partial f}{\partial x^k} d x^k$. Результат применения операции $\nabla_X \kappa$ функции f совпадает с производной этой функции по направлению поля $X: \nabla_X f=X(f)=d f(X)$.

3. Операция ∇_X удовлетворяет правилу Лейбница, а именно, для любых тензорных полей T типа (p, q) и S типа (r, s) имеет место равенство:

$$\nabla_X(T \otimes S)=\left(\nabla_X T\right) \otimes S+T \otimes\left(\nabla_X S\right)$$

в частности, в локальных координатах:

$$\nabla_k\left(T_{j_1 \ldots j_q}^{i_1 \ldots i_p} S_{\beta_1 \ldots \beta_s}^{\alpha_1 \ldots \alpha_r}\right)=\left(\nabla_k T_{j_1 \ldots j_q}^{i_1 \ldots i_p}\right) S_{\beta_1 \ldots \beta_s}^{\alpha_1 \ldots \alpha_r}+T_{j_1 \ldots j_q}^{i_1 \ldots i_p} \nabla_k S_{\beta_1 \ldots \beta_s}^{\alpha_1 \ldots \alpha_r}.$$

5. Операция ∇ перестановочна с операцией свертки: если T - тензорное поле типа (p, q), то для любых a и b таких, что $1 \leq a \leq p u 1 \leq b \leq q$, выполнено

$$\nabla C_b^a(T)=C_b^a(\nabla T) \quad u \quad \nabla_X C_b^a(T)=C_b^a\left(\nabla_X T\right)$$

Доказательство. Свойство (1) следует из закона преобразования символов Кристоффеля, который специально был подобран нами так, чтобы ковариантное дифференцирование было тензорной операцией. Провести соответствующую выкладку оставляется в качестве обязательного упражнения (эта выкладка, фактически, была проделана в предыдущей лекции при построении евклидовой связности). Свойства (2) и (3) немедленно вытекают из определения ковариантного дифференцирования.

Доказательство свойства (4) мы проведем для случая тензоров типа $(1,1)$, оставляя более громоздкий общий случай в качестве упражнения. Итак, пусть $T_{j_1}^{i_1}$ и $S_{j_2}^{i_2}$ - компоненты тензорных полей T и S типа $(1,1)$ в локальных координатах $(x^1, \ldots, x^n)$, и $P_{j_1 j_2}^{i_1 i_2}=T_{j_1}^{i_1} S_{j_2}^{i_2}$ - компоненты их тензорного произведения. Тогда, по определению операции ковариантного дифференцирования, в координатах ($x^1, \ldots, x^n$) имеем:

$$\begin{aligned}
\nabla_k P_{j_1 j_2}^{i_1 i_2} &= \frac{\partial P_{j_1 j_2}^{i_1 i_2}}{\partial x^k}+P_{j_1 j_2}^{\alpha i_2} \Gamma_{\alpha k}^{i_1}+P_{j_1 j_2}^{i_1 \alpha} \Gamma_{\alpha k}^{i_2}-P_{\alpha j_2}^{i_1 i_2} \Gamma_{j_1 k}^{\alpha}-P_{j_1 \alpha}^{i_1 i_2} \Gamma_{j_2 k}^{\alpha}= \\
&=\frac{\partial T_{j_1}^{i_1}}{\partial x^k} S_{j_2}^{i_2}+\frac{\partial S_{j_2}^{i_2}}{\partial x^k} T_{j_1}^{i_1}++T_{j_1}^{\alpha} S_{j_2}^{i_2} \Gamma_{\alpha k}^{i_1}+T_{j_1}^{i_1} S_{j_2}^{\alpha} \Gamma_{\alpha k}^{i_2}-T_{\alpha}^{i_1} S_{j_2}^{i_2} \Gamma_{j_1 k}^{\alpha}-T_{j_1}^{i_1} S_{\alpha}^{i_2} \Gamma_{j_2 k}^{\alpha}= \\
&=S_{j_2}^{i_2}\left(\frac{\partial T_{j_1}^{i_1}}{\partial x^k}+T_{j_1}^{\alpha} \Gamma_{\alpha k}^{i_1}-T_{\alpha}^{i_1} \Gamma_{j_1 k}^{\alpha}\right)+T_{j_1}^{i_1}\left(\frac{\partial S_{j_2}^{i_2}}{\partial x^k}+S_{j_2}^{\alpha} \Gamma_{\alpha k}^{i_2}-S_{\alpha}^{i_2} \Gamma_{j_2 k}^{\alpha}\right)= \\
&=S_{j_2}^{i_2} \nabla_k T_{j_1}^{i_1}+T_{j_1}^{i_1} \nabla_k S_{j_2}^{i_2}
\end{aligned}$$

что и требовалось.

Докажем теперь свойство (5). Для простоты, приведем соответствующую выкладку для

случая тензорного поля T типа $(2,2)$ и свертки C_2^1. Пусть $(x^1,\ldots,x^n)$ - некоторые локальные координаты, и $T_{j_1 j_2}^{i_1 i_2}$ компоненты тензорного поля T в этих координатах. Сначала возьмем тснзорнос поле $C_2^1(T)$, компонснты которого имсют вид $T_{j_1\beta}^{\beta i_2}$ и ковариантно продифференцируем. Компоненты получающегося в результате тензорного поля типа $(1,2)$ имеют вид:

$$\nabla_k \left(C_2^1(T)\right)_{j_1}^{i_2} = \frac{\partial T_{j_1\beta}^{\beta i_2}}{\partial x^k} + T_{j_1\beta}^{\beta\alpha}\Gamma_{\alpha k}^{i_2} - T_{\alpha\beta}^{\beta i_2}\Gamma_{j_1 k}^{\alpha}$$

С другой стороны, если мы сначала ковариантно продифференцируем тензорное поле T, а затем возьмем свертку, то компоненты полученного тензора будут иметь вид:
$C_2^1(\nabla T)_{j_1 k}^{i_2} = \nabla_k T_{j_1\beta}^{\beta i_2} == \frac{\partial T_{j_1\beta}^{\beta i_2}}{\partial x^k} + T_{j_1\beta}^{\alpha i_2}\Gamma_{\alpha k}^{\beta} + T_{j_1\beta}^{\beta\alpha}\Gamma_{\alpha k}^{i_2} - T_{\alpha\beta}^{\beta i_2}\Gamma_{j_1 k}^{\alpha} - T_{j_1\alpha}^{\beta i_2}\Gamma_{\beta k}^{\alpha}$.

Легко видеть, что последнее выражение совпадает с предыдущим. В общем случае выкладка аналогична: слагаемые, соответствующие сворачиваем индексам, сокращаются. Доказательство утверждения закончено.

Замечание. Иногда полезно записать правило Лейбница для ковариантного дифференцирования, не прибегая ни к координатам, ни к производной по направлению. Отметим, однако, что соотношение $\nabla(T\otimes S) = (\nabla T)\otimes S + T\otimes\nabla S$ места не имеет, так как нижний индекс, появляющийся при дифференцировании, расположен у первого и второго слагаемых в разных положениях. Верное равенство выглядит так:

$$\nabla(T\otimes S) = {}_\pi((\nabla T)\otimes S) + T\otimes\nabla S,$$

где π - перестановка, ставящая ($q+1$)-ый индекс на последнее место (здесь T - тензор типа (p,q)).

15.8.2 8.2. "Uniqueness" of the tensor differentiation operation

Оказывается, свойства (1)-(5) из утверждения 8.1 в однозначно определяют «тензорное» дифференцирование тензорных полей любого типа на многообразии как ковариантнос диффсрснцированис в нскоторой аффинной связности. А именно, имеет место следующий результат.

Теорема 8.1. Пусть M - гладкое многообразие. Пусть на тензорных полях на M определена операция ∇, удовлетворяющая свойствам (1)(5) из утверждения 8.1:

1. Результат ∇T применения операции ∇ к пензорному полю ∇T типа (p,q) является тензорным полем типа ($p,q+1$).

2. Операция ∇ линейна.

3. Результат применения операции $\nabla\kappa$ скалярной функиии f совпадает с дифферениалом этой функиии.

4. Операция ∇ удовлетворяет правилу Лейбница:

$$\nabla(T\otimes S) = {}_\pi(\nabla T\otimes S) + T\otimes\nabla S,$$

где π - перестановка, ставящая $(q+1)$-ый индекс на последнее место (здесь T - тензор типа (p,q)).
5) Операция ∇ перестановочна с операцией свертки.

Тогда на M существует такая аффинная связность, что операция ∇ совпадает с операцией ковариантного дифференцирования относительно этой связности.

Доказательство. Нам нужно построить на многообразии M аффинную связность. Для этого мы рассмотрим произвольную систему ($x^1, \ldots, x^n$) локальных координат на M, заданную в окрестности U, возьмем вектор ∂_{x^i} из канонического базиса и продифференцируем с помощью имеющейся у нас операции ∇ векторное поле ∂_{x^i}, определенное в U. В результате у нас, по определению, получится тензорное поле типа $(1,1)$, которое в каждой точке окрестности U можно разложить по элементам $\partial_{x^\alpha} \otimes dx^\beta$ канонического базиса пространства тензоров $\mathbb{T}^1_1$. В итоге получим:

$$\nabla \partial_{x^i} = \Gamma^\alpha_{i\beta} \partial_{x^\alpha} \otimes dx^\beta$$

где через $\Gamma^\alpha_{i\beta}$ обозначены гладко зависящие от $(x^1, \ldots, x^n)$ коэффициенты этого разложения, определенные в окрестности U.

Пусть теперь в окрестности U имеются другие локальные координаты, которые мы обозначим через ($x^{1'}, \ldots, x^{n'}$). Тогда в U соотношения

$$\nabla \partial_{x^{i'}} = \Gamma^{\alpha'}_{i'\beta'} \partial_{x^{\alpha'}} \otimes dx^{\beta'}$$

определяют другую систему гладких функций $\Gamma^{\alpha'}_{i'\beta'}$. Найдем связь между $\Gamma^{\alpha'}_{i'\beta'}$ и $\Gamma^\alpha_{i\beta}$. В силу свойств (3) и (4) операции ∇, получим:

$$\nabla \partial_{x^{i'}} = \nabla \left(\frac{\partial x^i}{\partial x^{i'}} \partial_{x^i} \right) = \nabla \left(\frac{\partial x^i}{\partial x^{i'}} \right) \otimes \partial_{x^i} + \frac{\partial x^i}{\partial x^{i'}} \nabla \partial_{x^i} =$$
$$= \frac{\partial}{\partial x^k} \left(\frac{\partial x^i}{\partial x^{i'}} \right) dx^k \otimes \partial_{x^i} + \frac{\partial x^i}{\partial x^{i'}} \Gamma^\alpha_{i\beta} \partial_{x^\alpha} \otimes dx^\beta = \left(\frac{\partial}{\partial x^\beta} \left(\frac{\partial x^\alpha}{\partial x^{i'}} \right) + \frac{\partial x^i}{\partial x^{i'}} \Gamma^\alpha_{i\beta} \right) \partial_{x^\alpha} \otimes dx^\beta$$

(здесь мы воспользовались, во-первых, тем, что если второй сомножитель в тензорном произведении $T \otimes S$ не имеет нижних индексов, то перестановка π из правила Лейбница (4) - тривиальна, и, во-вторых, коммутативностью тензорного произведения вектора и ковектора: $V \otimes \xi = \xi \otimes V$ для любых $V \in \mathbb{T}^1_0$ и $\xi \in \mathbb{T}^0_1$).

С другой стороны,

$$\nabla \partial_{x^{i'}} = \Gamma^{\alpha'}_{i'\beta'} \partial_{x^{\alpha'}} \otimes dx^{\beta'} = \Gamma^{\alpha'}_{i'\beta'} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\beta'}}{\partial x^\beta} \partial_{x^\alpha} \otimes dx^\beta$$

поэтому можно приравнять коэффициенты при одинаковых базисных элементах:

$$\Gamma^{\alpha'}_{i'\beta'} \frac{\partial x^\alpha}{\partial x^{\alpha'}} \frac{\partial x^{\beta'}}{\partial x^\beta} = \frac{\partial}{\partial x^\beta} \left(\frac{\partial x^\alpha}{\partial x^{i'}} \right) + \frac{\partial x^i}{\partial x^{i'}} \Gamma^\alpha_{i\beta}.$$

Перенося матрицы Якоби в одну сторону, получим:

$$\Gamma^{\alpha'}_{i'\beta'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial}{\partial x^\beta} \left(\frac{\partial x^\alpha}{\partial x^{i'}} \right) + \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^\beta}{\partial x^{\beta'}} \frac{\partial x^i}{\partial x^{i'}} \Gamma^\alpha_{i\beta}.$$

Поэтому, окончательно,

$$\Gamma^{\alpha'}_{i'\beta'} = \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial^2 x^\alpha}{\partial x^{i'} \partial x^{\beta'}} + \frac{\partial x^{\alpha'}}{\partial x^\alpha} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^j}{\partial x^{j'}} \Gamma^\alpha_{ji}.$$

Но последнее выражение - это в точности закон преобразования символов Кристоффеля аффинной связности. Итак, доказана следующая лемма.

Лемма 8.1. В сделанных предположениях, соответствие, сопоставляющее каждым локальным координатам $(x^1, \ldots, x^n)$ набор функций $\Gamma^\alpha_{i\beta}$, определенных из соотношений

$$\nabla \partial_{x^i} = \Gamma^\alpha_{i\beta} \partial_{x^\alpha} \otimes dx^\beta,$$

задает на многообразии M некоторую аффинную связность.
Теперь, после того как мы построили по операции ∇ аффинную связность на M, доказательство утверждения сводится к проверке того, что результат применения операции ∇ к произвольному тензорному полю совпадает с результатом ковариантного дифференцирования этого тензорного поля относительно построенной аффинной связности. Это можно сделать в три этапа.

1. Случай векторного поля. Пусть T - произвольное векторное поле на M. Тогда в локальных координатах ($x^1, \ldots, x^n$) поле T представимо в виде $T^i \partial_{x^i}$. Тогда

$$\begin{aligned}\nabla T &= \nabla \left(T^i \partial_{x^i}\right) = \nabla T^i \otimes \partial_{x^i} + T^i \nabla \partial_{x^i} = \\ &= \frac{\partial T^i}{\partial x^\beta} \partial_{x^i} \otimes dx^\beta + T^i \Gamma^\alpha_{i\beta} \partial_{x^\alpha} \otimes dx^\beta = \left(\frac{\partial T^i}{\partial x^\beta} + T^k \Gamma^i_{k\beta}\right) \partial_{x^i} \otimes dx^\beta\end{aligned}$$

Но это в точности означает, что тензорное поле ∇T совпадает с тензорным полем, полученным ковариантным дифференцированием поля T в построенной нами аффинной связности Γ.
2. Случай ковекторного поля. Пусть теперь ξ - произвольное ковекторное поле. Рассмотрим на многообразии M локальные координаты ($x^1, \ldots, x^n$), и построим скалярную функцию, свернув ковектор ξ с координатным векторным полем ∂_{x^i}. Эта скалярная функция, очевидно, равна i-ой компоненте ξ_i ковектора ξ в координатах $(x^1, \ldots, x^n)$. Тогда, в силу свойства (3) операции ∇ имеем:

$$\nabla \left(C_1^1 \left(\xi \otimes \partial_{x^i}\right)\right) = \nabla \xi_i = \frac{\partial \xi_i}{\partial x^k} dx^k$$

С другой стороны, в силу свойств (4) и (5) выражение слева можно переписать так:

$$\begin{aligned}&\nabla \left(C_1^1 \left(\xi \otimes \partial_{x^i}\right)\right) = C_1^1 \left(\nabla \left(\xi \otimes \partial_{x^i}\right)\right) = C_1^1 \left(\nabla \xi \otimes \partial_{x^i}\right) + C_1^1 \left(\xi \otimes \nabla \partial_{x^i}\right) = \\ &= C_1^1 \left((\nabla \xi)_{\alpha\beta} dx^\alpha \otimes dx^\beta \otimes \partial_{x^i}\right) + C_1^1 \left(\xi_k dx^k \otimes \Gamma^\alpha_{i\beta} \partial_{x^\alpha} \otimes dx^\beta\right)\end{aligned}$$

Выполнив операцию свертки, получим:

$$\nabla \left(C_1^1 \left(\xi \otimes \partial_{x^i}\right)\right) = (\nabla \xi)_{i\beta} dx^\beta + \Gamma^\alpha_{i\beta} \xi_\alpha dx^\beta.$$

Итак,

$$\frac{\partial \xi_i}{\partial x^\beta} dx^\beta = (\nabla \xi)_{i\beta} dx^\beta + \Gamma^\alpha_{i\beta} \xi_\alpha dx^\beta$$

откуда

$$(\nabla \xi)_{i\beta} = \frac{\partial \xi_i}{\partial x^\beta} - \Gamma^\alpha_{i\beta} \xi_\alpha$$

Таким образом, тензорное поле $\nabla \xi$ совпадает с результатом ковариантного дифференцирования ковекторного поля ξ относительно построенной аффинной связности. В частности, если $\xi = dx^i$ - базисный ковектор, то

$$\nabla dx^i = -\Gamma^i_{\alpha\beta} dx^\alpha \otimes dx^\beta$$

3. Общий случай. Для завершения доказательства утверждения осталось убедиться в том, что результат применения операции ∇ к произвольному тензорному полю типа (p, q) совпадает с результатом ковариантного дифференцирования этого поля относительно построенной аффинной связности. Для этого достаточно представить тензорное поле T в карте с координатами ($x^1, \ldots, x^n$) в виде

$$T = T^{i_1 \dots i_p}_{j_1 \dots j_q} \partial_{x^{i_1}} \otimes \dots \otimes \partial_{x^{i_p}} \otimes dx^{j_1} \otimes \dots \otimes dx^{j_q},$$

и воспользоваться правилом Лейбница и уже известным нам видом операции ∇ на векторах и ковекторах. Мы оставляем соответствующую выкладку в качестве обязательного упражнения. Теорема доказана.

Замечание. Теорему 8.1 можно рассматривать как некую теорему единственности тензорного дифференцирования. Фактически, она означает, что сложные и громоздкие свойства аффинной связности на самом деле вытекают из простых естественных предположений о свойствах (1)-(5) операций тензорного дифференцирования. Попросту говоря, если хотим эти простые свойства, значит, волей-неволей получаем некоторую аффинную связность, символы Кристоффеля и т.д. Подчеркнем однако, что аффинных связностей на каждом гладком многообразии много. Поэтому в заголовке данного пункта слово «единственность» стоит в кавычках.

15.8.3 8.3. Riemannian connectivity

Аффинная связность на гладком многообразии - это некоторая дополнительная структура, которая, вообще говоря, никак не связана с другими структурами, возможно имеющимися на этом многообразии, такими например, как риманова метрика. Тем не менее, оказывается, что наличие метрики позволяет среди широкого класса аффинных связностей на многообразии выделить одну, которая, в некотором смысле, лучше всего ведет себя по отношению к метрике.

Определение. Пусть M - риманово многообразие с римановой метрикой g. Аффинная связность Γ на M называется согласованной с метрикой g, если ковариантная производная тензорного поля g относительно этой связности равна нулю: $\nabla g = 0$. Симметричная согласованная с метрикой аффинная связность называется римановой.

Теорема 8.2. Пусть M - риманово многообразие с метрикой g. Тогда на M существует единственная риманова связность. Если на M фиксированы произвольные локальные координаты $(x^1, \dots, x^n)$, то символь Кристоффеля римановой связности могут быть вычислены так:

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\alpha} \left(\frac{\partial g_{j\alpha}}{\partial x^k} + \frac{\partial g_{k\alpha}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\alpha} \right)$$

Доказательство. Докажем сначала единственность римановой связности. Для этого предположим, что Γ - риманова связность, и вычислим ее символы Кристоффеля в локальных координатах $(x^1, \dots, x^n)$, исходя из равснства $\nabla g = 0$ и симметричности связности Γ. Последнсе условие в локальных координатах переписывается в виде

$$\frac{\partial g_{ij}}{\partial x^k} = g_{\alpha j} \Gamma^\alpha_{ik} + g_{i\alpha} \Gamma^\alpha_{jk}$$

Фиксировав произвольную тройку индексов (i, j, k), запишем систему уравнений, получающуюся из предыдущего равенства циклической перестановкой индексов:

$$\frac{\partial g_{ij}}{\partial x^k} = g_{\alpha j} \Gamma^\alpha_{ik} + g_{i\alpha} \Gamma^\alpha_{jk}$$
$$\frac{\partial g_{jk}}{\partial x^i} = g_{\alpha k} \Gamma^\alpha_{ji} + g_{j\alpha} \Gamma^\alpha_{ki}$$
$$\frac{\partial g_{ki}}{\partial x^j} = g_{\alpha i} \Gamma^\alpha_{kj} + g_{k\alpha} \Gamma^\alpha_{ij}$$

Сложим первое и третье уравнения и вычтем из них второе. Тогда, так как $\Gamma^i_{jk} = \Gamma^i_{kj}$ в силу симметрии римановой связности, получим в итоге равенство

$$\frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ki}}{\partial x^j} = 2g_{i\alpha}\Gamma^\alpha_{jk}$$

справедливое для любых индексов (i, j, k). Таким образом, для каждой пары фиксированных индексов (j, k) и каждого индекса $i, 1 \leq i \leq n$, мы имеем линейное уравнение на неизвестный вектор ($\Gamma^1_{jk}, \ldots, \Gamma^n_{jk}$). Эти уравнения объединяются в систему линейных уравнений, причем матрица этой линейной системы - это невырожденная матрица ($g_{i\alpha}$) римановой метрики. Решая эту систему, т.е. умножая на матрицу ($g^{\alpha\beta}$) обратную к $(g_{i\alpha})$, получаем требуемое равенство. Итак, мы доказали,

что если риманова связность существует, то ее компоненты однозначно вычисляются в локальных координатах через компоненты метрики, поэтому риманова связность единственна.

Докажем теперь существование римановой связности. Для этого сопоставим каждой локальной системе координат $(x^1, \ldots, x^n)$ на M набор величин Γ^i_{jk}, имеющих вид

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\alpha} \left(\frac{\partial g_{j\alpha}}{\partial x^k} + \frac{\partial g_{k\alpha}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\alpha} \right).$$

Эти величины, очевидно, симметричны по нижним индексам. Поэтому, в силу транзитивности закона преобразования символов Кристоффеля, в каждой карте мы, тем самым, задали некоторую симметричную связность. Более того, каждая построенная связность является римановой, так как $\nabla g = 0$ в силу выбора символов Кристоффеля. Осталось проверить, что связности из разных карт согласованы, т.е. связаны по закону преобразования символов Кристоффеля.

Пусть $\left(x^{1'}, \ldots, x^{n'}\right)$ и $(x^1, \ldots, x^n)$ - системы локальных координат двух пересекающихся карт U' и U соответственно. Рассмотрим в пересечении $U \cap U'$ две системы функций от координат $(x^1, \ldots, x^n)$: компоненты $\left\{\Gamma^i_{jk}\right\}$ римановой связности, построенной нами в карте U, и функции

$$\Delta^i_{jk} = \Gamma^{i'}_{j'k'} \frac{\partial x^i}{\partial x^{i'}} \frac{\partial x^{j'}}{\partial x^j} \frac{\partial x^{k'}}{\partial x^k} + \frac{\partial^2 x^{j'}}{\partial x^j \partial x^k} \frac{\partial x^i}{\partial x^{j'}},$$

пересчитанные из компонент $\left\{\Gamma^{i'}_{j'k'}\right\}$ римановой связности, построенной нами в карте U', по правилам преобразования символов Кристоффеля. Но так как соотношение $\nabla g = 0$ является тензорным, оно справедливо в области $U \cap U'$ как в координатах ($x^1, \ldots, x^n$), так и в координатах $\left(x^{1'}, \ldots, x^{n'}\right)$, откуда вытекает, что функции Δ^i_{jk} являются компонентами римановой связности в координатах $(x^1, \ldots, x^n)$, и, в силу уже доказанной единственности, совпадают с Γ^i_{jk}, что и требовалось. Утверждение доказано.

Замечание. В дальнейшем, говоря о ковариантном дифференцировании на римановом многообразии мы всегда, если не оговорено противное, будем иметь ввиду ковариантное дифференцирование относительно римановой связности.

Пример. Пусть $M = \mathbb{R}^n$ и $(z^1, \ldots, z^n)$ - евклидовы координаты. Компоненты евклидовой метрики в них постоянны и равны δ_{ij}. Поэтому компоненты символов Кристоффеля соответствующей римановой связности в координатах $(z^1, \ldots, z^n)$ равны нулю. Из формул преобразования компонент символов Кристоффеля вытекает, что построенная связность совпадает с уже известной нам евклидовой связностью в $\mathbb{R}^n$.

Пример. Пусть $M^n \subset \mathbb{R}^{n|1}$ - регулярная гиперповерхность в $\mathbb{R}^{n|1}$. Для касательных векторных полей на M определим операцию ковариантного дифференцирования по i-ой координате на поверхности так:

$$\nabla_i X = \left(\frac{\partial X}{\partial u^i}\right)^T$$

где X - касательное векторное поле, и через $(\cdot)^T$ обозначена операция ортогонального проецирования на касательную плоскость к поверхности. Из материала прошлого семестра (деривационные формулы) вытекает, что эта операция является ковариантным дифференцированием относительно римановой связности, порожденной на поверхности M индуцированной метрикой (т.е. первой квадратичной формой).

В дальнейшем нам также будет полезно следующее утверждение.

Лемма 8.2. Пусть X, Y и Z - произвольные векторные поля на римановом многообразии $M, u\nabla$ - ковариантное дифференцирование относительно римановой связности на M. Тогда

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Доказательство. Действительно, если $(x^1, \ldots, x^n)$ - произвольные локальные координаты, то

$$\nabla_X \langle Y, Z\rangle = X^k \nabla_k \left(g_{ij} Y^i Z^j\right).$$

Воспользовавшись формулой Лейбница, получим

$$\nabla_X \langle Y, Z\rangle = X^k \left(\nabla_k \left(g_{ij}\right) Y^i Z^j + g_{ij} \nabla_k \left(Y^i\right) Z^j + g_{ij} Y^i \nabla_k \left(Z^j\right)\right).$$

Первое слагаемое в этой сумме равно нулю в силу согласованности связности Γ с метрикой, поэтому окончательно:

$$\nabla_X \langle Y, Z\rangle = g_{ij} \left(X^k \nabla_k Y^i\right) Z^j + g_{ij} Y^i \left(X^k \nabla_k Z^j\right) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Лемма доказана.

Из теоремы 8.2 и теоремы Уитни вытекает следующий важный результат.

Теорема 8.3. Пусть M - произвольное (компактное) гладкое многообразие. Тогда на M существует аффинная связность.

Доказательство. Действительно, с помощью теоремы Уитни многообразие M можно вложить в $\mathbb{R}^N$ для подходящего N. Тогда многообразие M можно превратить в риманово, введя на нем индуцированную из $\mathbb{R}^N$ метрику. Но теперь, воспользовавшись теоремой 8.2, можно ввести на M риманову связность, которая, конечно, является аффинной связностью. Теорема доказана.

15.8.4 8.4. Euclidean coordinates for the metric

В предыдущей лекции мы определили евклидовы координаты на многообразии в терминах аффинной связности. На римановом многообразии это можно сделать с помощью римановой метрики.

Определение. Пусть M - риманово многообразие с метрикой g. Локальные координаты ($x^1, \ldots, x^n$) на M называются евклидовыли относительно метрики g, если в этих координатах компоненты метрики g постоянны.

Ясно, что если ($x^1, \ldots, x^n$) - евклидовы координаты на M, то от них линейной заменой можно перейти к координатам, в которых компоненты метрического тензора равны δ_{ij}.

Утверждение 8.2. Система координат ($x^1, \ldots, x^n$) на римановом многообразии M с метрикой g является евклидовой относительно метрики g, если и только если она является евклидовой относительно римановой связности Γ на многообразии M.

Доказательство. Действительно, из явного вида символов Кристоффеля римановой связности вытекает, что если координаты ($x^1, \ldots, x^n$) евклидовы для метрики, то символы Кристоффеля в этих координатах тождественно равны нулю, так как компоненты метрического тензора в координатах $(x^1, \ldots, x^n)$ постоянны. Обратно, если $\Gamma^i_{jk}(x^1, \ldots, x^n) = 0$, то условие римановости связности Γ записывается в виде

$$(\nabla g)_{ijk} = \frac{\partial g_{ij}}{\partial x^k} = 0,$$

что означает постоянность компонент метрического тензора в координатах $(x^1, \ldots, x^n)$. Утверждение доказано.

15.8.5 Problems

Задача 8.1. Показать, что функции

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\alpha} \left(\frac{\partial g_{j\alpha}}{\partial x^k} + \frac{\partial g_{k\alpha}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\alpha} \right).$$

при замене координат преобразуются как символы Кристоффеля, пользуясь только тензорным законом.

Задача 8.2. Пусть M - риманово многообразие, и T - произвольное векторное поле на M. Определим скалярную функцию, называемую дивергениией $\operatorname{div} T$ поля T, положив

$$\operatorname{div} T = C^1_1 \nabla T.$$

Показать, что в произвольных локальных координатах ($x^1, \ldots, x^n$) имеет место равенство

$$\operatorname{div} T = \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g} T^i}{\partial x^i} = \frac{\partial T^i}{\partial x^i} + T^i \frac{\partial}{\partial x^i} (\ln \sqrt{g}),$$

где через g обозначен определитель матрицы римановой метрики. Сравнить это определение с определением, данным выше для векторных полей в $\mathbb{R}^3$.

15.8.6 Extra Topics

8.1. Риманова связность индуцированной метрики. Пусть W - произвольное риманово многообразие, и $\bar{\nabla}$ - операция ковариантного дифференцирования относительно римановой связности на W. Далее, пусть M - подмногообразие в W, на котором рассматривается индуцированная из W риманова метрика, и ∇ ковариантное дифференцирование относительно римановой связности на M.

Утверждение 8.3. В сделанных обозначениях, для произвольных касательных векторных полей X и Y на M имеет место равенство

$$\nabla_X Y = \left(\bar{\nabla}_X Y \right)^T$$

где через $(\cdot)^T$ обозначена операчия ортогонального проецирования пространства $T_P W$ на подпространство $T_P M$ (в метрике многообразия W).

Доказательство. Рассмотрим на многообразии M операцию тензорного дифференцирования, обладающую свойствами (1)-(5) из теоремы 8.1, которая на векторных полях определяется формулой

$$\nabla_X Y = \left(\bar{\nabla}_X Y \right)^T$$

По теореме 8.1, в результате получится операция ковариантного дифференцирования в некоторой аффинной связности. Покажем, что это - риманова связность на M.

Лемма 8.3. Предположим, что на римановом многообразии M задана такая аффинная связность Γ, что для произвольных векторных полей X, Y и Z на M выполнено

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle .$$

Тогда связность Γ согласована с метрикой, т.е. $\nabla g = 0$.
Доказательство. Действительно, по условию, для произвольных векторных полей X, Y и Z, имеем

$$X^k \nabla_k \left(g_{ij}\right) Y^i Z^j = \nabla_X \langle Y, Z\rangle - -g_{ij} \nabla_k \left(Y^i\right) Z^j - g_{ij} Y^i \nabla_k \left(Z^j\right) = 0,$$

откуда $\nabla g = 0$, что и требовалось.
Рассмотрим произвольные поля X, Y и Z на $M \subset W$, и продолжим их в малую окрестность $U \subset W$ произвольным образом. Тогда в этой окрестности

$$\bar{\nabla}_X \langle Y, Z\rangle = \left\langle \bar{\nabla}_X Y, Z\right\rangle + \left\langle Y, \bar{\nabla}_X Z\right\rangle .$$

Но $\bar{\nabla}_X \langle Y, Z\rangle = \nabla_X \langle Y, Z\rangle = X(\langle Y, Z\rangle)$. Далее, каждый вектор $A \in T_P W$ можно однозначно разложить в сумму его ортогональной проекции $(A)^T$ на подпространство $T_P M$ и нормальной составляющей, поэтому для произвольных векторов $A \in T_P W$ и $B \in T_P M$ выполнено равенство $\langle A, B\rangle = \left\langle (A)^T, B\right\rangle$. Итак,

$$\nabla_X \langle Y, Z\rangle = \left\langle \left(\bar{\nabla}_X Y\right)^T, Z\right\rangle + \left\langle Y, \left(\bar{\nabla}_X Z\right)^T\right\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

поэтому, по лемме 8.3, построенная связность согласована с метрикой. Симметричность построенной связности также вытекает из симметричности римановой связности на W. Доказательство закончено.

15.9 9: Parallel Transfer and Geodetic

В данной лекции мы рассмотрим два важных приложения аффинной связности: параллельный перенос, позволяющий сравнивать значение тензорного поля в разных точках многообразия, и геодезические - аналог геодезических на поверхностях.

15.9.1 9.1. Translation (geometry)

Во многих конкретных задачах возникает необходимость сравнивать тензоры, определенные в разных точках гладкого многообразия. Даже такое естественное выражение как «скорость движения материальной точки по кривой постоянна» означает, что мы каким-то образом сравниваем касательные векторы в разных точках кривой, т.е. элементы разных касательных пространств к многообразию. В случае евклидового пространства $\mathbb{R}^n$ мы на интуитивном уровне отождествляем все касательные пространства с самим пространством $\mathbb{R}^n$. Другими словами, мы совмещаем начала координат всех этих линейных пространств, выполняя «параллельный перенос в $\mathbb{R}^n$ », т.е. сдвиг пространства на фиксированный вектор. Возникает естественное желание обобщить понятие параллельного переноса на случай произвольного гладкого многообразия. Для этого рассмотрим сначала постоянное векторное поле T на $\mathbb{R}^n$. Его значения в разных точках пространства $\mathbb{R}^n$ одинаковы и получаются друг из друга параллельным переносом в $\mathbb{R}^n$. Ясно, что поле на R^n постоянно, если и только если его компоненты не зависят от координат, т.е. $\frac{\partial T^i}{\partial x^j} = 0$, где $(x^1, \ldots, x^n)$ - стандартные координаты в $\mathbb{R}^n$, а T^i - компоненты поля T в этих координатах. Оказывается, это

свойство можно естественно обобщить на случай многообразий с аффинной связностью, используя ковариантное дифференцирование.

Пусть M - связное гладкое многообразие с аффинной связностью Г. Пусть γ - гладкая кривая на M. Говорят, что тензорное поле T на М параллельно вдоль кривой γ относительно аффинной связности Г, если $\nabla_{\gamma'} T = 0$. Ниже, для сокращения формул, мы рассматриваем случай векторного поля, хотя все приведенные ниже результаты остаются справедливыми и в общем случае, после надлежащей переформулировки.

Пусть $(x^1, \ldots, x^n)$ - локальные координаты на M, и $x^i(t), i = 1, \ldots, n$ - координатное представление кривой γ в этих координатах. Тогда условие параллельности векторного поля T вдоль γ записывается в этих координатах так:

$$0 = (\nabla_{\gamma'} T)^i = \frac{dx^\alpha}{dt} \nabla_\alpha T^i = \frac{dx^\alpha}{dt} \left(\frac{\partial T^i}{\partial x^\alpha} + T^\beta \Gamma^i_{\beta\alpha} \right) = \frac{dT^i}{dt} + T^\beta \Gamma^i_{\beta\alpha} \frac{dx^\alpha}{dt}.$$

Итак, доказана следующая лемма.

Лемма 9.1. Векторное поле T параллельно вдоль кривой γ на многообразии М с аффинной связностью Г если и только если в любой локальной системе координат компоненты поля T, ограниченные на кривую γ, удовлетворяют следующей системе дифференциальных уравнений:

$$\frac{dT^i}{dt} + T^\beta \Gamma^i_{\beta\alpha} \frac{dx^\alpha}{dt} = 0, \quad i = 1, \ldots, n$$

Система уравнений из леммы 9.1 называется уравнениями параллельного переноса. Отметим, что это - система линейных дифференциальных уравнений первого порядка на неизвестные функции $T^i(t)$.

Определим теперь операцию параллельного переноса вектора вдоль кривой. Пусть P и Q - произвольные точки связного гладкого многообразия M с аффинной связностью Г. Рассмотрим на M произвольную гладкую кривую $\gamma : [0, 1] \to M$, такую что $\gamma(0) = P$, и $\gamma(1) = Q$, и пусть $a \in T_P M$ - произвольный касательный вектор в точке P к многообразию M. Запишем систему уравнений параллельного переноса вдоль γ (мы для простоты предполагаем, что вся кривая γ лежит в одной карте с локальными координатами ($x^1, \ldots, x^n$))

$$\frac{dT^i}{dt} + T^\beta \Gamma^i_{\beta\alpha} \frac{dx^\alpha}{dt} = 0, \quad i = 1, \ldots, n,$$

и рассмотрим для этой системы уравнений задачу Коши с начальными условиями $T^i(0) = a^i$. Как известно из теории обыкновенных дифференциальных уравнений, на отрезке $[0, 1]$ существует единственное решение $T(t)$ этой задачи Коши. В частности, определен вектор $b = T(1) \in T_Q M$.

Определение. В сделанных обозначениях, вектор b называется результатом параллельного переноса вектора а из точки P в точку Q вдоль кривой γ относительно аффинной связности Г.

Замечание. Мы определили операцию параллельного переноса по гладкой кривой, лежащей в одной карте. Однако, эта операция легко обобщается на случай кусочно гладкой кривой, лежащей в нескольких картах (всегда можно предполагать, что таких карт конечное число). А именно, нужно последовательно переносить вектор по лежащим в одной карте гладким участкам, беря в качестве следующего начального условия результат предыдущего перенесения.

Упражнение 9.1. Как зависит операция параллельного переноса от замены параметризации кривой?

Из линейности системы уравнений параллельного переноса, единственности решения задачи Коши для нее и леммы 7.6 и предыдущего упражнения вытекает следующий результат.

Утверждение 9.1. Каждая кривая γ, соединяющая точки P и Q гладкого многообразия M с аффинной связностью, определяет отображение $\mathcal{A}_\gamma$ касательного пространства T_PM в касательное пространство T_QM, переводящее вектор a из T_PM в вектор $b \in T_QM$, являющийся результатом параллельного переноса вдоль кривой γ. Отпображение $\mathcal{A}_\gamma$ линейно, невырождено и не зависит от параметризации кривой γ.

Замечание. Операция параллельного перенесения существенно зависит от кривой, вдоль которой эта операция выполняется. Это естественно для случая многообразий. Действительно, если бы например на сфере S^2 существовала бы некая естественная операция параллельного перенесения векторов, не зависящая от кривой, то можно было бы легко построить на S^2 гладкое касательное векторное поле не обращающееся в нуль ни в одной точке сферы («причесать ежика без пробора»). Последнее невозможно (см. теорему 13.3).

Упражнение 9.2. Запишите уравнения параллельного переноса для случая произвольного тензорного поля, сформулируйте и докажите аналог утверждения 9.1.

15.9.2 9.2. Parallel transfer in Riemann connectivity

Параллельный перенос в римановой связности обладает рядом интересных дополнительных свойств.

Утверждение 9.2. Пусть M - риманово многообразие с соответствующей римановой связностью, и $\gamma(t)$ - некоторая гладкая кривая, соединяющая пару точек из M. Тогда операция параллельного переноса вдоль γ сохраняет скалярное произведение векторов: если $a(t)$ и $b(t)$ параллельные вдоль γ векторные поля на M, то $\langle a(t), b(t)\rangle = \mathrm{const}$.

Доказательство. Рассмотрим функцию $f(t) = \langle a(t), b(t)\rangle$, определенную на кривой γ. Тогда, в силу леммы 8.2 и параллельности полей a и b имеем:

$$\frac{df}{dt} = \nabla_{\gamma'}\langle a, b\rangle = \langle \nabla_{\gamma'} a, b\rangle + \langle a, \nabla_{\gamma'} b\rangle = 0,$$

что и требовалось доказать.

Следствие 9.1. Пусть γ - гладкая кривая, соединяющая пару точки P и Q на римановом многообразии M. Тогда линейное отображение $\mathcal{A}_\gamma : T_PM \to T_QM$, задаваемое операцией параллельного переноса вдоль γ относительно римановой связности, является ортогональным, т.е. сохраняет скалярное произведение, заданное в касательных пространствах римановой метрикой.

Для осуществления параллельного переноса часто полезным оказывается следующее очевидное утверждение.

Утверждение 9.3. Изометрии переводят параллельные векторные поля в параллельные. А именно, пусть $F : M \to N$ - изометрия римановых многообразий, γ - кривая на M, соединяющая точки P и $Q, u\tilde{\gamma} = F \circ \gamma$ - образ кривой γ на многообразии N. Если T - параллельное вдоль γ векторное поле на M, то $dF(T)$ - параллельное вдоль $\tilde{\gamma}$ векторное поле на N.

Доказательство. Это очевидно, так как изометрия не меняет метрический тензор, и, следовательно, не меняет уравнения параллельного переноса.

15.9.3 9.3. Definition and simplest properties of geodetic

Прямые в евклидовом пространстве и геодезических на поверхностях, рассмотренные нами в прошлом семестре, обладают следующим общим свойством: (ковариантная)

производная вектора скорости такой кривой вдоль нее самой равна нулю. Это свойство легко переносится на случай многообразий с аффинной связностью.

Определение. Пусть M - гладкое многообразие с аффинной связностью Γ, и ∇ - соответствующее ковариантное дифференцирование. Гладкая кривая γ называется геодезической на M в связности Γ, если во всех точках кривой γ выполнено следующее равенство: $\nabla_{\gamma'}\gamma' = 0$, т.е. поле скорости γ' кривой γ параллельно вдоль самой кривой γ относительно связности Γ.

Пусть $(x^1, \ldots, x^n)$ - локальные координаты на многообразии M, и $x^i(t), i = 1, \ldots, n$, - координатное представление кривой γ. Тогда, условие $\nabla_{\gamma'}\gamma' = 0$ можно записать так:

$$(\nabla_{\gamma'}\gamma')^i = \frac{d^2x^i}{dt^2} + \Gamma^i_{\alpha\beta}\frac{dx^\alpha}{dt}\frac{dx^\beta}{dt} = 0, \quad i = 1, \ldots, n$$

(сравните с уравнениями параллельного переноса). Полученная система уравнений называется уравнениями геодезических в связности Γ. Отметим, что точно такая же система уравнсний уже выписывалась нами в случае поверхностей. В частности, мы уже знаем следующее свойство решений этой системы.

Утверждение 9.4. Пусть $\gamma(t)$ - геодезическая, $ut = t(s)$ - замена параметра кривой γ. Кривая $\tilde{\gamma}(s) = \gamma(t(s))$ снова будет геодезической, если и только если эта замена линейна, т.е. $t(s) = as + b$, дде $a, b \in \mathbb{R}$.

Далее, как и в случае поверхностей, уравнения геодезических представляют собой систему обыкновенных дифференциальных уравнений

второго порядка. Поэтому для них справедлива стандартная теорема существования и единственности решения задачи Коши. Из этой теоремы немедленно вытекает теорема существования и единственности геодезической на многообразии.

Следствие 9.2. Пусть M - гладкое многообразие с аффинной связностью, P - произвольная точка из $M, uV \in T_PM$ - произвольный касательный вектор κM в точке P. Тогда на M существует геодезическая γ, выходящая из P и такая, что ее вектор скорости в точке P равен V. Более того, такая геодезическая γ единственна в том смысле, что любые две такие геодезических совпадают в пересечении областей определения.

Пусть теперь M - риманово многообразие с метрикой g, и Γ - риманова связность на M. Геодезические в римановой связности обладают рядом интересных свойств, аналогичных свойствам геодезических на поверхностях. Когда говорят о геодезических на римановом многообразии, то, обычно, имеют ввиду геодезические относительно римановой связности.

Следующие свойства геодезических на римановых многообразиях доказывается точно также, как соответствующие свойства геодезических на поверхностях.

Утверждение 9.5. Пусть γ - геодезическая на римановом многообразии M. Тогда модуль ее вектора скорости γ' не зависит от t. Если $F : M \to N$ - изометрия римановых многообразий, то F переводит геодезические в геодезические.

Следующий результат позволяет иногда находить геодезические не решая дифференциальных уравнений.

Следствие 9.3. Пусть существует такая изометрия $\nu : M \to Mpu$ манова многообразия M, что множество ее неподвижных точек представляет собой регулярную кривую γ. Тогда γ - геодезическая (после подходящей перепараметризации).

Доказательство. Действительно, пусть $P = \gamma(t_0)$ - произвольная точка из γ, и $\xi = \dot{\gamma}(t_0) \neq 0$ - вектор скорости кривой γ в точке P. Рассмотрим геодезическую $\gamma_1(s)$, такую что $\gamma_1(s_0) = P$ и $\dot{\gamma}_1(s_0) = \xi$. Так как γ - множество неподвижных точек изометрии ν, то изометрия сохраняет точку P и вектор ξ. Поэтому образ $\gamma_2 = \nu(\gamma_1)$ геодезической γ_1 - это геодезическая, проходящая через ту же точку P с тем же вектором скорости ξ. Теорема существования и единственности дает $\gamma_2 = \gamma_1$ (в некоторой окрестности параметра t_0), откуда образ γ_1 принадлежит образу γ. Поэтому в окрестности каждой точки кривая

γ, после соответствующей перепараметризации, является геодезической. Доказательство закончено.

Пример. Проверим с помощью следствия 9.3, что на плоскости Лобачевского геодезическими являются прямые Лобачевского и только они. Рассмотрим модель Пуанкаре. Напомним, что для каждой прямой ℓ на плоскости Лобачевского существует изометрия φ_ℓ, переводящая ее в прямую $y = 0$. Преобразование $\nu : z = x + iy \mapsto \bar{z} = x - iy$ является изометрией, и прямая $y = 0$ - это множество всех неподвижных точек изометрии ν. Композиция $\varphi_\ell^{-1} \circ \nu \circ \varphi_\ell$ это изометрия, множество неподвижных точек которой совпадает с прямая ℓ, что и требовалось.

Аналогично проверяется, что геодезические на сфере S^n - это большие круги (сечения двумерными плоскостями, проходящими через центр) и только они (укажите соответствующие изометрии).

Параллельный перенос вдоль геодезической на римановом многообразии обладает следующим свойством, особенно полезным в двумерном случае.

Утверждение 9.6. Если a - параллельное векторное поле вдоль γ, то угол между векторами $a(t)$ и $\gamma'(t)$ не зависит от t.

Доказательство. Покажем, что функция $f(t) = \langle a(t), \gamma'(t)\rangle$, определенная вдоль кривой γ, постоянна. Действительно,

$$\frac{df}{dt} = \nabla_{\gamma'} \langle a(t), \gamma'(t)\rangle = \langle \nabla_{\gamma'} a(t), \gamma'(t)\rangle + \langle a(t), \nabla_{\gamma'}\gamma'(t)\rangle = 0,$$

где первое слагаемое равно нулю по определению параллельного векторного поля, а второе - по определению геодезической. Кроме того, модули векторных полей $a(t)$ и $\gamma'(t)$ постоянны, откуда и вытекает требуемое. Доказательство закончено.

Следствие 9.4. Пусть M^2 - ориентируемое риманово двумерное многообразие, и γ - геодезическая на M, соединяющая точки P и Q. Тогда вектор $b \in T_Q M$ является результатом параллельного переноса вектора $a \in T_P M$ вдоль γ, если и только если $\|a\| = \|b\|$, угол между a $u\gamma'(P)$ равен углу между b и $\gamma'(Q)$, и ориентации базисов $(\gamma'(P), a)\, u\, (\gamma'(Q), b)$ согласованы.

15.9.4 9.4. Normal coordinates

На произвольном многообразии M с аффинной связностью с помощью геодезических можно определить так называемые нормальные координаты, которые оказываются удобными для многих вычислений. Пусть P - произвольная точка многообразия M, и V - произвольный вектор из касательного пространства $T_P M$ к M в точке P. Тогда, в силу следствия 9.2, существует и единственна геодезическая $\gamma_V(s)$, удовлетворяющая начальным условиям $\gamma_V(0) = P$ и $\gamma_V'(0) = V$ (здесь штрихом обозначено дифференцирование по s).

Определим экспонениальное отображение $\exp_P$ некоторой окрестности нулевого вектора касательного пространстве $T_P M$ в многообразие M так:

$$\exp_P : V \mapsto \gamma_V(1)$$

Ясно, что отображение $\exp_P$ - гладкое в области определения (в силу теоремы о гладкой зависимости решения задачи Коши от начальных условий). Оказывается, отображение $\exp_P$ является также регулярным в некоторой окрестности нуля.

Утверждение 9.7. Пусть P - произвольная точка многообразия Mc аффинной связностью. Тогда отображение $\exp_P$ задает локальные координаты в некоторой окрестности точки P. Другими словами, отображение $\exp_P$ гладко и взаимно однозначно

отображаст нскоторую окрестность нуля из T_PM на окрестность точки P в M, причем матрица Якоби этого отображения невырождена.

Доказательство. Для доказательства утверждения достаточно проверить, что матрица Якоби отображения $\exp_P$ невырождена в точке 0 из T_PM. Действительно, если это так, то матрица Якоби невырождена и в некоторой окрестности нуля, и тогда утверждение вытекает из теоремы о неявной функции.

Итак, по определению, $\exp_P(V) = \gamma_V(1)$. Фиксируем в окрестности точки P какие-нибудь координаты $(x^1, \ldots, x^n)$, и пусть $x^i(s, V)$ координатное представление геодезической $\gamma_V(s)$. Если мы фиксируем какой-нибудь базис в T_PM, и в этом базисе $V = (v^1, \ldots, v^n)$, то координатное представление отображения $\exp_P$ в координатах $(v^1, \ldots, v^n)$ на T_PM и ($x^1, \ldots, x^n$) на M имеет, очевидно, вид

$$x^i = x^i(1, V) = x^i\left(1, v^1, \ldots, v^n\right)$$

Нам понадобится следующая лемма.

Лемма 9.2. Пусть t - произвольное достаточно малое вещественное число. Тогда

$$\gamma_{tV}(s) = \gamma_V(ts)$$

в общей области определения.

Доказательство. При фиксированном t, вектор скорости геодезической $\gamma_V(ts)$ в точке $P = \gamma_V(0)$ имеет вид

$$\left.\frac{d\gamma_V(ts)}{ds}\right|_{s=0} = \dot{\gamma}_V(0)t = tV$$

поэтому геодезическая $\gamma_V(ts)$ удовлетворяет тем же начальным условиям, что и геодезическая $\gamma_{tV}(s)$. Теперь утверждение леммы вытекает из следствия 9.2.

Вернемся к доказательству утверждения. Поскольку, в силу леммы 9.2, $\gamma_{tV}(s) = \gamma_V(ts)$, в координатах имеем

$$x^i(s, tV) = x^i(st, V), \tag{*}$$

Продифференцируем соотношение (*) по t и положим $t = 0$. Дифференцируя левую часть, получим:

$$\left.\frac{\partial x^i(s, tV)}{\partial t}\right|_{t=0} = \left.\frac{\partial x^i(s, tV)}{\partial v^j}\right|_{t=0} v^j = \frac{\partial x^i(s, 0)}{\partial v^j} v^j.$$

С другой стороны, дифференцируя правую часть соотношения $(*)$, имеем:

$$\left.\frac{\partial x^i(st, V)}{\partial t}\right|_{t=0} = \left.\frac{\partial x^i(st, V)}{\partial s}\right|_{t=0} s = \frac{\partial x^i(0, V)}{\partial s} s = s\left(\dot{\gamma}_V(0)\right)^i = sv^i.$$

Приравняв полученные выражения и положив в них $s = 1$, получим окончательно:

$$\frac{\partial x^i(1, 0)}{\partial v^j} v^j = v^i$$

для произвольного вектора V. Поэтому матрица Якоби $\left(\frac{\partial x^i(1,0)}{\partial v^j}\right)$ отображения $\exp_P$ в точке $V = 0$ равна единичной матрице и, в частности, невырождена. Утверждение доказано.

Определение. Пусть в касательном пространстве T_PM к многообразию M с аффинной связностью Γ фиксирован произвольный базис (e_i), и $(v^1, \ldots, v^n)$ линейные координаты

в T_PM, соответствующие этому базису. Локальные координаты в окрестности точки P на M, порожденные отображением

$$V = \left(v^1, \ldots, v^n\right) \mapsto \exp_P(V)$$

называются нормальньми координатами с центром в точке P, порожденными базисом (e_i). Сама окрестность точки P, в которой заданы нормальные координаты, а также соответствующая окрестность точки 0 в T_PM, называется нормальной окрестностью. Геодезические γ_V называют радиальньлми. Если многообразие M и связность Γ римановы, то базис (e_i), обычно, выбирают ортонормированным.

Из взаимной однозначности отображения $\exp_P$ в некоторой окрестности точки P вытекает следующий результат.

Следствие 9.5. Любые две достаточно близкие точки многообразия аффинной связностью можно соединить геодезической.

Упражнение 9.3. Пусть $\gamma_V(t), t \in [0,1]$, - радиальная геодезическая на римановом многообразии. Рассмотрим отрезок γ этой геодезической, соответствующий значениям параметра $t \in [t_1, t_2] \subset [0,1]$. Вычислите длину кривой γ.

15.9.5 Problems

Задача 9.1. Показать, что если в некоторой связности Γ, заданной на римановом многообразии M, параллельный перенос вдоль любой кривой сохраняет скалярные произведения векторов, то связность Γ - риманова.

Задача 9.2. Описать геодезические на прямом цилиндре, круговом конусе, стандартной сфере, плоскости Лобачевского.

Задача 9.3. Проверить, что если гиперповерхность $M \subset \mathbb{R}^n$ содержит некоторую прямую $\ell \subset \mathbb{R}^n$, то, после перепараметризации, прямая $\ell \subset M$ является геодезической на M.

Задача 9.4. Построить нормальные координаты на стандартной двумерной сфере. Как устросна максимальная нормальная окрестность точки сфсры?

Задача 9.5. Пусть M - риманово многообразие, g - риманова метрика на нем, и P - произвольная его точка. Фиксируем в касательном пространстве T_PM ортонормированный базис, и пусть ($y^1, \ldots, y^n$) - соответствующие нормальные координаты в окрестности точки P на M. Показать, что в этих координатах $g_{ij}(P) = \delta_{ij}$ и $\Gamma^i_{jk}(P) = 0$.

Задача 9.6. Пусть $SL(n) \subset \mathbb{R}^{n^2}$ - множество вещественных ($n \times n$) матриц с единичным определителем. Описать геодезические на $SL(n)$, проходящие через точку $E \in SL(n)$, где E - единичная матрица. Тот же вопрос для $SO(n)$.

