Nonequilibrium Superconductivity: Part I

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This draft is not aimed for distribution.

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Preface

Examples and Solved Problems

Well-known and Important Theories

Old, Fundamental Theories (starts with articles from Larkin, Landenberg)

Special Effects: Useful, Selected Articles

Amazing facts

(I'll reveal it later)

Puzzles for motivation

(I'll reveal it later)

Part I
Nonequilibrium Superconductivity in a Nutshell

1 Main Formulas and Ideas (?)

1.1 My Main Formulas of Nonequilibrium SC

2 Method of Pseudospins and Lax Vector

2.1 Quantum quench phase diagrams of an s-wave BCS-BEC condensate by Yuzbashyan, Dzero, Gurarie, Foster

2.1.1 Main ideas and formulas

(??)

2.1.2 Models and approximations

Models and approximations

$$\hat{H}_{2\mathrm{ch}} = \sum_{\mathbf{p},\sigma=\uparrow,\downarrow} \epsilon_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}\sigma} \hat{a}_{\mathbf{p}\sigma} + \sum_{\mathbf{q}} \left(\omega + \frac{q^2}{4m}\right) \hat{b}^\dagger_{\mathbf{q}} \hat{b}_{\mathbf{q}} + g \sum_{\mathbf{pq}} \left(\hat{b}^\dagger_{\mathbf{q}} \hat{a}_{\frac{\mathbf{q}}{2}+\mathbf{p},\uparrow} \hat{a}_{\frac{\mathbf{q}}{2}-\mathbf{p},\downarrow} + \hat{b}_{\mathbf{q}} \hat{a}^\dagger_{\frac{\mathbf{q}}{2}-\mathbf{p},\downarrow} \hat{a}^\dagger_{\frac{\mathbf{q}}{2}+\mathbf{p},\uparrow} \right). \tag{2.1}$$

$$\gamma = \frac{g^2 \nu_F}{\epsilon_F} \tag{2.2}$$

$$\hat{H}_{1\mathrm{ch}} = \sum_{\mathbf{p},\sigma=\uparrow,\downarrow} \epsilon_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}\sigma} \hat{a}_{\mathbf{p}\sigma} - \frac{\lambda}{\nu_F} \sum_{\mathbf{pp'q}} \hat{a}^\dagger_{\frac{\mathbf{q}}{2}-\mathbf{p},\downarrow} \hat{a}^\dagger_{\frac{\mathbf{q}}{2}+\mathbf{p},\uparrow} \hat{a}_{\frac{\mathbf{q}}{2}+\mathbf{p}',\uparrow} \hat{a}_{\frac{\mathbf{q}}{2}-\mathbf{p}',\downarrow}, \tag{2.3}$$

where

$$\lambda = \frac{g^2 \nu_F}{\omega} = \frac{\gamma \varepsilon_F}{\omega}. \tag{2.4}$$

$$\hat{H}_{2\mathrm{ch}} = \sum_{\mathbf{p}} 2\epsilon_{\mathbf{p}} \hat{s}^z_{\mathbf{p}} + \omega \hat{b}^\dagger \hat{b} + g \sum_{\mathbf{p}} \left(\hat{b}^\dagger \hat{s}^-_{\mathbf{p}} + \hat{s}^+_{\mathbf{p}} \hat{b} \right), \tag{2.5}$$

where

$$\hat{s}^-_{\mathbf{p}} = \hat{a}_{\mathbf{p}\uparrow} \hat{a}_{-\mathbf{p}\downarrow}, \;\; \hat{s}^z_{\mathbf{p}} = \frac{1}{2}\left(\hat{a}^\dagger_{\mathbf{p}\uparrow} \hat{a}_{\mathbf{p}\uparrow} + \hat{a}^\dagger_{-\mathbf{p}\downarrow} \hat{a}_{-\mathbf{p}\downarrow} - 1\right) \tag{2.6}$$

$$\begin{aligned} \frac{d\hat{\vec{s}}_{\mathbf{p}}}{dt} = \hat{\vec{B}}_{\mathbf{p}} \times \hat{\vec{s}}_{\mathbf{p}}, \quad \frac{d\hat{b}}{dt} = -i\omega \hat{b} - ig\hat{J}_-, \\ \hat{\vec{J}} = \sum_{\mathbf{p}} \hat{\vec{s}}_{\mathbf{p}}, \quad \hat{\vec{B}}_{\mathbf{p}} = 2g\hat{\vec{b}} + 2\epsilon_{\mathbf{p}} \hat{\mathbf{z}}, \end{aligned} \tag{2.7}$$

$$\dot{\vec{s}}_{\mathbf{p}} = \vec{B}_{\mathbf{p}} \times \vec{s}_{\mathbf{p}}, \quad \dot{b} = -i\omega b - igJ_-,$$
$$\vec{J} = \sum_{\mathbf{p}} \vec{s}_{\mathbf{p}}, \quad \vec{B}_{\mathbf{p}} = 2g\vec{b} + 2\epsilon_{\mathbf{p}}\hat{\mathbf{z}}, \tag{2.8}$$

$\vec{s}_{\mathbf{p}} = \langle \hat{\vec{s}}_{\mathbf{p}} \rangle$

$$H_{2\text{ch}} = \sum_{\mathbf{p}} 2\epsilon_{\mathbf{p}} s^z_{\mathbf{p}} + \omega \bar{b} b + g \sum_{\mathbf{p}} \left(\bar{b} s^-_{\mathbf{p}} + b s^+_{\mathbf{p}} \right), \tag{2.9}$$

$$\left\{ s^a_{\mathbf{p}}, s^b_{\mathbf{k}} \right\} = -\varepsilon_{abc} \delta_{\mathbf{pk}} s^c_{\mathbf{p}}, \quad \{b, \bar{b}\} = i, \tag{2.10}$$

$$H_{1\text{ch}} = \sum_{\mathbf{p}} 2\epsilon_{\mathbf{p}} s^z_{\mathbf{p}} - \frac{\lambda}{\nu_F} \sum_{\mathbf{p},\mathbf{p}'} s^-_{\mathbf{p}} s^+_{\mathbf{p}'} \tag{2.11}$$

$$\Delta(t) = -g\langle \hat{b}(t) \rangle = -gb(t) \equiv \Delta_x(t) - i\Delta_y(t). \tag{2.12}$$

$$\Delta_{1\text{ch}}(t) = \frac{\lambda}{\nu_F} \sum_{\mathbf{p}} \langle \hat{a}_{\mathbf{p}\uparrow}(t) \hat{a}_{-\mathbf{p}\downarrow}(t) \rangle = \frac{\lambda}{\nu_F} \sum_{\mathbf{p}} s^-_{\mathbf{p}}. \tag{2.13}$$

Ground state

$$\Delta(t) = \Delta_0 e^{-2i\mu t}, \tag{2.14}$$

$$s^-_{\mathbf{p}} = \frac{\Delta_0 e^{-2i\mu t}}{2E(\varepsilon_{\mathbf{p}}; \Delta_0, \mu)}, \quad s^z_{\mathbf{p}} = -\frac{\varepsilon_{\mathbf{p}} - \mu}{2E(\varepsilon_{\mathbf{p}}; \Delta_0, \mu)}, \tag{2.15}$$

$$E(\varepsilon; \Delta, \mu) \equiv \sqrt{(\varepsilon - \mu)^2 + \Delta^2}. \tag{2.16}$$

(check it, but maybe it is easy)

$$|J_-| = \frac{(\omega - 2\mu)\Delta_0}{g^2}, \tag{2.17}$$

$$\frac{(\omega - 2\mu)}{g^2} = \sum_{\mathbf{p}} \frac{1}{2E(\varepsilon_{\mathbf{p}}; \Delta_0, \mu)}. \tag{2.18}$$

$$n = \bar{b} b + \sum_{\mathbf{p}} \left(s^z_{\mathbf{p}} + \frac{1}{2} \right), \tag{2.19}$$

$$2n = \frac{2\Delta_0^2}{g^2} + \sum_{\mathbf{p}} \left(1 - \frac{\varepsilon_{\mathbf{p}} - \mu}{E(\varepsilon_{\mathbf{p}}; \Delta_0, \mu)} \right). \tag{2.20}$$

$$\varepsilon_F = 1. \tag{2.21}$$

$$\nu(\varepsilon) = \nu_F f(\varepsilon), \tag{2.22}$$

$$\sum_{\mathbf{p}} A_{\mathbf{p}} \to \nu_F \int_0^\infty A(\varepsilon) f(\varepsilon) d\varepsilon. \tag{2.23}$$

$$2n = \int_0^1 2\nu(\varepsilon)d\varepsilon = \frac{4}{d}\nu_F, \tag{2.24}$$

$$\frac{4}{d} = \frac{2\Delta_0^2}{\gamma} + \int_0^\infty \left(1 - \frac{\varepsilon - \mu}{\sqrt{(\varepsilon-\mu)^2 + \Delta_0^2}}\right) f(\varepsilon)d\varepsilon, \tag{2.25}$$

$$\frac{2\omega - 4\mu}{\gamma} = \int_0^{\varepsilon_\Lambda} \frac{f(\varepsilon)d\varepsilon}{\sqrt{(\varepsilon-\mu)^2 + \Delta_0^2}}, \tag{2.26}$$

Eqs. (8.25) and (8.26) contain two independent parameters not counting the cutoff. For example, we can choose γ and ω and determine μ and Δ_0 from these equations, or choose γ and Δ_0 and determine μ and ω etc., see Fig. 8.1 for a plot of $\mu(\Delta_0)$ for various γ in 3d. Note also that $\Delta_0^2 = g^2\bar{b}b$ is proportional to the number of bosons and is therefore limited by the total number of particles. Eq. (8.25) implies

$$\Delta_0 \leq \sqrt{\frac{2\gamma}{d}} = \Delta_{\max}. \tag{2.27}$$

Quench setup and initial conditions

$$\begin{aligned} s_{\mathbf{p}}^-(t=0) &= \frac{\Delta_{0i}}{2E(\varepsilon_{\mathbf{p}}; \Delta_{0i}, \mu_i)}, \\ s_{\mathbf{p}}^z(t=0) &= -\frac{\varepsilon_{\mathbf{p}} - \mu_i}{2E(\varepsilon_{\mathbf{p}}; \Delta_{0i}, \mu_i)}, \end{aligned} \tag{2.28}$$

$$|\Psi(t)\rangle = |\psi(t)\rangle \otimes \left(\hat{b}^\dagger\right)^{n(t)}|0\rangle, \tag{2.29}$$

$$|\psi(t)\rangle = \prod_{\mathbf{p}} \left[\overline{u}_{\mathbf{p}}(t) + \overline{v}_{\mathbf{p}}(t)\hat{a}^\dagger_{\mathbf{p}\uparrow}\hat{a}^\dagger_{-\mathbf{p}\downarrow}\right]|0\rangle. \tag{2.30}$$

$$i\frac{\partial}{\partial t}\begin{pmatrix} u_{\mathbf{p}}(t) \\ v_{\mathbf{p}}(t) \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{p}} & \Delta(t) \\ \bar{\Delta}(t) & -\epsilon_{\mathbf{p}} \end{pmatrix}\begin{pmatrix} u_{\mathbf{p}}(t) \\ v_{\mathbf{p}}(t) \end{pmatrix}, \tag{2.31}$$

$$\frac{s_{\mathbf{p}}^-}{s_{\mathbf{p}}} = 2u_{\mathbf{p}}\overline{v}_{\mathbf{p}}, \quad \frac{s_{\mathbf{p}}^z}{s_{\mathbf{p}}} = |v_{\mathbf{p}}|^2 - |u_{\mathbf{p}}|^2, \tag{2.32}$$

$$\dot{\Delta} = -i\omega_f\Delta + i\gamma\int_0^\infty s^-(\varepsilon)f(\varepsilon)d\varepsilon. \tag{2.33}$$

Main results

And in general, the idea of the article, that with this formalism one can obtain all similar plots for different cases.

Figure 2.1: Detuning quench phase diagrams for two-channel model (8.1) in 2d for an assortment of resonance widths γ. Each point represents a single quench labeled by Δ_{0i} (vertical axis) and Δ_{0f} (horizonal axis) – pairing gaps the system would have in the ground state for initial and final detunings. At large times the system ends up in one of three steady states shown as regions I, II (including II'), and III. For quenches in region I the order parameter vanishes, $\Delta(t) \to 0$. In II $\Delta(t) \to \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}$ and in III $|\Delta(t)|$ oscillates persistently. Subregions II and II' differ in the sign of μ_∞ (out of equilibrium analog of the chemical potential): $\mu_\infty > 0$ in II and $\mu_\infty < 0$ in II'. The diagonal, $\Delta_{0i} = \Delta_{0f}$, is the no quench line. To the left of it are strong to weak coupling quenches, to the right – weak to strong. $\Delta_{\max} = \varepsilon_F\sqrt{\gamma}$ in 2d is the maximum possible ground state gap and $\Delta_{0\times}$ is the ground state gap corresponding to zero chemical potential, i.e. $\Delta_{0\times}$ is given by Eq. (8.25) for $\mu = 0$.

2.1.3 Method

2.2 Other, Little Articles in scope of Pseudospins and Lax Vector

2.2.1 Dynamical vanishing of the order parameter in a fermionic condensate by Yuzbashyan, Dzero

Main ideas and formulas

(??)

2.2.2 Onset of superconductivity in a voltage-biased NSN microbridge by Serbyn, Skvortsov

Main ideas and formulas

(??)

2.2.3 Relaxation and persistent oscillations of the order parameter in the non-stationary BCS theory by Yuzbashyan, Tsyplyatyev, Altshuler

Main ideas and formulas

(??)

2.2.4 Solitons and Rabi Oscillations in a Time-Dependent BCS Pairing Problem by Barankov, Levitov, Spivak

Main ideas and formulas

We use just equations for u, v, because kinetic equation and TDGL equation is not valid for the case when gap adjusts fast to the change of quasiparticle change.

$$\tau_\epsilon \gg \tau_\Delta \tag{2.34}$$

$$\mathcal{H} = \sum_{\mathbf{p},\sigma} \epsilon_{\mathbf{p}} a^+_{\mathbf{p},\sigma} a_{\mathbf{p},\sigma} - \frac{\lambda(t)}{2} \sum_{\mathbf{p},\mathbf{q}} a^+_{\mathbf{p}\uparrow} a^+_{-\mathbf{p}\downarrow} a_{-\mathbf{q}\downarrow} a_{\mathbf{q}\uparrow}\,, \tag{2.35}$$

$$|\Psi(t)\rangle = \prod_{\mathbf{p}} \left(u_{\mathbf{p}}(t) + v_{\mathbf{p}}(t) a^+_{\mathbf{p},\uparrow} a^+_{-\mathbf{p},\downarrow}\right) |0\rangle \tag{2.36}$$

$$\Delta(t) = \lambda \sum_{\mathbf{p}} u_{\mathbf{p}}(t) v^*_{\mathbf{p}}(t) \tag{2.37}$$

$$i\partial_t \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{p}} & \Delta \\ \Delta^* & -\epsilon_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} \tag{2.38}$$

$u^{(0)}_{\mathbf{p}}(t) = e^{-i\epsilon_{\mathbf{p}} t}\theta(\epsilon_{\mathbf{p}})$, $v^{(0)}_{\mathbf{p}}(t) = e^{i\epsilon_{\mathbf{p}} t}\theta(-\epsilon_{\mathbf{p}})$
$\Delta(t) \propto e^{\gamma t} e^{-i\omega t}$,

$$\delta u_{\mathbf{p}}(t) = \frac{\Delta(t) v^{(0)}_{\mathbf{p}}(t)}{i\gamma - 2\epsilon_{\mathbf{p}} + \omega}, \quad \delta v_{\mathbf{p}}(t) = \frac{\Delta^*(t) u^{(0)}_{\mathbf{p}}(t)}{i\gamma + 2\epsilon_{\mathbf{p}} - \omega} \tag{2.39}$$

$$1 = \lambda \sum_{\mathbf{p}} \frac{\operatorname{sgn} \epsilon_{\mathbf{p}}}{2\epsilon_{\mathbf{p}} - \zeta}, \quad \zeta = \omega + i\gamma \tag{2.40}$$

$$\tau_\epsilon \gg \gamma^{-1} \quad (\epsilon \simeq \Delta_0), \tag{2.41}$$

$r_1^2 + r_2^2 + r_3^2 = 1$

$$r_1 + ir_2 = 2u_{\mathbf{p}} v_{\mathbf{p}}^*; \quad r_3 = |u_{\mathbf{p}}|^2 - |v_{\mathbf{p}}|^2 \tag{2.42}$$

$$\sigma^+_{\mathbf{p}} = a^+_{\mathbf{p}\uparrow} a^+_{-\mathbf{p}\downarrow}\,, \quad \sigma^-_{\mathbf{p}} = a_{-\mathbf{p}\downarrow} a_{\mathbf{p}\uparrow} \tag{2.43}$$

$$\mathcal{H} = \sum_{\mathbf{p}}{}' \epsilon_{\mathbf{p}} \sigma^z_{\mathbf{p}} - 2\lambda \sum_{\mathbf{p},\mathbf{q}}{}' \sigma^+_{\mathbf{p}} \sigma^-_{\mathbf{q}} \tag{2.44}$$

$$\mathcal{H}_{\mathbf{p}} = \mathbf{b}_{\mathbf{p}} \cdot \sigma_{\mathbf{p}}, \quad \mathbf{b}_{\mathbf{p}} = (-\Delta', -\Delta'', \epsilon_{\mathbf{p}}) \tag{2.45}$$

$$\Delta \equiv \Delta' + i\Delta'' = \lambda \sum_{\mathbf{q}} \langle \sigma^+_{\mathbf{q}} \rangle \tag{2.46}$$

$$\dot{\sigma}_{\mathbf{p}} = i[\mathcal{H}_{\mathbf{p}}, \sigma_{\mathbf{p}}] = 2\mathbf{b}_{\mathbf{p}} \times \sigma_{\mathbf{p}} \tag{2.47}$$

Figure 2.2: Bloch dynamics $\dot{\mathbf{r}}_{\mathbf{p}} = 2\mathbf{b}^{\text{eff}}_{\mathbf{p}} \times \mathbf{r}_{\mathbf{p}}$ for 10^3 spins with a constant density of states, with $\mathbf{b}^{\text{eff}}_{\mathbf{p}}$ defined by (14.20),(14.23). Initial conditions $r_{3,\mathbf{p}} = \tanh(\frac{1}{2}\beta\epsilon_{\mathbf{p}})$, $(r_1 + ir_2)_{\mathbf{p}} = e^{i\phi_{\mathbf{p}}}(1 - r_3^2)^{1/2}$, with $\beta^{-1} \equiv T = 0.1\Delta_0$ and random uncorrelated $0 < \phi_{\mathbf{p}} < 2\pi$ were used to simulate thermal noise.

$$\mathbf{b}_{\mathbf{p}} \to \mathbf{b}^{\text{eff}}_{\mathbf{p}} = \mathbf{b}_{\mathbf{p}} - \gamma \mathbf{b}_{\mathbf{p}} \times \mathbf{r}_{\mathbf{p}} \tag{2.48}$$

$$\dot{r}_1 = -\xi_{\mathbf{p}} r_2, \quad \dot{r}_2 = \xi_{\mathbf{p}} r_1 + 2\Omega r_3, \quad \dot{r}_3 = -2\Omega r_2 \tag{2.49}$$

$$r_1 = A\Omega, \quad r_2 = B\dot{\Omega}, \quad r_3 = C\Omega^2 - D \tag{2.50}$$

$$C^2 \xi^2_{\mathbf{p}} \Omega^2 + C^2 \dot{\Omega}^2 + (C\Omega^2 - D)^2 = 1 \tag{2.51}$$

$$\dot{\Omega}^2 + (\Omega^2 - \Delta^2_-)(\Omega^2 - \Delta^2_+) = 0, \quad \Delta_- \leq \Delta_+ \tag{2.52}$$

$$(D^2 - 1)/C^2 = \Delta^2_- \Delta^2_+, \quad 2D/C = \xi^2_{\mathbf{p}} + \Delta^2_- + \Delta^2_+ \tag{2.53}$$

$$1 = \lambda \sum_{\mathbf{p}} \frac{\xi_{\mathbf{p}} \operatorname{sgn} \epsilon_{\mathbf{p}}}{\left((\xi^2_{\mathbf{p}} + \Delta^2_- + \Delta^2_+)^2 - 4\Delta^2_- \Delta^2_+\right)^{1/2}} \tag{2.54}$$

2.2.5 Synchronization in the BCS Pairing Dynamics as a Critical Phenomenon by Barankov, Levitov

Main ideas and formulas

(??)

3 Nonequilibrium Green's Function Method

3.1 Other, Little Articles in scope of Nonequilibrium Green's Function Method

3.1.1 Collisionless relaxation of the energy gap in superconductors by Volkov, Kogan

On the basis of equations derived from the Eliashberg equations, and in a time interval small in comparison with the electron energy-relaxation times

$$\tau_{ph} \approx \hbar\Theta_D^2/T^3, \qquad \tau_{ee} \approx \hbar\varepsilon_F/T^2,$$

the behavior of a small initial perturbation of Δ in the absence of fields is analyzed. We should believe that these times are such.

Part II
My Theory of Nonequilibrium Superconductivity

Part III
Examples and Solved Problems

4 Problems about Creation of a Model of Non-equilibrium Phenomena

(this is very important, since the first big question is what is going on??)

5 Problems about Transformation in Derivations

(a lot of articles use it, so I'll write it here)

6 Problems about Non-equilibrium Phenomena

(real problems)

7 Questions about Non-equilibrium Phenomena for Different Cases in Life

7.0.1 Questions to Check Knowledge

(there are a lot of them, I'll write later)

Part IV
Method of Pseudospins and Lax Vector

8 Quantum quench phase diagrams of an s-wave BCS-BEC condensate by Yuzbashyan, Dzero, Gurarie, Foster

(how exactly this article is useful????)
(I have a problem with intuition about what is going on here???)
We study the dynamic response of an s-wave BCS-BEC (atomic-molecular) condensate to detuning quenches within the two channel model beyond the weak coupling BCS limit.

At long times after the quench, the condensate ends up in one of three main asymptotic states (nonequilibrium phases), which are qualitatively similar to those in other fermionic condensates defined by a global complex order parameter.
(which ones????)

In phase I the amplitude of the order parameter vanishes as a power law, in phase II it goes to a nonzero constant, and in phase III it oscillates persistently.
(why these are phases? just regimes: zero gap, nonzero gap, oscillations (???? what does it mean physically??))
We construct exact quench phase diagrams that predict the asymptotic state (including the many-body wavefunction) depending on the initial and final detunings and on the Feshbach resonance width.
Outside of the weak coupling regime, both the mechanism and the time dependence of the relaxation of the amplitude of the order parameter in phases I and II are modified.
Also, quenches from arbitrarily weak initial to sufficiently strong final coupling do not produce persistent oscillations (in which regime??) in contrast to the behavior in the BCS regime.
The most remarkable feature of coherent condensate dynamics in various fermion superfluids is an effective reduction in the number of dynamic degrees of freedom as the evolution time goes to infinity.
(how is it???)
As a result, the long time dynamics can be fully described in terms of just a few new collective dynamical variables governed by the same Hamiltonian only with “renormalized" parameters.
(how is it???)
Combining this feature with the integrability of the underlying (e.g. the two channel) model, we develop and consistently present a general method that explicitly obtains the exact asymptotic state of the system.

8.1 Introduction

The problem of a superconductor driven out of equilibrium by a sudden perturbation goes back many decades. Early studies[Anderson(1958), Galaiko(1972), Volkov and Kogan(1973), Galperin et al.(1981), Littlewood and Varma(1982), Shumeiko(1990)] addressed small deviations from equilibrium using linearized equations of motion. (add formulas, how exactly??? where was the linearization?)

An important result was obtained by Volkov and Kogan [Volkov and Kogan(1973)], who discovered a power law oscillatory attenuation of the Bardeen-Cooper-Schriffer (BCS) order parameter for non-equilibrium initial conditions close to the superconducting ground state. (add formulas, how exactly??? not clear, what the setup is?)

In the last decade it was realized that even large deviations from equilibrium are within the reach of appropriate theoretical methods. Recent studies, motivated by experiments in cold atomic fermions, focused on quantum quenches – nonequilibrium conditions created by a sudden change in the superconducting coupling strength.

Barankov et. al. [Barankov et al.(2004)] in a paper that set off a surge of modern research in this long-standing problem [Amin et al.(2004), Andreev et al.(2004), Barankov and Levitov(2004), Szymanska et al.(2005), Warner and Leggett(2005), Yuzbashyan et al.(2005a), Yuzbashyan et al.(2005b), Yuzbashyan et al.(2005c), 19, Barankov and Levitov(2006), Yuzbashyan and Dzero(2006), Dzero et al.(2007), Barankov and Levitov(2007), Tomadin et al.(2008), Nahum and Bettelheim(2008), 36, Faribault et al.(2009)] (which problem authors mean???) in the context of quantum gases, found that for initial conditions close to the unstable normal state, the order parameter exhibits large unharmonic periodic oscillations. (think about their paper!)

Subsequently, Yuzbashyan et. al.[19] developed an analytical method to predict the state of the system at large times based on the integrability of the underlying BCS model. (write to the first part, what this method is?)

This work extended Volkov and Kogan's result (write, how exactly extended???) to large deviations from equilibrium and showed that the oscillation frequency is twice the *nonequilibrium* asymptotic value of the order parameter, a conclusion confirmed by recent terahertz pump pulse experiments in $Nb_{1\text{-}x}Ti_xN$ films [Matsunaga et al.(2013), 46]. (how exactly? analyze these articles)

Later studies [Barankov and Levitov(2006), Yuzbashyan and Dzero(2006)] mapped out the full quantum quench "phase diagram" for weakly coupled s-wave BCS superconductors finding that three distinct regimes occur depending on the strength of the quench: Volkov and Kogan like behavior, persistent oscillations, and exponential vanishing of the order parameter. (analyze this paper! write the connection to Volkov, Kogan's article!)

Most recent research[Papenkort et al.(2007), Papenkort et al.(2009), Krull et al.(2013), 77] fueled by experimental breakthroughs[98, Beck et al.(2013), Matsunaga et al.(2013)] investigates non-adiabatic dynamics of s-wave BCS superconductors in response to fast electromagnetic perturbations. (this is something that I don't understand at all. why nonadiabatic??? a lot of articles are here...)

Closely related subjects developing in parallel are exciton dynamics[Brierley et al.(2011)], collective neutrino oscillations[Pehlivan et al.(2011), Raffelt et al.(2013)], quenched p-wave superfluids[Foster et al.(2013), Foster et al.(2014)] etc. (have no idea what are they?)

Most existing work addressed the dynamics in the BCS regime and, in particular, quenches such that the interaction strength is weak both before and after the quench. This was so that the system always remains in the BCS regime, since the physics of the condensate beyond this regime was not sufficiently well understood. (strange statement, beyond BCS regime depends on a model, since BCS is just an approximation. how at all people are assured that in cold atoms, there is indeed the BCS regime???)

However, a superfluid made up of cold atoms can be as well quenched from the BCS to the Bose-Einstein Condensation (BEC) regime (how exactly?), or within the BEC regime. With few exceptions [36, Foster et al.(2013), Foster et al.(2014)], these types of quenches are not

adequately studied in the existing literature. (I don't know, hot is is related to quenches before? this paper answers this question)

Our paper aims to close this gap and analyze all possible interaction quenches throughout the BCS-BEC crossover in a paired superfluid, including BCS to BEC, BEC to BCS and BEC to BEC quenches.

We fully determine the steady state of the system at large times after the quench – the asymptote of the order parameter as well as the approach to the asymptote, the many-body wave function, and certain observables, such as the radio-frequency absorption spectrum and the momentum distribution (momentum distribution?).

In the BCS limit, we recover previous results. Beyond this limit the dynamics is quantitatively and sometimes qualitatively different.

For example, the power law in the Volkov and Kogan like attenuation changes in the BEC regime, exponential vanishing is replaced with a power law, and persistent oscillations first change their form and then disappear altogether after a certain threshold for quenches from any initial (e.g. arbitrarily weak) to sufficiently strong final coupling. (write below, how exactly? we can compare these results, from this article and from VK-s.)

We believe an experimental verification of the predictions of this work is within a reach of current experiments in cold atomic systems. (why??)

The long time dynamics can be determined explicitly due to a remarkable reduction mechanism at work (???), so that at large times the system is governed by an effective interacting Hamiltonian with just a few classical collective spin or oscillator degrees of freedom.

In a sense, the system "flows in time" to a much simpler Hamiltonian.

This observation combined with the integrability of the original Hamiltonian (see below) lead to a method originally proposed in Ref. [19] for obtaining the long time asymptote (steady state) of integrable Hamiltonian dynamics in the continuum (thermodynamic) limit.

Here we improve this method as well as provide its comprehensive and self-contained review including many previously unpublished results and steps. We do so in the context of the s-wave BCS (one channel) and inhomogeneous Dicke (two channel) models (what is a Dicke model??), but with some modifications the same method also applies to all known integrable pairing models [Gaudin(1983), Richardson(2002), Ibanez et al.(2009), Dunning et al.(2010), Rombouts et al.(2010), Dukelsky et al.(2004), Ortiz et al.(2005)] , such as $p + ip$ superfluids[Foster et al.(2013), Foster et al.(2014)], integrable fermion or boson pairing models with nonuniform interactions[J. Dukelsky(2001), L. Amico(2001)], Gaudin magnets (central spin models), and potentially can be extended to a much broader class of integrable nonlinear equations. (is it really applicable?? sounds that I'll not be able to connect them, but I can check later)

The purpose of this paper is therefore twofold. First, it serves as an encyclopedia (of course it is not an encyclopedia and not even close to it) of quantitatively exact predictions, new and old, for the quench dynamics of real s-wave BCS-BEC condensates in two and three spatial dimensions.

Readers primarily interested in this aspect of our work will find most of the relevant information in the Introduction, Sect. 8.7, and Conclusion.

In particular, Sect. 8.1.4 concisely summarizes our main results and provides a guide to other sections that contain further results and details.

Our second goal is to develop and thoroughly review (I am not sure that it is review, I currently saw just brief statements) a method for determining the far from equilibrium dynamics in a certain class of integrable models.

We refer readers interested in learning about the method to Sect. 8.2. Also, from this

viewpoint, Sects. 8.3 and 8.4 should be considered as applications of our approach and Sect. 8.5 as a related development.

A major experimental breakthrough with ultracold atoms was achieved in 2004, when they were used to emulate s-wave superconductors with an interaction strength that can be varied at will [Regal et al.(2004), Zwierlein et al.(2004)]. (read these papers!)

Experimental control parameter is the detuning ω – the binding energy of a two fermion bound state (molecule). This parameter determines the strength of the effective interaction between fermions and can be varied both slowly and abruptly with the help of a Feshbach resonance. Moreover, it is straightforward to make time resolved measurements of the subsequent evolution of the system. Thus cold atoms provide a natural platform to study quenches in superfluids and in a variety of other setups[Calabrese and Cardy(2006), Kollath et al.(2007)].

At large ω we have fermionic atoms with weak effective attraction that form a paired superfluid, an analog of the superconducting state of electrons in a metal. As ω is decreased, the atoms pair up into bosonic molecules which then Bose condense. It was argued for a long time that both the paired superfluid and the Bose-condensed molecules are in the same phase of the fermionic gas, named the BCS-BEC condensate [Leggett(1980), Nozières and Schmitt-Rink(1985)]. As ω decreases, the strength of the effective interaction (coupling) between fermions increases from weak to strong and the system undergoes a BCS-BEC crossover. At $\omega \gg 2\varepsilon_F$, where ε_F is the Fermi energy, the system is deep in the BCS regime, while at large negative ω it is deep in the BEC regime. It is not known how to recreate such a crossover in a conventional solid-state superconductor since the interaction strength cannot be easily adjusted.

In a quantum quench setup the system is prepared in the ground state at a detuning ω_i. At $t = 0$ the detuning is suddenly changed, $\omega_i \to \omega_f$. At $t > 0$ the system evolves with a new Hamiltonian $H(\omega_f)$. The main goal is to determine the state of the system at large times, $t \to \infty$.

8.1.1 Models and approximations

We consider two closely related models in this paper in both two and three dimensions. The first one is the well-known two channel model that describes two species of fermionic atoms interacting via an s-wave Feshbach resonance (what is Feshbach resonance???)

$$\hat{H}_{2\text{ch}} = \sum_{\mathbf{p},\sigma=\uparrow,\downarrow} \epsilon_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}\sigma} \hat{a}_{\mathbf{p}\sigma} + \sum_{\mathbf{q}} \left(\omega + \frac{q^2}{4m}\right) \hat{b}^\dagger_{\mathbf{q}} \hat{b}_{\mathbf{q}} + g \sum_{\mathbf{pq}} \left(\hat{b}^\dagger_{\mathbf{q}} \hat{a}_{\frac{\mathbf{q}}{2}+\mathbf{p},\uparrow} \hat{a}_{\frac{\mathbf{q}}{2}-\mathbf{p},\downarrow} + \hat{b}_{\mathbf{q}} \hat{a}^\dagger_{\frac{\mathbf{q}}{2}-\mathbf{p},\downarrow} \hat{a}^\dagger_{\frac{\mathbf{q}}{2}+\mathbf{p},\uparrow} \right). \quad (8.1)$$

(derive this model in another note!! I don't know how to do it)

It is convenient to think of the two types of fermions of mass m and energy $\epsilon_{\mathbf{p}} = p^2/2m$ as spin up and down, created and annihilated by operators $\hat{a}^\dagger_{\mathbf{p}\sigma}$ and $\hat{a}_{\mathbf{p}\sigma}$.

The interaction term converts two fermions into a bosonic molecule and vise versa at a rate controlled by the parameter g.

Molecules are created and annihilated by $\hat{b}^\dagger_{\mathbf{q}}$ and $\hat{b}_{\mathbf{q}}$ and have a binding energy ω.

The parameter g is set by the type of atoms and the specifics of a particular Feshbach resonance and cannot be changed in a single experiment; ω can be varied at will by varying the magnitude of the magnetic field applied during the experiment.

This model describes atoms in the BCS regime when ω is large, which undergo a crossover to the BEC regime as ω is decreased.

A parameter with dimensions of energy important for our analysis of this model is $g^2\nu_F$, where ν_F is the bulk density of states (proportional to the total volume) at the Fermi energy ϵ_F. A well known parameter

$$\gamma = \frac{g^2\nu_F}{\epsilon_F} \tag{8.2}$$

controls whether the resonance is narrow $\gamma \ll 1$ or broad $\gamma \gg 1$. (why?) This parameter is the dimensionless atom-molecule interaction strength or, equivalently, the resonance width.

A very convenient feature of the narrow resonance is that regardless of the regime of the system, controlled by ω, the system is adequately described with mean field theory [Gurarie and Radzihovsky(2007)] (why??). This is already clear from the form of the Hamiltonian: small γ implies that interaction g is small.

Broad resonances on the other hand correspond to large g. Under those conditions it is possible to integrate out the molecules $\hat{b}_{\mathbf{q}}$ to arrive at a simpler Hamiltonian [Gurarie and Radzihovsky(2007)] (how??) describing fermions interacting via a short range attractive interaction with variable strength

$$\hat{H}_{1\text{ch}} = \sum_{\mathbf{p},\sigma=\uparrow,\downarrow} \epsilon_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{p}\sigma}\hat{a}_{\mathbf{p}\sigma} - \frac{\lambda}{\nu_F}\sum_{\mathbf{p}\mathbf{p}'\mathbf{q}} \hat{a}^\dagger_{\frac{\mathbf{q}}{2}-\mathbf{p},\downarrow}\hat{a}^\dagger_{\frac{\mathbf{q}}{2}+\mathbf{p},\uparrow}\hat{a}_{\frac{\mathbf{q}}{2}+\mathbf{p}',\uparrow}\hat{a}_{\frac{\mathbf{q}}{2}-\mathbf{p}',\downarrow}, \tag{8.3}$$

where

$$\lambda = \frac{g^2\nu_F}{\omega} = \frac{\gamma\varepsilon_F}{\omega}. \tag{8.4}$$

This is the single (one) channel or BCS model, which is the second model we analyze in this paper. It also describes the BCS-BEC crossover (what is crossover?) as ω is decreased (λ is increased). However, while in the BCS and (to some extent) in the BEC regimes corresponding to large and small λ, respectively, mean field theory holds in equilibrium, for the intermediate values of λ (neither large nor small) the mean field theory is known to break down.

A special value of λ in the middle of the regime unaccessible to the mean field theory already in equilibrium is called the unitary point. It corresponds to the interaction strength where molecules are about to be formed. Non-condensed molecules play an important role in the description of the unitary point and its special properties are a subject of many studies in the literature[Nozières and Schmitt-Rink(1985), Burovski et al.(2006)].

Just as earlier work on the far from equilibrium superconductivity, we analyze the quench dynamics in the mean field approximation where no molecules are transferred into or out of the BCS-BEC condensate after the quench, i.e. the dynamics of the condensate is decoupled from the non-condensed modes. We analyze the validity of this approximation for nonequilibrium steady states produced by quenches in two channel model in Appendix 8.9. We find that the situation is similar to that in equilibrium [Gurarie and Radzihovsky(2007)]. In the case of a broad Feshbach resonance, mean field is expected to hold for quenches where both initial and final detunings are far from the unitary point. A quench into the unitary point is a very interesting problem addressed by some publications before [Bulgac and Yoon(2009)], but the method we employ here is not applicable to this case.

Nevertheless, a variety of quenches are still accessible to our description even when the resonance is broad, including BCS$\rightarrow$ BCS, BCS $\rightarrow$ BEC, BEC $\rightarrow$ BCS, and BEC $\rightarrow$ BEC, where BCS and BEC stand for the value of the interaction strength far weaker or far stronger than that at the unitary point. In the case of BCS-BEC superfluids formed with interactions

generated by narrow Feshbach resonances, the mean field theory treatment is valid even at the threshold of the formation of the bound state and throughout the BCS-BEC crossover. Here we consider quenches of the detuning ω for both narrow and broad resonances within mean field. Note that in the case of the one channel model we expect the mean field on the BEC side to be valid only in the far BEC limit where the ground state essentially consists of non-interacting Bose condensed molecules[Levinsen and Gurarie(2006)].

In the mean field treatment the condensate is described by the $\mathbf{q} = 0$ (I have no idea, why is it so???) part of the Hamiltonian (8.1), which is decoupled from $\mathbf{q} \neq 0$ terms in this approximation. The Hamiltonian therefore becomes

$$\hat{H}_{2\text{ch}} = \sum_{\mathbf{p}} 2\epsilon_{\mathbf{p}} \hat{s}^z_{\mathbf{p}} + \omega \hat{b}^\dagger \hat{b} + g \sum_{\mathbf{p}} \left(\hat{b}^\dagger \hat{s}^-_{\mathbf{p}} + \hat{s}^+_{\mathbf{p}} \hat{b} \right), \tag{8.5}$$

where

$$\hat{s}^-_{\mathbf{p}} = \hat{a}_{\mathbf{p}\uparrow} \hat{a}_{-\mathbf{p}\downarrow}, \;\; \hat{s}^z_{\mathbf{p}} = \frac{1}{2} \left(\hat{a}^\dagger_{\mathbf{p}\uparrow} \hat{a}_{\mathbf{p}\uparrow} + \hat{a}^\dagger_{-\mathbf{p}\downarrow} \hat{a}_{-\mathbf{p}\downarrow} - 1 \right) \tag{8.6}$$

are Anderson pseudospin-1/2 operators [Anderson(1958)] and

$$\hat{b} = \hat{b}_{\mathbf{q}=0}.$$

(okay, maybe if I think for 10 min, I'll derive it...)

Hamiltonian (8.5) is also known as inhomogeneous Dicke or Tavis-Cummings model. In a quantum quench problem we need to solve Heisenberg equations of motion for this Hamiltonian for given initial conditions

$$\begin{aligned} \frac{d\hat{\vec{s}}_{\mathbf{p}}}{dt} &= \hat{\vec{B}}_{\mathbf{p}} \times \hat{\vec{s}}_{\mathbf{p}}, \quad \frac{d\hat{b}}{dt} = -i\omega \hat{b} - ig\hat{J}_-, \\ \hat{\vec{J}} &= \sum_{\mathbf{p}} \hat{\vec{s}}_{\mathbf{p}}, \quad \hat{\vec{B}}_{\mathbf{p}} = 2g\hat{\vec{b}} + 2\epsilon_{\mathbf{p}} \hat{\mathbf{z}}, \end{aligned} \tag{8.7}$$

where $\hat{\vec{b}} = \hat{b}_x \hat{\mathbf{x}} + \hat{b}_y \hat{\mathbf{y}}$, $\hat{b}_x$ and $-\hat{b}_y$ are Hermitian and anti-Hermitian parts of the operator $\hat{b} = \hat{b}_x - i\hat{b}_y$, and $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are coordinate unit vectors.

The second step in the mean field treatment of the two-channel model is to replace Heisenberg operator $\hat{b}(t)$ in the first equation of motion in (8.7) with its time-dependent quantum mechanical average, $\hat{b}(t) \to \langle \hat{b}(t) \rangle \equiv b(t)$, which is expected to be exact in thermodynamic limit as long as the $\mathbf{q} = 0$ state is macroscopically occupied at all times.

This replacement can be shown to be exact in equilibrium using the exact solution for the spectrum of inhomogeneous Dicke model [Richardson(1977), Gaudin(1983)] and numerically for the time dependent problem [Sträter et al.(2012)] (to understand why is it so, I need to read these articles for 3 days at least).

Upon this replacement equations of motion become linear in operators and taking their quantum mechanical average, we obtain

$$\begin{aligned} \dot{\vec{s}}_{\mathbf{p}} &= \vec{B}_{\mathbf{p}} \times \vec{s}_{\mathbf{p}}, \quad \dot{b} = -i\omega b - igJ_-, \\ \vec{J} &= \sum_{\mathbf{p}} \vec{s}_{\mathbf{p}}, \quad \vec{B}_{\mathbf{p}} = 2g\vec{b} + 2\epsilon_{\mathbf{p}} \hat{\mathbf{z}}, \end{aligned} \tag{8.8}$$

where $\vec{s}_{\mathbf{p}} = \langle \hat{\vec{s}}_{\mathbf{p}} \rangle$. These are Hamiltonian equations of motion for a classical (what is meant by "classical") Hamiltonian

$$H_{2\text{ch}} = \sum_{\mathbf{p}} 2\epsilon_{\mathbf{p}} s^z_{\mathbf{p}} + \omega \bar{b} b + g \sum_{\mathbf{p}} \left(\bar{b} s^-_{\mathbf{p}} + b s^+_{\mathbf{p}} \right), \tag{8.9}$$

which describes a set of angular momenta (classical spins or vectors) coupled to a harmonic oscillator. Here, $\bar{b}$ denotes the complex conjugate of b. These dynamical variables obey the following Poisson brackets:

$$\left\{s^a_{\mathbf{p}}, s^b_{\mathbf{k}}\right\} = -\varepsilon_{abc}\delta_{\mathbf{pk}}s^c_{\mathbf{p}}, \quad \{b, \bar{b}\} = i, \tag{8.10}$$

where a, b, and c stand for spatial indecies x, y, and z.

Similar steps in the case of the single channel model (8.3) lead to a classical spin Hamiltonian

$$H_{1\text{ch}} = \sum_{\mathbf{p}} 2\epsilon_{\mathbf{p}} s^z_{\mathbf{p}} - \frac{\lambda}{\nu_F}\sum_{\mathbf{p},\mathbf{p}'} s^-_{\mathbf{p}} s^+_{\mathbf{p}'} \tag{8.11}$$

together with the corresponding equations of motion.

An important characteristic of the system both in and out of equilibrium is the superfluid order parameter or the gap function defined in the two channel model as

$$\Delta(t) = -g\langle\hat{b}(t)\rangle = -gb(t) \equiv \Delta_x(t) - i\Delta_y(t). \tag{8.12}$$

In the one channel limit this expression turns into

$$\Delta_{1\text{ch}}(t) = \frac{\lambda}{\nu_F}\sum_{\mathbf{p}}\langle\hat{a}_{\mathbf{p}\uparrow}(t)\hat{a}_{-\mathbf{p}\downarrow}(t)\rangle = \frac{\lambda}{\nu_F}\sum_{\mathbf{p}} s^-_{\mathbf{p}}. \tag{8.13}$$

The magnitude $|\Delta(t)|$ of the order parameter is known as the Higgs or amplitude mode for its similarity with the Higgs boson[Pekker and Varma(2014), Barankov and Levitov(2007)] (I don't still understand this relation to Higgs boson. I am not sure that there are answers in these articles, but once I'll try to find it out) and its time-dependent phase represents a Goldstone mode. Note however that out of equilibrium the gap function does not entirely determine the state of the system. It specifies the effective magnetic field acting on each spin according to Eq. (8.8), but there is still a certain freedom in how the spin moves in this field. For example, even for a constant field the spin can precess around it making an arbitrary constant angle with its direction.

In above models we took a free single particle spectrum, $\varepsilon_{\mathbf{p}} = p^2/2m$ and labeled states with momenta $\mathbf{p}$. This choice is not essential for our analysis. We can as well consider an arbitrary spectrum ε_i. The pairing is then between pairs of time reversed (time reversed) states[Anderson(1959), Kurland et al.(2000)], see also the first two pages in Ref. [Yuzbashyan et al.(2005a)] for more details.

For example, in Hamiltonian (8.5) this results in relabeling $\hat{\vec{s}}_{\mathbf{p}} \to \hat{\vec{s}}_i$, $\hat{a}_{\mathbf{p}\uparrow}\hat{a}_{-\mathbf{p}\downarrow} \to \hat{a}_{i\uparrow}\hat{a}_{i\downarrow}$, $\hat{a}^\dagger_{\mathbf{p}\uparrow}\hat{a}_{\mathbf{p}\uparrow} \to \hat{a}^\dagger_{i\uparrow}\hat{a}_{i\uparrow}$ etc., where the state $|i\downarrow\rangle$ is the time reversed counterpart of $|i\uparrow\rangle$.

(suddenly index i appears, is it the same momenta index??)

Our results below depend only on the density of the single particle states $\nu(\varepsilon)$ (why??)in the continuum limit regardless of whether these states are characterized by momenta $\mathbf{p}$ or any other set of quantum numbers i.

8.1.2 Ground state

In the ground state

$$\Delta(t) = \Delta_0 e^{-2i\mu t}, \tag{8.14}$$

where the magnitude Δ_0 is time-independent. Apart from an overall rotation about the z-axis with frequency 2μ (why 2μ?), the ground state is a static solution of the equations of motion

Figure 8.1: Ground state chemical potential μ (I am very confused, I thought it is fixed always. read about it a lot!) for two channel model in 3d in units of the Fermi energy ε_F as a function of the ground state gap Δ_0 for various resonance width γ. $\mu(\Delta_0)$ is calculated from Eqs. (8.25) and (8.26). Note that in the two channel model Δ_0 is bounded from above by $\Delta_{\max}$.

that minimizes $H_{2\text{ch}}$. The minimum is achieved when each spin is directed against its effective magnetic field, i.e. (check it, I don't see it yet)

$$s_{\mathbf{p}}^{-} = \frac{\Delta_0 e^{-2i\mu t}}{2E(\varepsilon_{\mathbf{p}};\Delta_0,\mu)}, \quad s_{\mathbf{p}}^{z} = -\frac{\varepsilon_{\mathbf{p}}-\mu}{2E(\varepsilon_{\mathbf{p}};\Delta_0,\mu)}, \tag{8.15}$$

where

$$E(\varepsilon;\Delta,\mu) \equiv \sqrt{(\varepsilon-\mu)^2+\Delta^2}. \tag{8.16}$$

Note that the length of the spin $s_{\mathbf{p}} = 1/2$. This is because the ground state is a tensor product of single spin-1/2 wave functions and $\vec{s}_{\mathbf{p}} = \langle \hat{\vec{s}}_{\mathbf{p}} \rangle$. (why?? see another paper that really explains what is going on???)

The equation of motion (8.8) for b yields (check it, but maybe it is easy)

$$|J_-| = \frac{(\omega-2\mu)\Delta_0}{g^2}, \tag{8.17}$$

which implies a self-consistency equation for Δ_0

$$\frac{(\omega-2\mu)}{g^2} = \sum_{\mathbf{p}} \frac{1}{2E(\varepsilon_{\mathbf{p}};\Delta_0,\mu)}. \tag{8.18}$$

Further, the Hamiltonian (8.9) conserves

$$n = \bar{b}b + \sum_{\mathbf{p}} \left(s_{\mathbf{p}}^{z} + \frac{1}{2} \right), \tag{8.19}$$

which is the average total number of bosons and fermion pairs. This number is related to Δ_0 and the chemical potential μ as

$$2n = \frac{2\Delta_0^2}{g^2} + \sum_{\mathbf{p}} \left(1 - \frac{\varepsilon_{\mathbf{p}}-\mu}{E(\varepsilon_{\mathbf{p}};\Delta_0,\mu)} \right). \tag{8.20}$$

(why?????)

The Fermi energy ε_F is the chemical potential of the fermionic atoms at zero temperature in the absence of any interaction, when only fermions are present. It provides an overall energy scale and it is convenient to measure all energies in units of the Fermi energy. Thus, from now on, we set everywhere below

$$\varepsilon_F = 1. \tag{8.21}$$

Below we often switch from discrete to continuum (thermodynamic limit) formulations. (I don't understand these two pictures, for me it is one picture. I'll think more about it later)

In the former version, there are N discrete single-particle energy levels $\varepsilon_{\mathbf{p}}$ with certain degeneracy each.

Any quantity $A_{\mathbf{p}}$ we consider in this paper depends on $\mathbf{p}$ only through $\varepsilon_{\mathbf{p}}$, $A_{\mathbf{p}} = A(\varepsilon_{\mathbf{p}})$ (why???).

For example, all spins $\vec{s}_{\mathbf{p}}$ on a degenerate level $\varepsilon_{\mathbf{p}}$ are parallel at all times and effectively merge into a single vector. There are N such vectors, so we count N distinct classical spins.

In thermodynamic limit, energies $\varepsilon_{\mathbf{p}}$ form a continuum on the positive real axis, i.e. are described by a continuous variable ε with a density of states $\nu(\varepsilon)$ that depends on the dimensionality of the problem

$$\nu(\varepsilon) = \nu_F f(\varepsilon), \tag{8.22}$$

where ν_F is the *bulk* density of states (proportional to the system volume) at the Fermi energy, $f(\varepsilon) = 1$ in 2d, and $f(\varepsilon) = \sqrt{\varepsilon}$ in 3d. Summations over $\mathbf{p}$ turn into integrations,

$$\sum_{\mathbf{p}} A_{\mathbf{p}} \to \nu_F \int_0^\infty A(\varepsilon) f(\varepsilon) d\varepsilon. \tag{8.23}$$

With only fermions present, the total particle number is

$$2n = \int_0^1 2\nu(\varepsilon) d\varepsilon = \frac{4}{d}\nu_F, \tag{8.24}$$

where $d = 2, 3$ is the number of spatial dimensions. Interaction redistributes this number between fermions and bosons as in Eq. (8.20).

Combining Eqs. (8.20) $\left(2n = \frac{2\Delta_0^2}{g^2} + \sum_{\mathbf{p}} \left(1 - \frac{\varepsilon_{\mathbf{p}} - \mu}{E(\varepsilon_{\mathbf{p}}; \Delta_0, \mu)}\right)\right)$ and (8.24) and taking the continuum limit, we obtain

$$\frac{4}{d} = \frac{2\Delta_0^2}{\gamma} + \int_0^\infty \left(1 - \frac{\varepsilon - \mu}{\sqrt{(\varepsilon - \mu)^2 + \Delta_0^2}}\right) f(\varepsilon) d\varepsilon, \tag{8.25}$$

where γ is the dimensionless resonance width defined in Eq. (8.2).

Similarly, Eq. (8.18) $\left(\frac{(\omega - 2\mu)}{g^2} = \sum_{\mathbf{p}} \frac{1}{2E(\varepsilon_{\mathbf{p}}; \Delta_0, \mu)}\right)$ becomes in the thermodynamic limit

$$\frac{2\omega - 4\mu}{\gamma} = \int_0^{\varepsilon_\Lambda} \frac{f(\varepsilon) d\varepsilon}{\sqrt{(\varepsilon - \mu)^2 + \Delta_0^2}}, \tag{8.26}$$

where ε_Λ is the high energy cutoff. In 3d it can be eliminated by an additive renormalization of the detuning ω, see e.g. Ref. [Gurarie and Radzihovsky(2007)]. (what have they done???)

This however does not affect our results for the quench dynamics as they depend on the difference between the initial and final values of the detuning.

Eqs. (8.25) and (8.26) contain two independent parameters not counting the cutoff. For example, we can choose γ and ω and determine μ and Δ_0 from these equations, or choose γ

and Δ_0 and determine μ and ω etc., see Fig. 8.1 for a plot of $\mu(\Delta_0)$ for various γ in 3d. Note also that $\Delta_0^2 = g^2\bar{b}b$ is proportional to the number of bosons and is therefore limited by the total number of particles. Eq. (8.25) implies

$$\Delta_0 \leq \sqrt{\frac{2\gamma}{d}} = \Delta_{\max}. \tag{8.27}$$

8.1.3 Quench setup and initial conditions

In a quantum quench setup we prepare the system in a ground state at a certain detuning ω_i, i.e. the initial state is

$$\begin{aligned} s^-_{\mathbf{p}}(t=0) &= \frac{\Delta_{0i}}{2E(\varepsilon_{\mathbf{p}};\Delta_{0i},\mu_i)}, \\ s^z_{\mathbf{p}}(t=0) &= -\frac{\varepsilon_{\mathbf{p}}-\mu_i}{2E(\varepsilon_{\mathbf{p}};\Delta_{0i},\mu_i)}, \end{aligned} \tag{8.28}$$

where Δ_{0i}, μ_i are the ground state values determined by Eqs. (8.25) and (8.26) with $\omega = \omega_i$. We then quench the detuning $\omega_i \to \omega_f$ and evolve the system with the two-channel Hamiltonian (8.9) starting from the initial state (8.28) at $t = 0$.

The state of the system is fully determined by the many-body wavefunction, which in the mean field treatment is at all times a product state of the form

$$|\Psi(t)\rangle = |\psi(t)\rangle \otimes \left(\hat{b}^\dagger\right)^{n(t)}|0\rangle, \tag{8.29}$$

where $n(t) = |b(t)|^2$ and $|\psi(t)\rangle$ is the fermionic part of the wave function:

$$|\psi(t)\rangle = \prod_{\mathbf{p}} \left[\overline{u}_{\mathbf{p}}(t) + \overline{v}_{\mathbf{p}}(t)\hat{a}^\dagger_{\mathbf{p}\uparrow}\hat{a}^\dagger_{-\mathbf{p}\downarrow}\right]|0\rangle. \tag{8.30}$$

Bogoliubov amplitudes $u_{\mathbf{p}}(t), v_{\mathbf{p}}(t)$ obey the Bogoliubov de Gennes (BdG) equations

$$i\frac{\partial}{\partial t}\begin{pmatrix} u_{\mathbf{p}}(t) \\ v_{\mathbf{p}}(t) \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{p}} & \Delta(t) \\ \bar{\Delta}(t) & -\epsilon_{\mathbf{p}} \end{pmatrix}\begin{pmatrix} u_{\mathbf{p}}(t) \\ v_{\mathbf{p}}(t) \end{pmatrix}, \tag{8.31}$$

with the normalization condition $|u_{\mathbf{p}}|^2 + |v_{\mathbf{p}}|^2 = 1$. Apart from an overall time-dependent phase (which is important for certain observables), these equations are equivalent to the classical spin equations of motion (8.8) and spins are related to the amplitudes as

$$\frac{s^-_{\mathbf{p}}}{s_{\mathbf{p}}} = 2u_{\mathbf{p}}\overline{v}_{\mathbf{p}}, \quad \frac{s^z_{\mathbf{p}}}{s_{\mathbf{p}}} = |v_{\mathbf{p}}|^2 - |u_{\mathbf{p}}|^2, \tag{8.32}$$

where $s_{\mathbf{p}}$ is the length of the spin. For quench initial conditions $s_{\mathbf{p}} = 1/2$, as explained below Eq. (8.16).

Each quench is uniquely characterized by three parameters – the resonance width $\gamma = g^2\nu_F$, the initial ω_i and final ω_f values of the detuning in units of the Fermi energy. Indeed, ω_i and γ determine Δ_{0i} and μ_i and thus the initial condition, while the equations of motion (8.7) in the thermodynamic limit depend only on ω_f and γ. To see the latter, note that model parameters enter the equation of motion for spin $\vec{s}_{\mathbf{p}} \equiv \vec{s}(\varepsilon_{\mathbf{p}})$ only through $\Delta = -gb$, while the equation of motion for the bosonic field b can be equivalently written as

$$\dot{\Delta} = -i\omega_f\Delta + i\gamma\int_0^\infty s^-(\varepsilon)f(\varepsilon)d\varepsilon. \tag{8.33}$$

Instead of ω_i, ω_f we find it more convenient to characterize the quench by Δ_{0i}, Δ_{0f} – the ground state gaps corresponding to these values of the detuning. As discussed below Eq. (8.26), for a given γ, the detuning ω uniquely determines Δ_0 and vice versa. Note that Δ_{0f} has nothing to do with the time-dependent gap function $\Delta(t)$ and in particular with the large time asymptote $\Delta(t \to \infty)$. Whenever $\Delta(t)$ goes to a constant at large times, we denote this constant Δ_∞.

8.1.4 Main results

Our main result is a complete description of the long time dynamics of two and one channel models (8.9) and (8.11) in two and three spatial dimensions following a quench of the detuning $\omega_i \to \omega_f$ (coupling $\lambda_i \to \lambda_f$ in the one channel model) in the thermodynamic limit. A key effect that makes such a description possible is a drastic reduction in the number of effective degrees of freedom as $t \to \infty$. It turns out that the large time dynamics can be expressed in terms of just a few new collective spins plus the oscillator in the two channel case that are governed by the same Hamiltonians (8.9) and (8.11) only with new effective parameters replacing $\varepsilon_{\mathbf{p}}$ and ω. The number of collective spins is $m = 0, 1$ or 2 and $m = -1, 0$ or 2 for one and two channel models, respectively, depending on the quench. The difference is due to the presence of the oscillator degree of freedom in the latter case. For example, $m = -1$ means that the effective large time Hamiltonian H_{red} not only has no spins, but also the oscillator b is absent, i.e. $H_{\text{red}} = 0$. This reduction effect combined with integrability of classical Hamiltonians (8.9) and (8.11) allows us to determine the state of the system (its many-body wave function) at $t \to \infty$. We explain this method in detail in Sect. 8.2. This subsection provides a summary of main results obtained with the help of this method.

In Sects. 8.3 and 8.4, we construct exact quench phase diagrams shown in Figs. 8.2 - 8.5. Depending on the values of ω_i and ω_f either system reaches one of three distinct steady states labeled by I, II (including subregion II') and III that can be thought about as nonequilibrium phases with second order phase transition lines between them ($t \to \infty$ limit of the order parameter $\Delta(t)$ is continuous along lines separating different regions). These steady states correspond to $m = 0, 1$ or 2 collective spins, respectively, for one channel model and to $m = -1, 0$ or 2 in the case of two channels.

Each point in the quench phase diagrams represents a particular quench specified by a pair of values $(\Delta_{0i}, \Delta_{0f})$. Here Δ_0 is the gap that the system would have in the ground state at detuning ω, which is a known function of ω. Values Δ_{0i} and Δ_{0f} – ground state gaps for $\omega = \omega_i$ and ω_f, respectively – uniquely determine ω_i and ω_f at fixed resonance width γ. Note that Δ_{0f} is *not* the magnitude of the actual steady state gap function $|\Delta(t)|$. Each quench $\omega_i \to \omega_f$ (or $\lambda_i \to \lambda_f$) therefore maps to a single point $(\Delta_{0i}, \Delta_{0f})$ and vise versa.

Steady states I, II, and III reached by the system at $t \to \infty$ can be described in terms of the superfluid order parameter $\Delta(t)$. In region I of phase diagrams in Figs. 8.2 - 8.5 the gap function vanishes at large times, $\Delta(t) \to 0$, see Fig. 8.6.

In region II (including subregion II') the magnitude of the order parameter asymptotes to a nonzero constant Δ_∞ as illustrated in Fig. 8.7,

$$\Delta(t) \to \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}, \tag{8.34}$$

where $\Delta_\infty, \mu_\infty$ are functions of ω_i, ω_f (or, equivalently, of Δ_{0i} and Δ_{0f}), and γ to be determined below, and φ is a contstant phase. Plots of Δ_∞ and μ_∞ as functions of Δ_{0f} for fixed Δ_{0i} are shown in Figs. 8.9, 8.18, and 8.19. The quantity μ_∞ plays the role of the out-of-equilibrium chemical potential. Subregions II and II' of region II correspond to $\mu_\infty > 0$ and $\mu_\infty < 0$, respectively.

In region III of quench phase diagrams the amplitude of the order parameter oscillates persistently at large times as shown in Fig. 8.8,

$$\Delta(t) \to \sqrt{\Lambda^2(t) + h_1} e^{-i\Phi(t)}, \tag{8.35}$$

where

$$\Lambda(t) = \Delta_+ \mathrm{dn}\left[\Delta_+(t - t_0), k'\right], \quad k' = \frac{\Delta_-}{\Delta_+}, \tag{8.36}$$

dn is the Jacobi elliptic function and t_0 is an integration constant. The magnitude of the order parameter oscillates periodically between $\Delta_b = (\Delta_-^2 + h_1)^{1/2}$ and $\Delta_a = (\Delta_+^2 + h_1)^{1/2}$. The phase contains linear and periodic parts[not(a)]

$$\Phi(t) = 2\mu t - \int \frac{\kappa dt}{\Lambda^2(t) + h_1}. \tag{8.37}$$

Constants $h_1, \Delta_+, \Delta_-, \mu$, and κ are known functions of Δ_{0i}, Δ_{0f} (or ω_i, ω_f), and γ to be specified below, see also Figs. 8.9 and 8.10 and refer to Sect. 8.2.4 for more information about the periodic solution.

Figure 8.2: Detuning quench phase diagrams for two-channel model (8.1) in 2d for an assortment of resonance widths γ. Each point represents a single quench labeled by Δ_{0i} (vertical axis) and Δ_{0f} (horizonal axis) – pairing gaps the system would have in the ground state for initial and final detunings. At large times the system ends up in one of three steady states shown as regions I, II (including II'), and III. For quenches in region I the order parameter vanishes, $\Delta(t) \to 0$. In II $\Delta(t) \to \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}$ and in III $|\Delta(t)|$ oscillates persistently. Subregions II and II' differ in the sign of μ_∞ (out of equilibrium analog of the chemical potential): $\mu_\infty > 0$ in II and $\mu_\infty < 0$ in II'. The diagonal, $\Delta_{0i} = \Delta_{0f}$, is the no quench line. To the left of it are strong to weak coupling quenches, to the right – weak to strong. $\Delta_{\max} = \varepsilon_F\sqrt{\gamma}$ in 2d is the maximum possible ground state gap and $\Delta_{0\times}$ is the ground state gap corresponding to zero chemical potential, i.e. $\Delta_{0\times}$ is given by Eq. (8.25) for $\mu = 0$.

Figure 8.3: Same as Fig. 8.2 but in three spatial dimensions.

Figure 8.4: Interaction (λ) quench phase diagram for one-channel model (8.11) in two dimensions. Otherwise same as Fig. 8.2

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Figure 8.5: Interaction quench phase diagram for one-channel model (8.11) in 3d. Otherwise same as Fig. 8.2. Consider e.g. quenches from fixed infinitesimal coupling λ_i (small Δ_{0i}) to various final couplings λ_f. Increasing λ_f (Δ_{0f}) we move through gapless (I), gapped (II), then oscillating (III) steady states. As λ_f increases further oscillations disappear and we again end up in a steady state characterized by constant asymptotic $|\Delta(t)|$ (II').

Figure 8.6: $|\Delta(t)|$ in region I for 3d two-channel model, $\gamma = 1$, obtained from numerical evolution of $N = 5024$ spins following a detuning quench $\omega_i \to \omega_f$. Here $\Delta_{0i} = 0.27\Delta_{\max}$, $\Delta_{0f} = 4.30 \times 10^{-2}\Delta_{\max}$ [cf. Fig. 8.3(b)]. From these two values all other parameters obtain, e.g. $\mu_i = 0.90\varepsilon_F$, and $\omega_f - \omega_i = 1.97\varepsilon_F$.

Figure 8.7: $|\Delta(t)|$ in regions II (top) and II' (bottom) for 3d two-channel model, $\gamma = 1$, obtained from numerical evolution of $N = 5024$ spins after quenching the detuning ω. $\Delta_{0i} = 0.27\Delta_{\max}, \mu_i = 0.90\varepsilon_F$ in both panels, same as in Fig. 8.6. The final detuning corresponds to: (a) $\Delta_{0f} = 0.56\Delta_{\max} = 2.07\Delta_{0i}$ and (b) $\Delta_{0f} = 0.97\Delta_{\max} = 3.59\Delta_{0i}$. See also Fig. 8.3(b).

Figure 8.8: Amplitude (Higgs mode) and phase $\Phi(t)$ of the order parameter $\Delta(t)$ in region III of Fig. 8.3(c) after detuning quench from deep BCS to BEC in 3d two-channel model for $\gamma = 10$. Numerical evolution with 5024 spins vs. Eqs. (8.35) and (8.37). $\Delta_{0i} = 3.20 \times 10^{-3}\Delta_{\max}$, $\Delta_{0f} = 0.45\Delta_{\max}$, and $\delta\omega = -5.86\gamma$.

Figure 8.9: Limiting values of $|\Delta(t)|$ for 3d two-channel model at large times after a detuning quench as functions of Δ_{0f} (or, equivalently, of final detuning ω_f) at fixed small $\Delta_{0i} = 0.05\Delta_{\max}$ (fixed initial detuning deep in the BCS regime). This corresponds to moving along a horizontal line (not shown) in Fig. 8.3(a) and (c) going through regions I where $|\Delta(t)| \to 0$, II where $|\Delta(t)| \to \Delta_\infty > 0$, III where $|\Delta(t)|$ oscillates periodically between Δ_a and Δ_b, and into region II' where again $|\Delta(t)| \to \Delta_\infty > 0$. Note that persistent oscillations appear and then disappear again as we decrease $\omega_f - \omega_i$ (i.e. increase Δ_{0f} at fixed Δ_{0i}). The same behavior is observed in the 3d one-channel model, see Fig. 8.5.

Figure 8.10: Parameter h_1 in Eq. (8.35) for asymptotic $|\Delta(t)|$ in phase III as a function of Δ_{0f} at fixed small $\Delta_{0i} = 0.05\Delta_{\max}$, same as in Fig. 8.9. For quenches within weak coupling limit $h_1 = 0$, so nonzero h_1 quantifies deviations from this limit. Note that one must have $h_1 \geq -\Delta_-^2$, so that the expression under the square root in Eq. (8.35) is nonnegative.

Figure 8.11: Spin distribution $\cos\theta_{\mathbf{p}}$ as a function of $\varepsilon_{\mathbf{p}}$ (in units of Fermi energy) at large times after the quench in 3d two channel model. In phases I and II, $-\cos\theta_{\mathbf{p}}/2$ is the projection of the spin $\vec{s}_{\mathbf{p}}$ onto its effective magnetic field (z-axis in phase I) around which it precesses. In equilibrium $\cos\theta_{\mathbf{p}} = \pm 1$ (1 in the ground state) for all momenta and in phase I $\cos\theta_{\mathbf{p}} = -1$ and 1 correspond to doubly occupied and unoccupied states, respectively. Quench parameters are $\gamma = 1$ and: (a) $\Delta_{0i} = 0.05\Delta_{\max}, \Delta_{0f} = 0.002\Delta_{\max}$ (BCS to deep BCS quench in phase I); (b) $\Delta_{0i} = 0.78\Delta_{\max}, \Delta_{0f} = 0.001\Delta_{\max}$ (BEC to deep BCS quench in phase I). In both cases $\mu_\infty \approx \varepsilon_F$. Note the Fermi-like shape of the distribution function in (a). Note that $\cos\theta_{\mathbf{p}} \to 1$ as $\varepsilon_{\mathbf{p}} \to \infty$, as it should, indicating that states at very high energies are empty.

Previous studies of the BCS dynamics[Volkov and Kogan(1973), Barankov et al.(2004), 19, Barankov and Levitov(2006), Yuzbashyan and Dzero(2006), Dzero et al.(2007)] were performed in the weak coupling regime when both Δ_{0i} and Δ_{0f} are much smaller than a characteristic high energy scale (Fermi energy for cold gases and Debye energy for conventional superconductors). This limit corresponds to an infinitesimal vicinity of the origin $\Delta_{0i} = \Delta_{0f} = 0$ in our quench phase diagrams in Figs. 8.2 - 8.5 . The weak coupling limit is universal in that it is independent of the resonance width and dimensionality and thus is the same in all diagrams. Critical lines separating regions I and II, and II and III are straight lines in this case coming out of the origin with slopes

$$\frac{\Delta_{0i}}{\Delta_{0f}} = e^{\pm\pi/2}. \tag{8.38}$$

Further, $h_1 = 0$ in Eq. (8.35) and Δ_∞, $\Delta_\pm$ take a simpler form given by Eqs. (8.150) – (8.152), and (8.154).

There are several qualitatively new effects beyond the weak coupling regime. At smaller resonance width $\gamma < \gamma_c = 16/\pi^2$, gapless region I terminates below $\Delta_{\max}$ at $\Delta_{0i} = \gamma\pi/4$ along the vertical axis in 2d. This means that as initial coupling gets stronger (Δ_{0i} increases), even quenches to arbitrarily weak final coupling (small Δ_{0f}) do not result in vanishing $\Delta(t)$ at large times in contrast to the weak coupling regime where quenches with sufficiently large Δ_{0f}/Δ_{0i}

always do. The I-II critical line also displays an interesting backwards bending behavior for $\gamma < \gamma_c = 16/\pi^2$, see the inset in Fig. 8.2(b) and Eqs. (8.161) and (8.157).

Region III of persistent oscillations terminates at a threshold value of $\Delta_{0f} < \Delta_{\max}$ in 3d, see Figs. 8.3 and 8.5. This means that even quenches from an infinitesimally weak initial coupling ($\lambda_i = 0+$ in the one channel model, which corresponds to a vicinity of the normal state) to final couplings stronger then a certain threshold value produce no oscillations and $|\Delta(t)|$ instead goes to a constant. At finite but small initial gap Δ_{0i} (e.g. along the dashed line in Fig. 8.3) there is a reentrant behavior in both 2d and 3d as the final coupling (Δ_{0f}) increases when first there are no oscillations, then they appear, and then disappear again. The threshold value of Δ_{0f} where the critical line separating regions II and III terminates is given by Eq. (8.170) (plotted as a function of the resonance width in Fig. 8.22) and Eq. (8.193) for one and two channel models, respectively. For more details about quench diagrams, such as the shape of the critical lines, various thresholds and termination points, values of parameters (e. g. $\Delta_\infty, \mu_\infty, \Delta_+$, and Δ_-) characterizing asymptotic $\Delta(t)$, see Sects. 8.3 and 8.4.

The large time asymptote of $\Delta(t)$ does not fully specify the steady state. One also needs to know the Bogoliubov amplitudes $u_{\mathbf{p}}(t \to \infty), v_{\mathbf{p}}(t \to \infty)$. We calculate them in Sect. 8.2.4 in all three steady states. In terms of spin vectors, this translates into steady state spin distribution. Even in regions I and II where $|\Delta(t)|$ goes to a constant, the steady state of the system is far from any equilibrium state. Time-independent $|\Delta(t)|$ means that in a frame that rotates around z-axis with frequency $2\mu_\infty$ the magnetic field $\vec{B}_{\mathbf{p}}$ that acts on spin $\vec{s}_{\mathbf{p}}$ in Eq. (8.7) is constant. In equilibrium $\vec{s}_{\mathbf{p}}$ aligns with $\vec{B}_{\mathbf{p}}$ or $-\vec{B}_{\mathbf{p}}$ (ground state). In steady states I and II it instead rotates around $\vec{B}_{\mathbf{p}}$ making a constant angle with it. Let $\theta_{\mathbf{p}}$ be the angle between $\vec{s}_{\mathbf{p}}$ and $-\vec{B}_{\mathbf{p}}$ (negative z-axis in steady state I), so that in the ground state $\theta_{\mathbf{p}} = 0$. Out of equilibrium $\theta_{\mathbf{p}}$ determines the steady state spin distribution function and is given by Eq. (8.134). This expression for $\cos\theta(\varepsilon_{\mathbf{p}})$ applies in all three steady states, but its interpretation in region III is slightly different and will be explained below. A plot of the distribution function $\cos\theta_{\mathbf{p}}$ is shown in Fig. 8.11. We explore the asymptotic states produced by detuning or interaction quenches in detail in Sect. 8.2.4. In Sect. 8.7 we provide further insight into their physical nature and discuss their experimental signatures.

We perform detailed analysis of linearized equations of motion that goes much beyond previous work even in the weak coupling regime and yields a range of new results. Small quenches of the detuning correspond to a small neighborhood of the diagonal in quench diagrams in Figs. 8.2 - 8.5, i.e. they fall within region II where $|\Delta(t)| \to \Delta_\infty$ and Eq. (8.34) applies. We show that within linear approximation $\Delta_\infty = \Delta_{0f}$ and $\mu_\infty = \mu_f$, i.e. there are no corrections to these equations linear in the change of detuning or, equivalently, in $\delta\Delta_0 = \Delta_{0f} - \Delta_{0i}$. This is in fact a general result that has been overlooked by previous work – to first order in deviations from the ground state $\Delta(t)$ always asymptotes to its ground state form for the Hamiltonian with which the system evolves at $t > 0$. Note however that when quadratic correction is taken into account one gets $\Delta_\infty < \Delta_{0f}$. For example, in the weak coupling regime we find

$$\Delta_\infty = \Delta_{0f} - \frac{(\delta\Delta_0)^2}{6\Delta_{0f}}. \tag{8.39}$$

We obtain an exact expression for $\Delta(t)$, Eqs. (8.231), (8.232), and (8.233), valid at all times and arbitrary coupling strength for both one and two channel models. In weak coupling regime this expression simplifies so that

$$|\Delta(t)| = \Delta_{0f} - 2\delta\Delta_0 \int_0^\infty \frac{dx}{\pi} \frac{\cos\left[2\Delta_0 t \cosh(\pi x/2)\right]}{1+x^2}, \tag{8.40}$$

From here short and long times asymptotes follow. At short times the order parameter amplitude rises or falls sharply as

$$|\Delta(t)| = \Delta_{0i} + \frac{\delta\Delta_0}{|\ln(\Delta_0 t)|}. \tag{8.41}$$

And the long time behavior in the weak coupling limit is

$$|\Delta(t)| = \Delta_{0f} - \frac{2\delta\Delta_0}{\pi^{3/2}} \frac{\cos(2\Delta_0 t + \pi/4)}{\sqrt{\Delta_0 t}}. \tag{8.42}$$

At stronger coupling in region II (but not II') the long time asymptote is still given by Eq. (8.42), only the coefficient in front of the second term on the right hand side is more involved.

Regions II and II' differ in the sign of the phase frequency μ_∞, $\mu_\infty > 0$ in II and $\mu_\infty < 0$ in II'. We will see below that frequency (Fourier) spectrum of quench dynamics in regions II and II' is $E_\infty(\varepsilon_{\mathbf{p}}) = \sqrt{(\varepsilon_{\mathbf{p}} - \mu_\infty)^2 + \Delta_\infty^2}$, so that the Fourier transform of a dynamical quantity reads $\int_0^\infty A(\varepsilon) e^{-2iE_\infty(\varepsilon)t} d\varepsilon$. For $\mu_\infty > 0$ the phase has a stationary point on the integration path at $\varepsilon = \mu_\infty$, while for $\mu_\infty < 0$ it is absent. As a result, the long time behavior in three dimensions in region II' changes

$$|\Delta(t)| = \Delta_{0f} \left(1 - c\frac{\delta\omega}{\gamma} \frac{\cos(2E_{\min} t + \pi/4)}{(2|\mu|t)^{3/2}}\right), \tag{8.43}$$

where $E_{\min} = \sqrt{\mu^2 + \Delta_0^2}$, c is of order one, and $\delta\omega = \omega_f - \omega_i$. The same expression holds for the one channel model after a replacement $\delta\omega/\gamma \to 1/\lambda_f - 1/\lambda_i$. Oscillation frequency $E_{\min}$ and $1/t^{3/2}$ decay are in agreement with Ref. [36] and reflect the fact that in the absence of a stationary point, the long time asymptote is dominated by the end point of integration at $\varepsilon = 0$, $E(0) = E_{\min}$, and the density of states in 3d vanishes as $\sqrt{\varepsilon}$ at small ε.

In two dimensions linear analysis yields a different approach to the asymptote in region II'

$$|\Delta(t)| = \Delta_{0f} \left(1 - \frac{\delta\omega}{\gamma} \frac{\sin(2E_{\min} t)}{|\mu| t \ln^2 t}\right), \tag{8.44}$$

because of a constant density of states and $\ln\varepsilon$ divergence of the Fourier amplitude of $|\Delta(t)|$ at small ε (see below). We also determine the time-dependent phase of the order parameter $\Phi(t)$ in all cases corresponding to Eqs. (8.40) – (8.44), asymptotes of individual spins $\vec{s}_{\mathbf{p}}(t)$ as $t \to \infty$, and many other new results for the linearized dynamics in Sect. 8.5.

Finally, we extend some of the above results for the long time behavior of $|\Delta(t)|$ to the nonlinear regime, though unlike the linear analysis these results are not rigorous. In region II

$$|\Delta(t)| = \Delta_\infty + c' \frac{\cos(2\Delta_\infty t + \pi/4)}{\sqrt{\Delta_\infty t}}, \tag{8.45}$$

where c' is a dimensionless coefficient. This answer holds for both one and two channel models in either dimension.

For region II' we argue that the answer depends on dimensionality similarly to the linear analysis and

$$|\Delta(t)| = \Delta_\infty \left(1 - c_1 \frac{\sin(2E_\infty^{\min} t)}{t \ln^2 t}\right) \quad \text{in 2d}, \tag{8.46}$$

$$|\Delta(t)| = \Delta_\infty \left(1 - c_2 \frac{\cos(2E_\infty^{\min} t + \pi/4)}{t^{3/2}}\right) \quad \text{in 3d}, \tag{8.47}$$

where $E_\infty^{\min} = \sqrt{\mu_\infty^2 + \Delta_\infty^2}$.

The approach to the gapless steady state (region I) is expected to be

$$|\Delta(t)| = \frac{c_4}{t \ln^r t} \quad \text{in 2d,} \tag{8.48}$$

where $r = 1$ or $r = 2$, and

$$|\Delta(t)| = \frac{c_3}{t^{3/2}} \quad \text{in 3d.} \tag{8.49}$$

We discuss these nonlinear large time asymptotes in more detail in Sect. 8.6.

8.2 Method

Here we describe a method that allows one to determine the asymptotic state of the system at long times. Both the quantum (8.5) and classical (8.9) two-channel models are integrable meaning that there are as many nontrivial conservation laws as there are degrees of freedom. There is an exact Bethe Ansatz type solution for the quantum spectrum [Gaudin(1983)]. In the classical case integrability implies a formal inexplicit solution of the equations of motion in terms of certain multivariable special (hyperelliptic) functions[Yuzbashyan et al.(2005c)] that can be helpful for understanding certain general features of the dynamics. Evaluating specific dynamical quantities of interest for realistic initial conditions with this solution is however roughly equivalent to just solving the equations of motion numerically. But the latter could be as well done directly without the formal exact solution. This is a typical situation in the standard theory of nonlinear integrable systems.

Fortunately, it was realized that at least for the BCS type models the large time dynamics dramatically simplifies in the thermodynamic limit, so that the number of evolving degrees of freedom effectively drops to just a few spins. Building on this insight, Yuzbashyan et. al. [19] were able to develop a method that goes beyond the standard theory and explicitly predicts the long time dynamics in the thermodynamic limit.

The main idea of this method is as follows. First, we construct a special class of *reduced* solutions of the classical equations of motion for the two-channel model such that the dynamics reduces to that of just few effective spins. Then, we choose a suitable reduced solution and fix its parameters so that its integrals of motion match those for a given quench in the thermodynamic limit. Reduced solutions have only few additional arbitrary constants and cannot generally satisfy all of the quench initial conditions (8.28). There are $2N + 2$ initial conditions (two angles per spin plus two initial conditions for the oscillator mode b) and only $N+1$ correspond to the integrals of motion.

Next, exploiting the fact that for fixed $\Delta(t)$ BdG equations (8.31) are linear in the amplitudes $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$, we derive the most general $t \to \infty$ asymptotic solution that has the same $\Delta(t)$ as the reduced one. It has the same integrals as the quench dynamics by construction and, in addition, $N + 1$ arbitrary independent constants to match the remaining initial conditions. We *conjecture* that so constructed asymptotic solution is the true large time asymptote of the actual quench dynamics. To verify this *few spin conjecture* it is sufficient to show that the large time asymptote of the actual $\Delta(t)$ matches that of the reduced (and therefore general asymptotic) solution. We do so numerically in the nonlinear case and analytically for infinitesimal quenches when the dynamics can be linearized.

We consider the two channel model in this and the following sections and then obtain similar results for the one channel (BCS) model in Sect. 8.4 by taking the broad resonance, $\gamma \to \infty$, limit.

8.2.1 Integrability and Lax vector construction

An object called Lax vector plays a key role in our approach. It encodes all the information about the integrals of motion and turns out to be especially useful in analyzing the quench

dynamics in the thermodynamic limit.

(how to guess about the Lax vector???)

The Lax vector is defined as

$$\vec{L}(u) = \sum_{\mathbf{p}} \frac{\vec{s}_{\mathbf{p}}}{u - \varepsilon_{\mathbf{p}}} - \frac{(\omega - 2\mu)}{g^2}\hat{\mathbf{z}} + \frac{2}{g^2}\left[(u-\mu)\hat{\mathbf{z}} - \vec{\Delta}\right], \tag{8.50}$$

where u is an auxiliary complex variable and $\vec{\Delta} \equiv \Delta_x \hat{\mathbf{x}} + \Delta_y \hat{\mathbf{y}}$. Poisson brackets of components of $\vec{L}(u)$ satisfy the following Gaudin algebra:

$$\left\{L^a(u), L^b(v)\right\} = \varepsilon_{abc}\frac{L^c(u) - L^c(v)}{u - v}. \tag{8.51}$$

(check. maybe it is not that hard?)

This implies an important equality

$$\left\{\vec{L}^2(u), \vec{L}^2(v)\right\} = 0. \tag{8.52}$$

Explicit evaluation of $\vec{L}^2(u)$ yields

$$\vec{L}^2(u) = \frac{(2u - \omega)}{g^4} + \frac{4\mathcal{H}_b}{\omega g^2} + \sum_{\mathbf{p}}\left(\frac{2\mathcal{H}_{\mathbf{p}}}{g^2(u - \varepsilon_{\mathbf{p}})} + \frac{s_{\mathbf{p}}^2}{(u - \varepsilon_{\mathbf{p}})^2}\right), \tag{8.53}$$

(check also) where

$$\begin{aligned} \mathcal{H}_{\mathbf{p}} &= g^2 \sum_{\mathbf{q}\neq\mathbf{p}} \frac{\vec{s}_{\mathbf{p}} \cdot \vec{s}_{\mathbf{q}}}{(\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{q}})} + (2\epsilon_{\mathbf{p}} - \omega)s_{\mathbf{p}}^z + g\left(\bar{b}s_{\mathbf{p}}^- + bs_{\mathbf{p}}^+\right), \\ \mathcal{H}_b &= \bar{b}b + \sum_{\mathbf{p}} s_{\mathbf{p}}^z. \end{aligned} \tag{8.54}$$

It follows from Eq. (8.52) that these spin Hamiltonians mutually Poisson commute (anticommute?), i.e.

$$\{\mathcal{H}_{\mathbf{p}}, \mathcal{H}_{\mathbf{p}'}\} = \{\mathcal{H}_{\mathbf{p}}, \mathcal{H}_b\} = 0. \tag{8.55}$$

Moreover, the two-channel Hamiltonian (8.9) is

$$H_{2\text{ch}} = \omega\mathcal{H}_b + \sum_{\mathbf{p}} \mathcal{H}_{\mathbf{p}}. \tag{8.56}$$

This implies that $\mathcal{H}_{\mathbf{p}}$ and $\mathcal{H}_b$ are conserved by $H_{2\text{ch}}$ and establishes the integrability of the two-channel Hamiltonian. Note that $\vec{L}^2(u)$ is also conserved for any value of u and serves as a generator of the integrals of motion for the two-channel model. The same construction works in the quantum case as well; one only needs to promote classical dynamical variables to corresponding quantum operators and replace Poisson brackets with commutators.

Equations of motion can be conveniently and compactly written in terms of the Lax vector as

$$\dot{\vec{L}} = \left(-2\vec{\Delta} + 2u\hat{\mathbf{z}}\right) \times \vec{L}. \tag{8.57}$$

Comparing the residues at the poles at both sides of this equation, we see that it is equivalent the equations of motion for spins (8.8).

The square of the Lax vector is of the form

$$\vec{L}^2(u) = \frac{Q_{2N+2}(u)}{g^4 \prod_{\varepsilon_{\mathbf{p}}}(u - \varepsilon_{\mathbf{p}})^2}, \tag{8.58}$$

where N is the total number of distinct single particle energies $\varepsilon_{\mathbf{p}}$, the product is similarly over nondegenerate values of $\varepsilon_{\mathbf{p}}$, and $Q_{2N+2}(u)$ is a polynomial in u of degree $2N+2$. The roots of this *spectral* polynomial (or equivalently of $\vec{L}^2(u)$) play an important role in further analysis of the asymptotic behavior. Note that since $\vec{L}^2(u)$ is conserved, so are its roots. They thus constitute a set of integrals of motion alternative to Eq. (8.54). Since $\vec{L}^2(u) \geq 0$ for real u, its roots come in complex conjugate pairs.

8.2.2 Reduced solutions

Let us look for special solutions of equations of motion (8.57) such that the Lax vector factorizes into time-dependent and independent parts

$$\vec{L}^{\text{red}}(u) = \sum_{\mathbf{p}} \frac{\vec{\sigma}_{\mathbf{p}}}{u-\varepsilon_{\mathbf{p}}} - \frac{(\omega-2\mu)}{g^2}\hat{\mathbf{z}} + \frac{2}{g^2}\left[(u-\mu)\hat{\mathbf{z}} - \vec{\Delta}\right] = \left(1 + \sum_{\mathbf{p}} \frac{d_{\mathbf{p}}}{u-\epsilon_{\mathbf{p}}}\right)\vec{L}_m(u), \quad (8.59)$$

where $\vec{\sigma}_{\mathbf{p}}$ (not to be confused with Pauli matrices) denote spins in this solution that can have arbitrary length to distinguish them from spins $\vec{s}_{\mathbf{p}}$ for the quench dynamics that have length $1/2$. Further, $d_{\mathbf{p}}$ are time-independent constants to be determined later and $\vec{L}_m(u)$ is the Lax vector for an effective m-spin system

$$\vec{L}_m(u) = \sum_{j=0}^{m-1} \frac{\vec{t}_j}{u-\eta_j} - \frac{(\omega'-2\mu)}{g^2}\hat{\mathbf{z}} + \frac{2}{g^2}\left[(u-\mu)\hat{\mathbf{z}} - \vec{\Delta}\right]. \quad (8.60)$$

Here $\vec{t}_j$ are new collective spin variables placed at new arbitrary "energy levels" η_j. Note that the bosonic field $\vec{b}$ and therefore $\vec{\Delta}$ are the same in the original and reduced models.

Substituting Eq. (8.59) into the equations of motion (8.57), we see that $\vec{L}_m(u)$ satisfies the same equation of motion. This means that variables $\vec{t}_j$ obey Bloch equations (8.8) with $\varepsilon_{\mathbf{p}} \to \eta_j$ and $\omega \to \omega'$, and are therefore governed by the same Hamiltonian

$$H^{\text{red}}_{2\text{ch}} = \sum_{j=0}^{m-1} 2\eta_j t_j^z + \omega'\bar{b}b + g\sum_{j=0}^{m-1}(\bar{b}t_j^- + bt_j^+). \quad (8.61)$$

We will need at most $m=1$ for analyzing the quench dynamics, so we will be able solve the equations of motions for $\vec{t}_j$ directly.

Matching the residues at $u=\varepsilon_{\mathbf{p}}$ on both sides of Eq. (8.59), we express original spins in terms of $\vec{t}_j$

$$\vec{\sigma}_{\mathbf{p}} = d_{\mathbf{p}}\vec{L}_m(\varepsilon_j), \quad (8.62)$$

Constants $d_{\mathbf{p}}$ are determined from the above equation using $\vec{\sigma}_{\mathbf{p}}^{\,2} = \sigma_{\mathbf{p}}^2$, where $|\sigma_{\mathbf{p}}|$ is the length of spin $\vec{\sigma}_{\mathbf{p}}$. Note that $\sigma_{\mathbf{p}}$ can be of either sign (for future convenience). We have

$$d_{\mathbf{p}} = -\frac{\sigma_{\mathbf{p}}}{\sqrt{\vec{L}_m^2(\varepsilon_{\mathbf{p}})}}. \quad (8.63)$$

It is important to note that $\sigma_{\mathbf{p}}$ are arbitrary constants at this point. We will determine them later so that the integrals of motion for the reduced solution match those for quench dynamics.

To satisfy Eq. (8.59), we also need to match the residues at $u=\eta_k$ and the $u\to\infty$ asymptotic. This leads to the following $m+1$ equations:

$$\begin{aligned} &1 + \sum_{\mathbf{p}} \frac{d_{\mathbf{p}}}{\eta_k - \varepsilon_{\mathbf{p}}} = 0 \quad k = 0,\ldots,m-1 \\ &\omega = \omega' - 2\sum_{\mathbf{p}} d_{\mathbf{p}}. \end{aligned} \quad (8.64)$$

Equations (8.64) constrain the coefficients of the spectral polynomial

$$Q_{2m+2}(u) = g^4 \vec{L}_m^2(u) \prod_{k=0}^{m-1} (u - \eta_k)^2, \quad m \geq 0 \tag{8.65}$$

of the m-spin system. Indeed, using Eq. (8.63), we can cast these constraints into the following form:

$$\begin{aligned} \sum_{\mathbf{p}} \frac{\sigma_{\mathbf{p}} \varepsilon_{\mathbf{p}}^{r-1}}{\sqrt{Q_{2m+2}(\varepsilon_{\mathbf{p}})}} &= -\frac{\delta_{rm}}{g^2}, \quad r = 1, \ldots, m \\ \omega' = \omega + 2 \sum_{k=0}^{m-1} \eta_k &+ \sum_{\mathbf{p}} \frac{2\sigma_{\mathbf{p}} g^2 \varepsilon_{\mathbf{p}}^m}{\sqrt{Q_{2m+2}(\varepsilon_{\mathbf{p}})}}. \end{aligned} \tag{8.66}$$

Here $m \geq 0$. These equations can be viewed as equations for determining the lengths of the collective spins $\vec{t}_j$.

We thus constructed a class of solutions such that the dynamics reduces to that of a smaller number of spins. These *few spin solutions* however do not match the quench initial conditions, but, as we will see, the long time asymptote of $\Delta(t)$ after the quench coincides with $\Delta(t)$ of an appropriately chosen few spin solution. Specifically, $m = -1, 0$, and 1 are realized depending on the magnitude and the sign of the change in the detuning ω. Let us therefore consider these particular cases.

m=-1 spin solutions

$m = -1$ refers to the case when there are no collective spins and $b = 0$, i.e. the oscillator (which can be viewed as an infinite length limit of a spin) is effectively absent as well. In other words, $H_{\text{red}} = 0$ and $\vec{L}_m(u) = \frac{2u-\omega'}{g^2}\hat{\mathbf{z}}$. Eq. (8.62) then implies that all spins in the reduced solution are along the z-axis pointing in either positive or negative direction. It is convenient to redefine the sign of $\sigma_{\mathbf{p}}$ (only for $m = -1$) so that $\vec{\sigma}_{\mathbf{p}} = -\sigma_{\mathbf{p}}\hat{\mathbf{z}}$. We see directly from the equations of motion (8.8) that this configuration together with $b = 0$ is indeed a solution, a stationary one in the present case.

m=0 spin solutions

In this case the reduced problem consists of a free classical oscillator as there are no collective spins, i.e. $H_{\text{red}} = \omega' \bar{b} b$. Equations of motion reduce to $\dot{b} = -i\omega' b$. Therefore

$$\Delta(t) = -gb = ce^{-2i\mu t}, \tag{8.67}$$

where c is a complex constant and we defined $\mu = \omega'/2$.

Expressions for the original spins follow from the reduced Lax vector

$$\vec{L}_m(u) = -\frac{2}{g^2}\left(\vec{\Delta} - (u - \mu)\hat{z}\right), \tag{8.68}$$

Eqs. (8.62) and (8.63) imply

$$\vec{\sigma}_{\mathbf{p}} = \frac{\sigma_{\mathbf{p}}}{E(\varepsilon_{\mathbf{p}}; \Delta, \mu)}\left(\vec{\Delta} - (\varepsilon_{\mathbf{p}} - \mu)\hat{\mathbf{z}}\right), \tag{8.69}$$

where $E(\varepsilon_{\mathbf{p}}; \Delta, \mu) = \sqrt{(\varepsilon_{\mathbf{p}} - \mu)^2 + |\Delta|^2}$. We see that the ground state (8.15) is a one spin solution with $c = \Delta_0$ and $\sigma_{\mathbf{p}} = 1/2$ (to minimize the energy). Excited states are also one spin solutions with different parameters.

There is only one (last) constraint among Eqs. (8.66) for $m = 0$, which we recognize as a generalization of the gap equation (8.18) ($\frac{(\omega-2\mu)}{g^2} = \sum_{\mathbf{p}} \frac{1}{2E(\varepsilon_{\mathbf{p}};\Delta_0,\mu)}$).

m=1 spin solutions

This example is substantially more involved then the previous two. Now there is one collective spin $\vec{t}$ coupled to an oscillator,

$$H_{\rm red} = 2\eta t^z + \omega' \bar{b} b + g(\bar{b} t^- + b t^+), \tag{8.70}$$

making the dynamics rather nontrivial. Our main goal presently is to derive a differential equation for $|\Delta(t)| = g|b(t)|$ and to relate its coefficients to the spectral polynomial $Q_4(u)$ of the reduced $m = 1$ problem given in general by Eq. (8.65).

$H_{\rm red}$ conserves $\bar{b}b + t^z$. It follows that t^z can be expressed through $|b|^2$ as $t^z = c_1\Omega^2 + c_2$, where $c_{1,2}$ are constants and we introduced a notation

$$\Delta = \Omega e^{-i\Phi}. \tag{8.71}$$

Eq. (8.62) then implies that the z-component of the original spins in the reduced solution can be similarly expressed through $|\Delta|$ as

$$\sigma^z_{\mathbf{p}} = a_{\mathbf{p}}\Omega^2 + b_{\mathbf{p}}. \tag{8.72}$$

Note that constants $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ are inversely proportional to $\sqrt{\vec{L}^2_m(\varepsilon_{\mathbf{p}})}$ and therefore to $\sqrt{Q_4(\varepsilon_{\mathbf{p}})}$. It turns out that an efficient strategy to derive an equation for Ω and relate its coefficients to those of $Q_4(u)$ is somewhat indirect. First, we use equations of motion for $\vec{\sigma}_{\mathbf{p}}$ together with Eq. (10.3) to obtain an equation for Ω and expressions for $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$. Identifying $\sqrt{Q_4(\varepsilon_{\mathbf{p}})}$ in the latter with the help of Eq. (8.63), we relate the coefficients.

Bloch equations (8.8) for spins in the reduced solution, $\vec{s}^{\rm red}_p \equiv \vec{\sigma}_{\mathbf{p}}$, can be written as

$$\dot{\sigma}^z_{\mathbf{p}} = -i(\sigma^-_{\mathbf{p}}\bar{\Delta} - \sigma^+_{\mathbf{p}}\Delta), \quad \dot{\sigma}^-_{\mathbf{p}} = -2i\sigma^z_{\mathbf{p}}\Delta - 2i\varepsilon_{\mathbf{p}}\sigma^-_{\mathbf{p}}. \tag{8.73}$$

Substituting $\sigma^z_{\mathbf{p}}$ from Eq. (10.3) into the first equation, we obtain

$$\sigma^-_{\mathbf{p}} e^{i\Phi} - \sigma^+_{\mathbf{p}} e^{-i\Phi} = 2i a_{\mathbf{p}}\dot{\Omega}. \tag{8.74}$$

Multiplying the second equation in (8.73) by $e^{i\Phi}$ and adding the resulting equation to its complex conjugate, we get

$$\frac{d}{dt}\left(\sigma^-_{\mathbf{p}} e^{i\Phi} + \sigma^+_{\mathbf{p}} e^{-i\Phi}\right) = 4a_{\mathbf{p}}\varepsilon_{\mathbf{p}}\dot{\Omega} - 2a_{\mathbf{p}}\dot{\Phi}\dot{\Omega}, \tag{8.75}$$

where we also used Eq. (8.74). Integrating this and adding the result to Eq. (8.74), we obtain

$$\sigma^-_{\mathbf{p}} e^{i\Phi} = 2a_{\mathbf{p}}\varepsilon_{\mathbf{p}}\Omega - a_{\mathbf{p}}A + i a_{\mathbf{p}}\dot{\Omega}, \tag{8.76}$$

where $A = \int dt \dot{\Phi}\dot{\Omega}$. Eq. (8.76) implies

$$|\sigma^-_{\mathbf{p}}|^2 = (2a_{\mathbf{p}}\varepsilon_{\mathbf{p}}\Omega - a_{\mathbf{p}}A)^2 + a^2_{\mathbf{p}}\dot{\Omega}^2. \tag{8.77}$$

Eqs. (8.77) and (10.3) combined with the conservation of the length of the spin, $(\sigma^z_{\mathbf{p}})^2 + |\sigma^-_{\mathbf{p}}|^2 = \sigma^2_{\mathbf{p}}$, yield a differential equation for Ω

$$(a_{\mathbf{p}}\Omega^2 + b_{\mathbf{p}})^2 + (2a_{\mathbf{p}}\varepsilon_{\mathbf{p}}\Omega - a_{\mathbf{p}}A + c_{\mathbf{p}})^2 + a^2_{\mathbf{p}}\dot{\Omega}^2 = \sigma^2_{\mathbf{p}} \tag{8.78}$$

Dividing the last equation by $a^2_{\mathbf{p}}$ and rearranging, we obtain

$$\begin{aligned} \dot{\Omega}^2 + \Omega^4 + \Omega^2\left(2\frac{b_{\mathbf{p}}}{a_{\mathbf{p}}} + 4\varepsilon^2_{\mathbf{p}}\right) - 4\varepsilon_{\mathbf{p}}A\Omega \\ + A^2 + \frac{b^2_{\mathbf{p}} - \sigma^2_{\mathbf{p}}}{a^2_{\mathbf{p}}} = 0 \end{aligned} \tag{8.79}$$

It turns out that A is a certain function of Ω. To see this, let $x_{\mathbf{p}}$ be a set of numbers such that $\sum_{\mathbf{p}} x_{\mathbf{p}} = 0$, multiply Eq. (8.79) by $x_{\mathbf{p}}$ and sum over $\mathbf{p}$. This yields

$$A = 2\mu\Omega + \frac{\kappa}{\Omega}, \tag{8.80}$$

where μ and κ are arbitrary real constants. Substituting Eq. (8.80) into Eq. (8.79), we obtain

$$\begin{gathered}\dot{\Omega}^2 + \Omega^4 + 2\Omega^2\left[\frac{b_{\mathbf{p}}}{a_{\mathbf{p}}} + 2\xi_{\mathbf{p}}^2\right] + \\ \frac{\kappa^2}{\Omega^2} + \frac{b_{\mathbf{p}}^2 - \sigma_{\mathbf{p}}^2}{a_{\mathbf{p}}^2} - 4\kappa\xi_{\mathbf{p}} = 0,\end{gathered} \tag{8.81}$$

where $\xi_{\mathbf{p}} = \varepsilon_{\mathbf{p}} - \mu$. Note that the same equation obtains in the reduced problem with $a_{\mathbf{p}} \to c_1$, $b_{\mathbf{p}} \to c_2$ etc. It follows that coefficients must be $\mathbf{p}$-independent, i.e.

$$\frac{b_{\mathbf{p}}}{a_{\mathbf{p}}} + 2\xi_{\mathbf{p}}^2 = 2\rho, \quad \frac{b_{\mathbf{p}}^2 - \sigma_{\mathbf{p}}^2}{a_{\mathbf{p}}^2} - 4\kappa\xi_{\mathbf{p}} = 4\chi, \tag{8.82}$$

where ρ and χ are $\mathbf{p}$-independent constants. We find

$$\begin{gathered} b_{\mathbf{p}} = -2(\xi_{\mathbf{p}}^2 - \rho)a_{\mathbf{p}}, \\ a_{\mathbf{p}} = \frac{-\sigma_{\mathbf{p}}}{2\sqrt{(\xi_{\mathbf{p}}^2 - \rho)^2 - \kappa\xi_{\mathbf{p}} - \chi}}. \end{gathered} \tag{8.83}$$

As mentioned above $a_{\mathbf{p}}$ and $b_{\mathbf{p}}$ are inversely proportional to $\sqrt{Q_4(\varepsilon_{\mathbf{p}})}$. Eq. (8.83) therefore implies

$$Q_4(u) = [(u-\mu)^2 - \rho]^2 - \kappa(u-\mu) - \chi, \tag{8.84}$$

while the differential Eq. (8.81) for Ω reads

$$\dot{\Omega}^2 + \Omega^4 + 4\rho\Omega^2 + \frac{\kappa^2}{\Omega^2} + 4\chi = 0. \tag{8.85}$$

This equation can be solved in terms of elliptic function. Let $w = \Omega^2$. We have

$$\dot{w}^2 + 4w^3 + 16\rho w^2 + 16\chi w + 4\kappa^2 \equiv \dot{w}^2 + 4P_3(w) = 0. \tag{8.86}$$

Further, let $P_3(w) = (w-h_1)(w-h_2)(w-h_3)$, where $h_3 \geq h_2 \geq h_1$, and define

$$\omega = \Lambda^2 + h_1, \quad \Delta_+^2 = h_3 - h_1, \quad \Delta_-^2 = h_2 - h_1. \tag{8.87}$$

We get

$$\dot{\Lambda}^2 = (\Delta_+^2 - \Lambda^2)(\Lambda^2 - \Delta_-^2) \tag{8.88}$$

with the solution

$$\Lambda = \Delta_+ \mathrm{dn}\left[\Delta_+(t-t_0), k'\right], \quad k' = \frac{\Delta_-}{\Delta_+}, \tag{8.89}$$

where dn is the Jacobi elliptic function and t_0 is an arbitrary integration constant.

It also follows from Eq. (8.80) and the definition of A below Eq. (8.76) that the phase of the order parameter is determined as

$$\dot{\Phi} = \frac{dA}{d\Omega} = 2\mu - \frac{\kappa}{\Lambda^2 + h_1}, \quad \Delta = \sqrt{\Lambda^2 + h_1}e^{-i\Phi}. \tag{8.90}$$

8.2.3 Matching integrals of motion

Given the quench initial conditions, we can evaluate all integrals of motion. This is equivalent to evaluating $\vec{L}^2(u)$ in the initial state as it is conserved and contains all the integrals as residues at $u = \varepsilon_{\mathbf{p}}$. It turns out that in the thermodynamic limit it is possible to find a reduced (few spin) solution that has the same $\vec{L}^2(u)$, i.e. exactly the same integrals as the quench dynamics.

In the thermodynamic limit single-particle energies $\varepsilon_{\mathbf{p}}$ form a continuum on the positive real axis and $\vec{L}^2(u)$ therefore has a continuum of poles at $u > 0$. Additionally, $\vec{L}^2(u)$ also has a continuum of roots along the $u > 0$ half line as we show in Appendix 8.10. Thus $\sqrt{\vec{L}^2(u)}$ has a branch cut along $u > 0$ in the continuum limit. There can also be several *isolated* roots whose imaginary parts remain finite in this limit. Isolated roots play an important role in the dynamics; we will determine them below and see that there are at most four such roots (two pairs of complex conjugate roots) for our quench problem.

Eq. (8.59) implies

$$1 + \int d\varepsilon' \frac{d(\varepsilon')\nu(\varepsilon')}{u - \varepsilon'} = -z(u)\sqrt{\frac{\vec{L}^2(u)}{\vec{L}_m^2(u)}}, \quad z(u) = \pm 1, \tag{8.91}$$

where $\vec{L}^2(u)$ is evaluated for the quench initial conditions. Our task is to find the parameters for the reduced problem – $d(\varepsilon)$ and $\vec{L}_m^2(u)$ – so that this equation holds. Then the reduced problem has the same integrals of motion as the quench dynamics.

Both sides of Eq. (8.91) have a branch cut along the positive real axis and tend to 1 as $u \to \infty$ for an appropriate choice of the sign $z(\infty)$. Further, provided the isolated roots of $\vec{L}^2(u)$ coincide with the roots of $\vec{L}_m^2(u)$, there are no more branching points and both sides are analytic away from the shared branch cut at $u > 0$. If we further ensure that the left and the right hand sides of Eq. (8.91) have the same jump across the branch cut, then their difference is an entire function that vanishes at infinity. It is therefore identically zero by Liouville's theorem from complex analysis and Eq. (8.91) holds.

To equate jumps across the branch cut, we take $u \to \varepsilon \pm i0$, apply the well-known formula $1/(x \pm i0) = \mathcal{P}(1/x) \mp i\pi\delta(x)$, and subtract one result from another. This fixes $d(\varepsilon)$,

$$d(\varepsilon) = -\frac{z(\varepsilon)}{2i\pi\nu(\varepsilon)} \frac{\sqrt{\vec{L}^2(\varepsilon_-)} - \sqrt{\vec{L}^2(\varepsilon_+)}}{\sqrt{\vec{L}_m^2(\varepsilon)}}, \tag{8.92}$$

where $\varepsilon_\pm = \varepsilon \pm i0$. According to expression (8.63) for $d_{\mathbf{p}} \equiv d(\varepsilon_{\mathbf{p}})$ this is equivalent to fixing the lengths of the spins, $|\sigma_{\mathbf{p}}| \equiv |\sigma(\varepsilon_{\mathbf{p}})|$, in the few spin solution so that

$$\sigma(\varepsilon) = z(\varepsilon) \frac{\sqrt{\vec{L}^2(\varepsilon_-)} - \sqrt{\vec{L}^2(\varepsilon_+)}}{2i\pi\nu(\varepsilon)}. \tag{8.93}$$

Thus, the few spin solution with this $\sigma(\varepsilon)$ and $\vec{L}_m^2(u)$ whose roots are the same as the isolated roots of $\vec{L}^2(u)$ has the same integrals of motion as the quench problem.

8.2.4 Asymptotic solution for the quench dynamics

There are altogether $2(N+1)$ initial conditions – two angles for each classical spin and two initial conditions for the oscillator. So far, we constructed a reduced m-spin solution that matches $N+1$ integrals of motion. This satisfies $N+1$ initial conditions. The dynamics of the

reduced m-spin Hamiltonian contains $2(m+1)$ constants, $m+1$ of which (integrals of motion for $H_{\rm red}$) are already fixed since we fixed $\vec{L}^2_m(u)$. The remaining $m+1$ constants are not sufficient to match the remaining $N \to \infty$ initial conditions for the quench dynamics at finite m. This is resolved as follows. We use the known m-spin solution to derive a general asymptotic (i.e. valid at $t \to \infty$) solution of the equations of motion for spins $\vec{s}_{\mathbf{p}}$ with the same $\Delta(t)$ and the same integrals of motion as the m-spin solution. Integrals of motion therefore are those for the quench dynamics. In addition, this general solution contains the correct number $N+1$ of independent constants. We therefore conjecture that this is the true solution for the quench dynamics at large times after the quench. By construction, to verify this *few spin conjecture*, it is sufficient to show that the true asymptote of $\Delta(t)$ coincides with $\Delta(t)$ in the m-spin solution because given $\Delta(t)$ we obtain the most general asymptotic solution of equations of motion.

As discussed above Eq. (8.33), each quench is characterized by three parameters – the resonance width γ and the final ω_f and initial ω_i values of the detuning. We determine in the next section that $\vec{L}^2(u)$ for the quench dynamics can have 0, 1, or 2 pairs of isolated complex roots for any γ depending on ω_i and ω_f. These by construction must also be all the roots of $\vec{L}^2_m(u)$, which has $m+1$ pairs of complex conjugate roots according to Eq. (8.65). Cases relevant for the quench phase diagram are therefore $m = -1, 0$, and 1.

It is worthwhile to consider the $m = -1$ case separately in some detail to illustrate this procedure. Suppose $\vec{L}^2(u)$ evaluated for the quench initial condition has no complex (isolated) roots away from the real axis. Then, there is an $m = -1$ spin solution constructed above that in the $N \to \infty$ limit has the same values of the integrals of motion as the spin dynamics. Spins in this solution are all along the z-axis, $\vec{\sigma}_{\mathbf{p}} = -\sigma_{\mathbf{p}}\hat{z}$, and $\Delta(t) = -gb(t) = 0$. It is a particular solution of the equations of motion (8.8) such that $b(t) = 0$.

The general solution of the *spin part* of the equations of motion in Eq. (8.8) with $b(t) = 0$ is: spins $\vec{s}_{\mathbf{p}}$ precess around the z-axis (or equivalently around the reduced spins $\sigma_{\mathbf{p}}$) with frequencies $2\varepsilon_{\mathbf{p}}$, i.e.

$$s^z_{\mathbf{p}} = \frac{\sigma^z_{\mathbf{p}}}{\sigma_{\mathbf{p}}}\frac{\cos\theta_{\mathbf{p}}}{2}, \quad s^-_{\mathbf{p}} = \frac{\sin\theta_{\mathbf{p}}}{2}e^{i\alpha_{\mathbf{p}}(t)}, \tag{8.94}$$

where $\theta_{\mathbf{p}}$ is the angle $\vec{s}_{\mathbf{p}}$ makes with $-\hat{\mathbf{z}}$ and $\alpha_{\mathbf{p}} = -2\varepsilon_{\mathbf{p}}t + \delta_{\mathbf{p}}$. Equivalently, this can be expressed as

$$\vec{s}_{\mathbf{p}} = \frac{\vec{\sigma}_{\mathbf{p}}}{\sigma_{\mathbf{p}}}\frac{\cos\theta_{\mathbf{p}}}{2} + \vec{s}^{\perp}_{\mathbf{p}}, \tag{8.95}$$

where $\vec{s}^{\perp}_{\mathbf{p}}$ is the component transverse to $\vec{\sigma}_{\mathbf{p}}$, which rotates around $\vec{\sigma}_{\mathbf{p}}$ with frequency $2\varepsilon_{\mathbf{p}}$. Note that the length of spin $\vec{s}_{\mathbf{p}}$ is $1/2$ as it should be for the quench initial conditions.

This spin configuration has N additional constants $\delta_{\mathbf{p}}$, but it does not satisfy the equation of motion for $b(t)$ in Eq. (8.8) because $b(t) = 0$, while $J_-(t) = \sum_{\mathbf{p}} s^-_{\mathbf{p}} = \sum_{\mathbf{p}} f_{\mathbf{p}} e^{-2\varepsilon_{\mathbf{p}}t} \neq 0$, where $2f_{\mathbf{p}} = \sin\theta_{\mathbf{p}} e^{i\delta_{\mathbf{p}}}$. However, in the thermodynamic limit $J_-(t) = \int f(\varepsilon)\nu(\varepsilon)e^{-2\varepsilon t} \to 0$ as $t \to \infty$ and this solution becomes self-consistent.

Next, we set $2\sigma_{\mathbf{p}} = \cos\theta_{\mathbf{p}}$ and substitute $\vec{s}_{\mathbf{p}} = \vec{\sigma}_{\mathbf{p}} + \vec{s}^{\perp}_{\mathbf{p}}$ into the Lax vector,

$$\vec{L}(u) = \vec{L}^{\rm red}(u) + \sum_{\mathbf{p}} \frac{\vec{s}^{\perp}_{\mathbf{p}}}{u - \varepsilon_{\mathbf{p}}}. \tag{8.96}$$

The second term vanishes by the Riemann-Lebesgue lemma (dephases) as $t \to \infty$ in the thermodynamic limit for u away from the real axis similarly to $J_-(t)$ and therefore $\vec{L}(u) \to \vec{L}^{\rm red}(u)$. Constants $\sigma_{\mathbf{p}}$ are given by Eq. (8.93) to match the integrals of motion. Then, the solution given by Eq. (8.94) with $2\sigma_{\mathbf{p}} = \cos\theta_{\mathbf{p}}$ has the same integrals of motion as the quench dynamics and the right number of additional constants to match the remaining initial conditions. As explained above, to verify that this is indeed the true asymptote of the quench dynamics, we

only need to show that asymptotic $\Delta(t)$ coincides with $\Delta(t)$ of the $m = -1$ spin solution, i.e. that $\Delta(t) \to 0$ at large times after the quench whenever $\vec{L}^2(u)$ has no isolated complex roots (region I in quench phase diagrams above). We confirm this numerically, see e.g. Figs. 8.5, 8.12, and Refs. [19, Yuzbashyan and Dzero(2006)]. There is also a justification of this statement based on the general theory of integrable Hamiltonian dynamics. It works for both $m = -1$ and $m = 0$ and we present at the end of the $m = 0$ case below Eq. (8.114).

To summarize: if $\vec{L}^2(u)$ has no isolated complex roots for given (quench) initial conditions, then $\Delta(t) \to 0$ at large times in the thermodynamic limit and the steady state spin configuration is

$$s^z_{\mathbf{p}} = -\frac{\cos\theta_{\mathbf{p}}}{2}, \quad s^-_{\mathbf{p}} = \frac{\sin\theta_{\mathbf{p}}}{2} e^{-2i\varepsilon_{\mathbf{p}}t + i\delta_{\mathbf{p}}}, \tag{8.97}$$

where

$$\cos\theta(\varepsilon) = z(\varepsilon)\frac{\sqrt{\vec{L}^2(\varepsilon_-)} - \sqrt{\vec{L}^2(\varepsilon_+)}}{i\pi\nu(\varepsilon)}, \tag{8.98}$$

and $\theta_{\mathbf{p}} \equiv \theta(\varepsilon_{\mathbf{p}})$. This expression evaluates explicitly for quench initial conditions; the answer is given by Eq. (8.134). The sign $z(\varepsilon) = \pm 1$ is fixed by requiring that $\cos\theta(\varepsilon)$ be smooth and spins $\vec{s}_{\mathbf{p}}$ point in the negative z-direction at $\varepsilon_{\mathbf{p}} \to \infty$ (so that corresponding single particles states be empty).

Figure 8.12: Order parameter $\Delta(t)$ vanishes whenever the square of the Lax vector $\vec{L}^2(u)$ has no isolated roots. Panel (a) shows real, Re$[c]$, and imaginary, Im$[c_m]$, parts of the roots c_m and panel (b) shows the corresponding $|\Delta(t)|$ for a detuning quench in a 3d 2-channel model with $\gamma = 0.1$ and $N = 1024$ spins. There are $N+1$ pairs of complex conjugate continual roots whose imaginary parts scale as $1/N$ so that in the $N \to \infty$ limit they form a continuum on the real axis. Here $\Delta_{0i} = 0.34\Delta_{\max}$, $\Delta_{0f} = 8.1 \times 10^{-3}\Delta_{\max}$, $\mu_i = 0.91\varepsilon_F$, and $\delta\omega = 3.45\gamma$.

The logic for $m \geq 0$ is similar, but the calculation is a bit more involved. To derive the analog of Eq. (8.94), it is convenient to work with the Bogoliubov de Gennes equations (8.31).

In addition, there is an equation of motion for b in Eq. (8.8), which can be viewed as a self-consistency condition. In terms of $\Delta = -gb$ and Bogoliuobov amplitudes it reads

$$\dot{\Delta} = -i\omega\Delta + ig^2 \sum_{\mathbf{p}} 2 s_{\mathbf{p}} u_{\mathbf{p}} \bar{v}_{\mathbf{p}} \tag{8.99}$$

The reduced m-spin solution is a particular solution $(U_{\mathbf{p}}, V_{\mathbf{p}})$ of the BdG equations that also satisfies the above self-consistency condition (with $s_{\mathbf{p}} \to \sigma_{\mathbf{p}}$). It is straightforward to check that $(\bar{V}_{\mathbf{p}}, -U_{\mathbf{p}})$ is also a solution of the BdG equations with the same $\Delta(t)$. Since for any fixed $\Delta(t)$ these equations are linear in the amplitudes, their most general normalized solution with this $\Delta(t)$ is a linear combination of these two independent solutions

$$\begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} = \cos\frac{\theta_{\mathbf{p}}}{2} \begin{pmatrix} U_{\mathbf{p}} \\ V_{\mathbf{p}} \end{pmatrix} + \sin\frac{\theta_{\mathbf{p}}}{2} \begin{pmatrix} \bar{V}_{\mathbf{p}} \\ -\bar{U}_{\mathbf{p}} \end{pmatrix}. \tag{8.100}$$

The coefficients are made real by dropping an unimportant overall time-independent phase and including the relative phase into the common phase of $U_{\mathbf{p}}$ and $V_{\mathbf{p}}$. At this point $\theta_{\mathbf{p}}$ is an arbitrary angle. This solution does not generally satisfy the self-consistency condition (8.99) at finite t, but, as we will see below, becomes self-consistent as $t \to \infty$.

Let us now determine the spins corresponding to this solution. Eq. (8.100) implies

$$\begin{aligned} |v_{\mathbf{p}}|^2 - |u_{\mathbf{p}}|^2 &= \left(|V_{\mathbf{p}}|^2 - |U_{\mathbf{p}}|^2\right)\cos\theta_{\mathbf{p}} - \\ &\quad \sin\theta_{\mathbf{p}}\left(\bar{U}_{\mathbf{p}}\bar{V}_{\mathbf{p}} + U_{\mathbf{p}}V_{\mathbf{p}}\right), \\ u_{\mathbf{p}}\bar{v}_{\mathbf{p}} &= U_{\mathbf{p}}\bar{V}_{\mathbf{p}}\cos\theta_{\mathbf{p}} + \frac{\sin\theta_{\mathbf{p}}}{2}\left(\bar{V}_{\mathbf{p}}^2 - U_{\mathbf{p}}^2\right). \end{aligned} \tag{8.101}$$

True spins $\vec{s}_{\mathbf{p}}$ are related to $u_{\mathbf{p}}$, $v_{\mathbf{p}}$ through Eq. (8.32) with $s_{\mathbf{p}} = 1/2$. Spins $\vec{\sigma}_{\mathbf{p}}$ are similarly related to $U_{\mathbf{p}}, V_{\mathbf{p}}$. Let

$$\begin{aligned} U_{\mathbf{p}} &= |U_{\mathbf{p}}| \exp\left[i\frac{\alpha_{\mathbf{p}} - \phi_{\mathbf{p}}}{2}\right], \\ V_{\mathbf{p}} &= |V_{\mathbf{p}}| \exp\left[i\frac{\alpha_{\mathbf{p}} + \phi_{\mathbf{p}}}{2}\right]. \end{aligned} \tag{8.102}$$

We can express the absolute values of the amplitudes and their relative phase through the spin components

$$|V_{\mathbf{p}}|^2 = \frac{1}{2} + \frac{\sigma_{\mathbf{p}}^z}{2\sigma_{\mathbf{p}}},\ |U_{\mathbf{p}}|^2 = \frac{1}{2} - \frac{\sigma_{\mathbf{p}}^z}{2\sigma_{\mathbf{p}}},\ e^{-i\phi_{\mathbf{p}}} = \frac{\sigma_{\mathbf{p}}^-}{|\sigma_{\mathbf{p}}^-|}, \tag{8.103}$$

while their common phase $\alpha_{\mathbf{p}}$ needs to be determined separately from the BdG equations.

We obtain in this notation

$$\begin{aligned} s_{\mathbf{p}}^z &= \frac{\sigma_{\mathbf{p}}^z}{\sigma_{\mathbf{p}}}\frac{\cos\theta_{\mathbf{p}}}{2} - \frac{|\sigma_{\mathbf{p}}^-|}{\sigma_{\mathbf{p}}}\frac{\sin\theta_{\mathbf{p}}}{2}\cos\alpha_{\mathbf{p}}, \\ s_{\mathbf{p}}^- &= \frac{\sigma_{\mathbf{p}}^-}{\sigma_{\mathbf{p}}}\frac{\cos\theta_{\mathbf{p}}}{2} + \frac{\sin\theta_{\mathbf{p}}}{2}e^{-i\phi_{\mathbf{p}}}\left(\frac{\sigma_{\mathbf{p}}^z}{\sigma_{\mathbf{p}}}\cos\alpha_{\mathbf{p}} - \right. \\ &\qquad \left. i\sin\alpha_{\mathbf{p}}\right). \end{aligned} \tag{8.104}$$

Note that $\sigma_{\mathbf{p}}^z/\sigma_{\mathbf{p}}$ and $\sigma_{\mathbf{p}}^-/\sigma_{\mathbf{p}}$ are components of the unit vector along the spin in the reduced solution $\vec{\sigma}_{\mathbf{p}}$. Geometrically Eq. (8.104) says that $\vec{s}_{\mathbf{p}}$ makes a constant angle $\theta_{\mathbf{p}}$ (or $\pi - \theta_{\mathbf{p}}$ for negative $\sigma_{\mathbf{p}}$) with $\vec{\sigma}_{\mathbf{p}}$ and rotates around it with an angular velocity $\dot{\alpha}_{\mathbf{p}}$,

$$\vec{s}_{\mathbf{p}} = \frac{\vec{\sigma}_{\mathbf{p}}}{\sigma_{\mathbf{p}}}\frac{\cos\theta_{\mathbf{p}}}{2} + \vec{s}_{\mathbf{p}}^{\perp}. \tag{8.105}$$

To see this, consider a body set of axis for $\vec{\sigma}_{\mathbf{p}}$. Take z' along $\vec{\sigma}_{\mathbf{p}}$, x'-axis along the intersection of the zz' plane with the plane perpendicular to $\vec{\sigma}_{\mathbf{p}}$, and y' normal to $x'z'$ to form a right-handed coordinate system as usual. Then $\alpha_{\mathbf{p}}$ is the angle between $\vec{s}^{\perp}_{\mathbf{p}}$ and the x'-axis and Eq. (8.104) follows.

The contribution of the second terms on the right hand side of Eqs. (8.104) and (8.105) (terms containing $\alpha_{\mathbf{p}}$) to $\vec{L}(u)$ at u away from the real axis and to $J_-(t)$ vanishes (dephases) at large times at least for $m = 0$ and 1 in the thermodynamic limit same as in the $m = -1$ case considered above. For this to be true it is sufficient that $\alpha_{\mathbf{p}}$ contain a dispersing linear in t term, i.e.

$$\alpha_{\mathbf{p}} = -2e_{\mathbf{p}}t + F_{\mathbf{p}}(t), \tag{8.106}$$

where $e_{\mathbf{p}}$ is a continuous non-constant function of $\varepsilon_{\mathbf{p}}$ and $F_{\mathbf{p}}(t)$ is a bounded function of t. Note that for $m = -1$, $e_{\mathbf{p}} = \varepsilon_{\mathbf{p}}$ and $F_{\mathbf{p}}(t) = \delta_{\mathbf{p}} = \text{const}$.

To derive the asymptotic state, we follow the same procedure as for $m = -1$ above. We set $2\sigma_{\mathbf{p}} = \cos\theta_{\mathbf{p}}$, where $\cos\theta_{\mathbf{p}} \equiv \cos\theta(\varepsilon_{\mathbf{p}})$ is given by Eq. (8.98). Then, $\vec{L}(u) \to \vec{L}^{\text{red}}(u)$, $\Delta(t)$ is described by this m-spin solution at large times and satisfies the self-consistency condition (8.99), the asymptotic spin configuration (8.104) and the m-spin problem have the same integrals of motion as the quench dynamics. The remaining $N+1$ constants required to match the initial conditions are in $\alpha_{\mathbf{p}}$ (see below) and in the phase of $\Delta(t)$.

To determine $\alpha_{\mathbf{p}}$, rewrite the BdG eqs as

$$i\partial_t(\ln U_{\mathbf{p}}) = \varepsilon_{\mathbf{p}} + \Delta\frac{V_{\mathbf{p}}}{U_{\mathbf{p}}}, \quad i\partial_t(\ln V_{\mathbf{p}}) = -\varepsilon_{\mathbf{p}} + \bar{\Delta}\frac{U_{\mathbf{p}}}{V_{\mathbf{p}}}. \tag{8.107}$$

Adding these equations and using Eqs. (8.102) and (8.103), we get after some algebra

$$\dot{\alpha}_{\mathbf{p}} = -\frac{\sigma_{\mathbf{p}}(\bar{\Delta}\sigma^-_{\mathbf{p}} + \Delta\sigma^+_{\mathbf{p}})}{|\sigma^-_{\mathbf{p}}|^2}. \tag{8.108}$$

m=0

Suppose $\vec{L}^2(u)$ has a single pair of isolated complex roots at $u = \mu_\infty \pm i\Delta_\infty$. The 0-spin expression (8.67) for $\Delta(t)$ reads

$$\Delta(t) = \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}. \tag{8.109}$$

The notation Δ_∞ and μ_∞ anticipates that this is also the long time asymptote for the quench dynamics. Eq. (8.69) implies

$$\frac{\sigma^-_{\mathbf{p}}}{\sigma_{\mathbf{p}}} = \frac{\Delta(t)}{E^\infty_{\mathbf{p}}}, \quad \frac{\sigma^z_{\mathbf{p}}}{\sigma_{\mathbf{p}}} = -\frac{\xi_p}{E^\infty_{\mathbf{p}}}, \tag{8.110}$$

where $E^\infty_{\mathbf{p}} = E(\varepsilon_{\mathbf{p}}; \Delta_\infty, \mu_\infty) = \sqrt{(\varepsilon_{\mathbf{p}} - \mu_\infty)^2 + \Delta^2_\infty}$ and $\xi_{\mathbf{p}} = \varepsilon_{\mathbf{p}} - \mu_\infty$.

Eq. (8.108) obtains $\dot{\alpha}_{\mathbf{p}} = -2E^\infty_{\mathbf{p}}$. We see that $\alpha_{\mathbf{p}}$ is of the form (8.106) and therefore the large time asymptote of $\Delta(t)$ according to the few spin conjecture is given by Eq. (8.109). The asymptotic spin configuration is then Eq. (8.104) with $\cos\theta_{\mathbf{p}} \equiv \cos\theta(\varepsilon_{\mathbf{p}})$ given by Eq. (8.98). Explicitly, using Eq. (8.110) and $\alpha_{\mathbf{p}} = -2E^\infty_{\mathbf{p}}t - \delta_{\mathbf{p}}$, we obtain

$$\begin{aligned} s^z_{\mathbf{p}} &= -\frac{\xi_{\mathbf{p}}}{2E^\infty_{\mathbf{p}}}\cos\theta_{\mathbf{p}} - \frac{\Delta_\infty}{2E^\infty_{\mathbf{p}}}\sin\theta_{\mathbf{p}}\cos(2E^\infty_{\mathbf{p}}t + \delta_{\mathbf{p}}), \\ s^-_{\mathbf{p}}e^{2i\mu_\infty t + 2i\varphi} &= \frac{\Delta_\infty}{2E^\infty_{\mathbf{p}}}\cos\theta_{\mathbf{p}} - \frac{\sin\theta_{\mathbf{p}}}{2}e^{2iE^\infty_{\mathbf{p}}t + i\delta_{\mathbf{p}}} - \\ &\quad \left(\frac{\xi_p}{E^\infty_{\mathbf{p}}} - 1\right)\frac{\sin\theta_{\mathbf{p}}}{2}\cos(2E^\infty_{\mathbf{p}}t + \delta_{\mathbf{p}}). \end{aligned} \tag{8.111}$$

In a reference frame rotating with frequency $2\mu_\infty$ around z-axis, $\Delta(t) \to \Delta_\infty$ meaning that magnetic field acting on spin $\vec{s}_{\mathbf{p}}$ is time-independent. In this frame $\vec{s}_{\mathbf{p}}$ rotates around the field or, equivalently, around the reduced spin $\vec{\sigma}_{\mathbf{p}}$ with frequency $2E^\infty_{\mathbf{p}}$ as described by Eq. (8.111).

We can also determine the Bogoliubov amplitudes corresponding to the 0-spin solution from Eqs. (8.102) and (8.103)

$$\begin{aligned} U_{\mathbf{p}} &= \sqrt{\frac{1}{2} + \frac{\xi_{\mathbf{p}}}{2E^\infty_{\mathbf{p}}}} e^{-iE^\infty_{\mathbf{p}} t - i\mu_\infty t - i\varphi}, \\ V_{\mathbf{p}} &= \sqrt{\frac{1}{2} - \frac{\xi_{\mathbf{p}}}{2E^\infty_{\mathbf{p}}}} e^{-iE^\infty_{\mathbf{p}} t + i\mu_\infty t + i\varphi}. \end{aligned} \tag{8.112}$$

These in turn determine the "real" asymptotic amplitudes according to Eq. (8.100) and therefore the many-body wavefunction (8.29), which allows one to calculate various few-particle Green's functions.

As before, to verify the few spin conjecture in the present case it is enough to check that the large time asymptote of $\Delta(t)$ after the quench is given by Eq. (8.109) as long as $\vec{L}^2(u)$ has one pair of isolated complex conjugate roots (regions II and II' in quench phase diagrams above). We do so numerically, see e.g. Figs. 8.7, 8.13, and Refs. [19, Yuzbashyan and Dzero(2006)]. The large time asymptote of $|\Delta(t)|$ is in excellent agreement with Δ_∞ derived as the imaginary part of the isolated root, see e.g. Fig. 2 in Ref. [Yuzbashyan and Dzero(2006)]. This is however guaranteed by conservation laws without reliance on the few spin conjecture. Indeed, suppose we find $\Delta(t) \to \tilde{\Delta}_\infty e^{-2i\tilde{\mu}_\infty t - 2i\tilde{\varphi}}$. Starting with this, one can retrace the steps that lead to Eq. (8.111) backwards and show that $\vec{L}^2(u)$ has a single pair of isolated complex conjugate roots at $\tilde{\mu}_\infty \pm i\tilde{\Delta}_\infty$. In other words, $\tilde{\mu}_\infty = \mu_\infty, \tilde{\Delta}_\infty = \Delta_\infty$, and the constant φ is arbitrary in the 0-spin solution, so we can always set $\tilde{\varphi} = \varphi$. Let us prove this somewhat differently using Bloch rather than BdG equations.

Going to a reference frame rotating around z-axis with frequency $2\tilde{\mu}_\infty$ eliminates time dependence in the asymptotic $\Delta(t)$. In this frame, the effective magnetic field acting on each spin $\vec{s}_{\mathbf{p}}$ in Eq. (8.8) is $\vec{B}_{\mathbf{p}} = -2\tilde{\Delta}_\infty \hat{\mathbf{x}} + 2(\epsilon_{\mathbf{p}} - \tilde{\mu}_\infty)\hat{\mathbf{z}}$ and is time-independent. The spin therefore rotates around the field making a constant angle (call it $\pi - \theta_{\mathbf{p}}$) with it. It is straightforward to determine spin components in this situation. They are given by Eq. (8.111) with $\mu_\infty \to \tilde{\mu}_\infty$, $\Delta_\infty \to \tilde{\Delta}_\infty$, and absent $e^{2i\mu_\infty t + 2i\varphi}$ on the left hand side in the rotating frame.

Next, we evaluate Lax vector (8.50) for this spin configuration. For u away from the real axis, summations over $\mathbf{p}$ can be safely replaced with integrations in the continuum limit and contributions from oscillating terms on the right hand side of Eqs. (8.111) vanish at $t \to \infty$. The same cancelation occurs in the gap equation of motion (8.33), so that it becomes Eq. (8.200) that we will later also need in a different context. Using this gap equation to simplify the expression for $\vec{L}(u)$, we obtain

$$\vec{L}(u) = \left[\tilde{\Delta}_\infty \hat{\mathbf{x}} - (u - \tilde{\mu}_\infty)\hat{\mathbf{z}}\right] L_\infty(u), \tag{8.113}$$

where

$$L_\infty(u) = \frac{2}{g^2} - \sum_{\mathbf{p}} \frac{1}{2(u - \varepsilon_{\mathbf{p}})E^\infty_{\mathbf{p}}}. \tag{8.114}$$

We see that $\vec{L}^2(u) = [\tilde{\Delta}^2_\infty - (u - \tilde{\mu}_\infty)^2]L^2_\infty(u)$ has a pair of isolated roots at $u = \tilde{\mu}_\infty \pm i\tilde{\Delta}_\infty$, i.e. the parameters of the asymptotic $\Delta(t)$ must coincide with those of an isolated root.

Finally, there is a general argument explaining why the actual quench dynamics at $t \to \infty$ should be described by the above asymptotic solutions derived from -1 and 0 spin solutions at least when $\vec{L}^2(u)$ has none or only one isolated root pair ($m = -1$ and 0). The general

motion of a classical Hamiltonian integrable model with N degrees of freedom is quasi-periodic with N independent frequencies, $\vec{\omega} = (\omega_1, \ldots, \omega_N)$, that are determined solely by the values of its integrals of motion[Arnold(1978), Tabor(1989)]. There are two types of (quasi)periodic motion: libration and rotation[H. Goldstein(2002)]. Let us explain this terminology with a one-dimensional example. In libration, the coordinate returns to its initial value after each period, such as e.g. the coordinate of a harmonic oscillator. In rotation, it increases each time by a fixed amount, such as e.g. the angle of a rotating pendulum. Dynamical variables of libration type can be decomposed in a multi-dimensional Fourier series as follows,

$$Q(t) = \sum_{\vec{m}} c_{\vec{m}} e^{i\vec{\omega}\cdot\vec{m}t}, \tag{8.115}$$

where $\vec{m} = (m_1, \ldots, m_N)$ is a vector with integer components. Dynamical variables of rotation type contain an additional linear term, i.e.

$$Q(t) = c_0 t + \sum_{\vec{m}} c_{\vec{m}} e^{i\vec{\omega}\cdot\vec{m}t}, \tag{8.116}$$

see e.g. Ref. [H. Goldstein(2002)] for further details. In our case, the absolute value of the order parameter, $|\Delta(t)|$ is of libration type, while its phase is of rotation type.

The frequency spectra of asymptotic solutions constructed above are $\omega(\varepsilon_{\mathbf{p}}) = 2\varepsilon_{\mathbf{p}}$ for $m = -1$ and $\omega(\varepsilon_{\mathbf{p}}) = 2\sqrt{(\varepsilon_{\mathbf{p}} - \mu_\infty)^2 + \Delta_\infty^2}$ for $m = 0$. Important for us is that the spectra are continuous with no isolated frequencies in the thermodynamic limit. Since setting $2\sigma_{\mathbf{p}} = \cos\theta_{\mathbf{p}}$ ensures the quench dynamics has the same integrals as this solution (lives on the same invariant torus), it also must have an identical frequency spectrum. Assuming $|\Delta(t)|$ is continuously distributed over the spectrum as a collective variable, i.e. the discrete summation in Eq. (8.115) turns into a continuous Fourier transform, it must dephase at large times, $|\Delta(t)| \to$ const. Under the same assumption, the phase of the order parameter according to Eq. (8.116) must tend to a linear in time function as $t \to \infty$. Therefore, $\Delta(t)$ at large times is of the form $\Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}$. Since finite Δ_∞ also implies an isolated root at $\mu_\infty \pm i\Delta_\infty$ while for $m = -1$ there are no isolated roots by definition, we must have $\Delta_\infty = 0$, i.e. $\Delta(t) \to 0$ in this case.

We also prove the few spin conjecture for infinitesimal quenches in Sect. 8.5.4 independently of above arguments and numerics.

Figure 8.13: Roots of $\vec{L}^2(u)$ (top) and $|\Delta(t)|$ for a detuning quench in a 3d 2-channel model for $N = 1024$ spins, $\gamma = 1.0$. There is one pair of isolated roots $c_\pm = \mu_\infty \pm i\Delta_\infty$ whose imaginary part remains finite in the large N limit and $N-1$ continual roots c_m close to the real axis ($\mathrm{Im}[c_m]$ is zoomed in by 10). Observe $|\Delta(t)| \to \Delta_\infty$ in agreement with the few spin conjecture. Here $\Delta_{0i} = 0.18\Delta_{\max}, \Delta_{0f} = 0.78\Delta_{\max}$, and $\delta\omega = -2.26\gamma$.

m=1

Suppose we found that for some initial condition (quench parameters) $\vec{L}^2(u)$ has two pairs of isolated complex conjugate roots $c, \bar{c}, c', \bar{c}'$. Given c and c', the above method allows us to determine the long time asymptote of $\Delta(t)$, asymptotic spin configuration and time-dependent Bogoliubov amplitudes $u_{\mathbf{p}}(t), v_{\mathbf{p}}(t)$ for the dynamics of the two-channel model (8.9) starting from this initial condition at $t = 0$.

By construction c, c' are also the roots of $\vec{L}_m^2(u)$ furnishing the spectral polynomial for the reduced problem $Q_4(u) = (u-c)(u-\bar{c})(u-c')(u-\bar{c}')$ and therefore the parameters μ, ρ, κ, χ through Eq. (8.84). We further obtain from Eq. (8.90)

$$\Delta(t) = \sqrt{\Lambda^2 + h_1} \exp\left(-2i\mu t - i\int \frac{\kappa dt}{\Lambda^2 + h_1}\right), \tag{8.117}$$

where Λ is the Jacobi elliptic function dn,

$$\Lambda = \sqrt{h_3 - h_1}\mathrm{dn}\left[\sqrt{h_3 - h_1}(t - t_0), \sqrt{\frac{h_3 - h_2}{h_3 - h_1}}\right], \tag{8.118}$$

t_0 is a constant, and $h_3 \geq h_2 \geq h_1$ are the roots of the third order polynomial $P_3(w) = w^3 + 4\rho w^2 + 4\chi w + \kappa^2$. The amplitude $|\Delta(t)|$ oscillates between a minimum $\Delta_b = \sqrt{h_2}$ and a maximum $\Delta_a = \sqrt{h_1}$. Plots of Δ_a, Δ_b, and h_1 for various quenches are shown in Figs. 8.9 and

8.10. As we will now see, the parameter h_1 also quantifies the deviation from the weak coupling limit where $h_1 = 0$.

Figure 8.14: Roots of $\vec{L}^2(u)$ (top) and $|\Delta(t)|$ for a detuning quench in a 3d 2-channel model with $N = 1024$ spins and $\gamma = 1.0$. There are two pairs of isolated roots $(c, \bar{c})$ and $(c', \bar{c}')$ and $N - 2$ continual roots close to the real axis. The large time asymptote of $|\Delta(t)|$ is described by Eq. (8.117), where parameters h_i are extracted from the isolated roots in agreement with the few spin conjecture. The phase of $\Delta(t)$ is also in excellent agreement, see e.g. Figs. 8.8 and 8.15. Quench parameters are: $\Delta_{0i} = 2.68\Delta_{\max}$, $\Delta_{0f} = 0.76\Delta_{\max}$, and $\delta\omega = -4.13\gamma$.

Figure 8.15: Magnitude and phase of $\Delta(t)$ in region III (two pairs of isolated roots) after detuning quench from deep BCS to BEC in 3d two-channel model for $\gamma = 1$. Numerical evolution with 5024 spins against Eq. (8.117). Parameters h_1, h_2, etc. are obtained from isolated roots of $\vec{L}^2(u)$ as described in the text. $\Delta_{0i} = 2.65 \times 10^{-2}\Delta_{\max}, \Delta_{0f} = 0.80\Delta_{\max}, \mu_i = 1.00\varepsilon_F, \delta\omega = -4.59\gamma$.

Of interest is the particular case when the parameter $\kappa = 0$. As we will see below, this is realized for quenches deep within the weak coupling BCS regime in the broad resonance limit when the two-channel model is equivalent to the BCS Hamiltonian (8.11). $\kappa = 0$ implies $h_1 = 0$, $4\chi = h_2 h_3$, $4\rho = -h_2 - h_3$, and $Q_4(u) = [(u-\mu)^2 - \rho]^2 - \chi$. Let $h_3 = \Delta_+^2, h_2 = \Delta_-^2$ in accordance with the notation of Eq. (8.87). The roots of $Q_4(u)$ in this case take a simple form with shared real part. Namely, they are

$$\mu \pm i\frac{\Delta_+ \pm \Delta_-}{2}, \tag{8.119}$$

and the expression (8.117) simplifies as well,

$$\Delta(t) = \Delta_+ \mathrm{dn}[\Delta_+(t-t_0)]e^{-2i\mu t - 2i\varphi}. \tag{8.120}$$

This expression for $\Delta(t)$ and the corresponding $m = 1$ spin solution were constructed in Ref. [Barankov et al.(2004)].

General expression for the reduced spins obtain from Eqs. (10.3), (8.76), and (8.83),

$$\begin{aligned} \frac{\sigma_{\mathbf{p}}^z}{\sigma_{\mathbf{p}}} &= -\frac{|\Delta|^2 - 2\xi_{\mathbf{p}}^2 + 2\rho}{2\sqrt{Q_4(\varepsilon_{\mathbf{p}})}}, \\ \frac{\sigma_{\mathbf{p}}^-}{\sigma_{\mathbf{p}}} &= -\frac{2\xi_{\mathbf{p}}\Delta - 2\mu\Delta + i\dot{\Delta}}{2\sqrt{Q_4(\varepsilon_{\mathbf{p}})}}, \end{aligned} \tag{8.121}$$

where $\xi_{\mathbf{p}} = \varepsilon_{\mathbf{p}} - \mu$ and Δ is given by Eq. (8.117). Bogoliubov amplitudes corresponding to the 1-spin solution can now be derived from Eq. (8.107). The imaginary and real parts of the right hand sides determine the absolute values of the amplitudes and their phases, respectively,

$$\begin{aligned} U_{\mathbf{p}} &= \frac{\sqrt{2c_{\mathbf{p}}^{+} - |\Delta|^2}}{2Q_4^{1/4}(\varepsilon_{\mathbf{p}})} e^{-i\mu t + i\xi_{\mathbf{p}} t} \exp\left[i \int \frac{\kappa - 4\xi_{\mathbf{p}} c_{\mathbf{p}}^{+}}{2c_{\mathbf{p}}^{+} - |\Delta|^2} dt \right], \\ V_{\mathbf{p}} &= \frac{\sqrt{2c_{\mathbf{p}}^{-} + |\Delta|^2}}{2Q_4^{1/4}(\varepsilon_{\mathbf{p}})} e^{i\mu t - i\xi_{\mathbf{p}} t} \exp\left[i \int \frac{\kappa + 4\xi_{\mathbf{p}} c_{\mathbf{p}}^{-}}{2c_{\mathbf{p}}^{-} + |\Delta|^2} dt \right], \end{aligned} \tag{8.122}$$

where $c_{\mathbf{p}}^{\pm} = \sqrt{Q_4(\varepsilon_p)} \pm (\xi_{\mathbf{p}}^2 - \rho)$.

The common phase of the amplitudes $\alpha_{\mathbf{p}}$ is the sum of their phases in the above equations, i.e.

$$\alpha_{\mathbf{p}} = \int \left[\frac{\kappa - 4\xi_{\mathbf{p}} c_{\mathbf{p}}^{+}}{2c_{\mathbf{p}}^{+} - |\Delta|^2} + \frac{\kappa + 4\xi_{\mathbf{p}} c_{\mathbf{p}}^{-}}{2c_{\mathbf{p}}^{-} + |\Delta|^2} \right] dt \tag{8.123}$$

The integrand is a periodic function of time. Therefore, $\alpha_{\mathbf{p}}$ is of the form (8.106), which is seen e.g. by expanding the expression under the integral in Fourier series. The linear part $e_{\mathbf{p}} t$ comes from the zeroth harmonics. We only need to show that $e_{\mathbf{p}}$ is a non-constant (dispersing) function of $\varepsilon_{\mathbf{p}}$. For this, we expand the integrand for large $\varepsilon_{\mathbf{p}}$, $e_{\mathbf{p}} = \varepsilon_{\mathbf{p}} + O(1)$. Therefore, $e_{\mathbf{p}}$ is indeed dispersing and the contribution of second terms on the right hand sides of Eq. (8.104) to $\vec{L}(u)$ and $J_{-}(t)$ dephases similarly to $m = -1, 0$ cases. By few spin conjecture the asymptotic behavior of $\Delta(t)$ is then given by Eqs. (8.117). The asymptotic spin configuration obtains by substituting Eqs. (8.123) and (8.121) into Eq. (8.104), where $\cos\theta_{\mathbf{p}} \equiv \cos(\varepsilon_{\mathbf{p}})$ is given by Eq. (8.98) and $e^{-i\phi_{\mathbf{p}}} = \sigma_{\mathbf{p}}^{-}/|\sigma_{\mathbf{p}}^{-}|$ straightforwardly derives from the second equation in (8.121).

As before, to verify the few spin conjecture it is sufficient to check that $\Delta(t)$ at large times after the quench is described by Eq. (8.117) whenever $\vec{L}^2(u)$ has two pairs of isolated roots. We do this numerically, see Figs. 8.14, 8.8, 8.15, and 8.16. In these plots we compare $\Delta(t)$ from direct numerical evolution of 5024 spins to Eq. (8.117), where parameters h_1, h_2, h_3, and μ obtain from the isolated roots of $\vec{L}^2(u)$. Note there are no fitting parameters apart from an overall shift t_0 along the time axis.

Figure 8.16: Post-quench $|\Delta(t)|$ for 3d 2-channel model in region III where $\vec{L}^2(u)$ has two pairs isolated roots. Numerical evolution with 5024 spins against Eq. (8.117). $\gamma = 0.1$, $\Delta_{0i} = 0.035\Delta_{\max}$ in all three panels. $\Delta_{0f}/\Delta_{\max} = 0.54, 0.67$, and 0.85 in (a), (b), and (c), respectively.

8.3 Quench phase diagram and asymptotic spin distribution for the 2-channel model

We established in the previous section that the long time dynamics of the system after a quench are determined by the isolated complex roots of $\vec{L}^2(u)$. We now proceed to evaluate the roots and thus construct the quench phase diagram – identify all possible steady states for quenches throughout the BCS-BEC crossover. We find that depending on the quench parameters $\vec{L}^2(u)$ has either zero, one, or two pairs of complex conjugate roots and the long time behavior is therefore that described in Sections 8.2.4, 8.2.4, or 8.2.4, respectively. Imaginary and real parts of the roots determine the parameters of the asymptotic behavior. For example, in the Volkov and Kogan regime (region II in our quench phase diagrams) where $\Delta(t \to \infty) \to \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}$, the roots are $\mu_\infty \pm i\Delta_\infty$. We first derive general equations for the roots, lines separating distinct regimes, and the asymptotic distribution function, and then consider various

cases, such as two and three dimensions, wide (one channel) and narrow resonance limits, and deep BCS and BEC regimes.

After the quench the system evolves with the Hamiltonian (8.9) where $\omega = \omega_f$ starting from the spin configuration (8.28) which is the ground state for $\omega = \omega_i$. Since $\vec{L}^2(u)$ is conserved, we can evaluate it at any t. It is convenient to do so at $t = 0$. The Lax vector at $t = 0$ obtains by plugging the initial condition into the definition (8.50)

$$\vec{L}(u)|_{t=0} = [\Delta_{0i}\hat{\mathbf{x}} - (u - \mu_i)\hat{\mathbf{z}}]\, L_0(u) - \frac{\delta\omega}{g^2}\hat{\mathbf{z}}, \tag{8.124}$$

where $\delta\omega = \omega_f - \omega_i$ and

$$L_0(u) = -\frac{2}{g^2} + \sum_{\mathbf{p}} \frac{1}{2(u - \varepsilon_{\mathbf{p}})E_i(\varepsilon_{\mathbf{p}})}, \tag{8.125}$$

$E_i(\varepsilon_{\mathbf{p}}) = E(\varepsilon_{\mathbf{p}}; \Delta_{0i}, \mu_i) = \sqrt{(\varepsilon_{\mathbf{p}} - \mu_i)^2 + \Delta_{0i}^2}$ and we also used the gap equation (8.18) ($\frac{(\omega-2\mu)}{g^2} = \sum_{\mathbf{p}} \frac{1}{2E(\varepsilon_{\mathbf{p}};\Delta_0,\mu)}$).

Taking the square of the above expression for $\vec{L}(u)$ and equating it to zero, we obtain an equation for the roots

$$(u - \mu_i \mp i\Delta_{0i})\left[\frac{2}{g^2} - \sum_{\mathbf{p}} \frac{1}{2(u - \varepsilon_{\mathbf{p}})E_i(\varepsilon_{\mathbf{p}})}\right] = \frac{\delta\omega}{g^2}. \tag{8.126}$$

Suppose first the single particle levels $\varepsilon_{\mathbf{p}}$ are discrete and there are $N \gg 1$ distinct $\varepsilon_{\mathbf{p}}$. Then, this is a polynomial equation with $N + 1$ pairs of complex conjugate roots. Most of the pairs are close to the real axis, at distances of the order of the spacing between $\varepsilon_{\mathbf{p}}$, which is inversely proportional to N (system volume) and goes to zero in the thermodynamic limit. In thermodynamic limit most of the roots of $\vec{L}^2(u)$ coalesce to the real axis merging with its poles to form a branch cut along the real axis. We fully verify this picture in this section and in Appendix 8.10. Here we consider the roots whose imaginary part remains finite as $N \to \infty$ and in Appendix 8.10 we evaluate the roots with vanishing imaginary parts to order $1/N$.

Consider first the ground state. This corresponds to $\delta\omega = 0$ in Eq. (8.126) and $\vec{L}^2(u) = [(u - \mu)^2 + \Delta_0^2]\, L_0^2(u)$. There is a pair of complex roots at $c_\pm = \mu \pm i\Delta_0$. The remaining $2N$ roots solve $L_0(u) = 0$ and are double degenerate and real, see Fig. 8.17. This is because $L_0(u)$ goes from $+\infty$ to $-\infty$ as u goes from the left vicinity of one pole at $u = \varepsilon_{\mathbf{p}}$ to the right vicinity of the next one along the real axis always crossing zero in between consecutive $\varepsilon_{\mathbf{p}}$. In the thermodynamic limit, spacings between $\varepsilon_{\mathbf{p}}$ vanish and real zeroes and poles merge into a continuos line. For $\delta\omega \neq 0$ the real roots acquire imaginary parts each degenerate root splitting into a complex conjugate pair as shown in Figs. 8.12, 8.13, and 8.14. The imaginary parts however scale as $1/N$.

We first take the continuum limit in Eq. (8.126) for u away from the real axis. Then, only isolated complex roots remain and we find that there are only zero, one or two pairs of such roots depending on $\delta\omega$. At $\delta\omega = 0$ there are two isolated complex conjugate roots at $u = \mu_i \pm i\Delta_{0i}$. One pair of roots persists for sufficiently small $|\delta\omega|$, but beyond a certain threshold the number of isolated roots changes as we now demonstrate. The continuum limit of Eq. (8.126) reads

$$\frac{2}{u - \mu_i \mp i\Delta_{0i}}\frac{\delta\omega}{\gamma} + \int_0^\infty \frac{f(\varepsilon)d\varepsilon}{(u - \varepsilon)E_i(\varepsilon)} = \frac{4}{\gamma}, \tag{8.127}$$

where as always we measure energies in units of ε_F and $f(\varepsilon)$ is the dimensionless density of states defined in Eq. (8.22).

As $\delta\omega$ is decreased or increased the single pair of roots can collapse to the real axis or a new pair of isolated roots can emerge from it. The threshold (critical) value of $\delta\omega$ when this occurs

Figure 8.17: Roots of $\vec{L}^2(u)$ for the ground state of 3d 2-channel model for $N = 54$ spins and $\gamma = 1.0$. There are N doubly degenerate real roots c_m (shown as circles and squares), $N-1$ of them located in between discretized energy levels $\varepsilon_{\mathbf{p}} \to \varepsilon_m$, and two isolated complex roots $c_\pm = \mu_i \pm \Delta_{0i}$. Here $\Delta_{0i} = 0.1\varepsilon_F$.

is determined by looking for roots of Eq. (8.127) with an infinitesimal imaginary part. Replace $u \to u \pm i\delta$ in Eq. (8.127) and use $(u - \varepsilon \pm i\delta)^{-1} = P(u-\varepsilon)^{-1} \mp i\pi\delta(u-\varepsilon)$ to separate its real and imaginary parts. The latter yields critical values of $\delta\omega$ when the number of roots changes

$$\frac{|\delta\omega|}{\gamma} = \frac{\pi f(u) E_i(u)}{2\Delta_{0i}}, \tag{8.128}$$

where u is real positive and obtains from the real part of Eq. (8.127)

$$⨍_0^\infty \frac{f(\varepsilon)d\varepsilon}{(u-\varepsilon)E_i(\varepsilon)} + \mathrm{sgn}(\delta\omega)\frac{\pi(u-\mu_i)f(u)}{E_i(u)\Delta_{0i}} = \frac{4}{\gamma}. \tag{8.129}$$

Dashed integral indicates principal value.

Last two equations determine critical lines in quench phase diagrams shown in Figs. 8.2, 8.3, 8.20, and 8.21. We construct the diagrams in $\left(\Delta_{0f}, \Delta_{0i}\right)$ plane – ground state gaps at final and initial detunings ω_i and ω_f. The resonance width (dimensionless interaction strength) γ is fixed throughout the diagram. Δ_{0i}, Δ_{0f}, and γ uniquely determine μ_i, ω_i, and ω_f through ground state Eqs. (8.25) and (8.26). Each point in this plane represents a particular quench of the detuning $\omega_i \to \omega_f$. We choose Δ_{0i} (or equivalently the ratio μ_i/Δ_{0i}) and the sign of $\delta\omega$ and solve Eq. (8.129) for real u. Eq. (8.128) then yields the final detuning ω_f and therefore Δ_{0f}. We thus obtain a critical line, Δ_{0f} as a function of Δ_{0i}, in the $\left(\Delta_{0f}, \Delta_{0i}\right)$ plane. The number of isolated root pairs changes by one as one crosses this line.

It turns out there is one critical line for either sign of $\delta\omega$. There are therefore three nonequilibrium phases or regimes – qualitatively different long time behaviors, indicated as regions I, II (including subregion II'), and III in Figs. 8.2, 8.3, 8.20, and 8.21. Region II contains the $\Delta_{0f} = \Delta_{0i}$ or, equivalently, $\omega_f = \omega_i$ line which corresponds to no quench, i.e. to the system remaining in the ground state at all times. Therefore, in region II Eq. (8.127) yields a single pair of isolated complex roots $u = \mu_\infty \pm i\Delta_\infty$. This in turn implies that $\Delta(t) \to \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}$ as $t \to \infty$. For all quenches in region II the system thus goes into the asymptotic state described in Section 8.2.4.

Negative $\delta\omega$ corresponds to $\Delta_{0f} > \Delta_{0i}$. As we cross the critical line going from region II into region III the number of isolated root pairs changes by one. It can be shown both analytically and numerically by analyzing Eq. (8.127) that this number increases, i.e. there are two pairs of complex conjugate isolated roots in region III. For quenches in this part of the diagram the large time asymptote of $\Delta(t)$ is given by Eq. (8.117) and the large time state of the system is

that obtained in Section 8.2.4. Plots of Δ_∞ and μ_∞ as functions of Δ_{0f} at two fixed values of Δ_{0i} are shown in Figs. 8.18 and 8.19.

Figure 8.18: $\Delta(t) \to \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}$ as $t \to \infty$ after a detuning quench $\omega_i \to \omega_f$ in 3d 2-channel model in region II of the quench phase diagram in Fig. 8.3. Δ_∞ extracted from the single isolated root pair of the Lax vector norm is shown as a function of Δ_{0f} (ground state gap for ω_f) at two fixed values of Δ_{0i} (ground state gap for the initial detuning ω_i). Note that $\Delta_\infty > \Delta_{0f}$ for BEC to BCS quenches $\Delta_{0i} = 0.99\Delta_{\max}$ for $\gamma = 0.1$.

Similarly, as we enter region I from region II, $\Delta_\infty \to 0$ and the single pair of isolated roots collapses to the real axis at the critical line. There are hence no isolated roots in region I and therefore $\Delta(t) \to 0$ for quenches in this regime and the system goes into the gapless steady state detailed at the beginning of Section 8.2.4.

Of interest is the line along which the real part of the root pair $\mu_\infty \pm i\Delta_\infty$ in region II vanishes, i.e. $\mu_\infty = 0$ (the line separating subregions II and II' in quench phase diagrams). This can be thought of as a nonequilibrium extension of the BCS-BEC crossover going from a positive to a negative chemical potential. Out of equilibrium, as we will see below, the change of sign of μ_∞ affects the approach of $\Delta(t)$ to its asymptote. For example, in 3d the approach changes from $1/t^{1/2}$ in II to $1/t^{3/2}$ in II'. Setting $u = \pm i\Delta_\infty$ in Eq. (8.127) and separating the real and imaginary parts, we obtain equations determining this line

$$\begin{aligned} \frac{\mu_i}{\Delta_{0i} - \Delta_\infty} \operatorname{Im} \mathcal{F} + \operatorname{Re} \mathcal{F} &= \frac{4}{\gamma} \\ \frac{\delta\omega}{\gamma} \frac{2(\Delta_\infty - \Delta_{0i})}{\mu_i^2 + (\Delta_\infty - \Delta_{0i})^2} &= \operatorname{Im} \mathcal{F}, \end{aligned} \tag{8.130}$$

where

$$\mathcal{F} = \int_0^\infty \frac{f(\varepsilon) d\varepsilon}{(i\Delta_\infty - \varepsilon) E_i(\varepsilon)}. \tag{8.131}$$

Figure 8.19: $\Delta(t) \to \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}$ as $t \to \infty$ after a detuning quench $\omega_i \to \omega_f$ in 3d 2-channel model in region II of the quench phase diagram in Fig. 8.3 where μ_∞ plays the role of the out of equilibrium analog of the chemical potential. Here μ_∞ is extracted from the single isolated root pair of the Lax vector norm and is shown as a function of Δ_{0f} (ground state gap for ω_f) at two fixed values of Δ_{0i} (ground state gap for the initial detuning ω_i). Note that μ_∞ behaves similarly to the ground state chemical potential in Fig. 8.1.

Eq. (8.130) determines the $\mu_\infty = 0$ line via a procedure similar to that for critical lines separating region I from II and II from III. For a given Δ_{0i}, the first equation yields Δ_∞. We then find $\delta\omega$ and consequently ω_f and Δ_{0f} from the second equation.

Note the intersection of the $\mu_\infty = 0$ line with the $\Delta_{0i} = \Delta_{0f}$ (no quench) line. Along the latter line we also have $\Delta_\infty = \Delta_{0i}$ and therefore at the intersection point $\mu_i = \mu_f = 0$ or the first term in the first equation in Eq. (8.130) would blow up. In equilibrium $\mu = 0$ corresponds to a certain ground state gap $\Delta_0 = \Delta_{0\times}$, which obtains from Eq. (8.25) and provides a characteristic energy scale for the crossover from the BCS to BEC regime. Vanishing of μ_i and μ_f at the intersection point implies that straight lines $\Delta_{0i} = \Delta_{0\times}$, $\Delta_{0f} = \Delta_{0\times}$, and $\Delta_{0i} = \Delta_{0f}$ and the $\mu_\infty = 0$ line must cross at the same point, which is indeed seen in all quench phase diagrams in Figs. 8.2, 8.3, 8.20, and 8.21.

Let us also obtain an explicit expression for the asymptotic spin distribution function Eq. (8.98) in all three regimes. Eq. (8.124) implies

$$\vec{L}^2(u) = \Delta_{0i}^2 L_0^2(u) + \left[(u-\mu_i)L_0(u) + \frac{\delta\omega}{g^2}\right]^2. \tag{8.132}$$

In the thermodynamic limit

$$L_0(u) = -\frac{2}{g^2} + \int_0^\infty \frac{f(\varepsilon)d\varepsilon}{2(u-\varepsilon)E_i(\varepsilon)}, \tag{8.133}$$

We evaluate $L_0(\varepsilon_\pm)$ using $(\varepsilon - \varepsilon' \pm i\delta)^{-1} = P(\varepsilon - \varepsilon')^{-1} \mp i\pi\delta(\varepsilon - \varepsilon')$. This results in

$$\begin{aligned}\cos\theta(\varepsilon) = &\frac{z(\varepsilon)}{i\pi f(\varepsilon)}\sqrt{A_-^2\Delta_{0i}^2 + \left[(\varepsilon-\mu_i)A_- + \frac{\delta\omega}{\gamma}\right]^2} \\ &-\frac{z(\varepsilon)}{i\pi f(\varepsilon)}\sqrt{A_+^2\Delta_{0i}^2 + \left[(\varepsilon-\mu_i)A_+ + \frac{\delta\omega}{\gamma}\right]^2},\end{aligned} \tag{8.134}$$

where

$$A_\mp = -\frac{2}{\gamma} \pm \frac{i\pi f(\varepsilon)}{2E_i(\varepsilon)} + \int_0^\infty \frac{f(\varepsilon')d\varepsilon'}{2(\varepsilon-\varepsilon')E_i(\varepsilon')}. \tag{8.135}$$

The integral here is the same as in Eq. (8.129). We evaluate it in elementary functions in 2d, in weak coupling BCS regime, and in BEC regime in Sects. 8.3.1 and 8.3.2 below, see also Eqs. (8.303) through (8.306) for explicit expressions. Note $\cos\theta(\varepsilon) = 1$ for $\delta\omega = 0$ (no quench) as it should. Representative plots of the spin distribution function for two quenches appear in Fig. 8.11. For future use we also write down the first two terms in large ε expansion of Eq. (8.134)

$$\cos\theta(\varepsilon) \approx 1 - \left(\frac{\delta\omega}{\gamma}\right)^2 \frac{2\Delta_{0i}^2}{E_i^2(\varepsilon)[H^2(\varepsilon) + \pi^2 f^2(\varepsilon)]}, \tag{8.136}$$

which are also independently the first two terms in its small $\delta\omega$ expansion. The function $H(\varepsilon)$ is defined in Eq. (8.307).

Next, we consider two and three dimensions separately as well as various special cases such as wide (single channel limit) and narrow resonance, deep BCS and BEC regimes.

Figure 8.20: Detuning quench phase diagrams for 2-channel model in 2d for various resonance widths γ obtained from Eqs. (8.139) and (8.140). Each point represents a single quench labeled by Δ_{0i} and Δ_{0f} – pairing gaps the system would have in the ground state for initial and final detunings. At large times the system ends up in one of three steady states shown as regions I, II (including II'), and III. For quenches in region I the order parameter vanishes. In II $\Delta(t) \to \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}$ and in III $|\Delta(t)|$ oscillates persistently. Subregions II and II' differ in the sign of μ_∞ (out of equilibrium analog of the chemical potential): $\mu_\infty > 0$ in II and $\mu_\infty < 0$ in II'. The diagonal, $\Delta_{0i} = \Delta_{0f}$, is the no quench line. $\Delta_{0\times}$ is the ground state gap corresponding to zero chemical potential, i.e. $\Delta_{0\times}$ is given by Eq. (8.25) for $\mu = 0$.

8.3.1 2d

In 2d the dimensionless density of states $f(\varepsilon) = 1$ and all integrals above in this section can be evaluated in terms of elementary functions. It is convenient to introduce a notation

$$x = \frac{\mu_i}{\Delta_{0i}}, \quad v = \frac{u - \mu_i}{\Delta_{0i}}. \tag{8.137}$$

Eq. (8.127) reads

$$\ln\left[-\frac{(v+x)(v+\sqrt{1+v^2})}{\sqrt{1+x^2}\sqrt{1+v^2}-xv+1}\right] = -\frac{2\delta\omega(v\mp i)}{\gamma\sqrt{1+v^2}}+\frac{4\Delta_{0i}}{\gamma}\sqrt{1+v^2}. \tag{8.138}$$

The critical lines separating the three asymptotic regimes are determined by Eqs. (8.129) and (8.128), which become

$$\frac{|\delta\omega|}{\gamma} = \frac{\pi}{2}\sqrt{1+v^2}, \tag{8.139}$$

$$\ln\left[\frac{(v+x)(v+\sqrt{1+v^2})}{\sqrt{1+x^2}\sqrt{1+v^2}-xv+1}\right] = -\mathrm{sgn}(\delta\omega)\pi v+\frac{4\Delta_{0i}}{\gamma}\sqrt{1+v^2}, \tag{8.140}$$

where v is real and $v > -x$. It is straightforward to analyze Eq. (8.140) graphically and to find v and thus the critical lines numerically.

Positive $\delta\omega$ mean $\Delta_{0f} > \Delta_{0i}$ and the corresponding v determine the critical line separating regions I and II. In this case, for γ above a certain threshold γ_c to be determined below, there is a single root for any Δ_{0i}. This means that a horizontal $\Delta_{0i} = \text{const}$ line intersects the I-II line once for any value of the const and region I therefore extends all the way up to $\Delta_{0i} = \sqrt{\gamma} = \Delta_{\max}$ as seen in Figs. 8.2(c), 8.20(b) and 8.20(c). When $\gamma < \gamma_c$, the number of roots for positive $\delta\omega$ changes from one to two and then to zero as Δ_{0i} increases. The I-II line then displays peculiar reentrant behavior, see the inset in Fig. 8.2(b).

Negative $\delta\omega$ mean $\Delta_{0f} < \Delta_{0i}$. The roots v in this case yield the II-III critical line. There are two roots for Δ_{0i} below a certain threshold and no roots above it, implying that a horizontal $\Delta_{0i} = \text{const}$ line intersects the II-III critical line twice for a sufficiently small value of the const.

The shape of the critical lines as well as the complex roots of Eq. (8.138) can be determined analytically when the initial and/or final value of the detuning ω is deep in the BCS or BEC regime. The BCS limit corresponds to detuning $\omega \to +\infty$. For the ground state this implies $\mu \to \varepsilon_F = 1$, $\Delta_0 \to 0$. The gap equation (8.26) then yields

$$\ln\frac{4\varepsilon_\Lambda}{\Delta_0^2} = \frac{2\omega-4}{\gamma}. \tag{8.141}$$

Deep BEC regime obtains when $\omega \to -\infty$. In this case $\mu \to -\infty$ in the ground state. The gap and chemical potential equations switch roles in the sense that the former determines the chemical potential and the latter the ground state gap. Eq. (8.26) becomes

$$\ln\frac{\varepsilon_\Lambda}{|\mu|} = \frac{2\omega+4|\mu|}{\gamma} \tag{8.142}$$

and Eq. (8.25) reads in this limit

$$\Delta_0 = \left(\frac{1}{\gamma}+\frac{1}{4|\mu|}\right)^{-1/2}. \tag{8.143}$$

First, we consider quenches originating deep in the BCS regime, i.e. $\omega_i \to +\infty$ and therefore $\Delta_{0i} \to 0$, $\mu_i \to 1$. Such initial states correspond to $x \to +\infty$. Eq. (8.140) becomes

$$\ln\left[\frac{(v+x)(\sqrt{1+v^2}+v)}{x(\sqrt{1+v^2}-v)}\right] = -\,\mathrm{sgn}(\delta\omega)\pi v. \tag{8.144}$$

The roots are: $v \to 0$ for either sign of $\delta\omega$ and $v \to -x + 0$ for $\delta\omega < 0$. This translates into

$$u \approx \begin{cases} \mu_i, & \delta\omega > 0, \\ \mu_i \text{ or } +0, & \delta\omega < 0. \end{cases} \tag{8.145}$$

For $v \to 0$ Eq. (8.139) yields $\delta\omega/\gamma = \pm\pi/2$. Therefore, both Δ_{0f} and Δ_{0i} are deep in the BCS regime. The gap equation Eq. (8.141) implies $\Delta_0 \propto \exp(-\omega/\gamma)$ and hence

$$\frac{\Delta_{0i}}{\Delta_{0f}} = e^{\pm\pi/2}. \tag{8.146}$$

This result has been already obtained in Refs. [Barankov and Levitov(2006), Yuzbashyan and Dzero(2006)], which studied quenches within the single channel model in the weak coupling (BCS) limit. Weak coupling means small Δ_{0i} and Δ_{0f}, which corresponds to a vicinity of the origin, $\Delta_{0i} = \Delta_{0f} = 0$, in our phase diagrams. Eq. (8.146) is the slope of the I-II and II-III critical lines at the origin in Figs. 8.2, 8.3, 8.20, 8.21.

As we will see below, Eq. (8.146) also holds in 3d. This is expected on general grounds because in the BCS limit superconducting correlations come from a narrow energy window around the Fermi energy. Main contribution to integrals determining the roots comes from these energies. The density of states is then well approximated by a constant rendering the 2d and 3d cases equivalent.

The second root at $\delta\omega < 0$, $v \to -x + 0$, yields $\delta\omega/\gamma \approx -\pi x/2$. This means that the initial state is deep in the BCS regime, while $\omega_f \to -\infty$ and the ground state at ω_f is in the BEC limit. Further, $\mu_i \to \varepsilon_F = 1$, so $x \approx 1/\Delta_{0i}$. Subtracting Eq. (8.141) from Eq. (8.142), we obtain

$$\ln \frac{\Delta_{0i}^2}{4|\mu_f|} = -\frac{\pi}{\Delta_{0i}} + \frac{4|\mu_f|}{\gamma} + \frac{4}{\gamma}. \tag{8.147}$$

Here we assume γ is finite and treat the single channel limit $\gamma \to \infty$ separately below. Since $1/\Delta_{0i}$ term diverges much faster then the logarithm in the above equation, we get $4|\mu_f| \approx \pi\gamma/\Delta_{0i}$. Eq. (8.143) now obtains

$$\frac{\Delta_{0f}}{\Delta_{\max}} = 1 - \frac{\Delta_{0i}}{2\pi}, \tag{8.148}$$

This equation shows that the II-III critical line terminates at $(\Delta_{0f}, \Delta_{0i}) = (\Delta_{\max}, 0)$ linearly with a slope $\Delta_{0i}/(\Delta_{0f} - \Delta_{\max}) = -2\pi/\sqrt{\gamma}$.

Simpler expressions can also be derived for complex roots for quenches within the BCS regime, i.e. in the vicinity of the of the origin in the phase diagrams. By Eq. (8.145) the real parts of the roots in this regime $\mathrm{Re}[u] \approx \mu_i \approx \varepsilon_F$. Then, v is purely imaginary and also $|v| \ll x$ because $\mathrm{Im}[u]$ is related to the asymptotic value of order parameter amplitude, which is much smaller then ε_F. Eq. (8.138) becomes

$$\ln \left[\frac{v + \sqrt{1+v^2}}{v - \sqrt{1+v^2}} \right] = -\frac{v \mp i}{\sqrt{1+v^2}} \frac{2\delta\omega}{\gamma}. \tag{8.149}$$

This equation is symmetric with respect to complex conjugation and with respect to $v \to -v$. The latter symmetry reflects emergence of the particle-hole symmetry in the BCS limit. Note that when there is only one root, these two symmetries together require that it be purely imaginary.

Let $v = -i \cosh\phi$ in Eq. (8.149), where ϕ is either purely real or purely imaginary, so that v is purely imaginary. Eq. (8.149) yields depending on the sign choice on the right hand side

$$\phi = -\frac{\delta\omega}{\gamma} \coth(\phi/2), \tag{8.150}$$

$$\phi = -\frac{\delta\omega}{\gamma} \tanh(\phi/2). \tag{8.151}$$

Note that in this regime $\delta\omega/\gamma = \ln(\Delta_{0i}/\Delta_{0f})$. It is straightforward to analyze these equations graphically and to determine when they have solutions. We summarize the results.

Region I: $\Delta_{0i}/\Delta_{0f} > e^{\pi/2}$. There are no isolated roots and hence $\Delta(t) \to 0$ at large times.

Region II: $e^{-\pi/2} < \Delta_{0i}/\Delta_{0f} < e^{\pi/2}$. There is a single pair of isolated roots at $\mu_\infty \pm i\Delta_\infty$,

$$\mu_\infty = \varepsilon_F, \quad \Delta_\infty = \Delta_{0i}\cosh\phi, \tag{8.152}$$

where ϕ is real for $\delta\omega < 0$ and imaginary for $\delta\omega > 0$ and is the solution of Eq. (8.150). One can show $\Delta_\infty \le \Delta_{0f}$ for any $\delta\omega$, where the equality is achieved only at $\delta\omega = 0$. The long-time dynamics is that described in Section 8.2.4.

It is instructive to evaluate Δ_∞, the asymptotic value of the magnitude of the gap, for infinitesimal quenches, when $|\Delta_{0f} - \Delta_{0i}| \ll \Delta_{0i}$. Expanding Eqs. (8.150) and (8.152) in small ϕ, we obtain after some calculation

$$\Delta_\infty = \Delta_{0f} - \frac{(\Delta_{0f} - \Delta_{0i})^2}{6\Delta_{0f}}. \tag{8.153}$$

Note that within linear analysis $\Delta_\infty = \Delta_{0f}$. As we show in Sect. 8.5, this is a general feature of linearized dynamics around the ground state regardless of coupling strength or initial conditions: $|\Delta(t)|$ tends to its ground state value corresponding to the Hamiltonian with which the system evolves at $t > 0$.

Region III: $\Delta_{0i}/\Delta_{0f} < e^{-\pi/2}$. There are two pairs of complex conjugate roots,

$$\varepsilon_F \pm i\Delta_{0i}\cosh\phi_1, \quad \varepsilon_F \pm i\Delta_{0i}\cosh\phi_2, \tag{8.154}$$

where ϕ_1 is the solution of Eq. (8.150) and ϕ_2 is the solution of Eq. (8.151); ϕ_2 is real when $\delta\omega/\gamma = \ln(\Delta_{0i}/\Delta_{0f}) \le -2$ and imaginary otherwise. We see that the roots are indeed of the form Eq. (8.119). The asymptotic state is that of Section 8.2.4, while $\Delta(t)$ takes the simplified form Eq. (8.120).

Just as Eq. (8.146) the above results starting with Eq. (8.149) are universal in that they hold for quenches within the BCS regime independent of the dimensionality and also hold for the single channel model.

Next, consider quenches originating deep in the BEC, which corresponds to $\mu_i \to -\infty, \Delta_{0i} \to \sqrt{\gamma}$, and $x \to -\infty$. Since $v > -x$ in Eq. (8.140), we also have $v \to \infty$ provided a real root exists. Eq. (8.140) for $\delta\omega > 0$ simplifies to

$$\ln\left[\frac{v+x}{|x|}\right] = v\left(\frac{4\Delta_{0i}}{\gamma} - \pi\right) \tag{8.155}$$

For $4\Delta_{0i}/\gamma < \pi$, there is a single root at $v \to -x$, which corresponds to $u \approx 0$. Since $\Delta_{0i} \le \sqrt{\gamma} = \Delta_{\max}$, the condition $\Delta_{0i} < \pi\gamma/4$ can be fulfilled only if $\gamma > \gamma_c$, where

$$\gamma_c = \frac{16}{\pi^2}. \tag{8.156}$$

For $\gamma \ge \gamma_c$ Eq. (8.140) at $\delta\omega > 0$ has a single root for any Δ_{0i} and, in particular, for $\Delta_{0i} \to \Delta_{\max}$. This means that the I-II critical line extends all the way up to $\Delta_{0i} = \Delta_{\max}$ terminating at $(\Delta_{0i}, \Delta_{0f}) = (\Delta_{\max}, 0)$.

It is interesting to work out the shape of the I-II critical line near its termination point. First, let $\gamma > \gamma_c$. Since $v \approx -x$, Eq. (8.139) implies $\delta\omega/\gamma \approx \pi|x|/2$. Using Eqs. (8.142) and (8.143) to determine Δ_{0i} and μ_i and Eq. (8.141) for Δ_{0f}, we get

$$\begin{gathered}\frac{\Delta_{0f}}{\Delta_{\max}} = \frac{1}{\sqrt{2\varepsilon}}\exp\left(-\frac{\alpha}{2\varepsilon}\right), \\ \varepsilon = \frac{\Delta_{\max} - \Delta_{0i}}{\Delta_{\max}}, \quad \alpha = \sqrt{\frac{\gamma}{\gamma_c}} - 1.\end{gathered} \tag{8.157}$$

This behavior is seen in Figs. 8.2(c), 8.20(b) and 8.20(c). Note the difference between $\gamma = 5$ and $\gamma = 50$ in Figs. 8.20(b) and 8.20(c) that correspond to $\alpha \approx 0.8$ and $\alpha \approx 4.6$, respectively.

Next, let $\gamma < \gamma_c$. In this case, the I-II critical line goes up, then bends backward reaching a maximum, goes down, and terminates on the Δ_{0i} axis below $\Delta_{\max}$, see e.g. the inset in Fig. 8.2(b). Near the termination point μ_i and ω_i are finite since $\Delta_{0i} < \Delta_{\max}$, while $\omega_f \to \infty$ since $\Delta_{0f} \to 0$. Eq. (8.139) implies $v \to \infty$ and $\delta\omega/\gamma \approx \pi v/2$. In this limit Eq. (8.140) becomes

$$\ln\left[\frac{2v}{\sqrt{1+x^2}-x}\right] = v\left(\frac{4\Delta_{0i}}{\gamma} - \pi\right). \tag{8.158}$$

We see that v diverges as $\Delta_{0i} \to \pi\gamma/4 = \pi\sqrt{\gamma}\Delta_{\max}/4 \equiv \Delta_{\text{th}}$. Therefore, the I-II critical line terminates at $(\Delta_{0i}, \Delta_{0f}) = (\Delta_{\text{th}}, 0)$. For Δ_{0i} above Δ_{th} and below a certain upper value, which we do not determine explicitly, Eq. (8.140) has two roots. For Δ_{0i} below Δ_{th} there is one root.