15.9.6 Extra Topics

9.1. Пример параллельного переноса на плоскости Лобачевского. Рассмотрим модель верхней полуплоскости плоскости Лобачевского L^2, на которой в стандартных координатах (x, y) задана метрика Лобачевского $ds^2 = \left(dx^2 + dy^2\right)/y^2$. Пусть γ - гладкая кривая, заданную параметрически так: $x(t) = t, y(t) = y_0 > 0$ (это горизонтальная прямая). Выпишем уравнения параллельного переноса вдоль кривой γ в римановой связности. Символы Кристоффеля римановой связности в нашем случае, как легко сосчитать, имеют вид:

$$\Gamma^1_{12} = \Gamma^1_{21} = \Gamma^2_{22} = -\Gamma^2_{11} = -\frac{1}{y}, \quad \Gamma^2_{12} = \Gamma^2_{12} = \Gamma^1_{11} = \Gamma^1_{22} = 0.$$

Поэтому уравнения параллельного переноса на компоненты (T^1, T^2) поля T имеют вид:

$$\begin{cases} \frac{dT^1}{dt} + T^\alpha \Gamma^1_{\alpha\beta} \frac{dx^\beta}{dt} = \frac{dT^1}{dt} - \frac{1}{y}\left(T^1 \frac{dy}{dt} + T^2 \frac{dx}{dt}\right) = 0, \\ \frac{dT^2}{dt} + T^\alpha \Gamma^2_{\alpha\beta} \frac{dx^\beta}{dt} = \frac{dT^2}{dt} + \frac{1}{y}\left(T^1 \frac{dx}{dt} - T^2 \frac{dy}{dt}\right) = 0. \end{cases}$$

После подстановки конкретного вида кривой $\gamma(t) = \{t, y_0\}$, получим:

$$\begin{cases} \frac{dT^1}{dt} = \frac{1}{y_0} T^2, \\ \frac{dT^2}{dt} = -\frac{1}{y_0} T^1. \end{cases}$$

Эта система липейшых дифферешциальшых урависпий легко решастся. Ответ можно записать в следующем виде:

$$T^1(t) = A\cos(t/y_0) + B\sin(t/y_0), \quad T^2(t) = -A\sin(t/y_0) + B\cos(t/y_0).$$

Таким образом, параллельное вдоль кривой γ векторное поле равномерно вращается, причем тем быстрее, чем меньше y_0, т.е. чем горизонтальная прямая γ расположена «ближе» к абсолюту.

9.2. Еще примеры параллельного переноса. Следующие несколько примеров демонстрируют стандартные методы, часто позволяющие вычислять результат параллельного переноса, не решая дифференциальные уравнения.

Пусть M - поверхность кругового конуса, заданная в $\mathbb{R}^3$ уравнением $x^2 + y^2 = z^2$, где $z > 0$. Обозначим через γ плоское сечение конуса M плоскостью $z = h > 0$. Пусть V - касательный вектор к конусу в точке $P \in \gamma$, направленный к его вершине. На какой угол повернется вектор V при параллельном переносе по кривой γ от P до P ?

Рис. 3. Развертка конуса на плоскость и параллельный перенос

Чтобы выяснить это, воспользуемся утверждением 9.3. Конус M, после разрезания по лучу OP, см. рис. 3, изометричен углу POP на евклидовой плоскости. На евклидовой плоскости параллельный перенос не меняет компоненты вектора, поэтому вдоль всего пути γ мы получаем вектор, параллельный исходному вектору V в обычном (евклидовом) смысле. В результате, обойдя вокруг конуса по окружности γ мы получим вектор V', который, как видно из рисунка, составляет с вектором V угол, равный по величине углу POP на плоскости.

Имеет место следующее полезное утверждение.

Утверждение 9.8. Пусть две поверхности $M_1 u M_2$ в $\mathbb{R}^3$ касаются по кривой γ (т.е., напомним, в точках кривой у них совпадают касательные плоскости). Тогда результат параллельного переноса вектора вдоль кривой γ не зависит от выбора поверхности.

Доказательство. Так как ковариантная производная $\nabla_{\gamma'(t)} T$ поля T вдоль кривой γ на поверхности $M_i, i = 1, 2$, совпадает с результатом проецирования на касательную

плоскость $T_{\gamma(t)}M_i$ вектора $T'(t)$, а касательные плоскости $T_{\gamma(t)}M_1$ и $T_{\gamma(t)}M_2$ совпадают, то совпадают и уравнения параллельного переноса вдоль γ на поверхностях M_1 и M_2, что и требовалось.

Упражнение 9.4. Пусть γ - плоское сечение двумерной сферы, заданной в $\mathbb{R}^3$ уравнением $x^2+y^2+z^2=1$, плоскостью $z=h$, где $0 \leq h < 1$. Пусть a - касательный вектор к сфере в точке $P \in \gamma$, направленный вдоль меридиана к северному полюсу. На какой угол повернется вектор a при параллельном переносе по кривой γ от P до P ? Указание: рассмотреть конус, касающийся сферы вдоль параллели γ.

Упражнение 9.5. Пусть M - поверхность вращения графика положительной функции $y=f(x)$ вокруг оси Ox, и пусть γ - сечение поверхности M ортогональной оси вращения плоскостью $x=x_0$. Пусть a - касательный вектор к M в точке $P \in \gamma$, направленный вдоль меридиана. На какой угол повернется вектор a при параллельном переносе по кривой γ от P до P ?

9.3. Подмногообразия риманова многообразия. Пусть W - риманово многообразие, и M - подмногообразие в W с индуцированной из W метрикой. Напомним,

что риманова связность ∇ на поверхности M, согласованная с индуцированной метрикой, может быть получена из римановой связности $\bar{\nabla}$, согласованной с метрикой объемлющего многообразия W, так:

$$\nabla_X Y = \left(\bar{\nabla}_X Y\right)^T$$

где $(\cdot)^T$ - ортогональное (в метрике на W) проецирование касательного пространства $T_P W$ на его подпространство $T_P M \subset T_P W$, являющееся касательным пространством к подмногообразию M. Поэтому уравнение геодезической γ на M имеет вид:

$$\left(\bar{\nabla}_{\dot{\gamma}} \dot{\gamma}\right)^T = 0.$$

Итак, мы получаем эквивалентное определение геодезической на подмногообразии: кривая γ на подмногообразии M риманова многообразия W является геодезической на M в индуцированной метрике, если и только если ее вектор ковариантного ускорения как кривой в W (т.е. относительно ковариантного дифференцирования в объемлющем многообразии) перпендикулярен подмногообразию M в каждой точке.

Следствие 9.6. Если γ - геодезическая на римановом многообразии W лежит в подмногообразии $M \subset W$, то γ - геодезическая на M в индучированной метрике.

Следствие 9.7. Если два подмногообразия риманова многообразия касаются вдоль кривой γ, являющейся геодезической на одном из подмногообразий (в индучированной метрике), то γ также является геодезической и на другом подмногообразии (в его индучированной метрике).

9.4. Геодезические на плоском торе. Напомним, что тор можно получить из параллелограмма склейкой противоположных сторон «без переворота». Фиксируем параллелограмм П, совместим одну из его вершин с началом координат на плоскости, и пусть e_1 и e_2 - векторы, соответствующие сторонам параллелограмма, выходящим из этой вершипы. Эти вскторы задают действие группы $\mathbb{Z}^2$ па плоскости $\mathbb{R}^2$ сдвигами на векторы $n_1 e_1 + n_2 e_2, (n_1, n_2) \in \mathbb{Z}^2$. Заметим, что при таком действии параллельные стороны параллелограмма П переходят друг в друга «без переворота», поэтому факторпространство $\mathbb{R}^2/\mathbb{Z}^2$ естественно отождествляется с тором. Обозначим через π естественную проекцию $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\mathbb{Z}^2 = T^2$. Эта проекция, очевидно, является локальным диффеоморфизмом. Если x - произвольная точка тора T^2, и p - произвольная точка из ее прообраза, т.е. $\pi(p)=x$, то весь прообраз $\pi^{-1}(x)$ представляет собой множество

$$\left\{n_1 e_1 + n_2 e_2 + p \mid (n_1, n_2) \in \mathbb{Z}^2\right\}.$$

Болес того, прообраз малой окрестности U точки x состоит из попарно нспересекающихся окрестностей точек прообраза, причем любые две таких окрестности отличаются на сдвиг на вектор вида $n_1e_1 + n_2e_2$. Поэтому на торе можно ввести риманову метрику, отождествив окрестность U с любой из окрестностей прообраза. Эта метрика в подходящих координатах имеет евклидов вид и называется плоской метрикой на торе. Отметим, что разные параллелограммы задают на торе, вообще говоря, разные, т.е. не изометричные, плоские метрики.

Упражнение 9.6. Какие параллелограммы задают изометричные плоские метрики на торе? Описать пространство всевозможных попарно неизометричных плоских метрик на торе.

Ясно, что проекция $\pi : \mathbb{R}^2 \to T^2$ является локальной изометрией. Поэтому геодезические па плоском торе T^2 - это в точности те кривые, в которые просцируются прямые на плоскости. Чтобы понять, как выглядит соответствующая кривая на торе, рассмотрим разбиение $\mathcal{P}$ плоскости на параллелограммы, полученные сдвигами исходного параллелограмма на всевозможные векторы вида $n_1e_1 + n_2e_2$. Каждый

параллелограмм разбиения $\mathcal{P}$ проецируется на тор. Пусть $\ell \subset \mathbb{R}^2$ - прямая на плоскости. Рассмотрим всевозможные параллелограммы из $\mathcal{P}$, через которые проходит прямая ℓ, и совместим их все с исходным параллелограммом П. В результате в параллелограмме получится набор отрезков, параллельных прямой ℓ. При этом, если направление прямой ℓ может быть задано как целочисленная линейная комбинация векторов e_1 и e_2, то различных отрезков будет конечное число, и мы получим замкнутую геодезическую на торе T^2. В противном случае, все отрезки будут попарно различны, и соответствующая незамкнутая геодезическая на торе будет образовывать его всюду плотное подмножество. В частности, произвольная геодезическая на плоском торе вернется в сколь угодно малую окрестность своей начальной точки (через достаточно большое время).

Упражнение 9.7. Пусть P и Q - произвольные точки плоского тора T^2. Показать, что их соединяет бесконечно много попарно неизометричных геодезических.

9.5. Геодезические на матричных группах. Матричные группы представляют собой важный класс римановых многообразий с симметриями. В качестве базового примера возьмем группу ортогональных матриц размера ($n \times n$), т.е. множество матриц

$$\mathrm{O}(n) = \left\{ A \in \mathbb{R}^{n^2} \mid AA^T = E \right\}$$

рассматриваемое как подмножество пространства всех матриц $\mathbb{R}^{n^2}$, в котором фиксировано стандартное скалярное произведение. Заметим, что если $A = (a_{ij})$ и $B = (b_{ij})$ - произвольные матрицы, заданные своими компонентами, то, рассматривая эти матрицы как векторы из $\mathbb{R}^{n^2}$, их скалярное произведение можно записать в виде

$$\langle A, B \rangle = \sum_{i,j=1}^{n} a_{ij} b_{ij} = \operatorname{tr} AB^T.$$

Множество $M = \mathrm{O}(n)$ задается регулярной системой из $\frac{n(n+1)}{2}$ уравнений и является поэтому $\frac{n(n-1)}{2}$-мерным гладким подмногообразием в $\mathbb{R}^{n^2}$, на котором индуцируется метрика из объемлющего евклидова пространства. Отметим, что сопряжение с любой ортогональной матрицей G, т.е. отображение $f_G : \mathbb{R}^{n^2} \to \mathbb{R}^{n^2}$, $f_G(A) = GAG^{-1}$, переводит многообразие M в себя (что очевидно) и сохраняет скалярное произведение, т.к.

$$\begin{aligned}\langle f_G(A), f_G(B) \rangle &= \operatorname{tr} f_G(A) f_G(B)^T = \operatorname{tr} GAG^{-1} \left(GBG^{-1}\right)^T = \\ &= \operatorname{tr} GAG^{-1} \left(G^{-1}\right)^T B^T G^T = \operatorname{tr} GAB^T G^{-1} = \operatorname{tr} AB^T = \langle A, B \rangle\end{aligned}$$

где мы воспользовались определением ортогональной матрицы и тензорностью следа. Поэтому f_G порождает изометрию многообразия M.

Напомним, что касательное пространство $T_E M$ совпадает с множеством кососимметрических матриц. Рассмотрим в $\mathbb{R}^{n^2}$ кривую $\gamma_X(t) = \exp(tX)$, где $X \in T_E M$ и $\exp(tX)$ - обычная экспонента от матрицы, и убедимся, что γ_X - геодезическая на M (в подходящей параметризации). Проверим сначала, что $\gamma_X(t) \subset M$. Для этого покажем, что $\gamma_X(t)\gamma_X(t)^T = E$. Положим $A(t) = \gamma_X(t)\gamma_X(t)^T$ и вычислим производную $\dot{A}(t)$. Имеем:

$$\dot{A}(t) = \dot{\gamma}_X(t)\gamma_X(t)^T + \gamma_X(t)\dot{\gamma}_X(t)^T = X\gamma_X(t)\gamma_X(t)^T + \gamma_X(t)\left(X\gamma_X(t)\right)^T = 0$$

в силу косой симметрии матрицы X и перестановочности матриц X и $\exp(tX)$, откуда $A(t) = A(0) = E$. Итак, кривая $\gamma_X(t)$ лежит в M. Осталось заметить, что эта кривая состоит из неподвижных точек изометрии $f_{\exp X}$, у которой нет других неподвижных точек в окрестности E, и воспользоваться следствием 9.3.

15.10 10. Extreme properties of geodetic

Геодезические на римановых многообразиях обладают экстремальными свойствами, а именно, они представляют собой локально кратчайшие кривые: каждый достаточно мальий отрезок геодезической является кратчайшей сладкой кривой, соединяющей свои концевые точки. Чтобы убедиться в этом, мы начнем с изучения свойств функции длины кривых на римановом многообразии.

15.10.1 10.1. Derivative of the curve length function when it varies

Пусть M - риманово многообразие. Рассмотрим на M произвольную регулярную кривую $\gamma : [t_1, t_2] \to M$, и обозначим через $P = \gamma(t_1)$ и $Q = \gamma(t_2)$ ее концевые точки. Далее, пусть G - некоторое подмножество плоскости $\mathbb{R}^2$ с координатами (τ, t), ограниченное прямыми $\tau = \pm\tau_0$ и двумя кривыми $t = t_1(\tau)$ и $t = t_2(\tau)$, где $t_1(\tau) < t_2(\tau)$ для любого $\tau \in [-\tau_0, \tau_0]$, и $t_1(0) = t_1$, а $t_2(0) = t_2$, см. рис. 4. Гладкое отображение ф области G в многообразие M называется вариацией кривой γ, если $\Phi(0,t) = \gamma(t)$. Отображение Φ задает однопараметрическое семейство кривых $\gamma_\tau(t) = \Phi(\tau, t), t \in [t_1(\tau), t_2(\tau)]$, где τ - параметр семейства, при этом $\gamma_0(t) = \gamma(t)$. Кривые $p(\tau) = \Phi(\tau, t_1(\tau))$ и $q(\tau) = \Phi(\tau, t_2(\tau))$, по которым движутся концевые точки кривых γ_τ, называются концевьлми кривыми. Если $p(\tau) = P$ и $q(\tau) = Q$ для любого τ, то отображение Φ называется вариацией с закрепленными концами.

В каждой точке $\gamma(t)$ кривой γ определим вектор

$$E(t) = d\Phi|_{(0,t)}\left(\frac{\partial}{\partial\tau}\right).$$

Соответствующее векторное поле, определенное вдоль кривой γ, называется полем вариации Φ. Поле вариации состоит из векторов скоростей

кривых $\nu_t(\tau)$ при $\tau = 0$, задаваемых отображением Φ так: $\nu_t(\tau) = \Phi(\tau, t)$. Отметим, что в концевых точках поле вариации вовсе не совпадает с векторами скоростей концевых кривых. А именно, имеет место следующая лемма (здесь и ниже точкой обозначается дифференцирование по t, а штрихом - дифференцирование по τ).

Лемма 10.1. Вектора скоростей концевых кривых в начальный момент $\tau = 0$ удовлетворяют следующим соотношениям:

$$p'(0) = t_1'(0)\dot{\gamma}(t_1) + E(t_1), \quad q'(0) = t_2'(0)\dot{\gamma}(t_2) + E(t_2).$$

Рис. 4. Область определения вариации Φ

Доказательство. Проверим, например, первое соотношение. По определению имеем:

$$p'(\tau) = \frac{d\Phi\left(\tau, t_1(\tau)\right)}{d\tau} = \frac{\partial\Phi\left(\tau, t_1(\tau)\right)}{\partial\tau} + \frac{\partial\Phi\left(\tau, t_1(\tau)\right)}{\partial t}\frac{dt_1(\tau)}{d\tau}$$

Подставив значение $\tau = 0$ и вспомнив, что $t_1(0) = t_1$, получим:

$$p'(0) = \frac{\partial\Phi\left(0, t_1(0)\right)}{\partial\tau} + \frac{\partial\Phi\left(0, t_1(0)\right)}{\partial t}t_1'(0) = E\left(t_1\right) + \dot{\gamma}\left(t_1\right)t_1'(0).$$

Доказательство закончено.

Рассмотрим функцию $\ell(\tau)$, определенную на отрезке $[-\tau_0, \tau_0]$ и равпуіо длиюе кривой γ_τ. Вычислим производлую этой фупкции в $\tau = 0$. Дополнительно предположим для простоты, что кривая $\gamma(t)$ натурально параметризована. Запишем функцию $\ell(\tau)$ в виде

$$\ell(\tau) = \int_{t_1(\tau)}^{t_2(\tau)} \|\dot{\gamma}_\tau\|\, dt$$

и воспользуемся теоремой о дифференцировании интеграла, зависящего от параметра. Получим:

$$\frac{d\ell(\tau)}{d\tau} = t_2'(\tau)\left\|\dot{\gamma}_\tau\left(t_2(\tau)\right)\right\| - t_1'(\tau)\left\|\dot{\gamma}_\tau\left(t_1(\tau)\right)\right\| + \int_{t_1(\tau)}^{t_2(\tau)} \|\dot{\gamma}_\tau(t)\|'\, dt$$

Вычислим производную $\|\dot{\gamma}_\tau(t)\|'$. Для этого нам понадобится следующая лемма.

Лемма 10.2. В сделанных выше обозначениях, имеет место следующее равенство:

$$\nabla_\tau\dot{\gamma}_\tau(t) = \nabla_t\nu_t'(\tau)$$

Доказательство. Пусть на многообразии фиксированы локальные координаты ($x^1, \ldots, x^n$). Обозначим через $x^i(\tau, t)$ координатное представление вариации Φ. Тогда k-ая компонента поля $\nabla_\tau\dot{\gamma}_\tau(t)$ имеет вид:

$$\frac{\partial x^\alpha}{\partial\tau}\nabla_\alpha\dot{x}^k = \frac{\partial x^\alpha}{\partial\tau}\left(\frac{\partial\dot{x}^k}{\partial x^\alpha} + \Gamma^k_{\beta\alpha}\dot{x}^\beta\right) = \frac{\partial}{\partial x^\alpha}\left(\frac{\partial x^k}{\partial t}\right)\frac{\partial x^\alpha}{\partial\tau} + \Gamma^k_{\beta\alpha}\dot{x}^\beta\left(x^\alpha\right)' =$$
$$= \frac{\partial^2 x^k}{\partial t\partial\tau} + \Gamma^k_{\alpha\beta}\dot{x}^\alpha\left(x^\beta\right)' = \frac{\partial x^\alpha}{\partial t}\nabla_\alpha\left(x^k\right)'$$

т.е. совпадает с k-ой компонентой поля $\nabla_t\nu_t'(\tau)$ (здесь мы воспользовались симметрией связности Γ). Доказательство закончено.

Теперь, с помощью леммы 10.2, найдем:

$$\|\dot{\gamma}_\tau(t)\|' = \nabla_\tau \sqrt{\langle \dot{\gamma}_\tau, \dot{\gamma}_\tau \rangle} = \frac{1}{2} \frac{\nabla_\tau \langle \dot{\gamma}_\tau, \dot{\gamma}_\tau \rangle}{\sqrt{\langle \dot{\gamma}_\tau, \dot{\gamma}_\tau \rangle}} = \frac{\langle \nabla_\tau \dot{\gamma}_\tau, \dot{\gamma}_\tau \rangle}{\sqrt{\langle \dot{\gamma}_\tau, \dot{\gamma}_\tau \rangle}} = \frac{\langle \nabla_t \nu'_t, \dot{\gamma}_\tau \rangle}{\sqrt{\langle \dot{\gamma}_\tau, \dot{\gamma}_\tau \rangle}}.$$

Подставим полученное выражение в формулу для производной функции длины $\ell(\tau)$, положим $\tau = 0$, воспользуемся предположением о том, что кривая $\gamma(t)$ натурально парамстризована, и тсм, что $E(t) = \nu'_t(0)$ по определению. Получим:

$$\frac{d\ell(0)}{d\tau} = t'_2(0) - t'_1(0) + \int_{t_1}^{t_2} \langle \nabla_t E(t), \dot{\gamma} \rangle \, dt$$

Далее,

$$\nabla_t \langle E(t), \dot{\gamma} \rangle = \langle \nabla_t E(t), \dot{\gamma} \rangle + \langle E(t), \nabla_t \dot{\gamma} \rangle \,,$$

поэтому выражение для производной переписывается в виде

$$\frac{d\ell(0)}{d\tau} = t'_2(0) - t'_1(0) + \int_{t_1}^{t_2} \left(\nabla_t \langle E(t), \dot{\gamma} \rangle - \langle E(t), \nabla_t \dot{\gamma} \rangle \right) dt =$$
$$= t'_2(0) - t'_1(0) + \langle E\left(t_2\right), \dot{\gamma}\left(t_2\right) \rangle - \langle E\left(t_1\right), \dot{\gamma}\left(t_1\right) \rangle - \int_{t_1}^{t_2} \langle E(t), \nabla_t \dot{\gamma} \rangle \, dt$$

Теперь, представив $t'_i(0)$ в виде $t'_i(0) \langle \dot{\gamma}\left(t_i\right), \dot{\gamma}\left(t_i\right) \rangle, i = 1, 2$, заметим, что

$$t'_i(0) + \langle E\left(t_i\right), \dot{\gamma}\left(t_i\right) \rangle = \langle t'_i(0) \dot{\gamma}\left(t_i\right) + E\left(t_i\right), \dot{\gamma}\left(t_i\right) \rangle \,.$$

Наконец, воспользовавшись леммой 10.1 и определением производной вдоль кривой, перепишем выражение для производной в виде

$$\frac{d\ell(0)}{d\tau} = \langle q'(0), \dot{\gamma}\left(t_2\right) \rangle - \langle p'(0), \dot{\gamma}\left(t_1\right) \rangle - \int_{t_1}^{t_2} \langle E(t), \nabla_{\dot{\gamma}} \dot{\gamma} \rangle \, dt$$

Итак, доказано следующее важное утверждение.

Утверждение 10.1. Пусть $\Phi(\tau, t)$ - произвольная вариачия гладкой кривой $\gamma : [t_1, t_2] \to M$, и $p(\tau)$ и $q(\tau)$ - концевые кривые вариации Φ. Тогда первая производная длины $\ell(\tau)$ кривой $\gamma_\tau(t) = \Phi(\tau, t)$ в начальный момент времени $\tau = 0$ существует и имеет вид

$$\frac{d\ell(0)}{d\tau} = \langle q'(0), \dot{\gamma}\left(t_2\right) \rangle - \langle p'(0), \dot{\gamma}\left(t_1\right) \rangle - \int_{t_1}^{t_2} \langle E(t), \nabla_{\dot{\gamma}} \dot{\gamma} \rangle \, dt.$$

Из утверждения 10.1 вытекает несколько важных следствий.

Следствие 10.1 (Вариация с закрепленными концами). Если вариация $\Phi(\tau, t)$ кривой $\gamma : [t_1, t_2] \to M$ имеет закрепленные концъ, то производлая длины $\ell(\tau)$ кривой γ_τ в пачальлый момент $\tau = 0$ имеет вид

$$\frac{d\ell(0)}{d\tau} = - \int_{t_1}^{t_2} \langle E(t), \nabla_{\dot{\gamma}} \dot{\gamma} \rangle \, dt$$

Следствие 10.2 (Стационарность геодезических). Любая геодезическая γ на римановом многообразии M является кривой стационарной длины по отношению κ произвольной вариации кривой γ с закрепленными концами. Обратно, любая натурально параметризованная кривая, длина которой стационарна по отношению к произвольным вариациям с закрепленными концами, является геодезической.

Следствие 10.3. Если кривая γ - геодезическая, то для произвольной ее вариации $\Phi(\tau, t)$ производная функиии длины $\ell(\tau)$ в начальный момент времени $\tau = 0$ имеет вид

$$\frac{d\ell(0)}{d\tau} = \langle q'(0), \dot{\gamma}(t_2)\rangle - \langle p'(0), \dot{\gamma}(t_1)\rangle,$$

где $p(\tau)$ и $q(\tau)$ - концевые кривые вариации Φ.

15.10.2 10.2. Gaussian lemma and local minimality of geodetic

Пусть в некоторой нормальной окрестности точки P риманова многообразия M фиксированы нормальные координаты. Рассмотрим в касательном пространстве T_PM сферу $S^n(\varepsilon)$ радиуса ε, целиком лежащую в нормальной окрестности точки $0 \in T_PM$. Напомним, что сфера $S^n(\varepsilon)$, по определению, состоит из касательных векторов v нормы ε, где норма вектора вычисляется с помощью римановой метрики. Образ $\exp_P(S^n(\varepsilon))$ сферы $S^n(\varepsilon)$ при экспоненциальном отображении называется геодезической сферой с центром в точке P и радиуса ε.

Утверждение 10.2 (Лемма Гаусса). Пусть Q - произвольная точка геодезической сферы Σ с центром в точке P. Тогда иеликом лежащая в нормальной окрестности точки P геодезическая, соединяющая точки P и Q, единственна. Более того, геодезическая γ приходит на геодезическую сферу Σ под прямьлм услом, тн.е. γ перпендикулярна любой регулярной кривой, проходящей через Q и лежащей в Σ.

Доказательство. Первая часть утверждения непосредственно следует из утверждения 9.7, т.е. из взаимной однозначности отображения $\exp_P$ в нормальной окрестности. Докажем вторую часть утверждения.

Пусть $\gamma(t)$ геодезическая, соединяющая точки P и Q и лежащая в нормальной окрестности точки P, и пусть $\sigma(\tau), \tau \in [-\tau_0, \tau_0]$, - произвольная регулярная кривая на геодезической сфере Σ, и $\sigma(0) = Q$. Рассмотрим семейство геодезических $\gamma_\tau(t)$, целиком лежащих в нормальной окрестности точки P и соединяющих P с точкой $\sigma(\tau) \in \Sigma$. В силу первой части утверждения, каждая геодезическая $\gamma_\tau(t)$ однозначно определена. Семейство геодезических $\gamma_\tau(t)$ определяет вариацию $\Phi(\tau, t) = \gamma_\tau(t)$ геодезической $\gamma(t) = \gamma_0(t)$, поскольку решение уравнений геодезических гладко зависит от начальных условий. При этом, по определению, одна концевая кривая $p(\tau)$ вариации Φ состоит из одной точки P, а другая $q(\tau)$, - совпадает с кривой $\sigma(\tau)$. Воспользуемся формулой первой вариации длины, см. следствие 10.3. Поскольку длины всех геодезических $\gamma_\tau(t)$ равны между собой и равны радиусу сферы Σ, имеем:

$$0 = \frac{d\ell(0)}{d\tau} = \langle \sigma'(0), \dot{\gamma}\rangle,$$

где вектор скорости $\dot{\gamma}$ вычисляется в точке Q, т.е. геодезическая γ перпендикулярна кривой σ, лежащей на сфере Σ. Доказательство закончено.

Упражнение 10.1. Пусть γ - лежащая в нормальной окрестности геодезическая, соединяющая точку P с точкой $Q \in \Sigma_\varepsilon$, где Σ_ε - геодезическая сфера с центром в точке P. Найдите длину кривой γ.

Докажем одно важное следствие леммы Гаусса. Рассмотрим две геодезические сферы $\Sigma(\varepsilon)$ и $\Sigma(\varepsilon')$ с центром в точке P. Пусть $Q \in \Sigma(\varepsilon)$ и $Q' \in \Sigma(\varepsilon')$ - две произвольные точки на этих сферах и γ - соединяющая их гладкая кривая, целиком лежащая в нормальной окрестности точки P

Следствие 10.4. Длина $\ell(\gamma)$ кривой γ не меньше абсолютной величины $|\varepsilon - \varepsilon'|$ разности радиусов геодезических сфер. При этом равенство достигается, если и только если кривая γ совпадает (с точностью до параметризации) с геодезической, проходящей через центр P обеих сфер.

При этом равенство достигается, если и только если $\frac{\partial \Psi}{\partial s} = 0$, т.е. если $\xi(s) = \xi_0 = \text{const}$. Последнее означает, что $\gamma(s) = \exp_P(t(s)\xi_0)$, т.е. кривая γ совпадает с радиальной геодезической, с точностью до параметризации. Следствие доказано.

Следствие 10.5. Пусть точка Q лежит в геодезическом шаре с центром в точке P достаточно малого радиуса ε (настолько малого, что отображение $\exp_P$ является на этом шаре диффеоморфизмом). Положим $Q = \exp_P \xi$. Тогда геодезическая $\gamma(t) = \exp_P(t\xi)$, является единственной кривой наименьшей длинья, соединяющей точки P и Q.

Доказательство. Для доказательства достаточно в следствии 10.4 перейти к пределу при $\varepsilon' \to 0$, и заметить, что любая кривая, выходящая из точки P и не лежащая целиком в нормальной окрестности этой точки, длиннее радиуса любой геодезической сферы, лежащей в этой окрестности.

Кривая γ на римановом многообразии, соединяющая точки P и Q, называется кратчайшей, если длина любой другой кривой, соединяющей те же точки, не меньше чем длина кривой γ. Мы уже видели, что для любых достаточно близких точек риманова многообразия существует единственная кратчайшая их соединяющая. Эта кратчайшая является геодезической. Оказывается, имеет место также и следующее утверждение.

Следствие 10.6. Пусть γ - некоторая кратчайшая, соединяющая произвольные точки P и Q риманова многообразия M. Тогда γ - геодсзическая на M (в подходящей параметризации).

Доказательство. Это немедленно вытекает из следствия 10.5. Действительно, раз γ - кратчайшая, то она является таковой и для любых двух своих достаточно близких точек. Поэтому каждый маленький кусочек кривой γ является геодезической, следовательно и вся кривая γ - тоже. Следствие доказано.

Упражнение 10.2. Любые ли две точки на римановом многообразии M можно соединить кратчайшей кривой?

15.10.3 Problems

Задача 10.1. Привести пример риманова многообразия и двух точек на нем, которые не соединяет ни одна кратчайшая и ни одна геодезическая.

Задача 10.2. Показать, что если M - замкнутое риманово многообразие, то любые его две точки можно соединить кратчайшей геодезической.

Задача 10.3. Пусть M - риманово многообразие с не пустым краем ∂M. Iусть P и Q - точки из M, и γ - кратчайшая кривая, соединяющая P и Q. Верно ли, что γ - геодезическая?

Задача 10.4. Описать геодезические сферы и радиальные геодезические на евклидовой плоскости, плоскости Лобачевского, сфере, прямом цилиндре. Сколько существует геодезических, проходящих через фиксированную пару точек этих многообразий? А кратчайших геодезических?

Задача 10.5. Вывести формулы для функции $\ell(t)$ длины прямолинейного отрезка $[a(t), b(t)]$ в пространстве $\mathbb{R}^n$, если его концы движутся по гладким кривым $a(t)$ и $b(t), t \in [0, 1]$. Найти производную функции $\ell(t)$ при $t = 0$. Изучить отдельно случай $a(0) = b(0)$.

15.10.4 Extra Topics

10.1. Уравнение геодезических в гамильтоновой форме. Напомним, см. дополнительный материал к лекции 6, что кокасательное расслоение T^*M к произвольному многообразию M представляет собой симплектическое многообразие, поэтому можно записать систему уравнений Гамильтона для произвольной функции H на нем. Пусть M - риманово многообразие, $(x^1, \ldots, x^n)$ - локальные координаты на нем,

и $(x^1, \ldots, x^n, p_1, \ldots, p_n)$ - соответствующая карта на T^*M. Определим функцию Гамильтона H на T^*M, положив $H(x,p) = \frac{1}{2}g^{ij}p_ip_j$, где g_{ij} - компоненты римановой метрики в координатах $(x^1, \ldots, x^n)$.

Упражнение 10.3. Проверьте, что заданная в локальных координатах функция Гамильтона H определена корректно.

Соответствующая система уравнений Гамильтона имеет вид

$$\dot{x}^i = \frac{\partial H}{\partial p_i} = g^{i\alpha}p_\alpha,$$
$$\dot{p}_i = -\frac{\partial H}{\partial x^i} = -\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^i}p_\alpha p_\beta,$$

где точкой, как обычно, обозначено дифференцирование по t. Из первого уравнения получим $p_\alpha = g_{\alpha\beta}\dot{x}^\beta$, т.е. "импульс p - это ковектор, соответствующий скорости $\dot{x}$." Исключим из второго уравнения импульсы p. Имеем:

$$-\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^i}g_{\alpha k}\dot{x}^k g_{\beta j}\dot{x}^j = \left(g_{i\beta}\dot{x}^\beta\right)^{\cdot} = \ddot{x}^\beta g_{i\beta} + \dot{x}^\beta\frac{\partial g_{i\beta}}{\partial x^k}\dot{x}^k.$$

Чтобы избавиться от производной обратной матрицы метрики, воспользуемся следующей тривиальной леммой.

Лемма 10.3. Имеет место следующее соотношение:

$$\frac{\partial g^{\alpha\beta}}{\partial x^i}g_{\beta k} = -g^{\alpha\beta}\frac{\partial g_{\beta k}}{\partial x^i}.$$

С помощью леммы 10.3 перепишем левую часть последнего уравнения в виде

$$-\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^i}g_{\alpha k}\dot{x}^k g_{\beta j}\dot{x}^j = \frac{1}{2}g^{\alpha\beta}\frac{\partial g_{\beta j}}{\partial x^i}g_{\alpha k}\dot{x}^k\dot{x}^j = \frac{1}{2}\frac{\partial g_{kj}}{\partial x^i}\dot{x}^k\dot{x}^j,$$

поэтому само уравнение можно записать так:

$$\frac{1}{2}\frac{\partial g_{kj}}{\partial x^i}\dot{x}^k\dot{x}^j = \ddot{x}^\beta g_{i\beta} + \dot{x}^\beta\frac{\partial g_{i\beta}}{\partial x^k}\dot{x}^k$$

Разбивая последнее слагаемое правой части на два равных, перенося все в правую часть и умножая на матрицу $g^{i\alpha}$ получаем

$$\ddot{x}^\alpha + \dot{x}^k\dot{x}^j\frac{1}{2}g^{i\alpha}\left(\frac{\partial g_{ij}}{\partial x^k} + \frac{\partial g_{ij}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i}\right) = 0.$$

Осталось заметить, что

$$\frac{\partial g_{ij}}{\partial x^k}\dot{x}^k\dot{x}^j = \frac{\partial g_{ik}}{\partial x^j}\dot{x}^k\dot{x}^j$$

поэтому наше уравнение - это в точности уравнение геодезических на M. Поскольку все наши преобразования можно проделать в обратном порядке, получаем следующее утверждение.

Утверждение 10.3. Пусть M - произвольное риманово многообразие. Каждое решение $(x(t), p(t))$ построенной выше гамильтоновой системь на кокасательном расслоении T^*M проецируется в аеодезическую $x(t)$ на M. Обратно, если $x(t)$ произвольная аеодезическая, $\dot{x}(t)$ - ее поле скоростей, а $p(t)$ - ковекторное поле, полученное из $\dot{x}(t)$ поднятием индекса с помощью римановой метрики, то пара $(x(t), p(t))$ - решение гамильтоновой системы уравнений на T^*M с пределенным выше гамильтонианом H.

15.11 11: Curvature tensor

Вопрос о существовании евклидовых координат для данной связности (метрики) уже затрагивался нами выше. В качестве одного из препятствий к построению таких координат, мы указали тензор кручения аффинной связности. В данном разделе мы определим и изучим так называемый тензор кривизны. Оказывается в терминах тензоров кривизны и кручения можно сформулировать критерий существования евклидовых координат. Мы дадим здесь два определения тензора кривизны.

15.11.1 11.1. Coordinate determination of the curvature tensor

Если на многообразии M с аффинной связностью Γ существуют евклидовы координаты ($z^1, \ldots, z^n$), то в этих координатах операция ковариантного дифференцирования в связности Γ совпадает, по определению, с операцией частного дифференцирования (так дело обстоит, например в $\mathbb{R}^n$ с евклидовой связностью и евклидовыми координатами $(z^1, \ldots, z^n)$). Поэтому для любого векторного поля T на M имеет место равенство

$$\nabla_p \left(\nabla_q T^i\right) = \frac{\partial^2 T^i}{\partial z^q \partial z^p} = \frac{\partial^2 T^i}{\partial z^p \partial z^q} = \nabla_q \left(\nabla_p T^i\right)$$

Таким образом, в евклидовых координатах (если такие существуют) вторые ковариантные производные не зависят от порядка. Отметим, что равенство

$$\nabla_p \left(\nabla_q T^i\right) = \nabla_q \left(\nabla_p T^i\right)$$

- тензорное, поэтому оно или имеет, или не имеет место во всех системах координат сразу.

Итак, пусть теперь M - многообразие с аффинной связностью Γ, и T - векторное поле на M. Вычислим $\nabla_p \left(\nabla_q T^i\right)$ в локальных координатах $(x^1, \ldots, x^n)$. Получим:

$$\begin{aligned}
&\nabla_p \left(\nabla_q T^i\right) = \nabla_p \left(\frac{\partial T^i}{\partial x^q} + \Gamma^i_{\alpha q} T^\alpha\right) = \\
&= \frac{\partial}{\partial x^p}\left(\frac{\partial T^i}{\partial x^q} + \Gamma^i_{\alpha q} T^\alpha\right) + \Gamma^i_{\beta p}\left(\frac{\partial T^\beta}{\partial x^q} + \Gamma^\beta_{\alpha q} T^\alpha\right) - \Gamma^\beta_{qp}\left(\frac{\partial T^i}{\partial x^\beta} + \Gamma^i_{\alpha\beta} T^\alpha\right) = \\
&\qquad = \underline{\frac{\partial^2 T^i}{\partial x^q \partial x^p}} + \frac{\partial \Gamma^i_{\alpha q}}{\partial x^p} T^\alpha \\
&\qquad\quad + \underline{\Gamma^i_{\alpha q} \frac{\partial T^\alpha}{\partial x^p} + \Gamma^i_{\beta p} \frac{\partial T^\beta}{\partial x^q}} + \\
&\qquad\quad + \Gamma^i_{\beta p} \Gamma^\beta_{\alpha q} T^\alpha - \Gamma^\beta_{qp} \frac{\partial T^i}{\partial x^\beta} - \Gamma^\beta_{qp} \Gamma^i_{\alpha\beta} T^\alpha
\end{aligned}$$

где подчеркнутые слагаемые симметричны по p и q. Ясно, что $\nabla_p \left(\nabla_q T^i\right)$ получается из предыдущего выражения перестановкой индексов p и q.

Поэтому

$$\begin{aligned}
\left(\nabla_p \nabla_q - \nabla_q \nabla_p\right) T^i = &\left(\frac{\partial \Gamma^i_{\alpha q}}{\partial x^p} - \frac{\partial \Gamma^i_{\alpha p}}{\partial x^q} + \Gamma^i_{\beta p} \Gamma^\beta_{\alpha q} - \Gamma^i_{\beta q} \Gamma^\beta_{\alpha p}\right) T^\alpha + \\
&+ \left(\Gamma^\beta_{pq} - \Gamma^\beta_{qp}\right)\left(\frac{\partial T^i}{\partial x^\beta} + \Gamma^i_{\alpha\beta} T^\alpha\right).
\end{aligned}$$

Отметим, что во втором слагаемом стоят уже известные нам компоненты тензора кручения Ω аффинной связности Γ. Обозначим множитель, стоящий перед T^α через $R^i_{\alpha pq}$. Тогда последнее выражение перепишется в виде

$$(\nabla_p \nabla_q - \nabla_q \nabla_p)\, T^i = R^i_{\alpha pq} T^\alpha + \Omega^\beta_{pq} \nabla_\beta T^\alpha, \qquad (*)$$

где

$$R^i_{\alpha pq} = \frac{\partial \Gamma^i_{\alpha q}}{\partial x^p} - \frac{\partial \Gamma^i_{\alpha p}}{\partial x^q} + \Gamma^i_{\beta p}\Gamma^\beta_{\alpha q} - \Gamma^i_{\beta q}\Gamma^\beta_{\alpha p}.$$

Из того обстоятельства, что все слагаемые в равенстве (*), за исключением $R^i_{\alpha pq} T^\alpha$ - тензорные, и T^α - компоненты произвольного векторного поля, вытекает следующий результат.

Лемма 11.1. Соответствие, сопоставляющее произвольным локальным координатам $(x^1, \ldots, x^n)$ на многообразии M набор функиий $R^i_{\alpha pq}$ задает на M тензорное поле R типа $(1,3)$.

Определение. Тензорное поле R типа $(1,3)$, компоненты которого в локальных координатах $(x^1, \ldots, x^n)$ на многообразии M с аффинной связностью Γ задаются в виде

$$R^i_{\alpha pq} = \frac{\partial \Gamma^i_{\alpha q}}{\partial x^p} - \frac{\partial \Gamma^i_{\alpha p}}{\partial x^q} + \Gamma^i_{\beta p}\Gamma^\beta_{\alpha q} - \Gamma^i_{\beta q}\Gamma^\beta_{\alpha p}.$$

называется тензором кривизны аффинной связности Γ.

Утверждение 11.1. Если на многообразии M существуют евклидовы координаты в окрестности точки P для связности Γ, то тензор кривизны этой связности равен нулю в этой окрестности. Если же тензор кривизны связности Γ в окрестности некоторой точки P отличен от нуля, то в окрестности точки P нельзя выбрать евклидовы координаты (в смысле связности Γ).

Доказательство. Это очевидно: тензор кривизны в евклидовых координатах равен нулю и, поэтому, если евклидовы координаты существуют, то тензор кривизны тождественно равен нулю в произвольных координатах.

Замечание. На самом деле, верно и обратное утверждение: если тензор кривизны симметричной связности Γ равен нулю, то существуют евклидовы координаты. Однако доказательство этого утверждения лежит за рамками нашего курса и сводится к разрешимости системы уравнений в частных производных (см. в следующей лекции комментарии для случая римановых многообразий).

Очевидно, что тензор кривизны обладает следующей косой симметрией: $R^i_{\alpha pq} = -R^i_{\alpha qp}$. Именно этим объясняется пробел, который мы ставим между первым и остальными двумя нижними индексами тензора кривизны - он позволяет визуально выделить кососимметричные индексы.

Другие симметрии тензора кривизны удобнее изучать в терминах его инвариантного определения. Чтобы дать это инвариантное определение, нам понадобится понятие коммутатора векторных полей на многообразии.

15.11.2 11.2. Vector field commutator

Пусть M - гладкое многообразие, и V - произвольное векторное поле на M. Тогда определено отображение пространства $\mathcal{F}(M)$ гладких функций на M в себя, переводящее произвольную гладкую функцию f из $\mathcal{F}(M)$ в функцию $V(f)$, значение которой в точке P есть производная по направлению вектора $V(P)$ от f в точке P. Пусть теперь имеется два векторных поля V и W. Рассмотрим отображение $[V, W]$ пространства $\mathcal{F}(M)$ в себя, заданное так:

$$[V,W]: f \mapsto V(W(f)) - W(V(f)), \quad f \in \mathcal{F}(M).$$

Имеет место следующая лемма.

Лемма 11.2. В сделанных выше обозначениях, существует единственное сладкое векторное поле A на M, такое что $A(f) = [V,W](f)$ для произвольной сладкой функции $f \in \mathcal{F}(M)$. Если $(x^1, \ldots, x^n)$ - локальные координаты на M, то компоненты векторного поля A вычисляются через компоненты полей V и W в этих координатах по следующим формулам:

$$a^\alpha = \frac{\partial w^\alpha}{\partial x^\beta} v^\beta - \frac{\partial v^\alpha}{\partial x^\beta} w^\beta.$$

Доказательство. Нам нужно проверить, что отображение $[V,W]$ является дифференцированием в каждой точке многообразия M (напомним, что касательный вектор можно определить как дифференцирование). Для произвольных функций f и g из $\mathcal{F}(M)$ и произвольных чисел α и β

имеем:

$$\begin{aligned}
[V,W](\alpha f + \beta g) &= V(\alpha W(f) + \beta W(g)) - W(\alpha V(f) + \beta V(g)) = \\
&= \alpha(V(W(f)) - W(V(f))) + \beta(V(W(g)) - W(V(g))) = \\
&= \alpha[V,W](f) + \beta[V,W](g),
\end{aligned}$$

поэтому операция $[V,W]$ линейна на $\mathcal{F}(M)$. Проверим правило Лейбница.

$$\begin{aligned}
[V,W](fg) &= V(W(f)g + fW(g)) - W(V(f)g + fV(g)) = \\
&= V(W(f))g + W(f)V(g) + V(f)W(g) + fV(W(g)) - \\
&- W(V(f))g - V(f)W(g) - W(f)V(g) - fW(V(g)) = \\
&= [V,W](f)g + f[V,W](g)
\end{aligned}$$

Итак, мы показали, что $[V,W]$ - дифференцирование. Поэтому она задает векторное поле на M, которое мы обозначим через A.

Пусть $(x^1, \ldots, x^n)$ - локальные координаты на M. Вычислим в этих координатах $[V,W](f)$. Имеем:

$$\begin{aligned}
[V,W](f) &= V(W(f)) - W(V(f)) = V\left(\frac{\partial f}{\partial x^\alpha} w^\alpha\right) - W\left(\frac{\partial f}{\partial x^\alpha} v^\alpha\right) = \\
&= \frac{\partial}{\partial x^\beta}\left(\frac{\partial f}{\partial x^\alpha} w^\alpha\right) v^\beta - \frac{\partial}{\partial x^\beta}\left(\frac{\partial f}{\partial x^\alpha} v^\alpha\right) w^\beta = \\
&= \frac{\partial^2 f}{\partial x^\alpha \partial x^\beta} w^\alpha v^\beta + \frac{\partial f}{\partial x^\alpha}\frac{\partial w^\alpha}{\partial x^\beta} v^\beta - \frac{\partial^2 f}{\partial x^\beta \partial x^\alpha} v^\alpha w^\beta - \frac{\partial f}{\partial x^\alpha}\frac{\partial v^\alpha}{\partial x^\beta} w^\beta = \\
&= \frac{\partial f}{\partial x^\alpha}\left(\frac{\partial w^\alpha}{\partial x^\beta} v^\beta - \frac{\partial v^\alpha}{\partial x^\beta} w^\beta\right)
\end{aligned}$$

Положим $a^\alpha = \frac{\partial w^\alpha}{\partial x^\beta} v^\beta - \frac{\partial v^\alpha}{\partial x^\beta} w^\beta$. Тогда

$$[V,W](f) = \frac{\partial f}{\partial x^\alpha} a^\alpha = A(f)$$

Таким образом, $(a^1, \ldots, a^n)$ - это компоненты векторного поля A в координатах $(x^1, \ldots, x^n)$. Лемма доказана.

Определение. Векторное поле $[V,W]$ называется коммутатором векторных полей V и W.

Приведем несколько простейших свойств коммутатора.
Утверждение 11.2. Операция коммутирования векторных полей обладает следующими свойствами.

1. Косая симметрия: для любых полей X и Y имеет место равенство $[X,Y]=-[Y,X]$.
2. Линейность над $\mathbb{R}^1$: для любых полей X,Y и Z и любых чисел $\alpha u\beta$ имеет место равенство
$$[X,\alpha Y+\beta Z]=\alpha[X,Y]+\beta[X,Z].$$
3. Правило Лейбница: для любых двух полей X и Y и гладкой функиии $f\in\mathcal{F}(M)$ имеют место равенства
$$[X,fY]=f[X,Y]+X(f)Y,\quad [fX,Y]=f[X,Y]-Y(f)X$$
4. Тождество Якоби: для любых полей X,Y и Z имеет место равенство
$$[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0.$$
5. Если $(x^1,\ldots,x^n)$ - произвольная система локальньх координат на многообразии, и $\partial_{x^i}u\partial_{x^j}$ - координатные векторные поля, то $[\partial_{x^i},\partial_{x^j}]=0$.

Доказательство. Первые два свойства очевидны. Докажем правило Лейбница. Имеем:
$$\begin{aligned}&[X,fY](h)=X(fY(h))-fY(X(h))=\\&\quad=X(f)Y(h)+fX(Y(h))-fY(X(h))=(f[X,Y]+X(f)Y)(h)\end{aligned}$$
что и требовалось (мы несколько раз пользовались правилом Лейбница для дифференцирования функций). Второе тождество проверяется аналогично.

Для проверки тождества Якоби достаточно заметить, что
$$\begin{aligned}[X,[Y,Z]](h)&=X([Y,Z](h))-[Y,Z](X(h))=\\&=X((YZ-ZY)(h))-(YZ-ZY)(X(h))=\\&=(XYZ-XZY-YZX+ZYX)(h)\end{aligned}$$
а остальные два слагаемых получаются из этого циклической перестановкой полей X,Y, и Z. Поэтому
$$\begin{aligned}[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]&=(XYZ-XZY-YZX+ZYX)+\\+(YZX-YXZ-ZXY+XZY)&+(ZXY-ZYX-XYZ+YXZ)=0.\end{aligned}$$
Отметим, что это доказательство в точности совпадает с доказательством тождества Якоби для линейных операторов, известным из линейной алгебры.

Последнее свойство очевидно, так как $\partial_{x^i}(h)=\frac{\partial h}{\partial x^i}$.

Пункт 5 утверждения 11.2 представляет самостоятельный интерес. Пусть в области U гладкого n-мерного многообразия M задано семейство векторных полей $\{X_1,\ldots,X_n\}$, такое что в каждой точке $P\in U$ векторы $\{X_i(P)\}$ образуют базис в касательном пространстве T_PM. Возникает естественный вопрос: существует ли в U система координат ($x^1,\ldots,x^n$), такая что $\partial_{x^i}=X_i$? Оказывается. такие координаты существуют не всегда, и пункт 5 утверждения 11.2 - это необходимое условие

существования таких координат. Покажем, что это условие является так же и достаточным.

Утверждение 11.3. Пусть в области U гладкого n-мерного многообразия M задано семейство векторных полей $\{X_1, \ldots, X_n\}$, такое что в каждой точке $P \in U$ векторы $\{X_i(P)\}$ образуют базис в касательном пространстве $T_P M$. Если коммутаторы $[X_i, X_j]$ векторных полей равны нулю, то в некоторой области $U' \subset U$ существуют координатьь $(x^1, \ldots, x^n)$, для которых $\partial_{x^i} = X_i$.

Доказательство. Пусть $(y^1, \ldots, y^n)$ - локальные координаты в области U, и ∂_{y^i} - соответствующие координатные векторные поля. Мы хотим сделать такую замену координат $x^i = x^i(y^1, \ldots, y^n)$, что $\partial_{x^i} = X_i$. Как известно,

$$\partial_{x^i} = \frac{\partial y^\alpha}{\partial x^i} \partial_{y^\alpha}, \quad X_i = X_i^\alpha \partial_{y^\alpha}$$

откуда условие $\partial_{x^i} = X_i$ равносильно системе уравнений

$$\frac{\partial y^\alpha}{\partial x^i} = X_i^\alpha, \quad 1 \le i, \alpha \le n$$

на неизвестные функции $y^i(x^1, \ldots, x^n)$. Поскольку все происходит в одной карте, которая, как можно предполагать, гомеоморфна диску, необходимое и достаточное условие разрешимости этой системы уравнений (по лемме Пуанкаре) состоит в следующем:

$$\frac{\partial^2 y^\alpha}{\partial x^i \partial x^j} = \frac{\partial^2 y^\alpha}{\partial x^j \partial x^i} \quad \Leftrightarrow \quad \frac{\partial}{\partial x^j} X_i^\alpha = \frac{\partial}{\partial x^i} X_j^\alpha.$$

Функции X_i^α - суть компоненты векторного поля X_i в исходных координатах ($y^1, \ldots, y^n$), поэтому

$$\frac{\partial}{\partial x^j} X_i^\alpha = \frac{\partial y^\beta}{\partial x^j} \frac{\partial}{\partial y^\beta} X_i^\alpha = X_j^\beta \frac{\partial}{\partial y^\beta} X_i^\alpha$$

откуда

$$\frac{\partial}{\partial x^j} X_i^\alpha = \frac{\partial}{\partial x^i} X_j^\alpha \quad \Leftrightarrow \quad X_j^\beta \frac{\partial}{\partial y^\beta} X_i^\alpha = X_i^\beta \frac{\partial}{\partial y^\beta} X_j^\alpha \quad \Leftrightarrow \quad [X_i, X_j] = 0$$

что и требовалось.

Нам также будет полезен следующий результат.

Лемма 11.3. Для симметричной аффинной связности имеет место тождество

$$[X, Y] = \nabla_X Y - \nabla_Y X$$

где X и Y - произвольные векторные поля.

Доказательство. Вычислим компоненту правой части проверяемого равенства в локальных координатах $(x^1, \ldots, x^n)$. Имеем:

$$\begin{aligned}(\nabla_X Y - \nabla_Y X)^k &= X^i \left(\frac{\partial Y^k}{\partial x^i} + \Gamma^k_{\alpha i} Y^\alpha \right) - Y^i \left(\frac{\partial X^k}{\partial x^i} + \Gamma^k_{\alpha i} X^\alpha \right) = \\ &= X^i \frac{\partial Y^k}{\partial x^i} - Y^i \frac{\partial X^k}{\partial x^i} + \left(\Gamma^k_{\alpha i} X^i Y^\alpha - \Gamma^k_{\alpha i} X^\alpha Y^i \right)\end{aligned}$$

Слагаемое в скобках равно нулю в силу симметричности связности. Поэтому

$$(\nabla_X Y - \nabla_Y X)^k = X^i \frac{\partial Y^k}{\partial x^i} - Y^i \frac{\partial X^k}{\partial x^i} = ([X, Y])^k$$

что и требовалось.

15.11.3 11.3. Invariant determination of the curvature tensor for symmetric connectivity

Пусть M - гладкое многообразие с симметричной аффинной связностью Γ, и X, Y и Z - векторные поля на M. Построим оператор кривизны R, положив

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z.$$

Если $\mathcal{X}(M)$ - пространство векторных полей на M, то оператор кривизны - это отображение $R : \mathcal{X}(M) \times \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$.
Лемма 11.4. Отображение R трилинейно относительно умножения на функции и, поэтому, задает тензорное поле типа (1,3), совпадающее с тензором кривизны.
Доказательство. Пусть f - гладкая функция на многообразии M. Покажем сначала, что $R(X,Y)(fZ) = fR(X,Y)Z$. Действительно,

$$\begin{aligned}
&R(X,Y)(fZ) = \nabla_X \nabla_Y (fZ) - \nabla_Y \nabla_X (fZ) - \nabla_{[X,Y]}(fZ) = \\
=&\nabla_X \left((\nabla_Y f) Z + f \nabla_Y Z \right) - \nabla_Y \left((\nabla_X f) Z + f \nabla_X Z \right) - \left(\nabla_{[X,Y]} f \right) Z - f \nabla_{[X,Y]} Z = \\
=&\left(\nabla_X \nabla_Y f \right) Z + \left(\nabla_Y f \right) \nabla_X Z + \left(\nabla_X f \right) \nabla_Y Z + f \nabla_X \nabla_Y Z - \\
&- \left(\nabla_Y \nabla_X f \right) Z - \left(\nabla_X f \right) \nabla_Y Z - \left(\nabla_Y f \right) \nabla_X Z - f \nabla_Y \nabla_X Z - \\
&- \left(\nabla_{[X,Y]} f \right) Z - f \nabla_{[X,Y]} Z = \\
=&f \left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - f \nabla_{[X,Y]} Z \right) + \\
&+ Z \left(\nabla_X \nabla_Y f - \nabla_Y \nabla_X f - \nabla_{[X,Y]} f \right).
\end{aligned}$$

Второе слагаемое в этой сумме равно нулю, так как

$$\nabla_X \nabla_Y f - \nabla_Y \nabla_X f = X(Y(f)) - Y(X(f)) = [X,Y](f) = \nabla_{[X,Y]} f,$$

поэтому окончательно:

$$R(X,Y)(fZ) = f \left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - f \nabla_{[X,Y]} Z \right) = fR(X,Y)(Z),$$

что и требовалось.
Покажем теперь, что $R(fX,Y)Z = fR(X,Y)Z$. Действительно:

$$R(fX,Y)Z = \nabla_{fX} \nabla_Y Z - \nabla_Y \nabla_{fX} Z - \nabla_{[fX,Y]} Z.$$

Пользуясь очевидным равенством $\nabla_{fX} Z = f \nabla_X Z$ и правилом Лейбница для коммутатора получим:

$$\begin{aligned}
&R(fX,Y)Z = f \nabla_X \nabla_Y Z - \nabla_Y \left(f \nabla_X Z \right) - \nabla_{f[X,Y] - Y(f)X} Z = \\
&= f \nabla_X \nabla_Y Z - \left(\nabla_Y f \right) \nabla_X Z - f \nabla_Y \nabla_X Z - f \nabla_{[X,Y]} Z + Y(f) \nabla_X Z = \\
&= f \left(\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - f \nabla_{[X,Y]} Z \right),
\end{aligned}$$

что и требовалось. Равенство $R(X,fY)Z = fR(X,Y)Z$ проверяется точно так же. Итак, мы проверили линейность отображения R по каждому аргументу.