The shape of the I-II critical line as it approaches the termination point for $\gamma < \gamma_c$ obtains from Eq. (8.158). Let $\varepsilon = (\Delta_{0i} - \Delta_{\text{th}})/\Delta_{\text{th}} \ll 1$. Eq. (8.158) implies

$$v \approx \frac{1}{\pi\varepsilon}\ln\left[\frac{2(\sqrt{1+x^2}+x)}{\pi\varepsilon}\right]. \tag{8.159}$$

The gap equation (8.26) yields in 2d

$$\sqrt{1+x^2} + x = \frac{2(\gamma - \Delta_{0i}^2)}{\gamma\Delta_{0i}}. \tag{8.160}$$

Since $\omega_f \to \infty$ corresponds to the BCS limit, we have $\Delta_{0f} \propto e^{-\omega_f/\gamma} \propto e^{-\pi v/2}$. Combining this with the last two equations and using $\Delta_{0i} \approx \Delta_{\text{th}} = \pi\gamma/4$, we get

$$\Delta_{0f} = C\exp\left(-\frac{1}{2\varepsilon}\ln\left[\frac{\gamma_c - \gamma}{\gamma\varepsilon}\right]\right), \tag{8.161}$$

where C is independent of ε.

The I-II critical line for $\gamma < \gamma_c$ is shown in Figs. 8.2(a), 8.2(b), and 8.20(a) that correspond to $\Delta_{\text{th}}/\Delta_{\max} \approx 0.25, 0.78$, and 0.18, respectively. Δ_{th} appears somewhat larger in these plots since exponentially small, but finite Δ_{0f} in Eq. (8.161) is not noticeable – the critical line effectively goes down along the Δ_{0i} axis. In the same way the I-II critical line appears to terminate below $\Delta_{\max}$ in Fig. 8.20 for $\gamma = 50$ due to exponential smallness of Δ_{0f} in Eq. (8.157).

Figure 8.21: Detuning quench phase diagrams for 2-channel model in 3d for various resonance widths γ obtained from Eqs. (8.163) and (8.164). Otherwise same as Fig. 8.20.

8.3.2 3d

3d diagrams for various values of resonance width γ are shown in Figs. 8.3 and 8.21. Overall they are qualitatively similar to 2d diagrams. A notable difference is that in 3d region III of the oscillating order parameter $\Delta(t)$ for sufficiently large γ terminates at $\Delta_{0f} < \Delta_{\max} = \sqrt{2\gamma/3}$. This means that quenches from infinitesimally weak to sufficiently strong coupling produce no oscillations. Also, in contrast to the 2d case, the critical line separating the gapless region I in principle always extends all the way up to $\Delta_{0i} = \Delta_{\max}$ and terminates at $\Delta_{0f} = \Delta_{0f}^{\mathrm{I-II}} > 0$. This is however not noticeable at small γ because in this case the value of $\Delta_{0f}^{\mathrm{I-II}}$ is exponentially small.

In 3d the dimensionless density of states $f(\varepsilon) = \sqrt{\varepsilon}$ and Eq. (8.127) becomes

$$\int_{-x}^{\infty} \frac{dy\sqrt{\Delta_{0i}(x+y)}}{(v-y)\sqrt{y^2+1}} = -\frac{2\delta\omega}{\gamma(v \pm i)} + \frac{4\Delta_{0i}}{\gamma}, \tag{8.162}$$

where $y = \varepsilon/\Delta_{0i} - x$, and x and v are defined in Eq. (8.137). Similarly, Eqs. (8.129) and (8.128) determining critical lines read

$$\frac{|\delta\omega|}{\gamma} = \frac{\pi}{2}\sqrt{\Delta_{0i}(x+v)(v^2+1)}, \tag{8.163}$$

$$\int_{-x}^{\infty} \frac{dy\sqrt{\Delta_{0i}(x+y)}}{(v-y)\sqrt{y^2+1}} + \mathrm{sgn}(\delta\omega)\frac{\pi v\sqrt{\Delta_{0i}(x+v)}}{\sqrt{v^2+1}} = \frac{4\Delta_{0i}}{\gamma}, \tag{8.164}$$

The integral here is a complete elliptic integral. Substitution $y = 1/t - x$ reduces it to one of the Carlson elliptic integrals with known asymptotic behaviors in various regimes [Carlson and Gustafson(1994), DLMF()]. We however find it more convenient to evaluate the limiting behaviors by a direct analysis of the integral.

First, we consider initial states deep in the BCS regime, i.e. $\omega_i \to +\infty$, which implies $\Delta_{0i} \to 0$, $\mu_i \to 1$, and $x \to 1/\Delta_{0i} \to +\infty$. To evaluate the integral in Eqs. (8.162) and (8.164) in this regime, we split the integration range into three intervals: $(-x, -y_\Lambda), (-y_\Lambda, y_\Lambda), (y_\Lambda, \infty)$, where y_Λ is such that $1 \ll y_\Lambda \ll x$. Let the corresponding integrals be I_1, I_2 and I_3. To the leading order in $1/y_\Lambda$ and y_Λ/x we can replace $\sqrt{y^2+1} \to |y|$ in I_1, I_3 and $\sqrt{x+y} \to \sqrt{x}$ in I_2. The resulting integrals evaluate in terms of elementary functions

$$\begin{aligned} I_1 + I_3 = {} & \frac{2\sqrt{x}}{v}\ln\frac{4x}{y_\Lambda} - \\ & \frac{\sqrt{x+v}}{v}\ln\frac{4x(\sqrt{x+v}+\sqrt{x})^2}{y_\Lambda^2 - 4x(\sqrt{x+v}-\sqrt{x})^2}, \end{aligned}$$

$$I_2 = \frac{\sqrt{x}}{\sqrt{1+v^2}}\ln\frac{(\sqrt{1+v^2}\sqrt{1+y_\Lambda^2}+vy_\Lambda)(v+y_\Lambda)}{(\sqrt{1+v^2}\sqrt{1+y_\Lambda^2}-vy_\Lambda)(v-y_\Lambda)},$$

where we used $1 \ll y_\Lambda \ll x$ to simplify expressions. The dependence on y_Λ should of course cancel from $I_1 + I_2 + I_3$ to the leading order in $1/y_\Lambda$ and y_Λ/x.

The gap equation (8.26) in the BCS regime is handled similarly by splitting the integral into three resulting in

$$\frac{\omega}{\gamma} - \frac{2}{\gamma} = \sqrt{\varepsilon_\Lambda} - 2 + \ln\frac{8}{\Delta_0}. \tag{8.165}$$

Suppose the final detuning is also in the BCS regime. The above equation then implies

$$\frac{\delta\omega}{\gamma} = \ln\frac{\Delta_{0i}}{\Delta_{0f}}, \tag{8.166}$$

same as in 2d. Because $\delta\omega/\gamma$ must remain of order one as $x \to +\infty$, it follows from Eq. (8.163) that v is also of order one for quenches within the BCS regime. Therefore $|v| \ll y_\Lambda$ in the above expressions for $I_1 + I_2$ and I_3. We obtain $|I_1 + I_2| \ll 1$ and

$$I_1 + I_2 + I_3 \approx I_3 \approx \frac{1}{1+v^2}\ln\left[\frac{v+\sqrt{1+v^2}}{v-\sqrt{1+v^2}}\right]. \tag{8.167}$$

Eq. (8.162) now turns into the 2d Eq. (8.149), Eq. (8.163) yields $|\delta\omega|/\gamma = \pi/2$ and therefore Eq. (8.146). Thus quenches within the BCS regime in 3d are identical to those in 2d and all results from Eq. (8.149) to Eq. (8.154) also hold in 3d. As we already commented above, this is expected since in the BCS regime superconductivity comes from the vicinity of the Fermi energy making the dependence of the density of states on the energy and thus the dimensionality inessential.

The horizontal Δ_{0i} = const line for infinitesimal values of the const intersects the II-III critical line twice, once near the origin, and the second time near the termination point of the II-III critical line. The former intersection corresponds to small v, as we saw above, and the latter to v of order x. To determine the termination point we therefore take $|v| \gg y_\Lambda$ in the above expressions for $I_1 + I_3$ and I_2. Eq. (8.162) becomes

$$\begin{aligned} \frac{\sqrt{x+v}}{\sqrt{x}} \ln \frac{(\sqrt{x+v}+\sqrt{x})^2}{-(\sqrt{x+v}-\sqrt{x})^2} = \\ -2\left[\frac{\delta\omega}{\gamma} + \ln\frac{8}{\Delta_{0i}} - \frac{2v\Delta_{0i}}{\gamma}\right] \pm \frac{2i\delta\omega}{v\gamma}. \end{aligned} \tag{8.168}$$

The real root of this equation is $v \approx -x \approx -1/\Delta_{0i}$ yielding

$$\frac{\delta\omega}{\gamma} = -\ln\frac{8}{\Delta_{0i}} - \frac{2}{\gamma}. \tag{8.169}$$

Combining this with Eq. (8.165), taking the limit $\Delta_{0i} \to 0$, and plugging into the gap equation (8.26), we obtain

$$4 + \frac{4\mu_f}{\gamma} = \int_0^\infty \left[\frac{1}{\varepsilon} - \frac{1}{\sqrt{(\varepsilon-\mu_f)^2 + \Delta_{0f}^2}}\right]\sqrt{\varepsilon}d\varepsilon, \tag{8.170}$$

where we sent the cutoff ε_Λ to infinity. Eq. (8.170) together with the chemical potential equation (8.25) determine the value of $\Delta_{0f}^{\text{II-III}}$ where the II-III critical line terminates on the Δ_{0f} axis. $\Delta_{0f}^{\text{II-III}}$ is a function of γ only, see Fig. 8.22.

Figure 8.22: Termination point of the II-III critical line as a function of resonance width γ in units of Fermi energy for 3d 2-channel model. This line encloses region III of persistent oscillations in Figs. 8.3 and 8.21. It starts at the origin and ends at $\Delta_{0f}^{\text{II-III}}$ along the Δ_{0f} axis. This reflects an interesting phenomenon: there are no persistent oscillations for quenches to couplings stronger then a certain threshold (i.e. quenches to detunings ω_f such that the corresponding ground state gaps $\Delta_{0f} \geq \Delta_{0f}^{\text{II-III}}$) no matter how weak the initial coupling is (i.e. for any initial detuning). At $\gamma \to \infty$ (1-channel limit) $\Delta_{0f}^{\text{II-III}}$ saturates at $1.49\varepsilon_F$ in agreement with Eq. (8.194).

Figure 8.23: Unlike 2d, in 3d Δ_{0f} tends to a finite value $\Delta_{0f}^{\text{I-II}}$ along the I-II critical line as the initial detuning $\omega_i \to -\infty$ ($\Delta_{0i} \to \Delta_{\max}$) for quenched 2-channel model, see e.g. Fig. 8.21. The gapless regime thus persists even for quenches from arbitrarily large negative ω_i to finite ω_f. Here we compare $\Delta_{0f}^{\text{I-II}}$ (in units of the Fermi energy) as a function of the resonance width γ extrapolated from actual phase diagrams with that obtained from Eqs. (3.51) and (3.52). Note that $\Delta_{0f}^{\text{I-II}}$ is exponentially small at small γ, so that the I-II critical line *appears* to close earlier at zero Δ_{0f} in Fig. 8.21(a).

We also note that it follows from the above analysis that, just as in 2d, for initial states deep in the BCS regime there are three roots: $v \to 0$ for either sign of $\delta\omega$ and $v \to -x + 0$ for $\delta\omega < 0$. Therefore, Eq. (8.145) holds in 3d as well.

Second, consider quenches from deep BEC to larger detuning $\omega_f > \omega_i$, i.e. $\omega_i \to -\infty, \delta\omega > 0, \mu_i \to -\infty, x \to -\infty, \Delta_{0i} \to \Delta_{\max}$. Since $y \geq |x| \gg 1$ in Eq. (8.164), we can replace $\sqrt{y^2+1} \to y$. The principal value integral evaluates to $-\pi\sqrt{|x|}/v$ and Eq. (8.164) becomes

$$-\frac{\pi\sqrt{|x|}}{v} + \pi\sqrt{v - |x|} = \frac{4\sqrt{\Delta_{0i}}}{\gamma}, \tag{8.171}$$

where we also took into account that we need $v \geq |x|$ so that Eq. (8.163) yields real $\delta\omega$. The solution for large $|x|$ is

$$\sqrt{v - |x|} \approx \frac{4\sqrt{\Delta_{0i}}}{\pi\gamma} + \frac{1}{\sqrt{|x|}}. \tag{8.172}$$

Eq. (8.163) now yields

$$\frac{\delta\omega}{\gamma} \approx \frac{2|\mu_i|}{\gamma} + \frac{\pi}{2}\sqrt{|\mu_i|} + \frac{32\Delta_{\max}^2}{\pi^2\gamma^3}, \tag{8.173}$$

where we replaced $\Delta_{0i}^2 \to \Delta_{\max}^2 = 2\gamma/3$ up to terms of order $|\mu_i|^{-1/2}$. The overall correction to this expression is also proportional to $|\mu_i|^{-1/2}$ at large $|\mu_i|$.

Similar simplifications occur in the gap equation (8.26). We replace the square root with $\varepsilon - \mu_i$ to obtain

$$\frac{\omega_i}{\gamma} \approx \sqrt{\varepsilon_\Lambda} - \frac{2|\mu_i|}{\gamma} - \frac{\pi}{2}\sqrt{|\mu_i|}.$$

Last two equations determine ω_f and from the gap equation (8.26) for $\omega = \omega_f$ we obtain

$$\frac{128}{3\pi^2\gamma^2} - \frac{4\mu_f}{\gamma} = \int_0^\infty \left[\frac{\sqrt{\varepsilon}}{\sqrt{(\varepsilon-\mu_f)^2 + \Delta_{0f}^2}} - \frac{1}{\sqrt{\varepsilon}}\right] d\varepsilon, \tag{8.174}$$

where we eliminated the cutoff similar to Eq. (8.170). This equation combined with Eq. (8.25) determines the termination point $(\Delta_{0i}, \Delta_{0f}) = (\Delta_{\max}, \Delta_{0f}^{\text{I-II}})$ of the I-II critical line. The plot of $\Delta_{0f}^{\text{I-II}}$ as a function of γ is shown in Fig. 8.23.

Note that, in contrast to the 2d case, this critical line formally always extends up to $\Delta_{0i} = \Delta_{\max}$ and $\Delta_{0f}^{\mathrm{I-II}}$ does not vanish as $\Delta_{0i} \to \Delta_{\max}$. This means that the gapless regime persists even for quenches to finite final detunings from initial states lying arbitrarily deep in the BEC regime. But for small γ the value of $\Delta_{0f}^{\mathrm{I-II}}$ is exponentially small and the critical line *appears* to have closed at smaller Δ_{0i}, see Figs. 8.3(a) and 8.21(a). Small γ implies large left hand side in Eq. (8.174) and therefore the final state deep in the BCS regime. In this regime $\mu_f \to 1$ and the integral in Eq. (8.174) is twice the right hand side of Eq. (8.165) without $\sqrt{\varepsilon_\Lambda}$ resulting in

$$\Delta_{0f}^{\mathrm{I-II}} = 8 \exp\left[-\frac{64}{3\pi^2\gamma^2} + \frac{2}{\gamma} - 2\right]. \tag{8.175}$$

We see from Fig. 8.23 that $\Delta_{0f}^{\mathrm{I-II}}$ becomes noticeable for $\gamma \gtrsim 0.45$. For smaller γ the gapless region I appears to close at smaller Δ_{0i} and zero Δ_{0f}. Fig. 8.23 also shows that Eq. (8.175) provides a reasonable estimate of $\Delta_{0f}^{\mathrm{I-II}}$ even for large γ, which will be useful in our analysis of the 1-channel model below.

8.4 One channel model

In this section we collect for reference purposes analogous results for the asymptotic steady state after a quench $\lambda_i \to \lambda_f$ in the one channel model given by Eqs. (8.3) and (8.5).

As explained in Sect. 8.1.1, the one channel model obtains in the broad resonance limit via replacements

$$\frac{\omega}{\gamma} = \frac{\omega}{g^2\nu_F} \to \frac{1}{\lambda}, \quad \gamma = g^2\nu_F \to \infty \tag{8.176}$$

(in units of ε_F). Our task is to go over equations of previous sections performing these replacements. All essential reasoning and methods are the same.

Chemical potential and gap Eqs. (8.25) and (8.26) now read

$$\frac{4}{d} = \int_0^\infty \left(1 - \frac{\varepsilon - \mu}{\sqrt{(\varepsilon-\mu)^2 + \Delta_0^2}}\right) f(\varepsilon) d\varepsilon, \tag{8.177}$$

and

$$\frac{2}{\lambda} = \int_0^{\varepsilon_\Lambda} \frac{f(\varepsilon) d\varepsilon}{\sqrt{(\varepsilon-\mu)^2 + \Delta_0^2}}, \tag{8.178}$$

respectively.

The Lax vector becomes

$$\vec{L}(u) = \sum_{\mathbf{p}} \frac{\vec{s}_{\mathbf{p}}}{u - \varepsilon_{\mathbf{p}}} - \frac{\hat{\mathbf{z}}}{\lambda\nu_F}. \tag{8.179}$$

Gaudin algebra, i.e. Eqs. (8.51) and (8.52), as well as the Lax equation of motion (8.57) are the same. The numerator of the conserved $\vec{L}^2(u)$ is now a polynomial of degree $2N$,

$$\vec{L}^2(u) = \frac{Q_{2N}(u)}{(\lambda\nu_F)^2 \prod_{\mathbf{p}} (u - \varepsilon_{\mathbf{p}})^2}, \tag{8.180}$$

where N is the number of nondegenerate $\varepsilon_{\mathbf{p}}$.

Reduced solutions are constructed in the same way with minor modifications. Specifically, the expressions for $\vec{L}^{\mathrm{red}}(u)$ in terms of $\vec{\sigma}_{\mathbf{p}}$ and $\vec{L}_m(u)$ in terms of $\vec{t}_j$ are replaced in Eqs. (8.59) and (8.60) with the corresponding 1-channel Lax vectors according to Eq. (8.179). The Hamiltonian governing the collective spin variables $\vec{t}_j$ is

$$H_{1\mathrm{ch}}^{\mathrm{red}} = \sum_{j=0}^{m-1} 2\eta_j t_j^z - \lambda\nu_F \sum_{j,k=0}^{m-1} t_j^- t_k^+. \tag{8.181}$$

Eqs. (8.62) and (8.63) as well as constraints (8.64) are the same, except that the last equation relating ω and ω' is absent. In terms of the m-spin spectral polynomial $Q_{2m}(u)$ the constraints become

$$\sum_{\mathbf{p}} \frac{\sigma_{\mathbf{p}} \varepsilon_{\mathbf{p}}^{r-1}}{\sqrt{Q_{2m}(\varepsilon_{\mathbf{p}})}} = -\frac{\delta_{rm}}{(\lambda \nu_F)^2}, \quad r = 1, \ldots, m. \tag{8.182}$$

Further, since the degree of the m-spin spectral polynomial is $2m$ rather then $2(m+1)$, an m-spin solution of the 2-channel model becomes an $(m+1)$-spin solution of the 1-channel model. This name change reflects the fact that the oscillator mode b in the 2-channel model is effectively an additional spin, which was not counted as such.

All remaining equations in Sect. 8.2, i.e. Eqs. (8.67) through (8.123), are identical for the one channel model, except Eq. (8.70) is replaced with Eq. (8.181) for $m = 2$ and the self-consistency condition (8.99) is now given by Eq. (8.13).

Equations determining isolated roots, critical lines, and $\mu_\infty = 0$ line for the 1-channel model are Eq. (8.127), Eqs. (8.129) and (8.128), and Eq. (8.130), respectively, with replacements

$$\frac{\delta\omega}{\gamma} \to \frac{1}{\lambda_f} - \frac{1}{\lambda_i} \equiv \beta, \quad \frac{1}{\gamma} \to 0. \tag{8.183}$$

Asymptotic spin distribution – the constant angle the spin $\vec{s}(\varepsilon)$ makes with the spin $\vec{\sigma}(\varepsilon)$ in the corresponding m-spin solution – is

$$\begin{aligned} \cos\theta(\varepsilon) = &\frac{z(\varepsilon)}{i\pi f(\varepsilon)} \sqrt{A_-^2 \Delta_{0i}^2 + [(\varepsilon - \mu_i)A_- + \delta\beta]^2} \\ &- \frac{z(\varepsilon)}{i\pi f(\varepsilon)} \sqrt{A_+^2 \Delta_{0i}^2 + [(\varepsilon - \mu_i)A_+ + \delta\beta]^2}, \end{aligned} \tag{8.184}$$

where

$$A_\pm = \pm \frac{i\pi f(\varepsilon)}{2E_i(\varepsilon)} + \fint_0^\infty \frac{f(\varepsilon')d\varepsilon'}{2(\varepsilon - \varepsilon')E_i(\varepsilon')}. \tag{8.185}$$

Eq. (8.184) is in excellent agreement with the actual spin distribution obtained from direct simulation of spin dynamics[Yuzbashyan and Dzero(2006)], see Fig. 3 therein.

8.4.1 Quench phase diagram

Quench phase diagrams for 1-channel model in 2d and 3d are shown in Figs. 8.4 and 8.5. There is only one diagram in each case extending to positive infinity in both Δ_{0i} and Δ_{0f} directions because $\gamma \to \infty$ and therefore $\Delta_{\max} \to \infty$.

As we commented below Eqs. (8.146) and (8.154), the weak coupling part of the diagrams (the region of small Δ_{0i} and Δ_{0f} near the origin) is independent of the dimensionality and is exactly the same for 1-channel model. In other words, all results contained in Eqs. (8.149) through (8.154) and the surrounding text apply to the one channel model in both 2d and 3d; one only needs to replace $\delta\omega/\gamma \to \delta\beta$.

When either the initial or final coupling is outside the deep BCS regime, we need to treat 2d and 3d cases separately.

2d

It is straightforward to take the broad resonance limit in Eqs. (8.138) to (8.144). In particular, the critical lines are determined by taking this limit in Eqs. (8.139) and (8.140)

$$|\delta\beta| = \frac{\pi}{2}\sqrt{1 + v^2}, \tag{8.186}$$

$$\ln\left[\frac{(v+x)(v+\sqrt{1+v^2})}{\sqrt{1+x^2}\sqrt{1+v^2}-xv+1}\right]=-\mathrm{sgn}(\delta\beta)\pi v. \tag{8.187}$$

Eq. (8.145) describing quenches originating in deep BCS remains as is, except the sign of $\delta\omega$ translates into the sign of $\delta\beta$. The two roots $u\approx\mu_i$ for either sign of $\delta\beta$ correspond to quenches also terminating in deep BCS, so they are in the universal regime given by Eqs. (8.149) through (8.154), which is shared by both models regardless of the dimensionality.

The analysis for the root $u\approx+0$ at $\delta\beta<0$ leading to Eq. (8.148) requires some modifications. $\gamma\to\infty$ limit in Eqs. (8.147) and (8.143) yields $4|\mu_f|=\Delta_{0i}^2e^{\pi/\Delta_{0i}}$, $\Delta_{0f}=\sqrt{4|\mu_f|}$ and finally

$$\Delta_{0f}=\Delta_{0i}e^{\pi/2\Delta_{0i}},\quad \Delta_{0i}\to 0. \tag{8.188}$$

This equation gives the asymptotic form of the II-III critical line in the $(\Delta_{0i},\Delta_{0f})$ plane in Fig. 8.4. We see that this line never terminates in the 2d 1-channel model.

Finally, let us work out the shape of the I-II critical line for large Δ_{0i}, i.e. for quenches originating deep in the BEC regime. Eq. (8.155) becomes

$$\ln\left[\frac{v+x}{|x|}\right]=-\pi v. \tag{8.189}$$

Now there is always a single root $v\to -x$ ($u\approx 0$). Eq. (8.186) implies

$$\delta\beta=\frac{1}{\lambda_f}-\frac{1}{\lambda_i}=\frac{\pi|x|}{2}=\frac{\pi|\mu_i|}{2\Delta_{0i}}. \tag{8.190}$$

We also need the gap equation in BCS and BEC limits and the chemical potential equation in the BEC limit. Sending γ to infinity in Eqs. (8.141), (8.142) and (8.143), we obtain

$$\ln\frac{4\varepsilon_\Lambda}{\Delta_{0i}^2}=\frac{2}{\lambda_i},\quad \ln\frac{\varepsilon_\Lambda}{|\mu_f|}=\frac{2}{\lambda_f},\quad \Delta_{0i}=\sqrt{4|\mu_i|}. \tag{8.191}$$

Combining these equations with Eq. (8.190), we get

$$\Delta_{0f}=\Delta_{0i}e^{-\pi\Delta_{0i}/8},\quad \Delta_{0i}\to\infty. \tag{8.192}$$

We see that Δ_{0f} exponentially vanishes along the I-II critical line (gapless regime closes) as Δ_{0i} increases. The vertical range of Fig. 8.4 is not enough to fully display this behavior, though we see that I-II line does incline towards the Δ_{0i} axis at large Δ_{0i}.

3d

In addition to quenches that fall within the universal weak coupling regime described in Eqs. (8.149) to (8.154) and the corresponding text, let us derive the termination point of the II-III critical line and analyze the I-II line at large Δ_{0i}.

First, we consider the II-III line. The termination point is given by Eq. (8.170). In the $\gamma\to\infty$ limit we have

$$4=\int_0^\infty\left[\frac{1}{\varepsilon}-\frac{1}{\sqrt{(\varepsilon-\mu_f)^2+\Delta_{0f}^2}}\right]\sqrt{\varepsilon}d\varepsilon. \tag{8.193}$$

Chemical potential equation (8.177) provides another relation between μ_f and Δ_{0f}. Numerical solution of these two equations is

$$\mu_f^{\mathrm{II-III}}\approx -1.4602\varepsilon_F,\quad \Delta_{0f}^{\mathrm{II-III}}\approx 1.4875\varepsilon_F. \tag{8.194}$$

This value of $\Delta_{0f}^{\text{II-III}}$ agrees with Fig. 8.22. Unlike 2d, in 3d region III encloses a finite area, resembling a dome in between the origin and the point $(\Delta_{0i}, \Delta_{0f}) = (0, \Delta_{0f}^{\text{II-III}})$.

Next, we turn to the critical line separating the gapless region I from region II. For finite γ we analyzed the termination point $(\Delta_{0i}, \Delta_{0f}) = (\Delta_{\max}, \Delta_{0f}^{\text{I-II}})$ of this line at the end of Sect. 8.3.2. In the single channel case, $\Delta_{\max} \to \infty$, so the I-II line does not close. As $\Delta_{0i} \to \infty$, the value of Δ_{0f} for a point on this line tends to $\Delta_{0f}^{\text{I-II}}$, which is determined by the $\gamma \to \infty$ limit of Eq. (8.174),

$$0 = \int_0^\infty \left[\frac{1}{\sqrt{(\varepsilon - \mu_f)^2 + \Delta_{0f}^2}} - \frac{1}{\varepsilon} \right] \sqrt{\varepsilon} d\varepsilon, \tag{8.195}$$

together with Eq. (8.177). The solution of these equations is

$$\mu_f \approx 0.5906 \varepsilon_F, \quad \Delta_{0f}^{\text{I-II}} \approx 0.6864 \varepsilon_F. \tag{8.196}$$

8.5 Transient Dynamics: Linear analysis

Here we solve the dynamics for small deviations from the ground state. Linear analysis for the one channel model in the weak coupling BCS regime was performed by Volkov and Kogan[Volkov and Kogan(1973)], see also Ref. [Yuzbashyan and Dzero(2006)]. Gurarie[36] extended this study to strongly coupled superconductors. Both these studies of the linearized dynamics conclude that

$$\Delta(t) \to \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi} \tag{8.197}$$

as $t \to \infty$, but the approach to this asymptote is different. Our analysis adds several new results to this prior work. We demonstrate that within linear analysis the amplitude of the order parameter asymptotes to its ground state value for the Hamiltonian with which the system evolves after nonequilibrium conditions are created, i.e. $\Delta_\infty = \Delta_{0f}$, a point that seems to have been missed by the earlier work. Also, $\mu_\infty = \mu_f$ – the ground state chemical potential. In other words, $\Delta_\infty - \Delta_0$ and $\mu_\infty - \mu_f$ are second order in the deviation. This is a general result that holds for both one and two channel models and is independent of the type of perturbation that drives the system out of equillibrium.

Further, we solve linearized equations of motion using the machinery of the exact solution [Yuzbashyan et al.(2005c), Yuzbashyan et al.(2005a)], which provides much more detailed information. For example, we also determine the short time behavior, normal modes, full explicit long time form $\Delta(t)$ and individual spins with all prefactors and phases etc. unavailable to conventional linear analysis. Note that in quench phase diagrams constructed above small quenches correspond to the vicinity of the diagonal $\Delta_{0i} = \Delta_{0f}$, see e.g Figs. 8.20 and 8.21.

8.5.1 Asymptotic $\Delta(t)$ and spins

Consider an infinitesimal quench of the detuning $\delta\omega = \omega_f - \omega_i$. More generally, $\delta\omega$ can be any small parameter that measures the deviation from the ground state in the two or one channel model. We work to linear order in $\delta\omega$. Suppose $\Delta(t) \to \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}$. For the detuning or interaction quenches this follows from the few spin conjecture and quench phase diagrams derived above and we also verify it independently below. Let us go to a reference frame that rotates with frequency $2\mu_\infty$ around the z-axis. In this frame $\Delta(t) = \Delta_\infty$ and the magnetic field $\vec{B}_{\mathbf{p}} = (-2\Delta_\infty, 0, 2\varepsilon_{\mathbf{p}} - 2\mu_\infty)$ acting on spin $\vec{s}_{\mathbf{p}}$ in Eq. (8.8) is time-independent. Note that transformation to the rotating frame results in shifts to $\varepsilon_{\mathbf{p}}$ and ω_f. Then, the spin rotates around $\vec{B}_{\mathbf{p}}$ making a constant angle $\pi - \theta_{\mathbf{p}}$ with it. This is in fact the asymptotic

solution described in Sect. 8.2.4,

$$\vec{s}_{\mathbf{p}}(t) = \frac{\vec{n}_{\mathbf{p}}}{2}\cos\theta_{\mathbf{p}} + \vec{s}^{\perp}_{\mathbf{p}}(t), \tag{8.198}$$

where $\vec{n}_{\mathbf{p}}$ is a unit vector along $-\vec{B}_{\mathbf{p}}$,

$$n^x_p = \frac{\Delta_\infty}{E^\infty_{\mathbf{p}}}, \quad n^y_{\mathbf{p}} = 0, \quad n^z_{\mathbf{p}} = -\frac{\varepsilon_{\mathbf{p}} - \mu_\infty}{E^\infty_{\mathbf{p}}}. \tag{8.199}$$

Eq. (8.8) with $\dot{b} = 0$ further implies $\Delta_\infty = -gb = g^2 J_-/(\omega_f - 2\mu_\infty)$. The contribution of $\vec{s}^{\perp}_{\mathbf{p}}$ to J_- dephases as $t \to \infty$. The latter is therefore $\sum_{\mathbf{p}} n^x_p/2$ – the sum of components of $\vec{s}_{\mathbf{p}}$ along $\vec{B}_{\mathbf{p}}$ projected onto the xy-plane,

$$\Delta_\infty = \frac{g^2}{\omega_f - 2\mu_\infty}\sum_{\mathbf{p}} \frac{\Delta_\infty \cos\theta_{\mathbf{p}}}{2\sqrt{(\varepsilon_{\mathbf{p}} - \mu_\infty)^2 + \Delta^2_\infty}}. \tag{8.200}$$

In the ground state $\vec{s}_{\mathbf{p}}$ is aligned with $-\vec{B}_{\mathbf{p}}$, i.e. $\theta_{\mathbf{p}} = 0$. This implies that $\theta_{\mathbf{p}}$ must be proportional to $\delta\omega$ and therefore corrections to $\cos\theta_{\mathbf{p}} = 1$ are second order in $\delta\omega$. But for $\cos\theta_{\mathbf{p}} = 1$, Eq. (8.200) is the ground state gap equation (8.18) $\left(\frac{(\omega-2\mu)}{g^2} = \sum_{\mathbf{p}} \frac{1}{2E(\varepsilon_{\mathbf{p}};\Delta_0,\mu)}\right)$ for $\omega = \omega_f$. Moreover, applying the same argument to J_z and Eq. (8.19), we find that Δ_∞ and μ_∞ also satisfy the ground state chemical potential equation (8.20). It follows that for small oscillations around the ground state one always has

$$\Delta_\infty = \Delta_{0f}, \quad \mu_\infty = \mu_f. \tag{8.201}$$

For the same reason the non-oscillatory part of $\vec{s}_{\mathbf{p}}$ (zeroth harmonic) in the steady state is the same as in the ground state at $\omega = \omega_f$, i.e. is given by Eq. (8.15) with $\Delta_0 \to \Delta_{0f}$ and $\mu \to \mu_f$.

The same is true for the one channel model. Note also that infinitesimal quenches in the BCS regime conform to this conclusion, see Eq. (8.153). Moreover, this result generalizes to finite spin dynamics, where, as we show below, zeroth harmonics of $\Delta(t)$ and $\vec{s}_{\mathbf{p}}$ to linear order in $\delta\omega$ coincide with the $\omega = \omega_f$ ground state values.

8.5.2 Normal modes and finite size dynamics

Now we turn to the linear analysis per se. At this point it is convenient to rewrite summations over $\mathbf{p}$ as summations over single particle energies. We adopt the following model of discrete spectrum. Let us discretize the magnitude of the momentum, $p \to p_k$. The corresponding energies are $\varepsilon_k = p_k^2/2m$ with degeneracy $N_k = N(\varepsilon_k)$ – the number of states in a momentum shell between p_k and p_{k+1}, which is a smooth function of ε_k. The level spacing $\delta_k = \varepsilon_{k+1} - \varepsilon_k$ is also assumed to depend on ε_k smoothly. We include this dependence in N_k, so without loss of generality we take it to be constant, $\delta_k = \delta$. Our final results depend only on the density of states $\nu(\varepsilon_k) = N_k/\delta$ – the number of states per unit energy. Equivalently, ε_i can represent levels of some other single particle potential, e.g. a 3d harmonic oscillator potential, see the discussion at the end of Sect. 8.1.1. All quantities and equations, including spins $\vec{s}_{\mathbf{p}}$, Hamiltonians, equations of motion and initial conditions, considered in this paper depend on $\mathbf{p}$ only through $\varepsilon_{\mathbf{p}}$. For any such quantity $A_{\mathbf{p}} = A(\varepsilon_{\mathbf{p}})$

$$\sum_{\mathbf{p}} A_{\mathbf{p}} = \sum_{k=1}^{N} N_k A_k \to \int \nu(\varepsilon) d\varepsilon, \tag{8.202}$$

where $A_k = A(\varepsilon_k)$. In particular, the Lax vector (8.50) reads

$$\vec{L}(u) = \sum_{k=1}^{N} \frac{N_k \vec{s}_k}{u - \varepsilon_k} - \frac{(\omega - 2\mu)}{g^2}\hat{\mathbf{z}} + \frac{2}{g^2}\left[(u-\mu)\hat{\mathbf{z}} - \vec{\Delta}\right]. \tag{8.203}$$

A convenient tool for linear analysis of the dynamics are the separation variables introduced in Refs. [Yuzbashyan et al.(2005a)] and [Yuzbashyan et al.(2005c)] for the one and two channel models, respectively. As we will see, in linearized dynamics these variables are simply the normal modes. Separation variables u_j are defined as the solutions of $L_-(u_j) \equiv L_x(u_j) - iL_y(u_j) = 0$, i.e.

$$L_-(u) = \frac{2b}{g} + \sum_{k=1}^{N} \frac{N_k s_k^-}{u - \varepsilon_k} = 0. \tag{8.204}$$

Because $u = u_j$ are the zeroes of the rational function $L_-(u)$ and $u = \varepsilon_k$ are its poles, we can also write it as

$$L_-(u) = \frac{2b}{g}\frac{\prod_j (u - u_j)}{\prod_k (u - \varepsilon_k)}. \tag{8.205}$$

Matching the residues at $u = \varepsilon_k$ and $u = \infty$ in Eqs. (8.204) and (8.205), we express the spins in terms of u_j,

$$N_k s_k^- = \frac{2b}{g}\frac{\prod_j (\varepsilon_k - u_j)}{\prod_{m\neq k}(\varepsilon_k - \varepsilon_m)}, \tag{8.206}$$

$$J_- = \sum_k N_k s_k^- = \frac{2b}{g}\sum_k (\varepsilon_k - u_k). \tag{8.207}$$

Equations of motion in terms of new variables are

$$\begin{aligned} \dot{u}_k &= -\frac{2i\sqrt{Q_{2N+2}(u_k)}}{\prod_{m\neq k}(u_k - u_m)}, \\ \dot{b} &= -2ib\left(\frac{\omega}{2} + \sum_k (\varepsilon_k - u_k)\right), \end{aligned} \tag{8.208}$$

see Ref. [Yuzbashyan et al.(2005c)] for a detailed derivation. Here $Q_{2N+2}(u)$ is the spectral polynomial defined in Eq. (8.58).

Roots of $Q_{2N+2}(u)$ are the same as roots of $\vec{L}^2(u)$ determined by Eq. (8.126). In our new notation

$$(u - \mu \mp i\Delta_0)\left[\frac{2}{g^2} - \sum_k \frac{N_k}{2(u-\varepsilon_k)E(\varepsilon_k)}\right] = \frac{\delta\omega}{g^2}, \tag{8.209}$$

where $E(\varepsilon_k) = \sqrt{(\varepsilon_k - \mu)^2 + \Delta_0^2}$. Here and everywhere below in this subsection μ and Δ_0 without a subscript indicate ground state values μ_i and Δ_{0i} for the initial detuning $\omega = \omega_i$. In the ground state $\vec{L}^2(u) = [(u-\mu_i)^2 + \Delta_{0i}^2]\, L_0^2(u)$. There is a pair of complex roots $c_\pm = \mu_i \pm i\Delta_{0i}$ and $2N$ real double degenerate roots x_k that solve

$$L_0(x) = -\frac{2}{g^2} + \sum_k \frac{N_k}{2(x-\varepsilon_k)E(\varepsilon_k)} = 0, \tag{8.210}$$

A plot of $L_0(x)$ reveals that x_k are located between consecutive ε_k, i.e. $\varepsilon_k < x_k < \varepsilon_{k+1}$.

Since $\vec{L}^2(x_k) = L_x^2(x_k) + L_y^2(x_k) + L_z^2(x_k) = 0$ in the ground state and x_k is real, all components of $\vec{L}(x_k)$ must vanish, $L_x(x_k) = L_y(x_k) = L_z(x_k) = 0$. It follows that $L_-(x_k) = 0$ meaning that the separation variables are frozen in the real double roots, $u_k = x_k$. After a quench they start to move from these initial positions, $u_k(t) = x_k + \delta u_k$, where δu_k vanishes at $t = 0$ and is proportional to $\delta\omega$ for an infinitesimal quench. For $\delta\omega \neq 0$ real double roots of $Q_{2N+2}(u)$ split into pairs of complex conjugate roots $c_k = x_k + \delta c_k$ and $\bar{c}_k = x_k + \delta\bar{c}_k$. Therefore, the expression for $Q_{2N+2}(u_k)$,

$$\begin{aligned} Q_{2N+2}(u_k) = (u_k - c_k)(u_k - \bar{c}_k)\times \\ (u_k - c_+)(u_k - c_-)\prod_{m\neq k}(u_k - c_m)(u_k - \bar{c}_m), \end{aligned} \tag{8.211}$$

to lowest nonzero order in $\delta\omega$ becomes

$$\begin{aligned} Q_{2N+2}(u_k) = (\delta u_k - \delta c_k)(\delta u_k - \delta\bar{c}_k)\Omega_k^2\times \\ \prod_{m\neq k}(x_k - x_m)^2, \end{aligned} \tag{8.212}$$

$\Omega_k = \sqrt{(x_k - \mu)^2 + \Delta_0^2}$, not to be confused with function $\Omega(t)$ in Sect. 8.2.2. Similarly, the denominator of the equation of motion (8.208) for u_k to the lowest order $\prod_{m\neq k}(u_k - u_m) = \prod_{m\neq k}(x_k - x_m)$, so this equation reads

$$\delta\dot{u}_k = \pm 2i\Omega_k\sqrt{(\delta u_k - \delta c_k)(\delta u_k - \delta\bar{c}_k)}. \tag{8.213}$$

Corrections to the roots due to the quench obtain by setting $u = x_k + \delta c_k$ in Eq. (8.209) and linearizing in δc_k. Separating real and imaginary parts, $\delta c_k = a_k + ib_k$, we have

$$a_k = \frac{\delta\omega(x_k - \mu)}{g^2\Omega_k^2 F_k}, \quad b_k = \frac{\delta\omega\Delta_0}{g^2\Omega_k^2 F_k}, \tag{8.214}$$

where

$$F_k = \sum_m \frac{N_k}{2(x_k - \varepsilon_m)^2 E(\varepsilon_m)}. \tag{8.215}$$

Let us also evaluate the correction to the complex root pair $c_\pm = \mu_i \pm i\Delta_{0i}$. Writing the perturbed roots as $\mu' \pm i\Delta'$, we obtain from Eq. (8.209) to linear order in $\delta\omega$

$$\begin{aligned} \mu' - \mu_i &= \frac{\delta\omega}{g^2}\frac{\beta_k}{\alpha_k^2 + \beta_k^2}, \\ \Delta' - \Delta_{0i} &= -\frac{\delta\omega}{g^2}\frac{\alpha_k}{\alpha_k^2 + \beta_k^2}, \end{aligned} \tag{8.216}$$

where α_k and β_k are defined in Eq. (8.313). Comparing this with first order shifts in the ground state gap and chemical potentials that readily derive from Eqs. (8.315), we conclude that

$$\mu' = \mu_f, \quad \Delta' = \Delta_{0f}, \tag{8.217}$$

as it should be according to Sect. 8.2.4, see the text following Eq. (8.114) and also below.

Eq. (8.213) is a harmonic oscillator equation, which yields

$$\delta u_k(t) = a_k(1 - \cos 2\Omega_k t) + il_k \sin 2\Omega_k t, \tag{8.218}$$

where

$$l_k = \pm\sqrt{a_k^2 + b_k^2} = \frac{\delta\omega}{g^2\Omega_k F_k}. \tag{8.219}$$

In deriving Eq. (8.218) we took into account the initial condition $\delta u_k(0) = 0$ and used expressions (8.214). We set the sign in the last equation in Eq. (8.219) to be plus, which we will justify later in this subsection.

Eq. (8.218) shows that $u_k(t)$ are the normal modes of small oscillations around the ground state and that the normal frequencies are $2\Omega_k = 2\sqrt{(x_k - \mu)^2 + \Delta_0^2}$, where x_k are the roots of Eq. (8.210). Eq. (8.218) also shows that in linear analysis separation variable $u_k(t)$ moves on an ellipse with semi-axes a_k and $\sqrt{a_k^2 + b_k^2}$ around the roots $c_k, \bar{c}_k$. The latter are the focal points of the ellipse. The function $\sqrt{Q_{2N+2}(u)}$ entering equations of motion for separation variables has branch cuts connecting pairs of conjugate roots c_k and $\bar{c}_k$, so one can also say that separation variables move on ellipses around brunch cuts without crossing any of them, see Fig. 8.24.

Figure 8.24: As a result of a quench doubly degenerate roots of $\vec{L}^2(u)$ in Fig. 8.17 split into pairs of complex conjugate roots c_m (not all $N = 54$ pairs of roots are shown). In linear analysis separation variables move periodically on ellipses around the brunch cuts of $(\vec{L}^2(u))^{-1/2}$ connecting complex conjugate c_m without crossing any of the brunch cuts. Each separation variable has its own distinct frequency and corresponds to a normal mode of small oscillations around the ground state. Here $\Delta_{0f} = 0.12\varepsilon_F$, $\delta\omega/\gamma = -0.1$, and other parameters are the same as in Fig. 8.17.

Next, we determine deviations of the spins $\delta\vec{s}_k(t)$ and the order parameter $\delta\Delta(t)$ from their initial ground state configuration (8.14) and (8.15). We go to a rotating reference frame,

$$s_k^- \to s_k^- e^{-2i\mu t}, \quad b \to b e^{-2i\mu t}, \tag{8.220}$$

to get rid of the time dependence in the unperturbed dynamical variables. This shifts $\omega \to \omega - 2\mu$ in the equation of motion (8.208) and now $\dot{b} = 0$ in the ground state *before* the quench, i.e. for $\omega = \omega_i$. Linearizing Eq. (8.206), we obtain a decomposition of spin deviations in terms of the normal modes

$$\frac{\delta s_k^-(t)}{s_k^-(0)} = \frac{\delta\Delta(t)}{\Delta_0} - \sum_j \frac{\delta u_j}{\varepsilon_k - x_j}. \tag{8.221}$$

Similarly, the second equation in (8.208) linearized and integrated in the rotating frame *after* the quench, i.e. with $\omega = \omega_f$, yields

$$\begin{aligned} \frac{\Delta(t)}{\Delta_0} &= 1 - \sum_k l_k \frac{1 - \cos 2\Omega_k t}{\Omega_k} - \\ i\delta\omega t &+ 2it \sum_k a_k - i \sum_k \frac{a_k \sin 2\Omega_k t}{\Omega_k}, \end{aligned} \tag{8.222}$$

where we took into account $\Delta(t) = -gb(t)$, $\Delta(0) = \Delta_0$ and expressions (8.218). The $i\delta\omega t$ appears because for unperturbed u_k the bracketed term in the second equation in (8.208) vanishes for $\omega = \omega_i$, while after the quench $\omega = \omega_f$.

Linearizing spin equations of motion (8.8) directly and plugging expressions (8.221) and (8.222), one can verify that the correct sign in the last equation in Eq. (8.219) is indeed plus, even though there is probably a simpler way to show this.

The imaginary part in Eq. (8.222) comes from the phase of the order parameter, so we write

$$\begin{aligned}\Delta(t) = \left(\Delta_0 - \Delta_0 \sum_k l_k \frac{1-\cos 2\Omega_k t}{\Omega_k}\right) \times \\ \exp\left[-i\delta\omega t + 2it\sum_k a_k - i\sum_k \frac{a_k \sin 2\Omega_k t}{\Omega_k}\right].\end{aligned} \tag{8.223}$$

This coincides with Eq. (8.222) to first order in $\delta\omega$. Moreover, we know from Eq. (8.201) that the linear part of the phase (zeroth harmonic in the derivative of the phase) is $-2\mu_f t$ in the continuum limit, where μ_f is the ground state chemical potential at detuning ω_f. Similarly, the zeroth harmonic in the amplitude of $\Delta(t)$ is equal to Δ_{0f}. It turns out that this is true even in the discrete case, i.e.

$$\begin{aligned}\Delta_0 - \Delta_0 \sum_k \frac{l_k}{\Omega_k} = \Delta_{0f}, \\ 2\mu + \delta\omega - 2\sum_k a_k = 2\mu_f,\end{aligned} \tag{8.224}$$

where we restored the phase of $\Delta(t)$ to the original reference frame according to Eq. (8.220). Recall that in this subsection μ and Δ_0 without a subscript indicate ground state values μ_i and Δ_{0i} for the initial detuning $\omega = \omega_i$. With the help of Eqs. (8.214) and (8.219) these relations become

$$\begin{aligned}\sum_k \frac{x_k - \mu}{\Omega_k^2 F_k} = \frac{g^2}{2} - g^2 \frac{\delta\mu}{\delta\omega}, \\ \sum_k \frac{\Delta_0}{\Omega_k^2 F_k} = -g^2 \frac{\delta\Delta_0}{\delta\omega},\end{aligned} \tag{8.225}$$

where $\delta\mu = \mu_f - \mu$ and $\delta\Delta_0 = \Delta_{0f} - \Delta_0$. These are in fact identities, as we prove in Appendix 8.11. Thus,

$$\begin{aligned}\Delta(t) = \left(\Delta_{0f} + \Delta_0 \sum_k l_k \frac{\cos 2\Omega_k t}{\Omega_k}\right) \times \\ \exp\left[-2i\mu_f t - i\sum_k \frac{a_k \sin 2\Omega_k t}{\Omega_k}\right],\end{aligned} \tag{8.226}$$

in the original reference frame.

An expression for $s_k^-(t)$ obtains similarly from Eqs. (8.221) and (8.222) with the help of

identity (8.316),

$$s_k^-(t) = s_{kf}^-\left(1 + \sum_j \frac{l_j \cos 2\Omega_j t}{\Omega_j} + \right.$$
$$\left. \sum_j \frac{a_j \cos 2\Omega_j t}{\varepsilon_k - x_j} - i \sum_j \frac{l_j \sin 2\Omega_j t}{\varepsilon_k - x_j}\right) \times \tag{8.227}$$
$$\exp\left[-2i\mu_f t - i\sum_j \frac{a_j \sin 2\Omega_j t}{\Omega_j}\right],$$

where

$$s_{kf}^- \equiv s_f^-(\varepsilon_k) = \frac{\Delta_{0f}}{2\sqrt{(\varepsilon_k - \mu_f)^2 + \Delta_{0f}^2}}. \tag{8.228}$$

The last term in round brackets in Eq. (8.227) can be as well included into the phase – to linear order the two versions are equivalent. The present form is more convenient for the long time analysis below. We see that again non-oscillatory parts of the magnitude and phase of $s_k^-(t)$ and magnitude of s_k^z are the same as in the ground state for final detuning $\omega = \omega_f$.

Finally, the expression for $s_k^z(t)$ follows the conservation of $|\vec{s}_k| = 1/2$, $s_k^z = \pm\sqrt{1/4 - |s_k^-|^2}$ expanded to the linear order in $\delta\omega$,

$$s_k^z(t) = s_{kf}^z\left(1 - \frac{\Delta_{0f}^2}{(\varepsilon_k - \mu_f)^2}\sum_j \frac{l_j \cos 2\Omega_j t}{\Omega_j} - \right.$$
$$\left. \frac{\Delta_{0f}^2}{(\varepsilon_k - \mu_f)^2}\sum_j \frac{a_j \cos 2\Omega_j t}{\varepsilon_k - x_j}\right), \tag{8.229}$$

where

$$s_{kf}^z \equiv s_f^z(\varepsilon_k) = -\frac{\varepsilon_k - \mu_f}{2\sqrt{(\varepsilon_k - \mu_f)^2 + \Delta_{0f}^2}}. \tag{8.230}$$

8.5.3 Continuum limit

In $N \to \infty$ limit, $x_k \to \varepsilon_k$ and summations in above expressions for $s_k^z(t), s_k^-(t)$, and $\Delta(t)$ turn into integrations. With the help of Eqs. (8.214), (8.219), (8.202), and (8.308), Eq. (8.226) obtains (as always in units of the Fermi energy ε_F)

$$\frac{\Delta(t)}{\Delta_{0f}} = \Big(1 + X_1(t)\Big)\exp\left[-2i\mu_f t - iX_2(t)\right], \tag{8.231}$$

where

$$X_1(t) = \frac{\delta\omega}{\gamma}\int_0^\infty \frac{2\cos\left[2E(\varepsilon)t\right] f(\varepsilon)d\varepsilon}{E(\varepsilon)\left[\pi^2 f^2(\varepsilon) + H^2(\varepsilon)\right]}, \tag{8.232}$$

$$X_2(t) = \frac{\delta\omega}{\gamma}\int_0^\infty \frac{2(\varepsilon - \mu)\sin[2E(\varepsilon)t] f(\varepsilon)d\varepsilon}{E^2(\varepsilon)[\pi^2 f^2(\varepsilon) + H^2(\varepsilon)]}, \tag{8.233}$$

$E(\varepsilon) = \sqrt{(\varepsilon - \mu)^2 + \Delta_0^2}$ and $H(\varepsilon)$ is defined in Eq. (8.307). In deriving this expression, we also used, $\Omega_k \to E(\varepsilon_k)$, $\nu(\varepsilon) = \nu_F f(\varepsilon), g^2\nu_F = \gamma$, and $\delta = N_k/\nu(\varepsilon_k)$. Eq. (8.231) is in excellent agreement with numerical results, see e.g. Fig. 8.25.

Figure 8.25: Comparison of Eq. (8.231) with $|\Delta(t)|$ computed by numerically evolving $N = 5024$ spins in a 3d 2-channel model after a detuning quench. Here $\gamma = 1.0, \Delta_{0i} = 0.122\Delta_{\max}, \Delta_{0f} = 0.126\Delta_{\max}$, and both Eq. (8.231) and the spin chain in the numerics are cutoff at $\varepsilon_\Lambda = 10\varepsilon_F$.

Expressions (8.229) and (8.227) for $s_k^-(t)$ and $s_k^z(t)$ contain two extra summations as compared to $\Delta(t)$. These are handled as in Appendix 8.11 by splitting each sum into two parts – over x_j inside and outside a small interval around ε_k. The same method works for summations over x_j because according to Eq. (8.307) $\varrho(\varepsilon)$ is a smooth function and therefore x_j are locally equally spaced with spacing δ just as ε_k. The second and the third sums in round brackets in Eq. (8.227) are

$$Y_1(\varepsilon,t) = \frac{\delta\omega}{\gamma}\fint_0^\infty \frac{2(\varepsilon'-\mu)\cos[2E(\varepsilon')t]f(\varepsilon')d\varepsilon'}{(\varepsilon-\varepsilon')E(\varepsilon')[\pi^2 f^2(\varepsilon')+H^2(\varepsilon')]} - \frac{\delta\omega}{\gamma}\frac{2(\varepsilon-\mu)H(\varepsilon)\cos[2E(\varepsilon)t]}{E(\varepsilon)[\pi^2 f^2(\varepsilon)+H^2(\varepsilon)]}, \tag{8.234}$$

$$Y_2(\varepsilon,t) = \frac{\delta\omega}{\gamma}\fint_0^\infty \frac{2\sin[2E(\varepsilon')t]f(\varepsilon')d\varepsilon'}{(\varepsilon-\varepsilon')[\pi^2 f^2(\varepsilon')+H^2(\varepsilon')]} - \frac{\delta\omega}{\gamma}\frac{2H(\varepsilon)\sin[2E(\varepsilon)t]}{\pi^2 f^2(\varepsilon)+H^2(\varepsilon)}, \tag{8.235}$$

respectively. Thus

$$\frac{s^-(\varepsilon,t)}{s_f^-(\varepsilon)} = \Big(1 + X_1(t) + Y_1(\varepsilon,t) - iY_2(\varepsilon,t)\Big)\times \exp\left[-2i\mu_f t - iX_2(t)\right], \tag{8.236}$$

$$\frac{s^z(\varepsilon,t)}{s_f^z(\varepsilon)} = 1 - \frac{\Delta_{0f}^2}{(\varepsilon-\mu_f)^2}\Big(X_1(t) + Y_1(\varepsilon,t)\Big). \tag{8.237}$$

Functions X_1 and X_2 are related via differentiation. Define

$$\widetilde{X}_1(t) = \int_0^\infty K(\varepsilon)e^{2i\widetilde{E}(\varepsilon)t}d\varepsilon, \tag{8.238}$$

where $\widetilde{E}(\varepsilon) = \sqrt{(\varepsilon-\widetilde{\mu})^2 + \Delta_0^2}$ and

$$K(\varepsilon) = \frac{\delta\omega}{\gamma}\frac{2f(\varepsilon)}{E(\varepsilon)\left[\pi^2 f^2(\varepsilon)+H^2(\varepsilon)\right]}. \tag{8.239}$$

Then,

$$X_1(t) = \mathrm{Re}\widetilde{X}_1(t)\Big|_{\widetilde{\mu}=\mu} \tag{8.240}$$

$$X_2(t) = \frac{1}{2t}\mathrm{Re}\frac{\partial \widetilde{X}_1(t)}{\partial \widetilde{\mu}}\Big|_{\widetilde{\mu}=\mu} \tag{8.241}$$

A similar relationship holds for Y_1 and Y_2.

8.5.4 Validity of the few spin conjecture

We are now in the position to prove the few spin conjecture for infinitesimal quenches independently of either numerics or arguments of Sect. 8.2. At $t \to \infty$ integrals in Eqs. (8.232) and (8.233) vanish by the Riemann-Lebesgue lemma. Therefore,

$$\Delta(t) \to \Delta_{0f} e^{-2i\mu_f t}. \tag{8.242}$$

According to the few spin conjecture this asymptotic behavior of $\Delta(t)$ occurs when there is a single isolated root pair at $\mu_f \pm i\Delta_{0f}$. Eq. (8.217) shows that our $\vec{L}^2(u)$ does have this pair of roots. Moreover, the remaining $2N$ roots are given by Eq. (8.214) and we explicitly see from Appendix 8.10 that their imaginary parts scale as $1/N$ at large N and that they merge into a continuum of roots on the real axis in $N \to \infty$ limit. Thus, there is indeed a single isolated root pair at $\mu_f \pm i\Delta_{0f}$ in the thermodynamic limit.

8.5.5 Weak coupling limit

Simpler expressions obtain in the weak coupling (BCS) limit when Δ_0 is much smaller then other energy scales (Fermi energy in gases and Debye energy in metals). This limit describes superconductivity in metals and applies to recent experiments on non-adiabatic BCS dynamics[Matsunaga et al.(2013), 98]. In our quench phase diagrams (Figs. 8.2 – 8.4 etc.) weak coupling regime corresponds to a small neighborhood of the origin.

At weak coupling $\mu \approx \varepsilon_F = 1$. Integrals (8.232) and (8.233) are dominated by energies close to the Fermi energy, $|\varepsilon - \mu| \sim \Delta_0$, where $f(\varepsilon) \approx 1$ independent of dimensionality. It is convenient to change the integration variable to $\xi = \varepsilon - \mu$ and extend the integration to the entire real axis. $X_2(t)$ vanishes by particle-hole symmetry (integrand is odd in ξ). The error due to these approximations is proportional to Δ_0/ε_F, which vanishes in weak coupling limit. Eq. (8.231) implies

$$|\Delta(t)| = \Delta_{0f} - 4\delta\Delta_0 \int_0^\infty \frac{\cos\left[2E(\xi)t\right] d\xi}{E(\xi)\left[\pi^2 + H^2(\xi)\right]}, \tag{8.243}$$

where $E(\xi) = \sqrt{\xi^2 + \Delta_0^2}$, $\delta\Delta_0 = \Delta_{0f} - \Delta_{0i}$, and

$$H(\varepsilon) = \ln\left[\frac{E(\xi) - \xi}{E(\xi) + \xi}\right]. \tag{8.244}$$

In deriving Eq. (8.243) we used the weak coupling gap formula $\Delta_0 \propto \exp(-\omega/\gamma)$ and Eqs. (8.305) and (8.307) [Note that at relevant energies $4E(\varepsilon)/\gamma \propto \Delta_0/\varepsilon_F \to 0$]. We also used the fact that the integrand is even in ξ to convert the integration range from $(-\infty, \infty)$ to $(0, \infty)$.

The phase of the order parameter defined through

$$\Delta(t) = |\Delta(t)| e^{-i\Phi(t)} \tag{8.245}$$

is simply $\Phi(t) = 2\varepsilon_F t$. Let us also note that in terms of $\xi = \Delta_0 \sinh(\pi x/2)$ Eq. (8.243) reads

$$|\Delta(t)| = \Delta_{0f} - 2\delta\Delta_0 \int_0^\infty \frac{dx}{\pi} \frac{\cos\left[2\tau \cosh(\pi x/2)\right]}{1+x^2}, \tag{8.246}$$

where $\tau = \Delta_0 t$.

8.5.6 Long time behavior of $\Delta(t)$: BCS side

Integrands in Eqs. (8.232) and (8.233) are highly oscillatory. The argument of the cosine is stationary at $\varepsilon = \mu$, $E'(\mu) = 0$. For $\mu > 0$ the stationary point is inside the integration range. For $\mu < 0$ there are no stationary points on the integration path. This leads to qualitatively different behavior of $\Delta(t)$ on the BCS ($\mu > 0$) and BEC ($\mu < 0$) sides.

Consider first the BCS regime. We evaluate $\widetilde{X}_1(t)$ in Eq. (8.238) in stationary phase approximation

$$\widetilde{X}_1(t) = K(\widetilde{\mu})\sqrt{\frac{\pi\Delta_0}{t}} e^{2i\Delta_0 t + i\pi/4} + O(1/t), \tag{8.247}$$

where we used $\widetilde{E}(\widetilde{\mu}) = \Delta_0$, $\widetilde{E}''(\widetilde{\mu}) = 1/\Delta_0$. With the help of Eq. (8.240) we obtain from Eq. (8.231) for the order parameter amplitude

$$|\Delta(t)| = \Delta_{0f} + \sqrt{\pi} K(\mu)\Delta_0^2 \frac{\cos(2\Delta_0 t + \pi/4)}{\sqrt{\Delta_0 t}}. \tag{8.248}$$

The phase of the order parameter obtains with the help of Eq. (8.241)

$$\Phi(t) = 2\mu_f t + \sqrt{\pi} K'(\mu)\Delta_0^2 \frac{\cos(2\Delta_0 t + \pi/4)}{2(\Delta_0 t)^{3/2}}. \tag{8.249}$$

Coefficients $K(\mu), K'(\mu)$ are given by Eqs. (8.239), (8.307), and (8.303). Simpler expressions for $G(\varepsilon)$ are available in 2d and in weak coupling (BCS) limit, see Eqs. (8.304) and (8.305). For example, in the BCS limit ($\Delta_0/\varepsilon_F \to 0$)

$$\Delta(t) = \left(\Delta_{0f} - \frac{2\delta\Delta_0}{\pi^{3/2}} \frac{\cos(2\Delta_0 t + \pi/4)}{\sqrt{\Delta_0 t}}\right) e^{-2i\mu_f t}, \tag{8.250}$$

where $\delta\Delta_0 = \Delta_{0f} - \Delta_{0i}$ and we additionally used $\Delta_0 \propto \exp(-\omega/\gamma)$. Note that the second term in Eq. (8.249) is proportional to Δ_0/ε_F. This expression for $\Delta(t)$ holds in the BCS limit for both one and two channel models in two and three dimensions. Eq. (8.250) for $\mu_f = 0$ appeared in Ref. [Yuzbashyan and Dzero(2006)] without derivation.

Let us also mention that long times for which asymptotes of the order parameter derived in this section apply in practice (e.g. in numerical simulations) mean t such that $1/\Delta_0 \ll t \ll 1/\delta$. At times of order of the inverse level spacing $1/\delta$ partial recurrences occur, see Fig. 8.26. Oscillations with frequency $2\Delta_0$ and $1/\sqrt{t}$ decay in the weak coupling limit of the one channel model were identified by Volkov and Kogan[Volkov and Kogan(1973)].

Figure 8.26: Finite size effects, such as partial recurrences in $|\Delta(t)|$, develop at times of order of the inverse level spacing $\delta \propto 1/N$ between discretized single-particle energy levels ε_k. Long time behaviors derived in our paper apply at times $t\delta \ll 1$. In other words, we take the thermodynamic limit first and large time limit second. Two detuning quenches in 3d 2-channel model are shown for $N = 5024$ and: (a) $\gamma = 0.5, \Delta_{0i} = 3.0 \times 10^{-2}\Delta_{\max}, \Delta_{0f} = 2.9 \times 10^{-4}\Delta_{\max}, \delta = 3.4 \times 10^{-3}\Delta_{\max}$, and (b) $\gamma = 0.1, \Delta_{0i} = 0.97\Delta_{\max}, \Delta_{0f} = 0.99\Delta_{\max}, \delta = 8.0 \times 10^{-3}\Delta_{\max}$.

8.5.7 Long time behavior of $\Delta(t)$: BEC side

In the absence of stationary points integrals of the type of Eq. (8.238) are dominated by the end point, $\varepsilon = 0$ here. Normally, they vanish as $1/t$ at large t, but in the present case $K(0) = 0$ in both 2d and 3d, so they vanish faster. Unlike the BCS side, the long time behavior on the BEC side is not universal in that it depends on the form of $K(\varepsilon)$ at small ε, i.e. on the density of states and on the asymptotic spin distribution. As a result, for example, it is different in two and three dimensions.

We first integrate by parts to obtain

$$\widetilde{X}_1(t) = -\frac{1}{2it}\int_0^\infty \left(\frac{K(\varepsilon)}{\widetilde{E}'(\varepsilon)}\right)' e^{2i\widetilde{E}(\varepsilon)t} d\varepsilon. \tag{8.251}$$

In 2d the dimensionless density of states $f(\varepsilon) = 1$ and it follows from Eqs. (8.239), (8.307), and (8.304) that $K(\varepsilon) \propto 1/\ln^2\varepsilon$. We evaluate the large t asymptote of this integral by splitting the integration range into three: $(0, 1/\Lambda t), (1/\Lambda t, \Lambda/t)$, and $(\Lambda/t, \infty)$, where Λ is such that $1 \ll \ln\Lambda \ll \ln t$. In the first integral we expand the integrand in small ε, which leads to an integral $\int_0^{1/\Lambda t} d(\ln\varepsilon)/\ln^3\varepsilon$ and

$$\widetilde{X}_1(t) = \frac{\delta\omega}{\gamma}\frac{ie^{2i\widetilde{E}(0)t}}{\widetilde{E}'(0)E(0)}\frac{1}{t\ln^2 t}. \tag{8.252}$$

The other two integrals vanish as $1/t\ln^3 t$ and are therefore negligible. Eqs. (8.231), (8.240), (8.241) yield the amplitude and the phase of the order parameter

$$|\Delta(t)| = \Delta_{0f}\left(1 - \frac{\delta\omega}{\gamma}\frac{\sin(2E_{\min}t)}{|\mu|t\ln^2 t}\right), \tag{8.253}$$

$$\Phi(t) = 2\mu_f t - \frac{\delta\omega}{\gamma}\frac{\cos(2E_{\min}t)}{E_{\min}t\ln^2 t}, \tag{8.254}$$

where $E_{\min} = \sqrt{\mu^2 + \Delta_0^2}$.

In 3d $f(\varepsilon) = \sqrt{\varepsilon}$ and $K(\varepsilon) \propto \sqrt{\varepsilon}$ at small ε. This follows from Eqs. (8.239), (8.307), and (8.303) and is, for example, readily verified in the strong coupling limit with the help of the last expression in Eq. (8.306). We split the integration range in Eq. (8.251) into two: $(0, 1/\Lambda)$ and $(1/\Lambda, \infty)$, where $t \gg \Lambda \gg 1$. In the first integral we can expand in small t, which results in a Gaussian integral that behaves as $1/\sqrt{t}$ at large t. The second integral vanishes faster as $t \to \infty$. We thus determine the following (exact) large time asymptote:

$$\widetilde{X}_1(t) = -\frac{\pi^{1/2}}{(2t)^{3/2}}\frac{\delta\omega}{\gamma}\frac{e^{2i\widetilde{E}(0)t+i\pi/4}}{[-\widetilde{E}'(0)]^{3/2}E(0)H^2(0)}. \tag{8.255}$$

With the help of Eqs. (8.240) and (8.241) we finally derive

$$|\Delta(t)| = \Delta_{0f}\left(1 - c\frac{\delta\omega}{\gamma}\frac{\cos(2E_{\min}t + \pi/4)}{(2|\mu|t)^{3/2}}\right), \tag{8.256}$$

$$\Phi(t) = 2\mu_f t - c\frac{|\mu|}{E_{\min}}\frac{\delta\omega}{\gamma}\frac{\sin(2E_{\min}t + \pi/4)}{(2|\mu|t)^{3/2}}. \tag{8.257}$$

The coefficient c depends on μ, Δ_0, and γ. It is known exactly from Eq. (8.255), but involves $G(0)$ which in 3d is an elliptic integral according to Eq. (8.303). In the strong coupling BEC limit, $\mu \to -\infty$, $G(\varepsilon)$ is independent of ε and takes a simple form (8.306). In this case

$$c = \sqrt{\frac{\pi|\mu|}{\varepsilon_F}}\left(\frac{4|\mu|}{\gamma\varepsilon_F} + \pi\sqrt{\frac{|\mu|}{\Delta_0}}\right)^{-2}, \tag{8.258}$$

where we restored the original energy units.

8.5.8 Long time behavior of spins

Let us also work out the long time behavior of individual spins given by Eqs. (8.236) and (8.237) and compare it to the asymptotic spin distribution, Eqs. (8.111) and (8.134), obtained earlier. The latter result is based on the few spin conjecture, so the agreement with linear analysis provides yet another (though redundant because we already proved the few spin conjecture for infinitesimal quenches in Sect. 8.5.4) check.

Functions $X_{1,2}$ vanish as $t \to \infty$, while the large time limit of $Y_{1,2}$ derives from the following identity:

$$\lim_{t\to\infty} \fint_0^\infty \frac{d\varepsilon' F(\varepsilon')e^{\pm 2iE(\varepsilon')t}}{\varepsilon' - \varepsilon} = \pm i\pi\alpha F(\varepsilon)e^{\pm 2iE(\varepsilon)t}, \tag{8.259}$$

where α is the sign of $tdE(\varepsilon')/d\varepsilon'$ at $\varepsilon' = \varepsilon$ and $F(\varepsilon')$ is an arbitrary bounded continuous function.

Applying this identity to Eqs. (8.234) and (8.235) and substituting resulting expressions into Eqs. (8.236) and (8.237), we obtain

$$\begin{aligned}\frac{s_\infty^-(\varepsilon,t)e^{2i\mu_f t}}{s_f^-(\varepsilon)} &= 1-\frac{2\delta\omega}{\gamma}\frac{\exp\left[-2iE(\varepsilon)t-i\phi\right]}{\sqrt{\pi^2 f^2(\varepsilon)+H^2(\varepsilon)}}\\ &-\frac{2\delta\omega}{\gamma}\left(\frac{\xi}{E(\varepsilon)}-1\right)\frac{\cos\left[2E(\varepsilon)t+\phi\right]}{\sqrt{\pi^2 f^2(\varepsilon)+H^2(\varepsilon)}},\end{aligned} \tag{8.260}$$

$$\frac{s_\infty^z(\varepsilon,t)}{s_f^z(\varepsilon)} = 1+\frac{2\delta\omega}{\gamma}\frac{\Delta_{0f}^2}{\xi E(\varepsilon)}\frac{\cos\left[2E(\varepsilon)t+\phi\right]}{\sqrt{\pi^2 f^2(\varepsilon)+H^2(\varepsilon)}}, \tag{8.261}$$

where $\xi=\varepsilon-\mu$ and ϕ is defined through

$$\begin{aligned}\cos\phi &= \frac{H(\varepsilon)}{\sqrt{\pi^2 f^2(\varepsilon)+H^2(\varepsilon)}},\\ \sin\phi &= \frac{\pi f(\varepsilon)\,\mathrm{sgn}(t\xi)}{\sqrt{\pi^2 f^2(\varepsilon)+H^2(\varepsilon)}}.\end{aligned} \tag{8.262}$$

In our case $t>0$, but we still kept it under the sign function to ensure proper behavior under time reversal, see Eq. (8.269). Eqs. (8.260) and (8.261) match Eq. (8.111) with

$$\theta(\varepsilon)\approx\sin\theta(\varepsilon)=\frac{2\delta\omega}{\gamma}\frac{\Delta_0}{E(\varepsilon)\sqrt{\pi^2 f^2(\varepsilon)+H^2(\varepsilon)}}. \tag{8.263}$$

(Not that in the present case $\Delta_\infty=\Delta_{0f}$ and $\mu_\infty=\mu_f$.) This indeed agrees with Eq. (8.136) obtained from the few spin conjecture.

8.5.9 Short time behavior

Here we analyze the short time behavior of $|\Delta(t)|$ for quenches within the universal weak coupling regime. For large quenches from weaker to stronger coupling, when $\Delta_{0f}/\Delta_{0i}\gg 1$, or from the normal state (zero initial coupling) in this regime $|\Delta(t)|$ grows as $e^{\Delta_{0f}t}$. This exponential growth reflects the instability of the normal state in the presence of superconducting interactions[Barankov et al.(2004), 19]. At the same time, even for small quenches $|\Delta(t)|$ rises or falls sharply at short times, see Figs. 8.7 and 8.25. Sharp growth is seen in experiment too, though most of it is probably due to a different mechanism[98].

A direct small t expansion of the cosine in Eq. (8.232) diverges at high energies. Cutting off the integral at ε_Λ (Debye energy in the case of metals), one obtains[Warner and Leggett(2005)] $\delta|\Delta(t)|\propto\delta\Delta_0(\varepsilon_\Lambda t)^2$. This is cutoff dependent and applies only to ultra-short times $t\ll 1/\varepsilon_\Lambda$ that vanish as the cutoff is sent to infinity. We are interested in times $1/\varepsilon_\Lambda\ll t\ll 1/\Delta_0$.

Consider Eq. (8.246). The argument of the cosine is small for $x\ll x_0$, where x_0 is determined by $e^{\frac{\pi x_0}{2}}=1/\tau$, i.e. $x_0=\frac{2}{\pi}\ln(1/\tau)$. Let us divide the domain of the integration into three intervals: $[0,x_0-a]$, $[x_0-a,x_0+a]$, and $[x_0+a,\infty)$ and let the corresponding integrals be I_1, I_2, and I_3, respectively. The auxiliary parameter a, $1\ll a\ll x_0$, is such that $1/a\to 0, a/x_0\to 0$ as $x_0\to\infty$. For example, one can take $a=\sqrt{x_0}$. Expanding the cosine in small τ in I_1 and integrating, we obtain

$$I_1=\frac{1}{2}-\frac{1}{\pi x_0}-\frac{a}{\pi x_0^2}+o(a/x_0^2). \tag{8.264}$$

In I_2 we replace $x^2+1\to x_0^2$ up to terms of order a/x_0. After this, a substitution $y=\exp\pi x/2$ transforms it into the cosine integral $\int dy\cos y/y$ with known behavior, leading to

$$I_2=\frac{a}{\pi x_0^2}+o(a/x_0^2). \tag{8.265}$$

And integrating by parts in I_3, we see that it is proportional to $e^{-\pi a/2}/x_0^2$, which is negligibly small. Thus,

$$I_1 + I_2 + I_3 = \frac{1}{2} - \frac{1}{2|\ln(\tau)|} + o\left(\frac{1}{|\ln(\tau)|}\right). \tag{8.266}$$

Note the cancelation of the auxiliary parameter a. Finally, plugging this into Eq. (8.246), we derive the short time behavior of the gap function amplitude

$$|\Delta(t)| = \Delta_{0i} + \frac{\Delta_{0f} - \Delta_{0i}}{|\ln(\Delta_0 t)|}. \tag{8.267}$$

8.6 Approach to the asymptote in the nonlinear case

Here we discuss the approach of $\Delta(t)$ to its large time asymptote in the nonlinear case. We will consider regimes I and II – the gapless phase and the phase where $\Delta(t) \to \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}$. Rather then rigorously deriving the $t \to \infty$ asymptote in its entirety as we did for the linearized dynamics, we present an argument based only on our knowledge of the frequency spectrum that works under certain general assumptions about relevant Fourier amplitudes.

As $t \to \infty$ spins tend to their steady state form, $\vec{s}(\varepsilon, t) \to \vec{s}_\infty(\varepsilon, t)$, where $\vec{s}_\infty(\varepsilon, t)$ is given by Eqs. (8.94) and (8.111) in regimes I and II, respectively. In phase II, in a reference frame rotating with frequency $2\mu_\infty$ around z-axis, $\vec{s}_\infty(\varepsilon)$ rotates with a constant frequency $2E_\infty(\varepsilon) = 2\sqrt{(\varepsilon - \mu_\infty)^2 + \Delta_\infty^2}$. As mentioned above, an integrable model with N degrees of freedom is characterized by N incommensurate frequencies [Arnold(1978)] that are determined by the integrals of motion and are fixed throughout its time evolution. The Fourier decomposition of any dynamical quantity can have only these basic frequencies in its spectrum. In particular,

$$|\Delta(t)| = \Delta_\infty + \int_0^\infty F(\varepsilon) \cos[2E_\infty(\varepsilon)t] f(\varepsilon) d\varepsilon \tag{8.268}$$

with some unknown function $F(\varepsilon)$.

Terms containing $\sin[2E_\infty(\varepsilon)t]$ are absent by time-reversal symmetry [cf. Eq. (8.231)] of the equations of motion (8.8) and (8.33) [see also Eq. (8.73)]

$$s^z(-t) = s^z(t),\ s^+(-t) = s^-(t),\ \bar{\Delta}(-t) = \Delta(t), \tag{8.269}$$

where we suppressed ε-dependence of spins for compactness. These relations hold at all times as long as the initial condition at $t = 0$ satisfies them, which our initial state (8.28) does.

A common practice in previous work is to attempt to determine the approach of $|\Delta(t)|$ to its asymptotic value Δ_∞ from the steady state spins $\vec{s}_\infty(\varepsilon, t)$. Consider the one channel case for simplicity. Continuum version of Eq. (8.13) at $t = \infty$ is

$$\Delta_\infty(t) = \lambda \int_0^\infty s_\infty^-(\varepsilon, t) f(\varepsilon) d\varepsilon. \tag{8.270}$$

The constant part of $s_\infty^-(\varepsilon, t)$ yields Δ_∞, while the contribution of the oscillating part integrated over ε vanishes (dephases) as $t \to \infty$. One can further determine the large time asymptote of Eq. (8.270) similarly to how we evaluated the large time behavior of Eq. (8.231). This is however *not* the correct asymptote of the actual $\Delta(t)$. Not only it does not yield the correct coefficient of the time-dependent part of $\Delta(t)$ [such as the coefficient c in Eq. (8.256)], but also the actual time dependence can be different.

At finite t there is a correction to the steady state value of the spin, $\vec{s}(\varepsilon,t) = \vec{s}_\infty(\varepsilon,t) + \delta\vec{s}(\varepsilon,t)$, so that the actual order parameter is

$$\Delta(t) = \lambda \int_0^\infty s_\infty^-(\varepsilon,t) f(\varepsilon) d\varepsilon + \lambda \int_0^\infty \delta s^-(\varepsilon,t) f(\varepsilon) d\varepsilon. \tag{8.271}$$

Even though $\delta s^-(\varepsilon,t)$ is small as compared to the oscillating part of $s_\infty^-(\varepsilon,t)$ at large times, this is no longer true after integrating these quantities over ε. Consider, for example, Eq. (8.236). We showed in Sect. 8.5.8 that $s_\infty^-(\varepsilon,t)$ comes from functions $Y_{1,2}(\varepsilon,t)$. But we see from Eq. (8.231) that the integral of these functions over ε vanishes and as a result they do not contribute to $\Delta(t)$ The correction $\delta s^-(\varepsilon,t)$ on the other hand comes from both $X_{1,2}(t)$ and $Y_{1,2}(\varepsilon,t)$. It is this contribution from $X_{1,2}(t)$ to $\delta s^-(\varepsilon,t)$ that actually determines $\Delta(t)$. Thus there is a partial cancelation between the two integrals in Eq. (8.271) and the true large time behavior of $\Delta(t)$ can only be determined by keeping both.

Nevertheless, $\Delta_\infty(t)$ being a legitimate dynamical quantity has the right frequency spectrum and also contains the dimensionless density of states $f(\varepsilon)$. So, it still produces a correct large time dependence when, for example, the latter is set by a stationary point as in Eq. (8.248) or by the behavior of $f(\varepsilon)$ at small ε as in Eq. (8.256). The situation on the BEC side in 2d is different. The $\ln^2 t$ dependence in the denominator of Eq. (8.253) comes from $K(\varepsilon) \propto 1/\ln^2\varepsilon$ behavior of the Fourier amplitude at small ε, see Eq. (8.238) and the text below Eq. (8.251). This is in turn a consequence of $K(\varepsilon) \propto H^{-2}(\varepsilon)$ and $H(\varepsilon) \propto \ln\varepsilon$, which follow from Eqs. (8.239), (8.307), and (8.304). Were we to evaluate the large time asymptote of $|\Delta(t)|$ using Eq. (8.270), we would obtain $1/(t\ln t)$ instead of $1/(t\ln^2 t)$. To see this, note that Eq. (8.260) implies that the oscillating part of $s_\infty^-(\varepsilon,t)$ is proportional to $H^{-1}(\varepsilon)$, i.e. to $1/\ln\varepsilon$, at small ε and apply the same steps as in the text below Eq. (8.251). The $1/\ln\varepsilon$ dependence cancels in Eq. (8.271) due to the second term on the right hand side. We note also that Eqs. (8.111) and (8.134) imply $s_\infty^-(\varepsilon,t) \propto 1/\ln\varepsilon$ in all of region II in 2d, not just in the linear approximation.

Similar considerations apply in analyzing Eq. (8.268). Let us work out the large time behavior of $|\Delta(t)|$ in steady states I, II, and II' separately.

8.6.1 Regime II

In steady states II and II' $\Delta(t) \to \Delta_\infty e^{-2i\mu_\infty t - 2i\varphi}$. For quenches in region II $\mu_\infty > 0$, so it can be viewed as a nonequilibrium extension of the BCS regime. The frequency spectrum $2E_\infty(\varepsilon)$ has a stationary point at $\varepsilon = \mu_\infty$, $E'_\infty(\mu_\infty) = 0$, which in regime II lies within the integration range. The large time behavior of Eq. (8.268) obtains with the help of stationary point method [cf. Eq. (8.248)]

$$|\Delta(t)| = \Delta_\infty + \sqrt{\pi} F(\mu_\infty)\Delta_\infty^2 \frac{\cos(2\Delta_\infty t + \pi/4)}{\sqrt{\Delta_\infty t}}. \tag{8.272}$$

The only assumption about $F(\varepsilon)$ here is that it is smooth. This is an extension of Eq. (8.248) to nonlinear regime. In the weak coupling BCS limit this result was published in Ref. [19]. In this limit Δ_∞ is given by Eq. (8.152) and generally it obtains from Eqs. (8.138) and (8.162) in 2d and 3d, respectively, and Eq. (8.137) as the imaginary part of u. Here we see that expression (8.272) holds throughout the entire region II for both one and two channel models.

8.6.2 Regime II'

Regime II' has the same asymptotic $\Delta(t)$ as II by definition only with $\mu_\infty < 0$. There are now no stationary points on the integration path. The approach to the asymptote is therefore

determined by the behavior of $F(\varepsilon)f(\varepsilon)$ near the end points, $\varepsilon = 0$ in this case. We *assume* this behavior is the same as in linear analysis, since we expect the time dependence to have the same functional form throughout a given regime. According to Sect. 8.5.7, this means finite nonzero $F(0)$ in 3d and $F(\varepsilon) \propto 1/\ln^2 \varepsilon$ for $\varepsilon \ll 1$ in 2d.

Expanding Eq. (8.134) in small ε and using Eq. (8.111), we see that the spin components at $t \to \infty$ do behave the same as in linear analysis, though this in itself does not prove our assumption. Moreover, the asymptotic spin distribution (8.134) is continuous across critical lines separating various regimes, so the same small ε form holds in gapless region I as well.

As long as our assumptions about $F(\varepsilon)$ are correct, the analysis of the integral in Eq. (8.270) is the same as that in Sect. 8.5.7 resulting in

$$|\Delta(t)| = \Delta_\infty \left(1 - c_1 \frac{\sin(2E_\infty^{\min} t)}{t \ln^2 t}\right) \quad \text{in 2d,} \tag{8.273}$$

and

$$|\Delta(t)| = \Delta_\infty \left(1 - c_2 \frac{\cos(2E_\infty^{\min} t + \pi/4)}{t^{3/2}}\right) \quad \text{in 3d,} \tag{8.274}$$

at large times, where $E_\infty^{\min} = \sqrt{\mu_\infty^2 + \Delta_\infty^2}$ and c_1 and c_2 are real coefficients that depend on Δ_{0i}, Δ_{0f}, and γ.

8.6.3 Gapless regime

Finally, we turn to regime I. Now $\Delta(t) \to 0$ at $t \to \infty$. Spins $\vec{s}_\infty(\varepsilon)$ rotate with frequencies 2ε around z-axis, so that the Fourier transform of the order parameter magnitude is of the form

$$|\Delta(t)| = \int_0^\infty F(\varepsilon) \cos(2\varepsilon t) f(\varepsilon) d\varepsilon, \tag{8.275}$$

and $\sin(2\varepsilon t)$ term vanishes by time-reversal symmetry (8.269).

In 3d we similarly assume finite and nonzero $F(0)$. Steps outlined below Eq. (8.254) in Sect. 8.5.7 now lead to the following large time behavior:

$$|\Delta(t)| = \frac{c_3}{t^{3/2}}. \tag{8.276}$$

In 2d we speculate that $F(\varepsilon) \propto 1/\ln^r \varepsilon$ at small ε, where r is either 1 or 2. As discussed before in this Section, $s_\infty^-(\varepsilon, t) \propto 1/\ln \varepsilon$ in 2d, so that $\Delta_\infty(t) \propto 1/(t \ln t)$. The $1/\ln \varepsilon$ term however cancels from $F(\varepsilon)$ at least in linear analysis and it ends up being proportional to $1/\ln^2 \varepsilon$ instead. In the gapless case we allow for a possibility that such a cancelation does not occur. The analysis of the integral in Eq. (8.275), analogous to that leading to Eq. (8.256), then yields

$$|\Delta(t)| = \frac{c_4}{t \ln^r t}. \tag{8.277}$$

The gapless regime contains the $\Delta_{0i} = \Delta_{0f} = 0$ point, the origin of quench phase diagrams. It therefore includes the weak coupling limit $\Delta_{0i}/\varepsilon_F \to 0$ and $\Delta_{0f}/\varepsilon_F \to 0$. Eq. (8.275) becomes in this limit [see Sect. 8.5.5]

$$|\Delta(t)| = \int_{-\infty}^\infty F(\xi) \cos(2\xi t) d\xi, \tag{8.278}$$

where $F(\xi)$ is even in ξ. Now there can be no power law in t contribution at large t coming from integration limits. Instead, $|\Delta(t)|$ vanishes exponentially[Barankov and Levitov(2006),

Yuzbashyan and Dzero(2006)] as $A(t)e^{-2\alpha\Delta_{0i}t}$ independent of dimensionality, where $\alpha \sim 1$ and $A(t)$ is a decreasing power law, $A(t) \sim \Delta_{0i}$ at $t \sim 1/\Delta_{0i}$. Recall that throughout this paper we have been using units where $\varepsilon_F = 1$. To convert to arbitrary units in Eqs. (8.276) and (8.277), one needs to replace $t \to \varepsilon_F t$. Guided by linear analysis we further assume that coefficients c_4 and c_5 are of order Δ_{0f}, which we take to be comparable to Δ_{0i}. It is clear that at any finite $\Delta_{0i}/\varepsilon_F \ll 1$ power laws in Eqs. (8.276) and (8.277) coming from the lower integration limit will eventually win over the exponential decay. The comparison of $e^{-2\alpha\Delta_{0i}t}$ with $(\varepsilon_F t)^{-1}$ shows that the weak coupling result is valid at times such that $\ln(\varepsilon_F/\Delta_{0i}) \gg \Delta_{0i}t \gg 1$, while for $\Delta_{0i}t \gg \ln(\varepsilon_F/\Delta_{0i})$ it has to be replaced with Eqs. (8.276) and (8.277).

8.7 Experimental signatures

Far from equilibrium states of fermionic superfluids described in this paper can be observed in different systems with various experimental techniques.

Matsunaga et. al.[Matsunaga et al.(2013), 46] directly measured the time-dependent amplitude $|\Delta(t)|$ induced by an ultrafast electromagnetic perturbation in $Nb_{1\text{-}x}Ti_x\mathbf{N}$ films using terahertz pump – terahertz probe spectroscopy. The underlying system is a BCS superconductor [weak coupling regime of one channel model (8.3)] and for perturbation strength below certain threshold its non-adiabatic dynamics falls within region II of our quench phase diagrams. Even though we considered BCS interaction quenches in one channel model in this paper, it is clear from our arguments that our results apply more generally to any kind of non-adiabatic global perturbation. Therefore, we expect $|\Delta(t)|$ to be described by Eq. (8.272) derived originally in nonlinear regime by Yuzbashyan et.al. [19]. These experiments indeed measure damped oscillations with frequency $2\Delta_\infty$, where Δ_∞ is the asymptotic value of $|\Delta(t)|$ even when the system is deep in the nonlinear regime and Δ_∞ is much different from the ground state gap. The power law approach however appears to be faster than $1/t^{1/2}$.

In this paper we primarily focused on detuning or interaction quenches in cold fermions. Experiments addressing superfluidity in these systems include measurements of the molecular condensate fraction[Regal et al.(2004), Zwierlein et al.(2004)], radio frequency absorption spectra[22], and observation of vortices[Zwierlein et al.(2005)]. Signatures of "far from equilibrium phases" I, II, and III – gapless, gapped (Volkov-Kogan), and oscillatory – in these experiments can be derived from the many-body wavefunction $\Psi(t)$ determined above.

The pseudospin (fermionic) part of $\Psi(t)$ is a direct product of spin-1/2 wavefunctions $\prod_{\mathbf{p}}(\bar{u}_{\mathbf{p}}|\downarrow\rangle + \bar{v}_{\mathbf{p}}|\uparrow\rangle)$ found in Sect. 8.2.4. In the gapless steady state

$$\begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} = \cos\frac{\theta_{\mathbf{p}}}{2}\begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-i\varepsilon_{\mathbf{p}}t} + \sin\frac{\theta_{\mathbf{p}}}{2}\begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{i\varepsilon_{\mathbf{p}}t - i\delta_{\mathbf{p}}}, \tag{8.279}$$

where $\cos\theta_{\mathbf{p}} \equiv \cos\theta(\varepsilon_{\mathbf{p}})$ is given by Eq. (8.134) in all three phases. The second term represents an occupied pair of states $\pm\mathbf{p}$ (pseudospin up), the first – empty (pseudospin down). $\Psi(t)$ in the gapless phase is a coherent superposition of eigenstates of a free Fermi gas with different energies reflecting the fact that $\Delta(t) \to 0$ implies vanishing of interactions between fermions on the mean-field level. Effectively the system is governed by a non-interacting Hamiltonian at $t \to \infty$. It nevertheless retains superconducting correlations. For example, in the weak coupling regime its superfluid density is half that in the ground state and in phase II[Yuzbashyan and Dzero(2006)]. Phase I is therefore a nonequilibrium gapless superfluid.

In the gapped steady state Eqs. (8.100) and (8.112) imply

$$\begin{pmatrix} u_{\mathbf{p}} e^{i\mu_\infty t} \\ v_{\mathbf{p}} e^{-i\mu_\infty t} \end{pmatrix} = \cos\frac{\theta_{\mathbf{p}}}{2} \overbrace{\begin{pmatrix} |U_{\mathbf{p}}| \\ |V_{\mathbf{p}}| \end{pmatrix} e^{-iE^\infty_{\mathbf{p}} t}}^{\text{ground state pair}} + \sin\frac{\theta_{\mathbf{p}}}{2} \overbrace{\begin{pmatrix} |V_{\mathbf{p}}| \\ -|U_{\mathbf{p}}| \end{pmatrix} e^{iE^\infty_{\mathbf{p}} t}}^{\text{excited pair}}, \tag{8.280}$$

where

$$|U_{\mathbf{p}}| = \sqrt{\frac{1}{2} + \frac{\xi_{\mathbf{p}}}{2E^\infty_{\mathbf{p}}}}, \quad |V_{\mathbf{p}}| = \sqrt{\frac{1}{2} - \frac{\xi_{\mathbf{p}}}{2E^\infty_{\mathbf{p}}}}, \tag{8.281}$$

$\xi_{\mathbf{p}} = \varepsilon_{\mathbf{p}} - \mu_\infty$, and we dropped the nonessential constant phase φ. Bogoliubov amplitudes $|U_{\mathbf{p}}|$ and $|V_{\mathbf{p}}|$ are the same as in the BCS ground state[Schrieffer(1989)] with gap Δ_∞ and chemical potential μ_∞. The two wavefunctions on the right hand side of Eq. (8.280) are the two orthonormal eigenstates of the BdG Hamiltonian

$$H_{\text{BdG}} = \begin{pmatrix} \xi_{\mathbf{p}} & \Delta_\infty \\ \Delta_\infty & -\xi_{\mathbf{p}} \end{pmatrix}. \tag{8.282}$$

The first one is a Cooper pair wavefunction in the BCS ground state and corresponds to an alignment of the pseudospin $\vec{s}_{\mathbf{p}}$ antiparallel to the effective magnetic field. The second one is an excited state of the Cooper pair ($\vec{s}_{\mathbf{p}}$ parallel to the effective magnetic field) termed an *excited pair* in the original BCS work[20]. It is interesting to note that these excitations of the condensate in superconducting metals carry no charge and spin, so non-adiabatic dynamics considered here provides a unique venue for creating and measuring them[Dzero et al.(2007)]. The steady state in phase II therefore is a coherent mixture of ground state and excited pairs – a superposition of eigenstates of the BCS Hamiltonian with gap Δ_∞ and chemical potential μ_∞.

A similar interpretation of the oscillatory state obtains by Fourier transforming the amplitudes (8.122)

$$\begin{pmatrix} u_{\mathbf{p}} e^{i\tilde{\mu} t} \\ v_{\mathbf{p}} e^{-i\tilde{\mu} t} \end{pmatrix} = \sum_{n=-\infty}^{\infty} \left\{ \cos\frac{\theta_{\mathbf{p}}}{2} \begin{pmatrix} a_{\mathbf{p}n} \\ b_{\mathbf{p}n} \end{pmatrix} e^{-i(e_{\mathbf{p}} - n\omega_\Delta)t} + \sin\frac{\theta_{\mathbf{p}}}{2} \begin{pmatrix} \bar{b}_{\mathbf{p}n} \\ -\bar{a}_{\mathbf{p}n} \end{pmatrix} e^{i(e_{\mathbf{p}} - n\omega_\Delta)t} \right\}, \tag{8.283}$$

where ω_Δ is the oscillation frequency of $|\Delta(t)|$, $\tilde{\mu}$ and $-2e_{\mathbf{p}}$ are the zeroth harmonics of the phase of $\Delta(t)$ and the common phase of the amplitudes, see Eqs. (8.102) and (8.106), and we again dropped the constant phase φ. This expression derives by first going to a frame rotating with frequency $2\tilde{\mu}$ to get rid of the linear term in the phase of $\Delta(t)$. This makes $e^{-i\phi_{\mathbf{p}}}$, the term involving the relative phase, periodic according to Eq. (8.103) and it does not contribute to the momentum dependent phases on the right hand side. Phase III therefore can be understood as a superposition of generalized excited and ground state pairs with dispersions $\pm e_{\mathbf{p}}$ and quanta of the amplitude (Higgs) mode $|\Delta(t)|$. As noted in Sect. 8.2.4, $e_{\mathbf{p}} \to \varepsilon_{\mathbf{p}}$ at large $\varepsilon_{\mathbf{p}}$.

The knowledge of the steady state allows us to compute far from equilibrium correlation and Green's functions in all three phases. For example[not(b)],

$$\begin{aligned} i\mathcal{G}_{\mathbf{p},>}(t,t') &= \langle \hat{a}_{\mathbf{p}\uparrow}(t)\, \hat{a}^{\dagger}_{\mathbf{p}\uparrow}(t')\rangle = \bar{u}_{\mathbf{p}}(t)u_{\mathbf{p}}(t'), \\ -i\mathcal{G}_{\mathbf{p},<}(t,t') &= \langle \hat{a}^{\dagger}_{\mathbf{p}\uparrow}(t')\, \hat{a}_{\mathbf{p}\uparrow}(t)\rangle = \bar{v}_{\mathbf{p}}(t')v_{\mathbf{p}}(t), \\ \mathcal{G}^{+}_{\mathbf{p}}(t,t') &= \langle \hat{a}^{\dagger}_{-\mathbf{p}\downarrow}(t)\, \hat{a}^{\dagger}_{\mathbf{p}\uparrow}(t')\rangle = v_{\mathbf{p}}(t)\bar{u}_{\mathbf{p}}(t'). \end{aligned} \tag{8.284}$$

With these we can evaluate various observables such as the superfluid density mentioned earlier in this section. Note also that the steady state momentum distribution $n_p^{\infty}(t)dp$ is simply related to the z-component of the pseudospin according to Eq. (8.6). Taking into account that $\mathbf{p}$ and $-\mathbf{p}$ are both included in $s^z_{\mathbf{p}}$ and integrating over the angles, we have

$$n_p^{\infty}(t) = 2p^2(2s^z_{\mathbf{p}} + 1). \tag{8.285}$$

Expressions for $s^z_{\mathbf{p}}$ in phases I, II, and II appear in Eq. (8.97), Eq. (8.111), and Eqs. (8.104) and (8.121), respectively.

Finally, let us discuss the signatures of nonequilibrium phases in radio frequency (RF) spectroscopy[Törmä and Zoller(2000), Regal and Jin(2003), et al.(2003), Kinnunen et al.(2004), Schunk et al.(2008), Basu and Mueller(2008), Schirotzek et al.(2008)]. Recall that in an atomic Fermi gas the pairing occurs between atoms in two different hyperfine states, $|\uparrow\rangle \equiv |1\rangle$ and $|\downarrow\rangle \equiv |2\rangle$. The RF photon transfers atoms from one of these states, say $|2\rangle$, to the third hyperfine state $|3\rangle$ that does not interact with $|1\rangle$ and $|2\rangle$. In an unpaired Fermi gas where atoms $|2\rangle$ are free, the RF absorption spectrum has a peak at the atomic transition energy $\omega = \mathcal{E}_{23}$. In the paired ground state, the peak shifts to $\omega > \mathcal{E}_{23}$ by an amount equal to the minimum binding energy of Cooper pairs[Törmä and Zoller(2000)].

The RF response of steady states I, II, and III was calculated in Ref. [Dzero et al.(2007)] for quenches within the BCS regime and in Ref. [Foster et al.(2013)] for quenched p-wave superfluids. The calculation in the present case is identical[not(c)], so we will not reproduce it here. The RF spectrum of phase I is similar to that of the normal state – a peak at $\omega = \mathcal{E}_{23}$. In phase II there are two peaks – at $\omega > \mathcal{E}_{23}$ and $\omega < \mathcal{E}_{23}$, which come from the ground state and excited pairs, respectively, see Eq. (8.280). The first peak corresponds to a process in which an RF photon breaks a ground state pair, the second – excited pair. The RF response of phase III similarly reflects the structure of the corresponding steady state wavefunction (8.283). There are two series of peaks spaced by ω_Δ, the frequency of oscillations of $|\Delta(t)|$, coming from processes where an RF photon breaks a ground state (excited) pair and absorbs or emits several quanta of the amplitude (Higgs) mode $|\Delta(t)|$.

8.8 Conclusion

In this paper we studied the coherent dynamics of an isolated BCS-BEC condensate in two and one channel (BCS) models in two and three spatial dimensions. Our main focus was on detuning quenches $\omega_i \to \omega_f$ (interaction quenches $\lambda_i \to \lambda_f$ in the one channel model). We constructed exact quench phase diagrams and predicted the order parameter dynamics $\Delta(t)$ and the full time-dependent wavefunction $\Psi(t)$ of the system at large times for any pair of values (ω_i, ω_f). In contrast to most previous work, we considered quenches beyond the weak coupling limit of BCS to BCS quenches. We add to this BCS to BEC and BEC to BCS quenches across the Feshbach resonance as well as quenches on the BEC side. We showed that the weak coupling limit is universal in that it is model and dimension independent. Outside of this limit, there are several qualitatively new features, the two channel model having richer quench phase diagram as it contains an extra parameter – dimensionless resonance width γ. All results for the one channel model obtain from the two channel ones by taking the broad resonance, $\gamma \to \infty$, limit.

We find the same three main nonequilibrium phases (asymptotic states) as in the weak coupling regime. Interestingly, this seems to be a universal, model independent feature of quench dynamics of fermionic condensates, at least when there is a global complex order parameter, so that the Cooper pairs interact only through this collective mode. The same three phases occur, for example, in p-wave superconductors[Foster et al.(2013), Foster et al.(2014)], spin-orbit coupled superfluids[Dong et al.(2014)], and s-wave superconductors with energy-dependent interaction[Barankov and Levitov(2007)]. One can speculate that similar universality according to the order parameter type exists among quench phase diagrams of multicomponent superfluids, such as three fermion species with pairing interactions or multi-band superconductors.

The above three main phases are: phase I where $\Delta(t)$ vanishes, phase II where $\Delta(t) \to \Delta_\infty e^{-2i\mu_\infty t}$ up to a constant phase factor, and phase III where $|\Delta(t)|$ oscillates persistently. It turns out μ_∞ plays the role of a nonequilibrium analog of the chemical potential. For quenches within the weak coupling regime $\mu_\infty \approx \varepsilon_F$, while for quenches to deep BEC $\mu_\infty \to -\infty$. Some of the new effects as one moves beyond the weak coupling regime are as follows. The oscillatory approach of $|\Delta(t)|$ to a constant (Volkov-Kogan behavior) changes from $1/\sqrt{t}$ for $\mu_\infty > 0$ to $1/t^{3/2}$ in three dimensions and $1/(t \ln^2 t)$ in two dimensions for $\mu_\infty < 0$, and the oscillation frequency changes from $2\Delta_\infty$ to $2\sqrt{\mu_\infty^2 + \Delta_\infty^2}$. For resonance width below a certain threshold, the asymptotic gap amplitude Δ_∞ can be much larger than Δ_{0f} – the ground state gap at final detuning ω_f. Similarly, exponential vanishing of $|\Delta(t)|$ in phase I gives way to a power law behavior. Persistent oscillations in phase III are first suppressed for stronger quenches and then disappear altogether. For example, in three dimensional one channel model there is a critical coupling λ_c, such that even quenches from an infinitesimally small λ_i to $\lambda_f > \lambda_c$ produce no such oscillations. As λ_f approaches λ_c from below, the oscillation amplitude first increases, then decreases and finally vanishes at $\lambda_f = \lambda_c$.

The post-quench asymptotic state of the condensate is a coherent superposition of ground state and excited pairs at each momentum (multiple bands of such pairs shifted by the oscillation frequency of $|\Delta(t)|$ in phase III). These are two orthogonal eigenstates of a Cooper pair in the self-consistent field, and, for instance, the BCS ground state is a direct product of ground state pair wavefunctions. Our steady state in phases I and II is a direct product of such time-dependent superpositions. In the Anderson pseudospin language, ground state (excited) pairs correspond to the alignment of pseudospin antiparallel (parallel) to the magnetic field. Even though we refer to these states as ground state or excited pairs, we should stress that they are not the same as similar states of Cooper pairs in the ground or excited states of the BCS Hamiltonian since the self-consistent field is different. Excited pairs are elusive excitations in superconductors – it is difficult to couple to them as they carry no charge or spin. Non-adiabatic dynamics of the BCS-BEC condensate provides an opportunity to access them, e.g. in the RF absorption spectrum.

Our treatment of the dynamics of the BCS-BEC condensate neglects the coupling to the non-condensed modes (mean field approximation) – molecules with nonzero momenta $\mathbf{q}$ in the two channel model. We check the validity of this approximation for the two channel model by estimating the rates of the decoherence processes due to these terms for post-quench steady states in phase II and comparing them to the typical timescale on which the quench dynamics occurs. Our preliminary results indicate that the mean field approach is justified for quenches sufficiently far from the $\mu_\infty = 0$ line in the quench phase diagrams, e.g. quenches within deep BEC, deep BCS, or across the resonance from deep BCS to deep BEC and vice versa. A more thorough study of these effects is necessary to fully clarify the situation.

In mean field various pairing Hamiltonians, e.g. one and two channel models considered here, chiral p-wave BCS, a certain class of d-wave BCS models[Richardson(2002)], are equivalent to integrable classical spin (or spin-oscillator) chains with long-range interactions. The most remarkable general feature of their dynamics is a reduction in the number of effective degrees of

freedom as $t \to \infty$. Consider e.g. the one channel model. As explained above, its dynamics in the thermodynamic limit at long times after the quench can be described in terms of just a few – zero (phase I), one (phase II), or two (phase III) – new collective classical spin variables. In other words, the number of spin at long times reduces from infinity to zero, one, or two. Moreover, the new spins time-evolve with the same Hamiltonian only with "renormalized" parameters. For example, in phase I the effective Hamiltonian at large times is simply $H = 0$, and in phase II it is $H = 2\mu_\infty S_z - gS_-S_+$, where $\vec{S}$ is the collective spin of length $|\vec{S}| = \Delta_\infty/g$, and g is the original BCS coupling constant. The order parameter $\Delta(t)$ coincides with that of the few spin problem, while the original spins relate to the collective ones in a more involved fashion.

It is this feature of the dynamics together with the integrability of the underlying model that allowed us to explicitly determine the exact post-quench asymptotic state of the system. In this paper we presented for the first time a comprehensive, consistent overview of a general method to explicitly evaluate the large time asymptotic solution in classical integrable systems that support this kind of reduction. We are not aware of any similar method for other integrable nonlinear models, the rather different soliton resolution conjecture[Duyckaerts et al.(2013)] being the closest analog we were able to identify.

An interesting open question is whether a similar reduction in the number of degrees of freedom in the course of time evolution occurs also in non-integrable pairing models. This can explain the aforementioned universality of the quench phase diagrams among systems characterized by a global complex order parameter. It seems non-accidental indeed that the non-integrable spin-orbit coupled superfluid[Dong et al.(2014)] has the same three main post-quench phases and that, moreover, $\Delta(t)$ in phase III is given by an elliptic function dn. Presumably a generalization of this method to non-integrable models would rely on more general considerations without recourse to integrability-specific techniques and thus would clarify the underlying physical mechanism. It would also make a number of interesting problems, such as e.g. the competition between chiral and anti-chiral components in p-wave superconductors upon switching on superconducting interactions and more generally the dynamical interplay among various components in a multicomponent superfluid, potentially amenable to in depth analysis.

8.9 Pair-breaking rates

In this appendix, we perform a preliminary analysis of the validity of neglecting $\mathbf{q} \neq 0$ terms far from equilibrium in the Hamiltonian (8.1). So far we have studied the quench dynamics of the condensate decoupled from these non-condensed modes. There are two kinds of relevant processes due to the $\mathbf{q} \neq 0$ terms: (i) excitation of molecules out of the condensate and (ii) excitation of fermionic quasiparticles through two-particle collisions. We estimate characteristic timescales of both processes in the post-quench steady state. We find that sufficiently far from the $\mu_\infty = 0$ line in our quench phase diagrams (see Figs. 8.3 and 8.21) these timescales are much larger than the characteristic time of the quench dynamics. This means that dropping $\mathbf{q} \neq 0$ terms is indeed justified at times it takes for the quench dynamics to develop and reach the steady state. At much later times, after the quench dynamics plays out, these terms set in, presumably leading to decoherence and eventual thermalization of our (isolated) system. We note also that the $\mu_\infty = 0$ line can be very roughly interpreted as a far from equilibrium generalization of the unitarity point. Quenches away from this line are from BCS or BEC initial detuning to far BCS and BEC side including quenches across the resonance.

In what follows we consider a three-dimensional condensate and, for simplicity, we content ourselves with steady states in phase II (including II') where pairing amplitude asymptotes to a constant, $|\Delta(t \to \infty)| = \Delta_\infty$.

8.9.1 Steady state molecular production

Here we compute the rate at which molecules with non-zero momentum are produced in steady state II where initially all molecules have zero momentum. To the lowest order in the interaction, the corresponding scattering amplitudes are[Phillips(2003)]

$$\mathcal{A}_b(\mathbf{p}_1,\mathbf{p}_2)\delta(E_{\text{fin}}-E_{\text{in}}) = \int_{-\infty}^{\infty} \langle\Psi_{\text{fin}}|\hat{V}(t)|\Psi_{\text{in}}\rangle dt, \tag{8.286}$$

where $|\Psi_{\text{in}}\rangle$ and E_{in} are the steady state wavefunction and energy. $|\Psi_{\text{fin}}\rangle$ obtains from $|\Psi_{\text{in}}\rangle$ by destroying two pairs and creating a molecule with momentum $\mathbf{q}=\mathbf{p}_1+\mathbf{p}_2$ and two unpaired atoms with momenta $\mathbf{p}_1$ and $\mathbf{p}_2$. The energy of the final state is

$$E_{\text{fin}} = E_{\text{in}} + \zeta_{\mathbf{q}} \pm E^{\infty}_{\mathbf{p}_1} \pm E^{\infty}_{\mathbf{p}_2}, \tag{8.287}$$

where plus (minus) corresponds to a ground (excited) pair and

$$\zeta_{\mathbf{q}} = \frac{q^2}{4m} + \omega_f - 2\mu_\infty, \tag{8.288}$$

is the energy of the molecule. The interaction $\hat{V}(t)$ is described by the last term in Eq. (8.1)

$$\begin{aligned}\hat{V}(t) = g\sum_{\mathbf{p}_1,\mathbf{p}_2}\Big[&\hat{b}^\dagger_{\mathbf{p}_1+\mathbf{p}_2}(t)\hat{a}_{\mathbf{p}_1\uparrow}(t)\hat{a}_{\mathbf{p}_2\downarrow}(t)+\\ &\hat{b}_{\mathbf{p}_1+\mathbf{p}_2}(t)\hat{a}^\dagger_{\mathbf{p}_2\downarrow}(t)\hat{a}^\dagger_{\mathbf{p}_1\uparrow}(t)\Big].\end{aligned} \tag{8.289}$$

Since our initial state does not contain molecules with non-zero momentum, only the first term in Eq. (8.289) contributes to the matrix element (8.286). One also needs to keep in mind that our steady state contains superpositions of a ground state pair with energy $-E^{\infty}_{\mathbf{p}}$ and an excited pair with energy $+E^{\infty}_{\mathbf{p}}$ for each $\mathbf{p}$. Eqs. (8.280) and (8.286) then yield the following four scattering amplitudes[not(b)]:

$$\begin{aligned}\mathcal{A}_b^{(--)}(\mathbf{p}_1,\mathbf{p}_2) &= g\cos\frac{\theta_{\mathbf{p}_2}}{2}\cos\frac{\theta_{\mathbf{p}_1}}{2}|V_{\mathbf{p}_2}||V_{\mathbf{p}_1}|,\\ \mathcal{A}_b^{(+-)}(\mathbf{p}_1,\mathbf{p}_2) &= g\sin\frac{\theta_{\mathbf{p}_2}}{2}\cos\frac{\theta_{\mathbf{p}_1}}{2}|U_{\mathbf{p}_2}||V_{\mathbf{p}_1}|,\\ \mathcal{A}_b^{(++)}(\mathbf{p}_1,\mathbf{p}_2) &= g\sin\frac{\theta_{\mathbf{p}_2}}{2}\sin\frac{\theta_{\mathbf{p}_1}}{2}|U_{\mathbf{p}_2}||U_{\mathbf{p}_1}|,\end{aligned} \tag{8.290}$$

where - (+) describes breaking a ground state (excited) pair, and $\mathcal{A}_b^{(-+)}(\mathbf{p}_1,\mathbf{p}_2)=\mathcal{A}_b^{(+-)}(\mathbf{p}_2,\mathbf{p}_1)$.

Molecular production rate per atom at zero temperature obtains from these amplitudes and Fermi's Golden Rule[Phillips(2003)]

$$\begin{aligned}\tau^{-1}_{\text{mol}} = \frac{2\pi}{N_f}\sum_{\mathbf{p}_1\mathbf{p}_2\alpha\beta}&|\mathcal{A}_b^{(\alpha\beta)}(\mathbf{p}_1,\mathbf{p}_2)|^2\times\\ &\delta\left(\zeta_{\mathbf{p}_1+\mathbf{p}_2}-\alpha E^{\infty}_{\mathbf{p}_2}-\beta E^{\infty}_{\mathbf{p}_1}\right).\end{aligned} \tag{8.291}$$

In this expression N_f is the total number of fermions in the absence of molecules and we took into account that there are no molecules with non-zero momentum in our steady state.

Let us specialize to quenches into either deep BCS ($\omega_f\to+\infty$) or deep BEC ($\omega_f\to-\infty$). We expect a much higher rate in the latter case, because in the BCS regime $\zeta_{\mathbf{q}}\to+\infty$ requiring

excited pairs of extremely high energy to create a molecule. For quenches to far BEC side $\mu_\infty \to -\infty$, while Δ_∞ remains finite regardless of the initial detuning, see e.g. Figs. 8.18 and 8.19. It follows that $E^\infty_{\mathbf{p}} \approx \xi_{\mathbf{p}} = |\mu_\infty| + p^2/2m$ and Eq. (8.200) implies $\omega_f \approx 2\mu_\infty$. For $\alpha = \beta = -1$ the argument of the delta-function in Eq. (8.291) is always positive, i.e. energy conservation cannot be satisfied meaning that the ground state pairs do not contribute to the rate. Similarly, if $\alpha = \beta = 1$ (two excited pairs)

$$\begin{aligned} &\zeta_{\mathbf{p}_1+\mathbf{p}_2} - E^\infty_{\mathbf{p}_1} - E^\infty_{\mathbf{p}_2} \approx \frac{(\mathbf{p}_1+\mathbf{p}_2)^2}{4m} + \omega_f - \\ &\frac{p_1^2+p_2^2}{2m} = \omega_f - \frac{(\mathbf{p}_1-\mathbf{p}_2)^2}{4m} < 0. \end{aligned} \tag{8.292}$$

Therefore, only scattering processes involving one fermion from an excited and another from a ground state pair contribute. Expression (8.291) for the rate in this case is

$$\begin{aligned} \tau^{-1}_{\text{mol}} \approx \frac{4\pi g^2}{N_f} \sum_{\mathbf{p}_1,\mathbf{p}_2} \sin^2\frac{\theta_{\mathbf{p}_2}}{2}\cos^2\frac{\theta_{\mathbf{p}_1}}{2}|U_{\mathbf{p}_2}|^2|V_{\mathbf{p}_1}|^2 \times \\ \delta\left(\frac{3p_1^2 + 2\mathbf{p}_1\cdot\mathbf{p}_2 - p_2^2}{4m}\right), \end{aligned} \tag{8.293}$$

Next, we go from summations to integrations, integrate over the angle between $\mathbf{p}_1$ and $\mathbf{p}_2$, and change integration variables from momenta to energies, which results in

$$\begin{aligned} \tau^{-1}_{\text{mol}} \approx \frac{3\gamma}{2\varepsilon_F}\int_0^\infty d\varepsilon_2 \sin^2\frac{\theta(\varepsilon_2)}{2}|U(\varepsilon_2)|^2 \times \\ \int_{\varepsilon_2/9}^{\varepsilon_2} d\varepsilon_1 \cos^2\frac{\theta(\varepsilon_1)}{2}|V(\varepsilon_1)|^2. \end{aligned} \tag{8.294}$$

We replace the cosine with one, use $|V(\varepsilon_1)|^2 \approx \Delta^2_\infty/4(\varepsilon_1+|\mu_\infty|)^2$, which follows from Eq. (8.281) together with $|U(\varepsilon_1)|^2 \approx 1$, and integrate over ε_1. According to Eq. (8.136), the probability of finding an excited pair

$$\sin^2\frac{\theta(\varepsilon_2)}{2} \to \frac{\Delta^2_{0i}(\delta\omega)^2}{16E_i^4(\varepsilon_2)} \quad \text{as } \varepsilon_2 \to \infty. \tag{8.295}$$

A larger rate obtains for finite ω_i than for ω_i close to ω_f. In this case $\delta\omega \approx 2\mu_\infty$ and $\sin^2[\theta(\varepsilon_2)/2]$ appreciably differs from zero at energies about $\sqrt{\Delta_{0i}|\mu_\infty|}$. We obtain

$$\tau^{-1}_{\text{mol}} \sim \frac{\gamma\Delta^2_\infty\Delta_{0i}}{\varepsilon_F|\mu_\infty|} \to 0. \tag{8.296}$$

In deriving Eq. (8.295) we assumed finite resonance width γ. A separate estimate for the broad resonance limit for quenches to deep BEC finds a rate that also vanishes, but as $\gamma^{-1/3}|\mu_\infty|^{-1/2}$.

This result for the molecular production rate should be compared with the typical time scale τ_{dyn} of the quench dynamics for quenches to the far BEC side. Eqs. (8.256) and (8.274) imply

$$\tau^{-1}_{\text{dyn}} \sim |\mu_\infty|. \tag{8.297}$$

We see that indeed $\tau_{\text{dyn}} \ll \tau_{\text{mol}}$.

8.9.2 Two-particle collisions

Next, we estimate the relaxation rate due to two-particle collisions. In contrast to the molecular production, we find that here the contribution coming from just the ground state pairs is of the same order of magnitude or larger than that from collisions that involve excited pairs. We therefore consider ground state pairs only and take the probability of finding such a pair at a given momentum $\mathbf{p}$ to be $\cos^2(\theta_{\mathbf{p}}/2) \approx 1$. Let us analyze quenches to the far BCS side of the Feshbach resonance from any initial detuning. In this case, $\omega_f \gg \mu_\infty \approx \epsilon_F$, see e.g. Fig 8.19. The total scattering amplitude for this case has been studied in Ref. [Galaiko(1972)] [see Eq. (71) therein], which also estimates the corresponding rate as

$$\tau_{\text{in}}^{-1} \sim \left(\frac{g^2\nu_F}{\omega_f}\right)^2 \frac{\Delta_\infty^2}{\epsilon_F} = \gamma^2\epsilon_F \left(\frac{\Delta_\infty}{\omega_f}\right)^2. \tag{8.298}$$

In fact, this is the well-know Fermi liquid result for the quasiparticle lifetime. Indeed, $\lambda = g^2\nu_F/\omega_f$ is the strength of the effective interaction between fermions [see Eq. (8.4)] and Δ_∞ is the typical excitation energy – the energy scale at which spins deviate appreciably from their ground state positions.

Eq. (8.298) has to be compared with the characteristic timescale of the dynamics for quenches to the far BCS side. According to Eq. (8.272) this timescale is

$$\tau_{\text{dyn}} \sim \frac{1}{\Delta_\infty}. \tag{8.299}$$

We see that $\tau_{\text{dyn}} \ll \tau_{\text{in}}$ for any finite resonance width γ since $\omega_f \to \infty$ in deep BCS. In the broad resonance limit too $\tau_{\text{dyn}}/\tau_{\text{in}} = \lambda^2\Delta_\infty/\varepsilon_F \ll 1$. This is because at large γ quenches to far BCS in phase II are only possible from initial detunings also on the far BCS side, see e.g. Figs. 8.3(c) and 8.5. It then follows from Eq. (8.152) that $\Delta_\infty \leq \Delta_{0f} \ll \varepsilon_F$.

A preliminary analysis for quenches to the far BEC side shows that, at least for a finite resonance width γ and sufficiently large $|\omega_f|$, one still has $\tau_{\text{dyn}} \ll \tau_{\text{in}}$. Thus, neglecting two-particle collisions is justified at times it takes the quench dynamics to fully develop and reach its asymptote.

8.10 Finite size corrections to the roots

As mentioned in Sect. 8.3, in the thermodynamic limit $\vec{L}^2(u)$ for quench initial conditions has a continuum of roots along the positive real axis. Here we verify this and determine finite size corrections to these roots.

Roots of $\vec{L}^2(u)$ are determined by Eq. (8.126) or, equivalently, by Eq. (8.209) in notation explained in the beginning Sect. 8.5.2, which we employ here as well. The level spacing δ is of order $1/N$. Thermodynamic limit means $N \to \infty$, so ε_k become continuous with density $\nu(\varepsilon)$.

Let us look for a pair of complex conjugate roots close to ε_m writing it as $c_m = \varepsilon_m + \varsigma_m\delta$. We take $\varsigma_m \equiv \varsigma(\varepsilon_m)$ to be of order 1, to be confirmed below. Note that ς_m is generally complex. Our goal is to evaluate c_m to first order in $1/N$. We split the summation in Eq. (8.209) into two parts – over ε_k in a small interval $(\varepsilon_m - \Delta\varepsilon, \varepsilon_m + \Delta\varepsilon)$ and over remaining ε_k. The interval is however sufficiently large so that it contains many ε_k. Specifically, $\Delta\varepsilon \to 0$, but $\Delta\varepsilon/\delta \to \infty$ in thermodynamic limit. For example, $\Delta\varepsilon = \delta\sqrt{N}$ fulfills these conditions. The latter summation becomes a principal value integral in the $N \to \infty$ limit, while the former one to leading order

in $1/N$ reads

$$\frac{N(\varepsilon_m)}{2E(\varepsilon_m)\delta}\sum_{p=0}^{\infty}\left[\frac{1}{p+\varsigma_m}-\frac{1}{p+1-\varsigma_m}\right]= \frac{\pi\nu(\varepsilon_m)}{2E(\varepsilon_m)}\cot\pi\varsigma_m. \tag{8.300}$$

The first sum is from $\varepsilon_k<\varepsilon_m$, the second – from $\varepsilon_k>\varepsilon_m$. Here it is important that the degeneracy $N_k\equiv N(\varepsilon_k)$ and the spacing between ε_k vary smoothly with ε_k. As long as this is the case, we can include any variation of the spacing into N_k.

Thus, Eq. (8.209) to leading order in $1/N$ becomes

$$\frac{2}{g^2}-\fint_0^{\infty}\frac{\nu(\varepsilon')d\varepsilon'}{2(\varepsilon_m-\varepsilon')E(\varepsilon')}-\frac{\nu(\varepsilon_m)}{2E(\varepsilon_m)}\cot\pi\varsigma_m= \frac{\delta\omega}{g^2}\frac{\varepsilon_m-\mu\pm i\Delta_0}{E^2(\varepsilon_m)}. \tag{8.301}$$

Recalling that $\nu(\varepsilon)=\nu_F f(\varepsilon)$ and $g^2\nu_F=\gamma$ in units of Fermi energy, we obtain

$$\pi\cot\pi\varsigma(\varepsilon)=\frac{4E(\varepsilon)}{\gamma f(\varepsilon)}-\frac{G(\varepsilon)}{f(\varepsilon)}-\frac{2\delta\omega}{\gamma}\frac{\varepsilon-\mu\pm i\Delta_0}{E(\varepsilon)f(\varepsilon)}, \tag{8.302}$$

where

$$G(\varepsilon)=E(\varepsilon)\fint_0^{\infty}\frac{f(\varepsilon')d\varepsilon'}{(\varepsilon-\varepsilon')E(\varepsilon')}. \tag{8.303}$$

This principal value integral is the same as in Eq. (8.129). We evaluated it in elementary functions for various cases in Sects. 8.3.1 and 8.3.2. Specifically, in 2d

$$G_{\rm 2d}(\varepsilon)=\ln\left[\frac{\varepsilon\left(\varepsilon-\mu+E(\varepsilon)\right)}{E(\varepsilon)\sqrt{\mu^2+\Delta_0^2}+\mu^2+\Delta_0^2-\mu\varepsilon}\right], \tag{8.304}$$

in weak coupling (BCS) limit, $\mu\approx\varepsilon_F\gg\Delta_0$, for energies not too far from the Fermi energy, in both 2d and 3d

$$G_{\rm wc}(\varepsilon)=\ln\left[\frac{E(\varepsilon)+\varepsilon-\mu}{E(\varepsilon)-\varepsilon+\mu}\right], \tag{8.305}$$

in strong coupling (BEC) limit in 2d and 3d,

$$G_{\rm sc}^{\rm 2d}(\varepsilon)=\ln\frac{\varepsilon}{|\mu|},\quad G_{\rm sc}^{\rm 3d}(\varepsilon)=-\pi\sqrt{\frac{|\mu|}{\Delta_0}}. \tag{8.306}$$

Ground state continual roots $x_k=\varepsilon_k+\varrho_k\delta$ obtain by setting $\delta\omega=0$ in Eq. (8.302), i.e.

$$\pi\cot\pi\varrho(\varepsilon)=\frac{4E(\varepsilon)}{\gamma f(\varepsilon)}-\frac{G(\varepsilon)}{f(\varepsilon)}\equiv\frac{H(\varepsilon)}{f(\varepsilon)}. \tag{8.307}$$

The quantity $F_k\equiv F(\varepsilon_k)$ defined in Eq. (8.215) evaluates similarly to Eq. (8.300),

$$F(\varepsilon)=-\frac{N(\varepsilon)}{2E(\varepsilon)\delta^2}\frac{\partial}{\partial\varrho}\sum_{p=0}^{\infty}\left[\frac{1}{p+1-\varrho}-\frac{1}{p+\varrho}\right]= \frac{\nu(\varepsilon)}{2E(\varepsilon)\delta}\frac{\pi^2f^2(\varepsilon)+H^2(\varepsilon)}{f^2(\varepsilon)}. \tag{8.308}$$

8.11 Identities

In this appendix we prove Eq. (8.225). To this end, consider a function

$$R(u) = L_0(u)\left[(u-\mu)^2 + \Delta_0^2\right], \tag{8.309}$$

where $L_0(u)$ is given by Eq. (8.210). Since zeroes of $L_0(u)$ are x_k and its poles are ε_k it alternatively can be written

$$R(u) = -\frac{2}{g^2}\frac{\prod_{k=1}^N (u-x_k)}{\prod_{k=1}^N (u-\varepsilon_k)}\left[(u-\mu)^2 + \Delta_0^2\right], \tag{8.310}$$

Eq. (8.225) follows by matching two leading terms in $1/u$ expansions of function $1/R(u)$ obtained with the help of these two alternative forms.

Because $1/R(u)$ is a rational function with poles at $u = x_k$ and $\mu \pm i\Delta_0$, we have

$$\begin{gathered}\frac{1}{R(u)} = \sum_k \frac{1}{(u-x_k)L_0'(x_k)\Omega_k^2} + \\ \frac{1}{2i\Delta_0(u-c_+)L_0(c_+)} - \frac{1}{2i\Delta_0(u-c_-)L_0(c_-)},\end{gathered} \tag{8.311}$$

where $c_\pm = \mu \pm i\Delta_0$ and we took into account that the square bracket in Eq. (8.309) evaluated at $u = x_k$ is equal to Ω_k^2. Note also that $L_0'(x_k) = -F_k$, see Eq. (8.215). Eq. (8.210) implies

$$L_0(c_\pm) = -\beta_k \mp i\alpha_k, \tag{8.312}$$

where

$$\begin{aligned}\alpha_k &= \sum_k \frac{N_k\Delta_0}{2\left[E(\varepsilon_k)\right]^{3/2}}, \\ \beta_k &= \frac{2}{g^2} + \sum_k \frac{N_k(\varepsilon_k-\mu)}{2\left[E(\varepsilon_k)\right]^{3/2}}.\end{aligned} \tag{8.313}$$

The leading term in $1/u$ expansion of $1/R(u)$ according to Eq. (8.310) is $-2/(g^2u^2)$. Therefore, the coefficient at $1/u$ in Eq. (8.311) vanishes and that at $1/u^2$ is $-2/g^2$. This yields

$$\begin{aligned}\sum_k \frac{\Delta_0}{F_k\Omega_k^2} &= \frac{\alpha_k}{\alpha_k^2+\beta_k^2}, \\ \sum_k \frac{x_k-\mu}{F_k\Omega_k^2} &= \frac{g^2}{2} - \frac{\beta_k}{\alpha_k^2+\beta_k^2}.\end{aligned} \tag{8.314}$$

Gap and chemical potential equations (8.18) $\left(\frac{(\omega-2\mu)}{g^2} = \sum_{\mathbf{p}} \frac{1}{2E(\varepsilon_{\mathbf{p}};\Delta_0,\mu)}\right)$ and (8.20) in the notation of Sect. 8.5.2 read

$$\begin{aligned}\frac{\omega-2\mu}{g^2} &= \sum_k \frac{N_k}{2E(\varepsilon_k)}, \\ 2n &= \frac{2\Delta_0^2}{g^2} + \sum_k N_k\left(1 - \frac{\varepsilon_k-\mu}{E(\varepsilon_k)}\right).\end{aligned} \tag{8.315}$$

Differentiation of these equations with respect to ω obtains $\delta\mu/\delta\omega$ and $\delta\Delta_0/\delta\omega$ and comparison of the resulting quantities with the right hand side of Eq. (8.314) proves Eq. (8.225).

Another identity used in Sect. 8.5.2 derives by noting that according to Eq. (8.310) $1/R(\varepsilon_k) = 0$. Setting $u = \varepsilon_k$ in Eq. (8.311), we obtain after some algebra

$$\sum_j \frac{1}{(\varepsilon_k - x_m)F_m\Omega_m^2} = \frac{\alpha_k(\varepsilon_k-\mu) - \Delta_0\beta_k}{2\Delta_0(\alpha_k^2+\beta_k^2)\left[E(\varepsilon_k)\right]^2}. \tag{8.316}$$

9 Nonequilibrium dynamics and thermodynamics of a degenerate Fermi gas across a Feshbach resonance by Andreev, Gurarie, Radzihovsky

Abstract

We consider a two-species degenerate Fermi gas coupled by a diatomic Feshbach resonance. We show that the resulting superfluid can exhibit a form of coherent BEC-to-BCS oscillations in response to a nonadiabatic change in the system's parameters, such as for example a sudden shift in the position of the Feshbach resonance. In the narrow resonance limit, the resulting soliton-like collisionless dynamics can be calculated analytically. In equilibrium the thermodynamics can be accurately computed across the full range of BCS-BEC crossover, with corrections controlled by the ratio of the resonance width to the Fermi energy.

Recent experimental advances to control interactions in trapped degenerate gases by tuning (via magnetic field) through a Feshbach resonance (FR), led to observation of long-sought after molecular Bose-Einstein condensation in bosonic [Donley et al(2002)] and fermionic atomic gases [Regal et al.(2004), Zwierlein et al(2004)]. Such tunability allows studies of these systems in previously unexplored regimes, as exemplified by recent JILA and MIT experiments [Regal et al.(2004), Zwierlein et al(2004)], that appear to observe paired fermionic condensates along the crossover between the BCS regime of weakly-paired, strongly overlapping Cooper pairs, and the BEC regime of tightly bound, weakly-interacting diatomic molecules. An even more exciting possibility, unavailable in other related (e.g., superconductors and superfluids in condensed matter) systems, is the nonadiabatic switching of system's parameters, thereby allowing access to highly coherent and nonequilibrium quantum states of matter. For bosonic ^{85}Rb atoms, this was recently spectacularly realized in experiments by Donley *et al* [Donley et al(2002)]. Using short magnetic field pulses that briefly bring the system close to a nearby FR, they observed coherent oscillations in the atomic condensate that can be interpreted as Rabi oscillations between atomic and molecular condensates [Holland and Kokkelmans(2002)].

Stimulated by these experimental advances, in this paper we study a zero-temperature collisionless dynamics of a two-species degenerate atomic Fermi gas near a FR, that can be tuned through the Fermi sea. The goal is to understand the evolution of the system following a nonadiabatic change in an externally-controlled system's parameter, such as the detuning of the FR, ω_0, relative to the Fermi energy, ϵ_F.

Our main result is the demonstration that such a system exhibits an integrable soliton-like solutions, corresponding to collective coherent oscillations of the Fermi gas between BCS- and BEC-like paired superfluid states. This dynamics is a paired-fermions analog of the atomic-to-molecular condensate Rabi oscillations observed by Donley *et al* in trapped bosonic gases. In the narrow Feshbach resonance limit we calculate the frequency and amplitude of these oscillations and find their analytic form. Our work directly builds on the recent discovery by Barankov, Levitov, and Spivak [Barankov et al()] of integrable dynamics in the BCS model following a sudden change in the negative s-wave scattering length.

Complementing our study of dynamics, we also analyze the thermodynamics along the full range of the BCS-BEC crossover [Holland et al(2001), Timmermans et al(2001)]. We show that for a FR that is narrow compared to the Fermi energy, and a background scattering length a_{bg} that is short compared to the interatomic spacing, $n^{-1/3}$, low-temperature thermodynamics can be accurately computed analytically [lar()].

Our starting point is the Hamiltonian [tra()]

$$H = \sum_{p,\sigma} \epsilon_p \, \hat{a}^\dagger_{p\sigma} \hat{a}_{p\sigma} + \sum_p \left(\epsilon_0 + \frac{\epsilon_p}{2}\right) \hat{b}^\dagger_p \hat{b}_p$$

$$+\sum_{p,q} \frac{g}{\sqrt{V}} \left(\hat{b}_q\, \hat{a}^\dagger_{p+q\uparrow}\hat{a}^\dagger_{-p\downarrow} + \hat{b}^\dagger_q\, \hat{a}_{-p\downarrow}\hat{a}_{p+q\uparrow}\right) \tag{9.1}$$

describing fermionic atoms, created by $\hat{a}^\dagger_{p\sigma}$ with momentum p, "spin" $\sigma =\uparrow,\downarrow$, and kinetic energy $\epsilon_p = p^2/2m$, that are coupled to diatomic molecular (resonant) states created by $\hat{b}^\dagger_q$. The position and the width (molecular lifetime) of the FR are respectively controlled by the bare detuning energy ϵ_0 and coupling g, the former easily experimentally tunable by a magnetic field. We have neglected nonresonant atom-atom and molecule-molecule interactions that we expect near a FR to be subdominant to the resonant scattering.

In what follows we focus on molecules with zero center of mass momentum, $\hat{b}_0$, neglecting molecules $\hat{b}_{q\neq 0}$ excited above the molecular condensate. For equilibrium phenomena, this approximation is justified at low temperatures and weak interactions (small g), for which condensate depletion is small. However, for nonequilibrium dynamics of interest to us, the validity of this approximation is a more delicate issue, as one might generically expect such bosonic finite momentum excitations to be induced by a nonadiabatic shift in the FR. Nevertheless, physically we expect that for an initially homogeneous condensate [tra()] and weak interactions the dynamics will be dominated by $b_0(t)$. However, on sufficiently long time scales the $b_{q\neq 0}$ excitations and particle collisions should decohere and damp out the collective BEC-BCS oscillations studied here, allowing a slow relaxation to a new equilibrium state for the shifted FR. Determining this relaxational dynamics is beyond the scope of the present work. We thus replace $\hat{b}_q$ in Eq. (12.1) by $\hat{b}_0\delta_{\mathbf{q},0} \equiv \hat{b}\,\delta_{\mathbf{q},0}$. Expected macroscopic occupation ($\langle\hat{b}^\dagger\hat{b}\rangle \gg 1$) of the molecular level ϵ_0 allows us to neglect quantum molecular fluctuations and replace operators $\hat{b}(t), \hat{b}^\dagger(t)$ by the corresponding c-numbers $b(t), b^*(t)$.

We now look for time dependent fermionic operators in terms of reference, time-independent fermions $\hat{\alpha}_{p\uparrow}, \hat{\alpha}_{-p\downarrow}$, related to $\hat{a}_{p,\sigma}(t)$ through the Bogoliubov amplitudes $u_p(t)$, $v_p(t)$ by $\hat{a}_{p\uparrow} = u^*_p\,\hat{\alpha}_{p\uparrow} + v_p\,\hat{\alpha}^\dagger_{-p\downarrow}$, $\hat{a}_{-p\downarrow} = u^*_p\,\hat{\alpha}_{-p\downarrow} - v_p\,\hat{\alpha}^\dagger_{p\uparrow}$ with a constraint $u^*_p u_p + v^*_p v_p = 1$, that ensures fermionic anticommutation relations. The Heisenberg dynamics is then encoded in the equations of motion for $u_p(t)$, $v_p(t)$, and $b(t)$

$$i\partial_t u_p = -\epsilon_p u_p + \frac{gb^*}{\sqrt{V}}\, v_p,\; i\partial_t v_p = \epsilon_p v_p + \frac{gb}{\sqrt{V}}\, u_p, \tag{9.2}$$

$$i\partial_t b = \epsilon_0 b + \frac{g}{\sqrt{V}} \sum_p u^*_p v_p. \tag{9.3}$$

The dynamical evolution equations (15.22) and (9.3) preserve pair correlations of fermions. Accordingly, we choose the initial state to be of the BCS type, $|\Omega(0)\rangle = \prod_p \left(u_p(0) + v_p(0)\; a^\dagger_{p\uparrow}a^\dagger_{-p\downarrow}\right)|0\rangle$. As a result of the time evolution the fermion wave function preserves the same form with time dependent factors $u_p(t)$ and $v_p(t)$.

Following Ref. [Barankov et al()] we utilize Anderson's spin analogy for the BCS problem [Anderson(1958)], and look for a vector-representation of these equations in terms of the variables

$$S_p = S^x_p + iS^y_p = 2v^*_p u_p, \quad S^z_p = v_p v^*_p - u_p u^*_p, \tag{9.4}$$

that satisfy $|\mathbf{S}_p|^2 = S^*_p S_p + S^z_p S^z_p = 1$. Equations (15.22) become

$$\partial_t S_p = 2i\epsilon_p\, S_p - \frac{2ig}{\sqrt{V}}\, b^* S^z_p,\; \partial_t S^*_p = -2i\epsilon_p\, S^*_p + \frac{2ig}{\sqrt{V}} b\, S^z_p,$$
$$\partial_t S^z_p = \frac{ig}{\sqrt{V}}\left(b^* S^*_p - b\, S_p\right), \tag{9.5}$$

and have a form of Bloch equations for a spin precessing in an effective (p-dependent) field $(-\frac{2g}{\sqrt{V}}\,\mathrm{Re}\, b, \frac{2g}{\sqrt{V}}\,\mathrm{Im}\, b, -2\epsilon_p)$, whose azimuthal dynamics is in turn self-consistently determined

by molecular ($b(t)$) evolution, Eq. (9.3). It is now straightforward to check that these equations can be solved by an ansatz (for the BCS limit, in the absence of molecules suggested in Ref. [Barankov et al()])

$$\begin{aligned} S_p(t) &= e^{i\omega t}\left(-(2\epsilon_p-\omega)C_p\Omega(t)+iC_p\dot{\Omega}(t)\right), \\ S_p^z(t) &= C_p\Omega^2(t)-D_p,\ \frac{g\,b(t)}{\sqrt{V}}=\Omega(t)\,e^{-i\omega t}, \end{aligned} \tag{9.6}$$

provided that the real function $\Omega(t)$ satisfies

$$\dot{\Omega}^2+\left(\Omega^2-\Delta_-^2\right)\left(\Omega^2-\Delta_+^2\right)=0, \tag{9.7}$$

and that C_p and D_p satisfy

$$\frac{D_p^2-1}{C_p^2}=\Delta_-^2\Delta_+^2,\ 2\frac{D_p}{C_p}=(2\epsilon_p-\omega)^2+\Delta_-^2+\Delta_+^2. \tag{9.8}$$

Here Δ_+, Δ_-, and ω are parameters characterizing the periodic instanton-like solution $\Omega(t)$ expressible in terms an elliptic integral. It follows from Eq. (9.7) that $b(t)/\sqrt{V}$ oscillates between the values Δ_-/g and Δ_+/g (we choose $\Delta_-<\Delta_+$) with a period of oscillations given by

$$T=\frac{2}{\Delta_+}K\left(1-\frac{\Delta_-^2}{\Delta_+^2}\right), \tag{9.9}$$

where $K(m)$ is the complete elliptic integral of the first kind.

A direct substitution of this ansatz into Eq. (9.3), shows that the solution, Eq. (9.6) is compatible with the evolution of $b(t)$ provided that the following two conditions are satisfied:

$$\epsilon_0-\omega = \frac{g^2}{2}\int\frac{d^3p}{(2\pi)^3}\,(2\epsilon_p-\omega)\,C_p, \tag{9.10}$$

$$1 = -\frac{g^2}{2}\int\frac{d^3p}{(2\pi)^3}\,C_p. \tag{9.11}$$

Eqs. (48.37), (9.11) determine ω and Δ_+ in terms of the experimentally controlled (initial conditions) parameters ϵ_0 and Δ_-. As we will see below, the first of these equations is a nonequlibrium generalization of the BCS gap equation. The second one simply reflects the conservation of the total particle number, $\frac{dN}{dt}=0$, with

$$N=2b^*b+V\int\frac{d^3p}{(2\pi)^3}\left(S_p^z+1\right). \tag{9.12}$$

We note that Eqs. (9.8) only determine C_p and D_p up to a p-dependent sign, that one would expect to be fixed by the initial fermion momentum distribution, encoded in $|\Omega(0)\rangle$. With the exception of a filled Fermi sea initial condition, $\Omega(0)=\Delta_-=0$, the solution encoded in C_p and D_p, Eq. (9.8), does *not* correspond to initial conditions ($u_p(0)$ and $v_p(0)$) characteristic of a ground state of the Hamiltonian (12.1) for any value of bare detuning ϵ_0. Nevertheless we fix the sign of C_p so as to most closely match $n_p=\frac{1}{2}(S_p^z+1)$ to the initial fermion momentum distribution. For a large positive detuning this corresponds to a Fermi-Dirac step function with discontinuity at the chemical potential μ, that separates the hole-like and particle-like states. Combining this criterion with Eqs.(9.8), we find

$$C_p=\frac{2\ \mathrm{sign}(\epsilon_p-\mu)}{\sqrt{\left[(2\epsilon_p-\omega)^2+{\Delta_-}^2+{\Delta_+}^2\right]^2-4\Delta_-^2\Delta_+^2}}, \tag{9.13}$$

where, as in equilibrium problem, μ is implicitly determined by the conserved total particle number N, Eq. (9.12), that reduces to $N = V\int \frac{d^3p}{(2\pi)^3}\,(1-D_p)$.

Equation (48.37) involves a linearly-divergent integral [Gorkov and Melik-Barkhudarov(1961)], arising from an unphysical aspect of the model Eq. (12.1), that the atomic modes with an arbitrarily large energy interact with the molecular ones with equal strength g. In a more realistic model, the momentum dependence of the coupling g would cutoff this divergence. As usual, our ignorance of this high energy physics can be buried in a (uv cutoff-dependent) relation between the parameter ϵ_0 appearing in the Hamiltonian and the position of the physical Feshbach resonance, ω_0.

To see this, we calculate the two-atom scattering amplitude within the model Hamiltonian Eq. (12.1). It is completely determined by the self-energy of the molecules, given by

$$\begin{aligned}\Sigma(E) &= \int \frac{d\omega}{2\pi}\frac{d^3p}{(2\pi)^3}\frac{ig^2}{(E-\omega-\epsilon_p+i0^+)\,(\omega-\epsilon_p+i0^+)}\\ &= -i\frac{g^2}{4\pi}m^{\frac{3}{2}}\sqrt{E}-g^2\int\frac{d^3p}{(2\pi)^3}\frac{m}{p^2},\end{aligned} \tag{9.14}$$

that diverges in exactly the same way as the integral in Eq. (48.37). Since $\Sigma(E)$ enters the retarded molecular propagator $G_b(E) = (E-\epsilon_0-\Sigma(E))^{-1}$ in combination with ϵ_0, we can trade in ϵ_0 for the physical ("renormalized") FR detuning ω_0 according to $\omega_0 = \epsilon_0 - g^2\int\frac{d^3p}{(2\pi)^3}\frac{m}{p^2}$. This leads to the two-atom scattering amplitude $f = -\frac{1}{\sqrt{m}}\frac{\gamma}{E-\omega_0+i\gamma\sqrt{E}}$, that is consistent with the generic form based on unitarity [Landau and Lifshitz(1981)] and is identical to that of the Fano-Anderson model (equivalent to our model at the two-body scattering level) [Fano(1961)]. We see that the scattering phase changes by π as the energy changes from below to above the physical ("renormalized") FR, ω_0, with finite width for positive detuning, $\omega_0 > 0$, controlled by $\gamma = g^2m^{3/2}/4\pi$. In contrast for negative detuning, $\omega_0 < 0$, the scattering amplitude has a real pole at negative energy (a bound state in the open channel), that corresponds to a real molecular state [cla()].

With above renormalization of detuning, the nonequilibrium gap equation (48.37) becomes

$$\omega_0-\omega = \frac{g^2}{2}\int\frac{d^3p}{(2\pi)^3}\left[(2\epsilon_p-\omega)\,C_p-\frac{2m}{p^2}\right], \tag{9.15}$$

where all the integrals are now convergent. This together with the atom conservation condition, (9.11), the total atom number Eq. (9.12) (that determines μ), the detuning ω_0, and the initial molecular density Δ_-, allows us to determine the condensate frequency ω, and the molecular density maximum Δ_+, that controls the period of oscillations, in accordance with Eq. (9.9). Although the complete solution requires a numerical evaluation of the integrals, here we will focus on two analytically tractable regimes: (i) $\Delta_+-\Delta_- \ll \Delta_+$ that corresponds to small amplitude oscillations of the condensate about the BCS-BEC ground state, and (ii) $\Delta_- \ll \Delta_+$, that corresponds to the evolution of the filled Fermi sea, following a large downward shift in detuning.

Let us first consider the regime of small oscillations, $\Delta_+-\Delta_- \ll \Delta_+$ about a BCS-BEC ground state. Limiting our analysis to $\omega > 0$, we find that Eq. (9.11) constrains ω to be close to 2μ. This together with the condition Eq. (9.15) gives (at $\Delta_- = \Delta_+ = \Delta$, and $\mu = \omega/2$)

$$\omega_0-2\mu = \frac{g^2}{2}\int\frac{d^3p}{(2\pi)^3}\left[\frac{1}{\sqrt{(\epsilon_p-\mu)^2+\Delta^2}}-\frac{1}{\epsilon_p}\right]. \tag{9.16}$$

This coincides with the BCS-BEC gap equation, that can be derived in the equilibrium treatment of this problem [Andreev et al.(2004)Andreev, Gurarie, and Radzihovsky]. It relates

the condensate density Δ/g to the Feshbach resonance ω_0, with the chemical potential μ determined by the total atom number equation, Eq. (9.12). Simple analysis of these equations shows that for a large positive detuning, $\omega_0 \gg \epsilon_F$, molecules are strongly suppressed, leading the chemical potential to "stick" to ϵ_F, and to a conventional atomic BCS ground state, with $\Delta(\omega_0) \approx 8e^{-2}\epsilon_F e^{-(\omega_0-2\epsilon_F)/g^2\nu(\epsilon_F)}$ ($\nu(\epsilon) = m^{\frac{3}{2}}\epsilon^{\frac{1}{2}}/\sqrt{2}\pi^2$ is the atomic density of states). In this far off-resonance BCS regime the accuracy of the mean-field treatment of the equilibrium problem is controlled by the ratio of the width of the FR to Fermi energy, namely by the dimensionless parameter γ^2/ϵ_F [lar()].

As the detuning ω_0 is lowered toward and below ϵ_F, the chemical potential begins to track the detuning, $\mu(\omega_0) \approx \omega_0/2 - \mathcal{O}[g^2\nu(\omega_0/2)]$, with atoms from states between ϵ_F and $\mu(\omega_0)$ converting into Bose-condensed molecules. The density of these tightly bound molecules, that coexist with BCS's Fermi sea determine the gap, which displays a rounded mean-field behavior $\Delta_{equil}(\omega_0) \approx \sqrt{\frac{2^{5/2}}{3\pi}\gamma(\epsilon_F^{3/2} - \mu(\omega_0)^{3/2})}$. In the $g \to 0$ limit, ω_0 crossing of ϵ_F is a genuine quantum transition, with an upper-critical dimension of $d_{uc} = 2$, and is therefore mean-field in 3d [Uzunov(1981)]. A finite atom-molecule coupling g rounds the transition into a smooth crossover near ϵ_F, that for small g (i.e., narrow FR, $\gamma^2 \ll \epsilon_F$) is therefore also accurately described by the mean-field theory summarized by Eqs.(9.12,9.16). Clearly, no additional anomalies appear when the 2-body FR ($\omega_0 = 0$) is crossed, since by that point nearly all atoms are bound up into well-ordered Bose-condensed molecules. One interesting observation is that in this $\mu < 0$ BEC regime the remaining dilute fermionic atoms are paired but are nondegenerate, and therefore realize a "strongly-coupled" BCS state [Read and Green(2000)]. This picture of the ground state (that can be easily extended to a finite temperature[Andreev et al.(2004)Andreev, Gurarie, and Radzihovsky]) is qualitatively consistent with recent observations by Regal, et al.[Regal et al.(2004)], which find that molecules appear at about 0.5 Gauss above the experimentally determined 2-body FR (in our interpretation corresponding to $\omega_0 \approx 2\epsilon_F$).

The nonequilibrium solution, Eq. (9.6) then describes small oscillations about this equilibrium BCS-BEC state, with the period $T = \pi/\Delta$ given by Eq. (9.9). Because $\omega \approx 2\mu$ it is easy to see that for a sufficiently small oscillation amplitude, the atomic momentum distribution function $n_p(t) = \frac{1}{2}(S_p^z(t) + 1)$ only changes near the Fermi surface, $\epsilon_p = \mu$.

The other interesting limit, $\Delta_- \to 0$, $\Delta_+ = \Delta$ describes oscillations between a Fermi sea (a normal "metal", with $\mu = \epsilon_F$) of atoms and bosonic molecules, following a nonadiabatic shift of detuning from ∞ down to ω_0. In this regime, Eq. (9.11),(9.15) reduce to

$$\begin{aligned} \omega_0 - \omega &= \frac{g^2}{2}\int \frac{d^3p}{(2\pi)^3}\left[\frac{\left(\epsilon_p - \frac{\omega}{2}\right)\operatorname{sign}\left(\epsilon_p - \epsilon_F\right)}{\left(\epsilon_p - \frac{\omega}{2}\right)^2 + \frac{\Delta^2}{4}} - \frac{1}{\epsilon_p}\right], \\ 1 &= -\frac{g^2}{4}\int \frac{d^3p}{(2\pi)^3}\frac{\operatorname{sign}\left(\epsilon_p - \epsilon_F\right)}{\left(\epsilon_p - \frac{\omega}{2}\right)^2 + \frac{\Delta^2}{4}}. \end{aligned} \tag{9.17}$$

Simple integration of the atom-number conservation (second) equation above gives

$$\Delta \approx g^2\nu(\omega/2)\tan^{-1}[(2\epsilon_F - \omega)/\Delta]. \tag{9.18}$$

The behavior then depends qualitatively on whether the detuning ω_0 is shifted to the BCS ($\omega_0/2 >> \epsilon_F$) or to the BCS-BEC ($0 < \omega_0/2 < \epsilon_F$) regime, i.e., above or below the Fermi surface. In the former case Eq. (9.18) reduces to $2\epsilon_F - \omega \approx \Delta^2/g^2\nu(\omega/2) \ll \Delta$, with an exponentially suppressed BCS-like gap $\Delta \propto e^{-\frac{\omega_0-2\epsilon_F}{g^2\nu(\epsilon_F)}}$ given by the gap (first) equation in Eq. (9.17). Hence in this regime, for narrow FR, $\omega \approx 2\epsilon_F$ and oscillations are confined to the vicinity of the Fermi surface.

In contrast, for the detuning shift *below* the Fermi surface, $0 < \omega_0/2 < \epsilon_F$, it is now Eq. (9.18) that determines the molecular density. It gives $\Delta \approx g^2\nu(\omega/2)/2 = \gamma\sqrt{\omega} \times \mathcal{O}(1)$,

while the gap equation gives $\omega_0 - \omega = g^2\nu(\epsilon_F) \times \mathcal{O}(1) \ll \omega_0$, enforcing the molecular energy ω to stick to the detuning ω_0. We note, that, although $\Delta \approx \gamma\sqrt{\omega_0}$ is much larger (scales as g^2 rather than exponentially suppressed in $1/g^2$) than for the detuning into the BCS regime, it is much smaller than the equilibrium $\Delta_{equil}(\omega_0)$, found as the solution to the equilibrium gap equation Eq. (9.16) in this regime. This suppression of the condensate oscillations is due to energy conservation between the atoms in the Fermi sea and the molecules. For a narrow FR resonance, $\gamma\sqrt{\omega_0} \ll \omega_0 < \epsilon_F$ it is only a small fraction of atoms in the Fermi sea that are in the immediate vicinity (set by the resonance width $\gamma\sqrt{\omega_0}$) of ω_0, that can resonantly bind into molecules. The resulting resonant atom-molecule interconversion leads to a narrow oscillatory depletion of the Fermi sea, illustrated in Fig.9.1, with a period of oscillations that diverges as $T \propto \frac{2}{\Delta_+} \log\left(\frac{\Delta_+}{\Delta_-}\right)$, in the limit of the "normal" Fermi sea ($\Delta_- \to 0$) as the initial condition.

Figure 9.1: Atomic momentum distribution function n_p displaying oscillations between a Fermi-sea and Bose-condensed molecules. The atomic depletion in n_p of width $\Delta \approx \gamma\sqrt{\omega_0}$ appears at the molecular energy determined by the detuning ω_0.

Analysis of the gap and atom-conservation equations shows that the amplitude of the atom-molecule oscillations vanishes as the molecular energy $\omega \approx \omega_0$ approaches zero. This is again enforced by the energy conservation that in the absence of other degrees of freedom (e.g., molecules above a condensate, $b_{q\neq 0}$) forbids conversion of molecules at negative energy ω_0 into atoms at positive energy ϵ_p. We leave analysis of the dynamics that incorporates these additional degrees of freedom to future research.

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10 Dynamical vanishing of the order parameter in a fermionic condensate by Yuzbashyan, Dzero

Abstract

We analyze the dynamics of a condensate of ultra-cold atomic fermions following an abrupt change of the pairing strength. At long times, the system goes to a non-stationary steady state, which we determine exactly. The superfluid order parameter asymptotes to a constant value. We show that the order parameter *vanishes* when the pairing strength is decreased below a certain critical value. In this case, the steady state of the system combines properties of normal and superfluid states – the gap and the condensate fraction vanish, while the superfluid density is nonzero.

Recently, several remarkable experiments have demonstrated Cooper pairing in cold atomic Fermi gases[[1]–[4]]. Key signatures of a paired state – condensation of Cooper pairs[1, 2] and the pairing gap[4] have been observed. In addition, trapped gases provide a unique tool to explore

aspects of fermion pairing normally inaccessible in superconductors. One of the most exciting prospects is a study of far from equilibrium coherent dynamics of fermionic condensates[[15]–[9]], made possible due to the precise experimental control over interactions between atoms [10, 11]. The dynamics can be initiated by quickly changing the pairing strength with external magnetic field.

Here, we determine the time evolution of a fermionic condensate in response to a sudden change of interaction strength. Initially, the gas is in equilibrium at zero temperature on the BCS side of the Feshbach resonance ((???? what is it???)) with a coupling constant $g_i > 0$. At $t = 0$ the coupling is suddenly changed to a smaller value $g_f > 0$ on the same side of the resonance, $g_i \to g_f$, Fig. 10.1 (inset). Ground states of the system at the old, g_i, and new, g_f, values of the coupling are characterized by corresponding BCS gaps, Δ_i and Δ_f, respectively. We consider the case $\Delta_i \geq \Delta_f$. It has been shown previously that following the change of coupling, the time-dependent order parameter $\Delta(t)$ asymptotes to a constant value[9], $|\Delta(t)| \to \Delta_\infty$ on a timescale $\tau_\Delta = 1/\Delta_i$. Here we evaluate Δ_∞ in terms of Δ_i and Δ_f.

We show that when the coupling is decreased below a certain critical value, Δ_∞ *vanishes*, Fig. 10.1. On a τ_Δ timescale the system goes to a steady *non-stationary* state that combines properties of normal and superfluid states in a peculiar way. For example, the gap vanishes, while the superfluid density remains finite. Provided the system is continuously cooled, the BCS ground state with a gap Δ_f is reached on the energy relaxation timescale τ_ϵ, which is typically much larger than τ_Δ. Experimental signatures of the novel state include the absence of the gap in rf absorbtion spectrum and zero condensate fraction after a fast projection onto the Bose-Einstein Condensation (BEC) side (see below).

Figure 10.1: Time evolution of the BCS order parameter $\Delta(t)$ following an abrupt change of the coupling constant. The coupling is changed by varying the magnetic field on the BCS side of the Feshbach resonance (inset). The order parameter exponentially decays to zero (solid line) at times $t \sim 1/\Delta_i$ and the system goes into a *gapless* steady state. If the system is continuously cooled, the order parameter recovers to its equilibrium value Δ_f (dashed) at times longer than the quasiparticle relaxation time τ_ϵ.

At times $t \ll \tau_\epsilon$, dynamics of the condensate in the weak coupling regime can be described by the BCS model. Here we are interested in the thermodynamic limit, in which case one can use the BCS mean-field approach[23] ((??? what Anderson's work has to do with it???)).

Using Anderson's pseudospin representation[23], one can describe the mean-field evolution by a *classical* spin Hamiltonian[23, 15, 6]

$$H = \sum_j 2\epsilon_j s_j^z - g \sum_{j,k} s_j^+ s_k^-, \tag{10.1}$$

where ϵ_j are single-particle energies relative to the Fermi level and $s_j^\pm = s_j^x \pm i s_j^y$. The summation in Eq. (10.1) is over $|\epsilon_j| < E_F$, where E_F is the Fermi energy. Dynamical variables $\mathbf{s}_j$ are vectors of fixed length, $|\mathbf{s}_j| = 1/2$. The BCS order parameter is $\Delta(t) = \Delta_x - i\Delta_y = g\sum_j s_j^-$. Equations of motion for classical spins $\mathbf{s}_j$ are

$$\dot{\mathbf{s}}_j = \mathbf{b}_j \times \mathbf{s}_j, \quad \mathbf{b}_j = (-2\Delta_x, -2\Delta_y, 2\epsilon_j)\,. \tag{10.2}$$

((??? so, to understand it, I need to understand Anderson's theory))

Components of spins are related to Bogoliubov amplitudes u_j and v_j

$$2s_j^z = |v_j|^2 - |u_j|^2, \;\; s_j^- = \bar{u}_j v_j, \tag{10.3}$$

At $t = 0$ the system is in the ground state with gap Δ_i. The ground state is obtained by aligning each spin $\mathbf{s}_j$ antiparallel to its "magnetic" field $\mathbf{b}_j$ [23] in order to minimize the total energy (10.1) for coupling constant $g = g_i$

$$s_j^x(0) = \frac{\Delta_i}{2\sqrt{\epsilon_j^2 + \Delta_i^2}}, \;\; s_j^z(0) = -\frac{\epsilon_j}{2\sqrt{\epsilon_j^2 + \Delta_i^2}}, \tag{10.4}$$

and $s_j^y(t = 0) = 0$.

At $t = 0$ the coupling is changed, $g_i \to g_f$, and the initial spin configuration (10.4) is no longer an equilibrium for $t > 0$. To determine the time evolution of the system, one has to solve Eqs. (11.3) with initial conditions (10.4).

We start with a linear analysis. Solving Eqs. (11.3) (with $g = g_f$) linearized around the spin configuration (10.4), we obtain up to terms of order $\delta\Delta/\Delta_f$

$$\Delta(t) = \Delta_f - 8\delta\Delta \int_0^\infty d\epsilon \frac{\cos\omega(\epsilon)t}{\omega(\epsilon)\left[\pi^2 + h^2(\epsilon)\right]}, \tag{10.5}$$

where $\delta\Delta = \Delta_f - \Delta_i$, $\omega(\epsilon) = 2\sqrt{\epsilon^2 + \Delta_f^2}$, and $h(\epsilon) = \sinh^{-1}(\epsilon/\Delta_f)$. In Eq. (10.5), besides the continuum limit, we took the weak coupling limit $E_F/\Delta_i \to \infty$.

The long time behavior of the order parameter in the linear approximation is obtained from Eq. (10.5) by stationary phase method (see also Ref. [24]),

$$\Delta(t) = \Delta_f - \frac{2\delta\Delta}{\pi^{3/2}\sqrt{\Delta_f t}} \cos\left(2\Delta_f t + \frac{\pi}{4}\right), \tag{10.6}$$

At times $t \gg \tau_\Delta$ the gap approaches a constant value $\Delta_\infty = \Delta_f$ to order $\delta\Delta/\Delta_f$.

Even though the gap is constant, the state of the system is non-stationary[17]. According to Eq. (11.3), at large times each spin $\mathbf{s}_j \equiv \mathbf{s}(\epsilon_j)$ precesses in its own constant field $\mathbf{b}_j = (-2\Delta_\infty, 0, 2\epsilon_j)$. For example, for the x-component of spins we derive from the linearized equations of motion,

$$\begin{aligned} s_x(\epsilon) = & \frac{\Delta_f}{2\sqrt{\epsilon^2 + {\Delta_f}^2}} \\ & - \frac{\delta\Delta\epsilon}{(\epsilon^2 + \Delta_f^2)\sqrt{\pi^2 + h^2(\epsilon)}} \cos\left[\omega(\epsilon)t + \phi(\epsilon)\right], \end{aligned} \tag{10.7}$$

However, the gap $\Delta(t) = g\sum_j s_j^x(t)$ contains oscillations with many different frequencies. At large times they go out of phase and cancel out in the continuum limit.

In the non-linear case it can be shown that the gap decays to a constant Δ_∞ by a similar mechanism[9]. To determine Δ_∞, we use the exact solution for the dynamics of the BCS model[6]. Consider the following vector function of an auxiliary parameter u: $\mathbf{L}(u) = -\hat{\mathbf{z}}/g + \sum_j \mathbf{s}_j/(u-\epsilon_j)$, where $\hat{\mathbf{z}}$ is a unit vector along the z-axis. Using Eq. (11.3), we obtain $d\mathbf{L}^2(u)/dt = 0$, i.e. $\mathbf{L}^2(u)$ is conserved for any u.

To determine the steady state gap Δ_∞, we evaluate $\mathbf{L}^2(u)$ at $t = 0$ for the initial spin configuration (10.4) and at $t \gg \tau_\Delta$, when each spin $\mathbf{s}(\epsilon)$ precesses in a constant field. Matching the two expressions, we derive for $\Delta_f \leq \Delta_i$

$$\Delta_f = \Delta_i \exp(-\kappa \tan\kappa/2), \qquad \Delta_\infty = \Delta_i \cos\kappa, \tag{10.8}$$

where $0 \leq \kappa \leq \pi/2$. This equation determines Δ_∞ in terms of Δ_i and Δ_f, see Fig. 10.2.

We make several observations. The steady state gap reaches its maximum $\Delta_\infty = \Delta_f$ at $\Delta_f/\Delta_i = 1$. Otherwise, $\Delta_\infty < \Delta_f$. Expanding around the maximum, we find $\Delta_\infty = \Delta_f - (\delta\Delta)^2/6\Delta_f$ up to terms of higher order in $\delta\Delta/\Delta_f$. The linear term vanishes in agreement with Eq. (10.6).

Figure 10.2: After a sudden change of the BCS coupling constant, the order parameter $\Delta(t)$ (inset) saturates to a constant value Δ_∞. The plot shows the steady state gap Δ_∞ in units of Δ_i as a function of the ratio Δ_f/Δ_i. The exact result given by Eq. (10.8) is compared to numerical solution of Eq. (11.3). Note that $\Delta_\infty = 0$ for $\Delta_f \leq 0.2\Delta_i$.

Most interestingly, we see from Eq. (10.8) that the order parameter vanishes in the steady state, $\Delta_\infty = 0$, when $\kappa = \pi/2$, i.e. for $(\Delta_f/\Delta_i)_c = e^{-\pi/2} \approx 0.21$. If Δ_f/Δ_i is below this critical ratio, Eq. (10.8) has no solutions – the system goes into a gapless steady state, $\Delta_\infty = 0$, for $\Delta_f \leq 0.21\Delta_i$. We also note from Eq. (10.8) and Fig. 10.2 that Δ_∞ has a cusp at $\Delta_f/\Delta_i = e^{-\pi/2}$, i.e. the transition to the gapless state is second order.

To gain further insight into properties of the gapless state, we need to know the spin configuration in this state. At large times spin $\mathbf{s}(\epsilon)$ rotates in a constant field $\mathbf{b}(\epsilon) = -2\Delta_\infty\hat{\mathbf{x}} + 2\epsilon\hat{\mathbf{z}}$, where $\hat{\mathbf{x}}$ and $\hat{\mathbf{z}}$ are unit vectors along the x and z axes, respectively. In the gapless case, $\mathbf{b}(\epsilon) = 2\epsilon\hat{\mathbf{z}}$. The spin configuration can be characterized by the angle $\theta(\epsilon)$ between $\mathbf{s}(\epsilon)$ and

$-\mathbf{b}(\epsilon)$. This angle can be determined by matching the conserved quantity $\mathbf{L}^2(u)$ at $t = 0$ and $t \gg \tau_\Delta$,

$$\sin^2\theta(\epsilon) = \frac{G(\epsilon)}{2\pi^2} - \sqrt{\frac{G^2(\epsilon)}{4\pi^4} - \frac{4\beta^2\Delta_i^2}{\pi^2(\epsilon^2+\Delta_i^2)}}\ , \tag{10.9}$$

where $\beta = \ln(\Delta_i/\Delta_f)$ and

$$\begin{aligned} G(\epsilon) =& \pi^2 + 4\left[\sinh^{-1}(\epsilon/\Delta_i)\right]^2 \\ &+ \frac{8\beta\epsilon}{\sqrt{\epsilon^2+\Delta_i^2}}\sinh^{-1}(\epsilon/\Delta_i) + 4\beta^2 \end{aligned} \tag{10.10}$$

Figure 10.3: Plot of the average occupation number $\langle n(\epsilon)\rangle$ as a function of energy in the gapless state for various values of $\Delta_f/\Delta_i < e^{-\pi/2}$. The exact result given by Eq. (10.9) is compared to numerical solution of Eq. (11.3). Note that the distribution function around the Fermi level is smeared over a width $\delta\epsilon \propto \Delta_i$. The inset shows a spin $\mathbf{s}(\epsilon_+)$ at energy $0 < \epsilon_+ \ll \Delta_i$ just above and below the critical point.

Note from Eq. (11.3) that in the steady state the component of spin $\mathbf{s}(\epsilon)$ along the field, $s_\parallel(\epsilon) = -\cos\theta(\epsilon)/2$, is constant, while the one perpendicular to the field, $|\mathbf{s}_\perp| = \sin\theta(\epsilon)/2$ rotates with a frequency $\omega(\epsilon) = 2\sqrt{\epsilon^2+\Delta_\infty^2}$. The approach to the steady state can be studied by linearizing Eq. (11.3) around this state. In particular, we obtain the asymptotic behavior of $\Delta(t)$

$$\frac{\Delta(t)}{\Delta_\infty} = 1 + a\frac{\cos(2\Delta_\infty t + \pi/4)}{\sqrt{\Delta_\infty t}}, \quad \frac{\Delta_f}{\Delta_i} > e^{-\pi/2} \tag{10.11}$$

$$\frac{\Delta(t)}{\Delta_i} = A(t)e^{-2\alpha\Delta_i t} + B(t)e^{-2\Delta_i t}, \quad \frac{\Delta_f}{\Delta_i} < e^{-\pi/2} \tag{10.12}$$

Here $\alpha = -\cos p$ and $\pi/2 \le p \le \pi$ is the solution of $p = \ln(\Delta_i/\Delta_f)\cot(p/2)$. The parameter α has a property $\alpha \to 0$ when $\Delta_f/\Delta_i \to e^{-\pi/2}$ and $\alpha \to 1$ when $\Delta_f/\Delta_i \to 0$. The coefficient a in Eq. (11.1) is time-independent, $A(t)$ and $B(t)$ in Eq. (10.12) are decaying power laws, $A(t), B(t) \propto 1/t^\nu$ with $1/2 \le \nu \le 2$.

We conclude that the decay law of $\Delta(t)$ changes from power law to exponential as we cross the critical point. Above the critical point spins rotate with frequencies $\omega(\epsilon) = 2\sqrt{\epsilon^2+\Delta_\infty^2}$.

The inverse square root decay and oscillations with frequency $2\Delta_\infty$ in Eq. (11.1) are due to the square root singularity in the spectral density $d\epsilon/d\omega \propto \omega/\sqrt{\omega^2 - 4\Delta_\infty^2}$, cf. Eqs. (10.5,10.6). Below the critical point Δ_∞ vanishes, $\omega(\epsilon) = 2\epsilon$, and the square root anomaly disappears.

What happens to the spin configuration as Δ_f/Δ_i is varied across the critical point? Consider a spin at energy $\epsilon_+ \ll \Delta_i$ just above the Fermi energy. Using Eq. (10.9), we obtain $\sin\theta(\epsilon_+) \approx \sin\theta(0) = 2\ln(\Delta_i/\Delta_f)/\pi$ for $\Delta_f > e^{-\pi/2}\Delta_i$ and $\sin\theta(0) = 1$ otherwise. The field $\mathbf{b}(\epsilon_+) = -2\Delta_\infty\hat{\mathbf{x}} + 2\epsilon_+\hat{\mathbf{z}}$ is along the x axis above the critical point ($\Delta_\infty \gg \epsilon_+$) and along the z axis below ($\Delta_\infty = 0$). At $\Delta_i = \Delta_f$ we have $\sin\theta(0) = 0$, i.e. the spin is parallel to the x axis. As Δ_f/Δ_i decreases, the x component of the spin also decreases until it vanishes at the critical point $\Delta_f = e^{-\pi/2}\Delta_i$. Just above it the spin lies in the yz plane and rotates around the x axis. Below the critical point it rotates in the xy plane around the z axis (Fig. 10.3).

In the gapless steady state each spin rotates around the z axis. Its z component is time-independent, $s_z(\epsilon) = -\cos\theta(\epsilon)/2$. It is related to the average occupation number per fermion species $\langle\hat{n}(\epsilon)\rangle$ at energy ϵ as $\langle\hat{n}(\epsilon)\rangle = s_z(\epsilon) + 1/2$ [23], i.e. $\langle\hat{n}(\epsilon)\rangle = (1 - \cos\theta(\epsilon))/2$. The distribution function $\langle\hat{n}(\epsilon)\rangle$ is smeared near the Fermi energy over a width $\delta\epsilon \propto \Delta_i$ (Fig. 10.3). Note that in this respect the gapless state is similar to the BCS ground state – the smearing over a width $\delta\epsilon \propto \Delta_i$ due to interactions is also present in the ground state distribution (10.4). In the normal state $s_z(\epsilon) = -\mathrm{sgn}\,\epsilon/2$ and $\langle\hat{n}(\epsilon)\rangle = \Theta(-\epsilon)$.

Given relation (10.3), one can reconstruct the time-dependent condensate wave function, and evaluate normal and anomalous correlation functions, e.g.

$$\langle\hat{c}_{\epsilon\sigma}(t)\hat{c}^\dagger_{\epsilon\sigma}(t')\rangle = e^{i\epsilon(t'-t)}\cos^2\tfrac{\theta(\epsilon)}{2} \tag{10.13}$$

$$\langle\hat{c}^\dagger_{\epsilon\uparrow}(t)\hat{c}^\dagger_{\epsilon\downarrow}(t')\rangle = e^{i\epsilon(t'+t)}\cos\tfrac{\theta(\epsilon)}{2}\sin\tfrac{\theta(\epsilon)}{2} \tag{10.14}$$

where $\hat{c}_{\epsilon\sigma}$ ($\hat{c}^\dagger_{\epsilon\sigma}$) annihilates (creates) a fermion of one of the two species $\sigma = \uparrow, \downarrow$ on energy level ϵ. Note that even though Δ_∞ vanishes, anomalous averages do not.

Next, we determine the superfluid density n_s in the gapless state. Consider a degenerate Fermi gas in an axially symmetric trap slowly rotating around the symmetry axis. The density n_s of the superfluid component can be defined as the rigidity of the superfluid with respect to an infinitesimal twist in the boundary conditions[14], $\hat{\psi}_\sigma(\mathbf{r}) \to e^{-i\alpha\phi}\hat{\psi}_\sigma(\mathbf{r})$, where ϕ is the azimuthal angle with respect to the symmetry axis, $\hat{\psi}_\sigma(\mathbf{r}) = \sum_j \hat{c}_{j\sigma}\varphi_j(\mathbf{r})$, and $\varphi_j(\mathbf{r})$ are the single particle wave functions. The twist generates a term $-\alpha\hat{J}$ in the Hamiltonian, where $\hat{J}$ is the current operator. The resulting supercurrent $J_s = n_s\alpha/m$ can be expressed in terms of correlation functions (10.13,10.14) using the standard linear response theory. We find $n_s = n/2$, where n is the particle density in the normal state. A similar calculation for nonzero Δ_∞ yields $n_s = n$ – the superfluid density in the gapped steady state is the same as in the BCS ground state. Since n_s is the second derivative of the free energy with respect to α, the jump from n to $n/2$ at the critical point agrees with the second order character of the transition. The reduction in n_s in the state with zero Δ_∞ is consistent with vanishing of the gap in this state[15].

We see that the superfluid density is finite in the gapless steady state. This reflects the existence of pair correlations (10.14) between fermions, a property that in equilibrium has been directly linked to nonzero n_s [16]. This situation is similar to the phenomenon of gapless superconductivity in metals[17], where the superconducting state is also characterized by non-vanishing anomalous Green's functions and zero spectral gap. A gapless state in superconductors is usually a consequence of strong perturbations - as the perturbation is increased, the superconducting metal first goes into a gapless state with finite n_s before it becomes normal.

A crude qualitative understanding of the dynamical transition to the gapless steady state can be derived from the following thermodynamic argument. After the change of coupling the initial state (10.4) has energy $\mathcal{E}(\Delta_f, \Delta_i) > 0$ relative to the ground state with gap Δ_f. In thermal equilibrium the system would have the same energy at a certain temperature $T_0(\Delta_f, \Delta_i)$. Let us

keep Δ_i fixed and vary Δ_f. At $\Delta_f = \Delta_i$, $T_0 = 0$, while the critical temperature $T_c \approx 0.57\Delta_f > 0$. As Δ_f increases both T_0 and T_c grow, but T_c grows faster and T_0 never catches up with it. On the other hand, when Δ_f decreases, T_c also decreases, while T_0 grows. Evaluating $T_0(\Delta_f, \Delta_i)$, to the first order in the coupling constant we find that $T_0 \geq T_c$ for $\Delta_f/\Delta_i \leq 0.52 + 0.17\lambda$ and $T_0 < T_c$ otherwise. Here $\lambda = g/d$ is the dimensionless BCS coupling constant and $d = \langle \epsilon_{j+1} - \epsilon_j \rangle$ is the mean level spacing. We see that decreasing the coupling beyond a certain value provides enough energy for the transition to the normal state to occur in thermal equilibrium.

Let us discuss experimental manifestations of the dynamical transition to the gapless steady state. After the initial sweep ($g_i \to g_f$) on the BCS side of the resonance such that $\Delta_f/\Delta_i < e^{-\pi/2}$, the gapless state is reached on a timescale $\tau_\Delta = 1/\Delta_i$. The pairing gap measured in Ref. [4] is the energy cost ΔE of breaking a Cooper pair. This corresponds[23] to removing the spin closest to the Fermi level in (10.1). The minimum energy cost is $\Delta E = \min |\mathbf{b}(\epsilon) \cdot \mathbf{s}(\epsilon)| = \Delta_\infty \cos\theta(0)$. Using Eq. (10.9), we obtain $\Delta E = \Delta_\infty \sqrt{1 - 4\ln^2(\Delta_i/\Delta_f)/\pi^2}$. In the steady state, $\Delta E = 0$, i.e. no gap in the rf absorbtion spectrum will be observed. Molecular condensate fraction [1, 2] *vanishes* in the gapless state, since $N_{k=0}/N \propto \Delta_\infty^2$ [18].

In conclusion, we determined the dynamics of the paired state of cold atomic fermions following an abrupt lowering of the pairing strength, $g_i \to g_f$. On a short $1/\Delta_i$ timescale, where Δ_i is the equilibrium gap for the initial coupling g_i, the system goes to a non-stationary steady state with a constant gap Δ_∞ and a distribution function given by Eqs. (10.8,10.9). When the coupling is reduced so that Δ_f/Δ_i is smaller than the critical value of $e^{-\pi/2}$, the steady state gap vanishes. The decay law of the time-dependent order parameter $\Delta(t)$ changes from power law to exponential as Δ_f/Δ_i goes through the critical point, Eqs. (11.1,10.12). The gapless state combines features of normal and superfluid states. In particular, the gap and the condensate fraction vanish, while the superfluid density is nonzero.

11 Relaxation and persistent oscillations of the order parameter in the non-stationary BCS theory by Yuzbashyan, Tsyplyatyev, Altshuler

(this is a little article that gives overview and then shows solutions in different regimes. I am not sure that relations of this approach to others are discussed)

Abstract

We determine the limiting dynamics of a fermionic condensate following a sudden perturbation for various initial conditions.

We demonstrate that possible initial states of the condensate fall into two classes.

In the first case, the order parameter asymptotes to a *constant* value. The approach to a constant is oscillatory with an inverse square root decay. This happens, e.g., when the strength of pairing is abruptly changed while the system is in the paired ground state and more generally for any *nonequilibrium* state that is in the same class as the ground state.

In the second case, the order parameter exhibits persistent oscillations with several frequencies. This is realized for nonequilibrium states that belong to the same class as excited stationary states.

(don't understand)

Our *classification* of initial states extends the concept of excitation spectrum to nonequilibrium regime and allows one to predict the evolution without solving equations of motion.

The response of a fermionic condensate to fast external perturbations presents a long-standing problem[[11]–[36]] (why is it not solved?).

The main difficulty is to describe the time evolution in the non-adiabatic (non-adiabatic??? never thought that it is important) regime when a nonequilibrium state of the condensate is created on a time scale shorter than the Cooper instability time $\tau_\Delta = 1/\Delta_0$, where Δ_0 is the equilibrium BCS gap. In this case the evolution of the system cannot be described in terms of a quasiparticle spectrum or a single time-dependent order parameter $\Delta(t)$ [1]. One has to account for the dynamics of individual Cooper pairs, making it a complex many-body problem.

The non-adiabatic regime can be accessed experimentally in ultra-cold Fermi gases, where the strength of pairing between fermions can be rapidly changed[13]. Non-adiabatic measurements can be also performed in quantum circuits utilizing superconducting qubits (nanoscale superconductors). Here nonequilibrium conditions can be generated by fast voltage pulses on a time scale comparable to τ_Δ[14].