Покажем теперь, что R задает тензорное поле типа $(1,3)$. Пусть ($x^1, \ldots, x^n$) - локальные координаты на M. Запишем векторные поля X, Y и Z в виде $X = X^\alpha \partial_{x^\alpha}, Y = Y^\beta \partial_{x^\beta}$ и $Z = Z^\gamma \partial_{x^\gamma}$ подставим полученные выражения в оператор кривизны. В силу доказанной линейности, имеем:

$$R(X,Y)Z = R\left(X^\alpha \partial_{x^\alpha}, Y^\beta \partial_{x^\beta} \right) \left(Z^\gamma \partial_{x^\gamma} \right) = X^\alpha Y^\beta Z^\gamma R\left(\partial_{x^\alpha}, \partial_{x^\beta} \right) \left(\partial_{x^\gamma} \right).$$

Величины $R\left(\partial_{x^\alpha}, \partial_{x^\beta}\right) \partial_{x^\gamma}$ являются векторами, поэтому могут быть записаны в виде

$$R\left(\partial_{x^\alpha}, \partial_{x^\beta}\right) \partial_{x^\gamma} = T^i_{\gamma\alpha\beta} \partial_{x^i}.$$

Сопоставим координатам $(x^1, \ldots, x^n)$ набор величин $T^i_{\gamma\alpha\beta}$. Из доказанной линейности относительно умножения на функции вытекает, что при замене координат эти величины меняются по тензорному закону, т.е. являются компонентами тензорного поля. Лемма доказана.

Покажем теперь, что тензорное поле T, соответствующее оператору кривизны, совпадает с определенным нами выше тензором кривизны. Чтобы вычислить компоненты поля T в координатах ($x^1, \ldots, x^n$), следует найти значение $R\left(\partial_{x^\alpha}, \partial_{x^\beta}\right) \partial_{x^\gamma}$ оператора кривизны на векторах канонического базиса. Поскольку коммутатор $[\partial_{x^\alpha}, \partial_{x^\beta}]$ координатных полей равен нулю, имеем:

$$R\left(\partial_{x^\alpha}, \partial_{x^\beta}\right) \partial_{x^\gamma} = \nabla_{\partial_{x^\alpha}} \nabla_{\partial_{x^\beta}} \partial_{x^\gamma} - \nabla_{\partial_{x^\beta}} \nabla_{\partial_{x^\alpha}} \partial_{x^\gamma}$$

Проделав вычисления, аналогичные приведенным в разделе о координатном определении тензора кривизны и воспользовавшись симметрией аффинной связности, получим

$$R\left(\partial_{x^\alpha}, \partial_{x^\beta}\right) \partial_{x^\gamma} = \left(\frac{\partial \Gamma^i_{\gamma\beta}}{\partial x^\alpha} - \frac{\partial \Gamma^i_{\gamma\alpha}}{\partial x^\beta} + \Gamma^i_{k\alpha} \Gamma^k_{\gamma\beta} - \Gamma^i_{k\beta} \Gamma^k_{\gamma\alpha} \right) \partial_{x^i}$$

откуда, компоненты $T^i_{\gamma\alpha\beta}$ совпадают с компонентами тензора кривизны, что и требовалось.

Замечание. Отметим, что $\nabla_{\partial_{xp}} \nabla_{\partial_{xq} q} \partial_{x^k}$ - это векторное поле, а $\nabla_p \nabla_q \partial_{x^k}$ - тензор типа $(1,2)$, поэтому их приравнивать нельзя и, значит, нельзя воспользоваться вычислениями для $\nabla_p \nabla_q \partial_{x^k}$ непосредственно.

15.11.4 Problems

Задача 11.1. Если A - квадратная вещественная матрица размера ($n \times n$), то обозначим через V_A векторное поле в $\mathbb{R}^n$, заданное так: $V_A(x) = Ax, x \in \mathbb{R}^n$. Показать, что $[V_A, V_B] = V_{[B,A]}$, где слева стоит коммутатор векторных полей, а справа векторное поле, порожденное коммутатором матриц.

Задача 11.2. Пусть на сфере $S^3 \subset \mathbb{R}^4$, стандартно вложенной в $\mathbb{R}^4$, т.е. $\sum (x^i)^2 = 1$, заданы векторные поля X и Y, которые в произвольной точке (x^1, x^2, x^3, x^4) $\in S^3$ имеют вид

$$X = \left(x^2, -x^1, x^4, -x^3\right), \quad Y = \left(x^3, x^4, -x^1, -x^2\right).$$

Найти $[X, Y]$.

Задача 11.3. Пусть $N \subset M$ - подмногообразие гладкого многообразия N, и пусть X и Y - гладкие поля на объемлющем многообразии M, касающиеся N в точках из N. Тогда определены ограничения $X|_N$ и $Y|_N$ полей на подмногообразие. Показать, что $[X, Y]|_N = [X|_N, Y_N]$, т.е. "коммутатор можно считать в объемлющем пространстве."

Задача 11.4. Вычислить тензор кривизны одномерного многообразия с произвольной аффинной связностью.

Задача 11.5. Пусть $\bar{\nabla}$ - ковариантное дифференцирование в $\mathbb{R}^3$ в стандартной евклидовой связности. Рассмотрим новую операцию, заданную на векторах так:

$$\nabla_X Y = \bar{\nabla}_X Y + \frac{1}{2} X \times Y$$

Показать, что это - ковариантное дифференцирование в некоторой связности. Вычислить ее тензоры кривизны и кручения.

15.11.5 Extra Topics

11.1. Скобки Пуассона. Пусть $M = M^{2n}$ - симплектическое многообразие с симплектической структурой ω. Скобкой Пуассона двух гладких функций f и g на M называется гладкая функция на M, заданная так:

$$\{f, g\} = \omega(\operatorname{sgrad} g, \operatorname{sgrad} f) = (\operatorname{sgrad} f)g$$

В произвольных локальных координатах ($x^1, \ldots, x^{2n}$) на M значение скобки Пуассона может быть вычислено по формуле

$$\{f, g\} = \omega^{ij} \frac{\partial g}{\partial x^i} \frac{\partial f}{\partial x^j}$$

где, напомним, через ω^{ij} обозначены компоненты матрицы, обратной к матрице (ω_{ij}) компонент формы ω.

Утверждение 11.4. Операция взятия скобки Пуассона билинейна над $\mathbb{R}$, кососимметрична и удовлетворяет тождеству Якоби

$$\{h, \{f, g\}\} + \{g, \{h, f\}\} + \{f, \{g, h\}\} = 0$$

Доказательство. Первые два свойства очевидны из определения. Третье свойство докажем для случая $M = \mathbb{R}^{2n}, \omega = \sum dx^i \wedge dy^i$. (Напомним, что в силу теоремы Дарбу это предположение не меняет общности рассуждений.) Заметим, что выражение вида $\{h, \{f, g\}\}$ записывается в локальных координатах в виде суммы, каждое слагаемое которой содержит вторую производную функции f или g. Поэтому, если записать левую часть тождества Якоби в координатах, то вторые производные функции f входят в выражения $\{h, \{f, g\}\}$ и $\{g, \{h, f\}\}$. С другой стороны, пользуясь представлением скобки Пуассона в виде производной одной функции вдоль косого градиента другой и косой симметрией, имеем:

$$\begin{aligned}\{h, \{f, g\}\} + \{g, \{h, f\}\} = -\{h, \{g, f\}\} + \{g, \{h, f\}\} = -\{h, (\operatorname{sgrad} g)f\} + \{g, (\operatorname{sgrad} h)f\} = \\ = -(\operatorname{sgrad} h \operatorname{sgrad} g)f + (\operatorname{sgrad} g \operatorname{sgrad} h)f = [\operatorname{sgrad} g, \operatorname{sgrad} h]f\end{aligned}$$

Но полученное выражение, равное результату дифференцирования функции f вдоль векторного поля, не содержит вторых производных функции f. Поэтому вторые производные функции f входят в левую часть тождества Якоби с нулевым коэффициентом. Значит все слагаемые тождества Якоби равны нулю, что и требовалось.

Следствие 11.1. Для произвольных гладких функиий f и g на симплектическом многообразии имеет место следующее тождество

$$\operatorname{sgrad}\{f, g\} = [\operatorname{sgrad} f, \operatorname{sgrad} g].$$

Доказательство. Для произвольной функции h, в силу определения косого градиента и скобки Пуассона, а также в силу ее косой симметрии и тождества Якоби, имеем

$$\begin{aligned}(\operatorname{sgrad}\{f, g\})h = \omega(\operatorname{sgrad} h, \operatorname{sgrad}\{f, g\}) = \{\{f, g\}, h\} = -\{h, \{f, g\}\} = \\ = -\{g, \{f, h\}\} + \{f, \{g, h\}\} = [\operatorname{sgrad} f, \operatorname{sgrad} g](h),\end{aligned}$$

что и требовалось.

Рассмотрим на симплектическом многообразии M гамильтонову систему уравнений

$\dot{x} = \operatorname{sgrad} H$ с гамильтонианом H. Напомним, что функция f называется первьлм интегралом системы уравнений, если f постоянна на решениях этой системы. Наличие первых интегралов позволяет уменьшить количество неизвестных функций, перейдя к рассмотрению уравнений на поверхности уровня первого интеграла.

Утверждение 11.5. Фужкия f является первым интегралом гамильтоновой системы $\dot{x} = \operatorname{sgrad} H$, если и только если $\{f, H\} = 0$.

Доказательство. Для доказательства достаточно заметить, что производная функции f вдоль решения системы $\dot{x} = \operatorname{sgrad} H$ равна $\operatorname{sgrad} H(f) = \{H, f\}$. Предложение доказано.

15.12 12: The case of Riemann connectivity (Riemann tensor)

Пусть M - риманово многообразие с римановой связностью. Тензор кривизны римановой связности часто называют тензором Римана соответствующего риманова многообразия.

15.12.1 12.1. New symmetries of the Riemann tensor

Мы начнем с доказательства тождества Якоби, которому удовлетворяет тензор кривизны произвольной симметричной аффинной связности.

Утверждение 12.1 (Тождество Якоби). Пусть M - многообразие с симметричной аффинной связностью, и R - тензор кривизны этой связности. Тогда для любых векторных полей X, Y и Z имеет место равенство

$$R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0$$

или, в локальных координатах, $R^i_{jpq} + R^i_{pqj} + R^i_{qjp} = 0$.

Доказательство. Заметим, что в силу полилинейности оператора кривизны R тождество Якоби достаточно проверить для координатных векторов, коммутаторы которых равны нулю. Поэтому тождество Якоби сводится к равенству

$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z + \nabla_Y \nabla_Z X - \nabla_Z \nabla_Y X + \nabla_Z \nabla_X Y - \nabla_X \nabla_Z Y,$$

где X, Y и Z - координатные векторные поля. Последнее равенство очевидно, так как в силу леммы 11.3

$$\nabla_X \nabla_Y Z - \nabla_X \nabla_Z Y = \nabla_X \left(\nabla_Y Z - \nabla_Z Y\right) = \nabla_X [Y, Z] = 0.$$

Утверждение доказано.

Если многообразие M - риманово, и Γ - риманова связность, то тензор кривизны этой связности обладает дополнительными симметриями.

Утверждение 12.2. Пусть M - риманово многообразие, Γ - риманова связность и R - соответствующий тензор кривизны.

- Для любых векторных полей X, Y, Z и W имеет место равенство $\langle R(X,Y)Z, W\rangle + \langle R(X,Y)W, Z\rangle = 0$, где $\langle \cdot, \cdot \rangle$ - скалярное произведение в римановой метрике. В локальных координатах имеем: $R_{ijpq} = -R_{jipq}$, где $R_{ijpq} = g_{i\alpha} R^\alpha_{jpq}$.
- Для любых векторных полей X, Y, Z и W имеет место равенство $\langle R(X,Y)Z, W\rangle = \langle R(Z,W)X, Y\rangle$, или в локальных координатах: $R_{ijpq} = R_{pqij}$

Доказательство. Для доказательства первого свойства рассмотрим оператор $R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ и изучим, как он действует на функциях. Очевидно

$$R(X,Y)(f) = X(Y(f)) - Y(X(f)) - [X,Y](f) = 0$$

для произвольной функции f. Поэтому требуемое равенство можно переписать в виде

$$0 = R(X,Y)(\langle Z,W\rangle) = \langle R(X,Y)Z,W\rangle + \langle Z,R(X,Y)W\rangle,$$

т.е. как аналог правила Лейбница для оператора $R(X,Y)$, действующего на скалярное произведение. Вычислим левую часть этого выражения.

$$\begin{aligned}
&R(X,Y)(\langle Z,W\rangle) = \nabla_X\left(\langle\nabla_Y Z,W\rangle + \langle Z,\nabla_Y W\rangle\right) - \\
&\quad - \nabla_Y\left(\langle\nabla_X Z,W\rangle + \langle Z,\nabla_X W\rangle\right) - \left\langle\nabla_{[X,Y]}Z,W\right\rangle - \left\langle Z,\nabla_{[X,Y]}W\right\rangle = \\
&= \langle\nabla_X\nabla_Y Z,W\rangle + \langle\nabla_Y Z,\nabla_X W\rangle + \langle\nabla_X Z,\nabla_Y W\rangle + \\
&\quad + \langle Z,\nabla_X\nabla_Y W\rangle - \langle\nabla_Y\nabla_X Z,W\rangle - \langle\nabla_X Z,\nabla_Y W\rangle - \\
&- \langle\nabla_Y Z,\nabla_X W\rangle - \langle Z,\nabla_Y\nabla_X W\rangle - \left\langle\nabla_{[X,Y]}Z,W\right\rangle - \left\langle Z,\nabla_{[X,Y]}W\right\rangle.
\end{aligned}$$

После сокращения получаем

$$\begin{aligned}
R(X,Y)(\langle Z,W\rangle) = &\langle\nabla_X\nabla_Y Z,W\rangle - \langle\nabla_Y\nabla_X Z,W\rangle - \left\langle\nabla_{[X,Y]}Z,W\right\rangle + \\
&+ \langle Z,\nabla_X\nabla_Y W\rangle - \langle Z,\nabla_Y\nabla_X W\rangle - \left\langle Z,\nabla_{[X,Y]}W\right\rangle = \\
&= \langle R(X,Y)Z,W\rangle + \langle Z,R(X,Y)W\rangle
\end{aligned}$$

что и доказывает первое свойство.

Второе свойство вытекает из уже доказанных нами симметрий тензора кривизны. Действительно, умножив тождество Якоби для полей X, Y и Z скалярно на W, мы получим равенство

$$\underbrace{\langle R(X,Y)Z,W\rangle}_{0} + \underbrace{\langle R(Y,Z)X,W\rangle}_{1} + \underbrace{\langle R(Z,X)Y,W\rangle}_{2} = 0.$$

Аналогично, сделав циклические перестановки всех четырех аргументов, найдем

$$\begin{aligned}
&\underbrace{\langle R(Y,Z)W,X\rangle}_{1} + \underbrace{\langle R(Z,W)Y,X\rangle}_{3} + \underbrace{\langle R(W,Y)Z,X\rangle}_{4} = 0, \\
&\underbrace{\langle R(Z,W)X,Y\rangle}_{3} + \underbrace{\langle R(W,X)Z,Y\rangle}_{5} + \underbrace{\langle R(X,Z)W,Y\rangle}_{2} = 0, \\
&\underbrace{\langle R(W,X)Y,Z\rangle}_{5} + \underbrace{\langle R(X,Y)W,Z\rangle}_{0} + \underbrace{\langle R(Y,W)X,Z\rangle}_{4} = 0.
\end{aligned}$$

Заметим теперь, что слагаемые, помеченные $0,1,3$ и 5 отличаются знаком в силу тождества $\langle R(X,Y)Z,W\rangle = -\langle R(X,Y)W,Z\rangle$, а слагаемые, помеченные 2 и 4 , совпадают в силу того же тождества и косой симметрии $R(X,Y) = R(Y,X)$. Поэтому, обозначив слагаемое, помеченное цифрой k через $\#k$, и сложив все четыре уравнения, получим

$$\begin{aligned}
(\#0+\#1+\#2) + (-\#1+\#3+\#4) + (-\#3+\#5+\#2) + (-\#5-\#0+\#4) = \\
= 2(\#2+\#4) = 0,
\end{aligned}$$

т.е.

$$\langle R(Z,X)Y,W\rangle + \langle R(W,Y)Z,X\rangle = \langle R(Z,X)Y,W\rangle - \langle R(Y,W)Z,X\rangle = 0,$$

что и требовалось. Утверждение доказано.

Поскольку евклидовы координаты в смысле римановой связности это, как мы уже знаем,

то же самое, что евклидовы координаты для римановой метрики, тензор кривизны на римановом многообразии отвечает за возможность выбора локальных координат, в которых метрика постоянна. А именно, имеет место следующее утверждение.

Утверждение 12.3. Если тензор кривизны на римановом многообразии не равен нулю, то на этом многообразии нельзя ввести координаты, в которых компоненты метрики были бы постоянны.

Замечание. Отметим, что при этом тензор кривизны достаточно вычислить в любой системе координат, поэтому это утверждение действительно эффективно.

Замечание. На самом деле имеет место и обратное утверждение: если тензор кривизны риманова многообразия равен нулю, то в окрестности каждой точки существуют координаты, в которых метрика постоянна.

Тензор кривизны - это очень сложный объект: он имеет $(\dim M)^4$ компонент (на самом деле, в силу симметрий независимых компонент меньше, см. ниже).

Упражнение 12.1. Сколько независимых компонент имеет тензор кривизны n-мерного риманова многообразия? Разобрать отдельно случаи $n = 2$ и $n = 3$.

Часто изучают различные свертки тензора кривизны, которые, оказывается тоже несут в себе определенную геометрическую информацию. Особенно часто встречаются две таких свертки.

Тензором Ричии римановой связности называется тензор $C_2^1(R)$, где R - тензор кривизны этой связности. Компоненты тензора Риччи $\tilde{R}$, очевидно, выглядят так: $\tilde{R}_{ij} = R^{\alpha}_{i_{\alpha}j} = g^{\alpha\beta}R_{\beta i\alpha j}$. Из последней формулы ясно, что тензор Риччи симметричен. Далее, скалярной кривизной
R риманова многообразия называется полная свертка тензора Риччи, у которого предварительно поднимают индекс: $R = g^{\alpha\beta}\tilde{R}_{\alpha\beta} = g^{\alpha\beta}R^{i}_{\alpha i\beta}$. Один пример использования этих сверток мы разберем ниже.

Нам также полезно будет найти явный вид компонент тензора кривизны римановой связности в терминах компонент римановой метрики. Ясно, что если ($x^1, \ldots, x^n$) - локальные координаты на римановом многообразии M, то компонента R_{ijpq} тензора кривизны, у которого опущен верхний индекс, может быть вычислена так:

$$R_{ijpq} = \langle R\left(\partial_p, \partial_q\right)\partial_j, \partial_i\rangle$$

где $\partial_k \equiv \partial_{x^k}$ - вектора из канонического базиса. Так как координатные векторные поля коммутируют, см. утверждение 11.2, можно переписать это выражение в виде

$$R_{ijpq} = \left\langle \nabla_{\partial_p}\nabla_{\partial_q}\partial_j, \partial_i\right\rangle - \left\langle \nabla_{\partial_q}\nabla_{\partial_p}\partial_j, \partial_i\right\rangle.$$

Вычислим первое слагаемое (второе получается из него перестановкой индексов). Для этого, воспользовавшись тем, что связность согласована с метрикой, перепишем это слагаемое в виде

$$\left\langle \nabla_{\partial_p}\nabla_{\partial_q}\partial_j, \partial_i\right\rangle = \nabla_{\partial_p}\left(\left\langle \nabla_{\partial_q}\partial_j, \partial_i\right\rangle\right) - \left\langle \nabla_{\partial_q}\partial_j, \nabla_{\partial_p}\partial_i\right\rangle.$$

Так как в силу леммы 7.4

$$\nabla_{\partial_q}\partial_j = \Gamma^{\alpha}_{jq}\partial_{\alpha}$$

а $\langle \partial_i, \partial_j\rangle = g_{ij}$ - компонента римановой метрики, имеем:

$$\left\langle \nabla_{\partial_p}\nabla_{\partial_q}\partial_j, \partial_i\right\rangle = \frac{\partial}{\partial x^p}\left(\Gamma^{\alpha}_{jq}g_{i\alpha}\right) - g_{\alpha\beta}\Gamma^{\alpha}_{jq}\Gamma^{\beta}_{ip}$$

Далее, вспомним, что символы Кристоффеля римановой связности выражаются через компоненты метрики так:

$$\Gamma^i_{jk} = \frac{1}{2} g^{i\alpha} \left(\frac{\partial g_{j\alpha}}{\partial x^k} + \frac{\partial g_{k\alpha}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^\alpha} \right).$$

Поэтому

$$\Gamma^\alpha_{jq} g_{i\alpha} = \frac{1}{2} \left(\frac{\partial g_{ji}}{\partial x^q} + \frac{\partial g_{qi}}{\partial x^j} - \frac{\partial g_{jq}}{\partial x^i} \right),$$

откуда

$$\frac{\partial}{\partial x^p} \left(\Gamma^\alpha_{jq} g_{i\alpha} \right) = \frac{1}{2} \left(\frac{\partial^2 g_{ji}}{\partial x^q \partial x^p} + \frac{\partial^2 g_{qi}}{\partial x^j \partial x^p} - \frac{\partial^2 g_{jq}}{\partial x^i \partial x^p} \right).$$

Таким образом,

$$\left\langle \nabla_{\partial_p} \nabla_{\partial_q} \partial_j, \partial_i \right\rangle = \frac{1}{2} \left(\frac{\partial^2 g_{ji}}{\partial x^q \partial x^p} + \frac{\partial^2 g_{qi}}{\partial x^j \partial x^p} - \frac{\partial^2 g_{jq}}{\partial x^i \partial x^p} \right) - g_{\alpha\beta} \Gamma^\alpha_{jq} \Gamma^\beta_{ip}.$$

Поэтому

$$\begin{aligned} R_{ijpq} &= \left\langle \nabla_{\partial_p} \nabla_{\partial_q} \partial_j, \partial_i \right\rangle - \left\langle \nabla_{\partial_q} \nabla_{\partial_p} \partial_j, \partial_i \right\rangle = \\ &= \frac{1}{2} \left(\frac{\partial^2 g_{qi}}{\partial x^j \partial x^p} - \frac{\partial^2 g_{jq}}{\partial x^i \partial x^p} - \frac{\partial^2 g_{pi}}{\partial x^j \partial x^q} + \frac{\partial^2 g_{jp}}{\partial x^i \partial x^q} \right) - g_{\alpha\beta} \left(\Gamma^\alpha_{jq} \Gamma^\beta_{ip} - \Gamma^\alpha_{jp} \Gamma^\beta_{iq} \right) \end{aligned}$$

Таким образом, доказано следующее утверждение.

Утверждение 12.4. Пусть M - риманово многообразие. Тогда компоненты тензора кривизны римановой связности на многообразии M в координатах $(x^1, \ldots, x^n)$ могут быть вычислены по формуле $R_{ijpq} == \frac{1}{2} \left(\frac{\partial^2 g_{qi}}{\partial x^j \partial x^p} - \frac{\partial^2 g_{jq}}{\partial x^i \partial x^p} - \frac{\partial^2 g_{pi}}{\partial x^j \partial x^q} + \frac{\partial^2 g_{jp}}{\partial x^i \partial x^q} \right) - g_{\alpha\beta} \left(\Gamma^\alpha_{jq} \Gamma^\beta_{ip} - \Gamma^\alpha_{jp} \Gamma^\beta_{iq} \right)$, где g_{ij} - компоненты римановой метрики, а Γ^i_{jk} - символы Кристоффеля в координатах $(x^1, \ldots, x^n)$.

15.12.2 12.2. 2D surface curvature tensor

В качестве иллюстрации, мы вычислим тензор кривизны римановой связности для регулярной двумерной поверхности M^2 в евклидовом пространстве $\mathbb{R}^3$ (метрика на M^2, естественно, индуцируется из $\mathbb{R}^3$). В частности, мы получим еще одно доказательство знаменитой теоремы Гаусса о независимости гауссовой кривизны от изометрии поверхности.

Итак, пусть $M^2 \subset \mathbb{R}^3$ - регулярная двумерная поверхность, и P произвольная точка на ней. Введем в $\mathbb{R}^3$ специальную декартову систему координат: поместим начало ее в точку P, оси x и y расположим в касательной плоскости $T_P M$, а ось z - по нормали к поверхности. Тогда, очевидно, поверхность M в окрестности точки P представляет собой график некоторой гладкой функции $z = f(x, y)$, причем $df(P) = 0$. Поэтому в координатах (x, y) на поверхности M матрица первой квадратичной формы имеет вид

$$(g_{ij}) = \begin{pmatrix} 1 + f_x f_x & f_x f_y \\ f_x f_y & 1 + f_y f_y \end{pmatrix},$$

где нижним индексом обозначено частное дифференцирование по соответствующей координате, причем в точке P метрика (g_{ij}) - это единичная матрица, а символы Кристоффеля $\Gamma^i_{jk} = 0$, так как все частные производные $\frac{\partial g_{ij}}{\partial x^k}$ в точке P равны нулю.

Теперь все готово для вычисления тензора кривизны. В случае двумерного многообразия тензор кривизны R имеет, вообще говоря, 16 компонент R_{ijpq}. Однако, в силу косых симметрий тензора кривизны, среди

этих компонент отличны от нуля лишь те, у которых первые два и последние два индекса различны. Таких компонент четыре, причем они отличаются друг от друга знаками:

$$R_{1212} = -R_{2112} = R_{2121} = -R_{1221}.$$

В силу утверждения 12.4 и равенства символов Кристоффеля в точке P нулю, компонента R_{1212} может быть вычислена так:
$R_{1212} = \frac{1}{2}\left(\frac{\partial^2 g_{21}}{\partial y \partial x} - \frac{\partial^2 g_{22}}{\partial x \partial x} - \frac{\partial^2 g_{11}}{\partial y \partial y} + \frac{\partial^2 g_{21}}{\partial x \partial y}\right) = \frac{1}{2}\left(2\frac{\partial^2 g_{21}}{\partial y \partial x} - \frac{\partial^2 g_{22}}{\partial x \partial x} - \frac{\partial^2 g_{11}}{\partial y \partial y}\right)$,
где $x^1 = x$ и $x^2 = y$. Воспользовавшись явным видом метрики g_{ij}, найдем:

$$\begin{aligned} R_{1212} &= \frac{1}{2}\left(2\left(f_x f_y\right)_{xy} - \left(1 + (f_y)^2\right)_{xx} - \left(1 + (f_x)^2\right)_{yy}\right) = \\ &= \left(f_{xx} f_y + f_x f_{yx}\right)_y - \left(f_x f_{xy}\right)_y - \left(f_y f_{yx}\right)_x \end{aligned}$$

Наконец, запишем значение компоненты R_{1212} в точке P, в которой, напомним, все первые частные производные функции f обращаются в нуль, поэтому соответствующие слагаемые можно не писать:

$$R_{1212} = \int_{xx}\int_{yy} + \int_{xy}\int_{xy} - \int_{xy}\int_{xy} - \int_{yx}\int_{yx} = \int_{xx}\int_{yy} - \int_{yx}\int_{yx} = K(P)$$

где через $K(P)$ обозначена гауссова кривизна поверхности M в точке P. (Напомним, что гауссова кривизна вычисляется как отношение определителей первой и второй квадратичных форм. Определитель первой квадратичной формы в точке P равен единице. Вторая квадратичная форма поверхности M в точке P в координатах (x, y) совпадает с матрицей Гессе функции f.) Итак, компонента R_{1212} в координатах (x, y) совпадает с гауссовой кривизной поверхности. Вычислим теперь скалярную кривизну $R(P)$ поверхности M в точке P. По определению,

$$R(P) = g^{\alpha\beta} R^i_{\alpha i \beta} = g^{\alpha\beta} g^{ij} R_{j\alpha i\beta}$$

В силу косых симметрий тензора $R_{j\alpha i\beta}$ имеем:

$$\begin{aligned} R(P) &= g^{22} g^{11} R_{1212} + g^{12} g^{12} R_{2112} + g^{22} g^{11} R_{2121} + g^{12} g^{12} R_{1221} = \\ &= R_{1212}\left(g^{22} g^{11} - g^{12} g^{12} + g^{22} g^{11} - g^{12} g^{12}\right) = \frac{2R_{1212}}{\det\left(g_{ij}\right)} \end{aligned}$$

В частности, в точке P имеем

$$R(P) = 2K(P)$$

Отметим, что последнее равенство - это равенство скаляров, поэтому оно не зависит от выбора системы координат. Итак, нами доказан следующий результат.

Утверждение 12.5. Пусть M - регулярная поверхность в $\mathbb{R}^3$. Тогда ее скалярная кривизна равна удвоенной гауссовой кривизне.

Следствие 12.1 (Теорема Гаусса). Гауссова кривизна двумерной регулярной поверхности полностью определяется первой квадратичной формой этой поверхности. В частности, гауссова кривизна не меняется при изометриях.

Доказательство. Это очевидно, так как компоненты тензора кривизны римановой связности полностью определяются компонентами метрического тензора.

15.12.3 12.3. Independent components of the Riemann tensor

В предыдущем разделе мы убедились, что тензор Римана двумерного многообразия имеет лишь одну независимую компоненту, а именно R_{1212}. Это означает, что все остальные компоненты или равны нулю, или вычисляются по R_{1212} (на самом деле, не нулевые компоненты или совпадают с $R_{12\ 12}$, или отличаются знаком, см. выше).

Сосчитаем теперь количество независимых компонент тензора Римана в общем случае n-мерного риманова многообразия. Для этого воспользуемся известными нам симметриями этого тензора, см. утверждения 12.1 и 12.2. Во-первых, тензор Римана R_{ijkl} кососимметричен по первой ij и последней kl парам индексов. Это означает, что каждая из этих пар принимает $N = C_n^2$ независимых значений. Далее, тензор Римана симметричен относительно перестановки первой и последней пар индексов. Другими словами, если $I = ij$ и $K = kl$ - мультииндексы, то $R_{IK} = R_{KL}$. Таким образом, у нас имеется симметричная «матрица», индексы I и K которой принимают N значений. Такая «матрица» имеет $N(N+1)/2$ независимых компонент. Итак, мы учли все известные симметрии тензора Римана, кроме тождества Якоби, и получили

$$\frac{N(N+1)}{2} = \frac{n(n-1)/2(n(n-1)/2+1)}{2} = \frac{(n^2-n)\,(n^2-n+2)}{8}$$

компопент.

Итак, нам остается учесть тождества Якоби. Вообще говоря, тождество Якоби следует записывать для каждой четверки индексов. Каждое такое тождество представляет собой линейное уравнение на компоненты тензора Римана. Вычислим, сколько среди этих уравнений нетривиальных и независимых. Имеет место следующая лемма.

Лемма 12.1. Если среди четверки индексов i, j, k, l есть пара совпадающих, то тождество Якоби вытекает из других симметрий тензора Римана. Если две четверки индексов $i, j, k, lui_1, j_1, k_1, l_1$ получены друг из друга перестановкой, то соответствующие тождества Якоби равносильны.

Доказательство. Докажем первое утверждение леммы. Если, например, совпадают первые два индекса, то имеем:
$R_{iikl} + R_{ikli} + R_{ilik} = 0 + R_{ikli} + R_{ilik} = R_{ikli} + R_{ikil} = R_{ikli} - R_{ikli} = 0.$
Остальные случаи разбираются аналогично.
Второе утверждение леммы достаточно проверить для транспозиций. Сравним, например, тождества Якоби

$$R_{ijkl} + R_{iklj} + R_{iljk} = 0 \quad \text{и} \quad R_{jikl} + R_{jkli} + R_{jlik} = 0,$$

отличающиеся транспозицией (ij). Так как, в силу известных симметрий, $R_{jikl} = -R_{ijkl}, R_{jkli} = R_{lijk} = -R_{iljk}$, а $R_{jlik} = R_{ikjl} = -R_{iklj}$, имеем:

$$R_{jikl} + R_{jkli} + R_{jlik} = -R_{ijkl} - R_{iljk} - R_{iklj},$$

т.е. второе выписанное нами тождество вытекает из первого. Остальные случаи разбираются аналогично. Лемма доказана.

Из доказанной нами Леммы 12.1 вытекает, что среди тождеств Якоби следует учитывать в точности те, которые отвечают различным четверкам попарно различных индексов. Таких четверок, а значит и тождеств, тождеств имеется C_n^4. Таким образом, число независимых компонент тензора Римана равно

$$\frac{(n^2-n)\,(n^2-n+2)}{8} - \frac{n(n-1)(n-2)(n-3)}{24} = \frac{n^2\,(n^2-1)}{12}.$$

Мы доказали следуютций результат.
Утверждение 12.6. Количество независимых компонент тензора Pu мана n-мерного риманова многообразия равно $n^2\left(n^2-1\right)/12$.

Упражнение 12.2. В частном случае $n=3$, число независимых компонент тензора Римана равно 6 , т.е. их ровно столько же, сколько компонент у тензора Риччи в этом случае (напомним, тензор Риччи - это симметричный тензор типа $(0,2)$). Показать, что компоненты тензора Римана в трехмерном случае выражаются через компоненты тензора Риччи.
М. Громов в своих замечательных лекциях о тензоре Римана [5] приводит следующее рассуждение, объясняющее правдоподобность теоремы, обратной к утверждению 12.3: если тензор кривизны риманова многообразия равен нулю, то в окрестности каждой точки существуют координаты, в которых метрика постоянна. Громов предлагает воспользоваться соображсниями размсрности. А имснно, пусть $(y^1,\ldots,y^n)$ -

некоторые координаты на n-мерном римановом многообразии в окрестности точки P, и (g_{ij}) - компоненты метрического тензора в этих координатах. Наша цель - выбрать на многообразии такие новые координаты $x^i=x^i\left(y^1,\ldots,y^n\right)$, что в них компоненты метрического тензора будут постоянны. Хорошо известно, что линейной заменой координат можно добиться того, что в точке P компоненты метрики будут иметь вид δ_{ij}. При этом, очевидно, линейная замена определяется первыми производными $\frac{\partial x^i}{\partial y^j}$ в точке P. Далее, нетривиальным, но тоже хорошо известным является тот факт, что можно подобрать такую замену координат, что и первые производные компонент метрического тензора (в новых координатах) окажутся равными нулю в точке P. Соответствующая конструкция была описана выше, однако эта возможность может быть предсказана из размерностных соображений. А именно, различных первых производных $\frac{\partial g_{ij}}{\partial x^k}$ метрики имеется $M=n^2(n+1)/2$ штук ($n(n+1)/2$ функций и n дифференцирований). Первые производные метрики определяются, очевидно, вторыми производными $\frac{\partial x^i}{\partial y^j \partial y^k}$ координат, которых в нашем распоряжении имеется тоже $M=n^2(n+1)/2$ штук (n функций и $n(n+1)/2$ дифференцирований). Таким образом, управляя M параметрами мы зануляем M величин.

Следующий шаг состоит в попытке обратить в нуль вторые производные метрики, управляя третьими производными координат. Вторых производных от метрики $n^2(n+1)^2/4$ штук ($n(n+1)/2$ функций и столько же дифференцирований), третьих производных от координат $-nC_{n+2}^3$ штук (n функций и C_{n+2}^3 дифференцирований). Таким образом, количество производных больше имеющегося у нас в распоряжении количества параметров на

$$\frac{n^2(n+1)^2}{4}-\frac{(n+2)(n+1)n^2}{6}=\frac{n^2\left(n^2-1\right)}{12},$$

что в точности равно числу независимых компонент тензора Римана. Оказывается (еще раз подчеркнем, что мы и не думали этого доказывать), как раз обращение в нуль компонент тензора Римана позволяет обратить в нуль вторые производные метрики, а вслед за этим сделать метрику постоянной в целой окрестности точки P.

15.12.4 Problems

Задача 12.1. Вычислить скалярную кривизну сферы радиуса R, плоскости Лобачевского, прямого кругового цилиндра.

[5] М. Громов, Знак и геометрический смысл кривизны, Ижевск, Издательский дом «Удмуртский университет», 1999.

Задача 12.2. Пусть S^n - стандартная сфера радиуса R в евклидовом пространстве $\mathbb{R}^{n+1}$. Показать, что ее тензор кривизны может быть задан формулой

$$R(X,Y)Z = \frac{1}{R^2}(\langle Y,Z\rangle X - \langle X,Z\rangle Y)$$

Задача 12.3. Риманово многообразие называется пространством Эйнштейна, если $\tilde{R}_{ij} = \lambda(x)g_{ij}$, т.е. метрический тензор пропорционален тензору Риччи. Показать, что любое двумерное риманово многообразие является пространством Эйнштейна.

Задача 12.4. На трехмерном римановом многообразии выразить компоненты тензора кривизны через компоненты тензора Риччи, метрику и скалярную кривизну.

15.12.5 Extra Topics

12.1. Тензор Римана трехмерного риманова многообразия. Как следует из утверждения 12.6 , тензор кривизны трехмерного риманова многообразия имеет 6 независимых компонент. Напомним, что тензор Риччи $\tilde{R}_{ij}$ представляет собой симметричную билинейную форму, поэтому тоже имеет 6 независимых компонент. Покажем, что в трехмерном случае компоненты тензора Римана выражаются через компоненты тензора Риччи.

По определению, $\tilde{R}_{ij} = g^{\alpha\beta}R_{\alpha i\beta j}, 1 \le i \le j \le 3$. Посмотрим на эти 6 соотношений как на систему линейных уравнений на неизвестные $R_{\alpha i\beta j}$. Из симметрий тензора Римана вытекает, что достаточно найти компоненты $R_{\alpha i\beta j}$ тензора Римана, для которых $\alpha < i, \beta < j$ и пара (α, i) не превосходит пары (β, j) в естественном лексикографическом порядке (т.е. или $\alpha < \beta$, или $\alpha = \beta$ и $i \le j$). Таких компонент, как мы знаем, 6. Вот их полный список:

$$R_{1212}, \quad R_{1213}, \quad R_{1223}, \quad R_{1313}, \quad R_{1323}, \quad R_{2333}.$$

Итак, у нас имеется 6 линейных уравнений на 6 неизвестных. Компоненты матрицы этой линейной системы выражаются через компоненты метрического тензора g^{ij}. А именно,

$$\begin{aligned}
R_{11} &= g^{22}R_{1212} + 2g^{23}R_{1213} + g^{33}R_{1313},\\
R_{22} &= g^{11}R_{1212} - 2g^{13}R_{1223} + g^{33}R_{2323},\\
R_{33} &= g^{11}R_{1313} + 2g^{12}R_{1323} + g^{22}R_{2323},\\
R_{12} &= -g^{12}R_{1212} - g^{13}R_{1213} + g^{23}R_{1223} + g^{33}R_{1323}\\
R_{13} &= -g^{12}R_{1213} - g^{22}R_{1223} - g^{13}R_{1313} - g^{23}R_{1323}\\
R_{23} &= g^{11}R_{1213} + g^{12}R_{1223} - g^{13}R_{1323} - g^{23}R_{2323}.
\end{aligned}$$

Определитель этой системы равен $2\left(\det g^{ij}\right)^2$, поэтому не равен нулю. Таким образом, система всегда разрешима, что и позволяет выразить компоненты тензора Римана через компоненты метрики и тензор Риччи.

15.13 13: Degree of Display

В этой и следующей лекции мы приведем несколько иллюстраций, демонстрирующих связь дифференциальной геометрии и топологии. Один пример результатов подобного рода у нас уже был, а именно, теория когомологий де Рама. Здесь мы рассмотрим ряд конструкций, связанных с понятием степени отображения, к определению которого мы и переходим.

15.13.1 13.1. Definition and main properties of the degree

Пусть M и N - гладкие связные замкнутые (т.е., напомним, компактные и без края) ориентированные многообразия одной и той же размерности n, и $f : M \to N$ - гладкое отображение. Пусть $y \in N$ регулярное значение отображения f. Последние, напомним, означает, что отображение f регулярно в каждой точке x прообраза $f^{-1}(y)$ точки y при отображении f. Другими словами, если $(x^1, \ldots, x^n)$ и $(y^1, \ldots, y^n)$ - локальные координаты в окрестности точки $x \in f^{-1}(y)$ и точки y соответственно, и $y^i(x^1, \ldots, x^n)$ - координатное представление отображения f в этих координатах, то определитель матрицы Якоби $\frac{\partial y^i}{\partial x^k}$ в точке x отличен от нуля. Отметим, что если прообраз точки y пуст, то y - регулярное значение по определению.

Из компактности M вытекает (проверьте!), что прообраз $f^{-1}(y)$ состоит из конечного числа точек. Отметим также, что в силу ориентируемости рассматриваемых многообразий, можно считать, что знак определителя матрицы Якоби $\frac{\partial y^i}{\partial x^k}$ в точке x не зависит от выбора локальных координат (для этого следует фиксировать ориентации на обоих многообразиях). Из сказанного вытекает корректность следующего определения.

Определение. Пусть $f : M \to N$ - гладкое отображение гладких компактных связных замкнутых ориентируемых многообразий одинаковой размерности, и $y \in N$ - некоторое его регулярное значение. Степенью отображения f по отношению κ регулярному значению y называется число

$$\deg f(y) = \sum_{x_\alpha \in f^{-1}(y)} \operatorname{sign} \det \left(\frac{\partial y^i}{\partial x^k}(x_\alpha) \right).$$

Важность понятия степени определяется следующим результатом.

Теорема 13.1. Пусть $f : M \to N$ - гладкое отображение гладких связных замкнутых ориентируемьх многообразий одинаковой размерности. Тогда степень отображения f не зависит от выбора регулярного значения и не меняется при гомотопиях отображения.

Доказательство. Нам понадобится следующая лемма.

Лемма 13.1. Пусть P и P' - две произвольные точки связного многообразия M. Тогда существует гладкое семейство диффеоморфизмов $\varphi_t : M \to M, t \in [0,1]$, такос что φ_0 - тождсственнос отображение, а $\varphi_1(P) = P'$. Более того, если многообразие M ориентировано, то диффеоморфизмы φ_t сохраняют ориентацию.

Доказательство. Пусть $n = \dim M$. В силу связности многообразия M, точки P и P' можно соединить простым регулярным путем, поэтому можно выбрать открытое множество $U \subset M$, диффеоморфное $\mathbb{R}^n$ и содержащее обе точки P и P' (для этого достаточно взять малую окрестность упомянутого пути). Более того, в U можно выбрать такую локальную систему координат $(x^1, \ldots, x^n)$, что $P = (0, \ldots, 0)$, $P' = (1, 0, \ldots, 0)$, а выбранный путь - это первая координатная линия.

Рассмотрим открытые подмножества U' и U'' в U, обладающие следующими свойствами: $[P, P'] \subset U' \subset U'' \subset \bar{U}'' \subset U$, и замыкание $\bar{U}''$ компактно. Тогда, как известно, существует гладкая функция ψ на U, равная 1 в U' и нулю вне U''. В окрестности U, во введенных координатах $(x^1, \ldots, x^n)$, рассмотрим векторное поле $X = \psi \partial_{x^1}$, и пусть φ'_t - однопараметрическая группа диффеоморфизмов, соответствующая X, т.е. $X = d\varphi'_t / dt|_{t=0}$, существование которой известно из теории обыкновенных дифференциальных уравнений. [6] По определению, при $t = 0$ диффеоморфизм φ'_t является тождественным преобразованием, а $\varphi'_1(P) = P'$. Более того, все φ'_t являются тождественными отображениями вне U''. Продолжим семейство диффеоморфизмов φ'_t вне окрестности U тождественным преобразованием. Тем самым мы построим искомое семейство φ_t. Осталось заметить, что в локальных координатах ($x^1, \ldots, x^n$) отображение φ_t - это просто сдвиг вдоль координаты x^1, поэтому оно

сохраняет ориентацию. Доказательство закончено.

Вернемся к доказательству теоремы. Из леммы 13.1 вытекает, что ее первое утверждение является следствием второго. Действительно, пусть y и y' - регулярные значения отображения f. Тогда, в силу леммы 13.1, существует однопараметрическое семейство диффеоморфизмов φ_t многообразия N, такое что $\varphi_0 = 1_N$, и $\varphi_1(y) = y'$. Ясно, что точка y является регулярной для всех отображений $\varphi_t \circ f$. Из второго утверждения теоремы вытекает, что $\deg f(y) = \deg \varphi_t \circ f(y)$. Поэтому, в частности, $\deg f(y) = \deg \varphi_1 \circ f(y)$. Осталось заметить, что якобиан отображения

$\varphi_1 \circ f$ в каждой точке из $(\varphi_1 \circ f)^1(y)$ равен произведению положительного якобиана отображения φ_1 в точке y и якобиана отображения f в соответствующей точке из $f^{-1}(y')$, откуда $\deg(\varphi_1 \circ f)(y) = \deg f(y')$, что и требовалось. Итак, для завершения доказательства теоремы достаточно проверить инвариантность степени отображения при гомотопиях этого отображения.

Пусть f и g - два гомотопных отображения многообразия M в многообразие N, и $F : [0,1] \times M \to N$ - гладкая гомотопия, их соединяющая: $f = F|_{\{0\}\times M}$, а $g = F|_{\{1\}\times M}$. Пусть $y \in N$ - регулярное значение обоих отображений (такое найдется в силу теоремы Сарда). Из теоремы о неявной функции вытекает, что каждая достаточно близкая к y точка $y' \in N$ также является регулярным значением для обоих отображений. Кроме того, для любого достаточно близкого к y регулярного значения y' имеем: $\deg f(y') = \deg f(y)$, а $\deg g(y') = \deg g(y)$. Далее, снова по теореме Сарда, в любой окрестности точки y существует точка y', являющаяся регулярным значением для отображения F. Итак, выберем y' так, чтобы y' была регулярным значением F, f и g, и выполнялись условия $\deg f(y') = \deg f(y)$, и $\deg g(y') = \deg g(y)$. Ясно, что доказательство достаточно провести для такой точки y'.

По теореме о неявной функции, прообраз $F^{-1}(y')$ является одномерным подмногообразием в $[0,1] \times M$. Более того, легко показать, что край $\partial F^{-1}(y')$ лежит в крае многообразия $[0,1] \times M$, т.е. в объединении $(\{0\} \times M) \cup (\{1\} \times M)$. Так как прообраз $\partial F^{-1}(y')$ - замкнутое, и, значит, компактное подмножество в M, этот прообраз состоит из конечного набора замкнутых кривых, диффеоморфных окружности, и незамкнутых кривых, диффеоморфных отрезку.

Обозначим через I_k незамкнутые кривые из $\partial F^{-1}(y')$. Концы a_k и b_k отрезков I_k лежат на крае многообразия $[0,1] \times M$ и составляют множество $f^{-1}(y') \cup g^{-1}(y')$. Более того, так как точки a_k и b_k являются регулярными для отображений f и g, векторы скоростей каждой кривой I_k в этих ее концевых точках не касаются края многообразия $[0,1] \times M$ (иначе дифференциал соответствующего отображения f или g переводит их в нуль).

Для завершения доказательства теоремы, покажем, что если концы a_k и b_k отрезка I_k лежат в одной компоненте связности края многообразия $[0,1] \times M$, то знаки якобианов ограничения отображения F на эту компоненту (т.е. знаки якобианов одного из отображений f и g) в точках a_k и b_k противоположны, а если a_k и b_k лежат в разных компонентах связности, то знаки якобианов отображений f и g в этих точках одинаковы. Это завершит доказательство теоремы, так как в первом случае точки a_k и b_k дают нулевой вклад в степень соответствующего отображения f или g, а во втором случае - дают одинаковый вклад в степени обоих отображений f и g.

Для каждого I_k рассмотрим на нем стандартную координату φ, та-

кую что точка a_k соответствует $\varphi = 0$, а точка b_k соответствует $\varphi = 1$. На

[06] Напомним, что диффеоморфизмы φ'_t получаются так. Пусть $\dot{x}(t) = X(x)$ - система обыкновенных дифференциальных уравнений, и $x = x(t, x_0)$ - решение задачи Коши для пачальшых условий $x(0) = x_0$. Тогда отображение $x_0 \mapsto x(t, x_0)$ и ссть диффеоморфизм φ'_t. Гладкость и взаимная однозначность отображений φ'_t - следствия теорем существования, единственности и гладкой зависимости от начальных условий решения задачи Коши.

многообразии $[0,1] \times M$ рассмотрим системы координат следующего вида. Пусть t - стандартная координата на отрезке $[0,1]$, а $(x^1, \ldots, x^n)$ - координаты на M из ориентированного атласа, определенные в некотором открытом множестве $U \subset M$. Тогда $(t, x^1, \ldots, x^n)$ - это координаты на $[0,1] \times U$, порождающие ориентированный атлас на $[0,1] \times M$.

Пусть a_k и b_k лежат в одной связной компоненте края многообразия $[0,1] \times M$. Предположим для определенности, что a_k и b_k лежат на компоненте $\{0\} \times M$. Пусть $U_\alpha \subset M$ и $U_\beta \subset M$ - некоторые окрестности точек a_k и b_k соответственно, а $(x_\alpha^1, \ldots, x_\alpha^n)$ и $\left(x_\beta^1, \ldots, x_\beta^n\right)$ - координаты в U_α и U_β из ориентированного атласа на M. Как и выше, рассмотрим на $[0,1] \times U_\alpha$ и $[0,1] \times U_\beta$ координаты $(t, x_\alpha^1, \ldots, x_\alpha^n)$ и $\left(t, x_\beta^1, \ldots, x_\beta^n\right)$. Тогда в некоторой окрестности точки a_k в $[0,1] \times M$ кривая I_k имеет вид $\gamma_\alpha(\varphi) = (t(\varphi), x_\alpha^1(\varphi), \ldots, x_\alpha^n(\varphi))$, а в некоторой окрестности точки b_k вид $\gamma_\beta(\varphi) = \left(t(\varphi), x_\beta^1(\varphi), \ldots, x_\beta^n(\varphi)\right)$. Ясно, что базисы $\left\{\partial_t, \partial_{x_\alpha^1}, \ldots, \partial_{x_\alpha^n}\right\}$ и $\left\{\gamma_\alpha'(0), \partial_{x_\alpha^1}, \ldots, \partial_{x_\alpha^n}\right\}$, определенные в точке a_k, имеют одинаковые ориентации, так как $dt/d\varphi(0) > 0$, в то время как базисы $\left\{\partial_t, \partial_{x_\beta^1}, \ldots, \partial_{x_\beta^n}\right\}$ и $\left\{\gamma_\beta'(1), \partial_{x_\beta^1}, \ldots, \partial_{x_\beta^n}\right\}$, определенные в точке b_k, имеют противоположные ориентации, так как $dt/d\varphi(1) < 0$.

Рассмотрим вдоль I_k такую систему из n векторных полей $\{X^p(\varphi)\}$, что в точке a_k семейство векторов $\{X^p(0)\}$ совпадает с базисом $\{\partial_{x_\alpha^p}\}$ касательного пространства к компоненте края $\{0\} \times M$, в точке b_k векторы $\{X^p(1)\}$ образуют некоторый базис в касательном пространстве к этой же компоненте, и в каждой точке кривой I^k система векторов, получающаяся из $\{X^p(\varphi)\}$ добавлением вектора скорости кривой I_k, линейно независима, т.е. образует базис $E(\varphi)$ в касательном пространстве к многообразию $[0,1] \times M$. Ясно, что ориентации всех базисов $E(\varphi)$ семейства одинаковы, в частности, одинаково ориентированы базисы $E(0) = \left\{\gamma_\alpha'(0), \partial_{x_\alpha^1}, \ldots, \partial_{x_\alpha^n}\right\}$ и $E(1) = \left\{\gamma_\beta'(1), X^1(1), \ldots, X^n(1)\right\}$. С другой стороны, t-компоненты векторов $\gamma_\alpha'(0)$ и $\gamma_\beta'(1)$ имеют противоположные знаки, поэтому базисы $\{X^p(1)\}$ и $\left\{\partial_{x_\beta^i}\right\}$ имеют противоположную ориентацию.

Остается заметить, что отображение F в каждой точке кривой I_k переводит семейство векторов $\{X^p(\varphi)\}$ в некоторый базис касательного пространства к многообразию N в точке y', причем ориентации всех полученных в y' базисов одинаковы. (Последнее вытекает из непрерывной зависимости этих базисов от параметра φ.) Таким образом, мы видим, что базисы $\left\{df\left(\partial_{x_\alpha^i}\right)\right\}$ и $\{df\left(X^p(1)\right)\}$ одинаково ориентированы, и их ориентация противоположна ориентации базиса $\left\{df\left(\partial_{x_\beta^i}\right)\right\}$. Следовательно, знаки якобиана отображения f в точках a_k и b_k противоположны.

Случай когда точки a_k и b_k лежат в разных компонентах края многообразия $[0,1] \times M$ разбирается аналогично, и оставляется в качестве

обязательного упражнения. Доказательство теоремы закончено.

Доказанная тсорсма 13.1 позволяст коррсктно опредслить степень отображения $\deg f$ как $\deg f(y)$ для произвольного регулярного значения y.

Замечание. Для неориентируемых многообразий можно определить степень отображения, складывая якобианы по модулю 2 .

Приведем теперь некоторые примеры использования степени отображения.

15.13.2 13.2. Fundamental theorem of algebra

Мы сформулируем и докажем Основную теорему алгебры в следующем упрощенном виде.

Теорема 13.2. Каждый комплексный многочлен степени $n \geq 1$ имеет по крайней мере один корень.

Пусть $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ - произвольный комплексный полином степени n. Очевидно, он задает гладкое (даже комплексно-аналитическое) отображение из плоскости в плоскость: $w = P(z)$. Однако, чтобы использовать понятие степени, нам нужно отображение компактных многообразий, поэтому мы продолжим P до отображения f_P сферы S^2 в себя так. Напомним, во первых, что сфера S^2 диффеоморфна комплексной проективной прямой $\mathbb{CP}^1$ (см. также дополнительный материал). Рассмотрим два экземпляра $\mathbb{CP}^1$ и фиксируем на каждом из них однородные координаты $(\ z_0 : z_1\ )$ и $(\ w_0 : w_1\ )$ соответственно. Рассмотрим карты $z_0 \neq 0$ с координатой $z = z_1/z_0$ и $w_0 \neq 0$ с координатой $w = w_1/w_0$ и зададим в этих картах отображение f_P, положив $w = P(z)$.

Покажем, что заданное так отображение действительно продолжается до гладкого отображения $f_P : \mathbb{CP}^1 \to \mathbb{CP}^1$. Для этого заметим, что

$$w = \frac{w_1}{w_0} = P(z) = \frac{z_1^n + a_{n-1}z_1^{n-1}z_0 + \cdots + a_1 z_1 z_0^{n-1} + a_0 z_0^n}{z_0^n},$$

и зададим f_P в однородных координатах так:

$$\begin{aligned} w_1\left(z_0 : z_1\right) &= z_1^n + a_{n-1}z_1^{n-1}z_0 + \cdots + a_1 z_1 z_0^{n-1} + a_0 z_0^n \\ w_0\left(z_0 : z_1\right) &= z_0^n \end{aligned}$$

В картах $\left(z_0 \neq 0, z = z_1/z_0\right), \left(w_0 \neq 0, w = w_1/w_0\right)$ отображение f_P, как мы уже видели, имеет вид $w = P(z)$ и, потому, является комплексноаналитическим. Отметим, что эта пара карт покрывает обе сферы $\mathbb{CP}^1$,

за исключением точек $z_0 = 0$ и $w_0 = 0$, соответствующих бесконечности. В окрестностях бесконечностей рассмотрим другие карты $z' = z_0/z_1$ и $w' = w_0/w_1$. Ясно, что

$$w' = \frac{w_0}{w_1} = \frac{z_0^n}{z_1^n + a_{n-1}z_1^{n-1}z_0 + \cdots + a_0 z_0^n} = \frac{\left(z'\right)^n}{1 + a_{n-1}z' + \cdots + a_0\left(z'\right)^n}$$

поэтому при $|z'| < \varepsilon$, для достаточно малого ε, координатное представление $w' = w'\left(z'\right)$ также является комплексно-аналитическим. Мы доказали следующую лемму.

Лемма 13.2. Построенное выше отображение $f_P : \mathbb{CP}^1 \to \mathbb{CP}^1$, совпадающее в конечной области с полиномом P, является комплексно-аналитическим.

Вычислим теперь степень отображения f_P. Для этого рассмотрим гладкую гомотопию F_t, заданную в однородных координатах так:

$$\begin{aligned} w_1\left(z_0 : z_1, t\right) &= z_1^n + t\left(a_{n-1}z_1^{n-1}z_0 + \cdots + a_1 z_1 z_0^{n-1} + a_0 z_0^n\right) \\ w_0\left(z_0 : z_1, t\right) &= z_0^n \end{aligned}$$

По теореме 13.1, все отображения $F_t : \mathbb{CP}^1 \to \mathbb{CP}^1$ имеют равные степени. При этом, в конечной области отображение F_1 совпадает с многочленом $P(z)$, а F_0 имеет вид $w(z) = z^n$. Вычислим степень отображения F_0. Для этого рассмотрим его значение $w = 1$. Нам следует проверить, что 1 - регулярное значение отображения F_0, найти прообразы (это просто, прообразы значения 1 суть примитивные корни n-ой степени из 1 , которых имеется ровно n штук), и определить знак якобиана в каждом прообразе.

Лемма 13.3. Пусть $f : \mathbb{C} \to \mathbb{C}$ - комплексно-аналитическое отображение. Тогда его якобиан неотрицателен и равен $|f'(z)|^2$, где штрих означает дифференцирование по комплексной переменной z.

Доказательство. Пусть $w = f(z)$ - гладкое отображение плоскости в себя, заданное коплекснозначной функцией комплексного переменного. Положив $z = x + iy$ и $w = u + iv$, запишем отображение f в следующем виде: $f(x, y) = u(x, y) + iv(x, y)$, где $u(x, y)$ и $v(x, y)$

- вещественная и мнимая части величины $f(x,y)$. Тогда якобиан J отображения f имеет, очевидно, вид

$$J = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

С другой стороны, $x = (z+\bar{z})/2, y = -i(z-\bar{z})/2$, откуда

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$$

и условие аналитичности функции f, т.е. условие $\partial f/\partial \bar{z} = 0$, принимает вид

$$\frac{\partial f}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)(u+iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = 0$$

что равносильно условиям Коши-Римана

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Кроме того, воспользовавшись условиями Коши-Римана, получаем
$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)(u+iv) = \frac{1}{2}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + \frac{i}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$,
поэтому

$$J = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = \left|\frac{\partial f}{\partial z}\right|^2 = \left|\frac{df}{dz}\right|^2 \geq 0$$

где последнее равенство имеет место, так как

$$\frac{\partial f}{\partial z} = \frac{df}{dz}$$

Лемма доказана.
Теперь проверка регулярности значения $w = 1$ тривиальна, знаки якобианов определены, и, тем самым, доказана следующая лемма.

Лемма 13.4. Степень каждого из отображений $F_t, t \in [0,1]$, равна n.
Осталось заметить, что так как степень отображения F_t отлична от нуля, то множество $F_t^{-1}(y)$ не пусто для каждого $y \in \mathbb{C}$. (Действительно, если $F_t^{-1}(y_0)$ пусто, то y_0 - регулярное значение и $\deg F_t = 0$, противоречие.) В частности, не пусто и множество $F_1^{-1}(0)$, т.е. уравнение $F_1(z) = 0$ имеет корни, что и требовалось.

15.13.3 13.3. Theorem "about what"

Пусть X - касательное векторное поле на многообразии M. Точка $P \in M$ называется особой для X, если $X(P) = 0$.

Приведем результат, известный в математическом фольклоре как «Теорема о еже». Эта теорема позволяет решить следующую задачу: существует ли на сфере S^2 касательное векторное поле без особых точек, т.е. можно ли расположить «иголки» (касательные векторы) сферического «ежа» так, чтобы в результате «прически» не возникло «пробора» (т.е. особых точек)? Имеет место следующий результат.

Теорема 13.3. На сфере S^{2k} четной размерности не существует касательного векторного поля без особых точек.

Доказательство. Рассмотрим сферу S^{2k} стандартно вложенную в $\mathbb{R}^{2k+1}$, и предположим, что на ней существует касательное векторное поле Y без особых точек.

Тогда векторное поле $X(P) = Y(P)/\|Y(P)\|$ также является гладким касательным векторным полем на S^{2k}.

В каждой точке $P \in S^{2k}$ рассмотрим два вектора: вектор внешней нормали $n(P)$ и противоположный вектор $-n(P)$. Так как $X(P) \neq 0$, то однозначно определена двумерная аффинная плоскость $\Pi(P) \subset \mathbb{R}^{2k+1}$, проходящая через P и натянутая на линейно независимые векторы $X(P)$ и $n(P)$. Эта плоскость, очевидно, гладко зависит от P и содержит вектор $-n(P)$. Рассмотрим семейство $X_t(P)$ векторов, лежащих в плоскости $\Pi(P)$, заданное так

$$X_t(P) = X(P)\cos t + n(P)\sin t, \quad t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

Заметим, что при каждом t поле X_t задает отображение $X_t : S^{2k} \to S^{2k}$ так: $P \mapsto X_t(P)$. Таким образом, мы построили гладкую гомотопию X_t отображений из S^{2k} в S^{2k}, причем $X_{\frac{\pi}{2}}$ - это тождественное отображение сферы в себя, а $X_{-\frac{\pi}{2}}$ - это «антиподальное» отображение, переводящсе P в $-P$. Из общей тсорсмы 13.1 вытскаст, что эти отображсния имеют равные степени. Степень тождественного отображения равна 1.

Лемма 13.5. Степень «антиподального» отображения $a : S^n \to S^n$, определенного формулой $a(P) = -P, P \in S^n \subset \mathbb{R}^{n+1}$, равна $(-1)^{n+1}$.

Доказательство. Рассмотрим отображение $A : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$, заданное так: $A(P) = -P$. Отображение A линейно, его определитель равен $(-1)^{n+1}$, поэтому A меняет ориентацию пространства $\mathbb{R}^{n+1}$, если и только если число $(n+1)$ нечетно. Отображение a совпадает с ограничением A на S^n.

Рассмотрим замкнутый шар $B^{n+1} \subset \mathbb{R}^{n+1}$. Это ориентируемое многообразие с краем S^n. Ориентируем B^{n+1} в соответствии с ориентацией $\mathbb{R}^{n+1}$. Далее, фиксируем на сфере S^n ориентацию, согласованную с ориентацией шара B^{n+1}. Но, так как A переводит внешнюю нормаль к сфере во внешнюю нормаль, то ориентации $A\left(S^n\right) = a\left(S^n\right)$ и $A\left(B^{n+1}\right)$ также согласованы. Поэтому, если A сохраняет ориентацию B^{n+1}, то и a сохраняет ориентацию сферы S^n. Обратно, если A меняет ориентацию шара, то ориентация сферы, оставаясь согласованной с ориентацией шара-образа, меняется под действием отображения a на противоположную. Лемма доказана.

Таким образом, если размерность сферы четная, то степень «антиподального отображения»равна -1 . Полученное противоречие завершает доказательство.

15.13.4 Problems

Задача 13.1. Показать, что если $f : M \to N$ - гладкое отображение гладких компактных многообразий одинаковой размерности, и $y \in N$ - регулярное значение отображения f, то прообраз $f^{-1}(y)$ состоит из конечного числа точек.

Задача 13.2. Покажите, что если найдется такая точка $y \in N$, что прообраз $f^{-1}(y)$ пуст, то степень отображения f равна нулю.

Задача 13.3. Найти критические точки и степень отображения $S^2 \to S^2$, заданного в конечной области формулойа (а) $z \mapsto z^3 - 3z^2 - 9z + 27$; (б) $z \mapsto \left(z^2+4\right)/(z+1)$.

Задача 13.4. Вычислить степень отображения $f : SO(2) \to SO(2)$, заданного формулой $f(A) = A^k$. Тот же вопрос для SO(3), для SU(2).

Задача 13.5. Пусть $f : S^{2k} \to S^{2k}$. Показать, что найдется такая точка $x \in S^{2k}$, что или $x = f(x)$, или $x = -f(x)$.

Задача 13.6. Построить примеры гладких касательных векторных полей на сферах S^1, S^3, S^7. Существует ли такое поле на $\mathbb{R}\mathrm{P}^2$? А на T^2 ?

15.13.5 Extra Topics

13.1. Диффеоморфизм сферы и комплексной проективной прямой. Напомним, что проективная прямая $\mathbb{CP}^1$ представляет собой множество комплексных одномерных подпространств пространства $\mathbb{C}^2$. Если (z^1, z^2) - декартовы (комплексные) координаты в $\mathbb{C}^2$, то каждая такая прямая может быть задана своим направляющим (комплексным) вектором, который определен с точностью до пропорциональности. Как и в вещественном случае, удобно задавать такие направляющие векторы так называемыми однородными координатами, т.е. упорядоченными парами вида ($z_0 : z_1$) с отношением эквивалентности $(z_0 : z_1) \sim (\lambda z_0 : \lambda z_1)$ для любого ненулевого комплексного числа λ.

Итак, пусть ($z_0 : z_1$) однородные координаты на $\mathbb{CP}^1$. Тогда, чтобы превратить $\mathbb{CP}^1$ в гладкое многообразие, можно определить две карты U_0 и U_1, положив

$$U_i = \{(z_0 : z_1) \mid z_i \neq 0\},$$

где $i = 0, 1$, и координатные гомеоморфизмы $\varphi_i : U_i \to \mathbb{C}\,(w_i)$, где через w_i обозначена комплексная координата на соответствующем экземпляре комплексной плоскости, положив $\varphi_0\,(z_0 : z_1) = z_1/z_0$ и $\varphi_1\,(z_0 : z_1) = z_0/z_1$. Поскольку $\varphi_1 \circ \varphi_0^{-1}\,(w_0) = 1/w_0$, функции перехода являются комплексно-аналитическими в общей области определения, откуда $\mathbb{CP}^1$ - одномерное комплексно-аналитическое многообразие.

Перейдем к сфере S^2 стандартно вложенной в $\mathbb{R}^3$. Рассмотрим на ней две карты стереографической проекции $\sigma_N : S^2 \backslash \{N\} \to \mathbb{C}(z)$ из северного и $\sigma_S : S^2 \backslash \{S\} \to \mathbb{C}(w)$ из южного полюсов на проходящую через центр сферы горизонтальную плоскость (на плоскостях проекции введены комплексные координаты z и w соответственно). Если $z = \sigma_N(P)$ и $w = \sigma_S(P)$ - проекции одной и той же точки $P \in S^2$ из разных полюсов, (x^1, x^2, x^3) - декартовы координаты точки P в $\mathbb{R}^3$, а $r = \sqrt{(x^1)^2 + (x^2)^2}$ - расстояние от точки P до оси Ox^3, то

$$\frac{|z|}{r} = \frac{1}{1 - x^3}, \quad \frac{|w|}{r} = \frac{1}{1 + x^3},$$

откуда, учитывая, что для точек сферы $r = \sqrt{1 - (x^3)^2}$, заключаем, что

$$|z| - \frac{\sqrt{1 - (x^3)^2}}{1 - x^3} - \sqrt{\frac{1 + x^3}{1 - x^3}}, \quad |w| - \frac{\sqrt{1 - (x^3)^2}}{1 + x^3} - \sqrt{\frac{1 - x^3}{1 + x^3}}.$$

Так как аргументы чисел z и w, очевидно, совпадают, заключаем, что $z = 1/w$. Кстати, мы показали, что S^2 - комплексно-аналитическое многообразие.

Осталось рассмотреть отображение $F : S^2 \to \mathbb{CP}^1$ заданное во введенных нами картах так: $F(z) = w_0$ и $F(w) = w_1$. Так как функции перехода и карты одинаковы, отображение F определено корректно (написанные формулы согласованы на пересечении карт). Построенное отображение задает искомый диффеоморфизм.

13.2. Проверка гладкости гомотопии F_t. В доказательстве основной теоремы алгебры мы рассмотрели гомотопию $F_t : [0, 1] \times \mathbb{CP}^1 \to \mathbb{CP}^1$, записанную в однородных координатах $(z_0 : z_1)$ и $(w_0 : w_1)$ в виде

$$\begin{aligned} w_1\,(z_0 : z_1, t) &= z_1^n + t\left(a_{n-1} z_1^{n-1} z_0 + \cdots + a_1 z_1 z_0^{n-1} + a_0 z_0^n\right) \\ w_0\,(z_0 : z_1, t) &= z_0^n \end{aligned}$$

и показали, что $\deg F_t = n$. Почему нельзя рассмотреть аналогичную гомотопию G_t

$$\begin{aligned} w_1\,(z_0 : z_1, t) &= t\left(z_1^n + a_{n-1} z_1^{n-1} z_0 + \cdots + a_1 z_1 z_0^{n-1}\right) + a_0 z_0^n \\ w_0\,(z_0 : z_1, t) &= z_0^n \end{aligned}$$

и, заметив, что $\deg G_t = 0$ доказать «альтернативную теорему», гласящую что алгебраические уравнения не имеют корней? Дело в том, что похожие, на первый взгляд, гомотопии F_t и G_t отличаются тем, что F_t - гладкая, а G_t - нет.

Чтобы проверить гладкость отображения его нужно честно записать в картах. Как и в доказательстве теоремы 13.2, введем в рассмотрение карты с координатами $z = z_1/z_0$ и $z' = z_0/z_1$ и координатами $w = w_1/w_0$ и $w' = w_0/w_1$. Тогда отображение F_t имеет вид

$$w = F_t(z) = z^n + t\left(a_{n-1}z^{n-1} + \cdots + a_1 z + a_0\right)$$
$$w' = F_t\left(z'\right) = \frac{1}{1 + t\left(a_{n-1}z' + \cdots + a_1\left(z'\right)^n 1 + a_0\left(z'\right)^n\right)}$$

Первая формула задает гладкую (даже полиномиальную) гомотопию в координатах z и w, покрывающих сферы целиком, за исключением бесконечности, соответствующей $z' = 0$. Вторая формула также задает гладкую гомотопию при $|z'| < \varepsilon$ (где ε настолько мал, что знаменатель не обратцается в нуль) в координатах z' и w', т.е. как раз в окрестности бесконечности. Тем самым, гомотопия F_t действительно гладкая.

Выполняя те же вычисления для гомотопии G_t имеем:

$$w = G_t(z) = t\left(z^n + a_{n-1}z^{n-1} + \cdots + a_1 z\right) + a_0$$
$$w' = G_t\left(z'\right) = \frac{1}{t\left(1 + a_{n-1}z' + \cdots + a_1\left(z'\right)^{n-1}\right) + a_0\left(z'\right)^n}$$

Первая формула по-прежнему задает гладкую гомотопию в конечной области, а вторая - имеет особенность по крайней мере при $z' = 0$ и $t = 0$. Поэтому гомотопия G_t не гладкая.

15.14 14: Other Applications of Display Degree

В данном разделе мы рассмотрим еще несколько применений степени отображения.

15.14.1 14.1. Degree and Integral

Пусть $f : M \to N$ - гладкое отображение гладких замкнутых (т.е. компактных и без края) связных ориентированных многообразий одинаковой размерности n, и ω - внешняя форма степени n на многообразии N. Тогда имеет место следующая полезная теорема.

Теорема 14.1. В сделанных предположениях,

$$\int_M f^*\omega = (\deg f)\int_N \omega.$$

Доказательство. Пусть $y \in N$ - регулярное значение отображения f. Тогда существует окрестность U точки y в N, состоящая из регулярных значений отображения f. Пусть $f^{-1}(y) = \{x_1, \ldots, x_m\}$ - полный прообраз точки y. Тогда окрестность U может быть выбрана настолько малой, что

$$f^{-1}(U) = U_1 \cup \cdots \cup U_m, \quad x_\alpha \in U_\alpha$$

причем окрестности U_α точек x_α попарно не пересекаются. Фиксируем локальные координаты $(y^1, \ldots, y^n)$ в окрестности U на N, и $(x_\alpha^1, \ldots, x_\alpha^n)$ - локальные координаты в окрестности U_α на M. Так как все точки из U_α - регулярные точки отображения f, то по теореме об обратной функции ограничение отображения f на U_α - диффеоморфизм. Пусть $\varphi\left(y^1, \ldots, y^n\right) dy^1 \wedge \cdots \wedge dy^n$ - координатное представление формы ω в координатах $(y^1, \ldots, y^n)$, и $y = y\left(x_\alpha\right)$ - координатное представление отображения f в координатах

$(x_\alpha^1, \ldots, x_\alpha^n)$ и $(y^1, \ldots, y^n)$. Тогда, по теореме о замене переменных под знаком интеграла, имеем:

$$\int_U \varphi(y) dy^1 \cdots dy^n = \int_{U_\alpha} \varphi\left(y\left(x_\alpha\right)\right) \left|\det \frac{\partial y^i}{\partial x_\alpha^k}\right| dx_\alpha^1 \cdots dx_\alpha^n =$$
$$= \operatorname{sign} \det \left(\frac{\partial y^i}{\partial x_\alpha^k}\right) \int_{U_\alpha} \varphi\left(y\left(x_\alpha\right)\right) \left(\det \frac{\partial y^i}{\partial x_\alpha^k}\right) dx_\alpha^1 \cdots dx_\alpha^n$$

откуда,

$$\operatorname{sign} \det \left(\frac{\partial y^i}{\partial x_\alpha^k}\right) \int_U \varphi(y) dy^1 \cdots dy^n = \int_{U_\alpha} \varphi\left(y\left(x_\alpha\right)\right) \det \left(\frac{\partial y^i}{\partial x_\alpha^k}\right) dx_\alpha^1 \cdots dx_\alpha^n$$

Воспользовавшись видом формы $f^*\omega$ и аддитивностью интеграла по непересекающимся областям, получим, что

$$\int_{f^{-1}(U)} f^*\omega = \left(\sum_\alpha \operatorname{sign} \det \left(\frac{\partial y^i}{\partial x_\alpha^k}\right)\right) \int_U \omega = (\deg f) \int_U \omega,$$

т.е. искомая формула имеет место в окрестности каждого регулярного значения. Далее, по теореме Сарда, множество критических значений отображения f это множество меры 0 , поэтому его вклад в интеграл $\int_N \omega$ равен нулю. С другой стороны, прообраз этого множества, т.е. множество критических точек отображения f, также дает нулевой вклад в интеграл $\int_M f^*\omega$, так как в этих точках форма $f^*\omega$ равна нулю. Таким образом, теорема вытекает из предыдущего равенства и аддитивности интеграла.

15.14.2 14.2. Gauss-Bonnet theorem

Пусть $M^{n-1} \subset \mathbb{R}^n$ - ориентируемая гиперповерхность в $\mathbb{R}^n$, и пусть N - поле единичных нормалей к поверхности M. Рассмотрим гауссово отображение $N : M \to S^{n-1}$, ставящее в соответствие каждой точке $P \in M$ вектор $N(P)$, и пусть $\Omega \in \Omega^{n-1}\left(S^{n-1}\right)$ - форма объема на стандартной сфере S^{n-1} радиуса 1. Обозначим через K гауссову кривизну гиперповерхности M, а через $d\sigma$ форму объема на M для индуцированной из $\mathbb{R}^n$ метрики.

Утверждение 14.1. В сделанных выше предположениях, при подходящем выборе ориентации поверхности, имеем:

$$N^*\Omega = K d\sigma$$

Доказательство. Пусть $P \in M$ - произвольная точка. Рассмотрим в $\mathbb{R}^n$ евклидовы координаты ($x^1, \ldots, x^n$) с началом в точке P, направив ось x^n в направлении вектора $N(P)$. Тогда поверхность M в окрестности точки P можно представить в виде графика некоторой функции $x^n = f\left(x^1, \ldots, x^{n-1}\right)$, причем, как мы уже видели выше, $f_{x^i}(P) = 0$. В координатах ($x^1, \ldots, x^{n-1}$) матрица индуцированной метрики на M в точке P равна единичной, т.е. δ_{ij}, поэтому $d\sigma = dx^1 \wedge \cdots \wedge dx^{n-1}$. Гауссова кривизна K в точке P равна определителю матрицы $(f_{x^i x^j}(P))$ вторых частных производных.

На сфере S^{n-1} в окрестности точки $N(P) = (0, \ldots, 1)$ выберем координаты ($u^1, \ldots, u^{n-1}$), совпадающие с евклидовыми координатами горизонтальной плоскости объемлющего евклидова пространства. Тогда в точке $N(P)$ в этих координатах метрика также евклидова, поэтому форма объема на S^{n-1} в точке $N(P)$ имеет вид $\Omega = du^1 \wedge \cdots \wedge du^{n-1}$.

Далее, вектор нормали к поверхности M в окрестности точки P задается так:

$$N = \frac{(-f_{x^1}, \cdots, -f_{x^{n-1}}, 1)}{\sqrt{1 + f_{x^1}^2 + \ldots + f_{x^{n-1}}^2}}$$

поэтому гауссово отображение $N : M \to S^{n-1}$ записывается в локальных координатах $(x^1, \ldots, x^{n-1})$ и $(u^1, \ldots, u^{n-1})$ в виде

$$u^i\left(x^1, \ldots, x^{n-1}\right) = \frac{-f_{x^i}}{\sqrt{1 + f_{x^1}^2 + \ldots + f_{x^{n-1}}^2}},$$

и, так как $f_{x^i}(P) = 0$, то дифференциал $dN(P)$ отображения N в точке P имеет вид

$$dN(P) = \left(-f_{x^i x^j}\right), \qquad \text{т.е.} \qquad \frac{\partial u^i}{\partial x^j}(P) = -\frac{\partial^2 f}{\partial x^i \partial x^j}(P)$$

Следовательно,

$$N^*\Omega = \det\left(\frac{\partial u^i}{\partial x^j}\right) dx^1 \wedge \cdots \wedge dx^{n-1} =$$
$$= \det\left(-\frac{\partial^2 f}{\partial x^i \partial x^j}\right) dx^1 \wedge \cdots \wedge dx^{n-1} = (-1)^{n-1} K d\sigma$$

Таким образом, в случае четной размерности, т.е. $n - 1 = 2k$, мы получаем требуемое равенство, а в случае $n = 2k$ нужно сменить ориентацию поверхности M на противоположную, что приведет к изменению знака гауссовой кривизны. Утверждение доказано.

Следствие 14.1 (Теорема Гаусса-Бонне). Пусть M - замкнутая связная ориентируемая поверхность в $\mathbb{R}^3$. Тогда

$$\int_M K d\sigma = 4\pi\lambda$$

где λ - некоторое челое число (равное степени гауссова отображения поверхности M) .

Доказательство. Пусть $N : M \to S^2$ - гауссово отображение поверхности M. Как и выше, обозначим через K гауссову кривизну для M, а через $d\sigma$ и Ω - формы площади на M и S^2 соответственно. Тогда

$$\int_M K d\sigma = \int_M N^*\Omega = (\deg N) \int_{S^2} \Omega = 4\pi \cdot \deg N$$

Положив $\lambda = \deg N$, получим требуемое.

Отметим, что двумерность поверхности в теореме Гаусса-Бонне не существенна. Для случая гиперповерхности изменится только множитель перед $\deg N$. Этот множитель, который мы обозначим через c_{n-1}, равен объему стандартной евклидовой сферы S^{n-1}.

Следствие 14.2. Пусть M - замкнутая связная ориентируемая гиперповерхность в $\mathbb{R}^n$. Тогда

$$\int_M K d\sigma = c_{n-1}\lambda$$

где λ - некоторое челое число (равное, с точностью до знака, степени гауссова отображения поверхности M).

15.14.3 14.3. Special points of vector fields

Пусть U - область в $\mathbb{R}^n$, такая что замыкание $\bar{U}$ области U компактно, и граница ∂U этой области - замкнутое $(n-1)$-мерное многообразие. Пусть X - некоторое векторное поле в области $W \supset U$. Напомним, что точка $P \in U$ называется особой для X, если $X(P) = 0$. Особая точка P поля X называется изолированной, если в некоторой ее окрестности нет других особых точек.

Отметим, что векторное поле вне своих особых точек порождает отображение N в S^{n-1}, ставящее в соответствие каждой неособой точке P точку $X(P)/\|X(P)\|$ из S^{n-1}.

Предположим, что поле X не имеет на ∂U особых точек. Тогда отображение N определено во всех точках границы ∂U.

Определение. Степень отображения $N : \partial U \to S^{n-1}$ называется индексом векторного поля X в области U и обозначается через $\operatorname{ind} X$.

Пусть $P \in U$ - изолированная особая точка. Рассмотрим замкнутую шаровую окрестность $B \subset U$ точки P, не содержащую других особых точек. Тогда отображение N определено во всех точках сферы ∂B.

Определение. Степень отображения $N : \partial B \to S^{n-1}$ называется индексом особой точки P поля X и обозначается через $\operatorname{ind}_P X$.

Теорема 14.2. Пусть поле X имеет в области U лишь изолированные особые точки и не имеет особых точек на границе области U. Тогда индекс поля X равен сумме индексов его особых точек.

Доказательство. Пусть P_i - особые точки поля X, и B_i - непересекающиеся шаровые окрестности точек P_i, причем каждая B_i не содержит особых точек, отличных от P_i. Обозначим через V область $U \setminus (\cup_i B_i)$. Тогда отображение N определено на всем на V.

Рассмотрим на S^{n1} стандартную форму объема Ω. Так как форма Ω замкнута, то $(n-1)$-форма $N^*(\Omega)$ на V также замкнута, поэтому по формуле Стокса

$$\int_{\partial V} N^*(\Omega) = \int_V dN^*(\Omega) = 0$$

С другой стороны,

$$\int_{\partial V} N^*(\Omega) = \int_{\partial U} N^*(\Omega) + \sum_i \int_{\partial B_i} N^*(\Omega) = 0,$$

где ориентации многообразия ∂U и сфер ∂B_i согласованы с ориентацией области V. Применяя теорему 14.1 и учитывая ориентации сфер ∂B_i, получаем:

$$\int_{\partial U} N^*(\Omega) = c_{n-1} \cdot \deg N|_{\partial U} = c_{n-1} \operatorname{ind} X,$$
$$\int_{\partial B_i} N^*(\Omega) = c_{n-1} \cdot \deg N|_{\partial B_i} = -c_{n-1} \operatorname{ind}_{P_i} X,$$

откуда

$$\operatorname{ind} X = \sum_i \operatorname{ind}_{P_i} X$$

что и требовалось.

15.14.4 14.4. Brauer's theorem

В качестве приложения теории индекса векторных полей, докажем гладкий вариант теоремы Брауэра о неподвижной точке (справедливой и в непрерывном случае).

Следствие 14.3 (Теорема Брауера). Пусть D^n - замкнутый шар $\mathbb{R}^n$. Любое гладкое отображение $\varphi : D^n \to D^n$ имеет неподвижную точку.

Доказательство. Предположим противное, т.е. существует такое отображение $\varphi : D^n \to D^n$, у которого нет неподвижных точек. Тогда на D^n имеется векторное поле X без особых точек, определенное так: $X(P) = \varphi(P) - P$. Ясно, что индекс этого векторного поля в D^n равен нулю.

С другой стороны, поле X, ограниченное на $S^{n-1} = \partial D^n$ также нигде не обращается в нуль и направлено строго внутрь шара D^n, поэтому существует гомотопия, переводящее поле $X|_{S^{n-1}}$ в поле внутренних нор- малей сферы S^{n-1}. Следовательно, индекс поля X совпадает со степенью антиподального отображения сферы S^{n-1} на себя, переводящего каждую точку в противоположную ей. Но у такого отображения степень равна ± 1. Это противоречие и завершает доказательство.

15.14.5 Problems

Задача 14.1. Найти степень векторного поля X на гиперповерхности M (т.е. степень соответствующего гауссова отображения), где

$$X = \left(x^2 + y^2 - 1, \frac{x^2}{4} + 4y^2 - 1\right), \quad M = \left\{(x, y) \mid 5x^2 - 6xy + 5y^2 - 3 = 0\right\}.$$

Задача 14.2. Доказать, тто стспспи любых двух вскторіных полей па произвольной двумерной замкнутой поверхности одинаковы. Верно ли это в произвольной размерности?

Задача 14.3. Доказать, что на всяком связном компактном замкнутом многообразии существует гладкое векторное поле с одной особой точкой. Найти индекс этой особой точки.

Задача 14.4. Построить пример гладкого векторного поля с одной особой точкой на (а) сфере; (б) на торе; (в) на сфере с g ручками; (г) на проективной плоскости; (д) на бутылке Клейна; (е) на сфере с μ пленками Мебиуса.

15.14.6 Extra Topics

14.1. Теорема Гаусса-Бонне и эйлерова характеристика. На самом деле, степень гауссова отображения двумерной поверхности не зависит ни от индуцированной на поверхности метрики, ни от способа вложения поверхности в $\mathbb{R}^3$. Она зависит лишь от топологии поверхности. Можно показать, что $\deg N$ совпадает с половиной эйлеровой характеристикой поверхности M.

Нетрудно вычислить степень гауссова отображения для стандартного вложения сферы M_g^2 с g ручками в евклидово пространство. Действительно, рассмотрим вложение M_g^2 в $\mathbb{R}^3$, при котором ручки поверхности расположены вдоль вертикальной оси, см. рис. 5.

Рассмотрим гауссово отображение $N : M_g^2 \to S^2$, и, в качестве его регулярного значения, выберем северный полюс сферы. Прообраз северного полюса состоит из $g + 1$ точки, см. рис. 5 , причем в одной точке (верхней на рисунке) гауссова кривизна поверхности положительна, а в остальных - отрицательна. Из утверждения 14.1 вытекает, что знак гауссовой кривизны совпадает со знаком якобиана гауссова отображения. Поэтому $\deg N = +1 - g$ и, значит, равна половине эйлеровой характеристики поверхности.

Отметим, что мы провели вычисления для конкретного (удобного для наших целей) вложения поверхности. Пусть теперь имеется два вложения $F_i : M_g^2 \to \mathbb{R}^3$,
$i = 0, 1$. Тогда имеется два гауссовых отображения $N_i : M_g^2 \to S^2$, степень которых, вообще говоря, различна. Однако, если существует гладкая гомотопия $F_t, t \in [0, 1]$,

одного вложения в другое в классе вложений (такие гомотопии называются изото*пи* ями), то существует и гладкая гомотопия $N_t, t \in [0,1]$ гауссовых отображений, откуда вытекает равенство их степеней. Таким образом, для доказательства независимости степени гауссова отображения от вложения достаточно доказать изотопичность любых двух вложений. Последний факт, однако, весьма нетривиален и его доказательство выходит за рамки нашего курса.

15.15 15: Elements of the Calculus of Variations

В предыдущих лекциях мы сначала определили геодезические как кривые с параллельным полем скоростей, а затем показали, что они обладают другим важным свойством: являются локально кратчайшими кривыми, т.е. кривыми, каждый достаточно малый фрагмент которых имеет наименьшую длину среди всех кривых, соединяющих граничные точки такого фрагмента. Еще одно эквивалентное определение геодезических: это такие кривые, которые являются критическими точками для функции длины, т.е. если мы продеформируем такую кривую, то ее длина в первом приближении не изменится. Эти идеи лежат в основе целого направления в математике, которое называется вариационным исчислением. В данной лекции мы кратко расскажем о нем с геометрической точки зрения.

15.15.1 15.1. Classical variation problem

Пусть M - гладкое n-мерное многообразие, $P \in M$ - произвольная точка, T_PM - касательное пространство, TM - касательное расслоение к многообразию M и $\pi : TM \to M$ - стандартная проекция, ставящая в соответствие каждому вектору v в некотором T_PM точку $P \in M$. Напомним, что каждые локальные координаты ($x^1, \ldots, x^n$) в окрестности U точки P порождают локальные координаты $(x^1, \ldots, x^n, y^1, \ldots, y^n)$ в окрестности $\pi^{-1}(U)$, где y^i - компоненты касательного вектора в каноническом базисе $\{\partial_{x^i}\}$.

Пусть $L : TM \times \mathbb{R} \to \mathbb{R}$ - некоторая непрерывная функция. Такие функции L называются лагранжианами. Если задан лагранжиан L, то по нему можно построить отображение Φ_L, ставящее в соответствие каждой кривой $\gamma(t)$ из многообразия M число $\Phi_L(\gamma)$. Покажем, как это делается.

Прежде всего, для каждой гладкой кривой $\gamma : [a,b] \to M$ определим кривую $\tilde{\gamma} : [a,b] \to TM \times \mathbb{R}$, положив $\tilde{\gamma}(t) = (\gamma(t), \dot{\gamma}(t), t)$. Кривая $\tilde{\gamma}$ называется поднятием кривой γ. Если L - лагранжиан, то вычислив L в точках кривой $\tilde{\gamma}$, получим непрерывную функцию на отрезке $[a,b]$. Интеграл от этой функции по $[a,b]$ и есть число $\Phi_L(\gamma)$:

$$\Phi_L(\gamma) = \int_a^b L(\tilde{\gamma}(t))dt = \int_a^b L(\gamma(t), \dot{\gamma}(t), t)dt$$

Отображение Φ_L называется (классическим одномерным) вариационным функционалом, соответствующим лагранжиану L.

Пример. Пусть M - риманово многообразие. Положим

$$L(x,y,t) = \sqrt{g_{ij}(x)y^iy^j},$$

где $g_{ij}(x)$ - компоненты римановой метрики в координатах $(x^1, \ldots, x^n)$. Легко видеть, что вариационный функционал Φ_L, соответствующий этому лагранжиану, ставит в соответствие гладкой кривой γ на M ее длину. Построенный функционал называется функционалом длинь.

Если $L(x,y,t) = \frac{1}{2}g_{ij}(x)y^iy^j$, то функционал Φ_L называется функционалом энергии.

Замечание. Часто координаты y^i обозначают через $\dot{x}^i$, чтобы подчеркнуть геометрический смысл этих координат - компонент векторов скоростей кривых.

Выше, выводя формулу первой вариации длины кривой, мы уже определяли гладкую вариацию кривой. Здесь мы ограничимся более частным опредсленисм. Гладкая гомотопия $\gamma_s, s \in [-1,1]$, отображения $\gamma : [a,b] \to M$, т.е. отображение $\Gamma : [-1,1] \times [a,b] \to M, \Gamma(s,t) = \gamma_s(t)$, называется гладкой деформацией или вариацией кривой γ. Говорят, то деформация γ_s неподвижна на концах, если $\gamma_s(a) = \gamma_0(a)$ и $\gamma_s(b) = \gamma_0(b)$ для любого $s \in [-1,1]$. Напомним, что полем деформации или полем вариации $E(t)$ называется поле вдоль кривой $\gamma(t)$, определяющееся по деформации $\gamma_s(t)$ так: $E(t) = d\gamma_s(t)/\, ds|_{s=0} = d\Gamma|_{(0,t)}(\partial_s)$. Если деформация γ неподвижна на концах, то $E(a) = E(b) = 0$.

Пусть γ - некоторая гладкая кривая. Предположим, что лагранжиан L является гладкой функцией в некоторой окрестности поднятия $\tilde{\gamma}$ кривой γ. Тогда для любой достаточно малой деформации γ_s кривой γ, т.е. такой деформации, что все $\tilde{\gamma}_s$ лежат в области гладкости функции L, определена гладкая функция $\Phi_L(\gamma_s)$ параметра s.

Определение. Кривая γ называется экстремалью функционала Φ_L, если для любой достаточно малой деформации γ_s этой кривой, неподвижной на концах, точка $s = 0$ является стационарной для функции $f(s) = \Phi(\gamma_s)$, т.е. выполняется соотношение

$$f'(0) = \left.\frac{d\Phi(\gamma_s)}{ds}\right|_{s=0} = 0$$

Задача о поиске экстремалей одномерного вариационного функционала называется классической вариационной задачей, решением которой занимается классическое вариационное исчисление, возникшее в работах Эйлера и Лагранжа.

Теорема 15.1. Кривая γ является экстремалью вариационного функичонала Φ_L, если и только, если вдоль γ выполняется следующая система дифференциальных уравнений Эйлера-Лагранжа:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0, \quad i = 1, \ldots, n$$

Доказательство. Вычислим производную по s функции $\Phi_L(\gamma_s)$ в начальный момент времени. Обозначим через $x^i(s)$ координатные функции кривой $\gamma_s(t)$. Имеем

$$\frac{d}{ds}\int_a^b L(x_s(t), \dot{x}_s(t), t)\, dt = \int_a^b \frac{d}{ds} L(\, x_s(t), \dot{x}_s(t), t)\, dt$$
$$= \int_a^b \left(\frac{\partial L}{\partial x^i}\frac{dx_s^i}{ds} + \frac{\partial L}{\partial \dot{x}^i}\frac{d\dot{x}_s^i}{ds}\right) dt$$

Заметим, что на каждой кривой $\gamma_s(t)$ выполнено,

$$\frac{d\dot{x}_s^i}{ds} = \frac{d}{ds}\frac{d}{dt}x_s^i = \frac{d}{dt}\frac{d}{ds}x_s^i$$

что позволяет продолжить предыдущее равенство так:

$$\int_a^b \left(\frac{\partial L}{\partial x^i}\frac{dx_s^i}{ds} + \frac{\partial L}{\partial \dot{x}^i}\frac{d\dot{x}_s^i}{ds}\right) dt =$$
$$= \int_a^b \left(\frac{\partial L}{\partial x^i}\frac{dx_s^i}{ds} + \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\frac{dx_s^i}{ds}\right) - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\frac{dx_s^i}{ds}\right) dt$$

Полагая в предыдущем выражении $s = 0$ и вспоминая, что $E(t) = dx_s/ds$ при $s = 0$, имеем:

$$\left.\frac{d\Phi\left(\gamma_s\right)}{ds}\right|_{s=0}=\int_a^b\left(\frac{\partial L}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right)E(t)dt+\left.\frac{\partial L}{\partial \dot{x}^i}E(t)\right|_a^b=$$
$$=\int_a^b\left(\frac{\partial L}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right)E(t)dt$$

В силу произвольности деформации γ_s, поле $E(t)$ также может быть выбрано любым, зануляющимся на концах отрезка $[a,b]$, поэтому условие экстремальности кривой γ равносильно тому, что

$$\xi^i(t)=\frac{\partial L}{\partial x^i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right)=0,\quad i=1,\ldots,n$$

вдоль кривой γ (иначе, если хотя бы одно из этих уравнений не равно нулю при некотором $t=t_0$, то в качестве $E(t)$ можно взять поле $f(t)\xi(t)=f(t)\left(\xi^1(t),\ldots,\xi^n(t)\right)$, где $f(t)$ - гладкая неотрицательная функция, равная 1 в точке $t=t_0$, и равная нулю в точках a и b : для деформации с таким полем $\eta(t)$ производная функции $\Phi_L\left(\gamma_s\right)$ отлична от нуля при $s=0$). Доказательство закончено.

15.15.2 15.2. Examples of Lagrangians

В качестве первого примера рассмотрим функционал энергии на римановом многообразии M. Пусть g_{ij} - риманова метрика, и

$$L\left(x^1,\ldots,x^n,\dot{x}^1,\ldots,\dot{x}^n,t\right)=\frac{1}{2}g_{ij}(x)\dot{x}^i\dot{x}^j$$

- соответствующий лагранжиан, задающий функционал энергии. Отметим, что функция L гладкая везде. Найдем экстремали этого функционала. Для этого запишем систему уравнений Эйлера-Лагранжа. Имеем:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right)-\frac{\partial L}{\partial x^\alpha}=\frac{d}{dt}\left(g_{\alpha j}\dot{x}^j\right)-\frac{1}{2}\frac{\partial g_{ij}}{\partial x^\alpha}\dot{x}^i\dot{x}^j=$$
$$=g_{\alpha j}\ddot{x}^j+\frac{\partial g_{\alpha j}}{\partial x^i}\dot{x}^i\dot{x}^j-\frac{1}{2}\frac{\partial g_{ij}}{\partial x^\alpha}\dot{x}^i\dot{x}^j=$$
$$=g_{\alpha j}\ddot{x}^j+\frac{1}{2}\left(\frac{\partial g_{\alpha j}}{\partial x^i}+\frac{\partial g_{\alpha i}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^\alpha}\right)\dot{x}^i\dot{x}^j=0,\quad \alpha=1,\ldots,n$$

Сворачивая эту систему с $g^{k\alpha}$, получаем

$$\ddot{x}^k+\frac{1}{2}g^{k\alpha}\left(\frac{\partial g_{\alpha j}}{\partial x^i}+\frac{\partial g_{i\alpha}}{\partial x^j}-\frac{\partial g_{ij}}{\partial x^\alpha}\right)\dot{x}^i\dot{x}^j=\ddot{x}^k+\Gamma_{ij}^k\dot{x}^i\dot{x}^j=0,\quad k=1,\ldots,n$$

а это и есть система уравнений, задающая геодезические. Итак, мы доказали следующую теорему.

Теорема 15.2. Кривая γ на римановом многообразии является геодезической тогда и только тогда, когда она - экстремаль функичонала энергии.

Пусть теперь $L\left(x^1,\ldots,x^n,\dot{x}^1,\ldots,\dot{x}^n,t\right)=\sqrt{g_{ij}\dot{x}^i\dot{x}^j}$ - лагранжиан функционала длины. Отметим, что функция L гладкая везде, где вектор $\left(\dot{x}^1,\ldots,\dot{x}^n\right)$ отличен от нуля. Иными словами, лагранжиан L является гладкой функцией в некоторой окрестности поднятия кривой γ, если и только если γ - регулярная кривая.

Найдем условие экстремальности регулярной кривой γ. Запишем уравнения Эйлера-Лагранжа. Имеем:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^\alpha}\right) - \frac{\partial L}{\partial x^\alpha} = \frac{d}{dt}\left(\frac{g_{\alpha j}\dot{x}^j}{\sqrt{g_{ij}\dot{x}^i\dot{x}^j}}\right) - \frac{\frac{\partial g_{ij}}{\partial x^\alpha}\dot{x}^i\dot{x}^j}{2\sqrt{g_{ij}\dot{x}^i\dot{x}^j}} = 0, \quad \alpha = 1, \ldots, n.$$

Если кривая γ параметризована пропорционально натуральному параметру, то условие экстремальности равносильно написанному выше условию экстремальности кривой для функционала энергии, т.е. условию того, что кривая γ - геодезическая. Итак, доказана следующая теорема.

Теорема 15.3. Регулярная кривая γ, параметризованная пропорционально натуральному параметру, является экстремалью функичонала длинь, если и только если она - геодезическая.

Заметим, что так как функционал длины не зависит от параметризации, то мы тем самым полностью описали его экстремали. А именно,

Следствие 15.1. Регулярная кривая $\gamma(t)$ является экстремалью функционала длины, если и только если после перепараметризации $t(s)$, превращающей параметр t в натуральный параметр s, кривая $\gamma(t(s))$ геодезическая.

15.15.3 15.3. Conservation laws

Начнем с еще одного примера. Рассмотрим механическую систему, состоящую из материальной точки массы m, движущейся под действием поля консервативных сил f. Последнее означает, что существует такая функция $U(x)$, называемая потенциалом, что $f = -\operatorname{grad} U$. Кинетическая энергия T задается в виде $\frac{m\dot{x}^2}{2}$, а полная энергия H равна $T + U$. Уравнение Ньютона, описывающее движение этой материальной точки, имеет вид $f = m\ddot{x}$. При этом выполняется закон сохранения энергии: вдоль траектории движения энергия H постоянна.

Оказывается, траектории движения материальной точки в этой системе можно описывать как экстремали вариационного функционала Φ_L, соответствующего лагранжиану $L = T - U$. Действительно, уравнения Эйлера-Лагранжа имеют вид:

$$\frac{d}{dt}(m\dot{x}) + \operatorname{grad} U = 0$$

что эквивалентно уравнениям Ньютона. Здесь $f = -\operatorname{grad} U = \partial L/\partial x$ задает поле сил, а $m\dot{x} = \partial L/\partial \dot{x}$ - импульс материальной точки. Далее, полную энергию H можно записать в виде

$$H = T + U = m\dot{x}^2 - (T - U) = \frac{\partial L}{\partial \dot{x}}\dot{x} - L$$

Пусть L - произвольный лагранжиан. В каждой точке $P \in M$ для локальных координат ($x^1, \ldots, x^n$) определен набор величин $p_i = \partial L/\partial \dot{x}^i$, а также величина $H = \frac{\partial L}{\partial \dot{x}^\alpha}\dot{x}^\alpha - L(x, \dot{x}, t)$

Лемма 15.1. Величины p_i являются компонентами ковекторного векторного поля в координатах $(x^1, \ldots, x^n)$, а величина H - скалярной функиией.

Доказательство. Из уравнений Эйлера-Лагранжа вытекает, что

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^i}\right) = 0$$

что и требовалось.

В качестве примера использования законов сохранения, докажем теорему Клеро. Пусть M - двумерная поверхность в $\mathbb{R}^3$, полученная вращением вокруг оси z некоторой кривой, лежащей в плоскости xz и не пересекающей ось z. Напомним, что кривые, полученные

пересечением поверхности M с плоскостями, проходящими через ось z, называются меридианами, а с плоскостями, параллельными плоскости xy - параллелями. Пусть $\gamma(t)$ - геодезическая на M, и $\psi(t)$ - угол между геодезической γ в точке $\gamma(t)$ и параллелью, проходящей через эту точку. Пусть $r(t)$ - расстояние от точки $\gamma(t)$ до оси z. Тогда имеет место следующий результат.

Теорема 15.6 (Клеро). В сделанных выше предположениях, величина

$$r(t)\cos\psi(t)$$

постоянна вдоль геодезической $\gamma(t)$.

Доказательство. В самом деле, вычислив метрику на поверхности вращения, легко проверить, что элемент длины, т.е. лагранжиан, задающий функционал длины, имеет вид $ds = L = \sqrt{(1+r_z^2(z))\,\dot{z}^2 + r^2(z)\dot{\varphi}^2}$, т.е. L явно не зависит от φ, поэтому имеет место закон сохранения импульса

$$\frac{\partial L}{\partial\dot{\varphi}} = \frac{r^2\dot{\varphi}}{L} = \text{ const.}$$

Так как каждая геодезическая γ параметризована пропорционально натуральному параметру, величина $L = \|\dot{\gamma}\|$ постоянна вдоль γ. Поэтому последнее соотношение равносильно условию $r^2\dot{\varphi} = \text{const}$, что, в свою очередь, равносильно утверждению теоремы (проверьте). Доказательство закончено.

Замечание. На самом деле, законы сохранения энергии и импульса позволяют полностью проинтегрировать уравнение геодезических на поверхности вращения $M \subset \mathbb{R}^3$ (сделайте это).

15.15.4 15.4. Multivariate variational problems

Заменим теперь кривую, на которой вычислялся вариационный функционал, на некоторую поверхность, заданную отображением области $\Omega \subset \mathbb{R}^k$ в m-мерное многообразие M. Для простоты будем предполагать, что

граница $\partial\Omega$ - несвязное объединение конечного числа гладких многообразий размерности $k-1$, т.е. Ω - гладкое многообразие с краем.

Рассмотрим сначала пространство

$$W = \cup_{P\in M}\underbrace{T_PM\times\cdots\times T_PM}_{k\text{ раз}}$$

и отметим, что на нем можно ввести структуру гладкого $(m\cdot k)$-мерного многообразия, действуя по аналогии с касательным расслоением. В частности, если $\pi : W \to M$ стандартная проекция, сопоставляющая набору касательных векторов $(v_1,\ldots,v_k)$ из T_PM точку P, то локальные координаты ($x^1,\ldots,x^m$) в окрестности точки p порождают локальные координаты

$$\left(x^1,\ldots,x^m,y_1^1,\ldots,y_1^m,\ldots,y_k^1,\ldots,y_k^m\right)$$

на W, где $(y_i^1,\ldots,y_i^m)$ - компоненты вектора v_i из соответствующего экземпляра T_PM в каноническом базисе $\left\{\partial_{x_j}\right\}$, порожденном координатами $(x^1,\ldots,x^m)$.

Многомерным лагранжианом L будем называть непрерывную функцию $L : W\times\mathbb{R}^k \to \mathbb{R}$.

Если u^α - стандартные координаты в $\mathbb{R}^m \supset \Omega$, то лагранжиан будем записывать в координатах так:

$$L = L(x, y, u) = L\left(x^1, \ldots, x^m, y_1, \ldots, y_k, u^1, \ldots, u^k\right)$$

где $y_i = (y_i^1, \ldots, y_i^m)$.
Если $f : \Omega \to M$ - гладкое отображение, то поднятием $\tilde{f}$ назовем отображение из Ω в $W \times \mathbb{R}^m$, заданное так:

$$\tilde{f}(u) = \left(f(u), \frac{\partial f}{\partial u}(u), u\right).$$

Многомерным вариационным функционалом, соответствующим лагранжиану L, назовем отображение, которое каждому f ставит в соответствие число

$$\Phi_L(f) = \int_\Omega L(\tilde{f}(u))du$$

Деформацией или вариачией отображения $f : \Omega \to M$ называется гладкая гомотопия этого отображения, т.е. такое гладкое отображение $F : [-1, 1] \times \Omega \to M$, что $F(s, u) = f_s(u)$, а полем вариации - векторное поле $E(u)$ вдоль f, определенное так: $E(u) = df_s/\, ds(u)|_{s=0} = dF|_{(0,u)}(\partial_s)$. Говорят, что деформация f_s неподвижна на крае, если точки из $\partial\Omega$ остаются неподвижными. В последнем случае поле E равно нулю на $\partial\Omega$.

Определение. Пусть лагранжиан L гладок в окрестности образа поднятия $\tilde{f}$. Тогда f называется экстремалью функционала Φ_L, если

$$\left.\frac{d\Phi_L(f_s)}{ds}\right|_{s=0} = 0$$

для любой достаточно малой деформации f_s отображения f, неподвижной на крае.
(Классическая) многомерная вариационная задача - это задача о поиске экстремалей многомерного вариационного функционала.

Следующая теорема доказывается аналогично теореме 15.1 (проверьте).
Теорема 15.7. Отображение f является экстремалью функционала Φ_L, если и только если вдоль f выполняется следующая система дифференциальных уравнений Эйлера-Лагранжа:

$$\sum_{\alpha=1}^{k} \frac{\partial}{\partial u^\alpha}\left(\frac{\partial L}{\partial y_\alpha^i}\right) - \frac{\partial L}{\partial x^i} = 0, \quad i = 1, \ldots, m$$

Пример. Пусть область Ω лежит на плоскости $\mathbb{R}^2$ с координатами (u^1, u^2), и $f : \Omega \to \mathbb{R}^3$ - погружение. Пусть (x^1, x^2, x^3) - стандартные координаты в $\mathbb{R}^3$, и отображение f задается тремя координатными функциями $x^i(u^1, u^2)$. Если $f(u^1, u^2)$ радиус-вектор точки поверхности f с координатами (u^1, u^2), то индуцированная метрика имеет вид $g_{ij} = \langle f_{u^i}, f_{u^j}\rangle$. Рассмотрим функцию Лагранжа $L : W \times \mathbb{R}^2 \to \mathbb{R}$

$$L\left(x^1, x^2, x^3, y_1, y_2, u\right) = \frac{\langle y_1, y_1\rangle + \langle y_2, y_2\rangle}{2}$$

Так как $\tilde{f} = (f(u), f_{u_1}(u), f_{u_2}(u), u)$, то Значение функционала $\Phi_L(f)$ имеет вид:

$$\Phi_L(f) = \frac{1}{2}\int_\Omega (g_{11} + g_{22})\, du^1 du^2$$

Этот функционал называется функиионалом энергии или функиионалом Дирихле. Найдем его экстремали, для чего запишем уравнения Эйлера-Лагранжа. Рассматриваемый лагранжиан не зависит явно от x^i. При $i = 1$ имеем

$$\sum_{\alpha=1}^{2} \frac{\partial}{\partial u^\alpha}\left(\frac{\partial L}{\partial y_\alpha^1}\right) = \frac{\partial y_1^1}{\partial u^1} + \frac{\partial y_2^1}{\partial u^2} = x_{u^1u^1}^1 + x_{u^2u^2}^1$$

При остальных i вычисления аналогичны. Итак, система уравнений ЭйлераЛагранжа принимает вид

$$x_{u^1u^1}^i + x_{u^2u^2}^i = 0, \quad i = 1, 2, 3.$$

Таким образом, координаты радиус-вектора экстремали функционала Дирихле являются гармоническими функциями. Напомним, что такие радиус-векторы (и соответствующие поверхности) называются гармоническими. Итак, мы доказали следующий результат.

Утверждение 15.1. Экстремали функционала Дирихле - гармонические поверхности.

Пример. В предыдущем примере изменим лагранжиан, положив

$$L\left(x^1, x^2, x^3, y_1, y_2, u\right) = \frac{1}{2}\sqrt{\langle y_1, y_1\rangle \langle y_2, y_2\rangle - \langle y_1, y_2\rangle^2}.$$

Тогда величина

$$\Phi_L(f) = \frac{1}{2}\int_\Omega \sqrt{g_{11}g_{22} - g_{12}^2}du^1du^2$$

пропорциональна площади поверхности f. Вариационный функционал Φ_L называется поэтому функиионалом площади и иногда обозначается через vol_2. Оказывается, экстремали этого функционала - это минимальные поверхности, т.е. поверхности нулевой средней кривизны (докажите).

15.15.5 Problems

Задача 15.1. Докажите, что экстремали многомерной вариационной задачи являются решениями уравнения Эйлера-Лагранжа.