Here we consider a BCS condensate that is out of equilibrium at $t = 0$ and study its time evolution for $t > 0$. Given the state of the system at $t = 0$, we predict the dynamics with no need for actually solving equations of motion. We show that physically meaningful initial states fall into two broad categories. In the first scenario, $|\Delta(t)|$ asymptotes to a constant value $\Delta_\infty < \Delta_0$. The approach to Δ_∞ is oscillatory with a $1/\sqrt{t}$ decay,

$$\frac{|\Delta(t)|}{\Delta_\infty} = 1 + a\frac{\cos(2\Delta_\infty t + \phi)}{\sqrt{\Delta_\infty t}}, \tag{11.1}$$

where the constants a and ϕ depend on details of the initial state. This scenario is realized, for example, when the pairing strength is abruptly changed, while the system is in the paired ground state. In the second scenario, $|\Delta(t)|$ oscillates persistently with several incommensurate frequencies. The number of frequencies as well as the limiting dynamics of individual pairs can be inferred directly from the initial state.

We propose a topological *classification* of initial states, which extends the concept of excitation spectrum to the nonequilibrium regime. If a state is in the same class as the paired ground state, Eq. (11.1) applies. Other states are topologically distinct, in which case persistent oscillations occur.

Our approach explains differences (which?) between previous studies of condensate dynamics.

A linear analysis around the BCS ground state yields[12] damped oscillations with a frequency $2\Delta_0$. Eq. (11.1) generalizes this result to the nonlinear case and a wide range of initial conditions (check that it is so. what about initial conditions???).

An oscillatory decay following a change in the coupling strength was observed numerically[19, 18] (what is going on in these papers? they confirmed the Volkov, Kogan results?).

We will see that this is due to the fact that initial states of Refs. [12, 19, 18] are in the same class (class???) as the BCS ground state.

Undamped periodic oscillations of $|\Delta(t)|$ have been found in Refs. [13, 14, 15]. They were also seen numerically for initial states close to a normal state[17]. (okay, but what is the difference with Volkov's statement of the problem?)

In contrast, Ref. [36] also starts from the normal state (meaning close to normal?), but obtains a saturation to $\Delta_\infty = \Delta_0/2$. It turns out[15] that this occurs if the initial state is a *paired* state with a small seed gap $\Delta_{in} \ll \Delta_0$ (this is very strange, why there is such

dependence on the initial state??? is it an artificial state?). Quasiperiodic oscillations of the order parameter[8, 9] can also be realized (see below).

In the non-dissipative regime, one can use the BCS model to describe the dynamics of the condensate. Here we are interested in the thermodynamic limit. Then, the BCS mean-field is valid as long as the order parameter is nonzero (if the ord.par. is zero and we are close to the Tc, BCS model should also work, no? at high T, there maybe will be effects that are not described by BCS. Which by the way??).

There are several equivalent ways to derive mean-field evolution equations. Using Anderson's pseudospin representation[11], one can describe the mean-field evolution by a *classical* spin Hamiltonian[8]

$$H_{BCS} = \sum_j 2\epsilon_j s_j^z - g \sum_{j,k} s_j^+ s_k^-, \tag{11.2}$$

(still, don't understand its meaning) where ϵ_j are the single-particle energies and $s_j^\pm = s_j^x \pm i s_j^y$. Dynamical variables $\mathbf{s}_j$ are vectors of fixed length, $|\mathbf{s}_j| = 1/2$. The BCS order parameter is $\Delta(t) = \Delta_x - i\Delta_y = g\sum_j s_j^-$ (why def with "-" sign? is it important?).

Mean-field equations of motion $\dot{\mathbf{s}}_j = \{H_{BCS}, \mathbf{s}_j\}$ follow from Hamiltonian (11.2) and the angular momentum Poisson brackets (why do we use Poisson br-s??? never thought about it), $\left\{s_j^x, s_k^y\right\} = -\delta_{jk} s_j^z$ etc.,

$$\dot{\mathbf{s}}_j = \mathbf{b}_j \times \mathbf{s}_j, \qquad \mathbf{b}_j = (-2\Delta_x, -2\Delta_y, 2\epsilon_j)\,. \tag{11.3}$$

The general non-stationary BCS problem can now be formulated as follows: given an initial distribution of spins, determine its evolution.

(there is the long standing question "why they use at all Anderson's formalism? I haven't learn it yet, I'll answer later")

Solution of Eq. (11.3) for each of the dynamical variables $\mathbf{s}_j(t)$ and $\Delta(t)$ is a superposition of harmonic oscillations with certain frequencies. (what is it and why??)

In the thermodynamic limit the frequency spectrum in general consists of continuum and discrete parts (and how is it in not thermodynamic limit?).

Consider, e.g., the absolute value of the order parameter $|\Delta(t)|$. Suppose the discrete part contains k incommensurate frequencies (contains what???).

Let their contribution to $|\Delta(t)|$ be $F_k(t)$,

$$|\Delta(t)| = F_k(t) + \int_{-D}^{D} d\omega A(\omega) \cos[\omega t + \phi(\omega)], \tag{11.4}$$

(why??)where D is the ultraviolet cutoff (Debye frequency in conventional BCS).

The integral in Eq. (11.4) vanishes as $t \to \infty$ for any well behaved $A(\omega)$. Hence, $|\Delta(t)| \to F_k(t)$ for $t \to \infty$, i.e. $|\Delta(t)|$ oscillates with k frequencies for $k \geq 1$ and $|\Delta(t)| \to \Delta_\infty$ if $k = 0$.

Given the initial state, the structure of the frequency spectrum can be determined using integrability of BCS dynamics[8, 9]. Frequencies of an integrable system depend only on its integrals of motion[28], which can be evaluated at $t = 0$. Frequencies can thus be determined without solving equations of motion.

Here we use the method of Ref. [9]. It is convenient to introduce the following vector function (Lax vector) of an auxiliary (spectral) parameter u:

$$\mathbf{L}(u) = -\frac{\hat{\mathbf{z}}}{g} + \sum_j \frac{\mathbf{s}_j}{u - \epsilon_j}, \tag{11.5}$$

where $\hat{\mathbf{z}}$ is a unit vector along the z-axis. (still haven't figure out what is the idea of Lax vector, but okay, it is a question for another article)

The square of the Lax vector is conserved by the evolution. The frequency spectrum is related to branch cuts of $w(u) = \sqrt{\mathbf{L}^2(u)}$. Note that the numerator of $\mathbf{L}^2(u)$ is a polynomial of degree $2n$, where n is the total number of levels ϵ_j. Since $\mathbf{L}^2(u) \geq 0$, all $2n$ roots come in complex conjugate pairs. For finite n, all roots are typically distinct leading to n isolated cuts connecting pairs of conjugate roots. This situation is described by the general solution[8] for finite n. One can also have a situation when $2(n-m)$ roots are real and therefore double degenerate. This leaves m branch cuts corresponding to m pairs of complex conjugate roots. The dynamics can now be described in terms of only $m < n$ effective spins governed by Hamiltonian (11.2) with m spins and m new effective energy levels. These *m-spin* solutions contain only m incommensurate frequencies. One frequency corresponds to a uniform rotation of all spins around the z-axis. Thus, $|\Delta(t)|$ contains $m-1$ frequencies [9].

In the thermodynamic limit some roots of $\mathbf{L}^2(u)$ merge into continuous lines and give rise to the continuum part of the spectrum, while isolated pairs of roots correspond to the discrete part.

Thus, the number of discrete frequencies in $|\Delta(t)|$ is the number of isolated pairs of roots less one, $k = m - 1$ (think about it, I don't understand the idea yet).

At large times $|\Delta(t)|$ exhibits persistent oscillations with k frequencies[17] and is described by an m-spin solution[16]. The number k is a topological property of the initial state. It is the number of handles on the Riemann surface of the function $\sqrt{\mathbf{L}^2(u)}$.

Discrete part of the frequency spectrum turns out to be related to discontinuities of the spin distribution $\mathbf{s}(\epsilon)$ as a function of ϵ. To see this, consider first stationary states. There are two types of such states. The BCS ground state and excited states with a constant $\Delta \neq 0$ are obtained by aligning each spin $\mathbf{s}_j$ self-consistently along its effective magnetic field $\mathbf{b}_j$. These states can be termed *anomalous* stationary states. Choosing the x-axis so that Δ is real, we obtain

$$2s_j^z = -\frac{e_j\epsilon_j}{\sqrt{\epsilon_j^2+\Delta^2}} \quad 2s_j^x = -\frac{e_j\Delta}{\sqrt{\epsilon_j^2+\Delta^2}}, \tag{11.6}$$

where $e_j = 1$ if the spin is antiparallel to the field and $e_j = -1$ otherwise. The self-consistency condition $\Delta = g\sum_j s_j^x$ yields the BCS gap equation. The state with all $e_j = 1$ is the BCS ground state. The state $e_k = -1$ and $e_{j\neq k} = 1$ is a state with a single excited pair[11] of energy $2\sqrt{\epsilon_k^2+\Delta_0^2}$. Using Eqs. (11.5, 11.6) and the gap equation, we derive

$$\mathbf{L}_s(u) = -(\Delta\hat{\mathbf{x}} + u\hat{\mathbf{z}})L_s(u), \tag{11.7}$$

$$L_s(u) = \sum_j \frac{e_j}{2(u-\epsilon_j)\sqrt{\epsilon_j^2+\Delta^2}} \tag{11.8}$$

We see that $\mathbf{L}_s^2(u) = (u^2+\Delta^2)L_s^2(u)$ has a pair of isolated roots at $u = \pm i\Delta$. All other roots are determined by the equation $L_s(u) = 0$ and are double degenerate.

First, consider the ground state, $e_j = 1$. Since $L_s(u) \to \pm\infty$ as $u \to \epsilon_j \pm 0$ for each j, all roots of $L_s(u)$ are real and located between consecutive ϵ_j. In the thermodynamic limit they merge into a line from $-D$ to D as shown in Fig. 11.1a. Note that the existence of a double real root between ϵ_j and ϵ_{j+1} relies on $e_j = e_{j+1}$. Further, Eq. (11.6) implies that when $e_j = e_{j+1}$, the components of spins $\mathbf{s}(\epsilon)$ are continuous at $\epsilon = \epsilon_j$. Now let $e_j = -e_{j+1}$. In this case the real root between ϵ_j and ϵ_{j+1} can disappear. Thus, discontinuities (jumps) in the spin distribution generate isolated complex roots (see Fig. 11.3a for an example). Since spins far from the Fermi

level are not flipped, the total number of jumps is even. In general, for $2p$ jumps $L_s(u)$ can have up to p pairs of isolated roots.

One can study equations of motion (11.3) linearized around anomalous stationary states. Setting $\delta\mathbf{s}_j = \mathbf{A}_j e^{i\omega_k t}$, we solve for the normal modes. The eigenvalues turn out to be $\omega_j = 2\sqrt{u_j^2 + \Delta^2}$, where u_j are the roots of $\mathbf{L}_s^2(u)$. For the ground state $x_j = \epsilon_j$ up to finite size corrections (cf. Ref. [11]).

Next, consider few examples of initial states far from equilibrium. Let the system be in an anomalous stationary state for $t < 0$. Suppose at $t = 0$ the coupling changes abruptly from g' to g. Using Eq. (11.5), one can show that the change in g results in a smooth deformation of the root distribution. Lines of roots deform into lines. On the other hand, doubly degenerate roots become non-degenerate. A state that had p pairs of degenerate isolated roots in addition to a pair of roots $\pm i\Delta$ now has $m = 2p + 1$ pairs of non-degenerate roots, i.e. $2p + 1$ cuts of $\sqrt{\mathbf{L}^2(u)}$. As shown above, in this case $|\Delta(t)|$ exhibits persistent oscillations with $k = m-1 = 2p$ frequencies. This behavior is illustrated in Fig. 11.3c.

Let the initial state be the ground state with coupling g'. Then, the line of double real roots splits into two complex conjugate lines (Fig. 11.1b). There is only one pair of isolated roots as in the ground state. Therefore, $k = 0$ and $|\Delta(t)|$ *asymptotes to a constant* Δ_∞ at $t \gg \tau_\Delta$, as illustrated in Fig. 11.1c. According to Eq. (11.3), at large times spin $\mathbf{s}_j$ rotates in a constant magnetic field $\mathbf{b}_j = (2\Delta_\infty, 0, 2\epsilon_j)$ with a frequency $\omega(\epsilon_j) = 2\sqrt{\epsilon_j^2 + \Delta_\infty^2}$. Using this, one derives Eq. (11.1). The $1/\sqrt{t}$ decay law is set by the square root singularity in the spectral density[15]. Note that, even though $|\Delta(t)|$ asymptotes to a constant, the final state of the system is non-stationary. In the final state each spin precesses with its own frequency[36].

There is another type of stationary states – *normal* states. In these states each spin is aligned along the z-axis, $s_j^z = z_j/2 = \pm 1/2$. The Fermi state is $z_j = -\operatorname{sgn}\epsilon_j$ (levels below the Fermi energy are occupied, above – empty). States with other z_j correspond to particle-hole excitations of the Fermi gas. For example, a state $z_k = \operatorname{sgn}\epsilon_k < 0$ has a pair of fermions removed from the level ϵ_k. The Lax vector for normal states is $\mathbf{L}_n(u) = L_n(u)\hat{\mathbf{z}}$, where

$$L_n(u) = -\frac{1}{g} + \sum_j \frac{z_j}{2(u - \epsilon_j)} \tag{11.9}$$

All roots of $\mathbf{L}^2(u) = L_n^2(u)$ are thus doubly degenerate. Note the absence of a branch cut of $\sqrt{\mathbf{L}^2(u)}$ connecting the points $u = \pm i\Delta$. Further analysis is similar to that for anomalous states. The Fermi state has a single jump in the $s_z(\epsilon)$ at the Fermi level. This results in a pair of complex conjugate isolated roots (Fig. 11.2a), which can be determined from the equation $L_n(u) = 0$. In the thermodynamic limit, we obtain $u = \pm i\Delta_0/2$. The rest of the roots are real and form a line from $-D$ to D. For a general normal state with $2p+1$ jumps[18] in $s_z(\epsilon)$, one can have up to $p+1$ complex conjugate pairs of roots. Each root is doubly degenerate. Linearizing Eqs. (11.3) around a normal state, we obtain normal frequencies $\omega_j = 2u_j$, where $L_n(u_j) = 0$. In particular, the Fermi state has a single unstable mode that corresponds to $u_j = \pm i\Delta_0/2$. This mode grows as $e^{\Delta_0 t}$ indicating the pairing instability of the Fermi state[2, 3]. Remaining frequencies are real and correspond to the precession of spins at their natural frequencies, $\omega_k = 2\epsilon_k$ up to finite size corrections.

Let the system be in or close to a normal state at $t = 0$. First, we consider the Fermi state. Since, within mean-field, normal states are unstable equilibria, a small perturbation is needed to start off the dynamics. One can start e.g. from a non-stationary spin distribution close to the Fermi state[17]. A typical deviation splits all double degenerate roots as illustrated in Fig. 11.2b. Real roots split into two complex conjugate lines close to the real axis. Their contribution to $|\Delta(t)|$ is small in the deviation from the Fermi state and decaying. Degenerate complex roots at $u = \pm i\Delta_0/2$ split into $m = 2$ isolated cuts close to each other. Since $k = m - 1 = 1$ in Eq. (11.4), $|\Delta(t)|$ will exhibit undamped periodic oscillations, as shown in Fig 11.2c. Its

functional form is described by a 2-spin solution[16]. The roots close to the real axis indicate that the initial spin distribution does not match the 2-spin solution exactly (see the discussion of m-spin solutions above).

The dynamics in normal states can also be triggered by quantum fluctuations. In Ref. [36] this is modelled by adding a small *external* field $-2\Delta_{QF}\hat{\mathbf{x}}$ to Eq. (11.3). This violates conservation of $\mathbf{L}^2(u)$. We have $d\mathbf{L}^2(u)/dt = 2\Delta_{QF}L_y/g$. Being applied for a short time t^*, the external field drives the system out of the normal state. Treating the evolution of $\mathbf{L}^2(u)$ perturbatively, we find that the new positions of the roots are determined by the equation $L_n(u) = \pm 2i\Delta_{QF}t^*/g$. Degenerate complex roots split along the real axis into two cuts by $2\Delta_{QF}^2 t^* d/g$, where $d = \langle \epsilon_{j+1} - \epsilon_j \rangle$ is the level spacing. The resulting root configuration shown in Fig. 11.2b contains two cuts. Thus, $k = 1$ and $|\Delta(t)|$ exhibits periodic oscillations, see Fig. 11.2c.

We see that for initial conditions close to the Fermi state a periodic solution is "dynamically selected"[17]. Other initial conditions "select" damped oscillations or multi-frequency undamped oscillations. In more conventional terms, this corresponds to a basic fact that the evolution of an integrable system depends on initial conditions. All diferent behaviors are captured by the general solution[8]. Here we systematically classified possible initial states and specialized the general solution to each type of initial conditions, i.e. we developed "selection rules" for the BCS dynamics. Which behavior is realized in a particular experimental setup depends on the initial state of the condensate. In this respect, the periodic solution is somewhat special: if we start from the ground state with a small $\Delta_{in} \ll \Delta_0$, the order parameter $|\Delta(t)|$ asymptotes to a constant value Δ_∞, see Eq. (11.1). In the thermodynamic limit, the damping disappears only when $\Delta_{in} = 0$[21].

Excited normal states have several jumps in the spin distribution and can therefore display oscillations with more than one frequency. Consider, e.g., a state where pairs in energy interval from $-\epsilon_a$ to $-\epsilon_b$ have been removed. The initial spin distribution is $s_z(\epsilon) = -\mathrm{sgn}[\epsilon(\epsilon+\epsilon_a)(\epsilon+\epsilon_b)]$. The $2p+1 = 3$ jumps[18] result in $p+1 = 2$ pairs of isolated double degenerate complex roots. As a small perturbation splits these roots into $m = 2p+2 = 4$ cuts, $|\Delta(t)|$ oscillates with $k = m-1 = 2p+1 = 3$ incommensurate frequencies, see Fig. 11.4.

In conclusion, we have shown how the non-stationary BCS dynamics can be predicted from the initial state of the condensate. We classified initial states by their integrals of motion – roots of $\mathbf{L}^2(u)$. For states with a root diagram as in the paired ground state, the order parameter $|\Delta(t)|$ displays damped oscillations described by Eq. (11.1). Other states are of the same type as excited stationary states of the BCS Hamiltonian. In these cases $|\Delta(t)|$ oscillates persistently with few incommensurate frequencies. The number of frequencies is related to the number of jumps in the pseudospin distribution of the corresponding stationary state. For the situation most relevant to the experiments on cold fermions – an abrupt change of the coupling – we predict damped oscillations with $1/\sqrt{t}$ decay.

Figure 11.1: **a)** Roots u_j of $\mathbf{L}^2(u)$ for the BCS ground state of $n = 100$ spins with equilibrium gap Δ_0'. Axes: $\mathrm{Im}(u_j)/\Delta_0'$ and $\mathrm{Re}(u_j)/d$, where d is the level spacing. Insets: spin distribution $s^z(\epsilon)$ and $s^x(\epsilon)$ ($s^y(\epsilon) = 0$). The absence of discontinuities in the spin distribution ($2p = 0$) implies a single pair of isolated roots at $\pm i\Delta_0'$ (red circles connected by a dashed line). Remaining $2n - 2 = 198$ roots (solid green circles) are real, doubly degenerate, and located between consecutive energy levels ϵ_j (ticks on the real axis).

b) At $t = 0$ the coupling constant is abruptly increased so that the corresponding ground state gap is $\Delta_0 = 2.4\Delta_0'$, while the spin configuration remains the same as in part a). The line of double real roots deforms into two complex conjugate lines. The Riemann surface of $\sqrt{\mathbf{L}^2(u)}$ has $m = 2p + 1 = 1$ isolated cuts (red circles connected by a dashed line). This nonequilibrium state has the same number of isolated roots as the BCS ground state and, therefore, is in the same topological class. The frequency spectrum of $|\Delta(t)|$ is thus continuous: $k = m - 1 = 2p = 0$, i.e. no discrete frequencies in Eq. (11.4).

c) Time evolution of $|\Delta(t)|$ after the change $\Delta_0' \to \Delta_0$. In the absence of discrete frequencies, the order parameter $|\Delta(t)|$ asymptotes to a constant value Δ_∞ (see Eq. (11.1)).

Figure 11.2: **a)** Roots u_j of $\mathbf{L}^2(u)$ for the Fermi state of $n = 100$ spins with ground state gap Δ_0. Axes: $\mathrm{Im}(u_j)/\Delta_0$ and $\mathrm{Re}(u_j)/d$, where d is the level spacing. Insets: spin distribution $s^z(\epsilon) = -\mathrm{sgn}\,\epsilon/2$ ($s^x(\epsilon) = s^y(\epsilon) = 0$). There is $2p+1 = 1$ discontinuity (a jump in $s^z(\epsilon)$ at the Fermi level) in the spin distribution resulting in $p+1 = 1$ pair of imaginary doubly degenerate roots $u_j = \pm i\Delta_0/2$ (blue crosses). Remaining $2n-4 = 196$ roots (solid green circles) are real, doubly degenerate and located between consecutive energy levels ϵ_j (ticks on the real axis). Since all spins are along the z-axis, this state is stationary (see Eq. (11.3)). The normal frequencies are $\omega_j = 2u_j$. The mode corresponding to $u_j = \pm i\Delta_0/2$ is unstable, it grows as $e^{\Delta_0 t}$.

b) Initially the system is in the Fermi state. A small "external magnetic field" $-2\Delta_{QF}\hat{\mathbf{x}} = -5.4 \times 10^{-2}\Delta_0\hat{\mathbf{x}}$ is added to Eq. (11.3) for a time $t^* = 1/\Delta_0$. The $p+1 = 1$ pair of double roots $u_j = \pm i\Delta_0/2$ splits into $m = 2p+2 = 2$ pairs of isolated roots (red circles connected by dashed lines); the line of real roots splits into two complex conjugate lines. This nonequilibrium state has the same number of isolated roots as the Fermi state and, therefore, belongs to the same topological class. The Riemann surface of $\sqrt{\mathbf{L}^2(u)}$ has $m = 2p+2 = 2$ isolated cuts. The frequency spectrum of $|\Delta(t)|$ has $k = m-1 = 2p+1 = 1$ discrete frequency in Eq. (11.4).

c) Time evolution of $|\Delta(t)|$ for the initial state described in part b). Since there is one discrete frequency, $|\Delta(t)|$ exhibits periodic oscillations. Its functional form is described by an $(m = 2)$–spin solution (see the text below Eq. (11.5)).

Figure 11.3: **a)** Roots u_j of $\mathbf{L}^2(u)$ of $n = 100$ spins in an excited stationary state. This *anomalous* state (see the text) has been obtained from the ground state with the gap Δ_0' by flipping spins in energy interval $(-0.37\Delta_0', 0)$. It has a gap $\Delta = 0.4\Delta_0'$. Axes: $\mathrm{Im}(u_j)/\Delta_0'$ and $\mathrm{Re}(u_j)/d$, where d is the level spacing. Insets: spin distribution $s^z(\epsilon)$ and $s^x(\epsilon)$ ($s^y(\epsilon) = 0$). Spin flips result in $2p = 2$ discontinuities in the spin distribution. Therefore, there is a $p = 1$ pair of isolated doubly degenerate roots (blue crosses) in addition to a pair of roots $u_j = \pm i\Delta$ (red circles connected by a dashed line). The remaining $2n - 6 = 194$ roots (solid green circles) are real, doubly degenerate, and located between consecutive energy levels ϵ_j (ticks on the real axis).

b) At $t = 0$ the coupling constant is abruptly increased so that the corresponding ground state gap is $\Delta_0 = 1.55\Delta_0'$, while the spin configuration remains the same as in part a). The $p = 1$ pair of isolated double roots splits into $2p = 2$ pairs of isolated roots. Together with a pair of roots coming from $u_j = \pm i\Delta$, there are $m = 2p + 1 = 3$ pairs of isolated roots (red circles connected by dashed lines). The line of double real roots deforms into two complex conjugate lines. The Riemann surface of $\sqrt{\mathbf{L}^2(u)}$ has $m = 2p + 1 = 3$ isolated cuts. This nonequilibrium state has the same number of isolated roots as the excited state in part a) and therefore is in the same topological class. The frequency spectrum of $|\Delta(t)|$ has $k = m-1 = 2p = 2$ discrete frequencies.

c) Time evolution of $|\Delta(t)|$ after the change $\Delta_0' \to \Delta_0$. Since there are $k = m - 1 = 2p = 2$ discrete frequencies in Eq. (11.4), the order parameter $|\Delta(t)|$ displays oscillations with two basic frequencies. Its functional form is described by an $(m = 3)$–spin solution (see the text below Eq. (11.5)).

Figure 11.4: **a)** Roots u_j of $\mathbf{L}^2(u)$ for an excited stationary state of $n = 100$ spins with ground state gap Δ_0. Axes: $\mathrm{Im}(u_j)/\Delta_0$ and $\mathrm{Re}(u_j)/d$, where d is the level spacing. This *normal* state (see the text) has been obtained from the Fermi state by flipping spins in energy interval $(-0.5\Delta_0, -\Delta_0)$. Insets: spin distribution $s^z(\epsilon)$ ($s^x(\epsilon) = s^y(\epsilon) = 0$). There are $2p + 1 = 3$ discontinuities (three jumps in $s^z(\epsilon)$) in the spin distribution resulting in $p + 1 = 2$ pairs of isolated double roots $u_j = \mu_{1,2} \pm i\gamma_{1,2}$ (blue crosses). Remaining $2n - 8 = 192$ roots (solid green circles) are real, doubly degenerate and located between consecutive energy levels ϵ_j (ticks on the real axis). Since all spins are along the z-axis, this state is stationary (see Eq. (11.3)). The normal frequencies are $\omega_j = 2u_j$. There are two unstable modes corresponding to $u_j = \mu_{1,2} \pm i\gamma_{1,2}$, which grow as $e^{2\gamma_1 t}$ and $e^{2\gamma_2 t}$.

b) Initially the system is in the stationary state described in part a). A small "external magnetic field" $-2\Delta_{QF}\hat{\mathbf{x}} = -5.4 \times 10^{-2}\Delta_0\hat{\mathbf{x}}$ is added to Eq. (11.3) for a time $t^* = 1/\Delta_0$. The $p + 1 = 2$ pairs of degenerate roots $u_j = \mu_{1,2} \pm i\gamma_{1,2}$ split into $m = 2p + 2 = 4$ pairs of isolated roots (red circles connected by a dashed line), while the line of real roots splits into two complex conjugate lines. This nonequilibrium state has the same number of isolated roots as the stationary state in part a) and, therefore, belongs to the same topological class. The Riemann surface of $\sqrt{\mathbf{L}^2(u)}$ has $m = 2p+2 = 4$ isolated cuts. Insets: Fourier spectra of $\mathrm{Re}\Delta(t)$ and $\mathrm{Im}\Delta(t)$ display $m = 2p + 2 = 4$ basic frequencies. One frequency (ν_0) corresponds to a uniform rotation of all spins around the z-axis and cancels out in $|\Delta(t)|$. Thus, the frequency

spectrum of $|\Delta(t)|$ has $k = m - 1 = 2p + 1 = 3$ discrete frequencies.

c) Time evolution of $|\Delta(t)|$ for the initial state described in part b). Since $k = m - 1 = 2p + 1 = 3$ in Eq. (11.4), $|\Delta(t)|$ exhibits quasiperiodic oscillations with three basic frequencies (see insets in Fig. 11.4b). Its functional form is described by an $(m = 4)$–spin solution (see the text below Eq. (11.5)).

12 Nonequilibrium dynamics of weakly and strongly paired superconductors by Gurarie

(maybe the article reports a good analytical method of calculations of what is already known???)

Abstract

We study small oscillations of the order parameter in weakly and strongly paired superconductors driven slightly out of equilibrium, in the collisionless approximation.

While it was known for quite some time that the amplitude of the oscillations in a weakly paired superconductor decays as $t^{-1/2}$, we show that in a superconductor sufficiently strongly paired so that its fermions form bound states usually referred to as molecules, these oscillations decay as $t^{-3/2}$.

The transition between these two regimes happens when the chemical potential of the superconductor vanishes, thus the behavior of the oscillations can be used to distinguish weakly and strongly paired superconductors.

These results are obtained in the mean field approximation which may not be reliable in the crossover region between the strong and weak pairing, so we also obtain identical results within the two-channel model, which can be tuned to be reliable throughout the entire crossover, although it then describes a special type of interactions between the fermions which may be difficult to observe experimentally.

(so the author is not sure that his results are relevant??)

Finally, we interpret the result in the strongly paired superconductor as the probability of the molecular decay as a function of time.

The study of quantum quenches, or the evolution of a quantum system when its parameters suddenly change, acquired prominence due to recent studies in Refs. [Calabrese and Cardy(2006), Kollath et al.(2007)].

Yet a particular case of it, the study of small oscillations of the order parameter in a perturbed superconductor, started more than 30 years ago in Ref. [Volkov and Kogan(1973)].

This paper showed that if a small perturbation is applied to a superconductor leading to a deviation of its gap from its equilibrium value, then this perturbation will evolve as $\cos(2\Delta_0 t)/\sqrt{t}$, where Δ_0 is the equilibrium value of the gap.

This work was continued in Ref. [Barankov et al.(2004)]. These authors showed that large oscillations of the order parameter in a superconductor do not decay but in fact continue forever (as long as the collisionless approximation remains valid).

Subsequently with the help of the exact integrability of this problem [Dukelsky et al.(2004)] it was shown that the exist a critical strength of the perturbation such that if a perturbation beyond that strength is applied, it excites oscillations which continue forever while perturbations below that strength still decay, Ref. [Barankov and Levitov(2006), Yuzbashyan and Dzero(2006)].

At the same time, advances in atomic physics allow now to create superconductors out of ultracold fermionic atoms whose interactions can be externally controlled. If the interactions are weak, these artificial superconductors behave just like the usual superconductors of condensed matter physics (termed BCS superconductors).

But the interactions can also be adjusted to be strong, in which case these superconductors become more akin to a Bose condensate (BEC) of diatomic bosonic molecules. As the interactions are tuned, the superconductor is said to undergo a BCS-BEC crossover (which is accompanied by the chemical potential changing from positive to negative values). The possibility of this crossover was discussed in a number of papers throughout the last four decades (see Refs. [Eagles(1969), Leggett(1980), Nozières and Schmitt-Rink(1985)]), and it finally was observed a few years ago (see Refs. [Regal et al.(2004), Zwierlein et al.(2004)]).

The unprecedented control over the interactions that the ultracold gases provide allows to change them quickly thus easily creating an initial out-of-equilibrium perturbation by a sudden change of their strength.

A natural question which arises in this regard concerns the fate of the oscillations, decaying and persistent, as the superconductor undergoes the BCS-BEC crossover.

Refs. [Andreev et al.(2004), Barankov and Levitov(2004)] explored the large nondecaying oscillations of a tunable superconductor using the ansatz of Ref. [Barankov et al.(2004)].

However, this method was found to break down as the superconductor is tuned to the BEC regime, and it was not possible to tell whether this was an artifact of the technique or an indication that an oscillating solution is indeed not possible in a strongly paired superconductor (see, however, Ref. [Holland et al.(2001)]).

To remedy this situation, in this paper we consider small oscillations of the order parameter in a tunable superconductor. We show that as long as the oscillations remain small, they always decay regardless of the strength of interactions in the superconductor. However, while in the BCS superconductor the amplitude of these oscillations decay as $1/\sqrt{t}$, in the BEC regime they decay as $1/t^{3/2}$. The transition between the two regimes happens exactly where the chemical potential μ is equal to zero.

The theory we develop is based on the mean field (collisionless) approximation. Thus one can question its validity near the unitary point [Giorgini et al.(2008)] which lies between the BCS and BEC regimes of the superconductors (at positive chemical potential) and is known to go beyond the applicability of the mean field theory [Levinsen and Gurarie(2006)] (see also Ref. [Bulgac and Yoon(2009)] for recent studies of the dynamics near this point). Therefore, we repeat the calculation in the two channel model [Timmermans et al.(1999)] describing BCS-BEC crossover with narrow Feshbach resonance [Gurarie and Radzihovsky(2007)] where mean field theory is applicable throughout and recover the same result. This shows that an example of the BCS-BEC crossover exists whose order parameter oscillations unambiguously obey the scenario discussed here.

In the “deep BEC" side of the crossover, the superconductor becomes the Bose-Einstein condensate of the diatomic bosonic molecules. In this regime, the result $1/t^{3/2}$ has a very simple interpretation. Indeed, as we will see in this paper, the order parameter of such superconductor is proportional to the probability of finding two fermions of opposite spins at the same position in space. As the superconductor is perturbed, the wave function of a pair of fermions no longer coincides with their bound state, but rather consists of the linear combination of a bound and excited states. The part of the wave function in the excited states moves off to infinity so that the probability amplitude to find two particles in the same spot decreases as $1/t^{3/2}$ reaching a

limiting finite value at large t which in turn follows from the behavior of a three dimensional propagator of a free particle. In fact, in this deep BEC regime this is true not only with a small but also with a relatively large initial perturbation (the precise criteria are developed below), in which case this picture makes it clear that the order parameter decays to a value smaller than the equilibrium value, just like in the BCS regime with certain perturbations [Yuzbashyan and Dzero(2006)]. However, to fully analyze the case of large perturbations to the superconductor and to see whether nondecaying solutions are possible in the BEC regime, one needs to take advantage of the integrability of the equations of motion, something which was done successfully for the weakly paired superconductor and whose strongly paired superconductor applications are left for future work.

(what is the method which they use???)

In what follows, we describe the derivation of these results. Consider spin-1/2 fermions interacting via a short range attractive s-wave interaction of strength λ. Within the mean field approximation, these fermions can be studied on a strictly classical level. We accomplish this by introducing the Cooper pair number $n_p = \left\langle \hat{a}^\dagger_{\uparrow p} a_{\uparrow p} + \hat{a}^\dagger_{\downarrow p} a_{\downarrow p} \right\rangle /2$. The evolution of this number obeys the following classical Hamiltonian, a classical version of the Anderson-Richardson (or reduced BCS) Hamiltonian describing these kind of fermions,

$$H = 2\epsilon_p n_p - \frac{\lambda}{V} \sum_{p,q} \sqrt{n_p(1-n_p)n_q(1-n_q)} \cos(\phi_p - \phi_q), \tag{12.1}$$

where $\epsilon_p = p^2/(2m)$ is the free fermions dispersion, ϕ_p is the phase variable canonically conjugate to n_p, and V is the space volume. The equations of motion of this Hamiltonian have the following solution, corresponding to a stationary superconductor

$$\phi^0_p = -2\mu t, n^0_p = \frac{1}{2}\left(1 - \frac{\xi_p}{E_p}\right), \tag{12.2}$$

where as always μ is the chemical potential, t is time, $\xi_p = \epsilon_p - \mu$, $E_p = \sqrt{\xi_p^2 + \Delta_0^2}$, and finally Δ_0 is the equilibrium gap of the superconductor which can be found from the gap equation $1 = \frac{\lambda}{2V}\sum_p \frac{1}{E_p}$. Now consider the initial conditions for the motion described by Eq. (12.1) consisting of a small perturbation to the stationary solution Eq. (48.37). In fact, it is most physical to take as initial conditions Eq. (48.37) for a slightly different value of $\Delta = \Delta_0 + \delta\Delta_0$ as well as the slightly different value of μ (corresponding to a superconductor whose interactions were slightly perturbed in the initial moment of time). Then it is straightforward to see, with the help of the particle conservation condition $\sum_p \delta n_p = 0$, that the initial conditions read

$$n_p = n^0_p + \delta n^0_p, \;\; \delta n^0_p = \left(\xi_p - \frac{\tilde{f}_0}{f_0}\right) \frac{\Delta_0 \delta\Delta_0}{2E_p^3}, \tag{12.3}$$

where f_0 and $\tilde{f}_0$ are defined below in Eq. (12.8). We then expand the Hamiltonian Eq. (12.1) about the stationary solution to obtain the quadratic Hamiltonian for the deviations δn_p, $\delta\phi_p$. Subsequently we construct the solution to the equations of motion of this Hamiltonian by using the method of Green's functions. The expanded Hamiltonian takes the form

$$\delta H = \frac{1}{2}\sum_{p,q} \left[\delta\phi_p \Phi_{pq} \delta\phi_q + \delta n_p K_{pq} \delta n_q\right], \tag{12.4}$$

where

$$\Phi_{pq} = \frac{\Delta_0^2}{E_p}\delta_{pq} - \frac{\lambda}{2V}\frac{\Delta^2}{E_p E_q}, K_{pq} = \frac{4E_p^3}{\Delta_0^2}\delta_{pq} - \frac{2\lambda\xi_p\xi_q}{V\Delta_0^2}. \tag{12.5}$$

This represents a collection of harmonic oscillators, labelled by the index p. Now it is possible to construct a retarded Green's function corresponding to these oscillators, with the end result

$$\delta n_p(t) = i \sum_q \int_{-\infty+i0}^{\infty+i0} \frac{\Omega d\Omega}{2\pi} G_{pq}(\Omega)\, e^{-2i\Omega t}\, \delta n_q^0, \tag{12.6}$$

where

$$G_{pq}(\Omega) = \frac{\delta_{pq}}{\Omega^2 - E_p^2} + \frac{\lambda}{2V} \frac{f\left(\Omega^2 \xi_p \xi_q + E_q^2\left(\Omega^2 - \Delta_0^2\right)\right) - \tilde{f}\left(\xi_p E_q^2 + \xi_q \Omega^2\right)}{\Omega^2 E_p \left(\Omega^2 - E_p^2\right)\left(\Omega^2 - E_q^2\right)\left[\tilde{f}^2 + f^2\left(\Delta_0^2 - \Omega^2\right)\right]}.$$

Here we introduced the functions

$$f = \frac{\lambda}{2V} \sum_p \frac{1}{E_p\left(\Omega^2 - E_p^2\right)}, \quad \tilde{f} = \frac{\lambda}{2V} \sum_p \frac{\xi_p}{E_p\left(\Omega^2 - E_p^2\right)}. \tag{12.7}$$

In what follows we will also need these same functions evaluated at $\Omega = 0$, or

$$f_0 = -\frac{\lambda}{2V} \sum_p \frac{1}{E_p^3}, \quad \tilde{f}_0 = -\frac{\lambda}{2V} \sum_p \frac{\xi_p}{E_p^3}. \tag{12.8}$$

The derivation of Eq. (12.7) is technical and not particularly instructive, so it will be published elsewhere.

Armed by the explicit expression for the fluctuations $\delta n_p(t)$ we can now discuss how to calculate the time dependent gap function. It is easiest to study the square of its absolute value,

$$|\Delta(t)|^2 = \frac{\lambda^2}{V^2} \sum_{p,q} \sqrt{n_p(1-n_p) n_q(1-n_q)} \cos(\phi_p - \phi_q). \tag{12.9}$$

In turn, this quantity can be decomposed into the sum of Δ_0^2, the square of the unperturbed gap, and the perturbation $\delta\,|\Delta|^2$, given by

$$\delta\,|\Delta(t)|^2 = \frac{2\lambda}{V} \sum_p \xi_p\, \delta n_p(t). \tag{12.10}$$

We combine this with the initial conditions Eq. (12.3) as well as with Eqs. (12.6) and (12.7) to find that

$$\delta\,|\Delta(t)|^2 = \frac{i\Delta_0 \delta\Delta_0}{\pi} \int_{-\infty+i0}^{\infty+i0} \Omega d\Omega\, e^{-2i\Omega t} \frac{\tilde{f}^2 + f^2\left(\Delta_0^2 - \Omega^2\right) - f f_0 \Delta_0^2 - f \tilde{f}_0^2 / f_0}{\Omega^2\left[\tilde{f}^2 + f^2\left(\Delta_0^2 - \Omega^2\right)\right]}.$$

This equation represents the main result of this paper.

It reduces to the main result of Ref. [Volkov and Kogan(1973)] at weak pairing.
(produce it!!!!!!! read the Kogan1973)
So, it represents the generalization of their result to a superconductor of an arbitrary pairing strength.
The integration here goes over the straight line slightly above the real axis, as shown on Fig. 12.1 with a dashed line.

Figure 12.1: The contour of integration in Eq. (12.11) in the complex plane of Ω. Dashed line shows the initial contour, while solid black line, together with a dashed-dotted semicircle, is the deformed contour. The dashed-dotted part of the new contour can be deformed away to infinity and does not contribute.

We expect the functions f and $\tilde{f}$ to have singularities on the real axis where $|\Omega|$ hits the minimum of E_p and cuts at larger $|\Omega|$. It may seem that there is also a singularity at $\Omega = 0$, but it is actually fictitious as the numerator of Eq. (12.11) vanishes at $\Omega \to 0$. At $t < 0$ the contour can be closed in the upper half plane resulting in zero for the integral as expected. At positive time $t > 0$ we can deform the contour as shown on Fig 12.1 by a solid line with arrows. Then at large positive times, the main contribution to the integral comes from the vicinity of the singular points, or the turning points of the new contour.

To evaluate f and $\tilde{f}$, we replace summation over p in Eq. (12.7) by the integration over $Vd^3p/(2\pi)^3$. In the BCS regime, it is standard to pass to the integration over the variable $\xi = \epsilon_p - \mu$ so that $d^3p/(2\pi)^3 \approx \nu d\xi$, where $\nu = \sqrt{m^3\mu}/(\pi^2\sqrt{2})$ is the density of states, and then extend the integration over ξ all the way to the entire real axis. Then $\tilde{f} = 0$ as its integral is antisymmetric in ξ. At the same time, f is given by

$$f = i\frac{\nu\lambda}{\Omega\sqrt{\Delta_0^2 - \Omega^2}} \ln\left[\sqrt{1 - \frac{\Omega^2}{\Delta_0^2}} + i\frac{\Omega}{\Delta_0}\right]. \tag{12.11}$$

It is then easy to see that the integrand in Eq. (12.11) has a singularity as Ω approaches Δ_0 which goes as $1/\sqrt{\Omega - \Delta_0}$ in agreement with the discussion above (Δ_0 coincides with the minimum of E_p). Introducing the variable $s = \Omega - \Delta_0$ and taking into account that at $t \gg 1/\Delta_0$ only small s contributes to the integral, we find after adding both left and right contours (which are complex conjugate of each other)

$$\delta\,|\Delta(t)|^2 \sim \mathrm{Re}\; e^{-2i\Delta_0 t}\int_0^\infty \frac{dse^{-ist}}{\sqrt{s}} \sim \frac{\cos(2\Delta_0 t)}{\sqrt{t}}. \tag{12.12}$$

Alternatively, in the BEC regime where $\mu < 0$, the integration measure in Eq. (12.7) must be kept as $Vd^3p/(2\pi)^3$ and both functions f and $\tilde{f}$ are not zero. The minimum of E_p is equal to $E_{\min} = \sqrt{\mu^2 + \Delta_0^2}$. The functions f and $\tilde{f}$ are now finite when $|\Omega|$ reaches $E_{\min}$, but they are not regular at that point. Generally they behave as $f \sim \text{const} + i\sqrt{\Omega - E_{\min}}$. The origin of such a drastically different behavior lies in the fact that the minimum of E_p is reached when $p = 0$. Then, the potential divergence in the integral at $\Omega = E_{\min}$ is removed by the p^2 coming from the measure of integration p^2dp. All this can be checked explicitly in the deep BEC regime where $|\mu| \gg \Delta_0$ and the analytic expressions behaving precisely in this way can be obtained by direct integration.

Now if we substitute this into Eq. (12.11) we find that the constant piece is single valued as Ω goes from the upper to lower branch of the contour on Fig. 12.1 and therefore its contribution

from each of the contours cancels. So it is the square root part which contributes. This gives at large times $t \gg 1/E_{\min}$

$$\delta\left|\Delta\right|^2 \sim \mathrm{Re}\; e^{-2itE_{\min}} \int ds\, e^{-ist}\sqrt{s} \sim \frac{\cos(2tE_{\min})}{t^{\frac{3}{2}}}. \tag{12.13}$$

As we see, the transition from the behavior Eq. (48.13) to Eq. (12.13) occurs exactly at $\mu = 0$ as this is the point where the minimum of the spectrum E_p shifts to $p = 0$.

One can worry if this derivation really works for a superconductor in the vicinity of the unitary point where the mean field theory breaks down. To have a controllable theory, we repeated this calculation for the two channel model, based on the equations of motion introduced in Ref. [Andreev et al.(2004)]. That model has an additional coupling g such that if $g^2m^2/n^{1/3} \gg 1$ (here n is the density), then the two channel model is equivalent to Eq. (12.1) considered earlier in this paper. In the opposite limit, the two channel model still undergoes a crossover but with the mean field theory applicable throughout. The answer for the two channel model can be worked out using the same methods as the ones described here. It looks almost identical to Eq. (12.11), except in all occurences of $\tilde{f}$ and $\tilde{f}_0$ one needs to substitute $\tilde{f} \to \tilde{f} - 2\lambda/g^2$, $\tilde{f}_0 \to \tilde{f}_0 - 2\lambda/g^2$. This replacement does not change any of the arguments presented here for the one channel model, thus all the conclusions remain valid.

Finally, we observe that in the deep BEC regime all $n_p^0 \ll 1$. Then the Hamiltonian Eq. (12.1) can be significantly simplified, reducing in this limit to the Schrödinger equation of one pair of fermions in a delta-function potential, as discussed at length in Ref. [Nozières and Schmitt-Rink(1985)]. Replacing $1 - n_p \approx 1$, and introducing the pair wave function $\psi_p = \sqrt{n_p}\, e^{i\phi_p}$, we find $H = 2\epsilon_p \left|\psi_p\right|^2 - \frac{\lambda}{V}\left|\sum_p \psi_p\right|^2$ leading to

$$i\partial_t\psi = 2\epsilon_p\psi_p - \frac{\lambda}{V}\sum_q \psi_q. \tag{12.14}$$

The "gap function" in this language is $\Delta = \lambda\sum_p \psi_p/V$ which has an obvious meaning of the probability of finding two opposite-spin fermions at the same point in space. We can solve the linear equation (12.14) with arbitrary initial conditions directly. The solution for the gap function reads

$$\begin{aligned}\Delta(t) &= \frac{\lambda}{V}\sum_p \int_{-\infty+i0}^{\infty+i0} d\Omega\, e^{-i\Omega t}\frac{1}{1+\Pi}\frac{\psi_p^0}{\Omega - 2\epsilon_p},\\ \Pi &= \frac{\lambda}{V}\sum_p \frac{1}{\Omega - 2\epsilon_p} = \mathrm{const} + \frac{\lambda m^{\frac{3}{2}}\sqrt{-\Omega}}{4\pi}\end{aligned} \tag{12.15}$$

with ψ_p^0 being the initial value of ψ_p. To compute the integral over Ω, we deform the contour of integration in a way similar to that shown on Fig. 12.1. Unlike Eq. (12.11), the integrand here has a simple pole at a negative value of $\Omega = \Omega_b$ where $\Pi(\Omega_b) = -1$, but similarly to Eq. (12.11) in the BEC regime, it has cut at $\Omega > 0$ where it behaves as $\mathrm{const} + i\sqrt{\Omega}$.

The pole corresponds to the bound state and results in a contribution to $\Delta(t)$ corresponding to the bound molecule (that is, the equilibrium solution). The cut produces the decaying behavior going precisely as $1/t^{3/2}$, thus overall $\left|\Delta(t)\right|^2 \approx \mathrm{const} + \cos(\Omega_b t)/t^{3/2}$ (for comparison with Eq. (12.13) observe that $\Omega_b = 2\mu$ in this regime). However now we see that this decay is correct for arbitrary initial conditions, as long as the applicability conditions $\left|\psi_p^0\right| \ll 1$ are satisfied. The value $\left|\Delta(t)\right|^2$ decays to is smaller than the equilibrium value of the gap function, since it is reduced by the overlap, smaller than 1, of the initial state and the bound state. The only initial distributions out of reach of this method are the ones where $\left|\psi_p^0\right|$ are no longer much smaller than 1.

In conclusion, we showed that the amplitude of the small oscillations of the order parameter in a superconductor decays as $1/t^{1/2}$ in the BCS ($\mu > 0$) and as $1/t^{3/2}$ in the BEC ($\mu < 0$) regimes. Applicability of this result to the Fermi gas close to the unitary point remains an open question. It would also be very interesting to see if, with modern experimental techniques, these oscillations can be detected in real superconductors and whether this theory can be generalized to superconductors with the d-wave symmetry of the gap.

13 Synchronization in the BCS Pairing Dynamics as a Critical Phenomenon by Barankov, Levitov

13.1 Theory

Abstract

Fermi gas with time-dependent pairing interaction hosts several different dynamical states. (which???)

Coupling between the collective BCS pairing mode and individual Cooper pair states can make the latter either synchronize or dephase. (why???)

We describe transition from phase-locked undamped (phase-locked undamped?) oscillations to Landau-damped (Landau-damped) dephased oscillations in the collisionless, dissipationless regime as a function of coupling strength.

In the dephased regime, we find a second transition at which the long-time asymptotic pairing amplitude vanishes.

Using a combination of numerical and analytical methods we establish a continuous (type II) character of both transitions. (and what is the coupling constant, at which there is a transition?)

(maybe in this article they show the proper way how to deal with Anderson's formalism? and these are the results if one decides to do it???)

Introduction

Figure 13.1: (this plot is about initial gap, but in the abstract it was about the value of the coupling constant... are they about the same?) Three regimes of the pairing dynamics *vs.* the initial gap value Δ_s (why it is allowed that $\Delta_s = 100\Delta_0$???why B region is righter than the red curve???). In synchronized phase (A), $\Delta_s < \Delta_{AB}$, the pairing amplitude oscillates between Δ_- and Δ_+. In the dephased regime (B, C), the pairing amplitude saturates to a constant value, Δ_a, when $\Delta_{AB} \leq \Delta_s < \Delta_{BC}$, and decreases to zero at $\Delta_s \geq \Delta_{BC}$. Dashed line: The stationary gap value $\Delta(T_*)$ reached in a closed system after equilibration.

Recent discovery of BCS pairing in fermionic vapors [1, 2], made possible by control of interactions in trapped cold gases [3], has renewed interest in quantum collective phenomena [4].

(why????)

Advanced detection techniques and long coherence times in vapors enable time-resolved studies of new collective modes, such as spin waves [5] and the BCS pairing mode [6].

(another articles with a lot of theory about it...)

Interaction between a collective mode and constituting particles is key for our understanding of dynamics in various systems, from plasma to quantum gases.

(not explained, why???)

One of the most surprising of these phenomena is Landau damping which occurs in a collisionless regime via direct dissipationless energy transfer from the collective mode to single particles. Its nondissipative and thus reversible character [7] leads to a variety of regimes, notably to quenching of the damping, first explored in plasma physics [8].

(I need to read about it, never heard about it.)

Remarkably, a linearly damped mode can regrow and transform to a stationary oscillatory Bernstein-Greene-Kruskal mode. This fascinating prediction was confirmed experimentally only recently [9].

(????? never read about it before)

Naturally, the richness of these nonlinear phenomena makes it tempting to look for their analogs in cold gases. Collisionless damping in cold gases was considered, in the linear regime, for (why in all these models we can neglect collisions?) optical excitations [10], spin waves [11, 12], and excitations in optical lattices [13].

Motivated by the work on fermion superfluidity [1, 2, 6], here we focus on the pairing dynamics of fermions [14, 15, 16, 17] induced by a sudden change of interaction.

The collisionless regime becomes practical in this case due to long elastic collision times $\tau_{\rm el} \gg \tau_\Delta = \hbar/\Delta$ [14], where Δ is the BCS gap (is it explained there? I am not sure...).

The pairing mode of a small amplitude oscillates at a frequency $2\Delta/\hbar$ and exhibits collisionless dephasing [24]. (read Volkov's paper again!)

These conclusions were extended recently to the nonlinear regime [19] (where does this regime originate?).

This behavior changes drastically as the perturbation increases. The main result of this work, as summarized in Fig.13.1, is prediction of a dynamical transition resulting from competition between synchronization and collisionless dephasing, taking place as a function of the initial pairing gap, Δ_s.

We found three qualitatively different regimes (A, B, and C) with the critical points at $\Delta_{AB} = e^{-\pi/2}\Delta_0$ and $\Delta_{BC} = e^{\pi/2}\Delta_0$, where Δ_0 is the equilibrium pairing amplitude in the final BCS state. Below the A-B transition, $\Delta_s < \Delta_{AB}$, individual Cooper pair states synchronize and the pairing amplitude oscillates between Δ_+ and Δ_- without damping. In contrast, in the interval $\Delta_{AB} < \Delta_s < \Delta_{BC}$ the pairing amplitude is Landau-damped and exhibits decaying oscilation, saturating at an asymptotic value, Δ_a, with non-monotonic dependence on Δ_s. A second transition occurs at $\Delta_s \geq \Delta_{BC}$, where the dynamics becomes overdamped, and $\Delta(t)$ decreases to zero without oscillations. The oscillation amplitude and the asymptotic value Δ_a vanish continuously at the critical points A-B and B-C, as in a type II transition. We demonstrate that these results are consistent with the spectral analysis [20] based on the integrability of the problem.

We also address the behavior on a long time scale, $t \gtrsim \tau_{\rm el}$, after dissipation sets in. We find that energy relaxation in a closed system, such as an atom trap, makes it evolve to a new equilibrium state. Both the temperature T_* and the gap $\Delta(T_*)$ exhibit a nonmonotonic dependence on the initial conditions (Fig.1).

The Model

In our analysis of the BCS problem we employ the well known pseudospin formulation [23] in which spin 1/2 operators $s^\pm_{\mathbf{p}} = s^x_{\mathbf{p}} \pm i s^y_{\mathbf{p}}$ describe Cooper pairs $(\mathbf{p}, -\mathbf{p})$. The BCS Hamiltonian takes the form

$$\mathcal{H} = -\sum_{\mathbf{p}} 2\epsilon_{\mathbf{p}} s^z_{\mathbf{p}} - \lambda(t) \sum_{\mathbf{p},\mathbf{q}} s^-_{\mathbf{p}} s^+_{\mathbf{q}}, \tag{13.1}$$

where $\epsilon_{\mathbf{p}} = \mathbf{p}^2/2m - \mu$ is the free particle spectrum with μ the Fermi energy. Here we consider the time evolution induced by an instantaneous change of interaction from λ_s at $t < 0$ to λ at $t > 0$. In the spin formulation, Eq.(14.19), the dynamics is of a Bloch form

$$\frac{d\mathbf{r}_{\mathbf{p}}}{dt} = 2\mathbf{b}_{\mathbf{p}} \times \mathbf{r}_{\mathbf{p}}, \quad \mathbf{b}_{\mathbf{p}} = -(\Delta_x, \Delta_y, \epsilon_{\mathbf{p}}), \tag{13.2}$$

where $\mathbf{r}_{\mathbf{p}} = 2\langle \mathbf{s}_{\mathbf{p}} \rangle$ are classical vectors, and the effective magnetic field $\mathbf{b}_{\mathbf{p}}$ depends on the pairing amplitude Δ. The latter is defined self-consistently:

$$\Delta = \Delta_x + i\Delta_y = \frac{\lambda(t)}{2} \sum_{\mathbf{p}} r^+_{\mathbf{p}}, \quad r^+_{\mathbf{p}} = r^x_{\mathbf{p}} + i r^y_{\mathbf{p}}. \tag{13.3}$$

We first present numerical results for the dynamics (13.2), (13.3). The Runge-Kutta method of the 4th order was used with $N = 10^4, 10^5$ equally spaced discrete energy states within a band $W = 50\Delta_0$ with a constant density of states $\nu(E_F)$. As an initial state we take

$$r^+_{\mathbf{p}}(0) = \frac{\Delta_s}{\sqrt{\Delta_s^2 + \epsilon^2_{\mathbf{p}}}}, \quad r^z_{\mathbf{p}}(0) = \frac{\epsilon_{\mathbf{p}}}{\sqrt{\Delta_s^2 + \epsilon^2_{\mathbf{p}}}}. \tag{13.4}$$

which describes the $T = 0$ paired ground state [23].

Without loss of generality we set $\Delta(t) = \Delta_x$, since the phase of Δ is a constant of motion due to the particle-hole symmetry of the model. The interaction constants λ_s and λ define, via the self-consistency relation (13.3), the initial and final equilibrium BCS gap values, $\Delta_s = We^{-1/g_s}$, $g_s = \lambda_s \nu \ll 1$, $\Delta_0 = We^{-1/g}$, $g = \lambda\nu$, which we use to parameterize the system.

Figure 13.2: (a): The pairing amplitude $\Delta(t)$ for the initial state (13.4) with $\Delta_s = 0.05\Delta_0$ as recorded from the simulation, oscillating between $\Delta_+ = 0.97\Delta_0$ and $\Delta_- = 0.31\Delta_0$; Synchronization (b,c): the phase $\phi_{\mathbf{p}}$ time dependence, Eq.(13.6), for $\epsilon_{\mathbf{p}} = 0, \Delta_0, 2\Delta_0$, and frequency $\omega_{\mathbf{p}}$ *vs.* $\epsilon_{\mathbf{p}}$, Eq.(13.6).

We observe three qualitatively different dynamical regimes. The initial states with a relatively small gap give rise to undamped oscillations (Fig.13.3a). In this case $\Delta(t)$ oscillates non-harmonically between Δ_- and Δ_+ (the regime A in Fig.13.1). Synchronization of different Cooper pair states results from their interaction with the mode singled out by BCS instability of the initial state, similar to the evolution from the normal state [14].

Desynchronization takes place at $\Delta_s \geq \Delta_{AB} = 0.21\Delta_0$ giving rise to two different regimes exhibiting dephasing, underdamped and overdamped (B and C, Fig. 13.1). The former, illustrated in Fig.13.3a, is simplest to understand for a small initial deviation, $\Delta_s \simeq \Delta_0$ [24], by linearizing Bloch equations about the equilibrium state. The analysis predicts damped oscillations at long times:

$$\Delta(t) = \Delta_a + A(t)\sin(2\Delta_a t + \alpha), \quad A(t) \propto t^{-1/2}. \tag{13.5}$$

The power-law decay of $A(t)$ was explained in Ref. [24] by interaction of the collective mode with the continuous spectrum of excitations with energies above $2\Delta_a$ and linked to the linear Landau damping. In the spin formulation, the dephasing results from the Larmor frequency of spin precession $\mathbf{b}_{\mathbf{p}}$ being a continuous function of $\epsilon_{\mathbf{p}}$. An extension of this argument to the nonlinear regime was proposed recently in Ref. [19] which, however, did not clarify the range of its validity. The dephased time evolution similar to (13.5) was also reported in Refs. [16, 17, 22].

In the overdamped regime, $\Delta_s \geq \Delta_{BC} = 4.81\Delta_0$ (Fig.13.3b), $\Delta(t)$ decays to zero without oscillations. This behavior can be understood in the limit $\Delta_s/\Delta_0 \gg 1$, i.e. when the coupling in the initial paired state in (14.19) is suddenly completely turned off. For different spins precessing freely and independently one obtains

$$r^+_{\mathbf{p}}(t) = e^{-i2\epsilon_{\mathbf{p}}t} r^+_{\mathbf{p}}(0), \quad \Delta(t \gg \Delta_s^{-1}) \propto (\Delta_s t)^{-1/2} e^{-2\Delta_s t}.$$

The fast dephasing can also be understood by noting that the energy distribution in (13.4) corresponds to an effective temperature $T \sim \Delta_s$ which exceeds T_c for Δ_0 (see below).

To fully exhibit phase locking in the synchronized regime which abruptly disappears in the dephased regime, we now explore the phase dynamics. It is convenient to measure precession angles relative to time-independent $\tilde{\mathbf{b}}_{\mathbf{p}} = -(\Delta_a, 0, \epsilon_{\mathbf{p}})$, where Δ_a is the asymptote $\Delta(t \to \infty)$ in the regimes B, C, and the average value of oscillating $\Delta(t)$ in A (dash-dotted line in Fig. 13.1). The angle and frequency of precession are defined by

$$n^+_{\mathbf{p}} = n^x_{\mathbf{p}} + i n^y_{\mathbf{p}} \propto e^{-i\phi_{\mathbf{p}}(t)}, \quad \omega_{\mathbf{p}}(t) = d\phi_{\mathbf{p}}/dt, \tag{13.6}$$

where the vectors $n_{\mathbf{p}}$ are obtained from $r_{\mathbf{p}}$ by a rotation about the y axis which maps $\hat{z}$ onto $\tilde{\mathbf{b}}_{\mathbf{p}}$: $n^y_{\mathbf{p}} = r^y_{\mathbf{p}}$, $n^x_{\mathbf{p}} + i n^z_{\mathbf{p}} = e^{i\theta_{\mathbf{p}}}(r^x_{\mathbf{p}} + i r^z_{\mathbf{p}})$, with the rotation angle defined by $\tan\theta_{\mathbf{p}} = \Delta_a/|\epsilon_{\mathbf{p}}|$. The phase evolution, which becomes linear at long times $\tau \gg \Delta_0^{-1}$ (Figs.13.2,13.3), can be characterized by average frequency (phase slope) $\omega_{\mathbf{p}} = \langle d\phi_{\mathbf{p}}/dt\rangle = (\phi_{\mathbf{p}}(\tau) - \phi_{\mathbf{p}}(0))/\tau$. While in the regime A different $\mathbf{p}$ states phase lock (Fig.13.2b,c), in the regimes B, C the frequencies $\omega_{\mathbf{p}}$ have dispersion (Fig.13.3c,d). The latter reproduces quasiparticle spectrum, $\omega_{\mathbf{p}} = 2(\epsilon^2_{\mathbf{p}} + \Delta_a^2)^{1/2}$ with the long-time asymptote Δ_a which vanishes in the overdamped regime C.

We observe a qualitative change in behavior, with $\omega_{\mathbf{p}}$ dispersing in the regions B, C and phase locking in the region A. However, the oscillation amplitude $\frac{1}{2}(\Delta_+ - \Delta_-)$ in A and the asymptotic amplitude Δ_a in B vanish continuously at the critical points $\Delta = \Delta_{AB}, \Delta_{BC}$ (Fig.13.1), indicating a type II transition.

Figure 13.3: Dephased dynamics (a): Simulated $\Delta(t)$ for the initial states (13.4) with $\Delta_s = 0.21\Delta_0$, $4.5\Delta_0$ with the asymptotic values $\Delta_a = 0.81\Delta_0$, $0.12\Delta_0$; Overdamped dynamics (b): Same as in (a) with $\Delta_s = 4.81\Delta_0$ and $\Delta_a = 0$; (c): The phase $\phi_{\mathbf{p}}$ time dependence, Eq.(13.6), for energies $\epsilon_{\mathbf{p}} = 0, \Delta_0, 2\Delta_0$ (bottom to top) for $\Delta_s = 0.21\Delta_0$; (d): The frequency $\omega_{\mathbf{p}}$ *vs.* $\epsilon_{\mathbf{p}}$ for $\Delta_s = 0.21\Delta_0$ (dashed line) and $\Delta_s = 4.81\Delta_0$ (solid line).

Until now, we considered the dissipationless dynamics at times shorter than the quasiparticle relaxation time, $t \lesssim \tau_{\rm el}$. Using the energy balance argument, one can determine the system state at long times, $t \gg \tau_{\rm el}$. To account for system equilibration, one needs to consider the full

many-body Hamiltonian which enables elastic scattering of individual quasiparticles, omitted in Eq.(14.19). For a closed system, such as an atomic trap, the final temperature T_* and the gap Δ_* can be determined from the energy conservation condition $E_{t\gg\tau_{\rm el}}(T_*) = E_0$, where E_0 is the energy immediately after interaction switching,

$$E_0 = \sum_{\mathbf{p}} (\epsilon_{\mathbf{p}} - \tilde{\epsilon}_{\mathbf{p}}) + \frac{\Delta_s^2}{\lambda_s}\left(2 - \frac{\lambda}{\lambda_s}\right), \tag{13.7}$$

with the spectrum $\tilde{\epsilon}_{\mathbf{p}} = (\epsilon_{\mathbf{p}}^2 + \Delta_s^2)^{1/2}$, and $E_{t\gg\tau_{\rm el}}(T_*)$ is the energy of the final state:

$$E_{t\gg\tau_{\rm el}}(T_*) = \sum_{\mathbf{p}} \left[\epsilon_{\mathbf{p}} - (1 - 2n_{\mathbf{p}})\,\tilde{\epsilon}_{\mathbf{p}}(\Delta_*)\right] + \frac{\Delta_*^2}{\lambda}. \tag{13.8}$$

Here $n_{\mathbf{p}} = 1/\left(1 + e^{\tilde{\epsilon}_{\mathbf{p}}(\Delta_*)/T_*}\right)$ describes equilibrium with $T = T_*$, $\tilde{\epsilon}_{\mathbf{p}}(\Delta_*) = (\epsilon_{\mathbf{p}}^2 + \Delta_*^2)^{1/2}$. After integrating over $\epsilon_{\mathbf{p}}$, we arrive at the equation for T_*:

$$F\left(\frac{\Delta_*}{2T_*}\right) = 1 - \left(\frac{\Delta_s}{\Delta_*}\right)^2 + \alpha\left(\frac{\Delta_s}{\Delta_*}\right)^2 \ln\left(\frac{\Delta_s}{\Delta_0}\right)^2, \tag{13.9}$$

where $F(u) = 2\int_0^\infty dx \cosh 2x\,[1 - \tanh(u\cosh x)]$, and $\alpha = 1 - g\ln(\Delta_s/\Delta_0)$. From Eq.(13.9) we obtain T_* and the equilibrium gap $\Delta_* = \Delta(T_*)$. They are approximately constant, with $T_* \approx 0.72T_c$, $\Delta(T_*) \approx 0.81\Delta_0$ in the regime A, are described by a non-monotonic function in B, and vanish in C. The system turns normal in the final state for $\Delta_s \geq f(g)\Delta_0$, $f(g) = 2.2 + 0.86g + O(g^2)$. Numerical solution of Eq.(13.9) at $g = 0.26$ is displayed in Fig.13.1 (dashed line).

Figure 13.4: The pairing amplitude $\Delta(t)$ as a function of time for the initial state (13.4) with $\Delta_s = 0.01\Delta_0$ as recorded from numerical simulation. At long time, numerical $\Delta(t)$ matches the analytic form in Eq.(13.13), oscillating non-harmonically between $\Delta_+ = 0.997\Delta_0$ and $\Delta_- = 0.093\Delta_0$.

(so, what is the Lax vector after all??? see cited papers about it)

It is instructive to compare our results to the spectral analysis based on the integrability of the BCS Hamiltonian [23, 24, 20]. There is an infinite number of commuting integrals of motion, $R_{\mathbf{p}} = \mathbf{L}_{\mathbf{p}}\mathbf{s}_{\mathbf{p}}$, parameterized by $\mathbf{p}$, where following Ref. [20] we employ the Lax vector,

$$\mathbf{L}_{\mathbf{p}} = \hat{z} + \lambda \sum_{\mathbf{p}'\neq\mathbf{p}} \frac{\mathbf{s}_{\mathbf{p}'}}{\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}'}}. \tag{13.10}$$

We will need $\mathbf{L}_\mathbf{p}^2 = 4(\mathbf{L}_\mathbf{p}\mathbf{s}_\mathbf{p})^2$ which is also conserved (due to the Pauli matrices algebra combined with Eq.(13.10)). The mean-field expressions are obtained by substituting the averages $\langle \mathbf{s}_\mathbf{p} \rangle$ instead of $\mathbf{s}_\mathbf{p}$.

The spectral polynomial defining the evolution of individual states is proportional to the square of the Lax vector [20], $Q(\epsilon) = \mathbf{L}^2(\epsilon) \prod_\mathbf{p} (\epsilon - \epsilon_\mathbf{p})^2$, where $\mathbf{L}(\epsilon_\mathbf{p}) = \mathbf{L}_\mathbf{p}$. The pairs of complex roots of the spectral equation $Q(\epsilon) = 0$ uniquely determine the long-time dynamics of the system [20]. Evaluation of $\mathbf{L}^2(y)$ for the initial state (13.4) is straightforward:

$$\mathbf{L}^2(y)/g^2 = \left(\ln \frac{\Delta_s}{\Delta_0} + yG(y) \right)^2 + G^2(y), \tag{13.11}$$

where $y = \epsilon/\Delta_s$ and

$$G(y) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{dx}{y - \sinh x} = \frac{1}{2\sqrt{1+y^2}} \ln \frac{y + \sqrt{1+y^2}}{y - \sqrt{1+y^2}},$$

where y is complex. One can show that the roots of $\mathbf{L}^2(y)$ lie on the imaginary axis. Upon variation of Δ_s/Δ_0 they disappear at $y = 0$. Expanding about $y = 0$, we obtain

$$\mathbf{L}^2(y)/g^2 = \left(\ln \frac{\Delta_s}{\Delta_0} - \frac{\pi}{2} \right) \left(\ln \frac{\Delta_s}{\Delta_0} + \frac{\pi}{2} \right) + O(y), \tag{13.12}$$

Thus $\mathbf{L}^2(y)$ has no complex roots at $\Delta_s/\Delta_0 \geq e^{\pi/2}$, one pair of roots $y = \pm iu$ when $e^{-\pi/2} \leq \Delta_s/\Delta_0 \leq e^{\pi/2}$, with another pair appearing at $\Delta_s/\Delta_0 \leq e^{-\pi/2}$.

There is a direct correspondence between this behavior of the roots and the dynamical regimes A, B, and C observed numerically. The pairing amplitude is subject to fast dephasing and tends to zero when $\mathbf{L}^2(y)$ does not have complex roots. A pair of complex roots $y_a = \pm i\Delta_a/\Delta_s$ defines the long-time asymptote $\Delta(t) \approx \Delta_a$. Two pairs of roots, $y = \pm iu_1$ and $y = \pm iu_2$, correspond to the parameters $\Delta_\pm = (u_1 \pm u_2)\Delta_s$ of the elliptic function which defines the asymptotic behavior:

$$\Delta(t) = \Delta_+ \mathrm{dn}\left[\Delta_+(t - \tau_0), k\right], \quad k = 1 - \Delta_-^2/\Delta_+^2, \tag{13.13}$$

where the time lag τ_0 is a half of the period. As illustrated in Fig.13.4, Eq.(13.13) agrees well with $\Delta(t)$ found numerically. Thus the spectral analysis is in accord with the simulation of Bloch dynamics. It confirms the existence of the three regimes and also provides the exact values $\Delta_{AB} = e^{-\pi/2}\Delta_0$ and $\Delta_{BC} = e^{\pi/2}\Delta_0$.

Finally, we estimate the change of scattering length required to cross the A-B and B-C transitions. Using the BCS gap in a weakly interacting Fermi gas[25], $\Delta = 0.49 E_F e^{-1/g}$, $g = \frac{2}{\pi} k_F |a|$, we see that the conditions $\Delta_s/\Delta_0 = e^{\pm\pi/2}$, written as $1/g - 1/g_s = \pm\pi/2$, translate into $1/a - 1/a_s = \pm k_F$. At weak coupling this corresponds to a small change of scattering length, $\delta a/a \approx \pm k_F a$, easily achievable for magnetically tunable Feshbach resonance.

13.2 My Questions about Synchronization in the BCS Pairing Dynamics

Are these results confirmed experimentally?

(??? good question)

Part V
Time-Dependent BdG Equations

14 Solitons and Rabi Oscillations in a Time-Dependent BCS Pairing Problem by Barankov, Levitov, Spivak

(so, finally, do we need a kinetic equation in general for such time scales?? gap adjusts fast, but will this "stationary" gap be final if we don't take into account scatterings of quasiparticles???)

In summary, this work provides an exact solution for the BCS pairing formation problem. In the nonadiabatic regime (nearly nothing is explained here about nonadiabaticity, what is it??), the dynamics is dissipationless and thus nonlinear (why nonlinearity follows from dissipation?). Soliton train solutions are obtained analytically and demonstrated to be generic and robust by a simulation.

(are they observed in experiments?)

Abstract

Motivated by recent efforts to achieve cold fermions pairing near a Feshbach resonance (what exactly motivates them???), we consider the dynamics of formation of the Bardeen-Cooper-Schrieffer (BCS) state.

At times shorter than the quasiparticle energy relaxation time (still haven't figure out what do they mean about this relaxation??), after the interaction is turned on (after quench), the dynamics of the system is nondissipative (because it is a closed system, as it was before) (can it become dissipative, if we have a closed system?).

We show that this collective nonlinear evolution (collective nonlinear evolution?) of the BCS-Bogoliubov amplitudes u_p, v_p, along with the pairing function Δ, is an integrable (how is it??? BCS is integrable itself in mean field appr. right?) dynamical problem, and obtain a family of exact solutions in the form of single solitons and soliton trains (what is the formula for them?).

We interpret the collective oscillations as Bloch precession of Anderson pseudospins, where each soliton causes a pseudospin 2π Rabi rotation. (I don't understand this interpretation. what is the general idea here??)

Numerical simulations demonstrate robustness of the solitons with respect to noise and damping.

(what is Feshbach resonance????)

Some information about cold gases and nonequilibrium superconductivity

Dilute fermionic alkali gases cooled below degeneracy [1] are expected to host the paired BCS state [2, 3]. (why????)

One of the unique and attractive features of this system which makes it qualitatively different from superconducting metals is the possibility to control the strength of pairing interaction and change it by using the Feshbach resonances [4, 5, 6, 7, 8, 9]. (how exactly?)

Pairing enhancement near the resonances (what is the pairing enhancement????), as well as the high coherence of atomic systems (high coherence????), can allow to realize the strong coupling BCS regime (??? how the SC is different in it compared to a weak coupling??), as well as to explore the time dynamics of the paired state.

Since the characteristic energy scales in atomic vapors are relatively low (why are they low?), one can perform time-resolved measurements on the intrinsic microscopic time scales (it is not clear. if times of relaxation are fast, then we can't do time resolved measurements, no?), and explore a range of fundamentally new phenomena in the time dynamics of the paired state.

These new prospects helped to revive interest in some of the basic issues of the BCS pairing problem [10, 11, 12, 13, 14, 15]. (which issues??)

In particular, one of the important questions has to do with the dynamics and characteristic times of formation of the BCS state in cold gases [16]. (maybe times are discussed in Houbiers99??? I'll check later)

The dynamics of the superconducting state in metals, described by BCS theory, has been a subject of active research for a long time [17]. (I am not sure that there is something about it in Tinkham, I'll check later. maybe, there are important explanations)

Characteristic times

Generally, there are two important time scales to be considered: the quasiparticle energy relaxation time τ_ϵ, and the characteristic time of the order parameter change, τ_Δ, estimated as the inverse increment of Cooper instability [18, 19]. (???? I don't know why it is so??? I didn't find answer in Schmid's article. nothing is about it in Volkov Kogan and Larkin's articles, I'll continue later this search)

For $\tau_\epsilon \ll \tau_\Delta$ (gap changes slowly, and quasiparticles adjust to it immediately), quasiparticles quickly reach local equilibrium parameterized by a time-dependent order parameter $\Delta(t)$.

In this case the dynamics is described by the time-dependent Ginzburg-Landau (TDGL) equation for $\Delta(t)$, with the relaxation time scale *ca.* τ_Δ (during which the gap is changing, since only it is present in TDGL).

However, as noted by Gorkov and Eliashberg [20], this scenario applies only to relatively exotic situations. (why!!??? I'll read it later, I have this article)

For example, it works for in a proximity of a transition point, $\Delta^2/T_c \ll \tau_\epsilon^{-1}$, or a fast pair breaking (e.g., due to paramagnetic impurities). (why???? maybe it is written in Gorkov68?)

In the opposite limit (when gap changes fast and adjsts to the current distribution of quasiparticles), which takes place at the temperatures not too close to critical (and if it is close to critical???),

$$\tau_\epsilon \gg \tau_\Delta \tag{14.1}$$

the system dynamics is usually described by Boltzmann kinetic equation for quasiparticles and a self-consistent equation for $\Delta(t)$ connecting it with the quasiparticle distribution function locally in time [21, 22]. (work out Aronov74 and LarkinOvchinikov!!! sounds like they are about something else)

The validity of this approach (Boltzmann kinetic equation) requires, in addition to (14.1) ($\tau_\epsilon \gg \tau_\Delta$), that the variation in time of both the quasiparticle distribution and the external parameters (e.g., magnetic field or voltage) is sufficiently slow on the τ_Δ time scale (they also change slowly, only gap adjusts fast) (what if it is not so?).

The origin of this criterion is seen most easily if one notes that the order parameter can exhibit free oscillations with a frequency *ca.* τ_Δ^{-1} [23] (what did Anderson58 write about it?) (also, see below). (what is meant here???)

Thus, at a slow parameter variation the order parameter $\Delta(t)$ (meaning "slow variation of order parameter", bad formulation) and the quasiparticle spectrum will adiabatically (adiabatically is how???) follow the changes of the quasiparticle distribution (quasiparticle spectrum is $\xi = \sqrt{\xi^2 + \Delta^2}$, ξ follows the change of the gap), without exciting the oscillations of Δ (do we assume that there is some influence of spectrum to the gap?).

This approximation is relevant to the majority of situations in superconducting metals because of the relatively big value of Δ (relatively to what? to the amount of energy in excited states?) and the difficulty of changing the external parameters and making measurements on the time scale of τ_Δ (this is very not clear. explanation is not given).

From this point of view, the cold fermionic gases present a completely (not really completely) new situation. The energy relaxation (τ_ϵ) in these systems is quite slow (why???), while the external parameters, such as the detuning from Feshbach resonance, can change very quickly on the time scale of τ_Δ.

Thus the BCS correlations in this case build up in a coherent fashion while the system is out of thermal equilibrium (it is also not clear at all, what is meant here and why?).

In such a situation, theory must account not only for the order parameter evolution, but also for the full dynamics of individual Cooper pairs (what is known at all about dynamics of individual Cooper pairs?) and quasiparticles.

It seems to be desirable to understand better this 'fast BCS buildup' regime, since the lifetime of the gas samples is finite (what is it??), while the time-resolved measurements can easily be performed with resolution better than τ_Δ (because it is very slow).

In this article we consider the situation when the pairing interaction is switched on essentially instantly (quench), on a time scale $\tau_0 \ll \tau_\Delta, \tau_\epsilon$. We shall discuss only the spatially uniform situation, relevant for samples of finite size (this is strange, we don't use anyhow finite size, yes?), and explore the time evolution of the unpaired ideal Fermi gas (and it will become paired immediately), which is unstable with respect to Cooper pairing. At not too long times, $t \ll \tau_\epsilon$, the dynamics of the system is governed by nondissipative equations which conserve both the entropy and the total energy. (which equations??? why "nondissipative" if we always have closed systems)

A stationary superconducting state with the same energy as that of the initial state (and) would have more quasiparticles (which will come form the Cooper pairs), and thus would have entropy *higher* than the initial Fermi gas entropy. (not clear, why entropy is here?)

This argument suggests that the system can reach a stationary (not necessarily equilibrium) state only at a very long time of the order τ_ϵ. (no, we define τ_ϵ as this time, there is no argument given here)

On shorter times, which nevertheless can exceed τ_Δ), the system will exhibit a nonlinear (what is linear and nonlinear here?) time evolution. During this period of time *the concept of the quasiparticle spectrum is irrelevant* (no, it is still relevant. or we really work without it at all???) and theory can rely neither on the kinetic equation, nor on TDGL equation (why? it is not explained here) (so this is why we don't use kinetic equation and TDGL equation for quenches). (more intuition about it is needed!! how exactly these equation look like and how they are applicable?)

Below, we present an approach which describes the BCS state buildup (change of Δ) and

accounts for coherent dynamics of individual Cooper pairs. We shall focus on the zero temperature case, when $\tau_\Delta \simeq \Delta^{-1}$ (see arguments below from Abrahams66, to understand it, one needs to read that article), and show that the result of Cooper instability (increasing of Δ) is a periodic oscillation $\Delta(t)$ having the form of a soliton train. (what is the formula for it?)

This regime can be described by the BCS hamiltonian

$$\mathcal{H} = \sum_{\mathbf{p},\sigma} \epsilon_{\mathbf{p}} a^+_{\mathbf{p},\sigma} a_{\mathbf{p},\sigma} - \frac{\lambda(t)}{2} \sum_{\mathbf{p},\mathbf{q}} a^+_{\mathbf{p}\uparrow} a^+_{-\mathbf{p}\downarrow} a_{-\mathbf{q}\downarrow} a_{\mathbf{q}\uparrow}\,, \tag{14.2}$$

with the coupling turned on abruptly, $\lambda(t) = \lambda\theta(t - t_*)$.

The main result of this work is that the time-dependent problem (14.2) is integrable (write the formula that is assumed by this statement). We generalize the BCS solution, which is exact for the separable pairing Hamiltonian (14.2), and demonstrate that at $t > t_*$ the many-body state evolves as

$$|\Psi(t)\rangle = \prod_{\mathbf{p}} \left(u_{\mathbf{p}}(t) + v_{\mathbf{p}}(t) a^+_{\mathbf{p},\uparrow} a^+_{-\mathbf{p},\downarrow}\right) |0\rangle \tag{14.3}$$

(which is obvious because it is the BCS wavefunction)

The Bogoliubov mean field treatment, which gives a state of the form (14.3), relies on the 'infinite range' (it is not true, we assume always that there is a Debye window) form of the pairing interaction in (14.2) (i.e., equal coupling strength of all $(\mathbf{p}, -\mathbf{p})$, $(\mathbf{q}, -\mathbf{q})$). Since the latter does not depend on the system being in equilibrium, one can introduce a time-dependent mean field pairing function

$$\Delta(t) = \lambda \sum_{\mathbf{p}} u_{\mathbf{p}}(t) v^*_{\mathbf{p}}(t) \tag{14.4}$$

(it is important that we at all can work with this function with this definition. We assume that there is a dynamics of u and v and the gap always is calculated by this equation. The argument is that the Hamiltonian is written without assumption of equilibrium. It is important, because one might think that gap is not defined for nonequilibrium case, no, this person is wrong.)

The amplitudes $u_{\mathbf{p}}(t)$, $v_{\mathbf{p}}(t)$ can be obtained from the Bogoliubov-deGennes equation

$$i\partial_t \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{p}} & \Delta \\ \Delta^* & -\epsilon_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} \tag{14.5}$$

to be solved along with the self-consistency condition (14.4) ($\Delta(t) = \lambda \sum_{\mathbf{p}} u_{\mathbf{p}}(t) v^*_{\mathbf{p}}(t)$).

We recall that the unpaired state is a self-consistent, albeit unstable, solution of Eqs. (14.5) ($i\partial_t \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{p}} & \Delta \\ \Delta^* & -\epsilon_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix}$),(14.4) ($\Delta(t) = \lambda \sum_{\mathbf{p}} u_{\mathbf{p}}(t) v^*_{\mathbf{p}}(t)$) with $\Delta = 0$, $T = 0$: $u^{(0)}_{\mathbf{p}}(t) = e^{-i\epsilon_{\mathbf{p}} t}\theta(\epsilon_{\mathbf{p}})$, $v^{(0)}_{\mathbf{p}}(t) = e^{i\epsilon_{\mathbf{p}} t}\theta(-\epsilon_{\mathbf{p}})$ (check it, it should be easy). The stability analysis [19] (read Abrahams66 later, it is an important not trivial formula! it is not explained here) shows that the deviation from the unpaired state (appearance of the gap) grows as $\Delta(t) \propto e^{\gamma t} e^{-i\omega t}$,

$$\delta u_{\mathbf{p}}(t) = \frac{\Delta(t) v^{(0)}_{\mathbf{p}}(t)}{i\gamma - 2\epsilon_{\mathbf{p}} + \omega}, \quad \delta v_{\mathbf{p}}(t) = \frac{\Delta^*(t) u^{(0)}_{\mathbf{p}}(t)}{i\gamma + 2\epsilon_{\mathbf{p}} - \omega} \tag{14.6}$$

with the growth exponent γ and the constant ω given by

$$1 = \lambda \sum_{\mathbf{p}} \frac{\operatorname{sgn} \epsilon_{\mathbf{p}}}{2\epsilon_{\mathbf{p}} - \zeta}, \quad \zeta = \omega + i\gamma \tag{14.7}$$

The electron-hole symmetry near the Fermi level, $N(\epsilon_{\mathbf{p}}) = N(-\epsilon_{\mathbf{p}})$, makes the frequency shift ω vanish ($\omega = 0$). (???? why???)

Using the similarity between Eq. (14.7) ($1 = \lambda \sum_{\mathbf{p}} \frac{\operatorname{sgn} \epsilon_{\mathbf{p}}}{2\epsilon_{\mathbf{p}} - \zeta}$) and the BCS gap equation at $T = 0$, the exponent γ can be shown to be close to the BCS gap value Δ_0. (In fact, in the constant density of states approximation (what is the formula for it in this case?), the two quantities coincide: $\gamma = \Delta_0$. (why??)) Thus, we estimate the time $\tau_\Delta \simeq \Delta_0^{-1}$.

Our goal will be to provide insight in the evolution at later times, $t \geq \gamma^{-1}$ (it was just written that there will be just a stationary state, no evolution???). We rely on the nondissipative character of the Bogoliubov-deGennes dynamics (14.5) ($i\partial_t \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{p}} & \Delta \\ \Delta^* & -\epsilon_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix}$),(14.4) ($\Delta(t) = \lambda \sum_{\mathbf{p}} u_{\mathbf{p}}(t) v^*_{\mathbf{p}}(t)$), manifest, for instance, in the energy $E = \langle \Psi(t) | \mathcal{H} | \Psi(t) \rangle$ conservation throughout the evolution. Since in the initial, unpaired state, E is larger than the BCS condensation energy, the system cannot relax to equilibrium. (it is written below that we consider fast times, that is why)

Eqs. (14.5)($i\partial_t \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{p}} & \Delta \\ \Delta^* & -\epsilon_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix}$),(14.4) ($\Delta(t) = \lambda \sum_{\mathbf{p}} u_{\mathbf{p}}(t) v^*_{\mathbf{p}}(t)$) hold at times shorter than quasiparticle thermalization time, $\tau_\epsilon^{-1} \propto \max[T^2, \epsilon^2]$ in Fermi liquid. For the temperatures away from the BCS transition, the time τ_ϵ is very long compared to the instability growth time

$$\tau_\epsilon \gg \gamma^{-1} \quad (\epsilon \simeq \Delta_0), \tag{14.8}$$

indicating (how does it indicate???) a long *nondissipative nonlinear phase* that follows the linear instability phase. Below we show that the evolution governed by Eqs. (14.5)($i\partial_t \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{p}} & \Delta \\ \Delta^* & -\epsilon_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix}$),(14.4) ($\Delta(t) = \lambda \sum_{\mathbf{p}} u_{\mathbf{p}}(t) v^*_{\mathbf{p}}(t)$) is described by soliton-like pulses in the $|\Delta|$. We obtain a family of exact solutions of the form of single solitons and soliton trains, and compare it with simulations.

We briefly note that the importance of coherent dynamics of individual Cooper pairs (14.3) in the time evolution of a paired state has been understood some time ago in the context of the discussion of the validity of the time-dependent Ginzburg-Landau equation approach [18, 19] (have no idea which role do individual Cooper pair play??).

It has been pointed out by Gorkov and Eliashberg [20, 17] that the time-dependent pairing function Δ is generally insufficient to describe the evolution. (why?? read their articles!!!)

For that to be possible, the pair breaking and energy relaxation must be fast compared to the time of change of Δ (why??). This can be realized only close to the BCS transition, or in special situations, such as superconductors with magnetic impurities [20]. (????)

Usually, however, the Cooper pairs *are not slaved* (how is it?) to the time-dependent $\Delta(t)$, and the dynamics of each pair has to be treated individually, via Eq. (14.5)($i\partial_t \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{p}} & \Delta \\ \Delta^* & -\epsilon_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix}$).

The interaction switching, while nonadiabatic, must also be gentle enough not to overheat the fermions above $T \simeq \Delta_0$ (since $T_c \simeq \Delta_0$). The effective temperature T_{eff} after switching can be estimated from the total energy increase, giving

$$aT_{\text{eff}}^2 = \sum_{\omega,\, 1\ldots4} \hbar\omega n_1 n_2 (1 - n_3)(1 - n_4) |\lambda_\omega|^2 \delta(\hbar\omega - \delta E) \tag{14.9}$$

with $n_i = n(\epsilon_{\mathbf{p}_i})$, $\delta E = \epsilon_{\mathbf{p}_3} + \epsilon_{\mathbf{p}_4} - \epsilon_{\mathbf{p}_1} - \epsilon_{\mathbf{p}_2}$, $a = \frac{\pi^2}{6}\nu$. (this formula is very unclear. where does it come from????) ($\simeq \Delta_0$ appears without any explanation, what is it?)

For the interaction switching from 0 to λ over a characteristic time τ_0, the RHS of (14.9) is of the order of $\lambda^2\tau_0^{-3}$ (why??).

Thus the cold T_{eff} condition is $E_F\tau_0 \gg (\lambda n/\Delta_0)^{2/3}$ (why??), which is compatible with the nonadiabaticity requirement $\tau_0 \ll \tau_\Delta$.

Single soliton solution

(why do we solve like this at all, if there are solution train solution below?? is there an approximation here, that does not give us trains?)

At $T = 0$, a soliton solution of Eqs. (14.5)($i\partial_t \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{p}} & \Delta \\ \Delta^* & -\epsilon_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix}$),(14.4) ($\Delta(t) = \lambda \sum_{\mathbf{p}} u_{\mathbf{p}}(t) v^*_{\mathbf{p}}(t)$) can be constructed most naturally in terms of the variable (why??)

$$w_{\mathbf{p}} = \begin{cases} u_{\mathbf{p}}/v_{\mathbf{p}}, & \epsilon_{\mathbf{p}} > 0 \\ v_{\mathbf{p}}/u_{\mathbf{p}}, & \epsilon_{\mathbf{p}} < 0 \end{cases} \tag{14.10}$$

Consider first the case $\epsilon_{\mathbf{p}} > 0$. From Eq. (14.5) ($i\partial_t \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} = \begin{pmatrix} \epsilon_{\mathbf{p}} & \Delta \\ \Delta^* & -\epsilon_{\mathbf{p}} \end{pmatrix} \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix}$) (do we typically solve BdG like this??? Never did like this yet) we obtain

$$i\partial_t w_{\mathbf{p}} = 2\epsilon_{\mathbf{p}} w_{\mathbf{p}} + \Delta(t) - \Delta^*(t) w_{\mathbf{p}}^2 \tag{14.11}$$

where $\Delta(t)$ is a function of time determined from (14.4) ($\Delta(t) = \lambda \sum_{\mathbf{p}} u_{\mathbf{p}}(t) v^*_{\mathbf{p}}(t)$).

(I'll redo these calculations later)

Motivated by the stability analysis (14.6), (14.7) ($1 = \lambda \sum_{\mathbf{p}} \frac{\operatorname{sgn} \epsilon_{\mathbf{p}}}{2\epsilon_{\mathbf{p}} - \zeta}$) (maybe if I read the paper about the stability analysis, I'll understand this motivation?), we try the ansatz $\Delta(t) = e^{-i\omega t}\alpha^{-1}(t)$ with $\alpha(t)$ real, and

$$w_{\mathbf{p}}(t) = 2\epsilon_{\mathbf{p}} f(t) - i\dot{f}(t), \quad f(t) \equiv \frac{1}{\Delta^*} = e^{-i\omega t}\alpha(t) \tag{14.12}$$

(because there are two functions, $w_{\mathbf{p}}, \Delta$, so we express both of them in terms of α.) Substituting this in Eq.(14.11), we obtain an equation

$$i\xi_{\mathbf{p}}\dot{\alpha} + \ddot{\alpha} = \xi_{\mathbf{p}}(\xi_{\mathbf{p}}\alpha - i\dot{\alpha}) + \frac{1}{\alpha}\left(1 - (\xi_{\mathbf{p}}\alpha - i\dot{\alpha})^2\right) \tag{14.13}$$

with $\xi_{\mathbf{p}} = 2\epsilon_{\mathbf{p}} - \omega$ (check it, it should be easy).

Remarkably, the terms with $\xi_{\mathbf{p}}$ cancel (check it, it should be easy), and Eq. (14.13) takes the same form for all the states,

$$\alpha\ddot{\alpha} = \dot{\alpha}^2 + 1 \tag{14.14}$$

which justifies the ansatz (14.12).

By a variable substitution $\alpha = e^{\phi}$, Eq. (14.14) can be brought to the form $\ddot{\phi} = e^{-2\phi}$. Integrating the latter equation, obtain $\dot{\phi}^2 + e^{-2\phi} = \gamma^2$, with γ an integration constant. (how does it connect to the γ before???) (what is the physical meaning of γ?).

This yields $\dot{\alpha}^2 = \gamma^2\alpha^2 - 1$,

$$\alpha(t) = \frac{1}{\gamma}\cosh\gamma(t - t_0), \quad \Delta(t) = \frac{\gamma e^{-i\omega t}}{\cosh\gamma(t - t_0)} \tag{14.15}$$

(check it later)

The modulus $|\Delta|$ time dependence (Fig.**??**, upper left), growing first, then decreasing and taking the system back to the unpaired state, reflects the absence of dissipation (because in BdG, there was no dissipation, right??? about time scales of validity of this solution - I don't understand it). (it is not a soliton, why then we say that we have solitons???)

The time t_0 at which $|\Delta|$ peaks is set by the initial condition at large negative times $t \sim t_*$ (why??).

For $\epsilon_{\mathbf{p}} < 0$, $w_{\mathbf{p}} = v_{\mathbf{p}}/u_{\mathbf{p}}$, the form of Eq. (14.11) remains the same up to a sign change $\epsilon_{\mathbf{p}} \to -\epsilon_{\mathbf{p}}$ and the permutation $\Delta \leftrightarrow \Delta^*$. Accordingly, the ansatz for $w_{\mathbf{p}}$ in this case is $w_{\epsilon_{\mathbf{p}}<0}(t) = 2|\epsilon_{\mathbf{p}}|f(t) - i\dot{f}(t)$, $f(t) \equiv \Delta^{-1} = e^{i\omega t}\alpha(t)$. The resulting equation for $\alpha(t)$ is identical to Eq. (14.14). (check later, it should be easy)

The last step is to analyze the requirements on this solution due to the self-consistency condition (why do we do it??? before, there are solutions that are not self consistent yet??? I don't see it yet) (14.4) ($\Delta(t) = \lambda \sum_{\mathbf{p}} u_{\mathbf{p}}(t) v^*_{\mathbf{p}}(t)$). For that, we rewrite it in terms of $w_{\mathbf{p}}(t)$ as

$$\Delta(t) = \lambda \sum_{\epsilon_{\mathbf{p}}>0} \frac{w_{\mathbf{p}}(t)}{1+|w_{\mathbf{p}}(t)|^2} + \lambda \sum_{\epsilon_{\mathbf{p}}<0} \frac{w^*_{\mathbf{p}}(t)}{1+|w_{\mathbf{p}}(t)|^2} \tag{14.16}$$

(check it, should be easy) and note that both the right and the left hand side have the same time dependence (why??), and are equal to each other provided that the quantity $\zeta = \omega + i\gamma$ satisfies Eq. (14.7) ($1 = \lambda \sum_{\mathbf{p}} \frac{\operatorname{sgn}\epsilon_{\mathbf{p}}}{2\epsilon_{\mathbf{p}} - \zeta}$). (I don't understand, why? this may be not that easy to see...)

This means (why??) that the gap modulus $|\Delta(t)|$ peak value is equal to the instability exponent γ defined by (14.7) ($1 = \lambda \sum_{\mathbf{p}} \frac{\operatorname{sgn}\epsilon_{\mathbf{p}}}{2\epsilon_{\mathbf{p}} - \zeta}$). (still I don't understand where γ comes into the game? is there an explicit formula for it??)

(Why Bloch sphere is 2d here??)

Coherent BCS dynamics on the Bloch sphere (14.17). Trajectories of individual Cooper pair states for the soliton train solutions (Eq. (14.27) and insets) oscillating in the limits $\Delta_- \leq \Delta(t) \leq \Delta_+$ are shown. Note that each state completes a full 2π Rabi cycle per soliton. The red and blue curves correspond to the energies $\epsilon_{\mathbf{p}}$ above and below the Fermi level.

At $T = 0$, in the case of a constant density of states (how is it?), γ is equal to the equilibrium BCS gap Δ_0, while ω vanishes due to particle-hole symmetry. Thus, remarkably, the modulus $|\Delta(t)|$ in this case peaks exactly at Δ_0. (check it later, I don't understand it, but maybe it follows from what is written before)

Illustration in terms of trajectories on a sphere

(write in the note on SC discussion of Bloch sphere, maybe it is not connected with Anderson's spins, since we don't use them at this stage at all) To illustrate the collective dynamics in the soliton solution, we plot the trajectories $u_{\mathbf{p}}(t)$, $v_{\mathbf{p}}(t)$ on the Bloch sphere $r_1^2 + r_2^2 + r_3^2 = 1$ using the parameterization

$$r_1 + ir_2 = 2u_{\mathbf{p}}v_{\mathbf{p}}^*; \quad r_3 = |u_{\mathbf{p}}|^2 - |v_{\mathbf{p}}|^2 \tag{14.17}$$

(Fig. **??**, left). Here, as well as in the soliton train solutions discussed below (Eq. (14.27) (I still don't understand why there are solitons here??)), each state $(u_{\mathbf{p}}, v_{\mathbf{p}})$ completes a full Rabi cycle per soliton. (why is it so??? and why Rabi cycle correspond to circlues on the Bloch sphere??)

(why do we have Z and Y axes on an Bloch sphere plot???)

The trajectories, which are small loops for the pairs with large energies $\epsilon_{\mathbf{p}}$, turn into a big circle as $\epsilon_{\mathbf{p}}$ tends to the Fermi level (why??).

To gain further insight, we reformulate the Bogoliubov approach, following Anderson [23], in terms of pseudospins associated with individual Cooper pair states. (still don't understand, why it is so useful??)

'Pauli spin' operators $\sigma^{\pm}_{\mathbf{p}} \equiv \frac{1}{2}(\sigma^x_{\mathbf{p}} \pm i\sigma^y_{\mathbf{p}})$ can be assigned to each pair of fermions with opposite momenta as follows

$$\sigma^+_{\mathbf{p}} = a^+_{\mathbf{p}\uparrow} a^+_{-\mathbf{p}\downarrow}, \quad \sigma^-_{\mathbf{p}} = a_{-\mathbf{p}\downarrow} a_{\mathbf{p}\uparrow} \tag{14.18}$$

The identity $\sigma^z_{\mathbf{p}} \equiv [\sigma^+_{\mathbf{p}}, \sigma^-_{\mathbf{p}}] = a^+_{\mathbf{p}\uparrow} a_{\mathbf{p}\uparrow} - a_{-\mathbf{p}\downarrow} a^+_{-\mathbf{p}\downarrow}$ allows to map the problem (14.2) $\mathcal{H} = \sum_{\mathbf{p},\sigma} \epsilon_{\mathbf{p}} a^+_{\mathbf{p},\sigma} a_{\mathbf{p},\sigma} - \frac{\lambda(t)}{2} \sum_{\mathbf{p},\mathbf{q}} a^+_{\mathbf{p}\uparrow} a^+_{-\mathbf{p}\downarrow} a_{-\mathbf{q}\downarrow} a_{\mathbf{q}\uparrow}$ onto an interacting spin problem

$$\mathcal{H} = \sum_{\mathbf{p}}{}' \epsilon_{\mathbf{p}} \sigma^z_{\mathbf{p}} - 2\lambda \sum_{\mathbf{p},\mathbf{q}}{}' \sigma^+_{\mathbf{p}} \sigma^-_{\mathbf{q}} \tag{14.19}$$

where $\sum'_{\mathbf{p}}$ means a sum over the pairs of states $(\mathbf{p}, -\mathbf{p})$. (I hope that this is the typical Anderson's theory)

Since all the spins interact with each other equally, the mean field theory here is exact, just like for the BCS problem (so what is known about beyond mean field theory?).

The mean field hamiltonian for each spin is

$$\mathcal{H}_{\mathbf{p}} = \mathbf{b}_{\mathbf{p}} \cdot \sigma_{\mathbf{p}}, \quad \mathbf{b}_{\mathbf{p}} = (-\Delta', -\Delta'', \epsilon_{\mathbf{p}}) \tag{14.20}$$

(derive it! maybe it is not hard. work out Anderson's paper. but it should be well-known)

Here the z component of the effective field $\mathbf{b}_{\mathbf{p}}$, given by the single particle energy, is spin-specific (the number of the spin 9(if one enumerates them) depends on the p and the last component is also determined by p), while the transverse components, the same for all of the spins, satisfy

$$\Delta \equiv \Delta' + i\Delta'' = \lambda \sum_{\mathbf{q}} \langle \sigma^+_{\mathbf{q}} \rangle \tag{14.21}$$

which is analogous to the BCS gap relation.

In the ground state each spin is aligned with $\mathbf{b}_{\mathbf{p}}$, and the spins form a texture near the Fermi surface [23] (which texture? Read Anderson58), with spin rotation described by the Bogoliubov angle.

The dynamical problem of interest can be cast in the form of Bloch equations for the spins,

$$\dot{\sigma}_{\mathbf{p}} = i[\mathcal{H}_{\mathbf{p}}, \sigma_{\mathbf{p}}] = 2\mathbf{b}_{\mathbf{p}} \times \sigma_{\mathbf{p}} \tag{14.22}$$

with the field $\mathbf{b}_{\mathbf{p}}$ defined self-consistently by (14.20),(14.21) ($\mathcal{H}_{\mathbf{p}} = \mathbf{b}_{\mathbf{p}} \cdot \sigma_{\mathbf{p}}$; $\Delta \equiv \Delta' + i\Delta'' = \lambda \sum_{\mathbf{q}} \langle \sigma^+_{\mathbf{q}} \rangle$). Anderson [23] used Eq. (14.22), linearized about the texture ground state, to study collective excitations of a superconductor (see also [24]).

Linearized about the unpaired state, Eq. (14.22) yields an instability identical to (14.6), (14.7) ($1 = \lambda \sum_{\mathbf{p}} \frac{\operatorname{sgn} \epsilon_{\mathbf{p}}}{2\epsilon_{\mathbf{p}} - \zeta}$). (maybe after reading cited papers, I'll derive it, now I don't understand why is it so?)

Simulations

(Have no idea what is meant here, I need to read it more)

The Bloch dynamics is suitable for simulation (Fig. 14.1), since Eq. (14.22) ($\dot{\sigma}_{\mathbf{p}} = i[\mathcal{H}_{\mathbf{p}}, \sigma_{\mathbf{p}}] = 2\mathbf{b}_{\mathbf{p}} \times \sigma_{\mathbf{p}}$) is linear in spin operators and can be replaced by an equation for the expectation values (14.17). (so why it is convenient to simulate exactly this equation???) Typical $\Delta(t)$, observed in the presence of thermal noise, is an orderly sequence of the cosh solitons (Fig. 14.1, top), indicating robustness of soliton solution (*cf.* Ref.[25]). (what is in Galperin81???)

Figure 14.1: Bloch dynamics $\dot{\mathbf{r}}_{\mathbf{p}} = 2\mathbf{b}^{\text{eff}}_{\mathbf{p}} \times \mathbf{r}_{\mathbf{p}}$ for 10^3 spins with a constant density of states, with $\mathbf{b}^{\text{eff}}_{\mathbf{p}}$ defined by (14.20),(14.23). Initial conditions $r_{3,\mathbf{p}} = \tanh(\frac{1}{2}\beta\epsilon_{\mathbf{p}})$, $(r_1 + ir_2)_{\mathbf{p}} = e^{i\phi_{\mathbf{p}}}(1 - r_3^2)^{1/2}$, with $\beta^{-1} \equiv T = 0.1\Delta_0$ and random uncorrelated $0 < \phi_{\mathbf{p}} < 2\pi$ were used to simulate thermal noise.

The effect of damping can be studied by replacing

$$\mathbf{b}_{\mathbf{p}} \to \mathbf{b}^{\text{eff}}_{\mathbf{p}} = \mathbf{b}_{\mathbf{p}} - \gamma \mathbf{b}_{\mathbf{p}} \times \mathbf{r}_{\mathbf{p}} \tag{14.23}$$

with a dimensionless damping constant, $\gamma \sim 1/(\Delta_0 \tau_\epsilon)$. This destroys integrability and prompts relaxation to BCS equilibrium (Fig. 14.1, bottom).

Solution in terms of soliton trains

(what is different comparing to what was before?)

(how to guess about this solution??)

In our nonlinear dynamical problem, we have to solve the Bloch equations (14.22) ($\dot{\sigma}_{\mathbf{p}} = i[\mathcal{H}_{\mathbf{p}}, \sigma_{\mathbf{p}}] = 2\mathbf{b}_{\mathbf{p}} \times \sigma_{\mathbf{p}}$) together with the self-consistency relation (14.21) ($\Delta \equiv \Delta' + i\Delta'' = \lambda \sum_{\mathbf{q}} \langle \sigma^+_{\mathbf{q}} \rangle$).

As above, we assume the pairing amplitude time dependence of the form $\Delta(t) = e^{-i\omega t}\Omega(t)$, where $\Omega(t)$ is real and ω is a constant frequency shift to be fixed by the relation (14.21).

In the Larmor frame rotating with the frequency ω, Eq. (14.22), written for the spins average polarization $r_i = \langle \sigma^i_{\mathbf{p}} \rangle$ components, reads

$$\dot{r}_1 = -\xi_{\mathbf{p}} r_2, \quad \dot{r}_2 = \xi_{\mathbf{p}} r_1 + 2\Omega r_3, \quad \dot{r}_3 = -2\Omega r_2 \tag{14.24}$$

with $\xi_{\mathbf{p}} = 2\epsilon_{\mathbf{p}} - \omega$, as before. (check it, it can be not hard)

An exact solution can be obtained from the ansatz

$$r_1 = A\Omega, \quad r_2 = B\dot{\Omega}, \quad r_3 = C\Omega^2 - D \tag{14.25}$$

The first and the third equation (14.24) are satisfied by (14.25) provided $A = -\xi_{\mathbf{p}} B$ and $B = -C$ (check it, it can be not hard), while the second equation (14.24) is consistent with the normalization condition $r_1^2 + r_2^2 + r_3^2 = 1$, and thus yields

$$C^2 \xi^2_{\mathbf{p}} \Omega^2 + C^2 \dot{\Omega}^2 + (C\Omega^2 - D)^2 = 1 \tag{14.26}$$

Eq. (14.26) will take the same form for all the spins,

$$\dot{\Omega}^2 + (\Omega^2 - \Delta_-^2)(\Omega^2 - \Delta_+^2) = 0, \quad \Delta_- \leq \Delta_+ \tag{14.27}$$

provided that the constants D, C are chosen as

$$(D^2 - 1)/C^2 = \Delta_-^2 \Delta_+^2, \quad 2D/C = \xi_{\mathbf{p}}^2 + \Delta_-^2 + \Delta_+^2 \tag{14.28}$$

(so these $\Delta 2_-, \Delta_+$ are chosen such that they obey these equations. One can get explicit formulas for them from here) with the sign factor $\operatorname{sgn} \epsilon_{\mathbf{p}}$ (where is the sign factor?).

Eq. (14.27) defines an elliptic function $\Omega(t)$ oscillating periodically between Δ_- and Δ_+.

At $\Delta_- \ll \Delta_+$, the solution is a train of weakly overlapping solitons (14.15) $\Delta(t) = \frac{\gamma e^{-i\omega t}}{\cosh \gamma(t-t_0)}$ with $\Delta_+ = \gamma$ (Fig. **??**, left).

The real part of the self-consistency relation (14.21) ($\Delta \equiv \Delta' + i\Delta'' = \lambda \sum_{\mathbf{q}} \langle \sigma_{\mathbf{q}}^+ \rangle$),

$$1 = \lambda \sum_{\mathbf{p}} \frac{\xi_{\mathbf{p}} \operatorname{sgn} \epsilon_{\mathbf{p}}}{\left((\xi_{\mathbf{p}}^2 + \Delta_-^2 + \Delta_+^2)^2 - 4\Delta_-^2 \Delta_+^2\right)^{1/2}} \tag{14.29}$$

fixes one of the constants $\Delta_\pm$, the other one being a free parameter that controls the inter-soliton time separation. With the ratio Δ_-/Δ_+ varying from 0 to 1, the soliton frequency increases, and the weakly overlapping solitons (14.15) gradually become overlapping more strongly, turning into weak harmonic oscillations (Fig. **??**).

We note that the weak oscillation limit has been investigated by Volkov and Kogan [24] (I thought that in their work, there are decaying and not only oscillations???) with the help of linearized Gorkov equation, and earlier by Anderson [23] (really? he got these results???), who used pseudospin representation.

The nonexponential decay of the lianearized oscillations [18, 24] was interpreted as collisionless damping, related to strong mixing of the oscillations of Δ with the states of two excited quasiparticles slightly above the superconducting gap.

The imaginary part of Eq. (14.21) fixes the value of the frequency shift ω (we recall that $\omega \neq 0$ in the presence of charge asymmetry). At $\Delta_- \ll \Delta_+$, Eq. (14.29) turns into Eq. (14.7) ($1 = \lambda \sum_{\mathbf{p}} \frac{\operatorname{sgn} \epsilon_{\mathbf{p}}}{2\epsilon_{\mathbf{p}} - \zeta}$) which, as we found above, defines the amplitude of a single soliton. In the opposite limit, $\Delta_- \to \Delta_+$, Eq. (14.29) coincides with the BCS gap equation.

Final comments

There is an interesting relation between our problem and the self-induced transparency phenomenon [26] (self-induced transparency phenomenon??? I don't understand this paragraph, I'll read McCall67 maybe later and then think about it). In the latter, an optical pulse interacting with an ensemble of atoms can dissipate its energy by inducing resonant Rabi transitions in the atoms.

However, when the pulse duration is tuned so that the atoms complete a full Rabi 2π cycle as the pulse goes by, the pulse sustains itself and propagates without dissipation. The Bloch equations describing this phenomenon bear striking similarity to our Eqs. (14.22) ($\dot{\sigma}_{\mathbf{p}} = i[\mathcal{H}_{\mathbf{p}}, \sigma_{\mathbf{p}}] = 2\mathbf{b}_{\mathbf{p}} \times \sigma_{\mathbf{p}}$), while the atoms' polarization is employed [26] to provide feedback on the pulse instead of our BCS self-consistency relation.

Before concluding, we note that the dynamics at finite temperature, in the regime described by Eq. (14.1) ($\tau_\epsilon \gg \tau_\Delta$), remains an open problem. (is it solved now?)

In particular, we can not rule out the possibility of chaotic behavior (where it can arise?).

At $T = 0$ the problem has a relatively simple solution, periodic in time, because in this case the quasiparticles with low energies are strongly coupled to the oscillations of Δ. (aren't we have 0 quasiparticles at $T = 0$??)

In contrast, in the case $T\tau_\Delta \gg 1$ there are two groups of quasiparticles: those with energies $\epsilon_{\mathbf{p}} \sim \tau_\Delta^{-1}$, which fully participate in the oscillations, as above, and the quasiparticles with $\tau_\Delta^{-1} \ll \epsilon_{\mathbf{p}} \ll T$, coupled to $\Delta(t)$ much weaker and playing the role of a thermal bath, thereby providing dissipation. (why??? dissipation to what???? nothing is explained here)

15 Pure Goldstone mode in the quench dynamics of a confined ultracold Fermi gas in the BCS-BEC crossover regime by Kettmann et al.

Abstract

We present a numerical study of the dynamic response of a confined superfluid Fermi gas to a rapid change of the scattering length (i.e., an interaction quench). Based on a fully microscopic time-dependent density-matrix approach within the full Bogoliubov-de Gennes formalism that includes a 3D harmonic confinement we simulate and identify the emergence of a Goldstone mode of the BCS gap in a cigar-shaped ^{6}Li gas. By analyzing this Goldstone mode over a wide range of parameters, we show that its excitation spectrum is gapless and that its main frequency is not fixed by the trapping potential but that it is determined by the details of the quench. Thus, we report the emergence of a pure Goldstone mode of the BCS gap that –in contrast to situations in many previous studies– maintains its gapless excitation spectrum predicted by the Goldstone theorem. Furthermore, we observe that the size-dependent superfluid resonances resulting from the atypical BCS-BEC crossover have a direct impact on this Goldstone mode. Finally, we find that the interaction quench-induced Goldstone mode leads to a low-frequency in-phase oscillation of the single-particle occupations with complete inversion of the lowest-lying single-particle states which could provide a convenient experimental access to the pure gapless Goldstone mode.

15.1 Conclusion

In conlusion, we have calculated the dynamics of a confined ultracold ^{6}Li gas at $T = 0$ induced by an interaction quench on the BCS side of the BCS-BEC crossover. We used a full dynamical BdG approach to set up and solve the equations of motion for the single-particle occupations and coherences. In doing so, we have shown that the interaction quench excites a low-energy linear dynamics of the complex phase of the BCS gap, i.e., a Goldstone mode. We have analyzed this Goldstone mode over a wide range of parameters showing that its excitation spectrum is gapless and that its main frequency is not fixed by the trap frequencies but that it is determined by the details of the quench. Furthermore, we found that the atypical BCS-BEC crossover leads to resonances in the gapless Goldstone mode. Finally, we investigated the impact of the gapless Goldstone mode on the single-particle occupations. We have shown that it leads to an in-phase oscillation of the whole single-particle spectrum with a full inversion of the lowest-lying single-particle states which could provide an experimental access to the gapless homogeneous Goldstone mode.

15.2 Introduction

Due to their unique controllability ultracold Fermi gases provide an ideal system to test concepts of many-particle physics as well as particle theory. Adjustable interparticle interac-

tions provide the possibility to explore both the regime of weak attractive interactions where a superfluid Bardeen Cooper Schrieffer (BCS) phase emerges as well as the regime of weak repulsive interactions which lead to the formation of a Bose-Einstein condensate (BEC). Both regimes are connected by a smooth BCS-BEC crossover with strong interparticle interactions including a point of unitarity where the coupling strength diverges [Giorgini *et al.*(2008), Bloch *et al.*(2008)]. Furthermore, the emergence of a BCS phase is associated with a spontaneously broken $U(1)$ symmetry which makes ultracold Fermi gases a convenient candidate to study the fundamental concept of spontaneous symmetry breaking (SSB) [Weinberg(1996)].

Spontaneously broken gauge symmetries and the resulting two types of fundamental collective excitations –gapped amplitude/Higgs modes and gapless phase/Goldstone modes (see Fig. 15.1)– are of fundamental interest for several fields of physics like condensed matter and particle physics. Probably the most prominent application of the concept of SSB is the Higgs mechanism in particle physics [Higgs(1964)]. In condensed matter physics SSB occurs in several systems, for example in ferromagnets (see, e.g., [Burgess(2000)]), superfluid ^{3}He [Paulson *et al.*(1973), Lawson *et al.*(1973)] and BCS superconductors [Anderson(1958), Weinberg(1996)]. In these cases, the fundamental excitations –known as magnons (i.e., spin waves), second sound (i.e., heat waves) and plasmons– correspond to the Goldstone modes resulting from SSB.

Figure 15.1: (color online) Ginzburg-Landau free energy of a BCS phase of a homogeneous Fermi gas for $T < T_C$ (schematic). The rotational $U(1)$ symmetry is broken, when the gap Δ of the system gains a certain phase inside the rim of the potential. The resulting excitation modes are the Higgs-amplitude mode and the Goldstone mode, i.e., a phase oscillation of the gap.

In the field of ultracold Fermi gases the Higgs and the Goldstone mode have received great attention over the past years. The Higgs mode, i.e., the amplitude oscillation of the BCS gap, is difficult to address in experiment since it does not couple directly to external probes [Pekker and Varma(2015)]. This is why measurements of the Higgs mode have only recently been achieved for lattice superfluids [Bissbort *et al.*(2011), Endres *et al.*(2012)] (in the case of BCS superconductors an experimental access has been found via THz spectroscopy [Matsunaga *et al.*(2013), Matsunaga *et al.*(2014)]), while several theoretical studies on the Higgs mode have been reported [Barankov *et al.*(2004), Barankov and Levitov(2006), Yuzbashyan

et al.(2006), Dzero *et al.*(2007), Scott *et al.*(2012), Bruun(2014), Hannibal *et al.*(2015)]. In contrast, the Goldstone mode, i.e., the oscillation of the complex phase of the BCS gap, has been intensively studied both theoretically and experimentally (see, e.g., [Kinast *et al.*(2004a), Kinast *et al.*(2004b), Bartenstein *et al.*(2004), Altmeyer *et al.*(2007a), Altmeyer *et al.*(2007b), Riedl *et al.*(2008), Baranov and Petrov(2000), Bruun and Mottelson(2001), Bruun(2002), Hu *et al.*(2004), Heiselberg(2004), Stringari(2004), Grasso *et al.*(2005), Korolyuk *et al.*(2011)]).

However, most previous studies were based on the fact that the Goldstone mode couples to the real-space dynamics of the Fermi gas. Therefore, it can be excited by inducing a collective oscillation of the trapped cloud, which can be achieved, e.g., by various schemes of confinement change or by optical excitation. The dynamics in the Goldstone mode is then directly reflected in the collective oscillation of the cloud and can be observed via the latter. However, a coupling to the real-space oscillation of the cloud implies a coupling to the trapping potential. I.e., in the previous studies the frequency of the Goldstone mode was fixed by the frequencies of the trap which stands in direct contrast to the Goldstone theorem predicting a gapless excitation spectrum of the phase mode. Thus, the observation of the gapless phase mode resulting from the Goldstone theorem was so far obstructed by the coupling to the trapping potential via the collective oscillation of the cloud.

In contrast, we report the emergence of a pure Goldstone mode in the dynamics of a trapped ultracold Fermi gas by showing that its original gapless excitation spectrum can be recovered by exciting the cloud via an interaction quench. To this end, we observe that the frequency of the interaction quench-induced phase mode can be tuned over a wide range from zero to finite values by adjusting the details of the quench. We explain this by the circumstance that the excited Goldstone mode is homogeneous, i.e., that it does not introduce any phase gradients, and that it therefore does not couple to the trapping potential.

To do so, we study the dynamics of the BCS gap of a confined ultracold ^{6}Li gas at $T = 0$ on the BCS side of the BCS-BEC crossover [i.e., the Fermi wave vector k_F times the scattering length a is given by $0 > 1/(k_F a) > -1$] as well as in the BCS regime [i.e., $-1 > 1/(k_F a)$]. We calculate the dynamics of the Bogoliubov quasiparticles in the framework of the Bogoliubov-de Gennes (BdG) formalism and by that the dynamics of the BCS gap. The investigated system is in the low-density regime, i.e., we use a short-range s-wave interaction between particles with opposite spin. The confinement is modeled by a cigar-shaped 3D harmonic potential, which in good approximation describes the standard laser confinement used in experiment [Bloch *et al.*(2008)]. The system is excited by an instantaneous interaction quench, i.e., a rapid change of the interparticle interaction strength. This can experimentally be achieved, e.g., by an optical control of a Feshbach resonance [Clark *et al.*(2015)].

Our work is structured as follows. In section 36.2 we present the formalism we used within the context of this work, i.e., a full BdG approach as well as its simplification via the Anderson approximation. We will derive the equations of motion for the quasiparticle expectation values which allow for a calculation of the phase dynamics of the BCS gap. In section 36.3 we present the results obtained by the full BdG approach and show that they are well reproduced by Anderson's approximate solution. Based on the latter we will analyze the effect of the excitation parameters and of the trapping frequencies on the interaction-quench-induced Goldstone mode as well as the impact of this Goldstone mode on an experimentally accessible quantity, i.e., the single-particle excitations of the cloud. Finally, we conclude and summarize our findings in section 36.4.

15.3 Theoretical approach

To calculate the ground state as well as the dynamical properties of a BCS condensate in a trap we start from the Bogoliubov-de Gennes (BdG) Hamiltonian [De Gennes(1989), Datta

and Bagwell(1999)]:

$$H_{BdG} = \int \Psi_\uparrow^\dagger(\mathbf{r}) H_0 \Psi_\uparrow(\mathbf{r})\, \mathrm{d}^3 r + \int \Psi_\downarrow^\dagger(\mathbf{r}) H_0 \Psi_\downarrow(\mathbf{r})\, \mathrm{d}^3 r + \int \left[\Delta(\mathbf{r}) \Psi_\uparrow^\dagger(\mathbf{r}) \Psi_\downarrow^\dagger(\mathbf{r}) + \Delta^*(\mathbf{r}) \Psi_\downarrow(\mathbf{r}) \Psi_\uparrow(\mathbf{r})\right] \mathrm{d}^3 r. \tag{15.1}$$

Here the field operators $\Psi_\sigma^{(\dagger)}(\mathbf{r})$ describe the annihilation (creation) of Fermions –in this case atoms of ^{6}Li– with spin σ at the position $\mathbf{r}$ and the BCS gap is given by

$$\Delta(\mathbf{r}) = -g \left\langle \Psi_\downarrow(\mathbf{r}) \Psi_\uparrow(\mathbf{r}) \right\rangle . \tag{15.2}$$

The interaction strength $g = -\frac{4\pi\hbar^2 a}{m}$ is determined by the s-wave scattering length a and the mass of the particles m. The 3D harmonic trapping potential $V(\mathbf{r})$ is included in the one-particle Hamiltonian $H_0 = \frac{p^2}{2m} + V(\mathbf{r}) - \mu$ with the chemical potential μ, where we set the trapping frequencies to $f_x = f_y =: f_\perp \gg f_\parallel := f_z$, i.e., a cigar-shaped trap.
In the following, we will show how to diagonalize Eq. (36.4) and thus obtain its eigenstates and energies which describe the single-particle excitations of the BCS condensate.

15.3.1 The BdG ground state

To calculate the ground-state properties of the BCS condensate it is instructive to write the eigenvalue equation corresponding to Eq. (36.4) as the BdG equation [De Gennes(1989)]

$$\begin{pmatrix} H_0 & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H_0^* \end{pmatrix} \begin{pmatrix} u_M(\mathbf{r}) \\ v_M(\mathbf{r}) \end{pmatrix} = E_M \begin{pmatrix} u_M(\mathbf{r}) \\ v_M(\mathbf{r}) \end{pmatrix}. \tag{15.3}$$

This equation has the form of a one-particle Schrödinger equation[1], which implies that H_{BdG} describes non-interacting quasiparticles, i.e., the single-particle excitations of the BCS condensate. Therefore, Eq. (36.6) can be diagonalized which yields the corresponding single-particle wave functions $[u_M(\mathbf{r}), v_M(\mathbf{r})]$ and energies E_M. However, before we do so, we want to state that the single-particle states divide into two branches (one of positive energies $E_M = E_{m\alpha} > 0$ and one of negative energies $E_M = E_{m\beta} = -E_{m\alpha}$) that can be expressed by one another [Datta and Bagwell(1999)]. Accordingly, we simplify our formalism by writing the corresponding expressions –whenever possible– solely in terms of the positive-energy states and by dropping the index α. I.e., in the following we set $[u_m(\mathbf{r}), v_m(\mathbf{r})] := [u_{m\alpha}(\mathbf{r}), v_{m\alpha}(\mathbf{r})]$ and $E_m := E_{m\alpha}$. Furthermore, we transform into the excitation picture, i.e., we flip the β-branch $[(m\beta) \to (mb)]$ and leave the α-branch unchanged $[(m\alpha) = (ma)]$. This yields one twofold degenerate branch with $E_m = E_{ma} = E_{mb}$. The corresponding creation operators read

$$\gamma_{ma}^\dagger = \int \left[u_m(\mathbf{r}) \Psi_\uparrow^\dagger(\mathbf{r}) + v_m(\mathbf{r}) \Psi_\downarrow(\mathbf{r})\right] \mathrm{d}^3 r \tag{15.4}$$

$$\gamma_{mb}^\dagger = \int \left[u_m(\mathbf{r}) \Psi_\downarrow^\dagger(\mathbf{r}) - v_m(\mathbf{r}) \Psi_\uparrow(\mathbf{r})\right] \mathrm{d}^3 r. \tag{15.5}$$

[1] To be precise, the many-body nature of the BCS pairing is included in Eq. (36.6) via the self-consistent calculation of the BCS gap $\Delta(\mathbf{r})$ (see below).

We solve the BdG equation by expressing the single-particle states $[u_M(\mathbf{r}), v_M(\mathbf{r})]$ in terms of the bare atomic states, i.e., the eigenstates of the harmonic trap $\phi_i(\mathbf{r})$,

$$u_M(\mathbf{r}) = \sum_{i=1}^{N} u_M^{(i)} \phi_i(\mathbf{r}) \tag{15.6}$$

$$v_M(\mathbf{r}) = \sum_{i=1}^{N} v_M^{(i)} \phi_i(\mathbf{r}), \tag{15.7}$$

with $M \in \{ma, mb\}$. Here, we restrict the sum to atomic states from a window of width $\Delta\epsilon \sim 1\mu$ around the chemical potential (i.e., $0.5\mu < \epsilon < 1.5\mu$) to reduce the numerical effort[2]. Inserting this in the BdG equation, multipliying by $\phi_m(\mathbf{r})$ and integrating over $\mathbf{r}$ yields:

$$\begin{pmatrix} \epsilon_1 - \mu & 0 & \cdots & 0 & (\Delta)_{11} & \cdots & & (\Delta)_{1N} \\ 0 & \epsilon_2 - \mu & \ddots & \vdots & \vdots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & 0 & & & & \\ 0 & \cdots & 0 & \epsilon_N - \mu & (\Delta)_{N1} & \cdots & & (\Delta)_{NN} \\ (\Delta)^*_{11} & \cdots & & (\Delta)^*_{1N} & -\epsilon_1 + \mu & 0 & \cdots & 0 \\ \vdots & \ddots & & \vdots & 0 & -\epsilon_2 + \mu & \ddots & \vdots \\ & & & & \vdots & \ddots & \ddots & 0 \\ (\Delta)^*_{N1} & \cdots & & (\Delta)^*_{NN} & 0 & \cdots & 0 & -\epsilon_N + \mu \end{pmatrix} \begin{pmatrix} u_M^{(1)} \\ u_M^{(2)} \\ \vdots \\ u_M^{(N)} \\ v_M^{(1)} \\ v_M^{(2)} \\ \vdots \\ v_M^{(N)} \end{pmatrix} = E_M \begin{pmatrix} u_M^{(1)} \\ u_M^{(2)} \\ \vdots \\ u_M^{(N)} \\ v_M^{(1)} \\ v_M^{(2)} \\ \vdots \\ v_M^{(N)} \end{pmatrix}, \tag{15.8}$$

with $(\Delta)_{mn} := \int \mathrm{d}^3 r \phi^*_m(\mathbf{r}) \Delta(\mathbf{r}) \phi_n(\mathbf{r})$.[3] However, the BCS gap can be expressed in terms of the Bogoliubov transformation [Eqs. (15.4) and (15.5)] which yields

$$\begin{aligned} \Delta(\mathbf{r}) = -g \sum_{m,n} & v^*_m(\mathbf{r}) u_n(\mathbf{r}) \left\langle \gamma^\dagger_{ma} \gamma_{na} \right\rangle \\ & + u_m(\mathbf{r}) u_n(\mathbf{r}) \left\langle \gamma_{mb} \gamma_{na} \right\rangle \\ & - v^*_m(\mathbf{r}) v^*_n(\mathbf{r}) \left\langle \gamma^\dagger_{ma} \gamma^\dagger_{nb} \right\rangle \\ & + u_m(\mathbf{r}) v^*_n(\mathbf{r}) \left[\left\langle \gamma^\dagger_{nb} \gamma_{mb} \right\rangle - \delta_{mn} \right], \end{aligned} \tag{15.9}$$

and for the ground-state gap

$$\Delta_{\mathrm{GS}}(\mathbf{r}) = g \sum_m u_m(\mathbf{r}) v^*_m(\mathbf{r}). \tag{15.10}$$

Therefore, a diagonalization of Eq. (15.8) requires a self-consistent treatment together with Eq. (36.9). In doing so, we set the chemical potential $\mu = E_F$ with E_F the Fermi energy of the bare atomic system, i.e., we assume that the chemical potential is not effected by the BCS pairing. Strictly speaking this assumption is only valid in the deep BCS regime. However, our numerical data on the basis of the Anderson approximation (see section 15.3.3) show that it has no qualitative effect on the features studied in this work.

A final remark to Eq. (36.9): In the presented form the gap equation exhibits an ultraviolet-divergence, i.e., –strictly speaking– Eq. (36.9) needs to be regularized to ensure the convergence

[2] We have checked that the qualitative features investigated in this work are independent from the size of this window. However, the features shift quantitatively, e.g., the gaps and frequencies shift to larger values when increasing the window size.

[3] To calculate matrix elements of the form $\int \mathrm{d}^3 r \phi_m(\mathbf{r}) \phi_n(\mathbf{r}) \phi_k(\mathbf{r}) \phi_l(\mathbf{r})$ we used an analytical expression derived in [Lord(1949)].

of the sum over the BdG eigenstates [Bloch *et al.*(2008)]. However, in our case we restrict those sums to a rather narrow energy range around the Fermi level (see above) and this numerical cutoff remedies the need for a further regularization. This will be different in the calculations based on the Anderson approximation, as will be discussed below.

15.3.2 Quench dynamics

To calculate the dynamics of the BCS condensate we make use of the Bogoliubov transformation Eqs. (15.4) and (15.5). The quasiparticles resulting from that transformation are the single-particle excitations of the BCS phase, which are created, when the system is perturbed. All dynamical quantities investigated in the context of this work can be expressed in terms of expectation values of these quasiparticles. Therefore, we use Heisenberg's equation of motion for the quasiparticle operators to numerically calculate the dynamics of the quasiparticle expectation values.

To this end, we express the BdG Hamiltonian in terms of the single-particle operators for a general nonequilibrium situation where $\Delta(\mathbf{r},t) \neq \Delta_{\mathrm{GS}}(\mathbf{r})$, i.e., where the current value of the gap differs from the ground-state value $\Delta_{\mathrm{GS}}(\mathbf{r})$. Thus, inserting Eqs. (15.4) and (15.5) into Eq. (36.4) where $\Delta = \Delta(\mathbf{r},t) \neq \Delta_{\mathrm{GS}}(\mathbf{r})$ and identifiying Δ_{GS} via Eq. (36.9) yields

$$\begin{aligned}
H_{\mathrm{BdG}} = &\sum_m E_{ma}\left(\gamma^\dagger_{ma}\gamma_{ma} + \gamma^\dagger_{mb}\gamma_{mb} - 1\right) \\
&+ \sum_{m,n}\left[(\Delta - \Delta_{\mathrm{GS}})_{u^*_m v_n} + (\Delta^* - \Delta^*_{\mathrm{GS}})_{v^*_m u_n}\right]\gamma^\dagger_{ma}\gamma_{na} \\
&+ \sum_{m,n}\left[(\Delta - \Delta_{\mathrm{GS}})_{u^*_m u^*_n} - (\Delta^* - \Delta^*_{\mathrm{GS}})_{v^*_m v^*_n}\right]\gamma^\dagger_{ma}\gamma^\dagger_{nb} \\
&- \sum_{m,n}\left[(\Delta - \Delta_{\mathrm{GS}})_{v_m v_n} - (\Delta^* - \Delta^*_{\mathrm{GS}})_{u_m u_n}\right]\gamma_{mb}\gamma_{na} \\
&+ \sum_{m,n}\left[(\Delta - \Delta_{\mathrm{GS}})_{v_m u^*_n} + (\Delta^* - \Delta^*_{\mathrm{GS}})_{u_m v^*_n}\right]\gamma^\dagger_{mb}\gamma_{nb} \\
&- \sum_{m,n}\left[(\Delta - \Delta_{\mathrm{GS}})_{v_m u^*_n} + (\Delta^* - \Delta^*_{\mathrm{GS}})_{u_m v^*_n}\right],
\end{aligned} \tag{15.11}$$

with

$$\begin{aligned}
(\Delta - \Delta_{\mathrm{GS}})_{u_m v_n} := \\
\int \mathrm{d}^3 r\, u^*_m(\mathbf{r})\left[\Delta(\mathbf{r},t) - \Delta_{\mathrm{GS}}(\mathbf{r})\right] v_n(\mathbf{r}).
\end{aligned} \tag{15.12}$$

We insert this into Heisenberg's equation of motion

$$\frac{d}{dt}A_H = \frac{i}{\hbar}\left[H_H, A_H\right] + \left(\frac{\partial}{\partial t}A\right)_H, \tag{15.13}$$

where A_H is the corresponding operator in the Heisenberg picture and $\left(\frac{\partial}{\partial t}A\right)_H = 0$ for the quasiparticle operators since all our calculations are performed with fixed basis states $[u_m(\mathbf{r}), v_m(\mathbf{r})]$. For the required single-particle expectation values this yields the following

equations of motion:

$$
\begin{aligned}
i\hbar\frac{d}{dt}\langle\gamma^\dagger_{ma}\gamma_{na}\rangle = -(E_m - E_n)\langle\gamma^\dagger_{ma}\gamma_{na}\rangle + \sum_l \Big(\\
- \Big[(\Delta - \Delta_{\mathrm{GS}})_{u^*_l v_m} + (\Delta^* - \Delta^*_{\mathrm{GS}})_{v^*_l u_m}\Big]\langle\gamma^\dagger_{la}\gamma_{na}\rangle + \Big[(\Delta - \Delta_{\mathrm{GS}})_{u^*_n v_l} + (\Delta^* - \Delta^*_{\mathrm{GS}})_{v^*_n u_l}\Big]\langle\gamma^\dagger_{ma}\gamma_{la}\rangle \\
+ \Big[(\Delta - \Delta_{\mathrm{GS}})_{u^*_n u^*_l} - (\Delta^* - \Delta^*_{\mathrm{GS}})_{v^*_n v^*_l}\Big]\langle\gamma^\dagger_{ma}\gamma^\dagger_{lb}\rangle - \big[-(\Delta - \Delta_{\mathrm{GS}})_{v_l v_m} + (\Delta^* - \Delta^*_{\mathrm{GS}})_{u_l u_m}\big]\langle\gamma_{lb}\gamma_{na}\rangle\Big)
\end{aligned}
\tag{15.14}
$$

$$
\begin{aligned}
i\hbar\frac{d}{dt}\langle\gamma^\dagger_{mb}\gamma_{nb}\rangle = -(E_m - E_n)\langle\gamma^\dagger_{mb}\gamma_{nb}\rangle + \sum_l \Big(\\
- \Big[(\Delta - \Delta_{\mathrm{GS}})_{u^*_l v_m} + (\Delta^* - \Delta^*_{\mathrm{GS}})_{v^*_l u_m}\Big]\langle\gamma^\dagger_{lb}\gamma_{nb}\rangle + \Big[(\Delta - \Delta_{\mathrm{GS}})_{u^*_n v_l} + (\Delta^* - \Delta^*_{\mathrm{GS}})_{v^*_n u_l}\Big]\langle\gamma^\dagger_{mb}\gamma_{lb}\rangle \\
+ \Big[(\Delta - \Delta_{\mathrm{GS}})_{u^*_n u^*_l} - (\Delta^* - \Delta^*_{\mathrm{GS}})_{v^*_n v^*_l}\Big]\langle\gamma^\dagger_{la}\gamma^\dagger_{mb}\rangle - \big[-(\Delta - \Delta_{\mathrm{GS}})_{v_l v_m} + (\Delta^* - \Delta^*_{\mathrm{GS}})_{u_l u_m}\big]\langle\gamma_{nb}\gamma_{la}\rangle\Big)
\end{aligned}
\tag{15.15}
$$

$$
\begin{aligned}
i\hbar\frac{d}{dt}\langle\gamma_{mb}\gamma_{na}\rangle = -i\hbar\frac{d}{dt}\Big(\langle\gamma^\dagger_{na}\gamma^\dagger_{mb}\rangle\Big)^* = (E_m + E_n)\langle\gamma_{mb}\gamma_{na}\rangle + \sum_l \Big(\\
\Big[(\Delta - \Delta_{\mathrm{GS}})_{u^*_n v_l} + (\Delta^* - \Delta^*_{\mathrm{GS}})_{v^*_n u_l}\Big]\langle\gamma_{mb}\gamma_{la}\rangle + \Big[(\Delta - \Delta_{\mathrm{GS}})_{u^*_n u^*_l} - (\Delta^* - \Delta^*_{\mathrm{GS}})_{v^*_n v^*_l}\Big]\Big(\delta_{ml} - \langle\gamma^\dagger_{lb}\gamma_m \\
- \Big[(\Delta - \Delta_{\mathrm{GS}})_{u^*_l u^*_m} - (\Delta^* - \Delta^*_{\mathrm{GS}})_{v^*_l v^*_m}\Big]\langle\gamma^\dagger_{la}\gamma_{na}\rangle + \Big[-(\Delta - \Delta_{\mathrm{GS}})_{v_l u^*_m} + (\Delta^* - \Delta^*_{\mathrm{GS}})_{u_l v^*_m}\Big]\langle\gamma_{lb}\gamma_{na}\rangle\Big).
\end{aligned}
\tag{15.16}
$$

We solve these nonlinearly coupled equations of motion for the initial value problem defined by an instantaneous interaction quench. I.e., we start from the ground state corresponding to a scattering length a_i and instantaneously switch to a different value[4] $a_i \to a_f$. Therefore, during the quench at the time $t = 0$ the system has no time to relax to the new ground state corresponding to a_f but it remains in the old ground state corresponding to a_i. I.e., all quasiparticle expectation values of the form $\langle\gamma^\dagger_M\gamma_{M'}\rangle$ and $\langle\gamma_M\gamma_{M'}\rangle$ with $M \in \{ma, mb\}$ and $M' \in \{m'a, m'b\}$ which correspond to the old ground state vanish for $t = 0$. With this in mind, we invert the Bogoliubov transformation for the operators γ_M in the old basis before the quench and insert the resulting expressions for Ψ_σ and $\Psi^\dagger_\sigma$ into the quasiparticle operators in the new basis after the quench [Eqs. (15.4) and (15.5)]. This yields the initial values for the dynamics

$$
\begin{aligned}
\langle\gamma^\dagger_{ma}\gamma_{na}\rangle|_{t=0} = \sum_k \int \mathrm{d}^3r \Big[v_m(\mathbf{r})\tilde{u}_k(\mathbf{r}) - u_m(\mathbf{r})\tilde{v}_k(\mathbf{r})\Big] \\
\cdot \int \mathrm{d}^3r' \Big[v^*_n(\mathbf{r}')\tilde{u}^*_k(\mathbf{r}') - u^*_n(\mathbf{r}')\tilde{v}^*_k(\mathbf{r}')\Big]
\end{aligned}
\tag{15.17}
$$

$$
\begin{aligned}
\langle\gamma^\dagger_{ma}\gamma^\dagger_{nb}\rangle|_{t=0} = \sum_k \int \mathrm{d}^3r \Big[v_m(\mathbf{r})\tilde{u}_k(\mathbf{r}) - u_m(\mathbf{r})\tilde{v}_k(\mathbf{r})\Big] \\
\cdot \int \mathrm{d}^3r' \Big[v_n(\mathbf{r}')\tilde{v}^*_k(\mathbf{r}') + u_n(\mathbf{r}')\tilde{u}^*_k(\mathbf{r}')\Big]
\end{aligned}
\tag{15.18}
$$

$$
\langle\gamma_{mb}\gamma_{na}\rangle|_{t=0} = \Big(\langle\gamma^\dagger_{na}\gamma^\dagger_{mb}\rangle|_{t=0}\Big)^* \tag{15.19}
$$

[4] This assumption of an instantaneous quench is valid, since the experimentally achieved quench times ∼ns are way below the timescales of the gap dynamics ∼ms.

$$\langle\gamma_{mb}^{\dagger}\gamma_{nb}\rangle|_{t=0} = -\sum_k \int \mathrm{d}^3r \left[v_m^*(\mathbf{r})\tilde{v}_k(\mathbf{r}) + u_m^*(\mathbf{r})\tilde{u}_k(\mathbf{r})\right]$$
$$\cdot \int \mathrm{d}^3r' \left[v_n(\mathbf{r}')\tilde{v}_k^*(\mathbf{r}') + u_n(\mathbf{r}')\tilde{u}_k^*(\mathbf{r}')\right] + \delta_{mn}, \tag{15.20}$$

where $\tilde{u}_m$ and $\tilde{v}_m$ refer to the single-particle states before the quench and u_m and v_m to those after the quench.

With Eqs. (15.17)-(15.20) we can numerically integrate the equations of motion and thus calculate the gap dynamics for a gas of ^{6}Li in a 3D harmonic trap via Eq. (36.8). In section 15.4.1 we will present the corresponding results for the phase dynamics of the gap. However, before that we introduce Anderson's approximation which we will use to perform elaborate parameter scans and to calculate the gap dynamics for systems with rather large particle number that are numerically too complex to address within the full BdG approach. In doing so, we will restrict our explanations to the main aspects of the approximation. A detailed description of the corresponding formalism with all expressions derived from the above can be found in our previous work [Hannibal *et al.*(2015)].

15.3.3 Anderson's Approximation

In Anderson approximation the expansion of the quasiparticle wave function in Eqs. (15.6) and (15.7) is truncated such that $u_m(\mathbf{r}) = u_m\phi_m(\mathbf{r})$ and $v_m(\mathbf{r}) = v_m\phi_m(\mathbf{r})$. This strongly simplifies the formalism presented above. For the ground-state properties the diagonalization of Eq. (15.8) directly yields

$$E_m = \sqrt{(\varepsilon_m - \mu)^2 + (\Delta_{mm}^{\mathrm{GS}})^2} \tag{15.21}$$

and

$$u_m = \sqrt{\frac{1}{2}\left(1 + \frac{\varepsilon_m - \mu}{E_m}\right)} \qquad v_m = \sqrt{\frac{1}{2}\left(1 - \frac{\varepsilon_m - \mu}{E_m}\right)}. \tag{15.22}$$

which –in combination with Eq. (36.9)– leads to a BCS-like selfconsistency equation, that we solve numerically. In doing so, we use much larger energy windows in the sum over the states as compared to the full BdG approach since the numerical effort is strongly reduced in the case of the Anderson approximation. Thereby, we employ the regularization scheme introduced in [Hannibal *et al.*(2015)]. Only when directly comparing the full BdG equations with the Anderson approximation (Sec. 15.4.1) we use the same cutoffs in both calculations to improve the comparability of the respective calculations.

For the dynamical situation we furthermore assume that $(\Delta - \Delta_{\mathrm{GS}})_{x_m y_n} = (\Delta - \Delta_{\mathrm{GS}})_{x_m y_n}\,\delta_{mn}$ with $x_m, y_m \in \{u_m, v_m\}$, i.e., that the main implications of the Anderson approximation $\Delta_{mn}^{\mathrm{GS}} = \Delta_{mn}^{\mathrm{GS}}\delta_{mn}$ holds for the dynamical situation as well. That leads to a great simplification of the equations of motion.

However, strictly speaking Anderson's approximation is only valid if $\Delta_{mn}^{\mathrm{GS}} \ll \delta\varepsilon$ with $\delta\varepsilon$ the level spacing of the harmonic eigenenergies. In general, this only holds for weak coupling, i.e., deep in the BCS regime, and/or for strong confinements and thus large level spacing. Nevertheless, our numerical data show that the approximation reproduces all the main features investigated in this work even for moderate confinements in the BCS-BEC crossover regime (cf. section 15.4.1).

15.4 Results

In the following, we investigate the phase dynamics of the spatially averaged BCS gap[5]

$$\bar{\Delta}(t) = \frac{1}{V}\int \mathrm{d}^3 r \Delta(\mathbf{r}, t) \tag{15.23}$$

for an ultracold gas of ^{6}Li in a cigar-shaped harmonic trap. We use $V = l_x l_y l_z$ with l_α being the oscillator length in direction α as a normalization volume. We excite the system by interaction quenches $a_i \to a_f$.

In section 15.4.1 we will identify the emergence of a Goldstone mode in the phase dynamics of the BCS gap with one dominant low-frequency contribution. Furthermore, we will show that the results obtained within the Anderson approximation are in good agreement with the full BdG solution. In section 15.4.2 we will analyze this Goldstone mode over a wide range of parameters and we will show that its excitation spectrum is gapless and that its main frequency is not determined by the trap parameters but by the details of the excitation.

In section 15.4.3 we will investigate the influence of the confinement parameters on the phase dynamics and by that the effect of the superfluid resonances found in [Shanenko *et al.*(2012)]. Furthermore, in section 15.4.4 we will evaluate the impact of the interaction quench-induced Goldstone mode on the single-particle excitations of the cloud which could provide an experimental access to the gapless phase mode.

15.4.1 Phase dynamics of the gap

Figure 15.2 shows the dynamics of the phase $\varphi = \arg(\bar{\Delta})$ of the spatially averaged BCS gap for a system with $f_\perp = 1\,\mathrm{kHz}$ and $f_\parallel = 96\,\mathrm{Hz}$ excited by quenches with the strength $\delta\,[1/(k_F a)] = 1/(k_F a_f) - 1/(k_F a_i) = -0.1$ at different positions in the BCS-BEC crossover. The particle number is set to $N_P = 120$ and the expansion of the single-particle wave functions in Eqs. (15.6) and (15.7) is restricted to atomic states from a window of width $\Delta\epsilon := 0.92 E_F$ around the chemical potential (i.e., $\mu - 0.46 E_F \leq \varepsilon \leq \mu + 0.46 E_F$). This is the limitation of our current numerical setup for the full BdG approach.

The solid lines in Fig. 15.2 show the data obtained by the full equations of motion (15.14) - (15.16). First of all, one clearly observes that –for all quenches– the phase dynamics of the gap is strongly dominated by a linear decrease in time. Therefore, after the quench the system performes a constant phase "motion" of the gap which nicely corresponds to the simplified picture of a Mexican-hat potential introduced in Fig. 15.1: The potential is flat inside the rim which implies a constant phase velocity, i.e., a steady oscillation inside the rim where the frequency of the latter –i.e., the frequency of the Goldstone mode– is defined by the time-averaged slope $f_G := \frac{1}{2\pi}\frac{\Delta\varphi}{\Delta t}$ where Δt is large compared to the intrinsic time scales of the system.[6] The corresponding values are given by $f_G = 1.6\,\mathrm{Hz}$ for the weakest coupling strength and $f_G = 21.9\,\mathrm{Hz}$ for $1/(k_F a_f) = -0.9$ and $f_G = 118.7\,\mathrm{Hz}$ for $1/(k_F a_f) = -0.5$ (for illustrative purposes the curves for the two weaker coupling strengths are scaled by a factor 3 and 10, respectively). Thus, the interaction quench-induced "phase velocity" strongly increases when approaching the unitary point $1/(k_F a) = 0$.

Furthermore, a closer look at Fig. 15.2 reveals that a higher-frequency oscillation exists on top of the linear contribution. This contribution is strongest for the system with $1/(k_F a_f) =$

[5] In our case the phase of the gap φ is nearly homogeneous (see below), i.e., $\bar{\varphi} \approx \varphi(\mathbf{r})$.

[6] We want to remark that the appearance of a Goldstone mode with a fixed frequency is similar to the case of the AC Josephson effect. There, a difference in the chemical potentials μ_i on the two sides of a Josephson junction leads to the emergence of a Goldstone mode with the frequency given by $f_G \sim (\mu_2 - \mu_1)$ [Pekker and Varma(2015)]. In contrast, in our case the Goldstone mode is driven by the quench which drives the system instantaneously from an equilibrium into a non-equilibrium state.

-1.4 and much weaker –and thus not directly visible in Fig. 15.2– for the stronger-coupling cases. A more detailed analysis shows that the corresponding frequencies again increase when approaching the unitary point, i.e., this dynamics is fast for large and slow for small coupling strengths. Nevertheless, the corresponding range of frequencies coincides with that from the spectrum of the Higgs mode. This indicates that the Higgs and the Goldstone mode are weakly coupled, where we observe that the influence of the Higgs mode increases when approaching the BCS limit $1/(k_F a) \ll -1$ and when increasing the modulus of quench strength $|\delta\,[1/(k_F a)]\,|$. However, the Higgs mode was extensively studied in [Hannibal *et al.*(2015)]. Therefore, in this work we will not go into details about these contributions.

Figure 15.2: (color online) Phase dynamics for the full BdG solution (solid lines) and the Anderson approximate solution (dashed lines) for $\delta\,[1/(k_F a)] = -0.1$ and different final coupling strengths: $1/(k_F a_f) = -0.5$, $1/(k_F a_f) = -0.9$ (scaled by a factor of 3) and $1/(k_F a_f) = -1.4$ (scaled by a factor of 10); parameters: $f_\parallel = 96\,\mathrm{Hz}$, $f_\perp = 1\,\mathrm{kHz}$, $N_P = 120$.

The dashed lines in Fig. 15.2 show the Anderson approximate solution corresponding to section 15.3.3 (note that in order to improve the comparability here we have used an ungregularized Anderson solution with the same cutoff as in the calculations without Anderson approximation). One can see that the approximate solution gives an overall good qualitative agreement with the full dynamics: It shows the same linear decrease in time with a –in the stronger coupling cases not directly visible– higher-frequency contribution on top. Again, the slow linear decrease corresponds to the Goldstone mode while the higher-frequency compo-

nent results from the coupling to the Higgs mode[7]. However, one can observe as well that the quantitative deviations between the full and the approximate solution increase with increasing coupling strength: While –considering that the corresponding curves in Fig. 15.2 are scaled by a factor of 10– the frequency of the Goldstone mode, i.e., the slope of the linear contribution, matches very well for the case of $1/(k_F a_f) = -1.4$ the deviation of the two solutions becomes rather significant when increasing the coupling strength.

Figure 15.3: (color online) Frequency of the Goldstone mode resulting from the full BdG solution (solid line) and from Anderson's approximate solution (dashed line) for the same system as in Fig. 15.2 but varying $1/(k_F a_f)$ (to reduce the numerical effort we set $\Delta\epsilon = 0.58 E_F$).

This is illustrated in Fig. 15.3. There, the frequencies of the Goldstone mode resulting from the full and from the approximate solution are shown for a fixed quench strength $\delta\,[1/(k_F a)] = -0.1$ and varying $1/(k_F a_f)$. One can clearly observe that the approximate solution reproduces well the qualitative trend, i.e., an overall increase of the frequency of the Goldstone mode when approaching the unitary point. Indeed, when entering the BCS regime even the quantitative values match very well. However, one can see as well, that the deviation between both solutions increases with increasing coupling strength and becomes rather large in the crossover regime. This indicates that Anderson's approximation tends to break down –as expected– when approaching the unitary point.

Nevertheless, we want to emphasize that all features in the phase dynamics investigated in this work are fully reproduced by the Anderson approximation. Solely the shift in frequency

[7]The agreement of both solutions with respect to the Higgs mode may not look very convincing. However, already rather small deviations in the spectral composition of the Higgs mode result in large deviations in the time domain at larger times. I.e., the qualitative agreement of both solutions is in indeed good.

between the full and the approximate solution increases with increasing coupling strength. Therefore, all following calculations of this work are performed in Anderson approximation which allows for a drastical reduction of computational effort and thus for a detailed investigation of parameter dependencies and of larger systems at all. In doing so, from now on all sums will be restricted to states from a window of size $2E_F$ around the Fermi level instead of $0.92E_F$ as before to ensure a better quantitative convergence of the obtained frequencies (cf. [Hannibal *et al.*(2015)]).

Continuing in Anderson approximation we will now investigate the phase dynamics in closer detail, i.e., we will analyze the influence of the external parameters on the frequency f_G of the Goldstone mode and we will point out by which quantities it is determined. To do so, we will first investigate the influence of the coupling strength and of the details of the quench. In section 15.4.3 we will focus on the effect of the confinement.

15.4.2 Influence of the quench

Figure 15.2 already suggests that the frequency of the Goldstone mode f_G depends on the details of the quench instead of being fixed by the external parameters of the cloud. The phase dynamics changes from a very slow decrease for the system in the BCS regime with $1/(k_F a_f) = -1.4$ to a rather fast decrease in the crossover regime with $1/(k_F a_f) = -0.5$. However, in Fig. 15.2 the quench strength by means of $\delta\,[1/(k_F a)]$ was kept fixed. Only the position in the BCS-BEC crossover was varied. Accordingly, we will in the following investigate the influence of the quench strength on the phase dynamics for one particular final coupling strength $1/(k_F a_f)$ in Fig. 15.4 and for a wider range of $1/(k_F a_f)$ in Fig. 15.5. In doing so, we will show, that the excitation spectrum of the interaction quench-induced Goldstone mode is gapless, i.e., it is a Goldstone mode in the original sense of the Goldstone theorem. Furthermore, we will show that its frequency can be adjusted by changing the details of the quench and that it is determined by the initial values of the dynamics and by the gap of the system after the quench.

Figure 15.4: (color online) Frequency of the Goldstone mode f_G for quenches with varying strength to $1/(k_F a_f) = -0.9$; parameters: $f_\parallel = 96\,\mathrm{Hz}$, $f_\perp = 1\,\mathrm{kHz}$, $N = 1000$.

Figure 15.4 shows the frequency of the Goldstone mode for a system with $1/(k_F a_f) = -0.9$ in the same trap as in Fig. 15.2 but –since we now apply Anderson's approximation– for a larger particle number[8] of $N_P = 1000$. The initial coupling strength $1/(k_F a_i)$ is varied, i.e., the dependence of f_G on the quench strength is shown.

First of all, Fig. 15.4 demonstrates that the phase dynamics strongly depends on the quench strength: When approaching $1/(k_F a_i) = 1/(k_F a_f) = -0.9$, i.e., the point of quench strength $\delta\,[1/(k_F a)] = 0$, from either side [from larger or smaller values of $1/(k_F a_i)$] the frequency of the Goldstone mode continuously decreases to zero. Thus, the frequency of the Goldstone mode decreases with decreasing (modulus of the) quench strength and continuously vanishes for $\delta\,[1/(k_F a)] \to 0$. This implies that the excitation spectrum of the interaction quench-induced phase mode is indeed gapless as stated above.

Furthermore, we observe that the dependence of f_G on the quench strength in Fig. 15.4 is asymmetric: Negative quenches with $\delta\,[1/(k_F a)] < 0$, i.e., those on the left hand side of Fig. 15.4, lead to a stronger increase in f_G and thus to larger frequencies than positive quenches. This asymmetry is linked to the fact that the same excitation strength in terms of $1/(k_F a)$, i.e.,

[8] With our current numerical setup we can calculate the dynamics of single systems for up to $N_P \sim 10^4$ particles. However, there are no qualitative changes in the gap dynamics when increasing the particle number, i.e., here we restrict ourselves to rather small N_P to reduce the numerical effort.

the same $|\delta\,[1/(k_F a)]\,|$, results in different actual changes in the scattering length $|\delta a| = |a_f - a_i|$ depending on the position of the initial system in the crossover. Hence, negative quenches lead to much larger changes in the scattering length $|\delta a|$ –and therefore in the gap– than positive quenches. As we will show later on, this results in larger f_G. However, before we do so, we will demonstrate that the features found above hold for a wide range of quenches in the BCS-BEC crossover. Indeed, our numerical data indicate that the above found nature of the phase dynamics holds for all moderate quenches on the BCS side of the BCS-BEC crossover, i.e., for all quenches that can be associated with the phase II of the quantum quench phase diagram introduced in [Yuzbashyan *et al.*(2015)]. For quenches exceeding this range, the phase dynamics tends to become irregular. In particular, strong negative quenches which lead to a dynamical vanishing of the gap in the Higgs mode (phase I) exhibit a persistent but very irregular phase dynamics which makes the definition of a frequency of the Goldstone mode arbitrary. However, in this work we restrict ourselves to the investigation of the regular phase dynamics in the gapless Golstone mode, i.e., to quenches in the phase II.

To this end, Fig. 15.5 shows the dependence of f_G on the quench strength for varying $1/(k_F a_f)$, i.e., each horizontal line in Fig. 15.5 corresponds to a plot like in Fig. 15.4 but for a different coupling strength.

Figure 15.5: (color online) Frequency of the Goldstone mode f_G for different excitations $1/(k_F a_i) \to 1/(k_F a_f)$ corresponding to phase II of the quantum quench phase diagram (see main text); quenches corresponding to phase I are not shown; parameters: $f_\parallel = 96\,\mathrm{Hz}$, $f_\perp = 1\,\mathrm{kHz}$, $N = 1000$.

Indeed, one observes on the basis of Fig. 15.5 that the nature of the Goldstone mode

described above holds for all phase II quenches investigated here (we omit the quenches in the upper left corner corresponding to phase I). For each horizontal line in Fig. 15.5 we see a decrease of f_G with decreasing modulus of the quench strength $|\delta\,[1/(k_F a)]\,|$ as found above with a continously vanishing Goldstone mode when approaching $1/(k_F a_i) = 1/(k_F a_f)$. However, furthermore one can see, that the frequency of the Goldstone mode also depends on the vertical position of the quench in Fig. 15.5: It is largest for quenches at the bottom, i.e., for systems with large coupling strength. In fact, one can observe that quenches with the same $|\delta\,[1/(k_F a)]\,|$ but opposite directions, e.g., those correspoding to the points $[1/(k_F a_i), 1/(k_F a_f)] = (-1.0, -0.5)$ and $[1/(k_F a_i), 1/(k_F a_f)] = (-0.5, -1.0)$ lead to significantly different f_G. This indicates that f_G depends not only on the quench strength but also on the coupling strength of the system after the quench, i.e., a stronger coupling results in a larger f_G. In the following, we will explain both these features –the dependence on the quench strength and the dependence on the final coupling strength– by taking into account that the only parameters of the dynamics affected by the quench are the initial values of the excitation and the gap after the quench. We will isolate the effects of both quantities on the basis of Fig. 15.5 and Eqs. (15.17)-(15.20).

To do so, we at first want to state that the frequency of the Goldstone mode depends linearly on the initial values of the dynamics which in Anderson approximation read (cf. [Hannibal *et al.*(2015)] and Eqs. (15.17)-(15.20))

$$\begin{aligned}\left\langle \gamma_{ma}^\dagger \gamma_{ma} \right\rangle\big|_{t=0} &= (v_m \tilde{u}_m - u_m \tilde{v}_m)^2 =: x_m^{(0)} \\ \left\langle \gamma_{ma}^\dagger \gamma_{mb}^\dagger \right\rangle\big|_{t=0} &= (v_m \tilde{u}_m - u_m \tilde{v}_m)\,(v_m \tilde{v}_m + u_m \tilde{u}_m) \\ &=: y_m^{(0)},\end{aligned} \tag{15.24}$$

with $\tilde{u}_m$ and $\tilde{v}_m$ (u_m and v_m) being the Anderson amplitudes before (after) the quench [see Eq. (15.22)]. Indeed, our numerical data show that by artificially multiplying these initial values by a factor k the frequency f_G increases by the same factor[9], i.e.,

$$f_G\left(k \cdot \{x_m^{(0)}, y_m^{(0)}\}\right) = k f_G\left(\{x_m^{(0)}, y_m^{(0)}\}\right).$$

This means that –whatever the actual system parameters (a_f, N_P, $f_\parallel$, $f_\perp$) are– the frequency of the Goldstone mode can be tuned by adjusting the initial values of the dynamics. As Eqs. (15.22) and (15.24) imply, the latter depend on the gaps of the system before and after the quench which are defined by the quench.

A detailed evaluation of Eqs. (15.24) shows that the initial values are large when the difference between the gaps before and after the quench and thus between the amplitudes $(\tilde{u}_m, \tilde{v}_m)$ and (u_m, v_m) is large. This is the case for strong quenches, i.e., strong quenches result in large initial values which well coincides with the above findings for f_G. However, Eqs. (15.24) also show that the amplitude of the initial excitation is independent from the quench direction. When changing the quench direction, i.e., when interchanging $(\tilde{u}_m, \tilde{v}_m) \Leftrightarrow (u_m, v_m)$ in Eqs. (15.24), only the sign of the anomalous initial values $\left\langle \gamma_{ma}^\dagger \gamma_{mb}^\dagger \right\rangle\big|_{t=0}$ changes. Therefore, the initial values for quenches with the same $|\delta\,[1/(k_F a)]\,|$ but opposite directions have the same strength but –as observed above– the frequency is larger for the respective positive quench to the stronger coupling $1/(k_F a_f)$. This indicates that the frequency of the Goldstone mode also depends on the actual coupling strength of the system after the quench.

We can conclude the results of this paragraph: The excitation spectrum of the Goldstone mode of the BCS gap of a 3D confined ultracold Fermi gas excited by an interaction quench is gapless and the frequency of the Goldstone mode can be tuned in a wide range by adjusting the

[9]For the situation of strong interactions and strong quenches even an analytical expression can be found: $\omega_G = 2\pi f_G \approx \mathrm{Im}(\frac{d}{dt}\bar{\Delta})|_{t=0}/\bar{\Delta}_{\mathrm{GS}}$.

strength of the quench. Again, this coincides with the simplified picture of Fig. 15.1: However small the quench-induced initial "momentum" of the phase dynamics might be, it results in a constant "phase motion" inside the rim. This is –in simple words– the consequence of the Goldstone theorem, which therefore can well be observed in the dynamics after an interaction quench.

However, the fact that in our case the Goldstone mode is gapless and that its frequency can be adjusted by the strength of the excitation stands in contrast to the experimental and theoretical findings of Refs. [Kinast *et al.*(2004a), Kinast *et al.*(2004b), Bartenstein *et al.*(2004), Altmeyer *et al.*(2007a), Altmeyer *et al.*(2007b), Riedl *et al.*(2008), Baranov and Petrov(2000), Bruun and Mottelson(2001), Bruun(2002), Hu *et al.*(2004), Heiselberg(2004), Stringari(2004), Grasso *et al.*(2005), Korolyuk *et al.*(2011)]. There, the frequency of the phase dynamics was found to be fixed by the frequencies of the trapping potential. Nevertheless, this discrepancy can be explained by the circumstance that in the previous works the dynamics was induced by spatially inhomogeneous perturbations of the cloud, e.g., by confinement quenches or optical excitations. Such a spatial perturbation creates a motion of the superfluid in the trap with a time-dependent verlocity v_s. This directly induces a dynamics of the phase of the gap via [Giorgini *et al.*(2008)]

$$\mathbf{v_s} = \frac{\hbar}{2m}\nabla\arg\big[\Delta(\mathbf{r},t)\big]. \tag{15.25}$$

On the one hand, this means that –whenever a superfluid velocity v_s is excited– the real-space dynamics is directly linked to a dynamics in the Goldstone mode. In this sense, the latter can be observed through the motion of the cloud. On the other hand, the trapping potential governs the real-space dynamics of the cloud. Therefore, a direct coupling of the Goldstone mode to the real-space dynamics implies that the trap imprints its frequencies on the Goldstone mode. For inhomogeneous excitations f_G is thus pushed to the trap frequencies as found in the previous works.[10]

In our case the excitation is spatially homogeneous and does not –as our numerical data confirm– produce any significant phase gradients. Therefore, no real-space dynamics of the cloud is induced by the interaction quench which implies that the phase dynamics does not couple to the trap. Accordingly, the excitation spectrum of this *homogeneous* Goldstone mode remains gapless. In this sense the interaction quench-induced Goldstone mode remains pure.

15.4.3 Impact of the superfluid resonances

In this section we want to study the influence of the confinement on the frequency of the gapless Goldstone mode and by that the impact of the size-dependend superfluid resonances theoretically predicted in [Shanenko *et al.*(2012)]. To do so, we will investigate the dependence of the phase dynamics on the trapping frequency in x-y direction $f_\perp$ for a system with fixed $f_\parallel = 96\,\mathrm{Hz}$, $N_P = 1000$ atoms in the trap and a quench given by $1/(k_F a) = -0.8 \to -0.9$. The complementary situation, i.e., a fixed $f_\perp$ with varying $f_\parallel$ produces the same effects. Therefore, the influence of $f_\parallel$ will not be investigated separately.

Figure 15.6 shows the frequency of the gapless Goldstone mode for the above system over a wide range of $f_\perp$. One can see, that –on top of a global increasing trend[11]– f_G exhibits a series of local maxima for different values of $f_\perp$. The distance of the maxima increases with increasing $f_\perp$ while at the same time the maxima become more pronounced. I.e., the maxima in f_G occur less frequent but more pronounced when approaching higher values of $f_\perp$.

[10] Actually, both manifestations of the Goldstone mode –the gapped inhomogeneous and the gapless one– may exist at the same time depending on the nature of the excitation.

[11] This global increase is due to the circumstance that the density of the condensate and thus the gap increases when $f_\perp$ is increased with fixed $f_\parallel$ and N_P.

Furthermore, the dashed lines in Fig. 15.6 indicate the positions of integer system parameter $S = \mu/\hbar\omega_\perp$ with $\omega_\perp = 2\pi f_\perp$. These positions indicate trap parameters where the minimum of an atomic subband crosses the chemical potential (for a detailed description of the band structure see section 15.4.4). To be precise: The distance between two atomic subbands is $\hbar\omega_\perp$ and the minimum of the lowest subband is at $\varepsilon = \hbar\omega_\perp$. Therefore, the integer part of S is the number of subbands that have a minimum below/at the chemical potential, i.e., the number of subbands crossing the chemical potential. One can see that every resonance closely follows such a point of integer system parameter S.

An explanation for this behavior can be given on the basis of the atypical BCS-BEC crossover [Shanenko *et al.*(2012)]: On the one hand, the atomic states closest to the chemical potential contribute strongest to the pairing. On the other hand, the states with the lowest quantum numbers m_z which are located at the subband minima exhibit the strongest interaction matrix elements [Hannibal *et al.*(2015)]. Therefore, each time an atomic subband crosses the chemical potential ($S = 1, 2, 3, ..$) the pairing is enhanced and the system is shifted towards the unitary point. Following section 15.3.2, this results in a larger frequency of the gapless Goldstone mode.

However, the decrease of the impact of the resonances for increasing system parameter reflects the circumstance that for large S several subbands contribute to the pairing while only a small fraction of the corresponding atomic states exhibits an enhanced coupling due to the resonance. Thus, for increasing S the influence of the resonant states on the overall coupling decreases.

Figure 15.6: (color online) Frequency of the gapless Goldstone mode f_G for fixed $f_\parallel = 96\,\mathrm{Hz}$, $N_P = 1000$ and $1/(k_F a) = -0.8 \to -0.9$; upper label: system paramter $S = \mu/\hbar\omega_\perp$.

15.4.4 Goldstone mode in the single-particle excitations

In this section we study the impact of the phase dynamics of the BCS gap on an experimentally more relevant physical quantity, the single-particle excitations of the condensate. An experimental investigation of the single-particle excitations has already been reported in [Stewart *et al.*(2008)] via RF-spectroscopy. Thus, they could provide a convenient access to the quench dynamics investigated here. Indeed, we will show that the gapless Goldstone mode is directly visible in the dynamics of the single-particle occupations and that it leads to a full inversion of the lowest-lying single-particle states. We will demonstrate this by investigating the effect of the phase dynamics on individual occupations as well as on the whole single-particle band structure.

In doing so, we focus on a cloud with the confinement frequencies given by $f_\parallel = 56\,\mathrm{Hz}$ and $f_\perp = 4\,\mathrm{kHz}$, with $N_P = 1700$ atoms in the trap and with an excitation of $1/(k_F a) = -0.8 \to -0.9$. For such a cigar-shaped trap the atomic energies $\varepsilon_{m_x,m_y,m_z} := \varepsilon_m$ are strongly separated with respect to m_x and m_y and comparatively dense with respect to m_z, i.e., they form subbands. The single-particle energies of Eq. (15.21) inherit this band structure which can be seen in Fig. 15.7.

Figure 15.7: (color online) Single-particle energies for a strongly confined Fermi gas in a BCS phase with four atomic subbands crossing the chemical potential; Δ_i^S denotes the gap of the subbands with $m_x + m_y = i$ and (m_x, m_y) denotes the subband index (see main text); the marked states (1), (2), (3) are discussed below; parameters: $f_\parallel = 56\,\mathrm{Hz}$, $f_\perp = 4\,\mathrm{kHz}$, $N = 1700$ and $1/(k_F a) = -0.9$.

There, a plot of the single-particle energies against the quantum number m_z is shown for the system introduced above. One clearly observes several subbands each of which corresponds to certain sets of quantum numbers (m_x, m_y), where –due to the cylindrical symmetry of the system– each subband is $2(m_x+m_y+1)$ fold degenerate (the factor 2 results from the degeneracy of the two single-particle branches corresponding to the two spin configurations)[12]. Furthermore, the four subbands with the lowest sets of quantum numbers (m_x, m_y) show minima when the corresponding atomic subbands cross the chemical potential, i.e., at different values for m_z. The states located at the minima thus lie in close vicinity to the chemical potential and contribute strongly to the BCS pairing (the expectation values $\Delta_{mm}^{\mathrm{GS}}$ corresponding to these states, i.e., the subband gaps, will be denoted as Δ_i^S with $i = m_x + m_y$ being the subband index; see Fig. 15.7). The higher atomic subbands with $m_x + m_y \geq 4$ do not cross the chemical potential. The corresponding single-particle subbands therefore do not exhibit any minima.

Since the single-particle operators corresponding to the energies of Fig. 15.7 –i.e., those

[12]Actually, subbands with different and not just interchanged quantum numbers (m_x, m_y) are not exactly degenerate due to slightly different subband gaps Δ_m^S (on the order of 0.1 peV for the investigated systems). For example: The subbands $(0,3)$ and $(3,0)$ are exactly degenerate, whereas the subbands $(0,3)$ and $(1,2)$ are split by $\sim 0.5\,\mathrm{peV}$. However, in the presented plots and with the assumed experimental accuracy this splitting is not resolved.

of Eqs. (15.4) and (15.5)– are defined in the excitation picture all energy states of Fig. 15.7 are not occupied in the ground state before the quench. But, during the temporal evolution following the quench occupations of the order of 1 are created. We will show this explicitly for three particular single-particle states [marked as (1), (2) and (3) in Fig. 15.7], one close to the minimum of the subband (1,2) ($E_m = 5.2\,\mathrm{peV}$), one at the minimum of the subband (0,0) ($E_m = 4.7\,\mathrm{peV}$) and one at a higher energy in the subband (0,0) ($E_m = 8.4\,\mathrm{peV}$). Furthermore, we will identify the gapless Goldstone mode of the BCS gap in the corresponding dynamics.

The dynamics of the three single-particle occupations is shown in Fig. 15.8 (a) for the first 20 ms after the quench. We clearly observe that all occupations oscillate in phase with one dominant low frequency. The states (1) (blue line) and (2) (red line) have a large amplitude of the order of 1 while the amplitude of state (3) (green line) is much smaller. Furthermore, the three occupations each exhibit an individual weak higher-frequency component which has the largest frequency for the state (3) of high energy. However, we find that the amplitude of the higher-frequency component increases with decreasing the scattering length, i.e., when entering the BCS regime with $1/(k_F a) < -1$.

Figure 15.8: (color online) (a) Dynamics of three particular single-particle occupations, one state of higher energy (3) and two from subband minima (1), (2) (see Fig. 15.7). (b) Fourier transform of the functions in (a).

A comparison of the single-particle dynamics with the dynamics of the phase of the gap [Fig. 15.8 (a); dashed line] shows that the dominant low oscillation frequency originates from

the Goldstone mode of the gap: The phase of the gap shows the same linear dynamics as in section 15.3.2 with a rate corresponding to the low-frequency part of the single-particle occupations. Thus, the gapless Goldstone mode is directly visible in the excitation dynamics of the condensate.

To investigate the single-particle dynamics in closer detail, Figure 15.8 (b) shows the Fourier spectrum of the data of Fig. 15.8 (a). Again, we observe that the dominant low frequencies of the occupations and the phase of the BCS gap exactly match. But, the origin of the higher frequencies in the excitation dynamics can now be seen as well: Besides the dominant low-frequency components each spectrum exhibits a series of weak peaks at approximately twice the energy of the corresponding single-particle state. I.e., the higher-frequency components result from an eigenoscillation of the single-particle occupations. In Ref. [Hannibal *et al.*(2015)] a sum of all eigenoscillations was shown to result in the Higgs mode of the gap. The higher-frequency components can thus be understood as fragments of the Higgs mode of the BCS gap.

Figure 15.9: (color online) Single-particle occupations at different times after the interaction quench; parameters: See Fig. 15.7.

At last, –to analyze the impact of the gapless Goldstone mode on the whole single-particle spectrum– Figure 15.9 shows snapshots of the single-particle occupations plotted against the quantum number m_z and the excitation energy E_m, like they could be measured by angle- and momentum-resolved RF spectroscopy [Stewart *et al.*(2008)], for a series of time steps after the quench. The first snapshot corresponds to the time $t = 0.6\,\mathrm{ms}$ and thus directly follows the quench. Here, the excitations are rather weak and can hardly be seen. However, going on in time we observe that all occupations increase in phase until the time $t = 3.8\,\mathrm{ms}$, where the maximum occupation of all states is reached. Afterwards the occupations decrease until the initial situation is reached again. Thus, Fig. 15.9 is an illustration of the in-phase oscillation of all single-particle occupations due to the gapless Goldstone mode.

In addition, –neglecting the contributions from the eigenoscillation– the snapshot for $t =$

3.8 ms provides a map of the amplitude of the single-particle oscillations since here the dominant low-frequency part of all occupations exhibits its maximum value. On the basis of this amplitude map one observes, that the amplitude distribution shows a resonance behaviour: The amplitude is largest for states with low quantum number m_z and low energy E_m and decreases with increasing values of m_z and E_m. In fact, directly at the minimum of the subbands with $m_x + m_y = 3$ the oscillation amplitude is 4, decreasing by 1 for every next lower subband $m_x + m_y$. However, this dependence is due to the $(m_x + m_y + 1)$-fold degeneracy of the subbands (only the particle-like excitations of the single-particle branch a are shown). I.e., the oscillation amplitude of each individual single-particle occupation at a subband minimum is 1. Thus, the single-particle occupations at the subband minima exhibit a full inversion.

Concluding this section we can thus state: We have shown that the gapless Goldstone mode of an interaction-quenched ultracold Fermi gas directly couples to the single-particle occupations and leads to a full inversion of the lowest-lying states. An experimental access to the dynamical single-particle occupations would thus allow for a direct observation of the massless Goldstone Boson predicted by the Goldstone theorem. However, we want to remark that an application of RF-spectroscopy –a state-of-the-art experimental access to the single-particle excitations [Stewart *et al.*(2008), Stewart *et al.*(2010)]– to the dynamical situation is restricted to the observation of the inhomogeneous phase dynamics. It turns out that it does not contain any signature of the gapless homogeneous Goldstone mode (see appendix). Therefore, at least a modification of this experimental technique would be required to observe the gapless Goldstone mode via the single-particle excitations.

15.5 Gapless Goldstone mode and RF spectroscopy

One way to study the single-particle occupations in experiment is RF spectroscopy as was shown in Ref. [Stewart *et al.*(2008)]. There, a first direct measurement of the single-particle excitations via RF spectroscopy was achieved for a thermal superfluid gas of ultracold ^{40}K in the BCS-BEC crossover. An application of RF spectroscopy to the dynamical situation could thus allow for a direct observation of the Goldstone mode without coupling it to the trap. This could be achieved, e.g., via a pump-probe like experimental setup: By introducing a delay time t between the quench (the "pump pulse") and the actual RF measurement (the "probe pulse") a time-resolved single-particle spectrum like in Fig. 15.9 could be obtained. However, in the following we will show that RF signals give a direct measurement of the single-particle occupations only if no single-particle coherences are present. In contrast, we will demonstrate that –for our case of a coherent evolution of the condensate– the single-particle coherences cancel the signature of the gapless Goldstone mode in the RF signal and thus prohibit its direct observation via RF spectroscopy.

The basic principle of RF spectroscopy applied to ultracold Fermi gases is to optically excite the atoms from one of the two hyperfine states of the condensate, i.e., the state denoted as $|k\uparrow\rangle$, to a third hyperfine state which is not involved in the BCS pairing (following Ref. [Ketterle and Zwierlein(2008)] we will denote this state as $|k\sigma\rangle$ with $\sigma = 3$; depending on the atom species used the actual spin of the corresponding hyperfine state will be different though). Then, the resulting occupations of the third hyperfine state can be used –at least in the absence of coherences between the single-particle states– as a direct measure of the corresponding single-particle occupations (see below and [Ketterle and Zwierlein(2008)]).

As stated above, an RF excitation results in a simultaneous spin flip of all atoms in the cloud which is described by the operator [Ketterle and Zwierlein(2008)]

$$\hat{V} = V_0 \sum_k \left(c^\dagger_{k3} c_{k\uparrow} + c^\dagger_{k\uparrow} c_{k3} \right) , \tag{15.26}$$

with $c^\dagger_{k\sigma}$ ($c_{k\sigma}$) creating (annihilating) one atom in the state with quantum number k and spin index $\sigma \in \{\uparrow, \downarrow, 3\}$. Here, we assume the excitation to be orthogonal with respect to the quantum number k, i.e., we only consider transitions $|k\uparrow\rangle \to |k'3\rangle$ with $k = k'$. Strictly speaking this applies only to very large systems where k corresponds to the wave number and if we furthermore assume $\hbar k_{RF} \ll \hbar k$, i.e., that the momentum of the photons is much smaller than the momentum of the atoms. However, in Anderson Approximation transitions with $k \neq k'$ do not contribute to the RF signal since all nondiagonal single-particle expectation values vanish (see below). Thus, we can apply Eq. (15.26) to our current situation.