Задача 15.2. Докажите, что экстремальность отображения по отношению к многомерному вариационному функционалу равносильна выполнению системы дифференциальных уравнений Эйлера-Лагранжа.

Задача 15.3. Пусть $\Omega \subset \mathbb{R}^2, f : \Omega \to \mathbb{R}^3$ - погружение, и vol_2 обозначает функционал площади. Докажите, что для любой деформации f_s отображения f, неподвижной на границе $\partial\Omega$, выполняется

$$\left.\frac{d\,\mathrm{vol}_2\left(f_s\right)}{ds}\right|_{s=0} = -\int_\Omega \langle E, H\rangle,$$

где F_2 - поле вариации f_s, а H - вектор средний кривизны поверхности f, т.е. произведение средней кривизны на единичную нормаль.

Задача 15.4. Докажите, что минимальная поверхность в $\mathbb{R}^3$ задается гармоническим радиус-вектором (все координатные функции - гармонические), если и только если индуцированная метрика конформна (изотермична).

15.15.6 Literature

1. Мищенко А. С., Фоменко А. Т. Курс дифференциальной геометрии и топологии. - М.: Факториал Пресс, 2000.

2. Мищенко А. С., Фоменко А. Т. Краткий курс дифференциальной геометрии и топологии. - М.: Физматлит, 2004.

3. Новиков С. П., Фоменко А. Т. Элементы дифференциальной геометрии и топологии. - М.: Наука, 1987.

4. Постников М. М. Лекции по геометрии. Семестр 2: Линейная алгебра и дифференциальная геометрия. - М.: Наука, 1979; Семестр 3: Гладкие многообразия. - М.: Наука, 1987.

5. Дубровин Б. А., Новиков С. П., Фоменко А. Т. Современная геометрия: в 2 т. - М.: Наука, 1979, 1984.

6. Рашевский П. К. Курс дифференциальной геометрии. - М.: Гостехиздат, 1956.

7. Трофимов В. В. Введение в геометрию многообразий с симметриями. - М.: Изд-во МГУ, 1989.

8. Висильев А. М., Соловьев Ю. П. Дифференциальная геометрия. М.: Изд-во МГУ, 1988.

9. Мищенко А. С., Соловьев Ю. П., Фоменко А. Т., Сборник задач по дифференциальной геометрии и топологии. - М.: Физматлит, 2004.

10. Иванов А. О., Тужилин А. А., Лекции по классической дифференциальной геометрии. - М.: Логос, 2008.

Рис. 5. Сфера с тремя ручками

Part III
Problems of Differential Geometry

16 Catalog of Problems

(P - значит задачу давал рослый, когда я был на 3 курсе. потом мб переобозначу.)

16.1 General Questions

16.1.1 Questions on Understanding the Essence of the Diffhem

16.1.2 Typical Knowledge Test Questions and Tasks

как понимать, что такое $\gamma - \theta\phi^{\phi}$?

(потом разберусь, см. диалект, пока не актуально это понимать.)

Problems from MSU

Замечание: во всех задачах, где координатами являются x, y или x, y, z, они упорядочены естественным образом, т.е. x - первая координата, y - вторая, z - третья.

1. Найти компоненты тензора $T^i_j = \begin{cases} 0, i \neq j, \\ 2, i = j \end{cases}$ после замены базиса с матрицей $\begin{pmatrix} 1 & 5 \\ 1 & 3 \end{pmatrix}$.
2. В базисе e_1, e_2 тензор T имеет следующие компоненты: $T^1_2 = 3, T^1_1 = T^2_1 = T^2_2 = 0$. Найти компоненту T^2_1 тензора T в базисе $(f_1, f_2) = (e_1, e_2)\begin{pmatrix} 1 & 1 \\ 5 & 4 \end{pmatrix}$.
3. Найти компоненту T^{22}_{11} тензора $T = e^1 \otimes e^1 \otimes e_2 \otimes e_2 + 2e^2 \otimes e^2 \otimes e_1 \otimes e_1$ в базисе $f_1 = 3e_1, f_2 = e_2$.
4. Пусть T_{ijk} - кососимметрический тензор в пространстве $\mathbb{R}^3$, у которого компонента T_{123} равна 2 в базисе e_1, e_2, e_3. Вычислить его компоненты в базисе $f_1 = e_2, f_2 = -e_3, f_3 = e_1 + e_2$.
5. Вычислить компоненты тензора $T = \mathrm{Alt}(u \otimes v)$, где $u = 2e_1 + e_2, v = 2e_1$.
6. Вычислить компоненты тензора $T = \mathrm{Sym}(\xi \otimes \eta)$, где $\xi = -e^1, \eta = e^2 - e^1$.
7. Вычислить значение тензора $\xi \otimes \eta$, где $\xi = e^1 + e^2, \eta = e^1 - e^2$, на паре векторов $5e_1$ и $4e_2$.
8. Найти свертку тензора $2e_1 \otimes e^2 - 3e_2 \otimes e^2 + e_3 \otimes e^2 - e_3 \otimes e^3$.
9. Найти компоненты тензора $T = e_3 \otimes e_2 \otimes e_1 + e_1 \otimes e_1 \otimes e_2$ после применения операции перестановки индексов, заданной перестановкой $(1, 2, 3) \to (2, 1, 3)$.
10. Поднять первый индекс у тензора $T = 2e^1 \otimes e^2 + 3e^2 \otimes e^1$ с помощью скалярного произведения, заданного матрицей $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$.
11. Вычислить внешнее произведение форм $dx + 2dy$ и $x^2 dy \wedge dz + dz \wedge dx$.
12. Вычислить внешний дифференциал формы $\omega = x^2 dx + xy^3 dy$.
13. Найти значения параметра λ, для которых форма $3\lambda x dy \wedge dz + y dx \wedge dz$ будет замкнутой.
14. Найти прообраз формы $\omega = y^2 dx \wedge dy$ при отображении $(x, y) \mapsto \left(x^3, y^2\right)$ плоскости $\mathbb{R}^2(x, y)$ в себя.
15. Найти прообраз формы $\omega = \frac{xdy - ydx}{x^2 + y^2}$ при отображении окружности в плоскость $\varphi \mapsto (x, y)$, заданном формулами $x = 2\cos\varphi$ и $y = 2\sin\varphi$.
16. Вычислить интеграл от 1 -формы $dx + dy$ по нижней половине эллипса $x^2 + 2y^2 = 1$.

17. Вычислить коммутатор векторных полей $X = (y, 0)$ и $Y = \left(x^2 + y, x - y\right)$ на плоскости $\mathbb{R}^2(x, y)$.

18. Вычислить коммутатор векторных полей $X = (0, 0, 1)$ и $Y = \left(x - 3y, -y^3, z\right)$ в пространстве $\mathbb{R}^3(x, y, z)$.

19. Найти ковариантную производную функции $f = yx^2 + y^2$ относительно римановой связности для метрики $ds^2 = dx^2 + y^2 dy^2$.

20. Найти ковариантную производную векторного поля $v = \left(x^2, y\right)$ относительно римановой связности для метрики $ds^2 = 2dx^2 + 3dy^2$.

21. Вычислить символ Кристоффеля Γ^1_{22} для метрики $ds^2 = x^2\left(dx^2 + dy^2\right)$.

22. Для евклидовой метрики $dx^2 + dy^2$ вычислить символ Кристоффеля Γ^2_{11} в координатах $u = y, v = xy$.

23. На плоскости с координатами (x, y) задана аффинная связность с символами Кристоффеля $\Gamma^1_{11} = y$, $\Gamma^2_{11} = \Gamma^2_{22} = 1$ (остальные символы Кристоффеля равны нулю). Выписать уравнения параллельного переноса вдоль прямой $y = 3$.

24. На плоскости с координатами (x, y) задана аффинная связность с символами Кристоффеля $\Gamma^2_{11} = 2$, $\Gamma^1_{12} = \Gamma^1_{21} = y$ (остальные символы Кристоффеля равны нулю). Выписать уравнения геодезических.

16.1.3 Questions on Understanding Typical Details

16.2 Typical Technical Tasks

16.2.1 Problems About the Properties of Differential Shapes

16.2.2 Problems About Defining Surface Parameters (???)

(нарешаю тут потом это.)

ИТ-4.1. Вычислить вторую фундаментальную форму для поверхности, заданной неявной функцией $F(x, y, z)$.

ИТ-4.2. Вычислить среднюю и гауссову кривизны для поверхности, заданной неявной функцией $F(x, y, z)$.

ИТ-4.3. Вычислить среднюю и гауссову кривизны для гиперповерхности, заданной графиком: $x^n = f\left(x^1, \ldots, x^{n-1}\right)$.

ИТ-4.4. Вычислить среднюю и гауссову кривизны для поверхности, заданной неявной функцией $F\left(x^1, \ldots, x^n\right)$.

ИТ-4.5. Вычислить вторую фундаментальную форму следующих поверхностей вращения:

1) $r = (R\cos u\cos v, R\cos u\sin v, R\sin u)(c\oint epa)$;

2) $r = ((a + b\cos u)\cos v, (a + b\cos u)\sin v, b\sin u)(mop)$;

3) $r = \left(a\,\mathrm{ch}\,\frac{u}{a}\cos v, a\,\mathrm{ch}\,\frac{u}{a}\sin v, u\right)$ (катеноид).

ИТ-4.6. Вычислить среднюю и гауссову кривизны сферы, тора и катеноида.

ИТ-4.7. Вычислить среднюю и гауссову кривизны поверхности, полученной вращением графика функции $x = f(z) > 0$ вокруг оси z.

ИТ-4.8. Пусть S - некоторая данная поверхность. Отложим на нормалях к поверхности S в одном направлении отрезки постоянной длины. Концы отложенных отрезков описывают поверхность S^*, "параллельную" поверхности S. Если уравнение поверхности S есть $r = r(u, v)$, то уравнение S^*

$$\rho = r(u, v) + an(u, v),$$

где $n(u, v)$ - единичный вектор нормали к S. Выразить коэффициенты первой и второй фундаментальных форм поверхности S^* через коэффициенты первой и второй фундаментальных форм поверхности S.

ИТ-4.9. Выразить гауссову кривизну K^* поверхности S^*, "параллельной" поверхности S, через гауссову и среднюю кривизны поверхности S.

ИТ-4.10. Выразить среднюю кривизну H^* поверхности S^*, "параллельной" поверхности S, через гауссову и среднюю кривизны поверхности S.

16.2.3 Problems About Vector Operations in Curved Coordinates (??)

(придумаю задачи в духе "посчитать градиент/дивергенцию в таких-то искривленных координатах")

16.2.4 Problems About Understanding Diversity

Рослый-2020-4.4

Для стандартной метрики на S^2, записанной в координатах стереографической проекции ($z = x + iy$):

$$g = 4\frac{dx^2 + dy^2}{(1 + |z|^2)^2},$$

найти $\int\limits_{\mathbb{R}^2} \sqrt{g}$. То же в угловых координатах:

$$g = d\theta^2 + \sin^2\theta d\varphi^2.$$

Рослый-2020-4.5

Доказать, что $\dim T_pM = \dim M, \forall p \in M$.

Рослый-2020-4.6

Доказать изоморфизм

$$\Lambda^1(M) = \mathrm{Hom}_{C^\infty(M)}\left(\mathrm{Vect}(M), C^\infty(M)\right),$$

где $\mathrm{Hom}_{\mathcal{R}}$ обозначает гомоморфизм $\mathcal{R}$-модулей над каким-то кольцом $\mathcal{R}$. (Можно сначала рассмотреть случай $M = \mathbb{R}^n$, а потом использовать разбиение единицы.)

ИТ-12.1.

Показать, что объединение двух координатных осей не является многообразием.

ИТ-12.2.

Задать на двумерном торе $T^2 = S^1 \times S^1$ структуру двумерного топологического многообразия.

ИТ-12.3.

Привести пример различных гладких структур на прямой $\mathbb{R}^1$.

ИТ-12.4.

Проверить эквивалентность двух описанных выше атласов на стандартной сфере.

ИТ-12.5.

Рассмотрим на прямой $\mathbb{R}$ атлас, состоящий из двух карт: $(\mathbb{R}, \varphi : x \mapsto x)$ и $\left(\mathbb{R}, \psi : x \mapsto x^3\right)$. Так как φ и ψ, очевидно, гомеоморфизмы, мы задали на $\mathbb{R}$ структуру топологического многообразия. Показать, что оно не является гладким. Модифицировать предыдущий пример так, чтобы получилось многообразие класса C^2, но не C^3.

Solution

ИТ-12.6.

Показать, что комплексное проективное пространство $\mathbb{CP}^n$ является комплексно аналитическим многообразием комплексной размерности n.

Solution

ИТ-12.7.

Показать, что гладкие многообразия $\mathbb{R}^1(x)$ с картой $\left(\mathbb{R}^1, \varphi(x) = x\right)$ и $\mathbb{R}^1(y)$ с картой $\left(\mathbb{R}^1, \varphi(y) = y^3\right)$ диффеоморфны.

Solution

ИТ-12.8.

Показать, что любое некомпактное связное одномерное многообразие диффеоморфно $\mathbb{R}^1$, а любое компактное связное одномерное многообразие диффеоморфно окружности S^1.

Solution

ИТ-12.9.

Рассмотрим в $\mathbb{R}^4$ поверхность, заданную системой уравнений

$$\begin{cases} \left(x^1\right)^2 + \left(x^2\right)^2 = 1, \\ \left(x^3\right)^2 + \left(x^4\right)^2 = 1. \end{cases}$$

Показать, что эта поверхность в $\mathbb{R}^4$ является гладким многообразием размерности 2, и это многообразие диффеоморфно двумерному тору $T^2 = S^1 \times S^1$.

Solution

ИТ-12.10. Многообразия по группе

Показать, что следующие матричные группы являются гладкими многообразиями и определить их размерность:

$$\begin{aligned} GL(n,\mathbb{R}) &= \left\{X \in \mathbb{R}^{n^2} | \det X \neq 0\right\}, \\ O(n,\mathbb{R}) &= \left\{X \in \mathbb{R}^{n^2} | XX^T = E\right\}, \\ SO(n,\mathbb{R}) &= \left\{X \in \mathbb{R}^{n^2} | XX^T = E, \quad \det X = 1\right\}, \\ U(n,\mathbb{R}) &= \left\{X \in \mathbb{C}^{n^2} = \mathbb{R}^{2n^2} | X\bar{X}^T = E\right\}, \\ SU(n,\mathbb{R}) &= \left\{X \in \mathbb{C}^{n^2} = \mathbb{R}^{2n^2} | X\bar{X}^T = E, \quad \det X = 1\right\}. \end{aligned}$$

Исследовать эти матричные группы на компактность и линейную связность.

Solution

16.2.5 Problems About Killing Vectors and Symmetries

Leuven-cosmo-2023hw-1.1. Killing vectors

a) Let ξ be a Killing vector field. Show that $\nabla_\sigma \nabla_\rho \xi_\mu = R^\lambda{}_{\sigma\rho\mu}\xi_\lambda$.
b) For $U_\mu = (1,0,0,0)$, verify that (in mostly plus signature)

$$\xi_{\mu\nu} = a^2\left(g_{\mu\nu} + U_\mu U_\nu\right)$$

is a Killing tensor in a cosmological spacetime $\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right]$, and $a \equiv a(t)$ i.e., show that $\nabla_{(\sigma}\xi_{\mu\nu)} = 0$.

Solution to a)

We put the Killing vector equation $\nabla_\rho\xi_\sigma + \nabla_\sigma\xi_\rho = 0$ in the Riemann tensor equation $\nabla_\sigma\nabla_\rho\xi_\mu - \nabla_\rho\nabla_\sigma\xi_\mu = R^d{}_{\mu bc}\xi_d$ and write for different indices:

$$\begin{aligned} \nabla_\sigma\nabla_\rho\xi_\mu + \nabla_\rho\nabla_\mu\xi_\sigma &= R^d{}_{\mu\rho\sigma}\xi_d \\ \nabla_\mu\nabla_\sigma\xi_b + \nabla_\sigma\nabla_\rho\xi_\mu &= R^d{}_{b\sigma\mu}\xi_d \\ \nabla_\rho\nabla_\mu\xi_\sigma + \nabla_\mu\nabla_\sigma\xi_\rho &= R^d{}_{\sigma\mu\rho}\xi_d \end{aligned}$$

We add the first two equations and subtract the third, then

$$\nabla_\sigma\nabla_\rho\xi_\mu + \nabla_\rho\nabla_\mu\xi_\sigma + \nabla_\mu\nabla_\sigma\xi_\rho + \nabla_\sigma\nabla_\rho\xi_\mu - \nabla_\rho\nabla_\mu\xi_\sigma - \nabla_\mu\nabla_\sigma\xi_\rho = R^d{}_{\mu\rho\sigma}\xi_d + R^d{}_{\rho\sigma\mu}\xi_d - R^d{}_{\sigma\mu\rho}\xi_d.$$

Then from $R_{dabc} + R_{dbca} + R_{dcab} = 0$ we have $R_{dabc} + R_{dbca} = -R_{dcab}$, so:

$$\begin{aligned}
2\nabla_\sigma\nabla_\rho\xi_\mu &= \left(R^d{}_{\mu\rho\sigma} + R^d{}_{\rho\sigma\mu} - R^d{}_{\sigma\mu\rho}\right)\xi_d \\
&= \left(-R^d{}_{\sigma\mu\rho} - R^d{}_{\sigma\mu\rho}\right)\xi_d \\
&= -2R^d{}_{\sigma\mu\rho}\xi_d \qquad\qquad (R_{d\sigma ab} = -R_{d\sigma ba}) \\
&= 2R^d{}_{\sigma\rho\mu}\xi_d
\end{aligned}$$

So

$$\nabla_\sigma\nabla_\rho\xi_\mu = R^d{}_{\sigma\rho\mu}\xi_d$$

Solution to b)

For this metric

$$\Gamma^\mu_{00} = \Gamma^0_{0\beta} = 0, \quad \Gamma^0_{ij} = a\dot{a}\gamma_{ij}, \quad \Gamma^i_{0j} = \frac{\dot{a}}{a}\delta^i_j, \qquad \Gamma^i_{jk} = \frac{1}{2}\gamma^{il}\left(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}\right).$$

We have $K_{00} = a^2(-1+1) = 0$ and because of $\nabla_\sigma K_{\mu\nu} + \nabla_\nu K_{\sigma\mu} + \nabla_\mu K_{\nu\sigma} = 0$ and symmetricity of K_{ij}

$$K_{0\mu} = 0 \qquad \text{for all } \ \mu.$$

Also

$$\begin{aligned}
g_{ij} &= -a^2\gamma_{ij} \equiv -a^2\gamma_{ii}\delta_{ij}, \\
K_{ij} &= -a^4\gamma_{ij} \equiv -a^4\gamma_{ii}\delta_{ij}
\end{aligned}$$

We need to prove that

$$\nabla_{(\sigma}K_{\mu\nu)} = 0,$$

$$\nabla_\sigma K_{\mu\nu} \equiv \partial_\sigma K_{\mu\nu} - \Gamma^\lambda_{\sigma\mu}K_{\lambda\nu} - \Gamma^\lambda_{\sigma\nu}K_{\mu\lambda}.$$

Because $K_{\mu\nu}$ is symmetric, $\nabla_{(\sigma}K_{\mu\nu)} = 0$ is proportional to itself with any position of indices, so for proof we can choose just one option of them, which is the most comfortable for us.

We have different cases of indices:

• If all three indices are timelike ($\mu \equiv 0, \nu \equiv 0, \sigma \equiv 0$), every term in our expression is either $\partial_0 K_{00} = 0$ or proportional to $K_{0\lambda} = 0$. Hence, the whole term is zero.

Note that in general it doesn't follows from from $K_{00} = 0$ that $\nabla_\sigma K_{00} \equiv 0$, since in there could be metrics and $K_{0\mu} \neq 0$, for which $\nabla_\sigma K_{00} = -\Gamma^\lambda_{\sigma 0}K_{\lambda 0} - \Gamma^\lambda_{\sigma 0}K_{0\lambda} \neq 0$. But it is not our case.

• If only two indices are timelike ($\mu \equiv 0, \nu \equiv 0, \sigma \equiv i$), the same way because of $K_{0\mu} = 0$ all terms become zero, except the term of the form $\Gamma^\lambda_{00}K_{\lambda i}$. However, in FRLW $\Gamma^\mu_{00} = 0$, so the total result is still zero.

• If one index is timelike ($\mu \equiv 0, \nu \equiv i, \sigma \equiv j$):

$$\begin{aligned}
\nabla_0 K_{ij} &= \partial_0 K_{ij} - \Gamma^\lambda_{0i}K_{j\lambda} - \Gamma^\lambda_{0j}K_{i\lambda}; \\
\nabla_j K_{0i} &= \partial_j K_{0i} - \Gamma^\lambda_{j0}K_{\lambda i} - \Gamma^\lambda_{ji}K_{0\lambda}; \\
\nabla_i K_{0j} &= \partial_i K_{j0} - \Gamma^\lambda_{ij}K_{\lambda 0} - \Gamma^\lambda_{i0}K_{j\lambda};
\end{aligned}$$

Also $\Gamma^i_{0j} = \frac{\dot{a}}{a}\delta_{ij}$, so $\lambda \to k$ where ($k = 1, 2, 3$), so

$$\nabla_{(0}K_{ij)} = -\partial_0\left(a^4\right)\gamma_{ii}\delta_{ij} - 2(-a^4)\left(\Gamma^j_{0i}\gamma_{jj} + \Gamma^i_{0j}\gamma_{ii}\right) = -\gamma_{ii}\delta_{ij}\left[a^3\dot{a} - 2(-a^4)\left(\frac{\dot{a}}{a} + \frac{\dot{a}}{a}\right)\right] = 0$$

• If all indexes are spacelike:

$$\begin{aligned}
\nabla_k K_{ij} &= \partial_k K_{ij} - \Gamma^l_{ki}K_{jl} - \Gamma^l_{kj}K_{il}, \\
\nabla_j K_{ki} &= \partial_j K_{ki} - \Gamma^l_{jk}K_{li} - \Gamma^l_{ji}K_{kl}, \\
\nabla_i K_{kj} &= \partial_i K_{jk} - \Gamma^l_{ij}K_{lk} - \Gamma^l_{ik}K_{jl};
\end{aligned}$$

After using $K_{ij} = -a^4\gamma_{ii}\delta_{ij}$ we have

$$\nabla_{(i}K_{jk)} = -a^4\left(\partial_i\gamma_{jk} + \partial_k\gamma_{ij} + \partial_j\gamma_{ki}\right) - 2(-a^4)\left(\Gamma^k_{ij}\gamma_{kk} + \Gamma^j_{ik}\gamma_{jj} + \Gamma^i_{kj}\gamma_{ii}\right) \qquad \text{(no summ. by } k)$$

Since the metric is diagonal,

$$\Gamma^k_{ij} = \frac{1}{2}\gamma^{kk}\left(\partial_i\gamma_{jk} + \partial_j\gamma_{ik} - \partial_k\gamma_{ij}\right) \qquad k \text{ is fixed,}$$

$$\Gamma^k_{ij}\gamma_{kk} + \Gamma^j_{ik}\gamma_{jj} + \Gamma^i_{kj}\gamma_{ii} = \ldots = \frac{1}{2}\left(\partial_i\gamma_{jk} + \partial_k\gamma_{ij} + \partial_j\gamma_{ki}\right)$$

and because of it

$$\nabla_{(i}K_{jk)}.$$

So for all possibilities $K_{\mu\nu}$ is a Killing tensor.

16.2.6 Problems About the Display

Рослый-2020-4.7

Даны многообразия и гладкие отображения:

$$M_1 \xrightarrow{\varphi} M_2 \xrightarrow{\psi} M_3.$$

Для отображений обратного образа на гладких функциях $\varphi^* : C^\infty(M_2) \to C^\infty(M_1)$ и $\psi^* : C^\infty(M_3) \to C^\infty(M_2)$ показать, что

$$(\psi \circ \varphi)^* = \varphi^* \circ \psi^*.$$

Рослый-2020-4.8

Для гладкого отображения $\varphi : M_1 \to M_2$ записать в локальных координатах отображение касательных пространств

$$D_p\varphi : T_pM_1 \to T_{\varphi(p)}M_2.$$

Рослый-2020-4.9

Для гладкого отображения $\varphi : M_1 \to M_2$ и для $f \in C^\infty(M_2)$ показать, что

$$\varphi^*(df) = d\left(\varphi^* f\right).$$

То же для k-формы $\omega \in \Lambda^k(M_2)$:

$$\varphi^*(d\omega) = d\left(\varphi^*\omega\right).$$

Рослый-2020-4.10

Пусть $\varphi : M \to M$ - диффеоморфизм, и $f \in C^\infty(M), v \in \mathrm{Vect}(M), \alpha \in \Lambda^1(M)$. Доказать, что

$$\langle \varphi^* v, \varphi^*\alpha\rangle = \varphi^*\langle v, \alpha\rangle,$$
$$\varphi^*(v(f)) = \left(\varphi^* v\right)\left(\varphi^* f\right).$$

Рослый-2020-4.11

Пусть $f : M \to \mathbb{R}^k$ - гладкое отображение. Иначе говоря, нам даны гладкие функции $f^1, \ldots, f^k \in C^\infty(M)$. Показать, что в любой точке $p \in M$ имеем conm ношение

$$\mathrm{rk}(Df) = k \Leftrightarrow df^1 \wedge \ldots \wedge df^k \neq 0,$$

где Df - отображение касательных пространств.

Рослый-2020-4.12

Привести простейшие примеры многообразий, на которых есть глобальный голономный репер. В каких из этих случаев также имеется глобальная система координат.

Рослый-2020-4.13

Пусть $\{e_a\}$ - произвольный репер на многообазии $M, a = 1, \ldots, \dim M$, и

$$[e_a, e_b] = f^c_{ab} e_c.$$

Пусть $\{e^a\}$ - двойственный корепер (двойственный базис 1-форм). Показать, что тогда

$$de^c = -\frac{1}{2} f^c_{ab} e^a \wedge e^b$$

Рослый-2020-4.14*

Пусть $\alpha \in \Lambda^1(M)$ - поле 1-форм, нигде не обращающихся в ноль на M. Определим распределение $\mathcal{D} \subset TM$ коразмерности 1 в касательном расслоении:

$$\mathcal{D}_p := \{\xi \in T_p M | \alpha_p(\xi) = 0\}.$$

Доказать, что $\mathcal{D}$ инволютивно тогда, и только тогда, когда

$$\alpha \wedge d\alpha = 0.$$

Рослый-2020-4.15

Пусть $v, u \in \mathrm{Vect}(M)$ - два векторных поля на мноообразии M. Производная Ли вдоль векторного поля обозначается L_v. Показать, что

$$[L_v, L_u] = L_{[v,u]}.$$

Напомним, что $[v, u]$ обозначает коммутатор (или скобку Ли) векторньг полей и что при действии производной Ли, в частности, на векторные поля имеем $L_v u = [v, u]$.

Рослый-2020-4.16

Формой объема на многообразии M называется такая форма старшей степени $\mathrm{vol} \in \Lambda^n(M)$, где $n = \dim M$, которая нигде не обращается в ноль. Если на M задана форма объема vol, то для любого векторного поля $v \in \mathrm{Vect}(M)$ делена его дивиргенция $\mathrm{div}\, v \in C^\infty(M)$ - функция, однозначно определ 5 соотношением $L_v \mathrm{vol} = \mathrm{div}\, v \cdot \mathrm{vol}$. Имеет смысл запись

$$\mathrm{div}\, v = \frac{L_v \mathrm{vol}}{\mathrm{vol}}.$$

Чаще всего мы пользуемся таким определением на римановом многообразии по отношению к метрической форме объема $\mathrm{vol} = \sqrt{g}$, определенной равенством (21) в задаче 4.3. Тогда $\mathrm{div}_g\, v := L_v \sqrt{g}/\sqrt{g}$. Докажите, что в локальных координатах

$$\mathrm{div}_g\, v = \frac{1}{\sqrt{\det(g_{\mu\nu})}} \partial_\lambda \left(\sqrt{\det(g_{\mu\nu})} v^\lambda \right).$$

Рослый-2020-4.17

Докажите, что для $\forall \mathrm{vol} \in \Lambda^n(M)$ и $\forall v, u \in \mathrm{Vect}(M)$ имеем

$$\mathrm{div}[v, u] = v\, \mathrm{div}\, u - u\, \mathrm{div}\, v.$$

Рослый-2020-4.19

Привести пример такого тензорного поля $t \neq 0$ на многообразии M, что $\forall v \in \mathrm{Vect}(M) : L_v t = 0$.

Рослый-2020-4.20*

Пусть отображение $\varphi : \mathbb{R}^n \backslash \{0\} \to \mathbb{R}^n \backslash \{0\}$ задано формулой

$$\varphi(x) = \frac{x}{\|x\|}.$$

Рассмотрим $(n-1)$-форму

$$\omega = \varepsilon_{i_1 i_2 \ldots i_n} x^{i_1} dx^{i_2} \wedge \ldots \wedge dx^{i_n}.$$

Докажите, что $d(\varphi^* \omega) = 0$.

Рослый-2020-4.21*

Пусть $v_1, v_2 \in \mathrm{Vect}(M), \alpha \in \bigwedge^1(M)$. Напомним формулу

$$d\alpha\left(v_1, v_2\right) = v_1\left(\alpha\left(v_2\right)\right) - v_2\left(\alpha\left(v_1\right)\right) - \alpha\left(\left[v_1, v_2\right]\right).$$

Придумать и доказать аналогичную формулу для $d\omega\left(v_1, v_2, v_3\right)$ в случае, если $\omega \in \Lambda^2(M)$

Рослый-2020-4.22

На многообразии $\mathbb{R}^2$ рассмотрим три векторных поля, которые запишем в терминах комплексной координаты $z = x + iy$ (черта обозначает комплексное сопряжение):

$$\frac{\partial}{\partial z} + \frac{\partial}{\partial \bar{z}}, \quad z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}, \quad z^2\frac{\partial}{\partial z} + \bar{z}^2\frac{\partial}{\partial \bar{z}}$$

Какие из этих векторных полей являются полными на $\mathbb{R}^2$, а какие нет?

4.23*

На сфере S^2 рассмотрим три векторных поля, заданных формулой (45) в координатах стереографической проекции (в одной из двух карт).

- Покажите, что этим определяются векторные поля на всей сфере, то есть, что в другой карте с координатой $u = z^{-1}$ эти векторные поля регулярны при $u = 0$.

- Поскольку S^2 - компактное многообразие, каждое из этих векторных полей полное, то есть интегрируется до однопараметрической группы диффеоморфизмов. Покажите, что все вместе они порождают трехпараметрическую группу преобразований сферы S^2. Назовите эту группу.

- Преобразования этой группы вообще говоря не оставляют северный полюс $\{z = \infty\}$ на месте. Поэтому рассматриваемая группа не действует на $\mathbb{R}^2 = S^2 \backslash \{z = \infty\}$. Только ее подгруппа, сохраняющая $\{z = \infty\}$, действует на $\mathbb{R}^2$. Как из этих соображений можно сразу получить ответ на вопрос предыдущей задачи?

16.2.7 Problems About the Topology of Manifolds

5.5

Пусть на замкнутом ориентируемом многообразии дана форма старшей степени $\omega \in \Lambda^n(M)$, где $n = \dim M$, которая нигде не обращается в ноль. Покажите, что такая форма не может быть точной.

5.6

Пусть $M = S^2$. Двойственноть Пуанкаре дает нам изомнорфизм $H_k(M, \mathbb{R}) \cong H^{2-k}(M, \mathbb{R})$. Пусть $[\{p\}] \in H_0(M, \mathbb{R})$, где p - любая точка M, а $[M] \in H_2(M, \mathbb{R})$ - двумерный класс гомологий, отвечающий самому замкнутому многообразию M

- Показать, что класс гомологий $[\{p\}]$ не зависит от выбора точки p.
- Предъявить класс когомологий (его представитель) в $H^2(M, \mathbb{R})$, отвечающий 0-циклу $\{p\}$.
- Предъявить класс когомологий (его представитель) в $H^0(M, \mathbb{R})$, отвечающий 2-циклу M.

5.7*

Доказать, что на замкнутом многообразии каждое векторное поле является полным.

5.8*

На компактном многообразии любое бесконечное множество точек содержит нетривиальную (то есть не постоянную) сходящуюся последовательность (или, что то же самое, имеет предельную точку). Равносильное утверждение: на компактном многообразии любое замкнутое подмножество, состоящее из изолированных точек, конечно. Докажите все это.

5.9*

Рассмотрим двумерный тор $M = S^1 \times S^1$. Рассмотрим 1-цикл в M вида $S^1 \times \{p\}$, где p - любая точка второй окружности. Показать, что класс гомологий такого цикла не зависит от точки p, т.е. другой подобный цикл $S^1 \times \{p'\}$ гомологичен первому. Пусть $a \in H_1(M)$ - класс гомологий, представленный любым из этих 1-циклов. По двойственности Пуанкаре ему отвечает класс $[\alpha] \in H^1(M, \mathbb{Z}) \subset H^1(M, \mathbb{R})$. Найдите 1-форму α, представляющую этот класс когомологий. (Используйте любое удобное описание тора, любые координаты.)

16.3 Tasks of Other Topics

16.3.1 Tasks About Rings and Differentials

2.2*

$\mathcal{R}$ - кольцо, $\mathrm{Drv}(\mathcal{R})$ - пространство дифференцирований этого кольца. Каково определение $\mathrm{Drv}(\mathcal{R})$? Показать, что $\mathrm{Drv}(\mathcal{R})$ - алгебра Ли.

Пусть теперь $\mathcal{R}$ - это $\mathbb{Z}_2$-градуированное суперкоммутативное кольцо, $\mathrm{arv}(\mathcal{R})$ - пространство супердифференцирований этого кольца. Каково определение $\mathrm{Drv}(\mathcal{R})$? Показать, что $\mathrm{Drv}(\mathcal{R})$ - супералгебра Ли. (Надо, в частности, продемонстрировать знание того, что такое супералгебра Ли.)

2.3

V - векторное пространство размерности n. Показать, что

$$S^{\bullet}\left(V^{*}\right)=\{\text{ polynomial functions on V}\} \cong \mathbb{R}\left[\mathrm{x}^{1}, \ldots, \mathrm{x}^{\mathrm{n}}\right],$$

где первый изоморфизм колец канонический, а второй не канонический (почему?). Здесь $\mathbb{R}\left[x^{1}, \ldots, x^{n}\right]$ обозначает кольцо полиномов от n переменных с коэффициентами в поле $\mathbb{R}$. Кольцо (алгебру) $S^{\bullet}\left(V^{*}\right)$ полиномов с вещественными коэффициентами на V можно/нужно также обозначать $\mathbb{R}[V]$. Описать алгебру Ли дифференцирований колец (15).

16.3.2 Problems of Homologous Algebra (??)

Пусть $C^{\bullet}$ - ограниченный коцепной комплекс векторных пространств с дифференциалом d, такой что $\dim C^{k}<\infty, \forall k$, и $H^{\bullet}$ - его группы (векторные пространстве когомологий.

Рослый-2020-3.1

Показать, что

$$\sum_{k}(-1)^{k} \dim H^{k}=\sum_{k}(-1)^{k} \dim C^{k}$$

Как называется сумма в левой части выше?

Рослый-2020-3.2

Пусть $L: C^{\bullet} \rightarrow C^{\bullet}$ - линейное отображение, сохраняющее градуировку и дифференциал, $L \circ d=d \circ L$. Показать, что тогда L индуцирует отображение пространств когомологий, которое обозначим тем же символом:

$$L: H^{\bullet} \rightarrow H^{\bullet}.$$

Тогда величина

$$\Lambda_{L}=\sum_{k}(-1)^{k} \operatorname{tr}_{H^{k}} L$$

называется числом Лефшеца отображения L. В частности, при $L=\mathrm{id}_{C^{\cdot}}$ получается, что $\Lambda_{\mathrm{id}}=\chi\left(C^{\bullet}\right)$.

Показать, что при наших предположениях (конечномерность) выполняется равенство

$$\sum_{k}(-1)^{k} \operatorname{tr}_{H^{k}} L=\sum_{k}(-1)^{k} \operatorname{tr}_{C^{k}} L$$

Рослый-2020-3.3

В тех же обозначениях, что и в задаче 3.2 показать, что

$$\prod_{k}\left(\det_{H^{k}} L\right)^{(-1)^{k}}=\prod_{k}\left(\det_{C^{k}} L\right)^{(-1)^{k}}$$

16.3.3 Linear Algebra Problems for Diffhem

В данном разделе слова “векторное пространство” подразумевают “конечномерное”, если не сказано противное.

1.1

V - векторное пространство. Опишите каноноческий изомрфизм $(V^*)^* \cong V$.

1.2

V, W - векторные пространства, $\dim V = n, \dim W = m$. Какова размерность векторных пространств $\mathrm{Hom}(V, W)$ и $V \otimes W$?

Пусть $\{e_i\}$ - басис в V, а $\{f_j\}$ - басис в W. Тогда $\{e_i \otimes f_j\}$ - басис в $V \otimes W$.

1.3

Опишите канонические изоморфизмы конечномерных векторных пространств

$$\mathrm{Hom}(\mathbb{R}, W) = W, \quad \mathrm{Hom}\left(V^*, \mathbb{R}\right) = V,$$

$$\mathrm{Hom}(V, W) \cong V^* \otimes W.$$

1.4$V \subset W$ - векторные пространства, $V^\perp \subset W^*$ - аннулятор V. Показать / найти:

$$V^* \cong W^*/V^\perp$$
$$\left(V^\perp\right)^* =?$$
$$\dim V + \dim W/V = \dim W$$
$$\dim V + \dim V^\perp = \dim W$$

1.5

V - векторное пространство, $e_1, \ldots e_k \in V, \quad \dim V = n \geqslant k$, а $f^1, \ldots, f^k \in V^*$ таковы, что $\left\langle f^i, e_j \right\rangle = \delta^i_j$. Показать, что тогда и векторы $e_1, \ldots, e_k$, и ковекторы $f^1, \ldots, f^k$ линейно независимы.

1.6

V - векторное пространство, $v_1, \ldots v_k \in V, \dim V = n \geqslant k$. Показать, что $v_1 \wedge \ldots \wedge v_k \neq 0 \Leftrightarrow \{v_1, \ldots, v_k$ линейно независимы $\}$. В частности, при $n = k : \{v_1, \ldots, v_n \quad -$ базис в $V\} \Leftrightarrow v_1 \wedge \ldots \wedge v_n \neq 0$.

1.7

Даны векторные пространства и линейные отображения:

$$V_1 \overset{A}{\to} V_2 \overset{B}{\to} V_3.$$

Показать, что

$$(B \circ A)^* = A^* \circ B^*.$$

1.8

V - векторное пространство, V^* - двойственное пространство, $\Lambda^\bullet V$ - кольцо поливекторов, $\Lambda^\bullet V^*$ - кольцо внешних форм. (Это суперкоммутативные кольца.) $\Lambda^\bullet V$ и $\Lambda^\bullet V^*$ - двойственные пространства, то есть $\Lambda^\bullet\left(V^*\right) \cong \left(\Lambda^\bullet V\right)^*$, но нужна точность в определении спаривания. Точнее, это вопрос удобства, но, сделав какой-то выбор, лучше во всех формулах потом уже придерживаться выбранного определения. Поступим следующим образом. Сначала определим оператор внутреннего умножения на (свертку с) вектром $v \in V$ как супердифференцирование степени (-1) кольца внешних форм:

$$\iota_v : \Lambda^\bullet\left(V^*\right) \to \Lambda^\bullet\left(V^*\right),$$

так что для $\alpha \in \Lambda^1\left(V^*\right) = V^*$ будет $\iota_v \alpha = \alpha(v)$. Определим теперь спаривание

$$\Lambda^\bullet V \otimes \Lambda^\bullet V^* \to \mathbb{R}$$

таким образом, что для $v_1, \ldots, v_p \in V$ и $\omega \in \Lambda^p\left(V^*\right)$:

$$\langle v_1 \wedge \ldots \wedge v_p, \omega \rangle \equiv \omega\left(v_1, \ldots, v_p\right) := \iota_{v_p} \circ \ldots \circ \iota_{v_1}(\omega).$$

Тогда, например, для $v, v_1, v_2 \in V, \quad \alpha, \alpha_1, \alpha_2 \in V^*$ будет

$$\langle v, \alpha \rangle = \alpha(v),$$
$$\langle v_1 \wedge v_2, \alpha_1 \wedge \alpha_2 \rangle = \alpha_1 (v_1) \alpha_2 (v_2) - \alpha_1 (v_2) \alpha_2 (v_1),$$

При сделанных определениях показать, что линейное отображение внешнего умножения на вектор $v \in V$,

$$e_v : \Lambda^\bullet(V) \to \Lambda^\bullet(V)$$

Найти все (супер)дифференцирования степени $(-1), 0$ и (-2) кольца $\Lambda^\bullet V^*$. Аналогичные определения можно сделать для симметрической алгебры $S^\bullet (V^*)$.

1.9

V - векторное пространство, $v_1, \dots v_p \in V, \quad \dim V = n \geqslant p$, а $f^1, \dots, f^p \in V^*$. Доказать, что

$$\left(f^1 \wedge \ldots \wedge f^p\right)(v_1, \ldots, v_p) = \det\left(f^i\left(v_j\right)\right)$$

16.4 Tasks About Differential Geometry Applications (???)

Обсудим особые многообразия и изучим их свойства.

16.4.1 Problems About Diffgem in Mechanics

16.4.2 Problems About Diffhem in Field Theory

тоже мб пригодится.

16.4.3 Problems About Diffhem in String Theory

тоже мб пригодится.

Part IV
—— Special Differential Geometry in a Nutshell ——

17 Deeper Understanding of Basics

17.1 Lectures on Classical Differential Geometry by Ivanov, A.A. Tuzhilin

14.1 1 Curves in Euclidean space 449 14.1.1 1.1 Symbols and Reminders 449 14.1.2 1.2 Parametric curves 450 14.1.3 1.3 Graph Curves453 14.1.4 1.4 Implicit curves453 14.1.5 1.5 Local equivalence 454 14.1.6 1.6 Curve length, natural parameter 454 14.1.7 1.7 Regular Curve Curvature 456 14.1.8 1.8 Flat curves 457 14.1.9 1.9 Problems 459 14.1.10 1.10 Supplemental material 459 14.1.11 1.10.1 Peano curve in a triangle459 14.1.12 1.10.2 Polyline as a smooth parametric curve461 14.1.13 1.10.3 The length of the curve as the limit of the lengths of inscribed polylines 462 14.1.14 1.10.4 Osculating Circle463 14.1.15 1.10.5 On explicit solutions of the natural equation 464 14.1.16 1.10.6 Evolutes & Evolvents 464 14.1.17 1.10.7 Huygens Pendulum

4.2 2 Curves in three-dimensional space 468 14.2.1 2.1 Frenet's Formulas 468 14.2.2 2.2 Natural equations 470 14.2.3 2.4 Supplemental material472 14.2.4 2.4.1 Slope lines 472 14.2.5 3.2 Graphic Surfaces 473 14.2.6 3.3 Implicit surfaces 474 14.2.7 3.4 Local equivalence 475 14.2.8 3.5 Surfaces and Displays476 14.2.9 3.6 Surface curves476 14.2.10 3.7 Tangent Space477 14.2.11 3.9 First fundamental form 479 14.2.12 3.10 Lengths and angles480 14.2.13 3.11 Surface isometries 482 14.2.14 3.13.2 Rhumb line483 14.2.15 3.13.3 First quadratic shape and volume484 14.2.16 3.13.4 Euclidean Metric Isometries 485

14.3 4 Surfaces. The second fundamental form 487 14.3.1 4.1 Definition of the second form487 14.3.2 4.2 The geometric meaning of the second form is the curvatures of flat sections 489 14.3.3 4.3 Principal curvatures and principal directions 491 14.3.4 4.4 Mean and Gaussian curvatures 493 14.3.5 4.5 Bonn Theorem494 14.3.6 4.6 Problems 495 14.3.7 4.7 Supplemental material495 14.3.8 4.7.1 Other proof of claim 4.7 495 14.3.9 4.7.2 Second cylinder shape and Meunier's theorem 496 14.3.10 4.7.3 Main Cylinder Curvatures497 14.3.11 4.7.4 Laplace-Poisson theorem 498 14.3.12 4.7.5 Mean and Gaussian curvatures of a two-dimensional surface 499 14.3.13 4.7.6 On the choice of coordinates on the surface 499 14.3.14 4.7.9 Rotation surfaces of constant Gaussian curvature502 14.3.15 4.7.10 Case of arbitrary codimensionality 504

14.4 5 Elements of differential calculus on surfaces505 14.4.1 5.1 Weingarten-Gauss derivational formulas 505 14.4.2 5.2 Gaussian Theorem 508 14.4.3 5.3 Covariant derivative OF tangent vector FIELD 509 14.4.4 5.4 Supplemental material512 14.4.5 5.4.1 On multidimensional generalizations of the Gaussian theorem512 14.4.6 5.4.2 Other proof of claim 5.27512

14.5 6 Geodetic on surfaces 513 14.5.1 6.1 Definition and simplest properties of geodetic 513 14.5.2 6.2 Examples, Claireaux's theorem 514

14.6 7 Curvilinear coordinates in the area and on the surface 519 14.6.1 7.1 Determination of curvilinear coordinates 520 14.6.2 7.2 Examples of regular coordinate systems 521 14.6.3 7.3 Tangent Space to Area523 14.6.4 7.4 Euclidean metric in curvilinear coordinates525 14.6.5 7.5 Regular coordinates on the surface526 14.6.6 7.6 Stereographic coordinates of the sphere 526 14.6.7 7.7 Problems 527 14.6.8 7.8 Supplemental material527 14.6.9 7.8.1 Stereographic Projection Properties 527 14.6.10 7.8.2 Multidimensional stereographic projection528 14.6.11 7.8.3 Normal coordinates530 14.6.12 7.8.4 Semi-geodetic coordinates532

14.7 8 Riemann and pseudo-Riemann metrics533 14.7.1 8.1 Bilinear shape in the area 533 14.7.2 8.2 Pseudo-Euclidean spaces 534 14.7.3 8.3 Spacetime 535 14.7.4 8.4 Pseudosphere geometry 536 14.7.5 8.5 Problems 538 14.7.6 8.6 Supplemental material538 14.7.7 8.6.1 Two-dimensional surfaces of constant Gaussian curvature 538

14.8 9 Lobachevsky geometry542 14.8.1 9.1 Non-Euclidean geometries 542 14.8.2 9.1.1 Elliptical geometry543 14.8.3 9.1.2 Lobachevsky geometry (hyperbolic geometry) 543 14.8.4 9.2 Poincar´e model in a circle 544 14.8.5 9.3 Fractional-linear transformations 545 14.8.6 9.4 Recording the metric in a comprehensive form 548 14.8.7 9.5 Top half-plane model549 14.8.8 9.6 Lobachevsky plane isometries550 14.8.10 9.8 Supplemental material552 14.8.11 9.8.1 Stereo-

graphic Projection Properties 552 14.8.12 9.8.2 Metric in the upper half-plane model 553 14.8.13 9.8.3 Geodetic on the plane of Lobachevsky 554 14.8.14 9.8.4 Lobachevsky plane motions in the Poincar´e model in a circle554 14.8.15 9.8.5 Distance between points on the Lobachevsky plane 5 14.8.16 9.8.6 Triangles on the Lobachevsky plane 555

14.9 10 Topological Spaces and Continuous Mappings 559 14.9.1 10.1 Topological spaces559 14.9.2 10.1.1 Topology and topological space559 14.9.3 10.1.2 Metric Spaces 559 14.9.4 10.1.3 Topological interior and topological closure 561 14.9.5 10.2 Continuous mappings562 14.9.6 10.3 Standard Structures563 14.9.7 10.4 Problems 565 14.9.8 10.5 Supplemental material 565 14.9.9 10.5.1 Examples of metric spaces565 14.9.10 10.5.2 Properties of metric spaces566 14.9.11 10.5.3 Examples of topological spaces 566

14.1011 Classes of topological spaces567 14.10.1 11.1 connectedness567 14.10.2 11.2 Separability axioms569 14.10.3 11.3 Compact 570 14.10.4 11.4 Problems 572 14.10.5 11.5 Supplemental material 573 14.10.6 11.5.1 Functional separability 573 14.10.7 11.5.2 Unit breakdown 576

14.1112 Diversity577 14.11.1 12.1 Topological manifolds577 14.11.2 12.2 Functions and displays 578 14.11.3 12.3 Smooth manifolds 579 14.11.4 12.4 Examples of smooth manifolds 580 14.11.5 12.5 Smooth functions, smooth mappings, diffeomorphisms581 14.11.6 12.6 Setting manifolds by equations - the geometric meaning of the implicit function theorem583 14.11.7 12.7 Problems 585 14.11.8 12.8 Supplemental material 586 14.11.9 12.8.1 Pathological Examples 586 14.11.1012.8.2 Setting the structure of a smooth manifold on a set 587 14.11.1112.8.3 Homeomorphism RP(3) и SO(3)587

14.1213 Tangent space to diversity, differential588 14.12.1 13.1 Tangent vector definitions588 14.12.2 13.2 Tangential delamination 592 14.12.3 13.3 Definition of differential 593 14.12.4 13.4 Local Properties of Mappings 594 14.12.5 13.5 Problems 597 14.12.6 13.6.2 Local device of submersions and immersions 599

14.1314 Injections of manifolds into Euclidean space600 14.13.1 14.1 Existence of an attachment600 14.13.2 14.2 Sard's theorem 602 14.13.3 14.3 Whitney's "weak" theorem604 14.13.4 14.4 Problems 605 14.13.5 14.5 Supplemental material 606 14.13.6 14.5.1 To prove Sard's theorem606 14.13.7 14.5.2 Examples of dives607

14.1415 Additional structures: Riemannian metric, orientability 607 14.14.1 15.1 Riemannian metric, Riemannian manifolds608 14.14.2 15.2 Isometries 611 14.14.3 15.4 Problems 618 14.14.4 15.5 Supplemental material 619 14.14.5 15.5.1 Examples: Klein bottle 619 14.14.6 15.5.2 Examples: projective space RP3

14.1516 Classification of connected compact two-dimensional manifolds 620 14.15.1 16.1 Polygon gluing 620 14.15.2 16.2 Sphere sealing622 14.15.3 16.3 Classification theorem 625 14.15.4 16.3.1 Triangulations 625 14.15.5 16.3.2 Canonical polygon glues

18 Other, Special Methods

18.0.1 Thetrads and Frames of Reference

(мб расширю, пока такая заготовка)

18.0.2 Characteristic Classes

On Chern Class (?????)

(wiki)

In mathematics, in particular in algebraic topology, differential geometry and algebraic geometry, the Chern classes are characteristic classes associated with complex vector bundles. They have since become fundamental concepts in many branches of mathematics and physics, such as string theory, Chern–Simons theory, knot theory, Gromov–Witten invariants.

On construction (?????)

On applications (?????)

(мб дойду когда-то, пока не до этого)

О классе Понтрягина (?????)

Об Эйлеровом классе

О формах Черна-Саймонса

О сигнатуре многообразия

О теореме Гаусса-Бонне

Пусть Ω - компактное двумерное ориентированное риманово многообразие с гладкой границей $\partial\Omega$. Обозначим через K гауссову кривизну Ω и через k_g геодезическую кривизну $\partial\Omega$. Тогда

$$\int\limits_{\Omega} K d\sigma + \int\limits_{\partial\Omega} k_g ds = 2\pi\chi(\Omega),$$

где $\chi(\Omega)$ - эйлерова характеристика Ω. В частности, если у Ω нет границы, получаем

$$\int\limits_{\Omega} K d\sigma = 2\pi\chi(\Omega).$$

Если поверхность деформируется, то её эйлерова характеристика не меняется, в то время как гауссова кривизна может меняться поточечно. Тем не менее, согласно формуле Гаусса - Бонне, интеграл гауссовой кривизны остаётся тот же.

18.0.3 Other About Bundles, Homotopies

(потом мб отдельные разделы создам, пока так, потому что не доучивал еще)

On Chern-weil Homomorphism

(wiki)

It is a basic construction in Chern-Weil theory that computes topological invariants of vector bundles and principal bundles on a smooth manifold M in terms of connections and curvature representing classes in the de Rham cohomology rings of M. That is, the theory forms a bridge between the areas of algebraic topology and differential geometry. It was developed in the late 1940 s by Shiing-Shen Chern and André Weil, in the wake of proofs of the generalized Gauss-Bonnet theorem. This theory was an important step in the theory of characteristic classes. Let G be a real or complex Lie group with Lie algebra $\mathfrak{g}$, and let $\mathbb{C}[\mathfrak{g}]$ denote the algebra of $\mathbb{C}$-valued polynomials on $\mathfrak{g}$ (exactly the same argument works if we used $\mathbb{R}$ instead of $\mathbb{C}$.) Let $\mathbb{C}[\mathfrak{g}]^G$ be the subalgebra of fixed points in $\mathbb{C}[\mathfrak{g}]$ under the adjoint action of G; that is, the subalgebra consisting of all polynomials f such that $f\left(\operatorname{Ad}_g x\right) = f(x)$, for all g in G and x in $\mathfrak{g}$, Given a principal G-bundle P on M, there is an associated homomorphism of $\mathbb{C}$-algebras,

$$\mathbb{C}[\mathfrak{g}]^G \to H^*(M;\mathbb{C})$$

called the Chern-Weil homomorphism, where on the right cohomology is de Rham cohomology. This homomorphism is obtained by taking invariant polynomials in the curvature of any connection on the given bundle. If G is either compact or semi-simple, then the cohomology ring of the classifying space for G-bundles, BG, is isomorphic to the algebra $\mathbb{C}[\mathfrak{g}]^G$ of invariant polynomials:

$$H^*(BG;\mathbb{C}) \cong \mathbb{C}[\mathfrak{g}]^G.$$

(The cohomology ring of BG can still be given in the de Rham sense:

$$H^k(BG;\mathbb{C}) = \varinjlim \ker\left(d : \Omega^k\left(B_jG\right) \to \Omega^{k+1}\left(B_jG\right)\right) / \operatorname{im} d.$$

when $BG = \lim\limits_{\longrightarrow} B_jG$ and B_jG are manifolds.)

On construction (?????)

On applications (?????)

(мб дойду когда-то, пока не до этого)

18.1 Other Structures

(потом мб изменю структуру, пока всё ходовое тупо тут и всё)

18.1.1 Manifolds in General

ИТ-12. Об определении и задании многообразий

12.1 Топологические многообразия (Tong also is here)

Definition: An n-dimensional differentiable manifold is a Hausdorff topological space M such that

i) M is locally homeomorphic to $\mathbf{R}^n$. This means that for each $p \in M$, there is an open set $\mathcal{O}$ such that $p \in \mathcal{O}$ and a homeomorphism $\phi : \mathcal{O} \to U$ with U an open subset of $\mathbf{R}^n$.

ii) Take two open subsets $\mathcal{O}_\alpha$ and $\mathcal{O}_\beta$ that overlap, so that $\mathcal{O}_\alpha \cap \mathcal{O}_\beta \neq \emptyset$. We require that the corresponding maps $\phi_\alpha : \mathcal{O}_\alpha \to U_\alpha$ and $\phi_\beta : \mathcal{O}_\beta \to U_\beta$ are compatible, meaning that the map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha (\mathcal{O}_\alpha \cap \mathcal{O}_\beta) \to \phi_\beta (\mathcal{O}_\alpha \cap \mathcal{O}_\beta)$ is smooth (also known as infinitely differentiable or C^∞), as is its inverse. This is depicted in Figure 14.