Following [Ketterle and Zwierlein(2008)], we use

$$c_{k\uparrow} = u_k \gamma^\dagger_{ka} - v_k \gamma_{kb} \tag{15.27}$$

and we furthermore assume the third hyperfine state to be initially empty, i.e., $c_{k3}|\Psi(t)\rangle = 0$ with $|\Psi(t)\rangle$ the state of the condensate before the RF pulse. This yields:

$$\hat{V} = V_0 \sum_k c^\dagger_{k3} \left(u_k \gamma^\dagger_{ka} - v_k \gamma_{kb} \right). \tag{15.28}$$

Therefore, the RF excitation transfers single atoms to the state $|k3\rangle$ by creating a quasiparticle in the single-particle state ka and destroying one in state kb. However, we are interested in the occupations of state $|k3\rangle$ after the excitation, i.e., we have to investigate the transitions governed by the matrix elements

$$M_k = \langle f|V|\Psi(t)\rangle, \tag{15.29}$$

where $|\Psi(t)\rangle = |\gamma(t)\rangle|0\rangle_3$ is composed of the quasiparticle contribution $|\gamma(t)\rangle$ and the vacuum of the third hyperfine state $|0\rangle_3$. Since the exact quasiparticle configuration after the RF pulse is not relevant, the final state of the transition $|f\rangle$ needs to take into account all possible end states for the quasiparticles, i.e.,

$$|f\rangle = \sum_{k'} \left(u_{k'} \gamma^\dagger_{k'a} - v_{k'} \gamma_{k'b} \right) |\gamma(t)\rangle|k3\rangle. \tag{15.30}$$

Inserting this into Eq. (15.29) and keeping in mind that in Anderson approximation $\langle \gamma^\dagger_{ma/b} \gamma_{na/b} \rangle = \langle \gamma^\dagger_{ma} \gamma^\dagger_{nb} \rangle = 0$ for $m \neq n$ and that $\langle \gamma^\dagger_{ma} \gamma_{ma} \rangle = \langle \gamma^\dagger_{mb} \gamma_{mb} \rangle$ [Hannibal *et al.*(2015)] we obtain

$$M_k \sim (u_k^2 - v_k^2) \langle \gamma^\dagger_{ka} \gamma_{ka} \rangle - 2 u_k v_k \mathrm{Re} \left(\langle \gamma^\dagger_{ka} \gamma^\dagger_{kb} \rangle \right) + v_k^2. \tag{15.31}$$

Therefore, the population in the state $|k3\rangle$ created by the RF pulse is a direct measure of the single-particle occupations if the single-particle coherences $\langle \gamma^\dagger_{ka} \gamma^\dagger_{kb} \rangle$ vanish. This is the case for every thermal state of the condensate. I.e., –in that case– a momentum- and energy-resolved measurement of the occupation of the third hyperfine state maps the single-particle band structure (cf. [Stewart *et al.*(2008)]).

However, if single-particle coherences are present, they may interfere with the signal from the occupations and prohibit an observation of the latter. This is the case for our situation as can be seen by calculating M_k directly from Eq. (15.26) The quasiparticle part of the transition matrix elements M_k is basically given by the occupation of the atomic state $|k\uparrow\rangle$, i.e.,

$$M_k \sim \langle c^\dagger_{k\uparrow} c_{k\uparrow} \rangle \tag{15.32}$$

Therefore, the overall transition matrix element considering all transtitions to the third hyperfine state yields

$$M_{\text{total}} \sim \sum_k \langle c^\dagger_{k\uparrow} c_{k\uparrow} \rangle = N_{P\uparrow}. \tag{15.33}$$

However, $N_{P\uparrow}$ is a conserved quantity during the free quench dynamics, i.e., during the dynamics before the RF pulse. Therefore, M_{total} and thus the overall RF-induced occupations of the third hyperfine state do not depend on the actual time of the RF measurement. This implies that the contribution from the Goldstone mode in the occupations has to be canceled out by a corresponding contribution in the single-particle coherences since all single-paricle occupations oscillate in phase with respect to the gapless Goldstone mode. Therefore, every individual M_k is a constant with respect to the Goldstone mode. In that sense, particle conservation prohibits an observation of the gapless Goldstone mode via RF spectroscopy.

Indeed, our numerical data confirm that the contribution of the single-particle coherences to Eq. (15.31) exactly cancels out the signal of the Goldstone mode from the single-particle occupations. This confirms that the Goldstone mode in the present form is not visible via RF spectroscopy.

However, the same applies to the real-space dynamics of the condensate: For the same reasons as stated above the gapless homogeneous Goldstone mode is not visible in the atomic density $\rho(\mathbf{r}, t)$ of the cloud. With Eqs. (15.4)-(15.5) one directly obtains

$$\begin{aligned}\rho(\mathbf{r}, t) &= \sum_{\sigma} \langle \hat{\Psi}^{\dagger}_{\sigma}(\mathbf{r}, t) \hat{\Psi}_{\sigma}(\mathbf{r}, t) \rangle \\ &= 2 \sum_{k} \big[(u_k^2 - v_k^2) \langle \gamma^{\dagger}_{ka} \gamma_{ka} \rangle \\ &\qquad - 2 u_k v_k \mathrm{Re} \left(\langle \gamma^{\dagger}_{ka} \gamma^{\dagger}_{kb} \rangle \right) + v_k^2 \big] |\varphi_k(\mathbf{r})|^2, \end{aligned} \tag{15.34}$$

which has the same structure as Eq. (15.31). Therefore, the gapless homogeneous Goldstone mode does not couple to the real-space dynamics of the cloud as already stated in section 15.3.2. However, the resemblence of Eqs. (15.34) and (15.31) also implies that the Goldstone mode is visible in the RF signal for the case of inhomogeneous excitations. There, the symmetry between the single-particle excitations and coherences preventing the gapless homogeneous Goldstone mode from observation must be broken to allow for a collective oscillation of the cloud. Therefore, we state that in our case of a homogeneous excitation the Goldstone mode is visible neither in the single-particle excitations nor in the real-space dynamics of the cloud. But, for the same reason it must be visible in both quantities for the case of inhomogeneous excitations.

Part VI
Nonequilibrium Green's Function Method

16 I Green functions in the BCS theory by Kopnin

(move it to the main note on SC)

16.1 Preface

After more than four decades of existence, the microscopic theory of superconductivity due to Bardeen, Cooper, and Schrieffer (1957) (BCS) has established itself as one of the most beautiful theories in condensed matter physics. Based on quite simple principles, it gives surprisingly good description of many properties of superconductors. Before the discovery of high temperature superconductors, the BCS theory with a phonon mediated electron coupling could be regarded as an almost perfect piece of art. The high temperature superconductivity appeared to be a challenge for the microscopic theory (by the way, why it can't be explained by BCS?). Apparently, its complicated nature goes, strictly speaking, beyond the BCS model, and many attempts have been undertaken to build a microscopic picture of this mysterious phenomenon. However, no reliable and commonly accepted theory has emerged.

Instead, the last decade of intensive studies in the high temperature superconductivity revealed yet another important advantage of the classical BCS theory: Sometimes it works reasonably well even when it is not expected do so! It still remains the best available theory to describe the new superconductors.

It also appears that the BCS model originally designed to describe the s-wave pairing in conventional (low temperature) superconductors, can easily be generalized to deal with unconventional superfluids and superconductors. The first successful application was to the p-wave superfluidity in ^{3}He Fermi liquid.

Here it provides the theory of a very exciting state with much more complicated and rich structure of the superfluid order parameter (Leggett 1973, Wölfle and Vollhardt 1990, Volovik 1992 (what is in that articles?)).

Heavy-Fermionic and high-temperature d-wave superconductors are examples where the BCS model can also be used with a great chance for a success (Mineev and Samokhin 1999). This is why the interest in the BCS theory remains alive despite its quite respectable age.

The BCS model grew into a highly powerful theory of superconductivity also because of its formulation in terms of the Green functions by Gor'kov (1958) (currently, it is a complicated methods, which often is just not used because it is unclear how to use it. I am not sure that this statement is correct at all). The Green function technique constitutes a complete tool for solving almost any problem within the BCS theory. A very important and useful improvement in the Green function theory of superconductivity has been provided by the so called quasiclassical method initiated by Eilenberger (1968) (write this method in my note on SC. what is its main formulas and examples??). The method operates with the Green functions integrated over the energy near the Fermi surface of the normal state. This method is based on the fact that the energies involved in the superconducting phenomena are normally much smaller than the Fermi energy which is the scale of variations of the Green function in the normal state. In the quasiclassical scheme, the calculations are reduced to a more or less automatic action provided the problem is adequate for the model (abstract, unclear statement).

It is the quasiclassical version of the Green function formalism which is now most frequently used for calculations of various properties of superconductors (repetition of what is already written).

Despite the quasiclassical methods are widely used for practical purposes, their description is hard to find in the textbook literature. The review by Serene and Rainer (1983) gives an introduction to the quasiclassical approach (read it! what is there?). The book by Svidzinskii (1982) deals with the quasiclassical theory of static properties of weak links (read it! what is there? sounds useful for solving problems of junctions). Applications of quasiclassical methods to various problems in nonstationary superconductivity can only be found in original papers and specialized reviews such as, for example, a review by Larkin and Ovchinnikov (1986) (if one thinks that vortex motion is useless, then there is an argument to study it: it is an example of a quasiclassical method. what is there?). The aim of the present book is to provide a basic knowledge of the microscopic theory of nonstationary superconductivity including the most advanced quasiclassical methods. We assume that the reader is familiar with the main ideas of the BCS theory of superconductivity. There are many books on the principles of the BCS theory, and we do not intend to give a comprehensive list of them here. The personal preference of the author is with the book by de Gennes (1966) which contains all the fundamentals which we would need to proceed with the more recent microscopic theory.

We try to describe the quasiclassical method in such a way that a newcomer to the field could learn it in a short time. First, we discuss stationary, time-independent properties. We shall learn about the Green function, how to find it for a particular superconducting state, and how to calculate the superconducting properties once the Green function is known. The nonstationary theory of superconductivity requires more efforts, which necessarily use principles developed for stationary problems. In the theory of nonstationary phenomena, the basic concept is the distribution function of excitations. The specifics of superconductors is that the interaction of a superconductor with an external electromagnetic field normally causes a considerable distortion of the quasiparticle distribution on the energy scale relevant for the superconducting parameters. As a result, the behavior of nonequilibrium excitations becomes one of the most important factors which govern the response of the superconductor to applied fields. This is why the problem of formulating the correct description of the quasiparticle distribution is of the major concern throughout the book.

We consider several applications of both stationary and nonstationary quasiclassical theory. For example, we derive and analyze the time-dependent Ginzburg-Landau theory which is most frequently used to describe the dynamics of superconductors. We demonstrate how powerful this theory is within certain limits and, at the same time, we emphasize that it is far from being a complete story about the kinetics of superconductors.

The great deal of attention is paid to the dynamics of vortices in the mixed state of type II superconductors. There are two good reasons for the choice. First, it is well established now that the dynamics of vortices controls almost all the magnetic properties of type II superconductors, especially those of hightemperature superconductors. It also determines hydrodynamics of superfluids. Moreover, dynamics of superfluid vortices is now believed to have a close relation to other fields of physics such as high energy physics and cosmology (Achucarro and Vachaspati 2000, Shellard and Vilenkin 1994). Second, the vortex dynamics has been the major interest of the author's research during many years, and it is hard to resist the temptation to tell something on this fascinating subject. The reader is assumed to be familiar with the basic properties of vortices at least within the framework of the usual Ginzburg-Landau theory. Here

we concentrate on the motion of vortices under the action of a current passing through a superconductor in the so called flux flow regime. We shall see that, in presence of vortices, the superconductor is no longer "superconducting" in a practical sense: it offers a resistivity to the current! This is why the vortex dynamics plays an important role in the physics of superconductors.

Of course, there are very many interesting and important phenomena left beyond the contents of the book. For example, we do not consider Josephson junctions and weak links; some aspects of this problem can be found in review by Aslamazov and Volkov (1986), and in books by Likharev (1986) and Tinkham (1996).

We do not discuss propagation of sound through superconductors (see, for example, Bulyzhenkov and Ivlev 1976 a, 1976 b)(what happens with sound in SC?). Neither we consider thermoelectric phenomena (is it derivede using GF?), etc. Each of these topics deserves a book of its own. In this context, we mention the book by Geilikman and Kresin (1974) which treats acoustic, thermoelectric, and some other effects with the standard Boltzmann kinetic equation (read it later, what is there?). Nevertheless, the quasiclassical scheme is not included there. We hope, therefore, that our presentation provides a basis for description of these and many other properties of superconductors in a coherent way using the simple and more advanced quasiclassical method.

16.2 INTRODUCTION

We give a brief outline of general ideas of the theory of superconductivity and introduce the basic quantities that characterize the superconducting state. We discuss the Ginzburg-Landau theory which provides the simplest description of stationary superconductors and consider its application to the vortex state. The microscopic Bogoliubov-de Gennes theory is introduced together with the concept of quasiclassical approximation. We also formulate the typical problems of nonstationary theory and consider its simplest methods such as the kinetic equation approach and the time dependent Ginzburg-Landau model.

16.2.1 1.1 Superconducting variables

The most important characteristic of a superconductor is the superconducting order parameter Δ. The order parameter is proportional to the wave function of "superconducting electrons" which form a "Bose condensate". It is normalized in such a way that its modulus is equal to the energy gap in the electronic spectrum which usually appears after a transition into the superconducting state. The order parameter is a complex function $\Delta = |\Delta|e^{i\chi}$ where the phase χ is the same for all condensate particles if there is no current. In presence of a supercurrent, the phase χ acquires a spatial dependence slowly varying from one point in the superconductor to another. The existence of a coherent phase of the wave function for all superconducting particles is the very essence of superconductivity.

Another important characteristics is the energy spectrum of single-particle excitations in a superconducting system. For a homogeneous clean superconductor in absence of currents and magnetic field the energy spectrum has a gap

$$\epsilon_{\mathbf{p}} = \sqrt{\xi_{\mathbf{p}}^2 + |\Delta|^2} \tag{1.1}$$

such that all excitations have energies above $|\Delta|$. In eqn (1.1),

$$\xi_{\mathbf{p}} = E_n(\mathbf{p}) - E_F \tag{1.2}$$

while $E_n(\mathbf{p})$ is the electronic spectrum in the normal state, and E_F is the Fermi energy. The energy is counted from E_F because, as in any Fermi liquid, relevant excitations are concentrated near the Fermi level. The normal spectrum $E_n(\mathbf{p})$ of the metal can have a complicated dependence on the momentum $\mathbf{p}$ reflecting the band structure of the metal. In many cases, if we are not interested in the
particular band-structure effects, it is sufficient to consider a simple parabolic spectrum $E_n = \mathbf{p}^2/2m$.

The order parameter determines both thermodynamic and transport properties of a superconductor. To learn more of the order parameter as well as of other important superconducting quantities we start with the Ginzburg-Landau theory for a time-independent superconducting state.

1.1.1 Ginzburg-Landau theory

The Ginzburg-Landau (Ginzburg and Landau 1950) theory is a generalization of the Landau theory of second-order phase transitions (Landau and Lifshitz 1959). Consider the free energy of a superconductor. Assume that we can expand it in terms of Δ and its gradients:

$$\mathcal{F}_{sn} = \int \left[\alpha|\Delta|^2 + \frac{\beta}{2}|\Delta|^4 + \gamma \left| \left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right) \Delta \right|^2 \right] dV \tag{1.3}$$

The free energy expression is supplemented with the Maxwell equation for the magnetic field

$$\operatorname{curl} \mathbf{H} = \frac{4\pi}{c}\mathbf{j} \tag{1.4}$$

where

$$\mathbf{H} = \operatorname{curl} \mathbf{A} \tag{1.5}$$

The average of $\mathbf{H}$ gives the magnetic induction $\mathbf{B}$.
The gradient term in eqn (1.3) is the momentum operator in presence of the magnetic field

$$\mathbf{P}_s = -i\hbar\nabla - \frac{2e}{c}\mathbf{A} \tag{1.6}$$

It refers to a Cooper pair having the charge $2e$ (e is the electronic charge). Equation (1.6) implies a gauge invariance: the free energy of the system and the magnetic field do not change if one makes a simultaneous transformation

$$\chi \to \chi + f(\mathbf{r}), \mathbf{A} \to \mathbf{A} + \frac{\hbar c}{2e}\nabla f(\mathbf{r}) \tag{1.7}$$

where $f(\mathbf{r})$ is an arbitrary function of coordinates.
At the transition temperature $T = T_c$, the coefficient α changes its sign and becomes negative for $T < T_c$, while β and γ remain constant. Microscopic theory (Gor'kov 1959 a, 1959 c) gives

$$\alpha = -\nu(0)\frac{T_c - T}{T}, \beta = \frac{7\zeta(3)\nu(0)}{8\pi^2 T_c^2} \tag{1.8}$$

where $\nu(0)$ is the density of states at the Fermi level, and $\zeta(3) \approx 1.202$. We use the units with $k_B = 1$. We will derive eqn (1.8) later in Section 6.1.2. Equation (1.8) demonstrates that the free energy expansion in eqn (1.3) goes in powers of Δ/T_c which suggests that Δ should be much smaller than T_c.

The coefficient γ depends on purity of the sample. The purity is characterized by the parameter $T_c\tau/\hbar$, where τ is the electronic mean free time due to the scattering by impurities. Superconductors are called clean when this parameter is large, and they are dirty in the opposite case. We discuss the microscopic justification of the Ginzburg-Landau theory later. Here we just write down the expression for γ in two limiting cases. For dirty superconductors, it is

$$\gamma = \frac{\pi\nu(0)D}{8\hbar T_c} \tag{1.9}$$

where D is the diffusion coefficient. In the clean case

$$\gamma = \frac{\nu(0)7\zeta(3)v_F^2}{48\pi^2 T_c^2} \tag{1.10}$$

Here v_F is the velocity of electrons at the Fermi surface.
The total free energy is

$$\mathcal{F}_{\text{tot}} = \mathcal{F}_{sn} + \int \frac{H^2}{8\pi} dV \tag{1.11}$$

Variation of $\mathcal{F}_{\text{tot}}$ with respect to Δ, Δ^* and $\mathbf{A}$ gives

$$\begin{aligned}\delta\mathcal{F}_{\text{tot}} = \int \Bigg(& \left[\alpha\Delta + \beta|\Delta|^2\Delta + \gamma\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)^2 \Delta\right]\delta\Delta^* + \text{ c.c.} \\ & + \left[\frac{\operatorname{curl}\operatorname{curl}\mathbf{A}}{4\pi} - \frac{2e\gamma}{c}\left[\Delta^*\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\Delta + \text{ c.c. }\right]\right]\delta\mathbf{A} \\ & + \operatorname{div}\left[\hbar\gamma\delta\Delta^*\left(\hbar\nabla - \frac{2ie}{c}\mathbf{A}\right)\Delta + \text{ c.c. } + \frac{1}{4\pi}\delta\mathbf{A}\times\operatorname{curl}\mathbf{A}\right]\Bigg) dV\end{aligned}$$

The requirement of extremum of the free energy leads to the GL equations. Vanishing of $\delta\mathcal{F}_{\text{tot}}/\delta\Delta^*$ results in

$$\alpha\Delta + \beta|\Delta|^2\Delta + \gamma\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)^2\Delta = 0 \tag{1.12}$$

Condition $\delta\mathcal{F}_{\text{tot}}/\delta\mathbf{A} = 0$ gives the expression for supercurrent

$$\mathbf{j}_s = 2e\gamma\left[\Delta^*\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\Delta + \Delta\left(i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\Delta^*\right] \tag{1.13}$$

1.1.1.1 Discussion of the GL equations Consider first the equation (1.12) for the order parameter. In a homogeneous case without a current and a magnetic field the order parameter is

$$\Delta \equiv \Delta_\infty = |\alpha|/\beta = \left(\frac{8\pi^2}{7\zeta(3)}\right)^{\frac{1}{2}} T_c\left(1 - T/T_c\right)^{\frac{1}{2}} \tag{1.14}$$

We denote it by Δ_∞ for the reason which will be clear later. The ratio Δ_∞/T_c should be small. This implies that the GL theory works for temperatures close
to T_c, i.e., when $1 - T/T_c \ll 1$. Simultaneously we see that $\Delta \sim T_c$ when temperatures are well below T_c.

Equation (1.12) defines the length

$$\xi(T) = \sqrt{\gamma\hbar^2/|\alpha|} \propto \left(1 - T/T_c\right)^{-\frac{1}{2}} \tag{1.15}$$

which is a characteristic scale of variations of the order parameter. It is called the coherence length. The notation ξ for the coherence length is to be distinguished from $\xi_{\mathbf{p}}$ that denotes the energy variable in eqn (1.2). We keep both these notations because they are commonly accepted in the literature.

In the clean case $T_c\tau/\hbar \gg 1$ we have from eqn (1.10)

$$\xi(T) = \sqrt{7\zeta(3)/12}\xi_0 \left(1 - T/T_c\right)^{-\frac{1}{2}} \tag{1.16}$$

where

$$\xi_0 = \frac{\hbar v_F}{2\pi T_c} \tag{1.17}$$

is the "zero-temperature" coherence length. In the dirty case eqn (1.9) gives

$$\xi(T) = (\pi/2\sqrt{3})\sqrt{\xi_0 \ell}\left(1 - T/T_c\right)^{-\frac{1}{2}} \tag{1.18}$$

where $\ell = v_F\tau$ is the electron mean free path. The impurity parameter can be expressed through the ratio of ξ_0 and ℓ :

$$T_c\tau/\hbar = \ell/2\pi\xi_0 \tag{1.19}$$

so that a dirty limit corresponds to $\ell \ll \xi_0$ while a clean limit is for $\ell \gg \xi_0$.
Consider now expression (1.13) for the current. Using the definition of the momentum operator eqn (1.6) we introduce the superconducting velocity

$$2m\mathbf{v}_s = \mathbf{P}_s \tag{1.20}$$

for a Cooper pair with the mass $2m$. We use here the bare mass m of an electron. However, the bare electronic mass is not a good quantity for a metal where the normal-sate electronic spectrum may have a complicated form. In this case also the "superconducting velocity" is not a real velocity of Cooper pairs. Bearing this in mind we write the supercurrent as

$$\mathbf{j} = -\frac{e^2 N_s}{mc}\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right) = N_s e\mathbf{v}_s \tag{1.21}$$

where

$$\mathbf{v}_s = \frac{\mathbf{P}_s}{2m} = \frac{\hbar}{2m}\left(\nabla\chi - \frac{2e}{\hbar c}\mathbf{A}\right) \tag{1.22}$$

and

$$N_s = 8m\gamma|\Delta|^2 \tag{1.23}$$

is the density of "superconducting electrons".
The Maxwell equation (1.4) combined with eqn (1.21) gives

$$\operatorname{curl}\operatorname{curl}\mathbf{A} = -\frac{4\pi N_s e^2}{mc^2}\left(\mathbf{A} - \frac{\hbar c}{2e}\nabla\chi\right) \tag{1.24}$$

Equation (1.24) defines the characteristic length

$$\lambda_L = \left(\frac{mc^2}{4\pi N_s e^2}\right)^{\frac{1}{2}} \tag{1.25}$$

In a homogeneous case

$$\lambda_L = \left(\frac{c^2\beta}{32\pi e^2\gamma|\alpha|}\right)^{\frac{1}{2}} \propto (1 - T/T_c)^{-\frac{1}{2}}$$

It is called the London penetration length. It determines the characteristic scale of variations of the magnetic field. The supercurrent can be written as

$$\mathbf{j} = \frac{c^2}{16\pi e\lambda_L^2}\left[\Delta^*\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\Delta + \Delta\left(i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\Delta^*\right]\Delta_\infty^{-2} \tag{1.26}$$

Sometimes, it can be convenient to use the normalization of the order parameter such that it has the form of the wave function Ψ of superconducting electrons. The free energy becomes

$$\mathcal{F}_{sn} = \int\left[a|\Psi|^2 + \frac{b}{2}|\Psi|^4 + \frac{1}{2m}\left|\left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A}\right)\Psi\right|^2\right]dV \tag{1.27}$$

The constants a and b satisfy

$$\frac{|a|}{b} = \frac{mc^2}{16\pi e^2\lambda_L^2}$$

and determine the new order parameter magnitude $|\Psi_{GL}|^2 = |a|/b$ which is

$$|\Psi_{GL}|^2 = 2m\gamma\Delta_\infty^2$$

in terms of the previous definition of Δ. The coherence length is now expressed through the electronic mass

$$\xi^2 = \frac{\hbar^2}{2m|a|}$$

1.1.1.2 The Ginzburg-Landau parameter. Type I and type II superconductors The ratio of the two characteristic lengths is called the Ginzburg-Landau parameter

$$\kappa = \frac{\lambda_L(T)}{\xi(T)} = \left(\frac{\beta c^2}{32\pi\hbar^2 e^2\gamma^2}\right)^{\frac{1}{2}} \tag{1.28}$$

It is independent of T near T_c and is determined by the material characteristics. For clean superconductors with γ from eqn (1.10) it is

$$\kappa = \left[\frac{9\pi^4}{14\zeta(3)}\right]^{\frac{1}{2}}\left[\left(\frac{a_0 p_F}{2\pi\hbar}\right)\left(\frac{e^2/a_0}{E_F}\right)\right]^{\frac{1}{2}}\frac{\hbar c}{e^2}\frac{T_c}{E_F} \tag{1.29}$$

Here a_0 is the interatomic distance. Usually, it is of the order of $1/2\pi\hbar p_F$. The ratio of the Coulomb interaction energy of conducting electrons e^2/a_0 to the Fermi energy is of the order of unity for good metals, but it may become larger for systems with strong correlations between electrons. The last factor in eqn (1.29) is usually small: T_c/E_F is of the order of 10^{-3} for conventional superconductors, but it is of the order of $10^{-1} \div 10^{-2}$ for high temperature superconductors with $T_c \sim 100K$ and $E_F \sim 1000K$. The fine structure constant $e^2/\hbar c = 1/137$.

We see that the Ginzburg-Landau parameter is normally small $\kappa \ll 1$ for conventional clean superconductors, though, in some cases it may be of the order of 1 . On the contrary, for high temperature superconductors, which have a tendency to be strongly correlated systems with a not very low ratio of T_c/E_F, the parameter κ is usually very large. The Ginzburg-Landau parameter increases for dirty superconductors. According to eqn (1.9) it becomes

$$\kappa_{\text{dirty}} \sim \kappa_{\text{clean}}\ (\hbar/T_c\tau) \tag{1.30}$$

Therefore, dirty alloys normally have a large κ.
The value of the Ginzburg-Landau parameter divides all superconductors into two classes: type I and type II superconductors. Those with $\kappa < 1/\sqrt{2}$ belong to the type I, while those with $\kappa > 1/\sqrt{2}$ are the type II superconductors.

1.1.2 Example: Vortices in type II superconductors

Vortices are very important topological objects in superconductors and superfluids. As it was already mentioned in the Preface, vortices determine the most fundamental properties of superconductors and their responses to external d.c. and a.c. electromagnetic fields. They play a crucial role also in hydrodynamics of superfluids. We will study the dynamics of vortices in detail later in this book. Here we summarize the main features of vortices which can be deduced from the Ginzburg-Landau model. A more detailed Ginzburg-Landau theory of the vortex state in superconductors can be found, for example, in the book by Saint-James et al. (1969).
1.1.2.1 Transition from normal into the superconducting state The transition from normal into the superconducting state in a magnetic field is of the second
order in type II superconductors. Let us find the magnetic field when a nonzero order parameter first appears (Abrikosov 1957). Close to the transition point the order parameter is small, $\Delta \ll \Delta_\infty$. We linearize the GL equations in a small Δ :

$$\xi^2 \left(\nabla - \frac{2ie}{\hbar c}\mathbf{A}\right)^2 \Delta + \Delta = 0 \tag{1.31}$$

Let the magnetic field be along the z axis. The vector potential can be taken in the Landau gauge $\mathbf{A} = (0, Hx, 0)$. The order parameter depends on x and y. Now we have

$$\frac{\partial^2 \Delta}{\partial x^2} + \left(\frac{\partial}{\partial y} - \frac{2ieHx}{\hbar c}\right)^2 \Delta + \xi^{-2}\Delta = 0 \tag{1.32}$$

This is the Schrödinger equation for a charge in a magnetic field. We put

$$\Delta = e^{iky} Y(x)$$

and obtain the oscillator equation

$$\frac{\partial^2 Y}{\partial x^2} - \left(k - \frac{2eHx}{\hbar c}\right)^2 Y + \xi^{-2} Y = 0 \tag{1.33}$$

where the oscillator frequency ω_0 and energy E are

$$\omega_0 = \frac{2|e|H}{mc}, E = \frac{\hbar^2}{2m\xi^2}$$

respectively. The energy spectrum $E = \omega_0(n + 1/2)$ gives

$$\frac{\hbar}{2m\xi^2} = \frac{2|e|H}{mc}\left(n + \frac{1}{2}\right)$$

The highest magnetic field $H = H_{c2}$ is for $n = 0$:

$$H_{c2} = \frac{\hbar c}{2|e|\xi^2} = \frac{\Phi_0}{2\pi\xi^2} \tag{1.34}$$

where

$$\Phi_0 = \frac{\pi\hbar c}{|e|}$$

is the magnetic flux quantum (see below). H_{c2} is the upper critical magnetic field below which the transition into superconducting state occurs. It is proportional to $1 - T/T_c$ near the critical temperature. Comparing it with the thermodynamic critical field (de Gennes 1966) H_c we observe that $H_{c2} = \sqrt{2}\kappa H_c$. For type II superconductors with $\kappa > 1/\sqrt{2}$, the upper critical field $H_{c2} > H_c$.

The solution for the lowest energy level is a Gaussian function

$$Y = C \exp\left[-\frac{1}{2\xi^2}\left(x - \frac{\hbar ck}{2eH_{c2}}\right)^2\right] \tag{1.35}$$

The function in eqn (1.35) is centered at $x = \hbar ck/2eH$. Actually, the full solution is a linear combination of these functions with different k. One can construct a periodic solution in the form

$$\Delta = \sum_n C_n e^{iqny} \exp\left[-\frac{1}{2\xi^2}\left(x - \frac{\hbar cqn}{2eH_{c2}}\right)^2\right] \tag{1.36}$$

It is periodic in y with the period $y_0 = 2\pi/q$. It will be periodic in x, as well, if the coefficients C_n satisfy periodicity condition $C_{n+p} = C_n$. Then,

$$\Delta\left(x + \frac{p\hbar cq}{2eH_{c2}}, y\right) = e^{ipqy}\Delta(x, y)$$

The simplest case is realized when all the coefficients C_n are equal. The period in x is then $x_0 = \hbar cq/2eH_{c2}$. The solution eqn (1.36) forms a rectangular lattice with the area $S_0 = x_0 y_0 = \Phi_0/H_{c2} = 2\pi\xi^2$ which corresponds to exactly one flux quantum per unit cell. If q is chosen such that $x_0 = y_0$, we obtain a square lattice.

The $|\Delta|$-pattern has zeroes at the points $x = x_0/2 + x_0 n, y = y_0/2 + y_0 m$, surrounded by supercurrent flow lines. Indeed, the supercurrent is

$$j_x = -\frac{\hbar c^2}{16\pi\lambda_L^2 e\Delta_\infty^2}\left(i\Delta^*\frac{\partial\Delta}{\partial x} - i\Delta\frac{\partial\Delta^*}{\partial x}\right)$$
$$j_y = -\frac{\hbar c^2}{16\pi\lambda_L^2 e\Delta_\infty^2}\left[\Delta^*\left(i\frac{\partial\Delta}{\partial y} + \frac{2eH_{c2}x}{\hbar c}\Delta\right) - \Delta\left(i\frac{\partial\Delta^*}{\partial y} - \frac{2eH_{c2}x}{\hbar c}\Delta\right)\right]$$

To transform this further we use the identity which holds for the function of the type of eqn (1.36):

$$\frac{\partial\Delta}{\partial x} = \left(-i\frac{\partial}{\partial y} - \frac{2eH_{c2}x}{\hbar c}\right)\Delta. \tag{1.37}$$

With help of eqn (1.37) we get

$$j_x = -\frac{\hbar c^2}{16\pi\lambda_L^2 e\Delta_\infty^2}\frac{\partial|\Delta|^2}{\partial y} \tag{1.38}$$

$$j_y = \frac{\hbar c^2}{16\pi\lambda_L^2 e\Delta_\infty^2}\frac{\partial|\Delta|^2}{\partial x} \tag{1.39}$$

These expressions suggest that $|\Delta(x, y)|^2$ is a stream function, i.e., that the current flows along the lines of constant $|\Delta|$. If we place the node of $|\Delta|$ in the center of the Bravais unit

cell, the current along the boundary of a unit cell
is zero: due to periodicity, the lines of constant $|\Delta|$ are perpendicular to the boundary. We now calculate the contour integral along the unit cell boundary

$$\oint \left(\mathbf{A} - \frac{\hbar c}{2e} \nabla \chi \right) d\mathbf{r} = 0 \tag{1.40}$$

It vanishes because $\mathbf{A} - (\hbar c/2e)\nabla\chi = 0$ at the boundary, as we have just proven. We obtain

$$\Phi_0^{-1} \int_{S_0} \mathbf{H} \cdot d\mathbf{S} = (2\pi)^{-1} \delta\chi$$

where the integral is over the unit cell and $\delta\chi$ is the variation of the order parameter phase along this contour. It is $\delta\chi = 2\pi$ since the flux through the unit cell is equal to one flux quantum. Therefore, the phase of the order parameter acquires the increment of 2π after encircling the point where the order parameter is zero. The phase variation by 2π is also necessary for single-valuedness of the order parameter. Here we come to a vortex: A quantized vortex is a linear (in three dimensions) topological object which is characterized by a quantized circulation of the order parameter phase around this line. In principle, vortices with a phase circulation of an integer multiple of 2π are also possible. The axis of vortex circulation $\hat{\mathbf{z}}$ is parallel to $\mathbf{H}$ if the charge of carriers is positive and antiparallel to it in the opposite case: $\hat{\mathbf{z}} = \mathrm{sign}(e)\hat{\mathbf{H}}$. For electrons, $\hat{\mathbf{z}}$ is antiparallel to $\mathbf{H}$. We see that transition into a superconducting state in a magnetic field below H_{c2} gives rise to formation of vortices. Vortices in superconductors were theoretically predicted by Abrikosov (1957).

Let us consider an applied magnetic field H slightly below H_{c2} such that $H_{c2} - H \ll H_{c2}$. The solution of the GL equation is the function eqn (1.36) plus a small correction Δ_1. This correction is caused by (i) nonlinear term in the GL equation, (ii) local variations in $\mathbf{A}$ due to the supercurrent eqns (1.38), (1.39), and (iii) deviation of H from H_{c2}.

Using the Maxwell equation

$$j_x = \frac{c}{4\pi} \frac{\partial H_z}{\partial y}, j_y = -\frac{c}{4\pi} \frac{\partial H_z}{\partial x}$$

we obtain variations in the local field

$$\delta H_z = -\frac{\hbar c |\Delta|^2}{4\lambda_L^2 e \Delta_\infty^2} \tag{1.41}$$

Therefore, the vector potential becomes $\mathbf{A} = \mathbf{A}_0 + \mathbf{A}_1$ where

$$\mathbf{A}_1 = (0, (H - H_{c2})\, x, 0) + \delta\mathbf{A}$$

where $\delta\mathbf{A}$ is due to the magnetic field δH_z.
Since the non disturbed function of eqn (1.36) satisfies the linearized GL equation with $\mathbf{A} = \mathbf{A}_0$, we obtain for the correction Δ_1

$$\left(\nabla - \frac{2ie}{\hbar c} \mathbf{A} \right)^2 \Delta_1 + \xi^{-2} \Delta_1$$

$$= \frac{2ie}{\hbar c} \left(\nabla - \frac{2ie}{\hbar c} \mathbf{A} \right) (\mathbf{A}_1 \Delta) + \frac{2ie}{\hbar c} \mathbf{A}_1 \left(\nabla - \frac{2ie}{\hbar c} \mathbf{A} \right) \Delta + \xi^{-2} |\Delta|^2 \Delta / \Delta_\infty^2.$$

Now we apply the orthogonality condition: we multiply this equation by Δ from eqn (1.36) and integrate it over $dxdy$. After integration by parts using eqn (1.37), we obtain

$$\frac{2e}{\hbar c}\int |\Delta|^2 \left(H - H_{c2} + \delta H\right) dxdy + \Delta^{-2}\xi^{-2}\int |\Delta|^4 dxdy = 0$$

We introduce the average over the area occupied by the vortex array

$$\left\langle |\Delta|^2 \right\rangle_L = S_0^{-1}\int_{S_0} |\Delta|^2 dxdy$$

and obtain using eqn (1.41)

$$\Delta_\infty^2 \left\langle |\Delta|^2 \right\rangle_L \left(1 - \frac{H}{H_{c2}}\right) = \left(1 - \frac{1}{2\kappa^2}\right) \left\langle |\Delta|^4 \right\rangle_L$$

It is convenient to introduce the ratio

$$\beta_A = \frac{\left\langle |\Delta|^4 \right\rangle_L}{\left\langle |\Delta|^2 \right\rangle_L^2} \geq 1 \tag{1.42}$$

which is called the Abrikosov parameter. The value β_A is determined by the structure of the vortex lattice. Now we get

$$\left\langle |\Delta|^2 \right\rangle_L = \frac{2\kappa^2\Delta_\infty^2}{\left(2\kappa^2 - 1\right)\beta_A}\left(1 - \frac{H}{H_{c2}}\right) \tag{1.43}$$

Equation (1.43) shows that $|\Delta|^2$ has a small magnitude proportional to $1 - H/H_{c2}$ if $\kappa > 1/\sqrt{2}$. In the opposite case, the order parameter can not be small for H close to H_{c2} : it jumps to a finite value making the transition of the first order. This is exactly the condition which separates type I and type II superconductors: A superconductor can accommodate vortices and allow a magnetic field below H_{c2} to penetrate into it if $\kappa > 1/\sqrt{2}$. On the contrary, the order parameter is always finite and no magnetic field can exist in the superconductor if $\kappa < 1/\sqrt{2}$.

Using eqn (1.41) we can find the magnetization of the superconductor

$$M_z = \frac{B - H}{4\pi} = \frac{\langle \delta H_z \rangle}{4\pi} = -\frac{H_{c2} - H}{4\pi\beta_A\left(2\kappa^2 - 1\right)}$$

where B is the magnetic induction. One can also calculate the total free energy density. It becomes

$$F = \frac{B^2}{8\pi} - \frac{\left(H_{c2} - B\right)^2}{8\pi\left[1 + \left(2\kappa^2 - 1\right)\beta_A\right]} \tag{1.44}$$

It decreases with decreasing β_A. For a square lattice $\beta_A = 1.18$ while for a hexagonal lattice $\beta_A = 1.16$. A hexagonal lattice corresponds to the coefficients

$$C_{n+2} = C_n, C_1 = iC_0$$

It is the stable configuration in an isotropic environment. On the other hand, a square lattice is unstable: it corresponds to an extremum rather than to a
minimum of the vortex free energy (Saint-James et al. 1969). It can be stable, however, if the interaction with the corresponding underlying crystalline structure is strong enough.

1.1.2.2 Single vortex The previous case corresponds to the situation where vortices are closely packed together: the distance between them is of the order of the coherence length. We now consider the situation when vortices can be treated separately. A single vortex has the order parameter phase which changes by 2π after encircling its axis which we choose to be the z axis.

We take $\chi = \phi$ where ϕ is the azimuthal angle in the cylindrical frame (ρ, ϕ, z). We thus assume a cylindrical symmetry of the vortex and look for a solution in the form

$$\Delta = \Delta_\infty f(\rho) e^{i\phi}$$

The vector potential has only a ϕ-component: $\mathbf{A} = (0, A_\phi, 0)$. We have for $f(\rho)$

$$\xi^2 \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{4e^2 Q^2}{\hbar^2 c^2} \right) f + f - f^3 = 0 \tag{1.45}$$

Here we introduce the gauge invariant vector potential

$$\mathbf{Q} = \mathbf{A} - \frac{\hbar c}{2e} \nabla \chi \tag{1.46}$$

In our case $\mathbf{Q}$ has only an azimuthal component $A_\phi - \hbar c/2e\rho$.
Equation (1.24) becomes

$$\lambda_L^2 \operatorname{curl} \operatorname{curl} \mathbf{A} + f^2 \mathbf{Q} = 0.$$

For $\rho \neq 0$ it is

$$\frac{\partial^2 Q}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial Q}{\partial \rho} - \frac{Q}{\rho^2} - \frac{f^2 Q}{\lambda_L^2} = 0 \tag{1.47}$$

This equation can be solved in the limit $\kappa \gg 1$ where $\lambda_L \gg \xi$ such that one can put $f = 1$ for $\rho \gg \xi$. Equation (1.47) gives

$$Q = -\frac{\hbar c}{2e\lambda} K_1 \left(\rho / \lambda_L \right)$$

Here $K_1(z)$ is the first-order Bessel function of an imaginary argument. For $z \ll 1$ the function $K_1(z) = 1/z$, and it decreases exponentially for large z :

$$K_1(z) = \left(\frac{2}{\pi z} \right)^{\frac{1}{2}} \exp(-z)$$

The constant at Q is chosen in such a way that $A_\phi = Q + (\hbar c/2e\rho)$ is not diverging for $\rho \ll \lambda$. The magnetic field is

$$H_z = \operatorname{curl}_z \mathbf{Q} = \frac{1}{\rho}\frac{\partial(\rho Q)}{\partial \rho} = \frac{\hbar c}{2e\lambda_L^2} K_0 \left(\rho / \lambda_L \right)$$

where $K_0(z)$ is the zero-order Bessel function of imaginary argument. For $z \ll 1$ it is $K_0(z) = -\ln z$, and it decreases exponentially for large z in the same

Fig. 1.1. The order parameter and the magnetic field near the vortex. The vortex core, i.e., the region where Δ is suppressed has the size of order of the coherence length ξ; the magnetic field and currents decay on the scale on order λ_{L}.

way as K_1. Therefore, the magnetic field produced by a single vortex decays exponentially for $\rho \gg \lambda$ and it is

$$H_z \approx \frac{\Phi_0}{2\pi\lambda_L^2} \ln \frac{\lambda_L}{\rho}$$

for $\rho \ll \lambda$. For small ρ the logarithm is cut off at $\rho \sim \xi$ where f starts to decrease. Equation (1.45) for the order parameter magnitude f takes a simple form in the region $\rho \ll \lambda_L$ where $Q = -\hbar c/2e\rho$. It is

$$\xi^2 \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} - \frac{1}{\rho^2} \right) f + f - f^3 = 0. \tag{1.48}$$

The function f decreases as $f \propto \rho$ for $\rho \to 0$. At large distances $\rho \gg \xi$, the function $f \to 1$; it remains $f \approx 1$ also for longer distances of the order of λ_L where Q decays due to the screening currents.

The region near the vortex axis with the size of the order of ξ where the order parameter is decreased from its value in the bulk is called the vortex core; the order parameter magnitude $|\Delta|$ vanishes at the vortex axis. The vortex core is surrounded by supercurrents which, together with the magnetic field, decay away from the vortex core at distances of the order of λ_L. The schematic behavior of $|\Delta|$ and H_z is shown in Fig. 1.1.

The cores of neighboring vortices do not overlap when the distance between vortices is larger than ξ. The intervortex distance can be found from the condition that each vortex unit cell carries one magnetic flux quantum. If we replace a Bravais unit cell by a circle of a radius r_0, we have

$$\pi r_0^2 B = \Phi_0$$

which gives

$$r_0/\xi \sim \sqrt{H_{c2}/H}.$$

Therefore, vortices can be considered separately if $H \ll H_{c2}$.

1.1.3 Bogoliubov-de Gennes equations

The simple Ginzburg-Landau description introduces many important characteristics. We see here the order parameter Δ, the coherence length ξ, the magneticfield penetration depth λ_L, the superconducting velocity, etc. The Ginzburg-Landau theory provides a reasonable description of a superconductor in the vicinity of the critical temperature. To learn about properties of superconductors at lower temperatures, however, one needs to go to a more microscopic level of description. A comparatively simple approach is provided by the Bogoliubov-de Gennes theory (Bogoliubov et al. 1958, see also de Gennes 1966). Unfortunately, it has a manageable form for clean superconductors only. We briefly summarize here the main ideas of the Bogoliubov-de Gennes method. We derive the Bogoliubov-de Gennes equations later in Section 3.2.

The single-particle wave function has two components: the particle-like function u and the hole-like function v which satisfy the so called Bogoliubov-de Gennes equations

$$\begin{aligned} \left[\frac{1}{2m} \left(-i\hbar\nabla - \frac{e}{c}\mathbf{A} \right)^2 - E_F \right] u + \Delta v &= \epsilon_K u \\ \left[\frac{1}{2m} \left(-i\hbar\nabla + \frac{e}{c}\mathbf{A} \right)^2 - E_F \right] v - \Delta^* u &= -\epsilon_K v \end{aligned} \tag{1.49}$$

K denotes the set of quantum numbers. Here we encounter the excitation spectrum ϵ_K as an important characteristics of superconductors. For simplicity, we consider a parabolic spectrum of normal electrons. The wave function is normalized such that

$$\int (u_K u_{K'}^* + v_K v_{K'}^*)\, dV = \delta_{K,K'} \tag{1.50}$$

The order parameter itself is determined self-consistently from the BCS equation in the form (Bogoliubov et al. 1958)

$$\Delta_K = -\sum_{K'} U_{K,K'} \left(1 - 2n_{K'}\right) u_{K'} v_{K'}^* \tag{1.51}$$

Here $U_{K,K'}$ is an attractive pairing interaction. The sum is over the states of the system and ϵ_K is the energy spectrum which is found from eqn (1.49). A new quantity appears in eqn (1.51), namely the distribution function n_K of excitations. In equilibrium, it is the Fermi function

$$n_K = \frac{1}{e^{\epsilon_K/T} + 1}$$

We shall see later that the distribution function of excitations is of a crucial importance for nonequilibrium processes in superconductors.

If the magnitude of the order parameter is constant in space, we can look for a solution in the form

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_p e^{i\chi/2} \\ v_p e^{-i\chi/2} \end{pmatrix} \exp\left(\frac{i}{\hbar}\int \mathbf{p}\cdot d\mathbf{r}\right) \tag{1.52}$$

where χ is the order parameter phase. Substituting it into eqn (1.49) we find

$$p^2 = p_F^2 - p_s^2 \pm 2m\sqrt{(\epsilon_{\mathbf{p}} - \mathbf{p}_s\cdot\mathbf{v})^2 - |\Delta|^2} \tag{1.53}$$

with $\mathbf{v} = \mathbf{p}/m$. The energy spectrum becomes $\epsilon_K = \epsilon_{\mathbf{p}}$ where

$$\epsilon_{\mathbf{p}} = \tilde{\epsilon}_{\mathbf{p}} + \mathbf{p}_s\cdot\mathbf{v}; \quad \tilde{\epsilon}_{\mathbf{p}} = \sqrt{\tilde{\xi}_{\mathbf{p}}^2 + |\Delta|^2} \tag{1.54}$$

In this equation,

$$\mathbf{p}_s = \frac{1}{2}\mathbf{P}_s = \frac{\hbar}{2}\nabla\chi - \frac{e}{c}\mathbf{A}$$

is the Cooper pair momentum per particle, and

$$\tilde{\xi}_{\mathbf{p}} = \xi_{\mathbf{p}} + \mathbf{p}_s^2/2m \tag{1.55}$$

Usually, $\mathbf{p}_s^2/2m \ll |\Delta|$, so that the energy $\tilde{\epsilon}_{\mathbf{p}}$ is defined by eqn (1.1). Equation (1.54) determines the energy spectrum only for a constant magnitude of the order parameter. Moreover, one has to assume that the supercurrent (or $\mathbf{p}_s$) is also constant. The Bogoliubov-de Gennes wave functions are

$$u_{\mathbf{p}}^2 = \frac{1}{2}\left(1 + \frac{\tilde{\xi}_{\mathbf{p}}}{\tilde{\epsilon}_{\mathbf{p}}}\right), v_{\mathbf{p}}^2 = \frac{1}{2}\left(1 - \frac{\tilde{\xi}_{\mathbf{p}}}{\tilde{\epsilon}_{\mathbf{p}}}\right) \tag{1.56}$$

They are sometimes called the coherence factors.

In the momentum space, $\sum \to \int d^3p/(2\pi\hbar)^3$ and the self-consistency equation (1.51) becomes

$$\Delta_{\mathbf{p}} = -\int U_{\mathbf{p},\mathbf{p}'} \frac{\Delta_{\mathbf{p}'}}{2\tilde{\epsilon}_{\mathbf{p}'}} \left(1 - 2n_{\mathbf{p}'}\right) \frac{d^3p'}{(2\pi\hbar)^3} \tag{1.57}$$

Equation (1.57) determines the order parameter as a function of temperature in a spatially homogeneous situation.

1.1.4 Quasiclassical approximation

If the order parameter and/or current vary in space, eqns (1.54) and (1.56) are no longer valid. Solution of the Bogoliubov-de Gennes equation (1.49) would become an impossible task if not for a very important observation.

In almost all known superconductors, the Fermi momentum p_F is much larger than the characteristic wave vectors associated with the order parameter variations $\hbar/\xi_0$ where ξ_0 is the zero-temperature coherence length defined by eqn (1.17). This implies that the particle wave length is much shorter than the characteristic scale of variations of the order parameter. It allows one to use
a semi-classical approach for solving the Bogoliubov-de Gennes equations. One can write, for example,

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u_{\mathbf{p}} \\ v_{\mathbf{p}} \end{pmatrix} \exp\left(\frac{i}{\hbar}\int \mathbf{p}_F \cdot d\mathbf{r}\right) \tag{1.58}$$

where $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$ vary in space on distances of the order of ξ. The Bogoliubov-de Gennes equations take the form

$$\begin{aligned} \mathbf{v}_F \left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right) u_{\mathbf{p}} + \Delta v_{\mathbf{p}} &= \epsilon_p u_{\mathbf{p}} \\ \mathbf{v}_F \left(-i\hbar\nabla + \frac{e}{c}\mathbf{A}\right) v_{\mathbf{p}} - \Delta^* u_{\mathbf{p}} &= -\epsilon_p v_{\mathbf{p}} \end{aligned} \tag{1.59}$$

We denote $\mathbf{v}_F = \partial E_n/\partial \mathbf{p}$ and assign the index F to indicate that it is the velocity taken at the Fermi surface. Moreover, we neglect $\mathbf{p}_s^2/2m$ as compared to $|\Delta|$ because

$$\mathbf{p}_s^2/2m \sim \hbar^2/m\xi_0^2 \sim |\Delta|\hbar/p_F\xi_0$$

which is much smaller than $|\Delta|$ since $p_F\xi_0/\hbar \gg 1$. The self-consistency equation (1.51) becomes

$$\Delta_{\mathbf{p}}(\mathbf{r}) = -\int U_{\mathbf{p},\mathbf{p}'} u_{\mathbf{p}'}(\mathbf{r}) v^*_{\mathbf{p}'}(\mathbf{r}) \left(1 - 2n_{\mathbf{p}'}\right) \frac{d^3p'}{(2\pi\hbar)^3} \tag{1.60}$$

The possibility to separate and then exclude fast oscillating parts in the quasiparticle wave functions known as the quasiclassical approximation is provided by the relations between the magnitudes of the superconducting- and normal-state characteristic parameters of the superconducting material, namely $p_F\xi_0/\hbar \gg 1$. It is this fundamental property of most of superconductors which makes the BCS theory so successful in its practical applications. The major simplification is that the momentum dependence of the pairing potential is separated from the coordinate dependence of the wave functions $u_{\mathbf{p}}$ and $v_{\mathbf{p}}$ in the self-consistency equation (1.60). The momentum $\mathbf{p}$ enters the wave functions only as a parameter. Moreover, equations (1.59) are now of the first order each. Therefore, the mathematical complexity of the theory is considerably reduced.

The accuracy of the quasiclassical approximation depends on how well the inequality

$$\frac{\hbar}{p_F \xi_0} \sim \frac{\Delta}{E_F} \ll 1 \tag{1.61}$$

is satisfied. For conventional low-temperature superconductors, the parameter Δ/E_F is of the order of 10^{-3} thus the accuracy is very good. High temperature superconductors have $\Delta/E_F \sim 10^{-1}$ to 10^{-2} so that the quasiclassical approximation is less universal. However, it still has a reasonably solid base for validity, though description of some phenomena require a more careful analysis. Using this approximation, one can formulate a very powerful quasiclassical method which is indispensable for solving spatially non-homogeneous problems of the microscopic
theory of superconductivity. It also makes the basis for the modern microscopic theory of nonstationary phenomena in superconductors. The great advantage of the quasiclassical nonstationary theory is that it can incorporate, in a coherent way, various relaxation mechanisms including interaction with impurities in superconducting alloys. It is this quasiclassical method which is the main subject of the present book.

16.2.2 1.2 Nonstationary phenomena

Nonstationary theory considers behavior of superconductors in a.c. external fields (electromagnetic fields, sound waves, etc.). It treats transport phenomena such as electric or thermal conductivity, and thermopower. Problems associated with a d.c. electric field should also be considered within the nonstationary theory. Indeed, a d.c. electric field accelerates electrons which should then relax and give away their energy to the superconductor. This produces a time-dependent or dissipative state where the absorbed power creates a nonequilibrium distribution of excitations. The nonstationary theory should provide a consistent approach to the whole class of such problems using the superconducting characteristics discussed above.

1.2.1 Time-dependent Ginzburg-Landau theory

The simplest description of nonstationary processes in superconductors is provided by the so called time-dependent Ginzburg-Landau (TDGL) model which generalizes the usual Ginzburg-Landau (GL) theory to include relaxation processes. The TDGL model is widely used, it often gives a reasonable picture of superconducting dynamics. However, as distinct from its static counterpart, validity of the TDGL theory is much more limited. It is not enough to be just close to the critical temperature. The necessary condition requires also that deviations from equilibrium are small: the quasiparticle excitations should remain essentially in equilibrium with the heat bath. For real superconductors it is, in general, a very strong limitation. It can be normally fulfilled only for the so called gapless superconductivity. The latter corresponds to a situation where mechanisms working to destroy Cooper pairs are almost successful: the energy gap in the spectrum disappears, but the order parameter retains the phase coherence, and the supercurrent is finite. These mechanisms are: interaction with magnetic impurities which act differently on the electrons with opposite spins in a Cooper pair (see Section 6.2), inelastic (not conserving energy) interactions with phonons (see Section 11.2), etc. We shall consider these examples later. For instance, inelastic scattering by phonons is characterized by mean free time $\tau_{\rm ph}$. The condition for applicability of the TDGL theory is $\tau_{\rm ph}\Delta \ll \hbar$. Due to a comparatively large magnitude of $\tau_{\rm ph}$ (for example, $\tau_{\rm ph} \sim 10^{-9}$ s as compared to $\Delta/\hbar \sim 10^{11}$ s^{-1} for Al), this condition is only satisfied in a very narrow vicinity of T_c. In this Section we consider a phenomenological derivation of the TDGL model. Its microscopic justification will be given later in Chapter 11.

The TDGL model is constructed in the following way. In equilibrium, the GL energy eqn (1.3) has a minimum with respect to Δ and $\mathbf{A}$:

$$\frac{\delta \mathcal{F}_{sn}}{\delta \Delta} = 0; \frac{\delta \mathcal{F}_{sn}}{\delta \mathbf{A}} + \frac{1}{c}\mathbf{j}_s = 0 \tag{1.62}$$

If the superconductor is driven out of equilibrium, the order parameter should relax back to its equilibrium value. The rate of relaxation depends on the deviation from equilibrium:

$$-\Gamma \frac{\partial \Delta}{\partial t} = \frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} \tag{1.63}$$

where Γ is a positive constant. This equation, however, is not gauge invariant. For a time dependent state, the gauge invariance requires, in addition to eqn (1.7), that the equations describing the superconductor do not change under the transformation

$$\chi \to \chi + f, \mathbf{A} \to \mathbf{A} + \frac{\hbar c}{2e}\nabla f, \varphi \to \varphi - \frac{\hbar}{2e}\frac{\partial f}{\partial t} \tag{1.64}$$

where $f(\mathbf{r}, t)$ is an arbitrary function of coordinates and time. To preserve the gauge invariance of eqn (1.63) we add the scalar potential to the time derivative in the form

$$-\Gamma \left(\frac{\partial \Delta}{\partial t} + \frac{2ie\varphi}{\hbar}\Delta \right) = \frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} \tag{1.65}$$

The second equilibrium condition of eqn (1.62) is generalized in the following way. The current in eqn (1.62) being just a supercurrent in equilibrium is now replaced with $\mathbf{j}_s = \mathbf{j} - \mathbf{j}_n$ which accounts for the fact that a part of current can be produced by normal electrons in presence of an electric field:

$$\mathbf{j}_n = \sigma_n \mathbf{E} = -\sigma_n \left(\nabla \varphi + \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} \right) \tag{1.66}$$

Here σ_n is the conductivity in the normal state. As a result

$$\frac{1}{c}(\mathbf{j} - \mathbf{j}_n) = -\frac{\delta \mathcal{F}_{sn}}{\delta \mathbf{A}} \tag{1.67}$$

Finally, the TDGL equations become

$$-\Gamma \left(\frac{\partial \Delta}{\partial t} + \frac{2ie\varphi}{\hbar}\Delta \right) = -|\alpha|\Delta + \beta|\Delta|^2\Delta - \gamma\hbar^2 \left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right)^2 \Delta \tag{1.68}$$

The total current $\mathbf{j} = \mathbf{j}_s + \mathbf{j}_n$ has the form

$$\begin{aligned} \mathbf{j} =& \sigma_n \left(-\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \varphi \right) \\ &+ 2e\gamma \left[\Delta^* \left(-i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right) \Delta + \Delta \left(i\hbar\nabla - \frac{2e}{c}\mathbf{A} \right) \Delta^* \right] \end{aligned} \tag{1.69}$$

where $\mathbf{j}_s$ is defined by eqn (1.13).
Equations (1.68) and (1.69) define characteristic relaxation times of the order parameter

$$\tau_\Delta = \frac{\Gamma}{|\alpha|} \propto \left(1 - \frac{T}{T_c} \right)^{-1} \tag{1.70}$$

and of the vector potential (or current)

$$\tau_j = \frac{\beta \sigma_n}{8e^2\gamma|\alpha|} = \frac{\sigma_n}{8e^2\gamma\Delta_0^2} \propto \left(1 - \frac{T}{T_c} \right)^{-1} \tag{1.71}$$

The ratio of these two relaxation times

$$u = \frac{\tau_\Delta}{\tau_j} = \frac{8\Gamma e^2 \gamma}{\beta \sigma_n} \tag{1.72}$$

is independent of T.
Separating real and imaginary parts of eqn (1.68) we obtain

$$\tau_\Delta \frac{\partial |\Delta|}{\partial t} = |\Delta| - \frac{|\Delta|^3}{\Delta_\infty^2} + \xi^2 \left(\nabla^2 - \frac{4e^2}{\hbar^2 c^2} \mathbf{Q}^2 \right) |\Delta| \tag{1.73}$$

and

$$c\tau_\Delta |\Delta|^2 \Phi + \xi^2 \operatorname{div} \left(\mathbf{Q} |\Delta|^2 \right) = 0 \tag{1.74}$$

where we introduce one more gauge invariant (scalar) potential

$$\Phi = \varphi + \frac{\hbar}{2e} \frac{\partial \chi}{\partial t} \tag{1.75}$$

in addition to the vector potential $\mathbf{Q}$ defined in eqn (1.46).
Equation (1.74) can also be obtained by calculating the variation of the free energy with respect to the order parameter phase

$$\begin{aligned} 2\frac{\delta \mathcal{F}_{sn}}{\delta \chi} &\equiv i \left(\Delta \frac{\delta \mathcal{F}_{sn}}{\delta \Delta} - \Delta^* \frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} \right) \\ &= -i\hbar\gamma \left[\Delta \left(\nabla + \frac{2ie}{\hbar c} \mathbf{A} \right)^2 \Delta^* - \Delta \left(\nabla - \frac{2ie}{\hbar c} \mathbf{A} \right)^2 \Delta \right] \\ &= -\frac{\hbar}{2e} \operatorname{div} \mathbf{j}_s \end{aligned} \tag{1.76}$$

With help of eqn (1.65) we obtain from eqn (1.76)

$$\frac{\delta \mathcal{F}_{sn}}{\delta \chi} = -\Gamma |\Delta|^2 \left(\frac{\partial \chi}{\partial t} + \frac{2e\phi}{\hbar} \right) = -\frac{\hbar}{4e} \operatorname{div} \mathbf{j}_s \tag{1.77}$$

which is nothing but eqn (1.74).
According to the charge neutrality of a superconductor as a metal

$$\operatorname{div} (\mathbf{j}_n + \mathbf{j}_s) = 0 \tag{1.78}$$

eqn (1.77) gives

$$\frac{\hbar}{4e} \operatorname{div} \mathbf{j}_s = -\frac{\hbar}{4e} \operatorname{div} \mathbf{j}_n = \frac{2e\Gamma |\Delta|^2}{\hbar} \Phi \tag{1.79}$$

Writing

$$\mathbf{j}_n = -\sigma_n \left(\nabla \Phi + \frac{1}{c} \frac{\partial \mathbf{Q}}{\partial t} \right)$$

we obtain

$$l_E^2 \left(\nabla^2 \Phi + \frac{1}{c} \operatorname{div} \frac{\partial \mathbf{Q}}{\partial t} \right) = \frac{|\Delta|^2}{\Delta_\infty^2} \Phi \tag{1.80}$$

This equation defines the new characteristic length

$$l_E = \sqrt{\frac{\hbar\sigma_n}{8e^2\Gamma\Delta_\infty^2}} = \xi/\sqrt{u} \tag{1.81}$$

Since a d.c. electric field exists as long as Φ is nonzero, this length has a meaning of the electric-field penetration length, i.e., the distance over which a d.c. electric field decays into a superconductor. At the same time, a nonzero Φ initiates a conversion of a normal current into a supercurrent according to eqn (1.79). We discuss this in more detail later in Chapter 11.

1.2.2 Microscopic argumentation

Let us discuss how one can justify the TDGL model microscopically. The TDGL equation was first derived by Schmid (1966) and by Abrahams and Tsuneto (1966) using a Green function formalism. We consider here a more simple method based on the BCS equation (1.57). Assume that $\mathbf{p}_s = 0$ and put

$$\frac{d^3p}{(2\pi\hbar)^3} = \nu\left(\xi_{\mathbf{p}}\right) d\xi_{\mathbf{p}}$$

where $\nu\left(\xi_{\mathbf{p}}\right)$ is the density of states in the normal metallic state. We assume that $\nu\left(\xi_{\mathbf{p}}\right)$ is approximately constant and equal to the density of states $\nu(0)$ at the Fermi surface, where

$$\nu(0) = mp_F/2\pi^2\hbar^3$$

for a parabolic normal-state spectrum. This assumption will be discussed in detail later in Chapter 5 . Next, we transform to the energy integration in eqn (1.57) through $\xi_{\mathbf{p}} d\xi_{\mathbf{p}} = \epsilon_{\mathbf{p}} d\epsilon_{\mathbf{p}}$ using eqn (1.1). The function $\xi_{\mathbf{p}}$ changes its sign during integration over d^3p. To account for this, we note that $1 - 2n_{\mathbf{p}} = \tanh\left(\epsilon_{\mathbf{p}}/2T\right)$ is
an odd function of $\epsilon_{\mathbf{p}}$. Let us define $\xi_{\mathbf{p}}$ as an analytical function in the complex plane of the variable $\epsilon_{\mathbf{p}}$ with a cut from $-|\Delta|$ to $+|\Delta|$ (see Fig. 5.1). We obtain

$$\xi_{\mathbf{p}}\left(\epsilon_{\mathbf{p}}\right) = \sqrt{\epsilon_{\mathbf{p}}^2 - |\Delta|^2} \tag{1.82}$$

Now $\epsilon_{\mathbf{p}}$ can be both positive and negative provided $\epsilon_{\mathbf{p}}^2 > |\Delta|^2$. The function $\xi_{\mathbf{p}}$ changes its sign as $\epsilon_{\mathbf{p}}$ varies from $\epsilon_{\mathbf{p}} > |\Delta|$ to $\epsilon_{\mathbf{p}} < -|\Delta|$. Equation (1.82) correctly transforms into the normal state spectrum $\epsilon_{\mathbf{p}} = \xi_{\mathbf{p}}$ if $|\Delta| = 0$. The BCS equation (1.57) becomes

$$\frac{\Delta_{\mathbf{p}}}{\lambda} = \int V_{\mathbf{p},\mathbf{p}'}\Delta_{\mathbf{p}'}\frac{d\Omega_{p'}}{4\pi}\int\frac{1-2n_{\mathbf{p}'}}{2\xi_{\mathbf{p}'}}\Theta\left(\epsilon_{\mathbf{p}'}^2 - |\Delta_{\mathbf{p}'}|^2\right) d\epsilon_{\mathbf{p}'} \tag{1.83}$$

where we put $U_{\mathbf{p},\mathbf{p}'} = -|g|V_{\mathbf{p},\mathbf{p}'}$ and denote $\lambda = \nu(0)|g|$. The factor $V_{\mathbf{p},\mathbf{p}'}$ is the spherical harmonics of the pairing interaction. For s-wave pairing $V_{\mathbf{p},\mathbf{p}'} = 1$. The integration runs over positive and negative energies. It is limited from above by $|\epsilon| < \Omega_{\mathrm{BCS}}$. The upper BCS cutoff energy is determined by the particular pairing mechanism.

Consider a limit of small Δ and assume s-wave pairing for simplicity. With our definition of $\xi_{\mathbf{p}}$ as an analytical function, eqn (1.83) becomes

$$\frac{\Delta}{\lambda} = \int\left(\frac{\Delta}{\epsilon + i\delta} + \frac{\Delta}{\epsilon - i\delta}\right)[1 - 2n(\epsilon)]\frac{d\epsilon}{4} \tag{1.84}$$

Here we drop the subscript $\mathbf{p}$ at ϵ for brevity. If the order parameter depends on time the energy ϵ becomes shifted due the external frequency ω. To obtain the correct shift, we notice that ϵ is a single-particle energy while ω refers to the frequency of a Cooper pair. Therefore, ϵ should be shifted by $\hbar\omega/2$. According to the casualty principle, we require that $\Delta(\omega)$ is an analytical function in the upper half-plane of complex ω. As a result, eqn (1.84) takes the form

$$\frac{\Delta(\omega)}{\lambda} = \int \Delta(\omega) \left(\frac{1}{\epsilon + \hbar\omega/2 + i\delta} + \frac{1}{\epsilon - \hbar\omega/2 - i\delta} \right) [1 - 2n(\epsilon)] \frac{d\epsilon}{4}$$

Expanding it in a small ω we get

$$\begin{aligned} \frac{\Delta(\omega)}{\lambda} =& \Delta(\omega) \int_{-\Omega_{\rm BCS}}^{\Omega_{\rm BCS}} \frac{1 - 2n(\epsilon)}{2\epsilon} d\epsilon \\ &+ \frac{\hbar\omega\Delta(\omega)}{4} \int_{-\infty}^{\infty} \left(\frac{1}{\epsilon + i\delta} - \frac{1}{\epsilon - i\delta} \right) \frac{dn(\epsilon)}{d\epsilon} d\epsilon \end{aligned} \tag{1.85}$$

Since

$$\frac{dn(\epsilon)}{d\epsilon} = -\frac{1}{4T} \cosh^{-2}\left(\frac{\epsilon}{2T}\right)$$

the last line of eqn (1.85) gives

$$\frac{\pi\hbar i}{8T}\omega\Delta(\omega) \quad \text{or} \quad -\frac{\pi\hbar}{8T}\frac{\partial\Delta}{\partial t}$$

in the time representation. As we shall see later, the first line gives the usual GL part of the equation. Comparing this with eqn (1.68) we find the microscopic value of the relaxation parameter

$$\Gamma = \frac{\nu(0)\pi\hbar}{8T_c} \tag{1.86}$$

The derivation is based on simple intuitive arguments. However, it is not generally correct. Indeed, it treats nonstationary processes by including the external energy ω into the energy spectrum and ignoring the processes which are associated with relaxation. In fact, this derivation implicitly assumes that both the energy spectrum and the quasiparticle distribution remain the same as in equilibrium, i.e., that the relaxation is infinitely fast. We shall demonstrate later in Section 11.2, that this picture is appropriate only for gapless superconductors. The result eqn (1.86) itself is literally applicable only for gapless superconductors without magnetic impurities. We stress once again that general limitations of validity of the TDGL model come from non-equilibrium excitations: equation (1.85) becomes incorrect as soon as the deviation from equilibrium increases.

1.2.3 Boltzmann kinetic equation

Consider an alternative scheme of treating nonstationary processes which is capable of incorporating the quasiparticle relaxation. One would expect that it is the Boltzmann kinetic equation that is appropriate to describe a nonequilibrium dynamics. The Boltzmann kinetic approach has been discussed in detail by Betbeder-Matibet and Nozières (1969), and by Aronov et al. (1981, 1986). We give only a brief outline here. A derivation of the Boltzmann equation is given later in Section 15.2.

With an account for a non-equilibrium scalar potential Φ the energy of the excitations in eqn (1.54) is defined with

$$\tilde{\xi}_{\mathbf{p}} = \xi_{\mathbf{p}} + e\Phi$$

where Φ is introduced by eqn (1.75). We neglect the term $p_s^2/2m$ in eqn (1.55). The canonical form of the Boltzmann equation for the distribution function for a situation where

the order parameter, supercurrent, and external fields are slowly varying in space and in time has the form

$$\frac{\partial n_{\mathbf{p}}}{\partial t} + \mathbf{v}_g \cdot \nabla n_{\mathbf{p}} + \mathbf{f}\frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} = I\left\{n_{\mathbf{p}}\right\} \tag{1.87}$$

The group velocity and the elementary force are defined as

$$\mathbf{v}_g = \frac{\partial \epsilon_{\mathbf{p}}}{\partial \mathbf{p}} = \frac{\tilde{\xi}_{\mathbf{p}}}{\tilde{\epsilon}_{\mathbf{p}}}\mathbf{v}_F, \mathbf{f} = -\frac{\partial \tilde{\epsilon}_{\mathbf{p}}}{\partial \mathbf{r}} + \mathbf{f}_L, \mathbf{f}_L = \frac{e}{c}\mathbf{v}_F \times \mathbf{H} \tag{1.88}$$

The force $\mathbf{f}_L$ is the elementary Lorentz force acting on a particle in a magnetic field.

The r.h.s. of eqn (1.87) is the collision integral. The impurity part of the collision integral has the form (see Section 15.2)

$$\begin{aligned} I^{(\mathrm{imp})}\left\{n_{\mathbf{p}}\right\} = &-\frac{1}{\nu(0)\tau_{\mathrm{imp}}}\int \left(u_{\mathbf{p}}u_{\mathbf{p}'} - v_{\mathbf{p}}v_{\mathbf{p}'}\right)^2 \\ &\times \left(n_{\mathbf{p}} - n_{\mathbf{p}'}\right)\delta\left(\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}'}\right)\frac{d^3p'}{(2\pi)^3} \end{aligned} \tag{1.89}$$

where the Bogoliubov-de Gennes coherence factors are determined by eqn (1.56). The expression for the phonon collision integral is rather complicated and we do not write it here.

The electric filed enters the kinetic equation through the time derivative of energy according to

$$\frac{\partial n_{\mathbf{p}}}{\partial t} = \frac{\partial n_0}{\partial \epsilon_{\mathbf{p}}}\frac{\partial \epsilon_{\mathbf{p}}}{\partial t} = \frac{\partial n_0}{\partial \epsilon_{\mathbf{p}}}\left(e\mathbf{v}_F \cdot \mathbf{E} + e\mathbf{v}_{\mathbf{F}} \cdot \nabla\Phi + \frac{\partial|\Delta|}{\partial t}\frac{|\Delta|}{\tilde{\epsilon}_{\mathbf{p}}}\right)$$

because

$$\frac{\partial \mathbf{p}_s}{\partial t} = e\mathbf{E} + e\nabla\Phi$$

Solving the Boltzmann kinetic equation one can find the distribution function which should then be used in the BCS equation (1.57) for self-consistent determination of the order parameter.

Equation (1.87) is simple and transparent. However, its region of applicability is very limited because the order parameter and the supercurrent were assumed almost constant in space. In fact, attempts to use this equation together with the spectral characteristics of a superconductor for slow varying $|\Delta|$ and $\mathbf{p}_s$ may lead to incorrect results because the excitation spectrum eqn (1.54) is itself only defined for constant $|\Delta|$ and $\mathbf{p}_s$ and does not contain corrections which would appear if $|\Delta|$ and $\mathbf{p}_s$ become functions of coordinates. Moreover, the Boltzmann kinetic equation uses a concept of quasiparticle spectrum. The spectrum is well defined only for a clean superconductor such that $\Delta\tau \gg \hbar$. Dirty alloys can not be incorporated into the Boltzmann scheme. Again, we are in a situation where the use of a more general microscopic theory is vital.

The kinetic equation approach is not reduced to a TDGL scheme as T approaches T_c. In fact, these two descriptions do not overlap. Indeed, for the TDGL model it is important that the distribution function of excitations does not deviate from equilibrium. On the contrary, the kinetic equation approach implies that deviations from equilibrium do appear in a time-dependent state of a superconductor. The kinetic theory based on eqn (1.87) can give a hint on the limit of applicability of the TDGL model. If the order parameter gradients are slow on a length scale of order of the mean free path, the kinetic equation can be transformed into a diffusion-like equation which gives (compare with eqn (10.107) on page 223)

$$\left(D\nabla^2 - \frac{\epsilon_{\mathbf{p}}}{\tau_{\rm ph}\xi_{\mathbf{p}}}\right)\delta n_{\mathbf{p}} = \frac{\partial|\Delta|}{\partial t}\frac{|\Delta|}{\xi_{\mathbf{p}}}\frac{dn_0}{d\epsilon_{\mathbf{p}}} \tag{1.90}$$

where $\delta n_{\mathbf{p}} = n_{\mathbf{p}} - n_0$ and D is the diffusion constant. The inelastic relaxation is usually very slow. If the diffusion dominates, the deviation of distribution from equilibrium is

$$\delta n_{\mathbf{p}} \sim \left(Dk^2\right)^{-1}\frac{\partial|\Delta|}{\partial t}\frac{|\Delta|}{\xi_{\mathbf{p}}}\frac{dn_0}{d\epsilon_{\mathbf{p}}}$$

The corresponding contribution to eqn (1.85) is much larger than the TDGL term

$$\frac{\hbar\pi}{8T_c}\frac{\partial|\Delta|}{\partial t}$$

due to a long-range diffusion relaxation. In the opposite limit, where the order parameter varies over distances shorter than the mean free path, i.e., when $\xi \ll \ell$, one generally has

$$\delta n_{\mathbf{p}} \sim \tau\frac{\partial|\Delta|}{\partial t}\frac{|\Delta|}{\epsilon_{\mathbf{p}}}\frac{dn_0}{d\epsilon_{\mathbf{p}}}$$

The corresponding contribution is by a big factor $|\Delta|\tau \gg \hbar$ larger than the time-dependent Ginzburg-Landau term.

We observe from eqn (1.90) that a nonequilibrium correction to distribution is small compared to the TDGL term only for a hydrodynamic limit when $\tau_{\rm ph}|\Delta| \ll \hbar$. This is exactly the region of the so called gapless superconductivity associated with inelastic scattering of electrons. The conclusion is that the TDGL model is expected to work only for gapless superconductors.

16.2.3 1.3 Outline of the contents

All the simple theories considered above have a limited applicability. The TDGL model can treat nonstationary problems in case when the order parameter and magnetic field vary in space. Moreover, the effect of impurities can be easily taken into account. This model, however, works only for small deviations from equilibrium and only in a close vicinity of T_c. On the contrary, the kinetic scheme of eqn (1.87) can work at low temperatures and include large deviations from equilibrium. Unfortunately, it is not suitable for spatially inhomogeneous problems. In addition, it is not designed to deal with superconducting alloys where the mean free path is comparable to the coherence length, i.e., in cases when the excitation spectrum is not well defined.

A theory which is supposed to describe nonstationary properties of superconductors in the whole range of parameters and in a general, spatially inhomogeneous situation, which would include various types of relaxation processes in a self-consistent and universal way, - this theory should provide answers at least to the following problems. (i) How can one define the order parameter and other physical observables in a nonstationary state? As an important part of this general issue, there emerges a problem of calculating the energy spectrum (or other equivalent characteristics of a superconductor if the spectrum is not well defined) in case when the superconducting state is not spatially uniform and varies in time. (ii) What are the processes which determine the distribution of excitations under nonstationary conditions? And, last but not least, (iii) what
are the practical methods for solving nonstationary problems and for calculating the physical observables?

We shall discuss these problems throughout the book and describe a general approach which is able to deal with these problems in a coherent way. It is the Green function technique, and, in particular, the technique which uses the real-time Green functions. The important component

of the general theory is the quasiclassical method which provides the easiest way of practical implementation of the Green function technique. The theory is based on general principles of the BCS scheme. In addition, it includes interactions of electrons with impurities, phonons, and electron-electron interaction. We start with the general definitions of the Green functions. We describe the BCS theory and derive the Gor'kov equations. Next, we introduce the quasiclassical approach. The real-time Green functions are introduced using two different but completely equivalent methods: the Keldysh formalism and the method of analytical continuation of the imaginary-time Matsubara Green functions onto the real-time domain. We obtain the general expressions for the order parameter and other quantities and derive transport-like and kinetic equations for the distribution functions of excitations. For practical implementations of the general formalism, we derive the TDGL equations and consider other examples of nonstationary phenomena. A considerable part of the book is devoted to the vortex dynamics as a problem which incorporates almost all nonstationary phenomena existing in a superconducting state. Discussion of the vortex dynamics for each particular system starts with an introductory Section which elucidates the main processes characteristic for the system under consideration. Let us now turn to the general definition of the Green functions.

16.3 GREEN FUNCTIONS

We introduce the second quantization formalism and define the temperature and time dependent Green functions which will be used throughout the book.

(here, there are very important and deep ideas, which I'll work out in the note on cond matter, and delete from here)

16.3.1 2.1 Second quantization

The Green function formalism uses the method of second quantization. We summarize its basic ideas (Landau and Lifshitz 1959 a, Abrikosov al. 1965) to make the presentation self-consistent. We mostly follow the style of Abrikosov al. (1965) in this Chapter. The reader who is familiar with the general definition of the Green functions can proceed directly to the next Chapter. From now on, we use the units where $\hbar = 1$.

Consider first a system of N Bose particles which can occupy quantum states with wave functions $\phi_1(x), \phi_2(x), \ldots$. Here x is the set of general coordinates. We describe a state of such a system by a set of numbers $N_1, N_2, \ldots$, which show how many particles are in the states $1, 2, \ldots$. The wave function for the state with the occupation numbers $N_1, N_2, \ldots$ is symmetric with respect to transposition of particles:

$$\Phi_{N_1 N_2 \ldots} = \left(\frac{N_1! N_2! \ldots}{N!} \right)^{1/2} \sum_P \phi_{i_1}(x_1) \phi_{i_2}(x_2) \ldots \phi_{i_N}(x_N) . \tag{2.1}$$

Here i_m are labels of the states, and the sum is taken over all possible transpositions of the labels i_m. The prefactor takes care of normalization. We introduce the operator a_i which decreases the occupation number in the state i by 1 . According to eqn (2.1), its only nonzero matrix element is

$$(a_i)_{N_i}^{N_i - 1} = \sqrt{N_i} \tag{2.2}$$

The conjugated operator increases the occupation number by 1 :

$$\left(a^\dagger\right)_{N_i - 1}^{N_i} = \sqrt{N_i} \tag{2.3}$$

Therefore

$$a_i^\dagger a_i = N_i, a_i a_i^\dagger = N_i + 1 \tag{2.4}$$

so that

$$a_i a_k^\dagger - a_k^\dagger a_i = \delta_{ik}, \tag{2.5}$$

and

$$a_i a_k - a_k a_i = a_i^\dagger a_k^\dagger - a_k^\dagger a_i^\dagger = 0. \tag{2.6}$$

For Fermi particles, the occupation numbers may be either 0 or 1 only, and the wave function is antisymmetric with respect to all variables:

$$\Phi_{N_1 N_2 \ldots} = \frac{1}{\sqrt{N!}} \sum_P \delta_P \phi_{i_1}(x_1) \phi_{i_2}(x_2) \ldots \phi_{i_N}(x_N). \tag{2.7}$$

Here δ_P takes +1 or -1 depending on whether the state $i_1, i_2, \ldots$ is obtained by an even or odd transposition of some initial configuration, respectively. Creation and annihilation operators now have the matrix elements

$$(a_i)_1^0 = \left(a_i^\dagger\right)_0^1 = (-1)^{\sum_{l=1}^{i-1} N_l}. \tag{2.8}$$

Therefore,

$$a_i^\dagger a_i = N_i, a_i a_i^\dagger = 1 - N_i. \tag{2.9}$$

The commutation rules are

$$a_i a_k^\dagger + a_k^\dagger a_i = \delta_{ik} \tag{2.10}$$

$$a_i a_k + a_k a_i = a_i^\dagger a_k^\dagger + a_k^\dagger a_i^\dagger = 0 \tag{2.11}$$

Consider an operator of the form

$$\hat{L} = \sum_a L^{(1)}(x_a) + \sum_{a,b} L^{(2)}(x_a, x_b) + \ldots \tag{2.12}$$

where $L^{(1)}$ depends on a single coordinate (single-particle operator), $L^{(2)}$ depends on two coordinates (two-particle operator), etc. In terms of creation and annihilation operators, we can write

$$\hat{L} = \sum_{ik} L_{ik}^{(1)} a_i^\dagger a_k + \sum_{iklm} L_{lm}^{(2)ik} a_i^\dagger a_k^\dagger a_l a_m + \ldots \tag{2.13}$$

where the matrix elements are

$$L_{ik}^{(1)} = \int \phi_i^*(x) L^{(1)}(x) \phi_k(x) dx$$

and

$$L_{iklm}^{(2)} = \int \phi_i^*(x_1) \phi_k^*(x_2) L^{(2)}(x_1, x_2) \phi_l(x_2) \phi_m(x_1) dx_1 dx_2$$

The operator in eqn (2.13) acts on the occupation numbers of particles. This definition holds for both Bose and Fermi particles.

2.1.1 Schrödinger and Heisenberg operators

(I'll write it in the note on cond matter, but I still don't understand how we glue two quantizations into one formalism (wave function and operators))

One can introduce the operators

$$\psi(x) = \sum_i \phi_i(x) a_i, \psi^\dagger(x) = \sum_i \phi_i^*(x) a_i^\dagger \tag{2.14}$$

Operator $\psi^\dagger(x)$ creates a particle at the point x while $\psi(x)$ annihilates it. These operators are called Schrödinger "particle-field" operators. We have

$$\psi(x)\psi^\dagger(x') \pm \psi^\dagger(x')\psi(x) = \delta(x - x') \tag{2.15}$$

$$\psi(x)\psi(x') \pm \psi(x')\psi(x) = 0 \tag{2.16}$$

$$\psi^\dagger(x)\psi^\dagger(x') \pm \psi^\dagger(x')\psi^\dagger(x) = 0 \tag{2.17}$$

The upper sign is for Fermi particles while the lower is for Bose particles. The coordinate x may also include spin. The operator $\hat{L}$, which for definiteness is taken to be dependent on spatial coordinates, can be written as

$$\begin{aligned}\hat{L} = &\sum_\alpha \int \psi_\alpha^\dagger(\mathbf{r}) L^{(1)}(\mathbf{r}) \psi_\alpha(\mathbf{r}) d^3r \\ &+ \sum_{\alpha\beta} \iint \psi_\alpha^\dagger(\mathbf{r}) \psi_\beta^\dagger(\mathbf{r}') L^{(2)}(\mathbf{r}, \mathbf{r}') \psi_\beta(\mathbf{r}') \psi_\alpha(\mathbf{r}) d^3r d^3r' \ldots\end{aligned}$$

Here α and β are the spin indices.
Let the Hamiltonian be $\hat{\mathcal{H}}$. The wave function obeys the Schrödinger equation

$$i\frac{\partial \Phi}{\partial t} = \hat{\mathcal{H}}\Phi \tag{2.18}$$

Its solution can be symbolically written as

$$\Phi(t) = e^{-i\hat{\mathcal{H}}t}\Phi_H \tag{2.19}$$

where Φ_H does not depend on t. A time-dependent matrix element of an operator F is

$$F_{nm}(t) = \int \Phi_n^*(t) F \Phi_m(t) d^3r = \int \Phi_{Hn}^*(t) e^{i\hat{\mathcal{H}}t} F e^{-i\hat{\mathcal{H}}t} \Phi_{Hm}(t) d^3r \tag{2.20}$$

This is the matrix element of the operator

$$\tilde{F}(t) = e^{i\hat{\mathcal{H}}t} F e^{-i\hat{\mathcal{H}}t} \tag{2.21}$$

This operator is called the Heisenberg operator. Its convenience is in that now one can use time-independent wave functions Φ_H to calculate time-dependent matrix elements. The time dependence is transferred to the operators. The time derivative of $\tilde{F}$ is

$$\frac{\partial \tilde{F}}{\partial t} = i(\hat{\mathcal{H}}\tilde{F} - \tilde{F}\hat{\mathcal{H}}) \tag{2.22}$$

Note that the commutation rules for Heisenberg operators taken at the same time are the same as for the corresponding operators in Schrödinger representation, since the exponential factors cancel in products of two operators.

16.3.2 2.2 Imaginary-time Green function

2.2.1 Definitions

The imaginary-time (Matsubara) Green function is defined for imaginary time $t = -i\tau$ within the interval

$$-\frac{1}{T} < \tau_1 - \tau_2 < \frac{1}{T}$$

as follows:

$$G_{\alpha\beta}\left(\mathbf{r}_1, \tau_1; \mathbf{r}_2, \tau_2\right) = \sum^{(d)} \left[\exp\left(\frac{\Omega + \mu\hat{\mathcal{N}} - \hat{\mathcal{H}}}{T}\right) e^{(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})(\tau_1-\tau_2)}\psi_\alpha\left(\mathbf{r}_1\right) e^{-(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})(\tau_1-\tau_2)}\psi_\beta^\dagger\left(\mathbf{r}_2\right)\right] \tag{2.23}$$

for $\tau_1 > \tau_2$ and

$$G_{\alpha\beta}\left(\mathbf{r}_1, \tau_1; \mathbf{r}_2, \tau_2\right) = \mp \sum^{(d)} \left[\exp\left(\frac{\Omega + \mu\hat{\mathcal{N}} - \hat{\mathcal{H}}}{T}\right) e^{-(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})(\tau_1-\tau_2)}\psi_\beta^\dagger\left(\mathbf{r}_2\right) e^{(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})(\tau_1-\tau_2)}\psi_\alpha\left(\mathbf{r}_1\right)\right] \tag{2.24}$$

if $\tau_1 < \tau_2$. Here $\psi_\alpha(\mathbf{r})$ and $\psi_\beta^\dagger(\mathbf{r})$ are Schrödinger operators; the signs $\mp$ refer to Fermi and Bose particles, respectively. Operators $\hat{\mathcal{H}}$ and $\hat{\mathcal{N}}$ are the Hamiltonian of the system and the particle-number operator. Operator $\sum^{(d)}$ means the sum over all diagonal matrix elements both for all quantum states for a given number of particles and for all numbers of particles. Thus G is a function of T and the chemical potential $\mu; \Omega$ is the thermodynamic potential: $d\Omega = -\mathcal{S}dT - PdV - \mathcal{N}d\mu$. It is defined as

$$\Omega = \mathcal{F} - \mu\mathcal{N} \tag{2.25}$$

where $\mathcal{F}$ is the free energy,

$$\mathcal{F} = -T \ln \sum_n e^{-E_n/T} \tag{2.26}$$

Therefore, the sum

$$\sum^{(d)} \left[\exp\left(\frac{\Omega + \mu\hat{\mathcal{N}} - \hat{\mathcal{H}}}{T}\right) \ldots\right] = \langle \ldots \rangle_{st} \tag{2.27}$$

is the Gibbs statistical average. Note that

$$\langle 1 \rangle_{st} = 1$$

due to the definitions of eqns (2.25) and (2.26).
The phonon Green function is

$$D\left(\mathbf{r}_1, \tau_1; \mathbf{r}_2, \tau_2\right) = \sum^{(d)} \left[\exp\left(\frac{\Omega - \hat{\mathcal{H}}}{T}\right) e^{\hat{\mathcal{H}}(\tau_1-\tau_2)}\phi\left(\mathbf{r}_1\right) e^{-\hat{\mathcal{H}}(\tau_1-\tau_2)}\phi\left(\mathbf{r}_2\right)\right]$$

for $\tau_1 > \tau_2$ and

$$D\left(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2\right)=\sum^{(d)}\left[\exp\left(\frac{\Omega-\hat{\mathcal{H}}}{T}\right)e^{-\hat{\mathcal{H}}(\tau_1-\tau_2)}\phi\left(\mathbf{r}_2\right)e^{\hat{\mathcal{H}}(\tau_1-\tau_2)}\phi\left(\mathbf{r}_1\right)\right]$$

for $\tau_1<\tau_2$.
The function $G\left(\tau_1-\tau_2\right)$ experiences a jump at $\tau=\tau_1-\tau_2=0$. For Fermi particles,

$$\begin{aligned}[G(\tau)-G(-\tau)]_{\tau\to+0}&=\left\langle\left(\psi_\alpha\left(\mathbf{r}_1\right)\psi_\beta^\dagger\left(\mathbf{r}_2\right)+\psi_\beta^\dagger\left(\mathbf{r}_2\right)\psi_\alpha\left(\mathbf{r}_1\right)\right)\right\rangle_{st}\\&=\delta_{\alpha\beta}\delta\left(\mathbf{r}_1-\mathbf{r}_2\right)\end{aligned}\tag{2.28}$$

Let us introduce the "Heisenberg" particle operators which depend on time τ. In the representation with a given μ we have

$$\tilde{\psi}_\alpha(\mathbf{r},\tau)=e^{(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})\tau}\psi_\alpha(\mathbf{r})e^{-(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})\tau}$$

$$\tilde{\psi}_\alpha^\dagger(\mathbf{r},\tau)=e^{(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})\tau}\psi_\alpha^\dagger(\mathbf{r})e^{-(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})\tau}\tag{2.29}$$

$$\tilde{\phi}(\mathbf{r},\tau)=e^{\hat{\mathcal{H}}\tau}\phi(\mathbf{r})e^{-\hat{\mathcal{H}}\tau}\tag{2.30}$$

The Heisenberg operators satisfy the equations
(I am already lost, why is it so??)

$$\begin{aligned}\frac{\partial\tilde{\psi}_\alpha}{\partial\tau}&=(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})\tilde{\psi}_\alpha-\tilde{\psi}_\alpha(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})\\\frac{\partial\tilde{\psi}_\alpha^\dagger}{\partial\tau}&=(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})\tilde{\psi}_\alpha^\dagger-\tilde{\psi}_\alpha^\dagger(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})\end{aligned}\tag{2.31}$$

Using this definition, we can write

$$\begin{aligned}G_{\alpha\beta}\left(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2\right)&=\mathrm{Tr}\left[\exp\left(\frac{\Omega+\mu\hat{\mathcal{N}}-\hat{\mathcal{H}}}{T}\right)T_\tau\left(\tilde{\psi}_\alpha\left(\mathbf{r}_1,\tau_1\right)\tilde{\psi}_\beta^\dagger\left(\mathbf{r}_2,\tau_2\right)\right)\right]\\&\equiv\left\langle T_\tau\left(\tilde{\psi}_\alpha\left(\mathbf{r}_1,\tau_1\right)\tilde{\psi}_\beta^\dagger\left(\mathbf{r}_2,\tau_2\right)\right)\right\rangle_{st}\end{aligned}\tag{2.32}$$

Here T_τ means ordering in τ : the ψ-operators under T_τ are placed from left to right in order of decreasing "time" τ. For Fermi particles we have

$$T_\tau\left(\psi_1\psi_2\ldots\right)=\delta_P\psi_{i_1}\psi_{i_2}\ldots\tag{2.33}$$

In the r.h.s., the ψ-operators are put in the chronological order, and δ_P takes +1 or -1 depending on whether the transposition

$$1,2,\ldots\to i_1,i_2,\ldots$$

is even or odd. For example,

$$T_\tau\left(\tilde{\psi}_1\tilde{\psi}_2^\dagger\right)=\begin{cases}\tilde{\psi}_1\tilde{\psi}_2^\dagger, & \tau_1>\tau_2\\-\tilde{\psi}_2^\dagger\tilde{\psi}_1, & \tau_1<\tau_2\end{cases}$$

The definition eqn (2.32) can be generalized to the case when the Hamiltonian depends on time τ explicitly. The Heisenberg operators are defined as

$$\tilde{\psi}_\alpha(\mathbf{r},\tau) = \hat{S}^{-1}(\tau,0)\psi_\alpha(\mathbf{r},0)\hat{S}(\tau,0)$$
$$\tilde{\psi}^\dagger_\alpha(\mathbf{r},\tau) = \hat{S}^{-1}(\tau,0)\psi^\dagger_\alpha(\mathbf{r},0)\hat{S}(\tau,0) \tag{2.34}$$

where we introduce the S-matrix

$$\hat{S}(\tau,0) = T_\tau \exp\left[-\int_0^\tau (\hat{\mathcal{H}} - \mu\hat{\mathcal{N}})d\tau'\right] \tag{2.35}$$

and

$$\hat{S}^{-1}(\tau,0) = \bar{T}_\tau \exp\left[-\int_\tau^0 (\hat{\mathcal{H}} - \mu\hat{\mathcal{N}})d\tau'\right] \tag{2.36}$$

The operator $\bar{T}_\tau$ orders times as they appear in the integral, i.e., first comes the time τ then a time which is close to τ and lies between τ and 0 , etc. Therefore,

$$\hat{S}^{-1}(\tau,0)\hat{S}(\tau,0) = 1$$

We have for $\tau_1 > \tau_2$

$$\hat{S}\left(\tau_1,\tau_2\right) = T_\tau \exp\left[-\int_{\tau_2}^{\tau_1} (\hat{\mathcal{H}} - \mu\hat{\mathcal{N}})d\tau'\right] = \hat{S}\left(\tau_1,0\right)\hat{S}^{-1}\left(\tau_2,0\right)$$

Moreover

$$\hat{S} \equiv \hat{S}\left(\frac{1}{T},0\right) = T_\tau \exp\left[-\int_0^{1/T} (\hat{\mathcal{H}} - \mu\hat{\mathcal{N}})d\tau\right]$$

We also have

$$\exp\left(-\frac{\Omega}{T}\right) = \sum^{(d)}[\hat{S}] \tag{2.37}$$

The Green function is now defined as

$$G_{\alpha\beta}\left(\mathbf{r}_1,\tau_1;\mathbf{r}_2,\tau_2\right) = \frac{\sum^{(d)}\left[\hat{S}T_\tau\left(\tilde{\psi}_\alpha\left(\mathbf{r}_1,\tau_1\right)\tilde{\psi}^\dagger_\beta\left(\mathbf{r}_2,\tau_2\right)\right)\right]}{\sum^{(d)}[\hat{S}]}$$
$$\equiv \left\langle T_\tau\left(\tilde{\psi}_\alpha\left(\mathbf{r}_1,\tau_1\right)\tilde{\psi}^\dagger_\beta\left(\mathbf{r}_2,\tau_2\right)\right)\right\rangle_{st} \tag{2.38}$$

which coincides with eqn (2.32). It is very important to note that since the Hamiltonian of the system is also constructed of the ψ operators, it is the use of
imaginary time which allows us to place all the ψ operators in proper positions consistent with the T_τ operation. This is possible if the Green function is defined within the time interval $-1/T < \tau < 1/T$ where $\tau = \tau_1 - \tau_2$. It is exactly the reason why we consider an imaginary time: for a real time it is impossible to define unambiguously the order of the ψ-operators.

There exists an important relation which couples $G(\tau < 0)$ with $G(\tau > 0)$. Indeed, if we make a cyclic transposition under the trace operator $\sum^{(d)}$, we obtain

$$
\begin{aligned}
G(\tau<0) &= \mp \sum^{(d)} \left[\exp\left(\frac{\Omega+\mu\hat{\mathcal{N}}-\hat{\mathcal{H}}}{T}\right) e^{-(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})\tau} \psi_\beta^\dagger(\mathbf{r}_2)\, e^{(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})\tau} \psi_\alpha(\mathbf{r}_1) \right] \\
&= \mp \sum^{(d)} \left[\exp\left(\frac{\Omega}{T}\right) e^{(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})\tau} \psi_\alpha(\mathbf{r}_1)\, e^{-(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})(\tau+1/T)} \psi_\beta^\dagger(\mathbf{r}_2) \right] \\
&= \mp \sum^{(d)} \left[\exp\left(\frac{\Omega+\mu\hat{\mathcal{N}}-\hat{\mathcal{H}}}{T}\right) \right. \\
&\quad \left. \times e^{(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})(\tau+1/T)} \psi_\alpha(\mathbf{r}_1)\, e^{-(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})(\tau+1/T)} \psi_\beta^\dagger(\mathbf{r}_2) \right]
\end{aligned}
$$

Comparing it with eqn (2.23) for $\tau > 0$, we see that

$$G(\tau<0) = \mp G(\tau+1/T) \tag{2.39}$$

For phonons we get

$$D(\tau<0) = D(\tau+1/T) \tag{2.40}$$

Equation (2.39) is of a great importance. Indeed, let us make a Fourier transformation in τ :

$$G(\tau) = T\sum_n e^{-i\omega_n\tau} G(\omega_n) \tag{2.41}$$

$$G(\omega_n) = \frac{1}{2}\int_{-1/T}^{1/T} e^{i\omega_n\tau} G(\tau) d\tau \tag{2.42}$$

Because time τ is defined within a finite range, the frequency is discrete $\omega_n = n\pi T$. These frequencies are called the Matsubara frequencies. One has

$$
\begin{aligned}
G(\omega_n) &= \frac{1}{2}\int_0^{1/T} e^{i\omega_n\tau} G(\tau) d\tau + \frac{1}{2}\int_{-1/T}^{0} e^{i\omega_n\tau} G(\tau) d\tau \\
&= \frac{1}{2}\int_0^{1/T} e^{i\omega_n\tau} G(\tau) d\tau \mp \frac{1}{2}\int_{-1/T}^{0} e^{i\omega_n\tau} G(\tau+1/T) d\tau \\
&= \frac{1}{2}\left(1 \mp e^{-i\omega_n/T}\right)\int_0^{1/T} e^{i\omega_n\tau} G(\tau) d\tau
\end{aligned} \tag{2.43}
$$

Therefore, the Fourier components

$$G(\omega_n) = \int_0^{1/T} e^{i\omega_n\tau} G(\tau) d\tau \tag{2.44}$$

are nonzero only for

$$\omega_n = \begin{cases} (2n+1)\pi T & \text{for Fermi particles} \\ 2n\pi T & \text{for Bose particles} \end{cases} \tag{2.45}$$

The Green function determines all properties of the system. For example, the particle-number operator is

$$\hat{\mathcal{N}} = \sum_\alpha \int \psi_\alpha^\dagger(\mathbf{r}) \psi_\alpha(\mathbf{r}) d^3r \tag{2.46}$$

Thus, the particle density becomes

$$N = -\sum_{\alpha} G_{\alpha\alpha}(\mathbf{r}, \tau; \mathbf{r}, \tau + 0) \tag{2.47}$$

Using this equation one can find μ as a function of temperature, etc. The momentum density operator is

$$\hat{\mathbf{P}} = -\frac{i}{2} \sum_{\alpha} \left(\psi_{\alpha}^{\dagger} \nabla \psi_{\alpha} - \nabla \psi_{\alpha}^{\dagger} \psi_{\alpha} \right)$$

Therefore, the momentum density is

$$\mathbf{P} = \frac{i\hbar}{2} \left[(\nabla_1 - \nabla_2) \sum_{\alpha} G_{\alpha\alpha} \left(\mathbf{r}_1, \tau; \mathbf{r}_2, \tau + 0 \right) \right]_{\mathbf{r}_1 = \mathbf{r}_2} \tag{2.48}$$

2.2.2 Example: Free particles

For free non-interacting particles the statistical average is taken over the states of every particle independently of other particles. The energy is

$$E_n^{(0)} = \sum_{\mathbf{p},\alpha} n_{\mathbf{p}\alpha} \epsilon_0(\mathbf{p}), \Omega_0 = \sum_{\mathbf{p},\alpha} \Omega_{\mathbf{p}\alpha}^{(0)} \tag{2.49}$$

and the Schrödinger operators are

$$\psi_{\alpha}(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} a_{\mathbf{p}\alpha} e^{i\mathbf{p}\mathbf{r}}, \psi_{\alpha}^{\dagger}(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{p}} a_{\mathbf{p}\alpha}^{\dagger} e^{-i\mathbf{p}\mathbf{r}}$$

Since the Hamiltonian and the particle-number operators are

$$\hat{\mathcal{H}}_0 = \sum_{\mathbf{p},\alpha} \hat{n}_{\mathbf{p}\alpha} \epsilon_0(\mathbf{p}), \hat{\mathcal{N}} = \sum_{\mathbf{p},\alpha} \hat{n}_{\mathbf{p}\alpha} \tag{2.50}$$

the creation and annihilation operators satisfy

$$e^{\left(\hat{\mathcal{H}}_0 - \mu\hat{\mathcal{N}}\right)\tau} a_{\mathbf{p}\alpha} e^{-\left(\hat{\mathcal{H}}_0 - \mu\hat{\mathcal{N}}\right)\tau} = a_{\mathbf{p}\alpha} e^{-(\epsilon_0(\mathbf{p}) - \mu)\tau}$$

$$e^{(\mathcal{H}_0 - \mu\mathfrak{N})\tau} a_{\mathbf{p}\alpha}^{\dagger} e^{-(\mathcal{H}_0 - \mu\mathfrak{N})\tau} = a_{\mathbf{p}\alpha}^{\dagger} e^{(\epsilon_0(\mathbf{p}) - \mu)\tau} \tag{2.51}$$

We have for $\tau > 0$

$$G_{\alpha\beta}^{(0)}(\tau > 0) = \frac{1}{V} \sum^{(d)} \sum_{\mathbf{p}_1, \mathbf{p}_2} e^{i(\mathbf{p}_1\mathbf{r}_1 - \mathbf{p}_2\mathbf{r}_2)} \exp\left(\frac{\Omega_0 + \mu\hat{\mathcal{N}} - \hat{\mathcal{H}}_0}{T} \right) \left[e^{\left(\hat{\mathcal{H}}_0 - \mu\hat{\mathcal{N}}\right)\tau} a_{\mathbf{p}_1\alpha} e^{-\left(\hat{\mathcal{H}}_0 - \mu\hat{\mathcal{N}}\right)\tau} a_{\mathbf{p}_2\beta}^{\dagger} \right]$$

As a results,

$$G_{\alpha\beta}^{(0)}(\tau > 0) = \delta_{\alpha\beta} \frac{1}{V} \sum_{\mathbf{p}} e^{i\mathbf{p}(\mathbf{r}_1 - \mathbf{r}_2) - \tau[\epsilon_0(\mathbf{p}_1) - \mu]} \left\langle a_{\mathbf{p}_1\alpha} a_{\mathbf{p}_2\beta}^{\dagger} \right\rangle_{st}$$

since

$$\left\langle a_{\mathbf{p}_1\alpha} a_{\mathbf{p}_2\beta}^{\dagger} \right\rangle_{st} = \delta_{\alpha\beta} \delta_{\mathbf{p}_1\mathbf{p}_2} \left\langle a_{\mathbf{p}\alpha} a_{\mathbf{p}\alpha}^{\dagger} \right\rangle_{st}$$

The average for Fermi particles is

$$\left\langle a_{\mathbf{p}\alpha} a^{\dagger}_{\mathbf{p}\alpha} \right\rangle_{st} = 1 - n(\mathbf{p}), n(\mathbf{p}) = \left[\exp\left(\frac{\epsilon_0(\mathbf{p}) - \mu}{T} \right) + 1 \right]^{-1}.$$

For Bose particles it is

$$\left\langle a_{\mathbf{p}\alpha} a^{\dagger}_{\mathbf{p}\alpha} \right\rangle_{st} = 1 + n(\mathbf{p}), n(\mathbf{p}) = \left[\exp\left(\frac{\epsilon_0(\mathbf{p}) - \mu}{T} \right) - 1 \right]^{-1}.$$

In these equations, $n(\mathbf{p})$ is the average equilibrium occupation number or the distribution function which is different for Fermi and Bose particles.

We now take the limit $V \to \infty$ using

$$\frac{1}{V} \sum_{\mathbf{p}} \to \int \frac{d^3 p}{(2\pi)^3}$$

and obtain

$$G^{(0)}_{\alpha\beta}(\mathbf{r}, \tau > 0) = \delta_{\alpha\beta} \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{pr} - \tau[\epsilon_0(\mathbf{p}_1) - \mu]} [1 \mp n(\mathbf{p})] \tag{2.52}$$

where $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$. For $\tau < 0$ we write

$$\begin{aligned} G^{(0)}_{\alpha\beta}(\mathbf{r}, \tau < 0) &= \mp G^{(0)}_{\alpha\beta}(\mathbf{r}, \tau + 1/T) \\ &= \mp \delta_{\alpha\beta} \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{pr} - \tau[\epsilon_0(\mathbf{p}_1) - \mu]} n(\mathbf{p}) \end{aligned} \tag{2.53}$$

The phonon Green function can be calculated in a similar way. The Schrödinger operator is

$$\phi(\mathbf{r}) = \frac{1}{\sqrt{V}} \sum_{\mathbf{k}} \sqrt{\frac{\omega_0(\mathbf{k})}{2}} \left(b_{\mathbf{k}} e^{i\mathbf{kr}} + b^{\dagger}_{\mathbf{k}} e^{-i\mathbf{kr}} \right)$$

where $\omega_0(\mathbf{k})$ is the phonon energy. The result is

$$D^{(0)}(\mathbf{r}, \tau) = \frac{1}{2} \int \frac{d^3 k}{(2\pi)^3} \omega_0(\mathbf{k}) \left[(n(\mathbf{k}) + 1) e^{i\mathbf{kr} - \omega_0(\mathbf{k})|\tau|} + n(\mathbf{k}) e^{i\mathbf{kr} + \omega_0(\mathbf{k})|\tau|} \right] \tag{2.54}$$

where

$$n(\mathbf{k}) = \left[e^{\omega_0(\mathbf{k})/T} - 1 \right]^{-1}$$

It is useful to look at the free-particle Green function in the Fourier representation. Consider a Fermi particle. To calculate $G(\mathbf{p}, \omega_n)$ we take $G(\tau > 0)$ and use eqns (2.44) and (2.52). We have

$$\begin{aligned} G^{(0)}_{\alpha\beta}(\mathbf{p}, \omega_n) &= \int d^3 r \int_0^{1/T} d\tau G^{(0)}_{\alpha\beta}(\mathbf{r}, \tau) e^{-i\mathbf{pr} + i\omega_n \tau} \\ &= \delta_{\alpha\beta}[1 - n(\mathbf{p})] \int_0^{1/T} d\tau e^{i\omega_n \tau - [\epsilon_0(\mathbf{p}) - \mu]\tau} \\ &= \frac{\delta_{\alpha\beta}}{i\omega_n - \epsilon_0(\mathbf{p}) + \mu} \left[e^{(2n+1)\pi i - [\epsilon_0(\mathbf{p}) - \mu]/T} - 1 \right] [1 - n(\mathbf{p})] \\ &= -\frac{\delta_{\alpha\beta}}{i\omega_n - \epsilon_0(\mathbf{p}) + \mu}, \qquad \omega_n = (2n+1)\pi T \end{aligned} \tag{2.55}$$

For Bose particles we get

$$G^{(0)}_{\alpha\beta}(\mathbf{p},\omega_n) = -\frac{\delta_{\alpha\beta}}{i\omega_n - \epsilon_0(\mathbf{p}) + \mu}, \omega_n = 2n\pi T \tag{2.56}$$

For phonons we have

$$D^{(0)}(\mathbf{k},\omega_n) = \frac{\omega_0^2(\mathbf{k})}{\omega_0^2(\mathbf{k}) + \omega_n^2}, \omega_n = 2n\pi T \tag{2.57}$$

2.2.3 The Wick's theorem

Consider the statistical average

$$\left\langle T_\tau \left(\tilde{\psi}(1)\tilde{\psi}(2)\ldots\tilde{\psi}^\dagger(3)\tilde{\psi}^\dagger(4)\ldots \right) \right\rangle_{st}$$

taken over the states of non-interacting particles. It is the sum over all the quantum states. The wave functions of each state are normalized in such a way that the normalization constant is proportional to $1/\sqrt{V}$ where V is volume of the system. Therefore, the average has the form

$$\frac{1}{\sqrt{V}}\sum_n \frac{1}{\sqrt{V}}\sum_m \cdots \frac{1}{\sqrt{V}}\sum_{n'} \frac{1}{\sqrt{V}}\sum_{m'} \cdots \left\langle T_\tau \left(a_n(\tau_1)\, a_m(\tau_2)\ldots a^\dagger_{n'}(\tau_3)\, a^\dagger_{m'}(\tau_4)\ldots \right) \right\rangle_{st} \tag{2.58}$$

The matrix elements for the whole combination of a-operators are only nonzero if there are equal numbers of creation and annihilation operators belonging to the same state. For example, nonzero is the term

$$\frac{1}{V}\sum_n \frac{1}{V}\sum_m \cdots \left\langle T_\tau \left(a_n(\tau_1)\, a_m(\tau_2)\ldots a^\dagger_n(\tau_3)\, a^\dagger_m(\tau_4)\ldots \right) \right\rangle_{st} \tag{2.59}$$

where the states with $n \neq m$ are different from all other quantum states. Nonzero are also other terms which can be obtained from eqn (2.59) by transpositions of the indices n, m, etc. If there are four creation and annihilation operators belonging to the same state the nonzero terms have the form

$$\frac{1}{V}\frac{1}{V}\sum_n \frac{1}{V}\sum_m \cdots \left\langle T_\tau \left(a_n a_n a_m a_m \ldots a^\dagger_n a^\dagger_n a^\dagger_m a^\dagger_m \ldots \right) \right\rangle_{st} \tag{2.60}$$

where $n \neq m$. Expressions of the type of eqn (2.59) are different from all other nonzero terms like eqn (2.60) in that the former have as many summation as there are $1/V$ factors. All other terms have at least one extra factor $1/V$. Since the number of quantum states is proportional to the volume of the system, only the terms of the type of eqn (2.59) remain finite in the limit of a large volume of the system $V \to \infty$.

We come to the conclusion that, in the limit of $V \to \infty$, i.e., for a macroscopic system, the statistical average of a combination of ψ-operators is nonzero only if the pairs of creation and annihilation operators $a_n a^\dagger_n$ all belong to different quantum states (???).

This implies that the statistical average can be calculated for each pair of operators $a_n a^\dagger_n$ independently. Indeed, the Hamiltonian and the particle-number operator of non-interacting particles, which both consist of pairs of the $aa^\dagger$-operators, commute with any pair of $aa^\dagger$ if they all belong to different states. Therefore,

$$\sum^{(d)}\left[T_\tau \exp\left(\frac{\Omega+\mu\hat{\mathcal{N}}-\hat{\mathcal{H}}}{T}\right)a_n a_n^\dagger a_m a_m^\dagger\right]$$
$$=\sum^{(d)}\left[T_\tau \exp\left(\frac{\Omega_n+\mu N_n-H_n}{T}\right)a_n a_n^\dagger\right]\sum^{(d)}\left[T_\tau \exp\left(\frac{\Omega_m+\mu N_m-H_m}{T}\right)a_m a_m^\dagger\right] \tag{2.61}$$

since Hamiltonian is an additive function for non-interacting particles. The fact that all τ are in the interval $-1/T < \tau < 1/T$ allows us to put the exponents again before all the ψ-operators.

In other words, the statistical average of a T_τ-product of a large number of ψ-and $\psi^\dagger$-operators is equal to the sum of T_τ-products of all possible $\psi\psi^\dagger$ pair averages. The sign of each term in the sum is + if the corresponding sequence of
the ψ-operators is obtained from the initial sequence by an even transposition, and it is - if the required transposition is odd. This is the contents of the Wick's theorem. For example,

$$\left\langle T_\tau\left(\tilde{\psi}(1)\tilde{\psi}(2)\tilde{\psi}^\dagger(3)\tilde{\psi}^\dagger(4)\right)\right\rangle_{st}=\left\langle T_\tau\left(\tilde{\psi}(1)\tilde{\psi}^\dagger(4)\right)\right\rangle_{st}\left\langle T_\tau\left(\tilde{\psi}(2)\tilde{\psi}^\dagger(3)\right)\right\rangle_{st}$$
$$\mp\left\langle T_\tau\left(\tilde{\psi}(1)\tilde{\psi}^\dagger(3)\right)\right\rangle_{st}\left\langle T_\tau\left(\tilde{\psi}(2)\tilde{\psi}^\dagger(4)\right)\right\rangle_{st}.$$

The Wick's theorem is of crucial importance for the Green function technique. Indeed, the Hamiltonian of the system is usually constructed of various combinations of the particle operators. If the particles interact, the Hamiltonian contains the operators to powers higher than the second power. They come in the exponent, and a general expression for the Green function is thus hopelessly complicated. Here the Wick's theorem saves the situation. We can expand the exponent into the power series in the interaction Hamiltonian and obtain the sum of terms which are various even powers of the particle operators. According to the Wick's theorem, each term can be reduced to a product of several Green functions of non-interacting particles. As a result, we obtain an expression for the full Green function of interacting particles as a series of terms containing powers of the interaction strength and products of non-interacting Green functions (Feynman diagrams). This allows one to find the full Green function to any desirable accuracy in the interaction strength. In many cases, the summation of such a series can be performed analytically which results in the so called Dyson equation for the full Green function. The Dyson equation is the central result for the Green function technique because it provides the algorithm for calculation of the Green function and thus of any physical observable of a complicated system of interacting particles. A more detailed description of the diagram technique for various systems can be found in the book by Abrikosov al. (1965).

16.3.3 2.3 The real-time Green functions

2.3.1 Definitions

The real-time Green function is defined as

$$G_{\alpha\beta}\left(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2\right)=i\sum^{(d)}\left[\exp\left(\frac{\Omega+\mu\hat{\mathcal{N}}\left(t_0\right)-\hat{\mathcal{H}}\left(t_0\right)}{T}\right)T_t\left(\tilde{\psi}_\alpha\left(\mathbf{r}_1,t_1\right)\tilde{\psi}_\beta^\dagger\left(\mathbf{r}_2,t_2\right)\right)\right] \tag{2.62}$$

The Heisenberg operators are determined through the real-time S-matrix which is defined similarly to eqn (2.35)

$$\hat{S}\left(t,t_0\right)=T_t\exp\left[-i\int_{t_0}^{t}(\hat{\mathcal{H}}-\mu\hat{\mathcal{N}})dt'\right] \tag{2.63}$$

The real-time S matrix can be obtained from eqn (2.35) by a formal substitution $\tau\to it$.

The statistical average in eqn (2.62) is taken at a time instant t_0. This definition implies that, at the time t_0, the system was in thermodynamic equilibrium; the field operators evolve since then according to the Schrödinger equation or as determined by the S matrix in eqn (2.63). Note that the Hamiltonian can now be a function of time. The real-time Green function defines the density matrix and thus it is this function which determines the physical quantities for a time-dependent system. For example, the particle density is

$$N(\mathbf{r},t)=\pm i\lim_{t_1\to t_2-0}\sum_{\alpha}G_{\alpha\alpha}\left(\mathbf{r},t_1;\mathbf{r},t_2\right). \tag{2.64}$$

We define also the so called retarded G^R and advanced G^A Green functions:

$$G^R_{\alpha\beta}(1;2)=i\sum^{(d)}\left[\exp\left(\frac{\Omega+\mu\hat{\mathcal{N}}\left(t_0\right)-\hat{\mathcal{H}}\left(t_0\right)}{T}\right)\left(\tilde{\psi}_\alpha(1)\tilde{\psi}^\dagger_\beta(2)\pm\tilde{\psi}^\dagger_\beta(2)\tilde{\psi}_\alpha(1)\right)\right] \tag{2.65}$$

for $t_1>t_2$ and

$$G^R_{\alpha\beta}\left(\mathbf{r}_1,t_1;\mathbf{r}_2,t_2\right)=0$$

for $t_1<t_2$. Similarly,

$$G^A_{\alpha\beta}(1;2)=-i\sum^{(d)}\left[\exp\left(\frac{\Omega+\mu\hat{\mathcal{N}}\left(t_0\right)-\hat{\mathcal{H}}\left(t_0\right)}{T}\right)\left(\tilde{\psi}_\alpha(1)\tilde{\psi}^\dagger_\beta(2)\pm\tilde{\psi}^\dagger_\beta(2)\tilde{\psi}_\alpha(1)\right)\right] \tag{2.66}$$

for $t_1<t_2$ and

$$G^A_{\alpha\beta}(1;2)=0$$

for $t_1>t_2$.

2.3.2 Analytical properties

Consider the Fourier transformed retarded Green function

$$G^R\left(t_1,t_2\right)=\int G^R(\epsilon)e^{-i\epsilon(t_1-t_2)}\frac{d\epsilon}{2\pi} \tag{2.67}$$

We assume for simplicity that the Hamiltonian does not depend on time. The Fourier transformed Green function $G^R(\epsilon)$ is an analytical function of ϵ in the upper half-plane of complex ϵ. Indeed, in this case we can calculate the integral in eqn (2.67) for $t_1-t_2<0$ by shifting the contour of integration into the upper half-plane of ϵ. Since $G^R(\epsilon)$ is analytical, the integral vanishes, and we have $G^R\left(t_1,t_2\right)=0$ for $t_1<t_2$ as it should be according to the definition. Similarly, $G^A(\epsilon)$ is an analytical function of ϵ in the lower half-plane.

We now want to prove that the Matsubara functions are coupled to the retarded and advanced functions through

$$G\left(\omega_n\right)=G^R\left(i\omega_n\right),\omega_n>0 \tag{2.68}$$

$$G\left(\omega_n\right)=G^A\left(i\omega_n\right),\omega_n<0 \tag{2.69}$$

To do this we note that the matrix elements of the Heisenberg operators are

$$\begin{aligned}\tilde{\psi}_{nm}(\mathbf{r},t) &= \psi_{nm}(\mathbf{r})e^{i\omega_{nm}t}\\ \tilde{\psi}^{\dagger}_{nm}(\mathbf{r},t) &= \psi^{\dagger}_{nm}(\mathbf{r})e^{-i\omega_{nm}t}\end{aligned} \tag{2.70}$$

where

$$\omega_{nm} = E_n - E_m - \mu\left(N_n - N_m\right) \tag{2.71}$$

with $N_n = N_m \pm 1$. Therefore, for $t > 0$ according to eqn (2.65)

$$\begin{aligned}G^R(t) =& i\sum_{n,m} e^{(\Omega+\mu N_n - E_n)/T}\left(\psi_{nm}\left(\mathbf{r}_1\right)\psi^{\dagger}_{mn}\left(\mathbf{r}_2\right)\right)e^{i\omega_{nm}t}\\ &\pm i\sum_{n,m} e^{(\Omega+\mu N_m - E_m)/T}\left(\psi^{\dagger}_{mn}\left(\mathbf{r}_2\right)\psi_{nm}\left(\mathbf{r}_1\right)\right)e^{-i\omega_{mn}t}\\ =& i\sum_{n,m} e^{(\Omega+\mu N_n - E_n)/T}\left(\psi_{nm}\left(\mathbf{r}_1\right)\psi^{\dagger}_{mn}\left(\mathbf{r}_2\right)\right)e^{-i\omega_{mn}t}\\ &\times\left[1 \pm e^{(E_n - E_m - \mu N_n + \mu N_m)/T}\right]\end{aligned} \tag{2.72}$$

and $G^R(t) = 0$ for $t < 0$. For the Fourier transforms we have

$$\begin{aligned}G^R(\omega) &= \int_{-\infty}^{\infty} G^R(t)e^{i\omega t}dt\\ &= \sum_{n,m} e^{(\Omega+\mu N_n - E_n)/T}\left(\psi_{nm}\left(\mathbf{r}_1\right)\psi^{\dagger}_{mn}\left(\mathbf{r}_2\right)\right)\\ &\quad\times\left(1 \pm e^{-\omega_{mn}/T}\right)\left[i\pi\delta\left(\omega - \omega_{mn}\right) - \frac{1}{\omega - \omega_{mn}}\right]\end{aligned}$$

Therefore,

$$G^R(\omega) = \int_{-\infty}^{\infty}\frac{\rho(x)dx}{x - \omega - i\delta} \tag{2.73}$$

where $\delta \to 0$ and

$$\rho(\omega) = \sum_{n,m} e^{(\Omega+\mu N_n - E_n)/T}\left(\psi_{nm}\left(\mathbf{r}_1\right)\psi^{\dagger}_{mn}\left(\mathbf{r}_2\right)\right)\left(1 \pm e^{-\omega_{mn}/T}\right)\delta\left(\omega - \omega_{mn}\right)$$

During the derivation we use the identity

$$\int_0^{\infty} e^{i\alpha x}dx = \left[\int_0^{\infty} e^{(i\alpha-\delta)x}dx\right]_{\delta\to 0} = \left[\frac{i}{\alpha + i\delta}\right]_{\delta\to 0} = \pi\delta(\alpha) + \frac{i}{\alpha}$$

In the same way we get

$$G^A(\omega) = \int_{-\infty}^{\infty}\frac{\rho(x)dx}{x - \omega + i\delta} \tag{2.74}$$

For the Matsubara Green function we have for $\tau > 0$

$$G(\tau) = \sum_{n,m} e^{(\Omega+\mu N_n - E_n)/T}\left(\psi_{nm}\left(\mathbf{r}_1\right)\psi^{\dagger}_{mn}\left(\mathbf{r}_2\right)\right)e^{\omega_{nm}\tau} \tag{2.75}$$

This gives

$$
\begin{aligned}
G\left(\omega_k\right) &= \int_0^{1/T} G(\tau) e^{i\omega_n \tau} d\tau \\
&= \sum_{n,m} e^{(\Omega+\mu N_n - E_n)/T} \left(\psi_{nm}\left(\mathbf{r}_1\right) \psi_{mn}^{\dagger}\left(\mathbf{r}_2\right)\right) \frac{1 \pm e^{-\omega_{mn}/T}}{\omega_{mn} - i\omega_k} \\
&= \int_{-\infty}^{\infty} \frac{\rho(x) dx}{x - i\omega_k}
\end{aligned}
\tag{2.76}
$$

where $\omega_k = (2k+1)\pi T$ for Fermi particles and $\omega_k = 2k\pi T$ for Bose particles. Comparing eqns (2.73) and (2.74) with (2.76) we arrive at eqn (2.69).

16.4 THE BCS MODEL

We apply the Green function formalism to the Bardeen-Cooper-Schrieffer theory of superconductivity and derive the Gor'kov equations which make the basis for our analysis. We couple the Green functions to such physical quantities as the order parameter of the superconductor, the electric current, the electron density, and the thermodynamic potential. We derive the Bogoliubov-de Gennes equations from the Gor'kov equations. Several applications of the Gor'kov theory are considered.

16.4.1 3.1 BCS theory and Gor'kov equations

The BCS Hamiltonian contains the attractive interaction between electrons. This attraction is needed for electrons to form Cooper pairs which then condense into a superconducting state. We consider a spin-independent interaction. This leads to a zero-spin (spin-singlet) superconducting state which is what one usually has in real superconductors. The Fermi statistics of electrons requires a spin-singlet state to have an even parity with respect to the transposition of the particle coordinates or (which is the same) with respect to inversion of the relative momentum of particles. It means that the superconducting state should be of either s-wave or d-wave (or of other higher-order even) symmetry in the relative momentum of particles. We start our consideration with the s-wave superconducting state and will consider some specifics of a d-wave state later.

There can be various physical mechanisms of attraction between electrons. In the phonon model, for example, the attraction is mediated by an exchange of phonons. The pairing interaction usually works in a restricted energy range and vanishes for energy transfer larger than some cutoff value Ω_0. In case of phonons, the interaction is effective only when the exchange of energy is less than the characteristic Debye frequency Ω_D. If the interaction is relatively weak the characteristic energies of particles participating in the superconducting phenomena, are much smaller than both the Fermi energy and the cutoff frequency Ω_0. In this case, a point-like interaction between the electrons in the form

$$U\left(\mathbf{r}_1 - \mathbf{r}_2\right) = \frac{g}{2} \delta\left(\mathbf{r}_1 - \mathbf{r}_2\right)$$

is a reasonable approximation. An attraction corresponds to $g < 0$. The limit of small interaction corresponds to $|g|\nu \ll 1$ where ν is the density of states at the Fermi level in the normal state. It is called a weak-coupling approximation.

Let us derive the equations for the Green functions for the BCS model following Gor'kov (1958) and Abrikosov al. (1965). The BCS Hamiltonian is

$$\hat{\mathcal{H}}_{\mathrm{BCS}} = \int \left[-\psi_\alpha^\dagger \frac{\nabla^2}{2m} \psi_\alpha + \frac{g}{2} \psi_\beta^\dagger \psi_\alpha^\dagger \psi_\alpha \psi_\beta \right] d^3 r \tag{3.1}$$

The Hamiltonian and the particle number operator

$$\hat{\mathcal{N}} = \int \psi_\alpha^\dagger \psi_\alpha d^3 r$$

can be written through the Heisenberg operators which are defined in eqn (2.29). Since $\hat{\mathcal{N}}$ commutes with the Hamiltonian, the operators $\hat{\mathcal{H}}$ and $\hat{\mathcal{N}}$ in this representation will have the same form as in eqns (3.1), (2.46).

Using eqn (2.31) we can calculate the time derivative of the Heisenberg operators $\tilde{\psi}$. Commuting $\tilde{\psi}$ with the Hamiltonian and the particle-number operator taken at the same time we get

$$\begin{aligned} \frac{\partial \tilde{\psi}_\alpha(x)}{\partial \tau} &= \left(\frac{\nabla^2}{2m} + \mu \right) \tilde{\psi}_\alpha(x) - g \tilde{\psi}_\gamma^\dagger(x) \tilde{\psi}_\gamma(x) \tilde{\psi}_\alpha(x) \\ \frac{\partial \tilde{\psi}_\alpha^\dagger(x)}{\partial \tau} &= - \left(\frac{\nabla^2}{2m} + \mu \right) \tilde{\psi}_\alpha^\dagger(x) + g \tilde{\psi}_\alpha^\dagger(x) \tilde{\psi}_\gamma^\dagger(x) \tilde{\psi}_\gamma(x) \end{aligned} \tag{3.2}$$

Here $x = (\tau, \mathbf{r})$. The time derivative of the Green function is

$$\begin{aligned} \frac{\partial G_{\alpha\beta}(x_1, x_2)}{\partial \tau_1} =& \frac{\partial}{\partial \tau_1} (\Delta G) + \left\langle T_\tau \frac{\partial \tilde{\psi}_\alpha(x_1)}{\partial \tau_1} \tilde{\psi}_\beta^\dagger(x_2) \right\rangle_{st} \\ =& \frac{\partial}{\partial \tau_1} \left[\delta_{\alpha\beta} \delta(\mathbf{r}_1 - \mathbf{r}_2) \Theta(\tau_1 - \tau_2) \right] \\ &+ \left\langle T_\tau \left[\left(\frac{\nabla^2}{2m} + \mu \right) \tilde{\psi}_\alpha(x_1) \tilde{\psi}_\beta^\dagger(x_2) \right.\right. \\ &\left.\left. - g \tilde{\psi}_\gamma^\dagger(x_1) \tilde{\psi}_\gamma(x_1) \tilde{\psi}_\alpha(x_1) \tilde{\psi}_\beta^\dagger(x_2) \right] \right\rangle_{st} \\ =& \delta_{\alpha\beta} \delta(\mathbf{r}_1 - \mathbf{r}_2) \delta(\tau_1 - \tau_2) + \left(\frac{\nabla^2}{2m} + \mu \right) G_{\alpha\beta}(x_1, x_2) \\ &- g \left\langle T_\tau \tilde{\psi}_\gamma^\dagger(x_1) \tilde{\psi}_\gamma(x_1) \tilde{\psi}_\alpha(x_1) \tilde{\psi}_\beta^\dagger(x_2) \right\rangle_{st} \end{aligned}$$

(ΔG) denotes the Green function jump at $\tau_1 = \tau_2$ according to eqn (2.28). The last line can be simplified using the Wick's theorem:

$$\begin{aligned} &\left\langle T_\tau \tilde{\psi}_\gamma^\dagger(x_1) \tilde{\psi}_\gamma(x_1) \tilde{\psi}_\alpha(x_1) \tilde{\psi}_\beta^\dagger(x_2) \right\rangle_{st} \\ =& - \left\langle T_\tau \tilde{\psi}_\gamma(x_1) \tilde{\psi}_\gamma^\dagger(x_1) \right\rangle \left\langle T_\tau \tilde{\psi}_\alpha(x_1) \tilde{\psi}_\beta^\dagger(x_2) \right\rangle_{st} \\ &+ \left\langle T_\tau \tilde{\psi}_\alpha(x_1) \tilde{\psi}_\gamma^\dagger(x_1) \right\rangle \left\langle T_\tau \tilde{\psi}_\gamma(x_1) \tilde{\psi}_\beta^\dagger(x_2) \right\rangle_{st} \\ &- \left\langle T_\tau \tilde{\psi}_\alpha(x_1) \tilde{\psi}_\gamma(x_1) \right\rangle \left\langle T_\tau \tilde{\psi}_\gamma^\dagger(x_1) \tilde{\psi}_\beta^\dagger(x_2) \right\rangle_{st} \end{aligned} \tag{3.3}$$

Here we need some discussion: First, we note that the decomposition of the four-operator average into a product of two averages of pairs of the operators is strictly true for non-interacting particles according to the Wick's theorem. Here we perform the decomposition for real, and thus interacting, particles. There are two major types of interaction. Interactions of one type are such that do not result in the superconducting behavior. We neglect them. This is a model

approximation; the quality of the model should be considered separately for each particular case. For example, it is not correct, of course, for a strongly correlated electron system. However, for usual superconductors, where electrons form a Fermi liquid with weak correlations, the BCS model proved to be very realistic. The interaction of the second kind is the one which results in the superconducting pairing. What we should have done with this interaction is to expand the average of the four particle operators into the power series in the interaction strength g and then to average each term using the Wick's theorem. We would obtain the series of diagrams containing the non-superconducting Green functions. Performing summation of this series we would then obtain the full Green function as it appears in eqn (3.3).

The second major feature of the model is as follows. The averages in the third line in the r.h.s of eqn (3.3) do not disappear in the limit of a macroscopic volume $V \to \infty$ as it was assumed earlier during the derivation of the Wick's theorem. The reason is that now the average

$$\left\langle T_\tau \tilde{\psi}_\alpha (x_1) \tilde{\psi}_\gamma (x_1) \right\rangle_{st} \tag{3.4}$$

contains a macroscopic number of particles because of the Cooper pairing effect: due to a small attraction between electrons they form pairs which then condense into a ground state. This is the basic assumption of the BCS theory. The average of eqn (3.4) has the form of a wave function of the condensed Cooper pairs, i.e., of the superconducting electrons.

The first and the second terms in the r.h.s. of eqn (3.3) have the form

$$-\Sigma_{\gamma\gamma} (x_1) G_{\alpha\beta} (x_1, x_2) + \Sigma_{\alpha\gamma} (x_1) G_{\gamma\beta} (x_1, x_2)$$

where

$$\Sigma_{\alpha\beta}(x) = \left\langle T_\tau \tilde{\psi}_\alpha (x_1) \tilde{\psi}^\dagger_\beta (x_1) \right\rangle_{st} \equiv G_{\alpha\beta}(x, x)$$

is called the self-energy. The self energy leads to a renormalization of the chemical potential μ. In addition, it introduces a small imaginary part proportional to the small g. In the BCS model, this relaxation term is ignored, and the only effect of the interaction is assumed to be the pairing leading to a nonzero average of the type of $\langle \psi\psi \rangle$. In the phonon model to be discussed in Section 8.2, the self energy is of the order of T^3/Ω_D^2; it is much smaller than the characteristic superconducting energy in the so called weak coupling limit to which we restrict our consideration. However, the relaxation part can be important for nonstationary processes; we shall incorporate it where appropriate while considering nonstationary phenomena throughout the book.

Starting with the BCS model, we omit the self energy from the expression for the time derivative of the Green function. Finally, we obtain

$$\begin{aligned}
&\left(\frac{\partial}{\partial \tau_1} - \frac{\nabla_1^2}{2m} - \mu \right) G_{\alpha\beta} (x_1, x_2) \\
&- g \left\langle T_\tau \tilde{\psi}_\alpha (x_1) \tilde{\psi}_\gamma (x_1) \right\rangle_{st} \left\langle T_\tau \tilde{\psi}^\dagger_\gamma (x_1) \tilde{\psi}^\dagger_\beta (x_2) \right\rangle_{st} \\
=&\delta_{\alpha\beta} \delta (x_1 - x_2)
\end{aligned}$$

where $\delta (x_1 - x_2) = \delta (\tau_1 - \tau_2) \delta (\mathbf{r}_1 - \mathbf{r}_2)$. We define the new Green functions

$$F^\dagger_{\alpha\beta} (x_1, x_2) = \left\langle T_\tau \tilde{\psi}^\dagger_\alpha (x_1) \tilde{\psi}^\dagger_\beta (x_2) \right\rangle_{st} \tag{3.5}$$

$$F_{\alpha\beta} (x_1, x_2) = \left\langle T_\tau \tilde{\psi}_\alpha (x_1) \tilde{\psi}_\beta (x_2) \right\rangle_{st} \tag{3.6}$$

They are called the "anomalous" or Gor'kov functions. We also define

$$\Delta_{\alpha\beta}(x) = |g| F_{\alpha\beta}(x, x) \tag{3.7}$$

Here we take into account that $g < 0$. The equation for the Green function becomes

$$\left(\frac{\partial}{\partial \tau_1} - \frac{\nabla_1^2}{2m} - \mu\right) G_{\alpha\beta}\left(x_1, x_2\right) + \Delta_{\alpha\gamma}\left(x_1\right) F_{\gamma\beta}^{\dagger}\left(x_1, x_2\right) = \delta_{\alpha\beta}\delta\left(x_1 - x_2\right). \tag{3.8}$$

The function $F_{\alpha\beta}\left(x_1, x_2\right)$ is odd in transposition of the particle coordinates and spin indices because of the Fermi statistics of electrons. For a pairing interaction which has an even parity in the orbital space such as an s-wave (or d-wave) interaction, the Cooper pairing can only occur between the electrons with opposite spin projections into a singlet state and the pair wave function is antisymmetric in spin indices:

$$\Delta_{\alpha\beta}(x) = -\Delta_{\beta\alpha}(x)$$

We can write

$$\Delta_{\alpha\beta} = i\sigma_{\alpha\beta}^{(2)}\Delta(x); \Delta_{\alpha\beta}^{\dagger} = -i\sigma_{\alpha\beta}^{(2)}\Delta^*(x)$$

where

$$\sigma_{\alpha\beta}^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

is the Pauli spin matrix. We also denote

$$F_{\alpha\beta}^{\dagger}\left(x_1, x_2\right) = -i\sigma_{\alpha\beta}^{(2)} F^{\dagger}\left(x_1, x_2\right); F_{\alpha\beta}\left(x_1, x_2\right) = i\sigma_{\alpha\beta}^{(2)} F\left(x_1, x_2\right).$$

Since our interaction does not depend on spin, the Green function G is proportional to the unit matrix in the spin indices:

$$G_{\alpha\beta}\left(x_1, x_2\right) = \delta_{\alpha\beta} G\left(x_1, x_2\right)$$

As a result, we obtain from eqn (3.8)

$$\left(\frac{\partial}{\partial \tau_1} - \frac{\nabla_1^2}{2m} - \mu\right) G\left(x_1, x_2\right) + \Delta\left(x_1\right) F^{\dagger}\left(x_1, x_2\right) = \delta\left(x_1 - x_2\right). \tag{3.9}$$

This equation contains an unknown function $F^{\dagger}$ and the quantity $\Delta(x)$ which is expressed through another unknown function F. To find these functions we need more equations. Equation for $F^{\dagger}$ can be obtained from the second eqn (3.2) in a way similar to that used to derive eqn (3.9). We have

$$\left(\frac{\partial}{\partial \tau_1} + \frac{\nabla_1^2}{2m} + \mu\right) F^{\dagger}\left(x_1, x_2\right) + \Delta^*\left(x_1\right) G\left(x_1, x_2\right) = 0. \tag{3.10}$$

Equation for F contains the function

$$\bar{G}_{\alpha\beta}\left(x_1, x_2\right) = -\left\langle T_\tau \tilde{\psi}_\alpha^{\dagger}\left(x_1\right) \psi_\beta\left(x_2\right)\right\rangle = \delta_{\alpha\beta}\bar{G}\left(x_1, x_2\right) \tag{3.11}$$

It is easy to see that

$$\bar{G}\left(x_1, x_2\right) = G\left(x_2, x_1\right). \tag{3.12}$$

We have for F and $\bar{G}$:

$$-\left(\frac{\partial}{\partial\tau_1}+\frac{\nabla_1^2}{2m}+\mu\right)\bar{G}\left(x_1,x_2\right)+\Delta^*\left(x_1\right)F\left(x_1,x_2\right)=\delta\left(x_1-x_2\right) \tag{3.13}$$

and

$$\left(-\frac{\partial}{\partial\tau_1}+\frac{\nabla_1^2}{2m}+\mu\right)F\left(x_1,x_2\right)+\Delta\left(x_1\right)\bar{G}\left(x_1,x_2\right)=0. \tag{3.14}$$

Equations (3.9), (3.10), (3.13) and (3.14) form the system of generalized Dyson equations for the BCS model. These equations are known as the Gor'kov equations (Gor'kov 1958). The quantity Δ is called the order parameter. We have already encountered it in Chapter 1. It has exactly the same meaning as in the Landau theory of second-order phase transitions: it is zero in the nonsuperconducting (normal) state and nonzero in a superconducting state. It is proportional to the wave function of superconducting electrons and thus is a complex function. According to the definition of eqn (3.7), we have

$$\Delta(x)=|g|F(x,x);\Delta^*(x)=|g|F^\dagger(x,x) \tag{3.15}$$

This is a self-consistency equation: it couples the order parameter with the function F which has to be found from equations containing Δ.

The function $G\left(x_1,x_2\right)$ describes a particle moving from point x_1 to point x_2 while $\bar{G}\left(x_1,x_2\right)$ describes a particle moving from point x_2 to point x_1. The latter particle is equivalent to a hole. According to eqn (3.12), the fact that the Gor'kov equations contain $G\left(x_1,x_2\right)$ and $\bar{G}\left(x_1,x_2\right)$ on equal footing is a manifestation of the particle-hole symmetry of the BCS theory.

We can introduce the matrix Green function

$$\check{G}\left(x_1,x_2\right)=\begin{pmatrix} G\left(x_1,x_2\right) & F\left(x_1,x_2\right) \\ -F^\dagger\left(x_1,x_2\right) & \bar{G}\left(x_1,x_2\right)\end{pmatrix} \tag{3.16}$$

and the matrix operator

$$\check{G}^{-1}=\check{\tau}_3\frac{\partial}{\partial\tau}+\check{\mathcal{H}}$$

where

$$\check{\mathcal{H}}=\begin{pmatrix} -\frac{\nabla^2}{2m}-\mu & -\Delta \\ \Delta^* & -\frac{\nabla^2}{2m}-\mu\end{pmatrix},\quad \check{\tau}_3=\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}. \tag{3.17}$$

These are called the matrices in the Nambu space. The Hamiltonian $\check{\mathcal{H}}$ is now measured from the chemical potential μ. It is equivalent to $\mathcal{H}-\mu\mathcal{N}$ of the previous Chapter. The four equations for the four Green functions can be written in the compact matrix form

$$\check{G}^{-1}\left(x_1\right)\check{G}\left(x_1,x_2\right)=\check{1}\delta\left(x_1-x_2\right). \tag{3.18}$$

We can apply the operator to act on the second coordinate x_2. To find the corresponding equations, we multiply the derivatives of eqn (3.2) taken from the functions of the coordinate x_2 by $\psi\left(x_1\right)$ or $\psi^\dagger\left(x_1\right)$ from the left. After decoupling of the averages using the Wick's theorem we find for the BCS model

$$\check{G}\left(x_1,x_2\right)\check{G}^{-1}\left(x_2\right)=\check{1}\delta\left(x_1-x_2\right) \tag{3.19}$$

where the conjugated operator is

$$\overline{\check{G}}^{-1} = -\check{\tau}_3 \frac{\partial}{\partial \tau} + \overline{\mathcal{H}} \tag{3.20}$$

Here $\overline{\mathscr{H}}$ coincides with $\check{\mathcal{H}}$.

3.1.1 Magnetic field

In the presence of a magnetic field, the momentum operator should be replaced with

$$-i\nabla - \frac{e}{c}\mathbf{A}, \text{ or } i\nabla - \frac{e}{c}\mathbf{A} \tag{3.21}$$

when it acts on ψ or $\psi^\dagger$, respectively. For a stationary electromagnetic field, the scalar potential φ can be incorporated into the chemical potential μ. Therefore, it does not appear in stationary problems. The operator $\check{G}^{-1}$ becomes

$$\check{G}^{-1} = \check{\tau}_3 \frac{\partial}{\partial \tau} + \check{\mathcal{H}} \tag{3.22}$$

where now

$$\check{\mathcal{H}} = \begin{pmatrix} -\frac{1}{2m}\left(\nabla - \frac{ie}{c}\mathbf{A}\right)^2 - \mu c & -\Delta \\ \Delta^* & -\frac{1}{2m}\left(\nabla + \frac{ie}{c}\mathbf{A}\right)^2 - \mu \end{pmatrix}. \tag{3.23}$$

The conjugated operator is

$$\overline{\check{G}}^{-1} = -\check{\tau}_3 \frac{\partial}{\partial \tau} + \overline{\mathscr{H}}$$

where

$$\overline{\mathscr{H}} = \begin{pmatrix} -\frac{1}{2m}\left(\nabla + \frac{ie}{c}\mathbf{A}\right)^2 - \mu c & -\Delta \\ \Delta^* & -\frac{1}{2m}\left(\nabla - \frac{ie}{c}\mathbf{A}\right)^2 - \mu \end{pmatrix}. \tag{3.24}$$

The electron-number density is

$$N = -2G(\mathbf{r}, \tau; \mathbf{r}, \tau + 0) = -2\bar{G}(\mathbf{r}, \tau + 0; \mathbf{r}, \tau) \tag{3.25}$$

The electric current can be expressed through the Green functions using the momentum density of eqn (2.48):

$$\begin{aligned} \mathbf{j} &= -\frac{e}{m}\left[\left[\left(-i\nabla_1 - \frac{e}{c}\mathbf{A}\right) + \left(i\nabla_2 - \frac{e}{c}\mathbf{A}\right)\right] G\left(\mathbf{r}_1, \tau; \mathbf{r}_2, \tau + 0\right)\right]_{\mathbf{r}_1 = \mathbf{r}_2} \\ &= \frac{ie}{m}\left[\left(\nabla_1 - \nabla_2\right) G\left(\mathbf{r}_1, \tau; \mathbf{r}_2, \tau + 0\right)\right]_{\mathbf{r}_1 = \mathbf{r}_2} - \frac{Ne^2}{mc}\mathbf{A} \end{aligned} \tag{3.26}$$

or

$$\begin{aligned} \mathbf{j} &= -\frac{e}{m}\left[\left[\left(i\nabla_1 - \frac{e}{c}\mathbf{A}\right) + \left(-i\nabla_2 - \frac{e}{c}\mathbf{A}\right)\right] \bar{G}\left(\mathbf{r}_1, \tau + 0; \mathbf{r}_2, \tau\right)\right]_{\mathbf{r}_1 = \mathbf{r}_2} \\ &= -\frac{ie}{m}\left[\left(\nabla_1 - \nabla_2\right) \bar{G}\left(\mathbf{r}_1, \tau; \mathbf{r}_2, \tau + 0\right)\right]_{\mathbf{r}_1 = \mathbf{r}_2} - \frac{Ne^2}{mc}\mathbf{A} \end{aligned} \tag{3.27}$$

The factor 1/2 is canceled here due to summation over spin indices implied in the definition, eqn (2.48).

The fact that the spatial derivatives enter the inverse operators for the Green functions and the expressions for the electric current in combinations $-i\nabla - (e/c)\mathbf{A}$ and $i\nabla - (e/c)\mathbf{A}$ is a manifestation of the gauge invariance of the theory of superconductivity. Indeed, a change in the phase $\chi/2$ of the single-electron wave function ψ by an arbitrary function $f/2$ results in the f-change in the order-parameter phase; it can be compensated by the variation of the vector potential by $(c/2e)\nabla f$. Since the magnetic field $H = \operatorname{curl}\mathbf{A}$ does not change, the superconducting state is invariant under the transformation of eqn (1.7) on page 16. Therefore, the vector potential should always appear in the combination with the order-parameter phase $\mathbf{A} - (c/2e)\nabla\chi$ to preserve the gauge invariance. This gives a convenient test for checking whether the calculations which we are doing are correct.

3.1.2 Frequency and momentum representation

Equations for the Green functions are more convenient in the frequency representation. We denote $\tau = \tau_1 - \tau_2$ and write

$$\check{G}\left(\mathbf{r}_1, \mathbf{r}_2; \tau\right) = T \sum_n e^{-i\omega_n \tau} \check{G}_{\omega_n}\left(\mathbf{r}_1, \mathbf{r}_2\right)$$

$$\check{G}_{\omega_n}\left(\mathbf{r}_1, \mathbf{r}_2\right) = \int_0^{1/T} e^{i\omega_n \tau} \check{G}\left(\mathbf{r}_1, \mathbf{r}_2; \tau\right) d\tau$$

where $\omega_n = (2n+1)\pi T$. The operators are

$$\check{G}^{-1}_{\omega_n}(\mathbf{r}) = -i\omega_n \check{\tau}_3 + \check{\mathcal{H}} \tag{3.28}$$

$$\bar{G}^{-1}_{\omega_n}(\mathbf{r}) = i\omega_n \check{\tau}_3 + \check{\mathcal{H}} \tag{3.29}$$

The matrix Gor'kov equation in the frequency representation is

$$\check{G}^{-1}_{\omega_n}\left(\mathbf{r}_1\right) \check{G}_{\omega_n}\left(\mathbf{r}_1, \mathbf{r}_2\right) = \check{1}\delta\left(\mathbf{r}_1 - \mathbf{r}_2\right), \tag{3.30}$$

$$\check{G}_{\omega_n}\left(\mathbf{r}_1, \mathbf{r}_2\right) \check{G}^{-1}_{\omega_n}\left(\mathbf{r}_2\right) = \check{1}\delta\left(\mathbf{r}_1 - \mathbf{r}_2\right) \tag{3.31}$$

The self-consistency equation for the order-parameter becomes

$$\frac{\Delta(\mathbf{r})}{|g|} = T \sum_n F_{\omega_n}(\mathbf{r}, \mathbf{r}), \frac{\Delta^*(\mathbf{r})}{|g|} = T \sum_n F^{\dagger}_{\omega_n}(\mathbf{r}, \mathbf{r}) \tag{3.32}$$

since we have to take $\tau = \tau_1 - \tau_2 = 0$. The electron density is obtained a similar way

$$\begin{aligned} N &= -2T \lim_{\tau \to -0} \sum_n G_{\omega_n}(\mathbf{r}, \mathbf{r}) e^{-i\omega_n \tau} \\ &= -2T \lim_{\tau \to +0} \sum_n \bar{G}_{\omega_n}(\mathbf{r}, \mathbf{r}) e^{-i\omega_n \tau} \end{aligned} \tag{3.33}$$

The limits in eqn (3.33) need to be taken because $G\left(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2\right)$ is discontinuous at $(\mathbf{r}_1, t_1) = (\mathbf{r}_2, t_2)$ and the sums for $\tau = 0$ do not converge.

The electric current becomes

$$\mathbf{j} = \frac{ie}{m}\left[\left(\nabla_1 - \nabla_2\right) T \sum_n G_{\omega_n}\left(\mathbf{r}_1, \mathbf{r}_2\right)\right]_{\mathbf{r}_1 = \mathbf{r}_2} - \frac{Ne^2}{mc}\mathbf{A} \tag{3.34}$$

or

$$\mathbf{j} = -\frac{ie}{m}\left[(\nabla_1 - \nabla_2)\, T\sum_n \bar{G}_{\omega_n}(\mathbf{r}_1, \mathbf{r}_2)\right]_{\mathbf{r}_1=\mathbf{r}_2} - \frac{Ne^2}{mc}\mathbf{A} \tag{3.35}$$

The limiting procedure is not necessary for eqn (3.34) since the discontinuous part vanishes as a result of action of the differential operators.

Fig. 3.1. Contours of integration over complex frequency. Contours \$C_{+}^{(i)}\$ and $C_-^{(i)}$ encircle the poles of $\tanh(\epsilon/2T)$ in the upper (lower) half plane, respectively. Contours $C_\pm^{(r)}$ are obtained by shifting the respective contours $C_\pm^{(i)}$ onto the real frequency axis.

The order-parameter equation and the expressions for the current and electron density can be written in terms of the real-time retarded and advanced Green functions, as well. We start with the order-parameter equation. The sum over frequencies $\omega_n = (2n+1)\pi T$ can be presented as

$$\begin{aligned} T\sum_n F_{\omega_n} &= T\sum_{n>0} F_{\omega_n} + T\sum_{n<0} F_{\omega_n} \\ &= \int_{C_+^{(i)}} \tanh\left(\frac{\epsilon}{2T}\right) F_\epsilon^R \frac{d\epsilon}{4\pi i} + \int_{C_-^{(i)}} \tanh\left(\frac{\epsilon}{2T}\right) F_\epsilon^A \frac{d\epsilon}{4\pi i} \end{aligned}$$

The function $\tanh(\epsilon/2T)$ has the poles at $\epsilon = (2n+1)\pi iT$, i.e., at $\epsilon = i\omega_n$. The contour $C_+^{(i)}$ encircles all the poles of $\tanh(\epsilon/2T)$ in the upper complex halfplane of ϵ while the contour $C_-^{(i)}$ encircles all the poles of $\tanh(\epsilon/2T)$ in the lower complex half-plane of ϵ (see Fig. 3.1). In the upper half plane, the function F_ϵ^R is an analytical function and coincides with F_{ω_n} for $\epsilon = i\omega_n$ when $\omega_n > 0$. In the same way, the function F_ϵ^A is an analytical function in the lower half-plane and coincides with F_{ω_n} for $\epsilon = i\omega_n$ when $\omega_n < 0$. We can now shift the contours in such a way that $C_+^{(i)}$ transforms into $C_+^{(r)}$ and goes along the real axis of ϵ from $-\infty$ to $+\infty$ and $C_-^{(i)}$ transforms into $C_-^{(r)}$ which goes along the real axis of ϵ from $+\infty$ to $-\infty$. Finally, we have

$$T\sum_n F_{\omega_n} = \int_{-\infty}^{+\infty} \tanh\left(\frac{\epsilon}{2T}\right)\left(F_\epsilon^R - F_\epsilon^A\right)\frac{d\epsilon}{4\pi i} \tag{3.36}$$

The order-parameter equation becomes

$$\frac{\Delta}{|g|} = \int_{-\infty}^{+\infty} \tanh\left(\frac{\epsilon}{2T}\right)\left(F_\epsilon^R - F_\epsilon^A\right)\frac{d\epsilon}{4\pi i} \tag{3.37}$$

The electric current is

$$\mathbf{j} = \frac{ie}{m}\left[(\nabla_1 - \nabla_2)\int_{-\infty}^{+\infty} \tanh\left(\frac{\epsilon}{2T}\right)\left(G_\epsilon^R(\mathbf{r}_1,\mathbf{r}_2) - G_\epsilon^A(\mathbf{r}_1,\mathbf{r}_2)\right)\right]_{\mathbf{r}_1=\mathbf{r}_2}\frac{d\epsilon}{4\pi i} - \frac{Ne^2}{mc}\mathbf{A} \tag{3.38}$$

or

$$\mathbf{j} = -\frac{ie}{m}\left[(\nabla_1 - \nabla_2)\int_{-\infty}^{+\infty} \tanh\left(\frac{\epsilon}{2T}\right)\left(\bar{G}_\epsilon^R(\mathbf{r}_1,\mathbf{r}_2) - \bar{G}_\epsilon^A(\mathbf{r}_1,\mathbf{r}_2)\right)\right]_{\mathbf{r}_1=\mathbf{r}_2}\frac{d\epsilon}{4\pi i} - \frac{Ne^2}{mc}\mathbf{A}$$

The transformation to the real frequencies for the electron density can not be performed straightforwardly because of the discontinuity of the corresponding Green functions. The discontinuity results in a divergence of the integrals over the remote contours. Let us first separate the contribution to the density from the normal state. The remaining part of the Green functions δG then decreases quickly enough for large ϵ, and the integrals over remote contours vanish. As a result we obtain

$$N = N_0 - 2\int_{-\infty}^{+\infty} \tanh\left(\frac{\epsilon}{2T}\right)\delta\left[G_\epsilon^R(\mathbf{r},\mathbf{r}) - G_\epsilon^A(\mathbf{r},\mathbf{r})\right]\frac{d\epsilon}{4\pi i} \tag{3.39}$$

or

$$N = N_0 - 2\int_{-\infty}^{+\infty} \tanh\left(\frac{\epsilon}{2T}\right)\delta\left[\bar{G}_\epsilon^R(\mathbf{r},\mathbf{r}) - \bar{G}_\epsilon^A(\mathbf{r},\mathbf{r})\right]\frac{d\epsilon}{4\pi i}$$

where N_0 is the electron density in the normal state.
The retarded and advanced Green functions satisfy the same equations as the Matsubara functions with the replacement $\omega_n = -i\epsilon$:

$$\begin{aligned}\check{G}_\epsilon^{-1}(\mathbf{r}_1)\,\check{G}_\epsilon^{R(A)}(\mathbf{r}_1,\mathbf{r}_2) &= \hat{1}\delta(\mathbf{r}_1 - \mathbf{r}_2)\\ \check{G}_\epsilon^{R(A)}(\mathbf{r}_1,\mathbf{r}_2)\,\check{G}_\epsilon^{-1}(\mathbf{r}_2) &= \hat{1}\delta(\mathbf{r}_1 - \mathbf{r}_2)\end{aligned} \tag{3.40}$$

where the inverse operators are

$$\check{G}_\epsilon^{-1}(\mathbf{r}) = -\epsilon\check{\tau}_3 + \check{\mathcal{H}}; \bar{G}_\epsilon^{-1}(\mathbf{r}) = -\epsilon\check{\tau}_3 + \overline{\mathcal{H}} \tag{3.41}$$

The factor $\tanh(\epsilon/2T)$ appears as a result of the replacement of the summation over imaginary frequencies with integration over the real frequencies. It has a simple physical meaning. Indeed,

$$\tanh(\epsilon/2T) = 1 - 2n(\epsilon) \tag{3.42}$$

where

$$n(\epsilon) = \left[e^{\epsilon/T} + 1\right]^{-1}$$

is the Fermi distribution function expressed in terms of the energy ϵ measured from the chemical potential.

One can introduce the momentum representation of the Green function,

$$\check{G}\left(\mathbf{r}_1,\mathbf{r}_2\right)=\int\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3}\check{G}(\mathbf{p},\mathbf{p}-\mathbf{k})e^{i\mathbf{p}\mathbf{r}_1-i(\mathbf{p}-\mathbf{k})\mathbf{r}_2} \tag{3.43}$$

The particle-hole symmetry equation (3.12), in particular, implies

$$G_{\omega_n}\left(\mathbf{p}_1,\mathbf{p}_2\right)=\bar{G}_{-\omega_n}\left(-\mathbf{p}_2,-\mathbf{p}_1\right) \tag{3.44}$$

or

$$G_{\epsilon}^{R(A)}\left(\mathbf{p}_1,\mathbf{p}_2\right)=\bar{G}_{-\epsilon}^{A(R)}\left(-\mathbf{p}_2,-\mathbf{p}_1\right) \tag{3.45}$$

In the momentum-frequency representation, the order-parameter equation becomes

$$\frac{\Delta(\mathbf{k})}{|g|}=T\sum_n\int\frac{d^3p}{(2\pi)^3}F_{\omega_n}(\mathbf{p},\mathbf{p}-\mathbf{k}) \tag{3.46}$$

or

$$\frac{\Delta(\mathbf{k})}{|g|}=\int\frac{d^3p}{(2\pi)^3}\int_{-\infty}^{+\infty}\tanh\left(\frac{\epsilon}{2T}\right)\left[F_{\epsilon}^{R}(\mathbf{p},\mathbf{p}-\mathbf{k})-F_{\epsilon}^{A}(\mathbf{p},\mathbf{p}-\mathbf{k})\right]\frac{d\epsilon}{4\pi i} \tag{3.47}$$

The current is

$$\mathbf{j}(\mathbf{k})=-\frac{e}{m}T\sum_n\int\frac{d^3p}{(2\pi)^3}(2\mathbf{p}-\mathbf{k})G_{\omega_n}(\mathbf{p},\mathbf{p}-\mathbf{k})-\frac{Ne^2}{mc}\mathbf{A} \tag{3.48}$$

or

$$\mathbf{j}(\mathbf{k})=-\frac{e}{m}\int\frac{d^3p}{(2\pi)^3}\int\frac{d\epsilon}{4\pi i}\tanh\left(\frac{\epsilon}{2T}\right)(2\mathbf{p}-\mathbf{k})\left[G_{\epsilon}^{R}(\mathbf{p},\mathbf{p}-\mathbf{k})-G_{\epsilon}^{A}(\mathbf{p},\mathbf{p}-\mathbf{k})\right]-\frac{Ne^2}{mc}\mathbf{A} \tag{3.49}$$

The electron density

$$N=-2T\lim_{\tau\to-0}\sum_n\int\frac{d^3p}{(2\pi)^3}G_{\omega_n}(\mathbf{p},\mathbf{p}-\mathbf{k})e^{-i\omega_n\tau} \tag{3.50}$$

or

$$N=N_0-2\int\frac{d^3p}{(2\pi)^3}\int\frac{d\epsilon}{4\pi i}\tanh\left(\frac{\epsilon}{2T}\right)\delta\left[G_{\epsilon}^{R}(\mathbf{p},\mathbf{p}-\mathbf{k})-G_{\epsilon}^{A}(\mathbf{p},\mathbf{p}-\mathbf{k})\right] \tag{3.51}$$

3.1.3 Order parameter of a d-wave superconductor

Until now we assumed a point-like interaction between electrons independent of particle momenta. In general, the pairing interaction may have a dependence on directions of the momenta of participating particles. In such a case, the order parameter will also be momentum-dependent. In general, one can write

$$\Delta_{\mathbf{p}}(\mathbf{k})=-T\sum_n U\left(\hat{\mathbf{p}},\hat{\mathbf{p}}_1\right)F_{\omega_n}\left(\mathbf{p}_1,\mathbf{p}_1-\mathbf{k}\right)\frac{d^3p_1}{(2\pi)^3} \tag{3.52}$$

where $U\left(\hat{\mathbf{p}},\hat{\mathbf{p}}_1\right)$ is a Fourier transform of the (attractive) interaction matrix element. For simplicity we consider systems which have the inversion symmetry. For such systems, the order

parameter has a definite symmetry in the momentum $\mathbf{p}$ because of the general parity of Fermi particles. The full order parameter including the spin degrees of freedom is an odd function with respect to transposition of particles. If the pairing occurs into a spin-singlet state which is odd in spin indices, the remaining orbital part should be even in the transposition of the particle coordinates, i.e., in the substitution $\mathbf{p} \to -\mathbf{p}$. This requires that the potential $U(\mathbf{p}, \mathbf{p}_1)$ is even in $\mathbf{p} \to -\mathbf{p}$ and, of course, in $\mathbf{p}_1 \to -\mathbf{p}_1$, as well, since it is symmetric in $\mathbf{p} \to \mathbf{p}_1$.

One can expand the pairing potential in terms of orthogonal eigenfunctions of angular motion

$$U(\hat{\mathbf{p}}, \hat{\mathbf{p}}_1) = U_s + U_d(\hat{\mathbf{p}}, \hat{\mathbf{p}}_1) + \dots$$

The s-wave component which we denoted earlier as $U_s = -|g|$ is independent of the momentum directions. The next expansion term represents the so-called d-wave pairing potential. In some cases it can be favorable, for example, because of the crystalline symmetry. If the crystal has a symmetry axis around the axis c the d-wave potential has the from $U_d(\hat{\mathbf{p}}, \hat{\mathbf{p}}_1) = -|g_d| V_d(\hat{\mathbf{p}}, \hat{\mathbf{p}}_1)$ where

$$V_d(\hat{\mathbf{p}}, \hat{\mathbf{p}}_1) = 2\left(\hat{p}_x^2 - \hat{p}_y^2\right)\left(\hat{p}_{1x}^2 - \hat{p}_{1y}^2\right) = 2\cos(2\alpha)\cos(2\alpha_1) \tag{3.53}$$

where α is the angle between the momentum in the crystalline (ab) plane and, say, the a axis. The factor 2 is introduced for normalization. It is easy to see that the order parameter has a form

$$\Delta_{\mathbf{p}} = \Delta_0 \cos(2\alpha)$$

with the amplitude satisfying the self-consistency order parameter equation

$$\frac{\Delta_0(\mathbf{k})}{|g_d|} = T\sum_n 2\cos(2\alpha) F_{\omega_n}(\mathbf{p}, \mathbf{p}-\mathbf{k}) \frac{d^3p}{(2\pi)^3} \tag{3.54}$$

16.4.2 3.2 Derivation of the Bogoliubov-de Gennes equations

(is it a good derivation? I am confused, how to derive it???)

The Green functions can be expanded in terms of the eigenfunctions of the matrix Schrödinger equation. Let us introduce the functions $u(\mathbf{r})$ and $v(\mathbf{r})$ and construct a vector in the Nambu space

$$\mathcal{U} = \begin{pmatrix} u \\ -v \end{pmatrix}.$$

Let the functions u and v satisfy the matrix Schrödinger equation

$$\check{\mathcal{H}}\mathcal{U} = E_K \check{\tau}_3 \mathcal{U} \tag{3.55}$$

where E_K are the eigenvalues, i.e., energy spectrum of the Hamiltonian $\check{\mathcal{H}}$. The two equations

$$\begin{aligned} \left[-\frac{1}{2m}\left(\nabla - \frac{ie}{c}\mathbf{A}\right)^2 - \mu\right] u + \Delta v &= E_K u \\ \left[-\frac{1}{2m}\left(\nabla + \frac{ie}{c}\mathbf{A}\right)^2 - \mu\right] v - \Delta^* u &= -E_K v \end{aligned} \tag{3.56}$$

arising from matrix equation (3.55) are called the Bogoliubov-de Gennes equations. We have encountered them already in Section 1.1.3. The functions u and v coincide with the Bogoliubov-de Gennes wave functions (Bogoliubov 1958) discussed earlier in Section 1.1.3.

$$\mathcal{U}^\dagger = (u^*, v^*)\,.$$

It satisfies the conjugated equation

$$\mathcal{U}^\dagger \overline{\mathcal{H}} = \mathcal{U}^\dagger \check{\tau}_3 E_K \tag{3.57}$$

where $\overline{\mathcal{H}}$ acts on the left. In components,

$$\begin{aligned} \left[-\frac{1}{2m}\left(\nabla + \frac{ie}{c}\mathbf{A}\right)^2 - \mu\right] u^* + \Delta^* v^* &= E_K u^* \\ \left[-\frac{1}{2m}\left(\nabla - \frac{ie}{c}\mathbf{A}\right)^2 - \mu\right] v^* - \Delta u^* &= -E_K v^* \end{aligned} \tag{3.58}$$

which are the complex conjugated eqns (3.56).
Since u and v are eigenfunctions of a linear operator, the functions belonging to different states are orthogonal. We normalize them in such a way that

$$\int d^3r \mathcal{U}_K^\dagger(\mathbf{r}) \check{\tau}_3 \mathcal{U}_{K'}(\mathbf{r}) = \delta_{K,K'} \tag{3.59}$$

Moreover, the functions u and v constitute the full basis thus

$$\sum_K \check{\tau}_3 \mathcal{U}_K(\mathbf{r}_1) \mathcal{U}_K^\dagger(\mathbf{r}_2) = \check{1}\delta(\mathbf{r}_1 - \mathbf{r}_2) \tag{3.60}$$

Using the wave functions u and v we can write

$$\check{G}_\epsilon^{R(A)}(\mathbf{r}_1, \mathbf{r}_2) = \sum_K \frac{\mathcal{U}_K(\mathbf{r}_1)\mathcal{U}_K^\dagger(\mathbf{r}_2)}{E_K - \epsilon \mp i\delta} \tag{3.61}$$

It easy to check that the function defined by eqn (3.61) indeed satisfies eqn (3.40) with the operators defined by eqn (3.41). The infinitely small imaginary term
in the denominator is chosen in accordance with the analytical properties of the retarded and advanced functions.

We can see that the zeroes in the denominator, i.e., the poles of the Green functions G and F are at $\epsilon = E_K$; they thus correspond to the energy spectrum of excitations. Moreover, the quantity (where does this formula come from?)

$$\nu_s(\epsilon) = \frac{1}{4\pi i}\,\mathrm{Tr}\int \check{\tau}_3\left[\check{G}_\epsilon^R(\mathbf{r},\mathbf{r}) - \check{G}_\epsilon^A(\mathbf{r},\mathbf{r})\right] dV = \sum_K \delta(\epsilon - E_K) \tag{3.62}$$

is the density of states in a superconductor per one spin projection. The operator Tr is the trace over the Nambu indices.

16.4.3 3.3 Thermodynamic potential

Here we derive a useful relation between the thermodynamic potential and the Green functions for a BCS system (Kopnin 1987). We first remind the definition of the thermodynamic potential. According to eqn (2.37)

$$\Omega_{sn} = -T \ln \sum^{(d)} \hat{S} \tag{3.63}$$

where

$$\hat{S} = T_\tau \exp\left[-\int_0^{1/T} [\hat{\mathcal{H}} - \mu\hat{\mathcal{N}}] d\tau\right] \tag{3.64}$$

and T_τ is the time-ordering operator in the imaginary "time" τ. The sum in eqn (3.63) denotes the sum of the diagonal matrix elements over the quantum states. In the BCS theory formulated through the Green functions which satisfy the Gor'kov equations, the order parameter plays the part of an external field. In this context, the BCS Hamiltonian measured from the chemical potential should be written as

$$\begin{aligned}\hat{\mathcal{H}}_{\mathrm{BCS}} = \sum_\alpha \Bigg\{ \int d^3r \left[\tilde{\psi}^\dagger_\alpha(\mathbf{r}) \left(-\frac{1}{2m}\left(\nabla - \frac{ie}{c}\mathbf{A}\right)^2 - \mu\right) \tilde{\psi}_\alpha(\mathbf{r})\right] \\ - \frac{1}{2}\int d^3r \left[\tilde{\psi}^\dagger_\alpha(\mathbf{r})\tilde{\psi}^\dagger_\gamma(\mathbf{r})\Delta_{\gamma\alpha}(\mathbf{r}) + \Delta^*_{\alpha\gamma}(\mathbf{r})\tilde{\psi}_\gamma(\mathbf{r})\tilde{\psi}_\alpha(\mathbf{r})\right] + \frac{\Delta^*_{\alpha\gamma}(\mathbf{r})\Delta_{\gamma\alpha}(\mathbf{r})}{2|g|}\Bigg\}\end{aligned} \tag{3.65}$$

It plays the part of $\hat{\mathcal{H}} - \mu\hat{\mathcal{N}}$ in eqn (3.64). We consider an s-wave pairing for definiteness. Here $\tilde{\psi}_\alpha(\mathbf{r})$ is a single-particle operator. The sum is over the spin indices. The Hamiltonian can have also terms describing interactions with other external fields (including impurities) and inter-particle interactions. With the Hamiltonian (3.65), one can directly reproduce equations (3.9), (3.10), (3.13) and (3.14) for the Green functions. The last term is added to ensure that the thermodynamic potential has a minimum for the value of the order parameter which satisfies the self-consistency equation.

To see this, let us calculate the variation of the thermodynamic potential Ω with respect to the fields $\delta\hat{\Delta}, \delta\hat{\Delta}^*$, and $\mathbf{A}$:

$$\delta\Omega_{sn} = -\frac{T\sum^{(d)} \delta\hat{S}}{\sum^{(d)} \hat{S}} \tag{3.66}$$

The variation of the S-matrix is expressed through variations of the Hamiltonian (3.65) and contains the Green functions defined according to eqns (2.38) and (3.6):

$$F_{\alpha\beta}(x, x') = \frac{\sum^{(d)} \left(T_\tau \tilde{\psi}_\alpha(x)\tilde{\psi}_\beta(x')\hat{S}\right)}{\sum^{(d)} \hat{S}} \equiv \left\langle T_\tau \tilde{\psi}_\alpha(x)\tilde{\psi}_\beta(x')\right\rangle_{st} \tag{3.67}$$

$$G_{\alpha\beta}(x, x') = \frac{\sum^{(d)} \left(T_\tau \tilde{\psi}_\alpha(x)\tilde{\psi}^\dagger_\beta(x')\hat{S}\right)}{\sum^{(d)} \hat{S}} \equiv \left\langle T_\tau \tilde{\psi}_\alpha(x)\tilde{\psi}^\dagger_\beta(x')\right\rangle_{st} \tag{3.68}$$

One has from eqn (3.66)

$$\delta\Omega_{sn} = -T\sum_n \mathrm{Tr}\int \left(\left[\delta\check{H}\check{G}_{\omega_n}(\mathbf{r}, \mathbf{r}')\right]_{\mathbf{r}=\mathbf{r}'} + \frac{\delta(\Delta\Delta^*)}{|g|}\right) dV \tag{3.69}$$

where $\check{H}$ is the effective Hamiltonian

$$\check{H} = \begin{pmatrix} \frac{ie}{2mc}(\mathbf{A}\nabla + \mathbf{A}\nabla) - \frac{e^2}{2mc^2}A^2 & -\Delta \\ \Delta^* & -\frac{ie}{2mc}(\mathbf{A}\nabla + \mathbf{A}\nabla) - \frac{e^2}{2mc^2}A^2 \end{pmatrix}. \tag{3.70}$$

It is easy to see that a minimization of Ω_{sn} with respect to $\delta\Delta^*$ leads to the self-consistency equation (3.32).

Let us perform transformation into momentum representation. The variation of the thermodynamic potential can now be written as

$$\delta\Omega_{sn} = -T\sum_n \int \mathrm{Tr}\left[\delta\check{H}(-\mathbf{k})\check{G}_{\omega_n}(\mathbf{p}_+,\mathbf{p}_-)\right]\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3} + \frac{1}{|g|}\int \delta\left(\Delta(\mathbf{k})\Delta^*(-\mathbf{k})\right)\frac{d^3k}{(2\pi)^3}$$

Here we omit the spin indices and denote $\mathbf{p}_\pm = \mathbf{p} \pm \mathbf{k}/2$.
Including the magnetic energy, we find for the total potential

$$\delta\left(\Omega_{sn} + \int \frac{H^2}{8\pi}d^3r\right) = -T\sum_n \int \mathrm{Tr}\left[\delta\check{H}(-\mathbf{k})\check{G}_{\omega_n}(\mathbf{p}_+,\mathbf{p}_-)\right]\frac{d^3p}{(2\pi)^3}\frac{d^3k}{(2\pi)^3} + \int\left[\frac{1}{|g|}\delta\left(\Delta(\mathbf{k})\Delta^*(-\mathbf{k})\right) + \frac{1}{c}\mathbf{j}(\mathbf{k})\cdot\delta\mathbf{A}(-\mathbf{k})\right]\frac{d^3k}{(2\pi)^3} \tag{3.71}$$

The total thermodynamic potential has a minimum also with respect to variation of the vector potential $\mathbf{A}$ because of equation (3.34) for the current.

One can check that eqn (3.71) holds also for a non-trivial pairing interaction determined, for example, by eqn (3.53). One only needs to replace the order parameter Δ with

$$\int V(\mathbf{p},\mathbf{p}')\,\Delta_{\mathbf{p}'}\frac{d\Omega_{\mathbf{p}'}}{4\pi}$$

in the first term in the r.h.s. of eqn (3.71) and introduce the momentum-direction average $\int d\Omega_{\mathbf{p}}/4\pi$ together with $\Delta_{\mathbf{p}}(\mathbf{k})\Delta^*_{\mathbf{p}}(-\mathbf{k})$ in the last term in the r.h.s. of eqn (3.71).

16.4.4 3.4 Example: Homogeneous state

3.4.1 Green functions

Consider a homogeneous state where the order parameter does not depend on coordinates, and the magnetic field is absent. The simplest way to solve the Gor'kov equations, eqns (3.28), (3.30), is to use the momentum representation of eqn (3.43). We have for a homogeneous case

$$\check{G}_{\omega_n}(\mathbf{r}_1,\mathbf{r}_2) = \int e^{i\mathbf{p}(\mathbf{r}_1-\mathbf{r}_2)}\check{G}_{\omega_n}(\mathbf{p})\frac{d^3p}{(2\pi)^3}$$

For the Fourier transformed Green functions we obtain

$$\begin{aligned}(-i\omega_n + \xi_{\mathbf{p}})\,G + \Delta F^\dagger &= 1\\ (-i\omega_n - \xi_{\mathbf{p}})\,F^\dagger + \Delta^* G &= 0\end{aligned} \tag{3.72}$$

where $\xi_{\mathbf{p}} = p^2/2m - \mu$ according to eqn (1.2). In a metal, the chemical potential $\mu \approx E_F$. The solution of the Gor'kov equations is

$$G = \frac{\xi_{\mathbf{p}} + i\omega_n}{\xi_{\mathbf{p}}^2 + \omega_n^2 + |\Delta|^2},\; F^\dagger = \frac{\Delta^*}{\xi_{\mathbf{p}}^2 + \omega_n^2 + |\Delta|^2} \tag{3.73}$$

Similarly,

$$\bar{G} = \frac{\xi_{\mathbf{p}} - i\omega_n}{\xi_{\mathbf{p}}^2 + \omega_n^2 + |\Delta|^2}, F = \frac{\Delta}{\xi_{\mathbf{p}}^2 + \omega_n^2 + |\Delta|^2} \tag{3.74}$$

The retarded and advanced real-time functions are

$$G^{R(A)} = \frac{\xi_{\mathbf{p}} + \epsilon}{\xi_{\mathbf{p}}^2 - (\epsilon \pm i\delta)^2 + |\Delta|^2}, F^{\dagger R(A)} = \frac{\Delta^*}{\xi_{\mathbf{p}}^2 - (\epsilon \pm i\delta)^2 + |\Delta|^2}$$
$$\bar{G}^{R(A)} = \frac{\xi_{\mathbf{p}} - \epsilon}{\xi_{\mathbf{p}}^2 - (\epsilon \pm i\delta)^2 + |\Delta|^2}, F^{R(A)} = \frac{\Delta}{\xi_{\mathbf{p}}^2 - (\epsilon \pm i\delta)^2 + |\Delta|^2}$$

which comply with the particle-hole symmetry of eqns (3.44) and (3.45).
One can also write

$$G^{R(A)} = -\frac{\xi_{\mathbf{p}} + \epsilon}{(\epsilon - \epsilon_{\mathbf{p}} \pm i\delta)(\epsilon + \epsilon_{\mathbf{p}} \pm i\delta)} \tag{3.75}$$

where

$$\epsilon_{\mathbf{p}} = \sqrt{\xi_{\mathbf{p}}^2 + |\Delta|^2} \tag{3.76}$$

This equation coincides with eqn (1.1) in Chapter 1. The Green functions have poles at $\epsilon = \pm\epsilon_{\mathbf{p}}$. It means that $\pm\epsilon_{\mathbf{p}}$ is the energy spectrum of excitations in a superconductor. There are two branches: one with positive and another with negative energies. We also notice that energy of an excitation can not be less than $|\Delta|$: there are no excitations with energies below the energy gap $|\Delta|$.

3.4.2 Gap equation for an s-wave superconductor

Inserting our solution for F into the self-consistency equation we can find the order parameter as a function of temperature. We have

$$\frac{\Delta}{|g|} = T\sum_n \int \frac{d^3p}{(2\pi)^3} \frac{\Delta}{\xi_{\mathbf{p}}^2 + \omega_n^2 + |\Delta|^2} = \nu(0)2T\sum_{n\geq 0} \int d\xi_{\mathbf{p}} \frac{\Delta}{\xi_{\mathbf{p}}^2 + \omega_n^2 + |\Delta|^2} \tag{3.77}$$

We integrate over the momentum with the help of

$$\frac{d^3p}{(2\pi)^3} = \frac{p^2dp}{2\pi^2}\frac{d\Omega_{\mathbf{p}}}{4\pi} = \nu(0)d\xi_{\mathbf{p}}\frac{d\Omega_{\mathbf{p}}}{4\pi} \tag{3.78}$$

where $d\xi_{\mathbf{p}} = v_F dp$ and $d\Omega_{\mathbf{p}}$ is the increment of the solid angle in the momentum space.

$$\nu(0) = \frac{mp_F}{2\pi^2} \tag{3.79}$$

is the density of states for one spin projection in the normal state. Indeed, the characteristic values of $\xi_{\mathbf{p}}$ are of the order of Δ or T which is much smaller than the Fermi energy E_F. Therefore, the momentum is always close to the Fermi momentum, and we can replace p with p_F everywhere. At this stage, we explicitly employ the quasiclassical approximation introduced in Section 1.1.4. We see that the quasiclassical approximation results in a substantial simplification of calculations even at this initial level.