12.2 Функции и отображения

12.3 Гладкие многообразия

12.4 Простейшие примеры гладких многообразий

12.5 Гладкие функции, гладкие отображения, диффеоморфизмы

12.6 Задание многообразий уравнениями - геометрический смысл теоремы о неявной функции

12.7 Подмногообразия

ИТ-13 Касательное пространство к многообразию, дифференциал

13.1 Определение касательного вектора

13.2 Касательное расслоение

13.3 Определение дифференциала

13.4 Локальные свойства отображений

14 Вложения многообразий в евклидово пространство

14.1 Существование вложения

14.2 Теорема Сарда

14.3 Теорема Уитни

ИТ-16 Классификация связных двумерных компактных многообразий

16.1 Склейки многоугольников

16.2 Заклеивание сферы

16.3 Теорема классификации

16.3.1 Триангуляции

16.3.2 Канонические склейки многоугольников

16.3.3 Последний шаг, эйлерова характеристика

18.1.2 Symplectic Structure and Geometry

(надеюсь, для самой сути раздела хватит. структура лекций Рослого)

Теорема Дарбу.

Лагранжевы подмногообразия.

Гамильтоновы векторные поля и их гамильтонианы.

Скобки Пуассона, симплектические листы.

Действие группы симплектоморфизмами, гамильтонова редукция.

Квантовые аналоги этих понятий

18.1.3 Spin Structure on Manifolds (!?!?)

18.1.4 Topological Spaces

(По Иванову Тужилину)

Метрические и топологические пространства

10.1.1 Метрические пространства

10.1.2 Топологические пространства

10.2 Непрерывные отображения

Классы топологических пространств

11.1 Связность

11.2 Аксиомы отделимости

11.3 Компактность

18.1.5 Homologies

18.1.6 Super Geometry

(последняя тема Рослого)

19 Applications in Physics

(из Тахтаджаяна снова тоже многое выпишу!!!)

19.1 Gravity and Cosmology

19.1.1 spacetimes in General Relativity

(потом допишу)

19.1.2 Geometry of Friedmann's Universe

Robertson-walker Metric

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\gamma_{ij}\ \mathrm{d}x^i\ \mathrm{d}x^j$$

$$\Gamma^{\mu}_{00} = \Gamma^{0}_{0\beta} = 0$$

$$\Gamma^{0}_{ij} = a\dot{a}\gamma_{ij}, \qquad \Gamma^{i}_{0j} = \frac{\dot{a}}{a}\delta^{i}_{j}, \qquad \Gamma^{i}_{jk} = \frac{1}{2}\gamma^{il}\left(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}\right)$$

$$R_{00} = -3\frac{\ddot{a}}{a}, R_{ij} = -\left[\frac{\ddot{a}}{a} + 2\left(\frac{\dot{a}}{a}\right)^2 + 2\frac{k}{a^2}\right]g_{ij}.$$

$$R = -6\left[\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right].$$

Метрика:

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right]$$

or

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\mathrm{d}\chi^2 + S_k^2(\chi)\mathrm{d}\Omega^2\right]$$

$$\mathrm{d}s^2 = a^2(\tau)\left[\mathrm{d}\tau^2 - \left(\mathrm{d}\chi^2 + S_k^2(\chi)\mathrm{d}\Omega^2\right)\right] \quad \text{with conformal time } \mathrm{d}\tau := \frac{\mathrm{d}t}{a(t)}$$

Symmetry

$$a \to \lambda a, \quad r \to r/\lambda, \quad k \to \lambda^2 k.$$

$x^i := \{x^1, x^2, x^3\}$ - comoving coordinates
$x^i_{\text{phys}} = a(t)x^i$ - physical coordinates

$$v^i_{\text{phys}} := \frac{dx^i_{\text{phys}}}{dt} = a(t)\frac{dx^i}{dt} + \frac{da}{dt}x^i := v^i_{\text{pec}} + Hx^i_{\text{phys}} \qquad \text{(physical velocity)}$$

$$v^i_{\text{pec}} := a(t)\dot{x}^i \qquad \text{(peculiar velocity)}$$

$$H := \frac{\dot{a}}{a}$$

Уравнения Фридмана

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2},$$
$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P).$$

where ρ and P - the sum of all contributions to the energy density and pressure in the universe. По сути это уравнения Эйнштейна для метрики Фридмана $G_{\mu\nu} = 8\pi G T_{\mu\nu}$, если подставить (??? выведу еще раз!) Два уравнения зависимые: второе получается из первого, однако уравнение второго порядка требует больше начальных условий, так что если решили 1-е, то следует проверить, выполняется ли 2-е (?? встречу в задаче - напишу точнее.)

Аналогичные формы:

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}. \qquad H := \dot{a}/a$$

$$H^2 = \rho/\left(3M_{\text{pl}}^2\right) \qquad \text{(flat space)}$$

$$\rho_{\text{crit},0} = \frac{3H_0^2}{8\pi G} = 1.9\cdot 10^{-29}h^2 \text{ grams cm}^{-3}$$
$$= 2.8\cdot 10^{11}h^2 M_\odot \text{Mpc}^{-3}$$
$$= 1.1\cdot 10^{-5}h^2 \text{ protons cm}^{-3}.$$

After defining $\Omega_{I,0} := \frac{\rho_{I,0}}{\rho_{\text{crit},0}}$, $\Omega_{k,0} := -k/\left(a_0 H_0\right)^2$ we will have

$$H^2(a) = H_0^2 \left[\Omega_{r,0}\left(\frac{a_0}{a}\right)^4 + \Omega_{m,0}\left(\frac{a_0}{a}\right)^3 + \Omega_{k,0}\left(\frac{a_0}{a}\right)^2 + \Omega_{\Lambda,0}\right],$$

or with normalization

$$\frac{H^2}{H_0^2} = \Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_k a^{-2} + \Omega_\Lambda.$$

$$|\Omega_k| \leq 0.01, \quad \Omega_r = 9.4 \cdot 10^{-5}, \quad \Omega_m = 0.32, \quad \Omega_\Lambda = 0.68$$

Geodesics in Frw

$$\frac{dU^\mu}{ds} + \Gamma^\mu_{\alpha\beta} U^\alpha U^\beta = 0, \qquad \Gamma^\mu_{\alpha\beta} := \frac{g^{\mu\lambda}}{2}\left(\partial_\alpha g_{\beta\lambda} + \partial_\beta g_{\alpha\lambda} - \partial_\lambda g_{\alpha\beta}\right)$$

Or after $\frac{d}{ds} U^\mu\left(X^\alpha(s)\right) = \frac{dX^\alpha}{ds}\frac{\partial U^\mu}{\partial X^\alpha} = U^\alpha \frac{\partial U^\mu}{\partial X^\alpha}$,

$$U^\alpha\left(\frac{\partial U^\mu}{\partial X^\alpha} + \Gamma^\mu_{\alpha\beta} U^\beta\right) = 0.$$

$$P^\alpha \frac{\partial P^\mu}{\partial X^\alpha} = -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta. \qquad P^\mu := m U^\mu$$

The homogeneity of the FRW background implies $\partial_i P^\mu = 0$, so

$$P^0 \frac{dP^\mu}{dt} = -\Gamma^\mu_{\alpha\beta} P^\alpha P^\beta, = -\left(2\Gamma^\mu_{0j} P^0 + \Gamma^\mu_{ij} P^i\right) P^j,$$

so

$$P^j = 0 \Rightarrow \frac{dP^i}{dt} = 0.$$

for $\mu = 0$ because of $\Gamma^0_{0j} = 0$, we have $E\frac{dE}{dt} = -\Gamma^0_{ij} P^i P^j = -\frac{\dot{a}}{a} p^2$, so

$$\frac{\dot{p}}{p} = -\frac{\dot{a}}{a} \quad \Rightarrow \quad p \propto \frac{1}{a}.$$

$$p = E \propto \frac{1}{a} \quad \text{(massless particles)},$$

$$p = \frac{mv}{\sqrt{1-v^2}} \propto \frac{1}{a} \quad \text{(massive particles)}$$

О времени (????)

О конформном времени (?!)

$$\eta := \int dt/a = \int \frac{da}{a}\frac{1}{H}\frac{a_0}{a}.$$

Note that in these $c = 1$ units, the conformal time doubles as the comoving size of the horizon. In an open universe, it is also related to the development angle by

$$\chi = \sqrt{-K}\eta$$

Asymptotic relations are often useful for converting values. Before curvature or Λ domination, the conformal time

$$\begin{aligned}\eta &= \frac{2\sqrt{2}}{k_{eq}}\left(\sqrt{1+a/a_{eq}}-1\right)\\ &= 2\left(\Omega_0 H_0^2\right)^{-1/2}\left(a_{eq}/a_0\right)^{1/2}\left(\sqrt{1+a/a_{eq}}-1\right)\end{aligned}$$

Back, the coordinate time (????? this might be wrong)

$$t = \int \frac{da}{a}\frac{1}{H}.$$

It also takes on simple asymptotic forms, e.g.

$$t = \frac{2}{3}\left(\Omega_0 H_0^2\right)^{-1/2} a_0^{-3/2}\left[\left(a+a_{eq}\right)^{1/2}\left(a-2a_{eq}\right)+2a_{eq}^{3/2}\right]. \quad \text{RD/MD}$$

О расстояниях

О метриках в космологии

Negatively (-) and positively (+) curved space:

$$\mathrm{d}\ell^2 = a^2\left[\,\mathrm{d}\boldsymbol{x}^2 + k\frac{(\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x})^2}{1-k\boldsymbol{x}^2}\right] := a^2\gamma_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j$$

$$\gamma_{ij} := \delta_{ij} + k\frac{x_i x_j}{1-k\cdot x_k x^k}, \qquad k := \begin{cases} 0 & \text{Euclidean}\\ +1 & \text{spherical}\\ -1 & \text{hyperbolic}\end{cases}$$

this is from $\mathrm{d}\ell^2 = a^2\left[\,\mathrm{d}\boldsymbol{x}^2 \pm \mathrm{d}u^2\right], \quad \boldsymbol{x}^2 \pm u^2 = \pm 1$.

Or in polar coordinates:

$$\mathrm{d}\boldsymbol{x}^2 = \mathrm{d}r^2 + r^2\left(\,\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right),$$

$$\mathrm{d}\ell^2 = a^2\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\Omega^2\right],$$

where $\mathrm{d}\Omega^2 := \mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2$.

Or by χ, which simplifies radial part

$$\mathrm{d}\chi := \mathrm{d}r/\sqrt{1-kr^2}$$

$$\mathrm{d}\ell^2 = a^2\left[\,\mathrm{d}\chi^2 + S_k^2(\chi)\mathrm{d}\Omega^2\right], \qquad S_k(\chi) := \begin{cases} \operatorname{sh}\chi & k=-1\\ \chi & k=0\\ \sin\chi & k=+1\end{cases}.$$

On metric distance $d_m = S_k(\chi)$

$$\mathrm{d}s^2 = \mathrm{d}t^2 - a^2(t)\left[\mathrm{d}\chi^2 + S_k^2(\chi)\mathrm{d}\Omega^2\right],$$

$$S_k(\chi) := \begin{cases} R_0\operatorname{sh}(\chi/R_0) & k=-1\\ \chi & k=0\\ R_0\sin(\chi/R_0) & k=+1\end{cases}.$$

$$\chi(z) = \int_{t_1}^{t_0}\frac{\mathrm{d}t}{a(t)} = \int_0^z \frac{\mathrm{d}z}{H(z)}$$

On luminosity distance d_L (??????)

$$d_L = d_m(1+z)$$

$$F = \frac{L}{4\pi\chi^2}.$$

$$F = \frac{L}{4\pi d_m^2(1+z)^2} := \frac{L}{4\pi d_L^2},$$

On angular diameter distance

$$d_A = \frac{D}{\delta\theta} = \frac{d_L}{(1+z)^2}$$

$$d_A = \frac{d_m}{1+z}.$$

because $D = a\left(t_1\right) S_k(\chi)\delta\theta = \frac{d_m}{1+z}\delta\theta$

(вставлю мб картинку потом)

19.2 Mechanics and Quantum Mechanics

19.2.1 On the Basics of Mechanics

We know that

$$\dot{q}^i = \frac{\partial H}{\partial p_i} \qquad \dot{p}_i = -\frac{\partial H}{\partial q^i}$$

With a Poisson bracket $\{f, g\} := \frac{\partial f}{\partial q^j}\frac{\partial g}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial g}{\partial q^j}$ the time evolution of any function f can be written as

$$\dot{f} = \{f, H\}$$

which reproduces Hamilton's equations if we take $f = q^i$ or $f = p_i$.

Underlying this story is the mathematical structure of forms. The key idea is that we have a manifold M and a function H on M. We want a machinery that turns the function H into a vector field X_H. Particles then follow trajectories in phase space that are integral curves generated by X_H.

To achieve this, we introduce a symplectic two-form ω on an even-dimensional manifold M. This two form must be closed, $d\omega = 0$, and non-degenerate, which means that the top form $\omega \wedge \ldots \wedge \omega \neq 0$. A manifold M equipped with a symplectic two-form is called a symplectic manifold.

Any 2 -form provides a map $\omega : T_p(M) \to T_p^*(M)$. Given a vector field $X \in \mathfrak{X}(M)$, we can simply take the interior product with ω to get a one-form $\iota_X\omega$. However, we want to go the other way: given a function H, we can always construct a one-form dH, and we'd like to exchange this for a vector field. We can do this if the map $\omega : T_p(M) \to T_p^*(M)$ is actually an isomorphism, so the inverse exists. This turns out to be true provided that ω is non-degenerate. In this case, we can define the vector field X_H by solving the equation

$$\iota_{X_H}\omega = -dH$$

If we introduce coordinates x^μ on the manifold, then the component form of this equation is

$$X_H^\mu \omega_{\mu\nu} = -\partial_\nu H$$

We denote the inverse as $\omega^{\mu\nu} = -\omega^{\nu\mu}$ such that $\omega^{\mu\nu}\omega_{\nu\rho} = \delta^\mu_\rho$. The components of the vector field are then

$$X_H^\mu = -\omega^{\nu\mu}\partial_\nu H = \omega^{\mu\nu}\partial_\nu H$$

The integral curves generated by X_H obey the differential equation (2.9)

$$\frac{dx^\mu}{dt} = X_H^\mu = \omega^{\mu\nu}\partial_\nu H$$

These are the general form of Hamilton's equations. They reduce to our earlier form (2.33) if we write $x^\mu = (q^i, p_j)$ and choose the symplectic form to have block diagonal form $\omega^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

To define the Poisson structure, we first note that we can repeat the map (2.34) to turn any function f into a vector field X_f obeying $\iota_{X_f}\omega = -df$. But we can then feed these vector fields back into the original 2 -form ω. This gives us a Poisson bracket,

$$\{f, g\} = \omega\left(X_g, X_f\right) = -\omega\left(X_f, X_g\right)$$

Or, in components,

$$\{f, g\} = \omega^{\mu\nu}\partial_\mu f \partial_\nu g$$

There are many other ways to write this Poisson bracket structure in invariant form. For example, backtracking through various definitions we find

$$\{f, g\} = -\iota_{X_f}\omega\left(X_g\right) = df\left(X_g\right) = X_g(f)$$

The equation of motion in Poisson bracket structure is then

$$\dot{f} = \{f_\imath H\} = X_H(f) = \mathcal{L}_{X_H} f$$

which tells us that the Lie derivative along X_H generates time evolution.

We haven't yet explained why the symplectic two-form must be closed, $d\omega = 0$. You can check that this is needed so that the Poisson bracket obeys the Jacobi identity. Alternatively, it ensures that the symplectic form itself is invariant under Hamiltonian flow, in the sense that $\mathcal{L}_{X_H}\omega = 0$. To see this, we use (2.31)

$$\mathcal{L}_{X_H}\omega = \left(d\iota_{X_H} + \iota_{X_H} d\right)\omega = \iota_{X_H} d\omega$$

The second equality follows from the fact that $d\iota_{X_H}\omega = -d(dH) = 0$. If we insist that $d\omega = 0$ then we find $\mathcal{L}_{X_H}\omega = 0$ as promised.

19.2.2 Other Properties of Mechanics (!!!??)

О математической лагранжевой механике (действии, Нетер, Лежандре)

Определение. Пусть (U, φ) - координатная окрестность в M с локальными координатами $\boldsymbol{q} = (q^1, \ldots, q^n)$. Координаты

$$(\boldsymbol{q}, \boldsymbol{v}) = \left(q^1, \ldots, q^n, v^1, \ldots, v^n\right)$$

на окрестности TU пространства TM, где $\boldsymbol{v} = (v^1, \ldots, v^n)$ - координаты в слое, соответствующие базису $\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}$ пространства T_qM, называются стандартными координатами. (то есть просто обычная пара из обычных координат и скоростей)

Энергия лагранжевой системы (M, L) - это функция E на $TM \times \mathbb{R}$, определенная в стандартных координатах на TM формулой

$$E(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) = \sum_{i=1}^n \dot{q}^i \frac{\partial L}{\partial \dot{q}^i}(\boldsymbol{q}, \dot{\boldsymbol{q}}, t) - L(\boldsymbol{q}, \dot{\boldsymbol{q}}, t)$$

Лагранжева система (M, L) называется невырожденной, если для любой координатной окрестности U в M матрица гессе $H_L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \left\{\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}(\boldsymbol{q}, \dot{\boldsymbol{q}})\right\}_{i,j=1}^n$ обратима на TU.

Для инвариантной формулировки рассмотрим 1-форму θ_L, определенную в стандартных координатах, связанных с координатной окрестностью $U \subset M$, формулой

$$\theta_L = \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}^i} dq^i \equiv \frac{\partial L}{\partial \dot{\boldsymbol{q}}} d\boldsymbol{q}.$$

Лагранжева система (M, L) невырождена, если и только если 2-форма $d\theta_L$ на TM невырождена.

Через 1-форму θ_L интеграл Нётер I имеет вид

$$I = i_{X'}(\theta_L)$$

где X' - подъем векторного поля X с M на TM, заданный уравнением (1.5). Из (1.6) также мгновенно следует, что если X - бесконечно малая симметрия, то

$$\mathcal{L}_{X'}(\theta_L) = 0.$$

Стандартные координаты в (U, φ) координатной окрестности в M - координаты $(\boldsymbol{p}, \boldsymbol{q}) = (p_1,.., p_n, q^1,.., q^n)$ в окрестности $T^*U \simeq \mathbb{R}^n \times U$ кокасательного расслоения T^*M, если для $(p, q) \in T^*U$ и $f \in C^\infty(U)$

$$p_i(df) = \frac{\partial f}{\partial q^i}, \quad i = 1,.., n$$

По традиции первые n координат параметризуют слой T^*U, а последние n координат базу. (???? проверю еще раз, чтобы про импульс не было вопросов??)

Эквивалентным образом, стандартные координаты на T^*U можно однозначно охарактеризовать условием, что $\boldsymbol{p} = (p_1,.., p_n)$ - координаты в слое, соответствующие базису $dq^1,.., dq^n$ пространства T_q^*M, двойственному к базису $\frac{\partial}{\partial q^1},.., \frac{\partial}{\partial q^n}$ пространства T_qM.

Преобразованием Лежандра для лагранжиана L - послойное отображение $\tau_L : TM \to T^*M$ называется, если

$$\theta_L = \tau_L^*(\theta)$$

Кстати, инвариантно $\theta_L(v) := dL\left((i \circ \pi_*)\, v\right)$, где $v \in T(TM)$ (??????????).

В стандартных координатах преобразование Лежандра задается формулой

$$\tau_L(\boldsymbol{q}, \dot{\boldsymbol{q}}) = (\boldsymbol{p}, \boldsymbol{q}), \qquad \text{где} \qquad \boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$$

Отображение τ_L - локальный диффеоморфизм, если и только если лагранжиан L невырожден.

Пусть преобразование Лежандра $\tau_L : TM \to T^*M$ - диффеоморфизм. Гамильтониан $H : T^*M \to \mathbb{R}$, соответствующий лагранжиану $L : TM \to \mathbb{R}$, определяется как

$$H \circ \tau_L = E_L = \dot{\boldsymbol{q}} \frac{\partial L}{\partial \dot{\boldsymbol{q}}} - L.$$

В стандартных координатах

$$H(\boldsymbol{p}, \boldsymbol{q}) = (\boldsymbol{p}\dot{\boldsymbol{q}} - L(\boldsymbol{q}, \dot{\boldsymbol{q}}))\big|_{\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}}$$

где $\dot{\boldsymbol{q}}$ - функция $\boldsymbol{p}$ и $\boldsymbol{q}$, определяемая уравнением $\boldsymbol{p} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ по теореме о неявной функции. Кокасательное расслоение T^*M называется фазовым пространством лагранжевой системы (M, L).

Об основах математической гамильтоновой механики (уравнениях Гамильтона, гамильтоновом действии, уравнении Лиувилля, канонических преобразованиях)

Каждой функции $H : T^*M \to \mathbb{R}$ на фазовом пространстве T^*M соответствуют уравнения Гамильтона, соответствующее векторное поле $X_H \frac{\partial H}{\partial \boldsymbol{p}} \frac{\partial}{\partial \boldsymbol{q}} - \frac{\partial H}{\partial \boldsymbol{q}} \frac{\partial}{\partial \boldsymbol{p}}$ на T^*U, порождает векторное поле X_H на T^*M, называемое гамильтоновым векторным полем. Предположим теперь, что векторное поле X_H на T^*M полно, т. е. что его интегральные кривые определены во все моменты времени. Соответствующая однопараметрическая группа $\{g_t\}_{t\in\mathbb{R}}$ диффеоморфизмов T^*M, порожденная X_H, называется гамильтоновым фазовым потоком. Он определяется равенством $g_t(p,q) = (p(t), q(t))$, где $p(t), q(t)$ - решение уравнений Гамильтона, удовлетворяющее условию $p(0) = p, q(0) = q$.

Каноническая 1-форма Лиувилля θ на T^*M определяет симплектическую 2-форму $\omega = d\theta$ на T^*M. В стандартных координатах на T^*M она задается формулой

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq^i = d\boldsymbol{p} \wedge d\boldsymbol{q}$$

Она определяет изоморфизм $J : T^*(T^*M) \to T(T^*M)$ касательного и кокасательного расслоений к T^*M. Для любого $(p,q) \in T^*M$ линейное отображение $J^{-1} : T_{(p,q)}T^*M \to T^*_{(p,q)}T^*M$ задается уравнением

$$\omega(u_1, u_2) = J^{-1}(u_2)(u_1), \quad u_1, u_2 \in T_{(p,q)}T^*M$$

и $J^{-1}(X) = -i_X\omega$. В частности, в стандартных координатах $J(d\boldsymbol{p}) = \frac{\partial}{\partial \boldsymbol{q}}$ и $J(d\boldsymbol{q}) = -\frac{\partial}{\partial \boldsymbol{p}}$, так что $X_H = J(dH)$.

Теорема 2.1. Гамильтонов фазовый поток на T^*M сохраняет каноническую симплектическую форму.

Следствие 2.2. $\mathcal{L}_{X_H}(\theta) = d(-H + \theta(X_H))$, где θ - каноническая 1-форма Лиувилля.

Каноническая симплектическая форма ω на T^*M определяет форму объема $\frac{\omega^n}{n!} = \frac{1}{n!}\omega_1 \wedge .. \wedge \omega_n$ на T^*M, называемую лиувиллевой формой объема.

(Теорема Лиувилля). Гамильтонов фазовый поток на T^*M сохраняет лиувиллеву форму объема.

Подмногообразие $\mathscr{L}$ фазового пространства T^*M называется лагранжевым подмногообразием, если $\dim \mathscr{L} = \dim M$ и $\omega|_{\mathscr{L}} = 0$.

С каждой функцией H на фазовом пространстве T^*M ассоциирована т.н. форма Пуанкаре-Картана - 1-форма на расширенном фазовом пространстве $T^*M \times \mathbb{R}$ вида

$$\theta - Hdt = \boldsymbol{p}d\boldsymbol{q} - Hdt$$

Пусть $\gamma : [t_0, t_1] \to T^*M$ - гладкий параметризованный путь в T^*M, такой, что $\pi(\gamma(t_0)) = q_0$ и $\pi(\gamma(t_1)) = q_1$, где $\pi : T^*M \to M$ - каноническая проекция.

По определению подъем пути γ в расширенное фазовое пространство $T^*M \times \mathbb{R}$ - это путь $\sigma : [t_0, t_1] \to T^*M \times \mathbb{R}$, задаваемый формулой $\sigma(t) = (\gamma(t), t)$, и путь σ в $T^*M \times \mathbb{R}$ называется допустимым путем, если он является подъемом пути γ в T^*M. Пространство допустимых путей в $T^*M \times \mathbb{R}$ обозначается $\tilde{P}(T^*M)^{q_1,t_1}_{q_0,t_0}$. Вариация допустимого пути σ - это гладкое семейство допустимых путей σ_ε, где $\varepsilon \in [-\varepsilon_0, \varepsilon_0]$ и $\sigma_0 = \sigma$, а соответствующая бесконечно малая вариация - это

$$\delta\sigma = \left.\frac{\partial \sigma_\varepsilon}{\partial \varepsilon}\right|_{\varepsilon=0} \in T_\sigma \tilde{P}(T^*M)^{q_1,t_1}_{q_0,t_0}$$

Через эти понятия легко написать принцип наименьшего действия, там $S(\sigma) = \int\limits_\sigma (\boldsymbol{p}d\boldsymbol{q} - Hdt)$.

Принцип Мопертюи. Траектория $\gamma = (\boldsymbol{q}(\tau))$ замкнутой лагранжевой системы (M, L), соединяющая точки q_0 и q_1 и имеющая энергию E, является экстремалью функционала

$\int\limits_{\gamma} \boldsymbol{p} d\boldsymbol{q} = \int\limits_{\gamma} \frac{\partial L}{\partial \dot{\boldsymbol{q}}}(\boldsymbol{q}(\tau), \dot{\boldsymbol{q}}(\tau))\dot{\boldsymbol{q}}(\tau) d\tau$ на пространстве всех путей в конфигурачионном пространстве M, соединяющих точки q_0 и q_1 и параметризованных так, что $H\left(\frac{\partial L}{\partial \dot{\boldsymbol{q}}}(\tau), \boldsymbol{q}(\tau)\right) = E$. Функционал $S_0(\gamma) = \int\limits_{\gamma} \boldsymbol{p} d\boldsymbol{q}$ называется укороченным действием.

О наблюдаемых и скобках Пуассона

Гладкие вещественнозначные функции на фазовом пространстве T^*M называются классическими наблюдаемыми. Векторное пространство $C^\infty(T^*M)$ является $\mathbb{R}$-алгеброй - ассоциативной алгеброй над $\mathbb{R}$ с единицей, заданной постоянной функцией 1, и умножением, заданным поточечным произведением функций. Коммутативная алгебра $C^\infty(T^*M)$ называется алгеброй классических наблюдаемых. Предполагая, что гамильтонов фазовый поток g_t определен для всех моментов времени, временная эволюция любой наблюдаемой $f \in C^\infty(T^*M)$ дается уравнением

$$f_t(p,q) = f(g_t(p,q)) = f(p(t), q(t)), \quad (p,q) \in TM$$

можно переписать уравнение Гамильтона в сжатой форме:

$$\frac{df}{dt} = \{H, f\}$$

где подразумевается, что (2.7) - дифференциальное уравнение для семейства функций f_t на T^*M с начальным условием $f_t(p,q)|_{t=0} = f(p,q)$. Свойства билинейного отображения

$$\{,\} : C^\infty(T^*M) \times C^\infty(T^*M) \to C^\infty(T^*M)$$

Теорема 2.9. Отображение $\{,\}$ удовлетворяет следующим свойствам для любых $f, g, h \in C^\infty(T^*M)$:
(i) (Связь с симплектической формой) $\{f,g\} = \omega(J(df), J(dg)) = \omega(X_f, X_g)$;
(ii) (Кососимметричность) $\{f,g\} = -\{g,f\}$
(iii) (Правило Лейбница) $\{fg,h\} = f\{g,h\} + g\{f,h\}$;
(iv) (Тождество Якоби) $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$
Наблюдаемая $\{f,g\}$ называется канонической скобкой Пуассона наблюдаемых f и g. Отображение скобки Пуассона $\{\} :, C^\infty(T^*M) \times \times C^\infty(T^*M) \to C^\infty(T^*M)$ превращает алгебру классических наблюдаемых $C^\infty(T^*M)$ в алгебру Ли со скобкой Ли, задаваемой скобкой Пуассона. Ее важное свойство - скобка Ли является бидифференцированием по отношению к умножению в $C^\infty(T^*M)$. Алгебра классических наблюдаемых $C^\infty(T^*M)$ - это пример пуассоновой алгебры - коммутативной алгебры над $\mathbb{R}$, наделенной структурой алгебры Ли с тем свойством, что скобка Ли - дифференцирование по отношению к произведению

Об алгебре и переходе в квантовую механику

(важное указание!)

Об определениях симплектических и пуассоновых многообразий

Мы просто говорим, что многообразие (такое-то), если на нем задана (такая-то структура), вот и все. (??? напишу своими словами тут многое потом!)

Понятие симплектического многообразия - это обобщение примера кокасательного расслоения T^*M.

Определение. Невырожденная, замкнутая 2-форма ω на многообразии $\mathscr{M}$ называется симплектической формой, а пара $(\mathscr{M}, \omega)$ - симплектическим многообразием.

Поскольку симплектическая форма ω невырождена, симплектическое многообразие $\mathscr{M}$ с необходимостью четномерно, $\dim \mathscr{M} = 2n$. Не обращающаяся в ноль $2n$-форма ω^n определяет каноническую ориентацию на $\mathscr{M}$, и, так же как в случае $\mathscr{M} = T^*M$, $\frac{\omega^n}{n!}$

называется лиувиллевой формой объема. Существует также общее понятие лагранжева подмногообразия.

Лагранжево подмногообразие $\mathscr{L}$ симплектического многообразия $(\mathscr{M}, \omega)$ - многообразие, у которого $\dim \mathscr{L} = \frac{1}{2} \dim \mathscr{M}$ и ограничение симплектической формы ω на $\mathscr{L}$ равно 0.

Кэлерово многообразие - комплексное многообразие $\mathscr{M}$, если на нем определена эрмитова метрика, мнимая часть которой замкнутая $(1,1)$-форма. В локальных комплексных координатах $z = (z^1,.., z^n)$ на $\mathscr{M}$ эрмитова метрика записывается как

$$h = \sum_{\alpha,\beta=1}^{n} h_{\alpha\bar{\beta}}(\boldsymbol{z}, \overline{\boldsymbol{z}}) dz^\alpha \otimes d\bar{z}^\beta.$$

$$g = \operatorname{Re} h = \frac{1}{2} \sum_{\alpha,\beta=1}^{n} h_{\alpha\bar{\beta}}(\boldsymbol{z}, \overline{\boldsymbol{z}}) \left(dz^\alpha \otimes d\bar{z}^\beta + d\bar{z}^\beta \otimes dz^\alpha \right) \qquad \text{- риманова метрика.}$$

$$\omega = -\frac{1}{2} \operatorname{Im} h = \frac{i}{2} \sum_{\alpha,\beta=1}^{n} h_{\alpha\bar{\beta}}(\boldsymbol{z}, \overline{\boldsymbol{z}}) dz^\alpha \wedge d\bar{z}^{\boldsymbol{\beta}} \qquad \text{- симплектическая форма.}$$

Простейшее некомпактное кэлерово многообразие - это n-мерное комплексное векторное пространство $\mathbb{C}^n$ со стандартной эрмитовой метрикой. В комплексных координатах $z = (z^1,.., z^n)$ на $\mathbb{C}^n$ она дается формулой $h = d\boldsymbol{z} \otimes d\bar{z} = \sum\limits_{\alpha=1}^{n} dz^\alpha \otimes d\bar{z}^\alpha$

Симплектическое векторное пространство - это пара (V, ω), где V - векторное пространство над $\mathbb{R}$, а ω - невырожденная кососимметрическая билинейная форма на V. Любое симплектическое векторное пространство изоморфно прямому произведению фазовых плоскостей $\mathbb{R}^2$ с канонической симплектической формой $dp \wedge dq$. Вводя комплексные координаты $\boldsymbol{z} = \boldsymbol{p} + i\boldsymbol{q}$, мы получаем изоморфизм $V \simeq \mathbb{C}^n$, так что любое симплектическое векторное пространство допускает кэлерову структуру.

Теорема Дарбу о существовании формы Дарбу. Пусть $(\mathscr{M}, \omega)$ - $2n$-мерное симплектическое многообразие. Для любой точки $x \in \mathscr{M}$ существует содержащая x окрестность U с локальными координатами $(\boldsymbol{p}, \boldsymbol{q}) = (p_1,.., p_n, q^1,.., q^n)$, такая, что на U

$$\omega = d\boldsymbol{p} \wedge d\boldsymbol{q} = \sum_{i=1}^{n} dp_i \wedge dq^i.$$

Координаты $\boldsymbol{p}, \boldsymbol{q}$ называются каноническими координатами (координатами Дарбу). Доказательство проводится индукцией по n (??? как интуитивно это понимать??)

Невырожденная 2-форма ω определяет для любого $x \in \mathscr{M}$ изоморфизм $J : T_x^* \mathscr{M} \to T_x \mathscr{M}$ по формуле

$$\omega(u_1, u_2) = J^{-1}(u_2)(u_1), u_1, u_2 \in T_x \mathscr{M}$$

Гамильтонова система - это пара, состоящая из симплектического многообразия $(\mathscr{M}, \omega)$ (называемого фазовым пространством) и гладкой вещественнозначной функции H на $\mathscr{M}$ (называемой гамильтонианом). Движение точек фазового пространства описывается т.н. гамильтоновым векторным полем:

$$X_H = J(dH),$$

Гамильтонов фазовый поток сохраняет симплектическую форму. (!?!??!?) (! потренируюсь в теории доказывать это!)

Следствие 2.13. Векторное поле X на $\mathscr{M}$ - гамильтоново векторное поле, если и только если 1-форма $i_X(\omega)$ точна.

Симплектическое векторное поле X на симплектическом многообразии $(\mathscr{M}, \omega)$ - такое поле, для которого 1-форма $i_X(\omega)$ замкнута, что эквивалентно тому, что $\mathscr{L}_X \omega = 0$.

Скобка Пуассона на алгебре $C^\infty(\mathscr{M})$ классических наблюдаемых на симплектическом многообразии $(\mathscr{M}, \omega)$ - это билинейное отображение $\{,\} :, C^\infty(\mathscr{M}) \times C^\infty(\mathscr{M}) \to C^\infty(\mathscr{M})$, определенное формулой

$$\{f, g\} = \omega(X_f, X_g), \quad f, g \in C^\infty(\mathscr{M})$$

Следствие 2.15. Подпространство $\mathrm{Ham}(\mathscr{M})$ гамильтоновых векторных полей на $\mathscr{M}$ - подалгебра Ли алгебры $\mathrm{Vect}\,(\mathscr{M})$. Отображение $C^\infty(\mathscr{M}) \to \mathrm{Ham}(\mathscr{M})$, определенное как $f \mapsto X_f$, - гомоморфизм алгебр Ли с ядром, состоящим из локально постоянных функций на $\mathscr{M}$.

Теорема Пуассона: скобка Пуассона двух интегралов движения - интеграл движения.

Группа симметрий гамильтоновой системы $((\mathscr{M},\omega),H)$ - группа Ли G, если существует гамильтоново действие G на $\mathscr{M}$, такое, что

$$H(g\cdot x) = H(x), \quad g\in G, x\in \mathscr{M}.$$

Теорема 2.17 (Теорема Нётер с симметриями). Если G - группа симметрий гамильтоновой системы $((\mathscr{M},\omega),H)$, то функции $\Phi_\xi, \xi\in\mathfrak{g}$ - интегралы движения. Если действие G пуассоново, интегралы движения удовлетворяют (2.15).

Пусть G - конечномерная группа Ли, действующая на связном симплектическом многообразии $(\mathscr{M},\omega)$ симплектоморфизмами. Алгебра Ли $\mathfrak{g}$ группы G действует на $\mathscr{M}$ векторными полями

$$X_\xi(f)(x) = \left.\frac{d}{ds}\right|_{s=0} f\left(e^{-s\xi}\cdot x\right),$$

и линейное отображение $\mathfrak{g}\ni\xi\mapsto X_\xi\in\mathrm{Vect}(\mathscr{M})$ - гомоморфизм алгебр Ли,

$$[X_\xi, X_\eta] = X_{[\xi,\eta]}, \quad \xi,\eta\in\mathfrak{g}$$

G-действие называется гамильтоновым действием, если X_ξ - гамильтоновы векторные поля, т.е. для любого $\xi\in\mathfrak{g}$ существует функция $\Phi_\xi\in C^\infty(\mathscr{M})$, определенная с точностью до аддитивной постоянной, такая, что $X_\xi = X_{\Phi_\xi} = J\left(d\Phi_\xi\right)$. Оно называется пуассоновым действием, если можно выбрать функции Φ_ξ, такие, что линейное отображение $\Phi:\mathfrak{g}\to C^\infty(\mathscr{M})$ - гомоморфизм алгебр Ли,

$$\{\Phi_\xi,\Phi_\eta\} = \Phi_{[\xi,\eta]}, \quad \xi,\eta\in\mathfrak{g}.$$

Следствие 2.18. Пусть (M,L) - лагранжева система, такая, что преобразование Лежандра $\tau_L : TM\to T^*M$ - диффеоморфизи. Тогда если группа Ли G - симметрия (M,L), то G - группа симметрий соответствующей гамильтоновой системы $\left((T^*M,\omega), H = E_L\circ\tau_L^{-1}\right)$, и соответствующее G-действие на T^*M пуассоново. В частности, $\Phi_\xi = -I_\xi\circ\tau_L^{-1}$, где I_ξ - нётеровы интегралы движения для однопараметрических подгрупп G, порожденных $\xi\in\mathfrak{g}$.

Гамильтонова система $((\mathscr{M},\omega),H), \dim\mathscr{M} = 2n$, называется вполне интегрируемой, если у нее есть n независимых интегралов движения $F_1 = H,.., F_n$ в инволюции. Первое условие означает, что дифференциалы $dF_1(x),.., dF_n(x)\in T_x^*\mathscr{M}$ линейно независимы для почти всех $x\in\mathscr{M}$. Гамильтоновы системы с одной степенью свободы, такие, что dH имеет лишь конечное число нулей, являются полностью интегрируемыми. Полное разделение переменных в уравнении Гамильтона - Якоби (см. раздел 1.2.5) доставляет другие примеры вполне интегрируемых гамильтоновых систем.

Оператор эволюции $U_t : \mathcal{A}\to\mathcal{A}$ определяется фазовым потоком g_t для полного гамильтонова векторного поля $X_H = \{H,\cdot\}$ по формуле

$$U_t(f)(x) = f\left(g_t(x)\right), f\in\mathcal{A}.$$

Оператор эволюции автоморфизм. (??? что это значит???)

Допустим, что любое гамильтоново векторное поле на пуассоновом многообразии $(\mathscr{M},\{\},)$ полно. Тогда для любого $H\in\mathcal{A}$ соответствующий оператор эволючии U_t является автоморфизмом пуассоновых алгебр $\mathcal{A}$, т. е.

$$U_t(\{f,g\}) = \{U_t(f), U_t(g)\} \quad \text{для любых} \quad f,g\in\mathcal{A}.$$

Обратно, если кососимметрическое билинейное отображение $\{,\}: C^\infty(\mathscr{M})\times C^\infty(\mathscr{M})\to C^\infty(\mathscr{M})$ таково, что $X_H = \{H,\cdot\}$ - полные векторные поля для всех $H\in\mathcal{A}$, и соответствующие операторы эволючии U_t удовлетворяют (2.17), то $(\mathscr{M},\{\},)$ - пуассоново многообразие.

Глобальное сечение η расслоения $T\mathscr{M} \wedge T\mathscr{M}$ - тензор Пуассона (????), если и только если

$$\mathcal{L}_{X_f}\eta = 0 \quad \text{для любого} \quad f \in \mathcal{A}.$$

Определение. Центр пуассоновой алгебры $\mathcal{A}$ - это

$$\mathcal{Z}(\mathcal{A}) = \{f \in \mathcal{A} : \{f, g\} = 0 \quad \text{для любого} \quad g \in \mathcal{A}\}.$$

Невырожденное пуассоново многообразие $(\mathscr{M}, \{\})$, если центр пуассоновой алгебры классических наблюдаемых $\mathcal{A} = C^\infty(\mathscr{M})$ состоит из одних локально постоянных функций ($\mathcal{Z}(\mathcal{A}) = \mathbb{R}$ для связного $\mathscr{M}$).

О представлении Гамильтона и Лиувилля (??!??!?)

(мб соединю с разделом про наблюдаемые, пока понять нормально эти темы нужно сперва)

Математически состояние μ на алгебре $\mathcal{A} = C^\infty(\mathscr{M})$ классических наблюдаемых на фазовом пространстве $\mathscr{M}$ - это соответствие

$$\mathcal{A} \ni f \mapsto \mu_f \in \mathscr{P}(\mathbb{R}),$$

где $\mathscr{P}(\mathbb{R})$ - множество вероятностных мер на $\mathbb{R}$ - борелевских мер на $\mathbb{R}$, таких, что полная мера $\mathbb{R}$ равна 1. Для любого борелевского подмножества $E \subseteq \mathbb{R}$ величина $0 \leqslant \mu_f(E) \leqslant 1$ - это вероятность того, что в состоянии μ значение наблюдаемой f принадлежит E. По определению математическое ожидание наблюдаемой f в состоянии μ дается интегралом ЛебегаСтильтьеса

$$\mathrm{E}_\mu(f) = \int\limits_{-\infty}^{\infty} \lambda d\mu_f(\lambda),$$

где $\mu_f(\lambda) = \mu_f((-\infty, \lambda))$ - функция распределения меры $d\mu_f$. Соответствие $f \mapsto \mu_f$ должно удовлетворять следующим естественным свойствам.

S1. $|\mathrm{E}_\mu(f)| < \infty$ для $f \in \mathcal{A}_0$ - подалгебре ограниченных наблюдаемых.

S2. $\mathrm{E}_\mu(1) = 1$, где 1 - единица в $\mathcal{A}$.

S3. Для всех $a, b \in \mathbb{R}$ и $f, g \in \mathcal{A}$

$$\mathrm{E}_\mu(af + bg) = a\mathrm{E}_\mu(f) + b\mathrm{E}_\mu(g),$$

если и $\mathrm{E}_\mu(f)$, и $\mathrm{E}_\mu(g)$ существуют. S4. Если $f_1 = \varphi \circ f_2$, где $\varphi : \mathbb{R} \to \mathbb{R}$ - гладкая функция, то для любого борелевского подмножества $E \subseteq \mathbb{R}$

$$\mu_{f_1}(E) = \mu_{f_2}\left(\varphi^{-1}(E)\right).$$

(самое важное)

Определение. Множество состояний $\mathcal{S}$ для гамильтоновой системы с фазовым пространством $\mathscr{M}$ - это выпуклое множество $\mathscr{P}(\mathscr{M})$ всех вероятностных мер на $\mathscr{M}$. Состояния, соответствующие дираковским мерам $d\mu_x$, сконцентрированным в точках $x \in \mathscr{M}$, называются чистыии состояниями, а фазовое пространство $\mathscr{M}$ также называется пространством состояний [21]. Все остальные состояния называются смешанными состояниями. Процесс измерения в классической механике - это соответствие

$$\mathcal{A} \times \mathcal{S} \ni (f, \mu) \mapsto \mu_f = f_*(\mu) \in \mathscr{P}(\mathbb{R}),$$

сопоставляющее каждой наблюдаемой $f \in \mathcal{A}$ и состоянию $\mu \in \mathcal{S}$ вероятностную меру μ_f на $\mathbb{R}$ - трансфер меры $d\mu$ на $\mathscr{M}$ посредством f. Для любого борелевского подмножества $E \subseteq \mathbb{R}$ величина $0 \leqslant \mu_f(E) \leqslant 1$ это вероятность того, что для системы в состоянии μ результат измерения наблюдаемой f лежит в множестве E. Математическое ожидание наблюдаемой f в состоянии μ дается формулой (2.19).

Лемма 2.1. Чистье состояния - это единственные состояния, в которых любая наблюдаемая имеет нулевую дисперсию.

И у нас 2 представления: гамильтоново и лиувиллево (?? почему нет лагранжева)

Гамильтоново описание динамики. Состояния не зависят от времени, а временная эволюция наблюдаемых дается уравнениями движения Гамильтона:

$$\frac{d\mu}{dt} = 0, \quad \mu \in \mathcal{S}, \quad \text{и} \quad \frac{df}{dt} = \{H, f\}, \quad f \in \mathcal{A}.$$

Математическое ожидание наблюдаемой f в состоянии μ в момент времени t дается формулой

$$\mathrm{E}_\mu(f_t) = \int\limits_{\mathscr{M}} f \circ g_t d\mu = \int\limits_{\mathscr{M}} f(g_t(x))\,\rho(x)dx,$$

где $\rho(x) = \frac{d\mu}{dx}$ - производная Радона - Никодима. В частности, математическое ожидание f в чистом состоянии $d\mu_x$, соответствующем точке $x \in \mathscr{M}$, это $f(g_t(x))$. Представление Гамильтона повсеместно используется для механических систем, состоящих из нескольких взаимодействующих частиц.

Лиувиллево описание динамики. Наблюдаемые не зависят от времени

$$\frac{df}{dt} = 0, \quad f \in \mathcal{A},$$

а состояния $d\mu(x) = \rho(x)dx$ удовлетворяют уравнению Лиувилля

$$\frac{d\rho}{dt} = -\{H, \rho\}, \quad \rho(x)dx \in \mathcal{S}$$

Здесь производная Радона-Никодима $\rho(x) = \frac{a\mu}{dx}$ и уравнение Лиувилля понимаются в смысле обобщенных функций. Математическое ожидание наблюдаемой f в состоянии μ в момент времени t дается формулой

$$\mathrm{E}_{\mu_t}(f) = \int\limits_{\mathscr{M}} f(x)\rho(g_{-t}(x))\,dx$$

Обзор применений симплектических методов

(это также в дополнениях будет, но пока также и тут, потому что это очень очень типичный вопрос, так что какие-то про это и сделаю заготовки. мб 2/3 страницы на это уйдет.)

Основные формулы и определения диффгема для механики

(коротко очень, подробнее в диффгеме про это, тут выписка из него, чтобы под рукой они были просто)

$$\mathcal{L}_X = i_X \circ d + d \circ i_X = (d + i_X)^2$$
$$i_{[X,Y]} = \mathcal{L}_X \circ i_Y - i_Y \circ \mathcal{L}_X.$$

Для данного гладкого отображения многообразий $f : M \to N$ будем обозначать как $f_* : TM \to TN$ и $f^* : T^*N \to T^*M$ соответственно индуцированные отображения на касательном и кокасательном расслоениях.

19.2.3 Quantum Mechanics

19.3 Field Theories: Good Models

(there are a lot of them, see my note on sp field theories)

19.3.1 Electrodynamics

Основные формулы электродинамики

$$A = A_\mu(x)dx^\mu$$
$$F = \frac{1}{2}F_{\mu\nu}dx^\mu \wedge dx^\nu = \frac{1}{2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)dx^\mu \wedge dx^\nu$$

19.3.2 Field Theories

Уравнения и функционал Янга-Миллса

О полях Янга-Миллса по Каку

(по каку пишу)
Определим один-форму A выражением $A = A_\mu dx^\mu$ где A_μ есть векторное поле,
а дифференциалы dx^μ антикоммутируют:

$$dx^\mu \Lambda dx^v = -dx^v \Lambda dx^\mu,$$

$$dx^\mu \Lambda dx^\mu = 0$$

Определим оператор производной как

$$d = dx^\mu \partial_\mu$$

Заметим, что поскольку производные коммутируют,

$$[\partial_\mu, \partial_\mathrm{v}] = 0$$

То

$$d^2 = 0$$

где оператор d нильпотентен.
Теперь определим два-форму:

$$F = F_{\mu v} dx^\mu \Lambda dx^v$$

Заметим, что кривизна векторного поля является два-формой:

$$F = dA = dx^\mu \partial_\mu A_v dx^v = \frac{1}{2}\left(\partial_\mu A_v - \partial_v A_\mu\right)dx^\mu \Lambda dx^v$$

Поскольку оператор d нильпотентен, имеем

$$dF = d^2 A = 0$$

Таким образом, тождества Бьянки для теории Максвелла, выраженные на языке форм, суть не что иное,
как нильпотентность оператора d.
Форма ω называется замкнутой, если $d\omega = 0$ Форма ω называется точной, если $\omega = dQ$ для некоторой формы Q.
Таким образом, форма кривизны является точной, поскольку она может быть записана как дивергенция один-формы A.
Она также является замкнутой вследствие тождеств Бьянки.
Все это можно также объединить с локальной калибровочной группой, порожденной образующими λ_a. Пусть $A = A^a_\mu \lambda_a dx^\mu$ Тогда форма кривизны есть $F = dA + A\Lambda A$ Далее, калибровочная вариация поля Янга- Миллса при преобразовании $\Lambda = \Lambda^a \lambda_a$
равна

$$\delta A = d\Lambda + A \wedge \Lambda - \Lambda \wedge A$$

Подставляя вариацию поля A в кривизну F, получаем

$$\delta F = F \wedge \Lambda - \Lambda \wedge F$$

Итак, вариация действия равна нулю:

$$\delta \operatorname{Tr}\left(F^2\right) = 2 \operatorname{Tr}(F \wedge \Lambda) - 2 \operatorname{Tr}(\Lambda \wedge F) = 0$$

Теперь выпишем аномальный член FF, выразив его на языке форм Дивергенция аксиального тока также является квадратом двух величин, выражающих кривизну, что также дает полную производную.

На языке форм, это, как мы увидим, есть точная форма:

$$\operatorname{Tr}(F \wedge \tilde{F}) = \mathrm{d}\omega_3$$

где

$$\omega_3 = \operatorname{Tr}\left(AdA + \frac{2}{3}A^3\right)$$

ω_3 - это три-форма, которую мы назовем формой Черна-Саймонса.

В свою очередь ее калибровочная вариация равна еще одной форме, которая также является точной: $\delta\omega_3 = \operatorname{Tr}(d\Lambda \wedge dA) = d\omega_2$

$$\delta\omega_3 = \operatorname{Tr}(d\Lambda \wedge dA) = d\omega_2$$

где

$$\omega_2 = \operatorname{Tr}(\Lambda \wedge dA)$$

Заметим также, что эти тождества в равной мере применимы и к теориям Янга-Миллса, и к обшей теории относительности.

Для теории тяготения имеем калибровочную группу O(3, 1).

Другими словами, гравитационное поле обладает двумя калибровочными симметриями, а именно обшей ковариантностью по координатам x-пространства и локальными лоренцевыми преобразованиями касательного пространства.

Подробнее это будет объяснено ниже.

Уравнения самодуальности

(у Рослого в планах это было)

Топологический заряд (??)

(у Рослого в планах это было)

19.3.3 On Gravity

19.4 Field Theories: Mathematical Proposals

19.4.1 CFT

(??)

19.4.2 Supergravity

(??)

19.5 Other Physics

19.5.1 On Gravitational Lensing (??????)

(!!!! там очень много всего в плане диффгема, тут указания про это. мб пару разделов сделаю даже.)

19.5.2 On Thermodynamics

(пока отношусь к этому как к фантазиям математика, вряд ли на самом деле диффгем там нужен)

Основные формулы

$$dE = \bar{d}Q + \vec{d}W$$

Here dE is the infinitesimal change of energy in the system. The first law of thermodynamics, as written above, states that this decomposes into the heat flowing into the system $\bar{d}Q$ and the work done on the system W.

Why the stupid notation? Well, the energy $E(p, q)$ is a function over the state space M and this means that we can write the change of the energy $dE = \frac{\partial E}{\partial p}dp + \frac{\partial E}{\partial V}dV$. But there is no such function $Q(p, q)$ or $W(p, q)$ and, correspondingly, $\vec{d}Q$ and $\vec{d}W$ are not exact differentials. Indeed, we have $dW = -pdV$ and later, after we introduce the second law, we learn that $\bar{d}Q = TdS$, with T the temperature and S the entropy, both of which are functions over M.

Part V
Other Constructions of Differential Geometry

Обсудим в порядке вероятности применимости мой настоящий взгляд некоторые темы дифференциальной геометрии, которыми я пока не занимался и скорее всего никогда не займусь.

(тут мб очень много будет тем, потому что полно профессиональных книг, ими по большей части и занимаются математики.)

20 Catalog of geometric objects (??)

(не уверен, что тут их будет много.)

21 Some Other Models

(всё важное, но не большое и отдельное вкратце тут выписываю!)

21.1 Kähler Geometry (????)

(хз, что это, но многие что-то про нее говорят.)

21.2 Hyperbolic Geometry

21.2.1 Hypercycles and Horocycles

21.2.2 Poincaré Geometry

21.2.3 Lobachevsky Geometry

(??? хватит же такого раздела на всю суть о ней?)

21.2.4 The Beltrami–klein Model

(wiki)

21.2.5 The Poincare Half-plane Model

21.2.6 The Hyperboloid Model

21.2.7 On Other Models of Hyperbolic Spaces (??)

(see wiki, there are a lot of information)

21.2.8 Homogeneous Tilings on the Hyperbolic Plane

(см. вики на англ и рус.)

21.3 Typical Surfaces and Manifolds

21.3.1 (anti-) de Sitter Spaces

Двумерная гиперболическая метрика Пространства (анти-) де Ситтера Их группы и алгебры изометрии

(позже если будет потребность, то исследую их)

21.3.2 Beltrami Surface

Для полноты приведем примеры.

Обзор поверхности Бельтрами

Поверхность Бельтрами в $\mathbb{R}^3$ можно задать параметрически как

$$\begin{cases} x = \sin a \cos b \\ y = \sin a \sin b \\ z = \ln \tan \frac{a}{2} + \cos a \end{cases}$$

Здесь $0 \leq a \leq \frac{\pi}{2}, 0 \leq b \leq 2\pi$.

кривизна поверхности Бельтрами

Вычислим кривизну вложенного многообразия M на примере поверхности Бельтрами (псевдосферы).

Обзор применений поверзности Бельтрами

хз

21.3.3 Moebius Belt

Основное

Другие вопросы

Какие минимальные соотношения должны быть у прямоугольника, чтобы его свернуть в ленту Мебиуса? (???7)

(не знаю, не думал, ограничение на кривизну и деформации нужно как-то ввести еще, мб когда-то подумаю, это уже исследовано, можно почитать статью, мне не актуально.)

21.4 Symplectic and Poisson Manifolds

(????)

21.4.1 Calabi-Yau Manifolds

(??? write main about them, it is a very big topic)

21.4.2 Klein Bottle

обзор

свойства бутылки Клейна

интегрирование по бутылке Клейна (???)

21.5 Transformations.

21.5.1 Transformations.

(как и в топологии)
и приходим к $\phi_V \circ f \circ \phi_U^{-1} : \phi_U(U) \to \phi_V(V)$ - это вид отображения в координатах.
и тут такая же история,

Определение 21.1. *диффеоморфизм многообразий - когда туда и обратно гладко. гомеоморфизм - когда непрерывно.*

и тут теория про производные отображения, оно очень интересно задается, с использованием касательного пространства и звездочек, но пока я просто не тяну его, это темное для меня место, лучше иванова читать.
(вставить сюда и в теорпол секцию про звездочки и отображения!!!)

21.5.2 homotopy Invariants

мы научились тривиально характеоризовать многообразия, теперь введем более прикладные характеристики
(как к этому прийти?)
замкнутость, точность.
хз что дальше.
какая-то теорема о цепной гомотопии, пока эти понимания совершенно закрыты для меня.

когомологии де рама

см иванова и т.п.
в общем, нам очень нужна тут топология и речь идет о гомотопию.
ну лемма пуанкаре, ну определения.
я вообще потерян в этом, хотя это не сложно вообще.

21.5.3 Applications to Mechanics

да и диффурами.
очень интересный раздел

многообразия грассмана

следуя Зелкину откроем для себя их.
говорят, что это многообразие из k мерных подпространств евклидового пространства
а вообще следует понимать, что у зелкина до этого идут тривиальные главы, напоминание про группы ли, так что плюну я и пойду лучше просто изучать и углублять их, потом вернусь, слишком много диффгема.

связь с уравнением Риккати

21.6 Tensor Fields

(пока что не до этого, но изначально у Катанаева увидел я большой раздел про это. мб перекину куда-то вначало, вроде полезная тема)

Обзор применений (!!)

(спрошу у Володи)

теория тензорных полей

Рассмотрим многообразие $\mathbb{M}$, $\dim \mathbb{M} = n$.

В каждой точке $x \in \mathbb{M}$ имеется два n-мерных векторных пространства: касательное $\mathbb{T}_x(\mathbb{M})$ и кокасательное $\mathbb{T}^*_x(\mathbb{M})$ пространства.

Рассмотрим их тензорное произведение

$$\mathbb{T}^r_{s,x}(\mathbb{M}) := \underbrace{\mathbb{T}_x(\mathbb{M}) \otimes \ldots \otimes \mathbb{T}_x(\mathbb{M})}_{r} \otimes \underbrace{\mathbb{T}^*_x(\mathbb{M}) \otimes \ldots \otimes \mathbb{T}^*_x(\mathbb{M})}_{s}$$

то есть тензорное произведение r экземпляров касательного и s экземпляров кокасательного пространства.

Для определенности мы фиксировали порядок сомножителей.

Таким образом, в каждой точке многообразия мы построили векторное пространство размерности $\dim \mathbb{T}^r_{s,x}(\mathbb{M}) = n^{r+s}$

Определение расслоения, тензорного поля, ранга, базы

Определение 21.2. *Расслоение тензоров типа (r, s) на многообразии M - это объединение*

$$\mathbb{T}^r_s(\mathbb{M}) := \bigcup_{x \in \mathbb{M}} \mathbb{T}^r_{s,x}(\mathbb{M})$$

взятое по всем точкам многообразия.

сечение этого расслоения $T^r_s(x)$ - это тензорное полем типа (r, s) или r раз контравариантным и s раз ковариантным тензорным полем на многообразии M.

Ранг тензорного поля - 'это число $r + s$.

Базой этого расслоения является многообразие M, типичным слоем - векторное

Пространство где $\mathbb{R}^n$ - типичный слой касательного расслоения.

Слоем над $x \in \mathbb{M}$ является векторное пространство (2.50) (тем самым мы определили проекцию).

Дифференцируемая структура на расслоении тензоров задается дифференцируемыми структурами на базе и в типичном слое аналогично тому, как она была построена для касательного расслоения.

Координатные базисы в касательном и кокасательном пространствах, $e_\alpha = \partial_\alpha$ и $e^\alpha = dx^\alpha$, индуцируют координатный базис в тензорном произведении, который мы обозначим

$$e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_r} \otimes e^{\beta_1} \otimes \cdots \otimes e^{\beta_s}$$

(???) Напомним, что тензорное произведение векторов не является коммутативным,

$$e_\alpha \otimes e^\beta \neq e^\beta \otimes e_\alpha$$

поэтому порядок следования базисных векторов в произведении (2.51) фиксирован: сначала мы пишем базисные векторы касательного, а затем кокасательного пространства.

Рассмотрим произвольную карту $(\mathbb{U}, \varphi)$ на многообразии.

Тогда тензорное поле типа (r, s) в координатах имеет вид

$$T^r_s(x) = T^{\alpha_1 \ldots \alpha_r}_{\beta_1 \ldots \beta_s}(x) e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_r} \otimes e^{\beta_1} \otimes \ldots \otimes e^{\beta_s}$$

Нижние и верхние индексы называют, соответственно, ковариантными и контравариантнъии.

Общее число индексов $r + s$ равно рангу тензорного поля.

Замечание.

Для определенности, у компонент $T^{\alpha_1\dots\alpha_r}_{\beta_1\dots\beta_s}(x)$ мы сначала выписали все ковариантные индексы, а затем - все контравариантные.

Порядок индексов зафиксирован порядком сомножителей в правой части (2.50) и принятом нами соглашением для записи тензорного поля типа (1,1):

$$dx^\beta T^\alpha_\beta \partial_\alpha$$

Контравариантные индексы, так же, как и ковариантные, упорядочены между собой.

В разделе 4 будет введена операция опускания и подъема индексов с помощью метрики.

Она будет неоднозначной, если порядок контравариантных и ковариантных индексов не фиксирован.

Ниже мы построим тензорную алгебру для тензорных полей вида (2.52).

Аналогично можно построить тензорную алгебру для произвольного расположения сомножителей в правой части (2.50), когда касательные и кокасательные пространства чередуются в произвольном порядке.

Мы будем предполагать, что все индексы упорядочены определенным образом.

В этом случае обозначение векторного поля $T^r_s(x)$ является грубым, т.к.

учитывает только общее число ковариантных и контравари- антных индексов, а не их последовательность.

????

Введем обозначение для прямой суммы тензорных полей

$$\mathcal{T}(\mathbb{M}) := \bigoplus_{r,s=0}^{\infty} \mathcal{T}^r_s(\mathbb{M})$$

На множестве $\mathcal{T}(\mathbb{M})$ можно ввести поточечное тензорное умножение, которое двум тензорам типа (r,s) и (p,q) ставит в соответствие тензорное поле типа $r+p, s+q$.

А именно, зафиксируем точку $x \in \mathbb{M}$. Из универсального факторизационного свойства тензорного произведения следует, что существует единственное билинейное отображение из $\mathbb{T}^r_{s,x} \times \mathbb{T}^p_{q,x}$ в $\mathbb{T}^{r+p}_{s+q,x}$, которое отображает пару тензоров

$$(X_1 \otimes \dots \otimes X_r \otimes A_1 \otimes \dots \otimes A_s) \in \mathbb{T}^r_{s,x}$$
$$(Y_1 \otimes \dots \otimes Y_p \otimes B_1 \otimes \dots \otimes B_q) \in \mathbb{T}^p_{q,x}$$

в тензор типа $r+p, s+q$:

$$X_1 \otimes \dots \otimes X_r \otimes Y_1 \otimes \dots \otimes Y_p \otimes A_1 \otimes \dots A_s \otimes B_1 \otimes \dots \otimes B_q \in \mathbb{T}^{r+p}_{s+q,x}$$

Это отображение называется тензорным произведением тензоров в данной точке.

Определение.

Тензорным произведением тензорных полей $\mathcal{T}^r_s(\mathbb{M})$ и $\mathcal{T}^p_q(\mathbb{M})$ называется тензорное поле типа $\mathcal{T}^{r+p}_{s+q}$ (M), полученное поточечным тензорным произведением (2.54).

Чтобы получить выражение для компонент тензорного произведения в определенной карте, достаточно в качестве векторных X, Y и ковекторных A, B полей в определении тензорного произведения выбрать координатный базис.

Пусть в некоторой карте задано два тензорных поля:

$$K = K_{\beta_1\dots\beta_s}{}^{\alpha_1\cdots\alpha_r} e_{\alpha_1} \otimes e_{\alpha_r} \otimes e^{\beta_1} \cdots \otimes e^{\beta_s}$$
$$L = L^{\gamma_1\cdots\gamma_p}_{\delta_1\cdots\delta_q} e_{\gamma_1} \otimes e_{\gamma_p} \otimes e^{\delta_1} \cdots \otimes e^{\delta_q} \quad \in \mathcal{T}^r_q(\mathbb{M})$$

Тогда компоненты их тензорного произведения

$$(K \otimes L)^{\alpha_1\cdots\alpha_r\gamma_1\cdots\gamma_p}_{\beta_1\cdots\beta_s\delta_1\cdots\delta_q} = K^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s} L^{\gamma_1\cdots\gamma_p}_{\delta_1\cdots\delta_q}$$

просто равны произведению компонент каждого сомножителя, как чисел. Пример 2.7.2.

Произведение двух векторных полей $X \otimes Y$ дает контравариантный тензор второго ранга с компонентами

$$T(x) = T^{\alpha\beta}(x) e_\alpha \otimes e_\beta = X(x) \otimes Y(x) = X^\alpha(x) Y^\beta(x) e_\alpha \otimes e_\beta$$

21.7 Homotopy Invariants

(пока малый раздел, потом мб увеличу)

21.7.1 Theory

21.7.2 Applications Overview

(хз, и это очень важно)

22 Cohomology

22.1 Theory

22.1.1 Theory

22.2 Other About Cohomologies

22.2.1 Applications Overview

(хз, и это очень важно)

23 Homology (???)

(????)

24 Introduction to other sections

24.0.1 Adding Curved Space to Physical Theories (??)

(если это пойму, то буду гораздо чаще пользоваться диффгемом. пока не такой уровень к сожалению)

24.1 Super Geometry

мотивация

применения

24.2 Laminated Spaces

(просто тут заготовка на всякий случай, вряд ли такое нужно будет, а сейчас точно не нужно)

24.2.1 Theory

24.2.2 Applications Overview

(хз, и это очень важно)

24.2.3 General Theory of Stratified Spaces

(структура Хьюзмоллера, мб только через пару лет буду этим заниматься для саморазвития, мб никогда)

Расслоении, определенные группами преобразований

Определение и примеры главных расслоений

специальная конструкция

Категории главных расслоений

Расслоения, индуцированные главными расслоениями

Определение расслоенных пространств

9

Функториальные свойства расслоенных пространств

Тривиальные и локально тривиальные расслоенные пространства

Описание сечений расслоенных пространств

Нумерируемые главные расслоения над пространством (?)

Функтор kQ

Конструкция Милнора

Гомотопическая классификация нумерируемых главных расслоений

Гомотопическая классификация главных G-расслоений над клеточными разбиениями

24.3 K-theory

(или в др запись выкину, или с точки зрения диффгема про неё писать буду)

24.3.1 Theory

24.3.2 Revision

(хз)

24.4 Geometry Modeling and Numerical Methods

(по идее можно еще моделировать, захочу - раздел для этого создан.)

24.5 Paradoxes of Geometry and Topology in a Nutshell

24.5.1 Banach-Tarski paradox

A ball can be cut into a finite number of pieces and re-assembling the pieces will get two balls, each of equal size to the first. The von Neumann paradox is a two-dimensional version.

24.5.2 Paradoxical set

A set that can be partitioned into two sets, each of which is equivalent to the original.

24.5.3 Coastline paradox

the perimeter of a landmass is in general ill-defined.

24.5.4 Coin rotation paradox

a coin rotating along the edge of an identical coin will make a full revolution after traversing only half of the stationary coin's circumference. The Banach-Tarski paradox: A ball can be decomposed and reassembled into two balls the same size as the original.

24.5.5 Gabriel's Horn

or Torricelli's trumpet: A simple object with finite volume but infinite surface area. Also, the Mandelbrot set and various other fractals are covered by a finite area, but have an infinite perimeter (in fact, there are no two distinct points on the boundary of the Mandelbrot set that can be reached from one another by moving a finite distance along that boundary, which also implies that in a sense you go no further if you walk "the wrong way" around the set to reach a nearby point).

24.5.6 Hausdorff paradox

There exists a countable subset C of the sphere S such that S\C is equidecomposable with two copies of itself.

24.5.7 Hooper's paradox

An image with many pieces whose size is 32 m^2, but drops down to 30 m^2 when its pieces are rearranged

24.5.8 Nikodym set

A set contained in and with the same Lebesgue measure as the unit square, yet for every one of its points there is a straight line intersecting the Nikodym set only in that point.

24.5.9 Sphere eversion

A sphere can, topologically, be turned inside out.

Part VI
Applications of Differential Geometry

Обсудим встречавшиеся мне применения дифференциальной геометрии в разделах физики и математике.

25 Typical mathematical applications

(то, чего я касался)

25.1 Geometry in Differential Equations

(потом мб изучу, пока не актуально)

25.2 Geometry in Functional Analysis

(потом мб изучу, пока не актуально)

26 Typical physical applications

26.1 Geometry in Mechanics

Обсудим механику на языке дифференциальной геометрии.

(!!! соберу основы механики - и проработаю эти связки, потом - с Володей пообщаюсь и проверю свои знания. пока этим не владею. Там кст потом почитаю если нужно будет, много книг, нпр Добронравов, но это отдельное занятие, пока и близко не до этого.)

26.1.1 Mechanics in the Language of Forms

у нас была такая вот механика:
????
давайте перепишем это, используя наши новые идеи.
делаем это.
получаем это:
...
посмотрим, насколько это удобнее
...

26.1.2 Samples

лагранжева механика в двух словах

(тут пока от темы я ушел, потом буду прописывать)
Рассмотрим конфигурационное пространство M.
(x^α, v^α) это координаты на TM.

$$\bigcup_{p\in M} T_pM = TM$$

Функция Лагранжа это такая функция,

$$L : TM \times \mathbb{R} \to \mathbb{R}$$

которая удовлетворяет принципу наименьшего действия.

декартово произведение, потому что в общем случае функция Лагранжа зависит от времени.

введем множество кривых

$$\Pi_{p_1 t_1}^{p_0 t_0} = ..$$

далее определяется действие.

далее вариация.

26.1.3 Definitions and Construction of Mechanics

Обсудим откуда взялось то, что выше.

гамильтонова механика - основные формулы (??)

гамильтонова механика - объяснение

В классической механике фазовое пространство представляет собой симплектическое многообразие M, параметризованное координатами (x^α, p_β), где x^α - обобщённые координаты, а p_β - обобщённые импульсы.

Гамильтониан механической системы $H(x,p)$ является функцией на многообразии M; уравнения Гамильтона в координатах

$$\dot{x}^\alpha = \frac{\partial H}{\partial p_\alpha}, \quad \dot{p}_\alpha = -\frac{\partial H}{\partial x^\alpha}$$

Как нам известно, фазовое пространство наделено дополнительной структурой - скобками Пуассона:

$$\{f,g\}_{\mathbb{P}} = \frac{\partial f}{\partial x^\alpha}\frac{\partial g}{\partial p_\alpha} - \frac{\partial f}{\partial p_\alpha}\frac{\partial g}{\partial x^\alpha}$$

Эволюция наблюдаемой $f(x,p)$ на фазовом пространстве определяется скобками Пуассона согласно

$$\frac{df}{dt} = \{f,H\}_{\mathrm{P}}$$

Поэтому гамильтонову механику можно сформулировать в терминах дифференциальных форм и векторных полей.

Вкратце, необходим механизм, который превращает функцию $H : M \to \mathbb{R}$ в векторное поле v_H; при этом, траекториями частиц в фазовом пространстве M являются интегральные кривые, порождённые векторным полем v_H.

Этот механизм довольно простой.

Дифференциальная форма второй степени ω на симплектическом многообразии M даёт нам отображение

$$\omega : T_pM \to T_p^\star M,$$

которое на самом деле является изоморфизмом в силу невырожденности симплектической формы ω.

???????? В таком случае для гамильтониана $H : M \to \mathbb{R}$ существует единственное гамильтоново векторное поле v_H, определяемое из формулы

$$\imath_{v_H}\omega = -2dH$$

Уравнение (1.117) может быть переписано в терминах локальных координат x^μ на многообразии M следующим образом:

$$v_H^\mu \omega_{\mu\nu} = -\partial_\nu H$$

Действительно,

$$\imath_{v_H}\omega\left(\partial_\gamma\right) = v_H^\beta \omega_{\mu\nu} dx^\mu \wedge dx^\nu\left(\partial_\beta, \partial_\gamma\right) = v_H^\beta \omega_{\mu\nu}\left(\delta_\beta^\mu \delta_\gamma^\nu - \delta_\gamma^\mu \delta_\beta^\nu\right) = 2v_H^\beta \omega_{\beta\gamma} = -2\partial_\gamma H$$

Здесь мы воспользовались кососимметричностью компонент $\omega_{\mu\nu}\,(\omega_{\mu\nu} = -\omega_{\nu\mu})$ симплектической формы ω.

Определяя $\omega^{\mu\nu}$ из равенства $\omega^{\mu\nu}\omega_{\nu\sigma} = \delta^{\mu}_{\sigma}$, получаем, что координаты векторного поля

$$v^{\mu}_{H} = -\omega^{\nu\mu}\partial_{\nu}H = \omega^{\mu\nu}\partial_{\nu}H$$

А как известно из раздела про интегрирование векторных полей, интегральные кривые, порождённые векторным полем v_H, удовлетворяют дифференциальному уравнению

$$\frac{dx^{\mu}}{dt} = v^{\mu}_{H} = \omega^{\mu\nu}\partial_{\nu}H$$

отличие Гамильтоновой механики от лагранжевой

что в одном случае работаем на касательном пространстве, во втором - в кокасательном.

симплектоморфизм - преобразования, которые сохраняют симплектическую форму

(!! в них потом връеду)

$$\varphi^{*}\omega = \omega$$

теорема Лиувилля о сохранении фазового объема

Рассмотрим

$$L_{X_H}\omega = 0$$

где ω - симплектическая форма.

26.2 Geometry in Quantum Mechanics

26.3 Geometry in Typical Field Theory

(список тем из какого-то теорфизного курса, скорее всего от кафедры Белавина, потом посмотрю, пока учу основы. Мб это курс Рослого для умных)

26.3.1 Some Themes:

(потом подумаю ответы. что есть темы от сюда)

Скаляр Бранса-Дикке

хз

Конформная геометрия

хз

конформная группа и алгебра

хз

Оператор (звёздочка) Ходжа

конформная инвариантность Четырёхмерие: самодуальность тензора Вейля и/или Римана

хз

Уравнения и функционал Янга-Миллса Уравнения самодуальности Топологический заряд

хз

Гармонические формы и когомологии

хз

Характеристические классы(классы Черна, Понтрягина, эйлеров класс),выражение через кривизну

хз

Формы Черна-Саймонса

хз

Сигнатура многообразия

хз

Теорема Гаусса-Бонне

хз

26.3.2 Spinor Fields on a Riemannian Manifold

(я вообще хз, что это и зачем)

спин-структура

хз, но мб однажды открою

теорема Лихнеровича

хз вообще

Понятие индекса (фредгольмова) оператора

Теорема об индексе оператора Дирака

Симплектическая структура

Теорема Дарбу

Лагранжевы подмногообразия

Гамильтоновы векторные поля и их гамильтонианы

Скобки Пуассона

симплектические листы

Действие группы симплектоморфизмами

гамильтонова редукция

Квантовые аналоги этих понятий

(хз всё это)

26.3.3 covariant Derivative in Field Theory

Появляется как определение оператора дифференцирования

26.3.4 Electrodynamics in the Language of Forms

быстро остановимся на том, как вставить нашу теорию в другие разделы физики. дальнейшее развитие этого будет в соответствующих разделах

26.3.5 Maxwell's Equations from a Geometric Point of View

26.3.6 Electromagnetic Field in Differential Forms

(по Болибруху напишу, одновременно диффгем качать нужно.)

Напоминание сути некоторых тем диффгема в контексте электромагнитного поля

1

Напомним, как определяются ротор и дивергенция:

$$\operatorname{div}\vec{H} = \frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z}$$
$$\operatorname{rot}\vec{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}, \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}, \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)$$

Но действительно ли величины $\vec{E}, \vec{H}$ и го $\vec{v}$ являются векторами? Как они меняются при замене координат? Скоро мы увидим, что они не очень похожи на векторы и что есть другая, более удобная форма записи тех же самых уравнений, которая возникла позднее, когда появилась теория дифференциальных форм, введенная Пуанкаре и Эли Картаном.

Чтобы лучше понять, как меняются компоненты вектора при замене координат, рассмотрим несколько примеров.

Пример

1. Рассмотрим n-мерное пространство $\mathbb{R}^n$ (можно считать для простоты, что $n = 3$) и гладкую кривую $\varphi : \mathbb{R} \to \mathbb{R}^n$, задаваемую в некоторой системе координат $(x^1, \ldots, x^n)$ уравнениями $x^i = \varphi^i(t)$. При $t = 0$ касательный вектор к этой кривой в выбранной системе координат имеет компоненты $\xi^i = \dot{\varphi}_i(0)$ (точкой обозначается дифференцирование по t). Теперь рассмотрим новую систему координат $(\hat{x}^1, \ldots, \hat{x}^n)$, связанную со старой при

помоши гладких функций $\hat{x}^i = f^i\left(x^1, \ldots, x^n\right)$. Компоненты касательного вектора в новой системе координат будем обозначать ξ.. Тогда

$$\hat{\xi}^i = \left.\frac{d\hat{x}^i}{dt}\right|_{t=0} = \left.\frac{df^i\left(\varphi^1(t), \ldots, \varphi^n(t)\right)}{dt}\right|_{t=0} = \sum_{j=1}^{n} \left.\frac{\partial f^i}{\partial x^j}\right|_{t=0} \xi$$

по формуле производной сложной функции. Вот как, оказывается, связаны координаты касательного вектора в новой и старой системах координат.

Пример 2. Рассмотрим функцию φ от n переменных: $\varphi = \varphi\left(\hat{x}^1, \ldots, \hat{x}^n\right)$. Зачем мы записали ее сразу в новых координатах, станет ясно чуть позже. Рассмотрим

градиент этой функции (который также часто называют вектором)

$$\operatorname{grad}\varphi = \left(\frac{\partial\varphi}{\partial\hat{x}^1}, \ldots, \frac{\partial\varphi}{\partial\hat{x}^n}\right) = \left(\hat{\eta}^1, \ldots, \hat{\eta}^n\right)$$

Теперь попробуем сделать замену координат и вычислить компоненть градиента, только на этот раз уже не в новой, а в старой системе координат:

$$\eta^i = \frac{\partial\varphi\left(f^1\left(x^1, \ldots, x^n\right), \ldots, f^n\left(x^1, \ldots, x^n\right)\right)}{\partial x^i} = \sum_{j=1}^{n} \frac{\partial\varphi}{\partial\dot{x}^j}\frac{\partial f^j}{\partial x^i} = \sum_{j=1}^{n} \hat{\eta}^j \frac{\partial f^j}{\partial x^i}$$

Сравним формулы, полученные в этих двух примерах. В обеих этих формулах участвуют выражения вида $\partial f^i/\partial x^j$, но между ними есть существенное отличие. В первом случае мы выражаем новые координаты через старые, а во втором - наоборот. Мы приходим к выводу, что при замене координат компоненты градиента меняются совсем по другому закону, чем компоненты касательного вектора. Градиент скорее напоминает не вектор, а дифференциал функции:

$$d\varphi = \sum_{i=1}^{n} \frac{\partial\varphi}{\partial x^i} dx^i = \sum_{j=1}^{n} \frac{\partial\varphi}{\partial\hat{x}^j} d\hat{x}^j$$

И градиент функции, и касательный вектор являются тензорами, но тензорами разного типа.

2

Рассмотрим на $\mathbb{R}^n$ пространство векторных полей $\mathcal{D}\left(\mathbb{R}^n\right)$. Под векторным полем мы будем понимать выражение

$$\vec{v}\left(x^1, \ldots, x^n\right) = \sum_{j=1}^{n} v_j\left(x^1, \ldots, x^n\right) \frac{\partial}{\partial x^j}$$

где $v_j\left(x^1, \ldots, x^n\right)$ - гладкие функции, а через $\partial/\partial x^j$ обозначен вектор e_j стандартного базиса в $\mathbb{R}^n$ (эти обозначения оказываются очень удобными

при рассмотрении векторных полей на многообразиях). Такие поля еще иногда называют касательньими. Множество гладких функций на $\mathbb{R}^n$ мы будем обозначать $\mathcal{F}\left(\mathbb{R}^n\right)$.

Множество векторных полей является линейным пространством: векторные поля можно складывать (покоординатно) и умножать на числа. Более того, их можно умножать на любые гладкие функции, заданные на всем пространстве

$\mathbb{R}^n$. К сожалению, гладкие функции образуют не поле, а кольцо, поэтому говорят не о линейном пространстве векторных полей, а о модуле векторньсх полей над кольцом гладких функиий. Рассмотрим сопряженное пространство к $\mathcal{D}\left(\mathbb{R}^n\right)$ - пространство $\mathcal{F}$ - линейных функционалов $\alpha : \mathcal{D}\left(\mathbb{R}^n\right) \to \mathcal{F}(\mathbb{R})$, т. е.

таких отображений, что $\alpha(f\vec{v}) = f\alpha(\vec{v})$ для любой гладкой функции $f(x)$ и любого

векторного поля $\vec{v}$ и $\alpha\left(\vec{v}_1 + \vec{v}_2\right) = \alpha\left(\vec{v}_1\right) + \alpha\left(\vec{v}_2\right)$. Это сопряженное пространство также является модулем над кольцом гладких функций. Оно называется пространством дифференциальных 1-форм и обозначается $\Lambda^1\left(\mathbb{R}^n\right)$.

У этого модуля есть образующие, сопряженные к образующим $\mathcal{D}(\mathbb{R}^n)$, которые мы будем обозначать через dx^j :

$$dx^j\left(\frac{\partial}{\partial x^i}\right)=\delta_i^j=\left\{\begin{array}{ll}1, & i=j,\\ 0, & i\neq j.\end{array}\right.$$

Любая 1-форма может быть представлена в виде $\mathcal{F}$ -линейной комбинации этих образующих:

$$\alpha=\sum_{i=1}^n\alpha_i\left(x^1,\ldots,x^n\right)dx^i$$

где $\alpha_i\left(x^1,\ldots,x^n\right)$ - гладкие функции из $\mathcal{F}$. При изменении координат в исходном пространстве соответственно меняются координаты и в сопряженном пространстве. Компоненты 1-формы при этом меняются по тому же закону, что и компоненты дифференциала. Это означает, что

дифференциал функции $\int$ является частным случаем 1 -формы (что оправдывает введение таких обозначений для базиса сопряженного пространства), т.е. у нас имеется отображение $d:\mathcal{F}(\mathbb{R}^n)\to\Lambda^1(\mathbb{R}^n)$.

Не всякая дифференциальная 1-форма является образом некоторой функции при взятии дифференциала, но имеет место включение: образ d лежит в пространстве 1-форм. Таким образом, дифференциал является не вектором, а ковектором - так принято называть дифференциальные 1 -формы.

Можно ли расширить это определение, т. е. рассмотреть не только 1-формы, но и, например, 2-формы? Оказывается, это сделать совсем

нетрудно. Для этого нужно лишь вместо векторного поля рассмотреть пару векторных полей, и каждой паре векторных полей поставить в соответствие гладкую функцию. И если это отображение является $\mathcal{F}$ -линейным по каждому аргументу, то мы приходим к понятию 2-ковектора на множестве векторных полей: это просто $\mathcal{F}$ -билинейное отображение $\alpha:\mathcal{D}(\mathbb{R}^n)\times\mathcal{D}(\mathbb{R}^n)\to\mathcal{F}(\mathbb{R}^n)$

Пример 3. Рассмотрим два векторных поля $\vec{v}_1$ и $\vec{v}_2$ В $\mathbb{R}^3$ и отображение, сопоставляюшее паре векторов их скалярное произведение:

$$r:\mathcal{D}\left(\mathbb{R}^3\right)\times\mathcal{D}\left(\mathbb{R}^3\right)\to\mathcal{F}\left(\mathbb{R}^3\right),\quad r\left(\vec{v}_1,\vec{v}_2\right)=\left|\vec{v}_1\right|\cdot\left|\vec{v}_2\right|\cdot\cos\gamma$$

Легко проверить что это отображение $\mathcal{F}$ -билинейно; к тому же оно еше и симметрическое: если поменять местами $\vec{v}_1$ и $\vec{v}_2$, то результат не изменится. Это отображение можно записать и в матричной форме:

$$r\left(\vec{v}_1,\vec{v}_2\right)=\left(x^1y^1z^1\right)\left(\begin{array}{ccc}1&0&0\\0&1&0\\0&0&1\end{array}\right)\left(\begin{array}{c}x^2\\y^2\\z^2\end{array}\right)$$

где x^i,y^i,z^i - координаты вектора $\vec{v}_i$. Получаем симметрическую 2 -форму, которая называется евклидовой метрикой в $\mathbb{R}^3$.

Пример 4. А вот пример (тоже хорошо всем известный) антисимметрического 2-ковектора, или внешней форми. Рассмотрим в пространстве $\mathbb{R}^2$ два векторных поля $\vec{v}_1$ и $\vec{v}_2$ и поставим им в соответствие ориенmированную площцадо т)$\vec{v}_1,\vec{v}_2)\in\mathcal{F}(\mathbb{R}^2)$ параллелограмма, построенного на этих векторах. В координатах эта плошадь равна определителю

$$\left|\begin{array}{cc}x^1&y^1\\x^2&y^2\end{array}\right|=x^1y^2-x^2y^1$$

Это отображение также является линейным по каждому из аргументов, но уже обладает свойством

антикоммутативности: $\tau\left(\vec{v}_1,\vec{v}_2\right)=-\tau\left(\vec{v}_2,\vec{v}_1\right)$. Именно такие 2-ковекторы (их еще называют кососимметрическими) представляют для нас особый интерес. Определение 1. Отображение $\alpha:\underbrace{\mathcal{D}(\mathbb{R}^n)\times\ldots\times\mathcal{D}(\mathbb{R}^n)}_{k}\to\mathcal{F}(\mathbb{R}^n)$ на- Зыва ется внешней дифферениицальной формой степени k, если оно $\mathcal{F}$ -линейно по каждому аргументу и

$$\alpha\left(\vec{v}_1, \ldots, \vec{v}_i, \ldots, \vec{v}_j, \ldots, \vec{v}_n\right) = -\alpha\left(\vec{v}_1, \ldots, \vec{v}_j, \ldots, \vec{v}_i, \ldots, \vec{v}_n\right)$$

ЛлЯ любых i, j.

Пространство $\Lambda^k\left(\mathbb{R}^n\right)$ внешних дифференциальных форм степени k и станет объектом нашего дальнейшего изучения. Естественно, что всякий 1-ковектор можно рассматривать как внешнюю дифференциальную форму (впрочем, точно так же его можно рассматривать и как симме-

трическую форму). Для того, чтобы записывать формы в координатах, мы введем новую операцию на дифференциальных формах - внешнее произведение. Сначала определим ее для пары 1-форм. Эта операция будет тоже $\mathcal{F}$ -линейной, поэтому мы определим ее только на базисных 1-ковекторах, а затем распространим

на все 1-формы по $\mathcal{F}$ -линейности. Каждой паре 1-форм эта операция ставит в соответствие 2-форму, причем на базисных ковекторах она определяется следующим образом:

$$\left(dx^i \wedge dx^j\right)\left(\vec{v}_1, \vec{v}_2\right) = dx^i\left(\vec{v}_1\right) dx^j\left(\vec{v}_2\right) - dx^i\left(\vec{v}_2\right) dx^j\left(\vec{v}_1\right)$$

Легко проверить, что полученная форма кососимметрична. Замечательно, что таким образом можно получить весь базис в пространстве 2-форм. Поэтому любой элемент)и из $\Lambda^2\left(\mathbb{R}^n\right)$ записывается очень просто:

$$\omega = \sum_{1 \leqslant i_1 < i_2 \leqslant n} \alpha_{i_1 i_2}\left(x^1, \ldots, x^n\right) dx^{i_1} \wedge dx^{i_2}$$

Аналогично можно определить внешнее произведение для форм произвольной степени. Пусть α и $\beta - p$ - и q -формы соответственно. Тогда внешним произведением форм α и β называется $(p+q)$ -форма

$$\begin{aligned}&(\alpha \wedge \beta)\left(\vec{v}_1, \ldots, \vec{v}_{p+q}\right) = \\ &\quad = \tfrac{1}{p!q!} \sum \varepsilon(s) \alpha\left(\vec{v}_s - 1_{(1)}, \ldots, \vec{v}_s - 1_{(p)}\right) \times \beta\left(\vec{v}_s - 1_{(p+1)}, \ldots, \vec{v}_{s^{-1}(p+q)}\right)\end{aligned}$$

где суммирование ведется по всевозможным перестановкам $p+q$ элементов, а $\varepsilon(s)$ - знак перестановки S.

Приведенное определение инвариантно, т. е. не зависит от выбора системы координат. Но если система координат фиксирована, то из него сразу следует, что базис в пространстве k -форм состоит из элементов вида $dx^{i_1} \wedge \ldots \wedge dx^{i_k}, i_1 < i_2 < \ldots < i_k$. Поэтому произвольная форма α имеет вид

$$\alpha = \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} \alpha_{i_1 \ldots i_k}\left(x^1, \ldots, x^n\right) dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

Отметим еше одно замечательное свойство операции внешнего дифференцирования: $\alpha \wedge \beta = (-1)^{pq} \beta \wedge \alpha$.

Отсюда, в частности, следует, что если степень α нечетна, то $\alpha \wedge \alpha = 0$.

3

Заметим, что операция дифференцирования каждой функции сопоставляет 1-форму. А можно ли определить операцию дифференцирования Для произвольной формы?

Оказывается, это очень легко сделать. По определению полагаем

$$d\alpha = \sum_{1 \leqslant i_1 < \ldots < i_k \leqslant n} d\alpha_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_k}$$

Видно, что дифференциал повышает на единицу степень формы. Кроме того, из этого определения сразу следует, что $d^2\alpha = 0$. Действие дифференциала на внешние формы можно представить следуюшей диаграммой:

$$\mathcal{F}\left(\mathbb{R}^n\right) \xrightarrow{d} \Lambda^1\left(\mathbb{R}^n\right) \xrightarrow{d} \Lambda^2\left(\mathbb{R}^n\right) \xrightarrow{d} \ldots \xrightarrow{d} \Lambda^k\left(\mathbb{R}^n\right) \xrightarrow{d} \ldots$$

Этот объект называется комплёксом де Рама, и в нем содержится больШой объем информации о пространстве, на котором заданы дифференциальные формы.

В данном случае мы имеем дело с $\mathbb{R}^n$, которое устроено довольно просто, но если бы мы стартовали с другого пространства, скажем, со сферы, то по этому комплексу мы бы узнали очень многое о топологии пространства.

Нетрудно проверить, что введенная с помощью координат операция внешнего дифференцирования на самом деле от координат не зависит, т. е. является инвариантной (проверьте это самостоятельно).

Наконец, последняя операция, которую нам осталось ввести - операция внутреннего произведения. Пусть у нас есть векторное поле $\vec{v}$ и внешняя форма а порядка p.

Тогда мы можем построить форму степени $p-1$, которая называется внутренним произведением формы α и поля $\vec{v}$

$$(i_v\alpha)(\vec{v}_1,\ldots,\vec{v}_{p-1}) = \alpha(\vec{v},\vec{v}_1,\ldots,\vec{v}_{p-1})$$

Это определение также не зависит от выбора системы координат.

4

Попробуем теперь понять, что означают операторы ротора и дивергенции на инвариантном языке дифференциальных форм. Хотя мы часто писали дифференциальные формы в координатах, определение их было инвариантным, от координат не зависящим. А вот определения ротора и дивергенции зависят от выбора системы координат. Так давайте выясним, в чем же состоит эта зависимость.

Разберемся сначала с дивергенцией. Мы начинаем с векторного поля $\vec{v}$ на $\mathbb{R}^3$. По этому векторному полю мы построим 2-форму следующим образом. Возьмем форму объема $\tau = dx\wedge dy\wedge dz$ и рассмотрим внутреннее произведение $i_v\tau$ - это и есть обещанная 2 -форма. Из нее мы получим 3-форму путем взятия внешнего дифференциала. Но пространство 3 -форм на $\mathbb{R}^3$ имеет единственный базисный вектор, и это в точности форма объема $dx\wedge dy\wedge dz$. Значит, полученная нами форма есть некоторая гладкая функция, помноженная на форму объема. Проверим, что эта гладкая функция и будет дивергенцией векторного поля $\vec{v}$. Пусть поле $\vec{v}$ имеет вид

$$\vec{v} = v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}.$$

Тогда

$$\begin{aligned}
(i_v\tau)\left(\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right) &= (dx\wedge dy\wedge dz)\left(\vec{v},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right) = v_x,\\
(i_v\tau)\left(\frac{\partial}{\partial z},\frac{\partial}{\partial x}\right) &= (dx\wedge dy\wedge dz)\left(\vec{v},\frac{\partial}{\partial z},\frac{\partial}{\partial x}\right) = v_y,\\
(i_v\tau)\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) &= (dx\wedge dy\wedge dz)\left(\vec{v},\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) = v_z.
\end{aligned}$$

Отсюда

$$i_v\tau = v_x dy\wedge dz + v_y dz\wedge dx + v_z dx\wedge dy.$$

Теперь возьмем дифференциал:

$$\begin{aligned}
d(i_v\tau) &= \frac{\partial v_x}{\partial x}dx\wedge dy\wedge dz + \frac{\partial v_y}{\partial y}dy\wedge dz\wedge dx + \frac{\partial v_z}{\partial z}dz\wedge dx\wedge dy =\\
&= \left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right)dx\wedge dy\wedge dz.
\end{aligned}$$

Мы получили в точности дивергенцию, помноженную на форму объема. Таким образом, дивергенцию можно записать в очень компактном виде:

$$\operatorname{div}\vec{v} = \frac{d(i_v\tau)}{\tau}.$$

Что же здесь зависит от выбора системы координат? Внутреннее произведение и дифференциал инвариантны относительно замен координат, от координат зависит только форма объема. Выясним, при каких заменах дивергенция не меняется.

При замене координат $\hat{x}^i = f^i\left(x^1, x^2, x^3\right)$ получаем

$$\hat{\tau} = d\hat{x}^1 \wedge d\hat{x}^2 \wedge d\hat{x}^3 = J(f)\tau,$$

где $J(f)$ - определитель матрицы (df^i/dx^j), называемый якобианом замены. Поэтому

$$\frac{d\left(i_v\hat{\tau}\right)}{\hat{\tau}} = \frac{d\left(J(f)i_v\tau\right)}{J(f)\tau} = \frac{dJ(f)\wedge i_v\tau}{J(f)\tau} + \frac{d\left(i_v\tau\right)}{\tau}.$$

В силу произвольности поля $\vec{v}$ получаем, что дивергенция не меняется лишь при таких заменах, для которых

$$dJ(f) = 0, \quad \text{т. е.} \quad J(f) = \text{ const }.$$

Если же под дивергенцией поля с компонентами v_x, v_y, v_z понимать компоненту внешнего дифференциала 2-формы с компонентами v_x, v_y, v_z (это форма $i_v\tau$, выписанная выше), то такое определение инвариантно полностью, т. е. при этом дивергенция никак от выбора координат не зависит.

Рассмотрим теперь ротор векторного поля $\vec{v}$. Для этого введем отображение

$$\sigma : \mathcal{D}\left(\mathbb{R}^3\right) \to \Lambda^1\left(\mathbb{R}^3\right), \quad \sigma\left(v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right) = v_x dx + v_y dy + v_z dz.$$

Заметим, что

$$(d\circ\sigma)(\vec{v}) = \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right) dx\wedge dy + \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right) dy\wedge dz + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right) dz\wedge dx.$$

Рассмотрим отображение $x : \Lambda^2\left(\mathbb{R}^3\right) \to \mathcal{D}\left(\mathbb{R}^3\right)$, «обратное» к внутреннему произведению, т. е. такое, что $i_{\varkappa(\alpha)}\tau = \alpha$ для всех $\alpha \in \Lambda^2\left(\mathbb{R}^3\right)$, где τ по-прежнему форма объема в $\mathbb{R}^3$. Тогда

$$\operatorname{rot}\vec{v} = (x\circ d\circ\sigma)(\vec{v}).$$

Проверка полученного соотношения сводится к проверке тождества $i_{\operatorname{rot}}\tau = d(\sigma(\vec{v}))$, которая проводится точно так же, как вычисление формы $i_v\tau$ в случае дивергенции.

Проанализируем эту конструкцию. Нетрудно видеть, что задание σ равносильно определению отображения

$$r : \mathcal{D}\left(\mathbb{R}^3\right) \times \mathcal{D}\left(\mathbb{R}^3\right) \to \mathcal{F}\left(\mathbb{R}^3\right), \quad r(\vec{v}, \vec{w}) = (\sigma(\vec{v}))(\vec{w}).$$

Но из формулы для σ вытекает, что r совпадает с симметрической 2 -формой из примера 3, т.е. r является евклидовой метрикой в $\mathbb{R}^3$.

Заметим, что в определении отображения x существенную роль играет трехмерность пространства $\mathbb{R}^3$, так как степень формы $i_{\varkappa(\alpha)}\tau$ на единицу меньше размерности пространства, а форма α имеет степень два.

Отсюда следует вывод: операция взятия ротора векторного поля может быть определена инвариантно лишь на ориентированном трехмерном многообразии, снабженном евклидовой метрикой (проверьте самостоятельно этот вывод, вычислив те замены координат, при которых компоненты ротора векторного поля действительно преобразуются как компоненты векторного поля).

Этот вывод показывает, что ротор лучше определять не как векторное поле, а инвариантным образом: как внешний дифференциал ковекторного поля $v_x dx + v_y dy + v_z dz$.

$5.

Перейдем теперь непосредственно к анализу уравнений Максвелла. Если следовать традиционному взгляду, что составляющие электромагнитного поля - поля $\vec{E}, \vec{H}$ - это векторные поля, то анализ структуры операторов ротора и дивергенции показывает, что уравнения Максвелла могут быть записаны корректно лишь в трехмерном ориентированном пространстве, снабженном евклидовой метрикой.

Но тот факт, что ротор и дивергенция могут меняться при каких-то заменах координат, не отражает физику уравнений Максвелла, потому что природа этих уравнений никак не связана с наличием, скажем, метрики в пространстве или с выбором системы координат. Поэтому нам хотелось бы иметь какое-то инвариантное описание уравнений Максвелла. Все вышеперечисленное позволяет предположить, что электромагнитное поле есть дифференциальная форма степени два [1].

Эта 2-форма Ω, называемая тензором электромагнитного по$\varkappa$, определена на четырехмерном пространстве $\mathbb{R}^4$ с координатами (x, y, z, t), где t - время, и имеет вид:

$$\Omega = cE_x dx \wedge dt + cE_y dy \wedge dt + cE_z dz \wedge dt + \\ + H_x dy \wedge dz + H_y dz \wedge dx + H_z dx \wedge dy.$$

Вычислим дифференциал этой формы и приравняем его к нулю.

$$\begin{aligned} d\Omega =& c\frac{\partial E_x}{\partial y} dy \wedge dx \wedge dt + c\frac{\partial E_x}{\partial z} dz \wedge dx \wedge dt + c\frac{\partial E_y}{\partial x} dx \wedge dy \wedge dt + \\ &+ c\frac{\partial E_y}{\partial z} dz \wedge dy \wedge dt + c\frac{\partial E_z}{\partial x} dx \wedge dz \wedge dt + c\frac{\partial E_z}{\partial y} dy \wedge dz \wedge dt + \\ &+ \frac{\partial H_x}{\partial x} dx \wedge dy \wedge dz + \frac{\partial H_x}{\partial t} dt \wedge dy \wedge dz + \frac{\partial H_y}{\partial y} dy \wedge dz \wedge dx + \\ &+ \frac{\partial H_y}{\partial t} dt \wedge dz \wedge dx + \frac{\partial H_z}{\partial z} dz \wedge dx \wedge dy + \frac{\partial H_z}{\partial t} dt \wedge dx \wedge dy = 0 \end{aligned}$$

Соберем коэффициенты при $dx \wedge dy \wedge dz$:

$$\frac{\partial H_x}{\partial x} + \frac{\partial H_y}{\partial y} + \frac{\partial H_z}{\partial z} = 0$$

и мы получим второе уравнение Максвелла. Теперь соберем коэффициенты при $dx \wedge dy \wedge dt$:

$$c\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} + \frac{\partial H_z}{\partial t}\right) = 0.$$

Это есть z-компонента первого уравнения Максвелла. Собирая коэффициенты при $dy \wedge dz \wedge dt$ и $dz \wedge dx \wedge dt$, мы получим x - и y-компоненты этого уравнения соответственно.

Итак, первые два уравнения Максвелла мы можем записать очень просто:

$$d\Omega = 0.$$

В чем преимущество этой записи? Мы записали уравнения в инвариантной форме: ни дифференциальные формы, ни внешнее дифференцирование не зависят ни от координат, ни от ориентации, ни от метрики.

Вспомним теперь, что $d^2 = 0$. Отсюда следует, что $\operatorname{Im} d \subset \operatorname{Ker} d$. Но в пространстве $\mathbb{R}^4$ имеет место лемма Пуанкаре: $\operatorname{Im} d = \operatorname{Ker} d$. Из этой леммы следует, что всякая замкнутая форма (дифференциал от которой равен 0) является точной (т. е. является дифференциалом некоторой

1 Здесь уместно еще раз отметить, что традиционная форма записи уравнений Максвелла появилась до создания теории дифференциальных форм. формы). В частности, $\Omega = d\omega$, где ω есть 1 -форма на пространстве $\mathbb{R}^4$:

$$\omega = A_x dx + A_y dy + A_z dz + A_t dt.$$

Эта форма называется 4-потенциалом электромагнитного поля. Введем, как это обычно делается в физике, координаты $x^1 = x, x^2 = y$, $x^3 = z, x^4 = ct$. Тогда мы можем записать наши формы в виде

$$\Omega = F_{\mu\nu} dx^\mu \wedge dx^\nu, \quad \omega = A_\mu dx^\mu$$

(знак суммирования по повторяющимся верхнему и нижнему индексам опущен). Тот факт, что $\Omega = d\omega$, можно записать совсем просто:

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}, \quad \mu < \nu.$$

Именно так выглядят уравнения Максвелла во всех современных учебниках физики.

Заметим, что форма ω определяется по форме Ω неоднозначно: мы можем всегда прибавить к этой форме точную 1-форму, т. е. дифференциал функции: $\omega' = \omega + dS$. И вновь мы получим, что $d\omega' = \Omega$, так как квадрат дифференциала равен нулю. Это означает, что 4-потенциал определяется, как говорят, с точностью до калибровочных преобразований: $A'_\mu = A_\mu + \partial S/\partial x_\mu$. А поскольку в уравнения Максвелла входят только элементы $F_{\mu\nu}$, то, стало быть, выбор 4-потенциала не влияет на уравнения Максвелла.

Так обстоят дела в классической электродинамике. Но уже в квантовом случае движение частицы зависит не только от тензора электромагнитного поля, но и от 4-потенциала, и там калибровки имеют более важное значение.

Лекция 2

На предыдущей лекции мы записали фарадееву часть уравнений Максвелла (т. е. первые два уравнения) в виде

$$d\Omega = 0,$$

где $\Omega = F_{\mu\nu} dx^\mu \wedge dx^\nu, \mu < \nu$ (знак суммирования по повторяющимся индексам здесь и ниже, как правило, опускается) — тензор электромагнитного поля.

Затем, используя тот факт, что для пространства $\mathbb{R}^4$ имеет место лемма Пуанкаре (каждая замкнутая форма на этом пространстве является точной), мы переписали полученное уравнение в виде

$$\Omega = d\omega, \quad \omega = A_\mu dx^\mu$$

или

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x_\mu} - \frac{\partial A_\mu}{\partial x_\nu}, \quad \mu < \nu.$$

Форма ω называется 4-потенциалом электромагнитного поля и определяется с точностью до калибровочных преобразований $A'_\mu = A_\mu + \partial S/\partial x_\mu$.

Так обстоит дело, если пространство, на котором разворачиваются события, $-\mathbb{R}^4$. Но не всегда ареной физических событий бывает $\mathbb{R}^4$. Например, уравнение магнитного диполя Дирака описывает ситуацию, когда в одной точке пространства расположен магнитный заряд. Тогда все уравнения пишутся в пространстве $(\mathbb{R}^3 \backslash \{x\}) \times \mathbb{R}$. В качестве такого пространства может также выступать произведение трехмерного тора на $\mathbb{R}$ и т. д.

В этих случаях лемма Пуанкаре уже не всегда имеет место, т. е. глобальный 4-потенциал может не существовать. Однако, поскольку локально каждое многообразие устроено как евклидово пространство, в каждой карте свой 4-потенциал может быть построен. Но как связать вместе все эти локальные 4-потенциалы, как записать в инвариантной форме уравнения электромагнитного поля, заданного на всем пространстве и не зависящего от выбора карт (т. е. локальных систем координат)? Для этого и существует язык расслоений и связностей, к знакомству с которым мы теперь и приступаем.

§6.

Мы начнем с простого примера расслоения, который вам очень хорошо знаком.

Пример 1. Рассмотрим функцию $f : \mathbb{R}^4 \to \mathbb{R}^2$. Ее можно представить в виде графика в пространстве $\mathbb{R}^4 \times \mathbb{R}^2$. Это множество точек $s = (x, f(x))$, где $x = (x^1, x^2, x^3, x^4)$. Имеется естественная проекция π пространства $\mathbb{R}^4 \times \mathbb{R}^2$, состоящего из точек (x, y), на $\mathbb{R}^4 : (x, y) \mapsto x$ (рис. 1). Наша функция есть не что иное, как отображение s из базы - пространства $\mathbb{R}^4$ - В пространство $\mathbb{R}^4 \times \mathbb{R}^2$; при этом $\pi \circ s = \mathrm{id}$. Всякое отображение s, обладающее таким свойством, называется сечением. А сама конструкция, которую мы построили, называется (тривиальным) расслоением или прямым произведением. Теперь мы обобщим приведенную в примере конструкцию. Рассмотрим покрытие $\mathbb{R}^4$ открытыми множествами U_i. Пусть для каждого непустого пересечения $U_i \cap U_j$ задана некоторая функция со значениями в группе невырожденных матриц размера 2×2:

$$g_{ij} : U_i \cap U_j \to \mathrm{GL}(2; \mathbb{R}).$$

Если набор функций $\{g_{ij}\}$ обладает свойствами

1. $g_{ji}(x) = (g_{ij}(x))^{-1}$,
2. $g_{ij}(x) \cdot g_{jk}(x) \cdot g_{ki}(x) = I \quad (\text{ если } U_i \cap U_j \cap U_k \neq \varnothing)$,

Рис. 1

то этот набор называют склеивающим коциклом (здесь $(g_{ij}(x))^{-1}$ - это матрица, обратная к $g_{ij}(x); I$ - единичная матрица). При помощи склеивающего коцикла можно построить новый объект, который локально похож на сконструированное в предыдущем примере прямое произведение.

Рис. 2 «Кирпичиками» в нашей конструкции будут пространства $U_i \times \mathbb{R}^2$, которые мы будем склеивать по множествам $(U_i \cap U_j) \times \mathbb{R}^2$. Есть очень простой способ склейки: точка $(x, \vec{\xi})$ из $U_i \times \mathbb{R}^2$ склеивается (отождествляется) с точкой $(x, \vec{\xi})$ из $U_j \times \mathbb{R}^2$. Тогда интуитивно ясно, что мы получим в точности прямое произведение. Давайте поступим иначе - склеим цилиндры «с подкруткой»: точку $(x, \vec{\xi})$ отождествим с точкой $\left(x, g_{ij}\vec{\xi}\right)$ (рис. 2). Ограничения, наложенные на функции g_{ij}, обеспечивают корректность такой склейки. Свойство 1) означает, что если точка $(x, \vec{\xi})$ склеивается с некоторой точкой $(x, \vec{\eta})$, то точка $(x, \vec{\eta})$ склеивается с точкой $(x, \vec{\xi})$. Если у нас имеется пересечение трех

цилиндров, то мы склеиваем точку из первого цилиндра с точкой из второго цилиндра, ту, в свою очередь, — с точкой из третьего цилиндра, а ее с точкой из первого цилиндра. Свойство 2) гарантирует нам, что после таких склеек точка первого цилиндра перейдет в себя.

После склейки мы получим некоторое пространство P, сконструированное из этих цилиндриков, у которого будет естественная проекция π на базу $\mathbb{R}^4$. Причем эта проекция корректно определена для всех цилиндриков, и склейка ее не портит.

Полученный объект называют векторным расслоением с базой $\mathbb{R}^4$, пространством расслоения P, слоем $\mathbb{R}^2$ и структурной группой $\mathrm{GL}(2;\mathbb{R})$. Возьмем теперь другой склеивающий коцикл и построим соответствующее ему векторное расслоение P' с отображением проектирования π'. Если существует взаимно-однозначное гладкое в обе стороны линейное на каждом слое отображение F из P в P', такое что диаграмма

$$\begin{tikzcd} P \arrow[rr, "F"] \arrow[dr, "\pi"'] & & P' \arrow[dl, "\pi'"] \\ & R^4 & \end{tikzcd}$$

коммутативна (т. е. $\pi' \circ F = \pi$), то говорят, что расслоения P и P' эквивалентны. Расслоения, эквивалентные прямому произведению, называют тривиальными. Можно доказать, что если в качестве базы выступает пространство $\mathbb{R}^4$, то все расслоения будут тривиальными. Но ясно, что в приведенной конструкции конкретный вид базы (в качестве которой мы выбрали для простоты пространство $\mathbb{R}^4$) и размерность слоя (пространства $\mathbb{R}^2$ в нашем случае) ни при чем. В качестве базы может выступать любое многообразие, а в качестве слоя — евклидово пространство любой размерности.

Приведем пример нетривиального расслоения с базой S^1 (окружность).

Пример 2. Рассмотрим два экземпляра цилиндра $S^1 \times \mathbb{R}$ и выбросим из первого цилиндра прямую $\{x\} \times \mathbb{R}$, а из второго - прямую $\{y\} \times \mathbb{R}$, где x и y - различные точки окружности S^1. Получим два кирпичика $U_1 \times \mathbb{R}$ и $U_2 \times \mathbb{R}$, где $U_1 = S^1 \backslash \{x\}$, а $U_2 = S^1 \backslash \{y\}$. Пересечение $U_1 \cap U_2$ состоит из двух компонент V_1 и V_2. Если $z \in V_1$, то склеим точки (z,t) и (z,t). Если же $z \in V_2$, то склеим наши кирпичики следующим образом: точка (z,t) первого кирпичика склеится с точкой $(z,-t)$ второго, т. е. в нашем случае склеивающий коцикл состоит из одного элемента g_{12}, тождественно равного единице на V_1 и равного -1, если $z \in V_2$. Структурная группа является группой $\mathrm{GL}(1;\mathbb{R})$, т. е. группой ненулевых действительных чисел с операцией умножения.

Пространство этого расслоения - поверхность, похожая на лист Мёбиуса, только бесконечная. Она не гомеоморфна цилиндру $S^1 \times \mathbb{R}$, поскольку цилиндр - ориентируемое многообразие, а полученное нами пространство расслоения - нет. Значит, это расслоение нетривиально.

Сечение произвольного расслоения определяется точно так же, как и в разобранном случае прямого произведения: это такое отображение s из базы в пространство расслоения, что $\pi \circ s = id$. Сечения расслоения образуют модуль над кольцом гладких функций базы: их можно складывать (покоординатно и поточечно) и умножать на функции.

§7.

У прямого произведения есть следующее замечательное свойство: сечение прямого произведения можно дифференцировать вдоль любого векторного поля $\vec{v}$ базы. А именно, возьмем сечение $(x, f(x))$ и поставим ему в соответствие новое сечение $(x, \partial_v f)$, где $\partial_v f = df(\vec{v}))$. Выражение df есть векторнозначная 1 -форма вида $(df^1, \ldots, df^4)$; ее значение на векторном поле $\vec{v}$ - гладкая вектор-функция $((df^1(\vec{v}), \ldots, df^4(\vec{v}))$. А ее график (т. е. сечение $(x, df(\vec{v}))$) и будет искомой производной $(x, \partial_v f)$ исходного сечения вдоль $\vec{v}$.