The function under the integral in eqn (3.77) has poles at $\xi_{\mathbf{p}} = \pm i\sqrt{\omega_n^2 + |\Delta|^2}$. Shifting the contour of integration into the upper half-plane of the complex variable $\xi_{\mathbf{p}}$ (see Fig. 3.2) we find

$$\frac{1}{\lambda} = 2\pi T \sum_{n\geq 0} \frac{1}{\sqrt{\omega_n^2 + |\Delta|^2}} \tag{3.80}$$

where $\lambda = \nu(0)|g|$ is the interaction constant. Inspection of this equation shows that the summation over n diverges logarithmically at large frequencies. This

FIG. 3.2. Contour of integration over $d \xi_{\mathbf{p}}$. We denote $\zeta(\omega_n) = \sqrt{\omega_n^2 + |\Delta|^2}$.

is because the pairing interaction was assumed energy independent. As we have already discussed, however, the pairing interaction is restricted from above by a characteristic energy $\Omega_{\rm BCS}$. For a weak coupling approximation, the cutoff frequency is much larger than T_c but considerably smaller than E_F. The energy cutoff is equivalent to truncating the sum at a limiting number $N_0 = \Omega_{\rm BCS}/2\pi T \gg 1$:

$$\frac{1}{\lambda} = 2\pi T \sum_{n=0}^{N_0} \frac{1}{\sqrt{\omega_n^2 + |\Delta|^2}} \tag{3.81}$$

This equation determines the temperature dependence of the order parameter.
The order parameter vanishes at the critical temperature $T = T_c$. To find T_c we put $\Delta = 0$ in eqn (3.81) which becomes

$$\frac{1}{\lambda} = 2\pi T_c \sum_{n=0}^{N_0} \frac{1}{(2n+1)\pi T_c} = \sum_{n=0}^{N_0} \frac{1}{n+1/2} \approx \ln\left(\Omega_{\rm BCS}/2\pi T_c\right) + 2\ln 2 + C \tag{3.82}$$

where $C = 0.5772\ldots$ is the Euler constant. The critical temperature is found to be

$$T_c = \frac{2\Omega_{\rm BCS}\gamma}{\pi} e^{-1/\lambda} \tag{3.83}$$

where $\gamma = e^C \approx 1.78$. The critical temperature is thus much lower than Ω_0 in the weak-coupling approximation $\lambda \ll 1$. It is due to the divergence of the sum in eqn (3.81) that the order parameter does not vanish: a small λ is compensated by a large logarithm $\ln\left(\Omega_{\rm BCS}/T_c\right)$ which introduces a new energy scale $\Omega_{\rm BCS}$ for the temperature. Equation (3.83) also applies to a d-wave superconductor because we omitted the angle-dependent $\Delta_{\mathbf{p}}$ in the denominator of the self-consistency equation.

Equation (3.81) can be presented in terms of real frequencies. Using expressions for retarded and advanced functions, we obtain

$$\frac{1}{\lambda} = \int d\xi_{\mathbf{p}} \int_0^{\Omega_{\rm BCS}} \frac{d\epsilon}{2\pi i} \tanh\left(\frac{\epsilon}{2T}\right) \left(\frac{1}{\xi_{\mathbf{p}}^2 - (\epsilon + i\delta)^2 + |\Delta|^2} - \frac{1}{\xi_{\mathbf{p}}^2 - (\epsilon - i\delta)^2 + |\Delta|^2}\right). \tag{3.84}$$

Fig. 3.3. Contour of integration over $d \xi_{\mathbf{p}}$. After shifting the contour into the upper half-plane, the integrals are determined by poles.

Integration over $d\xi_{\mathbf{p}}$ can be performed by shifting the contours of integration into the upper half-plane of the complex variable $\xi_{\mathbf{p}}$. For $|\epsilon > |\Delta|$, the contribution to the integrals come from the pole $\xi_{\mathbf{p}} = \xi^R_{\mathbf{p}} = \sqrt{(\epsilon + i\delta)^2 - |\Delta|^2}$ for the retarded function and from the pole $\xi_{\mathbf{p}} = -\xi^A_{\mathbf{p}} = -\sqrt{(\epsilon - i\delta)^2 - |\Delta|^2}$ for the advanced function (see Fig. 3.3). If $|\epsilon| < |\Delta|$, the poles are at $\xi_{\mathbf{p}} = \pm i\sqrt{|\Delta|^2 - \epsilon^2}$ for both the retarded and advanced functions. Therefore, for $|\epsilon| < |\Delta|$, the retarded and advanced functions are equal to each other, and the integral vanishes. We have

$$\begin{aligned}\frac{1}{\lambda} &= \int_0^{\Omega_{\mathrm{BCS}}} \tanh\left(\frac{\epsilon}{2T}\right) \frac{d\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}} \Theta\left[\epsilon^2 - |\Delta|^2\right] \\ &= \int_{|\Delta|}^{\Omega_{\mathrm{BCS}}} \tanh\left(\frac{\epsilon}{2T}\right) \frac{d\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}} \end{aligned} \tag{3.85}$$

Equation (3.85) is nothing but eqn (1.57). To see this we use eqn (3.42) and express ξ_p in terms of ϵ through eqn (3.76). Equation (3.85) gives the same value of T_c as eqn (3.81), of course. It can also be used to get the order parameter value at zero temperature:

$$\frac{1}{\lambda} = \int_{|\Delta|}^{\Omega_{\mathrm{BCS}}} \frac{d\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}} = \ln\left(\frac{2\Omega_{\mathrm{BCS}}}{|\Delta|}\right). \tag{3.86}$$

Therefore, at $T = 0$,

$$|\Delta| = (\pi/\gamma)T_c \approx 1.76T_c. \tag{3.87}$$

16.4.5 3.5 Perturbation theory

3.5.1 Diagram technique

Let us assume that the operator $\check{G}^{-1}$ consists of a main part plus a small correction. For example, we can consider the case when the magnetic field and
the order parameter are small. We can separate the operator $\check{G}^{-1}$ into the part corresponding to a free particle

$$\check{G}_0^{-1} = \begin{pmatrix} -\frac{\partial}{\partial\tau} - \frac{1}{2m}\nabla^2 - \mu c & 0 \\ 0 & \frac{\partial}{\partial\tau} - \frac{1}{2m}\nabla^2 - \mu \end{pmatrix} \tag{3.88}$$

and a perturbation $\check{H}$ defined by eqn (3.70) so that $\check{G}^{-1} = \check{G}_0^{-1} + \check{H}$. There may be other possibilities; for example, Δ is not small but the magnetic field is small, etc. Let us look for the Green function in the form of an expansion in powers of perturbation. We write

$$\check{G} = \check{G}^{(0)} + \check{G}^{(1)} + \check{G}^{(2)} + \dots$$

where $\check{G}^{(0)}$ is the Green function of a free particle:

$$\check{G}_0^{-1}\check{G}^{(0)} = \check{1}\delta\left(x_1 - x_2\right)$$

and the correction $\check{G}^{(k)}$ satisfies the equation

$$\check{G}_0^{-1}\check{G}^{(k)} + \check{H}\check{G}^{(k-1)} = 0.$$

The full Green function can be written as a series

$$\begin{aligned}\check{G}\left(x, x'\right) =& \check{G}^{(0)}\left(x, x'\right) - \int d^4x_1 \check{G}^{(0)}\left(x, x_1\right)\check{H}\left(x_1\right)\check{G}^{(0)}\left(x_1, x'\right) \\ &+ \int d^4x_1 d^4x_2 \check{G}^{(0)}\left(x, x_1\right)\check{H}\left(x_1\right)\check{G}^{(0)}\left(x_1, x_2\right)\check{H}\left(x_2\right)\check{G}^{(0)}\left(x_2, x'\right) \\ &- \ldots\end{aligned}$$

or in the form of the Dyson equation

$$\check{G}\left(x, x'\right) = \check{G}^{(0)}\left(x, x'\right) - \int d^4x_1 \check{G}^{(0)}\left(x, x_1\right)\check{H}\left(x_1\right)\check{G}\left(x_1, x'\right). \tag{3.89}$$

The Dyson equation (3.89) is equivalent to the Gor'kov equation (3.18). The same expansion holds also for retarded and advanced functions, as can be seen from eqns (3.40).

The series expansion and the Dyson equation can also be represented as graphs. Let us ascribe a single line to the Green function $\check{G}^{(0)}$, a double line to the full Green function $\check{G}$. Two ends represent two coordinates x_1 and x_2. We denote the interaction $\check{H}$ by a wavy line with one coordinate. As a result, the full Green function can be presented either as an infinite series of diagrams, Fig. 3.4 (a), or as an graphical Dyson equation, Fig. 3.4 (b).

3.5.2 Electric current

The electric current can be calculated using eqn (3.38) and the Green function which can be found, for example, by means of the perturbation theory. We have

$$\mathbf{j} = \frac{ie\hbar}{m}\int_{-\infty}^{+\infty}\tanh\left(\frac{\epsilon}{2T}\right)\frac{d\epsilon}{4\pi i}\left[\left(\nabla_1 - \nabla_2\right)\left(G_\epsilon^R\left(\mathbf{r}_1, \mathbf{r}_2\right) - G_\epsilon^A\left(\mathbf{r}_1, \mathbf{r}_2\right)\right)\right]_{\mathbf{r}_1=\mathbf{r}_2} - \frac{Ne^2}{mc}\mathbf{A}.$$

Fig. 3.4. Diagrammatic representations for the Green function. (a) The perturbation series. (b) The Dyson equation.

In the momentum representation we have

$$\mathbf{j}(\mathbf{k}) = -\frac{2e}{m}\int_{-\infty}^{+\infty}\tanh\left(\frac{\epsilon}{2T}\right)\frac{d\epsilon}{4\pi i}\int\frac{d^3p}{(2\pi)^3}\mathbf{p}\left[G_\epsilon^R\left(\mathbf{p}_+, \mathbf{p}_-\right) - G_\epsilon^A\left(\mathbf{p}_+, \mathbf{p}_-\right)\right] - \frac{Ne^2}{mc}\mathbf{A} \tag{3.90}$$

The last term is called the diamagnetic term. It should disappear in the normal state if one neglects the Landau quantization in a magnetic field. We prove this using the Green function of the normal state. In the presence of magnetic field

$$\check{H} = \begin{pmatrix} -(e/c)\mathbf{v}\cdot\mathbf{A} & 0 \\ 0 & (e/c)\mathbf{v}\cdot\mathbf{A} \end{pmatrix}$$

in the leading approximation in $\mathbf{A}$. Since $\xi_{\mathbf{p}-\mathbf{k}} \approx \xi_p - \mathbf{v}_F\mathbf{k}$, we get

$$G_\epsilon^{R(A)}(\mathbf{p}_+, \mathbf{p}_-) = (2\pi)^3\delta(\mathbf{k})\frac{1}{\xi_\mathbf{p} - \epsilon \mp i\delta} - \frac{e}{c}\frac{\mathbf{v}\cdot\mathbf{A}}{(\xi_\mathbf{p} - \epsilon \mp i\delta)(\xi_\mathbf{p} - \mathbf{vk} - \epsilon \mp i\delta)}$$

The first term does not give a contribution to the current. Since $p \gg k$, we have

$$G_\epsilon^{R(A)}(\mathbf{p}_+, \mathbf{p}_-) = -\frac{e}{c}\frac{\mathbf{vA}}{\mathbf{vk}}\left(\frac{1}{\epsilon - \xi_\mathbf{p} \pm i\delta} - \frac{1}{\epsilon - \xi_\mathbf{p} + \mathbf{vk} \pm i\delta}\right).$$

We use the identity

$$\frac{1}{\epsilon - \xi_\mathbf{p} \pm i\delta} = \mp\pi i\delta(\epsilon - \xi_\mathbf{p}) + \frac{1}{\epsilon - \xi_\mathbf{p}}.$$

Therefore,

$$\begin{aligned}
&\int_{-\infty}^{+\infty} \tanh\left(\frac{\epsilon}{2T}\right)\left[G(\mathbf{p}_+, \mathbf{p}_-) - G_\epsilon^A(\mathbf{p}_+, \mathbf{p}_-)\right]\frac{d\epsilon}{4\pi i} \\
&= -\frac{e\mathbf{vA}}{2c}\frac{1}{\mathbf{vk}}\left[\tanh\left(\frac{\xi_\mathbf{p}}{2T}\right) - \tanh\left(\frac{\xi_\mathbf{p} - \mathbf{vk}}{2T}\right)\right] \\
&= -\frac{e\mathbf{vA}}{2c}\frac{1}{2T}\cosh^{-2}\left(\frac{\xi_\mathbf{p}}{2T}\right)
\end{aligned}$$

The corresponding contribution to the current is

$$\frac{e^2}{c}A_k\int\frac{d^3p}{(2\pi)^3}\frac{\mathbf{v}v_k}{2T}\cosh^{-2}\left(\frac{\xi_\mathbf{p}}{2T}\right) = \frac{e^2}{c}\mathbf{A}\frac{\nu(0)v_F^2}{3}\int d\xi_\mathbf{p}\frac{1}{2T}\cosh^{-2}\left(\frac{\xi_\mathbf{p}}{2T}\right) = \frac{Ne^2}{mc}\mathbf{A}$$

This cancels the diamagnetic term.

16.5 SUPERCONDUCTING ALLOYS

We learn how to incorporate scattering by random impurity atoms into the general Green function formalism of the theory of superconductivity. The crossdiagram technique is derived using the Born approximation for the scattering amplitude.

16.5.1 4.1 Averaging over impurity positions

We describe a simple and powerful method how one can incorporate the interaction between electrons and random impurity atoms into the general BCS scheme. The method has been developed by Abrikosov and Gor'kov (1958). It is assumed that physical properties of a superconductor containing a large amount of random impurities can be obtained by averaging over realizations of the disordered impurity potentials. Another assumption is the Born approximation which implies that the scattering potential is small compared to the characteristic atomic potential the latter being of the order of the Fermi energy of the host material.

Each impurity interacts with electrons via the potential $u(\mathbf{r}-\mathbf{r}_a)$ where $\mathbf{r}_a$ is the position of an impurity atom. Let us consider this interaction as perturbation and calculate the Green function. We assume for simplicity a homogeneous order parameter. Generalization for a spatially dependent order parameter is straightforward. The Green function has the form

$$\begin{aligned}\check{G}_{\omega_n}(\mathbf{r},\mathbf{r}') =& \check{G}^{(0)}_{\omega_n}(\mathbf{r}-\mathbf{r}') - \sum_a \int d^3r_1 \check{G}^{(0)}_{\omega_n}(\mathbf{r}-\mathbf{r}_1)\, u(\mathbf{r}_1-\mathbf{r}_a)\, \check{G}^{(0)}_{\omega_n}(\mathbf{r}_1-\mathbf{r}') \\ &+ \sum_{a,b} \int d^3r_1 d^3r_2 \check{G}^{(0)}_{\omega_n}(\mathbf{r}-\mathbf{r}_1)\, u(\mathbf{r}_1-\mathbf{r}_a) \\ &\times \check{G}^{(0)}_{\omega_n}(\mathbf{r}_1-\mathbf{r}_2)\, u(\mathbf{r}_2-\mathbf{r}_b)\, \check{G}^{(0)}_{\omega_n}(\mathbf{r}_2-\mathbf{r}') - \ldots\end{aligned}$$

Here $\check{G}^{(0)}$ refers to a superconductor without impurities. Note that the Green functions on the both side of each impurity potential have the same frequency ω_n because the impurity scattering is elastic. We introduce the Fourier transformed Green functions and find

$$\begin{aligned}\check{G}(\mathbf{r},\mathbf{r}') =& \int \frac{d^3p}{(2\pi)^3} \check{G}^{(0)}(\mathbf{p}) e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')} - \sum_a \int \frac{d^3p d^3p'}{(2\pi)^6} e^{i(\mathbf{p}\mathbf{r}-\mathbf{p}'\mathbf{r}')} e^{-i(\mathbf{p}-\mathbf{p}')\mathbf{r}_a} \check{G}^{(0)}(\mathbf{p}) u(\mathbf{p}-\mathbf{p}')\, \check{G}^{(0)}(\mathbf{p}') \\ &+ \sum_{a,b} \int \frac{d^3p d^3p' d^3p_1}{(2\pi)^9} e^{i(\mathbf{p}\mathbf{r}-\mathbf{p}'\mathbf{r}')} e^{-i(\mathbf{p}-\mathbf{p}_1)\mathbf{r}_a} e^{-i(\mathbf{p}_1-\mathbf{p}')\mathbf{r}_b} \check{G}^{(0)}(\mathbf{p}) u(\mathbf{p}-\mathbf{p}_1)\, \check{G}^{(0)}(\mathbf{p}_1)\, u(\mathbf{p}_1-\mathbf{p}')\, \check{G}^{(0)}(\mathbf{p}')\end{aligned} \tag{4.1}$$

We omit the index ω_n for brevity. This series can be presented graphically as lines separated by crosses, where each cross designates the scattering potential u, Fig. 4.1

FIG. 4.1. Graphic representation of interaction with impurities.

Since the impurity atoms are distributed randomly, summation over positions of impurity atoms can be substituted with integration over their coordinates, i.e., with the averaging over positions of impurities and multiplication by the number of impurity atoms

$$\sum_a \Rightarrow n_{\text{imp}} \int d^3r_a$$

where n_{imp} is the concentration of impurity atoms. The integration results in δ-functions of the corresponding differences of momenta in eqn (4.1). The firstorder correction in eqn (4.1) gives

$$n_{\text{imp}} \int \check{G}^{(0)}(\mathbf{p}) u(0) \check{G}^{(0)}(\mathbf{p}) e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')} \frac{d^3p}{(2\pi)^3} \tag{4.2}$$

In the second-order correction we separate first the terms with $\mathbf{r}_a \neq \mathbf{r}_b$, i.e., the terms where the impurity scattering occurs on different atoms. The average gives the δ-functions $\delta(\mathbf{p}-\mathbf{p}_1)$ and $\delta(\mathbf{p}'-\mathbf{p}_1)$. The second-order term becomes

$$n^2_{\text{imp}} \int \check{G}^{(0)}(\mathbf{p}) u(0) \check{G}^{(0)}(\mathbf{p}) u(0) \check{G}^{(0)}(\mathbf{p}) e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')} \frac{d^3p}{(2\pi)^3} \tag{4.3}$$

If we now collect the terms of all orders in u where scattering occurs on different atoms we obtain the effective correction to the Hamiltonian in the form

$$\check{H}_1 = \begin{pmatrix} n_{\text{imp}}u(0) & 0 \\ 0 & n_{\text{imp}}u(0) \end{pmatrix}.$$

This term is not important: it can be incorporated in the chemical potential. We shall omit this contribution in what follows.

Consider now the second-order term where scattering occurs at the same atom $\mathbf{r}_a = \mathbf{r}_b$. The sum gives

$$n_{\text{imp}} \int \frac{d^3p d^3p_1}{(2\pi)^6} e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')} \check{G}^{(0)}(\mathbf{p}) u\left(\mathbf{p}-\mathbf{p}_1\right) \check{G}^{(0)}\left(\mathbf{p}_1\right) u\left(\mathbf{p}_1-\mathbf{p}\right) \check{G}^{(0)}(\mathbf{p})$$

$$= \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')} \check{G}^{(0)}(\mathbf{p}) \left[n_{\text{imp}} \int \frac{d^3p_1}{(2\pi)^3} \left|u\left(\mathbf{p}-\mathbf{p}_1\right)\right|^2 \check{G}^{(0)}\left(\mathbf{p}_1\right) \right] \check{G}^{(0)}(\mathbf{p})$$

$$\equiv \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')} \check{G}^{(0)}(\mathbf{p}) \check{\Sigma}^{(0)}(\mathbf{p}) \check{G}^{(0)}(\mathbf{p}) \tag{4.4}$$

Here we introduce the "self energy"

$$\check{\Sigma}^{(0)}_{\omega_n}(\mathbf{p}) = n_{\text{imp}} \int \frac{d^3p_1}{(2\pi)^3} \left|u\left(\mathbf{p}-\mathbf{p}_1\right)\right|^2 \check{G}^{(0)}_{\omega_n}\left(\mathbf{p}_1\right) \tag{4.5}$$

Since both $\mathbf{p}$ and $\mathbf{p}_1$ are close to the Fermi surface, the scattering amplitude depends only on the angle θ between $\mathbf{p}$ and $\mathbf{p}_1$: $\left|u\left(\mathbf{p}-\mathbf{p}_1\right)\right|^2 = |u(\theta)|^2$.

The scattering amplitude determines the scattering mean-free time. In the normal state it is

$$\frac{1}{\tau_{\text{imp}}} = 2\pi\nu(0) n_{\text{imp}} \int |u(\theta)|^2 \frac{d\Omega_{\mathbf{p}}}{4\pi} \tag{4.6}$$

One can introduce the scattering cross section $\sigma_{\mathbf{p},\mathbf{p}_1}$. In the Born approximation

$$\left|u\left(\mathbf{p}-\mathbf{p}_1\right)\right|^2 = \frac{2v_F}{\nu(0)} \sigma_{\mathbf{p},\mathbf{p}_1}$$

The scattering mean free time becomes

$$\frac{1}{\tau_{\text{imp}}} = 4\pi v_F n_{\text{imp}} \int \sigma_{\mathbf{p},\mathbf{p}_1} \frac{d\Omega_{\mathbf{p}}}{4\pi}$$

For an isotropic scattering by impurities $|u(\theta)|^2 = \text{const}$, the self-energy is independent of the momentum direction:

$$\check{\Sigma}^{(0)}_{\omega_n} = n_{\text{imp}} |u|^2 \int \frac{d^3p_1}{(2\pi)^3} \check{G}^{(0)}_{\omega_n}\left(\mathbf{p}_1\right) = \frac{1}{2\pi\nu(0)\tau_{\text{imp}}} \int \frac{d^3p_1}{(2\pi)^3} \check{G}^{(0)}_{\omega_n}\left(\mathbf{p}_1\right) \tag{4.7}$$

Here the scattering mean free time is

$$\frac{1}{\tau_{\text{imp}}} = 2\pi\nu(0)|u|^2 n_{\text{imp}} \tag{4.8}$$

The corresponding contribution to the Green function can be shown on a diagram as a dashed line connecting two crosses belonging to the same atom (see Fig. 4.2).

Consider now higher-order terms in u. The terms where all the atoms are different have already been taken into account. The rest can be separated into two groups. The first contains

such scattering processes that each atom only participates two times. On a diagram this corresponds to various combinations of dashed lines connecting two crosses. The second group involves processes where each atom participates more than two times. One can show that the contribution from more than two crosses per atom is smaller than that from two crosses per

(a) G

(b)

FIG. 4.2. (a) Averaging over positions of one atom. (b) First-order self energy.

(a)

(b)

(c)

)

FIG. 4.3. Averaging over positions of two atoms. (a) and (b) Dashed lines do not cross. (c) Dashed lines cross which restricts the interval of the momentum directions.
atom by a factor u/E_F for each extra cross. Indeed, the diagram with three crosses gives

$$\sum_{a,b,c}\int\frac{d^3pd^3p'd^3p_1d^3p_2}{(2\pi)^{12}}e^{i(\mathbf{pr}-\mathbf{p'r'})}e^{-i(\mathbf{p}-\mathbf{p}_1)\mathbf{r}_a-i(\mathbf{p}_1-\mathbf{p}_2)\mathbf{r}_b-i(\mathbf{p}_2-\mathbf{p'})\mathbf{r}_c}$$
$$\times\check{G}^{(0)}(\mathbf{p})u\left(\mathbf{p}-\mathbf{p}_1\right)\check{G}^{(0)}\left(\mathbf{p}_1\right)u\left(\mathbf{p}_1-\mathbf{p}_2\right)\check{G}^{(0)}\left(\mathbf{p}_2\right)u\left(\mathbf{p}_2-\mathbf{p}'\right)\check{G}^{(0)}\left(\mathbf{p}'\right)$$

We put $\mathbf{r}_a=\mathbf{r}_b=\mathbf{r}_c=\mathbf{r}$ and integrate over d^3r. We get

$$n_{\text{imp}}\int\frac{d^3pd^3p_1d^3p_2}{(2\pi)^9}e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')}\check{G}^{(0)}_{\omega_n}(\mathbf{p})u\left(\mathbf{p}-\mathbf{p}_1\right)\check{G}^{(0)}\left(\mathbf{p}_1\right)u\left(\mathbf{p}_1-\mathbf{p}_2\right)\check{G}^{(0)}\left(\mathbf{p}_2\right)u\left(\mathbf{p}_2-\mathbf{p}\right)\check{G}^{(0)}(\mathbf{p})$$
$$=\int\frac{d^3p}{(2\pi)^3}e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')}\check{G}^{(0)}(\mathbf{p})\check{\Sigma}^{(3)}(\mathbf{p})\check{G}^{(0)}(\mathbf{p})$$

where

$$\check{\Sigma}^{(3)}_{\omega_n}(\mathbf{p})=n_{\text{imp}}\int\frac{d^3p_1d^3p_2}{(2\pi)^6}u\left(\mathbf{p}-\mathbf{p}_1\right)u\left(\mathbf{p}_1-\mathbf{p}_2\right)u\left(\mathbf{p}_2-\mathbf{p}\right)$$

Fig. 4.4. The condition that all the momenta are close to the Fermi surface imposes a restriction on the momentum directions.

$$\times\check{G}^{(0)}_{\omega_n}\left(\mathbf{p}_1\right)\check{G}^{(0)}_{\omega_n}\left(\mathbf{p}_2\right).$$

The estimate shows that $\Sigma^{(3)}\sim n_{\text{imp}}\,\nu^2(0)u^3(\mathbf{p})$. It has an extra factor $\nu(0)u(\mathbf{p})$ as compared to $\Sigma^{(0)}$. Since

$$u(\mathbf{p})=\int d^3re^{i\mathbf{pr}}u(\mathbf{r})\sim p_F^{-3}u(\mathbf{r})$$

the factor $\nu(0)u(\mathbf{p})\sim u/E_F$ The ratio $u/E_F\ll 1$ in the Born approximation. It is this approximation which we adopt for our consideration of the impurity scattering. Therefore, the processes with two crosses per atom are sufficient for us.

The processes where each atom participates only twice can be of two different types shown in Fig. 4.3. The first type has dashed lines which do not intersect, Fig. 4.3 a,b. The second type has intersecting lines, Fig. 4.3 c. For example, the diagram with four crosses gives

$$\sum_{a,b,c,d}\int\frac{d^3pd^3p'd^3p_1d^3p_2d^3p_3}{(2\pi)^{15}}e^{i(\mathbf{pr}-\mathbf{p}'\mathbf{r}')}$$
$$\times\, e^{-i(\mathbf{p}-\mathbf{p}_1)\mathbf{r}_a-i(\mathbf{p}_1-\mathbf{p}_2)\mathbf{r}_b-i(\mathbf{p}_2-\mathbf{p}_3)\mathbf{r}_c-i(\mathbf{p}_3-\mathbf{p}')\mathbf{r}_d}$$
$$\times\,\check{G}^{(0)}(\mathbf{p})u\left(\mathbf{p}-\mathbf{p}_1\right)\check{G}^{(0)}\left(\mathbf{p}_1\right)u\left(\mathbf{p}_1-\mathbf{p}_2\right)\check{G}^{(0)}\left(\mathbf{p}_2\right)u\left(\mathbf{p}_2-\mathbf{p}_3\right)$$
$$\times\,\check{G}^{(0)}\left(\mathbf{p}_3\right)u\left(\mathbf{p}_3-\mathbf{p}'\right)\check{G}^{(0)}\left(\mathbf{p}'\right)$$

The intersecting lines correspond to $\mathbf{r}_a=\mathbf{r}_c$ and $\mathbf{r}_b=\mathbf{r}_d$. After averaging we obtain

$$n_{\text{imp}}^2\int\frac{d^3pd^3p_1d^3p_2}{(2\pi)^9}e^{i\mathbf{p}(\mathbf{r}-\mathbf{r}')}\check{G}^{(0)}(\mathbf{p})\left|u\left(\mathbf{p}-\mathbf{p}_1\right)\right|^2$$
$$\times\,\check{G}^{(0)}\left(\mathbf{p}_1\right)\check{G}^{(0)}\left(\mathbf{p}_2\right)\left|u\left(\mathbf{p}_1-\mathbf{p}_2\right)\right|^2\check{G}^{(0)}\left(\mathbf{p}+\mathbf{p}_2-\mathbf{p}_1\right)\check{G}^{(0)}(\mathbf{p}).$$

All the momenta which are arguments of the Green functions should be close to the Fermi surface, $|\mathbf{p}_i|\approx p_F$. This condition imposes a restriction on the

FIG. 4.5. Graphic Dyson equation for interaction with impurities.

integration range over the angles of the momenta (see Fig. 4.4). For example, the range of the azimuthal angle $\delta\theta\sim\delta G/G\sim\delta\xi_{\mathbf{p}}/E_F$. The characteristic $\delta\xi_{\mathbf{p}}\sim 1/\tau_{\text{imp}}$, therefore, this restriction reduces the contribution from the intersecting diagram by the factor $1/\left(E_F\tau_{\text{imp}}\right)\sim 1/\left(p_F\ell\right)$. This factor is small for metals $1/\left(p_F\ell\right)\ll 1$. Therefore, we can neglect all intersecting diagrams.

On the contrary, the diagram without intersections corresponds to either $\mathbf{r}_a=\mathbf{r}_b$ and $\mathbf{r}_c=\mathbf{r}_d$ (Fig. 4.3 a) or $\mathbf{r}_a=\mathbf{r}_d$ and $\mathbf{r}_b=\mathbf{r}_c$ (Fig. 4.3 b). These diagrams do not have restrictions on the integration over the angles, therefore, they have to be taken into account. After summation of all the diagram, we obtain an expression of the type of eqn (4.4) where the function $\check{G}^{(0)}\left(\mathbf{p}_1\right)$ in the self energy and the function $\check{G}^{(0)}(\mathbf{p})$ from the right are replaced with the total Green function containing all the diagrams:

$$\check{G}_{\omega_n}(\mathbf{p})=\check{G}^{(0)}_{\omega_n}(\mathbf{p})+\check{G}^{(0)}_{\omega_n}(\mathbf{p})\check{\Sigma}_{\omega_n}(\mathbf{p})\check{G}_{\omega_n}(\mathbf{p}) \tag{4.9}$$

where the total self energy is

$$\check{\Sigma}_{\omega_n}(\mathbf{p})=n_{\text{imp}}\int\frac{d^3p_1}{(2\pi)^3}|u(\theta)|^2\check{G}_{\omega_n}\left(\mathbf{p}_1\right) \tag{4.10}$$

For isotropic scattering, we have

$$\check{\Sigma}_{\omega_n}=\frac{1}{2\pi\nu(0)\tau_{\text{imp}}}\int\frac{d^3p_1}{(2\pi)^3}\check{G}_{\omega_n}\left(\mathbf{p}_1\right) \tag{4.11}$$

The Dyson equation (4.9) is presented graphically in Fig. 4.5. It is equivalent to the equation for the total Green function in the form

$$\check{G}^{-1}_{\omega_n}\left(\mathbf{r}_1\right)\check{G}_{\omega_n}\left(\mathbf{r}_1,\mathbf{r}_2\right)=\hat{1}\delta\left(\mathbf{r}_1-\mathbf{r}_2\right) \tag{4.12}$$

with the operator

$$\check{G}_{\omega_n}^{-1}(\mathbf{r}_1) = \check{G}_{\text{clean}}^{-1}(\mathbf{r}_1) - \check{\Sigma}_{\omega_n}$$

where $\check{G}_{\text{clean}}^{-1}(\mathbf{r}_1)$ is defined by eqn (3.28). The matrix self-energy can be written in components:

$$\check{\Sigma} = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ -\Sigma_2^\dagger & \bar{\Sigma}_1 \end{pmatrix}. \tag{4.13}$$

The method of averaging over the impurity atoms which leads to the Dyson equation in the form of Fig. 4.5 is called the cross-diagram technique. Its characteristic feature is the absence of intersecting lines (impurity averaging) which connect different crosses (impurity atoms).

The Born approximation provides quite satisfactory description of interaction of electrons with impurity atoms in many cases. Sometimes, it is, however, not sufficient for describing fine details of electronic properties, especially in d-wave superconductors. Beyond the Born approximation, one has to take into account various processes with more than two crosses per atom. This results in a substitution of the Born amplitude $u(\mathbf{p})$ with the full scattering amplitude. For a strong scattering potential, the so called unitary limit, the exact amplitude may be substantially different from that in the Born approximation. We do not consider such problems here. Some examples of full scattering amplitude analysis have been discussed by Buchholtz and Zwicknagl (1981), Graf et al. (1995), and Graf et al. (1996).

16.5.2 4.1.1 Magnetic impurities

Considering interaction of electrons with impurities one has to distinguish between the scattering by non-magnetic impurities and that by impurity atoms having a finite magnetic moment (Abrikosov and Gor'kov 1960). We already noticed that magnetic impurities scatter differently the two electrons which make a Cooper pair because of different spin states of the constituent electrons in the potential of the magnetic impurity. Remind that the functions G, on one hand, and $F, F^\dagger$, on the other hand, have different spin structures; G is proportional to the unit matrix in the spin state, while F is proportional to the antisymmetric matrix $i\sigma_y$. Therefore, the scattering amplitudes contain different projections of the impurity spin operators. One can write, therefore,

$$\Sigma_1 = n_{\text{imp}} \int \frac{d^3p_1}{(2\pi)^3} |u_1(\theta)|^2 G_{\omega_n}(\mathbf{p}_1) \tag{4.14}$$

and similarly for $\bar{\Sigma}_1$ in terms of $\bar{G}$. On the other hand,

$$\Sigma_2 = n_{\text{imp}} \int \frac{d^3p_1}{(2\pi)^3} |u_2(\theta)|^2 F_{\omega_n}(\mathbf{p}_1) \tag{4.15}$$

with a similar expression for $\Sigma_2^\dagger$ in terms of $F^\dagger$.
The corresponding scattering times are

$$\frac{1}{\tau_1} = 2\pi\nu(0) n_{\text{imp}} \int \frac{d\Omega_{\mathbf{p}_1}}{4\pi} |u_1(\theta)|^2 \tag{4.16}$$

$$\frac{1}{\tau_2} = 2\pi\nu(0) n_{\text{imp}} \int \frac{d\Omega_{\mathbf{p}_1}}{4\pi} |u_2(\theta)|^2 \tag{4.17}$$

One can define the difference of these scattering rates as

$$\frac{2}{\tau_s} = \frac{1}{\tau_1} - \frac{1}{\tau_2} \tag{4.18}$$

where τ_s is the spin-flip scattering time. It can be shown (Abrikosov and Gor'kov 1960) that

$$\frac{1}{\tau_s} = 4\pi n_{\text{imp}} \nu(0) \frac{S(S+1)}{(2S+1)^2} \int \frac{d\Omega_{\mathbf{p}_1}}{4\pi} \left| u_+(\theta) - u_-(\theta) \right|^2 > 0 \tag{4.19}$$

Here $u_+$is the scattering amplitude for an electron in the state with the total spin $S + 1/2$, and $u_-$is the scattering amplitude for an electron in the state with the total spin $S - 1/2$; S is the spin of an impurity atom. This is the so called spin-flip time: the average time between collisions resulting in a change of the spin state of an electron.

The scattering amplitudes can be parametrized by two different scattering cross sections σ_1 and σ_2 associated with the diagonal and off-diagonal components (i.e., with either the functions G and $\bar{G}$ or the functions F and $F^\dagger$) of the matrix Green function $\check{G}$, respectively,

$$|u_{1,2}(\theta)|^2 = \frac{2v_F}{\nu(0)} \sigma_{1,2}(\theta)$$

16.5.3 4.2 Homogeneous state of an s-wave superconductor

Consider a homogeneous s-wave superconductor without magnetic impurities. We assume also that the magnetic field is absent. In the Fourier representation, the Gor'kov equations are

$$\left(-i\omega_n + \xi_{\mathbf{p}} - \Sigma_1\right) G_{\omega_n}(\mathbf{p}) + \left(\Delta + \Sigma_2\right) F^\dagger_{\omega_n}(\mathbf{p}) = 1 \tag{4.20}$$

$$\left(i\omega_n + \xi_{\mathbf{p}} - \bar{\Sigma}_1\right) F^\dagger_{\omega_n}(\mathbf{p}) - \left(\Delta^* + \Sigma_2^\dagger\right) G_{\omega_n}(\mathbf{p}) = 0 \tag{4.21}$$

We shall see later that $\Sigma_1 = -\bar{\Sigma}_1$. Moreover, for a homogeneous state, one can take Δ to be real and thus $F = F^\dagger$ and $\Sigma_2 = \Sigma_2^\dagger$. The solution becomes

$$G = \frac{i\omega_n + \xi_{\mathbf{p}} + \Sigma_1}{\xi_{\mathbf{p}}^2 + \left(\omega_n - i\Sigma_1\right)^2 + \left(\Delta + \Sigma_2\right)^2} \tag{4.22}$$

$$F = \frac{\Delta + \Sigma_2}{\xi_{\mathbf{p}}^2 + \left(\omega_n - i\Sigma_1\right)^2 + \left(\Delta + \Sigma_2\right)^2} \tag{4.23}$$

The self-energies are

$$\Sigma_1 = n_{\text{imp}} \int \frac{d^3 p_1}{(2\pi)^3} |u(\theta)|^2 G_{\omega_n}(\mathbf{p}_1)$$

$$\Sigma_2 = n_{\text{imp}} \int \frac{d^3 p_1}{(2\pi)^3} |u(\theta)|^2 F_{\omega_n}(\mathbf{p}_1) \tag{4.24}$$

The Green functions for a homogeneous state depend only on the magnitude of the momentum. We shall see that the momentum integrals here are determined by the region near the Fermi surface. Therefore,

$$\Sigma_1 = \nu(0) n_{\text{imp}} \int \frac{d\Omega_{\mathbf{p}}}{4\pi} |u(\theta)|^2 \int d\xi_{\mathbf{p}} G_{\omega_n}(\mathbf{p}) = \frac{1}{2\pi\tau_{\text{imp}}} \int d\xi_{\mathbf{p}} G_{\omega_n}(\mathbf{p})$$

$$\Sigma_2 = \nu(0) n_{\text{imp}} \int \frac{d\Omega_{\mathbf{p}}}{4\pi} |u(\theta)|^2 \int d\xi_{\mathbf{p}} F_{\omega_n}(\mathbf{p}) = \frac{1}{2\pi\tau_{\text{imp}}} \int d\xi_{\mathbf{p}} F_{\omega_n}(\mathbf{p})$$

since the Green functions do not depend on the momentum direction. We have

$$\int d\xi_{\mathbf{p}} G_{\omega_n}(\mathbf{p}) = \int d\xi_{\mathbf{p}} \frac{i\omega_n + \xi_{\mathbf{p}} + \Sigma_1}{\xi_{\mathbf{p}}^2 + (\omega_n - i\Sigma_1)^2 + (\Delta + \Sigma_2)^2}$$
$$= \int \frac{d\xi_{\mathbf{p}}}{2} \frac{1}{\xi_{\mathbf{p}} - i\sqrt{(\omega_n - i\Sigma_1)^2 + (\Delta + \Sigma_2)^2}}$$
$$+ \int \frac{d\xi_{\mathbf{p}}}{2} \frac{1}{\xi_{\mathbf{p}} + i\sqrt{(\omega_n - i\Sigma_1)^2 + (\Delta + \Sigma_2)^2}}$$
$$+ \int d\xi_{\mathbf{p}} \frac{i\omega_n + \Sigma_1}{\left(\xi_{\mathbf{p}} - i\sqrt{(\omega_n - i\Sigma_1)^2 + (\Delta + \Sigma_2)^2}\right)}$$
$$\times \frac{1}{\left(\xi_{\mathbf{p}} + i\sqrt{(\omega_n - i\Sigma_1)^2 + (\Delta + \Sigma_2)^2}\right)}$$

We shall see that Σ_1 is imaginary while Σ_2 is real. Therefore, the first two integrals result in real quantities which are determined by large ξ_p and thus depend only on the normal properties of the system. We can incorporate these contributions to the chemical potential and disregard them in further calculations. The third integral can be calculated by residues. It gives

$$\Sigma_1 = \frac{1}{2\pi\tau_{\text{imp}}} \int d\xi_{\mathbf{p}} G_{\omega_n}(\mathbf{p}) = \frac{i}{2\tau_{\text{imp}}} \frac{\omega_n - i\Sigma_1}{\sqrt{(\omega_n - i\Sigma_1)^2 + (\Delta + \Sigma_2)^2}} \tag{4.25}$$

For Σ_2, the normal-state part is absent since F decreases quickly enough as a function of ξ_p. Calculating it by residues, we find

$$\Sigma_2 = \frac{i}{2\tau_{\text{imp}}} \frac{\Delta + \Sigma_2}{i\sqrt{(\omega_n - i\Sigma_1)^2 + (\Delta + \Sigma_2)^2}} \tag{4.26}$$

Equations (4.25) and (4.26) imply that

$$1 + \frac{\Sigma_1}{i\omega_n} = 1 + \frac{\Sigma_2}{\Delta} = \eta_n \tag{4.27}$$

where the function η_n is

$$\eta_n = 1 + \frac{1}{2\tau_{\text{imp}}} \frac{1}{\sqrt{\omega_n^2 + \Delta^2}} \tag{4.28}$$

Finally, the Green functions become

$$G = \frac{i\omega_n\eta_n + \xi_{\mathbf{p}}}{\xi_{\mathbf{p}}^2 + \eta_n^2\left(\omega_n^2 + \Delta^2\right)}, F = \frac{\Delta\eta_n}{\xi_{\mathbf{p}}^2 + \eta_n^2\left(\omega_n^2 + \Delta^2\right)} \tag{4.29}$$

The gap equation takes the form

$$\frac{\Delta}{\lambda} = 2T \sum_{n\geq 0}^{N_0} \int d\xi_{\mathbf{p}} F_{\omega_n}(\mathbf{p}) = 2\pi T \sum_{n\geq 0}^{N_0} \frac{\Delta}{\sqrt{\omega_n^2 + \Delta^2}} \tag{4.30}$$

The function η drops out after the integration over $d\xi_{\mathbf{p}}$! The gap equation for a superconducting alloy with an s-wave pairing in a spatially homogeneous state is exactly the same as for a clean superconductor. This means that neither the critical temperature nor the order parameter is affected by impurities. This statement holds for an s-wave superconductor and is known as the Anderson theorem.

17 II Quasiclassical method by Kopnin

17.1 GENERAL PRINCIPLES OF THE QUASICLASSICAL APPROXIMATION

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The quasiclassical Green functions integrated over the energy near the Fermi surface are introduced. Eilenberger equations and the normalization condition are derived for the quasiclassical Green functions. It is shown how to reduce the Eilenberger equations to diffusion-like Usadel equations in case of superconducting alloys. The boundary conditions for the quasiclassical Green functions at a rough interface between a superconductor and an insulator are derived.

17.1.1 5.1 Quasiclassical Green functions

As we already discussed in Introduction, all presently known superconductors have the Fermi energy considerably larger than Δ. This condition is especially well satisfied for conventional, low-temperature superconductors where the ratio E_F/Δ can be as high as 10^3. For high-temperature superconductors, the ratio E_F/Δ is not that high; nevertheless, it still ranges by the order of magnitude from 10 to 10^2. The BCS theory appears to be so successful mostly because it benefits essentially from the situation when the superconducting order parameter magnitude is much smaller than the characteristic energy of the normal state, i.e., the Fermi energy. In such a case, the Fermi momentum is much larger than the inverse coherence length $\xi^{-1} \sim \Delta/v_F$ such that the condition eqn (1.61) is satisfied. This fact offers a possibility to develop a relatively simple and powerful method of dealing with the Green functions in the BCS theory based on the quasiclassical approximation. The quasiclassical method is now commonly accepted; it is widely used for practical calculations especially for problems where the order parameter and electromagnetic fields are functions of coordinates and time. In the present Chapter we introduce the quasiclassical method for stationary properties of superconductors. Nonstationary problems are considered in the following parts of the book.

The definitions of physical quantities involve the Green functions whose spatial coordinates are taken in the limit $\mathbf{r}_1 \to \mathbf{r}_2$. In the momentum representation, the limit $\mathbf{r}_1 \to \mathbf{r}_2$ corresponds to the integration over the momentum space. The definitions of physical quantities would now contain Green functions integrated with various powers of the quasiparticle momenta. For example, the order parameter of an s-wave superconductor contains the function $F(\mathbf{p}_+, \mathbf{p}_-)$integrated with the momentum to the zeroth power. The order parameter of a d-wave superconductor involves second powers of components of $\mathbf{p}$. The current is expressed through $G(\mathbf{p}_+, \mathbf{p}_-)$integrated with the first power of $\mathbf{p}$, etc.
The fastest function under these integrals is the Green function itself: as a function of $\xi_{\mathbf{p}}$, it varies strongly near the Fermi surface when $\xi_{\mathbf{p}}$ changes by an amount $\delta\xi_{\mathbf{p}}$ of the order of Δ. Since this variation $\delta\xi_{\mathbf{p}}$ is much smaller than E_F the magnitude of the quasiparticle momentum $\mathbf{p}$ remains close to the Fermi momentum. Indeed, the variation of the momentum magnitude is $\delta p \sim \delta\xi_{\mathbf{p}}/v_F \sim \xi^{-1}$ such that $\delta p/p_F \sim (p_F\xi)^{-1} \ll 1$. Here we use again the quasiclassical condition eqn (1.61), namely that $p_F\xi$ is a large number. It means that, in all the prefactors in front of the Green function, we can put the momentum magnitude to be at the Fermi surface. In particular, one can parameterize the momentum-space integral using the Fermi surface defined for the normal state:

$$\frac{d^3p}{(2\pi)^3} = d\xi_{\mathbf{p}} \frac{dS_F}{(2\pi)^3 v_F} \tag{5.1}$$

where $d\xi_{\mathbf{p}}/v_F$ is the momentum increment in the direction perpendicular to the Fermi surface while dS_F is the Fermi-surface area element. For simplicity, we shall consider mostly a spherical Fermi surface. If the Fermi surface is a sphere, $dS_F = p_F^2 d\Omega_{\mathbf{p}}$, and we recover eqn (3.78). Generalizations for more complicated Fermi surfaces are straightforward. Of course, in the superconducting state, the Fermi surface is not defined in a strict sense because a gap in the energy spectrum opens near the original position of Fermi surface in the normal state. However, if the energy gap is much smaller than the original Fermi energy, the latter can still be used as a reference energy for the variable

$$\xi_{\mathbf{p}} = E_n(\mathbf{p}) - E_F$$

Remind that E_n is the electronic spectrum in the normal state.
All the expressions for physical quantities contain now the Green functions integrated over $d\xi_{\mathbf{p}}$ near the Fermi surface and then multiplied with various functions of the particle momentum directions. Since these expressions do not use energies of the order of the Fermi energy, the question arises: Is it possible to develop such a formalism that operates only with the Green functions taken near the Fermi surface? Or, what would be even better, can one develop a method which uses only Green functions already integrated over $d\xi_{\mathbf{p}}$ near the Fermi surface? The answer is yes; we show how to do this in the present Chapter. The new formulation of the theory of superconductivity was elaborated first by Eilenberger (1968), Larkin and Ovchinnikov (1977), and Eliashberg (1971). It is much simpler to handle than the original approach which uses the full Green functions. Fundamentals of the quasiclassical method can be found in the review by Serene and Rainer (1983).

We turn now to definitions of the quasiclassical Green functions. Integrating over the energy near the Fermi surface we note that it is easy to define the integrals

$$\int \frac{d\xi_{\mathbf{p}}}{\pi i} F_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-), \int \frac{d\xi_{\mathbf{p}}}{\pi i} F^\dagger_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) \tag{5.2}$$

They converge rather quickly for $\xi_{\mathbf{p}} \gg T, \Delta$ since F and $F^\dagger$ decrease as $\xi_{\mathbf{p}}^{-2}$. This is due to the fact that the functions F and $F^\dagger$ exist only in the superconducting state. The integrals are, of course, determined by contributions from the quasiparticles having energies near the Fermi surface. These quasiparticle states correspond to the poles of the Green functions. We denote this fact by

$$\int \frac{d\xi_{\mathbf{p}}}{\pi i} F_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) = \oint \frac{d\xi_{\mathbf{p}}}{\pi i} F_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) \equiv f_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) \tag{5.3}$$

$$\int \frac{d\xi_{\mathbf{p}}}{\pi i} F^\dagger_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) = \oint \frac{d\xi_{\mathbf{p}}}{\pi i} F^\dagger_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) \equiv f^\dagger_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) \tag{5.4}$$

where $\oint$ shows that we take the contributions from poles close to the Fermi surface. The particle momentum lies at the Fermi surface, the only variable is the momentum direction; we use this fact and denote $\hat{\mathbf{p}}$ the unit vector in the direction of $\mathbf{p}$.

However, the full integral over $d\xi_{\mathbf{p}}$ diverges for the function G. This is because G contains the normal part which decays slowly for large $\xi_{\mathbf{p}}$. We can, however, introduce the integral

$$\oint \frac{d\xi_{\mathbf{p}}}{\pi i} G_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) \equiv g_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) \tag{5.5}$$

which only takes the contribution from poles near the Fermi surface. With help of eqn (3.78) the full integral over the magnitude of the momentum can be written as

$$\int \frac{p^2 dp}{2\pi^2} G_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) = \int \frac{p^2 dp}{2\pi^2} G^{(n)}_{\omega_n} + \nu(0) \oint d\xi_{\mathbf{p}} \left[G_{\omega_n} - G^{(n)}_{\omega_n} \right] \tag{5.6}$$

where $G^{(n)}_{\omega_n}$ is the zero-field Green function in the normal state. In the last line, the integral over $d\xi_{\mathbf{p}}$ converges; it is thus determined by poles near the Fermi surface. The accuracy of this approximation is again of the order of Δ/E_F.

Since

$$G^{(n)}_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) = G^{(n)}_{\omega_n}(\mathbf{p})(2\pi)^3 \delta(\mathbf{k})$$

and

$$G^{(n)}_{\omega_n}(\mathbf{p}) = \frac{1}{\xi_{\mathbf{p}} - i\omega_n} = \frac{1}{\xi_{\mathbf{p}}} + \pi i \operatorname{sign}(\omega_n)\, \delta(\xi_{\mathbf{p}})$$

we have

$$\oint \frac{d\xi_{\mathbf{p}}}{\pi i} G^{(n)}_{\omega}(\mathbf{p}) = \operatorname{sign}(\omega_n) \tag{5.7}$$

Therefore,

$$\int \frac{p^2 dp}{2\pi^2} G_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) = \int \frac{p^2 dp}{2\pi^2} G^{(n)}_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) - \nu(0)\pi i \operatorname{sign}(\omega_n)(2\pi)^3 \delta(\mathbf{k}) + \nu(0)\pi i g_\omega(\hat{\mathbf{p}}, \mathbf{k}) \tag{5.8}$$

so that

$$\int \frac{p^2 dp}{2\pi^2} G_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) = (2\pi)^3 \delta(\mathbf{k}) \mathcal{P} \int \frac{p^2 dp}{2\pi^2} \frac{1}{\xi_{\mathbf{p}}} + \nu(0)\pi i g_\omega(\hat{\mathbf{p}}, \mathbf{k}) \tag{5.9}$$

where $\mathcal{P}\int$ means the principle value integral.
In the same way we introduce

$$\oint \frac{d\xi_p}{\pi i} \bar{G}_\omega(\mathbf{p}_+, \mathbf{p}_-) \equiv \bar{g}_\omega(\hat{\mathbf{p}}, \mathbf{k}) \tag{5.10}$$

so that

$$\int \frac{p^2 dp}{2\pi^2} \bar{G}_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) = \int \frac{p^2 dp}{2\pi^2} \bar{G}^{(n)}_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-) + \nu(0)\pi i \operatorname{sign}(\omega_n)(2\pi)^3 \delta(\mathbf{k}) + \nu(0)\pi i \bar{g}_\omega(\hat{\mathbf{p}}, \mathbf{k}) \tag{5.11}$$

and

$$\int \frac{p^2 dp}{2\pi^2} \bar{G}_\omega(\mathbf{p}_+, \mathbf{p}_-) = (2\pi)^3 \delta(\mathbf{k}) \mathcal{P} \int \frac{p^2 dp}{2\pi^2} \frac{1}{\xi_{\mathbf{p}}} + \nu(0)\pi i \bar{g}_\omega(\hat{\mathbf{p}}, \mathbf{k}) \tag{5.12}$$

We can see that the large $-\xi_{\mathbf{p}}$ contribution to the full momentum integral of G only depends on the properties of the material in the normal state. All the superconducting properties are included into the functions g, f, etc., determined by a vicinity of the Fermi surface. The functions

$$g, f, f^\dagger, \text{ and } \bar{g}$$

are called quasiclassical Green functions. The large $-\xi_p$ contributions are important, for example, when we calculate the density of electrons in the superconducting state. We consider this issue in the next Section.

We have defined the quasiclassical function in a symmetric way with respect to incoming $\mathbf{p}+\mathbf{k}/2$ and outgoing $\mathbf{p}-\mathbf{k}/2$ momenta in eqns (5.3), (5.4), and (5.5). Making shift in the momentum $\mathbf{p}$, one observes that also, with the accuracy of the quasiclassical approximation,

$$g_{\omega_n}(\hat{\mathbf{p}},\mathbf{k}) = \oint \frac{d\xi_{\mathbf{p}}}{\pi i} G_{\omega_n}(\mathbf{p},\mathbf{p}-\mathbf{k}) \tag{5.13}$$

and similarly for f and $f^\dagger$.

Now, after the $\xi_{\mathbf{p}}$-integration, the Green function only depends on the momentum direction and on the center-of-mass coordinate Fourier-transformed into the $\mathbf{k}$ space. The coordinate dependence of the Green function contains

$$e^{i(\mathbf{p}+\mathbf{k}/2)\mathbf{r}_1 - i(\mathbf{p}-\mathbf{k}/2)\mathbf{r}_2} = e^{i\mathbf{p}(\mathbf{r}_1-\mathbf{r}_2)+i\mathbf{k}(\mathbf{r}_1+\mathbf{r}_2)/2} \tag{5.14}$$

The integration over $d\xi_p$ is the integration over the magnitude of the momentum; it is equivalent to the limit $\mathbf{r}_1 \to \mathbf{r}_2$ taken along the direction of the quasiparticle momentum $\mathbf{p}$. The result thus depends on the unit vector $\hat{\mathbf{p}}$.

The term "quasiclassical Green function" means that the fast oscillations of the Green function associated with variations of the relative coordinate $\mathbf{r}_1 - \mathbf{r}_2$ on a scale of p_F^{-1} are excluded. They are not relevant to the superconducting properties. Instead, there remains only a slow dependence on the center-of-mass coordinate

$$(\mathbf{r}_1 + \mathbf{r}_2)/2 = \mathbf{r}$$

on a scale of ξ^{-1} characteristic to the superconducting state. Sometimes we use a mixed Fourier-coordinate representation

$$\check{g}_{\omega_n}(\hat{\mathbf{p}},\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{kr}} \check{g}_{\omega_n}(\hat{\mathbf{p}},\mathbf{k}) \tag{5.15}$$

which deals with the functions of the center-of-mass coordinate and of the particle momentum direction.

To make the formulas more compact, we introduce the matrix notation

$$\check{g}_{\omega_n}(\hat{\mathbf{p}},\mathbf{k}) = \begin{pmatrix} g_{\omega_n} & f_{\omega_n} \\ -f^\dagger_{\omega_n} & \bar{g}_{\omega_n} \end{pmatrix} \tag{5.16}$$

where the matrices in the Nambu space $\check{g}$ are constructed out of g, f, etc., in the same way as the matrix $\check{G}$ is made of G, F, etc.

17.1.2 5.2 Density, current, and order parameter

Using the momentum-space volume element in the form of eqn (3.78) the electron density can be written as

$$\begin{aligned} N(\mathbf{k}) = &-2T \lim_{\tau\to-0} \sum_n \int \frac{d^3p}{(2\pi)^3} G_{\omega_n}(\mathbf{p}_+,\mathbf{p}_-)\, e^{-i\omega_n\tau} \\ = &-2T \lim_{\tau\to-0} \sum_n \Bigg[\int \frac{d^3p}{(2\pi)^3} G^{(n)}_{\omega_n}(\mathbf{p}_+,\mathbf{p}_-)\, e^{-i\omega_n\tau} \\ &-\nu(0)\pi i\, \mathrm{sign}\,(\omega_n)\,(2\pi)^3\delta(\mathbf{k}) + \nu(0)\pi i \int g_{\omega_n}(\hat{\mathbf{p}},\mathbf{k}) \frac{d\Omega_{\mathbf{p}}}{4\pi} \Bigg] \end{aligned}$$

where

$$\mathbf{p}_{\pm} = \mathbf{p} \pm \mathbf{k}/2$$

Introducing the normal-state electron density

$$N_0 = -2T \lim_{\tau \to -0} \sum_n \int \frac{d^3p}{(2\pi)^3} G^{(n)}_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-)\, e^{-i\omega_n \tau}$$

we find

$$N = N_0 (2\pi)^3 \delta(\mathbf{k}) - 2\nu(0)\pi i T \sum_n \left[\int g_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) \frac{d\Omega_{\mathbf{p}}}{4\pi} - \operatorname{sign}(\omega_n)\, (2\pi)^3 \delta(\mathbf{k}) \right]. \tag{5.17}$$

Note that the procedure of taking the limit is only important for the normalstate part of the Green function: there is no singularity in the last two terms for large ω_n. Indeed, the sum in eqn (5.17) is

$$\sum_n \operatorname{sign}(\omega_n) = 0 \tag{5.18}$$

because the function under the sum is odd. Therefore,

$$N(\mathbf{k}) = N_0 (2\pi)^3 \delta(\mathbf{k}) - 2\nu(0)\pi i T \sum_n \int g_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) \frac{d\Omega_{\mathbf{p}}}{4\pi} \tag{5.19}$$

Similarly,

$$\begin{aligned} N(\mathbf{k}) = &- 2T \lim_{\tau \to +0} \sum_n \int \frac{d^3p}{(2\pi)^3} \bar{G}_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-)\, e^{-i\omega_n \tau} \\ = &- 2T \lim_{\tau \to +0} \sum_n \left[\int \frac{d^3p}{(2\pi)^3} \bar{G}^{(n)}_{\omega_n}(\mathbf{p}_+, \mathbf{p}_-)\, e^{-i\omega_n \tau} \right. \\ &\left. + \nu(0)\pi i \operatorname{sign}(\omega_n)\, (2\pi)^3 \delta(\mathbf{k}) + \nu(0)\pi i \int \bar{g}_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) \frac{d\Omega_{\mathbf{p}}}{4\pi} \right] \\ = & N_0 (2\pi)^3 \delta(\mathbf{k}) - 2\nu(0)\pi i T \sum_n \int \bar{g}_{\omega}(\hat{\mathbf{p}}, \mathbf{k}) \frac{d\Omega_{\mathbf{p}}}{4\pi} \end{aligned} \tag{5.20}$$

Combining eqns (5.19) and (5.20) we get

$$N(\mathbf{k}) = N_0 (2\pi)^3 \delta(\mathbf{k}) - \nu(0)\pi i T \sum_n \int \left[g_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) + \bar{g}_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) \right] \frac{d\Omega_{\mathbf{p}}}{4\pi} \tag{5.21}$$

Note that the sums of the type of eqn (5.18) in eqn (5.21) cancel each other.
We shall see later that always

$$g_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) + \bar{g}_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) = 0 \tag{5.22}$$

in a static situation. This is a consequence of the particle-hole symmetry of a superconductor as a Fermi-liquid, i.e., the symmetry between "particles" defined as excitations in the part of the spectrum with $\xi_{\mathbf{p}} > 0$ and "holes" with $\xi_{\mathbf{p}} < 0$.

This property holds only for quasiparticle states near the Fermi surface and is in addition to the general relation (3.44) or (3.45). The particle-hole symmetry is conserved in expressions for

the order parameter, the current and the particle density within the quasiclassical approximation which assumes a linear dependence of the phase-space volume on $\xi_{\mathbf{p}}$ and uses a fixed value of the particle momentum magnitude $|\mathbf{p}| = p_F$. If the density of states and/or the momentum magnitude are energy dependent, the particle-hole symmetry is broken in expressions for the physical observables. In particular, the electron density in the quasiclassical approximation does not change after a transition to an equilibrium superconducting state:

$$N = N_0$$

The expression for the electric current can be transformed as follows:

$$\begin{aligned}
\mathbf{j}(\mathbf{k}) =& -2eT\sum_n \int \frac{d^3p}{(2\pi)^3}\mathbf{v}_F G_{\omega_n}(\mathbf{p}_+,\mathbf{p}_-) - \frac{Ne^2}{mc}\mathbf{A} \\
=& -2eT\sum_n \left[\int \frac{d^3p}{(2\pi)^3}\mathbf{v}_F G^{(n)}_{\omega_n}(\mathbf{p}_+,\mathbf{p}_-)\right. \\
& \left. -(2\pi)^3\delta(\mathbf{k})\nu(0)\pi i\,\mathrm{sign}\,(\omega_n)\int \frac{d\Omega_{\mathbf{p}}}{4\pi}\mathbf{v}_F\right] \\
& -\frac{Ne^2}{mc}\mathbf{A} - 2e\nu(0)\pi iT\sum_n \int \frac{d\Omega_{\mathbf{p}}}{4\pi}\mathbf{v}_F g_{\omega_n}(\hat{\mathbf{p}},\mathbf{k})
\end{aligned}$$

The diamagnetic term in this equation is canceled by the large- $\xi_{\mathbf{p}}$ contribution from the normal state Green function $G^{(n)}_{\omega}(\mathbf{p}_+,\mathbf{p}_-)$in the same way as it happened in Section 3.4. Finally,

$$\mathbf{j}(\mathbf{k}) = -2e\nu(0)\pi iT\sum_n \int \frac{d\Omega_p}{4\pi}\mathbf{v}_F g_{\omega_n}(\hat{\mathbf{p}},\mathbf{k}) \tag{5.23}$$

Similarly,

$$\begin{aligned}
\mathbf{j}(\mathbf{k}) =& 2eT\sum_n \int \frac{d^3p}{(2\pi)^3}\mathbf{v}_F \bar{G}_{\omega_n}(\mathbf{p}_+,\mathbf{p}_-) - \frac{Ne^2}{mc}\mathbf{A} \\
=& 2eT\sum_n \left[\int \frac{d^3p}{(2\pi)^3}\mathbf{v}_F \bar{G}^{(n)}_{\omega_n}(\mathbf{p}_+,\mathbf{p}_-) + (2\pi)^3\delta(\mathbf{k})\nu(0)\pi i\,\mathrm{sign}\,(\omega_n)\int \frac{d\Omega_p}{4\pi}\mathbf{v}_F\right] \\
& -\frac{Ne^2}{mc}\mathbf{A} + 2e\nu(0)\pi iT\sum_n \int \frac{d\Omega_{\mathbf{p}}}{4\pi}\mathbf{v}_F \bar{g}_{\omega_n}(\hat{\mathbf{p}},\mathbf{k}) \\
=& 2e\nu(0)\pi iT\sum_n \int \frac{d\Omega_{\mathbf{p}}}{4\pi}\mathbf{v}_F \bar{g}_{\omega_n}(\hat{\mathbf{p}},\mathbf{k})
\end{aligned} \tag{5.24}$$

Therefore,

$$\mathbf{j}(\mathbf{k}) = -e\nu(0)\pi iT\sum_n \int \frac{d\Omega_{\mathbf{p}}}{4\pi}\mathbf{v}_F\left[g_{\omega_n}(\hat{\mathbf{p}},\mathbf{k}) - \bar{g}_{\omega_n}(\hat{\mathbf{p}},\mathbf{k})\right] \tag{5.25}$$

The order parameter can also be written in terms of a $\xi_{\mathbf{p}}$-integrated Green function. For an s-wave superconductor,

$$\frac{\Delta(\mathbf{k})}{|g|} = T\sum_n \int \frac{d^3p}{(2\pi)^3}F_{\omega_n}(\mathbf{p}_+,\mathbf{p}_-) = \nu(0)\pi iT\sum_n \int f_{\omega_n}(\hat{\mathbf{p}},\mathbf{k})\frac{d\Omega_{\mathbf{p}}}{4\pi} \tag{5.26}$$

or

$$\frac{\Delta(\mathbf{k})}{\lambda} = \pi i T \sum_n \int f_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) \frac{d\Omega_{\mathbf{p}}}{4\pi} \tag{5.27}$$

Similarly,

$$\frac{\Delta^*(\mathbf{k})}{\lambda} = \pi i T \sum_n \int f^\dagger_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) \frac{d\Omega_{\mathbf{p}}}{4\pi} \tag{5.28}$$

For nontrivial pairing, the order parameter equations contains the corresponding normalized moment of the pairing potential. For example, in case of d-wave pairing, the order parameter is

$$\frac{\Delta_{\mathbf{p}}(\mathbf{k})}{\lambda} = \pi i T \sum_n \int V\left(\hat{\mathbf{p}}, \hat{\mathbf{p}}'\right) f_{\omega_n}\left(\hat{\mathbf{p}}', \mathbf{k}\right) \frac{d\Omega_{\mathbf{p}'}}{4\pi} \tag{5.29}$$

where $V\left(\hat{\mathbf{p}}, \hat{\mathbf{p}}'\right) = 2\left(p_x^2 - p_y^2\right)\left(p_x'^2 - p_y'^2\right)$ or $V\left(\hat{\mathbf{p}}, \hat{\mathbf{p}}'\right) = 4\left(p_x p_y\right)\left(p_x' p_y'\right)$, etc. For simplicity, we assume an s-wave pairing if not specified otherwise.

Using the quasiclassical representation, one can write the expression for the variation of the thermodynamic potential eqn (3.71) in the form

$$\begin{aligned}\delta\left(\Omega_{sn} + \int \frac{H^2}{8\pi} dV\right) = &- \nu(0)\pi i T \sum_n \int \mathrm{Tr}\left[\delta \check{H} \check{g}_{\omega_n}(\mathbf{p}, \mathbf{r})\right] \frac{d\Omega_{\mathbf{p}}}{4\pi} dV \\ &+ \frac{1}{|g|} \int \delta|\Delta|^2 dV + \frac{1}{c} \int \mathbf{j} \delta \mathbf{A} dV\end{aligned} \tag{5.30}$$

where now

$$\check{H} = \begin{pmatrix} -(e/c)\mathbf{v}_F \cdot \mathbf{A} & -\Delta \\ \Delta^* & (e/c)\mathbf{v}_F \cdot \mathbf{A} \end{pmatrix} \tag{5.31}$$

instead of eqn (3.70).

17.1.3 5.3 Homogeneous state

For further use we calculate the functions g, f, etc., for a homogeneous case. We have

$$f_{\omega_n} = \int \frac{d\xi_{\mathbf{p}}}{\pi i} \frac{\Delta}{\left(\xi_{\mathbf{p}} + i\sqrt{\omega_n^2 + |\Delta|^2}\right)\left(\xi_{\mathbf{p}} - i\sqrt{\omega_n^2 + |\Delta|^2}\right)} = \frac{\Delta}{i\sqrt{\omega_n^2 + |\Delta|^2}} \tag{5.32}$$

and, similarly,

$$f^\dagger_{\omega_n} = \int \frac{d\xi_{\mathbf{p}}}{\pi i} \frac{\Delta^*}{\left(\xi_{\mathbf{p}} + i\sqrt{\omega_n^2 + |\Delta|^2}\right)\left(\xi_{\mathbf{p}} - i\sqrt{\omega_n^2 + |\Delta|^2}\right)} = \frac{\Delta^*}{i\sqrt{\omega_n^2 + |\Delta|^2}} \tag{5.33}$$

(check all this!!!)

The integration is performed by shifting the contour of integration to either upper or lower half-plane of $\xi_{\mathbf{p}}$, see Fig. 3.2.

To calculate g we write

$$
\begin{aligned}
g_{\omega_n} =& \oint \frac{d\xi_{\mathbf{p}}}{\pi i} \frac{\xi_{\mathbf{p}} + i\omega_n}{\left(\xi_{\mathbf{p}} + i\sqrt{\omega_n^2 + |\Delta|^2}\right)\left(\xi_p - i\sqrt{\omega_n^2 + |\Delta|^2}\right)} \\
=& \frac{1}{2} \oint \frac{d\xi_{\mathbf{p}}}{\pi i} \left(\frac{1}{\xi_{\mathbf{p}} - i\sqrt{\omega_n^2 + |\Delta|^2}} + \frac{1}{\xi_{\mathbf{p}} + i\sqrt{\omega_n^2 + |\Delta|^2}} \right) \\
&+ \oint \frac{d\xi_{\mathbf{p}}}{\pi i} \frac{i\omega_n}{\left(\xi_{\mathbf{p}} + i\sqrt{\omega_n^2 + |\Delta|^2}\right)\left(\xi_{\mathbf{p}} - i\sqrt{\omega_n^2 + |\Delta|^2}\right)}
\end{aligned} \tag{5.34}
$$

The two terms in the second line cancel each other. Using the contour of Fig. 3.2 we obtain from the third line

$$
g_{\omega_n} = \frac{\omega_n}{\sqrt{\omega_n^2 + |\Delta|^2}} \tag{5.35}
$$

We find in exactly the same way

$$
\bar{g}_{\omega_n} = \oint \frac{d\xi_{\mathbf{p}}}{\pi i} \frac{\xi_{\mathbf{p}} - i\omega_n}{\left(\xi_{\mathbf{p}} + i\sqrt{\omega_n^2 + |\Delta|^2}\right)\left(\xi_{\mathbf{p}} - i\sqrt{\omega_n^2 + |\Delta|^2}\right)} = -\frac{\omega_n}{\sqrt{\omega_n^2 + |\Delta|^2}} \tag{5.36}
$$

We see from these equations that

$$
g_{\omega_n} + \bar{g}_{\omega_n} = 0 \tag{5.37}
$$

This is in agreement with the general relation eqn (5.22). Another useful relation is the so called normalization condition which is satisfied by the functions g, f, and $f^\dagger$:

$$
g_{\omega_n}^2 - f_{\omega_n} f_{\omega_n}^\dagger = 1 \tag{5.38}
$$

Equations (5.37) and (5.38) can be combined into one matrix equation

$$
\check{g}_{\omega_n} \check{g}_{\omega_n} = \check{1} \tag{5.39}
$$

Indeed, since

$$
\check{g}_{\omega_n} \check{g}_{\omega_n} = \begin{pmatrix} gg - ff^\dagger & (g + \bar{g})f \\ -f^\dagger(g + \bar{g}) & \bar{g}^2 - ff^\dagger \end{pmatrix},
$$

eqn (5.39) is satisfied because of eqns (5.37, 5.38). We shall prove later that eqn (5.39) holds also in a general spatially non-homogeneous case for any stationary situation.

17.1.4 5.4 Real frequency representation

Equations for the order parameter, the current, and the particle density can be presented in terms of real-frequency Green functions, as well, using the recipe

$$
\pi i T \sum_{\omega_n} \check{g}_{\omega_n} \Rightarrow \int \frac{d\epsilon}{4} \tanh\left(\frac{\epsilon}{2T}\right) \left[\check{g}_\epsilon^R - \check{g}_\epsilon^A\right] \tag{5.40}
$$

This transformation can also be used for the particle density because the normalstate contribution vanishes due to an odd parity of $\tanh(\epsilon/2T)$. The quasiclassical retarded and advanced functions $\check{g}_\epsilon^{R(A)}$ can be obtained from the Matsubara functions $\check{g}_{\omega_n}$ by an analytical continuation using the relation

$$\check{g}^R_{i\omega_n} = \check{g}_{\omega_n} \text{ for } \omega_n > 0, \tag{5.41}$$
$$\check{g}^A_{i\omega_n} = \check{g}_{\omega_n} \text{ for } \omega_n < 0. \tag{5.42}$$

The retarded function is the Matsubara function continued analytically from the positive imaginary axis while the advanced function is its continued from the negative imaginary axis.

The functions g^R and g^A, in particular, determine the density of states. Indeed, we know from eqn (3.62) that

$$\nu_s(\epsilon) = \frac{1}{2\pi i}\left[G^R_\epsilon(\mathbf{r},\mathbf{r}) - G^A_\epsilon(\mathbf{r},\mathbf{r})\right] = \frac{1}{2\pi i}\int \frac{d^3p}{(2\pi)^3}\left[G^R_\epsilon(\mathbf{p}_+,\mathbf{p}_-) - G^A_\epsilon(\mathbf{p}_+,\mathbf{p}_-)\right]$$

is the density of states per one spin projection. We observe now that

$$\nu_s(\epsilon) = \frac{1}{2}\nu(0)\left[g^R_\epsilon(\mathbf{p}) - g^A_\epsilon(\mathbf{p})\right] \tag{5.43}$$

is the density of states near the Fermi surface (per one spin) for a given momentum direction.

5.4.1 Example: Homogeneous state

First of all, we note that, in the normal state

$$g_{\omega_n} = \text{sign}\,(\omega_n)$$

Fig. 5.1. The plane of complex ϵ with the cut. It is this plane where the retarded and advanced functions for a homogeneous state are defined.

Therefore,

$$g^R_\epsilon = -g^A_\epsilon = 1 \tag{5.44}$$

in the normal sate. In a homogeneous superconductor, we obtain from eqn (5.35)

$$g^R_\epsilon = \frac{\epsilon}{\sqrt{(\epsilon + i\delta)^2 - |\Delta|^2}}, g^A_\epsilon = -\frac{\epsilon}{\sqrt{(\epsilon - i\delta)^2 - |\Delta|^2}}. \tag{5.45}$$

The infinitely small complex part $i\delta$ is introduced to demonstrate the analytical properties of the functions. We can also define $\sqrt{\epsilon^2 - |\Delta|^2}$ as an analytical function on the plane of complex ϵ with the cut connecting the points $-|\Delta|$ and $+|\Delta|$ (see Fig. 5.1). This is the definition which results in eqn (5.44) in the limit $|\Delta| \to 0$. With this definition, we have for $|\epsilon| > |\Delta|$

$$g^R_\epsilon = -g^A_\epsilon = \frac{\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}}. \tag{5.46}$$

To get the function g^R in the region $|\epsilon| < |\Delta|$, we continue the square root $\sqrt{\epsilon^2 - |\Delta|^2}$ to the upper edge of the cut; it becomes $i\sqrt{|\Delta|^2 - \epsilon^2}$. To get g^A, we continue the square root to the lower edge of the cut; it becomes $-i\sqrt{|\Delta|^2 - \epsilon^2}$. Therefore, we have for $|\epsilon| < |\Delta|$

$$g_\epsilon^R = g_\epsilon^A = \frac{-i\epsilon}{\sqrt{|\Delta|^2 - \epsilon^2}}. \tag{5.47}$$

The functions $g^{R(A)}$ are thus "even" in ϵ for $|\epsilon| > |\Delta|$ and odd in ϵ for $|\epsilon| > |\Delta|$. This contrasts to g_{ω_n} which is always odd in ω_n. Of course, the functions $g_\epsilon^{R(A)}$ and $\bar{g}_\epsilon^{R(A)}$ are coupled through eqn (5.37), i.e.,

$$\bar{g}_\epsilon^{R(A)} = -g_\epsilon^{R(A)} \tag{5.48}$$

Similarly,

$$f_\epsilon^R = -f_\epsilon^A = \frac{\Delta}{\sqrt{(\epsilon \pm i\delta)^2 - |\Delta|^2}}. \tag{5.49}$$

One has for $|\epsilon| > |\Delta|$

$$f_\epsilon^R = -f_\epsilon^A = \frac{\Delta}{\sqrt{\epsilon^2 - |\Delta|^2}} \tag{5.50}$$

and

$$f_\epsilon^R = f_\epsilon^A = \frac{-i\Delta}{\sqrt{|\Delta|^2 - \epsilon^2}} \tag{5.51}$$

for $|\epsilon| < |\Delta|$. The functions $f^{R(A)}$ are odd in ϵ for $|\epsilon| > |\Delta|$ and even in ϵ for $|\epsilon| < |\Delta|$. Retarded and advanced function also satisfy the normalization condition of eqn (5.39).

One can find the density of states in a homogeneous superconductor. One has from eqn (5.43)

$$\nu_s(\epsilon) = \nu(0)\frac{\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}}\Theta\left(\epsilon^2 - |\Delta|^2\right).$$

It diverges near the gap edge and is zero for $|\epsilon| < |\Delta|$: there are no states within the energy gap.

17.1.5 5.5 Eilenberger equations

To derive equations for the quasiclassical Green functions we consider the Gor'kov equations in the momentum representation

$$\left[\check{G}^{-1}\check{G}\right]_{\mathbf{p}+\mathbf{k}/2,\mathbf{p}-\mathbf{k}/2} = \check{1}. \tag{5.52}$$

Here we use the shortcut notation

$$[\check{A}\check{B}]_{\mathbf{p}_1,\mathbf{p}_2} = \int \check{A}\left(\mathbf{p}_1, \mathbf{p}'\right)\check{B}\left(\mathbf{p}', \mathbf{p}_2\right)\frac{d^3p'}{(2\pi)^3}. \tag{5.53}$$

We write the inverse operator as

$$\check{G}^{-1} = \check{G}_0^{-1} + \check{H} - \check{\Sigma}. \tag{5.54}$$

Here Σ is the self energy matrix with the components

$$\check{\Sigma} = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ -\Sigma_2^\dagger & \bar{\Sigma}_1 \end{pmatrix}. \tag{5.55}$$

The zero-order inverse operator is

$$\check{G}_0^{-1}(\mathbf{p}+\mathbf{k}/2) = \begin{pmatrix} -i\omega + \xi_{\mathbf{p}} + \frac{\mathbf{vk}}{2} + \frac{k^2}{8m} & 0 \\ 0 & i\omega + \xi_{\mathbf{p}} + \frac{\mathbf{vk}}{2} + \frac{k^2}{8m} \end{pmatrix}, \tag{5.56}$$

and the effective Hamiltonian is [compare with eqn (3.70)]

$$\check{H} = \begin{pmatrix} -\frac{e}{c}\mathbf{v}_F \cdot \mathbf{A}(\mathbf{k}) + e\varphi & -\Delta(\mathbf{k}) \\ \Delta^*(\mathbf{k}) & \frac{e}{c}\mathbf{v}_F \cdot \mathbf{A}(\mathbf{k}) + e\varphi \end{pmatrix}. \tag{5.57}$$

We include the scalar electromagnetic potential φ into the Hamiltonian eqn (5.57) because $\mu = E_F - e\varphi$ while we measure energy from the Fermi level:
$\xi_{\mathbf{p}} = E_n - E_F$. We neglect $\mathbf{k}$ as compared to $\mathbf{p}_F$ in the Hamiltonian since $k/p_F \sim 1/\left(p_F\xi\right)$ and omit the term

$$\frac{e^2}{mc^2}A^2 \sim v_F\frac{e}{c}A\frac{eH_{c2}\xi}{cp_F} \sim v_F\frac{e}{c}A\frac{1}{p_F\xi}$$

which is also small in the same parameter, eqn (1.61), that determines the accuracy of the quasiclassical approximation.

The left-sided Gor'kov equation becomes

$$\left[\left(\check{G}_0^{-1} + \check{H} - \check{\Sigma}\right)\check{G}\right]_{\mathbf{p}_+,\mathbf{p}_-} = \check{1}. \tag{5.58}$$

Similarly, the right-sided equation is

$$\left[\check{G}\left(\check{G}_0^{-1} + \check{H} - \check{\Sigma}\right)\right]_{\mathbf{p}_+,\mathbf{p}_-} = \check{1} \tag{5.59}$$

where

$$\check{G}_0^{-1}(\mathbf{p}-\mathbf{k}/2) = \begin{pmatrix} i\omega + \xi_{\mathbf{p}} - \frac{\mathbf{vk}}{2} + \frac{k^2}{8m} & 0 \\ 0 & -i\omega + \xi_{\mathbf{p}} - \frac{\mathbf{vk}}{2} + \frac{k^2}{8m} \end{pmatrix} \tag{5.60}$$

Let us now subtract two eqns (5.58) and (5.59). We have

$$\mathbf{v}_F\mathbf{k}\check{G} - i\omega_n\left(\check{\tau}_3\check{G} - \check{G}\check{\tau}_3\right) + [(\check{H} - \check{\Sigma})\check{G}] - [\check{G}(\check{H} - \check{\Sigma})] = 0. \tag{5.61}$$

The term with ξ_p drops out, being proportional to the unit matrix which commutes with any matrix. We can now integrate this equation over $d\xi_p$ near the Fermi surface to obtain

$$\mathbf{v}_F\mathbf{k}\check{g} - i\omega_n\left(\check{\tau}_3\check{g} - \check{g}\check{\tau}_3\right) + [\check{H}\check{g}] - [\check{g}\check{H}] = \check{I}. \tag{5.62}$$

With the quasiclassical accuracy, we can put everywhere $|\mathbf{p}| = p_F$ so that $\check{g} \equiv \check{g}_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k})$ and $\check{H} \equiv \check{H}(\hat{\mathbf{p}}, \mathbf{k})$. The product $[\check{H}\check{g}]$ is a usual product in the Fourier representation:

$$[\check{H}\check{g}] = \int \check{H}\left(\hat{\mathbf{p}}, \mathbf{k}_1\right)\check{g}\left(\hat{\mathbf{p}}, \mathbf{k} - \mathbf{k}_1\right)\frac{d^3k_1}{(2\pi)^3}$$

The "collision integral" is

$$\check{I} = [\check{\Sigma}\check{g}] - [\check{g}\check{\Sigma}]. \tag{5.63}$$

It is the matrix in the Nambu space with the components

$$\check{I} = \begin{pmatrix} I_1 & I_2 \\ -I_2^\dagger & \bar{I}_1 \end{pmatrix}.$$

The equation for the real-frequency Green functions can be obtained by continuation of all Green functions in eqn (5.62) onto the real frequency axis from
the positive or from the negative imaginary axis for retarder or advanced Green functions, respectively. We have

$$\mathbf{v}_F \mathbf{k} \check{g}^{R(A)} - \epsilon \left(\check{\tau}_3 \check{g}^{R(A)} - \check{g}^{R(A)} \check{\tau}_3 \right) + \left[\check{H} \check{g}^{R(A)} \right] - \left[\check{g}^{R(A)} \check{H} \right] = \check{I}^{R(A)} \tag{5.64}$$

where

$$\check{I}^{R(A)} = \left[\check{\Sigma}^{R(A)} \check{g}^{R(A)} \right] - \left[\check{g}^{R(A)} \check{\Sigma}^{R(A)} \right] \tag{5.65}$$

Equations (5.62) or (5.64) are the famous Eilenberger equation (Eilenberger 1968) for quasiclassical Green functions. Note that the scalar potential $e\varphi$ disappears from the Eilenberger equations in the stationary case because it enters $\check{H}$ with the unit matrix and commutes with any matrix $\check{g}$.

5.5.1 Self energy

The self energies can also be presented in terms of the $\xi_{\mathbf{p}}$-integrated functions. For example, the impurity self energy eqn (4.10) is

$$\check{\Sigma}(\mathbf{p}, \mathbf{k}) = n_{\text{imp}} \int \frac{d^3 p'}{(2\pi)^3} |u(\theta)|^2 \check{G} \left(\mathbf{p}'_+, \mathbf{p}'_- \right). \tag{5.66}$$

For isotropic scattering,

$$\check{\Sigma} = \frac{1}{2\pi\nu(0)\tau_{\text{imp}}} \int \check{G} \left(\mathbf{p}_+, \mathbf{p}_- \right) \frac{d^3 p}{(2\pi)^3} \tag{5.67}$$

The momentum integral can be written as

$$\int \check{G}_{\omega_n} \left(\mathbf{p}_+, \mathbf{p}_- \right) \frac{d^3 p}{(2\pi)^3} = \nu(0)\pi i \int \frac{d\Omega_{\mathbf{p}}}{4\pi} \check{g}_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) + \check{1}\mathcal{P} \int \frac{p^2 dp}{2\pi^2} \frac{1}{\xi_{\mathbf{p}}}$$

The last term gives equal positive contributions to both Σ_1 and $\bar{\Sigma}_1$; being proportional to the unit matrix, it can thus be incorporated into the chemical potential. It then drops out of the Eilenberger equation. One can write finally

$$\check{\Sigma}_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) = \pi i \nu(0) n_{\text{imp}} \int \frac{d\Omega_{\mathbf{p}_1}}{4\pi} \left| u \left(\hat{\mathbf{p}} - \hat{\mathbf{p}}_1 \right) \right|^2 \check{g}_{\omega_n} \left(\hat{\mathbf{p}}_1, \mathbf{k} \right) \tag{5.68}$$

or

$$\check{\Sigma}_{\epsilon}^{R(A)}(\hat{\mathbf{p}}, \mathbf{k}) = \pi i \nu(0) n_{\text{imp}} \int \frac{d\Omega_{\mathbf{p_1}}}{4\pi} \left| u \left(\hat{\mathbf{p}} - \hat{\mathbf{p}}_1 \right) \right|^2 \check{g}_{\epsilon}^{R(A)} \left(\hat{\mathbf{p}}_1, \mathbf{k} \right) \tag{5.69}$$

For isotropic scattering, we obtain

$$\check{\Sigma}_{\omega_n} = \frac{i}{2\tau_{\text{imp}}} \int \frac{d\Omega_{\mathbf{p}}}{4\pi} \check{g}_{\omega_n}(\hat{\mathbf{p}}, \mathbf{k}) = \frac{i}{2\tau_{\text{imp}}} \left\langle \check{g}_{\omega_n} \right\rangle \tag{5.70}$$

or

$$\check{\Sigma}_{\epsilon}^{R(A)} = \frac{i}{2\tau_{\text{imp}}} \int \frac{d\Omega_{\mathbf{p}}}{4\pi} \check{g}_{\epsilon}^{R(A)}(\hat{\mathbf{p}}, \mathbf{k}) = \frac{i}{2\tau_{\text{imp}}} \left\langle \check{g}_{\epsilon}^{R(A)} \right\rangle \tag{5.71}$$

We introduce the average over the Fermi surface

$$\langle A(\mathbf{p})\rangle = \int A(\mathbf{p})\frac{d\Omega_{\mathbf{p}}}{4\pi} \tag{5.72}$$

or, more generally,

$$\langle A(\mathbf{p})\rangle = \frac{1}{\nu(0)}\int A(\mathbf{p})\frac{dS_F}{(2\pi)^3 v_F} \tag{5.73}$$

for a non-spherical Fermi surface. The quantity $\nu(0)$ here is the integrated density of states at the Fermi surface

$$\nu(0) = \int \frac{dS_F}{(2\pi)^3 v_F}$$

For magnetic impurities one has from eqns (4.14) and (4.15)

$$\Sigma_1(\hat{\mathbf{p}},\mathbf{k}) = \pi i\nu(0)n_{\text{imp}}\int \frac{d\Omega_{\mathbf{p}_1}}{4\pi}\left|u_1\left(\hat{\mathbf{p}}-\hat{\mathbf{p}}_1\right)\right|^2 g_{\omega_n}\left(\hat{\mathbf{p}}_1,\mathbf{k}\right) \tag{5.74}$$

$$\Sigma_2(\hat{\mathbf{p}},\mathbf{k}) = \pi i\nu(0)n_{\text{imp}}\int \frac{d\Omega_{\mathbf{p}_1}}{4\pi}\left|u_2\left(\hat{\mathbf{p}}-\hat{\mathbf{p}}_1\right)\right|^2 f_{\omega_n}\left(\hat{\mathbf{p}}_1,\mathbf{k}\right) \tag{5.75}$$

The scattering amplitudes can be parametrized by two different scattering cross sections σ_1 and σ_2 associated with the diagonal and off-diagonal components (i.e., with either the functions g and $\bar{g}$ or the functions f and $f^\dagger$) of the matrix Green function $\check{g}$, respectively. As in eqns (5.74) and (5.75),

$$\Sigma_1\left(\mathbf{p}_F,\mathbf{k}\right) = \frac{i}{2}v_F n_{\text{imp}}\int d\Omega_{\mathbf{p}'}\sigma_1(\theta)g\left(\mathbf{p}'_F,\mathbf{k}\right)$$

$$\Sigma_2\left(\mathbf{p}_F,\mathbf{k}\right) = \frac{i}{2}v_F n_{\text{imp}}\int d\Omega_{\mathbf{p}'}\sigma_2(\theta)f\left(\mathbf{p}'_F,\mathbf{k}\right)$$

One can define the scattering times according to eqns (4.16), (4.17) together with the spin-flip time eqn (4.18). Let us introduce the cross sections

$$\sigma_{\mathbf{pp}'} = \frac{1}{2}\left[\sigma_1(\theta)+\sigma_{\mathbf{2}}(\theta)\right], \sigma^{(s)}_{\mathbf{pp}'} = \frac{1}{2}\left[\sigma_1(\theta)-\sigma_{\mathbf{2}}(\theta)\right]$$

The spin-flip scattering cross section is positive according to eqn (4.19) (Abrikosov and Gor'kov 1960), however, it is usually much smaller than the non-magnetic cross section $\sigma^{(s)}_{\mathbf{pp}'} \ll \sigma_{\mathbf{pp}'}$. We can write

$$\check{\Sigma}\left(\mathbf{p}_F,\mathbf{k}\right) = \frac{i}{2}v_F n_{\text{imp}}\int d\Omega_{\mathbf{p}'}\sigma_{\mathbf{pp}'}\check{g}\left(\mathbf{p}'_F,\mathbf{k}\right) + \frac{i}{2}v_F n_{\text{imp}}\int d\Omega_{\mathbf{p}'}\sigma^{(s)}_{\mathbf{pp}'}\check{\tau}_3\check{g}\left(\mathbf{p}'_F,\mathbf{k}\right)\check{\tau}_3 \tag{5.76}$$

The first term describes the scattering without changes in the spin state of electrons and coincides with the scattering by non-magnetic impurities while the
second term refers to the scattering accompanied by a spin-flip process. We shall attribute the notation $\check{\Sigma}_{\text{imp}}$ only to the non-magnetic part of the self energy, i.e., to the term in the first line of eqn (5.76), and denote the spin-flip self energy in the second line of eqn (5.76) by $\check{\Sigma}_s$.

The impurity collision integral for non-magnetic scattering eqn (5.63) or (5.65) is expressed through non-magnetic self energies defined by eqn (5.76). The collision integral vanishes after averaging over the momentum directions:

$$\left\langle \check{I}_{\text{imp}}(\mathbf{p}) \right\rangle = \left\langle \check{I}^{R(A)}_{\text{imp}}(\mathbf{p}) \right\rangle = 0. \tag{5.77}$$

This is because

$$\left\langle \left\langle \check{g}\left(\hat{\mathbf{p}}'\right) \sigma_{\mathbf{p}\mathbf{p}'} \right\rangle_{\mathbf{p}'} \check{g}(\hat{\mathbf{p}}) \right\rangle_{\mathbf{p}} = \left\langle \check{g}(\hat{\mathbf{p}}) \left\langle \sigma_{\mathbf{p}\mathbf{p}'} \check{g}\left(\hat{\mathbf{p}}'\right) \right\rangle_{\mathbf{p}'} \right\rangle_{\mathbf{p}} = \iint \frac{d\Omega_{\mathbf{p}} d\Omega_{\mathbf{p}'}}{(4\pi)^2} \check{g}(\hat{\mathbf{p}}) \sigma_{\mathbf{p}\mathbf{p}'} \check{g}\left(\hat{\mathbf{p}}'\right) \tag{5.78}$$

Equation (5.77) does not hold, however, for the spin-flip scattering integral:

$$\left\langle \check{I}_s(\mathbf{p}) \right\rangle \neq 0 \tag{5.79}$$

because the matrices $\check{g}$ and $\check{\tau}_3 \check{g} \check{\tau}_3$ do not commute.
Another useful relation is

$$\left\langle \sigma_{\mathbf{p}\mathbf{p}'} \right\rangle_{\mathbf{p}'} \hat{\mathbf{p}} - \left\langle \sigma_{\mathbf{p}\mathbf{p}'} \hat{\mathbf{p}}' \right\rangle_{\mathbf{p}'} = \left\langle \left[1 - \left(\hat{\mathbf{p}}\hat{\mathbf{p}}'\right)\right] \sigma_{\mathbf{p}\mathbf{p}'} \right\rangle_{\mathbf{p}'} \hat{\mathbf{p}} \equiv \sigma_{\text{tr}} \hat{\mathbf{p}} \tag{5.80}$$

where

$$\sigma_{\text{tr}} = \frac{1}{2} \int_0^{\pi} (1 - \cos\theta) \sigma(\theta) \sin\theta d\theta$$

is the transport cross section. To prove it we note that $\left\langle \hat{\mathbf{p}}' \sigma_{\mathbf{p}\mathbf{p}'} \right\rangle_{\mathbf{p}'}$ should be directed along $\hat{\mathbf{p}} : \left\langle \hat{\mathbf{p}}' \sigma_{\mathbf{p}\mathbf{p}'} \right\rangle_{\mathbf{p}'} = a\hat{\mathbf{p}}$. Multiplying this by $\hat{\mathbf{p}}$, we find $a = \left\langle \left(\hat{\mathbf{p}}\hat{\mathbf{p}}'\right) \sigma_{\mathbf{p}\mathbf{p}'} \right\rangle_{\mathbf{p}'}$ which proves eqn (5.80). We shall also use the transport mean free time defined as

$$\frac{1}{\tau_{\text{tr}}} = 4\pi \sigma_{\text{tr}} v_F n_{\text{imp}} \tag{5.81}$$

We have considered the impurity scattering in detail here. The phonon self energy can also be expressed through the quasiclassical Green functions. We shall discuss this later in Section 8.2.

5.5.2 Normalization

In the mixed Fourier-coordinate representation of eqn (5.15), the Eilenberger equation for $\check{g}(\hat{\mathbf{p}}, \mathbf{r})$ becomes

$$-i\mathbf{v}_F \nabla \check{g} - i\omega_n \left(\check{\tau}_3 \check{g} - \check{g} \check{\tau}_3\right) + \check{H}\check{g} - \check{g}\check{H} = \check{I}. \tag{5.82}$$

Here the product $\check{H}\check{g}$ is just the usual product of two matrices. All the functions here are functions of $\hat{\mathbf{p}}$ and $\mathbf{r}$. The matrix Eilenberger equation can be written in components. We have

$$-i\mathbf{v}_F \nabla g + \Delta f^{\dagger} - f\Delta^* = I_1$$

$$-i\mathbf{v}_F \nabla \bar{g} + \Delta^* f - f^{\dagger}\Delta = \bar{I}_1$$

$$-i\mathbf{v}_F \left(\nabla - \frac{2ie}{c}\mathbf{A}\right) f - 2i\omega_n f - \Delta\bar{g} + g\Delta = I_2$$

$$i\mathbf{v}_F \left(\nabla + \frac{2ie}{c}\mathbf{A}\right) f^{\dagger} - 2i\omega_n f^{\dagger} + \Delta^* g - \bar{g}\Delta^* = I_2^{\dagger}$$

We observe in particular that the Eilenberger equations together with the quasiclassical functions for a stationary superconductor are invariant under the gauge transformation of eqn (1.7).

From the first two equations we see immediately that g and $\bar{g}$ are not independent:

$$g + \bar{g} = \text{ const}$$

To find the constant we assume that our superconducting state transforms gradually into a homogeneous state without impurities at large enough distances along the direction of $\mathbf{v}_F$ from the region which is under consideration for a given problem. Moreover, we assume that magnetic field vanished at these distances. In a homogeneous case, the constant is zero according to eqn (5.37); this implies that $g + \bar{g} = 0$ everywhere along $\mathbf{v}_F$, i.e., in any inhomogeneous situation, as well. Using this condition, we see that there are only three independent functions, i.e., g, f, and $f^\dagger$ which satisfy three equations

$$-i\mathbf{v}_F \nabla g + \Delta f^\dagger - f\Delta^* = I_1 \tag{5.83}$$

$$-i\mathbf{v}_F \left(\nabla - \frac{2ie}{c}\mathbf{A}\right) f - 2i\omega_n f + 2g\Delta = I_2 \tag{5.84}$$

$$i\mathbf{v}_F \left(\nabla + \frac{2ie}{c}\mathbf{A}\right) f^\dagger - 2i\omega_n f^\dagger + 2\Delta^* g = I_2^\dagger \tag{5.85}$$

However, they still are not completely independent due to the normalization condition of eqn (5.38). It was obtained for a homogeneous superconductor. To prove it for a general case, we multiply the matrix Eilenberger equation eqn (5.82) by $\check{g}$ first from the left and then from the right, and add the two thus obtained equations. We have

$$-i\mathbf{v}_F \nabla(\check{g}\check{g}) - i\omega_n \left[\check{\tau}_3(\check{g}\check{g}) - (\check{g}\check{g})\check{\tau}_3\right] + (\check{H} - \check{\Sigma})(\check{g}\check{g}) - (\check{g}\check{g})(\check{H} - \check{\Sigma}) = 0.$$

The solution of this equation has the form

$$(\check{g}\check{g}) = A\check{1} + B\check{g} \tag{5.86}$$

where A and B are constants. At large distances, i.e., in a homogeneous state without impurities, we have $A = 1$ and $B = 0$. Therefore, they should have the same values also for a non-homogeneous case. This proves eqn (5.39) together with eqns (5.37) and (5.38).

Equations (5.83-5.85) can be written in a matrix form. Let us introduce the matrices

$$\hat{\nabla}\check{g} = \begin{pmatrix} \nabla g & \left(\nabla - \frac{2ie}{c}\mathbf{A}\right) f \\ -\left(\nabla + \frac{2ie}{c}\mathbf{A}\right) f^\dagger & -\nabla g \end{pmatrix}, \check{H}_0 = \begin{pmatrix} -i\omega_n - \Delta \\ \Delta^* i\omega_n \end{pmatrix}. \tag{5.87}$$

The Eilenberger equations take the form

$$-i\mathbf{v}_F \hat{\nabla}\check{g} + \check{H}_0\check{g} - \check{g}\check{H}_0 = \check{I}. \tag{5.88}$$

The Eilenberger equations for the real-time retarded and advanced functions are obtained from eqns (5.83) $-$(5.85) by substitution $i\omega_n = \epsilon$:

$$-i\mathbf{v}_F \nabla g^{R(A)} + \Delta f^{\dagger R(A)} - f^{R(A)}\Delta^* = I_1^{R(A)} \tag{5.89}$$

$$-i\mathbf{v}_F \left(\nabla - \frac{2ie}{c}\mathbf{A}\right) f^{R(A)} - 2\epsilon f^{R(A)} + 2g^{R(A)}\Delta = I_2^{R(A)} \tag{5.90}$$

$$i\mathbf{v}_F \left(\nabla + \frac{2ie}{c}\mathbf{A}\right) f^{\dagger R(A)} - 2\epsilon f^{\dagger R(A)} + 2\Delta^* g^{R(A)} = I_2^{\dagger R(A)} \tag{5.91}$$

Retarded and advanced Green functions also satisfy the normalization condition (5.39). Therefore, only two of the equations (5.89), (5.90) and (5.91) are independent.

17.1.6 5.6 Dirty limit. Usadel equations

The Eilenberger equations can be simplified considerably in the dirty limit (Usadel 1970), when the impurity scattering rate or the mean free path ℓ satisfies the condition

$$1/\tau \gg T_c, \text{ i.e., } \ell \ll \xi_0$$

This limit applies only for s-wave superconductors where Δ is independent of the momentum direction.

A strong scattering by impurities produces averaging over momentum directions. Therefore, in the first approximation, the Green function does not depend on the momentum directions. The first-order correction is linear in $\mathbf{v}_F$. We can thus write

$$\check{g} = g_0 + \hat{\mathbf{v}}_F \check{\mathbf{g}} \tag{5.92}$$

where $\check{g}_0$ does not depend on directions of $\mathbf{v}_F$, and $|\mathbf{g}| \ll g_0; \hat{\mathbf{v}}_F$ is the unit vector in the direction of momentum. Therefore

$$\langle \check{g} \rangle = \check{g}_0$$

The normalization condition gives

$$\check{g}_0 \check{g}_0 = \check{1}, \quad \check{g}_0 \check{\mathbf{g}} + \check{\mathbf{g}} \check{g}_0 = 0. \tag{5.93}$$

Consider a superconductor without magnetic impurities. The non-magnetic collision integral is determined by eqn (5.63). As we know, it vanishes after
averaging over the momentum directions, eqn (5.77). Let us average eqn (5.88) over directions of $\mathbf{v}_F$. We obtain

$$-i\frac{v_F}{3}\hat{\nabla}\check{\mathbf{g}} + \check{H}_0 \check{g}_0 - \check{g}_0 \check{H}_0 = 0. \tag{5.94}$$

We now multiply eqn (5.88) by $\hat{\mathbf{v}}_F$ and average it over momenta. We have

$$-iv_F \hat{\nabla}\check{g}_0 + \left(\check{H}_0 \check{\mathbf{g}} - \check{\mathbf{g}} \check{H}_0\right) - \frac{i}{2\tau_{\text{tr}}}\left(\check{g}_0 \check{\mathbf{g}} - \check{\mathbf{g}} \check{g}_0\right) = 0. \tag{5.95}$$

We use here eqns (5.80) and (5.81). In the dirty limit, we can neglect the terms with $\check{H}_0$. Multiplying this equation by $\check{g}_0$ and using the normalization condition eqn (5.93), we get

$$\check{\mathbf{g}} = -\ell_{\text{tr}} \check{g}_0 \hat{\nabla} \check{g}_0 \tag{5.96}$$

where $\ell_{\text{tr}} = v_F \tau_{\text{tr}}$.
Equation for the averaged functions becomes

$$iD\hat{\nabla}\left(\check{g}_0 \hat{\nabla} \check{g}_0\right) + \check{H}_0 \check{g}_0 - \check{g}_0 \check{H}_0 = 0 \tag{5.97}$$

where $D = v_F \ell_{\text{tr}}/3$ is the diffusion constant. In components, eqn (5.97) has the form

$$-iD\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\left[g_0\left(\nabla - \frac{2ie}{c}\mathbf{A}\right) f_0 - f_0 \nabla g_0\right] = 2\Delta g_0 - 2i\omega_n f_0 \tag{5.98}$$

and

$$-iD\left(\nabla + \frac{2ie}{c}\mathbf{A}\right)\left[g_0\left(\nabla + \frac{2ie}{c}\mathbf{A}\right) f_0^\dagger - f_0^\dagger \nabla g_0\right] = 2\Delta^* g_0 - 2i\omega_n f_0^\dagger \tag{5.99}$$

These equations are known as the Usadel equations.
We can easily get the expression for the current from eqn (5.96):

$$\begin{aligned}\mathbf{j} &= -\frac{e\nu(0)v_F^2}{3}\pi iT\sum_n \mathrm{Tr}\left(\tau_3\check{\mathbf{g}}\right) = \frac{\sigma_n}{e}\pi iT\sum_{n\geq 0}\mathrm{Tr}\left(\tau_3\check{g}_0\hat{\nabla}\check{g}_0\right)\\ &= \frac{\sigma_n}{e}\pi iT\sum_{n\geq 0}\left[f_0^\dagger\left(\nabla-\frac{2ie}{c}\mathbf{A}\right)f_0 - f_0\left(\nabla+\frac{2ie}{c}\mathbf{A}\right)f_0^\dagger\right]\end{aligned} \tag{5.100}$$

The normal-state Drude conductivity is

$$\sigma_n = 2\nu(0)De^2 \tag{5.101}$$

In a homogeneous case, $\Delta = \text{const}$, we have

$$\mathbf{j} = -\frac{i\sigma_n}{e}\left[\Delta^*\left(\nabla-\frac{2ie}{c}\mathbf{A}\right)\Delta - \Delta\left(\nabla+\frac{2ie}{c}\mathbf{A}\right)\Delta^*\right]\sum_{n\geq 0}\frac{\pi T}{\omega_n^2+|\Delta|^2} \tag{5.102}$$

17.1.7 5.7 Boundary conditions

To solve the Eilenberger equations one needs some boundary conditions. They are imposed either at the surfaces of the superconductor or in the bulk, i.e., at large distances from those regions where the spatial dependence of the Green functions is non-trivial. The boundary conditions in the bulk are more or less obvious: one has to match the obtained solution with the Green functions for a homogeneous situation. The boundary conditions at the surfaces depend on the characteristics of the particular surface and vary from one surface to another. In general, boundary conditions can be formulated in terms of the scattering T matrix (Serene and Rainer 1983). However, in their general form, these conditions are rather complicated. We discuss briefly few simple examples of quasiclassical conditions at various boundaries.

The boundary conditions for dirty superconductors reduce to the boundary conditions for diffusion-like Usadel equations (5.97). They are usually formulated in terms of continuity of quasiparticle currents. For example, at the interface between the superconductor and vacuum (non conducting medium), the current through the surface should vanish. According to eqn (5.100) this requires

$$\mathbf{n}\cdot\left(\nabla-\frac{2ie}{c}\mathbf{A}\right)f_0 = \mathbf{n}\cdot\left(\nabla+\frac{2ie}{c}\mathbf{A}\right)f_0^\dagger = 0$$

at the boundary. Here $\mathbf{n}$ is the unit vector normal to the surface.
For clean superconductors, the boundary conditions are more complicated. The whole set of conditions at interfaces between superconductors and metals or other conducting media was derived by Zaitsev (1984).

In this Section, we discuss the boundary conditions which can be imposed for a clean superconductor at the boundary with vacuum. We consider two kinds of surfaces. First is a completely smooth surface which is characterized by specular reflection of incident quasiparticles. Specular reflections require that the Green function of the reflected particle taken at the surface $\mathbf{r} = \mathbf{r}_w$ is equal to that before the reflection:

$$\check{g}\left(\mathbf{p}_\parallel, \mathbf{p}_\perp; \mathbf{r}_w\right) = \check{g}\left(\mathbf{p}_\parallel, -\mathbf{p}_\perp; \mathbf{r}_w\right) \tag{5.103}$$

where $\mathbf{p}_\parallel$and $\mathbf{p}_\perp$ are the parallel and perpendicular to the surface components of the particle momentum. Equation (5.103) automatically results in vanishing of the current through the surface. Another example is a diffusive surface.

5.7.1 Diffusive surface

Diffusive surface reflects the incident particles randomly. The conditions at a diffusive surface are much more complicated (Serene and Rainer 1983). First of all, they strongly depend on the particular characteristic of the surface, for example, on the degree of diffusivity. In certain cases, a diffusively scattering boundary with an non-conducting medium (vacuum) is modeled by an impurity layer which covers the surface. The thickness of the layer and the quasiparticle mean-free path inside the layer are assumed to be much shorter than the coherence length ξ_0. The specular boundary conditions are applied at the outer side of the layer with respect to the superconductor (superfluid). To describe various degrees of diffusivity, the ratio of the layer thickness to the mean-free path, $\rho = d/l$, varies from zero (specular wall) to infinity (diffusive wall). The equations for the Green functions ought to be solved first inside the impurity layer, and then the solutions should have to be matched to those in the bulk.

In this section we describe another approach where the limit $\rho \to \infty$ is accomplished analytically for a completely diffusive surface using the method developed by Ovchinnikov (1969). It gives the boundary conditions for the Green functions which are to be applied directly at the boundary of the superconductor.

For a derivation of the boundary conditions it is useful to expand the quasiclassical Green functions

$$\check{g} = \begin{pmatrix} g & f \\ -f^\dagger & -g \end{pmatrix} \tag{5.104}$$

into the three Pauli matrices in the Nambu space

$$\check{\tau}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \check{\tau}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \check{\tau}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5.105}$$

which constitute the vector $\check{\tau}$. We can write

$$\check{g} = \mathbf{n} \cdot \check{\tau} \tag{5.106}$$

where $\mathbf{n}$ are the three expansion coefficients constituting a vector. Generally, the components n_α are functions of the coordinates and of the directions of the quasiparticle momenta, $\hat{\mathbf{p}}$.

The vector n_α can be expanded into three constant basis vectors $\mathbf{n}(N)$, where $N = 1, 2, 3$, which satisfy the orthogonality conditions

$$[\mathbf{n}\,(N_1) \cdot \check{\tau}, \mathbf{n}\,(N_2) \cdot \check{\tau}]_+ = 2\delta_{N_1 N_2} \check{1} \tag{5.107}$$

where $[,]_\pm$is an anticommutator (commutator).
One can easily check that, for any two non-equal vectors $\mathbf{n}\,(N_1)$ and $\mathbf{n}\,(N_2)$, the commutator [$\mathbf{n}\,(N_1) \cdot \check{\tau}, \mathbf{n}\,(N_2) \cdot \check{\tau}$]_ is proportional to the product of $\check{\tau}$ and the third $\mathbf{n}$-vector:

$$[\mathbf{n}\,(N_1) \cdot \check{\tau}, \mathbf{n}\,(N_2) \cdot \check{\tau}]_- = 2i\mathbf{n}\,(N_3) \cdot \check{\tau} \tag{5.108}$$

Note that the numbers N_1, N_2, and N_3 constitute an even permutation of 1,2 , and 3.
We can now write the Green function as follows:

$$\check{g} = \sum_N C_N \cdot \mathbf{n}(N) \cdot \check{\tau} \tag{5.109}$$

Above, the coefficients C_N are functions of $\mathbf{r}$ and $\mathbf{p}$. Due to the normalization

$$\check{g}^2 = \check{1}$$

the coefficients C_N satisfy the condition

$$\sum_N C_N^2 = 1. \tag{5.110}$$

The function $\check{g}$ in eqns (5.109) can be either retarded (advanced) or Matsubara Green function.

The quasiparticle self-energy due to impurity scattering inside the layer is

$$\check{\Sigma} = \frac{icv_F}{2} \int \sigma_{\mathbf{pp}'} \check{g}\left(\mathbf{p}', \mathbf{r}\right) d\Omega_{\mathbf{p}'} \tag{5.111}$$

where c is the impurity concentration in the layer, and $\sigma_{\mathbf{pp}'}$ is the scattering cross-section. The mean-free time is

$$\frac{1}{\tau_{imp}} = cv_F \int \sigma_{\mathbf{pp}'} d\Omega_{\mathbf{p}'} \tag{5.112}$$

The Eilenberger equation for the Green function inside the layer has the form

$$-i\mathbf{v}_F \cdot \nabla \check{g}(\mathbf{p}, \mathbf{r}) = \check{\Sigma}\check{g}(\mathbf{p}, \mathbf{r}) - \check{g}(\mathbf{p}, \mathbf{r})\check{\Sigma}. \tag{5.113}$$

From the definition of the impurity self-energy one has, identically,

$$\int [\check{\Sigma}\check{g}(\mathbf{p}, \mathbf{r}) - \check{g}(\mathbf{p}, \mathbf{r})\check{\Sigma}] d\Omega_{\mathbf{p}'} \equiv 0 \tag{5.114}$$

Deep in the boundary layer, the Green function depends neither on the distance form the surface of the layer nor on the momentum direction due to isotropization in the course of the scattering. Therefore, it can be presented in the form $\check{g} = \mathbf{n}(1) \cdot \check{\tau}$ with the vector $\mathbf{n}(1)$ independent of $\mathbf{p}$ and of the distance from the surface of the layer. However, the vector $\mathbf{n}(1)$ may have a smooth dependence on the position at the layer surface, $\mathbf{r}_w$, if the boundary is not planar. We choose this vector $\mathbf{n}(1)$ as one of the basis vectors for the $\check{g}$-expansion.

One can present the Green function near the layer surface in the form

$$\check{g} = \mathbf{n}(1) \cdot \check{\tau} + C_2 \mathbf{n}(2) \cdot \check{\tau} + iC_3 \mathbf{n}(3) \cdot \check{\tau}, \tag{5.115}$$

where the vectors $\mathbf{n}(2)$ and $\mathbf{n}(3)$ are the other two constant basis vectors. Due to the normalization condition of eqn (4) we have

$$C_2^2 - C_3^2 = 0. \tag{5.116}$$

The coefficients C_2 and C_3 now contain all the information on the coordinate and momentum dependencies of the Green function.

The self-energy becomes in the boundary layer, according to eqn (5.115),

$$\check{\Sigma} = \frac{i}{2\tau_{imp}} \mathbf{n}(1) \cdot \check{\tau} + \frac{icv_F}{2} \left[\mathbf{n}(2) \cdot \check{\tau} \int \sigma_{\mathbf{pp}'} C_2\left(\mathbf{p}'\right) d\Omega_{\mathbf{p}'} + i\mathbf{n}(3) \cdot \check{\tau} \int \sigma_{\mathbf{pp}'} C_3\left(\mathbf{p}'\right) d\Omega_{\mathbf{p}'} \right] \tag{5.117}$$

Using eqn (5.117) and the commutation rules of eqns (5.107,5.108), one gets

$$\begin{aligned} [\check{\Sigma}, \check{g}]_- = & \frac{i}{\tau_{imp}} \left[iC_2 \mathbf{n}(3) \cdot \check{\tau} + C_3 \mathbf{n}(2) \cdot \check{\tau} \right] \\ & + \left[cv_F \int \sigma_{\mathbf{pp}'} C_2\left(\mathbf{p}'\right) d\Omega_{\mathbf{p}'} \right] \mathbf{n}(3) \cdot \check{\tau} \\ & - \left[icv_F \int \sigma_{\mathbf{pp}'} C_3\left(\mathbf{p}'\right) d\Omega_{\mathbf{p}'} \right] \mathbf{n}(2) \cdot \check{\tau} \end{aligned} \tag{5.118}$$

This results in the equations

$$(v_F)_z \frac{\partial}{\partial z}\left[C_2(\mathbf{p}') \pm C_3(\mathbf{p}')\right] = \mp \frac{1}{\tau_{imp}}\left[C_2(\mathbf{p}') \pm C_3(\mathbf{p}')\right] \pm c v_F \int \sigma_{\mathbf{p}\mathbf{p}'}\left[C_2(\mathbf{p}') \pm C_3(\mathbf{p}')\right] d\Omega_{\mathbf{p}'}$$

for the coefficients C_2 and C_3 in the layer. Here z is measured along the layersurface normal directed inwards the superconductor so that $z < 0$ corresponds to a position inside the layer.

Like it was done by Ovchinnikov (1969), we assume that the scattering crosssection $\sigma_{\mathbf{p}\mathbf{p}'}$ is nonzero only for $p_z p_z' > 0$. Using this assumption one can eliminate the solutions which increase exponentially as $z \to -\infty$, by requiring

$$(C_2 + C_3)_{p_z>0} = 0; (C_2 - C_3)_{p_z<0} = 0 \tag{5.119}$$

at the surface of the layer.
There are two more conditions for the coefficients $C_{2,3}$ which follow from eqns (5.114,5.119) upon integration over the momentum directions:

$$\int_{p_z<0} |p_z| (C_2 + C_3)\, d\Omega_{\mathbf{p}'} = 0, \quad \int_{p_z>0} p_z (C_2 - C_3)\, d\Omega_{\mathbf{p}'} = 0 \tag{5.120}$$

Equations (5.119) and (5.120) ensure that there is no current through the interface.
Using eqns (5.115), (5.119), and (5.120) one obtains for the basis vector

$$\mathbf{n}(1) \cdot \check{\tau} = \frac{1}{\pi}\int_{p_z>0} \hat{p}_z \check{g}(\mathbf{p}, \mathbf{r}_w)\, d\Omega_{\mathbf{p}'} = \frac{1}{\pi}\int_{p_z<0} |\hat{p}_z|\, \check{g}(\mathbf{p}, \mathbf{r}_w)\, d\Omega_{\mathbf{p}'} \equiv \langle \hat{p}_z \check{g} \rangle_+ \tag{5.121}$$

where $\hat{\mathbf{p}}$ is the unit vector in the momentum direction. Since

$$C_2 \mathbf{n}(2) \cdot \check{\tau} + i C_3 \mathbf{n}(3) \cdot \check{\tau} = \check{g} - \mathbf{n}(1) \cdot \check{\tau}, \tag{5.122}$$

and

$$C_3 \mathbf{n}(2) \cdot \check{\tau} + i C_2 \mathbf{n}(3) \cdot \check{\tau} = \frac{1}{2}[\mathbf{n}(1) \cdot \check{\tau}, \check{g}]_- \tag{5.123}$$

one can exclude the unknown vectors $\mathbf{n}(2)$ and $\mathbf{n}(3)$ using eqn (5.116):

$$C_2[\check{g} - \mathbf{n}(1) \cdot \check{\tau}] = \frac{C_3}{2}[\mathbf{n}(1) \cdot \check{\tau}, \check{g}]_-$$

Since the coefficients C_2 and C_3 are coupled by eqn (5.119): $C_2 = -C_3$ for $\hat{p}_z > 0$, and $C_2 = C_3$ for $\hat{p}_z < 0$, one obtains

$$2\left[\check{g} - \langle \hat{p}_z \check{g} \rangle_+\right] = -\operatorname{sign}(\hat{p}_z)\left[\langle \hat{p}_z \check{g} \rangle_+, \check{g}\right]_- \tag{5.124}$$

The conditions of eqn (5.124) can also be written directly for the functions g, f, and $f^\dagger$:

$$\begin{aligned} 2\left(g - \langle \hat{p}_z g \rangle_+\right) &= -\operatorname{sign}(\hat{p}_z)\left(f \langle \hat{p}_z f^\dagger \rangle_+ - f^\dagger \langle \hat{p}_z f \rangle_+\right) \\ f - \langle \hat{p}_z f \rangle_+ &= -\operatorname{sign}(\hat{p}_z)\left(\langle \hat{p}_z g \rangle_+ f - g \langle \hat{p}_z f \rangle_+\right) \\ f^\dagger - \langle \hat{p}_z f^\dagger \rangle_+ &= -\operatorname{sign}(\hat{p}_z)\left(g \langle \hat{p}_z f^\dagger \rangle_+ - \langle \hat{p}_z g \rangle_+ f^\dagger\right) \end{aligned} \tag{5.125}$$

These three equations determine the three functions g, f, and $f^\dagger$.
These conditions hold for both the real-frequency (retarded and advanced) and Matsubara Green functions. They are linear with respect to the functions for a given momentum direction. This fact may be very important for practical use. The coefficients, however, are the integrals of the functions over momentum directions, so the boundary conditions of eqns (5.125) are nonlinear in a strict sense.

17.2 QUASICLASSICAL METHODS IN STATIONARY PROBLEMS

(move it to the main note on SC)

We demonstrate the potentialities of the quasiclassical approach for selected problems in the theory of stationary superconductivity: We derive the Ginzburg-Landau equations, calculate the upper critical field of dirty superconductors at arbitrary temperatures, and discuss the gapless regime in superconductors with magnetic impurities. We consider effects of impurities on the density of states in d-wave superconductors. Finally, we find the energy spectrum of excitations in vortex cores of s-wave and d-wave superconductors.

17.2.1 6.1 s-wave superconductors with impurities

6.1.1 Small currents in a uniform state

In this Chapter we consider several useful applications of the quasiclassical methods to stationary properties of superconductors. We start with a homogeneous state of an s-wave superconductors. Without a magnetic field, the Green functions do not depend on coordinates and on the momentum direction. This fact tells us that

$$\check{\Sigma} = \frac{i}{2\tau}\check{g}$$

We denote the impurity scattering time simply by τ for brevity. Therefore, the self energies commute with the Green functions and drop out of the Eilenberger equations. As a result, the homogeneous state does not depend on impurity concentration. We know this already as the Anderson's theorem. This consideration clearly demonstrates how much simpler the quasiclassical method is as compared to the full Green function technique: without any calculations we have arrived to the conclusion which took two pages of algebra in Section 4.2. Eilenberger equations (5.84, 5.85) give

$$f = \frac{\Delta}{i\omega_n}g, f^\dagger = \frac{\Delta^*}{i\omega_n}g$$

Equation (5.83) vanishes. To find g and f we need to use the normalization condition eqn (5.38). We obtain

$$g = \frac{\omega_n}{\sqrt{\omega_n^2 + |\Delta|^2}}, \text{ and } f = \frac{\Delta}{i\sqrt{\omega_n^2 + |\Delta|^2}}$$

The sign was chosen by comparison with our previous result. For retarded and advanced functions we recover eqns (5.46), (5.47), (5.50), and (5.51).

Consider now a superconducting state where the magnetic field and currents are small, and the gradients are slow. The condition is

$$(2e/c)A \ll \xi_0^{-1} \text{ i.e., } H \ll \frac{c}{2e\xi^2} = H_{c2}$$

and $v_F k \ll T_c$. The latter means that variations are slow on the scale of the coherence length ξ_0. We also assume an isotropic scattering by impurities.

We solve the Eilenberger equations by perturbations. Let the Green function be

$$\check{g} = \check{g}_0 + \check{g}_1$$

where $\check{g}_0$ is the solution for a homogeneous state and $\check{g}_1$ is a small correction proportional to gradients and the vector potential. The first-order correction is proportional either to the dot product of ∇ and $\mathbf{v}_F$ or to the dot product of $\mathbf{A}$ and $\mathbf{v}_F$. It is thus linear in $\mathbf{v}_F$. We have therefore

$$\langle \check{g} \rangle = \check{g}_0$$

The normalization condition (5.38) gives

$$2g_0 g_1 = f_0 f_1^\dagger + f_0^\dagger f_1. \tag{6.1}$$

The corrections f_1 and $f_1^\dagger$ are found from eqns (5.84) and (5.85)

$$-i\mathbf{v}_F \left(\nabla - \frac{2ie}{c}\mathbf{A}\right) f_0 - 2i\omega_n f_1 + 2g_1\Delta - \frac{i}{\tau}\left(g_0 f_1 - g_1 f_0\right) = 0, \tag{6.2}$$

$$i\mathbf{v}_F \left(\nabla + \frac{2ie}{c}\mathbf{A}\right) f_0^\dagger - 2i\omega_n f_1^\dagger + 2\Delta^* g_1 - \frac{i}{\tau}\left(g_0 f_1^\dagger - g_1 f_0^\dagger\right) = 0. \tag{6.3}$$

We multiply the first equation by $f_0^\dagger$, the second equation by f_0 and add the two equations. Using the normalization condition and eqn (6.1), we obtain

$$\frac{i}{2}\mathbf{v}_F \left[f_0^\dagger \left(\nabla - \frac{2ie}{c}\mathbf{A}\right) f_0 - f_0 \left(\nabla + \frac{2ie}{c}\mathbf{A}\right) f_0^\dagger\right] = -\left[2i\omega_n g_0 - \left(\Delta f_0^\dagger + \Delta^* f_0\right) + \frac{i}{\tau}\right] g_1.$$

We find from this equation

$$g_1 = \frac{\mathbf{v}_F \left[\Delta^*(\nabla - (2ie/c)\mathbf{A})\Delta - \Delta(\nabla + (2ie/c)\mathbf{A})\Delta^*\right]}{4\left(\omega_n^2 + |\Delta|^2\right)\left[\sqrt{\omega_n^2 + |\Delta|^2} + (1/2\tau)\right]} \tag{6.4}$$

The current becomes

$$\mathbf{j} = -\frac{c}{4\pi\lambda_L^2}\left(\mathbf{A} - \frac{c}{2e}\nabla\chi\right) \tag{6.5}$$

where the London penetration length is found from

$$\lambda_L^{-2} = \frac{16\pi^2\nu(0)v_F^2 e^2|\Delta|^2 T}{3c^2}\sum_{n\geq 0}\frac{1}{\left(\omega_n^2 + |\Delta|^2\right)\left[\sqrt{\omega_n^2 + |\Delta|^2} + (1/2\tau)\right]} \tag{6.6}$$

Note that the order parameter does depend on the impurity concentration in presence of magnetic field or current and/or in presence of order-parameter gradients because the Green function now contains the mean free time τ.

The London penetration depth in the dirty case, $1/\tau \gg T_c$, is determined by

$$\lambda_L^{-2} = \frac{16\pi^2\sigma_n|\Delta|^2 T}{c^2}\sum_{n\geq 0}\frac{1}{\omega_n^2 + |\Delta|^2} \tag{6.7}$$

This sum can be evaluated using

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2 + x^2} = \frac{\pi}{4x} \tanh \frac{\pi x}{2} \tag{6.8}$$

We find

$$\lambda_L^{-2} = \frac{4\pi^2 \sigma_n}{c^2} |\Delta| \tanh \frac{|\Delta|}{2T}$$

λ_L is proportional to $1/\sqrt{\ell}$ and increases with a decrease in ℓ.
One can express the density of superconducting electrons through λ_L using eqn (1.25). In the clean limit, $1/\tau \ll T_c$, the superconducting density is

$$N_s = N|\Delta|^2 2\pi T \sum_{n\geq 0} \frac{1}{(\omega_n^2 + |\Delta|^2)^{3/2}} \tag{6.9}$$

In the dirty case we have

$$N_s = \frac{\pi m \sigma_n |\Delta|}{e^2} \tanh \frac{|\Delta|}{2T} \tag{6.10}$$

6.1.2 Ginzburg - Landau theory

The Ginzburg-Landau equations can be easily derived using the quasiclassical method. Let us assume that Δ and the magnetic field are small and spatial variations of all the quantities are slow. Under these conditions, we can expand the Eilenberger equations in small gradients and in Δ. Within the zero approximation in gradients, we have for an s-wave superconductor

$$\tilde{g}_0 = \frac{\omega_n}{\sqrt{\omega_n^2 + |\Delta|^2}}, \tag{6.11}$$

$$\tilde{f}_0 = \frac{\Delta}{i\sqrt{\omega_n^2 + |\Delta|^2}} \approx \frac{\Delta}{i\omega_n} - \frac{|\Delta|^2 \Delta}{2i\omega_n^3} \tag{6.12}$$

$$\tilde{f}_0^\dagger = \frac{\Delta^*}{i\sqrt{\omega_n^2 + |\Delta|^2}} \approx \frac{\Delta^*}{i\omega_n} - \frac{|\Delta|^2 \Delta^*}{2i\omega_n^3} \tag{6.13}$$

We need the function f up to the terms which are simultaneously of the third order in Δ and zero order in gradients, or of the second order in gradients and
of the first order in Δ. Within the first order in Δ, the function $g = 1$. Note that the function g contains only even powers of Δ. Therefore, a first-order correction in gradients for the function g is simultaneously of the second order in Δ.

The first-order corrections both in the gradient and in Δ are found from eqn (6.2)

$$f_1 = -\frac{\mathbf{v}_F \left(\nabla - \frac{2ie}{c}\mathbf{A}\right) f_0}{2\omega_n + 1/\tau}; \quad g_1 = 0 \tag{6.14}$$

The correction of the second-order in gradients and first order in Δ is found from

$$-i\mathbf{v}_F \left(\nabla - \frac{2ie}{c}\mathbf{A}\right) f_1 - 2i\omega_n f_2 + 2g_2\Delta - \frac{i}{\tau}\left(f_2 - \langle f_2 \rangle\right) = 0$$

Since g_2 is zero again, we get

$$f_2 = -\frac{\mathbf{v}_F(\nabla - 2ie\mathbf{A}/c) f_1}{2\omega_n + 1/\tau} + \frac{\langle f_2 \rangle}{(2\omega_n + 1/\tau)\,\tau} \tag{6.15}$$

Performing the angular averaging we solve the resulting equation for $\langle f_2 \rangle$ whence

$$\langle f_2 \rangle = \frac{v_F^2}{3} \frac{(\nabla - 2ie\mathbf{A}/c)^2 f_0}{(2\omega_n + 1/\tau)\, 2\omega_n} = -\frac{iD(\nabla - 2ie\mathbf{A}/c)^2 \Delta}{2\omega_n^2\,(2\omega_n\tau + 1)}$$

where we use

$$\int \frac{d\Omega_{\mathbf{p}}}{4\pi} v_i v_k = \frac{v^2}{3}\delta_{ik}$$

for a spherical Fermi surface. For the third order terms in Δ we do not need gradients. Therefore, the final expression for the averaged Green function is

$$\langle f \rangle = \frac{\Delta}{i\omega_n} - \frac{|\Delta|^2\Delta}{2i\omega_n^3} - \frac{iD(\nabla - 2ie\mathbf{A}/c)^2\Delta}{2\omega_n^2\,(2\omega_n\tau + 1)} \tag{6.16}$$

The gradient-independent terms do not contain τ in accordance with the Anderson theorem.

6.1.2.1 Order parameter

The self-consistency equation for the order parameter (5.27) gives

$$\frac{\Delta(\mathbf{r})}{\lambda} = 2\pi T \sum_{n=0}^{N_0(T)} \frac{\Delta(\mathbf{r})}{\omega_n} - \pi T|\Delta|^2\Delta \sum_{n=0}^{\infty} \frac{1}{\omega_n^3} + \pi T D \left(\nabla - \frac{2ie}{c}\mathbf{A}\right)^2 \Delta(\mathbf{r}) \sum_{n=0}^{\infty} \frac{1}{\omega_n^2\,(2\omega_n\tau + 1)} \tag{6.17}$$

In the first sum, we introduce the cut-off value as in eqn (3.81)

$$N_0(T) = \frac{\Omega_{\mathrm{BCS}}}{2\pi T} > \frac{\Omega_{\mathrm{BCS}}}{2\pi T_c} = N_0\,(T_c) \tag{6.18}$$

Therefore

$$\begin{aligned} 2\pi T \sum_{n=0}^{N_0(T)} \frac{1}{\omega_n} &= 2\pi T \sum_{n=0}^{N_0(T_c)} \frac{1}{\omega_n} + 2\pi T \sum_{N_0(T_c)}^{N_0(T)} \frac{1}{\omega_n} \\ &= \frac{1}{\lambda} + \sum_{N_0(T_c)}^{N_0(T)} \frac{1}{n + 1/2} = \frac{1}{\lambda} + \ln\left(\frac{T_c}{T}\right) \end{aligned} \tag{6.19}$$

where we use the expression for the critical temperature eqn (3.83) through the interaction constant. Next,

$$2\pi T \sum_{n=0}^{\infty} \frac{1}{\omega_n^3} = \frac{1}{4\pi^2 T^2} \sum_{n=0}^{\infty} \frac{1}{(n + 1/2)^3} = \frac{7\zeta(3)}{4\pi^2 T^2} \tag{6.20}$$

Here we use

$$\sum_{n=0}^{\infty} \frac{1}{(n + 1/2)^z} = (2^z - 1)\,\zeta(z)$$

where

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}$$

is the Riemann ζ function. The equation for the order parameter becomes

$$\Delta \ln\left(\frac{T_c}{T}\right) - \frac{7\zeta(3)}{8\pi^2 T^2}|\Delta|^2\Delta + \frac{\pi\tilde{D}}{8T_c}\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)^2 \Delta = 0 \tag{6.21}$$

where

$$\tilde{D} = D\Lambda\left(T_c\tau\right)$$

and

$$\Lambda(x) = \frac{8}{\pi^2}\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2[(2n+1)2\pi x + 1]} \tag{6.22}$$

The function

$$\Lambda\left(\tau T_c\right) = \begin{cases} 1 & \text{for } \tau T_c \ll 1, \\ 7\zeta(3)/2\pi^3\tau T_c & \text{for } \tau T_c \gg 1. \end{cases}$$

For clean superconductors $\tau T_c \gg 1$ the coefficient

$$\tilde{D} = \frac{7\zeta(3)v_F^2}{6\pi^3 T_c}$$

The condition of a small Δ is $\Delta \ll T$. This is satisfied for $T \to T_c$. Therefore, $\ln\left(T_c/T\right) \approx 1 - T/T_c$. We finally obtain

$$\left(1 - \frac{T}{T_c}\right)\Delta - \frac{7\zeta(3)}{8\pi^2 T_c^2}|\Delta|^2\Delta + \frac{\pi\tilde{D}}{8T_c}\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)^2 \Delta = 0. \tag{6.23}$$

Equation (6.23) is identical to the famous Ginzburg-Landau equation (1.12) for the order parameter. It was derived microscopically by Gor'kov (1959 a, 1959 c). Equation (6.23) determines the constants α, β, and γ introduced on page 16. The constants α and β were already defined by eqn (1.8). The constant γ is

$$\gamma = \frac{\pi\nu(0)\tilde{D}}{8T_c} \tag{6.24}$$

The equilibrium order parameter is

$$\Delta_\infty(T) = T_c\sqrt{\frac{8\pi^2}{7\zeta(3)}}\sqrt{1 - \frac{T}{T_c}} \tag{6.25}$$

The GL equation determines the temperature-dependent coherence length, i.e., characteristic scale of variations of Δ :

$$\xi(T) = \sqrt{\frac{\pi^3 T\Lambda D}{7\zeta(3)\Delta_\infty^2}} = \sqrt{\frac{\pi\tilde{D}}{8\left(T_c - T\right)}} \propto \left(1 - \frac{T}{T_c}\right)^{-1/2} \tag{6.26}$$

We have already encountered it on page 18. For clean superconductors

$$\xi(T) = \frac{v_F}{\sqrt{6}\Delta(T)} \sim \xi_0 \left(1 - \frac{T}{T_c}\right)^{-1/2}$$

where $\xi_0 = v_F/2\pi T_c$ is the zero-temperature coherence length. For dirty superconductors

$$\xi(T) \sim \sqrt{\xi_0 \ell_0} \left(1 - \frac{T}{T_c}\right)^{-1/2}$$

The condition of slow variations of the order parameter implies that

$$\xi(T) \gg \xi_0$$

This is always satisfied for $T \to T_c$.

6.1.2.2 Current

To find the current, we can use the results obtained earlier. We take eqn (6.5) and put $\Delta \to 0$ in the denominator. We reproduce eqn (1.13) which can be written as

$$\mathbf{j} = -\frac{\pi \tilde{D} \nu(0) e^2 |\Delta|^2}{c T_c} \left(\mathbf{A} - \frac{c}{2e} \nabla \chi\right) \tag{6.27}$$

using the microscopic value for γ. With this expression for the supercurrent we find, in particular, that the density of superconducting electrons is

$$N_s = 2N\left(1 - T/T_c\right)$$

in clean superconductors. For dirty superconductors

$$N_s = \frac{\pi N \tau |\Delta|^2}{2T_c} \tag{6.28}$$

This agrees, of course, with eqn (6.6).
Note that in addition to $\xi(T) \gg \xi_0$, the condition of slow variations implies also

$$\lambda_L(T) \gg \xi_0$$

It is fulfilled for both type I and type II superconductors when temperatures are close enough to T_c because λ_L increases as T approaches T_c.

6.1.2.3 Free energy

One can use equation (5.30) for the thermodynamic potential to find the Ginzburg-Landau free energy of a superconductor. For a constant chemical potential we have

$$\delta\Omega_{sn} = \nu(0) \int \left[-2\pi i T \sum_n \left(\delta\Delta \left\langle f^\dagger_{\omega_n} \right\rangle + \delta\Delta^* \left\langle f_{\omega_n} \right\rangle\right) + \frac{\delta\left(\hat{\Delta}\hat{\Delta}^*\right)}{\lambda} \right] d^3r \tag{6.29}$$

The same expression also holds for variation of the free energy for a constant particle density. Calculating the sums we find from eqn (6.16)

$$\mathcal{F}_{sn} = \nu(0) \int \left[-\left(1 - \frac{T}{T_c}\right) |\Delta|^2 + \frac{7\zeta(3)}{16\pi^2 T_c^2} |\Delta|^4 + \frac{\pi \tilde{D}}{8T_c} \left| \left(\nabla - \frac{2ie}{c} \mathbf{A} \right) \Delta \right|^2 \right] d^3r \tag{6.30}$$

It coincides, of course, with eqn (1.3) for given α, β, and γ.

6.1.3 The upper critical field in a dirty alloy

Consider the second-order phase transition of a dirty alloy from normal into the superconducting state in a decreasing magnetic field. As in Section 1.1.2, we are going to calculate the critical magnetic field below which the superconducting state first appears, i.e., the upper critical field H_{c2}. At this time, however, we do not restrict ourselves to temperatures close to T_c but shall use the quasiclassical method and demonstrate that it is able to treat this problem for arbitrary temperatures.

Near the transition, the order parameter is small. We can again expand the Green function in a small Δ. In the zero-order approximation, $g_0 = 1$, and

$$f = \frac{\Delta}{i\alpha_n}$$

where α_n has yet to be found. The Usadel eqn (5.98) gives

$$D \left(\nabla - \frac{2ie}{c} \mathbf{A} \right)^2 \Delta + 2 \left(\alpha_n - \omega_n \right) \Delta = 0 \tag{6.31}$$

This equation has exactly the same form as the Ginzburg-Landau equation for the order parameter near the upper critical magnetic field, eqn (1.31). We again take the vector potential in the form

$$\mathbf{A} = (0, Hx, 0)$$

where the magnetic field is nearly homogeneous and is directed along the z axis and put

$$\Delta = e^{iky} Y(x) \tag{6.32}$$

The equation for $Y(x)$ becomes the oscillator equation

$$\frac{\partial^2 Y}{\partial x^2} - \left(k - \frac{2eH_{c2}x}{c} \right)^2 Y + 2 \left(\frac{\alpha_n - \omega_n}{D} \right) Y = 0 \tag{6.33}$$

The solution which appears first with decreasing the magnetic field corresponds to the lowest level of eqn (6.33) and has the form of eqn (1.35) with

$$\alpha_n = \omega_n + \frac{DeH_{c2}}{c} \tag{6.34}$$

The full expression for Δ is a linear combination of functions (6.32) with various k. A periodic solution has the form of a vortex lattice, eqn (1.36), obtained by Abrikosov (1957).

The Green function becomes

$$f_0 = \frac{\Delta}{i \left(\omega_n + \epsilon_0 \right)} \tag{6.35}$$

where

$$\epsilon_0 = DeH_{c2}/c \tag{6.36}$$

Using the expression for T_c we can write the order parameter equation as

$$\begin{aligned}\frac{\Delta}{\lambda} &= \sum_{n=0}^{N_0(T_c)} \frac{\Delta}{n+\frac{1}{2}} + \sum_{N_0(T_c)}^{N_0(T)} \frac{\Delta}{n+\frac{1}{2}} + \sum_{n=0}^{\infty} \left[\frac{\Delta}{n+\frac{1}{2}+\frac{\epsilon_0}{2\pi T}} - \frac{\Delta}{n+\frac{1}{2}} \right] \\ &= \frac{\Delta}{\lambda} + \Delta \ln\left(\frac{T_c}{T}\right) + \Delta \left[\psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2}+\frac{\epsilon_0}{2\pi T}\right) \right]\end{aligned}$$

Here

$$\psi(z) = \frac{d}{dz}[\ln \Gamma(z)] \tag{6.37}$$

is the so called digamma function coupled to the Euler gamma function

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$$

Finally, we obtain the equation which determines the upper critical magnetic field (Maki 1969)

$$\ln\left(\frac{T_c}{T}\right) = \psi\left(\frac{1}{2}+\frac{\epsilon_0}{2\pi T}\right) - \psi\left(\frac{1}{2}\right) \tag{6.38}$$

For low temperatures, $T \to 0$, we can use the asymptotic expression for $\psi(z)$ for large z :

$$\psi(z) \approx \ln z - \frac{1}{2z}$$

Since $\psi(1/2) = -C - 2\ln 2$ we obtain

$$\epsilon_0 = \pi T_c / 2\gamma$$

where $\gamma = e^C \approx 1.78$. Equation (6.36) gives the upper critical field

$$H_{c2} = \frac{\pi c T_c}{2\gamma D e} \tag{6.39}$$

For high temperatures, $T \to T_c$, we have

$$\psi\left(\frac{1}{2}+\frac{\epsilon_0}{2\pi T}\right) - \psi\left(\frac{1}{2}\right) \approx \frac{d\psi(1/2)}{dz} \frac{\epsilon_0}{2\pi T_c}.$$

Since $\psi'(1/2) = \pi^2/2$, we obtain

$$H_{c2} = \frac{4cT}{\pi D e}\left(1 - \frac{T}{T_c}\right) = \frac{c}{2e\xi^2(T)} \tag{6.40}$$

This result coincides, of course, with the solution of the Ginzburg-Landau equation (1.34) since the coherence length in the dirty limit is

$$\xi^2(T) = \frac{\pi D}{8T_c}\left(1 - \frac{T}{T_c}\right)^{-1} \tag{6.41}$$

17.2.2 6.2 Gapless s-wave superconductivity

Consider an alloy with magnetic impurities (Abrikosov and Gor'kov 1960). Scattering on an impurity atom depends on the spin state of an electron. The scattering process breaks the spin coherence of paired electrons and leads to a suppression of superconductivity. The key point is that the self energies for g and f contain different scattering times since the functions g, on one hand, and f, $f^\dagger$, on the other hand, have different spin structures, For a homogeneous case we have from eqns (5.74) and (5.75)

$$\Sigma_1 = \frac{i}{2\tau_1} g_{\omega_n}, \text{ while } \Sigma_2 = \frac{i}{2\tau_2} f_{\omega_n}, \Sigma_2^\dagger = \frac{i}{2\tau_2} f_{\omega_n}^\dagger \tag{6.42}$$

where the mean free times are defined by eqns (4.16) and (4.17). The Eilenberger equation (5.84) gives

$$-2i\omega_n f + 2\Delta g - \frac{2i}{\tau_s} g f = 0$$

where the spin-flip time is determined by eqn (4.18). Therefore,

$$f = \frac{\Delta g}{i\left(\omega_n + g/\tau_s\right)} \tag{6.43}$$

From the normalization condition $g^2 - f^2 = 1$ we find

$$g^2 = \frac{\left(\omega_n + g/\tau_s\right)^2}{\left(\omega_n + g/\tau_s\right)^2 + \Delta^2} \tag{6.44}$$

and

$$f = \frac{\Delta}{i\sqrt{\left(\omega_n + g/\tau_s\right)^2 + \Delta^2}} \tag{6.45}$$

6.2.1 Critical temperature

The order-parameter equation (5.27) takes the form

$$\frac{1}{\lambda} = 2\pi T \sum_{n\geq 0} \frac{1}{\sqrt{\left(\omega_n + g/\tau_s\right)^2 + \Delta^2}} \tag{6.46}$$

Let us find the critical temperature. Putting $\Delta = 0$, we get $g = 1$ and

$$\begin{aligned}\frac{1}{\lambda} =& 2\pi T \sum_{n=0}^{N_0(T_c)} \frac{1}{\omega_n} + 2\pi T \sum_{n\geq 0} \left(\frac{1}{\omega_n + 1/\tau_s} - \frac{1}{\omega_n}\right) \\ =& 2\pi T \sum_{n=0}^{N_0(T_{c0})} \frac{1}{\omega_n} + 2\pi T \sum_{N_0(T_{c0})}^{N_0(T_c)} \frac{1}{\omega_n} + 2\pi T \sum_{n=0}^{\infty} \left(\frac{1}{\omega_n + 1/\tau_s} - \frac{1}{\omega_n}\right)\end{aligned} \tag{6.47}$$

Here T_{c0} is the critical temperature without magnetic impurities, eqn (3.83):

$$T_{c0} = \left(2\Omega_{\text{BCS}}\gamma/\pi\right) e^{-1/\lambda}$$

We obtain

$$\ln\left(\frac{T_{c0}}{T_c}\right) + \sum_{n=0}^{\infty}\left(\frac{1}{n+1/2+1/2\pi T_c\tau_s} - \frac{1}{n+1/2}\right) = 0$$

or

$$\ln\left(\frac{T_{c0}}{T_c}\right) + \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{1}{2\pi T_c\tau_s}\right) = 0 \tag{6.48}$$

It is the familiar equation: we have seen it on page 120 when we calculated the upper critical field for a dirty superconductor.

Let us consider a limiting case when the spin-flip scattering rate is large: $\tau_s T_c \ll 1$. In this limit, the critical temperature decreases down to zero, $T_c \ll T_{c0}$ and $1/2\pi T_c\tau_s$ increases. For large arguments, we can use the asymptotics

$$\psi(z) = \ln z - \frac{1}{2z} - \frac{1}{12z^2}$$

The value $\psi(1/2) = -\ln(4\gamma)$ where $\gamma \approx 1.78$. We have

$$\psi\left(\frac{1}{2} + \frac{1}{2\pi T_c\tau_s}\right) - \psi\left(\frac{1}{2}\right) = \ln\left(\frac{2\gamma}{\pi T_c\tau_s}\right) + \frac{(\pi T_c\tau_s)^2}{6}.$$

As the result, the critical temperature is

$$T_c^2 = \frac{6}{\pi^2\tau_s^2}\ln\left(\frac{\pi T_{c0}\tau_s}{2\gamma}\right) \tag{6.49}$$

The critical temperature vanishes when the scattering rate is

$$\frac{1}{\tau_s} = \pi T_{c0}/2\gamma$$

For low concentration of magnetic impurities, $1/\tau_s \to 0$, we have $\psi(1/2+x) - \psi(1/2) = \psi'(1/2)x$ and

$$1 - \frac{T_c}{T_{c0}} = \frac{\pi}{4T_{c0}\tau_s}$$

The critical temperature is slightly reduced from its value without magnetic impurities.

6.2.2 Gap in the energy spectrum

The real-frequency (retarded and advanced) Green functions are found from

$$g^2 = \frac{(\epsilon + ig/\tau_s)^2}{(\epsilon + ig/\tau_s)^2 - \Delta^2} \tag{6.50}$$

which is obtained from eqn (6.44) by replacing ω_n with $-i\epsilon$. Retarded function g^R is defined as the solution of eqn (6.50) which has no singularities in the upper half-plane of ϵ, while g^A should not have singularities in the lower half-plane.

Let us look for a solution in the form $g^R = g^A = -ia$ where a is a real function. If such a solution exists for a given energy ϵ, the density of states $g^R - g^A = 0$. This means that the corresponding energy is below the energy gap. We have for a

$$a^2 = \frac{(\epsilon + a/\tau_s)^2}{\Delta^2 - (\epsilon + a/\tau_s)^2} \tag{6.51}$$

This gives

$$\frac{\epsilon}{\Delta} = -\frac{a}{\tau_s \Delta} \pm \frac{a}{\sqrt{a^2+1}} \tag{6.52}$$

We choose the solution with the + sign since it gives the correct behavior in the case without magnetic impurities $1/\tau_s \to 0$. Indeed, without magnetic impurities we would have

$$g = -i\frac{\epsilon}{\sqrt{\Delta^2 - \epsilon^2}}$$

for $|\epsilon| < \Delta$. It results in the asymptotics $a \to +\infty$ for $\epsilon/\Delta \to 1 - 0$ which should be reproduced by eqn (6.52). Equation (6.52) for a takes the form

$$\epsilon = \Delta F(a)$$

where we define the function

$$F(a) = -\frac{a}{\tau_s \Delta} + \frac{a}{\sqrt{a^2+1}} \tag{6.53}$$

For $\tau_s \Delta > 1$, the function $F(a)$ has a positive maximum. Calculating the derivative we find that the maximum is reached at

$$a = \sqrt{(\tau_s \Delta)^{2/3} - 1}$$

The function at maximum is equal to

$$F_{\max} = \left[1 - \frac{1}{(\tau_s \Delta)^{2/3}}\right]^{3/2}$$

Therefore, a real solution for a exists if $\epsilon < \Delta F_{\max}$. Thus

$$\epsilon_0 = \Delta \left[1 - \frac{1}{(\tau_s \Delta)^{2/3}}\right]^{3/2}$$

is the gap in the energy spectrum. For $\epsilon > \epsilon_0$, there is no real solution for a, thus g becomes complex: $g^R = -ia + b$ and $g^A = -ia - b$ with a finite density of states $g^R - g^A = 2b$.

The energy gap disappears when the order parameter satisfies the gapless condition $\tau_s \Delta(T) \leq 1$. Note that there is a temperature range where superconductivity exists without an energy gap in the excitation spectrum. Indeed, superconductivity exists for temperatures below T_c if $\pi T_{c0} \tau_s / 2\gamma > 1$. For low enough τ_s^{-1}, the condition $\tau_s \Delta(T = 0) > 1$ is obviously fulfilled, and there is an energy gap for low temperatures. However, the gap vanishes at higher temperatures when $\tau_s \Delta(T)$ decreases down to 1 . In the opposite case, $\tau_s \Delta(T = 0) < 1$, the energy gap is absent for all temperatures.

We note that the vicinity of the upper critical field considered in the previous section also belongs to the gapless regime. Indeed, for the real frequency $\epsilon = 0$ one has from eqn (6.35)

$$f^R = f^A = \frac{\Delta}{i\epsilon_0}; \qquad g^R = -g^A = 1 + \frac{\Delta^2}{2\epsilon_0^2}$$

The density of states is thus finite. The gapless situation can be created also by an inelastic electron-phonon scattering within certain range of temperatures (see Section 11.2).

17.2.3 6.3 Aspects of d-wave superconductivity

Many problems of d-wave superconductivity can be successfully solved using the quasiclassical methods. Their applications for stationary and nonequilibrium properties of d-wave superconductors can be found in many publications (see, for example, Graf et al. 1995, Graf et al. 1996); they belong to a rapidly growing field of research. We consider in this Section only several selected examples.

6.3.1 Impurities and d-wave superconductivity

The order parameter of a d-wave superconductor satisfies the self-consistency equation (5.29). It has the form

$$\frac{\Delta_{\mathbf{p}}(\mathbf{k})}{\lambda} = 2\pi i T \sum_{n\geq 0} \left\langle V_d\left(\hat{\mathbf{p}}, \hat{\mathbf{p}}_1\right) f_{\omega_n}\left(\mathbf{p}_1 + \mathbf{k}/2, \mathbf{p}_1 - \mathbf{k}/2\right)\right\rangle_{\mathbf{p}_1} \tag{6.54}$$

in the quasiclassical approximation. Here we average over the directions of $\mathbf{p}_1$ at the Fermi surface according to eqn (5.73) having in mind that the Fermi surface can be non-spherical. For example, in d-wave superconductors which have uniaxial crystal symmetry, its shape is more close to a cylinder. The d-wave pairing potential has the structure of eqn (3.53). It is normalized in such a way that

$$\left\langle V_d\left(\hat{\mathbf{p}}, \hat{\mathbf{p}}_1\right) \Delta_{\mathbf{p}_1}\right\rangle_{\mathbf{p}_1} = \Delta_{\mathbf{p}} \tag{6.55}$$

If $V_d = \cos(2\alpha)\cos\left(2\alpha_1\right)$ the order parameter in a d-wave superconductor is

$$\Delta_{\mathbf{p}} = \Delta_0 \cos(2\alpha) \tag{6.56}$$

where the amplitude satisfies the equation

$$\frac{\Delta_0(\mathbf{r})}{\lambda} = 2\pi i T \sum_{n\geq 0} \left\langle 2\cos(2\alpha) f_{\omega_n}(\hat{\mathbf{p}}, \mathbf{r})\right\rangle \tag{6.57}$$

The order parameter has nodes, i.e., it vanishes for momentum directions $\alpha = \pi/4 + (\pi/2)n$. Note that

$$\langle \Delta_{\mathbf{p}} \rangle = 0 \tag{6.58}$$

We restrict ourselves to a homogeneous case. The quasiclassical Green functions are found from the Eilenberger equations. We assume an isotropic scattering by impurities. We have from eqn (5.84)

$$-2i\omega_n f + 2\Delta_{\mathbf{p}} g - \frac{i}{\tau}(\langle g\rangle f - \langle f\rangle g) = 0$$

As we shall see later, in a homogeneous state,

$$\langle f(\hat{\mathbf{p}})\rangle = 0 \tag{6.59}$$

Under this condition, the self energies do not drop out of the Eilenberger equation eqns (5.83) - (5.85) any more. Instead, we obtain with help of the normalization condition $g^2 - f^2 = 1$

$$f = \frac{\Delta_{\mathbf{p}}}{i\sqrt{\left(\omega_n + \langle g\rangle/2\tau\right)^2 + \left|\Delta_{\mathbf{p}}\right|^2}}, g = \frac{\omega_n + \langle g\rangle/2\tau}{\sqrt{\left(\omega_n + \langle g\rangle/2\tau\right)^2 + \left|\Delta_{\mathbf{p}}\right|^2}} \tag{6.60}$$

With this solution, it easy to see that eqn (6.59) holds.
Equation (6.60) resembles the expressions for an alloy with magnetic impurities. Indeed, properties of a d-wave superconductor with usual non-magnetic impurities are very much similar to those for magnetic s-wave alloys. First of all, the presence of impurities suppresses the d-wave superconductivity. The reason is that the scattering by impurities destroys the momentum coherence of the paired state in a way similar to that how the spin-dependent scattering destroys the spin coherence of the paired state in an s-wave superconductor. We thus can anticipate that there exists a critical scattering rate $1/\tau$ such that the critical temperature of the d-wave superconductor vanishes.

To find the critical temperature as a function of the scattering rate, we put $\Delta = 0$ in eqn (6.60) and insert thus obtained f into eqn (6.57). We have

$$\frac{1}{\lambda} = 2\pi T \sum_{n\geq 0} \frac{1}{\omega_n + 1/2\tau} \tag{6.61}$$

This is exactly the equation (6.47) for the critical temperature of a magnetic alloy with the substitution $1/\tau_s \to 1/2\tau$. Therefore, we obtain

$$\ln\left(\frac{T_{c0}}{T_c}\right) + \psi\left(\frac{1}{2}\right) - \psi\left(\frac{1}{2} + \frac{1}{4\pi T_c \tau}\right) = 0 \tag{6.62}$$

Here T_{c0} is the critical temperature without impurities

$$T_{c0} = (2\Omega_{\mathrm{BCS}}\gamma/\pi)\, e^{-1/\lambda}$$

The critical temperature vanishes for

$$\frac{1}{\tau} = \pi T_{c0}/\gamma$$

This shows that d-wave superconductivity can exists only in rather clean compounds.

6.3.2 Impurity-induced gapless excitations

Presence of impurities broadens the order-parameter nodes into gapless lowenergy states (Gor'kov and Kalugin 1985). In a strict sense, there is no energy gap in a d-wave alloy. To see that, let us find the real-frequency Green function. The retarded function is from eqn (6.60)

$$g^R = \frac{\epsilon + \frac{i}{2\tau}\left\langle g^R\right\rangle}{\sqrt{\left(\epsilon + \frac{i}{2\tau}\left\langle g^R\right\rangle\right)^2 - |\Delta_{\mathbf{p}}|^2}} \tag{6.63}$$

Let us find the average

$$\left\langle g^R\right\rangle = \int \frac{d\alpha}{2\pi} \frac{\epsilon + \frac{i}{2\tau}\left\langle g^R\right\rangle}{\sqrt{\left(\epsilon + \frac{i}{2\tau}\left\langle g^R\right\rangle\right)^2 - |\Delta_{\mathbf{p}}|^2}}$$

We have for $\epsilon = 0$

$$1 = \frac{1}{2\tau} \int \frac{d\alpha}{2\pi} \frac{1}{\sqrt{\Delta_0^2 |\cos(2\alpha)|^2 + \left(\frac{1}{2\tau}\left\langle g^R\right\rangle\right)^2}} \tag{6.64}$$

We shall see that $(1/2\tau)\left\langle g^R\right\rangle \ll \Delta_0$ therefore, the angles with a small $\cos(2\alpha)$ are important. We put $x = \pi/4 - \alpha$. Substituting $\cos(2\alpha) \approx 2x$ for small x, we get

$$\frac{1}{\pi\tau}\int_0^\infty \frac{dx}{\sqrt{\Delta_0^2 x^2 + \left(\langle g^R\rangle/4\tau\right)^2}} = 1 \tag{6.65}$$

since there are 4 nodes. Calculating the integral we find

$$\ln\left(\frac{4\tau\Delta_0}{\langle g^R\rangle}\right) = \pi\Delta_0\tau \tag{6.66}$$

The averaged Green function is

$$\left\langle g^R\right\rangle = 4\Delta_0\tau\exp\left(-\pi\Delta_0\tau\right) \tag{6.67}$$

It is a real quantity. Thus there exists a finite density of states at zero energy

$$\nu = 4\nu(0)\Delta_0\tau\exp\left(-\pi\Delta_0\tau\right).$$

It is exponentially small in the clean limit $\Delta_0\tau \gg 1$ in the Born approximation. However, the density of states becomes a power-law function if the impurity scattering is strong such that it should be treated within the full scattering amplitude (Graf et al. 1996).

We see that there is no energy gap: a small but finite density of states exists down to zero energy. The angle-resolved density of states for $\epsilon = 0$ is

$$\frac{1}{2}\left(g^R - g^A\right) = \frac{\epsilon_0}{\sqrt{\epsilon_0^2 + \left[\Delta_0\cos(2\alpha)\right]^2}}$$

where ϵ_0 is

$$\epsilon_0 = \left\langle g^R\right\rangle/2\tau = 2\Delta_0\exp\left(-\pi\Delta_0\tau\right)$$

One can say that, instead of gap nodes, there opens "an impurity band" with the bandwidth $2\epsilon_0$.

6.3.3 The Ginzburg-Landau equations

Here we consider an ideal case where the pairing potential has only a d-wave component. A more complete version of the Ginzburg-Landau equation including both d-wave and s-wave components of the pairing potential and the order
parameter is given by Heeb et al. (1996). Since a d-wave superconductor can only be clean, we have from eqns (6.12, 6.14, 6.15) for $\tau \to \infty$

$$f = \frac{\Delta_{\mathbf{p}}}{i\omega_n} - \frac{\left|\Delta_{\mathbf{p}}\right|^2\Delta_{\mathbf{p}}}{2i\omega_n} + \frac{1}{4i\omega_n}v_i v_k\hat{\nabla}_i\hat{\nabla}_k\Delta_{\mathbf{p}}$$

where

$$\hat{\nabla} = \nabla \mp \frac{2ie}{c}\mathbf{A}$$

for Δ and Δ^*, respectively. Result of integration over the Fermi surface depends on its shape. d-wave superconductors are usually anisotropic. We assume for simplicity that the Fermi surface has an axial symmetry around the crystal c axis. Performing integration in the self-consistency equation over the angle α within the (ab) plane we obtain

$$\frac{\Delta_0}{\lambda} = 2\pi T\sum_{n>0}\left\langle\frac{\Delta_0}{\omega_n} + \frac{1}{8\omega_n^3}\left[\left(v_{ab}^2\hat{\nabla}_{ab}^2 + 2v_c^2\hat{\nabla}_c^2\right)\Delta_0 - 3\left|\Delta_0\right|^2\Delta_0\right]\right\rangle$$

where v_{ab} and v_c are the Fermi velocity projections on the ab plane and the c axis, respectively. The average now is over the momentum projection on the z axis. Using the sums calculated earlier, we find the Ginzburg-Landau equation for uniaxial anisotropic d-wave superconductor

$$\alpha\Delta_0 + \beta\left|\Delta_0\right|^2\Delta_0 + \gamma_{ab}\left(\nabla_{ab} - \frac{2ie}{c}\mathbf{A}_{ab}\right)^2\Delta_0 + \gamma_c\left(\nabla_c - \frac{2ie}{c}\mathbf{A}_c\right)^2\Delta_0 = 0 \tag{6.68}$$

where α is defined by eqn (1.8). The constant β now is

$$\beta = \frac{21\nu(0)\zeta(3)}{32\pi^2 T_c^2}$$

while

$$\gamma_{ab} = \frac{7\zeta(3)\nu(0)\left\langle v_{ab}^2\right\rangle}{32\pi^2 T_c^2}, \gamma_c = \frac{7\zeta(3)\nu(0)\left\langle v_c^2\right\rangle}{16\pi^2 T_c^2}$$

Equation (6.68) is isotropic in the (ab) plane.
We can reconstruct the GL thermodynamic potential (free energy) using eqn (6.29):

$$\mathcal{F}_{sn} = \int\left[\alpha\left|\Delta_0\right|^2 + \frac{\beta}{2}\left|\Delta_0\right|^4 + \gamma_{ab}\left|\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)_{ab}\Delta_0\right|^2 + \gamma_c\left|\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)_c\Delta_0\right|^2\right]d^3r. \tag{6.69}$$

The supercurrent can be found from

$$\mathbf{j}_s = -c\frac{\delta\mathcal{F}}{\delta\mathbf{A}}$$

We obtain

$$\begin{aligned}\mathbf{j}_{ab} &= -2ie\gamma_{ab}\left[\Delta_0^*\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)_{ab}\Delta_0 - \Delta_0\left(\nabla + \frac{2ie}{c}\mathbf{A}\right)_{ab}\Delta_0^*\right]\\ j_c &= -2ie\gamma_c\left[\Delta_0^*\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)_c\Delta_0 - \Delta_0\left(\nabla + \frac{2ie}{c}\mathbf{A}\right)_c\Delta_0^*\right]\end{aligned} \tag{6.70}$$

in a full analogy to eqn (1.13).

17.2.4 6.4 Bound states in vortex cores

We turn now to another important problem associated with the structure of a vortex in the mixed state of clean type II superconductors. We already know from Section 1.1.2 that each vortex is a singular line at which the order parameter goes to zero; the order-parameter phase winds around this line by 2π. Each vortex carries one magnetic flux quantum $\Phi_0 = \pi\hbar c/|e|$. The magnetic field decays away from vortex at a distance of the order of the London penetration depth λ_L due to the screening supercurrent which circulates around the vortex within the distance on order λ_L. The order parameter recovers its bulk magnitude at a distance of the order of the coherence length ξ from the vortex axis. The region where the order parameter is essentially suppressed is called the vortex core. The structure of a single vortex is shown schematically in Fig. 1.1 for the case $\kappa \gg 1$.

The order-parameter magnitude profile $|\Delta(\rho)|$ forms a potential well near the vortex axis. Quasiparticles with energies below the bulk gap Δ_∞ become localized and occupy discrete levels

in the vortex core. The energy spectrum of excitations in the vortex core for a single vortex was first found by Caroli et al. (1964). These authors have calculated the low energy levels $\epsilon \ll \Delta_\infty$ by solving the Bogoliubov-de Gennes equations (3.56) (see also de Gennes 1966).

Here we consider the same problem using the quasiclassical scheme according to the method developed by Kramer and Pesch (1974). We assume that the Ginzburg-Landau parameter of the superconductor $\kappa \gg 1$; it is the condition under which vortices can be treated separately from each other. Indeed, the intervortex distance which is of the order of the radius of the Bravais unit cell r_0 is such that the flux quantum is $\Phi_0 = \pi r_0^2 B$ therefore, $r_0 \sim \xi\sqrt{H_{c2}/H}$. It is much longer than the core size ξ if $H \ll H_{c2}$. At the same time, magnetic field should be larger than the lower critical filed H_{c1} which is only possible if $H_{c1} \ll H_{c2}$. The latter is fulfilled for $\kappa \gg 1$ since $H_{c1} \sim H_{c2}/\kappa^2$.

The problem of quasiparticle states in the vortex core is important for many applications. One needs their energy spectrum, for example, to calculate the density of states in the mixed state of superconductors which determines the low-temperature behavior of the specific heat and of the London penetration depth, etc. We shall use the energy spectrum of the core states later for the vortex dynamics.

Fig. 6.1. The coordinate frame associated with the particle localized in the vortex core. The line AB is the quasiparticle trajectory passing by the vortex axis at an impact parameter b through the position point (ρ, ϕ) shown by the black dot.

As usually, the problem starts with calculating the quasiclassical Green functions. We need them for $\epsilon \ll \Delta_\infty$. Let us first choose the proper reference frame. We take the direction of the z axis of the cylindrical frame in such a way that the vortex has a positive circulation. The z axis is thus parallel the magnetic field for positive charge carriers, while it is antiparallel to it for negative carriers: $\hat{\mathbf{z}} = (\mathbf{H}/H)\,\mathrm{sign}(e)$. Let $\mathbf{v}_\perp$ be the projection of the quasiparticle (Fermi) velocity $\mathbf{v}_F$ on the (x, y) plane. The vector $\mathbf{v}_\perp$ makes an angle α with the x axis. The quasiparticle trajectory which passes through the position point is characterized by an impact parameter b with respect to the vortex axis. We take the impact parameter and the distance s along the trajectory as the new coordinate frame. The cylindrical coordinates of the position point (ρ, ϕ) are connected to the impact parameter and the coordinate along the trajectory through $\rho^2 = b^2 + s^2$ where (see Fig. 6.1)

$$b = \rho \sin(\phi - \alpha); s = \rho \cos(\phi - \alpha). \tag{6.71}$$

As a function of s, the angle ϕ is $\pi + \alpha$ for $s \to -\infty$, while $\phi = \pi/2 + \alpha$ for $s = 0$, and $\phi = \alpha$ for $s \to \infty$.

6.4.1 Superconductors with s-wave pairing

Let us consider first an s-wave superconductor. We put $\Delta = \Delta_0 \exp(i\phi)$ and

$$f = F_0 \exp(i\phi + i\eta); f^\dagger = F_0^\dagger \exp(-i\phi - i\eta), \tag{6.72}$$

where

$$\begin{aligned} \eta &= -\int_0^s \left(\frac{\partial \phi}{\partial s} - \frac{2e}{c} A_s \right) ds \\ &= \frac{\pi}{2} + \alpha - \phi(s) - \frac{2eb}{c} \int_0^s A_\phi \frac{ds'}{\rho'} \end{aligned} \tag{6.73}$$

Note that $\eta(s) = -\eta(-s)$. Moreover, $\eta \to \pm\pi/2$ for $s \to \pm s_0$ where $s_0 \gg \xi$.
Now we write F_0 and $F_0^\dagger$ in terms of the symmetric and antisymmetric functions $\theta(s) = -\theta(-s); \zeta(s) = \zeta(-s)$:

$$F_0 = -\theta(s) + i\zeta(s); F_0^\dagger = \theta(s) + i\zeta(s) \tag{6.74}$$

The normalization condition eqn (5.39) requires $g^2 + \theta^2 + \zeta^2 = 1$. Eilenberger equations (5.90) and (5.91) give

$$v_\perp \frac{\partial \zeta}{\partial s} + 2\epsilon\theta - 2i\Delta_0 g \sin\eta = 0 \tag{6.75}$$

$$v_\perp \frac{\partial \theta}{\partial s} - 2\epsilon\zeta - 2i\Delta_0 g \cos\eta = 0 \tag{6.76}$$

The boundary conditions for $s \to \pm\infty$ are:

$$f^{R(A)} = -i \frac{|\Delta_\infty| e^{i\phi}}{\sqrt{|\Delta_\infty|^2 - \epsilon^2}}, g^{R(A)} = -i \frac{\epsilon}{\sqrt{|\Delta_\infty|^2 - \epsilon^2}}$$

where $\epsilon \ll |\Delta_\infty|$. Therefore, $g^{R(A)}(s) \to 0, f^{R(A)} \to -ie^{i\phi}$ at large distances. This requires $\theta \to \pm 1, \zeta \to 0$. For energies close to the bound state energy ϵ_0, i.e., close to the pole $\epsilon - \epsilon_0(b) \ll \epsilon_0(b)$, the functions g and $f, f^\dagger$ are large near the vortex. We assume that $\zeta^2 \gg \theta^2 - 1$, so that $g = i\zeta$. The plus sign here is chosen to satisfy the condition of vanishing of g at large distances according to eqn (6.75).

The solution of eqns (6.75, 6.76) is

$$\zeta^{R(A)} = \frac{v_\perp \exp[-K(s)]}{2C(v_\perp)[\epsilon - \epsilon_0(b) \pm i\delta]} \tag{6.77}$$

$$\theta^{R(A)} = \frac{\int_0^s (\epsilon - \Delta_0 \cos\eta) \exp[-K(s')] ds'}{C(v_\perp)[\epsilon - \epsilon_0(b) \pm i\delta]} \tag{6.78}$$

Here

$$K(s) = \left| \frac{2}{v_\perp} \int_0^s \Delta_0 \sin\eta ds' \right| \tag{6.79}$$

The functions f and g have poles at $\epsilon = \epsilon_0(b)$, i.e., $\epsilon_0(b)$ is the energy of a bound state,

$$\epsilon_0(b) = C^{-1}(v_\perp) \int_0^\infty \Delta_0(s) \cos\eta \exp[-K(s)] ds \tag{6.80}$$

where

$$C\left(v_{\perp}\right)=\int_{0}^{\infty}\exp[-K(s)]ds \tag{6.81}$$

and

$$\cos\eta=\frac{b}{\rho}\left(1+\frac{2es}{c}\int_{0}^{s}A_{\phi}\frac{ds'}{\rho'}\right) \tag{6.82}$$

The Green functions are

$$g^{R(A)}=\frac{iv_{\perp}e^{-K}}{2C\left[\epsilon-\epsilon_{0}\pm i\delta\right]} \tag{6.83}$$

and $F_0=F_0^{\dagger}=g$.
For small impact parameters, $b\ll\xi$, one has $|s|=\rho$. Therefore,

$$\begin{aligned}\epsilon_{0}(b)=&C^{-1}\left(v_{\perp}\right)b\int_{0}^{\infty}\left[\left(\frac{1}{\rho}-\frac{2e}{c}A_{\phi}\right)\Delta_{0}+\frac{ev_{\perp}H}{c}\right]e^{-K(\rho)}d\rho\\=&C^{-1}\left(v_{\perp}\right)b\int_{0}^{\infty}\frac{\Delta_{0}}{\rho}e^{-K(\rho)}d\rho+\frac{\omega_{c}bp_{\perp}}{2}\end{aligned} \tag{6.84}$$

where

$$\omega_{c}=\frac{|e|H}{mc}$$

is the cyclotron frequency. Modulus of charge appears due to the choice of the z axis.

In the quasiclassical approximation, the energy $\epsilon_0(b)$ is a continuous function of the impact parameter. However, the impact parameter is coupled to the angular momentum μ so that $b=-\mu/p_{\perp}$; the minus sign appears because a positive impact parameter corresponds to a negative angular momentum as can be seen from Fig. 6.1. The angular momentum is quantized such that $\mu=m+1/2$ where m is an integer (de Gennes 1966). Therefore, the energy can be written in the form

$$\epsilon_{0}=-\omega_{0}\left(m+\frac{1}{2}\right) \tag{6.85}$$

where

$$\omega_{0}=p_{\perp}^{-1}\frac{\partial\epsilon_{0}}{\partial b}=p_{\perp}^{-1}C^{-1}\left(v_{\perp}\right)\int_{0}^{\infty}\frac{\Delta_{0}}{\rho}e^{-K(\rho)}d\rho+\frac{\omega_{c}}{2} \tag{6.86}$$

For a superconductor with a large Ginzburg-Landau parameter λ_L/ξ, the contributions from H is small for fields $H\ll H_{c2}$. Indeed, the first term in eqn (6.86) is of the order of

$$\omega_{0}\sim\frac{\Delta_{\infty}^{2}}{E_{F}}$$

which is much larger than the second term because $\omega_c/\omega_0\sim H/H_{c2}$. Equation (6.85) coincides with the result by Caroli et al. (1964). Equation (6.86) with the account of magnetic field was obtained by Hansen (1968).

The energy spectrum (6.85) depends on two quantum numbers: the angular momentum and the momentum along the vortex axis. In principle, the spectrum

Fig. 6.2. Energy levels $\epsilon_{n}(b)$ as function of the impact parameter for various radial quantum numbers n. The anomalous branch with $n = 0$ crosses zero of energy and is odd in b. Other branches with $n \neq 0$ are even in b and lie considerably higher than the anomalous branch.

should depend on three quantum numbers in a three dimensional problem. The third is the radial quantum number. It does not appear in eqn (6.85) because the energies $\epsilon \ll \Delta_\infty$ are only accessible for particles having the radial quantum number zero. The levels with nonzero radial quantum numbers are located at energies of the order of Δ_∞ and can not be calculated using this approach. The general feature of levels with $n \neq 0$ is that they make the spectral branches which are even functions of μ as distinct from the branch with $n = 0$. The latter is chiral: it is odd in μ and crosses zero of energy. We shall see that it is of a great importance for vortex dynamics. The energy spectrum $\epsilon_n(b)$ is shown schematically in Fig. 6.2.
6.4.1.1 Density of states Equation (6.83) can be used to calculate the density of states in the vortex core. Integrating the local density of states, eqn (5.43), over the vortex core area we find

$$\begin{aligned}\nu_s(\epsilon) &= \frac{1}{2}\nu(0)\int_{-\infty}^{\infty}\left(g^R - g^A\right) ds db = \pi\nu(0)v_\perp\int_{-\infty}^{\infty}\delta\left(\epsilon - \epsilon_0\right) db \\ &= \pi\nu(0)v_\perp\left(\frac{\partial\epsilon_0}{\partial b}\right)^{-1} = \frac{\pi\nu(0)}{m\omega_0}\end{aligned} \tag{6.87}$$

for a given direction of the momentum with respect to the z axis. Therefore, the density of states for low energies is finite

$$\nu_s(\epsilon) \sim \nu(0)\pi\xi_0^2$$

It is independent of energy and has the same order of magnitude as the density of states in the normal region of the radius of the vortex core.

6.4.2 d-wave superconductors

The quasiclassical method can also be used for a d-wave superconductor within the same scheme irrespectively of the fact that, in d-wave superconductors, the angular momentum is not conserved in a strict sense because the axial symmetry is broken. The point is that the trajectory of a particle localized in the core is a straight line in the quasiclassical approximation. Indeed, a momentum change due to the vortex potential is $\delta p/p_F \sim \Delta/E_F \ll 1$. A particle, being reflected by the vortex potential (Andreev reflection), does not change its momentum

direction thus remaining on the same trajectory. Therefore, the impact parameter is a well defined quantity even if the vortex is not axisymmetric.

We solve the quasiclassical Eilenberger equations for the Green functions g, f, and $f^\dagger$ for a single vortex using the same scheme as before. The boundary conditions are now slightly different. Indeed, at large distances from the vortex axis, the Green function is

$$f^R = -i\frac{\Delta_{\mathbf{p}}}{\sqrt{|\Delta_{\mathbf{p}}|^2 - \tilde{\epsilon}^2}} \tag{6.88}$$

For simplicity, we measure the angle α from one of the gap nodes such that $\Delta_{\mathbf{p}} = \Delta_0 \sin(2\alpha)e^{i\phi}$. In combination with eqn (6.74), it gives the boundary condition

$$\zeta \to 0; \theta \to \pm\,\mathrm{sign}[\sin(2\alpha)] \text{ for } s \to \pm\infty. \tag{6.89}$$

The solution of eqns (6.75) and (6.76) is

$$\zeta^{R(A)} = \frac{\mathrm{sign}[\sin(2\alpha)]v_\perp e^{-\tilde{K}}}{\int_{-\infty}^{\infty}\left[\epsilon - \Delta_0|\sin(2\alpha)|\cos\eta \pm i\delta\right]e^{-\tilde{K}}ds}$$

$$\theta^{R(A)} = 2\,\mathrm{sign}[\sin(2\alpha)]\left(\frac{\int_{-\infty}^{s}\left[\epsilon - \Delta_0|\sin(2\alpha)|\cos\eta\right]e^{-\tilde{K}}ds'}{\int_{-\infty}^{\infty}\left[\epsilon - \Delta_0|\sin(2\alpha)|\cos\eta \pm i\delta\right]e^{-\tilde{K}}ds} - \frac{1}{2}\right)$$

where

$$\tilde{K} = \frac{2}{v_\perp}\int_0^s \Delta_0|\sin(2\alpha)|\sin\eta ds' \tag{6.90}$$

As a result,

$$g^{R(A)} = \frac{iv_\perp e^{-\tilde{K}}}{\tilde{C}\left[\epsilon - \epsilon_0 \pm i\delta\right]} \tag{6.91}$$

where now

$$\epsilon_0(\alpha, b) = \tilde{C}^{-1}|\sin(2\alpha)|\int_{-\infty}^{\infty}\Delta_0\cos\eta e^{-\tilde{K}}ds \tag{6.92}$$

with

$$\tilde{C} = \int_{-\infty}^{\infty} e^{-\tilde{K}}ds \tag{6.93}$$

For impact parameters, $b \ll \xi$, one has $|s| = \rho$. Therefore,

$$\epsilon_0(b) = b\left(\tilde{C}^{-1}|\sin(2\alpha)|\int_0^\infty \Delta_0 e^{-\tilde{K}(\rho)}\frac{d\rho}{\rho} + p_\perp\omega_c/2\right) \tag{6.94}$$

Equation (6.94) can be simplified for the momentum directions near the gap nodes, where $\sin(2\alpha) \ll 1$. Significant values of $\tilde{K}$ are determined by $\rho \gg \xi$. Therefore, one can put $\Delta_0 = \Delta_\infty$:

$$\begin{aligned}\tilde{K}(s) &= \frac{2\Delta_\infty|\sin(2\alpha)|}{v_\perp}\int_0^s \frac{s'ds'}{\rho'} \\ &= \frac{2\Delta_\infty|\sin(2\alpha)|}{v_\perp}(\rho - b)\end{aligned}$$

The normalization constant becomes

$$\tilde{C} = 2e^{\gamma} \int_b^{\infty} e^{-\gamma\rho/b} \frac{\rho d\rho}{\sqrt{\rho^2 - b^2}} = be^{\gamma} K_1(\gamma) \tag{6.95}$$

where K_n is the Bessel function of an imaginary argument with $\gamma = 2\Delta_\infty b|\sin(2\alpha)|/v_\perp$. The energy takes the form

$$\epsilon_0(b) = \Delta_0|\sin 2\alpha| \frac{K_0(\gamma)}{K_1(\gamma)} + bp_\perp\omega_c/2. \tag{6.96}$$

As a result, for $\xi \ll b \ll \xi/|\sin 2\alpha|$, we get

$$\epsilon_0(\alpha, b) = \frac{2\Delta_\infty^2 \sin^2(2\alpha) Lb}{v_\perp} + \frac{p_\perp b\omega_c}{2} \tag{6.97}$$

where $L = \ln[1/|\sin(2\alpha)|]$ for $b \ll \xi$ and $L = \ln[\xi v_\perp/bv_F|\sin(2\alpha)|]$ for $\xi \ll b \ll \xi/|\sin(2\alpha)|$. The states with energies much below the gap at infinity, $\epsilon \ll \Delta_0|\sin 2\alpha|$ correspond to $b \ll \xi/|\sin 2\alpha|$.

Equation (6.97) defines the energy of a particle that moves along the trajectory passing by a vortex at an impact parameter b. However, it is not the true quantum mechanical state because its energy depends on the momentum direction α. For a fixed energy, the particle trajectory starts to precess. We can find the true quantum states for low energies using the semi-classical Bohr-Sommerfeld quantization rule (Kopnin and Volovik 1997). Let us define the angular momentum as $\mu = -b/p_\perp$. The angular momentum $\mu(\alpha)$ is expressed through the quasiparticle energy via $E = \epsilon_0(\alpha, b)$ which determines $\mu(\alpha)$ as a function of α. The quantization rule requires

$$\oint \mu(\alpha) d\alpha = 2\pi \left(m + \frac{1}{2}\right) \tag{6.98}$$

where m is an integer, $\frac{1}{2}$ appears because the single-particle wave function changes its sign after encircling a single-quantum vortex.

Consider an energy $\epsilon \ll \Delta_\infty \sqrt{H/H_{c2}}$. We have from eqn (6.97)

$$-\oint \frac{E d\alpha}{\omega_0(\alpha)} = 2\pi \left(m + \frac{1}{2}\right) \tag{6.99}$$

where

$$\omega_0(\alpha) = 8\tilde{\omega}_0\alpha^2 + \omega_c/2 \tag{6.100}$$

with $\tilde{\omega}_0 = (\Delta_\infty^2/v_\perp p_\perp)\, L$. The integral (6.99) converges and is determined by angles $\alpha \sim \sqrt{\omega_c E_F/\Delta_\infty^2} \sim \sqrt{H/H_{c2}}$. The characteristic impact parameters are of the order of $b \sim \epsilon/p_\perp\omega_c$, i.e., $b \ll r_0$ where $r_0 \sim \xi\sqrt{H_{c2}/H}$ is of the order of the distance between vortices. We obtain

$$E_0 = -\Omega_0(m + 1/2) \tag{6.101}$$

where

$$\Omega_0 = \sqrt{\tilde{\omega}_0\omega_c} \tag{6.102}$$

The logarithmic factor in $\tilde{\omega}$ becomes

$$L = \ln\left[(\Delta_\infty/\epsilon_0)\sqrt{H/H_{c2}}\right]$$

The states defined by eqn (6.101) have much smaller interlevel spacing as compared to those in an s-wave superconductor, eqn (6.85). Moreover, we see that only the states with very low

energies $\epsilon \ll \Delta_\infty\sqrt{H/H_{c2}}$ are truly localized while the states with energies $\epsilon \sim \Delta_\infty\sqrt{H/H_{c2}}$ and above are not localized in a strict quantum mechanical sense. Indeed, these particles spend most of their time having such momentum directions $\alpha \sim \sqrt{H/H_{c2}}$ that their energies are above the gap along this direction $|\Delta_{\mathbf{p}}| \sim \Delta_\infty\sqrt{H/H_{c2}}$. Thus a particle with an energy $\epsilon \sim \Delta_\infty\sqrt{H/H_{c2}}$ escapes from the vortex core along the gap nodes.

The singular behavior of the energy spectrum in the directions of gap nodes results in a non-trivial magnetic field dependence of the density of states in the vortex core. Indeed, eqn (6.87) determines now a density of states which depends not only on p_z but also on the momentum direction within the ab plane. Averaging it over the angles α we obtain

$$\langle\nu(\epsilon)\rangle = \frac{\nu(0)v_\perp}{2p_\perp}\int_0^{2\pi}\frac{d\alpha}{\omega_0(\alpha)} = \frac{\pi\nu(0)}{m\Omega_0} \tag{6.103}$$

The density of states per vortex is considerably larger than in an s-wave case. It can be understood as an increase in the effective size of the vortex core due to a larger extension of the wave function in the directions of the gap nodes. Moreover, the density of states depends on the magnetic field $\langle\nu(\epsilon)\rangle \propto \sqrt{H_{c2}/H}$. Multiplied with the number of vortices, it gives an energy-independent density of states proportional to $\sqrt{H/H_{c2}}$. This behavior of the low-energy density of states in d-wave superconductors in presence of vortices was first predicted by Volovik (1988, 1993).

17.3 QUASICLASSICAL METHOD FOR LAYERED SUPERCONDUCTORS

We generalize the quasiclassical scheme for layered superconductors. The Ginzburg-Landau theory for layered superconductors known as the LawrenceDoniach model is derived. Using this model we discuss the behavior of the upper critical field and investigate interaction of vortices with the underlying crystalline structure (intrinsic pinning).

17.3.1 7.1 Quasiclassical Green functions

The quasiclassical method described in Chapter 5 can be directly applied to anisotropic superconductors, as well as to two dimensional systems. However, its generalization for quasi-two-dimensional (layered) superconductors is more complicated. The difficulty arises because a quasi-two-dimensional system, in fact, is not fully quasiclassical: the quasiclassical approximation is not fulfilled when the coherence length in the direction perpendicular to the layers is comparable with the distance between the layers. Nevertheless, if the coherence length in other directions is still larger than the interatomic distance within the layers, one can construct a generalization of the above approach which sometimes can be successfully used for reducing the full Green function technique to a more manageable form.

Consider a system which consists of layers with a good conductivity in the crystallographic ab plane; these layers are stacked along the c axis with a weak conductivity in the c direction. A strong anisotropy of conductivity in a metallic state is associated with a very anisotropic normal-state electronic spectrum which can be written in the form

$$\xi_p = E_n^{(2)}(\mathbf{p}) + \psi\left(p_z\right) - E_F \tag{7.1}$$

where $\psi \ll E_n^{(2)}$. The bold letters in this Chapter denote two-dimensional vectors in the ab plane, and p_z is the momentum along the c axis chosen as the z coordinate axis. The function ψ can also depend on $\mathbf{p}$. If the distance between the layers is s, the energy $\psi\left(p_z\right)$ is periodic in p_z with the period $2\pi/s$. A small energy $\psi\left(p_z\right)$ is usually associated with a narrow energy band

produced by a weak tunneling between the conducting layers. It is thus natural to assume the spectrum in the form

$$\psi\left(p_z\right)=J\cos\left(p_z s\right) \tag{7.2}$$

Characteristic superconducting wave vectors along the c axis are $k_z \sim \xi_c^{-1}$ where ξ_c is the coherence length in the c direction. If it were $\xi_c \gg s$, we would have $k_z \ll p_z$ and the energy $\psi\left(p_z+k_z\right)$ could be expanded as

$$\psi\left(p_z+k_z\right)=\psi\left(p_z\right)+\frac{\partial \psi}{\partial p_z} k_z=\psi\left(p_z\right)+v_{Fz}\left(p_z\right) k_z$$

This would allow us to determine the Fermi momentum ($\mathbf{p}_F, p_{Fz}$) from the condition $E_n^{(2)}(\mathbf{p})+\psi\left(p_z\right)-E_F=0$ and define the quasiclassical Green functions in a usual way as

$$\check{g}\left(\mathbf{p}_F, p_{Fz} ; \mathbf{k}, k_z\right)=\oint \check{G}\left(\mathbf{p}, p_z ; \mathbf{p}-\mathbf{k}, p_z-k_z\right) \frac{d \xi_p}{\pi i}.$$

For such definition it is very important that all the components of both ($\mathbf{p}, p_z$) and ($\mathbf{p}-\mathbf{k}, p_z-k_z$) are close to the corresponding components of the Fermi momentum ($\mathbf{p}_F, p_{Fz}$). One then returns to the usual quasiclassical scheme with an anisotropic Fermi surface.

The situation changes when $\xi_c \sim s$. Now one can not determine a threedimensional Fermi momentum in such a way that both ($\mathbf{p}, p_z$) and ($\mathbf{p}-\mathbf{k}, p_z-k_z$) are close to the corresponding components ($\mathbf{p}_F, p_{Fz}$). Nevertheless, one can introduce a two-dimensional Fermi momentum $\mathbf{p}_F$ through the condition $\xi_{\mathbf{p}}^{(2)}=0$ where

$$\xi_{\mathbf{p}}^{(2)}=E_n\left(\mathbf{p}_F\right)-E_F$$

such that the momenta $\mathbf{p}$ and $\mathbf{p}-\mathbf{k}$ are close to $\mathbf{p}_F$. The Fermi momentum $\mathbf{p}_F$ belongs to a two-dimensional (cylindrical) Fermi surface and is specified by an angle α in the ab plane. Let us define the "quasiclassical" Green function as

$$\check{g}\left(\mathbf{p}_F, \mathbf{k} ; p_z, p_z^{\prime}\right)=\oint \check{G}\left(\mathbf{p}, p_z ; \mathbf{p}-\mathbf{k}, p_z^{\prime}\right) \frac{d \xi_{\mathbf{p}}^{(2)}}{\pi i}$$

The energy integration is carried out along the normal

$$\mathbf{v}_F=\frac{\partial \xi_{\mathbf{p}}^{(2)}}{\partial \mathbf{p}}$$

to the cylindrical Fermi surface. The function $\check{g}$ depends on the Fermi-momentum direction α, on a two-dimensional center-of-mass coordinate $\mathbf{r}$ through the wave vector $\mathbf{k}$, and on two momenta p_z and $p_z^{\prime}$ in the z direction. The difference from the traditional quasiclassical Green function is that the dependence on the center-of-mass coordinate $\left(z+z^{\prime}\right)/2$ through $p_z-p_z^{\prime}$ does not separate any more from the relative momentum dependence $\left(p_z+p_z^{\prime}\right)/2$. This is the price which one has to pay for the fact that the interlayer distance is comparable with the coherence length ξ_c.

The momentum-space integral becomes

$$\int \frac{d^3 p}{(2 \pi)^3}=s \nu^{(2)} \int d \xi_{\mathbf{p}}^{(2)} \int_{-\pi / s}^{\pi / s} \frac{d p_z}{2 \pi} \int \frac{d \alpha}{2 \pi}$$

Here $\nu^{(2)}$ is the two-dimensional density of states. It is

$$\nu^{(2)}=\frac{m}{2 \pi s}$$

for the spectrum $E_n^{(2)} = \mathbf{p}^2/2m$.
The order parameter equation (5.27) for the Matsubara representation becomes

$$\frac{\Delta(\mathbf{k}, k_z)}{\lambda^{(2)}} = \pi i T \sum_j \int_{-\pi/s}^{\pi/s} \frac{s dp_z}{2\pi} \int \frac{d\alpha}{2\pi} f_{\omega_j}(\mathbf{p}_F, \mathbf{k}; p_+, p_-) \tag{7.3}$$

where the interaction constant $\lambda^{(2)} = \nu^{(2)}|g|$ and

$$p_\pm = p_z \pm \frac{k_z}{2}$$

In the real frequency representation, one has

$$\begin{aligned}\frac{\Delta(\mathbf{k}, k_z)}{\lambda^{(2)}} = & \int_{-\pi/s}^{\pi/s} \frac{s dp_z}{2\pi} \int \frac{d\alpha}{2\pi} \int \frac{d\epsilon}{4} \tanh\left(\frac{\epsilon}{2T}\right) \\ & \times \left[f_\epsilon^R(\mathbf{p}_F, \mathbf{k}; p_+, p_-) - f_\epsilon^A(\mathbf{p}_F, \mathbf{k}; p_+, p_-)\right]\end{aligned} \tag{7.4}$$

The current in the ab plane has its usual form

$$\mathbf{j}(\mathbf{k}, k_z) = -2e\nu^{(2)}\pi i T \sum_j \int_{-\pi/s}^{\pi/s} \frac{s dp_z}{2\pi} \int \frac{d\alpha}{2\pi} \mathbf{v}_F g_{\omega_j}(\mathbf{p}_F, \mathbf{k}; p_+, p_-)$$

for the Matsubara representation or

$$\begin{aligned}\mathbf{j}(\mathbf{k}, k_z) = & -2e\nu^{(2)} \int \frac{d\epsilon}{4} \tanh\left(\frac{\epsilon}{2T}\right) \int_{-\pi/s}^{\pi/s} \frac{s dp_z}{2\pi} \int \frac{d\alpha}{2\pi} \mathbf{v}_F \\ & \times \left[g_\epsilon^R(\mathbf{p}_F, \mathbf{k}; p_+, p_-) - g_\epsilon^A(\mathbf{p}_F, \mathbf{k}; p_+, p_-)\right]\end{aligned} \tag{7.5}$$

for the real-frequency representation.
The current in the c direction is defined as follows

$$\begin{aligned}j_z(\mathbf{k}, k_z) = & -2e\nu^{(2)} \int \frac{d\epsilon}{4} \tanh\left(\frac{\epsilon}{2T}\right) \int_{-\pi/s}^{\pi/s} \frac{s dp_z}{2\pi} \int \frac{d\alpha}{2\pi} V_z \\ & \times \left[g_\epsilon^R(\mathbf{p}_F, \mathbf{k}; p_+, p_-) - g_\epsilon^A(\mathbf{p}_F, \mathbf{k}; p_+, p_-)\right]\end{aligned} \tag{7.6}$$

where

$$V_z = \frac{1}{2}\left[\frac{\partial\psi(p_+ - eA_z/c)}{\partial p_z} + \frac{\partial\psi(p_- - eA_z/c)}{\partial p_z}\right]$$

This definition complies with the condition of minimum of the thermodynamic potential $\delta\Omega/\delta A_z = 0$ where the variation of Ω is expressed through the Green
function according to eqn (5.30). For a layered system, the Hamiltonian should be taken in the form

$$\check{H}(\mathbf{r}, z) = \check{H}^{(2)}(\mathbf{r}, z) + \psi\left(p_z - \frac{e}{c}A_z\check{\tau}_3\right) \tag{7.7}$$

where

$$\check{H}^{(2)}(\mathbf{r}, z) = \begin{pmatrix} -(e/c)\mathbf{v}_F\mathbf{A}(\mathbf{r}, z) & -\Delta(\mathbf{r}, z) \\ \Delta^*(\mathbf{r}, z) & (e/c)\mathbf{v}_F\mathbf{A}(\mathbf{r}, z) \end{pmatrix}. \tag{7.8}$$

17.3.2 7.2 Eilenberger equations for layered systems

Equations for the retarded and advanced (or Matsubara) quasiclassical Green functions can be obtained in a way similar to that used earlier in Section 5.5. It is more convenient to formulate them for the functions in the z-coordinate representation

$$\check{g}\left(\mathbf{p}_F, \mathbf{r}; z, z'\right) = \int \check{g}\left(\mathbf{p}_F, \mathbf{k}; p_z, p_z'\right) e^{i\mathbf{k}\mathbf{r}+ip_z z-ip_z' z'} \frac{d^3k}{(2\pi)^3} \frac{dp_z dp_z'}{(2\pi)^2}$$

We obtain for the real-frequency functions

$$\begin{aligned}
&\left[-i\mathbf{v}_F \frac{\partial}{\partial \mathbf{r}} + \psi\left(\hat{p}_z - \frac{e}{c} A_z \check{\tau}_3\right) - \psi\left(\hat{p}_{z'}^{\dagger} - \frac{e}{c} A_{z'} \check{\tau}_3\right)\right] \check{g}_\epsilon\left(\mathbf{p}_F, \mathbf{r}; z, z'\right) \\
&- \epsilon\left[\check{\tau}_3 \check{g}_\epsilon\left(\mathbf{p}_F, \mathbf{r}; z, z'\right) - \check{g}_\epsilon\left(\mathbf{p}_F, \mathbf{r}; z, z'\right) \check{\tau}_3\right] \\
&+ \check{H}^{(2)}(\mathbf{r}, z) \check{g}_\epsilon\left(\mathbf{p}_F, \mathbf{r}; z, z'\right) - \check{g}_\epsilon\left(\mathbf{p}_F, \mathbf{r}; z, z'\right) \check{H}^{(2)}\left(\mathbf{r}, z'\right) \\
&- \check{\Sigma}_\epsilon(\mathbf{r}, z) \check{g}_\epsilon\left(\mathbf{p}_F, \mathbf{r}; z, z'\right) + \check{g}_\epsilon\left(\mathbf{p}_F, \mathbf{r}; z, z'\right) \check{\Sigma}_\epsilon\left(\mathbf{r}, z'\right) = 0
\end{aligned} \tag{7.9}$$

Here $\check{g}$ stands for $\check{g}^{R(A)}$,

$$\hat{p}_z = -i\frac{\partial}{\partial z}, \hat{p}_{z'}^{\dagger} = i\frac{\partial}{\partial z'}$$

and the z component of the vector potential $A_{z'}$ is a function of ($\mathbf{r}, z'$). The effective Hamiltonian is determined by eqn (7.8). It is a two-dimensional part of the "full" Hamiltonian of the type of eqn (5.57) used to derive the Eilenberger equations for the three dimensional case in Section 5.5. Σ is the self energy. For example, for isotropic impurity scattering it is

$$\check{\Sigma} = n_{\text{imp}} u^2 \int \check{G} \frac{d^3p}{(2\pi)^3} = \pi i \nu^{(2)} u^2 n_{\text{imp}} \int \check{g} \frac{s dp_z}{2\pi} \frac{d\alpha}{2\pi} = \frac{i}{2\tau} \int \check{g} \frac{s dp_z}{2\pi} \frac{d\alpha}{2\pi}$$

Let us derive now the normalization condition for quasiclassical Green functions in layered superconductors using the same approach as we did for three dimensional case on page 105. Multiplying eqn (7.9) with $\check{g}\left(\mathbf{p}_F, \mathbf{r}; z, z'\right)$ from the left and adding to the equation multiplied from the right we obtain that the combination

$$\int \check{g}_\epsilon^{R(A)}\left(\mathbf{p}_F, \mathbf{r}; z, z''\right) \check{g}_\epsilon^{R(A)}\left(\mathbf{p}_F, \mathbf{r}; z'', z'\right) dz''$$

again satisfies equation (7.9). Now we employ the same argumentation as for the three dimensional case. Let us assume that, at large distances (in the direction of the ab planes) from the region under consideration, the order parameter becomes independent of coordinates, and the magnetic field vanishes. We have in this case

$$g\left(p_z, p_z'\right) = \frac{\epsilon}{\sqrt{\epsilon^2 - \Delta^2}} 2\pi\delta\left(p_z - p_z'\right), f\left(p_z, p_z'\right) = \frac{\Delta}{\sqrt{\epsilon^2 - \Delta^2}} 2\pi\delta\left(p_z - p_z'\right)$$

It follows that, in the momentum representation, one has at large distances

$$\int \check{g}^{R(A)}\left(\mathbf{p}_F, \mathbf{r}; p_z, p_z''\right) \check{g}^{R(A)}\left(\mathbf{p}_F, \mathbf{r}; p_z'', p_z'\right) \frac{dp_z''}{2\pi} = \check{1} \cdot 2\pi\delta\left(p_z - p_z'\right) \tag{7.10}$$

Since the expression in eqn (7.10) is an integral of motion, we find that eqn (7.10) holds also everywhere including the region where Δ has its actual spatial dependence, and the magnetic filed is finite. Equation (7.10) in the coordinate representation reads

$$\int \check{g}_{\epsilon}^{R(A)}\left(\mathbf{p}_F, \mathbf{r}; z, z''\right) \check{g}_{\epsilon}^{R(A)}\left(\mathbf{p}_F, \mathbf{r}; z'', z'\right) dz'' = \check{1}\delta\left(z - z'\right) \tag{7.11}$$

Eilenberger equations (7.9) and the normalization eqn (7.10) are not local in the center-of-mass coordinate $(z + z')/2$. This reduces considerably the potentialities of the quasiclassical methods for quasi-two-dimensional systems.

A layered superconductor can also be described by introducing the coupling between two-dimensional layers via mechanisms other than the electron tunneling between layers based on eqn (7.1). For example, an interaction between layers which, by themselves, form a "good" two-dimensional quasiclassical environment can be mediated via scattering by impurities (Graf et al. 1993).

17.3.3 7.3 Lawrence-Doniach model

One important example when the Eilenberger equations (7.9) can be solved is the Lawrence-Doniach model (Lawrence and Doniach 1971) which is an analogue of the Ginzburg-Landau theory for layered superconductors. The basic assumptions are as follows. First, it is required that the temperature is close to the critical temperature T_c. This allows to expand the Green function in powers of Δ/T_c and in slow gradients in the ab plane in the same way as it was done to derive the usual Ginzburg-Landau theory in Section 6.1.2. The second requirement is that the corrugation of the Fermi surface is small compared to the critical temperature $\psi \ll T_c$. We shall also assume that the vector potential varies slowly at distances of the order of ξ_0 and s which is the case for $T \to T_c$. The approach which we describe below applies to either s-wave or d-wave pairing: In these cases the pairing interaction is independent of the momentum p_z, and the order parameter only depends on the center-of-mass coordinate $(z + z')/2$ through $p_z - p_z' = k_z$, i.e. $\Delta = \Delta(k_z)$. We shall see that this form of the p_z dependence is essential: the situation would be different if Δ depended on p_z and p_z' separately. For simplicity, we consider here only the s-wave case.

7.3.1 Order parameter

Assume an isotropic scattering by impurities and solve the Eilenberger equation in the momentum representation with respect to the coordinate z using the perturbation expansion in Δ/T_c and ψ/T_c :

$$g = g_0 + g_1; f = f_0 + f_1 + f_2$$

Denote

$$\hat{L} = -i\mathbf{v}_F\left(\nabla - \frac{2ie}{c}\mathbf{A}\right) + \psi\left(p_+ - \frac{e}{c}A_z\right) - \psi\left(p_- + \frac{e}{c}A_z\right)$$

We find using the normalization eqn (7.10)

$$g_0^{R(A)}\left(p_+,\ p_-\right) = \pm 2\pi\delta\left(k_z\right) \tag{7.12}$$

$$\begin{aligned} g_1^{R(A)}\left(p_+, p_-\right) = \frac{|\Delta|^2}{2\epsilon^2} + \frac{\Delta\left(k_z'\right)}{2\epsilon} f_1^{\dagger R(A)}\left(p_+ - k_z', p_-\right) \\ + f_1^{R(A)}\left(p_+, p_- + k_z'\right)\frac{\Delta\left(k_z'\right)}{2\epsilon} \end{aligned} \tag{7.13}$$

and

$$f_0^{R(A)}\left(p_+,p_-\right)=\pm\frac{\Delta\left(k_z\right)}{\epsilon} \tag{7.14}$$

$$f_1^{R(A)}\left(p_+,p_-\right)=\frac{\hat{L}f_0^{R(A)}\pm\left(i/2\tau\epsilon\right)\left\langle\hat{L}f_0^{R(A)}\right\rangle}{2\epsilon\pm i/\tau} \tag{7.15}$$

$$f_2^{R(A)}\left(p_+,p_-\right)=\frac{\hat{L}f_1^{R(A)}\pm\left(i/2\tau\epsilon\right)\left\langle\hat{L}f_1^{R(A)}\right\rangle}{2\epsilon\pm i/\tau}\pm\frac{|\Delta|^2\Delta}{2\epsilon^3} \tag{7.16}$$

The average over the Fermi surface for a layered system is defined as

$$\langle\cdots\rangle=\int\frac{sdp_z}{2\pi}\frac{d\alpha}{2\pi}(\cdots)$$

We obtain

$$\left\langle f^{R(A)}\right\rangle=\pm\left[\frac{\Delta}{\epsilon}+\frac{\left\langle\hat{L}^2\right\rangle\Delta}{2\epsilon^2(2\epsilon\pm i/\tau)}+\frac{|\Delta|^2\Delta}{2\epsilon^3}\right] \tag{7.17}$$

We assume here that the spectrum has the inversion symmetry $\psi\left(p_z\right)=\psi\left(-p_z\right)$ such that $\langle\hat{L}\rangle=0$.

Consider for simplicity an uniaxial superconductor isotropic in the ab plane. The order parameter equation (7.4) yields

$$\left[\frac{T_c-T}{T_c}\Delta-\frac{7\zeta(3)}{8\pi^2T_c^2}|\Delta|^2\Delta+\frac{\pi v_F^2\tau\Lambda}{16T_c}\left(\nabla-\frac{2ie}{c}\mathbf{A}\right)^2\Delta\right]_{k_z}$$
$$-\frac{\pi\tau\Lambda}{8T_c}\int\frac{sdp_z}{2\pi}\left[\psi\left(p_+-\frac{e}{c}A_z\right)-\psi\left(p_-+\frac{e}{c}A_z\right)\right]^2\Delta\left(k_z\right)=0 \tag{7.18}$$

where $\Lambda\left(T_c\tau\right)$ is defined by eqn (6.22). The first line is a Fourier transform at the momentum k_z.

In the z-coordinate representation, the order parameter is determined only on the conducting layers. We specify the layers with the number n such that $z_n=ns$. We have

$$\Delta_n(\mathbf{r})=s\int_{-\pi/s}^{\pi/s}\Delta\left(k_z,\mathbf{r}\right)e^{ik_zsn}\frac{dk_z}{2\pi};\Delta\left(k_z,\mathbf{r}\right)=\sum_n\Delta_n(\mathbf{r})e^{-ik_zsn} \tag{7.19}$$

Equations (7.18) and (7.19) completely determine the order parameter in a layered superconductor. However, eqn (7.18) can be simplified further using eqn (7.2). We have

$$\int_{-\pi/s}^{\pi/s}\frac{sdp_z}{2\pi}\left[\psi\left(p_z+\frac{k_z}{2}-\frac{e}{c}A_z\right)-\psi\left(p_z-\frac{k_z}{2}+\frac{e}{c}A_z\right)\right]^2$$
$$=4J^2\sin^2\left(\frac{sk_z-\chi_A}{2}\right)\int_{-\pi/s}^{\pi/s}\frac{sdp_z}{2\pi}\sin^2\left(p_zs\right)$$
$$=2J^2\sin^2\left(\frac{sk_z-\chi_A}{2}\right)$$

where

$$\chi_A=\frac{2e}{c}\int_{ns}^{(n+1)s}A_zdz=\frac{2e}{c}\int_{(n-1)s}^{ns}A_zdz$$

We use that the vector potential varies slowly at distances of the order of the interlayer distance. One has

$$\sin^2\left(\frac{sk_z-\chi_A}{2}\right)\Delta\left(k_z\right)=-\frac{1}{4}\sum_n\Delta_n e^{-ik_zsn}\left[e^{i(k_zs-\chi_A)}+e^{-i(k_zs-\chi_A)}-2\right]$$
$$=-\frac{1}{4}\sum_n\left[\Delta_{n+1}e^{-i\chi_A}+\Delta_{n-1}e^{i\chi_A}-2\Delta_n\right]e^{-ik_zsn}$$

Equation (7.18) in the coordinate representation becomes

$$\alpha\Delta_n+\beta\left|\Delta_n\right|^2\Delta_n-\gamma_{ab}\left(\nabla-\frac{2ie}{c}\mathbf{A}\right)^2\Delta_n$$
$$-\frac{\gamma_c}{s^2}\left(\Delta_{n+1}e^{-i\chi_A}+\Delta_{n-1}e^{+i\chi_A}-2\Delta_n\right)=0 \tag{7.20}$$

where

$$\alpha=\nu^{(2)}\frac{T-T_c}{T_c},\beta=\frac{7\zeta(3)\nu^{(2)}}{8\pi^2T_c^2},$$
$$\gamma_{ab}=\frac{\pi\nu^{(2)}v_F^2\tau\Lambda}{16T_c},\gamma_c=\frac{\pi\nu^{(2)}J^2s^2\tau\Lambda}{16T_c}$$

Equation (7.20) is the Lawrence-Doniach equation for layered superconductors. It has been derived microscopically by Klemm et al. (1975).

As in the three dimensional case, we put

$$\xi_0=\frac{v_F}{2\pi T_c}$$

The temperature dependent coherence length in the ab plane is

$$\xi_{ab}^2(T)=\frac{\gamma_{ab}}{|\alpha|}=(\xi_0\ell)\frac{\pi^2\Lambda}{8}\frac{T_c}{T_c-T}$$

The last term in the square brackets of eqn (7.20) is the discrete analogue of the second derivative. In the continuous limit, $s\to 0$, we can write

$$\Delta_{n+1}e^{-i\chi_A}+\Delta_{n-1}e^{+i\chi_A}-2\Delta_n=s^2\left(\frac{\partial}{\partial z}-\frac{2ie}{c}A_z\right)^2\Delta(z)$$

Comparing this with the first term in the square brackets, we can define the coherence length ξ_c through the ratio

$$\frac{\xi_c^2}{\xi_{ab}^2}=\frac{\gamma_c}{\gamma_{ab}}=\frac{J^2s^2}{v_F^2}$$

In the continuous limit, $s\ll\xi_c$, eqn (7.20) transforms into the Ginzburg-Landau equation for anisotropic uniaxial superconductor

$$\alpha\Delta+\beta|\Delta|^2\Delta-\gamma_{ab}\left(\nabla-\frac{2ie}{c}\mathbf{A}\right)^2\Delta-\gamma_c\left(\frac{\partial}{\partial z}-\frac{2ie}{c}A_z\right)^2\Delta=0 \tag{7.21}$$

It has exactly the same form as eqn (6.68) for an anisotropic d-wave superconductor.

7.3.2 Free energy and the supercurrent

Equations (7.13) and (7.16) allow us to reconstruct the Lawrence-Doniach free energy. Using eqn (3.71) or (5.30) with the definition eqn (7.7) we find

$$\begin{aligned}\mathcal{F}_{sn} = s\sum_n \int d^2r \Bigg[&\alpha\left|\Delta_n\right|^2 + \frac{\beta}{2}\left|\Delta_n\right|^4 \\ &+\gamma_{ab}\left|\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\Delta_n\right|^2 + \frac{\gamma_c}{s^2}\left|\Delta_{n+1}e^{-i\chi_A} - \Delta_n\right|^2\Bigg].\end{aligned} \tag{7.22}$$

Equation (7.20) is reproduced when one calculates the variation

$$\frac{\delta\mathcal{F}_{sn}}{\delta\Delta_n^*} = 0$$

The supercurrent can be found using the definition

$$\mathbf{j}_s = -c\frac{\delta\mathcal{F}_{sn}}{\delta\mathbf{A}}$$

The supercurrent density in the ab direction at the position $z_n = ns$ of the n-th layer has the usual form

$$\mathbf{j}_s = -2ie\gamma_{\alpha b}\left[\Delta_n^*\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\Delta_n - \Delta_n\left(\nabla + \frac{2ie}{c}\mathbf{A}\right)\Delta_n^*\right] \tag{7.23}$$

The current density in the c direction at the n-th layer becomes

$$j_s^{(c)} = -\frac{ie\gamma_c}{s}\left[\left(\Delta_{n+1}\Delta_n^* + \Delta_n\Delta_{n-1}^*\right)e^{-i\chi_A} - \left(\Delta_{n+1}^*\Delta_n + \Delta_n^*\Delta_{n-1}\right)e^{i\chi_A}\right] \tag{7.24}$$

or

$$\begin{aligned}j_s^{(c)} =&\frac{2e\gamma_c}{s}\Bigg[\left|\Delta_{n+1}\Delta_n\right|\sin\left(\chi_{n+1} - \chi_n - \frac{2es}{c}A_z\right) \\ &+ \left|\Delta_n\Delta_{n-1}\right|\sin\left(\chi_n - \chi_{n-1} - \frac{2es}{c}A_z\right)\Bigg]\end{aligned} \tag{7.25}$$

This expression corresponds to the Josephson current which is produced by Cooper pair tunneling between the layers.

For a highly layered case, when $s \sim \xi_c$, the maximum Josephson current in eqn (7.25) is

$$j_{\max} = 4e\gamma_c|\Delta|^2/s$$

It enters the definition of the penetration length λ_c for currents flowing in the c direction. Indeed, for small fields, one can expand eqn (7.25) in a small argument to get

$$\begin{aligned}j_s^{(c)} &= -\frac{8e^2\gamma_c|\Delta|^2}{c}\left[A_z - \frac{c}{2es}\left(\chi_{n+1} - \chi_n\right)\right] \\ &= -\frac{2ej_{\max}s}{c}\left[A_z - \frac{c}{2es}\left(\chi_{n+1} - \chi_n\right)\right].\end{aligned}$$

Together with the Maxwell equation

$$j_s^{(c)} = \frac{c}{4\pi}\,\mathrm{curlcurl}\,A_z$$

this determines

$$\lambda_c^{-2} = \frac{8\pi^2 s j_{\max}}{c\Phi_0} \tag{7.26}$$

which is called the Josephson length. If the Josephson coupling is small the penetration length λ_c^2 diverges.

When the order parameter is normalized in such a way that it is the wave function Ψ of superconducting electrons, the Lawrence-Doniach free energy eqn (7.22) becomes

$$\mathcal{F}_{sn} = s \int d^2r \sum_n F(n) \tag{7.27}$$

where the energy density at a layer n is

$$\begin{aligned} F(n) =& a|\Psi(n,\mathbf{r})|^2 + \frac{b}{2}|\Psi(n,\mathbf{r})|^4 + \frac{1}{2m_{ab}}\left|\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\Psi(n,\mathbf{r})\right|^2 \\ &+ \frac{1}{2m_c s^2}\left|\Psi(n+1,\mathbf{r})e^{-i\chi_A} - \Psi(n,\mathbf{r})\right|^2 \end{aligned} \tag{7.28}$$

This is similar to eqn (1.27). The constants a and b satisfy

$$\frac{|a|}{b} = \frac{m_{ab}c^2}{16\pi e^2 \lambda_{ab}^2} = |\Psi_{GL}|^2$$

where λ_{ab} is the London penetration length for currents flowing in the ab plane. The order parameter magnitude is

$$|\Psi_{GL}|^2 = 2m_{ab}\gamma_{ab}\Delta_\infty^2 \tag{7.29}$$

The coherence lengths are expressed through the "effective masses"

$$\xi_{ab}^2 = \frac{1}{2m_{ab}|a|}, \xi_c^2 = \frac{1}{2m_c|a|} \tag{7.30}$$

The effective mass in the ab plane for clean superconductors $\ell \gg \xi_{ab}$ has the usual meaning

$$m_{ab}^{-1} = \partial^2 E_n^{(2)}/\partial p^2$$

The "effective mass" m_c, however, can be written as $(\partial^2\psi/\partial p_z^2)^{-1}$ only in the continuous limit $\xi_c \gg s$.

The Lawrence-Doniach equations can also be derived using the model by Graf al. (1993) mentioned in the beginning of this Section. In this case, the effective mass m_c is expressed through the interlayer diffusion.

7.3.3 Microscopic derivation of the supercurrent

The expression for the supercurrent can also be obtained microscopically using our results for the Green functions eqn (7.13). The current along the ab plane has a standard form and can be obtained exactly in the same way as it was done for the usual Ginzburg-Landau equation in Section 6.1.2. The current along c direction can be calculated from the corresponding terms in the Green function, namely,

$$g_1^{R(A)}(p_+,p_-) = \pm\left[\psi\left(p_+ - \frac{e}{c}A_z\right) - \psi\left(p_- + k_z' + \frac{e}{c}A_z\right)\right]\frac{\Delta^*(k_z')\,\Delta(k_z - k_z')}{2\epsilon^2(2\epsilon \pm i/\tau)}$$
$$\mp\left[\psi\left(p_+ - k_z' + \frac{e}{c}A_z\right) - \psi\right.$$

We now insert this expression into the definition of current, eqn (7.6), and make the shift of integration variable $p_z - (e/c)A_z \to p_z$. The terms which are odd in $p_z \to -p_z$ drop out after integration over dp_z and we get

$$\begin{aligned} j_s^{(c)}(k_z) =& e\nu^{(2)} \int \frac{d\epsilon}{8} \tanh\left(\frac{\epsilon}{2T}\right) \left[\frac{1}{\epsilon^2(2\epsilon + i/\tau)} - \frac{1}{\epsilon^2(2\epsilon - i/\tau)}\right] \\ & \times \int_{-\pi/s}^{\pi/s} \frac{s dp_z}{2\pi} \int \frac{d\alpha}{2\pi} \left[\frac{\partial\psi(p_+)}{\partial p_z} + \frac{\partial\psi(p_-)}{\partial p_z}\right] \\ & \times \left[\psi\left(p_+ - k_z' + \frac{2eA_z}{c}\right) \Delta(k_z') \Delta^*(k_z - k_z') \right. \\ & \left. + \psi\left(p_- + k_z' + \frac{2eA_z}{c}\right) \Delta^*(k_z') \Delta(k_z - k_z')\right] \end{aligned}$$

Using eqn (7.2) we obtain

$$\begin{aligned} j_s^{(c)}(k_z) = & -e\nu^{(2)} J^2 s \int \frac{d\epsilon}{8} \tanh\left(\frac{\epsilon}{2T}\right) \left[\frac{1}{\epsilon^2(2\epsilon + i/\tau)} - \frac{1}{\epsilon^2(2\epsilon - i/\tau)}\right] \\ & \times \left[\sin(k_z' s - \chi_A) - \sin(k_z s - k_z' s + \chi_A)\right] \Delta(k_z') \Delta^*(k_z - k_z') \end{aligned}$$

In the Fourier representation of eqn (7.19),

$$\begin{aligned} \sin(k_z' s - \chi_A) \Delta(k_z') \Delta^*(k_z - k_z') &\to \frac{\Delta_n^*}{2i} \left[\Delta_{n+1} e^{-i\chi_A} - \Delta_{n-1} e^{i\chi_A}\right] \\ \sin(k_z s - k_z' s + \chi_A) \Delta^*(k_z - k_z') \Delta(k_z') &\to \frac{\Delta_n}{2i} \left[\Delta_{n+1}^* e^{i\chi_A} - \Delta_{n-1}^* e^{-i\chi_A}\right] \end{aligned}$$

Finally, we return to eqn (7.24).

17.3.4 7.4 Applications of the Lawrence-Doniach model

The Lawrence-Doniach model is widely used for layered superconductors; it is simple and quite general at the same time. It covers the whole class of anisotropic systems from three dimensional weakly anisotropic to highly anisotropic layered and even two dimensional superconductors. It is able to provide a reasonable description for many properties of the mixed state of layered superconductors. For example, layered superconductors can have two-dimensional analogues of the usual Abrikosov vortices, the so called pancake vortices connected with each other by strings of Josephson vortices. Pancake vortices have been first considered by Artemenko and Kruglov (1990) and by Clem (1991). Pancakes turned out to
be very important for high temperature superconductors most of which are highly anisotropic or even layered compounds. Other interesting features of layered superconductors is an unusual behavior of the upper critical magnetic field: For orientation parallel to the layers, it grows rapidly with lowering the temperature as $\xi_c(T)$ approaches the interlayer distance s. Yet another important property is the so called intrinsic pinning considered first by Tachiki and Takahashi (1989) and elaborated further by Ivlev and Kopnin (1989) and by Barone et al. (1990). Many problems specific for layered superconductors are discussed in the review by Blatter et al. (1994). Here we briefly outline two mentioned examples: the upper critical field and intrinsic pinning which will be used later when we consider the vortex dynamics in anisotropic and layered superconductors.

7.4.1 Upper critical field

Let the magnetic field be parallel to the layers. As in Section 1.1.2 we look for a second-order phase transition from the normal into the superconducting state which occurs with lowering

the field. We choose the y axis along the direction of the field with the gauge $\mathbf{A} = (0, 0, -Hx)$ and assume a solution in the form

$$\Delta_n(\mathbf{r}) = e^{iqns} Y(x)$$

The linearized equation (7.20) for $Y(x)$ becomes the Mathieu equation:

$$\xi_{ab}^2 \frac{\partial^2 Y}{\partial x^2} - \frac{2\xi_c^2}{s^2}\left[1 - \cos\left(qs + \frac{2eHsx}{c}\right)\right] Y + Y = 0 \tag{7.31}$$

There are two limiting cases: (i) Weakly layered (nearly continuous) limit, and (ii) highly layered case. Consider them separately.

7.4.1.1 Weakly layered case In the continuous anisotropic case the upper critical field is known to be (Kats 1969, 1970, Lawrence and Doniach 1971):

$$H_{c2} = \frac{c}{2e\xi_{ab}\xi_c} \tag{7.32}$$

This equation can be reproduced also from eqn (7.31). Since $x \sim \xi_{ab}$ we find that the argument of the cosine function in eqn (7.31) is

$$qs + \frac{sx}{\xi_{ab}\xi_c} \sim \frac{s}{\xi_c}$$

In the continuous limit, $s \ll \xi_c$, the cosine function in eqn (7.31) can be expanded in a small argument. We obtain

$$\xi_{ab}^2 \frac{\partial^2 Y}{\partial x^2} - \xi_c^2 \left(q + \frac{2eHx}{c}\right)^2 Y + Y = 0 \tag{7.33}$$

This is the oscillator equation (1.33). Again, the lowest-level solution is

$$Y = \exp\left[-\frac{1}{2\xi_{ab}^2}\left(x + \frac{cq}{2eH_{c2}}\right)^2\right] \tag{7.34}$$

where H_{c2} is determined by eqn (7.32).

7.4.1.2 Highly layered case As we shall see, this limit corresponds to a high upper critical field and is realized for a certain relation between the interlayer distance s and the coherence length ξ_c. We introduce the parameter

$$h = \frac{2eHs^2\xi_{ab}}{c\xi_c} \tag{7.35}$$

For a weakly layered superconductor, $h = s^2/\xi_c^2 \ll 1$. On the contrary, a highly layered case is realized for $h \gg 1$ when $H \to \infty$. For this limit, a solution of eqn (7.31) can be obtained as an expansion in powers of a small $1/h$. The lowest-level periodic solution corresponds to an even Mathieu function

$$Y = 1 + \frac{2}{h^2}\cos\left(\frac{2eHsx}{c}\right) + \frac{1}{2h^4}\cos\left(\frac{4eHsx}{c}\right) + \ldots$$

provided

$$h^2 = \frac{1}{1 - (s^2/2\xi_c^2)}$$

This relation determines the upper critical field

$$H_{c2} = \frac{\Phi_0}{2\pi s^2}\frac{\xi_c}{\xi_{ab}}\frac{1}{\sqrt{1-(s^2/2\xi_c^2)}} \tag{7.36}$$

This solution is valid as long as $h \gg 1$, or $\xi_c(T) \to s/\sqrt{2}$.
We observe that the upper critical field diverges as the temperature dependent coherence length $\xi_c(T)$ approaches $s/\sqrt{2}$ from above with lowering the temperature (Klemm 1975). In this limit, vortex cores fit in-between the superconducting layers, and the supercurrents do not destroy superconductivity. Divergence of the upper critical field is similar to the Little and Parks effect for a non-singly-connected superconductor where the critical temperature is a periodic function of the magnetic field instead of vanishing with an increasing field. Therefore, an anisotropic superconductor that is essentially a three-dimensional system with $\xi_c(T) \gg s$ at a higher temperature, can undergo a transition into effectively a two-dimensional superconductor with lowering the temperature if the zero-temperature coherence length $\xi_c(0)$ is short enough,. This occurs when $\xi_c(T)$ decreases and reaches $s/\sqrt{2}$.

7.4.2 Intrinsic pinning: Single vortex in a weak potential

In this Section we consider an important effect specific for layered systems, namely, an interaction of the order parameter with the underlying crystalline structure. More particularly, we discuss an example of vortices which are aligned parallel to the superconducting planes. The interaction with the inhomogeneous layered environment gives rise to the "intrinsic pinning": The vortex energy becomes dependent on the vortex position with respect to the planes and produces a pinning force which tries to keep vortices in between the superconducting layers (Tachiki and Takahashi 1989). We consider a nearly continuous limit $\xi_c \gg s$.

Fig. 7.1. Integration contours in the summation formula. The poles of $\cot(\pi n)$ are shown by dots.

Assume that the magnetic field is parallel to the layers and find the energy of a single vortex in a layered structure. We use the approach developed by Barone et al. (1990). To calculate the sum in eqn (7.27) we employ the summation formula

$$\sum_n F(n) = -\frac{1}{2i}\int_C \cot(\pi n)F(n)dn = \int_{-\infty}^{\infty} F(n)dn$$
$$+\frac{i}{2}\left(\int_{C_1}[\cot(\pi n)+i]F(n)dn + \int_{C_2}[\cot(\pi n)-i]F(n)dn\right)$$

The integration contour C goes above and below the poles $n = 0, \pm 1, \ldots$ of $\cot(\pi n)$ encircling them clockwise, while the contours C_1 and C_2 go above and below the poles, respectively, so that $C = C_1 + C_2$ (see Fig. 7.1). The total free energy becomes

$$\mathcal{F} = \int d^2r dz \left[\frac{H^2}{8\pi} + F(z)\right] + U_{pin} \tag{7.37}$$

where

$$U_{pin} = \frac{i}{2}\int d^2r \left(\int_{C_1} \left[\cot\frac{\pi z}{s} + i\right] F(z)dz + \int_{C_2} \left[\cot\frac{\pi z}{s} - i\right] F(z)dz \right) \tag{7.38}$$

The first term in eqn (7.37) is independent of the vortex position. However, the energy U_{pin} is a function of the vortex coordinate along the z axis.

For a weakly layered case, to the first approximation in $s \ll \xi_c$, a vortex solution has the same form as for an anisotropic superconductor. Indeed, let us take the y axis along the magnetic field and put

$$\Psi = \Psi_{GL} f(\tilde{\rho}) e^{i\tilde{\chi}}$$

where

$$\tilde{\rho} = \sqrt{x^2/\xi_{ab}^2 + z^2/\xi_c^2}$$

and

$$\tan\tilde{\chi} = \frac{x/\xi_{ab}}{z/\xi_c}.$$

Assume that $\kappa = \lambda_{ab}/\xi_{ab}$ is large, $\kappa \gg 1$. We can neglect the magnetic field in the equation for f and obtain from eqns (7.21), (7.29), and (7.30)

$$\frac{d^2 f}{d\tilde{\rho}^2} + \frac{1}{\tilde{\rho}}\frac{df}{d\tilde{\rho}} - \frac{f}{\tilde{\rho}^2} + f - f^3 = 0 \tag{7.39}$$

This equation coincides with eqn (1.48) in Section 1.1.2 which we derived for a single vortex in an isotropic superconductor. We see that the anisotropy of the vortex core scales with the coherence length in the corresponding directions.

To calculate the pinning energy, we insert the exact vortex solution of eqn (7.20) into the free energy expression. In the small position-dependent part of the energy in eqn (7.38) one can use the approximate solution eqn (7.39) for an anisotropic case. To calculate the integrals in eqn (7.38), we shift the integration contours C_1 and C_2 to the upper and lower complex half planes of the integration variable z, respectively. The integral is then determined by a singularity of the function $f(z)$ which is the closest to the real axis. The function $f(z)$ has poles on the imaginary $\tilde{\rho}$ axis:

$$f(\rho) = \pm\frac{\sqrt{2}}{\tilde{\rho} \mp i\rho_0} \tag{7.40}$$

This fact can be easily understood from a model expression for the vortex-core function $f(\tilde{\rho}) = \tanh(\tilde{\rho})$. It has poles at $i\rho_0 = i(\pi/2 + \pi k)$. The numerical solution (Barone et al. 1990) shows that the closest poles to the real axis have $\rho_0 = 2.51$ in eqn (7.40).

Let the vortex be located at the point $x = 0, z = z_0$. The vortex amplitude f is a function of the shifted variable

$$\tilde{\rho} = \sqrt{x^2/\xi_{ab}^2 + (z - z_0)^2/\xi_c^2} \tag{7.41}$$

and its phase is now

$$\tan\tilde{\chi} = \frac{x/\xi_{ab}}{(z - z_0)/\xi_c}.$$

To the leading approximation in $\tilde{\rho} - \rho_0$, the energy density is determined by the highest singularity which comes from the gradient term:

$$F(z) = \frac{4|a|\Psi_{GL}^2}{(\tilde{\rho} \mp i\rho_0)^4}$$

To calculate the integrals in eqn (7.38) we write

$$z = z_0 + \xi_c\sqrt{\tilde{\rho}^2 - (x/\xi_{ab})^2}, dz = \xi_c \frac{\tilde{\rho} d\tilde{\rho}}{\sqrt{\tilde{\rho}^2 - (x/\xi_{ab})^2}}$$

Consider first the integral over the contour C_1. At $\tilde{\rho} = \rho_0$, the imaginary part of $z = z' + iz''$ is large: $z''/s \gg 1$, therefore,

$$\cot\frac{\pi z}{s} = -i - 2i\exp\left(\frac{2i\pi z'}{s} - \frac{2\pi z''}{s}\right)$$

The integral over the contour C_1 gives

$$\begin{aligned}
&\frac{i}{2}\int dydx \int_{C_1}\left[\cot\frac{\pi z}{s} + i\right] F(z)dz \\
=&2i|a|\Psi_{GL}^2\xi_c L\int dx\int_{C_1}\left(\cot\left[\frac{\pi}{s}\left(z_0 + \xi_c\sqrt{\tilde{\rho}^2 - (x/\xi_{ab})^2}\right)\right] + i\right) \\
&\times \frac{\tilde{\rho}d\tilde{\rho}}{\sqrt{\tilde{\rho}^2 - (x/\xi_{ab})^2}}\frac{1}{(\tilde{\rho} - i\rho_0)^4} \\
=&\frac{32\pi^4|a|\Psi_{GL}^2\xi_c^4 L}{3s^3}\exp\left(\frac{2i\pi z_0}{s}\right)\int\exp\left(-\frac{2\pi\xi_c}{s}\sqrt{\rho_0^2 + (x/\xi_{ab})^2}\right)dx \\
=&\frac{32\pi^4|a|\Psi_{GL}^2\xi_{ab}\xi_c\sqrt{\rho_0}L}{3}\left(\frac{\xi_c}{s}\right)^{\frac{5}{2}}\exp\left(-\frac{2\pi\xi_c\rho_0}{s}\right)\exp\left(\frac{2\pi i z_0}{s}\right)
\end{aligned}$$

Here L is the vortex length along the y axis. During the integration we use that $x/\xi_{ab} \sim \sqrt{s/\xi_c} \ll 1$ and calculate the derivative only of the most rapidly varying exponential part of the function under the integral. Together with the integral over the contour C_2, this equation gives the energy per unit vortex length

$$U_{pin} = U_0\cos\left(\frac{2\pi z_0}{s}\right) \tag{7.42}$$

where

$$U_0 = \frac{4\pi\Phi_0^2\sqrt{\rho_0}}{3\lambda_{ab}^2\Gamma}\left(\frac{\xi_c}{s}\right)^{\frac{5}{2}}\exp\left(-\frac{2\pi\xi_c\sqrt{\rho_0}}{s}\right) \tag{7.43}$$

We see that the vortex prefers to sit in-between the layers $z_0 = s/2$ where its energy is minimal. If the vortex is not exactly in the middle, $z_0 = s/2 + \delta z$, a restoring (pinning) force appears. The pinning force per unit vortex length is

$$-\frac{\partial U_{pin}}{\partial\delta z} = -\frac{2\pi U_0}{s}\sin\left(\frac{2\pi u}{s}\right) \tag{7.44}$$

The pinning energy is exponentially small because the layered structure is weakly pronounced: the distance between the layers is small compared to the coherence length ξ_c.

18 III Nonequilibrium Superconductivity by Kopnin

18.1 NONSTATIONARY THEORY

We consider both the method of analytical continuation worked out by Eliashberg and the Keldysh diagram technique, which are designed for calculating the real-time Green functions of nonstationary superconductors. The Eliashberg phonon model of superconductivity is discussed. We derive equations for the Keldysh Green function for particles interacting with impurities, phonons and with each other.

18.1.1 8.1 The method of analytical continuation

This part of the book is devoted to introduction of general principles and tools of the microscopic nonstationary theory of superconductivity and is based on the general concepts we learned in the previous chapters. A physical system which evolves in time may be out of equilibrium with respect the heat bath, moreover, various parts of the system may be not in equilibrium with each other. To treat a non-equilibrium situation, one needs to know the distribution of non-equilibrium excitations in addition to their spectrum. Moreover, the spectrum itself can be distorted as the system gets far from equilibrium. The specifics of superconductors is that the excitations which affect the response of a superconductor to an external field can accumulate near the energy gap when their relaxation is relatively slow. The major problem of the nonstationary theory is thus to find the distribution function of these excitations. In a nonlinear case, one needs this distribution also to calculate the modified spectrum in a self-consistent way.

Within the Green function formalism of the microscopic theory of nonstationary superconductivity, all these different tasks are combined into a single basic problem of finding the time-dependent Green function of the system. However, we encounter a major complication on this way: there is no simple recipe how to calculate the real-time Green functions because the Dyson equations for them can not be constructed by simply repeating the previous argumentation. There are two general methods developed so far for calculating the time-dependent Green functions. The first is due to Keldysh (1964). It deals directly with the set of real-time Green functions defined on the time axis with special rules of time ordering of the particle-field operators. The second method has been developed by Gor'kov and Eliashberg (1968) and by Eliashberg (1971). It operates with the Matsubara Green functions which, as functions of frequencies, are analytically continued from imaginary onto the real frequency axis. We shall start with the analytical continuation technique. We demonstrate in Section 8.5 that both these methods are completely equivalent. In the present Chapter, we derive the equations for those real-time Green functions which are appropriate for the superconducting response. The following Chapters describe how to define and calculate the distribution function. We derive the kinetic equations for the distribution function; we solve them for various cases and show how to use the obtained distribution for further calculations.

As we mentioned already, there is no simple way how to calculate the realtime Green function. Instead, comparatively simple equations and the corresponding diagram technique exist for the Green functions which are defined for an imaginary time $t = -i\tau$, the so called Matsubara functions. This is because, in contrast to its imaginary counterpart, a real time can not be directly time-ordered with the inverse temperature $-i/T$, and thus the Wick's theorem can not be directly applied for the real-time field operators. However, solving the imaginary-time equations or, equivalently, calculating all the diagrams, one can obtain the Matsubara functions. Since the field operators in both real and imaginary time representations obey the same equations with the substitution $t \to -i\tau$, one expects that the real-time Green function

can be obtained from the Matsubara function by an analytical continuation from imaginary time onto the real time axis provided the necessary initial conditions are satisfied. We assume that all the interactions between particles are switched on at the time $t = -\infty$. Therefore, the Green functions at $t = -\infty$ coincide with those for free particles which are in equilibrium with each other and with the heat bath with a temperature T.

For practical purposes, the Matsubara Green functions are usually calculated as the Fourier transforms for discrete imaginary (Matsubara) frequencies $\epsilon_n = 2\pi i T(n + 1/2)$ for Fermi particles, and $\omega_k = 2\pi i T k$ for Bose particles and external fields. The analytical continuation in time is equivalent to the analytical continuation in frequencies ϵ and ω from imaginary onto the real frequency axis. To satisfy the initial conditions, one has to make the continuation in such a way that all the external field frequencies have an infinitesimal positive imaginary part. In this case, all diagrams containing the field operators vanish for $t \to -\infty$ according to the causality principle. In other words, one has to continue all the field frequencies from the upper half-plane of complex ω.

In principle, one can first solve the particular physical problem and find the Matsubara Green function for it. After that, one can continue the Matsubara function analytically and obtain the required Green function in the real time representation. This method, however, is not convenient. It operates with quantities which do not have a simple physical meaning and thus can cause a confusion, which, sometimes, leads to incorrect results. The modern microscopic theory, instead, works with the real-time Green functions from the very beginning. In this representation, we can always keep track of the real physical phenomena involved into the particular problem. The price for that is more complicated equations needed to find the real-time functions. This chapter describes how to derive and solve equations for the real-time Green functions of superconductors. We shall follow the approach suggested by Eliashberg (1971).

FIG. 8.1. Diagrammatic presentation of the Green function expansion in powers of the external field H (shown by wavy lines) up to the third order term.

All physical quantities like the current, the particle number, or the order parameter are expressed through sums of certain matrix elements of the Green function $\check{G}$ over the Matsubara frequencies:

$$T \sum_n \check{G}_{\epsilon_n, \epsilon_n - \omega_k}. \tag{8.1}$$

In our case, $\epsilon_n = 2\pi i T(n + 1/2)$ are Matsubara frequencies for Fermionic operators. The frequencies $\omega_k = 2\pi i T k$ refer to the "external" Bose fields and to the time dependent order parameter. Our goal is to find an analytical continuation of eqn (8.1) from the upper half plane of the external frequency ω. To make the analytical continuation in a general case, we consider the full function $\check{G}_{\epsilon_n, \epsilon_n - \omega_k}$ as a perturbation series expansion in terms of a generalized "field". We start with the Dyson equation for the Matsubara functions

$$\begin{aligned} \textstyle\sum_{\epsilon_m} \int \frac{d^3 k_1}{(2\pi)^3} \left[\check{G}^{-1}_{\epsilon_n} (\mathbf{p} - \mathbf{k}_1, \omega_1) - \check{\Sigma}_{\epsilon_n, \epsilon_m} (\mathbf{p}, \mathbf{k}_1) \right] \check{G}_{\epsilon_m, \epsilon_k} (\mathbf{p} - \mathbf{k}_1, \mathbf{p} - \mathbf{k}) \\ = \check{1} \cdot (2\pi)^3 \delta(\mathbf{k}) \delta_{\epsilon_n - \epsilon_k} \end{aligned} \tag{8.2}$$

Here $\omega_1 = \epsilon_n - \epsilon_m$ and

$$\check{G}_{\epsilon}^{-1}\left(\mathbf{p}-\mathbf{k}_1,\omega_1\right)=\begin{pmatrix}\xi_{\mathbf{p}}-\epsilon & 0\\ 0 & \xi_{\mathbf{p}}+\epsilon\end{pmatrix}(2\pi)^3\delta\left(\mathbf{k}_1\right)\delta_{\omega_1}+\check{H}_{\omega_1}\left(\mathbf{k}_1\right) \tag{8.3}$$

and $\check{H}$ is an "external field" matrix as defined by eqn (5.57). For the beginning, we do not consider the interaction with impurities and phonons.

8.1.1 Clean superconductors

Let us consider the N-th order term in the expansion of eqn (8.2) which we write as

$$\check{G}^{(N)}_{\epsilon_n,\epsilon_n-\omega}=(-1)^N G^{(0)}_{\epsilon_n}\check{H}_{\omega_1}G^{(0)}_{\epsilon_n-\omega_1}\check{H}_{\omega 2}G^{(0)}_{\epsilon_n-\omega_1-\omega_2}\ldots\check{H}_{\omega_N}G^{(0)}_{\epsilon_n-\omega} \tag{8.4}$$

where $\omega=\omega_1+\ldots+\omega_N$ and $G^{(0)}_{\epsilon_n}$ is the non-perturbed Green function of the normal state,

Fig. 8.2. The cuts (shown by thick solid lines) along the singular lines of the function $\check{G}_{z, z-\omega}^{(3)}$ on the plane of complex z. Contours around the points $z=2\pi iT(k+1/2)$ (circles) are unrolled into contours going above and below the cuts.

$$G^{(0)}_{\epsilon_n}=\frac{1}{\xi_p-\epsilon_n}$$

The diagrammatic representation of eqn (8.4) is shown in Fig. 8.1.The summations of the Matsubara frequencies ω_i and integration over the corresponding momenta $\mathbf{k}_i$ are implicitly assumed. The sum of the type of eqn (8.1) can be written as the contour integral

$$T\sum_n \check{G}^{(N)}_{\epsilon_n,\epsilon_n-\omega_k}=\oint \check{G}^{(N)}_{z,z-\omega_k}\tanh\left(\frac{z}{2T}\right)\frac{dz}{4\pi i} \tag{8.5}$$

taken around each of the poles of the function $\tanh(z/2T)$, i.e., around all $z=2\pi iT(n+1/2)$. Consider the poles of $\check{G}^{(N)}_{z,z-\omega_k}$ as functions of z. The poles are at $z=\xi_p, z-\omega_1=\xi_p,\ldots,z-\omega=\xi_p$. Since ξ_p is real, they are determined by equations $\operatorname{Im}\left(z-\sum_{j=0}^{l}\omega_j\right)=0$, where $l=0,\ldots,N$ and $\omega_0=0$.

Let us make $N+1$ horizontal cuts on the complex plane z passing through all $\operatorname{Im}\left(z-\sum_{j=0}^{l}\omega_j\right)=0$ as shown in Fig. 8.2. Since $\omega_j=2\pi iTk_j$, these cuts do not go through the poles of $\tanh(z/2T)$. We shall continue from positive imaginary values of all ω_j, therefore we can consider all k_j to be positive, and numerate the cuts from $\operatorname{Im} z=0$ to $\operatorname{Im} z=\operatorname{Im}\omega$ upwards in order of increasing k_j. Now we can unroll the contours going around each $z=2\pi iT(n+1/2)$ into the contours going along the lower and upper coasts of each of $N+1$

cuts. At
each cut labelled by l, the variable $z = \epsilon + i\,\mathrm{Im}\left(\sum_{j=0}^{l}\omega_j\right)$; the lower contour goes from right to left while the upper goes from left to right (see Fig. 8.2). As a result, we have

$$\oint \check{G}^{(N)}_{z,z-\omega}\tanh\left(\frac{z}{2T}\right)\frac{dz}{4\pi i} = \int \tanh\left(\frac{\epsilon}{2T}\right)\frac{d\epsilon}{4\pi i} \times \left[\delta_0\check{G}^{(N)} + \delta_1\check{G}^{(N)} + \ldots + \delta_N\check{G}^{(N)}\right]$$

Here

$$\delta_l\check{G}^{(N)} = \left[\check{G}^{(N)}_{z,z-\omega}\right]_{z=z_l+0} - \left[\check{G}^{(N)}_{z,z-\omega}\right]_{z=z_l-0}$$

is the jump of the Green function at the l-th cut where $z_l = \epsilon + i\,\mathrm{Im}\sum_{j=0}^{l}\omega_j$. Because of periodicity of $\tanh(z/2T)$ along the imaginary axis with the period $2\pi iT$, the argument of the hyperbolic tangent is just ϵ at each cut. Let us write one of the jumps explicitly. For example,

$$\delta_1\check{G}^{(N)} = (-1)^N G^{(0)}_{\epsilon+\omega_1}\check{H}_{\omega_1}\left[G^{(0)}_{\epsilon+i0} - G^{(0)}_{\epsilon-i0}\right]\check{H}_{\omega_2}G^{(0)}_{\epsilon-\omega_2}\ldots\check{H}_{\omega_N}G^{(0)}_{\epsilon+\omega_1-\omega}.$$

Here all ω_j are still discrete imaginary. When we unroll the contours, we make the analytical continuation in the variable z. At the cut $l = 1$ for example, we have, for a real $\epsilon, \mathrm{Im}\,(\epsilon+\omega_1) > \mathrm{Im}(\epsilon + i0) > 0$ while $\mathrm{Im}\,(\epsilon+\omega_1-\omega) < \ldots < \mathrm{Im}\,(\epsilon-\omega_2) < \mathrm{Im}(\epsilon - i0) < 0$. Therefore, we obtain

$$\delta_1\check{G}^{(N)} = (-1)^N G^{(0)R}_{\epsilon+\omega_1}\check{H}_{\omega_1}\left[G^{(0)R}_{\epsilon} - G^{(0)A}_{\epsilon}\right]\check{H}_{\omega_2}G^{(0)A}_{\epsilon-\omega_2}\ldots\check{H}_{\omega_N}G^{(0)A}_{\epsilon+\omega_1-\omega}. \tag{8.6}$$

Now we can consider all ω_j as real continuous values. The sums over discrete Matsubara frequencies $\omega_j = 2\pi iTk_j$ in equations of the type of eqn (8.6) transform into the integrals according to the rule

$$T\sum_{k_j} = \int\frac{d\omega_j}{2\pi}$$

We make the substitution of integration variable $\epsilon + \sum_{j=0}^{l}\omega_j \Rightarrow \epsilon$ at the l-th cut to restore the previous notation of incoming and outgoing frequencies and obtain

$$\begin{aligned}
&\oint \check{G}^{(N)}_{z,z-\omega_k}\tanh\left(\frac{z}{2T}\right)\frac{dz}{4\pi i}\\
&= \int\left[\tanh\left(\frac{\epsilon}{2T}\right)\left[\delta_0\check{G}^{(N)}\right]_{\epsilon,\epsilon-\omega} + \tanh\left(\frac{\epsilon-\omega_1}{2T}\right)\left[\delta_1\check{G}^{(N)}\right]_{\epsilon,\epsilon-\omega} + \right.\\
&\quad + \tanh\left(\frac{\epsilon-\omega_1-\omega_2}{2T}\right)\left[\delta_2\check{G}^{(N)}\right]_{\epsilon,\epsilon-\omega} + \ldots\\
&\qquad\qquad \left. + \tanh\left(\frac{\epsilon-\omega}{2T}\right)\left[\delta_N\check{G}^{(N)}\right]_{\epsilon,\epsilon-\omega}\right]\frac{d\epsilon}{4\pi i}
\end{aligned}$$

Here $\left[\delta_j\check{G}^{(N)}\right]_{\epsilon,\epsilon-\omega}$ are constructed according to the scheme shown in eqn (8.6) but with shifted frequencies:

$$\left[\delta_1\check{G}^{(N)}\right]_{\epsilon,\epsilon-\omega} = (-1)^N G^{(0)R}_{\epsilon}\check{H}_{\omega_1}\left[G^{(0)R}_{\epsilon-\omega_1} - G^{(0)A}_{\epsilon-\omega_1}\right]\check{H}_{\omega_2}G^{(0)A}_{\epsilon-\omega_1-\omega_2}\ldots\check{H}_{\omega_N}G^{(0)A}_{\epsilon-\omega} \tag{8.7}$$

Therefore, the sum in eqn (8.5) transforms into

$$\int\frac{d\epsilon}{4\pi i}\check{G}^{(N)K}_{\epsilon,\epsilon-\omega}$$

where

$$\check{G}^{(N)K}_{\epsilon,\epsilon-\omega} = \sum_{l=0}^{N} \left[\delta_l \check{G}^{(N)}\right]_{\epsilon,\epsilon-\omega} \tanh\left(\frac{\epsilon - \sum_{j=0}^{l}\omega_j}{2T}\right) \tag{8.8}$$

Writing all the jumps explicitly, we obtain

$$\begin{aligned}\check{G}^{(N)K}_{\epsilon,\epsilon-\omega} =& (-1)^N G^{(0)R}_{\epsilon} \check{H}_{\omega_1} G^{(0)R}_{\epsilon-\omega_1} \ldots \check{H}_{\omega_N} G^{(0)R}_{\epsilon-\omega} f^{(0)}(\epsilon-\omega) \\ &+ (-1)^{N-1} G^{(0)R}_{\epsilon} \check{H}_{\omega_1} G^{(0)R}_{\epsilon-\omega_1} \ldots \check{h}_{\epsilon-\omega+\omega_N}, \epsilon-\omega G^{(0)A}_{\epsilon-\omega} \\ &+ \ldots + (-1)^{N-1} G^{(0)R}_{\epsilon} \check{h}_{\epsilon,\epsilon-\omega_1} G^{(0)A}_{\epsilon-\omega_1} \ldots \check{H}_{\omega_N} G^{(0)A}_{\epsilon-\omega} \\ &- (-1)^N f^{(0)}(\epsilon) G^{(0)A}_{\epsilon} \check{H}_{\omega_1} G^{(0)A}_{\epsilon-\omega_1} \ldots \check{H}_{\omega_N} G^{(0)A}_{\epsilon-\omega}\end{aligned}$$

where

$$f^{(0)}(\epsilon) = \tanh\left(\frac{\epsilon}{2T}\right) \tag{8.9}$$

and

$$\check{h}_{\epsilon,\epsilon-\omega} = -\check{H}_{\omega}\left[f^{(0)}(\epsilon) - f^{(0)}(\epsilon-\omega)\right]. \tag{8.10}$$

Collecting now all orders of the perturbation expansion, we find that the analytical continuation gives

$$T\sum_n \check{G}_{\epsilon_n,\epsilon_n-\omega_k} \Rightarrow \int \frac{d\epsilon}{4\pi i} \check{G}^K_{\epsilon,\epsilon-\omega} \tag{8.11}$$

where we define the total time-dependent Green function

$$\check{G}^K_{\epsilon,\epsilon-\omega} = \check{G}^R_{\epsilon,\epsilon-\omega} f^{(0)}(\epsilon-\omega) - f^{(0)}(\epsilon)\check{G}^A_{\epsilon,\epsilon-\omega} + \check{G}^{(a)}_{\epsilon,\epsilon-\omega}. \tag{8.12}$$

It is this function which determines all the physical properties of a nonstationary system in a real time. We assign the superscript K to the total function to indicate that it coincides with the so called Keldysh Green function introduced by Keldysh (1964). We shall prove this later.

The retarded and advanced functions are defined in the usual way as

$$\begin{aligned}\check{G}^R_{\epsilon,\epsilon-\omega} &= \sum_{N=0}^{\infty}(-1)^N G^{(0)R}_{\epsilon} \check{H}_{\omega_1} G^{(0)R}_{\epsilon-\omega_1} \check{H}_{\omega_2} G^{(0)R}_{\epsilon-\omega_1-\omega_2} \ldots \check{H}_{\omega_N} G^{(0)R}_{\epsilon-\omega}, \\ \check{G}^A_{\epsilon,\epsilon-\omega} &= \sum_{N=0}^{\infty}(-1)^N G^{(0)A}_{\epsilon} \check{H}_{\omega_1} G^{(0)A}_{\epsilon-\omega_1} \check{H}_{\omega_2} G^{(0)A}_{\epsilon-\omega_1-\omega_2} \ldots \check{H}_{\omega_N} G^{(0)A}_{\epsilon-\omega}.\end{aligned}$$

We assume integration over all ω_l and $\mathbf{k}_l$. These functions only contain all retarded or all advanced zero-order functions $G^{(0)}$, respectively, and are obtained by summation of all the diagrams exactly as for the Matsubara functions. They satisfy the Gor'kov equations with real frequencies ϵ.

The anomalous function is

$$\check{G}^{(a)}_{\epsilon,\epsilon-\omega} = \sum_{N=0}^{\infty} \check{G}^{(N)(a)}_{\epsilon,\epsilon-\omega}$$

where

$$\begin{aligned}\check{G}^{(N)(a)}_{\epsilon,\epsilon-\omega} =&(-1)^{N-1}G^{(0)R}_{\epsilon}\check{h}_{\epsilon,\epsilon-\omega_1}G^{(0)A}_{\epsilon-\omega_1}\ldots\check{H}_{\omega_N}G^{(0)A}_{\epsilon-\omega}+\ldots\\ &+(-1)^{N-1}G^{(0)R}_{\epsilon}\check{H}_{\omega_1}\ldots G^{(0)R}_{\epsilon-\omega+\omega_N}\check{h}_{\epsilon-\omega+\omega_N,\epsilon-\omega}G^{(0)A}_{\epsilon-\omega}\end{aligned} \tag{8.13}$$

It necessarily contains at least one incoming retarded together with one outgoing advanced function $G^{(0)}$.

The regular functions enter eqn (8.12) in combinations with the equilibrium distribution, eqn (8.9), which is

$$f^{(0)}(\epsilon)=1-2n(\epsilon) \text{ where } n(\epsilon)=\frac{1}{e^{\epsilon/T}+1} \tag{8.14}$$

is the Fermi function. It is the anomalous function which describes the deviation of the system from equilibrium. The summation over all N gives

$$\begin{aligned}\check{G}^{(a)}_{\epsilon,\epsilon-\omega}=&\int\frac{d\epsilon' d\omega'}{(2\pi)^2}\frac{d^3k'd^3k''}{(2\pi)^6}\check{G}^{R}_{\epsilon,\epsilon'}\left(\mathbf{p},\mathbf{p}-\mathbf{k}'\right)\check{h}_{\epsilon',\epsilon'-\omega'}\left(\mathbf{k}''\right)\\ &\times\check{G}^{A}_{\epsilon'-\omega',\epsilon-\omega}\left(\mathbf{p}-\mathbf{k}'-\mathbf{k}''\right)\end{aligned} \tag{8.15}$$

This simple expression for $\check{G}^{(a)}_{\epsilon,\epsilon-\omega}$ is only valid if there are no interactions with impurities, phonons, and particle-particle interactions. We now consider these interactions in turn.

8.1.2 Impurities

Consider first impurity scattering. We shall not write down all possible diagrams, instead, we concentrate on one typical example which demonstrates the general structure of the result of the analytical continuation. Consider the term which is
of the 3 -rd order in the field $\check{H}_\omega$ and of the 2 -nd order in the impurity potential u. We have, similarly to the previous case,

$$\begin{aligned}\oint\check{G}^{(3,2)}_{z,z-\omega}\tanh\left(\frac{z}{2T}\right)\frac{dz}{4\pi i}=&\int\tanh\left(\frac{\epsilon}{2T}\right)\frac{d\epsilon}{4\pi i}\\ &\times\left[\delta_0\check{G}^{(3)}+\delta_1\check{G}^{(3)}+\delta_2\check{G}^{(3)}+\delta_3\check{G}^{(3)}\right]\end{aligned}$$

where, for example,

$$\begin{aligned}\delta_1\check{G}^{(3)}=&(-1)^3G^{(0)}_{\epsilon+\omega_1}\check{H}_{\omega_1}\left[G^{(0)}_{\epsilon+i0}uG^{(0)}_{\epsilon+i0}-G^{(0)}_{\epsilon-i0}uG^{(0)}_{\epsilon-i0}\right]\\ &\times\check{H}_{\omega_2}G^{(0)}_{\epsilon-\omega_2}uG^{(0)}_{\epsilon-\omega_2}\check{H}_{\omega_3}G^{(0)}_{z-\omega_2-\omega_3}.\end{aligned}$$

Note that the Green functions on both sides of the impurity potential U have the same frequencies because the impurity scattering is elastic and does not change the particle energy. Proceeding in the same way as before, we obtain for this particular term of the 3-rd order in $\check{H}_\omega$ and of the 2-nd order in U, after summation over all cuts:

$$\begin{aligned}&\check{G}^{(3,2)K}_{\epsilon,\epsilon-\omega}\\ =&-G^{(0)R}_{\epsilon}\check{H}_{\omega_1}G^{(0)R}_{\epsilon-\omega_1}uG^{(0)R}_{\epsilon-\omega_1}\check{H}_{\omega_2}G^{(0)R}_{\epsilon-\omega_1-\omega_2}uG^{(0)R}_{\epsilon-\omega_1-\omega_2}\check{H}_{\omega_3}G^{(0)R}_{\epsilon-\omega}f^{(0)}(\epsilon-\omega)\\ &+G^{(0)R}_{\epsilon}\check{H}_{\omega_1}G^{(0)R}_{\epsilon-\omega_1}uG^{(0)R}_{\epsilon-\omega_1}\check{H}_{\omega_2}G^{(0)R}_{\epsilon-\omega_1-\omega_2}uG^{(0)R}_{\epsilon-\omega_1-\omega_2}\check{h}_{\epsilon-\omega_1-\omega_2,\epsilon-\omega}G^{(0)A}_{\epsilon-\omega}\\ &+G^{(0)R}_{\epsilon}\check{H}_{\omega_1}G^{(0)R}_{\epsilon-\omega_1}uG^{(0)R}_{\epsilon-\omega_1}\check{h}_{\epsilon-\omega_1,\epsilon-\omega_1-\omega_2}G^{(0)A}_{\epsilon-\omega_1-\omega_2}uG^{(0)A}_{\epsilon-\omega_1-\omega_2}\check{H}_{\omega_3}G^{(0)A}_{\epsilon-\omega}\\ &+G^{(0)R}_{\epsilon}\check{h}_{\epsilon,\epsilon-\omega_1}G^{(0)A}_{\epsilon-\omega_1}uG^{(0)A}_{\epsilon-\omega_1}\check{H}_{\omega_2}G^{(0)A}_{\epsilon-\omega_1-\omega_2}uG^{(0)A}_{\epsilon-\omega_1-\omega_2}\check{H}_{\omega_3}G^{(0)A}_{\epsilon-\omega}\\ &\quad+f^{(0)}(\epsilon)G^{(0)A}_{\epsilon}\check{H}_{\omega_1}G^{(0)A}_{\epsilon-\omega_1}uG^{(0)A}_{\epsilon-\omega_1}\check{H}_{\omega_2}G^{(0)A}_{\epsilon-\omega_1-\omega_2}uG^{(0)A}_{\epsilon-\omega_1-\omega_2}\check{H}_{\omega_3}G^{(0)A}_{\epsilon-\omega}.\end{aligned}$$

After averaging over positions of impurity atoms, we can present this term as

$$\begin{aligned}\check{G}^{(3,2)K}_{\epsilon,\epsilon-\omega} =& \check{G}^{(3,2)R}_{\epsilon,\epsilon-\omega} f^{(0)}(\epsilon-\omega) - f^{(0)}(\epsilon)\check{G}^{(3,2)A}_{\epsilon,\epsilon-\omega} \\ &+ \check{G}^{(2,2)R}_{\epsilon,\epsilon-\omega_1-\omega_2}\check{h}_{\epsilon-\omega_1-\omega_2,\epsilon-\omega} G^{(0)A}_{\epsilon-\omega} + G^{(0)R}_{\epsilon}\check{h}_{\epsilon,\epsilon-\omega_1}\check{G}^{(2,2)A}_{\epsilon-\omega_1,\epsilon-\omega} \\ &+ \check{G}^{(1,0)R}_{\epsilon,\epsilon-\omega_1}\left(n_{imp}\int\frac{d^3p}{(2\pi)^3}|u(\theta)|^2 G^{(0)R}_{\epsilon-\omega_1}\check{h}_{\epsilon-\omega_1,\epsilon-\omega_1-\omega_2} G^{(0)A}_{\epsilon-\omega_1-\omega_2}\right) \\ &\times \check{G}^{(1,0)A}_{\epsilon,\epsilon-\omega}\end{aligned}$$

It is clear that, collecting all orders in $\check{H}_\omega$ and in the impurity potential, we again obtain eqn (8.12) where the anomalous function now is

$$\begin{aligned}\check{G}^{(a)}_{\epsilon,\epsilon-\omega}(\mathbf{p},\mathbf{p}-\mathbf{k}) = \int\frac{d\epsilon' d\omega'}{(2\pi)^2}\frac{d^3k' d^3k''}{(2\pi)^6}\check{G}^R_{\epsilon,\epsilon'}(\mathbf{p},\mathbf{p}-\mathbf{k}') \\ \times\left[\check{h}_{\epsilon',\epsilon'-\omega'}(\mathbf{k}'') + \check{\Sigma}^{(a)}_{\epsilon',\epsilon'-\omega'}(\mathbf{p},\mathbf{k}'')\right] \\ \times\check{G}^A_{\epsilon'-\omega',\epsilon-\omega}(\mathbf{p}-\mathbf{k}'-\mathbf{k}'',\mathbf{p}-\mathbf{k})\end{aligned} \tag{8.16}$$

Sometimes, we shall use a symbolic form

$$\check{G}^{(a)} = \check{G}^R\left(\check{h}+\check{\Sigma}^{(a)}\right)\check{G}^A \tag{8.17}$$

where the frequencies and momenta are not shown, and the integration over all internal variables is assumed. Equation (8.16) differs from eqn (8.15) by the presence of the anomalous impurity self-energy $\Sigma^{(a)}$ which necessarily contains both retarder and advanced functions

$$\check{\Sigma}^{(a)}_{\epsilon,\epsilon-\omega}(\mathbf{p},\mathbf{k}) = n_{\text{imp}}\int\frac{d^3p'}{(2\pi)^3}|u(\theta)|^2\check{G}^{(a)}_{\epsilon,\epsilon-\omega}(\mathbf{p}',\mathbf{p}'-\mathbf{k}) \tag{8.18}$$

Here θ is the angle between $\mathbf{p}$ and $\mathbf{p}'$. The self energy is obtained from $\check{G}^{(a)}_{\epsilon,\epsilon-\omega}$ in the same way as the self-energy $\check{\Sigma}^{R(A)}_{\epsilon,\epsilon-\omega}$ is obtained from the regular Green functions $\check{G}^{R(A)}_{\epsilon,\epsilon-\omega}$:

$$\check{\Sigma}^{R(A)}_{\epsilon,\epsilon-\omega}(\mathbf{p},\mathbf{k}) = n_{\text{imp}}\int\frac{d^3p'}{(2\pi)^3}|u(\theta)|^2\check{G}^{R(A)}_{\epsilon,\epsilon-\omega}(\mathbf{p}',\mathbf{p}'-\mathbf{k})$$

The regular functions are determined by the sum of diagrams of all orders in $\check{H}$ and $\check{\Sigma}^{R(A)}$, therefore, they satisfy the usual equations

$$\begin{aligned}\int\frac{d\omega_1}{2\pi}\frac{d^3k_1}{(2\pi)^3}\left[\check{G}^{-1}_{\epsilon}(\mathbf{p}-\mathbf{k}_1,\omega_1) - \check{\Sigma}^{R(A)}_{\epsilon,\epsilon-\omega_1}(\mathbf{p},\mathbf{k}_1)\right] \\ \times\check{G}^{R(A)}_{\epsilon-\omega_1,\epsilon-\omega}(\mathbf{p}-\mathbf{k}_1,\mathbf{p}-\mathbf{k}) = \check{1}\cdot(2\pi)^4\delta(\omega)\delta(\mathbf{k})\end{aligned} \tag{8.19}$$

where the inverse operator $\check{G}^{-1}_{\epsilon}(\mathbf{p},\omega_1)$ is determined by eqn (8.3). Equation (8.19) in the symbolic form is

$$\left(\check{G}^{-1}-\check{\Sigma}^{R(A)}\right)\check{G}^{R(A)} = \check{1} \tag{8.20}$$

One can obtain an equation for the anomalous function $\check{G}^{(a)}_{\epsilon,\epsilon-\omega}$ as well. Let us apply the operator in eqn (8.3) to eqn (8.16) from the left. We obtain, in a symbolic form,

$$\left(\check{G}^{-1}-\check{\Sigma}^{R}\right)\check{G}^{(a)} = \left(\check{h}+\check{\Sigma}^{(a)}\right)\check{G}^A \tag{8.21}$$

Using the definition of the total time-dependent function $\check{G}^K$ and equations for the retarded and advanced functions one can prove that the total function $\check{G}^K$ satisfies a homogeneous equation of the type of eqn (8.21). Indeed, let us take the combination

$$\check{G}^{(r)}_{\epsilon,\epsilon-\omega} = \check{G}^R_{\epsilon,\epsilon-\omega} f^{(0)}(\epsilon-\omega) - f^{(0)}(\epsilon)\check{G}^A_{\epsilon,\epsilon-\omega} \tag{8.22}$$

and apply to it the operator $\check{G}^{-1}_{\epsilon} - \check{\Sigma}^R_{\epsilon,\epsilon'}$ from the left. We get

$$\left(\check{G}^{-1} - \check{\Sigma}^R\right)\check{G}^{(r)} = f^{(0)}(\epsilon)\left[1 - \left(\check{G}^{-1} - \check{\Sigma}^A\right)\check{G}^A\right] - \check{h}\check{G}^A$$

$$+ \left[\check{\Sigma}^R f^{(0)}(\epsilon') - f^{(0)}(\epsilon)\check{\Sigma}^A\right]\check{G}^A \tag{8.23}$$

Adding eqns (8.23) and (8.23) together, we find

$$\left(\check{G}^{-1} - \check{\Sigma}^R\right)\check{G}^K - \check{\Sigma}^K\check{G}^A = 0. \tag{8.24}$$

Here

$$\check{\Sigma}^K_{\epsilon,\epsilon'} = \check{\Sigma}^R_{\epsilon,\epsilon'} f^{(0)}(\epsilon') - f^{(0)}(\epsilon)\check{\Sigma}^A_{\epsilon,\epsilon'} + \check{\Sigma}^{(a)}_{\epsilon,\epsilon'} \tag{8.25}$$

is the total self energy. It is clear that

$$\check{\Sigma}^K_{\epsilon,\epsilon-\omega}(\mathbf{p},\mathbf{k}) = n_{\text{imp}} \int \frac{d^3p'}{(2\pi)^3}|u(\theta)|^2 \check{G}^K_{\epsilon,\epsilon-\omega}(\mathbf{p}',\mathbf{p}'-\mathbf{k})$$

Similarly, applying the operator $\check{G}^{-1} - \check{\Sigma}^A$ from the right, we find

$$\check{G}^{(a)}\left(\check{G}^{-1} - \check{\Sigma}^A\right) = \check{G}^R\left(\check{h} + \check{\Sigma}^{(a)}\right) \tag{8.26}$$

and

$$\check{G}^K\left(\check{G}^{-1} - \check{\Sigma}^A\right) - \check{G}^R\check{\Sigma}^K = 0. \tag{8.27}$$

8.1.3 Order parameter, current, and particle density

The self-consistency order-parameter equation can easily be obtained using the general equation for the order parameter in terms of the Matsubara Green function and the rules we have derived here for calculating the sums over the Matsubara frequencies in terms of real-frequency integrations, eqn (8.11). We find for an s-wave pairing

$$\begin{aligned}\frac{\Delta(\mathbf{k},\omega)}{|g|} &= \int \frac{d^3p}{(2\pi)^3}\frac{d\epsilon}{4\pi i} F^K_{\epsilon_+,\epsilon_-} \quad (\mathbf{p}_+,\mathbf{p}_-)\\ \frac{\Delta^*(\mathbf{k},\omega)}{|g|} &= \int \frac{d^3p}{(2\pi)^3}\frac{d\epsilon}{4\pi i} F^{\dagger K}_{\epsilon_+,\epsilon_-} \quad (\mathbf{p}_+,\mathbf{p}_-)\end{aligned} \tag{8.28}$$

In the same way, the current takes the form

$$\begin{aligned}\mathbf{j}(\omega,\mathbf{k}) &= -\frac{2e}{m}\int \frac{d^3p}{(2\pi)^3}\frac{d\epsilon}{4\pi i}\mathbf{p}G^K_{\epsilon_+,\epsilon_-}(\mathbf{p}_+,\mathbf{p}_-) - \frac{Ne^2}{mc}\mathbf{A}\\ &= \frac{2e}{m}\int \frac{d^3p}{(2\pi)^3}\frac{d\epsilon}{4\pi i}\mathbf{p}\bar{G}^K_{\epsilon_+,\epsilon_-}(\mathbf{p}_+,\mathbf{p}_-) - \frac{Ne^2}{mc}\mathbf{A}\end{aligned} \tag{8.29}$$

The electron density becomes

$$N(\omega, \mathbf{k}) = N_0 - 2\int \frac{d^3p}{(2\pi)^3}\frac{d\epsilon}{4\pi i}\delta G^K_{\epsilon_+,\epsilon_-}(\mathbf{p}_+, \mathbf{p}_-)$$
$$= N_0 - 2\int \frac{d^3p}{(2\pi)^3}\frac{d\epsilon}{4\pi i}\delta \bar{G}^K_{\epsilon_+,\epsilon_-}(\mathbf{p}_+, \mathbf{p}_-) \tag{8.30}$$

18.1.2 8.2 The phonon model

8.2.1 Self energy

Consider how one can generalize this approach for the phonon model where the pairing between electrons occurs due to their interaction with phonons. In the phonon model by Eliashberg (1960), the order parameter itself is regarded not as a "field" but rather as a self-energy in the Gor'kov Green function F. The starting point is the equation for the Matsubara Green functions eqn (8.2) where $\epsilon_n = 2\pi i T(n + 1/2)$. The inverse operator $\check{G}^{-1}_{\epsilon_n}$ is defined as in eqn (8.3), the difference being that the "field" $\check{H}$ contains only the electromagnetic potentials. The phonon self energy has the same diagram structure as the impurity self energy shown in Fig. 4.2(b) where the dashed line stands for the phonon Green function D while the crosses denote the matrix element g_{ph} of the electronphonon interaction:

$$\check{\Sigma}^{(ph)}_{\epsilon,\epsilon-\omega}(\mathbf{p}, \mathbf{p}-\mathbf{k}) = T\sum_{\epsilon'}\int \frac{d^3p'}{(2\pi)^3} g^2_{ph} D_{\epsilon'-\epsilon}(\mathbf{p}'-\mathbf{p})\,\check{G}_{\epsilon',\epsilon'-\omega}(\mathbf{p}', \mathbf{p}'-\mathbf{k}) \tag{8.31}$$

Here $\epsilon' - \epsilon = 2\pi i T m$. For simplicity, we do not include impurities here. We consider phonons as "free" particles assuming that their interaction with electrons does not change their properties considerably (Migdal 1958). According to eqn (2.57), the phonon Green function is

$$D_{\epsilon'-\epsilon}(\mathbf{p}'-\mathbf{p}) = \frac{\omega^2_{\mathbf{p}'-\mathbf{p}}}{\omega^2_{\mathbf{p}'-\mathbf{p}} - (\epsilon'-\epsilon)^2} \tag{8.32}$$

Consider a contribution of the N-th order in field to the self energy in eqn (8.31) and transform the sum over the Matsubara frequency ϵ' into the contour integral. We have

$$\check{\Sigma}^{(N)}_{\epsilon,\epsilon-\omega} = g^2_{ph}\oint \frac{dz}{4\pi i}\tanh\left(\frac{z}{2T}\right)\left[D_{z-\epsilon}\check{G}^{(N)}_{z,z-\omega}\right]$$

We do not show integration over d^3p' for brevity. Singularities of the function in the square brackets are at the lines $\operatorname{Im}(z-\omega_j)$ where $\check{G}^{(N)}_{z,z-\omega}$ has its poles, and at the line $\operatorname{Im}(z-\epsilon)$ where the poles of D are located (see Fig. 8.3). We make cuts along these singular lines, and then unroll the contours along the cuts. Doing this, we put $z = \epsilon' + i\operatorname{Im}\omega_j$ at those cuts which correspond to the singular lines of $\check{G}^{(N)}_{z,z-\omega}$ and $z = \epsilon' + i\operatorname{Im}\epsilon_n$ at the cut which corresponds to the singular line of the function D. Note that both $\epsilon = 2\pi i T(n + 1/2)$ and $\omega_j = 2\pi i T k_j$ are still imaginary, but ϵ' is already a real variable. We obtain

$$\check{\Sigma}^{(N)}_{\epsilon,\epsilon-\omega} = g^2_{ph}\int \frac{d\epsilon'}{4\pi i}\left[\coth\left(\frac{\epsilon'}{2T}\right)\left(D^R_{\epsilon'} - D^A_{\epsilon'}\right)\check{G}^{(N)}_{\epsilon'+\epsilon,\epsilon'+\epsilon-\omega}\right]$$
$$+ g^2_{ph}\int \frac{d\epsilon'}{4\pi i}\tanh\left(\frac{\epsilon'}{2T}\right)\left[D_{\epsilon'-\epsilon}\delta_0\check{G}^{(N)}(\epsilon', \epsilon'-\omega)\right.$$

Fig. 8.3. The complex plane of frequencies with cuts along the singular lines of $\check{G}_{z, z-\omega}^{(2)}$ and of $D_{z-\epsilon}$.

$$+D_{\epsilon'-\epsilon+\omega_1}\delta_1\check{G}^{(N)}\left(\epsilon',\epsilon'-\omega\right)+\ldots$$
$$+D_{\epsilon'-\epsilon+\omega}\delta_N\check{G}^{(N)}\left(\epsilon',\epsilon'-\omega\right)\Big] \tag{8.33}$$

The notation $\check{G}^{(N)}\left(\epsilon',\epsilon'-\omega\right)$ shows the variables of the function $\check{G}$; in particular, one can see that it does not depend on ϵ.

Until now we did not include the interaction with phonons into the Green function $\check{G}^{(N)}_{\epsilon,\epsilon-\omega}$ under the integral in eqn (8.33). However, we can make an important observation. Since ϵ' is real, the singular lines of $\check{\Sigma}^{(N)}_{\epsilon,\epsilon-\omega}$ as a function of ϵ coincide with those of $\check{G}^{(N)}_{\epsilon,\epsilon-\omega}$. It means that we can do the same analytical continuation if we include the self energies into the internal function $\check{G}^{(N)}_{z,z-\omega}$ under the integral in eqn (8.33) and thus consider it as a total Green function which contains all orders of the interaction with phonons.

Consider first analytical continuation over ϵ which gives the regular self energies. Performing the continuation from the region $\operatorname{Im}\epsilon > \operatorname{Im}(\epsilon-\omega) > 0$ we get the retarded self energy. We make now shifts of the integration variable $\epsilon' \Rightarrow \epsilon' - \sum_{l=0}^{j}\omega_l$ at each cut belonging to $\check{G}^{(N)}$ and also the shift $\epsilon' \Rightarrow \epsilon' - \epsilon$ at the cut for D. We have

$$\check{\Sigma}^{(N)R}_{\epsilon,\epsilon-\omega} = g_{ph}^2\int\frac{d\epsilon'}{4\pi i}\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\left(D^R_{\epsilon'-\epsilon}-D^A_{\epsilon'-\epsilon}\right)\check{G}^{(N)R}_{\epsilon',\epsilon'-\omega}\right.$$
$$\left.+D^A_{\epsilon'-\epsilon}\check{G}^{(N)K}_{\epsilon',\epsilon'-\omega}\right]$$

where $\check{G}^{(N)K}_{\epsilon',\epsilon'-\omega}$ is defined by eqn (8.8). Collecting all orders in N, we obtain

$$\check{\Sigma}^{(ph)R}_{\epsilon,\epsilon-\omega}(\mathbf{p},\mathbf{p}-\mathbf{k}) = g_{ph}^2\int\frac{d\epsilon'}{4\pi i}\frac{d^3p'}{(2\pi)^3}$$
$$\times\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\left(D^R_{\epsilon'-\epsilon}-D^A_{\epsilon'-\epsilon}\right)\check{G}^R_{\epsilon',\epsilon'-\omega}+D^A_{\epsilon'-\epsilon}\check{G}^K_{\epsilon',\epsilon'-\omega}\right] \tag{8.34}$$

Similarly, continuing eqn (8.33) over ϵ from the region $\operatorname{Im}(\epsilon-\omega) < \operatorname{Im}\epsilon < 0$ and collecting all orders in N, we obtain the advanced self energy

$$\check{\Sigma}^{(ph)A}_{\epsilon,\epsilon-\omega}(\mathbf{p},\mathbf{p}-\mathbf{k}) = g_{ph}^2 \int \frac{d\epsilon'}{4\pi i}\frac{d^3p'}{(2\pi)^3}$$
$$\times \left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\left(D^R_{\epsilon'-\epsilon} - D^A_{\epsilon'-\epsilon}\right)\check{G}^A_{\epsilon',\epsilon'-\omega} + D^R_{\epsilon'-\epsilon}\check{G}^K_{\epsilon',\epsilon'-\omega}\right] \tag{8.35}$$

The phonon functions $D^{R(A)}(\mathbf{p}-\mathbf{p}')$ depend on the momentum transfer $\mathbf{p}-\mathbf{p}'$, while the electron functions have the variables $\check{G}_{\epsilon',\epsilon'-\omega}(\mathbf{p}',\mathbf{p}'-\mathbf{k})$. The total function $\check{G}^K_{\epsilon',\epsilon'-\omega}$ is determined by the same equations (8.12) and (8.16) as for the case with impurities

$$\check{G}^K_{\epsilon,\epsilon-\omega} = \sum_{N=0}^{\infty}\sum_{l=0}^{N}\left[\delta_l\check{G}^{(N)}\right]_{\epsilon,\epsilon-\omega}\tanh\left(\frac{\epsilon-\sum_{j=0}^{l}\omega_j}{2T}\right)$$
$$= \check{G}^R_{\epsilon,\epsilon-\omega}f^{(0)}(\epsilon-\omega) - f^{(0)}(\epsilon)\check{G}^A_{\epsilon,\epsilon-\omega} + \check{G}^{(a)}_{\epsilon,\epsilon-\omega}$$

The retarded and advanced functions are defined as sums of all the contributions which contain only the retarded G^R and $\check{\Sigma}^R$ or only the advanced ones, G^A and $\check{\Sigma}^A$, respectively. The equation for retarded and advanced functions can thus be derived by the analytical continuation of the corresponding equation for the Matsubara functions from the region $\operatorname{Im}\epsilon > \operatorname{Im}(\epsilon-\omega) > 0$ and $\operatorname{Im}(\epsilon-\omega) < \operatorname{Im}\epsilon < 0$, respectively:

$$\left(\check{G}^{-1} - \check{\Sigma}^{(ph)R}\right)\check{G}^R = \check{1}, \quad \left(\check{G}^{-1} - \check{\Sigma}^{(ph)A}\right)\check{G}^A = \check{1}. \tag{8.36}$$

The total self energy is defined exactly as in eqn (8.25):

$$\check{\Sigma}^K_{\epsilon,\epsilon-\omega} = \sum_{N=0}^{\infty}\sum_{l=0}^{N}\left[\delta_l\check{\Sigma}^{(N)}\right]_{\epsilon,\epsilon-\omega}\tanh\left(\frac{\epsilon-\sum_{j=0}^{l}\omega_j}{2T}\right)$$
$$= \check{\Sigma}^R_{\epsilon,\epsilon-\omega}f^{(0)}(\epsilon-\omega) - f^{(0)}(\epsilon)\check{\Sigma}^A_{\epsilon,\epsilon-\omega} + \check{\Sigma}^{(a)}_{\epsilon,\epsilon-\omega}. \tag{8.37}$$

Here $\check{\Sigma}^{(a)}$ necessarily contains both retarded and advanced functions.

We need to express now the total self energy in eqn (8.37) through the total Green functions. Consider one of the jumps in the N-th order self energy defined by eqn (8.33); it is

$$\left[\delta_l\check{\Sigma}^{(N)}\right]_{z,z-\omega}\tanh\left(\frac{z_l}{2T}\right)$$
$$= g_{ph}^2\int\frac{d\epsilon'}{4\pi i}\left[\coth\left(\frac{\epsilon'}{2T}\right)\left(D^R_{\epsilon'} - D^A_{\epsilon'}\right)\left[\delta_l\check{G}^{(N)}\right]_{\epsilon'+z,\epsilon'+z-\omega}\tanh\left(\frac{z_l}{2T}\right)\right]$$
$$+ g_{ph}^2\int\frac{d\epsilon'}{4\pi i}\tanh\left(\frac{\epsilon'}{2T}\right)\left[\tanh\left(\frac{z_l}{2T}\right)\delta_l D_{\epsilon'-z_l}\delta_l\check{G}^{(N)}(\epsilon',\epsilon'-\omega)\right]. \tag{8.38}$$

Here $z_l = z - \sum_{j=0}^{l}\omega_j$. We took into account that the jump of the phonon Green function at the cut l is nonzero only for the term with $\operatorname{Im}\left(\epsilon' - z + \sum_{j=0}^{l}\omega_j\right) = 0$.

The jump in the second line is taken at the cut where $\operatorname{Im}\left(\epsilon' + z - \sum_{j=0}^{l}\omega_j\right) = 0$. Putting $z = \epsilon + \sum_{j=0}^{l}\omega_j$ where ϵ is real and all ω_j are still imaginary, we obtain

$$\left[\delta_l\check{G}^{(N)}\right]_{\epsilon'+z,\epsilon'+z-\omega}\tanh\left(\frac{z_l}{2T}\right)$$
$$= \tanh\left(\frac{\epsilon}{2T}\right)G_{\epsilon'+\epsilon+\sum_j^l\omega_j}\ldots G_{\epsilon'+\epsilon+\omega_l}\check{H}_{\omega_l}$$
$$\times\left[G_{\epsilon'+\epsilon+i0} - G_{\epsilon'+\epsilon-i0}\right]\check{H}_{\omega_{l+1}}\ldots G_{\epsilon'+\epsilon-\omega+\sum_j^l\omega_j}. \tag{8.39}$$

Performing the shift of the integration variable $\epsilon + \epsilon' = \epsilon_1$, we have under the integral

$$\tanh\left(\frac{\epsilon}{2T}\right)\coth\left(\frac{\epsilon_1-\epsilon}{2T}\right)\left(D^R_{\epsilon_1-\epsilon}-D^A_{\epsilon_1-\epsilon}\right)\delta_l\check{G}\left(\epsilon_1,\epsilon_1-\omega\right)$$

The jump in the third line of eqn (8.38) is taken at the cut where $\mathrm{Im}\left(\epsilon' - z + \sum_{j=0}^{l}\omega_j\right) = 0$. Putting $z = \epsilon + \sum_{j=0}^{l}\omega_j$ in the third line of eqn (8.38) we get

$$\begin{aligned}\tanh\left(\frac{z_l}{2T}\right)\delta_l D_{\epsilon'-z+\sum_j^l\omega_j} &= \tanh\left(\frac{\epsilon}{2T}\right)\left(D_{\epsilon'-\epsilon-i0}-D_{\epsilon'-\epsilon+i0}\right)\\ &= \tanh\left(\frac{\epsilon}{2T}\right)\left(D^A_{\epsilon'-\epsilon}-D^R_{\epsilon'-\epsilon}\right)\end{aligned} \tag{8.40}$$

The full expression for the jump becomes

$$\begin{aligned}&\left[\delta_l\check{\Sigma}^{(N)}\right]_{z,z-\omega}\tanh\left(\frac{z_l}{2T}\right)\\ =&g^2_{ph}\int\frac{d\epsilon'}{4\pi i}\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\tanh\left(\frac{\epsilon}{2T}\right)-\tanh\left(\frac{\epsilon'}{2T}\right)\tanh\left(\frac{\epsilon}{2T}\right)\right]\\ &\times\delta_l\check{G}^{(N)}\left(\epsilon',\epsilon'-\omega\right)\left(D^R_{\epsilon'-\epsilon}-D^A_{\epsilon'-\epsilon}\right)\\ =&g^2_{ph}\int\frac{d\epsilon'}{4\pi i}\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\tanh\left(\frac{\epsilon'}{2T}\right)-1\right]\\ &\times\delta_l\check{G}^{(N)}\left(\epsilon',\epsilon'-\omega\right)\left(D^R_{\epsilon'-\epsilon}-D^A_{\epsilon'-\epsilon}\right)\end{aligned} \tag{8.41}$$

Here we use the identity

$$\tanh\left(\frac{\epsilon'}{2T}\right)\tanh\left(\frac{\epsilon}{2T}\right)=1-\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\left[\tanh\left(\frac{\epsilon'}{2T}\right)-\tanh\left(\frac{\epsilon}{2T}\right)\right]$$

Let us sum up all the jumps and collect all orders in N. We note that

$$\int\frac{d\epsilon'}{4\pi i}\sum_{N=0}^{\infty}\sum_l\delta_l\check{G}^{(N)}\left(\epsilon',\epsilon'-\omega\right)=\int\frac{d\epsilon'}{4\pi i}\left[\check{G}^R_{\epsilon',\epsilon'-\omega}-\check{G}^A_{\epsilon',\epsilon'-\omega}\right]$$

and

$$\begin{aligned}&\sum_{N=0}^{\infty}\tanh\left(\frac{\epsilon'}{2T}\right)\sum_l\delta_l\check{G}^{(N)}\left(\epsilon',\epsilon'-\omega\right)\\ =&\sum_{N=0}^{\infty}\sum_l\left[\delta_l\check{G}^{(N)}\right]_{\epsilon',\epsilon'-\omega}\tanh\left(\frac{\epsilon'-\sum_{j=0}^{l}\omega_j}{2T}\right)\\ =&\check{G}^K\left(\epsilon',\epsilon'-\omega\right).\end{aligned}$$

Finally, we arrive at the expression for the total self energy

$$\begin{aligned}\check{\Sigma}^K_{\epsilon,\epsilon-\omega}(\mathbf{p},\mathbf{p}-\mathbf{k})=&g^2_{ph}\int\frac{d\epsilon'}{4\pi i}\frac{d^3p'}{(2\pi)^3}\left(D^R_{\epsilon'-\epsilon}-D^A_{\epsilon'-\epsilon}\right)\\ &\times\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\check{G}^K_{\epsilon',\epsilon'-\omega}-\left(\check{G}^R_{\epsilon',\epsilon'-\omega}-\check{G}^A_{\epsilon',\epsilon'-\omega}\right)\right]\end{aligned} \tag{8.42}$$

Here the phonon functions $D\left(\mathbf{p}'-\mathbf{p}\right)$ depend on the momentum transfer while the electron function are of the form of $\check{G}\left(\mathbf{p}',\mathbf{p}'-\mathbf{k}\right)$.

8.2.2 Order parameter

In the phonon model, the order parameter is defined as a part of the self energy

$$\Delta_{\epsilon,\epsilon-\omega}(\mathbf{p},\mathbf{p}-\mathbf{k}) = \frac{g_{ph}^2}{2}\int \frac{d\epsilon'}{4\pi i}\frac{d^3p'}{(2\pi)^3}\left(D^R_{\epsilon'-\epsilon}+D^A_{\epsilon'-\epsilon}\right)F^K_{\epsilon',\epsilon'-\omega} \tag{8.43}$$

$$\Delta^*_{\epsilon,\epsilon-\omega}(\mathbf{p},\mathbf{p}-\mathbf{k}) = \frac{g_{ph}^2}{2}\int \frac{d\epsilon'}{4\pi i}\frac{d^3p'}{(2\pi)^3}\left(D^R_{\epsilon'-\epsilon}+D^A_{\epsilon'-\epsilon}\right)F^{\dagger K}_{\epsilon',\epsilon'-\omega} \tag{8.44}$$

The frequencies of the order of Δ, i.e., in the range $\epsilon \ll \Omega_D$ where Ω_D is the Debye frequency, are only important because the function F^K vanishes for $\epsilon \gg \Delta$. For such frequencies, one can put $\left(D^R + D^A\right)/2 = 1$. As a result, Δ is actually independent of ϵ and of the momentum $\mathbf{p}$. It can be written simply in the form of a BCS-type s-wave self-consistency equation

$$\Delta_\omega(\mathbf{k}) = g_{ph}^2\int \frac{d\epsilon}{4\pi i}\frac{d^3p}{(2\pi)^3}F^K_{\epsilon,\epsilon-\omega} \tag{8.45}$$

which coincides with eqn (8.28) if the square of the matrix element of the electronphonon interaction g_{ph}^2 is replaced with $|g|$. We see that the phonon model in the form of eqn (8.32) favors an s-wave pairing.

Separating Δ from the self energy we obtain

$$\Sigma_2^R = \tilde{\Sigma}_2^R + \Delta; \Sigma_2^A = \tilde{\Sigma}_2^A + \Delta$$

where

$$\begin{aligned}\tilde{\Sigma}_2^R =& g_{ph}^2\int \frac{d\epsilon'}{4\pi i}\frac{d^3p'}{(2\pi)^3}\left(D^R_{\epsilon'-\epsilon}-D^A_{\epsilon'-\epsilon}\right)\\ &\times\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)F^R_{\epsilon',\epsilon'-\omega}-\frac{1}{2}F^K_{\epsilon',\epsilon'-\omega}\right]\end{aligned}$$

and

$$\begin{aligned}\tilde{\Sigma}_2^A =& g_{ph}^2\int \frac{d\epsilon'}{4\pi i}\frac{d^3p'}{(2\pi)^3}\left(D^R_{\epsilon'-\epsilon}-D^A_{\epsilon'-\epsilon}\right)\\ &\times\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)F^A_{\epsilon',\epsilon'-\omega}+\frac{1}{2}F^K_{\epsilon',\epsilon'-\omega}\right]\end{aligned}$$

and similarly for $\tilde{\Sigma}_2^{\dagger R}$ and $\tilde{\Sigma}_2^{\dagger A}$.

The diagonal components of the self energy determine the renormalization of the chemical potential

$$\delta\mu = \frac{g_{ph}^2}{2}\int \frac{d\epsilon'}{4\pi i}\frac{d^3p'}{(2\pi)^3}\left(D^R_{\epsilon'-\epsilon}+D^A_{\epsilon'-\epsilon}\right)G^K_{\epsilon',\epsilon'-\omega} \tag{8.46}$$

We can separate the contribution from the normal state by writing $G^K = G_n^K + \left(G^K - G_n^K\right)$. For the difference $G^K - G_n^K$, the frequencies $\epsilon \sim \Delta$ are important. Therefore, we can neglect ϵ compared with Ω_D in the phonon Green functions for this term. For the first term, the frequencies $\epsilon \sim \Omega_D$ only participate. For such frequencies, the Green functions coincide with their values for the normal state with the accuracy $(\Delta/\Omega_D)^2$; the difference can thus be neglected in the weak coupling approximation. We have

$$\delta\mu = \delta\mu_{\text{norm}} + g_{ph}^2\int \frac{d\epsilon}{4\pi i}\frac{d^3p}{(2\pi)^3}\left(G^K_{\epsilon,\epsilon-\omega} - G^K_{\epsilon,\epsilon-\omega,\text{ norm}}\right) \tag{8.47}$$

where $\delta\mu$ is the renormalization of the chemical potential in the normal state due to the interaction with phonons. It only depends on the normal state properties and can be disregarded. The second term in the r.h.s. is $\frac{1}{2}g_{\rm ph}^2\delta N$ where δN is the difference between the densities of electrons in the superconducting and in the normal state according to eqn (8.30). In a good metal with a strong Coulomb interaction between electrons and ions of the crystalline lattice, the electronic density is equal to the ion density, and does not change at a transition to the superconducting state. If the Coulomb interaction is weak (or in case of electrically neutral systems), the density change is $\delta N \approx \nu(0)\delta\mu_{sn}$ where $\delta\mu_{sn}$ is the change in the chemical potential at the transition. In the weak coupling approximation,
where the electron-phonon interaction constant $\lambda = \nu(0)g_{ph}^2 \ll 1$, this term can also be neglected as compared to $\delta\mu_{sn}$ itself. As a result, the diagonal term eqn (8.46) can be included into the normal-state chemical potential and excluded from the consideration.

From now on we denote by $\check{\Sigma}^{R(A)}$ the new self energy matrix which does not contain the order parameter or the renormalization of the chemical potential

$$\begin{aligned}\check{\Sigma}^{(\mathrm{ph})R(A)}_{\epsilon,\epsilon-\omega}(\mathbf{p},\mathbf{p}-\mathbf{k}) =& g_{ph}^2\int\frac{d\epsilon'}{4\pi i}\frac{d^3p'}{(2\pi)^3}\left(D^R_{\epsilon'-\epsilon}-D^A_{\epsilon'-\epsilon}\right)\\ &\times\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\check{G}^{R(A)}_{\epsilon',\epsilon'-\omega}\mp\frac{1}{2}\check{G}^K_{\epsilon',\epsilon'-\omega}\right]\end{aligned} \tag{8.48}$$

where $\check{G}_{\epsilon',\epsilon'-\omega}\equiv\check{G}_{\epsilon',\epsilon'-\omega}(\mathbf{p}',\mathbf{p}'-\mathbf{k})$ and $D^{R(A)}_{\epsilon'-\epsilon}\equiv D^{R(A)}_{\epsilon'-\epsilon}(\mathbf{p}'-\mathbf{p})$. The self energies defined by eqn (8.48) describe inelastic relaxation processes and may be important for dynamical problems. In presence of impurities, the corresponding impurity self energies should be added to the phonon self energies.

As a result, the equations for the retarded and advanced Green functions now have the form of eqns (8.19) with the operator [compare with eqn (8.3)]

$$\check{G}^{-1}_{\epsilon}(\mathbf{p}-\mathbf{k}_1,\omega_1) = \begin{pmatrix}\xi_{\mathbf{p}}-\epsilon & 0\\ 0 & \xi_{\mathbf{p}}+\epsilon\end{pmatrix}(2\pi)^4\delta(\omega_1)\,\delta(\mathbf{k}_1)+\check{H}_{\omega_1} \tag{8.49}$$

with the effective Hamiltonian determined by eqn (3.70). It now includes both the electromagnetic potentials and the order parameter.

The equations for the anomalous and total Green functions are exactly the same as eqns (8.55) and (8.61). The only difference is that the self energy contains the phonon contribution in the form of eqns (8.42), (8.48) in addition to the impurity self energies.

18.1.3 8.3 Particle-particle collisions

Electron-electron collisions in usual superconductors are not very important for dynamic processes. This is in contrast with another Fermi superfluid, ^{3}He, where the particle-particle collisions are not only responsible for kinetics of excitations but also determine the pairing itself. In our discussion of superconductors, however, we consider the pairing within the BCS model; particle-particle collisions are only taken into account as long as they provide a relaxation mechanism for excitations.

For degenerate Fermi systems such as metals and ^{3}He, we can only consider pairwise interactions of excitations which result in a self energy shown in Fig. 8.4. The self energy diagrams contain two vertexes connected by three particle Green functions. In this Section we do not consider the particular expressions for the particle-particle self energy. Instead, we shall only discuss the procedure of analytical continuation for the self energy of particle-particle interaction. The particular expression for the relaxation parts of the self energies are given in Section 10.4.

FIG. 8.4. Diagram representation for the particle-particle self energy

We are interested in a sum in the form

$$\check{\Sigma}_{\epsilon,\epsilon-\omega} \propto T^2 \sum_{\epsilon_1,\epsilon_2} \check{G}_{\epsilon_1,\epsilon_1-\omega_1} \check{G}_{\epsilon_2,\epsilon_2-\omega_2} \check{G}_{\epsilon-\epsilon_1-\epsilon_2,\epsilon-\epsilon_1-\epsilon_2-\omega-\omega_1-\omega_2}. \tag{8.50}$$

Consider the contribution of the order N in the external field as a function of the complex variable ϵ for fixed imaginary frequencies of the field operator $\check{H}$. It has the singular lines determined by $\mathrm{Im}\,(\epsilon - \Omega_k) = 0$ which are located between the lowest line $\mathrm{Im}\,\epsilon = 0$ and the highest line $\mathrm{Im}(\epsilon - \omega) = 0$. The frequencies Ω_k are combinations of the external field frequencies which now depend on the particular distribution of the field vertices $\check{H}$ between the functions $\check{G}$ in the sum in eqn (8.50), i.e., between the internal lines of the diagram in Fig. 8.4. Consider the term which has the singular lines at $\mathrm{Im}\,(\epsilon - \Omega_{1i}) = 0, \mathrm{Im}\,(\epsilon - \Omega_{2k}) = 0$, and $\mathrm{Im}\,(\epsilon - \Omega_{3l}) = 0$ and make the cuts along these lines. We now transform the double sum over frequencies in eqn (8.50) into the double contour integral. We shall make this transformation in two stages. On the first stage, the sum over one frequency, say ϵ_2, is transformed into the contour integral while keeping ϵ_1 as a fixed imaginary frequency. On the second stage, the sum over ϵ_1 is transformed. During these transformations, the external frequency ϵ and all field frequencies are fixed at their imaginary values. The transformation of the particular term in eqn (8.50) gives

$$\begin{aligned}
&T^2 \sum_{\epsilon_1,\epsilon_2} \check{G}_{\epsilon_1} \check{G}_{\epsilon_2} \check{G}_{\epsilon-\epsilon_1-\epsilon_2} \Rightarrow \iint \frac{dz_1 dz_2}{(4\pi i)^2} \tanh\left(\frac{z_2}{2T}\right) \\
&\times \Bigg[\sum_{i,k} \left(\delta_i \check{G}_{z_1+\Omega_{1i}}\right) \left(\delta_k \check{G}_{z_2+\Omega_{2k}}\right) \check{G}_{\epsilon-\Omega_{1i}-\Omega_{2k}-z_1-z_2} \tanh\left(\frac{z_1}{2T}\right) \\
&- \sum_{k,l} \check{G}_{\epsilon+z_1-\Omega_{2k}-\Omega_{3l}} \left(\delta_k \check{G}_{z_2+\Omega_{2k}}\right) \left(\delta_l \check{G}_{-z_1-z_2+\Omega_{3l}}\right) \coth\left(\frac{z_1}{2T}\right) \\
&+ \sum_{k,l} \check{G}_{\epsilon+z_1-\Omega_{2k}-\Omega_{3l}} \left(\delta_k \check{G}_{z_2-z_1+\Omega_{2k}}\right) \left(\delta_l \check{G}_{-z_2+\Omega_{3l}}\right) \coth\left(\frac{z_1}{2T}\right) \\
&- \sum_{i,l} \left(\delta_i \check{G}_{z_1+\Omega_{1i}}\right) \check{G}_{\epsilon-z_1+z_2-\Omega_{1i}-\Omega_{3l}} \left(\delta_l \check{G}_{-z_2+\Omega_{3l}}\right) \tanh\left(\frac{z_1}{2T}\right) \Bigg].
\end{aligned} \tag{8.51}$$

Here we omitted the second index at each $\check{G}$ for brevity.

Making now the analytical continuation of eqn (8.51) in ϵ from the region $\mathrm{Im}(\epsilon - \omega) > 0$ and, after that, in all field frequencies ω_i from the upper halfplane, we obtain Σ^R. Making the continuation in ϵ from the region $\mathrm{Im}\,\epsilon < 0$ and, after that, in all field frequencies ω_i from the upper half-plane, we obtain Σ^A. We are interested in the relaxation part $\Sigma^R - \Sigma^A$. It is

$$\Sigma^R - \Sigma^A \Rightarrow \iint \frac{d\epsilon_1 d\epsilon_2}{(4\pi i)^2} \left[\check{G}_1^K \check{G}_2^K \left(\check{G}_3^R - \check{G}_3^A\right) + \check{G}_1^K \left(\check{G}_2^R - \check{G}_2^A\right) \check{G}_3^K + \left(\check{G}_1^R - \check{G}_1^A\right) \check{G}_2^K \check{G}_3^K + \left(\check{G}_1^R - \check{G}_1^A\right) \left(\check{G}_2^R - \check{G}_2^A\right) \left(\check{G}_3^R - \check{G}_3^A\right)\right] \tag{8.52}$$

Here $\check{G}_1$ has the frequencies $\epsilon_1, \epsilon_1 - \omega_1$, etc., with $\epsilon_3 = \epsilon - \epsilon_1 - \epsilon_2$, and $\omega_3 = \omega - \omega_1 - \omega_2$. Define the total self energy according to eqn (8.37) as

$$\begin{aligned}\check{\Sigma}^K_{\epsilon,\epsilon-\omega} &= \sum_{N=0}^{\infty} \sum_j \left[\delta_j \check{\Sigma}^{(N)}\right]_{\epsilon,\epsilon-\omega} \tanh\left(\frac{\epsilon - \Omega_j}{2T}\right) \\ &= \check{\Sigma}^R_{\epsilon,\epsilon-\omega} f^{(0)}(\epsilon - \omega) - f^{(0)}(\epsilon) \check{\Sigma}^A_{\epsilon,\epsilon-\omega} + \check{\Sigma}^{(a)}_{\epsilon,\epsilon-\omega}\end{aligned} \tag{8.53}$$

We obtain

$$\check{\Sigma}^K \Rightarrow \iint \frac{d\epsilon_1 d\epsilon_2}{(4\pi i)^2} \left[\check{G}_1^K \left(\check{G}_2^R - \check{G}_2^A\right) \left(\check{G}_3^R - \check{G}_3^A\right) + \left(\check{G}_1^R - \check{G}_1^A\right) \check{G}_2^K \left(\check{G}_3^R - \check{G}_3^A\right) + \left(\check{G}_1^R - \check{G}_1^A\right) \left(\check{G}_2^R - \check{G}_2^A\right) \check{G}_3^K + \check{G}_1^K \check{G}_2^K \check{G}_3^K\right] \tag{8.54}$$

18.1.4 8.4 Transport-like equations and the conservation laws

One can derive kinetic-type equations for the anomalous and the total Green functions. To do this, let us subtract eqn (8.26) from eqn (8.21). We obtain

$$\begin{aligned}&\left[\check{G}^{-1}\check{G}^{(a)}\right]_{\epsilon,\epsilon-\omega} - \left[\check{G}^{(a)}\check{G}^{-1}\right]_{\epsilon,\epsilon-\omega} \\ &- \left[\check{\Sigma}^R\check{G}^{(a)}\right]_{\epsilon,\epsilon-\omega} + \left[\check{G}^{(a)}\check{\Sigma}^A\right]_{\epsilon,\epsilon-\omega} - \left[\check{\Sigma}^{(a)}\check{G}^A\right]_{\epsilon,\epsilon-\omega} + \left[\check{G}^R\check{\Sigma}^{(a)}\right]_{\epsilon,\epsilon-\omega} \\ =&\left[\check{h}_{\epsilon,\epsilon-\omega'}\check{G}^A_{\epsilon-\omega',\epsilon-\omega}\right]_{\epsilon,\epsilon-\omega} - \left[\check{G}^R_{\epsilon,\epsilon-\omega+\omega'}\check{h}_{\epsilon-\omega+\omega',\epsilon-\omega}\right]_{\epsilon,\epsilon-\omega}\end{aligned} \tag{8.55}$$

The integration over internal variables is assumed in the square brackets of the type of eqn (5.53)

$$[AB]_{\epsilon,\epsilon_1;\mathbf{p},\mathbf{p}_1} = \int A_{\epsilon,\epsilon'}(\mathbf{p},\mathbf{p}') B_{\epsilon',\epsilon_1}(\mathbf{p}',\mathbf{p}_1) \frac{d\epsilon'}{2\pi} \frac{d^3p'}{(2\pi)^3} \tag{8.56}$$

The internal frequency variables are shown explicitly in the last line of eqn (8.55). The momentum variables are not shown.

Equation (8.55) can also be written as

$$\begin{aligned}&\left(\xi_{\mathbf{p}+\mathbf{k}/2} - \check{\tau}_3\epsilon_+\right)\check{G}^{(a)}_{\epsilon_+,\epsilon_-} - \check{G}^{(a)}_{\epsilon_+,\epsilon_-}\left(\xi_{\mathbf{p}-\mathbf{k}/2} - \check{\tau}_3\epsilon_-\right) \\ &- \left[\check{\Sigma}^R\check{G}^{(a)}\right]_{\epsilon_+,\epsilon_-} + \left[\check{G}^{(a)}\check{\Sigma}^A\right]_{\epsilon_+,\epsilon_-} - \left[\check{\Sigma}^{(a)}\check{G}^A\right]_{\epsilon_+,\epsilon_-} + \left[\check{G}^R\check{\Sigma}^{(a)}\right]_{\epsilon_+,\epsilon_-} \\ &+ \left[\check{H}\check{G}^{(a)}\right]_{\epsilon_+,\epsilon_-} - \left[\check{G}^{(a)}\check{H}\right]_{\epsilon_+,\epsilon_-} = \left[\check{h}\check{G}^A\right]_{\epsilon_+,\epsilon_-} - \left[\check{G}^R\check{h}\right]_{\epsilon_+,\epsilon_-}\end{aligned} \tag{8.57}$$

where $\epsilon_\pm = \epsilon \pm \omega/2$.

Subtracting eqn (8.27) from eqn (8.24) we find the equation for the total function

$$\left(\xi_{\mathbf{p}+\mathbf{k}/2}-\check{\tau}_3\epsilon_+\right)\check{G}^K_{\epsilon_+,\epsilon_-}-\check{G}^K_{\epsilon_+,\epsilon_-}\left(\xi_{\mathbf{p}-\mathbf{k}/2}-\check{\tau}_3\epsilon_-\right)$$
$$+\left[\check{H}\check{G}^K\right]_{\epsilon_+,\epsilon_-}-\left[\check{G}^K\check{H}\right]_{\epsilon_+,\epsilon_-}=\check{\mathcal{I}}^K_{\epsilon_+,\epsilon_-} \quad (8.58)$$

The "collision integral" is

$$\check{\mathcal{I}}^K_{\epsilon_+,\epsilon_-}=\left[\check{\Sigma}^R\check{G}^K-\check{G}^K\check{\Sigma}^A-\check{G}^R\check{\Sigma}^K+\check{\Sigma}^K\check{G}^A\right]_{\epsilon_+,\epsilon_-} \quad (8.59)$$

Equation (8.58) it can be written as

$$\mathbf{v}_F\mathbf{k}\check{G}^K_{\epsilon_+,\epsilon_-}-\epsilon_+\check{\tau}_3\check{G}^K_{\epsilon_+,\epsilon_-}+\check{G}^K_{\epsilon_+,\epsilon_-}\epsilon_-\check{\tau}_3$$
$$+\left[\check{H}\check{G}^K\right]_{\epsilon_+,\epsilon_-}-\left[\check{G}^K\check{H}\right]_{\epsilon_+,\epsilon_-}=\check{\mathcal{I}}^K_{\epsilon_+,\epsilon_-} \quad (8.60)$$

because $\xi_{\mathbf{p}+\mathbf{k}/2}-\xi_{\mathbf{p}-\mathbf{k}/2}=\mathbf{v}_F\mathbf{k}$. It can also be represented in a more general form

$$\left[\check{G}_0^{-1}\check{G}^K\right]-\left[\check{G}^K\check{G}_0^{-1}\right]=\check{\mathcal{I}}^K \quad (8.61)$$

Here

$$\check{G}_0^{-1}(\mathbf{p})=-i\frac{\partial}{\partial t}\check{\tau}_3+\begin{pmatrix} E_n\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)-E_F & -\Delta \\ \Delta^* & E_n\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)-E_F \end{pmatrix}+e\varphi\check{1}$$

is shown in the coordinate representation. The momentum operator is $\mathbf{p}=-i\hbar\nabla$ and $E_n(\mathbf{p})$ is the normal-state electronic spectrum. Equations (8.61, 8.25) or eqns (8.55, 8.18) together with eqn (8.19) for the regular functions make the basis for describing nonstationary properties of superconductors.

Equation (8.61) contains the conservation of particle number and the energy conservation. To get the particle conservation

$$e\frac{\partial N}{\partial t}+\operatorname{div}\mathbf{j}=0 \quad (8.62)$$

we multiply eqn (8.61) with $\widetilde{\tau}_3$ then take the trace and integrate it over frequency $d\epsilon/4\pi i$ and momentum $d^3p/(2\pi)^3$. The terms of the form of Δ^*F^K and $\Delta F^{\dagger K}$ cancel out due to the self consistency equation. One can also check that all the
collision integral vanish under this operation (for example, the impurity collision integral vanishes already after integration over the momentum directions).

To obtain the energy conservation, we multiply eqn (8.61) by ϵ, take the trace and integrate it over frequency $d\epsilon/4\pi i$ and momentum $d^3p/(2\pi)^3$. We find

$$\frac{\partial\mathcal{E}}{\partial t}+\operatorname{div}\mathbf{j}_\mathcal{E}=\mathbf{j}\cdot\mathbf{E} \quad (8.63)$$

The internal energy is

$$\mathcal{E}=-\int\epsilon\left[G^K_{\epsilon_+,\epsilon_-}(\mathbf{p}_+,\mathbf{p}_-)-\bar{G}^K_{\epsilon_+,\epsilon_-}(\mathbf{p}_+,\mathbf{p}_-)\right]\frac{d\epsilon}{4\pi i}\frac{d^3p}{(2\pi)^3}+\frac{|\Delta|^2}{|g|}-Ne\varphi \quad (8.64)$$

while the internal energy current is

$$\mathbf{j}_\mathcal{E}=-\frac{1}{m}\int\epsilon\left[\left(\mathbf{p}-\frac{e}{c}\mathbf{A}\right)G^K_{\epsilon_+,\epsilon_-}(\mathbf{p}_+,\mathbf{p}_-)\right.$$
$$\left.+\left(\mathbf{p}+\frac{e}{c}\mathbf{A}\right)\bar{G}^K_{\epsilon_+,\epsilon_-}(\mathbf{p}_+,\mathbf{p}_-)\right]\frac{d\epsilon}{4\pi i}\frac{d^3p}{(2\pi)^3}-\mathbf{j}\varphi \quad (8.65)$$

We assume that the electronic system is in equilibrium with phonons. Otherwise, there will be the additional energy source in the r.h.s. of eqn (8.63)

$$\left(\frac{\partial \mathcal{E}}{\partial t}\right)_{\text{ph}} = -i \int \epsilon \, \text{Tr} \, \check{\mathcal{I}}^{K(\text{ph})}_{\epsilon_+,\epsilon_-} \frac{d\epsilon}{4\pi i} \frac{d^3p}{(2\pi)^3}$$

due to the interaction with phonons. The electron-electron collision integral conserves the energy of interacting particles and thus drops out after integration over $d\epsilon$ with ϵ. The elastic impurity integral also conserves the energy; it vanishes already after integration over the momentum directions.

The total energy

$$\mathcal{E}_{\text{tot}} = \mathcal{E} + \frac{H^2}{8\pi}$$

obeys the conservation equation

$$\frac{\partial \mathcal{E}_{\text{tot}}}{\partial t} + \text{div} \, \mathbf{j}_{\mathcal{E}_{\text{tot}}} = 0$$

where the total-energy current is

$$\mathbf{j}_{\mathcal{E}_{\text{tot}}} = \mathbf{j}_{\mathcal{E}} + \frac{c}{4\pi}[\mathbf{E} \times \mathbf{H}]$$

18.1.5 8.5 The Keldysh diagram technique

8.5.1 Definitions of the Keldysh functions

We have already mentioned that it is the absence of the proper time-ordering procedure in the definitions of the real-time Green functions that produces practical difficulties with calculations of the real-time functions. However, one can establish the necessary time ordering for some auxiliary Green functions in such a way that the Wick's theorem works for them. After the Wick's theorem is restored, one can derive the Dyson equation (or to formulate the diagram technique) thus providing the algorithm of calculating these auxiliary Green functions. The next step is to relate the auxiliary functions with the true real-time Green functions or to express the physical observables through some of the auxiliary Green functions (Keldysh 1964). In this Section we discuss the correspondence between the method of the analytical continuation and the Keldysh technique. We shall see that the both methods lead to the same results: The equations for the Green functions which determine the physical observables are identical in the both cases. One can use any of these two methods. Since we have derived already the basic equations using the method of analytical calculation, we discuss the Keldysh approach only briefly for the most simple case. A more detailed description of the Keldysh technique can be found in the review by Rammer and Smith (1986).

Let us consider the statistical average of a single-particle-observable operator which is linear in the particle-field operators $\tilde{\psi}\tilde{\psi}^\dagger$ in the Heisenberg representation as introduced in Section 2.1. It contains the average

$$\sum^{(d)} \left[\exp\left(\frac{\Omega + \mu\hat{\mathcal{N}} - \hat{\mathcal{H}}(t_0)}{T} \right) \left(\tilde{\psi}_\alpha(\mathbf{r}_1, t_1) \, \tilde{\psi}^\dagger_\beta(\mathbf{r}_2, t_2) \right) \right] = \left\langle \tilde{\psi}_\alpha(\mathbf{r}_1, t_1) \, \tilde{\psi}^\dagger_\beta(\mathbf{r}_2, t_2) \right\rangle_{st}$$

where $\tilde{\psi}_\alpha(1)$ is the Heisenberg operator

$$\tilde{\psi}_\alpha(\mathbf{r},t) = \hat{S}^{-1}(t,t_0)\,\psi_\alpha(\mathbf{r},t_0)\,\hat{S}(t,t_0)$$

defined through the S-matrix introduced by eqn (2.63):

$$\hat{S}(t,t_0) = T_t \exp\left[-i\int_{t_0}^{t}(\hat{\mathcal{H}} - \mu\hat{\mathcal{N}})dt'\right]$$

It is assumed that the system was in equilibrium at $t = t_0$ and had a temperature T. We put $t_0 = -\infty$ assuming that the nonequilibrium interaction is turned on at $t_0 = -\infty$. We see that the operators depend on times which are confined to the interval ranging from $-\infty$ up to the largest time max $\{t_1, t_2\}$. Moreover, the time instants belonging to the operator $\tilde{\psi}$ are not ordered with respect to the instants which belong to the operator $\tilde{\psi}^\dagger$. We want to relate this average to the average of a sequence of operators whose times are fully ordered with respect to each other. The proper ordering is achieved along the time contour depicted in Fig. 8.5. It starts at $-\infty$ and runs up to the largest time max $\{t_1, t_2\}$ and then returns back to the time $-\infty$. We denote the ordering along this contour by T_c and define the auxiliary Green function

Fig. 8.5. The Keldysh time contours for real-time ordered Green functions.

$$G_{\alpha\beta}(1,2) = i\left\langle T_c\tilde{\psi}_\alpha(\mathbf{r}_1,t_1)\,\tilde{\psi}^\dagger_\beta(\mathbf{r}_2,t_2)\right\rangle_{st}. \tag{8.66}$$

The time ordering T_c orders the operators as follows

$$T_c\tilde{\psi}(\mathbf{r}_1,t_1)\,\tilde{\psi}^\dagger(\mathbf{r}_2,t_2) = \begin{cases} \tilde{\psi}(\mathbf{r}_1,t_1)\,\tilde{\psi}^\dagger(\mathbf{r}_2,t_2), t_1 >_c t_2 \\ \mp\tilde{\psi}^\dagger(\mathbf{r}_2,t_2)\,\tilde{\psi}(\mathbf{r}_1,t_1), t_1 <_c t_2 \end{cases}$$

The relation $t_1 >_c t_2$ means that t_1 is later than t_2 along the contour c. The contour c is shown in Fig. 8.5 (a) for $t_1 >_c t_2$. The upper sign refers to Fermi particles while the lower sign is for the Bose particles. The brackets denote the statistical average.

We introduce also the other auxiliary functions

$$G^>_{\alpha\beta}(1,2) = i\left\langle\tilde{\psi}_\alpha(1)\tilde{\psi}^\dagger_\beta(2)\right\rangle_{st}$$
$$G^<_{\alpha\beta}(1,2) = \mp i\left\langle\tilde{\psi}^\dagger_\beta(2)\tilde{\psi}_\alpha(1)\right\rangle_{st}$$

One can see that

$$G(1,2) = \begin{cases} G^>_{\alpha\beta}(1,2), t_1 >_c t_2 \\ G^<_{\alpha\beta}(1,2), t_1 <_c t_2 \end{cases}$$

We can extend the contour c up to $+\infty$ by introducing the identical unity

$$1 \equiv S(\max\{t_1,t_2\}, +\infty)\,S(+\infty, \max\{t_1,t_2\})$$

under the averaging in eqn (8.66). The contour c transforms into the contour C which consists of two parts: the contour C_1 runs from $-\infty$ to $+\infty$ and the contour C_2 which runs back from $+\infty$ to $-\infty$. The contour C is shown in Fig. 8.5 (b).

The physical observables are expressed through the function $G^{<}_{\alpha\beta}(1;2)$. For example, the Fermionic density is

$$N = i\sum_{\alpha}\left[G^{<}_{\alpha\alpha}(1;1)\right] \tag{8.67}$$

where the sum is over the spin indices. At the same time, it is the contour-ordered Green function G in eqn (8.66) that possesses a simple perturbative expansion. Indeed, we can prove the Wick's theorem for the average of the contour-ordered particle operators exactly in the same way as it was done for averages of the imaginary-time ordered operators. We want now to establish a connection between $G^{<}$and the time-ordered function G.

Let us now construct the matrix in the so called Keldysh space

$$\underline{\breve{G}} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \tag{8.68}$$

whose components are defined as follows

$$\begin{aligned} G_{11}(1,2) &= i\left\langle T_t\tilde{\psi}\left(\mathbf{r}_1,t_1\right)\tilde{\psi}^{\dagger}\left(\mathbf{r}_2,t_2\right)\right\rangle_{st} \\ G_{12}(1,2) &= G^{<}(1,2) \\ G_{21}(1,2) &= G^{>}(1,2) \\ G_{22}(1,2) &= i\left\langle \bar{T}_t\tilde{\psi}\left(\mathbf{r}_1,t_1\right)\psi^{\dagger}\left(\mathbf{r}_2,t_2\right)\right\rangle_{st} \end{aligned} \tag{8.69}$$

The time ordering operator T_t is the usual time ordering along the contour C_1 and the operator $\bar{T}_t$ orders along the contour C_2, i.e., it is the "inverse-time" ordering in the usual sense

$$\bar{T}_t\tilde{\psi}\left(\mathbf{r}_1,t_1\right)\tilde{\psi}^{\dagger}\left(\mathbf{r}_2,t_2\right) = \begin{cases} \tilde{\psi}\left(\mathbf{r}_1,t_1\right)\tilde{\psi}^{\dagger}\left(\mathbf{r}_2,t_2\right), & t_1 < t_2 \\ \mp\tilde{\psi}^{\dagger}\left(\mathbf{r}_2,t_2\right)\tilde{\psi}\left(\mathbf{r}_1,t_1\right), & t_1 > t_2 \end{cases}$$

It is convenient, however, to use another representation of the matrix Green function which is obtained from eqn (8.68) by a linear transformation. The point is that the four components in eqn (8.69) are not linearly independent. Indeed, we observe that

$$\begin{aligned} G_{11}(1,2) - G_{12}(1,2) &= G_{21}(1,2) - G_{22}(1,2) \equiv G^{R}(1,2) \\ G_{11}(1,2) - G_{21}(1,2) &= G_{12}(1,2) - G_{22}(1,2) \equiv G^{A}(1,2) \\ G_{21}(1,2) + G_{12}(1,2) &= G_{11}(1,2) + G_{22}(1,2) \equiv G^{K}(1,2) \end{aligned}$$

We introduce here the retarded and advanced Green functions G^R and G^A, respectively, defined in the usual way

$$G^{R}_{\alpha\beta}(1;2) = \begin{cases} i\left\langle\left(\tilde{\psi}_{\alpha}(1)\tilde{\psi}^{\dagger}_{\beta}(2) \pm \tilde{\psi}^{\dagger}_{\beta}(2)\tilde{\psi}_{\alpha}(1)\right)\right\rangle_{st}, & t_1 > t_2 \\ 0, & t_1 < t_2 \end{cases}$$

and

$$G^{A}_{\alpha\beta}(1;2) = \begin{cases} -i\left\langle\left(\tilde{\psi}_{\alpha}(1)\tilde{\psi}^{\dagger}_{\beta}(2) \pm \tilde{\psi}^{\dagger}_{\beta}(2)\tilde{\psi}_{\alpha}(1)\right)\right\rangle_{st}, & t_1 < t_2 \\ 0, & t_1 > t_2 \end{cases}$$

The function G^K is defined as

$$G^K(1,2) = G^>(1,2) + G^<(1,2) \tag{8.70}$$

It is called the Keldysh function. Performing the transformation

$$\breve{G} = L\breve{\tau}_3\underline{\breve{G}}L^+$$

where

$$L = \left(\breve{1} - i\breve{\tau}_2\right)/\sqrt{2},$$

we arrive at the matrix Green function

$$\breve{G} = \begin{pmatrix} G^R & G^K \\ 0 & G^A \end{pmatrix}. \tag{8.71}$$

Equation (8.71) is the standard representation accepted in the literature.
The physical observables can be expressed through the Keldysh function. Indeed, using the identity

$$\lim_{t_1\to t_2}\left[G^>_{\alpha\beta}(1,2) - G^<_{\alpha\beta}(1,2)\right] = i\delta\left(\mathbf{r}_1 - \mathbf{r}_2\right)\delta_{\alpha\beta}$$

which follows from the general commutation rules eqn (2.15) we find from eqn (8.70)

$$\lim_{t_1\to t_2} G^<_{\alpha\beta}(1,2) = \frac{1}{2}\lim_{t_1\to t_2} G^K_{\alpha\beta}(1,2) - i\delta\left(\mathbf{r}_1 - \mathbf{r}_2\right)\delta_{\alpha\beta} \tag{8.72}$$

Let us calculate the particle density using eqn (8.67). We write $N = N_0 + \delta N$ where N_0 is the density for non-interacting particles. We have

$$N = N_0 + \frac{i}{2}\sum_\alpha\left[\lim_{t_1\to t_2}\delta G^K_{\alpha\alpha}(1,2)\right]_{\mathbf{r}_1=\mathbf{r}_2}$$

In the frequency representation

$$G^K(1,2) = \int e^{-i\epsilon(t_1-t_2)-i\omega(t_1+t_2)/2}e^{i\mathbf{p}(\mathbf{r}_1-\mathbf{r}_2)+i\mathbf{k}(\mathbf{r}_1+\mathbf{r}_2)/2}$$
$$\times G^K_{\epsilon_+,\epsilon_-}\left(\mathbf{p}_+,\mathbf{p}_-\right)\frac{d\epsilon d\omega}{(2\pi)^2}\frac{d^3pd^3k}{(2\pi)^6}$$

we get

$$N(\omega,\mathbf{k}) = N_0 - \sum_\alpha\int\delta G^K_{\epsilon_+,\epsilon_-}\left(\mathbf{p}_+,\mathbf{p}_-\right)\frac{d\epsilon}{4\pi i}\frac{d^3p}{(2\pi)^3}$$

This expression coincides with eqn (8.30) obtained by the analytical continuation of the Matsubara Green function.

8.5.2 Dyson equation

Consider a series expansion for the simplest example of the interaction with the external potential U. The first-order diagram of the type shown in Fig. 8.1 gives for the contour-ordered Green function

$$G^{(1)}\left(1,1'\right) = -\int d^3r_2\int_C dt_2 G^{(0)}(1,2)U(2)G^{(0)}\left(2,1'\right)$$

The integral along the contour C can be separated into the integrals along C_1 and C_2. Interchanging the limits of integration on the lower contour C_2 we present $G^{(1)}$ as the sum of two terms which can be written as components in the Keldysh space

$$G_{ik}^{(1)}(1,1') = -\int d^3r_2 \int_{-\infty}^{+\infty} dt_2 G_{ij}^{(0)}(1,2) U_{jl}(2) G_{lk}^{(0)}(2,1') \tag{8.73}$$

where we introduce the matrix

$$\breve{U}(2) = U(2)\breve{\tau}_3 = \begin{pmatrix} U(2) & 0 \\ 0 & -U(2) \end{pmatrix}$$

and assume summation over repeated indices. Equation (8.73) can be written in the matrix form

$$\underline{G}^{(1)} = -\underline{\breve{G}}^{(0)} \breve{U} \underline{G}^{(0)}$$

where we assume integration over internal coordinates and time. It is now clear that collecting all orders of the perturbation expansion in U we arrive at the Dyson equation for the full Green function $\underline{\breve{G}}$

$$\underline{\breve{G}} = \underline{\breve{G}}^{(0)} - \underline{\breve{G}}^{(0)} \breve{U} \underline{\breve{G}}. \tag{8.74}$$

One can easily check that eqn (8.74) transforms into

$$\breve{G} = \breve{G}^{(0)} - \breve{G}^{(0)} U \breve{G} \tag{8.75}$$

for the matrices in the representation of eqn (8.71). The potential U is proportional to the unit matrix in the Keldysh space.

We introduce now the inverse operator for the Green function of non-interacting particles

$$G_0^{-1} = -i\frac{\partial}{\partial t} + E_n(\mathbf{p}) - E_F$$

Applying the inverse operator to eqn (8.75) we obtain the Dyson equation in the differential form

$$\left(G_0^{-1} + U\right) \breve{G} = \breve{1}$$

The interaction with impurities, phonons, and electron-electron interactions can also be included into the same scheme. These interactions introduce the corresponding self energies which also are Keldysh matrices:

$$\breve{\Sigma} = \begin{pmatrix} \Sigma^R & \Sigma^K \\ 0 & \Sigma^A \end{pmatrix}$$

The resulting Dyson equation takes the form

$$\left(G_0^{-1} - \breve{\Sigma}\right) \breve{G} = \breve{1} \tag{8.76}$$

We do not specify the rules how to construct the elements of these matrices within the Keldysh formalism: they are the same as those obtained using the method of analytical continuation. The detailed description of the Keldysh formulation of various interactions can be found in the review by Rammer and Smith (1986).

8.5.3 Keldysh functions in the BCS theory

We have four different Green functions in the BCS theory. Accordingly, the components of the matrix in the Keldysh space used in the BSC theory are matrices in the Nambu space. We define the Keldysh matrices

$$\breve{G} = \begin{pmatrix} \check{G}^R & \check{G}^K \\ 0 & \check{G}^A \end{pmatrix}, \quad \breve{\Sigma} = \begin{pmatrix} \check{\Sigma}^R & \check{\Sigma}^K \\ 0 & \check{\Sigma}^A \end{pmatrix}, \quad \breve{1} = \begin{pmatrix} 1 & 0 \\ 0 & \check{1} \end{pmatrix}.$$

Form eqns (8.20) and (8.24) or eqn (8.27) one can directly see that the total function $\check{G}^K$ introduced as a result of analytical continuation of the Matsubara Green functions is nothing but the Keldysh function because it satisfies the same equations. Indeed, instead of two equations (8.20) and one equation (8.24) we obtain one matrix equation

$$\left(\breve{1}\breve{G}^{-1} - \breve{\Sigma}\right)\breve{G} = \breve{1}$$

which is a generalization of eqn (8.76).
The expressions for the order parameter and for the current density in terms of the Keldysh functions are the same as obtained by the analytical continuation. Indeed, the δ-term in eqn (8.72) is absent for the F-component of the Keldysh function $\check{G}^K$. This directly results in eqn (8.28) for the order parameter. The δ term also disappears if one applies the momentum operator to the function G^K when calculating the current. As a result, eqn (8.29) is reproduced as well. We thus have confirmed that the method of analytical continuation and the Keldysh method are completely equivalent.

18.2 QUASICLASSICAL METHOD FOR NONSTATIONARY PHENOMENA

We apply the quasiclassical approximation to nonstationary problems in the theory of superconductivity and derive the Eliashberg equations for the quasiclassical Green functions. We express the Keldysh function in terms of a generalized two-component distribution function. As an example, we consider a nonlinear response of a superconductor to microwave irradiation: the order parameter becomes enhanced as a result of a nonequilibrium distribution of excitations under irradiation.

18.2.1 9.1 Eliashberg equations

The quasiclassical method based on the fundamental assumption that the Fermi momentum is larger than the coherence length can be applied for nonstationary problems, as well. As in Chapter 5, this assumption again allows us to consider the Green functions only in a vicinity of the Fermi surface. As in the static case, we introduce the quasiclassical matrix Green function integrated over the energy variable $d\xi_{\mathbf{p}}$. In addition to $\check{g}^{R(A)}$ we define

$$\check{g}^K_{\epsilon_+,\epsilon_-}(\mathbf{p}_F, \mathbf{k}) = \oint \frac{d\xi_{\mathbf{p}}}{\pi i} \check{G}^K_{\epsilon_+,\epsilon_-}(\mathbf{p}_+, \mathbf{p}_-) = \begin{pmatrix} g^K & f^K \\ -f^{\dagger K} & \bar{g}^K \end{pmatrix}$$

and similarly for $\check{g}^{(a)}$. The momentum $\mathbf{p}_F$ lies at the Fermi surface; its magnitude is fixed, thus the Green function only depends on the direction of the momentum. One can obtain equations for the quasiclassical functions using the corresponding transport-like equations derived in the previous Section in a full analogy to the static case.

We start the derivation of the quasiclassical equations with an important observation that the transport-like equation (8.60) does not contain $\xi_{\mathbf{p}}$ explicitly. Within the quasiclassical

approximation, the Hamiltonian $\check{H}$ contains $\mathbf{v}_F$ taken at the Fermi surface, therefore, it also depends only on the direction of the momentum. This is completely similar to the stationary case where we were able to make a transformation to the quasiclassical Green functions by performing an integration of the corresponding equation over the energy variable $d\xi_{\mathbf{p}}$. Integrating now the nonstationary equation (8.60) over $d\xi_{\mathbf{p}}$ we find

$$\mathbf{v}_F\mathbf{k}\check{g}^K_{\epsilon_+,\epsilon_-} - \epsilon_+\check{\tau}_3\check{g}^K_{\epsilon_+,\epsilon_-} + \check{g}^K_{\epsilon_+,\epsilon_-}\epsilon_-\check{\tau}_3 + \left[\check{H}\check{g}^K\right]_{\epsilon_+,\epsilon_-} - \left[\check{g}^K\check{H}\right]_{\epsilon_+,\epsilon_-} = \check{I}^K_{\epsilon_+,\epsilon_-}. \tag{9.1}$$

Here we introduce the quasiclassical collision integral

$$\check{I}^K_{\epsilon_+,\epsilon_-}(\mathbf{p}_F,\mathbf{k}) = \int \frac{d\xi_{\mathbf{p}}}{\pi i}\left[\check{\Sigma}^R\check{G}^K - \check{G}^K\check{\Sigma}^A \simeq G^R\check{\Sigma}^K + \check{\Sigma}^K\check{G}^A\right]_{\epsilon_+,\epsilon_-} \tag{9.2}$$

We again use the shortcut notation eqn (8.56) which now reads, for example,

$$[\check{H}\check{g}]_{\epsilon_+,\epsilon_-} = \int \frac{d^3k_1}{(2\pi)^3}\frac{d\omega_1}{2\pi}\check{H}_{\omega_1}(\mathbf{k}_1)\,\check{g}_{\epsilon_+-\omega_1,\epsilon_-}(\mathbf{v}_F,\mathbf{k}-\mathbf{k}_1) \tag{9.3}$$

Equation (9.1) was first obtained by Eliashberg (1971).
The integration over $d\xi_{\mathbf{p}}$ performed in eqn (9.1) for the coordinate representation means, as in Section 5.1, that we take the limit $|\mathbf{r}_1-\mathbf{r}_2|\to 0$. Doing so we average out the coordinate dependence of the order parameter and the vector potential over distances of order of p_F^{-1}, i.e., over distances of the atomic scale. In the momentum representation, the limit $\mathbf{r}_1-\mathbf{r}_2\to 0$ implies that we neglect terms of the first order in $k/p_F\sim 1/p_F\xi(T)$, i.e., terms like

$$\frac{\partial\check{G}}{\partial\mathbf{p}}\nabla\check{H} \tag{9.4}$$

in the expansion of the convolution $[\check{G}\check{H}]$ in powers of k/p_F and keep only zeroorder terms like the simple product $\check{G}(\mathbf{p}_+,\mathbf{p}_-)\check{H}(\mathbf{k})$. We shall discuss this procedure in more detail later in Section 10.1. The same approximation was used when we derived the Eilenberger equations for the stationary Green functions. These terms can indeed be neglected in the Eilenberger equations for the regular functions. However, they may become sometimes important in the equations for the Keldysh functions. This is due to the fact that some leading expansion terms vanish. We shall see later in Chapter 10 that one can not neglect the momentum derivatives for clean superconductors when the collision integrals are small such that the mean free path ℓ approaches $\xi(T)\,[E_F/\Delta]$. The accuracy of eqn (9.1) becomes poor already for clean superconductors with $\ell\gg\xi(T)$: some phenomena are lost in this description. The quasiclassical equation (9.1) is thus applicable for superconductors if the mean free path of quasiparticles is not very long. The limit of clean superconductors is discussed in more detail in Section 10.2.

The corresponding equation for the anomalous function can be obtained from eqn (8.55) in exactly the same way:

$$\begin{aligned}&\mathbf{v}_F\mathbf{k}\check{g}^{(a)}_{\epsilon_+,\epsilon_-} - \epsilon_+\check{\tau}_3\check{g}^{(a)}_{\epsilon_+,\epsilon_-} + \check{g}^{(a)}_{\epsilon_+,\epsilon_-}\check{\tau}_3\epsilon_- + \left[\check{H}\check{g}^{(a)} - \check{g}^{(a)}\check{H}\right]_{\epsilon_+,\epsilon_-}\\ =&\check{I}^{(a)}_{\epsilon_+,\epsilon_-} + \left[\check{h}\check{g}^A - \check{g}^R\check{h}\right]_{\epsilon_+,\epsilon_-}\end{aligned}$$

where

$$\check{I}^{(a)}_{\epsilon_+,\epsilon_-} = \int \frac{d\xi_{\mathbf{p}}}{\pi i}\left[\check{\Sigma}^R\check{G}^{(a)} - \check{G}^{(a)}\check{\Sigma}^A - \check{G}^R\check{\Sigma}^{(a)} + \check{\Sigma}^{(a)}\check{G}^A\right]_{\epsilon_+,\epsilon_-} \tag{9.5}$$

and $\check{h}_{\epsilon,\epsilon-\omega}$ are defined by eqn (8.10) on page 160 .

Of course, one can also obtain similar equations for the regular functions, as well. Subtracting the left-handed and right-handed equations

$$\left(\check{G}^{-1}-\check{\Sigma}^{R(A)}\right)\check{G}^{R(A)}=\check{1},\quad \check{G}^{R(A)}\left(\check{G}^{-1}-\check{\Sigma}^{R(A)}\right)=\check{1} \tag{9.6}$$

from each other, and integrating the result over $d\xi_{\mathbf{p}}$, we obtain

$$\mathbf{v}_F\mathbf{k}\check{g}^{R(A)}_{\epsilon_+,\epsilon_-}-\epsilon_+\check{\tau}_3\check{g}^{R(A)}_{\epsilon_+,\epsilon_-}+\check{g}^{R(A)}_{\epsilon_+,\epsilon_-}\check{\tau}_3\epsilon_-+\left[\check{H}\check{g}^{R(A)}-\check{g}^{R(A)}\check{H}\right]_{\epsilon_+,\epsilon_-}=\check{I}^{R(A)}_{\epsilon_+,\epsilon_-} \tag{9.7}$$

where the corresponding collision integral is

$$\check{I}^{R(A)}_{\epsilon_+,\epsilon_-}=\int\frac{d\xi_{\mathbf{p}}}{\pi i}\left[\check{\Sigma}^{R(A)}\check{G}^{R(A)}-\check{G}^{R(A)}\check{\Sigma}^{R(A)}\right]_{\epsilon_+,\epsilon_-} \tag{9.8}$$

9.1.1 Self energies

The self energies in eqns (9.2), (9.5), and (9.8) also contain only the $\xi_{\mathbf{p}}$-integrated Green functions; they depend only on the direction of the momentum whose magnitude is fixed by the Fermi surface. Indeed, for interaction with non-magnetic impurities, we have

$$\begin{aligned}\check{\Sigma}^K_{\epsilon,\epsilon-\omega}\left(\mathbf{p}_F,\mathbf{k}\right)&=n_{\text{imp}}\int\frac{d^3p'}{(2\pi)^3}|u(\theta)|^2\check{G}^K_{\epsilon,\epsilon-\omega}\left(\mathbf{p}',\mathbf{p}'-\mathbf{k}\right)\\&=\pi i\nu(0)n_{\text{imp}}\int\frac{d\Omega_{\mathbf{p}'}}{4\pi}|u(\theta)|^2\check{g}^K_{\epsilon,\epsilon-\omega}\left(\mathbf{p}'_F,\mathbf{k}\right)\end{aligned} \tag{9.9}$$

and

$$\check{\Sigma}^{R(A)}_{\epsilon,\epsilon-\omega}\left(\mathbf{p}_F,\mathbf{k}\right)=\pi i\nu(0)n_{\text{imp}}\int\frac{d\Omega_{\mathbf{p}'}}{4\pi}|u(\theta)|^2\check{g}^{R(A)}_{\epsilon,\epsilon-\omega}\left(\mathbf{p}'_F,\mathbf{k}\right) \tag{9.10}$$

in a complete analogy to eqn (5.69). The impurity self energy is written in the Born approximation. The Fourier transformed impurity potential squared is $|u(\theta)|^2=2v_F\sigma_{\mathbf{pp}'}/\nu(0)$ where $\sigma_{\mathbf{pp}'}$ is the scattering cross section. One can thus present the self energy as

$$\begin{aligned}\check{\Sigma}^K_{\epsilon,\epsilon-\omega}\left(\mathbf{p}_F,\mathbf{k}\right)&=\frac{i}{2}v_Fn_{\text{imp}}\int d\Omega_{\mathbf{p}'}\sigma_{\mathbf{pp}'}\check{g}^K_{\epsilon,\epsilon-\omega}\left(\mathbf{p}'_F,\mathbf{k}\right)\\\check{\Sigma}^{R(A)}_{\epsilon,\epsilon-\omega}\left(\mathbf{p}_F,\mathbf{k}\right)&=\frac{i}{2}v_Fn_{\text{imp}}\int d\Omega_{\mathbf{p}'}\sigma_{\mathbf{pp}'}\check{g}^{R(A)}_{\epsilon,\epsilon-\omega}\left(\mathbf{p}'_F,\mathbf{k}\right)\end{aligned}$$

Remind that the scattering rate in the normal sate is

$$\frac{1}{\tau_{\text{imp}}}=4\pi\left\langle\sigma_{\mathbf{pp}'}\right\rangle v_Fn_{\text{imp}}$$

The phonon self energy can also be expressed through the quasiclassical functions. It is

$$\begin{aligned}\check{\Sigma}^{(\text{ph})K}_{\epsilon,\epsilon-\omega}\left(\mathbf{p}_F,\mathbf{k}\right)=\nu(0)g^2_{ph}\int\frac{d\epsilon'}{4}\frac{d\Omega_{\mathbf{p}'}}{4\pi}\left(D^R_{\epsilon'-\epsilon}-D^A_{\epsilon'-\epsilon}\right)\\\times\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\check{g}^K_{\epsilon',\epsilon'-\omega}-\left(\check{g}^R_{\epsilon',\epsilon'-\omega}-\check{g}^A_{\epsilon',\epsilon'-\omega}\right)\right]\end{aligned}$$

The relaxation part of the regular phonon self energy takes the form

$$\begin{aligned}\check{\Sigma}^{(\text{ph})R(A)}_{\epsilon,\epsilon-\omega}\left(\mathbf{p}_F,\mathbf{k}\right)=&\nu(0)g^2_{ph}\int\frac{d\epsilon'}{4}\frac{d\Omega_{\mathbf{p}'}}{4\pi}\left(D^R_{\epsilon'-\epsilon}-D^A_{\epsilon'-\epsilon}\right)\\&\times\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\check{g}^{R(A)}_{\epsilon',\epsilon'-\omega}\mp\frac{1}{2}\check{g}^K_{\epsilon',\epsilon'-\omega}\right]\end{aligned}$$

The function $\check{g}^R_{\epsilon',\epsilon'-\omega}$ in these equations carries the momentum $\mathbf{p}'$, i.e., $\check{g}^R_{\epsilon',\epsilon'-\omega} \equiv \check{g}^R_{\epsilon',\epsilon'-\omega}(\mathbf{p}',\mathbf{k})$. Since the phonon energy is $\omega \sim T$ and satisfies the inequality $\omega \ll \Omega_D$ where $\Omega_D \sim sp_F$ is the Debye frequency, the phonon energy is $\omega_{\mathbf{pp}'} = 2sp_F|\sin(\theta/2)|$ where s is the velocity of sound. Because the relative change in the momentum during the emission of a phonon is small

$$|\mathbf{p}-\mathbf{p}'| \sim \Delta/s \ll p_F$$

the the scattering angle of a phonon $\theta \ll 1$. We have

$$\int \frac{d\Omega_{\mathbf{p}'}}{4\pi}\left[D^R_\omega(\theta) - D^A_\omega(\theta)\right] = \frac{\pi i}{2\left(sp_F\right)^2}\omega^2 \operatorname{sign}\omega \tag{9.11}$$

As a result,

$$\begin{aligned}\check{\Sigma}^{(\mathrm{ph})K}_{\epsilon,\epsilon-\omega}(\mathbf{p}_F,\mathbf{k}) =& \frac{\pi i\nu(0)g^2_{ph}}{2\left(sp_F\right)^2}\int\frac{d\epsilon'}{4}\left(\epsilon'-\epsilon\right)^2\operatorname{sign}\left(\epsilon'-\epsilon\right)\\ &\times\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\check{g}^K_{\epsilon',\epsilon'-\omega}(\mathbf{p},\mathbf{k})\right.\\ &\left.-\left(\check{g}^R_{\epsilon',\epsilon'-\omega}(\mathbf{p},\mathbf{k})-\check{g}^A_{\epsilon',\epsilon'-\omega}(\mathbf{p},\mathbf{k})\right)\right]\end{aligned} \tag{9.12}$$

$$\begin{aligned}\check{\Sigma}^{(\mathrm{ph})R(A)}_{\epsilon,\epsilon-\omega}(\mathbf{p}_F,\mathbf{k}) =& \frac{\pi i\nu(0)g^2_{ph}}{2\left(sp_F\right)^2}\int\frac{d\epsilon'}{4}\left(\epsilon'-\epsilon\right)^2\operatorname{sign}\left(\epsilon'-\epsilon\right)\\ &\times\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\check{g}^{R(A)}_{\epsilon',\epsilon'-\omega}(\mathbf{p},\mathbf{k})\right.\\ &\left.\mp\frac{1}{2}\check{g}^K_{\epsilon',\epsilon'-\omega}(\mathbf{p},\mathbf{k})\right]\end{aligned} \tag{9.13}$$

Note that the functions $\check{g}_{\epsilon'}(\mathbf{p},\mathbf{r})$ in eqns (9.12, 9.13) have the same momentum p as the self energy itself since the relative change in the momentum during interaction with the acoustic phonons is small.

The collision integral eqn (9.2) takes the form

$$\check{I}^K_{\epsilon_+,\epsilon_-}(\mathbf{p}_F,\mathbf{k}) = \left[\check{\Sigma}^R\check{g}^K - \check{g}^K\check{\Sigma}^A - \check{g}^R\check{\Sigma}^K + \check{\Sigma}^K\check{g}^A\right]_{\epsilon_+,\epsilon_-} \tag{9.14}$$

where all terms are expressed through quasiclassical functions.

The collision integral eqn (9.5) for the anomalous function is

$$\check{I}^{(a)}_{\epsilon_+,\epsilon_-} = \left[\check{\Sigma}^R\check{g}^{(a)} - \check{g}^{(a)}\check{\Sigma}^A - \check{g}^R\check{\Sigma}^{(a)} + \check{\Sigma}^{(a)}\check{g}^A\right]_{\epsilon_+,\epsilon_-} \tag{9.15}$$

For regular functions in eqn (9.7) we obtain

$$\check{I}^{R(A)}_{\epsilon_+,\epsilon_-} = \left[\check{\Sigma}^{R(A)}\check{g}^{R(A)} - \check{g}^{R(A)}\check{\Sigma}^{R(A)}\right]_{\epsilon_+,\epsilon_-} \tag{9.16}$$

9.1.2 Order parameter, current, and particle density

The order parameter, current, and the particle density can be expressed through the quasiclassical functions in the same way as for the static case. The order parameter equation becomes for a general pairing

$$\frac{\Delta_{\mathbf{p}}(\mathbf{k},\omega)}{\lambda} = \int V\left(\hat{\mathbf{p}},\hat{\mathbf{p}}'\right) f^{K}_{\epsilon_+,\epsilon_-}\left(\hat{\mathbf{p}}'_F,\mathbf{k}\right)\frac{d\Omega_{\mathbf{p}'}}{4\pi}\frac{d\epsilon}{4}$$
$$\frac{\Delta^*_{\mathbf{p}}(\mathbf{k},\omega)}{\lambda} = \int V\left(\hat{\mathbf{p}},\hat{\mathbf{p}}'\right) f^{\dagger K}_{\epsilon_+,\epsilon_-}\left(\hat{\mathbf{p}}'_F,\mathbf{k}\right)\frac{d\Omega_{\mathbf{p}'}}{4\pi}\frac{d\epsilon}{4} \tag{9.17}$$

The expression for current is

$$\begin{aligned}\mathbf{j}(\omega,\mathbf{k}) &= -2e\nu(0)\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\frac{d\epsilon}{4}\mathbf{v}_F g^{K}_{\epsilon_+,\epsilon_-}\left(\mathbf{p}_F,\mathbf{k}\right)\\ &= 2e\nu(0)\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\frac{d\epsilon}{4}\mathbf{v}_F \bar{g}^{K}_{\epsilon_+,\epsilon_-}\left(\mathbf{p}_F,\mathbf{k}\right)\end{aligned} \tag{9.18}$$

and the electron density has the form

$$N(\omega,\mathbf{k}) = N_0(2\pi)^4\delta(\mathbf{k})\delta(\omega) + \delta N(\omega,\mathbf{k})$$

where

$$\begin{aligned}\delta N(\omega,\mathbf{k}) &= -2\nu(0)\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\frac{d\epsilon}{4}g^{K}_{\epsilon_+,\epsilon_-}\left(\mathbf{p}_F,\mathbf{k}\right)\\ &= -2\nu(0)\int\frac{d\Omega_{\mathbf{p}}}{4\pi}\frac{d\epsilon}{4}\bar{g}^{K}_{\epsilon_+,\epsilon_-}\left(\mathbf{p}_F,\mathbf{k}\right)\end{aligned} \tag{9.19}$$

They are easily obtained from the corresponding equations in terms of the total Green functions and the quasiclassical expressions derived earlier for the static case. The diamagnetic term in the current, eqn (9.18), disappears in the quasiclassical representation since it is canceled by the normal-state contribution subtracted from the ξ-integrated function g as it has been shown earlier.

9.1.3 Normalization of the quasiclassical functions

Here we derive useful identities for the total quasiclassical Green functions. First of all, we demonstrate that the normalization condition

$$\int\frac{d\epsilon'}{2\pi}\frac{d^3k'}{(2\pi)^3}\check{g}^{R(A)}_{\epsilon,\epsilon'}\left(\mathbf{p}_F,\mathbf{k}\right)\check{g}^{R(A)}_{\epsilon',\epsilon-\omega}\left(\mathbf{p}_F,\mathbf{k}-\mathbf{k}'\right) = \check{1}(2\pi)^4\delta(\omega)\delta(\mathbf{k}) \tag{9.20}$$

holds in the nonstationary case, as well. To see this, we multiply eqn (9.7) by $\check{g}^{R(A)}_{\epsilon,\epsilon'}$ first from the left and then from the right and add the two equations. We obtain

$$\begin{aligned}&\mathbf{v}_F\mathbf{k}[\check{g}\check{g}]^{R(A)}_{\epsilon_+,\epsilon_-} - \epsilon_+\check{\tau}_3[\check{g}\check{g}]^{R(A)}_{\epsilon_+,\epsilon_-} + [\check{g}\check{g}]^{R(A)}_{\epsilon_+,\epsilon_-}\epsilon_-\check{\tau}_3\\ &+ \check{H}_{\omega_1}[\check{g}\check{g}]^{R(A)}_{\epsilon_+-\omega_1,\epsilon_-} - [\check{g}\check{g}]^{R(A)}_{\epsilon_+,\epsilon_-+\omega_1}\check{H}_{\omega_1}\\ =&\left[\check{\Sigma}^{R(A)}[\check{g}\check{g}]^{R(A)} - [\check{g}\check{g}]^{R(A)}\check{\Sigma}^{R(A)}\right]_{\epsilon_+,\epsilon_-}\end{aligned} \tag{9.21}$$

where the shortcut $[\check{g}\check{g}]$ again stands for the expression in the l.h.s. of eqn (9.20). We observe that $[\check{g}\check{g}]^{R(A)} = \check{1}\cdot$ const is a solution of eqn (9.21). We can use the same argumentation as for the static case. We assume that, at large distances and at times $t \to -\infty$, the system is in equilibrium and is homogeneous. Therefore the condition $[\check{g}\check{g}]^{R(A)} = \check{1}$, being fulfilled in equilibrium, should also hold in a general situation. We thus arrive at eqn (9.20).

Let us now multiply eqn (9.1) first by $\check{g}^R$ from the left and then by $\check{g}^A$ from the right. Next, we multiply eqn (9.7) for $\check{g}^R$ by $\check{g}^K$ from the right, and then eqn (9.7) for $\check{g}^A$ by $\check{g}^K$ from the left. We now add the four equations together and obtain with the help of eqn (9.20)

$$\begin{aligned}&\mathbf{v}_F\mathbf{k}\check{C}_{\epsilon_+,\epsilon_-} - \epsilon_+\check{\tau}_3\check{C}_{\epsilon_+,\epsilon_-} + \check{C}_{\epsilon_+,\epsilon_-}\epsilon_-\check{\tau}_3 + [\check{H}\check{C}]_{\epsilon_+,\epsilon_-} - [\check{C}\check{H}]_{\epsilon_+,\epsilon_-}\\ =&\left[\check{\Sigma}^R\check{C} - \check{C}\check{\Sigma}^A\right]_{\epsilon_+,\epsilon_-}\end{aligned} \tag{9.22}$$

where $\check{C} = \left[\check{g}^R\check{g}^K + \check{g}^K\check{g}^A\right]$. We prove now that $\check{C} = 0$. Indeed, the solution $\check{C} = 0$ satisfies eqn (9.22) and the initial conditions for an equilibrium system where

$$\check{g}^K_{\epsilon_+,\epsilon_-} = \left(\check{g}^R_\epsilon - \check{g}^A_\epsilon\right)\tanh\left(\frac{\epsilon}{2T}\right)(2\pi)\delta(\omega) \tag{9.23}$$

Therefore, in a general situation

$$\begin{aligned}&\int\frac{d\epsilon'}{2\pi}\frac{d^3k'}{(2\pi)^3}\left[\check{g}^R_{\epsilon,\epsilon'}\left(\mathbf{p}_F,\mathbf{k}'\right)\check{g}^K_{\epsilon',\epsilon-\omega}\left(\mathbf{p}_F,\mathbf{k}-\mathbf{k}'\right)\right.\\ &\left.+\check{g}^K_{\epsilon,\epsilon'}\left(\mathbf{p}_F,\mathbf{k}'\right)\check{g}^A_{\epsilon',\epsilon-\omega}\left(\mathbf{p}_F,\mathbf{k}-\mathbf{k}'\right)\right] = 0\end{aligned} \tag{9.24}$$

This result was first obtained by Larkin and Ovchinnikov (1975). Using eqn (9.20) we can see that eqn (9.24) also holds for the anomalous function $\check{g}^{(a)}$.

18.2.2 9.2 Generalized distribution function

Equation (9.24) allows us to present the Keldysh function in the form

$$\begin{aligned}\check{g}^K_{\epsilon_+,\epsilon_-}\left(\mathbf{p}_F,\mathbf{k}\right) = \int\frac{d\epsilon'}{2\pi}\frac{d^3k'}{(2\pi)^3}&\left[\check{g}^R_{\epsilon_+,\epsilon'}\left(\mathbf{p}_F,\mathbf{k}'\right)\check{f}_{\epsilon',\epsilon_-}\left(\mathbf{p}_F,\mathbf{k}-\mathbf{k}'\right)\right.\\ &\left.-\check{f}_{\epsilon_+,\epsilon'}\left(\mathbf{p}_F,\mathbf{k}+\mathbf{k}'\right)\check{g}^A_{\epsilon',\epsilon_-}\left(\mathbf{p}_F,\mathbf{k}'\right)\right]\end{aligned} \tag{9.25}$$

where $\check{f}_{\epsilon,\epsilon'}$ is another unknown matrix. Comparing this equation with eqn (9.23) we can see that $\check{f}_{\epsilon,\epsilon'}$ has the meaning of a generalized "distribution function" which describes the system in a nonequilibrium situation. Remind that the hyperbolic tangent in eqn (9.23) comes from the equilibrium distribution eqn (8.14).

We shall show that the matrix $\check{f}_{\epsilon,\epsilon'}$ has only two independent components. Consider the operators $\mathcal{L}_\pm\{\check{A}\}$ acting on a matrix $\check{A}$ which are defined by

$$\mathcal{L}_+\{\check{A}\} = \left[\check{g}^R\check{A}\right] + \left[\check{A}\check{g}^A\right], \mathcal{L}_-\{\check{A}\} = \left[\check{g}^R\check{A}\right] - \left[\check{A}\check{g}^A\right].$$

Here we again imply integration over internal frequencies and momenta. Using the definition of the $\mathcal{L}$ operators and the normalization condition eqn (9.20) for the regular functions we find

$$\mathcal{L}_+\mathcal{L}_- = \mathcal{L}_-\mathcal{L}_+ = 0 \tag{9.26}$$

Consider the eigen-matrices of these operators

$$\mathcal{L}_+\left\{\check{A}_+\right\} = \lambda_+\check{A}_+; \mathcal{L}_-\left\{\check{A}_-\right\} = \lambda_-\check{A}_-. \tag{9.27}$$

Each of these operators has four eigen-matrices belonging to four eigenvalues. It is easy to check that the eigen-matrix $\check{A}_+$belonging to a nonzero eigenvalue λ_+ of the operator $\mathcal{L}_+$is simultaneously an eigen-matrix of the operator $\mathcal{L}_-$with eigenvalue zero, and vice versa: the

eigen-matrix $\check{A}_-$belonging to a nonzero eigenvalue $\lambda_-$of the operator $\mathcal{L}_-$is simultaneously an eigen-matrix of the operator $\mathcal{L}_+$with eigenvalue zero. To see this, we apply the operator $\mathcal{L}_-$to the first equation (9.27) using eqn (9.26) and obtain $\mathcal{L}_-\left\{\check{A}_+\right\}=0$. We can also apply $\mathcal{L}_+$to the second equation (9.27) to get $\mathcal{L}_+\left\{\check{A}_-\right\}=0$. Applying the operator $\mathcal{L}_-$two times to the eigen-matrix $\check{A}_-$we have

$$2\left(\check{A}_- - \left[\check{g}^R\check{A}_-\check{g}^A\right]\right)=\lambda_-^2\check{A}_-$$

Applying the operator $\mathcal{L}_-$again we find $\lambda_-\left(\lambda_-^2-4\right)=0$. In the same way we get $\lambda_+\left(\lambda_+^2-4\right)=0$. Therefore, the four eigen values of the operator $\mathcal{L}_-$are $\lambda_-^{(1)}=+2, \lambda_-^{(2)}=-2$, and twice degenerate $\lambda_-^{(3,4)}=0$. The corresponding eigenmatrices $\check{A}_1,\check{A}_2,\check{A}_3,\check{A}_4$ are simultaneously the eigen-matrices of the operator $\mathcal{L}_+$with the eigen values $0,0,+2$, and -2 , respectively.

Let us now expand the matrix $\check{\mathscr{F}}_{\epsilon,\epsilon'}$ in the matrices $\check{A}_1,\check{A}_2,\check{A}_3,\check{A}_4$. We have

$$\check{f}_{\epsilon,\epsilon'}=\check{A}_1\tilde{f}^{(1)}+\check{A}_2\tilde{f}^{(2)}+\check{A}_3\tilde{f}^{(3)}+\check{A}_4\tilde{f}^{(4)}.$$

It is clear that the coefficients $\tilde{f}^{(3)}$ and $\tilde{f}^{(4)}$ will not appear in the total Green function $\check{g}^K=\mathcal{L}_-\{\check{f}\}$ because the matrices $\check{A}_3,\check{A}_4$ correspond to zero eigenvalues of the operator $\mathcal{L}_-$. Therefore, the matrix $\check{f}_{\epsilon,\epsilon'}$ in eqn (9.25) has only two independent components $\tilde{f}^{(1)}$ and $\tilde{f}^{(2)}$. The two remaining coefficients $\tilde{f}^{(3)}$ and $\tilde{f}^{(4)}$ can be chosen arbitrarily. For, example, one can choose them in such a way as to make the function $\check{f}_{\epsilon,\epsilon'}$ diagonal

$$\check{f}_{\epsilon,\epsilon'}=\check{1}f^{(1)}\left(\epsilon,\epsilon'\right)+\check{\tau}_3f^{(2)}\left(\epsilon,\epsilon'\right). \tag{9.28}$$

This form of presentation for the distribution function was introduced by Schmid and Schön (1975) and by Larkin and Ovchinnikov (1977) We note that, in equilibrium,

$$f^{(1)}\left(\epsilon,\epsilon'\right)=2\pi\delta\left(\epsilon-\epsilon'\right)f^{(0)}(\epsilon),\quad f^{(2)}=0$$

where, as in eqn (8.9),

$$f^{(0)}(\epsilon)=\tanh\frac{\epsilon}{2T}$$

We use this form of presentation of the distribution function throughout the book. Another choice for the free parameters $\tilde{f}^{(1)}$ and $\tilde{f}^{(2)}$ has been suggested by Shelankov (1985). It has an advantage in some cases when the relation to the Boltzmann kinetic equation is considered.

In Chapter 10 we discuss the generalized two-component distribution function in detail and derive the kinetic equations which determine $f^{(1)}$ and $f^{(2)}$. The obtained solutions will be used for various applications later in the book. In the rest of the present Chapter, we consider several examples of a direct implementation of the Eliashberg equations in their original form.

18.2.3 9.3 ?-wave superconductors with a short mean free path

Here we discuss an approach which can be used for superconductors with an impurity mean free path $\ell\ll\xi(T)$. This limit is realized for some s-wave superconducting alloys. For d-wave superconductors, where always $\ell\gg\xi_0$, this approach can also be used in a close vicinity of the critical temperature such that $\xi(T)$ already exceeds ℓ. We consider the d-wave case later in Section 11.3 while concentrating here on s-wave superconductors. The method is similar to the one used in Section 5.6 to derive the Usadel equations.

The gradients of the order parameter and of other physical quantities are generally of the order of $\xi^{-1}(T)$ and are thus small compared with ℓ^{-1}. We can use a "hydrodynamic" approximation, assuming that the Green functions are, to the first approximation, independent of the momentum direction, and restrict ourselves to the first correction in small gradients. Since these corrections are vectors, we write

$$\check{g} = \langle \check{g} \rangle + \hat{\mathbf{p}} \check{g}$$

for all (retarded, advanced, and Keldysh) Green functions. Here $\hat{\mathbf{p}}$ is the unit vector in the momentum direction, and $\langle \ldots \rangle$ denotes an average over the momentum directions in accordance with eqns (5.72) or (5.73). Both $\langle \check{g} \rangle$ and $\check{\mathbf{g}}$ are independent of the momentum direction. The vector part of the Green function is small compared to its averaged value $\langle \check{g} \rangle$.