Можно ли похожим образом научиться дифференцировать сечения произвольного расслоения? На сечение расслоения можно смотреть как на согласованную совокупность

(x, f_i) сечений кирпичиков $U_i \times \mathbb{R}^2$. Условие согласованности состоит в том, что эти локальные сечения склеиваются в глобальное, т. е. для любой точки $x \in U_i \cap U_j$ имеет место тождество $f_i(x) = g_{ij}(x) f_j(x)$.

Давайте продифференцируем каждое такое элементарное сечение обычным образом вдоль векторного поля. Нужно только проверить, что полученные локальные сечения вновь корректно склеиваются:

$$\partial_v f_i = \partial_v \left(g_{ij} f_j\right) = \left(\partial_v g_{ij}\right) f_j + g_{ij} \partial_v f_j.$$

Если бы члена $\left(\partial_v g_{ij}\right) f_j$ не было, сечения бы склеивались и при склеивании давали глобально определенное, без разрывов, сечение. Но если функции g_{ij} зависят от x, то это определение ни к чему хорошему не приводит. Заметим, что невязка этой формулы — член, линейный по f_j. Давайте чуть-чуть подправим определение производной вдоль векторного поля, добавив такую линейную по f часть, чтобы новые сечения склеились. А именно, определим новую производную, которая будет называться ковариантной производной вдоль векторного поля $\vec{v}$:

$$\nabla_v f_i = \partial_v f_i + A_v^i f_i$$

где A_v^i - матрица размера 2×2 (здесь суммирование по индексу i не производится). Мы постараемся подобрать матрицы A_v^i (свою для каждого кирпичика) так, чтобы в итоге после применения ковариантной производной все локальные сечения склеились в одно.

Распишем определение ∇_v:

$$\begin{gathered}\nabla_v f_i = \partial_v f_i + A_v^i f_i = \partial_v \left(g_{ij} f_j\right) + A_v^i \left(g_{ij} f_j\right) = \left(\partial_v g_{ij}\right) f_j + g_{ij} \partial_v f_j + A_v^i g_{ij} f_j = \\ = g_{ij} \left(g_{ij}^{-1} \left(\partial_v g_{ij}\right) f_j + \partial_v f_j + g_{ij}^{-1} A_v^i g_{ij} f_j\right) = g_{ij} \nabla_v f_j.\end{gathered}$$

Последнее равенство необходимо для того, чтобы локальные сечения склеились. Выражение $\nabla_v f_j$ должно содержать член $\partial_v f_j$, следовательно, сумма остальных членов должна равняться $A_v^j f_j$:

$$A_v^j = g_{ij}^{-1} \partial_v g_{ij} + g_{ij}^{-1} A_v^i g_{ij}$$

Отсюда

$$A_v^i = -\partial_v g_{ij} g_{ij}^{-1} + g_{ij} A_v^j g_{ij}^{-1}.$$

Итак, если удается найти матричные функции A_v^i, удовлетворяющие полученному условию, то можно ввести операцию дифференцирования сечений вдоль векторного поля $\vec{v}$.

Но как быть, если надо продифференцировать сечения вдоль другого векторного поля? Искать каждый раз подходящие матричные функции было бы довольно затруднительно. Кроме того, естественно было бы ожидать, что $\nabla_{v_1} f + \nabla_{v_2} f = \nabla_{v_1+v_2} f$, как в случае обычных функций. Поэтому давайте потребуем, чтобы на каждой окрестности U_i была задана матричная дифференциальная форма (т. е. матрица размера 2×2, все элементы которой являются 1-формами) ω_i так, что

$$\omega_i = -dg_{ij} g_{ij}^{-1} + g_{ij} \omega_j g_{ij}^{-1}$$

для любых пересекающихся U_i и U_j.

Тогда оказывается, что мы можем дифференцировать любое сечение вдоль любого векторного поля. Для этого нужно лишь положить

$$A_v^i = \omega_i(\vec{v})$$

Определение 2. Говорят, что в векторном расслоении задана связность, если для любого набора окрестностей и склеивающих коциклов, по которым можно построить это расслоение, задан набор дифференциальных матричных 1 -форм $\{\omega_i\}$, удовлетворяющих свойству $(*)$. Связность обозначается обычно значком ∇. Форма ω_i называется локальной формой связности.

Связность соотносится с ковариантной производной примерно так же, как дифференциал функции с частной производной. Если s - сечение расслоения, имеющее в цилиндре $U_i \times \mathbb{R}^2$ вид $(x, f_i(x))$, то

$$\nabla s = (x, df_i + \omega_i f_i)$$

Другими словами, если мы применим к сечению не ковариантное дифференцирование, а оператор связности, то мы получим сечение с коэффициентами в 1-формах. Проиллюстрируем это диаграммой:

$$\nabla : \{ \text{ сечение } \} \to \{ \text{ сечение } \} \otimes \Lambda^1 \left(\mathbb{R}^4\right).$$

Итак, связность в векторном расслоении — это просто возможность дифференцировать сечения. Связность является линейным отображением (т. е. $\nabla\left(s_1 + s_2\right) = \nabla s_1 + \nabla s_2$) и удовлетворяет правилу Лейбница: для любого $f \in \mathcal{F}\left(\mathbb{R}^4\right)$ и для любого сечения s имеем $\nabla(fs) = (df)s + f\nabla s$.

Это инвариантное определение связности: говорят, что на расслоении задана связность, если любому сечению ставится в соответствие $\{$ сечение $\}$ $\{$дифференциальная форма $\}$, и это отображение линейно и удовлетворяет правилу Лейбница. Легко показать что такое определение связности эквивалентно данному выше «координатному».

Рассмотрим снова цилиндр $U_i \times \mathbb{R}^2$. В нем есть стандартный базис сечений $\xi_1 = (x, e_1), \xi_2 = (x, e_2)$ (где e_1 и e_2 образуют стандартный базис в $\mathbb{R}^2$). Любое сечение $s = (x, f_i(x))$ над U_i может быть записано в виде $s = f_i^1\xi_1 + f_i^2\xi_2 = (\xi_1, \xi_2)\, f_i$ (где последнее выражение понимается как произведение строки на столбец). Для построенных сечений имеем

$$\nabla\xi_j = (x, \nabla e_j) = (x, \omega_i e_j) = \left(x, \omega_i^j\right),$$

где через ω_i^j обозначен j-й столбец матричной дифференциальной формы ω_i.

Следовательно, $(\nabla\xi_1, \nabla\xi_2) = (\xi_1, \xi_2)\,\omega_i$ (вектор-строка умножается на матрицу), поэтому на матрицу ω_i можно смотреть как на матрицу связности в базисе (ξ_1, ξ_2).

Теперь делается более понятным смысл выражения $(**)$. Его можно переписать в виде

$$\nabla f_i = df_i + \omega_i f_i.$$

Это действие связности на координатах сечения s (по отношению к базисным сечениям ξ_1, ξ_2).

Во всяком ли векторном расслоении можно ввести связность? Ответ на этот вопрос дает следующее утверждение, которое мы оставляем читателю в качестве упражнения.

В любом гладком векторном расслоении можно ввести связНосmb.

(Указание: Рассмотрите гладкое разбиение единицы $\{\lambda_i\}$, подчиненное покрытию $\{U_i\}$; так как над каждым U_i расслоение имеет вид прямого произведения, в ограничении расслоения на U_i можно ввести некоторую связность ∇_i; покажите, что линейная комбинация $\sum \lambda_i \nabla_i$ является корректно определенной связностью в исходном расслоении.)

Сечение s называется горизонтальным (по отношению к связности ∇), если $\nabla s = 0$.

Из $(**)$ следует, что в базисе ξ_1, ξ_2 локальных сечений расслоения над U_i координаты горизонтальных сечений удовлетворяют следующей системе дифференциальных уравнений:

$$df_i = -\omega_i f_i$$

Если расслоение тривиально, то в базисе ξ_1, ξ_2 глобальных сечений расслоения координаты горизонтальных сечений удовлетворяют следующей системе дифференциальных уравнений:

$$df = -\omega f.$$

В другом базисе сечений (при другой тривиализации расслоения) ξ_1', ξ_2' согласно $(*)$ уравнение горизонтальных сечений принимает вид

$$df' = -\omega' f', \quad \text{где} \quad f' = \Gamma y, \quad \omega' = -d\Gamma\Gamma^{-1} + \Gamma\omega\Gamma^{-1}$$

(и где в свою очередь $(\xi_1, \xi_2) = (\xi_1', \xi_2')\,\Gamma$). Заметим, что такие преобразования называются калибровочными преобразованиями формы связности.

Итак, связность в тривиальном расслоении задает систему дифференциальных уравнений с точностью до калибровочных преобразований.

Если же расслоение нетривиально, то связность задает согласованное (в смысле соотношений $(*)$) семейство систем дифференциальных уравнений.

§8.

Попробуем продолжить диаграмму дифференцирования сечений:

$$\{\text{ сечение }\} \xrightarrow{\nabla} \{\text{ сечение }\} \otimes \Lambda^1(\mathbb{R}^n) \xrightarrow{\tilde{\nabla}} \{\text{ сечение }\} \otimes \Lambda^2(\mathbb{R}^n),$$

где отображение $\widetilde{\nabla}$ каждому сечению с коэффициентами в 1-формах ставит в соответствие сечение с коэффициентами в 2-формах. Мы также потребуем, чтобы для него выполнялось правило Лейбница:

$$\tilde{\nabla}(f\alpha) = df \wedge \alpha + f\tilde{\nabla}\alpha, \quad \tilde{\nabla}(\beta s) = (d\beta)s - \beta \wedge \nabla s$$

для произвольной функции f, сечения s и 1 -формы β. Определим сквозное отображение $K = \widetilde{\nabla} \circ \nabla$.

Лемма. Отображение K является $\mathcal{F}$-линейным.

Д о к а з а т е л ь с т в о. Пусть f - функция, s - сечение. Тогда

$K(fs) = \widetilde{\nabla}(\nabla(fs)) = \widetilde{\nabla}((df)s + f\nabla s) = (d^2 f)\, s - df \wedge \nabla s + df \wedge \nabla s + fK(s) = fK(s)$, поскольку $d^2 f = 0$, а второй и третий члены сокращаются. Отображение K называется тензором кривизны связности. Локально над U_i в базисе ξ_1, ξ_2 тензор кривизны можно записать в виде

$$K(\xi_1, \xi_2) = (\xi_1, \xi_2)\, \Omega_i,$$

где Ω_i - матрица дифференциальных 2 -форм.

Вычислим эту матрицу в нашем базисе через ω_i:

$$\begin{gathered}\tilde{\nabla}(\nabla(\xi_1, \xi_2)) = \widetilde{\nabla}((\xi_1, \xi_2)\,\omega_i) = (\nabla\xi_1, \nabla\xi_2) \wedge \omega_i + (\xi_1, \xi_2)\, d\omega_i = \\ = (\xi_1, \xi_2)(\omega_i \wedge \omega_i) + (\xi_1, \xi_2)\, d\omega_i = (\xi_1, \xi_2)(\omega_i \wedge \omega_i + d\omega_i).\end{gathered}$$

Сравнивая коэффициенты при базисных векторах слева и справа, мы получаем уравнение $\Omega_i = d\omega_i + \omega_i \wedge \omega_i$. (Внешнее произведение $\omega_i \wedge \omega_i$ не обязано равняться нулю, поскольку ω_i - матричная форма.) Полученное уравнение называется структурным уравнением. Дифференцируя это уравнение, мы получим $d\Omega_i = \Omega_i \wedge \omega_i - \omega_i \wedge \Omega_i$. Это равенство носит название тождество Бьянки. Введя новый оператор $D\Omega_i = d\Omega_i - \Omega_i \wedge \omega_i + \omega_i \wedge \Omega_i$ (его называют ковариантной производной формы), данное выражение можно переписать в виде

$$D\Omega_i = 0.$$

\$9.

По аналогии с конструкцией векторного расслоения рассмотрим теперь главное расслоение. Для этого вместо $\mathbb{R}^2$ в наших кирпичиках возьмем $\mathrm{GL}(2;\mathbb{R})$, т. е. в качестве элементарных цилиндров будем рассматривать цилиндры $U_i \times \mathrm{GL}(2;\mathbb{R})$. Склеивать мы их будем точно так же, с помощью коцикла $\{g_{ij}\}$:

$$(x, G) \sim (x, g_{ij}G), \qquad \text{если } x \in U_i \cap U_j.$$

Только умножать матрицу g_{ij} нам придется уже не на вектор, а на другую матрицу G. Расслоение, у которого структурная группа и слой — это одно и то же пространство, называют главным расслоением.

В главном расслоении также можно ввести связность, формально воспользовавшись данным ранее определением (как совокупность согласованных с помощью соотношения (*) матричных дифференциальных форм). Но ее уже нельзя будет рассматривать как дифференцирование вектор-сечений. Геометрическая интерпретация связности в главном расслоении состоит в том, что связность выделяет в касательном пространстве

к каждой точке пространства расслоения подпространство так называемых горизонтальных векторов, гладко зависящее от точки. Объем данной брошюры не позволяет нам познакомиться подробнее с этой интерпретацией. Отметим лишь еще раз, что аналогично случаю векторного расслоения в главном расслоении можно ввести кривизну связности и выписать структурное уравнение и тождество Бьянки.

$10.

Вернемся вновь к уравнениям Максвелла. Оказывается, на электромагнитное поле можно смотреть как на связность в главном раслоении над $\mathbb{R}^4$ со структурной группой и слоем $\mathrm{U}(1) \subset \mathrm{GL}(1;\mathbb{C})$ ($\mathrm{U}(1)$ - это группа комплексных чисел, равных по модулю 1, т. е. геометрически это окружность S^1).

В этом расслоении функции склейки g_{ij} принимают значение в группе $\mathrm{U}(1)$, а формы связности и кривизны становятся скалярными, а не матричными. Поэтому для локальных форм связности и кривизны связности получаем: $\omega_i \wedge \omega_i = 0$ и $\Omega_i \wedge \omega_i - \omega_i \wedge \Omega_i = 0$. Структурное уравнение принимает при этом вид $\Omega_i = d\omega_i$, а тождество Бьянки $-d\Omega_i = 0$.

Поэтому локальную форму связности $\omega_i = A^i_\mu dx^\mu$ мы будем интерпретировать как локальный 4-потенциал в данной калибровке, т. е. в данной локальной тривиализации расслоения над U_i, локальную форму кривизны Ω_i - как тензор электромагнитного поля в данной калибровке, а тождество Бьянки превратится в фарадееву часть уравнений Максвелла.

При переходе от одной локальной формы связности к другой (т. е. при замене калибровки) получим согласно соотношению $(*)$ (и согласно тому, что функции склейки - скалярные):

$$\omega_i = -dg_{ij}g_{ij}^{-1} + g_{ij}\omega_j g_{ij}^{-1} = \omega_j + dS_{ij},$$

где $S_{ij} = d\log(-g_{ij})$, или в терминах компонент формы ω_i:

$$A^i_\mu = A^j_\mu + \frac{\partial S_{ij}}{\partial \mu}$$

Согласно этой формуле, компоненты локальной формы связности преобразуются точно так же, как компоненты 4-потенциала электромагнитного поля при калибровочных преобразованиях, что еще раз подтверждает правильность нашей интерпретации.

Итак, в случае главного расслоения все наши формулы могут быть описаны на двух языках: геометрическом и физическом. Составим соответствующий словарик:

Выбор локального цилиндра (локальная тривиализация)	$U_i \times G$ $(U_i \times S^1)$	Калибровка
Локальная форма связности	$\omega_i = A^i_\mu dx^\mu$	4-потенциал электромагнитного поля в данной калибровке
Связность	∇	Электромагнитное поле
Локальная форма кривизны связности	Ω_i	Тензор электромагнитного поля в данной калибровке
Тензор кривизны связности	K	Тензор электромагнитного поля
Тождество Бьянки	$D\Omega_i = 0$	Фарадеева часть уравнений Максвелла
Пространство расслоения	$P(\mathbb{R}^4 \times S^1)$	Фазовое пространство физической системы
База расслоения	$B(\mathbb{R}^4)$	Пространство-время
Структурная группа расслоения	$G(\mathrm{U}(1))$	Группа «вращений» в зарядовом пространстве.

Последняя строчка словарика требует пояснения. Дело в том, что структурная группа $\mathrm{U}(1)$ появляется в описании электромагнитного поля в связи с тем, что физическая ситуация не должна зависеть от выбора фазы (выбора направления в зарядовом пространстве), что соответствует инвариантности относительно калибровочных преобразований вида $g_{ij} = \exp(is(x))$ со значениями в $\mathrm{U}(1)$.

§11.

Случай электромагнитного поля — в каком-то смысле простейший из возможных, так как структурная группа расслоения в этом случае очень проста. Однако этим примером не исчерпывается описание физических полей в терминах расслоений и связностей.

Другой известный пример - это знаменитые поля Янга-Миллса, служащие для описания поведения нуклонов (протонов и нейтронов). Эти поля интерпретируются как связности в главном расслоении со структурной группой $G = \mathrm{SU}(2)$ (так обозначается группа всех невырожденных комплексных матриц размера 2×2, определитель которых равен 1). В этом случае все формы связности и кривизны являются матричными, и те упрощения, которые были возможны в случае электромагнитного поля, здесь не проходят. Однако выписанный нами словарик «работает» и для этих полей.

Исследуя подобного рода модели, физики научились вычислять такие величины, как масса протона и масса нейтрона, просто решая с помощью компьютера соответствующие уравнения. Но это тема совсем другой лекции.

Обзор приложений электромагнитное поле а дифференциальных формах

(очень многие укажу, но сперва дорасти до таких задач нужно, скорее всего только в ОТО они и используются, я хз, где еще?)

Другое про электромагнитное поле а дифференциальных формах

формулировка уравнений Максвелла в бескоординатном методе

$$\begin{aligned}\omega_E =& E_x(x,y,z)dy \wedge dz + E_y(x,y,z)dz \wedge dx + E_z(x,y,z)dx \wedge dy \\ \lambda_H =& H_x(x,y,z)dx + H_y(x,y,z)dy + H_z(x,y,z)dz\end{aligned}$$

Тогда дифференциал формы имеет вид:

$$d\omega_E = \frac{\partial E_x}{\partial x} dx \wedge dy \wedge dz + \frac{\partial E_y}{\partial y} dy \wedge dz \wedge dx + \frac{\partial E_z}{\partial z} dz \wedge dx \wedge dy$$

сгруппировав члены, имеем $d\omega_E = \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dx \wedge dy \wedge dz$

Дифференциал магнитного поля имеет вид

$$d\lambda_H = \frac{\partial H_x}{\partial y} dy \wedge dx + \frac{\partial H_x}{\partial z} dz \wedge dx + \frac{\partial H_y}{\partial x} dx \wedge dy + \frac{\partial H_y}{\partial z} dz \wedge dy + \frac{\partial H_z}{\partial x} dx \wedge dz + \frac{\partial H_z}{\partial y} dy \wedge dz$$

или

$$d\lambda_H = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right) dy \wedge dz + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right) dz \wedge dx + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right) dx \wedge dy$$

очевидные плюсы такого подхода

пока не вижу, потом мб увижу

решения

хз как решать

26.4 Geometry of Gravity and Relativity

Обсудим, каким образом дифференциальная геометрия используется в ОТО.
(потом мб как-то разделю, пока так вот)

26.4.1 Essence of Oto

Известно, что в криволинейных координатах инертная масса ведет себя также, как гравитационная, и если ввести 4 мерное пространство время, то гравитационные силы могут быть описаны за счет введения тензора кривизны.

и таким образом, получаем описание гравитации за счет изменения метрики пространства-времени.

26.4.2 Examples of Manifolds in Gr and Sr

(мб потом увеличу раздел, пока такой вот он.)

Examples of Manifolds in Gr and Sr

(вот соберу их и все, пока опыта мало, проблем тут полно, привести примеры четко многие.)

геометрия пространства Минковского

(выпишу из Мищенко Фоменко, потом добавлю в теорию поля)

геометрия пространства де Ситтера

потом займусь и свяжу с ОТО.

пока что я вообще хз.

Обсудим математические и всевозможные другие различные приложения дифференциальной геометрии.

(пока так, на всякий случай часть висит, мб удалю)

27 Other physical applications

(потом напишу мб еще какие-то)

27.1 Geometry in Special Field Theories

(список тем из какого-то теорфизного курса, скорее всего от кафедры Белавина, потом посмотрю, пока учу основы. Мб это курс Рослого для умных)

27.2 Geometry in Gravitational Lensing

(тут много про это будет, скоррелирую, кстати, с теорией выше.)

28 Other applications

28.1 philosophical Applications

смысл многообразий

типа прямо как наше видение мира, с разных сторон можно смотреть на него.

то се.

Part VII
Appendix

A Introduction and overview of the diffem

A.1 Another Motivation for Differential Geometry

A.1.1 Other Common Motivation

A Brief Introduction by Tu to manifolds

Undergraduate calculus progresses from differentiation and integration of functions on the real line to functions in the plane and in 3 -space. Then one encounters vectorvalued functions and learns about integrals on curves and surfaces. Real analysis extends differential and integral calculus from $\mathbb{R}^3$ to $\mathbb{R}^n$. This book is about the extension of calculus from curves and surfaces to higher dimensions.

The higher-dimensional analogues of smooth curves and surfaces are called manifolds. The constructions and theorems of vector calculus become simpler in the more general setting of manifolds; gradient, curl, and divergence are all special cases of the exterior derivative, and the fundamental theorem for line integrals, Green's theorem, Stokes's theorem, and the divergence theorem are different manifestations of a single general Stokes's theorem for manifolds.

Higher-dimensional manifolds arise even if one is interested only in the threedimensional space that we inhabit. For example, if we call a rotation followed by a translation an affine motion, then the set of all affine motions in $\mathbb{R}^3$ is a sixdimensional manifold. Moreover, this six-dimensional manifold is not $\mathbb{R}^6$.

We consider two manifolds to be topologically the same if there is a homeomorphism between them, that is, a bijection that is continuous in both directions. A topological invariant of a manifold is a property such as compactness that remains unchanged under a homeomorphism. Another example is the number of connected components of a manifold. Interestingly, we can use differential and integral calculus on manifolds to study the topology of manifolds. We obtain a more refined invariant called the de Rham cohomology of the manifold.

Our plan is as follows. First, we recast calculus on $\mathbb{R}^n$ in a way suitable for generalization to manifolds. We do this by giving meaning to the symbols dx, dy, and dz, so that they assume a life of their own, as differential forms, instead of being mere notations as in undergraduate calculus.

While it is not logically necessary to develop differential forms on $\mathbb{R}^n$ before the theory of manifolds-after all, the theory of differential forms on a manifold in Chapter 5 subsumes that on $\mathbb{R}^n$, from a pedagogical point of view it is advantageous to treat $\mathbb{R}^n$ separately first, since it is on $\mathbb{R}^n$ that the essential simplicity of differential forms and exterior differentiation becomes most apparent.

Another reason that we do not delve into manifolds right away is so that in a course setting the students without a background in point-set topology can read Appendix A on their own while studying the calculus of differential forms on $\mathbb{R}^n$.

Armed with the rudiments of point-set topology, we define a manifold and derive various conditions for a set to be a manifold. A central idea of calculus is the approximation of a nonlinear object by a linear object. With this in mind, we investigate the relation between a manifold and its tangent spaces. Key examples are Lie groups and their Lie algebras.

Finally, we do calculus on manifolds, exploiting the interplay of analysis and topology to show on the one hand how the theorems of vector calculus generalize, and on the other hand, how the results on manifolds define new C^∞ invariants of a manifold, the de Rham cohomology groups.

The de Rham cohomology groups are in fact not merely C^∞ invariants, but also topological invariants, a consequence of the celebrated de Rham theorem that establishes an isomorphism between de Rham cohomology and singular cohomology with real coefficients. To prove this theorem would take us too far afield. Interested readers may find a proof in the sequel [4] to this book.

Приложения в теоретической физике

(потом раскрою)

В принципе большая вероятность появления в теоретической физики новой опирающейся на дифференциальную геометрию теории

(и всё, приехали, если диффгем не сделан. минус пара месяцев в лучшем случае. поэтому лучше уж иметь заготовки, потому что чуть что их переучивать не нужно будет - плюс пара месяцев к скорости. нпр у Тонга, Мизнера, Вергелеса по ОТО полно всего, что сполна на диффгем опирается.)

Приложения в теории струн

(это очень мотивирует изучать диффгем, так что укажу потом их)

Множество применений в ОТО

(потом раскрою, тут очень многое есть)

Углубленные понимание теории поля

некоторые темы такие

физики очень часто говорят про дифференциальную геометрию

поэтому если не понимаешь диффгем, то общение с ними будет отнимать слишком много сил.
и это аргумент изучить ее, потому что по факту темы не сложные,
так что правильнее собрать эту теорию.

Углубленные понимание механики

до Арнольда пока не дошел, но однажды дойду

Продвинутые математические методы

Возможность описывать кривые и поверхности

именно диффгем дает этот аппарат.

Общая научная эрудиция

(потом напишу)

дифференциальная геометрия - фундаментальная часть математики

A.1.2 Motivation to Sections and to Study the Features of the Diffhem

Мотивация к изучению аффинных пространств

(хз)

Мотивация к углубленным типичным разделам

мотивация к когомологиям

пропишем, а то непонятно, зачем заниматься ими.

Мотивация к отдельным огромным разделам

Мотивация к K-теории

хз

Мотивация к углубленным расслоениям

хз вообще, какая-то там теория, не знаю, зачем она

Мотивация к алгебраической топологии

хз вообще.

Мотивация к бескоординатному языку

я вот хз, так и не въехал, зачем всё на нем проходить.

A.2 The Mindset of Geometry Professionals

A.2.1 A Look at Differential Geometry

На самом деле, скучная или не скучаня дифференциальная геометрия?

(тут важные рассуждения, ответ тем, кто говорит, что это скучно. если коротко, должен быть правильный подход, тогда она будет не скучной. допишу потом подробно.)

Дифференциальная геометрия как набор типичных конструкций, которые просто нужно понять и иногда они могут сыграть ключевую роль

Такие полезные конструкции, если заниматься некоторыми теориями. Их не много, просто отдельно нужно каждую понять и все, можно даже не задаваться вопросом о связях их друг с другом. Но если человек выбрал просидеть много времени в одной теории, которая не связана с диффгемом, то в таком случае она не нужна и можно не интересоваться ей.

Подход физика к геометрии

Очень часто физическая теория имеет слишком много математики, которую и нужно собирать в отдельную запись

(именно поэтому я и прописываю параллельно диффгем. нпр читаешь Вергелеса и Тонга - куча всего, что совсем не имеет отношения к ОТО, но поясняет происходящее. именно в этой записи всё и будет собираться в единую картину.)

Особенности геометрии

Наличие больших отдельных разделов

(то есть это целый мир, где полным полно конструкций, в которых запросто утонуть. так что единая запись, которая их объединяет крайне полезна! а то поток книг по 300-600 страниц с теорией, которую почти впервые видишь, которая раз в жизни будет нужна - вот реальность теоретической физики. запись по диффгему крайне спасает ориентироваться и связи между темами прорабатавыть.)

О важности различных тем

О важности производной Ли

(хз, отдельно ответ на этот вопрос хотел бы записать. так и не понял, вроде вообще не важна, но нужно лучше разобраться.)

Особенности объяснения геометрии незнающим

(??? пока не тот уровень, чтобы про это думать)

Диффгем познает тот, кто его начнет применять, а для этого нужна вмеру общая подготовка

У других шансы в 10 раз меньше. А кто занимается разделами физики, где нужен диффгем, тот каждый день и будет тренироваться. А диффгем полон мелочей, так что тут много очень дней нужно им заниматься, тогда только будет им владение. Поэтому выводим физику на отработанный по упрощенным темам уровень - и тогда диффгем естественно и будет добавлен в научную деятельность. Тогда он и возьмется. Без этого дойти до профессионального уровня, не рассматривая себя как будущего специалиста по диффгему, очень очень маловероятно.

A.2.2 Ways to Guess the Main Ideas of Geometry

(незаменимая часть нормального понимания предмета. пока хз.)

после введения двух операций, можно посмотреть на их коммутатор и антикоммутатор

Володя говорит, что так можно догадаться до производной Ли, например.

A.2.3 The Real Importance of Differential Geometry

(I'll discuss this question here)

Насколько нужен диффгем в теории поля?

(обсужу приложения. вроде как нужен, но не уверен, четко такой обзор-обсуждение не готов писать)

A.2.4 The Exceptional

A.2.5 Thinking to Effectively Learn Geometry

Любители заумничать мало чему научат, не следует у них учиться, пока совсем много сам не прошел

В диффгеме, как и много где в математике таких можно встретить. Очень тупиково у них чему-то учиться. Можно было бы что-то понять, однако это очень низкоприоритетное дело, в реальности никогда до этого не дойдет, и это очень хорошо.

Поэтому учим диффгем вмеру быстро по нормальным книгами/лекциям и не паримся, пока не появилась потребность в этом. А то иначе можно на пустом месте месяцы и просидеть, в итоге только начать чувствовать себя очень неприятно.

Необходимые темы для изучения основ

Базовая общая физика, если изучающий физик

Я хочу подчеркнуть, что геометрия второстепенная, так что если физик начинает её учить, не имея базу по физике, то это ему не нужно. Только имея базу по физике, познав геометрию, он сможет стать лучше как физик.

Базовая линейная алгебра

Начало дифференциальной геометрии опирается на линейную алгебру, так что без неё основы не станут понятны. Опускать основы - решение хуже, чем просто подучить линейную алгебру.

Основы топологии

По топологии нужно иметь зрелую минимальную картину предмета.
иначе все-таки отлеты из диффгема будут большие.
(пока не четко понимаю это)

Необходимые предметы для изучения продвинутых тем

Темы для многообразий (?)

Дополнительные темы для изучения геометрии

Способы изучения геометрии

(пока не уверен в нем)

Полезно читать дополнения к физическим книгам, используемых геометрию

там шикарные короткие дополнения, помню, потом буду тренироваться.

Обход типичных тупиков в изучении диффгема

Не забывайте о тренировках, решении задач и думании о темах!

Очень важно тренироваться и думать самому. Самая типичная ошибка - просто перестать вдумываться.
(раскрою потом подробнее, на 1/2 страницы указание про это планирую.)

Книга/курс непонятный/слишком много времени занимает? Удалите и выкиньте все от него и начните изучать заново, по другому, более авторитетному курсу!

Просто удаляешь это и начинаешь с нормальной теории. Так у меня после курса Рослого было, только я не сразу это понял, казалось, что все-таки можно многое понять. Можно. Но это не стоит того, это слишком заумно. Задачи нормальные порешать можно, вот хорошая идея. А саму теорию с какими-то математическими заворотами нужно просто выкинуть, если она не заходит. И все, если так сразу сделаешь - то и пойдешь обучение дальше, а иначе будешь долго стоять.)

Важно понимать, что кривую теорию не спасти, самое лучшее - полностью выкинуть и начать заново. Если это примерно понятно, но нет веры в это, то будет тянуть держаться за кривую теорию, и это отнимет просто недели и даст крайне мало. Так что нужно быть

уверенным: да, плохую теорию только удаляем и все! Уже нет такого, что нет других учебников!

(напишу подробнее, компактно, что рыться в сложно написанном - плохое всегда дело.)

О неверном видении геометрии

Можно чего угодно неверного понимать, но тут я соберу самые типичные заблуждения.

Неверно видеть, что геометрия - центр мира

только раздражает, когда кто-то так говорит, обоснования потом никогда не дождешься, хотя в какой-то небольшой степени это и правда.

(пока мысль не раскрыта.)

A.2.6 How one Should not Study Differential Geometry? (!!!)

(впишу разные соображения, чтобы не тратить время на разных обещающих обучить диффгему. Всякого бредового говорят они, потом оказывается, что просто не нужно было так обучать и всё.)

О неверных способах изучения геометрии

(мб свой опыт от Рослого пропишу, ибо я точно страдал херней, когда был у него на семинарах)

Ошибка - начинать с алгебраичного или слишком детального подхода

Потому что так ничего не поймешь. Запутаться тут легко, а вот начать применять - на самом деле сложно. И вообще, маловероятно, что нужен такой подход. К тому же скорее всего преподаватель, который так обучает, бестолковый, поэтому пользы от такого подхода скорее всего не будет.

Это я понял, когда очень малополезно потратил время на курс Рослого.

Ошибка - слушать лекции того, кто дает материал не в близком тебе подходе

Слушать другие структуры на начальных этапах бесполезно.

хотя и поняте "близкое" растяжимое.

просто забей на такие подходы и всё, итак в диффегеме заморочек выше крыши.

просто не выплыть, если пробовать далекие от твоего видения структуры собирать в порядок.

так что просто забывай всё, что знаешь от таких и начинай учиться по приятными источникам материала.

A.3 Acknowledgements

Currently, no one except me has worked on the sections of this note (with the exception of sections taken from books).

A.4 Literature

A.4.1 Main Literature

Основная обучающая

Tong D. Lectures on general relativity

много чисто диффгемовских конструкций, объем меньше, чем в учебниках по диффгему, так что она в приоритете!

[1] А.О.Иванов, А.А.Тужилин Лекции по классической дифференциальной геометрии часть 1, 2004

Полезные лекции об основах, много примеров и вроде очень крутые лекции. Про кривые и поверхности потом пропишу по ней структуру, пока что делать нужно основы. Нужно сперва Тонга пройти, потом будет очень хорошим дополнением.

(?? пока таких основных книг еще не знаю, потом точно добавлю.)

Основная профессиональная

[2] L Tu An Introduction to Manifolds

говорят, хорошая книга, но крайне большая. Потом буду смотреть её, как минимально необходимые книги будут пройдены.

[3] Катанаев М. О. Геометрические методы в математической физике

Большая книга, из которой многое обязательно возьму, пока этап сборки самых основ. Также нужно подготовиться к ней по умф и теорфизике. Уже некоторые выгрузки делал.

Основная по приложениям в физике

Тахтаджаян Л. А. Квантовая механика для математиков

Много диффгема и математической квантовой механики, которая дает ответы на многие математические вопросы. Нужно месяца 2 на изучение ее, если пройдены базовые курсы математики и квантовой механики.

Mikio Nakahara Geometry, Topology And Physics Second Edition

Книга, где много диффгема, а также приложений в физике, особенно теоретической. Вот и можно сидеть месяц только над ней, многое разобрать и понять.

A.4.2 Complementary and In-depth Literature

Другая обучающая

Thurston W.P. Geometry and topology of 3-manifolds

Тёрстон У. Трехмерная геометрия и топология 2001 310 страниц

Шикарная книга, очень многое хотел бы допонять по ней, автор очень авторитетный, так что будет время - подумаю. В интернете написано, что он умел представлять 4д пространство даже. Now I don't have time for this, it is an advanced and maybe very useless book.

https://empg.maths.ed.ac.uk/Activities/GT/ José Figueroa-O'Farrill Course of Gauge Theory

Very interesting and good lectures by a respectful professor. Maybe later I'll think a lot about them. Understanding of them will take at least a weak for me.

[4] Мищенко А. С, Фоменко А. Т. Краткий курс дифференциальной геометрии и топологии

Мб потом многое добавлю в запись, пока как-то будто можно и другими книгами обойтись.

[5] А.О.Иванов, А.А.Тужилин Лекции по классической дифференциальной геометрии часть 2

открывал, многое интересное, после основ буду дополнять ей структуру. Всякие профессиональные темы тут.

https://www.youtube.com/playlist?list=PLcZKkvA0SKb7eiK7QDWYIXz0J00GMXhQr Рослый Лекции по диффгему 2

То, что эти лекции записаны - единственное, что их спасает. Так что интересно их досматривать, но это только уже после того, как станет понятна более готовая для слушателя теория.

Литература по математике в помощь

Зорич Математический анализ ч. 2
Тоже очень много хорошего объяснения, просто там акцент на матане, а не на диффгеме. Добавлю тоже многое из нее, пока просто по ней в маатане пишу суть!
[6] Карасёв Р.Н. "Отдельные темы математического анализа"
Есть разделы с неплохим введением, но их мало и предполагается, что где-то эта теория используется, так что почитав эти разделы, читатель сразу продолжит думать над местом, где понадобились они.
[?] Иванов Г. Е. математический анализ 2
Не очень интересная книга, может, когда-то дополню запись соображениями из нее, пока вечно лень, но мб в целом будет полезно посидеть на ней тоже однажды.

Большие профессиональные книги по теоретической дифференциальной геометрии

Хьюзмоллер Расслоенные пространства и K-теории
Всякие закоулки диффгема, пока основы не сделаны, нет смысла открывать даже.

Статьи по геометрии для теории струн и конформной теории поля

[7] B. Craps, F. Roose, W. Troost, A. Van Proeyen What is Special Kähler Geometry?
Возможно, хорошая статья про кэлерову геометрию.

A.4.3 Literature on Applications

По теоретической и математической физике, использующей геометрию

Каку М. введение в теорию суперструн
очень крутые дополнения там, многое взял. (потом разберу)
[8] Арнольд В. И. "Математические методы классической механики"
Применения диффгема к механике, буду писать механику и диффгем по ней дорабатывать.

Литература о приложениях к другим разделам математики

Арнольд Геометрические методы в диффурах
мб дойду однажды, пока просто знаю, что есть такая книга
Зелкин (??)
там про грассмановы многообразия и какой-то еще пиздец

A.5 Overview of Differential Geometry

Обсудим основные элементы и особенности дифференциальной геометрии
(тут позже введение в другие разделы запишу)

A.5.1 Geometry in a Nutshell

Обсудим, что из себя представляет предмет наиболее кратко, выделяя самую суть.
(пока незрелый я, это пишу, как сама запись будет прописана)

появление Предмет в нашей картине мира

Что есть геометрия?

Многие те, кто знают диффгем, не могут на этот вопрос ответить, так что я пока что не нашел ответ.

один большой раздел

такой-то набор следствий

появление дифференциальной геометрии

A.5.2 review of Theoretical Approaches

что вообще в геометрии происходит?

A.5.3 Summary of Attachments

тут многое прописывать буду

A.5.4 Overview of Further Developments in Geometry

A.5.5 Brief Historical Overview

(интересно, но скорее всего никогда не напишу, потому что максимально не актуально)

A.5.6 Links With Other Mathematical Sciences

Связь с дифференциальными уравнениями

(почитаю Арнольда, пойму, пока я не знаю, нет смысла писать)
Связь огромная
Траектории - гладкие кривые, которые объекты диффгема.
Диффгем находит всю траекторию сразу.
Изначально у нас имеется многообразие и векторное поле.
Создаем векторное поле на многообразии.
Далее сдвигаемся и сравниваем с помощью производной Ли.

Связь с топологией

(пока связь с топологией не ясна)

О физике как части дифференциальной геометрии

(пообщаюсь с Володей, лучше пойму, как это так, проишу потом связки)

Важность понимания физических приложений

(пока я хз, почему это важно)

Важность понимания связей с механикой

(Володя кажется уже рассказывал, потом мб напишу)

A.6 Description of Record

A.6.1 Features of the Record as a Whole

описание записи в целом

О целесообразности выделять риманову геометрию в отдельный раздел

(так тонг делает, так что и я тоже сделал. но тут обсудим, есть ли смысл так делить. скорее всего, да... не знаю.)

Строгость формулировок

потом по идее все определения будут очень строгими, пока что не занимался этим.

A.6.2 Instructions

Часть о всей геометрии в двух словах

ей занимался больше всего, ибо такой мой уровень.

Часть о многообразиях

так до неё и нед дошел, хз, что там.

Часть про приложения в физике

по идее к теории поля и к ОТО подробно будет раскрыты они.
мб еще что-нибудь.

A.6.3 Legend

Обозначения ковариантной производной

есть такие:
$\nabla_{e_\alpha} := \nabla_\alpha$

Некоторые условные обозначения

(потом раскину)

$$a^\lambda_{;\mu} = (\nabla_\mu a)^\lambda$$

$$a^\lambda_{,\mu} = \partial_\mu a^\lambda$$

$u^\bullet$ - значит, что у u есть несколько компонент, и речь идет о каждой из них, стоящей на данное позиции.

метод формальных индексов

это индексные цифры Лашкевича
хз

Антисимметризация

пропишу тоже там квадратные скобки в индексах, типичное обозначение.

A.6.4 About Terminology

(мб напишу потом тоже)

A.7 Differential Geometry Puzzles

Обсудим в порядке интересности задачки и вопросы дифференциальной геометрии.
(с опытом соберу, пока не шарю. мб у Володи наберу интересные вопросы)

A.7.1 Selected Fun Puzzles

О приложениях дифференциальной геометрии в физике

A.7.2 Puzzles for Nerds

Обсудим головоломки, которые будут интересны тем, кому нравится копаться в деталях и кто отдельно изучал геометрию.

О том, в каких объектах мы работаем

Какие есть изометрии пространства Минковского, как R^4 с метрикой Минковского?

(я так и не понял, Володя понимает, потом подумаю, просто это не актуально максимально)
Не преобразования группы Пуанкаре
неверно это верно если ты Минковский будешь смотреть как аффинное пространства, и R^4 будет там действовать как группа параллельных переносов а вот до этого, исходя из честного определения оператора изометрии на евклидовом или банаховом пространстве это не следует нужно пополнение векторов точками без этого трансляций не будет а будет только группа Лоренца
именно поэтому изучаем аффинные пространства, чтобы дальше смотреть изометрии в общем смысле слова ну ладно, ты же не математик, тебе простительно

Почему мы работаем именно с кососимметричными функциями?

(не знаю, нужно переучить теорию, остался вопрос.)

Об определениях

что есть dx

Это базисная 1-форма
то се, пропишу потом

Что есть вектор?

потом запишу.

О том, что можно сконструировать, а что нельзя

На всяком ли многообразии можно задать метрику?

на всяком гладком - да.
берем локально конечное разбиение единицы, в каждом - задаем метрику.
вто и риманова структура.

B Mathematics for Diffhem

Обсудим математические конструкции, которые считаются используются в дифференциальной геометрии.

(!!! диффгем основывается на многих математических разделах, так что сперва их прописываю, потом тут в дополнениях буду указывать его!!! Это важно, потому что иначе тут под 200 страниц придется написать, что никуда не годится.)

B.1 Elements of Algebra for Diffhem

Линейная алгебра крайне важна для геометрии, особенно в начале её изучения. Приведем все необходимые такие элементы с указанием приложений.

(будут параграфы с приложениями)

(пока последовательность разделов рандомная)

B.1.1 About Duality

Обсудим двойственность - конструкцию, которая крайне часто встречается.

Двойственное пространство к векторному пространству

Алгебраически сопряженное

володя достал откуда-то, потом разберусь

О каноническом изоморфизме

раскрою, что есть он?

пока зависшая тема...

О строгом подходе к двойственности

или теория категорий тут будет, или еще что-то, что максимально строго на всякий случай раскроет эту тему.

О применении двойственности в геометрии

там вроде переход к кокасательному пространству за счет неё делается.

но я так-то хз, потом пропишу.

B.1.2 Polyline Shapes

Определение полилинейных форм

Градуированная алгебра полилинейных функций

(? и что)

Множество полилинейных функций на векторном пространстве $\mathcal{V}$ образует градуированную алгебру

$$\mathbf{T}_\star(\mathcal{V}) = \bigoplus_{q=0}^{\infty} \mathbf{T}_q(\mathcal{V})$$

относительно операции тензорного произведения форм:

$$(\alpha \otimes \beta)(v_1, \ldots, v_{q+m}) = \alpha(v_1, \ldots, v_q)\, \beta(v_{q+1}, \ldots, v_{q+m}), \alpha \in \mathbf{T}_q(\mathcal{V}), \beta \in \mathbf{T}_m(\mathcal{V})$$

Здесь через $\mathbf{T}_q(\mathcal{V})$ обозначено векторное пространство полилинейных форм степени q на $\mathcal{V}$, а под прямой суммой векторных пространств понимается внешняя прямая сумма: внешней прямой суммой пространств $\mathcal{V}_1, \ldots, \mathcal{V}_k$ называется векторное пространство $\mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_k$, составленное из всех последовательностей $(x_1, \ldots, x_k)$, где $x_i \in \mathcal{V}_i$, с покомпонентными операциями сложения и умножения на вещественные числа. Более подробно, операции в $\mathcal{V}_1 \oplus \ldots \oplus \mathcal{V}_k$ определяются следующим образом:

$$(x_1, \ldots, x_k) + (y_1, \ldots, y_k) = (x_1 + y_1, \ldots, x_k + y_k)$$
$$\lambda\,(x_1, \ldots, x_k) = (\lambda x_1, \ldots, \lambda x_k)$$

B.1.3 Bilinear Shapes

(пока взято у Володи)

определение квадратичных форм

$$q(\mathbf{x}) = \beta(\mathbf{x}, \mathbf{x})$$

$$q(\mathbf{x}) = \beta_{\mu\nu} x^\mu x^\nu$$

сопоставление квадратичной форме билинейной

Над полем $\mathbb{R}^n$ однозначно квадратичная форма восстанавливается по билинейной. Дальнейшие детали не важны (??? да??), но их обсуждение можно найти в записи по линейной алгебре.

$$q(\mathbf{x} + \mathbf{y}) = \beta(\mathbf{x} + \mathbf{y}, \mathbf{x} + \mathbf{y}) = q(\mathbf{x}) + q(\mathbf{y}) + 2\beta(\mathbf{x}, \mathbf{y})$$

$$\beta(\mathbf{x}, \mathbf{y}) = \frac{1}{2}[q(\mathbf{x} + \mathbf{y}) - q(\mathbf{x}) - q(\mathbf{y})]$$

О использовании квадратичных форм в диффгеме

я пока не уверен, где они нужны?
метрика?

Положительно определенные квадратичные формы

потом разберусь, зачем онинужны.

B.1.4 Tensor Algebra

Определене тензора

$$\alpha : \underbrace{\mathcal{V} \times \ldots \times \mathcal{V}}_{q \text{ штук}} \times \underbrace{\mathcal{V}^* \times \ldots \times \mathcal{V}^*}_{p \text{ штук}} \to \mathbb{C}$$

где в частных случаях может быть не $\mathbb{C}$, а другое любое поле.

$$(\alpha_1 + \alpha_2)\,(\mathbf{x}_1, \ldots, \mathbf{x}_q; f_1, \ldots, f_p) = \alpha_1\,(\mathbf{x}_1, \ldots, \mathbf{x}_q; f_1, \ldots, f_p) + \alpha_2\,(\mathbf{x}_1, \ldots, \mathbf{x}_q; f_1, \ldots, f_p)$$
$$(\lambda\alpha)\,(\mathbf{x}_1, \ldots, \mathbf{x}_q; f_1, \ldots, f_p) = \lambda\alpha\,(\mathbf{x}_1, \ldots, \mathbf{x}_q; f_1, \ldots, f_p)$$

свойства тензоров

тензорное произведение

Определение B.1. *Тензорное произведение это отображение из два тензоров в общем случае из разных пространств и в тензорное пространство их суммы,*
то есть:

$$\otimes : \mathbf{T}^p_q(\mathcal{V}) \times \mathbf{T}^r_m(\mathcal{V}) \to \mathbf{T}^{p+r}_{q+m}(\mathcal{V})$$

Можно записать тензорное произведене как:

$$\alpha \otimes \gamma \in \mathbf{T}^{p+r}_{q+m}(\mathcal{V})$$

или покомпонентно (?)

$$(\alpha \otimes \gamma)\left(\mathbf{x}_1, \ldots, \mathbf{x}_{q+m}; f_1, \ldots, f_{p+r}\right) = \alpha\left(\mathbf{x}_1, \ldots, \mathbf{x}_q; f_1, \ldots, f_p\right)\gamma\left(\mathbf{x}_{q+1}, \ldots, \mathbf{x}_{q+m}; f_{p+1}, \ldots, f_{p+r}\right)$$

ассоциативность тензорного произведения

Посмотрим на свойства.
Имеется ассоциативность

$$(\alpha \otimes \beta) \otimes \gamma = \alpha \otimes (\beta \otimes \gamma)$$

действительно
....
также

$$(\alpha \otimes \beta)(\mathbf{v}, \mathbf{w}) = \alpha(\mathbf{v})\beta(\mathbf{w}) = \alpha\left(\mathbf{e}_1\right)\beta\left(\mathbf{e}_2\right) = 1$$
$$(\beta \otimes \alpha)(\mathbf{v}, \mathbf{w}) = \beta(\mathbf{v})\alpha(\mathbf{w}) = \beta\left(\mathbf{e}_1\right)\alpha\left(\mathbf{e}_2\right) = 0$$

обоснование существования базиса (!!!???)

Оказывается, что конструкция ниже - базис.

$$\left\{\mathbf{e}_{\mu_1} \otimes \ldots \otimes \mathbf{e}_{\mu_p} \otimes \mathfrak{h}^{\nu_1} \otimes \ldots \otimes \mathfrak{h}^{\nu_q} : 1 \leq \mu_k, \nu_\ell \leq n\right\}$$

дальше важное большое доказательство
(надеюсь, у Володи оно верное)

$$\mathbf{e}_{\mu_1} \otimes \ldots \otimes \mathbf{e}_{\mu_p} \otimes \mathfrak{h}^{\nu_1} \otimes \ldots \otimes \mathfrak{h}^{\nu_q}\left(f_1, \ldots, f_p; \mathbf{x}_1, \ldots, \mathbf{x}_q\right) = f_1\left(\mathbf{e}_{\mu_1}\right) \ldots f_p\left(\mathbf{e}_{\mu_p}\right)\mathbf{h}^{\nu_1}\left(\mathbf{x}_1\right) \ldots \mathbf{h}^{\nu_q}\left(\mathbf{x}_q\right)$$

$$\mathbf{e}_{\mu_1} \otimes \ldots \otimes \mathbf{e}_{\mu_p} \otimes \mathfrak{h}^{\nu_1} \otimes \ldots \otimes \mathfrak{h}^{\nu_q}\left(\mathbf{h}^{\alpha_1}, \ldots, \mathfrak{h}^{\alpha_p}; \mathbf{e}_{\beta_1}, \ldots, \mathbf{e}_{\beta_q}\right) = \delta^{\alpha_1}_{\mu_1} \ldots \delta^{\alpha_p}_{\mu_p} \cdot \delta^{\nu_1}_{\beta_1} \ldots \delta^{\nu_q}_{\beta_q}$$

$$\mathcal{A} := a^{\mu_1 \ldots \mu_p}{}_{,\nu_1 \ldots \nu_q}\, \mathbf{e}_{\mu_1} \otimes \ldots \otimes \mathbf{e}_{\mu_p} \otimes \mathfrak{h}^{\nu_1} \otimes \ldots \otimes \mathfrak{h}^{\nu_q}$$

$$t = \tau^{\mu_1 \ldots \mu_p}\mathbf{e}_{\mu_1} \otimes \ldots \otimes \mathbf{e}_{\mu_p} \otimes \mathfrak{h}^{\nu_1} \otimes \ldots \otimes \mathfrak{h}^{\nu_q}$$

значит, базис
теперь про размерности подумаем...

преобразование тензора

Обсудим преобразование тензора.
Преобразуется по формуле:

$$\tau'^{\mu_1 \ldots \mu_\nu}_{\nu_1 \ldots \nu_q} = \mathcal{A}^{\beta_1}_{\nu_1} \ldots \mathcal{A}^{\beta_q}_{\nu_q}\left(\mathcal{A}^{-1}\right)^{\mu_1}_{\alpha_1} \ldots \left(\mathcal{A}^{-1}\right)^{\mu_p}_{\alpha_p}\tau^{\alpha_1 \ldots \alpha_p}_{\beta_1 \ldots \beta_q}$$

доказательство корректности тензорного произведения

примеры тензорных произведений в дифференциальной геометрии

свертка тензоров - обзор

Рассмотрим тензор $\tau \in \mathbf{T}^p_q(\mathcal{V})$
базис можно записать

$$\mathbf{e}_{\mu_1} \otimes \ldots \otimes \mathbf{e}_{\mu_p} \otimes \mathfrak{h}^{\nu_1} \otimes \ldots \otimes \mathfrak{h}^{\nu_q}$$

Свертка определяется как

$$\mathrm{tr}^{\mu_k}_{\nu_\ell}\, \tau \in \mathbf{T}^{p-1}_{q-1}(\mathcal{V})$$

$$\mathrm{tr}^{\mu_k}_{\nu_\ell}\, \tau = \tau^{\mu_1 \ldots \mu_{k-1} \lambda \mu_{k+1} \ldots \mu_p} \nu_{1 \ldots \nu_{\ell-1} \lambda \nu_{\ell+1} \ldots \nu_q}$$

$$\mathrm{tr}^{\mu}_{\nu}\, \tau' = \tau'^{\mu}_{\mu} = \mathcal{A}^{\beta}_{\mu} \left(\mathcal{A}^{-1}\right)^{\mu}_{\alpha} \tau^{\alpha}_{\beta} = \delta^{\beta}_{\alpha} \tau^{\alpha}_{\beta} = \tau^{\alpha}_{\alpha}$$

B.1.5 About Graduated Algebras

Часто встречается понятие градуировки.

градуированное кольцо

почему дифференцирование градуированного кольца является градуированным пространством?

B.2 Superalgebra

(??)

B.3 Elements of Topology

тоже часто топология нужна, так что её тут приведем.

B.3.1 Fundamentals of Topology

(тут все типичные вещи будут)

B.4 Elements of Mathematical Analysis

Обсудим классические конструкции анализа, которые нужны в геометрии.

B.4.1 Reverse Mapping Theorem

хз, где-то идейно её можно мб использовать.

Суть её

Применения в геометрии

хз, я пока ни матан, ни диффгем не прошел.

B.4.2 different Operators???

Инвариантные относительно диффеоморфизмов операции и дифференциальные операторы.

B.5 Other Mathematical Constructions

B.5.1 Category Theory Elements

(что будет нужно - то буду собирать тут)

О применениях теории категорий в геометрии

(абсолютно хз, нужно ли это или нет)

Функтор

Ковариантный функтор

см Володя.

Функтор двойственного сопряжения

хз

C Bibliography

References

[1] Иванов, А. О., Тужилин А. А.: *Лекции по классической дифференциальной геометрии ч 1.* Новая университетская библиотека, 2009.

[2] Tu, Loring W: *Manifolds.* In *An Introduction to Manifolds.* Springer, 2011.

[3] О., Катанаев М.: *Геометрические методы в математической физике: E-print.* arXiv preprint arXiv:1311.0733, 2016.

[4] Мищенко, А. С., Фоменко А. Т.: *Краткий курс дифференциальной геометрии и топологии.* Физматлит, 2004.

[5] Иванов, А. О., Тужилин А. А.: *Лекции по классической дифференциальной геометрии ч 2.* Новая университетская библиотека, 2009.

[6] Н., Карасёв Р.: *Отдельные темы математического анализа.*

[7] Craps, Ben, Roose, Frederik, Troost, Walter, and Proeyen, Antoine Van: *What is special kähler geometry?* Nuclear Physics B, 503(3):565–613, oct 1997. https://doi.org/10.1016%2Fs0550-3213%2897%2900408-2.

[8] И., Арнольд В.: *Математические методы классической механики.* Рипол Классик, 1979.

[9] Рослый: *Лекции по дифференциальной геометрии.*

[10] Nakahara, M.: *Geometry, topology and physics*. 2003. http://www.slac.stanford.edu/spires/find/hep/www?key=7208855.