Averaging of the normalization condition eqn (9.20) gives for the regular functions

$$\left[\left\langle \check{g}^{R(A)} \right\rangle \left\langle \check{g}^{R(A)} \right\rangle \right]_{\epsilon_+, \epsilon_-} = \check{1} (2\pi)^4 \delta(\mathbf{k}) \delta(\omega). \tag{9.29}$$

Averaging it with the momentum direction $\hat{\mathbf{p}}$ results in

$$\left[\left\langle \check{g}^{R(A)} \right\rangle \check{\mathbf{g}}^{R(A)} + \check{\mathbf{g}}^{R(A)} \left\langle \check{g}^{R(A)} \right\rangle \right]_{\epsilon_+, \epsilon_-} = 0 \tag{9.30}$$

For the Keldysh function we have from eqn (9.24)

$$\left[\left\langle \check{g}^{R} \right\rangle \left\langle \check{g}^{K} \right\rangle + \left\langle \check{g}^{K} \right\rangle \left\langle \check{g}^{A} \right\rangle \right]_{\epsilon_+, \epsilon_-} = 0 \tag{9.31}$$

and

$$\left[\left\langle \check{g}^{R} \right\rangle \check{\mathbf{g}}^{K} + \check{\mathbf{g}}^{K} \left\langle \check{g}^{A} \right\rangle + \check{\mathbf{g}}^{R} \left\langle \check{g}^{K} \right\rangle + \left\langle \check{g}^{K} \right\rangle \check{\mathbf{g}}^{A} \right]_{\epsilon_+, \epsilon_-} = 0. \tag{9.32}$$

The impurity collision integrals for non-magnetic scattering eqn (9.14) or (9.16) is expressed through non-magnetic self energies defined by eqn (5.76). The non-magnetic collision integrals vanishes after averaging over the momentum directions:

$$\left\langle \check{I}^{R(A)}_{\text{imp}}(\mathbf{p}) \right\rangle = \left\langle \check{I}^{K}_{\text{imp}}(\mathbf{p}) \right\rangle = 0.$$

Using this fact, we can simplify the quasiclassical equations in the same way as it has been done in Section 5.6. On the first stage, we average the kinetic equation over directions of the momentum. For the Keldysh function we get from eqn (9.1) in coordinate representation

$$\begin{aligned} &\frac{1}{3} v_F \left(-i \nabla \check{\mathbf{g}}^K_{\epsilon_+, \epsilon_-} - \frac{e}{c} \left(\check{\tau}_3 \left[\mathbf{A} \check{\mathbf{g}}^K \right]_{\epsilon_+, \epsilon_-} - \left[\check{\mathbf{g}}^K \mathbf{A} \right]_{\epsilon_+, \epsilon_-} \check{\tau}_3 \right) \right) \\ &- \epsilon_+ \check{\tau}_3 \left\langle \check{g}^K_{\epsilon_+, \epsilon_-} \right\rangle + \left\langle \check{g}^K_{\epsilon_+, \epsilon_-} \right\rangle \epsilon_- \check{\tau}_3 + \left[\check{H}_\Delta \left\langle \check{g}^K \right\rangle \right]_{\epsilon_+, \epsilon_-} - \left[\left\langle \check{g}^K \right\rangle \check{H}_\Delta \right]_{\epsilon_+, \epsilon_-} \\ = &\left\langle \check{I}^{(ph)K}_{\epsilon_+, \epsilon_-} \right\rangle + \left\langle \check{I}^{(s)K}_{\epsilon_+, \epsilon_-} \right\rangle \end{aligned} \tag{9.33}$$

Here

$$\check{H}_\Delta = \begin{pmatrix} 0 & -\Delta \\ \Delta^* & 0 \end{pmatrix}.$$

We note here an important difference between s-wave and d-wave superconductors. For the s-wave case, the average of $\check{H}_\Delta$ does not disappear because Δ is independent of the momentum directions. On the contrary, for a d-wave case, this average vanishes.

Since the largest part of the total collision integral associated with nonmagnetic impurities disappears after averaging we have to keep other terms. In the phonon and spin-flip collision

integrals, one can replace the full Green
functions by their averaged parts using smallness of the vector parts. We have, for example

$$\begin{aligned}\left\langle \check{I}^{(s)K}_{\epsilon_+,\epsilon_-}\right\rangle =&\frac{i}{2\tau_s}\left[\check{\tau}_3\left\langle\check{g}^R\right\rangle\check{\tau}_3\left\langle\check{g}^K\right\rangle-\left\langle\check{g}^R\right\rangle\check{\tau}_3\left\langle\check{g}^K\right\rangle\check{\tau}_3\right.\\&\left.-\left\langle\check{g}^K\right\rangle\check{\tau}_3\left\langle\check{g}^A\right\rangle\check{\tau}_3+\check{\tau}_3\left\langle\check{g}^K\right\rangle\check{\tau}_3\left\langle\check{g}^A\right\rangle\right]_{\epsilon_+,\epsilon_-}.\end{aligned}\tag{9.34}$$

Let us now multiply eqn (9.1) by the momentum unit vector $\hat{\mathbf{p}}$ and average over its directions. We obtain

$$\begin{aligned}&v_F\left(-i\nabla\left\langle\check{g}^K_{\epsilon_+,\epsilon_-}\right\rangle-\frac{e}{c}\left(\check{\tau}_3\left[\mathbf{A}\left\langle\check{g}^K\right\rangle\right]_{\epsilon_+,\epsilon_-}-\left[\left\langle\check{g}^K\right\rangle\mathbf{A}\right]_{\epsilon_+,\epsilon_-}\check{\tau}_3\right)\right)\\&-\epsilon_+\check{\tau}_3\check{\mathbf{g}}^K_{\epsilon_+,\epsilon_-}+\check{\mathbf{g}}^K_{\epsilon_+,\epsilon_-}\epsilon_-\check{\tau}_3+\left[\check{H}_\Delta\check{\mathbf{g}}^K\right]_{\epsilon_+,\epsilon_-}-\left[\check{\mathbf{g}}^K\check{H}_\Delta\right]_{\epsilon_+,\epsilon_-}\\=&\frac{i}{2\tau_{\rm tr}}\left[\left\langle\check{g}^R\right\rangle\check{\mathbf{g}}^K-\check{\mathbf{g}}^R\left\langle\check{g}^K\right\rangle+\left\langle\check{g}^K\right\rangle\check{\mathbf{g}}^A-\check{\mathbf{g}}^K\left\langle\check{g}^A\right\rangle\right]_{\epsilon_+,\epsilon_-}\\&+3\left\langle\hat{\mathbf{p}}\check{I}^{(ph)K}_{\epsilon_+,\epsilon_-}\right\rangle+3\left\langle\hat{\mathbf{p}}\check{I}^{(s)K}_{\epsilon_+,\epsilon_-}\right\rangle\end{aligned}$$

We use here eqn (5.80) and the definition eqn (5.81) of the transport mean free time. The relaxation rate $1/\tau_{\rm tr}$ is usually large compared with $1/\tau_s$. In the dirty limit it is also large compared to T and Δ, to say nothing about the electronphonon relaxation rate. Therefore, we can neglect the corresponding terms here and obtain

$$\begin{aligned}&v_F\left(-i\nabla\left\langle\check{g}^K_{\epsilon_+,\epsilon_-}\right\rangle-\frac{e}{c}\left(\check{\tau}_3\left[\mathbf{A}\left\langle\check{g}^K\right\rangle\right]_{\epsilon_+,\epsilon_-}-\left[\left\langle\check{g}^K\right\rangle\mathbf{A}\right]_{\epsilon_+,\epsilon_-}\check{\tau}_3\right)\right)\\=&\frac{i}{2\tau_{\rm tr}}\left[\left\langle\check{g}^R\right\rangle\check{\mathbf{g}}^K-\check{\mathbf{g}}^R\left\langle\check{g}^K\right\rangle+\left\langle\check{g}^K\right\rangle\check{\mathbf{g}}^A-\check{\mathbf{g}}^K\left\langle\check{g}^A\right\rangle\right]_{\epsilon_+,\epsilon_-}.\end{aligned}\tag{9.35}$$

This equation allows one to find the vector part of the Keldysh function. Let us multiply eqn (9.35) by $\left\langle\check{g}^R\right\rangle$ from the left and integrate over internal frequencies and momenta, then multiply eqn (9.35) again by $\left\langle\check{g}^A\right\rangle$ from the right and subtract the two equations. Using eqns (9.29)-(9.32) we obtain

$$\begin{aligned}&-iv_F\left[\left\langle\check{g}^R\right\rangle\left(\nabla\left\langle\check{g}^K\right\rangle\right)-\left(\nabla\left\langle\check{g}^K\right\rangle\right)\left\langle\check{g}^A\right\rangle\right]_{\epsilon_+,\epsilon_-}\\&-\frac{e}{c}v_F\left[\left(\left\langle\check{g}^R\right\rangle\check{\tau}_3+\check{\tau}_3\left\langle\check{g}^R\right\rangle\right)\mathbf{A}\left\langle\check{g}^K\right\rangle+\left\langle\check{g}^K\right\rangle\mathbf{A}\left(\left\langle\check{g}^A\right\rangle\check{\tau}_3+\check{\tau}_3\left\langle\check{g}^A\right\rangle\right)\right]_{\epsilon_+,\epsilon_-}\\&=\frac{i}{\tau_{\rm tr}}\left[\mathbf{g}^K-\left\langle\check{g}^R\right\rangle\mathbf{g}^K\left\langle\check{g}^A\right\rangle\right]_{\epsilon_+,\epsilon_-}\end{aligned}$$

The r.h.s. can also be written as

$$\frac{i}{\tau_{\rm tr}}\left[2\mathbf{g}^K-\mathbf{g}^R\left\langle\check{g}^R\right\rangle\left\langle\check{g}^K\right\rangle-\left\langle\check{g}^K\right\rangle\left\langle\check{g}^A\right\rangle\mathbf{g}^A\right]_{\epsilon_+,\epsilon_-}$$

Having found the vector part from this equation, we can then use it to determine the averaged functions from eqn (9.33). However, further transformations of these
equations can not be performed easily in a general case. We shall consider some examples later.

The equations for regular functions can be obtained in the same way. The averaged equation is

$$
\begin{aligned}
&\frac{1}{3}v_F\left(-i\nabla\check{\mathbf{g}}^{R(A)}_{\epsilon_+,\epsilon_-}-\frac{e}{c}\left(\check{\tau}_3\left[\mathbf{A}\check{\mathbf{g}}^{R(A)}\right]_{\epsilon_+,\epsilon_-}-\left[\check{\mathbf{g}}^{R(A)}\mathbf{A}\right]_{\epsilon_+,\epsilon_-}\check{\tau}_3\right)\right)\\
&-\epsilon_+\check{\tau}_3\left\langle\check{g}^{R(A)}_{\epsilon_+,\epsilon_-}\right\rangle+\left\langle\check{g}^{R(A)}_{\epsilon_+,\epsilon_-}\right\rangle\epsilon_-\check{\tau}_3\\
&+\left[\check{H}_\Delta\left\langle\check{g}^{R(A)}\right\rangle\right]_{\epsilon_+,\epsilon_-}-\left[\left\langle\check{g}^{R(A)}\right\rangle\check{H}_\Delta\right]_{\epsilon_+,\epsilon_-}\\
=&\left\langle\check{I}^{(ph)R(A)}_{\epsilon_+,\epsilon_-}\right\rangle+\left\langle\check{I}^{(s)R(A)}_{\epsilon_+,\epsilon_-}\right\rangle
\end{aligned}
\tag{9.36}
$$

where

$$
\left\langle\check{I}^{(s)R(A)}_{\epsilon_+,\epsilon_-}\right\rangle=\frac{i}{2\tau_s}\left[\check{\tau}_3\left\langle\check{g}^{R(A)}\right\rangle\check{\tau}_3\left\langle\check{g}^{R(A)}\right\rangle-\left\langle\check{g}^{R(A)}\right\rangle\check{\tau}_3\left\langle\check{g}^{R(A)}\right\rangle\check{\tau}_3\right]_{\epsilon_+,\epsilon_-}
\tag{9.37}
$$

The vector part is

$$
\begin{aligned}
&\ell_{\mathrm{tr}}\left(-i\nabla\left\langle\check{g}^{R(A)}_{\epsilon_+,\epsilon_-}\right\rangle-\frac{e}{c}\left(\check{\tau}_3\left[\mathbf{A}\left\langle\check{g}^{R(A)}\right\rangle\right]_{\epsilon_+,\epsilon_-}-\left[\left\langle\check{g}^{R(A)}\right\rangle\mathbf{A}\right]_{\epsilon_+,\epsilon_-}\check{\tau}_3\right)\right)\\
=&\frac{i}{2}\left[\left\langle\check{g}^{R(A)}\right\rangle\check{\mathbf{g}}^{R(A)}-\check{\mathbf{g}}^{R(A)}\left\langle\check{g}^{R(A)}\right\rangle\right]_{\epsilon_+,\epsilon_-}
\end{aligned}
\tag{9.38}
$$

where $\ell_{\mathrm{tr}}=v_F\tau_{\mathrm{tr}}$. The vector part of the regular functions can be found using the normalization of eqns (9.29)-(9.30). Multiplying eqn (9.38) by $\left\langle\check{g}^{R(A)}\right\rangle$ from the left, we get

$$
\begin{aligned}
\check{\mathbf{g}}^{R(A)}_{\epsilon_+,\epsilon_-}=&-\ell_{\mathrm{tr}}\left[\left\langle\check{g}^{R(A)}\right\rangle\nabla\left\langle\check{g}^{R(A)}\right\rangle\right]_{\epsilon_+,\epsilon_-}\\
&-i\ell_{\mathrm{tr}}\frac{e}{c}\left[\mathbf{A}\check{\tau}_3-\left\langle\check{g}^{R(A)}\right\rangle\check{\tau}_3\mathbf{A}\left\langle\check{g}^{R(A)}\right\rangle\right]_{\epsilon_+,\epsilon_-}
\end{aligned}
\tag{9.39}
$$

Sometimes, one uses the equations for anomalous functions, as well. We obtain in exactly the same way

$$
\begin{aligned}
&\frac{1}{3}v_F\left(-i\nabla\check{\mathbf{g}}^{(a)}_{\epsilon_+,\epsilon_-}-\frac{e}{c}\left(\check{\tau}_3\left[\mathbf{A}\check{\mathbf{g}}^{(a)}\right]_{\epsilon_+,\epsilon_-}-\left[\check{\mathbf{g}}^{(a)}\mathbf{A}\right]_{\epsilon_+,\epsilon_-}\check{\tau}_3\right)\right)\\
&-\epsilon_+\check{\tau}_3\left\langle\check{g}^{(a)}_{\epsilon_+,\epsilon_-}\right\rangle+\left\langle\check{g}^{(a)}_{\epsilon_+,\epsilon_-}\right\rangle\epsilon_-\check{\tau}_3+\left[\check{H}_\Delta\left\langle\check{g}^{(a)}\right\rangle\right]_{\epsilon_+,\epsilon_-}-\left[\left\langle\check{g}^{(a)}\right\rangle\check{H}_\Delta\right]_{\epsilon_+,\epsilon_-}\\
=&\left[\check{h}_\Delta\left\langle\check{g}^{A}\right\rangle\right]_{\epsilon_+,\epsilon_-}-\left[\left\langle\check{g}^{R}\right\rangle\check{h}_\Delta\right]_{\epsilon_+,\epsilon_-}+\frac{1}{3}\left(\left[\check{\mathbf{h}}_A\check{\mathbf{g}}^{A}\right]_{\epsilon_+,\epsilon_-}-\left[\check{\mathbf{g}}^{R}\check{\mathbf{h}}_A\right]_{\epsilon_+,\epsilon_-}\right)\\
&+\left\langle\check{I}^{(ph)(a)}_{\epsilon_+,\epsilon_-}\right\rangle+\left\langle\check{I}^{(s)(a)}_{\epsilon_+,\epsilon_-}\right\rangle
\end{aligned}
\tag{9.40}
$$

Here

$$
\check{h}_\Delta=-\left[\check{H}_\Delta\right]_{\epsilon-\epsilon'}\left[f^{(0)}(\epsilon)-f^{(0)}(\epsilon')\right]
$$

$$
\check{\mathbf{h}}_A=\check{\tau}_3\frac{e}{c}v_F\mathbf{A}_{\epsilon-\epsilon'}\left[f^{(0)}(\epsilon)-f^{(0)}(\epsilon')\right]
$$

The vector part of the equation is

$$
\begin{aligned}
&v_F\left(-i\nabla\left\langle\check{g}^{(a)}_{\epsilon_+,\epsilon_-}\right\rangle-\frac{e}{c}\left(\check{\tau}_3\left[\mathbf{A}\left\langle\check{g}^{(a)}\right\rangle\right]_{\epsilon_+,\epsilon_-}-\left[\left\langle\check{g}^{(a)}\right\rangle\mathbf{A}\right]_{\epsilon_+,\epsilon_-}\check{\tau}_3\right)\right)\\
=&\left[\check{\mathbf{h}}_A\left\langle\check{g}^{A}\right\rangle\right]_{\epsilon_+,\epsilon_-}-\left[\left\langle\check{g}^{R}\right\rangle\check{\mathbf{h}}_A\right]_{\epsilon_+,\epsilon_-}\\
&+\frac{i}{2\tau_{\mathrm{tr}}}\left[\left\langle\check{g}^{R}\right\rangle\check{\mathbf{g}}^{(a)}-\check{\mathbf{g}}^{(a)}\left\langle\check{g}^{A}\right\rangle-\check{\mathbf{g}}^{R}\left\langle\check{g}^{(a)}\right\rangle+\left\langle\check{g}^{(a)}\right\rangle\check{\mathbf{g}}^{A}\right]_{\epsilon_+,\epsilon_-}
\end{aligned}
\tag{9.41}
$$

18.2.4 9.4 Stimulated superconductivity

We consider here an example that demonstrates how the quasiparticle distribution affects the superconducting order parameter. This example is closely related with experiment and is instructive in that it treats a nonlinear response of a superconductor to an applied time-dependent perturbation. Consider a superconducting film placed in a microwave electromagnetic field. A part of electromagnetic energy is absorbed in the film. The intuitive picture is that, the superconducting being heated, its temperature grows and the order parameter decreases. However, this is not always the case. The real behavior is more tricky. The temperature, indeed, can increase. However, the overall rise in temperature depends on the cooling rate provided by the thermal contact of the film with the heat bath. If the heat contact is good, the temperature does not increase considerably. What happens is as follows. The microwave irradiation with a frequency of the order of the energy gap produces redistribution of energy between excitations and shifts the quasiparticle distribution to higher energies. The states with $\epsilon \sim \Delta$ become depleted. This causes an increase in the energy gap as compared to its value without irradiation. This effect is called stimulated superconductivity. It was first theoretically predicted by Eliashberg (1970) and then observed in many experiments. The detailed discussion of stimulated superconductivity can be found in the review by Eliashberg and Ivlev (1986).

We consider the simplest case of temperatures close to T_c, and restrict ourselves to dirty superconductors with the order parameter homogeneous in space. We first find the (averaged over time) distribution function taking into account absorption of energy from an external electromagnetic field with a frequency $\omega \sim \Delta$. For such frequencies, the order parameter performs only small oscillations near its average value (see Section 11.2), therefore we can take $\Delta = \text{const}$.

To find the average in time distribution function we use the kinetic equation for the anomalous function taken at zero frequency. Within the linear approximation in A^2 we have from eqn (9.40)

$$\begin{aligned}\operatorname{Tr}\left\langle \check{I}^{(ph)(a)}_{\epsilon,\epsilon}\right\rangle =& \frac{ev_F}{3c}\operatorname{Tr}\left(\check{\tau}_3\left[\mathbf{A}\check{\mathbf{g}}^{(a)}\right]_{\epsilon,\epsilon} - \left[\check{\mathbf{g}}^{(a)}\mathbf{A}\right]_{\epsilon,\epsilon}\check{\tau}_3\right)\\ &- \frac{1}{3}\operatorname{Tr}\left(\left[\check{\mathbf{h}}_A\check{\mathbf{g}}^A\right]_{\epsilon,\epsilon} - \left[\check{\mathbf{g}}^R\check{\mathbf{h}}_A\right]_{\epsilon,\epsilon}\right)\end{aligned} \tag{9.42}$$

We assume that magnetic impurities are absent. The vector parts of both the regular functions $\check{\mathbf{g}}^R$ and $\check{\mathbf{g}}^A$ and of the anomalous function $\check{\mathbf{g}}^{(a)}$ are proportional to the vector potential $\mathbf{A}$ and have a Fourier components at the frequency of the external field. Therefore, we obtain from eqn (9.41)

$$\begin{aligned}\operatorname{Tr}\left\langle \check{I}^{(ph)(a)}_{\epsilon,\epsilon}\right\rangle =& \frac{ev_F}{3c}\operatorname{Tr}\left(\check{\tau}_3\mathbf{A}_{-\omega}\check{\mathbf{g}}^{(a)}_{\epsilon+\omega,\epsilon} - \check{\mathbf{g}}^{(a)}_{\epsilon,\epsilon-\omega}\mathbf{A}_{-\omega}\check{\tau}_3\right)\\ &- \frac{1}{3}\operatorname{Tr}\left(\check{\mathbf{h}}_{A\epsilon,\epsilon-\omega}\check{\mathbf{g}}^A_{\epsilon-\omega,\epsilon} - \check{\mathbf{g}}^R_{\epsilon,\epsilon+\omega}\check{\mathbf{h}}_{A\epsilon+\omega,\epsilon}\right),\end{aligned}$$

where

$$\begin{aligned}\check{\mathbf{h}}_{A\epsilon,\epsilon-\omega} &= \check{\tau}_3\frac{e}{c}v_F\mathbf{A}_\omega\left(f^{(0)}(\epsilon) - f^{(0)}(\epsilon-\omega)\right)\\ \check{\mathbf{h}}_{A\epsilon+\omega,\epsilon} &= \check{\tau}_3\frac{e}{c}v_F\mathbf{A}_\omega\left(f^{(0)}(\epsilon+\omega) - f^{(0)}(\epsilon)\right)\end{aligned}$$

The vector components of regular Green functions are found from eqn (9.39). Since the averaged Green functions do not depend on time

$$\left\langle \check{g}^{R(A)}_{\epsilon,\epsilon'}\right\rangle = 2\pi\delta\left(\epsilon-\epsilon'\right)\left\langle \check{g}^{R(A)}_{\epsilon}\right\rangle$$

we find

$$\check{\mathbf{g}}^{R(A)}_{\epsilon_+,\epsilon_-} = i\tau_{\rm tr} v_F \frac{e}{c}\mathbf{A}_\omega \left[\left\langle \check{g}^{R(A)}_{\epsilon_+}\right\rangle \check{\tau}_3 \left\langle \check{g}^{R(A)}_{\epsilon_-}\right\rangle - \check{\tau}_3\right]$$

Making the corresponding shifts in the frequencies, we have

$$\check{\mathbf{g}}^{A}_{\epsilon-\omega,\epsilon} = i\ell_{\rm tr} \frac{e}{c}\mathbf{A}_{-\omega} \left[\left\langle \check{g}^{A}_{\epsilon-\omega}\right\rangle \check{\tau}_3 \left\langle \check{g}^{A}_{\epsilon}\right\rangle - \check{\tau}_3\right]$$
$$\check{\mathbf{g}}^{R}_{\epsilon,\epsilon+\omega} = i\ell_{\rm tr} \frac{e}{c}\mathbf{A}_{-\omega} \left[\left\langle \check{g}^{R}_{\epsilon}\right\rangle \check{\tau}_3 \left\langle \check{g}^{R}_{\epsilon+\omega}\right\rangle - \check{\tau}_3\right]$$

The vector part of the anomalous function is to be found from eqn (9.41) which gives

$$\left\langle \check{g}^{R}_{\epsilon_+}\right\rangle \check{\mathbf{h}}_{A\epsilon_+,\epsilon_-} - \check{\mathbf{h}}_{A\epsilon_+,\epsilon_-} \left\langle \check{g}^{A}_{\epsilon_-}\right\rangle = \frac{i}{2\tau_{\rm tr}} \left[\left\langle \check{g}^{R}_{\epsilon_+}\right\rangle \check{\mathbf{g}}^{(a)}_{\epsilon_+,\epsilon_-} - \check{\mathbf{g}}^{(a)}_{\epsilon_+,\epsilon_-} \left\langle \check{g}^{A}_{\epsilon_-}\right\rangle\right]$$

within a linear approximation in $\mathbf{A}^2$. Consider the relation

$$\left\langle \check{g}^{R}_{\epsilon_+}\right\rangle \check{\mathbf{g}}^{(a)}_{\epsilon_+,\epsilon_-} + \check{\mathbf{g}}^{(a)}_{\epsilon_+,\epsilon_-} \left\langle \check{g}^{A}_{\epsilon_-}\right\rangle + \left\langle \check{g}^{(a)}_{\epsilon_+}\right\rangle \check{\mathbf{g}}^{A}_{\epsilon_+,\epsilon_-} + \check{\mathbf{g}}^{R}_{\epsilon_+,\epsilon_-} \left\langle \check{g}^{(a)}_{\epsilon_-}\right\rangle = 0.$$

We can neglect the last two terms with the vector components of regular functions which are small in $\mathbf{A}^2$. We find from eqn (9.4)

$$\frac{i}{\tau_{\rm tr}} \check{\mathbf{g}}^{(a)}_{\epsilon_+,\epsilon_-} = \check{\mathbf{h}}_{A\epsilon_+,\epsilon_-} - \left\langle \check{g}^{R}_{\epsilon_+}\right\rangle \check{\mathbf{h}}_{A\epsilon_+,\epsilon_-} \left\langle \check{g}^{A}_{\epsilon_-}\right\rangle.$$

Making the corresponding shifts in frequencies, we have

$$\check{\mathbf{g}}^{(a)}_{\epsilon+\omega,\epsilon} = -i\tau_{\rm tr} \left(\check{\mathbf{h}}_{A\epsilon+\omega,\epsilon} - \left\langle \check{g}^{R}_{\epsilon+\omega}\right\rangle \check{\mathbf{h}}_{A\epsilon+\omega,\epsilon} \left\langle \check{g}^{A}_{\epsilon}\right\rangle\right)$$

$$\check{\mathbf{g}}^{(a)}_{\epsilon,\epsilon-\omega} = -i\tau_{\rm tr} \left(\check{\mathbf{h}}_{A\epsilon,\epsilon-\omega} - \left\langle \check{g}^{R}_{\epsilon}\right\rangle \check{\mathbf{h}}_{A\epsilon,\epsilon-\omega} \left\langle \check{g}^{A}_{\epsilon-\omega}\right\rangle\right)$$

Finally, inserting $\check{\mathbf{g}}^{(a)}$ together with $\check{\mathbf{g}}^{R(A)}$ into eqn (9.42) we obtain the equation for the distribution function

$$\begin{aligned}
&{\rm Tr}\left\langle \check{I}^{(ph)(a)}_{\epsilon,\epsilon}\right\rangle = iD\frac{e^2}{c^2}\mathbf{A}_\omega\mathbf{A}_{-\omega} \\
&\times \left[\left(f^{(0)}(\epsilon) - f^{(0)}(\epsilon-\omega)\right) {\rm Tr}\left[\check{\tau}_3 \left\langle \check{g}^{A}_{\epsilon-\omega}\right\rangle \check{\tau}_3 \left(\left\langle \check{g}^{R}_{\epsilon}\right\rangle - \left\langle \check{g}^{A}_{\epsilon}\right\rangle\right)\right]\right. \\
&\left. + \left(f^{(0)}(\epsilon+\omega) - f^{(0)}(\epsilon)\right) {\rm Tr}\left[\check{\tau}_3 \left\langle \check{g}^{R}_{\epsilon+\omega}\right\rangle \check{\tau}_3 \left(\left\langle \check{g}^{R}_{\epsilon}\right\rangle - \left\langle \check{g}^{A}_{\epsilon}\right\rangle\right)\right]\right]
\end{aligned}$$

For temperatures close to T_c, the phonon collision integral at zero frequency can be taken in a τ-approximation (see Section 10.4 for more detail)

$${\rm Tr}\left\langle \check{I}^{(ph)(a)}_{\epsilon,\epsilon}\right\rangle = \frac{2i}{\tau_{\rm ph}} \left(g^R_\epsilon - g^A_\epsilon\right) f_1$$

where $f_1 = f^{(1)} - f^{(0)}$ is the correction to the distribution function $f^{(1)}$ defined in eqn (9.28). As a result,

$$\begin{aligned}
&\left(g^R_\epsilon - g^A_\epsilon\right) f_1 = \tau_{\rm ph} D\frac{e^2}{c^2}\mathbf{A}_\omega\mathbf{A}_{-\omega} \\
&\times \left[\left(f^{(0)}(\epsilon) - f^{(0)}(\epsilon-\omega)\right) \left[g^A_{\epsilon-\omega}\left(g^R_\epsilon - g^A_\epsilon\right) + f^A_{\epsilon-\omega}\left(f^R_\epsilon - f^A_\epsilon\right)\right]\right. \\
&\left. + \left(f^{(0)}(\epsilon+\omega) - f^{(0)}(\epsilon)\right) \left[\left(g^R_\epsilon - g^A_\epsilon\right) g^R_{\epsilon+\omega} + \left(f^R_\epsilon - f^A_\epsilon\right) f^R_{\epsilon+\omega}\right]\right]
\end{aligned}$$

We omit the averaging brackets for brevity.
The regular functions are given by eqns (5.46), (5.47), and (5.50), (5.51). Therefore, we find for $\epsilon > 0$

$$
\begin{aligned}
f_1 = &- \alpha \frac{df^{(0)}}{d\epsilon} \left[\frac{\epsilon(\epsilon-\omega)+\Delta^2}{\epsilon\sqrt{(\epsilon-\omega)^2-\Delta^2}} \Theta(\epsilon-\Delta-\omega) \right. \\
&- \frac{\epsilon(\epsilon+\omega)+\Delta^2}{\epsilon\sqrt{(\epsilon+\omega)^2-\Delta^2}} \Theta(\epsilon-\Delta) \\
&\left. + \frac{\epsilon(\omega-\epsilon)-\Delta^2}{\epsilon\sqrt{(\omega-\epsilon)^2-\Delta^2}} \Theta(\epsilon-\Delta)\Theta(\omega-\Delta-\epsilon) \right]
\end{aligned} \tag{9.43}
$$

The term

$$
\alpha = \tau_{\text{ph}} D \frac{e^2}{c^2} \omega \mathbf{A}_\omega \mathbf{A}_{-\omega}
$$

in eqn (9.43) is proportional to the absorbed power. Indeed, since $2\mathbf{A}_\omega \mathbf{A}_{-\omega} = \overline{\mathbf{A}^2(t)}$ the dissipated energy per volume is

$$
\sigma_n \overline{E^2} = 2\nu(0) D e^2 \omega^2 \overline{\mathbf{A}^2(t)}/c^2
$$

multiplied by the inelastic relaxation time τ_{ph}. This energy is distributed among $\delta N \sim \nu(0)\Delta$ particles. Therefore, the quantity $(\omega/\Delta)\alpha$ is the absorbed energy per particle. The last term in eqn (9.43) only exists if $\omega > 2\Delta$. It corresponds to creation of new excitations over the energy gap Δ.

The nonequilibrium correction to the distribution function is negative. It means that the number of excitations with energies $\epsilon \sim \Delta$ decreases. It is equivalent to a decrease in an "effective temperature" of excitations. Let us study how this affects the order parameter. We define the function

$$
W(\Delta) = \frac{1}{2} \int_\Delta^\infty \frac{d\epsilon}{\sqrt{\epsilon^2-\Delta^2}} f_1
$$

We have for $\Delta \ll T$

$$
W(\Delta) = \frac{\alpha}{4T} P\left(\frac{\omega}{\Delta}\right)
$$

where the function $P(\omega/\Delta)$ is

$$
\begin{aligned}
P(w) = &w \int_1^\infty \frac{x(x+w)+1}{x(x+w)\sqrt{x^2-1}\sqrt{(x+w)^2-1}} dx \\
&- \Theta(w-2) \int_1^{w-1} \frac{(w-x)x-1}{x\sqrt{x^2-1}\sqrt{(w-x)^2-1}} dx
\end{aligned}
$$

with $w = \omega/\Delta$. The last term is nonzero for $\omega > \Delta/2$. As we mentioned, it corresponds to creation of quasiparticle excitations by the external irradiation. The function

$$
P(w) = \begin{cases} w \ln(8/ew), & w \ll 1 \\ \pi/w, & w \gg 1 \end{cases}
$$

It has a maximum at $w = 2$ where its first derivative is discontinuous.
Let us consider the equation for the order parameter. We obtain the Keldysh function from eqns (9.25) and (9.28) in the form

$$
\left\langle f^K_{\epsilon,\epsilon} \right\rangle = \left(\left\langle f^R_\epsilon \right\rangle - \left\langle f^A_\epsilon \right\rangle \right) \left(f^{(0)} + f_1 \right)
$$

Let us insert this into the order-parameter equation (9.17)

$$\frac{\Delta(\mathbf{k},\omega)}{\lambda} = \int \frac{d\epsilon}{4} \left(\langle f_\epsilon^R \rangle - \langle f_\epsilon^A \rangle \right) \left(f^{(0)} + f_1 \right)$$

The equilibrium part of the distribution function gives the usual stationary Ginzburg-Landau equation (6.23). The resulting equation has the form

$$-\frac{\pi D}{8T_c} \frac{4e^2 \mathbf{A}_\omega \mathbf{A}_{-\omega}}{c^2} + \frac{T_c - T}{T_c} + W'(\Delta) = 0 \tag{9.44}$$

where

$$W'(\Delta) = W(\Delta) - \frac{7\zeta(3)}{8\pi^2 T_c^2} |\Delta|^2$$

One can generally neglect the first term in eqn (9.44) when $\omega\tau_{\text{ph}} \gg 1$.

Fig. 9.1. The order parameter as a function of temperature. The long-dashed line $\Delta_{0}(T)$ shows the order parameter in absence of irradiation. The increasing part of $W'(\Delta)$ shown by short-dashed line is unstable. The arrows indicate the order parameter jumps with varying the temperature.

One can see that a nonzero order parameter exists even if $T > T_c$ because $W' > 0$ for not very high Δ. For the irradiation intensity such that $\omega/T \ll \alpha \ll 1$ the temperature dependence of the order parameter is determined mainly by the first term in W'. The maximum temperature $T_{\max}$ in this limit is determined by

$$\frac{T_{\max} - T_{c0}}{T_{c0}} = \frac{\alpha\omega}{2T} P(2) = \frac{\alpha\omega}{2T} \frac{\pi}{\sqrt{3}}$$

At this temperature, $\Delta = \omega/2$. If the temperature is increased further the order parameter jumps to zero. The increasing branch of $W(\Delta)$ is unstable. The order parameter approaches the equilibrium value with decreasing temperature. The temperature dependence $\Delta(T)$ is shown schematically in Fig. 9.1.

18.3 KINETIC EQUATIONS

We derive the kinetic equations for the two-component distribution function in a gauge invariant form. We write down the collision integrals for interaction of excitations with impurities, phonons, and with each other.

18.3.1 10.1 Gauge invariant Green functions

In this Section we introduce a more convenient description of a time-dependent state of a superconductor in terms of the gauge invariant Green functions. We know that the gauge invariance is the basic property of the superconducting state. The Eilenberger equations are gauge invariant and so are the regular (retarded and advanced) quasiclassical Green functions for a time-independent state of a superconductor which are determined by these equations. Of course, the gauge invariance holds for a nonstationary state, as well. However, the generalized distribution function introduced in Chapter 9 and the regular functions for a nonstationary state, treated separately, can become not gauge invariant if special precautions are not taken.

Remind that according to the general gauge invariance (see also page 31), the electromagnetic potentials can be changed according to

$$\mathbf{A} \to \mathbf{A} + \frac{\hbar c}{2e}\nabla f; \varphi \to \varphi - \frac{\hbar}{2e}\frac{\partial f}{\partial t} \tag{10.1}$$

where f is an arbitrary function, without changing the electromagnetic fields

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi; \mathbf{H} = \operatorname{curl}\mathbf{A}. \tag{10.2}$$

In superconductors, the transformation eqn (10.1) is accompanied by the transformation of the order-parameter phase

$$\chi \to \chi + f. \tag{10.3}$$

Therefore the combinations

$$\mathbf{Q} = \mathbf{A} - \frac{c}{2e}\nabla\chi, \Phi = \varphi + \frac{1}{2e}\frac{\partial\chi}{\partial t} \tag{10.4}$$

are also gauge invariant; they are called the gauge invariant potentials. Of course, the order-parameter magnitude is gauge invariant, too. Therefore, the equations for the Green functions and for the distribution function should contain,
in addition to the electric field $\mathbf{E}$ and magnetic field $\mathbf{H}$, the vector and scalar potentials in proper combinations with the spatial and time derivatives of the order-parameter phase such that they appear as the gauge invariant potentials eqn (10.4). Note that the time dependent Ginzburg-Landau model discussed earlier in Section 1.2.1 is an example of such gauge invariant theory because it only contain the potentials (10.4) and the order-parameter magnitude.

In this Section we define the Green functions and the generalized distribution functions in a gauge invariant way. We start with the transport-like equations obtained in Section 8.4 and derive the kinetic equations for the distribution function in a state which slowly varies in time. More particularly, we assume that the time variations of the superconducting parameters are such that the characteristic frequency is $\omega \ll \Delta$. These equations will make the basis for further discussions of nonequilibrium properties of superconductors.

Let us return back to the definitions of non-quasiclassical Green functions in Chapter 8. We remind the operator eqn (8.3) acting on the matrix Green function in the Nambu space. It can be written as

$$\check{G}^{-1} = \check{G}_0^{-1} + \check{H}. \tag{10.5}$$

where

$$\check{G}_0^{-1}(\mathbf{p}) = \begin{pmatrix} -i\partial/\partial t + E_n(\mathbf{p}) - E_F & 0 \\ 0 & i\partial/\partial t + E_n(\mathbf{p}) - E_F \end{pmatrix}, \tag{10.6}$$

and $\mathbf{p} = -i\nabla$ is the particle momentum operator. The Hamiltonian is

$$\check{H} = \begin{pmatrix} H(\mathbf{p}) & -\Delta_{\mathbf{p}} \\ \Delta_{\mathbf{p}}^* & H(-\mathbf{p}) \end{pmatrix} \tag{10.7}$$

where

$$H(\mathbf{p}) = -\frac{e}{2c}(\mathbf{vA} + \mathbf{Av}) + \frac{e^2 A^2}{2m^* c^2} + e\varphi \tag{10.8}$$

see eqn (5.57). The velocity is $\mathbf{v} = \partial\epsilon(\mathbf{p})/\partial\mathbf{p}$, and m^* is the effective mass of an electron. As distinct from eqn (3.70), the effective Hamiltonian contains the scalar potential φ which should be separated from a spatially independent part of the chemical potential, E_F.

The equation for the regular Green function can be written as

$$\left[\left(\check{G}^{-1}(\mathbf{p}_1) - \check{\Sigma}^{R(A)}\right) \otimes \check{G}^{R(A)}\right](\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \delta(\mathbf{r}_1 - \mathbf{r}_2)\,\delta(t_1 - t_2) \tag{10.9}$$

The full convolution in the coordinate space is

$$[\check{A} \otimes \check{B}](\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) = \int \check{A}(\mathbf{r}_1, t_1; \mathbf{r}', t')\,\check{B}(\mathbf{r}', t'; \mathbf{r}_2, t_2)\,d^3r'dt' \tag{10.10}$$

It is equivalent to the shortcut $[AB]$ used in the momentum-frequency representation. The equation of motion for the total Green function is obtained from eqn (8.61):

$$\left[\check{G}^{-1}(\mathbf{p}_1) \otimes \check{G}^K - \check{G}^K \otimes \check{G}^{-1}(\mathbf{p}_2)\right] = \check{\mathcal{I}}. \tag{10.11}$$

The collision-integral matrix is

$$\check{\mathcal{I}} = \begin{pmatrix} \mathcal{I}_1 & \mathcal{I}_2 \\ -\mathcal{I}_2^\dagger & \overline{\mathcal{I}}_1 \end{pmatrix} = \check{\Sigma}^R \otimes \check{G}^K - \check{G}^K \otimes \check{\Sigma}^A + \check{\Sigma}^K \otimes \check{G}^A - \check{G}^R \otimes \check{\Sigma}^K. \tag{10.12}$$

The regular Green functions satisfy eqn (10.11) with the collision integral

$$\check{\mathcal{I}}^{R(A)} = \check{\Sigma}^{R(A)} \otimes \check{G}^{R(A)} - \check{G}^{R(A)} \otimes \check{\Sigma}^{R(A)} \tag{10.13}$$

As before, we use the mixed coordinate-momentum representation

$$\check{G}_\epsilon(\mathbf{p}; \mathbf{r}, t) = \int \check{G}_\epsilon(\mathbf{p}; \mathbf{k}, \omega) e^{-i\omega t + \mathbf{kr}} \frac{d^3k d\omega}{(2\pi)^4} \tag{10.14}$$

in the center-of-mass momentum and frequency for the Green function defined through

$$\check{G}_{\epsilon_+, \epsilon_-}(\mathbf{p}_+, \mathbf{p}_-) \equiv \check{G}_\epsilon(\mathbf{p}; \mathbf{k}, \omega) \tag{10.15}$$

where $\mathbf{p}_\pm = \mathbf{p} \pm \mathbf{k}/2$ and $\epsilon_\pm = \epsilon \pm \omega/2$. We are going to derive equations for the function $\check{G}_\epsilon(\mathbf{p}; \mathbf{r}, t)$.

10.1.1 Equations of motion for the invariant functions

We denote the center-of-mass coordinate and time $\mathbf{r} = (\mathbf{r}_1 + \mathbf{r}_2)/2$ and $t = (t_1 + t_2)/2$ and introduce the modified Green functions

$$\begin{aligned} G(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) &= \tilde{G}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) \exp\left[i\psi(\mathbf{r}_1, t_1) - i\psi(\mathbf{r}_2, t_2)\right] \\ F(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) &= \tilde{F}(\mathbf{r}_1, t_1; \mathbf{r}_2, t_2) \exp\left[i\psi(\mathbf{r}_1, t_1) + i\psi(\mathbf{r}_2, t_2)\right] \end{aligned} \tag{10.16}$$

and same for $\bar{G}$ and $F^\dagger$ with $\psi \to -\psi$ in the corresponding exponents. Here

$$\psi\left(\mathbf{r}_1, t_1\right) = \frac{e}{c}\int_{\mathbf{r}_0}^{\mathbf{r}_1} \mathbf{A}\left(\mathbf{r}', t_1\right) d\mathbf{r}' - e\int_{t_0}^{t_1} \varphi\left(\mathbf{r}_1, t'\right) dt' \tag{10.17}$$

Equations (10.16) refer both to regular and Keldysh Green functions. The point ($\mathbf{r}_0, t_0$) is taken close, i.e., on a scale of p_F^{-1}, to the points ($\mathbf{r}_1, t_1$) and ($\mathbf{r}_2, t_2$). Integration in the first term is carried out along the shortest path connecting the points $\mathbf{r}_0$ and $\mathbf{r}_1$. We shall write the equation in the differential form at the point ($\mathbf{r}, t$) therefore, in the final equations, we can put ($\mathbf{r}_0, t_0$) $\to (\mathbf{r}, t)$.

Since the relative distances are always small, $|\mathbf{r}_1 - \mathbf{r}_2| \sim p_F^{-1} \ll \xi$, we can expand all the quantities in $|\mathbf{r}_1 - \mathbf{r}_2|/\xi$. For further derivation, we adopt one more assumption, namely, that the electric field $\mathbf{E} = -\nabla\varphi - (1/c)\partial\mathbf{A}/\partial t$ is small: $eE \ll \Delta/\xi$ or, equivalently, that the time variations are slow: $\omega \ll \Delta$. This means that we restrict ourselves to a linear response to the electric field. In return, we are able to expand in the relative time, $(t_1 - t_2)\,\omega \sim \omega/\Delta$, as well.

The equation of motion for the modified regular Green function can be obtained (Kopnin 1994) after substitution of eqn (10.16) into eqn (10.9) and expansion of the operators $\check{G}^{-1}$ and Σ in powers of $|\mathbf{r}_1 - \mathbf{r}_2|/\xi$ and $(t_1 - t_2)\,\omega$
which is performed in a standard way (compare with Serene and Rainer 1983, Aronov et al. 1981, Eckern and Schmid 1981). Simultaneously, we expand in φ and $\mathbf{E}$.

On the next step we go into the mixed momentum-coordinate representation performing the Fourier transformation with respect to the relative coordinates $\mathbf{r}_1 - \mathbf{r}_2$ and $t_1 - t_2$ to the momentum $\mathbf{p}$ and frequency ϵ, respectively. Equations for thus obtained modified functions $\tilde{\check{G}}_\epsilon(\mathbf{p}; \mathbf{r}, t)$ become

$$\tilde{\check{G}}_R^{-1}\tilde{\check{G}}^{R(A)} - \tilde{\Sigma}^{R(A)} \otimes \tilde{\check{G}}^{R(A)} = \check{1}, \tag{10.18}$$

or

$$\tilde{\check{G}}^{R(A)}\tilde{\check{G}}_L^{-1} - \tilde{\check{G}}^{R(A)} \otimes \tilde{\Sigma}^{R(A)} = \check{1} \tag{10.19}$$

Here we defined the gauge invariant operators

$$\begin{aligned}
\tilde{\check{G}}_R^{-1} &= \tilde{\check{G}}_0^{-1} - \frac{i}{2}\check{\tau}_3\frac{\partial}{\partial t} - \frac{i}{2}\mathbf{v}\cdot\nabla + \tilde{\check{H}}_d \\
\tilde{\check{G}}_L^{-1} &= \tilde{\check{G}}_0^{-1} + \frac{i}{2}\check{\tau}_3\frac{\partial}{\partial t} + \frac{i}{2}\mathbf{v}\cdot\nabla - \tilde{\check{H}}_d
\end{aligned} \tag{10.20}$$

where

$$\tilde{G}_0^{-1} = \begin{pmatrix} -\epsilon + \xi_{\mathbf{p}} & -\Delta_{\mathbf{p}} \\ \Delta_{\mathbf{p}}^* & \epsilon + \xi_{\mathbf{p}} \end{pmatrix}$$

The differential operator $\tilde{\check{H}}_d = \tilde{\check{H}}_\epsilon + \tilde{\check{H}}_p$ contains

$$\tilde{\check{H}}_\epsilon = \begin{pmatrix} e\mathbf{v}_F\mathbf{E} & -\hat{\partial}\Delta/\partial t \\ \hat{\partial}\Delta^*/\partial t & -e\mathbf{v}_F\mathbf{E} \end{pmatrix}\frac{\partial}{\partial\epsilon}, \tag{10.21}$$

and

$$\begin{aligned}
\tilde{\check{H}}_p = &\begin{pmatrix} e\mathbf{E} + (e/c)\left[\mathbf{v}_F \times \mathbf{H}\right] & -\hat{\nabla}\Delta \\ \hat{\nabla}\Delta^* & e\mathbf{E} - (e/c)\left[\mathbf{v}_F \times \mathbf{H}\right] \end{pmatrix}\frac{\partial}{\partial\mathbf{p}} \\
&+ \begin{pmatrix} 0 & \partial\Delta/\partial\mathbf{p} \\ -\partial\Delta/\partial\mathbf{p} & 0 \end{pmatrix}\nabla.
\end{aligned} \tag{10.22}$$

The gauge-invariant derivatives $\hat{\partial}/\partial t$ and $\hat{\nabla}$ are

$$\begin{aligned} \tfrac{\hat{\partial}\Delta}{\partial t} &= \left(\tfrac{\partial}{\partial t} + 2ie\varphi\right)\Delta, \quad & \hat{\nabla}\Delta &= \left(\nabla - \tfrac{2ie}{c}\mathbf{A}\right)\Delta \\ \tfrac{\hat{\partial}\Delta^*}{\partial t} &= \left(\tfrac{\partial}{\partial t} - 2ie\varphi\right)\Delta^*, \quad & \hat{\nabla}\Delta^* &= \left(\nabla + \tfrac{2ie}{c}\mathbf{A}\right)\Delta^* \end{aligned} \tag{10.24}$$

The operator $\partial/\partial t$ in $\tilde{\tilde{G}}^{-1}$ acts on the center-of-mass time t.

The convolution for the modified Green functions in our approximation is replaced with

$$\begin{aligned} \tilde{\tilde{G}}_1 \otimes \tilde{\tilde{G}}_2 =& \tilde{\tilde{G}}_1\tilde{\tilde{G}}_2 + \frac{i}{2}\left(\frac{\partial\tilde{\tilde{G}}_1}{\partial\epsilon}\frac{\partial\tilde{\tilde{G}}_2}{\partial t} - \frac{\partial\tilde{\tilde{G}}_1}{\partial t}\frac{\partial\tilde{\tilde{G}}_2}{\partial\epsilon}\right) \\ & - \frac{i}{2}\left(\frac{\partial\tilde{\tilde{G}}_1}{\partial\mathbf{p}}\nabla\tilde{\tilde{G}}_2 - \nabla\tilde{\tilde{G}}_1\frac{\partial\tilde{\tilde{G}}_2}{\partial\mathbf{p}}\right) \end{aligned} \tag{10.25}$$

The differential operator $\widetilde{\tilde{H}}_d$ in eqn (10.21), (10.22) and the derivatives $\partial/\partial\epsilon$ and $\partial/\partial\mathbf{p}$ in eqn (10.25) appear as a result of expansion of the convolution near the point $(\mathbf{r}_1 + \mathbf{r}_2)/2$ and $(t_1 + t_2)/2$ in small $\mathbf{r}_1 - \mathbf{r}_2$ and $t_1 - t_2$. To demonstrate this, let us consider the time-dependent part in the expansion of eqn (10.25). On can check that, in the frequency representation,

$$\begin{aligned} \check{G}_{\epsilon_+,\epsilon'}\check{G}_{\epsilon',\epsilon-} \equiv& \check{G}_{\epsilon+(\omega-\omega')/2}(\omega')\,\check{G}_{\epsilon-\omega'/2}(\omega-\omega') \\ =& \check{G}_\epsilon(\omega')\,\check{G}_\epsilon(\omega-\omega') - \frac{\omega'}{2}\check{G}_\epsilon(\omega')\frac{\partial\check{G}_\epsilon(\omega-\omega')}{\partial\epsilon} \\ & + \frac{\partial\check{G}_\epsilon(\omega')}{\partial\epsilon}\frac{\omega-\omega'}{2}\check{G}_\epsilon(\omega-\omega') \end{aligned} \tag{10.26}$$

where we put $\epsilon' = \epsilon + \omega/2 - \omega'$ and use eqn (10.15). Performing the Fourier transformation over the center-of-mass frequencies ω' and $\omega - \omega'$ we obtain the first line of eqn (10.25). The momentum part can be expanded similarly.

A transport-like equation for the modified regular Green function is obtained after subtracting eqn (10.19) from eqn (10.18):

$$\tilde{\tilde{G}}_R^{-1}\tilde{\tilde{G}}^{R(A)} - \tilde{\tilde{G}}^{R(A)}\tilde{\tilde{G}}_L^{-1} = \tilde{\tilde{\mathcal{I}}}^{R(A)} \tag{10.27}$$

Equation (10.11) for the modified Keldysh function can be transformed in the same way. We have

$$\tilde{\tilde{G}}_R^{-1}\tilde{\tilde{G}}^K - \tilde{\tilde{G}}^K\check{G}_L^{-1} = \check{\tilde{\mathcal{I}}}^K \tag{10.28}$$

The gauge invariant equation (10.28) is the basis for our further derivation. As an example, we write down two elements of the matrix eqn (10.28):

$$\begin{aligned} & \frac{\partial}{\partial t}\tilde{G}^K + \mathbf{v}\cdot\nabla\tilde{G}^K + e\mathbf{v}\cdot\mathbf{E}\frac{\partial\tilde{G}^K}{\partial\epsilon} + e\mathbf{E}\cdot\frac{\partial\tilde{G}^K}{\partial\mathbf{p}} + \frac{e}{c}[\mathbf{v}\times\mathbf{H}]\cdot\frac{\partial\tilde{G}^K}{\partial\mathbf{p}} \\ & + i\left(\Delta\tilde{F}^{\dagger K} - \tilde{F}^K\Delta^*\right) + \frac{1}{2}\left[\frac{\hat{\partial}\Delta}{\partial t}\frac{\partial\tilde{F}^{\dagger K}}{\partial\epsilon} + \frac{\partial\tilde{F}^K}{\partial\epsilon}\frac{\hat{\partial}\Delta^*}{\partial t}\right] \\ & - \frac{1}{2}\left[(\hat{\nabla}\Delta)\cdot\frac{\partial\tilde{F}^{\dagger K}}{\partial\mathbf{p}} + \frac{\partial\tilde{F}^K}{\partial\mathbf{p}}\cdot\left(\hat{\nabla}\Delta^*\right)\right] \\ & \qquad + \frac{1}{2}\left[\frac{\partial\Delta}{\partial\mathbf{p}}\cdot\nabla\tilde{F}^{\dagger K} + \frac{\partial\Delta^*}{\partial\mathbf{p}}\cdot\nabla\tilde{F}^K\right] = i\tilde{\mathcal{I}}_1 \end{aligned} \tag{10.29}$$

and

$$
\begin{aligned}
&-2i\epsilon\tilde{F}^K + \mathbf{v}\cdot\nabla\tilde{F}^K + e\mathbf{E}\cdot\frac{\partial\tilde{F}^K}{\partial\mathbf{p}} - i\Delta\left(\tilde{\bar{G}}^K - \tilde{G}^K\right)\\
&-\frac{1}{2}\frac{\hat{\partial}\Delta}{\partial t}\left(\frac{\partial\tilde{\bar{G}}^K}{\partial\epsilon} + \frac{\partial\tilde{G}^K}{\partial\epsilon}\right) + \frac{1}{2}\left(\frac{\partial\tilde{\bar{G}}^K}{\partial\mathbf{p}} + \frac{\partial\tilde{G}^K}{\partial\mathbf{p}}\right)\cdot\hat{\nabla}\Delta\\
&-\frac{1}{2}\left(\nabla\tilde{G}^K + \nabla\tilde{\bar{G}}^K\right)\cdot\frac{\partial\Delta}{\partial\mathbf{p}} = i\tilde{\mathcal{I}}_2
\end{aligned}
\tag{10.30}
$$

In eqns (10.29), (10.30) the arguments $\epsilon, \mathbf{p}, \mathbf{r}$, and t of the Green functions $\tilde{G}_\epsilon(\mathbf{p};\mathbf{r},t), \tilde{F}_\epsilon(\mathbf{p};\mathbf{r},t)$, etc., are omitted for brevity. Equations for $\bar{G}$ and $\tilde{F}^\dagger$ can be obtained by the transformation $\tilde{G}\to\bar{G}$, and $\tilde{F}\to\tilde{F}^\dagger$ under complex conjugation and the change of signs $\epsilon\to-\epsilon$, and $\mathbf{p}\to-\mathbf{p}$ in eqns (10.29) and (10.30). The same equations hold for the regular functions with the corresponding collision integrals in the right-hand side.

From eqns (10.27) and (10.28) it is clear that the functions $\tilde{\bar{G}}^K$ and $\tilde{\bar{G}}^{R(A)}$ are gauge invariant while the usual functions $\check{G}^K$ are "gauge covariant". Note that one can put locally $\tilde{F}_\epsilon(\mathbf{p};\mathbf{r},t) = F_\epsilon(\mathbf{p};\mathbf{r},t)$ and $\tilde{F}^\dagger_\epsilon(\mathbf{p};\mathbf{r},t) = F^\dagger_\epsilon(\mathbf{p};\mathbf{r},t)$ since we take the limit $\mathbf{r}_0\to\mathbf{r}, t_0\to t$ in eqn (10.16). However, the derivatives should be transformed according to eqn (10.16)

$$
\begin{aligned}
\nabla\tilde{F}_\epsilon(\mathbf{p};\mathbf{r},t) &= \left(\nabla - \frac{2ie}{c}\mathbf{A}\right)F_\epsilon(\mathbf{p};\mathbf{r},t)\\
\frac{\partial}{\partial t}\tilde{F}_\epsilon(\mathbf{p};\mathbf{r},t) &= \left(\frac{\partial}{\partial t} + 2ie\varphi\right)F_\epsilon(\mathbf{p};\mathbf{r},t)
\end{aligned}
\tag{10.31}
$$

and similarly for $\tilde{F}^\dagger_\epsilon$. At the same time, eqn (10.16) implies that $\tilde{G}$ is the function of the "particle" kinetic momentum $\tilde{\mathbf{p}} = \mathbf{p} - (e/c)\mathbf{A}$ and an energy $\epsilon - e\varphi$, while $\tilde{\bar{G}}$ is the function of the "hole" kinetic momentum $\tilde{\mathbf{p}} = \mathbf{p} + (e/c)\mathbf{A}$ and an energy $\epsilon + e\varphi$. We shall use the matrix gauge-invariant time derivative acting on a gauge covariant functions defined through

$$
\frac{\hat{\partial}\check{G}}{\partial t} = \begin{pmatrix} \frac{\partial}{\partial t}G & \left(\frac{\partial}{\partial t} + 2ie\varphi\right)F \\ -\left(\frac{\partial}{\partial t} - 2ie\varphi\right)F^\dagger & \frac{\partial}{\partial t}\bar{G} \end{pmatrix},
\tag{10.32}
$$

and similarly for $\hat{\nabla}\check{G}$. Therefore,

$$
\frac{\partial\tilde{G}}{\partial t} = \frac{\hat{\partial}\check{G}}{\partial t}; \quad \nabla\tilde{\bar{G}} = \hat{\nabla}\check{G}
$$

18.3.2 10.2 Quasiclassical kinetic equations

In the previous Chapter, we have neglected the terms with the momentum derivatives of the type of eqn (9.4) when deriving the Eliashberg equations from the transport-like equations. These terms appear now in eqns (10.29) and (10.30) which are of a higher accuracy in the quasiclassical parameter $1/p_F\xi$. Such terms have been also neglected during derivation of the Eilenberger equations in Chapter 5 . With this approximation, the Eliashberg and Eilenberger equations are local in the momentum $\mathbf{p}$ which reduces immensely the complexity of the full Green function formalism. However, as we have already mentioned on page 182, this approximation can be insufficient for the Keldysh Green function. We shall see later in this Section that, within this approximation, the distribution functions $f^{(1)}$ and $f^{(2)}$ enter the kinetic equations in such a way that the left-handside of one equation, for example, contains the time derivative

of $f^{(1)}$ and the coordinate derivative of $f^{(2)}$ while the function $f^{(1)}$ itself is contained only in the corresponding collision integral. On the other hand, some of the neglected contributions with the momentum derivatives do contain the function $f^{(1)}$ in the form of $\partial f^{(1)}/\partial \mathbf{p}$. If the collision term and/or the time derivative is larger than the momentum derivative, the latter can be safely neglected. The problem arises when the collision terms are small. If they are of the order of momentum derivatives, the latter can not be neglected any more. Therefore, in general, we have to keep the momentum derivatives in our equations. A complication which we immediately obtain is that there are many other terms of the order of $1/p_F\xi$ in the expansion of the full Green function transport equations in the quasiclassical parameter. Fortunately, not all of them are equally important.

To separate the important contributions of the first order in the quasiclassical parameter $1/p_F\xi$ we note that various terms in the transport equations of the type of eqn (10.29) for the full Green functions, which result from the expansion of the convolution $\check{H} \otimes \check{G}$ in the small parameter $1/p_F\xi$ are of two different types. First, there are corrections to already existing terms in the quasiclassical kinetic equations derived in Chapter 9 within the zero order in $1/p_F\xi$. Second, there are new contributions which appear only as the first approximation in $1/p_F\xi$. As we already mentioned, the new terms (though they are formally of the relative order of $1/p_F\xi$) enter the kinetic equations in the same way as the collision integrals, and, therefore, they have to be compared with the terms of the order of $1/\tau$. In clean superconductors with $1/\tau \ll \Delta$, the terms of the new structure may happen to be not negligible compared to the (small) collision integrals. Thus one can neglect the corrections of the first type and, at the same time, keep the corrections of the new structure. The possibility to separate the most important contributions from all the numerous terms of the expansion in $1/p_F\xi$ exists for clean superconductors with the mean free path $\ell \gg \xi(T)$. For dirty superconductors with $\ell \sim \xi(T)$, one has two options. If the quasiclassical accuracy is sufficient for the particular problem, the first-order terms in $1/p_F\xi$ including the momentum derivatives should be all neglected. If the quasiclassical approximation is not sufficient, one has to take into account other expansion terms in $1/p_F\xi$. This approach has been developed by Larkin and Ovchinnikov (1995) for the problems of the Hall effect in vortex dynamics (see Chapter 14). This approach is much more complicated, and we do not describe it here.

In what follows we shall work with the quasiclassical functions integrated over $d\xi_{\mathbf{p}}/\pi i$. The Keldysh Green function can be generally parametrized in terms of four linearly independent matrices. We can write

$$\widetilde{\check{g}}^K = \widetilde{\check{g}}^R \otimes \check{f}^{(-)} - \check{f}^{(-)} \otimes \widetilde{\check{g}}^A + \widetilde{\check{g}}^R \otimes \check{f}^{(+)} + \check{f}^{(+)} \otimes \widetilde{\check{g}}^A, \tag{10.33}$$

where $\check{f}$ is the matrix distribution function; it consists of the diagonal matrix $\check{f}(-)$

$$\check{f}_\epsilon^{(-)}(\mathbf{p};\mathbf{r},t) = \check{1} f^{(1)} + \check{\tau}_3 f^{(2)} \tag{10.34}$$

and another two-component matrix $\check{f}^{(+)}$. In the zero-order quasiclassical approximation (see Chapter 9), $\check{f}^{(+)} = 0$ due to the normalization of the quasiclassical Keldysh function. If we include non-quasiclassical corrections, the matrix $\check{f}^{(+)}$ becomes nonzero. The components of $\check{f}^{(+)}$are thus small in the quasiclassical parameter $(p_F\xi)^{-1}$. We shall write $f^{(1)} = f^{(0)} + f_1$ and $f^{(2)} \equiv f_2$ where f_1 and f_2 are the nonequilibrium corrections. The equilibrium distribution function $f^{(0)}(\epsilon)$ depends only on energy; $f_1(\epsilon, \mathbf{p}; \mathbf{r}, t)$ and $f_2(\epsilon, \mathbf{p}; \mathbf{r}, t)$ are corrections proportional to time derivatives of the order parameter and to the electric field. The functions f_1 and f_2 are respectively the odd and even in ϵ and $\mathbf{p}$ parts of the total distribution function. We have

$$\widetilde{\check{g}}^K = \left[\widetilde{\check{g}}^R \check{f}^{(-)} - \check{f}^{(-)}\widetilde{\check{g}}^A\right] + \left[\widetilde{\check{g}}^R \check{f}^{(+)} + \check{f}^{(+)}\widetilde{\check{g}}^A\right] - \frac{i}{2}\frac{df^{(0)}}{d\epsilon}\left[\frac{\partial\widetilde{\check{g}}^R}{\partial t} + \frac{\partial\widetilde{\check{g}}^A}{\partial t}\right]$$
$$- \frac{i}{2}\left[\frac{\partial\widetilde{\check{g}}^R}{\partial \mathbf{p}}\nabla\check{f}^{(-)} + \nabla\check{f}^{(-)}\frac{\partial\widetilde{\check{g}}^A}{\partial \mathbf{p}}\right] + \frac{i}{2}\left[\nabla\widetilde{\check{g}}^R\frac{\partial\check{f}^{(-)}}{\partial \mathbf{p}} + \frac{\partial\check{f}^{(-)}}{\partial \mathbf{p}}\nabla\widetilde{\check{g}}^A\right]. \tag{10.35}$$

In the second line we neglect the function $\check{f}^{(+)}$since it is already small in the quasiclassical parameter.

Now we make the next step in derivation: we integrate eqns (10.27) and (10.28) over $d\xi_{\mathbf{p}}/\pi i$. As we explained earlier, we omit the corrections of the order of $1/p_F\xi$ to the terms which do already exist in the zero-order approximation. Within this accuracy, we neglect the dependence of all the factors on $\xi_{\mathbf{p}}$, and put $\mathbf{v} = \mathbf{v}_F$. Moreover, we replace the derivative $\partial/\partial\mathbf{p} = \mathbf{v}_F\,(\partial/\partial\xi_{\mathbf{p}}) + \partial/\partial\mathbf{p}_F$ with just the second term. Here $\delta\mathbf{p}_F$ is the increment of the momentum $\mathbf{p}_F$ belonging to the Fermi surface $\xi_{\mathbf{p}}(\mathbf{p}) = 0$, i.e., $\delta\mathbf{p}_F$ is perpendicular to $\mathbf{v}_F$. We obtain

$$i\widetilde{\tau}_3\frac{\partial\widetilde{\check{g}}^{R(A)}}{\partial t} + i\mathbf{v}_F\nabla\widetilde{\check{g}}^{R(A)} = \check{H}_0\widetilde{\check{g}}^{R(A)} - \widetilde{\check{g}}^{R(A)}\check{H}_0$$
$$- \frac{i}{2}\left[\widetilde{\check{H}}_d\widetilde{\check{g}}^{R(A)} + \widetilde{\check{g}}^{R(A)}\widetilde{\check{H}}_d\right] - \check{I}^{R(A)} \tag{10.36}$$

and

$$i\check{\tau}_3\frac{\partial\widetilde{\check{g}}^K}{\partial t} + i\mathbf{v}_F\nabla\widetilde{\check{g}}^K = \check{H}_0\widetilde{\check{g}}^K - \widetilde{\check{g}}^K\check{H}_0$$
$$-\frac{i}{2}\left[\widetilde{\check{H}}_d\widetilde{\check{g}}^K + \widetilde{\check{g}}^K\widetilde{\check{H}}_d\right] - \check{I}^K \tag{10.37}$$

Here

$$\check{H}_0 = \begin{pmatrix} -\epsilon & -\Delta \\ \Delta^* & \epsilon \end{pmatrix}$$

and $\widetilde{\check{H}}_d$ is defined by eqns (10.21) and (10.22).
Using eqn (10.33) we find

$$-i\widetilde{\tau}_3\frac{\partial\widetilde{\check{g}}^K}{\partial t} - i\mathbf{v}_F\nabla\widetilde{\check{g}}^K + \check{H}_0\widetilde{\check{g}}^K - \widetilde{\check{g}}^K\check{H}_0$$
$$-\frac{i}{2}\left[\widetilde{\check{H}}_d\left(\widetilde{\check{g}}^R\check{f}^{(-)} - \check{f}^{(-)}\widetilde{\check{g}}^A\right) + \left(\widetilde{\check{g}}^R\check{f}^{(-)} - \check{f}^{(-)}\widetilde{\check{g}}^A\right)\widetilde{\check{H}}_d\right]$$
$$-\frac{i}{2}\left[\widetilde{\check{H}}_d\left(\widetilde{\check{g}}^R\underline{\check{f}}^{(-)} - \underline{\check{f}}^{(-)}\widetilde{\check{g}}^A\right) + \left(\widetilde{\check{g}}^R\check{f}^{(-)} - \underline{\check{f}}^{(-)}\widetilde{\check{g}}^A\right)\widetilde{\check{H}}_d\right] = \check{I}^K. \tag{10.38}$$

The operators $\partial/\partial\epsilon, \partial/\partial\mathbf{p}$, and ∇ in $\widetilde{\check{H}}_d$ act on the underlined functions. The components of $\check{f}^{(+)}$can again be neglected if they appear together with the operator $\widetilde{\check{H}}_d$.

Equations for the distribution functions $\check{f}^{(-)}$and $\check{f}^{(+)}$can be obtained from eqn (10.38) by projecting it onto the eigenfunctions of the operators

$$\mathcal{L}_-\{\check{A}\} = \check{g}^R\check{A} - \check{A}\check{g}^A \text{ and } \mathcal{L}_+\{\check{A}\} = \check{g}^R\check{A} + \check{A}\check{g}^A,$$

respectively. Here $\check{A}$ is the matrix equation (10.38).

10.2.1 Superconductors in electromagnetic fields

In many situations, the kinetic equations can be simplified considerably. The wide class of such phenomena includes, in particular, responses of superconductors to applied electromagnetic fields. These are the cases when the non-quasiclassical corrections in the equation for the function f_2 can be neglected. We specify the corresponding conditions later.

Using eqn (10.36) and its time derivative, we find in this case

$$\begin{aligned}
&-i\widetilde{\tau}_3\left(\widetilde{\check{g}}^R-\widetilde{\check{g}}^A\right)\frac{\partial f_1}{\partial t}-i\left(\widetilde{\check{g}}^R-\widetilde{\check{g}}^A\right)\mathbf{v}_F\nabla f_1-i\widetilde{\tau}_3\frac{\partial\delta\widetilde{g}^K}{\partial t}-i\mathbf{v}_F\nabla\delta\widetilde{g}^K\\
&+\left[\check{H}_0\delta\widetilde{g}^K-\delta\widetilde{g}^K\check{H}_0\right]-\frac{i}{2}\left[\widetilde{\check{H}}_d\underline{\delta\widetilde{g}^K}+\underline{\delta\widetilde{g}^K}\widetilde{\check{H}}_d\right]\\
&-\frac{i}{2}\left[\widetilde{\check{H}}_d\left(\widetilde{\check{g}}^R-\widetilde{\check{g}}^A\right)+\left(\widetilde{\check{g}}^R-\widetilde{\check{g}}^A\right)\widetilde{H}_d\right]\underline{f_1}\\
&+\frac{i}{2}\left[\widetilde{\check{H}}_\epsilon\left(\widetilde{\check{g}}^R+\widetilde{\check{g}}^A\right)-\left(\widetilde{\check{g}}^R+\widetilde{\check{g}}^A\right)\widetilde{\check{H}}_\epsilon\right]\underline{f_1}=\check{K}
\end{aligned}\tag{10.39}$$

where

$$\check{K}=\check{I}-\left(\check{I}^R-\check{I}^A\right)\left(f^{(0)}+f_1\right)+\frac{i}{2}\frac{\hat{\partial}}{\partial t}\left(\check{I}^R+\check{I}^A\right)\frac{\partial f^{(0)}}{\partial\epsilon}.\tag{10.40}$$

Within our approximation, we can omit the terms of order of $1/p_F\xi_0$ in the collision integral. We also define

$$\delta\widetilde{\check{g}}^K=\widetilde{\check{g}}^K-\left[\widetilde{\check{g}}^R_\epsilon-\widetilde{\check{g}}^A_\epsilon\right]f_1+\frac{i}{2}\frac{df^{(0)}}{d\epsilon}\left[\frac{\partial\widetilde{\check{g}}^R}{\partial t}+\frac{\partial\widetilde{\check{g}}^A}{\partial t}\right]$$

We now take the trace of eqn (10.39). The first line gives

$$-2i\left(g^R-g^A\right)\frac{\partial f_1}{\partial t}-2i\left(g^R-g^A\right)\mathbf{v}_F\nabla f_1-i\mathbf{v}_F\nabla\,\mathrm{Tr}\left[\delta\widetilde{g}^K\right]$$

We neglect the non-quasiclassical corrections in the first two terms for the reason which we discuss later. Consider the trace

$$\begin{aligned}
\mathrm{Tr}\,\delta\check{g}^K=&2\left(g^R-g^A\right)f_2+\mathrm{Tr}\left(\check{g}^R\check{f}^{(+)}+\check{f}^{(+)}\check{g}^A\right)\\
&-i\left[\frac{\partial g^R}{\partial\mathbf{p}}+\frac{\partial g^A}{\partial\mathbf{p}}\right]\nabla f_2+i\frac{\partial f_2}{\partial\mathbf{p}}\nabla\left(g^R+g^A\right)
\end{aligned}$$

All terms except for the first one contain small non-quasiclassical correction and can be neglected.

Putting this into eqn (10.39), we write the final equation for the gauge covariant functions using the transformation rules of eqn (10.23) and (10.24):

$$\begin{aligned}
&\left(e\mathbf{v}_F\cdot\mathbf{E}g_-+\frac{1}{2}\left[f_-\frac{\hat{\partial}\Delta^*}{\partial t}+f_-^\dagger\frac{\hat{\partial}\Delta}{\partial t}\right]\right)\frac{\partial f^{(0)}}{\partial\epsilon}+\mathbf{v}_F\cdot\nabla\left(f_2g_-\right)+g_-\frac{\partial f_1}{\partial t}\\
&-\left(\frac{1}{2}\left[(\hat{\nabla}\Delta)f_-^\dagger+\left(\hat{\nabla}\Delta^*\right)f_-\right]-\frac{e}{c}\left[\mathbf{v}_F\times\mathbf{H}\right]g_-\right)\frac{\partial f_1}{\partial\mathbf{p}}\\
&+\frac{1}{2}\left(\frac{\partial\Delta}{\partial\mathbf{p}}f_-^\dagger+\frac{\partial\Delta^*}{\partial\mathbf{p}}f_-\right)\nabla f_1=J_1
\end{aligned}\tag{10.41}$$

Here

$$\check{g}_{\pm} = \frac{1}{2}\left(\check{g}_{\epsilon}^{R} \pm \check{g}_{\epsilon}^{A}\right) \tag{10.42}$$

and

$$J_1 = \frac{i}{4}\,\mathrm{Tr}\,\check{K} \tag{10.43}$$

An equation for f_2 is obtained from eqn (10.39) after multiplying it by $\widetilde{\tau}_3$ and taking the trace. Neglecting the non-quasiclassical corrections in this equation we get

$$g_{-}\mathbf{v}_F\nabla f_1 + \frac{1}{2}\left[f_{+}\frac{\hat{\partial}\Delta^*}{\partial t} - f_{+}^{\dagger}\frac{\hat{\partial}\Delta}{\partial t}\right]\frac{\partial f^{(0)}}{\partial\epsilon} + i\left[\Delta^* f_{+} + \Delta f_{+}^{\dagger}\right] f_2 = J_2 \tag{10.44}$$

Here

$$J_2 = \frac{i}{4}\,\mathrm{Tr}\left[\check{\tau}_3\check{K}\right] \tag{10.45}$$

Equation (10.44) shows that the term $\mathrm{Tr}\left(\widetilde{\widetilde{g}}^{R} - \widetilde{\widetilde{g}}^{A}\right)\mathbf{v}_F\nabla f_1$ omitted in eqn (10.39) during the derivation of eqn (10.41) introduces corrections proportional to the quasiclassical parameter $(p_F\xi)^{-1}$ to the terms which are already present in eqn (10.41); it thus should be neglected.

Equation (10.41) can be further modified using the equation for the static Green functions

$$\mathbf{v}_F\nabla g_{\epsilon}^{R(A)} + i\left(\Delta f_{\epsilon}^{\dagger R(A)} - f_{\epsilon}^{R(A)}\Delta^*\right) = \int\left[I_1^{R(A)} - \bar{I}_1^{R(A)}\right]\frac{d\xi_{\mathbf{p}}}{2\pi} \tag{10.46}$$

The non-quasiclassical correction can be omitted here. Multiplying this equation by f_2 and subtracting it from eqn (10.41), we obtain

$$\begin{aligned}
&\left(e\mathbf{v}_F\cdot\mathbf{E}g_{-} + \frac{1}{2}\left[f_{-}\frac{\hat{\partial}\Delta^*}{\partial t} + f_{-}^{\dagger}\frac{\hat{\partial}\Delta}{\partial t}\right]\right)\frac{\partial f^{(0)}}{\partial\epsilon} + g_{-}\mathbf{v}_F\cdot\nabla f_2\\
&+ i\left[\Delta^* f_{-} - \Delta f_{-}^{\dagger}\right] f_2 - \left(\frac{1}{2}\left[(\hat{\nabla}\Delta)f_{-}^{\dagger} + \left(\hat{\nabla}\Delta^*\right) f_{-}\right] - \frac{e}{c}\left[\mathbf{v}_F\times\mathbf{H}\right] g_{-}\right)\frac{\partial f_1}{\partial\mathbf{p}}\\
&+ \frac{1}{2}\left(\frac{\partial\Delta}{\partial\mathbf{p}}f_{-}^{\dagger} + \frac{\partial\Delta^*}{\partial\mathbf{p}}f_{-}\right)\nabla f_1 = J_1'
\end{aligned} \tag{10.47}$$

where

$$J' = \frac{i}{4}\,\mathrm{Tr}\,\check{K}' \tag{10.48}$$

and

$$\check{K}' = \check{I} - \left(\check{I}^{R} - \check{I}^{A}\right)\left(f^{(0)} + f_1 + \check{\tau}_3 f_2\right) + \frac{i}{2}\frac{\hat{\partial}}{\partial t}\left(\check{I}^{R} + \check{I}^{A}\right)\frac{\partial f^{(0)}}{\partial\epsilon}. \tag{10.49}$$

In the literature, the function f_2' is used sometimes instead of the function f_2, it is defined according to

$$f_2 = f_2' + \frac{1}{2}\left(\frac{\partial\chi}{\partial t} + 2e\varphi\right)\frac{df^{(0)}}{d\epsilon} \tag{10.50}$$

where χ is the phase of the order parameter. Using the fact that the collision integral $J_1'^{(2)}$ vanishes for any function independent of the momentum-direction [see eqn (10.76)] we find instead of eqns (10.47)

$$\begin{aligned}
&\left(-\frac{e}{c}\mathbf{v}_F\frac{\partial\mathbf{Q}}{\partial t}g_- + \frac{1}{2}\left[f_-\frac{\tilde{\partial}\Delta^*}{\partial t} + f_-^\dagger\frac{\tilde{\partial}\Delta}{\partial t}\right]\right)\frac{\partial f^{(0)}}{\partial\epsilon} \\
&+ g_-\mathbf{v}_F\cdot\nabla f_2' + i\left[\Delta^* f_- - \Delta f_-^\dagger\right]f_2' \\
&-\left(\frac{1}{2}\left[(\hat{\nabla}\Delta)f_-^\dagger + \left(\hat{\nabla}\Delta^*\right)f_-\right] - \frac{e}{c}\left[\mathbf{v}_F\times\mathbf{H}\right]g_-\right)\frac{\partial f_1}{\partial\mathbf{p}} \\
&+\frac{1}{2}\left(\frac{\partial\Delta}{\partial\mathbf{p}}f_-^\dagger + \frac{\partial\Delta^*}{\partial\mathbf{p}}f_-\right)\nabla f_1 = J_1'\left\{f_1, f_2'\right\}
\end{aligned} \tag{10.51}$$

where the operator $\tilde{\partial}/\partial t$ acts only on the modulus of the order parameter. Equation (10.51) can be further transformed into

$$\begin{aligned}
&g_-\frac{\partial f_1}{\partial t} + \left(-\frac{e}{c}\mathbf{v}_F\cdot\frac{\partial\mathbf{Q}}{\partial t}g_- + \frac{1}{2}\left[f_-\frac{\tilde{\partial}\Delta^*}{\partial t} + f_-^\dagger\frac{\tilde{\partial}\Delta}{\partial t}\right]\right)\frac{\partial f^{(0)}}{\partial\epsilon} \\
&+\mathbf{v}_F\cdot\nabla\left(f_2'g_-\right) - \left(\frac{1}{2}\left[(\hat{\nabla}\Delta)f_-^\dagger + \left(\hat{\nabla}\Delta^*\right)f_-\right] - \frac{e}{c}\left[\mathbf{v}_F\times\mathbf{H}\right]g_-\right)\frac{\partial f_1}{\partial\mathbf{p}} \\
&+\frac{1}{2}\left(\frac{\partial\Delta}{\partial\mathbf{p}}f_-^\dagger + \frac{\partial\Delta^*}{\partial\mathbf{p}}f_-\right)\nabla f_1 = J_1\left\{f_1, f_2'\right\}
\end{aligned} \tag{10.52}$$

Equation (10.44) becomes

$$\begin{aligned}
&g_-\mathbf{v}_F\nabla f_1 + \frac{1}{2}\left[f_+\frac{\tilde{\partial}\Delta^*}{\partial t} - f_+^\dagger\frac{\tilde{\partial}\Delta}{\partial t}\right]\frac{\partial f^{(0)}}{\partial\epsilon} \\
&+ i\left[\Delta^* f_+ + \Delta f_+^\dagger\right]f_2' = J_2\left\{f_1, f_2'\right\}
\end{aligned} \tag{10.53}$$

These are the equations for the new function f_2'.

10.2.2 Discussion

The kinetic equations eqn (10.41) [or eqn (10.47)], and eqn (10.44) for the distribution functions f_1 and f_2 are the central result of this Chapter. The kinetic equations contain the momentum derivatives of $\partial f/\partial\mathbf{p}$ multiplied by "forces" which include the Lorentz force $\mathbf{f}_L = (e/c)\left[\mathbf{v}_F\times\mathbf{H}\right]$ and the forces arising due to spatial variations of the order parameter. Note that the electric field is included in the terms which describe the energy variation in time, since $\mathbf{E}$ is of the first order in time derivative. For this reason, it does not enter the force terms.

As we already discussed on page 204, the function f_1 appears in the l.h.s. of eqn (10.41) only as $\partial f_1/\partial t$ and in combinations with the momentum derivatives $\partial f_1/\partial\mathbf{p}$ or as $(\partial\Delta/\partial\mathbf{p})\nabla f_1$. For a slow variations in time, $\omega \ll 1/\tau$, the only terms with f_1 are those with the momentum derivatives of the distribution function and of the order parameter. Such terms only exist in the first-order approximation in $1/p_F\xi$. All other non-quasiclassical contributions are omitted: they turn to be corrections to the first line of eqn (10.41) which does already exist in the zero approximation.

From eqn (10.41) we note that the momentum derivatives are of the same order of magnitude as the collision integrals when

$$\frac{\Delta}{p_F \xi} \sim \frac{1}{\tau} \tag{10.54}$$

The relative importance of the terms with the momentum derivatives decreases as the mean free path gets shorter. For $\tau \ll E_F/\Delta^2$, their contribution to the distribution function is of the order of $\Delta^2\tau/E_F$ as compared to that determined by the collision integral. This correction becomes of the order of $1/p_F\xi$ when $\Delta\tau$ decreases down to values of the order of unity. For such short τ, the contributions from $\partial f_1/\partial \mathbf{p}$ and $\partial\Delta/\partial\mathbf{p}$ become comparable or less than the terms which we have neglected during the derivation of the quasiclassical kinetic equations.

Thus, it becomes illegal to keep the momentum derivatives $\partial f_1/\partial\mathbf{p}$ and $\partial\Delta/\partial\mathbf{p}$ in equation (10.41) for $\Delta\tau \sim 1$ and they should be neglected. Therefore, for dirty superconductors, eqns (10.41, 10.44) should be used with the momentum derivatives neglected. In the dirty case we thus obtain from eqn (10.41)

$$\left(e\mathbf{v}_F \cdot \mathbf{E} g_- + \frac{1}{2}\left[f_- \frac{\hat{\partial}\Delta^*}{\partial t} + f_-^\dagger \frac{\hat{\partial}\Delta}{\partial t}\right]\right) \frac{\partial f^{(0)}}{\partial \epsilon} + \mathbf{v}_F \cdot \nabla\left(f_2 g_-\right) + g_- \frac{\partial f_1}{\partial t} = J_1 \tag{10.55}$$

Equation (10.44) can be used as it stands. These equations can be also obtained directly from the Eliashberg equations (9.1) using the Keldysh function in the form of eqn (9.25) with the distribution function from eqn (9.28). The derivation of the quasiclassical set of equations (10.55) and (10.44) for dirty superconductors is described in the review by Larkin and Ovchinnikov (1986).

For clean superconductors with $\Delta\tau \gg 1$, on the contrary, the momentum derivatives should be included into the kinetic equations. They give small contributions in the so called moderately clean regime when $\Delta^{-1} \ll \tau \ll E_F/\Delta^2$. As τ increases further, their role becomes more important. Finally, in the collisionless limit (the so called superclean regime) $\tau \gg E_F/\Delta^2$, the momentum derivatives $\partial f/\partial\mathbf{p}$ and $\partial\Delta/\partial\mathbf{p}$ dominate.

One more observation is that, for clean superconductors interacting with electromagnetic fields, $f_2 \ll f_1$. Indeed, eqn (10.41) gives

$$f_1 \sim \tau\omega\Delta \frac{\partial f^{(0)}}{\partial \epsilon}$$

On the other hand, eqn (10.44) shows that the variation of f_1 is $\delta f_1 \sim \omega\partial f^{(0)}/\partial\epsilon$ along a trajectory part with a length of the order of ξ. This variation is much smaller than f_1 itself. Therefore, eqn (10.44) reduces simply to

$$g_- \mathbf{v}_F \nabla f_1 = 0 \tag{10.56}$$

i.e., the function f_1 is constant along the quasiparticle trajectory. As a result, eqn (10.41) implies that $f_2 \sim \omega\partial f^{(0)}/\partial\epsilon$, i.e., $f_2 \ll f_1$. It is this condition that allows one to neglect non-quasiclassical corrections to f_2 and to use eqn (10.56) in a clean superconductor.

Equation (10.56) holds for a trajectory which is not very long. The maximum length L is determined by the shortest of the three length scales: $L \ll \min\{\ell, (E_F/\Delta)\, l_\Delta, r_H\}$ where l_Δ is the characteristic scale of variations of the order parameter, and $r_H = v_F/\omega_c$ is the Larmor radius. If this condition is violated, one can not neglect the other terms in eqn (10.39) any more. This condition, in particular, determines the necessary requirement of validity of eqn (10.56); it is $L \gg \xi$ which is equivalent, of course, to the clean-limit condition $\ell \gg \xi$.

The non-quasiclassical corrections to eqn (10.44) which were omitted during the derivation may become important for problems associated with the heat transport, when there exists a temperature gradient in a superclean regime with
$\tau \sim E_F/\Delta^2$ and longer. In this case, on the contrary, the function f_2 is larger than f_1 and eqn (10.41) gives $g_- f_2 = \text{const}$. The second equation (10.44) should then be derived including the non-quasiclassical corrections. We shall not consider such problems here.

18.3.3 10.3 Observables in the gauge invariant representation

Using the modified gauge invariant Green functions, the current can be written as

$$\mathbf{j} = -e \int \tilde{\mathbf{v}}_F \left[\tilde{G}^K_\epsilon(\tilde{\mathbf{p}}; \mathbf{r}, t) - \tilde{\bar{G}}^K_\epsilon(\tilde{\mathbf{p}}; \mathbf{r}, t) \right] \frac{d^3p}{(2\pi)^3} \frac{d\epsilon}{4\pi i} \tag{10.57}$$

Here $\tilde{\mathbf{p}} = \mathbf{p} - (e/c)\mathbf{A}$ for $\tilde{G}$ and $\tilde{\mathbf{p}} = \mathbf{p} + (e/c)\mathbf{A}$ for $\tilde{\bar{G}}$ according to the definition of the gauge invariant functions. We see that the diamagnetic term does not appear explicitly for the gauge invariant representation in the quasiclassical limit. Performing shifts in the momentum under the integrals we can write the current through the gauge invariant quasiclassical Keldysh functions in the usual form

$$\mathbf{j} = -e\nu(0) \int \mathbf{v}_F \left[\tilde{g}^K_\epsilon(\mathbf{p}; \mathbf{r}, t) - \tilde{\bar{g}}^K_\epsilon(\mathbf{p}; \mathbf{r}, t) \right] \frac{d\Omega_\mathbf{p}}{4\pi} \frac{d\epsilon}{4} \tag{10.58}$$

The order parameter is

$$\frac{\Delta(\mathbf{r}, t)}{|g|} = \int \tilde{F}^K_\epsilon(\mathbf{p}; \mathbf{r}, t) \frac{d^3p}{(2\pi)^3} \frac{d\epsilon}{4\pi i} \tag{10.59}$$

where $|g|$ is the pairing interaction. The quasiclassical version of this expression is

$$\frac{\Delta(\mathbf{r}, t)}{\lambda} = \int \tilde{f}^K_\epsilon(\mathbf{p}; \mathbf{r}, t) \frac{d\Omega_\mathbf{p}}{4\pi} \frac{d\epsilon}{4} \tag{10.60}$$

where $\lambda = \nu(0)|g|$ is the pairing constant.
The electron density is (Eliashberg 1971)

$$\begin{aligned} N - N_0 &= -\int \left[G^K_\epsilon(\mathbf{p}; \mathbf{r}, t) + \bar{G}^K_\epsilon(\mathbf{p}; \mathbf{r}, t) \right] \frac{d^3p}{(2\pi)^3} \frac{d\epsilon}{4\pi i} \\ &= -2\nu(0)e\varphi - \nu(0) \int \left[g^K_\epsilon(\mathbf{p}_F; \mathbf{r}, t) + \bar{g}^K_\epsilon(\mathbf{p}_F; \mathbf{r}, t) \right] \frac{d\Omega_\mathbf{p}}{4\pi} \frac{d\epsilon}{4} \end{aligned}$$

in terms of the usual Green functions. The scalar potential appears in the r.h.s. due to a change in the normal-state density caused by a $e\varphi$ variation of the chemical potential. The electron density becomes

$$\begin{aligned} N - N_0 &= -2\nu(0)e\varphi - \nu(0) \int \left[\tilde{g}^K_{\epsilon - e\varphi}(\mathbf{p}_F; \mathbf{r}, t) + \tilde{\bar{g}}^K_{\epsilon + e\varphi}(\mathbf{p}_F; \mathbf{r}, t) \right] \frac{d\Omega_\mathbf{p}}{4\pi} \frac{d\epsilon}{4} \\ &= -\nu(0) \int \left[\tilde{g}^K_\epsilon(\mathbf{p}_F; \mathbf{r}, t) + \tilde{\bar{g}}^K_\epsilon(\mathbf{p}_F; \mathbf{r}, t) \right] \frac{d\Omega_\mathbf{p}}{4\pi} \frac{d\epsilon}{4} \end{aligned} \tag{10.61}$$

in terms of the gauge invariant Green functions. Here we use that

$$\tilde{g}^K_{\epsilon - e\varphi}(\mathbf{p}_F; \mathbf{r}, t) = \tilde{g}^K_\epsilon(\mathbf{p}_F; \mathbf{r}, t) - e\varphi \frac{\partial \tilde{g}^K_\epsilon}{\partial \epsilon}$$

and

$$\int \frac{\partial \tilde{g}_\epsilon^K}{\partial \epsilon} \frac{d\epsilon}{4} = 1$$

The latter follows from the observation that for $|\epsilon| \to \infty$

$$\check{g}_\epsilon^{R(A)} \to \pm\check{1}, f^{(1)} \to f^{(0)}, \text{ and } f^{(2)} \to 0$$

We can now replace the gauge invariant functions in equations (10.58), (10.60), and (10.61) with the usual (gauge covariant) functions. The only condition is that they have to be determined by the gauge invariant kinetic equations (10.41) and (10.44) together with the gauge invariant parameterization for the Keldysh function through f_1 and f_2.

The Keldysh function takes the form

$$\begin{aligned}\check{g}_\epsilon^K =& \left[\check{g}_\epsilon^R - \check{g}_\epsilon^A\right] f^{(0)} - \frac{i}{2}\frac{df^{(0)}}{d\epsilon}\left[\frac{\partial \check{g}_\epsilon^R}{\partial t} + \frac{\partial \check{g}_\epsilon^A}{\partial t}\right] \\ &+ \left[\check{g}_\epsilon^R \delta\check{f}^{(-)} - \delta\check{f}^{(-)}\check{g}_\epsilon^A\right] + \left[\check{g}_\epsilon^R \check{f}^{(+)} + \check{f}^{(+)}\check{g}_\epsilon^A\right] \\ &- \frac{i}{2}\left[\frac{\partial \check{g}_\epsilon^R}{\partial \mathbf{p}}\hat{\nabla}\check{f}^{(-)} + \hat{\nabla}\check{f}^{(-)}\frac{\partial \check{g}_\epsilon^R}{\partial \mathbf{p}}\right] + \frac{i}{2}\left[\hat{\nabla}\check{g}_\epsilon^R \frac{\partial \check{f}^{(-)}}{\partial \mathbf{p}} + \frac{\partial \check{f}^{(-)}}{\partial \mathbf{p}}\hat{\nabla}\check{g}_\epsilon^A\right]\end{aligned}$$

where $\delta\check{f}^{(-)} = \check{f}^{(-)} - \check{1}f^{(0)}$. We shall neglect the small non-quasiclassical corrections and write

$$\begin{aligned}\check{g}_{\epsilon_+,\epsilon_-}^K =& \left[\check{g}_\epsilon^R - \check{g}_\epsilon^A\right] f_\epsilon^{(0)} - \frac{i}{2}\frac{\hat{\partial}}{\partial t}\left(\check{g}_\epsilon^R + \check{g}_\epsilon^A\right)\frac{\partial f^{(0)}}{\partial \epsilon} \\ &+ \left(\check{g}_\epsilon^R - \check{g}_\epsilon^A\right) f_1 + \left(\check{g}_\epsilon^R \check{\tau}_3 - \check{\tau}_3 \check{g}_\epsilon^A\right) f_2\end{aligned} \tag{10.62}$$

In this approximation, the Keldysh function is parametrized by two distribution functions f_1 and f_2.

With eqn (10.62), the current and the order parameter have their usual form

$$\mathbf{j} = -e\nu(0)\int \mathbf{v}_F\left[g_\epsilon^K(\mathbf{p};\mathbf{r},t) - \bar{g}_\epsilon^K(\mathbf{p};\mathbf{r},t)\right]\frac{d\Omega_\mathbf{p}}{4\pi}\frac{d\epsilon}{4} \tag{10.63}$$

and

$$\frac{\Delta(\mathbf{r},t)}{\lambda} = \int f_\epsilon^K(\mathbf{p};\mathbf{r},t)\frac{d\Omega_\mathbf{p}}{4\pi}\frac{d\epsilon}{4} \tag{10.64}$$

The expressions for the internal energy and the internal energy current can also be written in terms of the gauge invariant functions. We have

$$\begin{aligned}\mathcal{E} =& -\int\left[(\epsilon - e\varphi)G_\epsilon^K(\mathbf{p};\mathbf{r},t) - (\epsilon + e\varphi)\bar{G}_\epsilon^K(\mathbf{p};\mathbf{r},t)\right]\frac{d\epsilon}{4\pi i}\frac{d^3p}{(2\pi)^3} + \frac{|\Delta|^2}{|g|} \\ =& -\int\left[(\epsilon - e\varphi)\tilde{G}_{\epsilon-e\varphi}^K(\mathbf{p};\mathbf{r},t) - (\epsilon + e\varphi)\tilde{\bar{G}}_{\epsilon+e\varphi}^K(\mathbf{p};\mathbf{r},t)\right]\frac{d\epsilon}{4\pi i}\frac{d^3p}{(2\pi)^3} + \frac{|\Delta|^2}{|g|} \\ =& -\int\epsilon\left[\tilde{G}_\epsilon^K(\mathbf{p};\mathbf{r},t) - \tilde{\bar{G}}_\epsilon^K(\mathbf{p};\mathbf{r},t)\right]\frac{d\epsilon}{4\pi i}\frac{d^3p}{(2\pi)^3} + \frac{|\Delta|^2}{|g|}\end{aligned}$$

In the quasiclassical representation, it becomes

$$\mathcal{E} = \mathcal{E}_n - \nu(0)\int\epsilon\left[\delta g_\epsilon^K(\mathbf{p};\mathbf{r},t) - \delta\bar{g}_\epsilon^K(\mathbf{p};\mathbf{r},t)\right]\frac{d\epsilon}{4}\frac{d\Omega_\mathbf{p}}{4\pi} + \frac{|\Delta|^2}{|g|} \tag{10.65}$$

Here $\mathcal{E}_n$ is the energy in the normal state, and δg is the deviation of the Green function from the normal state. We have separated the normal state contribution because the normal state energy can not be calculated within the quasiclassical approximation: the momentum integral is determined by $\xi_{\mathbf{p}}$ far from the Fermi surface. The energy current takes the form

$$\begin{aligned}\mathbf{j}_{\mathcal{E}} &= -\int \epsilon \mathbf{v} \left[\tilde{G}^K_\epsilon(\mathbf{p};\mathbf{r},t) + \tilde{\bar{G}}^K_\epsilon(\mathbf{p};\mathbf{r},t)\right] \frac{d\epsilon}{4\pi i}\frac{d^3p}{(2\pi)^3} \\ &= -\nu(0)\int \epsilon \mathbf{v}_F \left[g^K_\epsilon(\mathbf{p};\mathbf{r},t) + \bar{g}^K_\epsilon(\mathbf{p};\mathbf{r},t)\right] \frac{d\epsilon}{4}\frac{d\Omega_{\mathbf{p}}}{4\pi}\end{aligned} \tag{10.66}$$

10.3.1 The electron density and charge neutrality

The electron density eqn (10.61) takes the form

$$\begin{aligned}N - N_0 &= -\nu(0)\int \left[g^K_\epsilon(\mathbf{p};\mathbf{r},t) + \bar{g}^K_\epsilon(\mathbf{p};\mathbf{r},t)\right] \frac{d\Omega_{\mathbf{p}}}{4\pi}\frac{d\epsilon}{4} \\ &= -\nu(0)\int \left(\mathrm{Tr}\left[\check{g}^R_\epsilon - \check{g}^A_\epsilon\right] f^{(0)} + 4g_- f_2\right) \frac{d\Omega_{\mathbf{p}}}{4\pi}\frac{d\epsilon}{4}\end{aligned}$$

The combination $\mathrm{Tr}\left[\check{g}^R_\epsilon(\mathbf{p};\mathbf{r},t) - \check{g}^A_\epsilon(\mathbf{p};\mathbf{r},t)\right]$ is nonzero in a nonstationary situation. To find it, we use the normalization condition

$$\check{g}^R \otimes \check{g}^R \equiv \check{g}^R_\epsilon \check{g}^R_\epsilon + \frac{i}{2}\left(\frac{\partial \check{g}^R_\epsilon}{\partial \epsilon}\frac{\partial \check{g}^R_\epsilon}{\partial t} - \frac{\partial \check{g}^R_\epsilon}{\partial t}\frac{\partial \check{g}^R_\epsilon}{\partial \epsilon}\right) = \check{1}$$

We put $\check{g}^R = \check{g}^R_0 + \check{g}^R_1$ where $\check{g}^R_0$ is the stationary Green function and $\check{g}^R_1$ is a nonstationary correction and find

$$\check{g}^R_0\check{g}^R_1 + \check{g}^R_1\check{g}^R_0 = -\frac{i}{2}\left(\frac{\partial \check{g}^R_0}{\partial \epsilon}\frac{\partial \check{g}^R_0}{\partial t} - \frac{\partial \check{g}^R_0}{\partial t}\frac{\partial \check{g}^R_0}{\partial \epsilon}\right)$$

We multiply this equation by $\check{g}^R_0$ and take the trace

$$2\,\mathrm{Tr}\,\check{g}^R_1 = \frac{i}{2}\left(\check{g}^R_0\frac{\hat{\partial}\check{g}^R_0}{\partial t} - \frac{\hat{\partial}\check{g}^R_0}{\partial t}\check{g}^R_0\right)\frac{\partial \check{g}^R_0}{\partial \epsilon}$$

With help of the identity $g^2 - ff^\dagger = 1$ we obtain

$$\begin{aligned}\mathrm{Tr}\,\check{g}^R_1 = -\frac{i}{2}&\left[\frac{1}{f^R}\left(\frac{\partial}{\partial t} + 2ie\phi\right)f^R - \frac{1}{f^{\dagger R}}\left(\frac{\partial}{\partial t} - 2ie\phi\right)f^{\dagger R}\right]\frac{\partial g^R}{\partial \epsilon} \\ &+\frac{1}{2g^R}\left[\frac{1}{f^R}\frac{\partial f^R}{\partial t} + \frac{1}{f^\dagger R}\frac{\partial f^{\dagger R}}{\partial t}\right]\end{aligned} \tag{10.67}$$

and the same for the advanced function. Next, we note that according to the particle-hole symmetry of the Eilenberger equations (5.89) - (5.91), the static Green functions transform as

$$g^R \to -g^A, f^R \to f^A, f^{\dagger R} \to f^{\dagger A} \tag{10.68}$$

under the transformation $\epsilon \to -\epsilon, \mathbf{p} \to -\mathbf{p}$. Therefore, the first line of eqn (10.67) transforms into

$$-\frac{i}{2}\left[\frac{1}{f^A}\left(\frac{\partial}{\partial t} + 2ie\varphi\right)f^A - \frac{1}{f^{\dagger A}}\left(\frac{\partial}{\partial t} - 2ie\phi\right)f^{\dagger A}\right]\frac{\partial g^A}{\partial \epsilon}$$

while the second line transforms into

$$-\frac{1}{2g^A}\left[\frac{1}{f^A}\frac{\partial f^A}{\partial t}+\frac{1}{f^{\dagger A}}\frac{\partial f^{\dagger A}}{\partial t}\right]$$

As a result, we can write

$$\mathrm{Tr}\left(\check{g}^R_{\epsilon_+,\epsilon_-}-\check{g}^A_{\epsilon_+,\epsilon_-}\right)=\left(\frac{\partial\chi}{\partial t}+2e\varphi\right)\left(\frac{\partial g^R_\epsilon}{\partial\epsilon}-\frac{\partial g^A_\epsilon}{\partial\epsilon}\right)+A(\epsilon,\mathbf{p})$$

where $A(\epsilon,\mathbf{p})$ is an even function of $(\epsilon,\mathbf{p})$ while the first term is an odd function of these variables. The term with $A(\epsilon,\mathbf{p})$ vanishes after integration with the odd function $f^{(0)}=\tanh(\epsilon/2T)$. The electron density becomes

$$N=N_0-\nu(0)e\Phi\int_{-\infty}^{\infty}\left\langle\frac{\partial g_-}{\partial\epsilon}\right\rangle f^{(0)}d\epsilon-\nu(0)\int_{-\infty}^{\infty}\langle g_-f_2\rangle\,d\epsilon \tag{10.69}$$

where Φ is defined by eqn (10.4), and the averaging over momentum directions eqn (5.72) is implied. Integrating by parts in the first line and using that

$$g_-|_{\epsilon=\pm\infty}=\pm1;\, f^{(0)}(\epsilon=\pm\infty)=\pm1$$

we find

$$N=N_0-\nu(0)e\Phi+\nu(0)e\Phi\int_{-\infty}^{\infty}\langle g_-\rangle\frac{\partial f^{(0)}}{\partial\epsilon}d\epsilon-\nu(0)\int_{-\infty}^{\infty}\langle g_-f_2\rangle\,d\epsilon \tag{10.70}$$

With the definition eqn (10.50) of the function f_2', the electron density can be written as

$$N=N_0-\nu(0)e\Phi-\nu(0)\int_{-\infty}^{\infty}\langle g_-f_2'\rangle\,d\epsilon \tag{10.71}$$

In metals, all bulk charges are screened. It can be seen from the Poisson equation for the normal state

$$\mathrm{div}\,\mathbf{E}=-\nabla^2\varphi=4\pi e\left(N_e-N_i\right)$$

where N_e and N_i are densities of electrons and ions, respectively. In equilibrium, $N_e=N_i$. Introduction of an additional electronic charge results in a shift of the chemical potential $\delta\mu=-e\delta\varphi$ such that

$$\delta N=N_e-N_i=\frac{\partial N_e}{\partial\mu}\delta\mu=-2\nu e\delta\varphi$$

The Poisson equation becomes

$$\nabla^2\delta\varphi=8\pi e^2\nu\delta\varphi$$

It demonstrates that the potential φ together with δN decay at distances of the order of the Debye screening length

$$\lambda_D^2=\frac{1}{8\pi e^2\nu}$$

which is of the order of interatomic distance in good metals. Therefore, variations in the electronic charge density are practically zero at distances of the order of the coherence length, and we have to put the constraint $\delta N_e=0$ or

$$\mathrm{div}\,\mathbf{j}=0 \tag{10.72}$$

The latter follows from the continuity equation

$$\frac{\partial\left(eN_e\right)}{\partial t}+\operatorname{div}\mathbf{j}=0$$

Since $N_i=N_0$ where N_0 is the normal-state electron density, the charge neutrality condition requires

$$N=N_0 \tag{10.73}$$

18.3.4 10.4 Collision integrals

Collision integrals describe interaction of excitations with impurities, phonons and with each other. These interactions are responsible for establishing equilibrium in electronic subsystem in a superconductor assuming that the crystal lattice (phonons) together with impurities are themselves in equilibrium and form a heat bath. In practical superconducting compounds, impurity concentrations are usually such that the most effective relaxation is brought about through scattering of electrons by impurities; the electron-phonon relaxation rate is usually much smaller, to say nothing about electron-electron collisions. If not specified otherwise, we assume in what follows that the impurity collision integral is the largest source of relaxation.

In this Section we consider various collision integrals and derive expressions for them in terms of the distribution functions $f^{(1)}$ and $f^{(2)}$ which will be used later in our discussions. We note that all the collision integrals vanish for the equilibrium distribution $f^{(1)}=f^{(0)}$ and $f^{(2)}=0$. They are thus proportional to deviations from equilibrium. Therefore, one can use the stationary Green functions $\check{g}^{R(A)}$ in the collision integrals.

10.4.1 Impurities

10.4.1.1 Non-magnetic scattering We shall assume for simplicity that the impurity scattering is isotropic. Remind that the self energies for non-magnetic impurities are

$$\begin{aligned}\check{\Sigma}^{K}_{\epsilon,\epsilon-\omega}(\mathbf{k})&=\frac{i}{2\tau}\left\langle\check{g}^{K}_{\epsilon,\epsilon-\omega}\left(\mathbf{p}_F,\mathbf{k}\right)\right\rangle\\ \check{\Sigma}^{R(A)}_{\epsilon,\epsilon-\omega}(\mathbf{k})&=\frac{i}{2\tau}\left\langle\check{g}^{R(A)}_{\epsilon,\epsilon-\omega}\left(\mathbf{p}_F,\mathbf{k}\right)\right\rangle\end{aligned}$$

Here we use notation τ instead of τ_{imp} for brevity. The collision integrals $J^{(\text{imp})}_{1,2}$ are found from eqns (9.14) and (9.16) using the definitions eqns (10.43), (10.45), and (10.48):

$$\begin{aligned}&J^{(\text{imp})}_1=J^{(1)}_1\left\{f^{(1)}\right\}+J^{(2)}_1\left\{f^{(2)}\right\},\quad J'^{(\text{imp})}_1=J^{(1)}_1\left\{f^{(1)}\right\}+J'^{(2)}_1\left\{f^{(2)}\right\},\\ &J^{(\text{imp})}_2=J^{(1)}_2\left\{f^{(1)}\right\}+J^{(2)}_2\left\{f^{(2)}\right\}.\end{aligned}$$

where

$$\begin{aligned}J^{(1)}_1=&-\frac{1}{\tau}\left[\left(f^{(1)}\left\langle g_-\right\rangle-\left\langle f^{(1)}g_-\right\rangle\right)g_--\frac{1}{2}\left(f^{(1)}\left\langle f^{\dagger}_-\right\rangle-\left\langle f^{(1)}f^{\dagger}_-\right\rangle\right)f_-\right.\\ &\left.-\frac{1}{2}\left(f^{(1)}\left\langle f_-\right\rangle-\left\langle f^{(1)}f_-\right\rangle\right)f^{\dagger}_-\right]\end{aligned} \tag{10.74}$$

$$\begin{aligned}J^{(2)}_2=&-\frac{1}{\tau}\left[\left(f^{(2)}\left\langle g_-\right\rangle-\left\langle f^{(2)}g_-\right\rangle\right)g_--\frac{1}{2}\left(f^{(2)}\left\langle f^{\dagger}_+\right\rangle-\left\langle f^{(2)}f^{\dagger}_+\right\rangle\right)f_+\right.\\ &\left.-\frac{1}{2}\left(f^{(2)}\left\langle f_+\right\rangle-\left\langle f^{(2)}f_+\right\rangle\right)f^{\dagger}_+\right]\end{aligned} \tag{10.75}$$

and

$$J_1^{(2)} = -\frac{1}{2\tau}\left[\left(f^{(2)}\left\langle f_-^\dagger\right\rangle f_+ - \left\langle f^{(2)} f_+\right\rangle f_-^\dagger\right) - \left(f^{(2)}\left\langle f_-\right\rangle f_+^\dagger - \left\langle f^{(2)} f_+^\dagger\right\rangle f_-\right)\right],$$
$$J_2^{(1)} = -\frac{1}{2\tau}\left[\left(f^{(1)}\left\langle f_-\right\rangle - \left\langle f^{(1)} f_-\right\rangle\right) f_+^\dagger - \left(f^{(1)}\left\langle f_-^\dagger\right\rangle - \left\langle f^{(1)} f_-^\dagger\right\rangle\right) f_+\right].$$

In addition

$$J_1'^{(2)} = -\frac{1}{2\tau}\left[\left(f'^{(2)}\left\langle f_+\right\rangle - \left\langle f'^{(2)} f_+\right\rangle\right) f_-^\dagger - \left(f'^{(2)}\left\langle f_+^\dagger\right\rangle - \left\langle f'^{(2)} f_+^\dagger\right\rangle\right) f_-\right] \tag{10.76}$$

Note that the collision integrals J_1' and J_1 differ in two respects. First, the collision integral J_1 vanishes after integration over the momentum directions while the integral J_1' does not:

$$\left\langle J_1'^{(2)}\right\rangle \neq 0$$

At the same time $J_1'^{(2)}$ vanishes for any function $f'^{(2)}$ independent of the momentum directions while $J_1^{(2)}$ does not.

10.4.1.2 Scattering by magnetic impurities Remind that the self energy in eqn (5.76) for magnetic impurities is

$$\check{\Sigma}_s(\mathbf{k}) = \frac{i}{2\tau_s}\left\langle \check{\tau}_3 \check{g}\left(\mathbf{p}_F, \mathbf{k}\right) \check{\tau}_3\right\rangle \tag{10.77}$$

The spin-flip collision integrals become

$$\begin{aligned} J_1^{(s)(1)} = &-\frac{1}{\tau_s}\left[\left(f^{(1)}\left\langle g_-\right\rangle - \left\langle f^{(1)} g_-\right\rangle\right) g_- + \frac{1}{2}\left(f^{(1)}\left\langle f_-^\dagger\right\rangle - \left\langle f^{(1)} f_-^\dagger\right\rangle\right) f_- \right. \\ &\left. + \frac{1}{2}\left(f^{(1)}\left\langle f_-\right\rangle - \left\langle f^{(1)} f_-\right\rangle\right) f_-^\dagger\right] \end{aligned} \tag{10.78}$$

$$\begin{aligned} J_2^{(s)(2)} = &-\frac{1}{\tau_s}\left[\left(f^{(2)}\left\langle g_-\right\rangle - \left\langle f^{(2)} g_-\right\rangle\right) g_- + \frac{1}{2}\left(f^{(2)}\left\langle f_+^\dagger\right\rangle - \left\langle f^{(2)} f_+^\dagger\right\rangle\right) f_+ \right. \\ &\left. + \frac{1}{2}\left(f^{(2)}\left\langle f_+\right\rangle - \left\langle f^{(2)} f_+\right\rangle\right) f_+^\dagger\right] \end{aligned} \tag{10.79}$$

and

$$J_1^{(s)(2)} = -\frac{1}{2\tau_s}\left[\left(f^{(2)}\left\langle f_-\right\rangle f_+^\dagger - \left\langle f^{(2)} f_+^\dagger\right\rangle f_-\right) - \left(f^{(2)}\left\langle f_+^\dagger\right\rangle f_+ - \left\langle f^{(2)} f_+\right\rangle f_-^\dagger\right)\right]$$
$$J_2^{(s)(1)} = -\frac{1}{2\tau_s}\left[\left(f^{(1)}\left\langle f_-^\dagger\right\rangle - \left\langle f^{(1)} f_-^\dagger\right\rangle\right) f_+ - \left(f^{(1)}\left\langle f_-\right\rangle - \left\langle f^{(1)} f_-\right\rangle\right) f_+^\dagger\right]$$

10.4.2 Electron-phonon collision integral

For small deviations from the equilibrium distribution function and for $\omega \ll \Delta$, we can consider only the terms linear in $f^{(2)}$. The electron-phonon self energies have the form of eqns (9.12) and (9.13). One obtains for the collision integrals:

$$\begin{aligned} J_1^{(\mathrm{ph})} = &-\frac{\pi\lambda}{16\left(sp_F\right)^2}\int \frac{d\epsilon'}{2}\left(\epsilon'-\epsilon\right)^2 \operatorname{sign}\left(\epsilon'-\epsilon\right)\left[2\left(g_{\epsilon'}^R - g_{\epsilon'}^A\right)\left(g_\epsilon^R - g_\epsilon^A\right)\right. \\ &\left. - \left(f_{\epsilon'}^R - f_{\epsilon'}^A\right)\left(f_\epsilon^{\dagger R} - f_\epsilon^{\dagger A}\right) - \left(f_\epsilon^R - f_\epsilon^A\right)\left(f_{\epsilon'}^{\dagger R} - f_{\epsilon'}^{\dagger A}\right)\right] \\ &\times\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\left(f_\epsilon^{(1)} - f_{\epsilon'}^{(1)}\right) - f_\epsilon^{(1)} f_{\epsilon'}^{(1)} + 1\right] \end{aligned} \tag{10.80}$$

Here $\lambda = \nu(0) g_{\rm ph}^2$ is the electron-phonon interaction constant. Terms linear in $f^{(2)}$ do not appear in $J_1^{(\rm ph)}$. The second collision integral contains both $f^{(1)}$ and $f^{(2)}$. It has a form $J_2^{(\rm ph)} = J_2^{(\rm ph)(1)} + J_2^{(\rm ph)(2)}$. The first part is

$$\begin{aligned} J_2^{(\rm ph)(2)} = & -\frac{\pi\lambda}{16\left(sp_F\right)^2}\int\frac{d\epsilon'}{2}\left(\epsilon'-\epsilon\right)^2 \operatorname{sign}\left(\epsilon'-\epsilon\right)\left[2\left(g_{\epsilon'}^R - g_{\epsilon'}^A\right)\left(g_{\epsilon}^R - g_{\epsilon}^A\right)\right. \\ & \left. -\left(f_{\epsilon'}^{R} + {}_{\epsilon'}^{A}\right)\left(f_{\epsilon}^{\dagger R} + f_{\epsilon}^{\dagger A}\right) - \left(f_{\epsilon}^{R} + f_{\epsilon}^{A}\right)\left(f_{\epsilon'}^{\dagger R} + f_{\epsilon'}^{\dagger A}\right)\right] \\ & \times\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\left(f_{\epsilon}^{(2)} - f_{\epsilon'}^{(2)}\right) - f_{\epsilon}^{(1)} f_{\epsilon'}^{(2)} - f_{\epsilon}^{(2)} f_{\epsilon'}^{(1)}\right] \end{aligned} \tag{10.81}$$

The function $f^{(1)}$ here should be considered equilibrium: $f^{(1)} = \tanh(\epsilon/2T)$. The part $J_2^{(\rm ph)(1)}$ only depends on $f^{(1)}$

$$\begin{aligned} J_2^{(\rm ph)(1)} = & -\frac{\pi\lambda}{16\left(sp_F\right)^2}\int\frac{d\epsilon'}{2}\left(\epsilon'-\epsilon\right)^2 \operatorname{sign}\left(\epsilon'-\epsilon\right) \\ & \times\left[\left(f_{\epsilon'}^{R} - f_{\epsilon'}^{A}\right)\left(f_{\epsilon}^{\dagger R} + f_{\epsilon}^{\dagger A}\right) - \left(f_{\epsilon}^{R} + f_{\epsilon}^{A}\right)\left(f_{\epsilon'}^{\dagger R} - f_{\epsilon'}^{\dagger A}\right)\right] \\ & \times\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\left(f_{\epsilon}^{(1)} - f_{\epsilon'}^{(1)}\right) - f_{\epsilon}^{(1)} f_{\epsilon'}^{(1)} + 1\right] \end{aligned} \tag{10.82}$$

The functions $\check{g}_\epsilon(\mathbf{p},\mathbf{r})$ and $\check{g}_{\epsilon'}(\mathbf{p},\mathbf{r})$ in eqns (10.80) - (10.82) have the same momentum since the relative change in $\mathbf{p}$ during interaction with the acoustic phonons is small.

We note that the collision integrals $J_1^{(\rm ph)}$ and $J_2^{(\rm ph)(1)}$ vanish for the equilibrium $f^{(1)} = \tanh(\epsilon/2T)$. In turn, the collision integral $J_2^{(\rm ph)(2)}$ vanishes for any function of the form of $f^{(2)} = \text{const}\left(df^{(0)}/d\epsilon\right)$; the latter corresponds to the invariance with respect to variations in the chemical potential.

10.4.2.1 Simplifications for $T \to T_c$ Near the critical temperature when $\Delta \ll T$, one usually needs the distribution functions for energies $\epsilon \sim \Delta$, i.e., for $\epsilon \ll T$, while the deviation from equilibrium $f_1 \equiv f^{(1)} - f^{(0)}$ is small for $\epsilon' \sim T$. The phonon collision integral can be simplified considerably in this case.

For $\Delta \ll T$, and for $\epsilon \ll T, \epsilon' \sim T$ we have $g^{R(A)} \approx \pm 1$ and $f^{R(A)} \ll 1$. As a result

$$\begin{aligned} J_1^{(\rm ph)} = & -\frac{\pi\lambda}{4\left(sp_F\right)^2}\int\frac{d\epsilon'}{2}\left(\epsilon'-\epsilon\right)^2 \operatorname{sign}\left(\epsilon'-\epsilon\right) \\ & \times\left(g_{\epsilon}^R - g_{\epsilon}^A\right)\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\left(f_{\epsilon}^{(1)} - f_{\epsilon'}^{(1)}\right) - f_{\epsilon}^{(1)} f_{\epsilon'}^{(1)} + 1\right] \\ = & -\frac{1}{\tau_{\rm ph}} g_- f_1 \end{aligned} \tag{10.83}$$

where the electron-phonon mean free time is

$$\frac{1}{\tau_{\rm ph}} = \frac{\pi\lambda}{\left(sp_F\right)^2}\int_0^\infty \frac{\epsilon^2 d\epsilon}{\sinh(\epsilon/T)} = \frac{7\zeta(3)\pi\lambda T^3}{2\left(sp_F\right)^2} \tag{10.84}$$

The second collision integral becomes

$$\begin{aligned} J_2^{(\rm ph)} = & -\frac{\pi\lambda}{4\left(sp_F\right)^2}\int\frac{d\epsilon'}{2}\left(\epsilon'-\epsilon\right)^2 \operatorname{sign}\left(\epsilon'-\epsilon\right) \\ & \times\left(g_{\epsilon}^R - g_{\epsilon}^A\right)\left[\coth\left(\frac{\epsilon'-\epsilon}{2T}\right)\left(f_{\epsilon}^{(2)} - f_{\epsilon'}^{(2)}\right) - f_{\epsilon}^{(2)} f_{\epsilon'}^{(1)} - f_{\epsilon}^{(1)} f_{\epsilon'}^{(2)}\right] \end{aligned}$$

$$= -\frac{1}{\tau_{\rm ph}} g_- \frac{df^{(0)}}{d\epsilon} \left[\chi_\epsilon^{(2)} - \chi_{\epsilon'}^{(2)} \right] \tag{10.85}$$

Here

$$f_\epsilon^{(2)} = \chi_\epsilon^{(2)} \frac{df^{(0)}}{d\epsilon} \tag{10.86}$$

The electron-phonon relaxation rate, eqn (10.84), is $\tau_{\rm ph}^{-1} \sim \lambda T^3/\Omega_D^2$. Usually the Debye frequency is higher than the critical temperature, $T_c \ll \Omega_D$ and also $\lambda \lesssim 1$. The electron-phonon relaxation rate is thus small compared to the order parameter for temperatures which are not very close to T_c : $\tau_{\rm ph}^{-1} \ll \Delta(T)$. High-temperature superconductors can be exceptions because the critical temperature is roughly of the same order as Ω_D. Nevertheless, experimentally, the inequality $\tau_{\rm ph}^{-1} \ll \Delta(T)$ is believed to hold. This should be especially correct for d-wave superconductors where it is crucial for the very existence of d-wave superconductivity as we have pointed out in Section 6.3.

10.4.3 Electron-electron collision integral

Self-energies for the electron-electron collisions can be obtained from the expressions established (Eliashberg 1971) for the electron-electron interaction in Section 8.3. The transformation to the quasiclassical functions is made according to the general rules described earlier. The only difference is that the self energy contains three G functions and only two integrations over the momenta. However, we can put

$$\check{G} = \pi i \check{g} \delta\left(\xi_{\mathbf{p}}\right)$$

because we are only interested in energies close to the Fermi surface. We obtain

$$\begin{aligned} \Sigma_1^R - \Sigma_1^A =& \frac{\nu^2 \pi i}{16 p_F v_F} \int d\epsilon_1 d\epsilon_2 \int \frac{d\Omega_{\mathbf{p}_1} d\Omega_{\mathbf{p}_2}}{(4\pi)^2} \delta\left(\frac{p_3}{p_F} - 1\right) \\ & \times \left[A \left\{ g_1 g_2 \bar{g}_3 \right\}^{(r)} - B \left\{ f_1 f_2^+ g_3 \right\}^{(r)} \right] \end{aligned} \tag{10.87}$$

$$\begin{aligned} \Sigma_2^R - \Sigma_2^A =& \frac{\nu^2 \pi i}{16 p_F v_F} \int d\epsilon_1 d\epsilon_2 \int \frac{d\Omega_{\mathbf{p}_1} d\Omega_{\mathbf{p}_2}}{(4\pi)^2} \delta\left(\frac{p_3}{p_F} - 1\right) \\ & \times \left[B \left\{ g_1 \bar{g}_2 f_3 \right\}^{(r)} - A \left\{ f_1 f_2 f_3^+ \right\}^{(r)} \right] \end{aligned} \tag{10.88}$$

and

$$\begin{aligned} \Sigma_1 =& \frac{\nu^2 \pi i}{16 p_F v_F} \int d\epsilon_1 d\epsilon_2 \int \frac{d\Omega_{\mathbf{p}_1} d\Omega_{\mathbf{p}_2}}{(4\pi)^2} \delta\left(\frac{p_3}{p_F} - 1\right) \\ & \times \left[A \left\{ g_1 g_2 \bar{g}_3 \right\}^{(t)} - B \left\{ f_1 f_2^+ g_3 \right\}^{(t)} \right] \end{aligned} \tag{10.89}$$

$$\begin{aligned} \Sigma_2 =& \frac{\nu^2 \pi i}{16 p_F v_F} \int d\epsilon_1 d\epsilon_2 \int \frac{d\Omega_{\mathbf{p}_1} d\Omega_{\mathbf{p}_2}}{(4\pi)^2} \delta\left(\frac{p_3}{p_F} - 1\right) \\ & \times \left[B \left\{ g_1 \bar{g}_2 f_3 \right\}^{(t)} - A \left\{ f_1 f_2 f_3^+ \right\}^{(t)} \right] \end{aligned} \tag{10.90}$$

Here A and B are the matrix elements of the interaction potential; different signs appear because of different spin structure of the matrix elements between the

states $\psi^\dagger$ and ψ as compared to those between the states ψ and ψ. The curly brackets are defined as

$$\begin{aligned}\{g_1 g_2 \bar{g}_3\}^{(r)} =& g_1^K g_2^K \left(\bar{g}_3^R - \bar{g}_3^A\right) + g_1^K \left(g_2^R - g_2^A\right) \bar{g}_3^K + \left(g_1^R - g_1^A\right) g_2^K \bar{g}_3^K \\ &+ \left(g_1^R - g_1^A\right)\left(g_2^R - g_2^A\right)\left(\bar{g}_3^R - \bar{g}_3^A\right) \\ \{g_1 g_2 \bar{g}_3\}^{(t)} =& g_1^K g_2^K \bar{g}_3^K + g_1^K \left(g_2^R - g_2^A\right)\left(\bar{g}_3^R - \bar{g}_3^A\right) \\ &+ \left(g_1^R - g_1^A\right) g_2^K \left(\bar{g}_3^R - \bar{g}_3^A\right) \\ &+ \left(g_1^R - g_1^A\right)\left(g_2^R - g_2^A\right) \bar{g}_3^K\end{aligned}$$

The energies and momenta satisfy the conservation laws:

$$\epsilon = \epsilon_1 + \epsilon_2 + \epsilon_3; \mathbf{p} = \mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 \tag{10.91}$$

For a point interaction $V(\mathbf{r}) = V\delta(\mathbf{r})$ one has

$$A = -|V|^2; \quad B = |V|^2.$$

The electron-electron interaction constant is defined as $\lambda_{ee} = \nu V$. We introduce the operator

$$\hat{L}[\ldots] = \frac{\lambda_{ee}^2 \pi}{2 p_F v_F} \int d\epsilon_1 d\epsilon_2 \int \frac{d\Omega_{\mathbf{p}_1} d\Omega_{\mathbf{p}_2}}{(4\pi)^2} \delta\left(\frac{p_3}{p_F} - 1\right)[\ldots] \tag{10.92}$$

Self energies become

$$\Sigma_1^R - \Sigma_1^A = -\frac{i}{8}\hat{L}\left[\{g_1 g_2 \bar{g}_3\}^{(r)} + \left\{f_1 f_2^+ g_3\right\}^{(r)}\right] \tag{10.93}$$

$$\Sigma_2^R - \Sigma_2^A = \frac{i}{8}\hat{L}\left[\{g_1 \bar{g}_2 f_3\}^{(r)} + \left\{f_1 f_2 f_3^+\right\}^{(r)}\right] \tag{10.94}$$

$$\Sigma_1 = -\frac{i}{8}\hat{L}\left[\{g_1 g_2 \bar{g}_3\}^{(t)} + \left\{f_1 f_2^+ g_3\right\}^{(t)}\right] \tag{10.95}$$

$$\Sigma_2 = \frac{i}{8}\hat{L}\left[\{g_1 \bar{g}_2 f_3\}^{(t)} + \left\{f_1 f_2 f_3^+\right\}^{(t)}\right] \tag{10.96}$$

We write down only one electron-electron collision integral

$$J_1^{(\mathrm{e-e})} = J_1^{(1)}\left\{f^{(1)}\right\} + J_1^{(2)}\left\{f^{(2)}\right\}$$

To calculate $J_1^{(1)}\left\{f^{(1)}\right\}$ we can put $f^{(2)} = 0$. Using that $\mathrm{Tr}\,\hat{I}^{R(A)} = 0$ we obtain

$$J_1^{(1)}\left\{f^{(1)}\right\} = -\hat{L}\left[M_1 g_-(\epsilon, \mathbf{p}) g_-(\epsilon_1, \mathbf{p}_1) g_-(\epsilon_2, \mathbf{p}_2) g_-(\epsilon_3, \mathbf{p}_3) F(\epsilon, \epsilon_1, \epsilon_2, \epsilon_3)\right] \tag{10.97}$$

where

$$\begin{aligned}M_1 = 1 &- \frac{f_-(1) f_-^+(2) + f_-^+(1) f_-(2)}{2 g_-(1) g_-(2)} + \frac{f_-(0) f_-^+(3) + f_-^+(0) f_-(3)}{2 g_-(0) g_-(3)} \\ &- \frac{f_-(1) f_-(2) f_-^+(3) f_-^+(0) + f_-^+(1) f_-^+(2) f_-(3) f_-(0)}{2 g_-(1) g_-(2) g_-(3) g_-(0)}\end{aligned}$$

and

$$\begin{aligned}F(\epsilon, \epsilon_1, \epsilon_2, \epsilon_3) =& f^{(1)}(0) f^{(1)}(1) f^{(1)}(2) + f^{(1)}(0) f^{(1)}(1) f^{(1)}(3) \\ &+ f^{(1)}(0) f^{(1)}(2) f^{(1)}(3) - f^{(1)}(1) f^{(1)}(2) f^{(1)}(3) \\ &+ f^{(1)}(0) - f^{(1)}(1) - f^{(1)}(2) - f^{(1)}(3)\end{aligned}$$

Here the index 0 refers to $\epsilon, \mathbf{p}$, while the index 1 is for $\epsilon_1, \mathbf{p}_1$, etc.
The equilibrium distribution function $f^{(0)} = \tanh(\epsilon/2T)$ turns the function $F(\epsilon, \epsilon_1, \epsilon_2, \epsilon_3)$ to zero. If we put now

$$f^{(1)} = f^{(0)} + \frac{\partial f^{(0)}}{\partial \epsilon}\chi$$

where χ is a small correction, we obtain

$$\begin{aligned} F = &\left(1 + f^{(0)}(1)f^{(0)}(2) + f^{(0)}(1)f^{(0)}(3) + f^{(0)}(2)f^{(0)}(3)\right) \\ &\times \frac{\partial f^{(0)}}{\partial \epsilon}(\chi(0) - \chi(1) - \chi(2) - \chi(3)) \\ = &\frac{1}{2T}\frac{[\chi(0) - \chi(1) - \chi(2) - \chi(3)]}{\cosh(\epsilon/2T)\cosh(\epsilon_1/2T)\cosh(\epsilon_2/2T)\cosh(\epsilon_3/2T)} \end{aligned}$$

18.3.5 10.5 Kinetic equations for dirty s-wave superconductors

To make an example, we apply our kinetic equations to dirty s-wave superconductors such that the mean free time satisfies the condition

$$\tau\Delta(T) \ll 1$$

The starting equations have the form of eqns (10.55) and (10.44). We write them again

$$\left(e\mathbf{v}_F\mathbf{E}g_- + \frac{1}{2}\left[f_-\frac{\hat{\partial}\Delta^*}{\partial t} + f_-^\dagger\frac{\hat{\partial}\Delta}{\partial t}\right]\right)\frac{\partial f^{(0)}}{\partial \epsilon} + \mathbf{v}_F\nabla\left(f_2 g_-\right) = J_1 \tag{10.98}$$

$$g_-\mathbf{v}_F\nabla f_1 + \frac{1}{2}\left[f_+\frac{\hat{\partial}\Delta^*}{\partial t} - f_+^\dagger\frac{\hat{\partial}\Delta}{\partial t}\right]\frac{\partial f^{(0)}}{\partial \epsilon} + i\left[\Delta^* f_+ + \Delta f_+^\dagger\right]f_2 = J_2 \tag{10.99}$$

The collision integrals are given by eqns (10.74)-(10.75) for scattering by impurities, and by eqns (10.80)-(10.82) for electron-phonon relaxation.

We use the method employed earlier in Sections 5.6 and 9.3 for superconductors with a short mean free path. Let us first average eqn (10.98) over the
directions of $\mathbf{v}_F$. The non-magnetic impurity collision integral vanishes and we obtain

$$\begin{aligned} &\left(\frac{1}{3}ev_F\left(\mathbf{E}\cdot\mathbf{g}_-\right) + \frac{1}{2}\left[\langle f_-\rangle\frac{\hat{\partial}\Delta^*}{\partial t} + \left\langle f_-^\dagger\right\rangle\frac{\hat{\partial}\Delta}{\partial t}\right]\right)\frac{\partial f^{(0)}}{\partial \epsilon} \\ &+ \nabla\cdot\langle\mathbf{v}_F\left(f_2 g_-\right)\rangle = \left\langle J_1^{(\mathrm{ph})}\right\rangle + \left\langle J_1^{(\mathrm{s})}\right\rangle \end{aligned} \tag{10.100}$$

This equation contains the vector component of $\mathbf{f}_2$ in the left-hand side and the average component of f_1 in the right-hand side. Now we multiply eqn (10.99) with $\mathbf{v}_F$ and average it:

$$v_F^2\langle g_-\rangle\nabla f_1 = 3\left\langle\mathbf{v}_F J_2^{(\mathrm{imp})}\right\rangle \tag{10.101}$$

Here we keep the largest collision term due to scattering by non-magnetic impurities. The vector components of regular functions in eqn (10.101) are neglected as compared to the averaged components. This equation also contains the averaged f and the vector $\mathbf{f}_2$.

The second set of equations for vector $\mathbf{f}_1$ and averaged f_2 components is obtained as follows. First, we average eqn (10.99):

$$\langle g_- \rangle \frac{v_F}{3} \nabla \mathbf{f}_1 + \frac{1}{2} \left[\langle f_+ \rangle \frac{\hat{\partial} \Delta^*}{\partial t} - \left\langle f_+^\dagger \right\rangle \frac{\hat{\partial} \Delta}{\partial t} \right] \frac{\partial f^{(0)}}{\partial \epsilon}$$
$$+ i \left[\Delta^* \langle f_+ \rangle + \Delta \left\langle f_+^\dagger \right\rangle \right] f_2 = \left\langle J_2^{(\mathrm{ph})} \right\rangle + \left\langle J_2^{(\mathrm{s})} \right\rangle \tag{10.102}$$

Next, we multiply eqn (10.98) with $\mathbf{v}_F$ and average it:

$$e v_F^2 \mathbf{E} \langle g_- \rangle \frac{\partial f^{(0)}}{\partial \epsilon} + v_F^2 \nabla \langle f_2 g_- \rangle = 3 \left\langle \mathbf{v}_F J_1^{(\mathrm{imp})} \right\rangle \tag{10.103}$$

The four equations (10.100), (10.101), (10.102), and (10.103) determine the four functions $f_1, f_2, \mathbf{f}_1$, and $\mathbf{f}_2$.

10.5.1 Small gradients without magnetic impurities

The regular functions are determined by the Usadel equations (5.98), (5.99). Assume that the order-parameter gradients are small $Dk^2 \ll \Delta$. We also assume that the electron-phonon relaxation is slow such that $\tau_{\mathrm{ph}}^{-1} \ll \Delta$ which is usually the case for temperatures not extremely close to T_c. The Usadel equations give the expressions for the Green functions as in a homogeneous case

$$\begin{aligned} \langle g_- \rangle &= \tfrac{\epsilon}{\sqrt{\epsilon^2 - |\Delta|^2}} \Theta \left(\epsilon^2 - |\Delta|^2 \right), \\ \langle f_- \rangle = \left\langle f_-^\dagger \right\rangle^* &= \tfrac{\Delta}{\sqrt{\epsilon^2 - |\Delta|^2}} \Theta \left(\epsilon^2 - |\Delta|^2 \right), \end{aligned} \tag{10.104}$$

and

$$\begin{aligned} \langle g_+ \rangle &= \tfrac{-i\epsilon}{\sqrt{|\Delta|^2 - \epsilon^2}} \Theta \left(|\Delta|^2 - \epsilon^2 \right), \\ \langle f_+ \rangle = \left\langle f_+^\dagger \right\rangle^* &= \tfrac{-i\Delta}{\sqrt{|\Delta|^2 - \epsilon^2}} \Theta \left(|\Delta|^2 - \epsilon^2 \right), \end{aligned} \tag{10.105}$$

in the leading approximation in small gradients. As a result, we have from eqn (10.75) for $\epsilon^2 > |\Delta|^2$

$$\left\langle \mathbf{v}_F J_2^{(\mathrm{imp})} \right\rangle = -\frac{1}{\tau_{\mathrm{imp}}} \langle \mathbf{v}_F (g_- f_2) \rangle \langle g_- \rangle + \frac{v_F}{3\tau_{\mathrm{imp}}} \mathbf{g}_- \langle g_- f_2 \rangle .$$

We can neglect the second term in the r.h.s. since the vector part of the Green function is small compared to the averaged one. Equation (10.101) gives

$$\langle \mathbf{v}_F (g_- f_2) \rangle = -D \nabla f_1 \tag{10.106}$$

Using eqn (10.100) we find

$$D \nabla^2 f_1 + \left\langle J_1^{(\mathrm{ph})} \right\rangle = \frac{1}{2} \left[\langle f_- \rangle \frac{\hat{\partial} \Delta^*}{\partial t} + \left\langle f_-^\dagger \right\rangle \frac{\hat{\partial} \Delta}{\partial t} \right] \frac{\partial f^{(0)}}{\partial \epsilon} \tag{10.107}$$

The vector part of the Green function is small compared to the averaged component and can be neglected.

Equation (10.102) should be used together with eqn (10.103) which determines the vector part $\mathbf{f}_1$. The impurity collision integral $J_1^{(2)}$ vanishes because $f_+ f_- = 0$ for slow gradients. The part $J_1^{(1)}$ gives

$$\mathbf{f}_1 = -e\ell \langle g_- \rangle \mathbf{E} \frac{\partial f^{(0)}}{\partial \epsilon} - \ell \nabla \langle g_- f_2 \rangle \tag{10.108}$$

The function f_2 is obtained from eqn (10.102) where one inserts $\mathbf{f}_1$ from eqn (10.108). Note that the Green functions $g_\pm$and $f_\pm$in eqn (10.102) should be taken with the proper account of the phonon scattering rate since the last term in the l.h.s. is proportional to $\tau_{\rm ph}^{-1}$. The so obtained function f_2 should then be used in the charge neutrality condition to find the equation which determines the scalar potential φ.

10.5.2 Heat conduction

Equation (10.106) allows one to calculate the thermal conductivity. Using the definition of the internal energy current, eqn (10.66), we have

$$\mathbf{j}_{\mathcal{E}} = -\nu(0) \int \epsilon \mathbf{v}_F \left(g^K + \bar{g}^K \right) \frac{d\epsilon}{4} \frac{d\Omega_{\mathbf{p}}}{4\pi} = -\nu(0) \int \epsilon \left\langle \mathbf{v}_F \left(g_- f_2 \right) \right\rangle d\epsilon$$

We find from eqn (10.106)

$$\mathbf{j}_{\mathcal{E}} = 2\nu(0) D \int_\Delta^\infty \epsilon \nabla f^{(1)} d\epsilon$$

Let us assume that the distribution function is $f^{(1)} = f^{(0)} = \tanh(\epsilon/2T)$ where T is a slow function of coordinates. The gradient becomes

$$\nabla f_1 = -\frac{df^{(0)}}{d\epsilon} \frac{\epsilon}{T} \nabla T$$

The heat current takes the form $\mathbf{j}_{\mathcal{E}} = -\kappa \nabla T$ where the thermal conductivity is

$$\kappa = 2\nu(0) D \int_\Delta^\infty \frac{df^{(0)}}{d\epsilon} \frac{\epsilon^2}{T} d\epsilon \tag{10.109}$$

This expression for the thermal conductivity was first derived by Bardeen et al. (1959).

18.4 THE TIME-DEPENDENT GINZBURG-LANDAU THEORY

We find out the conditions when the time dependent Ginzburg-Landau model can be justified microscopically. We demonstrate that the TDGL model is exact for gapless superconductors. It is not justified, however, for systems with a finite energy gap. The role of nonequilibrium excitations is elucidated in the dynamics of superconductors. The charge imbalance and decay of a d.c. electric field in a superconductor is discussed.

18.4.1 11.1 Gapless superconductors with magnetic impurities

We already discussed the time-dependent Ginzburg-Landau (TDGL) model which generalizes the usual Ginzburg-Landau (GL) theory and provides a simple description of nonstationary processes in superconductors (see Section 1.2.1). Consider now how and under which conditions, the microscopic theory can justify the phenomenological TDGL model.

We start with the simplest example which requires a minimum of calculations if we use the preparatory work done in the previous Chapters. This is the case of dirty superconductors which have a large concentration of magnetic impurities such that $\tau_s T_c \ll 1$ (Gor'kov and Eliashberg 1968). Note that this is exactly the condition of gapless superconductivity. In this case also $\tau_s \Delta \ll 1$. The regular functions are found from eqn (6.45):

$$g^R = -g^A = 1, f^R = \frac{\Delta}{\epsilon + i/\tau_s}, f^A = -\frac{\Delta}{\epsilon - i/\tau_s}, \tag{11.1}$$

within the first-order approximation in Δ and the zero-order approximation in spatial gradients. The characteristic energy scale in eqn (11.1) is $\epsilon \sim \tau_s^{-1}$. Therefore, the external frequency $\omega \sim \Delta$ is small compared to the energy scale of the regular Green functions. One can thus safely use the kinetic equations obtained by expansion in small ω. The collision integrals for magnetic scattering in eqns (10.100), (10.102) become

$$J_1^{(s)(1)} = -\frac{1}{\tau_s}\left(f_1 - \langle f_1 \rangle\right), J_1^{(s)(2)} = -\frac{1}{\tau_s}\left(f_2 - \langle f_2 \rangle\right)$$

All other terms are small in $\tau_s^2 \Delta^2$.
Kinetic equations (10.100), (10.102) become

$$f_- \frac{\partial \Delta}{\partial t} \frac{\partial f^{(0)}}{\partial \epsilon} = J_1^{(s)} \tag{11.2}$$

and

$$-i \frac{\partial f^{(0)}}{\partial \epsilon} \left(\frac{\partial \chi}{\partial t} + 2e\varphi \right) \Delta f_+ + 2i\Delta f_+ f_2 = J_2^{(s)} \tag{11.3}$$

The phonon collision integrals are small as compared to the magnetic scattering and can be neglected. We see that the corrections f_1 and f_2 to the equilibrium $f^{(0)}$ are proportional to $\tau_s^2 \Delta^2$ and are thus small as compared to the terms with $f^{(0)}$ in the Keldysh function eqn (10.62). Therefore

$$f^K_{\epsilon_+,\epsilon_-} = \left(f^R_{\epsilon_+,\epsilon_-} - f^A_{\epsilon_+,\epsilon_-}\right) f^{(0)}_\epsilon - \frac{i}{2}\left(\frac{\partial}{\partial t} + 2ie\phi\right)\left(f^R_\epsilon + f^A_\epsilon\right) \frac{\partial f^{(0)}}{\partial \epsilon} \tag{11.4}$$

As a result, the equation for the order parameter takes the form

$$\begin{aligned} \frac{\Delta(\mathbf{k},\omega)}{\lambda} = \int \frac{d\Omega_\mathbf{p}}{4\pi} \frac{d\epsilon}{4} \Big[& \left(f^R_{\epsilon_+,\epsilon_-} - f^A_{\epsilon_+,\epsilon_-}\right) f^{(0)}_\epsilon \\ & - \frac{i}{2}\left(\frac{\partial}{\partial t} + 2ie\varphi\right)\left(f^R_\epsilon + f^A_\epsilon\right) \frac{\partial f^{(0)}}{\partial \epsilon} \Big] \end{aligned} \tag{11.5}$$

It is instructive to compare this equation with eqn (1.85). These two equations coincide if one replaces the imaginary part i/τ_s in eqn (11.1) for the functions $f^{R(A)}$ with an infinitely small $i\delta$. However, the expression (11.4) for the Keldysh function which has resulted in eqn (11.5) is not exact; it requires that the deviations from equilibrium $f_{1,2}$ are small. It is the gapless condition $\Delta\tau_s \ll 1$ that makes $f_{1,2}$ small. We thus confirm the conclusion of Section 1.2.2 that the TDGL scheme works only for gapless superconductors when deviations from equilibrium are negligible.

Equation (11.5) yields

$$\begin{aligned} \frac{\Delta(\mathbf{k},\omega)}{\lambda} = & \int \frac{d\Omega_\mathbf{p}}{4\pi} \frac{d\epsilon}{4} \left(f^R_\epsilon - f^A_\epsilon\right) f^{(0)}_\epsilon \\ & - \left(\frac{\partial}{\partial t} + 2ie\varphi\right) \Delta \int \frac{d\epsilon}{4\tau_s\left(\epsilon^2 + 1/\tau_s^2\right)} \frac{\partial f^{(0)}}{\partial \epsilon} \end{aligned} \tag{11.6}$$

Since $T \ll 1/\tau_s$, the derivative of the distribution function becomes $\partial f^{(0)}/\partial\epsilon = 2\delta(\epsilon)$. The last term in eqn (11.6) gives

$$\left(\frac{\partial}{\partial t} + 2ie\varphi\right)\Delta \int \frac{d\epsilon}{4\tau_s\left(\epsilon^2 + 1/\tau_s^2\right)}\frac{\partial f^{(0)}}{\partial\epsilon} = \frac{\tau_s}{2}\left(\frac{\partial}{\partial t} + 2ie\varphi\right)\Delta$$

The first term in the r.h.s. of eqn (11.6) gives the stationary part of the Ginzburg-Landau equation (Abrikosov and Gor'kov 1960). We have from eqn (11.6)

$$\left(\frac{\partial}{\partial t} + 2ie\varphi\right)\Delta = \frac{\tau_s}{3}\left[\pi^2\left(T_c^2 - T^2\right) - \frac{|\Delta|^2}{2}\right]\Delta + D\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)^2\Delta. \tag{11.7}$$

The expression for the current is obtained in exactly the same way as before. We have for the vector part of the distribution function from eqn (10.103)

$$\mathbf{f}_1 = -ev_F\tau\frac{\partial f^{(0)}}{\partial\epsilon}\mathbf{E}$$

Thus the normal current is $\mathbf{j}_n = \sigma_n\mathbf{E}$. The total current becomes

$$\mathbf{j} = \sigma_n\mathbf{E} - \frac{2\sigma_n\tau_s|\Delta|^2}{c}\mathbf{Q} \tag{11.8}$$

One can easily see that the system of equations (11.7) and (11.8) can be written in the form of general equations of relaxation dynamics, eqns (1.65), (1.67) used to derive the TDGL model in Section 1.2.1 with the superconducting free energy

$$\begin{aligned}\mathcal{F}_{sn} =& \nu(0)\int\left(\frac{\tau_s^2}{6}\left[\pi^2\left(T^2 - T_c^2\right)|\Delta|^2 + \frac{|\Delta|^4}{4}\right]\right.\\ &\left.+\frac{D\tau_s}{2}\left|\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\Delta\right|^2\right)dV\end{aligned}$$

The order-parameter relaxation constant as defined in Section 1.2.1 is

$$\Gamma = \nu(0)\tau_s/2$$

The relaxation times are

$$\tau_j = \frac{1}{4\pi^2\tau_s\left(T_c^2 - T^2\right)}, \tau_\Delta = \frac{3}{\pi^2\tau_s\left(T_c^2 - T^2\right)}$$

Their ratio $u = 12$.

We conclude therefore that gapless superconductors with large concentration of magnetic impurities can be exactly described by the TDGL model. The reason is that the distribution of excitations is very close to equilibrium. It is the small parameter $\Delta\tau_s$ which makes the deviation from equilibrium negligible. We shall see in the next example, that the TDGL model starts to fail as the deviation from equilibrium increases.

18.4.2 11.2 Generalized TDGL equations

The example which we discuss here is more interesting and more realistic in a sense that it does not require strong limitations necessary for gapless superconductivity. Of course, we need some restrictions to simplify the general kinetic equations. Consider a dirty superconducting alloy with the mean free path $\ell \ll \xi_0$ or $T_c\tau_{\text{imp}} \ll 1$. We assume that the temperature is close to T_c. Moreover, we assume that variations in space and time are slow:

$$\omega, Dk^2 \ll \tau_{\rm ph}^{-1} \tag{11.9}$$

In our example, the inelastic scattering by phonons is of major importance. Since the largest wave vector is $k^2 \sim \xi^{-2}(T)$ and $D\xi^{-2}(T) \propto \Delta^2/T_c$, the condition of slow spatial variations is certainly satisfied if $T_c - T \ll \tau_{\rm ph}^{-1}$. The gradients can satisfy eqn (11.9) also because the characteristic scale of the particular problem may happen to be large compared to $\xi(T)$ for one or another reason; the condition $T_c - T \ll \tau_{\rm ph}^{-1}$ can be lifted in this case. As a reward to the restriction of eqn (11.9) we obtain a possibility to derive a closed set of time dependent equations for the order parameter magnitude and phase which resemble the TDGL model. On the other hand, they are more general and allow us to follow the crossover from the simple TDGL scheme to a more realistic dynamics for systems with strong deviations from equilibrium. The TDGL-like equations in this case were derived by Watts-Tobin et al. (1981).

Consider first the regular Green functions. Within the leading approximation in spatial gradients and in the external frequency, one has from Eilenberger equation (5.84)

$$-\epsilon f^{R(A)} + \Delta g^{R(A)} = \Sigma_1^{R(A)} f^{R(A)} - g^{R(A)} \Sigma_2^{R(A)}. \tag{11.10}$$

The phonon self energies are

$$\begin{aligned}\Sigma_1^{R(A)} =& \frac{i\pi\lambda}{4\left(sp_F\right)^2} \int_{-\infty}^{\infty} \left(\epsilon' - \epsilon\right)^2 \operatorname{sign}\left(\epsilon' - \epsilon\right) \frac{d\epsilon'}{2} \\ & \times \left[\coth\left(\frac{\epsilon' - \epsilon}{2T}\right) g_{\epsilon'}^{R(A)} \mp \frac{1}{2} f_{\epsilon'}^{(0)} \left(g_{\epsilon'}^{R} - g_{\epsilon'}^{A}\right)\right]\end{aligned} \tag{11.11}$$

and

$$\begin{aligned}\Sigma_2^{R(A)} =& \frac{i\pi\lambda}{4\left(sp_F\right)^2} \int_{-\infty}^{\infty} \left(\epsilon' - \epsilon\right)^2 \operatorname{sign}\left(\epsilon' - \epsilon\right) \frac{d\epsilon'}{2} \\ & \times \left[\coth\left(\frac{\epsilon' - \epsilon}{2T}\right) f_{\epsilon'}^{R(A)} \mp \frac{1}{2} f_{\epsilon'}^{(0)} \left(f_{\epsilon'}^{R} - f_{\epsilon'}^{A}\right)\right]\end{aligned} \tag{11.12}$$

Energies $\epsilon' \sim T$ dominate in the integrals in eqns (11.11) and (11.12). At these energies, the functions $g^{R(A)} \approx \pm 1$ and $f^{R(A)} \sim \Delta/T \ll 1$, therefore,

$$\Sigma_1^{R(A)} = \pm \frac{i}{2\tau_{\rm ph}}$$

and $\Sigma_2 \sim (\Delta/T_c)\,\Sigma_1 \ll \Sigma_1$, where the electron-phonon mean free time $\tau_{\rm ph}$ is defined by eqn (10.84).

Equation (11.10) gives

$$\Delta g^{R(A)} - \left(\epsilon \pm \frac{i}{2\tau_{\rm ph}}\right) f^{R(A)} = 0$$

Using the normalization $\left[g^{R(A)}\right]^2 - f^{R(A)} f^{\dagger R(A)} = 1$ we obtain

$$g^{R(A)} = \pm \frac{\epsilon \pm i/2\tau_{\rm ph}}{\sqrt{\left(\epsilon \pm i/2\tau_{\rm ph}\right)^2 - \Delta^2}}, f^{R(A)} = \pm \frac{\Delta}{\sqrt{\left(\epsilon \pm i/2\tau_{\rm ph}\right)^2 - \Delta^2}} \tag{11.13}$$

We choose Δ to be real for brevity at this point, and put $f^{R(A)} = f^{\dagger R(A)}$ within the leading approximation.

Kinetic equations (10.100) takes the form

$$f_- \frac{\partial \Delta}{\partial t} \frac{\partial f^{(0)}}{\partial \epsilon} = J_1^{(\mathrm{ph})} \tag{11.14}$$

whence

$$f_1 = -\tau_{\mathrm{ph}} \frac{\partial \Delta}{\partial t} \frac{\partial f^{(0)}}{\partial \epsilon} \frac{f_-}{g_-} \tag{11.15}$$

Equation (10.102) becomes

$$-i \frac{\partial f^{(0)}}{\partial \epsilon} \left(\frac{\partial \chi}{\partial t} + 2e\varphi \right) \Delta f_+ + 2i\Delta f_+ f_2 = J_2^{(\mathrm{ph})} \tag{11.16}$$

We can neglect the spatial derivative in eqn (11.16) because it is small as compared to the other terms when $\epsilon \sim \Delta, \tau_{\mathrm{ph}}^{-1}$. Fortunately, it is this range of ϵ which gives the main contribution to the equation for the order parameter. Indeed, for higher energies $\epsilon \sim T$, the spatial gradient could not be ignored any more since the other terms in eqn (11.16) become also small. However, as we shall see in a moment, the function f_2 decreases rapidly as ϵ increases. It becomes small for $\epsilon \sim T$ so that the energy range $\epsilon \sim T$ does not contribute much to the integrals which contain f_2. Equation (11.16) yields after a little of algebra

$$f_2 = \frac{\partial f^{(0)}}{\partial \epsilon} e\Phi + f_2', f_2' = C \frac{\partial f^{(0)}}{\partial \epsilon} \left(1 - \frac{\Delta}{\epsilon} \frac{f_-}{g_-} \right) \tag{11.17}$$

where C is independent of ϵ. In contrast to f_2, the function f_2' is not expected to decay for $\epsilon \sim T$. Equation (11.17) has been derived using the argumentation as follows. We note that the first term in f_2 satisfies eqn (11.16) exactly while f_2' transforms into $C\partial f^{(0)}/\partial \epsilon$ for energies $\epsilon \sim T$ and larger. Since energies $\epsilon' \sim T$ dominate in the collision integral the expression (10.85) for J_2 takes the form

$$J_2 = -\frac{g_-}{\tau_{\mathrm{ph}}} \left[f_2'(\epsilon) - C \frac{\partial f^{(0)}}{\partial \epsilon} \right]$$

Remind that it is assumed $\epsilon \ll T$. Inserting this into eqn (11.16) we arrive at eqn (11.17).

The constant C is to be found from the charge neutrality condition. The change in the electron density eqn (10.71) due to nonequilibrium processes is

$$\delta N^{(\mathrm{nst})} = -\nu(0) e\Phi - \nu(0) \int_{-\infty}^{\infty} \langle g_- f_2' \rangle \, d\epsilon \tag{11.18}$$

The change in the electron density should be zero. The second term in the r.h.s. of eqn (11.18) is determined by $\epsilon \sim T$. For these energies, $g_- \approx 1$. Using eqn (11.17) we find

$$C = -e\Phi$$

Therefore

$$f_2 = e\Phi \frac{\partial f^{(0)}}{\partial \epsilon} \frac{\Delta}{\epsilon} \frac{f_-}{g_-} \tag{11.19}$$

The function f_2 decreases with ϵ and becomes small for $\epsilon \gg \Delta$. This behavior is exactly what we expect for f_2.

Consider now the self-consistency equation for the order parameter. We have

$$\frac{\Delta(\mathbf{k},\omega)}{\lambda} = \int \frac{d\Omega_{\mathbf{p}}}{4\pi}\frac{d\epsilon}{4} f^K_{\epsilon_+,\epsilon_-}$$

The Keldysh function in eqn (10.62) is

$$f^K_{\epsilon_+,\epsilon_-} = \left(f^R_{\epsilon_+,\epsilon_-} - f^A_{\epsilon_+,\epsilon_-}\right) f^{(0)}_\epsilon + f^{(\mathrm{nst})}_{\epsilon_+,\epsilon_-} \tag{11.20}$$

where the non-stationary part is

$$\begin{aligned} f^{(\mathrm{nst})}_{\epsilon_+,\epsilon_-} = &-\frac{i}{2}\left(\frac{\partial}{\partial t} + 2ie\varphi\right)\left(f^R_\epsilon + f^A_\epsilon\right)\frac{\partial f^{(0)}}{\partial \epsilon} \\ &+ \left(f^R_\epsilon - f^A_\epsilon\right) f_1 - \left(f^R_\epsilon + f^A_\epsilon\right) f_2 \end{aligned} \tag{11.21}$$

For the energy integration, we put

$$x = \frac{1}{2}\left[\sqrt{\left(\epsilon + \frac{i}{2\tau_{\mathrm{ph}}}\right)^2 - \Delta^2} + \sqrt{\left(\epsilon - \frac{i}{2\tau_{\mathrm{ph}}}\right)^2 - \Delta^2}\right]. \tag{11.22}$$

The two radicals are determined as analytical functions on the complex plane of ϵ with the cuts connecting $-\Delta \mp i/2\tau_{\mathrm{ph}}$ and $\Delta \mp i/2\tau_{\mathrm{ph}}$, respectively (compare with Fig. 5.1). The variable x is real. It increases from $-\infty$ to $+\infty$ as ϵ varies from $-\infty$ to $+\infty$ along the real axis. It is convenient to use the identities

$$\frac{dx}{d\epsilon} = g_-, \epsilon = x\sqrt{\frac{x^2 + \Delta^2 + 1/4\tau^2_{\mathrm{ph}}}{x^2 + 1/4\tau^2_{\mathrm{ph}}}}$$

and

$$\begin{aligned} \left(\frac{f_-}{g_-}\right)^2 &= \frac{x^2\Delta^2}{\left(x^2 + 1/4\tau^2_{\mathrm{ph}}\right)\left(x^2 + \Delta^2 + 1/4\tau^2_{\mathrm{ph}}\right)} \\ \frac{f_+}{g_-} &= -i\frac{\Delta/2\tau_{\mathrm{ph}}}{x^2 + 1/4\tau^2_{\mathrm{ph}}} \end{aligned}$$

which follow from eqn (11.22). Now it is easy to perform the integration. For example,

$$\begin{aligned} &\int_{-\infty}^{\infty}\frac{d\epsilon}{4}\left[-\frac{i}{2}\frac{\partial}{\partial t}\left(f^R_\epsilon + f^A_\epsilon\right)\frac{\partial f^{(0)}}{\partial\epsilon} + \left(f^R_\epsilon - f^A_\epsilon\right) f_1\right] \\ &= -\frac{1}{4}\frac{\partial\Delta}{\partial t}\int_{-\infty}^{\infty} 2\tau_{\mathrm{ph}}\frac{\partial f^{(0)}}{\partial\epsilon}\frac{\Delta^2 + 1/4\tau^2_{\mathrm{ph}}}{x^2 + \Delta^2 + 1/4\tau^2_{\mathrm{ph}}}dx \\ &= -\frac{\pi}{8T_c}\sqrt{1 + 4\tau^2_{\mathrm{ph}}\Delta^2}\frac{\partial\Delta}{\partial t} \end{aligned}$$

Here we use that the characteristic $x \ll T_c$. Similarly,

$$\begin{aligned} &\int_{-\infty}^{\infty}\frac{d\epsilon}{4}\left[\frac{1}{2}\left(\frac{\partial\chi}{\partial t} + 2e\varphi\right)\left(f^R_\epsilon + f^A_\epsilon\right)\frac{\partial f^{(0)}}{\partial\epsilon} - \left(f^R_\epsilon + f^A_\epsilon\right) f_2\right] \\ &= -\frac{ie\Phi\Delta}{8\tau_{\mathrm{ph}}}\int_{-\infty}^{\infty}\frac{1}{x^2 + \Delta^2 + 1/4\tau^2_{\mathrm{ph}}}dx = -\frac{\pi}{8T_c}\frac{2ie\Phi\Delta}{\sqrt{1 + 4\tau^2_{\mathrm{ph}}\Delta^2}} \end{aligned}$$

The stationary part of the Keldysh function defined by eqn (11.20) gives the usual Ginzburg-Landau contribution. Here we note that

$$\int_{-\infty}^{\infty} \frac{d\Omega_{\mathbf{p}}}{4\pi} \frac{d\epsilon}{4} \left(f^R_{\epsilon_+,\epsilon_-} - f^A_{\epsilon_+,\epsilon_-} \right) f^{(0)}_{\epsilon}$$

is even in ω, therefore, one can use here only the zero-order terms in ω. Finally, we obtain

$$\begin{aligned} &\frac{\pi}{8T_c} \left[\sqrt{1+4|\Delta|^2\tau_{\rm ph}^2} \frac{\partial|\Delta|}{\partial t} + \frac{2ie\Phi|\Delta|}{\sqrt{1+4|\Delta|^2\tau_{\rm ph}^2}} \right] \\ =&\frac{\pi D}{8T_c} \left(\nabla - \frac{2ie\mathbf{Q}}{c} \right)^2 |\Delta| + \frac{T_c - T}{T_c}|\Delta| - \frac{7\zeta(3)}{8\pi^2 T_c^2}|\Delta|^3. \end{aligned} \tag{11.23}$$

Here we restore the complex-valuedness of the order parameter.
Expression for the current is

$$\mathbf{j}(\omega, \mathbf{k}) = -e\nu(0) \int \frac{d\Omega_{\mathbf{p}}}{4\pi} \frac{d\epsilon}{4} \mathbf{v}_F \left(g^K_{\epsilon_+,\epsilon_-} - \bar{g}^K_{\epsilon_+,\epsilon_-} \right)$$

In the leading approximation in Δ, eqn (10.62) gives

$$g^K_{\epsilon_+,\epsilon_-} - \bar{g}^K_{\epsilon_+,\epsilon_-} = 4g_- f^{(0)} + 4g_- f_1$$

The stationary part of the Keldysh function eqn (11.20) gives the supercurrent. The vector part of the distribution function $f_1 = \langle f_1 \rangle + \hat{\mathbf{v}}_F \mathbf{f}_1$ is found from eqn (10.108). The characteristic frequencies in the integral for the current are $\epsilon \sim T$, therefore, one can neglect the function f_2 and put $g_- = 1$. Therefore

$$\mathbf{f}_1 = -e\ell \frac{\partial f^{(0)}}{\partial \epsilon} \mathbf{E}$$

where $\ell = v_F \tau$. The total current becomes

$$\mathbf{j} = \mathbf{j}_n + \mathbf{j}_s$$

Here the supercurrent has its usual form

$$\mathbf{j}_s = -\frac{\pi\sigma_n|\Delta|^2}{2cT} \mathbf{Q} \tag{11.24}$$

The normal current is again $\mathbf{j}_n = \sigma_n \mathbf{E}$.
Note that the generalized TDGL equation (11.23) can not be presented in the form of eqns (1.65), (1.67) because the relaxation of the order parameter magnitude is governed by a different process compared to the relaxation of the order-parameter phase. Nevertheless, the real part of eqn (11.23) can be separately written as

$$\frac{\pi}{4T} \sqrt{1+4|\Delta|^2\tau_{\rm ph}^2} \frac{\partial|\Delta|}{\partial t} = -\frac{\delta \mathcal{F}_{sn}}{\delta|\Delta|} \tag{11.25}$$

where we use expression (6.30) for the free energy of a dirty superconductor.
Relaxation of the order parameter phase determines conversion of normal current into a supercurrent and penetration of the electric field into a superconductor. Using eqns (11.23), (11.24), and the charge neutrality condition $\operatorname{div} \mathbf{j} = 0$ we can obtain the equation for gauge invariant potential Φ. The imaginary part of eqn (11.23) gives

$$\operatorname{div} \mathbf{j}_s = \frac{\pi\nu(0)e^2}{T} \frac{\Phi|\Delta|^2}{\sqrt{1+4|\Delta|^2\tau_{\rm ph}^2}} \tag{11.26}$$

From the charge neutrality we find

$$-\sigma\left(\frac{1}{c}\operatorname{div}\frac{\partial\mathbf{Q}}{\partial t}+\nabla^2\Phi\right)+\frac{\pi\nu(0)e^2}{T}\frac{\Phi|\Delta|^2}{\sqrt{1+4|\Delta|^2\tau_{\rm ph}^2}}=0 \tag{11.27}$$

This equation determines the relaxation length for the potential Φ, i.e., the electric field penetration length. For a constant Δ,

$$l_E^2=\frac{2DT}{\pi|\Delta|^2}\sqrt{1+4|\Delta|^2\tau_{\rm ph}^2} \tag{11.28}$$

The penetration length is $l_E\sim\xi(T)\left(1+4|\Delta|^2\tau_{\rm ph}^2\right)^{1/4}$; it can be much longer than $\xi(T)$ when $|\Delta|\tau_{\rm ph}\gg 1$. Equation (11.28) was first derived by Tinkham (1972), Schmid and Schön (1975), and by Artemenko and Volkov (1975) for $|\Delta|\tau_{\rm ph}\gg 1$. We discuss this phenomenon in more detail later in Section 11.4.

Equation (11.23) transforms into the usual TDGL equation when $\Delta\tau_{\rm ph}\ll 1$. This limit corresponds to the gapless superconductivity where the pair breaking is due to the inelastic scattering by phonons. One can see this from comparing eqn (11.13) with eqn (11.1) obtained for magnetic impurities. The phenomenological
relaxation parameter in this limit is defined by eqn (1.86). The relaxation time for current defined in Section 1.2.1 is

$$\tau_j=\frac{7\zeta(3)}{4\pi^3T_c}\left(1-\frac{T}{T_c}\right)^{-1} \tag{11.29}$$

The order-parameter relaxation time is

$$\tau_\Delta=\frac{\pi}{8T_c}\left(1-\frac{T}{T_c}\right)^{-1}$$

Ratio of the two times is

$$u=\frac{\pi^4}{14\zeta(3)}\approx 5.79$$

In a more realistic limit one has, however, $\Delta\tau_{\rm ph}\gg 1$. Equation (11.23) does not have a TDGL form any more. Indeed, the relaxation time for the order parameter becomes Δ-dependent:

$$\tau_\Delta=\frac{\pi|\Delta|\tau_{\rm ph}}{4\left(T_c-T\right)}$$

Moreover, it is much longer than the TDGL time and has a tendency to approach $\tau_{\rm ph}$ as temperature decreases away from T_c. A long order-parameter relaxation time means that the order parameter in a rapidly oscillating external field is unable to follow the instantaneous magnitude of the field. In fact, it oscillates slightly near an average value, the latter being determined by eqn (11.23) averaged over time. After averaging over fast oscillations, the time derivative drops out; on the other hand, one can replace $|\Delta|$ with its averaged value.

18.4.3 11.3 TDGL theory for d-wave superconductors

The unusual d-wave symmetry affects both thermodynamic and dynamic properties of superconductors. However, d-wave superconductors are expected to display more "conventional" type of behavior in the region of temperatures close to the critical temperature. From a thermodynamical point of view, this is the temperature range where the Ginzburg-Landau theory

is applicable for both s and d-wave superconductors. As far as dynamics is concerned, d-wave superconductors appear to be even more simple than the usual s-wave superconductors. Indeed, for s-wave superconductors, in addition to solving the equation for the order parameter one has to take care of nonequilibrium excitations which become extremely important due to a singular density of states near the energy gap. As we already know, a comparatively simple time-dependent Ginzburg-Landau (TDGL) theory is only available for several particular situations. The most important example is the gapless superconductivity realized in materials with large enough concentration of magnetic impurities. The energy gap is suppressed due to a spin-dependent scattering which breaks spin-singlet Cooper pairs. d-wave superconductors with non-magnetic impurities are in a sense similar to s-wave
superconductors with magnetic impurities: the energy gap is destroyed due to breaking of the momentum coherence of a d-wave Cooper pair by scattering at impurity atoms (see Section 6.3). In the limit $\tau\Delta(T) \ll 1$, all memory of the energy gap disappears completely. This is where one can expect a comparatively simple nonequilibrium dynamics.

The "gapless" behavior is a natural high-temperature limit for d-wave superconductors with impurities in contrast to s-wave superconductors where it is quite an exotic situation. The d-wave superconductivity can only exist in clean compounds with the impurity mean free path ℓ much larger than the zerotemperature coherence length ξ_0; this is equivalent to the condition $\tau_{\text{imp}}T_c \gg 1$. The "gapless" regime is thus realized in close vicinity of T_c where $1 - T/T_c \ll (T_c\tau_{\text{imp}})^{-2}$. In a sense, d-wave superconductors in close vicinity of the critical temperature display a much simpler dynamics than s-wave superconductors near T_c. Nevertheless, it is more rich than the conventional TDGL theory.

In the present Section we demonstrate that the general nonstationary microscopic theory of superconductivity in this temperature range can be reduced to a set of equations resembling the usual TDGL equations with one extra term responsible for relaxation of nonequilibrium excitations (Kopnin 1998 a). In an inhomogeneous situation, relaxation is diffusion dominated. If the diffusion occurs at distances of the order of the coherence length $\xi(T)$, the relaxation is comparatively fast and results in a dissipation of the same order of magnitude as in the usual TDGL theory. The extra term becomes more important when the diffusion slows down, i.e., when the scale of spatial variations is large. In a homogeneous situation, the extra term is determined by yet slower inelastic relaxation and thus can reach rather high values. The overall behavior is reminiscent of an s-wave superconducting alloy with a small concentration of magnetic impurities considered by Eliashberg (1968). To compare the results, one needs to replace the spin-flip relaxation time τ_s with $2\tau_{\text{imp}}$.

We start our derivation with the general self-consistency equation for the d-wave order parameter, eqn (5.29). For a time dependent problem, it has the form

$$\frac{\Delta_{\mathbf{p}}(\mathbf{k},\omega)}{\lambda} = \int \frac{d\epsilon}{4\pi i} \left\langle V_d\left(\hat{\mathbf{p}},\hat{\mathbf{p}}_1\right) f^K_{\epsilon_+,\epsilon_-}\left(\mathbf{p}_{1+},\mathbf{p}_{1-}\right)\right\rangle_{\mathbf{p}_1} \tag{11.30}$$

The angle brackets, as usually, denote an average over the Fermi surface according to eqn (5.73) having in mind that, in d-wave superconductors, the Fermi surface may be non-spherical. The d-wave pairing potential $V_d\left(\hat{\mathbf{p}},\hat{\mathbf{p}}_1\right)$ is normalized in such a way that

$$\left\langle V_d\left(\hat{\mathbf{p}},\hat{\mathbf{p}}_1\right)\Delta_{\mathbf{p}_1}\right\rangle_{\mathbf{p}_1} = \Delta_{\mathbf{p}} \tag{11.31}$$

To find the regular quasiclassical Green functions we use the Eilenberger equations

$$-i\mathbf{v}_F\left(\nabla - \frac{2ie}{c}\mathbf{A}\right) f_\epsilon - 2\epsilon f_\epsilon + 2\Delta_{\mathbf{p}} g_\epsilon - 2\Sigma_1 f_\epsilon + 2g_\epsilon\Sigma_2 = 0 \tag{11.32}$$

$$i\mathbf{v}_F\left(\nabla + \frac{2ie}{c}\mathbf{A}\right) f^\dagger_\epsilon - 2\epsilon f^\dagger_\epsilon + 2\Delta^*_{\mathbf{p}} g_\epsilon - 2\Sigma_1 f^\dagger_\epsilon + 2g_\epsilon\Sigma^\dagger_2 = 0 \tag{11.33}$$

for both retarded and advanced functions. The impurity self energies are

$$\Sigma_1 = \frac{i}{2\tau_{\text{imp}}}\langle g\rangle, \Sigma_2 = \frac{i}{2\tau_{\text{imp}}}\langle f\rangle \tag{11.34}$$

For the distribution function we use the kinetic equations (10.98) and (10.99) in the "dirty limit" since we assume that $\Delta\tau_{\text{imp}} \ll 1$. The collision integrals may include both impurity and inelastic (electron-phonon) scattering $J = J^{(\text{imp})} + J^{(\text{ph})}$. We assume that the inelastic mean free time is much larger that the impurity scattering time, $\tau_{\text{ph}} \gg \tau_{\text{imp}}$.

We consider here the so called "gapless" limit which is realized at temperatures close enough to the critical temperature. The corresponding condition is $\tau_{\text{imp}} T_c \left(1 - T/T_c\right)^{1/2} \ll 1$; this is equivalent to $\ell \ll \xi(T)$. In this gapless limit, one can perform expansion in the small order parameter $\Delta\tau_{\text{imp}} \ll 1$ as well as in small gradients and in the small vector potential. Since $\langle\Delta_{\mathbf{p}}\rangle = 0$, the Eilenberger equations in the first approximation give $g^{R(A)} = \pm 1$ and

$$f_\epsilon^{R(A)} = \pm\frac{\Delta_{\mathbf{p}}}{\epsilon \pm i/2\tau_{\text{imp}}}, f_\epsilon^{\dagger R(A)} = \pm\frac{\Delta_{\mathbf{p}}^*}{\epsilon \pm i/2\tau_{\text{imp}}} \tag{11.35}$$

In case of a small admixture of an s-wave component into the order parameter, eqn (11.35) is valid as long as $|\langle\Delta_{\mathbf{p}}\rangle| \ll |\Delta_{\mathbf{p}}|$.

The impurity collision integrals become

$$J_1^{(\text{imp})} = -\frac{1}{\tau_{\text{imp}}}\left(f_1 - \langle f_1\rangle\right), J_2^{(\text{imp})} = -\frac{1}{\tau_{\text{imp}}}\left(f_2 - \langle f_2\rangle\right) \tag{11.36}$$

The inelastic collision integral in eqn (10.98) is $J_1^{(\text{ph})} = -f_1/\tau_{\text{ph}}$.
We put

$$f_1 = \langle f_1\rangle + \tilde{f}_1, f_2 = \langle f_2\rangle + \tilde{f}_2 \tag{11.37}$$

where $\tilde{f}_1 \ll \langle f_1\rangle$ and $\tilde{f}_2 \ll \langle f_2\rangle$, and obtain from eqn (10.101) and (10.103)

$$\tilde{f}_1 = -\tau_{\text{imp}} e\mathbf{v}_F \cdot \mathbf{E}\frac{df^{(0)}}{d\epsilon} - \tau_{\text{imp}}\mathbf{v}_F \cdot \nabla\langle f_2\rangle, \tilde{f}_2 = -\tau_{\text{imp}}\mathbf{v}_F \cdot \nabla\langle f_1\rangle \tag{11.38}$$

Equation (10.100) gives

$$\frac{\epsilon}{2\left(\epsilon^2 + 1/4\tau_{\text{imp}}^2\right)}\frac{\partial\langle|\Delta_{\mathbf{p}}|^2\rangle}{\partial t}\frac{df^{(0)}}{d\epsilon} - D\nabla^2\langle f_1\rangle + \frac{1}{\tau_{\text{ph}}}\langle f_1\rangle = 0. \tag{11.39}$$

From now on we assume an isotropic Fermi surface for simplicity and put

$$\langle\tau_{\text{imp}} v_{Fi} v_{Fk}\rangle = \delta_{ik} D$$

where D is the diffusion constant.
We introduce a new function of coordinates $U(\mathbf{r})$ defined as

$$\langle f_1\rangle = -\frac{U(\mathbf{r})\epsilon}{2\tau\left(\epsilon^2 + 1/4\tau_{\text{imp}}^2\right)}\frac{df^{(0)}}{d\epsilon} \tag{11.40}$$

The function U satisfies the diffusion equation

$$D\nabla^2 U - \frac{1}{\tau_{\text{ph}}}U = -\tau_{\text{imp}}\frac{\partial\langle|\Delta_{\mathbf{p}}|^2\rangle}{\partial t} \tag{11.41}$$

The second averaged kinetic equation (10.102) gives

$$-\frac{\langle|\Delta_{\mathbf{p}}|^2\rangle}{\tau_{\text{imp}}\left(\epsilon^2+1/4\tau_{\text{imp}}^2\right)}\frac{df^{(0)}}{d\epsilon}e\Phi+\frac{\langle|\Delta_{\mathbf{p}}|^2 f_2\rangle}{\tau_{\text{imp}}\left(\epsilon^2+1/4\tau_{\text{imp}}^2\right)}$$
$$-D\nabla^2\langle f_2\rangle - De\,\text{div}\,\mathbf{E}\frac{df^{(0)}}{d\epsilon}=\left\langle J_2^{(\text{ph})}\right\rangle \tag{11.42}$$

Here χ is the order parameter phase.
Using the charge neutrality $\delta N^{(nst)}=0$ we can exclude f_2 from eqn (11.42). Indeed, since in eqn (10.69), $\partial g_-/\partial\epsilon\sim|\Delta|^2/T^3$, the distribution function should satisfy $f_2\ll e\Phi/T$ such that $\int_{-\infty}^{\infty}f_2 d\epsilon$ is zero within the leading approximation in Δ^2/T^2. We can thus omit f_2 from the first line of eqn (11.42) together with the inelastic collision integral and integrate this equation over $d\epsilon$. The term with $D\nabla^2\langle f_2\rangle$ in the second line of eqn (11.42) vanishes as compared to div $\mathbf{E}$. As a result, f_2 disappears completely, and we find

$$De\,\text{div}\,\mathbf{E}+\frac{\pi}{2T}\langle|\Delta_{\mathbf{p}}\Big|^2\Big\rangle e\Phi=0 \tag{11.43}$$

Equation (11.43) yields, in particular

$$\text{div}\,\mathbf{j}_s=\frac{\pi\nu(0)}{T}\langle|\Delta_{\mathbf{p}}\Big|^2\Big\rangle e^2\Phi, \tag{11.44}$$

which describes conversion of supercurrent into a normal current. It is similar to eqn (11.26).
Equation (11.43) introduces the characteristic electric field penetration length

$$l_E=\sqrt{\frac{2DT}{\pi\langle|\Delta_{\mathbf{p}}|^2\rangle}} \tag{11.45}$$

which describes the relaxation of the scalar potential Φ in a d-wave superconductor. We have already encountered a similar length on page 232 in this Section and also on page 33 in Section 1.2.1. This penetration length is of the order of $\sqrt{T_c\tau}\xi(T)$, i.e., is much longer than $\xi(T)$. We refer the reader to the following Section for more discussion of the behavior of the electric field.

Using eqn (11.21) together with eqn (11.40) we find for the nonstationary part of the Keldysh Green function

$$f^{(\text{nst})}_{\epsilon_+,\epsilon_-}=-\frac{1}{2\tau_{\text{imp}}\left(\epsilon^2+1/4\tau_{\text{imp}}^2\right)}\frac{df^{(0)}}{d\epsilon}\left[\frac{\hat{\partial}\Delta_{\mathbf{p}}}{\partial t}+\frac{2\epsilon^2\Delta_{\mathbf{p}}U}{\epsilon^2+1/4\tau_{\text{imp}}^2}\right]$$

As the result, we obtain for the order parameter

$$-\frac{\delta\mathcal{F}_{sn}}{\delta\Delta_{\mathbf{p}}^*}=\frac{\pi\nu(0)}{8T_c}\left(\frac{\partial\Delta_{\mathbf{p}}}{\partial t}+2ie\varphi\Delta_{\mathbf{p}}\right)+\frac{\pi\nu(0)}{8T_c}\Delta_{\mathbf{p}}U \tag{11.46}$$

The free energy in the left hand side of eqn (11.46) is given by the Ginzburg-Landau expression (6.69). Equations (11.46) together with eqn (11.41) coincide with the corresponding equations obtained by Eliashberg (1968) if one neglects the small reactive coefficient and puts there $\tau_s=2\tau_{\text{imp}}$.

The total current $\mathbf{j}=\mathbf{j}_s+\mathbf{j}_n$ is determined by the variation of $\mathcal{F}$ with respect to the vector potential. We find, using eqn (11.38)

$$-\frac{\delta\mathcal{F}_{sn}}{\delta\mathbf{A}}=\frac{1}{c}\left(\mathbf{j}-\mathbf{j}_n\right) \tag{11.47}$$

where the normal current can again be written as $\mathbf{j}_n = \sigma_n \mathbf{E}$ (remind that we assume an isotropic Fermi surface). The complex equation (11.46) contains the equations for the order parameter magnitude and phase. Together with eqn (11.47) and the current conservation $\operatorname{div} \mathbf{j} = 0$, the latter is equivalent to eqn (11.43).

Equation (11.46) differs from a simple TDGL equation by the additional term with U; it is determined by a diffusive relaxation of excitations driven out of equilibrium by external perturbations. It becomes particularly important in situations where the diffusion is slow, i.e., where the characteristic scale of spatial variations is larger than ξ. For example, in a spatially homogeneous case,

$$U = \tau_{\mathrm{ph}} \tau_{\mathrm{imp}} \frac{\partial \langle |\Delta_{\mathbf{p}}|^2 \rangle}{\partial t}$$

The U-term is much larger than the TDGL term as long as the temperature is not very close to T_c, i.e., when the condition $\tau_{\mathrm{imp}} \tau_{\mathrm{ph}} \Delta^2 \gg 1$ holds.

To summarize our discussion of the microscopic justification of the TDGL model we conclude that the microscopic theory supports the TDGL model only in a very limited situation where the deviation of the excitations from equilibrium is small. The best result, i.e., when the TDGL model occurs to be exact is achieved for the gapless limit which can be associated with various mechanisms such as scattering by magnetic impurities, phonons, or by non-magnetic impurities in unconventional superconductors. However, as soon as the distribution of excitations starts to deviate from equilibrium, the TDGL model breaks down. We have seen this both for the case when the relaxation of excitations is
dominated by electron-phonon scattering and when the relaxation is by diffusion. When the distribution is essentially non-equilibrium, one can not construct a simple model any more which involves a closed set of equations for only the order parameter magnitude and phase. The distribution function of excitations appears as a new important variable. In our examples, the function U in eqn (11.40) plays the role of such a variable. In general, one would need to invoke the full Green function technique described in the previous Chapters. Of course, to get the results one has to work hard and solve the whole set of equations which are basically the Eilenberger equations and the kinetic equations for the distribution function. Next step would be to self-consistently determine the order parameter and electromagnetic fields through the order-parameter and Maxwell equations. In the following part of the book we consider how one can carry out this program for a very important and interesting problem of the vortex dynamics. Before going to the vortex dynamics, however, we discuss one more aspect which can be elucidated using the TDGL model.

18.4.4 11.4 D.C. electric field in superconductors. Charge imbalance

Let us discuss the TDGL equations (1.80), (11.27), or (11.43) which all can be written in the form

$$\nabla^2 \Phi - \eta \left(\frac{|\Delta|}{\Delta_\infty} \right) l_E^{-2} \Phi = -\frac{1}{c} \operatorname{div} \frac{\partial \mathbf{Q}}{\partial t} \tag{11.48}$$

where $\eta\left(|\Delta|/\Delta_\infty\right)$ is a function of the ratio of the order parameter magnitude $|\Delta|$ and its homogeneous value Δ_∞; η is defined in such a way that $\eta = 1$ for $|\Delta|/\Delta_\infty = 1$. The function $\eta = f^2$ for the usual TDGL model, eqn (1.80). The magnitude of l_E and the specific form of η depend on parameters of the superconductor.

We already mentioned that l_E determines the length over which a d.c. electric field decays into a bulk superconductor. According to the definition of the gauge invariant potentials in eqns (1.46) and (1.75), the electric field is

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{Q}}{\partial t} - \nabla\Phi$$

A d.c. electric field is simply $\mathbf{E} = -\nabla\Phi$. Consider an interface between a normal metal and a superconductor, and assume that the superconductor occupies the space $x > 0$. If there is a current along the x axis flowing from the normal metal into the superconductor, the electric field $E = j/\sigma_n$ exists in the normal metal such that $d\varphi/dx = -j/\sigma_n$. However, in the bulk superconductor, a d.c. electric field can not exist: It would produce continuous acceleration of superconducting electrons and destroy superconductivity. It is eqn (11.48) that describes relaxation of E. We have for a d.c. electric field

$$\nabla^2\Phi = l_E^{-2}\Phi$$

whence

$$\Phi = \frac{jl_E}{\sigma_n}\exp\left(-\frac{x}{l_E}\right)$$

The constant is chosen to satisfy the continuity of current through the interface. According to this equation, the electric field and the normal current decay into the superconductor, while supercurrent increases until $j_s = j - j_n$ takes all the current flowing into the superconductor: a normal current

$$j_n(x) = -\sigma_n d\Phi/dx$$

converges into supercurrent over the distance l_E.
Because a d.c. electric field exists in a superconductor within the length on order l_E near its boundary with the normal metal, the interface offers an extra resistance to the current. The voltage across the conversion layer is jl_E/σ_n and thus the extra resistance is $\delta R = l_E/\sigma_n$. Of course, such a simple picture takes place only as long as the generalized TDGL model holds. In general, conditions at the interface itself also cause changes in the electric field which contribute to the extra resistance. This problem was considered by many researchers and the results are summarized in the review by Artemenko and Volkov (1979). We are not going into these details here but rather concentrate on another aspect of the problem.

Equation (11.48) tells us that, in a bulk superconductor, we have $\Phi = 0$ for a stationary electric field. In other words,

$$\varphi + \frac{1}{2e}\frac{\partial\chi}{\partial t} = 0 \tag{11.49}$$

which is the celebrated Josephson equation. We thus conclude that the Josephson equation holds only in equilibrium.

Another way of looking at eqn (11.49) is as follows. The quantity $\delta\mu_n = -e\varphi$ can be considered as a change in the chemical potential of a (normal) electron due to the presence of electromagnetic potential φ. The quantity $\partial\chi/\partial t$ is then the chemical potential of a Cooper pair. Equation (11.49) now tells that

$$\delta\mu_n - \delta\mu_s = 0 \tag{11.50}$$

where

$$\delta\mu_s = \frac{1}{2}\frac{\partial\chi}{\partial t}$$

is the chemical potential of a superconducting electron measured from the Fermi energy. Equation (11.50) implies that, under static conditions and in a bulk superconductor, normal and superconducting electrons are in equilibrium with each other.

Equilibrium $\Phi = 0$ may be violated if time-dependent processes take place in the superconductor and/or near its border. A nonequilibrium state with a nonzero Φ is known as a state with the charge imbalance (Tinkham 1972). This terminology comes from the following observation. In a static situation, $g_\epsilon^{R(A)} = -\bar{g}_\epsilon^{R(A)}$ which reflects the particle-hole symmetry. This causes, in particular, that always

$$\mathrm{Tr}\,\check{g}^K = 0$$

in a static case. This is the condition of a constant particle density: there are exactly as many particle-like excitations as there are hole-like ones; moving an electron from under the Fermi level to a position above it we do not create a new particle. The situation changes if $\Phi \neq 0$: now $\mathrm{Tr}\,\check{g}^K$ is nonzero if a special condition on Φ is not imposed. It means that the numbers of particle-like and hole-like excitations are not equal any more. To keep the overall particle density constant one needs to shift the chemical potential. This is called the charge imbalance: the particle-like and the hole-like charges are not equal. Of course, the overall charge density remains constant because we adjust Φ exactly in such a way as to maintain zero value of $\mathrm{Tr}\,\check{g}^K$. This effect is sometimes also called "branch mixing" because the relaxation of Φ is associated with equalizing the populations of the particle-like and hole-like branches of the excitation spectrum.

The process of relaxation of the charge imbalance with $\delta\mu_n \neq \delta\mu_s$ is associated with an exchange of particles between the normal part and the condensate in the superconductor. Any process which facilitates the exchange contributes to the relaxation. Inelastic processes are definitely the ones which work for such an exchange. During an inelastic scattering event, the particle energy changes together with the state of a given particle. It causes redistribution of particles between the normal part and the condensate thus contributing to the charge imbalance relaxation. This is why the electric field penetration length contains the electron-phonon relaxation time in eqn (11.28). In the gapless regime, the energy gap disappears, the exchange is very effective, and the only characteristic time is τ_Δ. The electric field penetration length squared is thus the diffusion constant multiplied with this time which makes $l_E \sim \xi$. The same happens during the spin-flip scattering process because the energy of an electron in the order-parameter field depends on its spin. In an s-wave superconductor, the scattering by impurities will not cause the exchange because an elastic scattering does not change the state of a particle if the energy gap is independent of the direction of the particle momentum. However, if in presence of a supercurrent, and/or for a d-wave superconductor, the energy does depend on the momentum direction, and impurity scattering does change the state of a particle. This is why eqn (11.45) determines l_E through τ_{imp} without characteristics of the inelastic relaxation. For a more detailed discussion, see the book by Tinkham (1996).

19 IV Vortex dynamics by Kopnin

19.1 Time–dependent Ginzburg–Landau analysis

Vortex dynamics is considered within the framework of the TDGL model. The balance of forces acting on a moving vortex is derived from the free energy considerations. The vortex

viscosity and the flux flow conductivity are calculated. A modification of the TDGL model is considered which allows us to account for the flux flow Hall effect.

19.1.1 12.1 Introduction

A very simple question that can arise in connection with the title of the present part of the book is as follows. What is special in the vortex dynamics and why do we want to study it? To answer this we note that the majority of known superconductors are type II superconductors whose London penetration length λ_L is longer than the coherence length ξ. The exact condition which separates type II from type I superconductors is $\kappa \equiv \lambda_L/\xi > 1/\sqrt{2}$, see Section 1.1.2, page 24. Almost all alloys and films made of conventional (low temperature) superconducting materials are of type II. All known high-temperature and heavy-Fermionic superconductors are also of type II. We can also mention superfluid ^{3}He where $\lambda_L = \infty$ since particles which make Cooper pairs are uncharged. If we place a type II superconductor in a magnetic field (or rotate superfluid ^{3}He) vortices appear in the range of fields $H_{c1} < H < H_{c2}$, and the so called mixed state arises, see Section 1.1.2. Since the lower critical magnetic field $H_{c1} \sim H_{c2}/\kappa^2$ (see, for example de Gennes 1966, Saint-James et al. 1969), the mixed state exists in a broad range of magnetic fields if the Ginzburg-Landau parameter κ is substantially larger than unity which is the case in high-temperature superconductors where κ can be as high as 100 .

In superconductors, each vortex carries exactly one magnetic-flux quantum $\Phi_0 = \pi c/|e|$. Being magnetically active, vortices determine magnetic properties of superconductors. In addition, they are mobile if the material is homogeneous and there are no defects which can attract vortices and "pin" them somewhere in a superconductor. In fact, a superconductor in the mixed state is not fully superconducting any more. Indeed, there is no complete Meissner effect: some magnetic filed penetrates into superconductor via vortices. In addition, regions with the normal phase appear. Indeed, since the order parameter turns to zero at the vortex axis (see Fig. 1.1) and is suppressed around each vortex axis within a vortex core region of the order of the coherence length, a finite low-energy density of states appears in the vortex cores. Moreover, mobile vortices move in
presence of an average (transport) current: the Lorentz force

$$\mathbf{F}_L = \frac{\Phi_0}{c}\left[\mathbf{j}_{\rm tr} \times \hat{\mathbf{H}}\right] \tag{12.1}$$

pushes vortices in a direction perpendicular to the current (here $\hat{\mathbf{H}}$ is the unit vector along the magnetic field). An electric field perpendicular to the vortex velocity is generated by a moving flux, and a voltage appears across the superconductor. Since vortices move at an angle to the transport current, there may be components of the electric filed both parallel and perpendicular to the current. The longitudinal component produces dissipation in a superconductor while the transverse one is responsible for the Hall effect. We see that there appears a finite resistivity, (the so called flux flow resistivity): a superconductor is not superconducting any more! This is certainly an important effect.

The magnitude and direction of the vortex velocity is determined by a balance of the Lorentz force and the forces acting on a moving vortex from the environment. In the absence of pinning these forces include friction (longitudinal with respect to the vortex velocity) and gyroscopic (transverse) forces (see Fig. 12.1). The friction force accounts for dissipation, i.e., for an effective longitudinal or Ohmic "flux flow" conductivity while the transverse force determines the Hall conductivity.

Experimental studies of flux flow effects began with the work by Kim et al. (1965). Since then enormous efforts have been undertaken to find out and understand the processes involved in the vortex dynamics and the vortex physics in general. One now uses a notion of "vortex

matter" to comprise all features which vortices introduce to physics of superconductivity. We can mention the reviews by Gor'kov and Kopnin (1975) and by Larkin and Ovchinnikov (1986) which deal with the vortex dynamics and a review by Blatter et al. (1994) which contains many basic concepts of the vortex physics especially those which are relevant to vortex lattices, vortex pinning, flux creep, etc.

In this part of the book, we concentrate on the theoretical description of the vortex dynamics based on the microscopic theory of nonstationary superconductivity. We consider the most representative examples among all numerous situations studied by many researchers during several decades of intensive work. We shall see that motion of vortices initiates almost all nonequilibrium processes and involves all relaxation mechanisms which work in superconductors. It is one more reason why it is important to understand the vortex dynamics. We start with the simplest theory which uses the TDGL model. The next Chapter deals with a more complicated physics in superconducting alloys when nonequilibrium excitations come seriously into play. The last two Chapters describe the most interesting and intriguing phenomena in clean superconductors, i.e., in those classes of materials which include, in particular, new high-temperature superconductors. We do not consider effects associated with the vortex pinning by random defects in a superconducting material assuming that vortices are free to move in a homogeneous environment. An exception will be made for the intrin-

Fig. 12.1. Forces on a moving vortex: the Lorenz force from the supercurrent $\mathbf{j}_{\mathrm{tr}}$ is balanced by a force from environment $\mathbf{F}_{\mathrm{env}}$. The latter has the longitudinal (friction) $\mathbf{F}_{\parallel}$and the transverse $\mathbf{F}_{\perp}$ components. The moving vortex generates an electric field $\mathbf{E}$ perpendicular to the vortex velocity $\mathbf{v}_L$.

sic pinning when vortices interact with the regular crystal structure in layered superconductors.

19.1.2 12.2 Energy balance

The TDGL model provides the simplest basis for dealing with nonstationary processes in superconductors. Using it, one can establish quite general relationships which are particularly useful for our goal of understanding the vortex dynamics. The first is the energy balance (Schmid 1966). Let us calculate the time derivative of the total GL free energy eqn (1.11). We have

$$\begin{aligned}\frac{\partial \mathcal{F}_{tot}}{\partial t} = &\int \left(\frac{\partial \Delta}{\partial t} \left[-|\alpha|\Delta^* + \beta|\Delta|^2\Delta^* - \gamma \left(\nabla + \frac{2ie}{c}\mathbf{A} \right)^2 \Delta^* \right] + c.c. \right. \\ &\left. + \frac{1}{c}\left(\mathbf{j} - \mathbf{j}_s\right) \frac{\partial \mathbf{A}}{\partial t} \right) dV \\ &+ \int \operatorname{div} \left[\gamma \frac{\partial \Delta}{\partial t} \left(\nabla + \frac{2ie}{c}\mathbf{A} \right) \Delta^* + c.c. + \frac{1}{4\pi} \frac{\partial \mathbf{A}}{\partial t} \times \mathbf{H} \right] dV.\end{aligned}$$

With help of the TDGL equations (1.68) and (1.66) we obtain

$$\begin{aligned}\frac{\partial \mathcal{F}_{tot}}{\partial t} = &\int \left[\frac{\delta \mathcal{F}_{sn}}{\delta \Delta} \left(\frac{\partial \Delta}{\partial t} + 2ie\phi\Delta \right) + \frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} \left(\frac{\partial \Delta^*}{\partial t} - 2ie\phi\Delta^* \right) \right. \\ &\left. -2ie\phi \left(\Delta \frac{\delta \mathcal{F}_{sn}}{\delta \Delta} - \Delta^* \frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} \right) + \frac{\mathbf{j}_n}{c} \frac{\partial \mathbf{A}}{\partial t} \right] dV \\ &+ \int \operatorname{div} \left[\gamma \frac{\partial \Delta}{\partial t} \left(\nabla + \frac{2ie}{c}\mathbf{A} \right) \Delta^* + c.c. + \frac{1}{4\pi} \left[\frac{\partial \mathbf{A}}{\partial t} \times \mathbf{H} \right] \right] dV \\ = &\int \left[\frac{\delta \mathcal{F}_{sn}}{\delta \Delta} \left(\frac{\partial \Delta}{\partial t} + 2ie\varphi\Delta \right) + c.c. + \mathbf{j}_n \left(\frac{1}{c} \frac{\partial \mathbf{A}}{\partial t} + \nabla\varphi \right) \right] dV \\ &+ \int \operatorname{div} \left\{ \left[\gamma \left(\frac{\partial \Delta}{\partial t} + 2ie\varphi\Delta \right) \left(\nabla + \frac{2ie}{\hbar c}\mathbf{A} \right) \Delta^* + c.c. \right] \right. \\ &\left. + \frac{1}{4\pi} \left[\frac{\partial \mathbf{A}}{\partial t} \times \mathbf{H} \right] - \mathbf{j}\varphi \right\} dV\end{aligned}$$

We use eqn (1.76) to transform the second line. Since

$$\operatorname{div}(\mathbf{j}\varphi) = \frac{c}{4\pi} \operatorname{div}[\varphi \operatorname{curl} \mathbf{H}] = -\frac{c}{4\pi} \operatorname{div}[\nabla\varphi \times \mathbf{H}]$$

we finally arrive at the free-energy balance equation

$$\frac{\partial \mathcal{F}_{tot}}{\partial t} + \int \operatorname{div} \mathbf{j}_{\mathcal{F}} = -\int W dV \tag{12.2}$$

Here the free-energy current density is

$$\mathbf{j}_{\mathcal{F}} = -\gamma \left[\left(\frac{\partial \Delta}{\partial t} + 2ie\varphi\Delta \right) \left(\nabla + \frac{2ie}{c}\mathbf{A} \right) \Delta^* + c.c. \right] + \frac{c}{4\pi}[\mathbf{E} \times \mathbf{H}] \tag{12.3}$$

and the dissipation function density has the form

$$W = 2\Gamma \left| \left(\frac{\partial \Delta}{\partial t} + \frac{2ie\varphi\Delta}{\hbar} \right) \right|^2 + \sigma_n \mathbf{E}^2 \tag{12.4}$$

One can identify two different terms in the dissipation function. The first contains the time derivative of the order parameter. It describes dissipation produced by relaxation of the order parameter. The second is due to normal currents and is similar to that in usual normal metals.

19.1.3 12.3 Moving vortex

Vortex moves when there is a transport current $\mathbf{j}_{\text{tr}}$ giving rise to the Lorentz force eqn (12.1). We can calculate the flux flow conductivity σ_f directly from the dissipation function, putting it to

$$W = \mathbf{j}_{\text{tr}} \langle \mathbf{E} \rangle_L = \sigma_f \langle \mathbf{E} \rangle_L^2 \tag{12.5}$$

where $\langle \mathbf{E} \rangle_L$ is the electric field averaged over the vortex array. The average is defined as

$$\langle \ldots \rangle_L = S_0^{-1} \int_{S_0} (\ldots) dS \tag{12.6}$$

where $S_0 = \Phi_0 / B$ is the area occupied by each vortex.

All the values in eqn (12.4) can be easily calculated if the vortex velocity $\mathbf{v}_L$ is small. To calculate the time derivative of the order parameter and of the vector potential we note that, for a small vortex velocity,

$$\Delta(\mathbf{r}) = \Delta_{st} (\mathbf{r} - \mathbf{v}_L t) + \Delta_1, \mathbf{A}(\mathbf{r}) = \mathbf{A}_{st} (\mathbf{r} - \mathbf{v}_L t) + \mathbf{A}_1 \tag{12.7}$$

where Δ_{st} and $\mathbf{A}_{st}$ are the values for a static vortex, and Δ_1 and $\mathbf{A}_1$ are small corrections. Therefore, within the main terms in the vortex velocity $\mathbf{v}_L$, we can write

$$\frac{\partial \Delta}{\partial t} = -(\mathbf{v}_L \cdot \nabla) \Delta_{st}, \frac{\partial \mathbf{A}}{\partial t} = -(\mathbf{v}_L \cdot \nabla) \mathbf{A}_{st} \tag{12.8}$$

The scalar potential φ is to be found from eqn (1.80). For small $\mathbf{v}_L$ it contains only the static vortex order parameter.

Let us calculate the average electric field. The local electric field is

$$\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla\varphi = -\frac{1}{c}\frac{\partial \mathbf{Q}}{\partial t} - \nabla\Phi - \frac{1}{2e}\left(\frac{\partial}{\partial t}\nabla\chi - \nabla\frac{\partial \chi}{\partial t}\right) \tag{12.9}$$

The last term here is nonzero for a moving vortex. Indeed,

$$\begin{aligned} -\frac{1}{2e}\left(\frac{\partial}{\partial t}\nabla\chi - \nabla\frac{\partial \chi}{\partial t}\right) &= \frac{1}{2e}\left[(\mathbf{v}_L \cdot \nabla)\nabla\chi - \nabla(\mathbf{v}_L \cdot \nabla)\chi\right] \\ &= -\frac{1}{2e}\left[\mathbf{v}_L \times \operatorname{curl}\nabla\chi\right] \\ &= -\frac{\Phi_0}{c}\left[\mathbf{v}_L \times \hat{\mathbf{z}}\right]\operatorname{sign}(e)\delta^{(2)}(\mathbf{r}) \end{aligned} \tag{12.10}$$

Here $\hat{\mathbf{z}}$ is the unit vector along the vortex axis in the positive direction of its circulation. The vortex circulation is determined by the sense of the phase increment, it coincides with the magnetic field direction for positive charge of carriers and is antiparallel to it for negative carriers: $\hat{\mathbf{z}} = \operatorname{sign}(e)\hat{\mathbf{H}}$, see Section 1.1.2. We now average eqn (12.9) over the vortex array. Since

$$\frac{\partial \mathbf{Q}}{\partial t} = -(\mathbf{v}_L \cdot \nabla)\mathbf{Q}_{st}$$

the average of this term vanishes because the gauge invariant vector potential $\mathbf{Q}_{st}$ is a bounded oscillating function (periodic for a periodic vortex array). Its value at the boundary of a region occupied by each vortex, i.e., at the boundary of the vortex unit cell is zero, see eqn (1.40) on page 23. The average of $\nabla\Phi$ vanishes, too, since Φ is also a bounded oscillating function periodic for a regular vortex lattice. It turns to zero at the boundary of the vortex

unit cell because of the Josephson relation, eqn (11.49). The only term which survives is the last term in eqn (12.9). Equation (12.10) gives

$$\langle \mathbf{E} \rangle_L = \frac{1}{c} \mathbf{B} \times \mathbf{v}_L \tag{12.11}$$

which is the Faraday's law.

Now we can calculate W. It is more instructive, however, to consider another useful relation which follows from the TDGL equations. It is expression for the force which acts on a moving vortex.

19.1.4 12.4 Force balance

Let us shift our vortex by a small arbitrary constant vector $\mathbf{d}$ and calculate the variation of the free energy caused by such a displacement. During the derivation we neglect surface contributions. We have

$$\begin{aligned}
\delta \mathcal{F}_{tot} = & \int \left(\frac{\delta \mathcal{F}_{sn}}{\delta \Delta} (\mathbf{d} \cdot \nabla) \Delta + \frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} (\mathbf{d} \cdot \nabla) \Delta^* \right. \\
& \left. + \frac{\delta \mathcal{F}_{sn}}{\delta \mathbf{A}} (\mathbf{d} \cdot \nabla) \mathbf{A} + \frac{\mathbf{H}}{4\pi} \operatorname{curl}[(\mathbf{d} \cdot \nabla) \mathbf{A}] \right) dV \\
= & \int \left[\frac{\delta \mathcal{F}_{sn}}{\delta \Delta} \mathbf{d} \left(\nabla - \frac{2ie}{c} \mathbf{A} \right) \Delta + \frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} \mathbf{d} \left(\nabla + \frac{2ie}{c} \mathbf{A} \right) \Delta^* \right. \\
& + \frac{2ie}{c} (\mathbf{d} \cdot \mathbf{A}) \left(\Delta \frac{\delta \mathcal{F}_{sn}}{\delta \Delta} - \Delta^* \frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} \right) \\
& \left. + \frac{1}{c} (\mathbf{j} - \mathbf{j}_s) (\mathbf{d} \cdot \nabla) \mathbf{A} \right] dV \\
= & \mathbf{d} \int \left[\frac{\delta \mathcal{F}_{sn}}{\delta \Delta} \left(\nabla - \frac{2ie}{c} \mathbf{A} \right) \Delta + c.c. + \frac{1}{c} [\mathbf{j}_n \times \mathbf{H}] \right] dV
\end{aligned} \tag{12.12}$$

where we use eqn (1.76) and the charge neutrality $\operatorname{div} \mathbf{j} = 0$.
Variation of the free energy determines the force which acts on a moving vortex from the environment

$$\delta \mathcal{F}_{\text{tot}} = - \int \mathbf{d} \cdot \mathbf{F}_{\text{env}} dV \tag{12.13}$$

It should be balanced by external Lorentz force due to the transport current. To include this force we add the free energy due to the transport current which is supplied by the external source. Its variation due to the vortex displacement is

$$\begin{aligned}
\delta \mathcal{F}_{\text{ext}} &= -\frac{1}{c} \int \mathbf{j}_{\text{tr}} (\mathbf{d} \cdot \nabla) \mathbf{A} dV \\
&= \int \left(\frac{1}{c} \mathbf{j}_{\text{tr}} [\mathbf{d} \times \mathbf{H}] - \frac{1}{c} \operatorname{div} [\mathbf{j}_{\text{tr}} (\mathbf{d} \cdot \mathbf{A})] + \frac{1}{c} (\mathbf{d} \cdot \mathbf{A}) \operatorname{div} \mathbf{j}_{\text{tr}} \right) dV
\end{aligned}$$

On the next step, we omit the surface term and use that $\operatorname{div} \mathbf{j}_{\text{tr}} = 0$. We find

$$\delta \mathcal{F}_{\text{ext}} = -\mathbf{d} \int \frac{1}{c} [\mathbf{j}_{\text{tr}} \times \mathbf{H}] \, dV \tag{12.14}$$

Conservation of $\mathcal{F}_{\text{tot}}$ plus the external energy implies its translation invariance

$$\delta\mathcal{F}_{\text{tot}} + \delta\mathcal{F}_{\text{ext}} = 0$$

It gives the balance of forces acting on a moving vortex of a unit length

$$\mathbf{F}_L - \int_{S_0} \left[\frac{\delta\mathcal{F}_{sn}}{\delta\Delta} \left(\nabla - \frac{2ie}{c}\mathbf{A} \right) \Delta + c.c. + \frac{1}{c} \left[\mathbf{j}_n \times \mathbf{H} \right] \right] dS = 0 \tag{12.15}$$

where $\mathbf{F}_L$ is the Lorentz force eqn (12.1). We use the fact that $\mathbf{d}$ is an arbitrary vector. In other words,

$$\begin{aligned} \frac{\Phi_0}{c} \left[\mathbf{j}_{\text{tr}} \times \hat{\mathbf{H}} \right] = \int_{S_0} \Bigg(& -\Gamma \left[\left(\frac{\partial\Delta^*}{\partial t} - 2ie\varphi\Delta^* \right) \left(\nabla - \frac{2ie}{c}\mathbf{A} \right) \Delta \right. \\ & \left. + \left(\frac{\partial\Delta}{\partial t} + 2ie\varphi\Delta \right) \left(\nabla + \frac{2ie}{c}\mathbf{A} \right) \Delta^* \right] \\ & + \frac{1}{c} \left[\mathbf{j}_n \times \mathbf{H} \right] \Bigg) dS \end{aligned} \tag{12.16}$$

Here the integration is over the unit cell of the vortex array.
To summarize, in eqns (12.12) or (12.16) the Lorentz force is balanced by the force produced by the interaction with the environment. The latter is proportional to the vortex velocity since the time derivative, electric field and the scalar potential are all proportional to $\mathbf{v}_L$.

Note that the dissipation function integrated over the unit cell can be again obtained if we multiply eqn (12.16) with the vortex velocity $\mathbf{v}_L$:

$$\begin{aligned} \frac{\Phi_0}{c} \left(\mathbf{j}_{\text{tr}} \cdot \left[\hat{\mathbf{H}} \times \mathbf{v}_L \right] \right) &= \int_{S_0} \Bigg[\left(\mathbf{v}_L \nabla\Delta - 2ie\varphi\Delta \right) \frac{\delta\mathcal{F}_{sn}}{\delta\Delta} + \text{ c.c.} \\ &\quad - \left(\frac{1}{c} \left(\mathbf{v}_L \mathbf{A} \right) - \varphi \right) \operatorname{div} \mathbf{j}_n + \frac{1}{c} \left(\mathbf{j}_n \left[\mathbf{H} \times \mathbf{v}_L \right] \right) \Bigg] dS \\ &= \int_{S_0} \Bigg\{ - \left[\left(\frac{\partial\Delta}{\partial t} + 2ie\varphi\Delta \right) \frac{\delta\mathcal{F}_{sn}}{\delta\Delta} + \text{ c.c. } \right] \\ &\quad + \mathbf{j}_n \left(\frac{1}{c} \nabla \left(\mathbf{v}_L \mathbf{A} \right) - \nabla\varphi \right) + \frac{1}{c} \left(\mathbf{j}_n \left[\mathbf{H} \times \mathbf{v}_L \right] \right) \Bigg\} dS \\ &= \int_{S_0} W dS \end{aligned}$$

The left hand side of this equation can be expressed through the effective flux flow conductivity. We write

$$\frac{\Phi_0}{c} \left(\mathbf{j}_{\text{tr}} \cdot \left[\hat{\mathbf{H}} \times \mathbf{v}_L \right] \right) = \mathbf{j}_{\text{tr}} \langle \mathbf{E} \rangle_L S_0 = \sigma_f \langle \mathbf{E} \rangle_L^2 S_0$$

since $S_0 = \Phi_0/B$. The right hand side is also proportional to $\langle \mathbf{E} \rangle_L^2$. The coefficient of proportionality determines σ_f.

19.1.5 12.5 Flux flow

12.5.1 Single vortex: Low fields

Consider first a single vortex provided $\kappa \gg 1$ and $H \ll H_{c2}$. In this case we can neglect the magnetic field and the vector potential compared to the gradient of the order parameter since

$$(2e/c)A \sim (2e/c)H\xi \ll (2e/c)H_{c2}\xi \sim 1/\xi.$$

We have from eqn (12.15)

$$\frac{\Phi_0}{c}\left[\mathbf{j}_{tr}\times\hat{\mathbf{H}}\right]=-\Gamma\int_{S_0}\left[\left(\frac{\partial\Delta^*}{\partial t}-2ie\varphi\Delta^*\right)\nabla\Delta+c.c.\right]dS$$
$$=2\Gamma\int_{S_0}\left[\left(\mathbf{v}_L\cdot\nabla|\Delta|\right)\nabla|\Delta|-2e|\Delta|^2\Phi\nabla\chi\right]dS \tag{12.17}$$

The order parameter magnitude $|\Delta|=\Delta_\infty f(\rho)$ for a static vortex satisfies eqn (1.48). To calculate the integrals in eqn (12.17) we also need to know the gauge invariant potential Φ. It satisfies eqn (11.48) whence

$$\nabla^2\Phi-l_E^{-2}f^2\Phi=0$$

since $\eta\left(|\Delta|/\Delta_\infty\right)=f^2$ for the usual TDGL model, eqn (1.80). The potential φ should be finite. Therefore, we require

$$\Phi\to-\frac{\hbar}{2e}\left(\mathbf{v}_L\cdot\nabla\right)\chi \tag{12.18}$$

for $\rho\to 0$. We put $\rho=\xi\tilde{\rho}$ and

$$\Phi=-\frac{v_{L\phi}\mu}{2e\xi} \tag{12.19}$$

where ϕ is the azimuthal angle in the cylindrical frame (ρ,ϕ,z). The function $\mu(\tilde{\rho})$ satisfies the equation

$$\left(\frac{d^2\mu}{d\tilde{\rho}^2}+\frac{1}{\tilde{\rho}}\frac{d\mu}{d\tilde{\rho}}-\frac{\mu}{\tilde{\rho}^2}\right)-uf^2\mu=0 \tag{12.20}$$

where $u=\xi^2/l_E^2$, see eqn (1.81). The condition eqn (12.18) requires $\mu\to 1/\tilde{\rho}$ for $\tilde{\rho}\to 0$. We see that a moving vortex induces a dipole-like scalar potential proportional to the vortex velocity.

The two terms under the integral in eqn (12.17) represent two different mechanisms of dissipation during the vortex motion exactly as the two corresponding terms in the dissipation function eqn (12.4). The first term is the Tinkham's mechanism (Tinkham 1964) of relaxation of the order parameter: Due to the motion of a vortex, the order parameter at a given point varies in time which produces a relaxation accompanied by dissipation. The second is the so called

Bardeen and Stephen (1965) contribution. It accounts for normal currents flowing through the vortex core.

Equation (12.17) gives

$$\frac{\Phi_0}{c}\left(\left[\mathbf{j}_{\mathrm{tr}}\times\hat{\mathbf{H}}\right]\cdot\mathbf{d}\right)=2\Gamma\int\left[\left(v_{L\rho}d_\rho\right)\left(\frac{d|\Delta|}{d\rho}\right)^2+\left(v_{L\phi}d_\phi\right)\frac{|\Delta|^2\mu}{\rho}\right]dS$$
$$=2\pi\Gamma\left(\mathbf{v}_L\cdot\mathbf{d}\right)\int\left[\left(\frac{d|\Delta|}{d\tilde{\rho}}\right)^2+\frac{|\Delta|^2\mu}{\tilde{\rho}}\right]\tilde{\rho}d\tilde{\rho} \tag{12.21}$$

Here we have restored the arbitrary vector $\mathbf{d}$ that appears in eqns (12.12) and (12.14) to make the calculations more convenient. The transport current is

$$\mathbf{j}_{\mathrm{tr}}=\frac{2\pi c\Gamma\Delta_\infty^2 a}{\Phi_0 B}\left[\mathbf{B}\times\mathbf{v}_L\right] \tag{12.22}$$

Here

$$a = \int \left[\left(\frac{df}{d\tilde{\rho}} \right)^2 + \frac{f^2 \mu}{\tilde{\rho}} \right] \tilde{\rho} d\tilde{\rho} \tag{12.23}$$

Expressing $\mathbf{v}_L$ through $\langle \mathbf{E} \rangle_L$ we obtain the flux flow conductivity

$$\sigma_f = \frac{2\pi c^2 \Gamma \Delta_\infty^2 a}{B \Phi_0}$$

With the microscopic values of Γ and Δ_∞ taken for dirty superconductors, we find

$$\sigma_f = \frac{ua}{2} \sigma_n \frac{H_{c2}}{B} \tag{12.24}$$

As we already mentioned, the first term in eqn (12.23) is due to the dissipation caused by relaxation of the order parameter after passage of a vortex, while the second one comes from the normal-current dissipation in the vortex core. In the TDGL model with $u \sim 1$, both mechanisms give comparable contributions to σ_f. The numerical calculations using the vortex order parameter obtained by solving the GL equation give for the first term (see the review by Gor'kov and Kopnin (1975) and references therein)

$$a_1 = \int \left(\frac{df}{d\tilde{\rho}} \right)^2 \tilde{\rho} d\tilde{\rho} \approx 0.274$$

which is independent of u. The second term, however, does depend on u. For example, for $u = 5.79$ we have

$$a_2 = \int f^2 \mu d\tilde{\rho} \approx 0.228$$

To calculate a_2 we need first to solve eqn (12.20). In this case, the total coefficient is $a \approx 0.502$. The flux flow conductivity becomes

$$\sigma_f \approx 1.45 \sigma_n \frac{H_{c2}}{B} \tag{12.25}$$

It is close to the expression $\sigma_f = \sigma_n H_{c2}/B$ known as the Bardeen-Stephen model (Bardeen and Stephen 1965). This simple prediction implies that the flux flow resistivity is just a normal-state resistivity times the fraction of space occupied by vortex cores.

12.5.2 Dense lattice: High fields

Consider now high fields, $H \to H_{c2}$. We put $\mathbf{A} = (0, Hx, 0)$, and choose the static potential in the form $\varphi = -E_x x$, where the electric field $E_x = -(v_y/c) H$ is homogeneous in the leading approximation. Remind the static order parameter in the vicinity of the upper critical field, see Section 1.1.2, eqn (1.36):

$$\Delta = \sum_n C_n e^{iqny} \exp \left[-\frac{1}{2\xi^2} \left(x - \frac{cqn}{2eH} \right)^2 \right] \tag{12.26}$$

Equation (12.16) gives

$$\frac{1}{c} [\mathbf{j}_{\text{tr}} \times \mathbf{B}] \mathbf{d} = -\Gamma \left\langle \left[\mathbf{d} \left(\nabla - \frac{2ie}{c} \mathbf{A} \right) \Delta \right] \left(\frac{\partial \Delta^*}{\partial t} - 2ie\varphi \Delta^* \right) + c.c. \right\rangle_L + \frac{1}{c} \langle [\mathbf{j}_n \times \mathbf{H}] \rangle_L \mathbf{d}$$

where $\mathbf{d}$ is again an arbitrary vector. We have

$$\begin{aligned}\frac{1}{c}\left[\mathbf{j}_{\mathrm{tr}}\times\mathbf{B}\right]\mathbf{d} =&\Gamma\left\langle\left[d_x\frac{\partial\Delta}{\partial x}+d_y\left(\frac{\partial\Delta}{\partial y}-\frac{2ieHx}{c}\Delta\right)\right]\right.\\ &\left.\times v_y\left[\frac{\partial\Delta^*}{\partial y}+\frac{2ieHx}{c}\Delta^*\right]+\text{ c.c. }\right\rangle_L\\ &+\frac{1}{c}\left[\mathbf{j}_n\times\mathbf{B}\right]\mathbf{d}\end{aligned}\tag{12.27}$$

We need to calculate the averages

$$\left\langle\frac{\partial\Delta}{\partial x}\left(\frac{\partial\Delta^*}{\partial y}+\frac{2ieHx}{c}\Delta^*\right)+c.c.\right\rangle_L\tag{12.28}$$

and

$$\left\langle\left(\frac{\partial\Delta}{\partial y}-\frac{2ieHx}{c}\Delta\right)\left(\frac{\partial\Delta^*}{\partial y}+\frac{2ieHx}{c}\Delta^*\right)+c.c.\right\rangle_L.\tag{12.29}$$

We make use of the identities

$$\begin{aligned}\frac{\partial\Delta}{\partial x}&=\mathrm{sign}(e)\left(-i\frac{\partial\Delta}{\partial y}-\frac{2eHx}{c}\Delta\right)\\ \frac{\partial\Delta^*}{\partial x}&=\mathrm{sign}(e)\left(i\frac{\partial\Delta^*}{\partial y}-\frac{2eHx}{c}\Delta^*\right),\end{aligned}\tag{12.30}$$

derived for the order parameter from eqn (12.26). The sign of the electronic charge appears because the upper critical filed contains the modulus of the electronic charge,

$$H_{c2}=\frac{\Phi_0}{2\pi\xi^2}=\frac{c}{2|e|\xi^2}$$

For the average in eqn (12.28) we obtain

$$\left\langle\frac{\partial\Delta}{\partial x}\left(\frac{\partial\Delta^*}{\partial y}+\frac{2ieHx}{c}\Delta^*\right)+\text{ c.c. }\right\rangle_L=\left\langle-i\frac{\partial\Delta}{\partial x}\frac{\partial\Delta^*}{\partial x}+\text{ c.c. }\right\rangle_L=0.$$

The second average is

$$\left\langle\left(\frac{\partial\Delta}{\partial y}-\frac{2ieHx}{c}\Delta\right)\left(\frac{\partial\Delta^*}{\partial y}+\frac{2ieHx}{c}\Delta^*\right)\right\rangle_L=\left\langle\frac{\partial\Delta}{\partial x}\frac{\partial\Delta^*}{\partial x}\right\rangle_L.$$

Therefore

$$\begin{aligned}&\left\langle\left(\frac{\partial\Delta}{\partial y}-\frac{2ieHx}{c}\Delta\right)\left(\frac{\partial\Delta^*}{\partial y}+\frac{2ieHx}{c}\Delta^*\right)\right\rangle_L\\ &=-\frac{1}{2}\left\langle\Delta\left[*\left(\frac{\partial}{\partial y}-\frac{2ieHx}{c}\right)^2\Delta+\frac{\partial^2\Delta}{\partial x^2}\right]\right\rangle_L=\frac{1}{2\xi^2}\left\langle|\Delta\Big|^2\right\rangle_L.\end{aligned}\tag{12.31}$$

We obtain from eqn (12.27)

$$\frac{1}{c}\left[\mathbf{j}_{\mathrm{tr}}\times\mathbf{B}\right]=\mathbf{v}_L\left(\Gamma\frac{\langle|\Delta|^2\rangle_L}{\xi^2}+\frac{\sigma_n B^2}{c^2}\right)\tag{12.32}$$

Finally

$$\mathbf{j}_{\mathrm{tr}} = \frac{1}{c}\left[\mathbf{B}\times\mathbf{v}_L\right]\left(\sigma_n + 4e^2\Gamma\xi^2\left\langle|\Delta|^2\right\rangle_L\right) = \sigma_f\mathbf{E} \tag{12.33}$$

where the flux flow conductivity is

$$\sigma_f = \sigma_n + 4e^2\Gamma\xi^2\left\langle|\Delta|^2\right\rangle_L \tag{12.34}$$

The average magnitude of the order parameter is from eqn (1.43)

$$\left\langle|\Delta|^2\right\rangle_L = \frac{2\kappa^2\Delta_\infty^2}{\left(2\kappa^2-1\right)\beta_A}\left(1-\frac{B}{H_{c2}}\right).$$

Therefore

$$\begin{aligned}\sigma_f &= \sigma_n + \frac{4e^2\Gamma\xi^2\Delta_\infty^2}{\beta_A\left[1-1/\left(2\kappa^2\right)\right]}\left(1-\frac{B}{H_{c2}}\right)\\ &= \sigma_n\left[1+\frac{u}{2}\frac{\left(1-B/H_{c2}\right)}{\beta_A\left[1-\left(1/2\kappa^2\right)\right]}\right]\end{aligned} \tag{12.35}$$

Equations (12.25) and (12.35) determine the flux flow conductivity in the limits of low and high magnetic fields. The effective resistivity $\rho_f = \sigma_f^{-1}$ is linear
in the magnetic field in both limits. However, the slope at low fields is smaller than that for $H \to H_{c2}$. Indeed,

$$\frac{H_{c2}}{\rho_n}\frac{\partial\rho_f}{\partial B} = \begin{cases} 2/ua, & H \ll H_{c2}\\ u/2\beta_A, & H \to H_{c2}\end{cases}$$

For practical values of $u \geq 5.79$, one has $2/ua < u/2\beta_A$.
It is interesting to note that the linearized TDGL equation has an analytical solution describing the moving vortex lattice (Schmid 1966). Indeed, the TDGL equation

$$\tau_\Delta\left(\frac{\partial\Delta}{\partial t} + 2ie\varphi\Delta\right) = \xi^2\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)^2\Delta + \Delta \tag{12.36}$$

with $\mathbf{A} = (0, Hx, 0)$ and $\phi = -E_x x$ has the solution within the first-order terms in E

$$\begin{aligned}\Delta = &\sum_n C_n \exp\left[iqn\left(y + \frac{cE_x t}{H}\right)\right]\\ &\times\exp\left[-\frac{1}{2\xi^2}\left(x-\frac{qnc}{2eH}\right)^2 + ie\tau_\Delta E_x\left(x-\frac{qnc}{2eH}\right)\right]\end{aligned} \tag{12.37}$$

It corresponds to a vortex lattice moving with the velocity $v_y = -cE_x/H$.

12.5.3 Direction of the vortex motion

From eqns (12.22) and (12.33) we observe that vortices move at the right angle to the transport current, and generate the electric field parallel to it. The Lorentz force from the transport current is balanced by purely dissipative friction force:

$$\frac{\Phi_0}{c}\left[\mathbf{j}_{\mathrm{tr}}\times\hat{\mathbf{H}}\right] + \eta\mathbf{v}_L = 0$$

where the vortex viscosity is expressed through σ_f :

$$\eta = \frac{B\Phi_0\sigma_f}{c^2} \tag{12.38}$$

There is no Hall effect and no gyroscopic force, i.e., the force which would be perpendicular to the vortex velocity: the transverse force $\mathbf{F}_\perp$ in Fig. 12.1 is absent. This is the result of a fully dissipative dynamics of the TDGL model. We discuss what can be the origin of a gyroscopic force and of the Hall effect in the TDGL model later in Section 12.9.

19.1.6 12.6 Anisotropic superconductors

Almost all high temperature superconductors are highly anisotropic, and some of them have even a well developed layered structure. The time dependent Ginzburg-Landau model can also be used to calculate the flux flow conductivity in such superconductors. One should note, however, that the TDGL scheme
works reasonably well only for dirty superconductors and in a very limited range of parameters. Therefore, the results of this Section do not directly apply to real high temperature superconductors which are, in fact, clean materials. Nevertheless, it is instructive to see what predictions can be expected within the TDGL model for anisotropic superconductors. In this Section we consider uniaxial anisotropic superconductors. Some aspects of the vortex dynamics in layered systems will be discussed in the following Section.

Applying the general TDGL scheme described on page 31 to the anisotropic Ginzburg-Landau equation (7.21) we obtain the corresponding order parameter equation in the form

$$\begin{aligned}-\Gamma\left(\frac{\partial\Delta}{\partial t}+2ie\varphi\Delta\right) =&\alpha\Delta+\beta|\Delta|^2\Delta-\gamma_{ab}\left(\nabla-\frac{2ie}{c}\mathbf{A}\right)^2\Delta\\ &-\gamma_c\left(\frac{\partial}{\partial z}-\frac{2ie}{c}A_z\right)^2\Delta\end{aligned}$$

The total current in the ab plane is

$$\mathbf{j}=\sigma_n^{(ab)}\mathbf{E}-2ie\gamma_{\alpha b}\left[\Delta^*\left(\nabla-\frac{2ie}{c}\mathbf{A}\right)\Delta-\Delta\left(\nabla+\frac{2ie}{c}\mathbf{A}\right)\Delta^*\right] \tag{12.39}$$

while the total current along c has the form

$$j_z=\sigma_n^{(c)}E_z-2ie\gamma_c\left[\Delta^*\left(\frac{\partial}{\partial z}-\frac{2ie}{c}A_z\right)\Delta-\Delta\left(\frac{\partial}{\partial z}+\frac{2ie}{c}A_z\right)\Delta^*\right] \tag{12.40}$$

where the z axis is directed along the crystallographic c axis.
We follow the approach developed by Ivlev and Kopnin (1991). Assume that the normal-state conductivity satisfies the condition

$$\frac{\sigma_n^{(ab)}}{\sigma_n^{(c)}}=\frac{\gamma_{ab}}{\gamma_c}=\frac{\xi_{ab}^2}{\xi_c^2}=\frac{m_c}{m_{ab}} \tag{12.41}$$

since both the gradient terms in the Ginzburg-Landau equation and the conductivity tensor are proportional to the same factor

$$\frac{\partial E_n}{\partial p_i}\frac{\partial E_n}{\partial p_j}$$

averaged over the Fermi surface with the inverse impurity scattering cross section, see Section 9.3. This is true for dirty superconductors. In clean systems, eqn (12.41) is expected to hold only for isotropic scattering by impurities.

Consider the orientation of the magnetic field at an angle θ to the crystallographic c axis. We chose the coordinate frame (x', y', z') with the $x' = x$ axis perpendicular to the plane comprising the c axis and the magnetic field, see Fig. 12.2. The y' axis makes the angle θ with the ab plane. The order parameter and

Fig. 12.2. The coordinate frame (x, y, z) is associated with the crystallographic axes; the frame ($x^{\prime}, y^{\prime}, z^{\prime}$) has the z' along the magnetic field.

the magnetic field depend only on x' and y'. The order parameter equation takes the form

$$-\Gamma\left(\frac{\partial\Delta}{\partial t}+2ie\varphi\Delta\right)=-|\alpha|\Delta+\beta|\Delta|^2\Delta-\gamma_{ab}\left(\frac{\partial}{\partial x'}-\frac{2ie}{c}A_{x'}\right)^2\Delta$$
$$-\gamma(\theta)\left(\frac{\partial}{\partial y'}-\frac{2ie}{c}A_{y'}\right)^2\Delta \tag{12.42}$$

where we define

$$\gamma(\theta)=\gamma_{ab}\cos^2\theta+\gamma_c\sin^2\theta.$$

The current along the x axis is given by eqn (12.39) while

$$j_{y'}=\sigma_n(\theta)E_{y'}$$
$$-2ie\gamma(\theta)\left[\Delta^*\left(\frac{\partial}{\partial y'}-\frac{2ie}{c}A_{y'}\right)\Delta-\Delta\left(\frac{\partial}{\partial y'}+\frac{2ie}{c}A_{y'}\right)\Delta^*\right] \tag{12.43}$$

We define also

$$\sigma_n(\theta)=\sigma_n^{(ab)}\cos^2\theta+\sigma_n^{(c)}\sin^2\theta=\sigma_n^{(ab)}\gamma(\theta)/\gamma_{ab}.$$

Comparison of eqn (12.42) with eqn (7.33) shows, in particular, that the upper critical magnetic field for this configuration is (Kats 1969, 1970, Lawrence and Doniach 1971)

$$H_{c2}(\theta)=\frac{\Phi_0}{2\pi\xi_{ab}\xi(\theta)}$$

where $\xi^2(\theta)/\xi_{ab}^2=\gamma(\theta)/\gamma_{ab}$.

12.6.1 Low fields

For $H_{c1}\ll H\ll H_{c2}$, we have essentially a single-vortex problem similarly to that in Section 12.5.1. Making the scaling of variables

$$x' = \xi_{ab}\tilde{x}, y' = \xi(\theta)\tilde{y}$$

and $\mathbf{A} = (H/H_{c2})\,\tilde{\mathbf{A}}$ one can see that the static single-vortex problem reduces to an isotropic situation of Section 1.1.2. A static vortex determined by eqns (12.42), (12.39), and (12.43) has an axial symmetry in the space

$$\tilde{\rho} = \left(\frac{x'}{\xi_{ab}}, \frac{y'}{\xi(\theta)}\right)$$

such that

$$\Delta = \Delta_\infty f(\tilde{\rho}) \exp(i\tilde{\phi})$$

where $\tilde{\rho}$ and $\tilde{\phi}$ are the radius and the azimuthal angle in the coordinate frame $(\tilde{x}, \tilde{y})$. The function $f(\tilde{\rho})$ satisfies eqn (7.39) on page 151.

To calculate the flux flow conductivity we can use eqn (12.17) which is also valid for anisotropic superconductors. To evaluate the integrals in the r.h.s. of eqn (12.17) we need to find the gauge invariant potential Φ. Let us put

$$\Phi = \frac{1}{2e}\left[\frac{v_{Lx}\sin\tilde{\phi}}{\xi_{ab}} - \frac{v_{Ly}\cos\tilde{\phi}}{\xi(\theta)}\right]\mu(\tilde{\rho}).$$

The anisotropy is scaled out with this transformation due to eqn (12.41). The function μ satisfies eqn (12.20) where now

$$u = \frac{8e^2\Gamma\gamma_{ab}}{\beta\sigma_n^{(ab)}}$$

We find from eqn (12.17)

$$\frac{\Phi_0}{c}\left(\left[\mathbf{j}_{\mathrm{tr}} \times \hat{\mathbf{H}}\right]\cdot\mathbf{d}\right) = 2\pi\Gamma\Delta_\infty^2\xi_{ab}\xi(\theta)a\left(\frac{v_{Lx}d_x}{\xi_{ab}^2} + \frac{v_{Ly}d_y}{\xi^2(\theta)}\right)$$

where a is determined by eqn (12.23).
Expressing the vortex velocity in terms of the average electric field

$$\mathbf{v}_L = (cE_{y'}/B, -cE_{x'}/B) \tag{12.44}$$

we obtain the flux flow conductivity components for currents along the x axis and in the (x', y') plane:

$$\frac{\sigma_f^{(xx)}}{\sigma_n^{(ab)}} = \frac{\sigma_f^{(yy)}(\theta)}{\sigma_n(\theta)} = \frac{ua}{2}\frac{H_{c2}(\theta)}{B}. \tag{12.45}$$

This important result shows that, for the given orientation of the magnetic field, the ratio of the flux flow conductivity to the normal-state conductivity in the same direction in the plane perpendicular to the applied magnetic filed scales as the upper critical field.

12.6.2 High fields

For $H \to H_{c2}(\theta)$ we calculate the flux flow conductivity using a solution of the linearized time dependent equation as it was done on page 254 . We assume $\varphi = -E_{x'}x' - E_{y'}y'$ and $A_{y'} = Hx', \quad A_{x'} = 0$. Within the first order terms in E we find instead of eqn (12.37)

$$\Delta = \sum_n C_n \exp\left[i\left(qn + 2eE_{y'}t\right)\left(y' + cE_{x'}t/H\right)\right]$$
$$\times \exp\left[-\frac{1}{2\xi_{ab}}\left(x - \frac{cE_{x'}t}{H} - \frac{cqn}{2eH}\right)^2\right]$$
$$\times \exp\left[\frac{\Gamma\xi_{ab}}{\gamma_{ab}}\left(ieE_{x'}\xi_{ab} - eE_{y'}\xi(\theta)\right)\left(x - \frac{cqn}{2eH}\right)\right]$$

The solution describes the vortex lattice moving with the velocity eqn (12.44). The coefficients C_n satisfy the periodicity conditions for an Abrikosov lattice and are normalized according to the nonlinear eqn (12.42). Calculating the components of current we find (remind that we use $\kappa \gg 1$) the same scaling as for low fields,

$$\frac{\sigma_f^{(xx)}}{\sigma_n^{(ab)}} = \frac{\sigma_f^{(yy)}(\theta)}{\sigma_n(\theta)} = 1 + \frac{u}{2}\frac{H_{c2}(\theta) - B}{\beta_A H_{c2}(\theta)}$$

For a general orientation of the transport current in the (x', y') plane, the electric field is not parallel to the current, of course. However, it does not give rise to the Hall effect because the direction of electric filed does not change if the magnetic filed is reversed, $\mathbf{E}(\mathbf{H}) = \mathbf{E}(-\mathbf{H})$: both the vortex circulation and the vortex velocity change their signs keeping $\mathbf{E}$ unchanged.

19.1.7 12.7 Flux flow in layered superconductors

Flux flow in layered superconductors is a very rich and complicated phenomenon. Moreover, it is not fully investigated so far. The fundamental difference with respect to homogeneous isotropic and anisotropic superconductors is a strong interaction between vortices and the crystal structure itself known as intrinsic pinning. This introduces new features to the vortex dynamics. Motion of vortices in presence of pinning (including intrinsic pinning) is associated with deformations of the vortex lattice and should be considered together with elastic properties of vortex arrays. The problem of vortex pinning is a special issue in the vortex physics. The reader can find some relevant references in the review by Blatter et al. (1994). In this Section we consider two particular examples when the vortex deformations are either not important or can be easily taken into account. The first example is motion of vortices in a highly layered superconductor in a magnetic filed inclined with respect to the layers. In this case a vortex makes sharp bends before it crosses superconducting layers and penetrates through them in a form of two dimensional pancake vortices. The second example treats straight

Fig. 12.3. Motion of vortices in layered superconductors. Only pancake vortices move and produce dissipation in a presence of transport current $\mathbf{j}$

vortices aligned parallel to the layers when they move in the direction perpendicular to the layers such that they experience an intrinsic pinning described in Section 7.4.2.

12.7.1 Motion of pancakes vortices

Consider a highly layered superconductor in a magnetic field which is inclined to the conducting layers at some angle θ. We can describe a magnetic field penetration in terms of two

systems of vortices (Kes et al. 1990, Ivlev et al. 1990). The component parallel to the layers B_{ab} produces Josephson vortices whose cores fit in between the layers so that superconductivity on the layers is not essentially affected. This is equivalent to the assumption of a very large value of the in-plane upper critical field $H_{c2}(\pi/2)$. These vortices are strongly pinned by an interaction with the crystal structure. The magnetic field component B_c penetrates by forming pancake vortices having normal cores on the layers and screening currents flowing in the layers. Combining these two vortex systems, one can say that, for a tilted magnetic field, Josephson vortices (produced by the component B_{ab}) cross the layers by making kinks in places where they meet pancake vortices (produced by B_c), see Fig. 12.3. If a transport current flows along the layers, the kinks, i.e., the pancake vortices will move along the planes thus producing dissipation. Since the density of pancake vortices is proportional to $B_c/H_{c2}(0)$, the corresponding flux flow conductivity is

$$\frac{\sigma_f^{(xx)}}{\sigma_n^{(ab)}} = \frac{ua}{2}\frac{H_{c2}(0)}{B|\cos\theta|}$$

according to eqn (12.45). The flux flow conductivity depends only on the magnetic field component along the c axis.

12.7.2 Intrinsic pinning

Consider the situation when the magnetic field and vortices are aligned parallel to the layers. The transport current flows in the plane of layers such that the Lorentz force eqn (12.1) is perpendicular to the layers. We choose the z axis to be along the crystallographic c direction. The vortex displacement δz along the z axis is opposed by the intrinsic pinning force. In this Section, we consider the simplest case of a weakly layered structure. For this limit, the pinning force is given by eqn (7.44) on page 152. Balancing the Lorentz force and the pinning force we find the maximum depinning current

$$j_p = \frac{2\pi c U_0}{\phi_0 s} \tag{12.46}$$

If the transport current along the x axis is smaller than j_p, a vortex can not overcome the pinning force and remains trapped in between the layers. For higher currents, it starts to move along the z direction. The vortex motion is opposed also by friction such that the full force balance is

$$\eta^{(xx)}\frac{\partial\delta z}{\partial t} = \frac{\Phi_0}{c}\left[j - j_p\sin\left(\frac{2\pi\delta z}{s}\right)\right]. \tag{12.47}$$

The vortex viscosity introduced in eqn (12.38)

$$\eta^{(xx)} = \frac{B\Phi_0\sigma_f^{(xx)}}{c^2} = \frac{2\pi\Gamma\Delta_0^2\xi_{ab}}{\xi_c}$$

is expressed through the corresponding flux flow conductivity $\sigma_f^{(xx)}$. It is the conductivity in eqn (12.45) for the magnetic field orientation $\theta = \pi/2$.

A vortex which moves according to eqn (12.47) needs a time

$$t_0 = \frac{\eta^{(xx)}cs}{\Phi_0}\sqrt{j^2 - j_p^2}$$

to cover the distance s. The averaged velocity is $v_z = s/t_0$ which gives rise to the electric field in the x direction $E_x = Bv_z/c$ such that

$$E_x = \sigma_f^{(xx)} \sqrt{j^2 - j_p^2}$$

The I-V curve becomes nonlinear in presence of pinning. A finite voltage appears when $j > j_c$. For high currents, $j \gg j_c$, the linear dependence is restored.

19.1.8 12.8 Flux flow within a generalized TDGL theory

12.8.1 Dirty superconductors

In this Section we consider a situation where the superconductor can not be described by a simple TDGL model but is still within the range of parameters such that the generalized TDGL scheme derived in Section 11.2 is applicable. We remind that it requires slow gradients and time derivatives such that

$$Dk^2, \omega \ll \tau_{\text{ph}}^{-1}$$

Since, in presence of vortices, the gradients are $k \sim \xi^{-1}(T)$, the condition of applicability is $\Delta^2/T \ll \tau_{\text{ph}}^{-1}$ or $T_c - T \ll \tau_{\text{ph}}^{-1}$. Note that, depending on temperature, the order parameter can be either outside of, $\Delta \gg \tau_{\text{ph}}^{-1}$, or inside, $\Delta \ll \tau_{\text{ph}}^{-1}$, the gapless region.

We start with eqn (12.15) and transform it as follows

$$\begin{aligned}\frac{\Phi_0}{c}\left[\mathbf{j}_{\text{tr}} \times \hat{\mathbf{H}}\right] &= \int_{S_0} \left[e^{i\chi} \nabla|\Delta| \frac{\delta \mathcal{F}_{sn}}{\delta \Delta} + e^{-i\chi} \nabla|\Delta| \frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} \right. \\ &\quad \left. - \frac{2ie}{\hbar c} \mathbf{Q} \left(\Delta \frac{\delta \mathcal{F}_{sn}}{\delta \Delta} - \Delta^* \frac{\delta \mathcal{F}_{sn}}{\delta \Delta^*} \right) + \frac{1}{c} \left[\mathbf{j}_n \times \mathbf{H}\right] \right] dS \\ &= \int_{S_0} \left[\nabla|\Delta| \frac{\delta \mathcal{F}_{sn}}{\delta |\Delta|} + \frac{1}{c} \mathbf{Q} \operatorname{div} \mathbf{j}_s + \frac{1}{c} \left[\mathbf{j}_n \times \mathbf{H}\right] \right] dS \end{aligned} \tag{12.48}$$

Using eqns (11.25) and (11.26) we can write the force balance equation (12.48) as

$$\begin{aligned}\frac{\Phi_0}{c}\left[\mathbf{j}_{\text{tr}} \times \hat{\mathbf{H}}\right] =& \nu(0) \int_{S_0} \left[- \left(\nabla|\Delta| \frac{\partial|\Delta|}{\partial t} \right) \frac{\pi}{4T} \sqrt{1 + 4|\Delta|^2 \tau_{\text{ph}}^2} \right. \\ & \left. + \frac{\pi e^2}{cT} \frac{\Phi |\Delta|^2}{\sqrt{1 + 4|\Delta|^2 \tau_{\text{ph}}^2}} \mathbf{Q} \right] dS + \int_{S_0} \left[\mathbf{j}_n \times \mathbf{H}\right] dS \end{aligned} \tag{12.49}$$

In this Section, we only consider the limit of low fields when vortices are well separated. One can neglect the magnetic field in eqn (12.49) and put

$$\mathbf{Q} = -\frac{c}{2e} \nabla \chi$$

Using eqn (11.27) we find for the function μ defined through eqn (12.19)

$$\frac{d^2\mu}{d\rho^2} + \frac{1}{\rho} \frac{d\mu}{d\rho} - \frac{\mu}{\rho^2} - \frac{\pi |\Delta|^2 \mu}{2DT \sqrt{1 + 4|\Delta|^2 \tau_{\text{ph}}^2}} = 0$$

This equation differs from eqn (12.20) by a more complicated Δ-dependence of the last term. The boundary condition is $\mu \to 1/\rho$ for $\rho \to 0$. Replacing $\partial|\Delta|/\partial t$ with $-\left(\mathbf{v}_L \cdot \nabla\right)|\Delta| = -v_{L\rho}(\partial|\Delta|/\partial\rho)$ in eqn (12.49) we obtain

$$\frac{\Phi_0}{c}\left[\mathbf{j}_{\rm tr}\times\hat{\mathbf{H}}\right] = \mathbf{v}_L\frac{\pi^2\nu(0)}{4T}\int_0^{r_0}\left[\left(\frac{\partial|\Delta|}{\partial\rho}\right)^2\sqrt{1+4|\Delta|^2\tau_{\rm ph}^2}\right.$$
$$\left.+\frac{\mu}{\xi\rho}\frac{|\Delta|^2}{\sqrt{1+4|\Delta|^2\tau_{\rm ph}^2}}\right]\rho d\rho \tag{12.50}$$

The integral is extended over the radius r_0 of the unit cell of the vortex lattice. Equation (12.50) determines the flux flow conductivity. Using $\mathbf{v}_L = c[\mathbf{E}\times\hat{\mathbf{H}}]/B$ we obtain

$$\sigma_f = \frac{c^2\nu(0)\pi^2\Delta_\infty^2 F(q)}{4T\Phi_0 B} \tag{12.51}$$

where $q = 2\Delta_\infty\tau_{\rm ph}$, and Δ_∞ is the order parameter magnitude far from the vortex. The function $F(q) = F_1(q) + F_2(q)$ is defined by two equations,

$$F_1(q) = \int_0^{r_0}\left(\frac{\partial f}{\partial\rho}\right)^2\sqrt{1+q^2f^2}\rho d\rho \tag{12.52}$$

$$F_2(q) = \int_0^{r_0}\frac{\mu}{\xi\rho}\frac{f^2}{\sqrt{1+q^2f^2}}\rho d\rho \tag{12.53}$$

where, as before, $f = |\Delta|/\Delta_\infty$. Using the dirty-limit value

$$\xi^2 = \frac{\pi^3 T_c D}{7\zeta(3)\Delta_\infty^2}$$

from eqns (1.9) and (1.15), we find

$$\sigma_f = \beta\sigma_n\frac{H_{c2}}{B} \tag{12.54}$$

where

$$\beta = \frac{uF(q)}{2}$$

and $u = \pi^4/14\zeta(3) \approx 5.79$.
In the gapless regime, $q \ll 1$, the functions $F_1(q)$ and $F_2(q)$ are of the same order so that $F(q) \to a$, and the result of the TDGL model eqn (12.24) with $\beta = ua/2$ is reproduced. A more realistic case, when the order parameter is outside the gapless region, corresponds to $q \gg 1$. For $q \gg 1$ when $T_c\tau_{\rm ph}\sqrt{1-T/T_c} \gg 1$, the electric field penetration length is large,

$$l_E = \sqrt{(q/uf)}\xi$$

The function μ under the integral in eqn (12.53) is $\mu \approx 1/\rho$ within a region larger than the vortex core. The term $F_2(q)$ diverges logarithmically and is cut off at distances $\rho \sim R_0$ such that

$$R_0 = \min\{l_E, r_0\} \tag{12.55}$$

where $r_0 \sim \xi(T)\sqrt{H_{c2}/B}$ is the intervortex distance,

$$F_2(q) = \frac{1}{q\xi}\ln\frac{R_0}{\xi} \tag{12.56}$$

However, the prefactor in eqn (12.56) is small. The largest contribution comes thus from $F_1(q)$ in eqn (12.52) which gives

$$F(q) \approx F_1(q) = q \int_0^\infty f \left(\frac{\partial f}{\partial \rho}\right)^2 \rho d\rho$$

The integral is of the order of unity. The flux flow conductivity is thus much higher than what is predicted by the Bardeen and Stephen model. Indeed,
$\beta \sim \sqrt{1-T/T_c}\,(T_c\tau_{\rm ph}) \gg 1$. It is determined by a relaxation of the order parameter (Tinkham's mechanism) via slow inelastic electron-phonon interactions while the contribution from normal currents in the vortex core is smaller by a factor on order q^{-2}. The factor β decreases proportionally to $\sqrt{1-T/T_c}$ when T approaches T_c.

12.8.2 d-wave superconductors

The same approach can be applied to calculate the flux flow conductivity in a ^{d}-wave superconductor (Kopnin 1998) described by eqns (11.46) and (11.47). Remind that d-wave superconductors should be clean, $T_c\tau_{\rm imp} \gg 1$. However, the applicability of eqns (11.46) and (11.47) requires that the temperatures are close to T_c such that $\ell \ll \xi(T)$ which is equivalent to $1-T/T_c \ll (\tau_{\rm imp}\,T_c)^2$. We obtain from eqn (12.48)

$$\begin{aligned}\frac{\Phi_0}{c}\left[\mathbf{j}_{\rm tr}\times\hat{\mathbf{H}}\right] =& \frac{\pi\nu(0)}{8T_c}\int_{S_0}\Big[2\langle\nabla|\Delta_{\mathbf{p}}|(\mathbf{v}_L\cdot\nabla)|\Delta_{\mathbf{p}}|\rangle_{\mathbf{p}} \\ &+ \frac{8e^2}{c}\mathbf{Q}\Phi\langle|\Delta_{\mathbf{p}}\Big|^2\Big\rangle_{\mathbf{p}} - U\nabla\langle|\Delta_{\mathbf{p}}|^2\Big\rangle_{\mathbf{p}}\Big]\,dS \\ &+ \frac{\sigma_n}{c}\int_{S_0}[\mathbf{E}\times\mathbf{H}]dS \end{aligned} \tag{12.57}$$

We keep the subscript $\mathbf{p}$ to distinguish the momentum average from the average $\langle\ldots\rangle_L$ taken over the vortex array.

The function U is found from eqn (11.41) under the condition that it does not increase with distance. Assuming that the diffusion is faster than the inelastic relaxation we find

$$\nabla U = \mathbf{v}_L\frac{\tau_{\rm imp}}{D}\left[\langle|\Delta_{\mathbf{p}}|^2\rangle_{\mathbf{p}} - \left\langle\langle|\Delta_{\mathbf{p}}|^2\rangle_{\mathbf{p}}\right\rangle_L\right] \tag{12.58}$$

In the low field limit $H \ll H_{c2}$ we have $\mathbf{Q} = -(c/2e)\nabla\chi$. Moreover, the scalar potential is $\Phi = (1/2e)\partial\chi/\partial t$ because $l_E \gg \xi$. For a vortex lattice with a symmetry not lower than tetragonal,

$$\int_{S_0}\langle\nabla_i|\Delta_{\mathbf{p}}|\nabla_k|\Delta_{\mathbf{p}}|\rangle_p dS = \frac{1}{2}\delta_{ik}\int_{S_0}\left\langle(\nabla|\Delta_{\mathbf{p}}|)^2\right\rangle_{\mathbf{p}} dS$$

$$\int_{S_0}(\nabla_i\chi\nabla_k\chi)\langle|\Delta_{\mathbf{p}}\Big|^2\Big\rangle_p dS = \frac{1}{2}\delta_{ik}\int_{S_0}(\nabla\chi)^2\langle|\Delta_{\mathbf{p}}\Big|^2\Big\rangle_{\mathbf{p}} dS$$

As a result, the flux flow conductivity becomes

$$\sigma_f = \sigma_n + \frac{\pi^2\nu(0)c^2\langle|\Delta_{\mathbf{p}}|^2_\infty\rangle_{\mathbf{p}} F_d}{4T_c\Phi_0 B} \tag{12.59}$$

Equation (12.59) looks exactly as eqn (12.51) with the difference that the factor F_d is now

$$F_d = \left(2\pi\langle|\Delta_{\mathbf{p}}|^2_\infty\rangle_{\mathbf{p}}\right)^{-1}\int_{S_0}\Big[\left\langle(\nabla|\Delta_{\mathbf{p}}|)^2\right\rangle_{\mathbf{p}} + (\nabla\chi)^2\langle|\Delta_{\mathbf{p}}\Big|^2\Big\rangle_{\mathbf{p}}$$

$$+\frac{\tau_{\rm imp}}{D}\left[\langle|\Delta_{\mathbf{p}}|^2\rangle_{\mathbf{p}}-\left\langle\langle|\Delta_{\mathbf{p}}|^2\rangle_{\mathbf{p}}\right\rangle_L\right]^2\Bigg]\,dS \tag{12.60}$$

instead of eqns (12.52), (12.53). The largest contribution comes from the second term under the integral which is logarithmically diverging at distances $r \sim R_0$ where R_0 is determined by eqn (12.55). Therefore, $F_d \approx \ln[R_0/\xi(T)]$. We had already such logarithmic divergence of F due to a long relaxation length l_E in eqn (12.56) of the previous Section. However, it was not important for the flux flow conductivity of dirty superconductors. On the contrary, it dominates in the present example. Writing eqn (12.51) in the form of eqn (12.54) we find that $\beta \sim F/T_c\tau_{\rm imp} \ll 1$ because $T_c\tau_{\rm imp} \gg 1$ for a clean superconductor.

12.8.3 Discussion: Flux flow conductivity

We can now summarize the results obtained for the flux flow conductivity within the TDGL model. The model identifies two principle sources for dissipation associated with a moving vortex. The first is due to variations of the order parameter with time as vortex passes through a superconductor. The simple TDGL model describes the process of the order parameter relaxation in terms of a characteristic relaxation time τ_Δ which is of the order of $(T_c - T)^{-1}$. The generalized TDGL scheme treats this process in a more detailed manner. We encounter a concept of nonequilibrium excitations which relax through their interactions with phonons. Deviation from equilibrium is created by order parameter variations which cause changes in the energy spectrum and thus produce redistribution of excitations away from equilibrium. The order parameter returns to its undisturbed value only together with the relaxing excitations. Its characteristic relaxation time depends now on the electron-phonon mean free time $\tau_{\rm ph}$:

$$\tau_\Delta \sim \tau_{\rm ph}\left(1-\frac{T}{T_c}\right)^{-1/2}$$

and is much slower than $(T_c - T)^{-1}$ when the parameters are outside the gapless region. This results in a larger dissipation and larger conductivity as compared to the simple TDGL model.

The second mechanism is associated with currents which flow through vortex cores. Near a vortex core, the supercurrent converges into a normal current according to eqn (11.26) through the current conservation (the charge neutrality)

$$\operatorname{div}\mathbf{j}_s + \operatorname{div}\mathbf{j}_n = 0$$

It is the normal current which dissipates energy. The conversion rate is determined by the r.h.s. of eqns (1.79) or (11.26), or by the r.h.s. of eqn (11.44) in the d-wave case. For a gapless regime, the conversion is fast. In our examples, the gapless regime in s-wave superconductors is associated with magnetic impurities or with an electron-phonon interaction very near the critical temperature. Here the dissipation caused by the normal currents is high. In a gapless situation for
a d-wave superconductor near T_c, it gives even the largest contribution to the flux flow conductivity. However, the conversion rate is slow in superconductors which are far from the gapless regime, see eqn (12.56) in the limit $\Delta_\infty\tau_{\rm ph} \gg 1$.

The obtained results confirm our general expectation that the vortex dynamics is governed by kinetics of excitations driven out of equilibrium by a moving vortex. In the following Chapter we consider an example when the kinetics of excitations creates even larger dissipation than that predicted by the generalized TDGL approach.

What the generalized TDGL model can not describe completely is the Hall effect associated with the transverse gyroscopic force on a moving vortex. To conclude the Chapter on the

TDGL model, we discuss how one should modify the TDGL scheme to incorporate at least some possible mechanisms responsible for the vortex Hall effect.

19.1.9 12.9 Flux flow Hall effect

We start with a very brief summary on the Hall effect in the normal state. A more detailed analysis of the Hall effect in normal metals can be found, for example, in the book by Abrikosov (1998). In presence of a magnetic field, the current in a normal metal has generally both longitudinal and transverse components with respect to the electric field

$$\mathbf{j}_n = \sigma_n \mathbf{E} + \sigma_n^H [\mathbf{E} \times \hat{\mathbf{H}}] \tag{12.61}$$

where σ_n^H is the normal-state Hall conductivity. The Hall angle is defined as

$$\tan \Theta_H = \sigma_n^H / \sigma_n.$$

The Hall conductivity has a simple form for a metal with electronic spectrum $E_n = \mathbf{p}^2/2m^*$. For low magnetic fields,

$$\sigma_n^H = \sigma_n \omega_c \tau \tag{12.62}$$

where

$$\omega_c = \frac{eH}{m^* c}$$

is the cyclotron frequency. The low field limit implies $\omega_c \tau \ll 1$.
Let us estimate the parameter $\omega_c \tau$ for magnetic fields less than H_{c2}. For dirty superconductors

$$\omega_c \tau \sim \frac{\tau}{2m^* \xi^2} \frac{H}{H_{c2}} \sim \frac{T_c}{E_F} \left(1 - \frac{T}{T_c}\right) \frac{H}{H_{c2}} \tag{12.63}$$

It is very small because, normally, $T_c/E_F \ll 1$. In this limit, the Hall conductivity is much smaller than the Ohmic component and the Hall angle is also small. We thus expect that the Hall effect in dirty superconductors can not be large. We shall see nevertheless, that it exhibits interesting features specific only
for the superconducting state. The most interesting property is that the Hall angle can change its sign after transition into a superconducting state, this is known as the Hall effect anomaly. The Hall anomaly has been observed in many experiments starting with the conventional Niobium and Vanadium type-II superconductors (Niessen et al. 1967, Noto et al. 1976) and later re-discovered also in high temperature superconductors (see, for example, Hagen et al. 1993). This section describes the aspects of the flux flow Hall effect that can be identified within the TDGL theory.

12.9.1 Modified TDGL equations

As we have seen, the usual TDGL theory gives zero Hall effect. This is a result of pure dissipative nature of the TDGL equations. To incorporate non-dissipative forces, we have to modify our TDGL equations (Kopnin et al. 1992, 1993, Dorsey 1992). This can be done in two ways. First, we write the normal current in the form of eqn (12.61). Second, we allow for an imaginary part of the relaxation constant Γ

$$\Gamma = \Gamma' + i\Gamma'' \tag{12.64}$$

The origin of an imaginary part of Γ is as follows (Aronov and Rapoport 1992). We know that a moving vortex induces a scalar potential proportional to the vortex velocity. The induced potential adds to the chemical potential of the superconductor: $\mu = \mu_0 - e\varphi$. On the other hand, the critical temperature depends on the chemical potential, therefore, the coefficient α in the GL equation becomes

$$\alpha = \alpha_0 - \frac{\partial \alpha}{\partial \mu} e\varphi = -\nu \left(1 - \frac{T}{T_c}\right) + \frac{\nu}{T_c} \frac{\partial T_c}{\partial \mu} e\varphi. \tag{12.65}$$

Since the critical temperature in the BCS theory is $T_c = \Omega_{\mathrm{BCS}} \exp(-1/\lambda)$ where Ω_{BCS} is the cutoff energy of pairing interaction, and $\lambda = g\nu$ is the coupling constant, we have

$$\frac{\partial T_c}{\partial \mu} = \frac{T_c}{\nu\lambda} \frac{\partial \nu}{\partial \mu}$$

Therefore, correction to α becomes

$$\delta\alpha = \frac{1}{\lambda} \frac{\partial \nu}{\partial \mu} e\varphi$$

To preserve the gauge invariance, we should write the correction in the form

$$\delta\alpha\Delta = -i \frac{1}{2\lambda} \frac{\partial \nu}{\partial \mu} \left(\frac{\partial \Delta}{\partial t} + 2ie\varphi\Delta \right) \tag{12.66}$$

which results in an imaginary part of the coefficient in front of the orderparameter time-derivative

$$\Gamma'' = -\frac{1}{2\lambda} \frac{\partial \nu}{\partial \mu} \tag{12.67}$$

The imaginary part Γ'' is small

$$\zeta = -\frac{\Gamma''}{\Gamma'} \sim \frac{1}{\lambda} \frac{T_c}{E_F}, \tag{12.68}$$

however, it has the correct order of magnitude to account for a Hall effect in dirty superconductors. For example, for usual superconductors, $\zeta \sim 10^{-2} \div 10^{-3}$ and $\zeta \sim 10^{-1} \div 10^{-2}$ for HTSC.

The imaginary part of Γ results in a modification of the equation for Φ. It takes the form

$$\begin{aligned} \nabla^2\Phi - \frac{8e^2\Gamma'|\Delta|^2}{\sigma_n}\Phi = &- \frac{1}{c} \operatorname{div} \frac{\partial \mathbf{Q}}{\partial t} + \frac{2e\Gamma''}{\sigma_n} \frac{\partial |\Delta|^2}{\partial t} \\ &- \frac{4\pi \left(\sigma_n^H / H\right)}{\sigma_n^2 c} \left[\mathbf{Ej} + \frac{\partial}{\partial t} \left(\frac{H^2}{8\pi} \right) \right]. \end{aligned} \tag{12.69}$$

The last line is obtained under the assumption that $\sigma_n^H \propto H$.

12.9.2 Hall effect: Low fields

We calculate the Ohmic and Hall components of conductivity in the same way as in the previous Section. The force balance is obtained from eqn (12.15):

$$\frac{\Phi_0}{c}\left[\mathbf{j}_{\rm tr}\times\hat{\mathbf{H}}\right]\cdot\mathbf{d}=-\int_{S_0}\left[(\mathbf{d}\nabla\Delta)\Gamma\left(\frac{\partial\Delta^*}{\partial t}-2ie\phi\Delta^*\right)+c.c.\right]dS$$
$$=2\Gamma'\int_{S_0}\left[(\mathbf{v}_L\nabla|\Delta|)\,(\mathbf{d}\nabla|\Delta|)-2e|\Delta|^2(\mathbf{d}\nabla\chi)\Phi\right]dS$$
$$+2\Gamma''\int_{S_0}\left[|\Delta|\,(\mathbf{v}_L\nabla|\Delta|)\,[\mathbf{d}\nabla\chi]+2e\Phi|\Delta|(\mathbf{d}\nabla|\Delta|)\right]dS$$

We put $\Phi=\Phi_1+\Phi_2$ where, as in eqn (12.19),

$$\Phi_1=-\frac{v_{L\phi}\mu_1}{2e\xi} \tag{12.70}$$

while

$$\Phi_2=-\zeta\frac{v_{L\rho}\mu_2}{2e\xi} \tag{12.71}$$

Equations for $\mu_1(\rho)$ and $\mu_2(\rho)$ follow from eqn (12.69) where one can neglect the second line, since $\sigma_n^H/\sigma_n\sim(H/H_{c2})\,\zeta\ll\zeta$ for $H\ll H_{c2}$. The function μ_1 satisfies equation (12.20) while μ_2 is to be found from

$$\frac{d^2\mu_2}{d\rho^2}+\frac{1}{\rho}\frac{d\mu_2}{d\rho}-\frac{\mu_2}{\rho^2}-\frac{f^2\mu_2}{l_E^2}=-\frac{\xi}{2l_E^2}\frac{df^2}{d\rho} \tag{12.72}$$

with the condition $\mu_2\to 0$ for both $\rho\to 0$ and $\rho\to\infty$.
The transport current becomes

$$\mathbf{j}_{\rm tr}=\frac{2\pi c\Gamma'\Delta_\infty^2}{\Phi_0}\left(a\left[\hat{\mathbf{H}}\times\mathbf{v}_L\right]+b\zeta\,{\rm sign}(e)\mathbf{v}_L\right) \tag{12.73}$$

The sign of the electron charge in the second term appears because the sense of the phase circulation is positive for positive charge of carriers when the direction
of magnetic field coincides with the vortex circulation, and it is negative for a negative charge of carriers. Expressing the vortex velocity through the average electric field we find

$$\mathbf{j}_{\rm tr}=\sigma_{\rm O}\mathbf{E}+\sigma_{\rm H}[\mathbf{E}\times\hat{\mathbf{H}}]$$

where the longitudinal (Ohmic) flux flow conductivity $\sigma_{\rm O}$ is given by eqn (12.24) with the constant a defined by eqn (12.23). The Hall conductivity is

$$\sigma_{\rm H}={\rm sign}(e)\sigma_n\frac{ub\zeta}{2}\left(\frac{H_{c2}}{B}\right) \tag{12.74}$$

The constant b is

$$b=\int_0^\infty\left[\frac{1}{2}\left(\xi+\rho\mu_1\right)\frac{df^2}{d\rho}-f^2\mu_2\right]\frac{d\rho}{\xi} \tag{12.75}$$

It is $b=0.27$ for $u=5.79$.

12.9.3 High fields

Equation (12.15) gives

$$\frac{1}{c}\left[(\mathbf{j}_{\rm tr}-\mathbf{j}_n)\times\mathbf{B}\right]=-\left\langle\Gamma^*\left[\mathbf{d}\left(\nabla-\frac{2ie}{\hbar c}\mathbf{A}\right)\Delta\right]\left[\frac{\partial\Delta^*}{\partial t}-\frac{2ie\phi}{\hbar}\Delta^*\right]+c.c.\right\rangle_L.$$

Using eqns (12.30) and eqn (12.31) we find

$$\begin{aligned}\mathbf{j}_{\rm tr} &= \mathbf{j}_n + 2|e|\Gamma'\left\langle|\Delta|^2\right\rangle_L \left(\left[\hat{\mathbf{H}}\times\mathbf{v}_L\right] + \mathrm{sign}(e)\zeta\mathbf{v}_L\right)\\ &= \left(\sigma_n + 4e^2\xi^2\Gamma'\left\langle|\Delta|^2\right\rangle_L\right)\mathbf{E}\\ &\quad + \left(\sigma_n^H + 4e^2\xi^2\Gamma'\zeta\,\mathrm{sign}(e)\left\langle|\Delta|^2\right\rangle_L\right)[\mathbf{E}\times\hat{\mathbf{H}}]\end{aligned} \tag{12.76}$$

The longitudinal (Ohmic) conductivity is the same as in eqn (12.34), while the Hall conductivity has the form

$$\begin{aligned}\sigma_{\rm H} &= \sigma_n^H + 4e^2\xi^2\Gamma'\zeta\,\mathrm{sign}(e)\left\langle|\Delta|^2\right\rangle\\ &= \sigma_n^H + \mathrm{sign}(e)\sigma_n\frac{\zeta u}{2}\frac{(1-B/H_{c2})}{\beta_A\left[1-(1/2\kappa^2)\right]}\end{aligned} \tag{12.77}$$

12.9.4 Discussion: Hall effect

The TDGL model gives, of course, an oversimplified description of the Hall effect. This model predicts a Hall angle $\Theta_H \sim \zeta \sim \lambda^{-1}(T_c/E_F)$. We shall see later that, in clean superconductors with $\ell \gg \xi(T)$, the mechanisms of the Hall effect are more complicated. The resulting Hall conductivity appears to be much larger than what we obtain from our simple TDGL model except for a relatively short mean free path $\ell \sim \xi(T)$. However, the contribution which we consider here is also present in clean superconductors. Indeed, eqn (12.65) is quite general and does not depend on the quasiparticle mean free path. To make a bridge between

Fig. 12.4. The Hall angle as a function of magnetic field. The sign of the Hall angle is not changed after a transition into the superconducting state if the signs of ω_{c} and of ζ are the same (curve 1). The sign reverses if ω_c and of ζ have different signs (curve 2). The thick dashed line is an extrapolation of the normal-state Hall angle into the superconducting region.

the general situation and the TDGL model we note that, in superconductors which are far from the gapless conditions like the one considered in Section 12.8, the electric field penetration length l_E is usually long compared to $\xi(T)$. In this case, eqn (12.72) gives $\mu_2 = 0$ while eqn (12.20) results in $\mu_1 = \xi/\rho$. In this limit, we obtain $b = 1$ and

$$\sigma_{\rm H} = \mathrm{sign}(e)\sigma_n\frac{u\zeta}{2}\left(\frac{H_{c2}}{B}\right).$$

Using $\Gamma'' = -(1/2\lambda)(d\nu/d\epsilon)$ and $\Gamma' = \pi\nu/8T_c$ with the expression for H_{c2} and the clean-limit value

$$\xi^2(T) = \pi^3 T_c D/7\zeta(3)\Delta_\infty^2$$

we find

$$\sigma_{\mathrm{H}} = \delta N_\Delta ec/B \tag{12.78}$$

We denote here

$$\delta N_\Delta = \left(\frac{1}{\lambda}\frac{d\nu}{d\mu}\right)\Delta_\infty^2 \tag{12.79}$$

the "virtual" variation in the electron density caused by the change in the electronic spectrum after transition into the superconducting state. Equation (12.78) was obtained by Feigel'man et al. (1995) and by van Otterlo et al. (1995). Of course, there is no real density change in a superconductor because of the charge neutrality: all the variations are compensated by the corresponding variations in the chemical potential. Equation (12.78) appears to be exact from the point of view of the microscopic theory (Kopnin 1996) in the limit $T \to T_c$, see Section 14.6.2.

Note one interesting feature of the obtained expressions for the Hall conductivities. Equations (12.74) and (12.77) show that the sign of the Hall effect in the superconducting state near the critical temperature is determined by

$$\mathrm{sign}(e\zeta) = \mathrm{sign}\left(e\frac{\partial\nu}{\partial\mu}\right)$$

which can be different from the sign (e/m^*) for a metal whose Fermi surface has both electron-like and hole-like pockets. If these signs are different, the Hall angle changes its sign after a transition into the superconducting state and the Hall anomaly appears (see Fig. 12.4).

The suggested mechanism takes into account the effect of vortex motion on the pairing interaction due to local variations in the chemical potential of superconducting electrons $\mu_s = (1/2)\partial\chi/\partial t$. This mechanism is of a thermodynamic nature and is not related directly to the kinetics of nonequilibrium excitations. We shall see later that interaction of vortices with excitations in clean superconductors can produce much larger flux flow Hall effect: excitations play a prominent role again.

19.2 VORTEX DYNAMICS IN DIRTY SUPERCONDUCTORS

The force exerted on a vortex from the environment is derived microscopically. The kinetic equation is solved for the distribution function of excitations driven out of equilibrium by the moving vortex; the flux flow conductivity in a dirty superconductor is calculated. The vortex viscosity appears to be much larger than what is predicted by the TDGL model.

19.2.1 13.1 Microscopic derivation of the force on moving vortices

The TDGL scheme can only be applied in very special situations under rather restricting conditions of gapless superconductivity. This Chapter treats the vortex dynamics in a more general case when nonstationary processes in a superconductor go beyond the TDGL model. We shall see that kinetics of nonequilibrium excitations makes the nonstationary behavior of a superconductor more complicated and diverse.

The complexity of the problem gives rise to the expectation that it would not be easy to calculate anything beyond the simple TDGL equations. Fortunately, this is not exactly the case, especially when we are interested in a linear response of a vortex array to applied perturbations. The major simplification is that the force on the moving vortex can be expressed through the characteristics of a static vortex and through the solutions to the kinetic equations which only contain the order parameter and magnetic field for a steady vortex array. The force thus does not contain distortions of the order parameter and of the magnetic field caused by the moving vortex. Restricting ourselves to the linear approximation in the vortex velocity we start with the derivation of the general expression for the force which acts on a moving vortex from the environment.

13.1.1 Variation of the thermodynamic potential

Let us consider a variation of the thermodynamic potential eqn (3.71) derived in Section 3.3. We have in the real frequencies representation

$$\int \delta\Omega_{tot} d^3r = -\nu(0)\,\mathrm{Tr}\int \left[\delta\check{H}(\mathbf{r})\check{g}_\epsilon^{(\mathrm{st})}(\mathbf{p},\mathbf{r})\right]\frac{d\Omega_p}{4\pi}\frac{d\epsilon}{4}d^3r + \int \frac{\delta\left(\Delta_\mathbf{p}(\mathbf{r})\Delta_\mathbf{p}^*(\mathbf{r})\right)}{|g|}\frac{d\Omega_\mathbf{p}}{4\pi}d^3r + \frac{1}{c}\int \mathbf{j}\cdot\delta\mathbf{A}d^3r \tag{13.1}$$

Here

$$\Omega_{tot} = \Omega_{sn} + \frac{H^2}{8\pi}$$

We remind that the Hamiltonian is

$$\check{H} = \begin{pmatrix} -(e/c)\mathbf{v}_F\cdot\mathbf{A} & -\Delta_\mathbf{p} \\ \Delta_\mathbf{p}^* & (e/c)\mathbf{v}_F\cdot\mathbf{A} \end{pmatrix}$$

in the quasiclassical approximation. The variation eqn (13.1) is taken at a constant chemical potential.

When the system is in a nonstationary state, the concept of thermodynamic potential can be used only if deviations from the equilibrium are small. The total Green function describing the nonstationary state in frequency representation is

$$\check{g}^K_{\epsilon_+,\epsilon_-} = \check{g}_\epsilon^{(\mathrm{st})}2\pi\delta(\omega) + \check{g}^{(\mathrm{nst})}_{\epsilon_+,\epsilon_-} \tag{13.2}$$

where $\epsilon_\pm = \epsilon \pm \omega/2$. It is the stationary part $\check{g}^{(\mathrm{st})}$ which enters the variation of the thermodynamic potential in eqn (13.1). We substitute it with the difference $\check{g}^K_{\epsilon_+,\epsilon_-} - \check{g}^{(\mathrm{nst})}_{\epsilon_+,\epsilon_-}$, and take into account that the order parameter and the current in the nonstationary state satisfy equations (9.17) and (9.18). We obtain

$$\int \delta\left(\Omega_{sn} + \frac{H^2}{8\pi}\right)d^3r = \nu(0)\,\mathrm{Tr}\int\left[\delta\check{H}(\mathbf{r})\check{g}_\epsilon^{(\mathrm{nst})}(\mathbf{p},\mathbf{r},t)\right]\frac{d\Omega_p}{4\pi}\frac{d\epsilon}{4}d^3r \tag{13.3}$$

The terms with $\check{g}^K$ disappear due to eqns (9.17) and (9.18). Remind that $\lambda = |g|\nu(0)$.

Equation (13.3) determines the variation of the thermodynamic potential with respect to Δ and $\mathbf{A}$. For example,

$$\frac{\delta\Omega_{sn}}{\delta\Delta_\mathbf{p}^*} = \nu(0)\int V(\mathbf{p},\mathbf{p}')\,f_\epsilon^{(\mathrm{nst})}(\mathbf{p}'_F,\mathbf{k};t)\,\frac{d\epsilon}{4}\frac{d\Omega_{p'}}{4\pi} \tag{13.4}$$

and

$$\frac{\delta}{\delta \mathbf{A}}\left(\Omega_{sn}+\frac{H^2}{8\pi}\right)=\frac{1}{c}\mathbf{j}^{(\mathrm{nst})} \tag{13.5}$$

where

$$\mathbf{j}^{(\mathrm{nst})}=-e\nu(0)\,\mathrm{Tr}\int \mathbf{v}_F\check{\tau}_3\check{g}_\epsilon^{(\mathrm{nst})}(\mathbf{p},\mathbf{r},t)\frac{d\Omega_p}{4\pi}\frac{d\epsilon}{4} \tag{13.6}$$

Equations (13.4) and (13.5) are microscopic counterparts of the corresponding TDGL equations. The right-hand sides of these equations are proportional to deviations from equilibrium; they vanish in a stationary state.

13.1.2 Force on vortices

Consider a variation of the total thermodynamic potential in the form

$$\delta\Omega_{tot}=\int\left[\frac{\delta\Omega_{tot}}{\delta\Delta}(\mathbf{d}\cdot\nabla)\Delta+c.c.+\frac{\delta\Omega_{tot}}{\delta\mathbf{A}}(\mathbf{d}\cdot\nabla)\mathbf{A}+\frac{\delta\Omega_{tot}}{\delta\mu}(\mathbf{d}\cdot\nabla)\mu\right]dV \tag{13.7}$$

where $\mathbf{d}$ is an arbitrary constant vector of an infinitesimal translation. The last term can be written as

$$-\int N\nabla\mu dV=-\int\nabla\left(N\mu_1\right)dV+\int\mu_1\nabla N dV$$

where $\mu_1=\mu-E_F$ is the deviation of the chemical potential from its value in the normal state. Equation (13.7) takes the form

$$\delta\mathcal{F}_{tot}=\int\left[\frac{\delta\Omega_{tot}}{\delta\Delta}(\mathbf{d}\cdot\nabla)\Delta+\text{ c.c. }+\frac{\delta\Omega_{tot}}{\delta\mathbf{A}}(\mathbf{d}\cdot\nabla)\mathbf{A}\right]dV \tag{13.8}$$

where

$$\mathcal{F}=\Omega+\int N\mu_1 dV$$

is the free energy which is a function of N instead of μ. The gradient of the particle density is omitted because the density is constant due to the charge neutrality. Equation (13.8) follows from the general property of thermodynamic functions: Variation of the free energy $\mathcal{F}$ for constant N is equal to the variation of Ω for a constant μ. The gradient of free energy density for a constant number of particles gives the density of force acting on the vortex from environment.

The r.h.s. of eqn (13.8) can be transformed in exactly the same way as we did it earlier for the derivation of the force balance within the TDGL model in Section 12.4. The force from environment per unit vortex length becomes

$$\begin{aligned}\mathbf{F}_{\mathrm{env}}=-\int_{S_0}\Bigg(&\left[\left(\nabla-\frac{2ie}{c}\mathbf{A}\right)\Delta\right]\frac{\delta\Omega_{sn}}{\delta\Delta}+\left[\left(\nabla+\frac{2ie}{c}\mathbf{A}\right)\Delta^*\right]\frac{\delta\Omega_{sn}}{\delta\Delta^*}\\&+\frac{1}{c}\left[\mathbf{j}^{(\mathrm{nst})}\times\mathbf{H}\right]\Bigg)d^2r\end{aligned} \tag{13.9}$$

The relation

$$2\frac{\delta\Omega_{sn}}{\delta\chi}\equiv i\left(\Delta\frac{\delta\Omega_{sn}}{\delta\Delta}-\Delta^*\frac{\delta\Omega_{sn}}{\delta\Delta^*}\right)=-\frac{1}{2e}\,\mathrm{div}\left(\mathbf{j}-\mathbf{j}^{(\mathrm{nst})}\right)$$

used during the derivation follows from eqn (13.5) and the requirement of the gauge invariance according to which the vector potential and the gradient of the order-parameter phase come always in the combination $\mathbf{A} - (c/2e)\nabla\chi$.

Using eqns (13.4) and (13.6) we can finally present eqn (13.9) in the form (Larkin and Ovchinnikov 1986)

$$\mathbf{F}_{\text{env}} = -\int_{S_0} d^2r \int \frac{d\epsilon}{4} \int \frac{dS_F}{(2\pi)^3 v_F} \operatorname{Tr}\left(\check{g}^{(\text{nst})}\hat{\nabla}\check{H}\right) \tag{13.10}$$

where

$$\hat{\nabla}\check{H} = \begin{pmatrix} (e/c)\,[\mathbf{H}\times\mathbf{v}_F] & -(\nabla - 2ie\mathbf{A}/c)\Delta \\ (\nabla + 2ie\mathbf{A}/c)\Delta^* & -(e/c)\,[\mathbf{H}\times\mathbf{v}_F] \end{pmatrix} \tag{13.11}$$

Equation (13.10) is written in the form which does not assume a spherical Fermi surface.

In presence of a transport current, the force $\mathbf{F}_{\text{env}}$ should be balanced by the Lorentz force. The corresponding term eqn (12.14) can be incorporated into the free energy exactly in the same way as we did it in Section 12.4. The force balance becomes

$$\frac{\Phi_0}{c}\left[\mathbf{j}_{\text{tr}} \times \hat{\mathbf{H}}\right] + \mathbf{F}_{\text{env}} = 0 \tag{13.12}$$

Consider now the nonstationary part of the Green function. We write the Keldysh function eqn (10.62) in frequency representation

$$\begin{aligned}\check{g}^K_{\epsilon_+,\epsilon_-} &= \left(\check{g}^R_{\epsilon_+,\epsilon_-} - \check{g}^A_{\epsilon_+,\epsilon_-}\right) f^{(0)}(\epsilon) - \frac{df^{(0)}}{d\epsilon}\frac{\hat{\omega}}{2}\left(\check{g}^R_{\epsilon_+,\epsilon_-} + \check{g}^A_{\epsilon_+,\epsilon_-}\right) \\ &\quad + \left(\check{g}^R_{\epsilon_+,\epsilon_-} - \check{g}^A_{\epsilon_+,\epsilon_-}\right) f_1(\epsilon) + \left(\check{g}^R_{\epsilon_+,\epsilon_-}\check{\tau}_3 - \check{\tau}_3\check{g}^A_{\epsilon_+,\epsilon_-}\right) f_2(\epsilon)\end{aligned}$$

where

$$\hat{\omega}\check{g} = \begin{pmatrix} \omega g_\epsilon(\omega) & (\omega - 2e\varphi) f_\epsilon(\omega) \\ -(\omega + 2e\varphi) f^\dagger_\epsilon(\omega) & \omega\bar{g}_\epsilon(\omega) \end{pmatrix}. \tag{13.13}$$

As in eqn (11.21) we define the nonstationary function as

$$\check{g}^{(\text{nst})}_{\epsilon_+,\epsilon_-} = \check{g}^K_{\epsilon_+,\epsilon_-} - \left(\check{g}^R_\epsilon - \check{g}^A_\epsilon\right) f^{(0)}(\epsilon) 2\pi\delta(\omega)$$

Using the particle-hole symmetry of the Green functions $\check{g}^{R(A)}_{\epsilon_+,\epsilon_-}$ of the type of eqn (10.68) with respect to the transformation $\epsilon \to -\epsilon$ and $\mathbf{p} \to -\mathbf{p}$ one can show that

$$\nu(0)\int dr \int \frac{d\epsilon}{4}\int \frac{d\Omega_p}{4\pi} f^{(0)}(\epsilon) \operatorname{Tr}\left[\left(\check{g}^R_{\epsilon_+,\epsilon_-} - \check{g}^A_{\epsilon_+,\epsilon_-}\right)\hat{\nabla}\check{H}\right] \tag{13.14}$$

is even in ω. Therefore, the first derivatives of Δ and $\mathbf{A}$ in time drop out of eqn (13.14). As a result, the nonequilibrium part of the total Green function can be written as

$$\begin{aligned}\check{g}^{(\text{nst})}_{\epsilon_+,\epsilon_-} &= -\frac{df^{(0)}}{d\epsilon}\frac{\hat{\omega}}{2}\left(\check{g}^R_{\epsilon_+,\epsilon_-} + \check{g}^A_{\epsilon_+,\epsilon_-}\right) \\ &\quad + \left(\check{g}^R_{\epsilon_+,\epsilon_-} - \check{g}^A_{\epsilon_+,\epsilon_-}\right) f_1(\epsilon) + \left(\check{g}^R_{\epsilon_+,\epsilon_-}\check{\tau}_3 - \check{\tau}_3\check{g}^A_{\epsilon_+,\epsilon_-}\right) f_2(\epsilon)\end{aligned}$$

within the first-order terms in ω. Performing the Fourier transformation back to the center-of-mass time, we get

$$\begin{aligned}\check{g}^{(\text{nst})} &= -\frac{i}{2}\frac{\hat{\partial}}{\partial t}\left(\check{g}^R_\epsilon + \check{g}^A_\epsilon\right)\frac{df^{(0)}}{d\epsilon} \\ &\quad + \left(\check{g}^R_\epsilon - \check{g}^A_\epsilon\right) f_1(\epsilon) + \left(\check{g}^R_\epsilon\check{\tau}_3 - \check{\tau}_3\check{g}^A_\epsilon\right) f_2(\epsilon)\end{aligned} \tag{13.15}$$

where the gauge invariant time derivative is defined by eqn (10.32).
Equations (13.10) and (13.15) make the basis for calculations of the force on a vortex within the microscopic theory. We shall use them for both dirty and clean superconductors in the following Sections. We observe that the force is expressed through the Green functions $\check{g}_\epsilon^R$ and $\check{g}_\epsilon^R$, as well as through the distribution functions f_1 and f_2. For the Green functions we can take their values for a steady vortex array because f_1 and f_2 are themselves proportional to deviation from equilibrium. In turn, f_1 and f_2 are solutions of kinetic equations which again contain only the steady-state Green functions. The problem thus reduces to two major steps. First, we need to find the Green functions for a stationary vortex. Second, we solve the resulting kinetic equations and find the distribution function of nonequilibrium excitations. The force can then be calculated performing the necessary operations in eqn (13.10).

19.2.2 13.2 Diffusion controlled flux flow

In this Section we consider dirty s-wave superconductors such that

$$\tau_{\text{imp}}\Delta(T) \ll 1 \tag{13.16}$$

for temperatures close to T_c. Note that the gradients are not assumed to be small any more in the sense of eqn (11.9). On the contrary, we concentrate on the situation when the relaxation by diffusion is faster than the inelastic relaxation:

$$D\xi^{-2}(T) \gg \tau_{\text{ph}}^{-1} \tag{13.17}$$

We remind also that the inelastic relaxation rate is much smaller than the critical temperature, $T_c \gg \tau_{\text{ph}}^{-1}$. The temperature range when the condition eqn (13.17) is fulfilled corresponds to

$$T_c \gg T_c - T \gg \tau_{\text{ph}}^{-1}$$

This condition is much less restrictive than what we have considered in the previous Chapter where, on the contrary, $D\xi^{-2}$ was assumed smaller than τ_{ph} and thus temperatures $T_c - T \ll \tau_{\text{ph}}^{-1}$ were required. The present conditions are more "natural" and can be easily realized in experiments.

Due to eqn (13.16) the stationary Green functions for dirty superconductors are determined by Usadel equations (5.97). For temperatures close to T_c, the diffusion terms in the Usadel equations can be neglected despite the fact that the gradients are of the order of $\xi^{-1}(T)$. Indeed,

$$D\xi^{-2}(T) \sim \left(1 - \frac{T}{T_c}\right)\Delta \ll \Delta$$

if $\tau_{\text{imp}}T_c \ll 1$, or

$$D\xi^{-2}(T) \sim \tau_{\text{imp}}\,\Delta^2 \ll \Delta$$

when $\tau_{\text{imp}}\,T_c \gg 1$ but still $\tau_{\text{imp}}\,\Delta(T) \ll 1$. The diffusion terms are thus small compared to Δ under conditions of eqn (13.16). We also neglect the small inelastic pair-breaking. As a result, we obtain the adiabatic expressions (10.104) and (10.105) for the regular functions $\langle g_-\rangle$and $\langle g_+\rangle$.

Under these conditions, we can use kinetic equations (10.107) and (10.108) derived for adiabatic Green functions. Neglecting the inelastic scattering we obtain

$$D\nabla^2 f_1 = \frac{1}{2}\left[\langle f_-\rangle \frac{\hat{\partial}\Delta^*}{\partial t} + \left\langle f_-^\dagger \right\rangle \frac{\hat{\partial}\Delta}{\partial t}\right] \frac{\partial f^{(0)}}{\partial \epsilon} \tag{13.18}$$

The boundary conditions for eqn (13.18) are formulated as follows. For $\epsilon^2 > |\Delta|^2_{\max}$, the quasiparticle Green functions extend over long distances from the vortex cores. For a periodic vortex array, the distribution function f_1 is translationally invariant with the period of the vortex array. Thus it should not increase at large distances from the vortex. This requires that the average gradient of f_1 vanishes

$$S_0^{-1}\int_{S_0} \nabla f_1 d^2 r = 0 \tag{13.19}$$

However, for particles with $\epsilon^2 < |\Delta|^2_{\max}$, large distances are not accessible. The boundary conditions are imposed at the surface determined by the condition

$$|\epsilon| = |\Delta(\mathbf{r})| \tag{13.20}$$

beyond which the particle can not penetrate. The conditions can be obtained from eqns (10.106) and (13.18). Integrating eqn (13.18) across the surface in eqn (13.20) one finds that the derivative $\nabla_n f_1$ along its normal is continuous at this surface. At the same time, eqn (10.106) implies that, at the side of the surface where $|\epsilon| < |\Delta(\mathbf{r})|$, the gradient vanishes because $g_- = 0$. Therefore, one should also have

$$\nabla_n f_1|_{|\epsilon|=|\Delta(\mathbf{r})|} = 0 \tag{13.21}$$

at the other side of the surface defined by eqn (13.20), where ∇_n is the derivative along the normal to the surface. Equations (13.19) and (13.21) are the required boundary conditions to eqn (13.18) (Larkin and Ovchinnikov 1986).

The expression for the force eqn (13.10) contains the nonstationary part of the current eqn (13.6) proportional to the energy integral of $g_-\mathbf{f}_1$. According to eqn (10.108), we have

$$\mathbf{j}^{(\mathrm{nst})} = \sigma_n \int \left[\mathbf{E}\left\langle g_-\right\rangle^2 \frac{\partial f^{(0)}}{\partial \epsilon} + e^{-1}\left\langle g_-\right\rangle \nabla \left\langle g_- f_2\right\rangle\right] \frac{d\epsilon}{2}$$

The integral diverges logarithmically at $\epsilon \to \Delta$ due to a square-root singularity in $\langle g_-\rangle$. This divergence is cut off at $\epsilon - \Delta \sim D\xi^{-2}$. However, the prefactor at the logarithm is proportional to Δ/T; it is thus small near the critical temperature. The main contribution, however, comes from the region of energies $\epsilon \sim T$. For such energies, $\langle g_-\rangle = 1$. At the same time, the function f_2 is proportional to Δ/T (compare with eqn (11.19)) and can be neglected. As a result, we again have

$$\mathbf{j}^{(\mathrm{nst})} = \mathbf{j}_n = \sigma_n \mathbf{E}$$

According to eqn (13.18), the distribution function is, by the order of magnitude,

$$f_1 \sim \frac{\xi^2}{D}\frac{df^{(0)}}{d\epsilon}\frac{v_L}{\xi}$$

and gives a much larger contribution to the nonstationary Green function than the first term in the r.h.s. of eqn (13.15). Therefore, neglecting f_2 we have

$$\check{g}^{(\mathrm{nst})} = 2\check{g}_- f_1(\epsilon)$$

Equation (13.10) becomes

$$
\begin{aligned}
\mathbf{F}_{\text{env}} =& -\frac{\nu(0)}{2}\int_{S_0} d^2r \int_{-\infty}^{\infty} d\epsilon f_1 \left(\hat{\nabla}\Delta^* \langle f_- \rangle + \hat{\nabla}\Delta \left\langle f_-^\dagger \right\rangle\right) \\
& -\frac{1}{c}\int_{S_0} \left[\mathbf{j}^{(\text{nst})} \times \mathbf{H}\right] d^2r \\
=& \nu(0)\int_{S_0} d^2r \int_{-\infty}^{\infty} d\epsilon f_1 \nabla \left[\sqrt{\epsilon^2 - |\Delta|^2}\right] \Theta\left(\epsilon^2 - |\Delta|^2\right) \\
& -\frac{1}{c}\int_{S_0} \left[\mathbf{j}^{(\text{nst})} \times \mathbf{H}\right] d^2r
\end{aligned}
\tag{13.22}
$$

The next step is to solve the kinetic equation (13.18) for f_1. Consider first the limit of low magnetic fields. For well-separated vortices, the order-parameter magnitude depends only on the distance from the vortex axis. Equation (13.18) for $\epsilon^2 > |\Delta|^2$ takes the form

$$
D\nabla^2 f_1 = -\frac{\partial}{\partial t}\sqrt{\epsilon^2 - |\Delta|^2}\frac{df^{(0)}}{d\epsilon} = \left(v_{L\rho}\frac{d}{d\rho}\right)\sqrt{\epsilon^2 - |\Delta|^2}\frac{df^{(0)}}{d\epsilon}
$$

We put

$$
f_1 = D^{-1} v_{L\rho} w(\rho)\frac{df^{(0)}}{d\epsilon}
$$

and obtain for $w(\rho)$

$$
\left(\frac{d^2}{d\rho^2} + \frac{1}{\rho}\frac{d}{d\rho} - \frac{1}{\rho^2}\right) w = \frac{d}{d\rho}\sqrt{\epsilon^2 - |\Delta|^2}
$$

The solution regular at $\rho \to 0$ is

$$
w = \rho^{-1}\int_0^\rho \left(\sqrt{\epsilon^2 - |\Delta|^2} - C\right)\rho' d\rho'
$$

The constant should be found from the boundary conditions. Denote $\mathbf{d}$ a constant arbitrary vector. We have

$$
(\mathbf{d}\cdot\nabla) f_1 = d_\rho \frac{df_1}{d\rho} + d_\phi \frac{1}{\rho}\frac{df_1}{d\phi}
$$

$$
\begin{aligned}
=& D^{-1}\frac{df^{(0)}}{d\epsilon}\left(d_\phi v_{L\phi} - d_\rho v_{L\rho}\right) \\
& \times \rho^{-2}\int_0^\rho \left(\sqrt{\epsilon^2 - |\Delta|^2} - C\right)\rho' d\rho' \\
& + D^{-1}\frac{df^{(0)}}{d\epsilon} d_\rho v_{L\rho}\left(\sqrt{\epsilon^2 - |\Delta|^2} - C\right)
\end{aligned}
\tag{13.23}
$$

For $\epsilon^2 > |\Delta|^2_{\max}$, the boundary condition has the form of eqn (13.19). We average eqn (13.23) over the angles ϕ. Since

$$
\left(d_\phi v_{L\phi} - d_\rho v_{L\rho}\right) = -\left[\left(d_x v_x - d_y v_y\right)\cos(2\phi) + \left(d_x v_y + d_y v_x\right)\sin(2\phi)\right]
$$

the first term in the r.h.s. of eqn (13.23) vanishes. The second term gives

$$
\int (\mathbf{d}\cdot\nabla) f_1 d\phi = \frac{\pi\left(\mathbf{d}\cdot\mathbf{v}_L\right)}{D}\frac{df^{(0)}}{d\epsilon}\left(\sqrt{\epsilon^2 - |\Delta|^2} - C\right)
$$

Integrating now over the radial distance, we have from eqn (13.19)

$$\int_0^{r_0} \left(\sqrt{\epsilon^2 - |\Delta|^2} - C\right) \rho d\rho = 0$$

where r_0 is the radius of the vortex lattice unit cell. Thus

$$C = \left\langle \sqrt{\epsilon^2 - |\Delta|^2} \right\rangle_L$$

where the average is taken over the vortex unit cell. The function $w(\rho)$ vanishes for $\rho = r_0$.

For $\epsilon^2 < |\Delta|^2_{\max}$ we use the boundary condition eqns (13.21). Equation (13.23) taken for the derivative along the ρ direction with $d_\phi = 0$ gives

$$C = -\frac{2\pi}{S_\epsilon^2} \int_0^{\rho_\epsilon} \sqrt{\epsilon^2 - |\Delta|^2} \rho' d\rho' \equiv -\left\langle \sqrt{\epsilon^2 - |\Delta|^2} \right\rangle_{S_\epsilon}$$

where $S_\epsilon = \pi \rho_\epsilon^2$ and ρ_ϵ is determined by $|\epsilon| = |\Delta(\rho_\epsilon)|$.

Now we calculate the force on a vortex. The balance of the Lorentz force and the force in eqn (13.22) reads

$$\begin{aligned}\mathbf{F}_L = &- \nu(0) \int_{-\infty}^{\infty} d\epsilon \int_{S_0} d^2 r f_1 \nabla \left[\sqrt{\epsilon^2 - |\Delta|^2}\right] \Theta\left(\epsilon^2 - |\Delta|^2\right) \\ &+ \frac{1}{c} \int_{S_0} \left[\mathbf{j}^{(\mathrm{nst})} \times \mathbf{H}\right] d^2 r\end{aligned}$$

Integrating the first term in the r.h.s. by parts we find that it is equal to

$$\begin{aligned}&\nu(0) \int_{|\epsilon|>|\Delta|_{\max}} d\epsilon \int_{S_0} d^2 r \left(\sqrt{\epsilon^2 - |\Delta|^2} - C\right) \nabla f_1 \\ &+\nu(0) \int_{|\epsilon|<|\Delta|_{\max}} d\epsilon \int_{S_0} d^2 r \sqrt{\epsilon^2 - |\Delta|^2} \Theta\left(\epsilon^2 - |\Delta|^2\right) \nabla f_1\end{aligned}$$

The surface term vanishes because $f_1 = 0$ at $\rho = r_0$ for $|\epsilon| > |\Delta|_{\max}$, and $\sqrt{\epsilon^2 - |\Delta|^2} = 0$ at $\rho = \rho_\epsilon$ for $|\epsilon| < |\Delta|_{\max}$. The energy integral is determined by $\epsilon \sim |\Delta|$ and the spatial integral converges at distances of the order of $\xi(T)$. The equation of the force balance becomes

$$\mathbf{F}_L = \frac{\nu(0)\alpha(H)}{2TD} \langle |\Delta|^2 \rangle_L^{3/2} \xi^2 \mathbf{v}_L + \frac{1}{c} \int_{S_0} \left[\mathbf{j}^{(\mathrm{nst})} \times \mathbf{H}\right] d^2 r$$

where we denote

$$\begin{aligned}\alpha(H) = &\int_{\tilde{\epsilon}>\tilde{\Delta}_{\max}} d\tilde{\epsilon} \int_{S_0} d^2 \tilde{r} \left[\sqrt{\tilde{\epsilon}^2 - \tilde{\Delta}^2} - \left\langle \sqrt{\tilde{\epsilon}^2 - \tilde{\Delta}^2} \right\rangle_L\right]^2 \\ &+ \int_{\tilde{\epsilon}<\tilde{\Delta}_{\max}} d\tilde{\epsilon} \int_{S_\epsilon} d^2 \tilde{r} \left[\left(\tilde{\epsilon}^2 - \tilde{\Delta}^2\right) + \left\langle \sqrt{\tilde{\epsilon}^2 - \tilde{\Delta}^2} \right\rangle_{S_\epsilon}^2\right]\end{aligned}$$

We put here

$$\epsilon = \tilde{\epsilon}\sqrt{\langle |\Delta|^2 \rangle_L}, |\Delta| = \tilde{\Delta}\sqrt{\langle |\Delta|^2 \rangle_L}, \rho = \tilde{\rho}\xi(T)$$

As a result,

$$\mathbf{j} - B^{-1} \left\langle \mathbf{j}^{(\mathrm{nst})} H \right\rangle = \mathbf{v}_L \frac{\sigma_n H_{c2} \alpha(H) \Lambda^2}{2\pi c} \left(\frac{\pi^3}{7\zeta(3)}\right)^2 \frac{T}{\Delta_\infty} \left\langle \frac{|\Delta|^2}{\Delta_\infty^2} \right\rangle_L^{3/2} \tag{13.24}$$

and, as usually,

$$\Delta_\infty^2 = \frac{8\pi^2 T_c\,(T_c - T)}{7\zeta(3)}$$

The coherence length is defined by eqn (6.26)

$$\xi^2(T) = \frac{\pi^3 T \Lambda D}{7\zeta(3)\Delta_\infty^2}$$

For high magnetic fields when vortices are closely packed, it is not easy to solve eqn (13.18) because the vortex lattice does not have an axial symmetry any more. To simplify calculations, we adopt the approximation of the round unit cell even for high vortex density. Within this approximation, we obtain the same solution as before even for high magnetic fields. Therefore, we shall use eqn (13.24) as an approximation for the whole range of magnetic fields.

For well separated vortices at low fields one can neglect the normal current in the l.h.s. of eqn (13.24). Indeed, it is of the order of $\sigma_n E$ while the term in the r.h.s. has an order of $(H_{c2}/B)\,\sigma_n E$. For fields close to the upper critical field
H_{c2}, one has $\mathbf{j}^{(\mathrm{nst})} = \sigma_n \mathbf{E}$. Putting $\mathbf{v}_L = c[\mathbf{E} \times \hat{\mathbf{H}}]/B$ one thus can write an interpolation expression

$$\mathbf{j} - \sigma_n \mathbf{E} = \sigma_n \frac{H_{c2}}{B} \frac{\alpha(H)\Lambda^2}{2\pi} \left(\frac{\pi^3}{7\zeta(3)}\right)^2 \frac{T_c}{\Delta_\infty} \left\langle \frac{|\Delta|^2}{\Delta_\infty^2} \right\rangle_L^{3/2} \mathbf{E}$$

The flux flow conductivity becomes

$$\frac{\sigma_f}{\sigma_n} = 1 + \frac{\alpha(H)\Lambda^2}{2\pi} \left[\frac{\pi^3}{7\zeta(3)}\right]^2 \frac{T_c}{\Delta_\infty} \frac{H_{c2}}{B} \left\langle \frac{|\Delta|^2}{\Delta_\infty^2} \right\rangle_L^{3/2} \tag{13.25}$$

This result was obtained by Gor'kov and Kopnin (1973 a) and by Larkin and Ovchinnikov (1986). The function $\alpha(H)$ is of the order of unity. To calculate it for well separated vortices at low fields we observe that $\langle |\Delta|^2 \rangle_L = \Delta_\infty^2$ and $\tilde{\Delta}_{\max} = 1$; moreover

$$\left\langle \sqrt{\tilde{\epsilon}^2 - \tilde{\Delta}^2} \right\rangle_L = \sqrt{\tilde{\epsilon}^2 - 1}$$

On the other hand, for high fields

$$\langle |\Delta|^2 \rangle_L = \frac{2\kappa^2 \Delta_\infty^2}{(2\kappa^2 - 1)\,\beta_A} \left(1 - \frac{B}{H_{c2}}\right)$$

The high-field flux flow conductivity becomes

$$\frac{\sigma_f}{\sigma_n} - 1 = \frac{\alpha(H)\Lambda^2}{\pi} \left[\frac{\pi^3}{7\zeta(3)}\right]^2 \frac{T_c}{\Delta_\infty} \left(\frac{1 - B/H_{c2}}{[1 - (1/2\kappa^2)]\,\beta_A}\right)^{3/2}$$

As a function of the magnetic field, this expression is valid until

$$1 - \frac{H}{H_{c2}} \gg \left(1 - \frac{T}{T_c}\right)$$

For higher fields, the superconductor becomes essentially gapless, and the regular TDGL contribution [the first term in the r.h.s. of eqn (13.15)] dominates.

Equation (13.25) can be written in a form similar to eqn (12.54)

$$\frac{\sigma_f}{\sigma_n} - 1 = \beta \frac{H_{c2}}{B} \tag{13.26}$$

where β is a function of temperature, magnetic field, and also of the electron mean free path. The numerical values for β are listed in the review by Larkin and Ovchinnikov (1986). In the dirty limit $T_c\tau \ll 1$, one has for $H \ll H_{c2}$

$$\beta = 4.04\left(1 - \frac{T}{T_c}\right)^{-1/2}$$

FIG. 13.1. The factor β as a function of temperature, $t=T / T_{c}$. The solid line is an interpolation between: (i) $\beta \approx 0.9$ at zero temperature, (ii) $\beta = 4.04\,(1 - T/T_c)^{-1/2}$ in the region close to T_c (thin line), and (iii) the TDGL result $\beta = 1.45$ for $T = T_c$. The temperature $t^* = T^*/T_c$ where β reaches its maximum is such that $1 - T^*/T_c \sim (T_c\tau_{\rm ph})^{-1}$.

and

$$\beta = 0.31\left(1 - \frac{T}{T_c}\right)^{-1/2}\left(\frac{1 - B/H_{c2}}{1 - (1/2\kappa^2)}\right)^{3/2}$$

for $H \to H_{c2}$. The factor $(1 - T/T_c)^{-1/2}$ follows form eqn (13.25) due to the term T_c/Δ_∞ in the right hand side.

13.2.1 Discussion

We note first of all that the flux flow conductivity in dirty superconductors appears to be much larger than what is predicted by the simple TDGL model or by the Bardeen-Stephen model. Indeed, for low inductions the factor β in eqn (13.26) is $\beta = 1$ within the Bardeen and Stephen model. The microscopic theory, however, predicts a large β near T_c which grows as

$$\beta \sim \left(1 - \frac{T}{T_c}\right)^{-1/2}$$

when T approaches T_c. This is the consequence of a decrease in the diffusion relaxation rate when the size of the vortex core increases near T_c with an increasing coherence length. When temperature is close enough to T_c the diffusion relaxation rate becomes comparable with the inelastic relaxation rate when $D\xi^{-2}(T) \sim \tau_{\rm ph}^{-1}$. This happens for temperature $(1 - T/T_c) \sim (T_c\tau_{\rm ph})^{-1}$ at the border of applicability of eqn (13.25). At this point, the increase in β saturates at

$$\beta \sim \left(1 - \frac{T}{T_c}\right)^{1/2} (T_c \tau_{\rm ph})$$

which has the same order of magnitude as the factor $\beta \sim F(q)$ in eqn (12.54). This is exactly the regime which is described by the generalized TDGL scheme discussed in Section 12.8. When the temperature approaches T_c even closer, the factor β starts to decrease as $(1 - T/T_c)^{1/2}$ down to values of the order of unity according to eqn (12.54).

We have only discussed a vicinity of the critical temperature. The lowtemperature behavior of the flow conductivity in dirty superconductors can also be considered within the same framework based on the kinetic equations derived in Section 10.5. This was done by Gor'kov and Kopnin (1973 b). The result approaches the Bardeen and Stephen model with $\beta \approx 0.9$. The overall behavior of the flux flow conductivity in a dirty superconductor as a function of temperature is shown schematically in Fig. 13.1. Experimental data for conventional dirty superconductors lie generally above the results of the Bardeen-Stephen model; on the other hand, they are below the predictions of the exact microscopic theory (see the reviews by Gor'kov and Kopnin 1975 and by Larkin and Ovchinnikov 1986). The reason is that the parameter $T_c \tau_{\rm ph}$ is not that large in practice, thus the increase in β is limited to lower values.

19.3 VORTEX DYNAMICS IN CLEAN SUPERCONDUCTORS

We discuss first the general features of the vortex dynamics in clean superconductors using the Boltzmann kinetic equation. We establish the conditions when the dissipative dynamics of vortices transforms into a Hamiltonian one. We emphasize the crucial importance of excitations localized in vortex cores. Next we use the quasiclassical Green function technique and the kinetic equations of the previous chapters to calculate the longitudinal and Hall components of the flux flow conductivity for s-wave superconductors. We discuss the forces on a vortex as functions of temperature and of purity of the superconductor.

19.3.1 14.1 Introduction

This Chapter is devoted to the vortex dynamics in clean superconductors. Clean systems offer more intriguing physics than dirty superconductors considered in the previous Section. For example, one of the fundamental problems can be formulated as follows: Speaking of clean superconductors one can, in particular, think of such a system where no relaxation processes are available, i.e., the mean free path of excitations is infinite. In this case, vortices should move without dissipation since there is no mechanism to absorb the energy. The vortex velocity should then be parallel to the transport current, which makes the electric field perpendicular to the current; the dissipation thus vanishes $\mathbf{j} \cdot \mathbf{E} = 0$. This contrasts with what we know from the previous chapters on dirty superconductors: the vortex motion is dissipative, and each moving vortex experiences a large friction; it generates an electric field parallel to the transport current which produces energy dissipation. Clearly, a crossover should occur from dissipative to non-dissipative vortex motion as the quasiparticle mean free path increases. The question is what is the condition which controls the crossover?

This question is of a fundamental importance for our understanding of the dynamics of superconductors and, in a more broad sense, for the understanding of dynamic properties of quantum condensed matter in general. To illustrate the problem one can consider a simple example as follows. One can argue that a time dependent non-dissipative superconducting state, similarly to any other quantum state, can be described by a Hamiltonian dynamics based on a time dependent Schrödinger equation. Such a description for superfluid helium II has been suggested by Pitaevskii (1961) and Gross (1961, 1963). The Gross-Pitaevskii equation

is essentially a nonlinear Schrödinger equation, it has the imaginary factor $-i\hbar$ in front of the time derivative of the condensate wave function $\partial\Psi/\partial t$.

On the other hand, the time dependent Ginzburg-Landau model which is a particular case of a more general Model F dynamics (Hohenberg and Halperin 1977) is believed to describe a relaxation dynamics of superconductors near the transition temperature. In contrast to the Gross-Pitaevskii equation, it has the time derivative $\partial\Psi/\partial t$ with a real factor in front of it. The question which we are interested in can be formulated as follows: What is the condition when the imaginary prefactor transforms into a real one?

It seems that there is no universal answer to this simple question in general. However, the problem of crossover from non-dissipative to dissipative behavior of a condensed matter state can be solved for the particular example of superconducting vortex dynamics. We have already seen in Section 12.9 that a relaxation constant in the time dependent Ginzburg-Landau model has in fact a small imaginary part which results in appearance of a small transverse component of the electric field with respect to the current. We shall see later that the transverse component of the electric field increases at the expense of the longitudinal component as the mean free path of excitations grows. The crossover condition, however, does not coincide simply with the condition which divides superconductors between dirty and clean ones. The criterion rather involves the spectrum of excitations localized in the vortex cores; the distance between their levels takes the part of the energy gap. The condition for a non-dissipative vortex motion requires that the relaxation rate of localized excitations is smaller than the distance between the levels. This implies a much longer mean free path of excitations than the condition for a superconductor to be just in a clean limit.

14.1.1 Boltzmann kinetic equation approach

A vortex moving in a clean superconductor experiences both friction and transverse forces. The transverse force comes from several different sources including the hydrodynamic Magnus force, the force produced by excitations scattered from the vortex, and the force associated with the momentum flow from the heat bath to the vortex through the localized excitations. The whole rich and exciting physics involved in the vortex dynamics can be successfully described by the general formalism developed in Chapter 10. In this Chapter we concentrate on isolated vortices such that their cores do not overlap, i.e., on the region of magnetic fields $H \ll H_{c2}$. It is this limit when the specifics of core states is more pronounced.

Let us first discuss a general picture using a simple approach based on the Boltzmann kinetic equation. For simplicity, we consider an s-wave superconductor. We remind that the profile of the order parameter $\Delta(\mathbf{r})$ near the vortex core produces a potential well where localized states with a discrete spectrum exist. The spectrum of excitations in the vortex core has been calculated in Section 6.4 (see also Fig. 6.2). The localized states correspond to energies $|\epsilon| < \Delta_\infty$. The spectrum has the so called anomalous branch with $n = 0$ whose energy varies from $-\Delta_\infty$ to $+\Delta_\infty$ as the particle impact parameter b changes from $-\infty$ to $+\infty$ and crosses $\epsilon = 0$, being an odd function of b. For low $\epsilon \ll \Delta_\infty$, the anomalous branch is $\epsilon_0 = -\omega_0\mu$ where $\mu = -bp_\perp$ is the angular momentum, and $p_\perp$ is the
momentum in the plane perpendicular to the vortex axis (Caroli et al. 1964). In a s-wave superconductor with an axisymmetric vortex, the angular momentum μ is quantized and so is the spectrum, ω_0 being the distance between the discrete levels in the vortex core. The spectrum also has branches with $n \neq 0$ which are separated from the one with $n = 0$ by energies of the order of Δ. They are even in b and symmetric in energy with respect to $\epsilon = 0$. We denote the separation between the levels with neighboring angular momenta through

$$\omega_n = p_\perp^{-1}\frac{\partial\epsilon_n}{\partial b} = -\frac{\partial\epsilon_n}{\partial\mu} \tag{14.1}$$

The interlevel spacing $\omega_n(b)$ is an even function of b for $n = 0$, and it is an odd function for $n \neq 0$.

We choose the direction of the z axis in such a way that the vortex has a positive circulation. The z axis is thus parallel the magnetic field for positive charge of carriers, and it is antiparallel to it for negative charge: $\hat{\mathbf{z}} = \hat{\mathbf{H}}\,\mathrm{sign}(e)$. Since the particle velocity $\mathbf{v}_\perp$ makes an angle α with the x axis, the cylindrical coordinates of the position point (ρ, ϕ) are connected with the impact parameter and the coordinate along the trajectory through $\rho^2 = b^2 + s^2$ where

$$b = \rho \sin(\phi - \alpha); s = \rho \cos(\phi - \alpha) \tag{14.2}$$

The coordinates are shown in Fig. 6.1.

The first step is as follows. We assume that the quasiclassical spectrum $\epsilon_n(b)$ of a particle plays the role of its effective Hamiltonian. We can thus invoke the Boltzmann equation in the canonical form

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \alpha} \cdot \frac{\partial \epsilon_n}{\partial \mu} - \frac{\partial \epsilon_n}{\partial \alpha} \cdot \frac{\partial f}{\partial \mu} = \left(\frac{\partial f}{\partial t}\right)_{\mathrm{coll}} \tag{14.3}$$

to describe the quasiparticle distribution (Stone 1996). We shall derive eqn (14.3) from our set of generalized kinetic equations in the following Chapter.

Second, we assume that the force acting on a vortex from the environment is exerted via excitations localized in the vortex core. We shall later demonstrate using the quasiclassical Green function technique that this is indeed the correct picture if the mean free path of quasiparticles is not extremely long, i.e., if $\omega_c \tau \ll 1$ where $\omega_c = eH/mc$ is the cyclotron frequency. This condition assures that the excitations with energies above the energy gap Δ_∞ are in equilibrium with the heat bath and do not influence considerably the vortex motion. According to this picture, the force on a moving vortex can be written as the momentum transfer from the localized excitations to the vortex

$$\mathbf{F}_{\mathrm{env}} = \frac{1}{2} \sum_n \int \frac{dp_z}{2\pi} \frac{d\alpha d\mu}{2\pi} \frac{\partial \mathbf{p}_n}{\partial t} f_1 = -\frac{1}{2} \sum_n \int \frac{dp_z}{2\pi} \frac{d\alpha d\mu}{2\pi} \frac{\partial \epsilon_n}{\partial b} \left[\hat{\mathbf{z}} \times \hat{\mathbf{v}}_\perp\right] f_1 \tag{14.4}$$

Here we make use of the Hamilton equation

$$\frac{\partial \mathbf{p}_n}{\partial t} = -\nabla \epsilon_n = -\frac{\partial \epsilon_n}{\partial b} \left[\hat{\mathbf{z}} \times \hat{\mathbf{v}}_\perp\right] \tag{14.5}$$

The second equality follows from the fact that the energy only depends on the particle impact parameter b. The normalization in eqn (14.4) is chosen such that the sum over the two spin states enters as $(1/2)\sum_s$. We shall derive eqn (14.4) microscopically in Section 14.5.

For a moving vortex, the force from the environment is balanced by the Lorentz force:

$$\mathbf{F}_{\mathrm{L}} = \frac{\Phi_0}{c} \left[\mathbf{j}_{\mathrm{tr}} \times \hat{\mathbf{z}}\right] \mathrm{sign}(e) \tag{14.6}$$

with the flux quantum $\Phi_0 = \pi c/|e|$. This form of the Lorentz force is the same as in eqn (12.1) but the vortex circulation axis $\hat{\mathbf{z}}$ is used. The force balance equation

$$\mathbf{F}_{\mathrm{L}} + \mathbf{F}_{\mathrm{env}} = 0 \tag{14.7}$$

determines the transport current in terms of the vortex velocity and thus allows to find the flux flow conductivity tensor.

14.1.2 Forces in s-wave superconductors

We consider a parabolic spectrum of quasiparticles in the normal state and thus a spherical Fermi surface. We shall simplify the collision integral in eqn (14.3) using the relaxation-time approximation

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = -\frac{f_1}{\tau_n} \tag{14.8}$$

Equation (14.3) is easy to solve. Let us take the distribution function in the form

$$f_1 = -\frac{\partial f^{(0)}}{\partial \epsilon}\left[\gamma_{\text{O}}\left([\mathbf{v}_L \times \mathbf{p}_\perp]\cdot\hat{\mathbf{z}}\right) + \gamma_{\text{H}}\left(\mathbf{v}_L\cdot\mathbf{p}_\perp\right)\right] \tag{14.9}$$

where the factors $\gamma_{\text{O,H}}$ are to be found. For an axi-symmetric s-wave vortex the energies ϵ_n do not depend on α and the term $\partial\epsilon_n/\partial\alpha$ vanishes. In the term with the time derivative, the energy ϵ_n contains a time dependence through $\mu(t) = [(\mathbf{r} - \mathbf{v}_L t)\times\mathbf{p}]\cdot\hat{\mathbf{z}}$ such that

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial \epsilon}\frac{\partial \epsilon_n}{\partial \mu}\frac{\partial \mu}{\partial t} = \frac{\partial f^{(0)}}{\partial \epsilon}\frac{\partial \epsilon_n}{\partial \mu}\left([\mathbf{p}_\perp\times\mathbf{v}_L]\cdot\hat{\mathbf{z}}\right) \tag{14.10}$$

With the Ansatz (14.9), the force eqn (14.4) splits into two terms $\mathbf{F}_{\text{env}} = \mathbf{F}_\parallel + \mathbf{F}_\perp$, with the friction $\mathbf{F}_\parallel$and transverse $\mathbf{F}_\perp$ forces given by

$$\mathbf{F}_\parallel = -\pi N\left\langle\left\langle\sum_n\int\omega_n\gamma_{\text{O}}\frac{\partial f^{(0)}}{\partial\epsilon}\frac{d\mu}{2}\right\rangle\right\rangle\mathbf{v}_L \tag{14.11}$$

$$\mathbf{F}_\perp = \pi N\left\langle\left\langle\int\omega_0\gamma_{\text{H}}\frac{\partial f^{(0)}}{\partial\epsilon}\frac{d\mu}{2}\right\rangle\right\rangle[\hat{\mathbf{z}}\times\mathbf{v}_L] \tag{14.12}$$

where N is the electron density, $\langle\langle\ldots\rangle\rangle$ is the average over the Fermi surface with the weight $\pi p_\perp^2$,

$$\langle\langle\ldots\rangle\rangle = V_F^{-1}\int\pi p_\perp^2 dp_z(\ldots) \tag{14.13}$$

and V_F is the volume encompassed by the Fermi surface. For an isotropic Fermi surface,

$$\langle\langle\ldots\rangle\rangle = \frac{3}{4}\int\sin^3\theta d\theta(\ldots)$$

where $p_z = p_F\cos\theta$. Only the spectral branch with $n = 0$ contributes to the transverse force $\mathbf{F}_\perp$ as all ω_n with $n \neq 0$ are odd functions of μ and thus drop out of the sum over n in eqn (14.12).

The Boltzmann equation (14.3) gives the factors $\gamma_{\text{O,H}}$ in the form

$$\gamma_{\text{O}} = \frac{\omega_n\tau_n}{\omega_n^2\tau_n^2+1},\quad \gamma_{\text{H}} = \frac{\omega_n^2\tau_n^2}{\omega_n^2\tau_n^2+1} \tag{14.14}$$

The longitudinal force $\mathbf{F}_\parallel$defines the friction coefficient in the vortex equation of motion and determines the Ohmic component of the conductivity σ_{O}. Expressing the vortex velocity $\mathbf{v}_L$ through the average electric field $\mathbf{E}$, as $\mathbf{v}_L = c[\mathbf{E}\times\hat{\mathbf{z}}]/B\,\text{sign}(e)$, we find

$$\sigma_{\text{O}} = \frac{N|e|c}{B}\left\langle\left\langle\sum_n\int\frac{\omega_n\tau_n}{\omega_n^2\tau_n^2+1}\frac{\partial f^{(0)}}{\partial\epsilon}\frac{d\epsilon}{2}\right\rangle\right\rangle. \tag{14.15}$$

The transverse force determines the Hall conductivity

$$\sigma_{\text{H}} = \frac{Nec}{B}\left\langle\left\langle\int\frac{\omega_0^2\tau_0^2}{\omega_0^2\tau_0^2+1}\frac{\partial f^{(0)}}{\partial\epsilon}\frac{d\epsilon}{2}\right\rangle\right\rangle \tag{14.16}$$

We emphasize that the force $\mathbf{F}_{\text{env}}$ is defined as the response of the whole environment to the vortex displacement. It is therefore the total force acting on the vortex from the ambient system, including all partial forces such as the longitudinal friction force and the non-dissipative transverse force. The transverse force, in turn, includes various parts which can be identified historically (Sonin 1987) as the Iordanskii force (Iordanskii 1964), the spectral flow force (Volovik 1986, Stone and Gaitan 1987, Kopnin et al. 1995), and the Magnus force. We shall discuss this later in more detail in Section 14.6.3.

The main conclusion is that the Ohmic and Hall conductivities depend on the purity of the sample through the parameter $\omega_0\tau$ (remind that $\omega_n \sim \omega_0$). One can distinguish two regimes: moderately clean $\omega_0\tau \ll 1$ and superclean $\omega_0\tau \gg 1$. Note that the moderately clean regime still requires that the superconductor is clean in the usual sense $\Delta_\infty\tau \gg 1$.

In the moderately clean limit where $\omega_0\tau \ll 1$, the conductivity roughly follows the Bardeen and Stephen expression at low temperatures though it exhibits an extra temperature-dependent factor Δ_∞/T_c on approaching T_c (Kopnin and Lopatin 1995),

$$\sigma_{\text{O}} \sim \sigma_n \frac{H_{c2}}{H}\frac{\Delta_\infty}{T_c}$$

The factor Δ_∞/T_c appears because the number of delocalized quasiparticles contributing to the vortex dynamics decreases near T_c. Note that the flux flow conductivity of a dirty superconductor has just the inverse extra factor T_c/Δ_∞ as compared to the Bardeen-Stephen model (see Section 13.2.1). We discuss this later in Section 14.6.2. The Hall conductivity and the Hall angle are small $\sigma_{\text{H}} \sim (\omega_0\tau)\,\sigma_{\text{O}}$ and $\tan\Theta_H \sim (\omega_0\tau)$, respectively.

In the superclean limit, on the contrary, the Ohmic conductivity is small. The vortex dynamics becomes non-dissipative. The friction force vanishes while the transverse force eqn (14.12) takes the form

$$\mathbf{F}_\perp = \pi N \tanh\frac{\Delta_\infty}{2T}\,[\hat{\mathbf{z}} \times \mathbf{v}_L]$$

We have accounted for the fact that

$$\int \omega_0 \frac{\partial f^{(0)}}{\partial \epsilon}\frac{d\mu}{2} = \tanh\frac{\Delta_\infty}{2T}$$

since $\epsilon_0(\mu)$ varies from $-\Delta_\infty$ to $+\Delta_\infty$. The force balance equation tell us that the transport current is coupled to the vortex velocity by

$$\mathbf{j}_{\text{tr}} = eN\tanh\frac{\Delta_\infty}{2T}\mathbf{v}_L$$

It shows that $N\tanh(\Delta_\infty/2T)$ electrons move together with the vortex. The factor $\tanh(\Delta_\infty/2T)$ accounts for the fact that delocalized excitations are at rest with respect to the heat bath according to our assumption made on page 285. At low temperatures, delocalized excitations are absent, and the current is proportional to the density of all the electrons. We shall discuss this in more detail in Section 14.6.3. The corresponding Hall conductivity is

$$\sigma_H = \frac{Nec}{B}\tanh\left(\frac{\Delta_\infty}{2T}\right)$$

We conclude that it is the parameter $\omega_0\tau$ which determines the crossover from dissipative to non-dissipative vortex dynamics. To get a more comprehensive picture of this phenomenon, we turn now to the microscopic description of the vortex dynamics in clean superconductors. We shall prove, in particular, that the simple picture discussed above provides the correct account of the dynamical processes associated with the moving vortex.

14.2 Spectral representation for the Green functions

We consider first some general properties of Green functions of clean superconductors which will be used later. For clean systems, the quasiparticle spectrum is well defined. According to the general properties of the Green functions, the combination $G^R - G^A$ is a sum of a spectral weight multiplied by δ-functions at energies corresponding to the quasiparticle spectrum. We show that the same holds also for quasiclassical Green functions taken on a discrete spectrum. We consider states in a vortex core for definiteness.

Let us expand the Green function $\check{G}_\epsilon^{R(A)}$ in the eigenfunctions of a quasiclassical particle with the momentum $p \sim p_F \gg \xi^{-1}$ moving along a definite trajectory. In case of a linear vortex, we specify the trajectory by the direction of particle velocity $\mathbf{v}_F$ and by the impact parameter b with respect to the vortex axis. Consider the Bogoliubov-de Gennes wave function $\mathcal{U}(s, b, z)$

$$\mathcal{U} = \begin{pmatrix} u \\ -v \end{pmatrix}, \quad \mathcal{U}^\dagger = \begin{pmatrix} u^* & v^* \end{pmatrix}$$

at a point with coordinates s and b in the coordinate frame (s, b) rotated by an angle α with respect to the (x, y) frame as shown in Fig. 6.1 and with z coordinate along the vortex axis. It satisfies the equation

$$\left[-\frac{1}{2m^*} \left[\left(\frac{\partial}{\partial s} \right)^2 + \left(\frac{\partial}{\partial b} \right)^2 \right] - E_F + \check{H} \right] \mathcal{U} = \epsilon \check{\tau}_3 \mathcal{U}.$$

For a two dimensional problem, we ignore the trivial z dependence for brevity. Remind that

$$\check{H} = \begin{pmatrix} -(e/c)\mathbf{v}_F \cdot \mathbf{A} & -\Delta \\ \Delta^* & (e/c)\mathbf{v}_F \cdot \mathbf{A} \end{pmatrix}.$$

We take a fixed point b_0 in the vicinity of b and put $b = b_0 + b_1$ where $p_F^{-1} \ll b_1 \ll \xi$. Next, we perform a Fourier transformation in the coordinate b_1. As a result we obtain a wave function $\mathcal{U}\left(b_0, p_b, s\right)$ which depends on the wave vector p_b and on the coordinate s; in addition, it also has a parametric dependence on b_0. The obtained wave function satisfies the equation

$$\left[\frac{1}{2m^*} \left[-\left(\frac{\partial}{\partial s} \right)^2 + p_b^2 \right] - E_F + \check{H}\left(b_0, s\right) \right] \mathcal{U} = \epsilon \check{\tau}_3 \mathcal{U}$$

Here we neglect a small correction b_1 in the Hamiltonian $\check{H}$. We now take the limit of small p_b and fix it within the interval $\xi^{-1} \ll p_b \ll \sqrt{p_F/\xi}$. This means that the particle momentum becomes parallel to the s axis. In other words, the s axis is now directed along the particle trajectory while b_0 is thus the impact parameter. The equation takes the form

$$\left[\frac{1}{2m^*} \left[-\left(\frac{\partial}{\partial s} \right)^2 \right] - E_F + \check{H}(b, s) \right] \mathcal{U} = \epsilon_n(b) \check{\tau}_3 \mathcal{U}. \tag{14.17}$$

The term $p_b^2/2m$ can be neglected as compared to $\check{H} \sim \Delta$. We also omit the index 0 at b. The wave function depends now on the distance along the trajectory s and on the parameters b and α where α is the angle between the particle momentum $\mathbf{p}_\perp$ and the x axis of the coordinate frame. They satisfy the orthogonality conditions eqns (3.59) and (3.60) which now read

$$\sum_n \check{\tau}_3 \mathcal{U}_n\left(s_1\right) \mathcal{U}_n^\dagger\left(s_2\right) = \check{1} \delta\left(s_1 - s_2\right) \tag{14.18}$$

$$\int ds \mathcal{U}_n^\dagger(s) \check{\tau}_3 \mathcal{U}_{n'}(s) = \delta_{n,n'} \tag{14.19}$$

We also denote the energy spectrum as $\epsilon_n(b)$; it is a function of the impact parameter b and also on the momentum along the z axis p_z as well as (possibly) on α. In addition, it can depend on the principal quantum number n and on other quantum numbers (for example, particle-hole isotopic spin, etc.) relevant for the particular problem. The impact parameter b is a good quantum number for a quasiclassical particle which moves in a potential Δ whose magnitude is much smaller than the kinetic energy E_F. Under these conditions, the trajectory is a straight line with a constant b even if the problem does not have an axial symmetry.

We can now expand the Green function of the particle in terms of these wave functions:

$$\begin{aligned}\check{G}_\epsilon^{R(A)}\left(\mathbf{v}, b; s_1, s_2\right) &= \begin{pmatrix} G^{R(A)} & F^{R(A)} \\ -F^{\dagger R(A)} & \bar{G}^{R(A)} \end{pmatrix} \\ &= -\sum_n \frac{\mathcal{U}_n\left(\mathbf{v}, b; s_1\right)\mathcal{U}_n^\dagger\left(\mathbf{v}, b; s_2\right)}{\epsilon - \epsilon_n(b) \pm i\delta}\end{aligned} \tag{14.20}$$

The Green function matrix obeys the equation

$$\left[E_n\left(-i\frac{\partial}{\partial s}\right) - E_F - \epsilon\check{\tau}_3 + \check{H}\right]\check{G}_\epsilon^{R(A)} = \check{1}\delta\left(s_1 - s_2\right) \tag{14.21}$$

where $E_n(\mathbf{p})$ is the normal-state electronic spectrum.
Next, we determine the relation between the Green function $\check{G}_\epsilon^{R(A)}$ and its quasi-classical counterpart $\check{g}^{R(A)}$. We represent the Green function (both retarded and advanced) as

$$\check{G}_\epsilon = e^{ip_\perp(s_1 - s_2)}\left[\check{a}_+\Theta\left(s_1 - s_2\right) - \check{a}_-\Theta\left(s_2 - s_1\right)\right] \tag{14.22}$$

where $\check{a}_\pm(s)$ are slow functions of $s = \left(s_1 + s_2\right)/2$; they vary over distances of the order of ξ as compared to atomic-scale variations of the exponents. From eqn (14.21) we find $\check{a}_+ + \check{a}_- = i/v_\perp$ where $v_\perp$ is the particle velocity in the plane perpendicular to the vortex axis. It is easy to check that the functions $\check{a}_\pm$satisfy the same Eilenberger equations as the quasi-classical Green functions $\check{g}^{R(A)}$. To do this we insert eqn (14.22) into eqns (14.21) and expand in small gradients of
$a_\pm$. Using the boundary conditions at large distances, we obtain (Gor'kov and Kopnin 1973 a)

$$\check{a}_\pm = \frac{i}{2v_\perp}(\check{1} \pm \check{g}).$$

Thus,

$$\check{G}_\epsilon^{R(A)}\left(s_1 \to s_2\right) = \frac{i}{2v_\perp}\left[\check{g}^{R(A)} + \check{1}\,\mathrm{sign}\left(s_1 - s_2\right)\right].$$

This relation was obtained by Kopnin and Lopatin (1995). Finally

$$\check{G}_\epsilon^R\left(s_1 \to s_2\right) - \check{G}_\epsilon^A\left(s_1 \to s_2\right) = \frac{i}{2v_\perp}\left(\check{g}^R - \check{g}^A\right). \tag{14.23}$$

Comparing eqns (14.20) and (14.23), one can write

$$\check{g}_-(\mathbf{p}, \mathbf{r}) = \sum_n \check{g}_n\left(p_z, \alpha, b; s\right)\delta\left[\epsilon - \epsilon_n\left(p_z, \alpha, b\right)\right] \tag{14.24}$$

where $\check{g}_n$ are the spectral weights for the quasiclassical Green function.

19.3.2 14.3 Useful identities

Here we concentrate on the Green functions in presence of vortices and derive several identities (Kopnin and Lopatin 1995, Blatter et al. 1999) which will be used in what follows. Consider the energies $|\epsilon| < \Delta_\infty$. We find from eqn (14.19)

$$\begin{aligned}\int \mathrm{Tr}\left[\check{\tau}_3 \check{g}_-\right] ds &= -iv_\perp \int \mathrm{Tr}\left[\check{\tau}_3 \left(\check{G}^R_\epsilon(s,s) - \check{G}^A_\epsilon(s,s)\right)\right] ds \\ &= 2\pi v_\perp \sum_n \delta\left(\epsilon - \epsilon_n\right) \int \mathrm{Tr}\left[\mathcal{U}^\dagger_n(s)\check{\tau}_3 \mathcal{U}_n(s)\right] ds \\ &= 2\pi v_\perp \sum_n \delta\left(\epsilon - \epsilon_n\right) \end{aligned} \tag{14.25}$$

We turn to derivation of another identity. Consider the matrix $\hat{\nabla}\check{H}$ defined by eqn (13.11) and calculate

$$\begin{aligned}&\mathrm{Tr}\left[\check{g}_-(\mathbf{d}\cdot\hat{\nabla}\check{H})\right] \\ =&f_-\mathbf{d}\cdot\left(\nabla + \frac{2ie}{c}\mathbf{A}\right)\Delta^* + f^\dagger_-\mathbf{d}\cdot\left(\nabla - \frac{2ie}{c}\mathbf{A}\right)\Delta + \frac{2e}{c}g_-\mathbf{d}\cdot\left[\mathbf{H}\times\mathbf{v}_F\right] \\ =&\left(f_-\mathbf{d}\cdot\nabla\Delta^* + f^\dagger_-\mathbf{d}\cdot\nabla\Delta\right) - \frac{2e}{c}g_-\mathbf{d}\cdot\nabla\left(\mathbf{v}_F\cdot\mathbf{A}\right) \\ &+ \frac{2ie}{c}\mathbf{d}\cdot\mathbf{A}\left(f_-\Delta^* - f^\dagger_-\Delta\right) + \frac{2e}{c}g_-\left(\mathbf{v}_F\cdot\nabla\right)\mathbf{d}\cdot\mathbf{A} \\ &= \mathrm{Tr}\left[\check{g}_-(\mathbf{d}\cdot\nabla\check{H})\right] + \frac{2e}{c}\left(\mathbf{v}_F\cdot\nabla\right)\left(\mathbf{d}\cdot\mathbf{A}g_-\right)\end{aligned} \tag{14.26}$$

where $\mathbf{d}$ is an arbitrary constant vector. In the fourth line we use the Eilenberger equation

$$\mathbf{v}_F\cdot\nabla g^{R(A)} - i\left(f^{R(A)}\Delta^* - f^{\dagger R(A)}\Delta\right) = 0 \tag{14.27}$$

neglecting the small collision integral. Since $g^R - g^A$ vanishes for large distances from the vortex if $\epsilon < \Delta_\infty$ we obtain

$$\int \mathrm{Tr}\left[(\mathbf{d}\cdot\hat{\nabla}\check{H})\check{g}_-\right] ds = \int \mathrm{Tr}\left[(\mathbf{d}\cdot\nabla\check{H})\check{g}_-\right] ds \tag{14.28}$$

We use again eqn (14.23) and find that the r.h.s. of eqn (14.28) is

$$\begin{aligned}&-iv_\perp \int \mathrm{Tr}\left[(\mathbf{d}\cdot\nabla\check{H})\left(\check{G}^R_\epsilon(s,s) - \check{G}^A_\epsilon(s,s)\right)\right] ds \\ =&2\pi v_\perp \sum_n \delta\left(\epsilon - \epsilon_n\right) \int \mathrm{Tr}\left[\mathcal{U}^\dagger_n\left(d_s\frac{\partial\check{H}}{\partial s} + d_b\frac{\partial\check{H}}{\partial b}\right)\mathcal{U}_n\right] ds \\ =&2\pi v_\perp d_b \sum_n \delta\left(\epsilon - \epsilon_n\right) \int \mathrm{Tr}\left[\mathcal{U}^\dagger_n\frac{\partial\check{H}}{\partial b}\mathcal{U}_n\right] ds \\ =&2\pi\left(\left[\mathbf{v}_\perp\times\mathbf{d}\right]\cdot\hat{\mathbf{z}}\right)\sum_n \frac{\partial\epsilon_n}{\partial b}\delta\left(\epsilon - \epsilon_n\right)\end{aligned}$$

This proves the identity

$$\int \mathrm{Tr}\left[(\hat{\nabla}\check{H})\hat{g}_-\right] ds = 2\pi\left[\hat{\mathbf{z}}\times\mathbf{v}_\perp\right]\sum_n \frac{\partial\epsilon_n(b)}{\partial b}\delta\left[\epsilon - \epsilon_n(b)\right] \tag{14.29}$$

It is interesting to note that the integral

$$\begin{aligned}
&\int \mathrm{Tr}\left[(\hat{\nabla}\check{H})\hat{g}_-\right] d^2r \\
=&2\pi\left[\hat{\mathbf{z}}\times\mathbf{v}_\perp\right]\sum_n \int_{-\infty}^{\infty} \frac{\partial\epsilon_n(b)}{\partial b}\delta\left[\epsilon-\epsilon_n(b)\right] db \\
=&2\pi\left[\hat{\mathbf{z}}\times\mathbf{v}_\perp\right]\sum_n \mathrm{sign}\left(\frac{\partial\epsilon_n(b)}{\partial b}\right)
\end{aligned} \tag{14.30}$$

gives $2\pi\left[\hat{\mathbf{z}}\times\mathbf{v}_\perp\right]N(\epsilon)$ where an integer $N(\epsilon)$ is the algebraic sum of numbers of spectrum branches which cross the energy ϵ as functions of b. Here $+1$ is assigned to the branch which crosses ϵ upwards and -1 is for the branch going in the opposite direction. Therefore, the branch which starts at $b=-\infty$ from an energy above ϵ, crosses ϵ even number of times and then returns at $b=\infty$ to an energy above ϵ, does not contribute to N. For $|\epsilon|<\Delta_\infty$ it is the anomalous branch with $n=0$ only which crosses any energy once, and, therefore, $N=1$.

Application of eqn (14.30) to energies $|\epsilon|>\Delta_\infty$ needs a discussion. First, an integral over the remote surface may appear in course of transformations involved in the derivation of eqn (14.28). We will see how to deal with the surface terms later. Second, for energies above the gap, the energy spectrum is continuous. One can, however, make it quasi-discrete by placing the system in a large box. Now we can count the energy branches which cross an energy ϵ once. This counting can be performed for impact parameters larger than the core radius. This implies that the number N does not depend on the actual structure of the vortex core: to calculate N one can use any convenient model for the core. The surface contributions are determined by large distances and are also independent of the actual core structure.

The simplest model is an "artificial" vortex with a core size much larger than ξ. In such a vortex the order parameter magnitude varies on distances much longer than ξ so that the Green functions can be found by expanding the Eilenberger equations (5.84), (5.85) in powers of small gradients. We get for $|\epsilon|>|\Delta(\rho)|$:

$$f_- = \left(f_0 - \frac{\partial g_0}{\partial|\Delta|}\frac{e}{c}\mathbf{v}_F\left(\mathbf{A}-\frac{c}{2e}\nabla\chi\right) - \frac{i}{2|\Delta|}\mathbf{v}_F\nabla g_0\right)e^{i\chi} \tag{14.31}$$

$$g_- = g_0 + \frac{\partial g_0}{\partial\epsilon}\frac{e}{c}\mathbf{v}_F\left(\mathbf{A}-\frac{c}{2e}\nabla\chi\right) \tag{14.32}$$

and $f_- = \left(f_-^\dagger\right)^*$. For $|\epsilon|<|\Delta(\rho)|$ the functions are $f_- = f_-^\dagger = g_- = 0$. Here

$$g_0 = \frac{\epsilon}{\sqrt{\epsilon^2-|\Delta|^2}}\Theta\left(\epsilon^2-|\Delta|^2\right), f_0 = \frac{|\Delta|}{\sqrt{\epsilon^2-|\Delta|^2}}\Theta\left(\epsilon^2-|\Delta|^2\right). \tag{14.33}$$

We now find

$$\begin{aligned}
&\frac{1}{2}\int_{S_0}\mathrm{Tr}\left[(\hat{\nabla}\check{H})\left(\hat{g}^R-\hat{g}^R\right)\right] d^2r \\
=&-\int_{S_0}\left[\mathbf{v}_\perp\times\left(\mathrm{curl}\left[\left(\frac{2e}{c}\mathbf{A}-\nabla\chi\right)g_0\right]+g_0\,\mathrm{curl}\,\nabla\chi\right)-2f_0\nabla|\Delta|\right] d^2r \\
=&[\hat{\mathbf{z}}\times\mathbf{v}_\perp]\left(g_0(\infty)\oint d\mathbf{r}\left(\frac{2e}{c}\mathbf{A}-\nabla\chi\right)+2\pi g_0(0)\right)
\end{aligned}$$

The last term in the second line vanishes being the full derivative of $\sqrt{\epsilon^2-|\Delta|^2}$ because $|\Delta|$ is periodic in the vortex lattice. The contour integral taken along the boundary of the unit

cell represents the surface term mentioned earlier (note that for large distances, $g_0 = g_-$). It vanishes because the magnetic flux through the unit cell is just the flux quantum Φ_0. Since $|\Delta(0)| = 0$, the function $g_0(0) = 1$, and we obtain

$$\frac{1}{2}\int_{S_0} \mathrm{Tr}\left[\left(\hat{\nabla}\check{H}\right)\left(\hat{g}^R - \hat{g}^R\right)\right] d^2r = 2\pi\left[\hat{\mathbf{z}} \times \mathbf{v}_\perp\right] \tag{14.34}$$

The right hand side of eqn (14.34) corresponds to the number $N = 1$ of the branches crossing ϵ once. Note that the same model of the vortex core would
result in $N = 1$ for energies $|\epsilon| < \Delta_\infty$, as well, in accordance with the statement that this topological property of the spectrum does not depend on the core structure.

19.3.3 14.4 Distribution function

In this Section we concentrate on s-wave superconductors. The case of d-wave superconductors will be discussed later. The distribution function is determined by the kinetic equations (10.41) and (10.56) which are derived in Section 15.5. We write down these equations again

$$\begin{aligned}&\left(e\mathbf{v}_F \cdot \mathbf{E} g_- + \frac{1}{2}\left[f_- \frac{\hat{\partial}\Delta^*}{\partial t} + f_-^\dagger \frac{\hat{\partial}\Delta}{\partial t}\right]\right)\frac{\partial f^{(0)}}{\partial \epsilon} + \mathbf{v}_F \cdot \nabla\left(f_2 g_-\right)\\ &-\left(\frac{1}{2}\left[(\hat{\nabla}\Delta) f_-^\dagger + \left(\hat{\nabla}\Delta^*\right) f_-\right] - \frac{e}{c}\left[\mathbf{v}_F \times \mathbf{H}\right] g_-\right)\frac{\partial f_1}{\partial \mathbf{p}} = J_1\end{aligned} \tag{14.35}$$

and

$$\mathbf{v}_F \nabla f_1 = 0 \tag{14.36}$$

In the first equation, we omitted $\partial f_1/\partial t$ for a steady vortex motion as well as $\partial\Delta/\partial\mathbf{p}$ which vanishes for an s-wave superconductor. We already know from Section 15.5 that $f_2 \ll f_1$. Equations (14.35) and (14.36) are the basis for our derivations.

14.4.1 Localized excitations

Consider first excitations with $|\epsilon| < \Delta_\infty$ localized in the vortex core. The constant function f_1 can be found by integrating eqn (14.35) along the trajectory:

$$\begin{aligned}&-\int\left(\frac{1}{2}\left[(\hat{\nabla}\Delta) f_-^\dagger + \left(\hat{\nabla}\Delta^*\right) f_-\right] - \frac{e}{c}\left[\mathbf{v}_F \times \mathbf{H}\right] g_-\right)\frac{\partial f_1}{\partial \mathbf{p}} ds\\ &+\frac{\partial f^{(0)}}{\partial \epsilon}\int\left(e\mathbf{v}_F \mathbf{E} g_- + \frac{1}{2}\left[f_- \frac{\hat{\partial}\Delta^*}{\partial t} + f_-^\dagger \frac{\hat{\partial}\Delta}{\partial t}\right]\right) ds = \int J_1^{(1)}\left\{f_1\right\} ds\end{aligned} \tag{14.37}$$

In the collision integral J, only the term with f_1 is important. The term with $g_- f_2$ vanishes at large distances from the vortex core.

For a moving vortex,

$$\frac{\partial\Delta}{\partial t} = -\left(\mathbf{v}_L \cdot \nabla\right)\Delta, \quad \frac{\partial\mathbf{A}}{\partial t} = -\left(\mathbf{v}_L \cdot \nabla\right)\mathbf{A}$$

up to the leading terms in the vortex velocity $\mathbf{v}_L$. The expression

$$e\mathbf{v}_F \mathbf{E} g_- + \frac{1}{2}\left[f_- \frac{\hat{\partial}\Delta^*}{\partial t} + f_-^\dagger \frac{\hat{\partial}\Delta}{\partial t}\right]$$

in the second line in the l.h.s. of eqn (14.37) can be written as

$$g_-\frac{e}{c}\left(\mathbf{v}_L\nabla\right)\left(\mathbf{v}_F\mathbf{A}\right)-\frac{\mathbf{v}_L}{2}\left[f_-\nabla\Delta^*+f_-^\dagger\nabla\Delta\right]$$
$$-eg_-\mathbf{v}_F\nabla\varphi-ie\varphi\left(f_-\Delta^*-f_-^\dagger\Delta\right)$$

and transformed into

$$\begin{aligned}&-\frac{e}{c}\mathbf{v}_L\left[\mathbf{H}\times\mathbf{v}_F\right]g_--\frac{\mathbf{v}_L}{2}\left[f_-\hat{\nabla}\Delta^*+f_-^\dagger\hat{\nabla}\Delta\right]\\&+\frac{ie}{c}\left(\mathbf{v}_L\cdot\mathbf{A}\right)\left(f_-\Delta^*-f_-^\dagger\Delta\right)+g_-\frac{e}{c}\left(\mathbf{v}_F\nabla\right)\left(\mathbf{v}_L\mathbf{A}\right)-e\mathbf{v}_F\nabla\left(\varphi g_-\right)\\=&-\frac{\mathbf{v}_L}{2}\operatorname{Tr}\left[\hat{\nabla}\check{H}\check{g}_-\right]-\mathbf{v}_F\nabla\left[g_-\left(e\varphi-\frac{e}{c}\mathbf{v}_L\mathbf{A}\right)\right].\end{aligned}\tag{14.38}$$

Here we used two times the Eilenberger equation (14.27).
The integral in the first line of eqn (14.37) is

$$\frac{1}{2}\int\operatorname{Tr}\left(\hat{g}_-\hat{\nabla}\check{H}\right)\frac{\partial f_1}{\partial\mathbf{p}}ds\tag{14.39}$$

The integration along the trajectory in eqn (14.37) can now be performed employing the identity of eqn (14.29) and the fact that g_-vanishes for $s\to\pm\infty$. We find

$$\left[\pi\left(\left[\mathbf{v}_\perp\times\mathbf{v}_L\right]\hat{\mathbf{z}}\right)\tfrac{\partial f^{(0)}}{\partial\epsilon}\tfrac{\partial\epsilon_n}{\partial b}+\pi\left(\left[\mathbf{v}_\perp\times\tfrac{\partial f_1}{\partial\mathbf{p}}\right]\hat{\mathbf{z}}\right)\tfrac{\partial\epsilon_n}{\partial b}\right]\delta\left(\epsilon-\epsilon_n\right)$$
$$+\int_{-\infty}^{\infty}J_1ds=0$$

We use that $\partial f_1/\partial\mathbf{p}$ is constant along the trajectory. Consider this in more detail. The distribution function f_1 depends on the variables p_z,α,b, where p_z is the momentum along the z axis. The momentum derivative $\partial f_1/\partial\mathbf{p}$ can be presented as

$$\frac{\partial f_1}{\partial\mathbf{p}}=\pm\frac{\partial f_1}{\partial p_\perp}\frac{\mathbf{v}_\perp}{v_\perp}\pm\frac{1}{p_\perp}\left(\frac{\partial f_1}{\partial\alpha}-s\frac{\partial f_1}{\partial b}\right)\frac{\hat{\mathbf{z}}\times\mathbf{v}_\perp}{v_\perp}\tag{14.40}$$

The plus sign is for quasi-electrons while the minus sign is for quasi-holes, since the directions of $\mathbf{v}_\perp$ and $\mathbf{p}_\perp$ are either the same or opposite for these two types of quasiparticles, respectively. The terms $\partial f_1/\partial p_\perp$ and $\partial f_1/\partial\alpha$ in eqn (14.40) are indeed constant along the trajectory but the term $s\partial f_1/\partial b$ is not. However, one can show that this term vanishes identically for an axisymmetric vortex in $s-$ wave superconductors. We shall prove this later in Section 15.3 in a more general form.

The collision integral can be also presented as a spectral sum

$$J_1=\sum_n J_n\delta\left(\epsilon-\epsilon_n\right)\tag{14.41}$$

because it contains terms proportional to either g_-or $f_-,f_-^\dagger$. The kinetic equation becomes

$$\pi\left(\left[\mathbf{v}_\perp\times\mathbf{v}_L\right]\hat{\mathbf{z}}\right)\frac{\partial f^{(0)}}{\partial\epsilon}\frac{\partial\epsilon_n}{\partial b}+\pi\left(\left[\mathbf{v}_\perp\times\frac{\partial f_1}{\partial\mathbf{p}}\right]\hat{\mathbf{z}}\right)\frac{\partial\epsilon_n}{\partial b}+\int_{-\infty}^{\infty}J_nds=0\tag{14.42}$$

where all terms are taken at $\epsilon=\epsilon_n$. The term $\partial f_1/\partial\mathbf{p}$ in eqn (14.42) should be now understood as

$$\left(\left[\mathbf{v}_\perp\times\frac{\partial f_1}{\partial\mathbf{p}}\right]\hat{\mathbf{z}}\right)=\pm\frac{v_\perp}{p_\perp}\frac{\partial f_1}{\partial\alpha}\tag{14.43}$$

The component in eqn (14.40) parallel to $\mathbf{v}_\perp$ drops out of eqn (14.42).
14.4.1.1 Low temperatures The integro-differential equation (14.42) can hardly be solved analytically in a general case. Some simplifications arise when temperatures are much below T_c. Only the level with $n = 0$ is excited at such temperatures. The corresponding solution was first obtained by Kopnin and Kravtsov (1976 a). For simplicity, we restrict ourselves to a spherical Fermi surface.

We assume that the scattering by impurities is the primary source of relaxation. This is the most natural assumption for practically all superconducting materials. Consider the impurity collision integral for energies $\epsilon \ll \Delta_\infty$. We have from eqn (10.74) for the state with $n = 0$

$$J_0 = -\frac{1}{\tau}\left[\left(f_1\left\langle g_-\right\rangle - \left\langle f_1 g_-\right\rangle\right) g_0 \right.$$
$$\left. -\frac{1}{2}\left(\left(f_1\left\langle f_-^\dagger\right\rangle - \left\langle f_1 f_-^\dagger\right\rangle\right) f_0 + \left(f_1\left\langle f_-\right\rangle - \left\langle f_1 f_-\right\rangle\right) f_0^\dagger\right)\right].$$

Remind that $\langle\cdots\rangle$ is the usual average over the Fermi surface. The Green functions for low energies were found in Section 6.4. According to eqn (6.83) the spectral weights are

$$g_0 = \frac{\pi v_\perp e^{-K}}{2C}, f_0 = g_0 e^{i\phi}\frac{is}{\rho}$$

The bound state energy is given by eqn (6.84).

$$\epsilon_0 = C^{-1} b \int_0^\infty \left(\Delta_0/\rho\right) e^{-K} d\rho \tag{14.44}$$

where

$$K(\rho) = \frac{2}{v_\perp}\int_0^\rho \Delta_0 d\rho, C = \int_0^\infty e^{-K} d\rho$$

and we neglect a small ω_c. Due to the δ-function in $\check{g}_-$, the integration over $d\alpha$ selects a narrow region of angles near $\alpha = \phi$ and $\alpha = \pi + \phi$ such that the impact parameter is small $b \sim \epsilon/\Delta_\infty$ and the distance is either $s \approx \rho$ or $s \approx -\rho$, respectively. Calculating the Fermi-surface averages we will perform the integration over $d\alpha$ using the rule

$$\int \frac{d\alpha}{2\pi} F(s;b) = \int \frac{db}{2\pi|s|}\left[F(s=\rho;b) + F(s=-\rho;b)\right]$$

As a result

$$\int \frac{d\alpha}{2\pi} g_0 \delta\left(\epsilon - \epsilon_0\right) = \frac{g_0}{\pi\rho\left(\partial\epsilon_0/\partial b\right)}; \int \frac{d\alpha}{2\pi} f_0 \delta\left(\epsilon - \epsilon_0\right) = 0$$

Let us take the distribution function in the form of eqn (14.9) and consider the collision integral. Performing the averaging, we have

$$\int \frac{d\alpha}{2\pi} f_1 g_0 \delta\left(\epsilon - \epsilon_0\right) = 0$$

and

$$\int \frac{d\alpha}{2\pi} f_1 f_0 \delta\left(\epsilon - \epsilon_0\right) = -\frac{i p_\perp g_0 e^{i\phi}}{\pi\rho\left(\partial\epsilon_0/\partial b\right)}\frac{\partial f^{(0)}}{\partial\epsilon}\left[\gamma_{\mathrm{O}} v_{L\phi} + \gamma_{\mathrm{H}} v_{L\rho}\right]$$

Here we omit contributions proportional to small b. Using

$$p_\perp v_{L\rho} = \left[\frac{s}{\rho}\left(\mathbf{v}_L \cdot \mathbf{p}_\perp\right) + \frac{b}{\rho}\left(\left[\mathbf{v}_L \times \mathbf{p}_\perp\right] \cdot \mathbf{z}\right)\right]$$
$$p_\perp v_{L\phi} = \left[-\frac{b}{\rho}\left(\mathbf{v}_L \cdot \mathbf{p}_\perp\right) + \frac{s}{\rho}\left(\left[\mathbf{v}_L \times \mathbf{p}_\perp\right] \cdot \mathbf{z}\right)\right]$$

we obtain for the collision integral

$$\int J_0 ds = -\frac{1}{\tau_{\text{imp}}}\int_{-\infty}^{\infty}\left[f_1\left\langle g_-\right\rangle g_0 + \frac{1}{2}\left(\left\langle f_1 f_-^\dagger\right\rangle f_0 + \left\langle f_1 f_-\right\rangle f_0^\dagger\right)\right] ds$$
$$= \frac{\pi v_\perp}{\tau_{\text{imp}}}\frac{\partial f^{(0)}}{\partial \epsilon}\left[\hat{T}\left\{\gamma_{\text{H}}\right\}\left(\mathbf{v}_L \cdot \mathbf{p}_\perp\right) + \hat{T}\left\{\gamma_{\text{O}}\right\}\left(\left[\mathbf{v}_L \times \mathbf{p}_\perp\right]\mathbf{z}\right)\right] \tag{14.45}$$

where $\hat{T}\left\{\gamma_{\text{H,O}}\right\}$ denotes the operator

$$\hat{T}\left\{\gamma_{\text{H,O}}\right\} = \frac{1}{2C}\left[\left\langle\frac{v_\perp}{C\left(\partial\epsilon_0/\partial b\right)}\right\rangle\gamma_{\text{H,O}} + \frac{1}{p_\perp}\left\langle\frac{p_\perp v_\perp \gamma_{\text{H,O}}}{C\left(\partial\epsilon_0/\partial b\right)}\right\rangle\right]\ln\left(\frac{\Delta}{\epsilon}\right)$$

The integral $\int J_0 ds$ is logarithmic; the divergence is cut off at $\rho \sim (\epsilon/\Delta)\xi$.
To solve the kinetic equation we insert this into eqn (14.42) and find the expressions for γ_{O} and γ_{H} in terms of the vortex velocity. These expressions also contain the averages

$$\left\langle\frac{p_\perp v_\perp \gamma_{\text{H,O}}}{C\left(\partial\epsilon_0/\partial b\right)}\right\rangle$$

Next, we average γ_{O} and γ_{H} with the weight $p_\perp v_\perp / C\left(\partial\epsilon_0/\partial b\right)$, solve the obtained equation for these averages, and exclude them from expressions for γ_{O} and γ_{H}. This completes the calculation of γ_{O} and γ_{H}.

This general scheme of calculations can be carried out explicitly if we make use of the so called Kramer-Pesch effect: the compression of the vortex core at

Fig. 14.1. The factors $\left\langle\left\langle\gamma_{\text{O}}\right\rangle\right\rangle$ and $\left\langle\left\langle\gamma_{\text{H}}\right\rangle\right\rangle$ as functions of the purity parameter x.

low temperatures. According Kramer and Pesch (1974) the size of the vortex core decreases as $\xi_1 \sim (T/\Delta)\xi_0$ for $T \ll \Delta$. In this case one has from eqn (14.44)

$$C\left(\partial\epsilon_0/\partial b\right) = \Delta_\infty \ln\left(\Delta_\infty/T\right)$$

Equation (14.42) yields a set of equations

$$\left(\sin\theta\langle\sin\theta\rangle\gamma_{\text{O}} + \left\langle\sin^2\theta\gamma_{\text{O}}\right\rangle\right) = \pi x\left(1-\gamma_{\text{H}}\right)/4$$
$$\left(\sin\theta\langle\sin\theta\rangle\gamma_{\text{H}} + \left\langle\sin^2\theta\gamma_{\text{H}}\right\rangle\right) = \pi x\gamma_{\text{O}}/4 \tag{14.46}$$

where we put $v_\perp = v_F \sin\theta$ and

$$x = \frac{4\Delta_\infty^2 \tau_{\rm imp} \ln\left(\Delta_\infty/T\right)}{\pi E_F} \tag{14.47}$$

One finds from equations (14.46)

$$\left\langle \sin^2\theta \gamma_{\rm H} \right\rangle = \frac{z/2}{(1+y)^2 + (2z/\pi x)^2} \tag{14.48}$$

$$\left\langle \sin^2\theta \gamma_{\rm O} \right\rangle = \frac{\pi x}{4}\left[1 - \frac{1+y}{(1+y)^2 + (2z/\pi x)^2}\right] \tag{14.49}$$

where

$$y(x) = \frac{4}{\pi}\left\langle \frac{\sin^3\theta}{\sin^2\theta + x^2}\right\rangle, z(x) = 2x^2\left\langle \frac{\sin^2\theta}{\sin^2\theta + x^2}\right\rangle.$$

We shall see later that these are the values

$$\langle\langle \gamma_{\rm O,H} \rangle\rangle \equiv \frac{3}{2}\left\langle \sin^2\theta \gamma_{\rm O,H}\right\rangle$$

which are needed to calculate the corresponding components of the flux flow conductivity. The function $y(x) = 1$ for $x \to 0$ and $y(x) \to 0$ for $x \to \infty$ while $z(x) = 2x^2$ for $x \to 0$ and $z(x) = 4/3$ for $x \to \infty$. Correspondingly,

$$\langle\langle \gamma_{\rm H} \rangle\rangle = \begin{cases} 3x^2/8, & x \to 0, \\ 1, & x \to \infty, \end{cases} \qquad \langle\langle \gamma_{\rm O} \rangle\rangle = \begin{cases} 3\pi x/16, x \to 0 \\ 4.41/x, x \to \infty \end{cases}$$

The factors $\langle\langle \gamma_{\rm O,H} \rangle\rangle$ are plotted in Fig. 14.1 as functions of the purity parameter x.

14.4.1.2 Relaxation-time approximation, arbitrary temperatures The solution of the kinetic equations obtained for low temperatures by Kopnin and Kravtsov (1976 a) can be generalized to arbitrary temperatures (Kopnin and Lopatin 1995). To do this we need to simplify the collision integral. Inspection of eqn (14.45) shows that the collision integral takes the form

$$\int_{-\infty}^{\infty} J_0 ds = -\frac{\pi v_\perp}{\tau_0} f_1$$

if one neglects the p_z dependence of $\gamma_{\rm O}$ and $\gamma_{\rm H}$. Here $\tau_0 \sim \tau$ is some effective relaxation time for the low-energy level. To generalize our consideration for temperatures not much below T_c we adopt a τ-approximation for the collision integral. We assume that

$$\int_{-\infty}^{\infty} J_n ds = -\frac{\pi v_\perp}{\tau_n} f_1 \tag{14.50}$$

for all states where $\tau_n(b) \sim \tau_{\rm imp}$ is the effective scattering time. Within the relaxation-time approximation the relaxation rate τ_n^{-1} can also include electronphonon and electron-electron collisions. The electron-phonon scattering can be important for very low concentration of impurities when $\ell_{\rm imp} \gg \ell_{\rm ph}$.

We also write the distribution function in a more general form

$$f_1 = -\frac{\partial f^{(0)}}{\partial \epsilon}\left[\frac{p_\perp \gamma_{\rm O}}{v_\perp}\left(\left[\mathbf{v}_L \times \mathbf{v}_\perp\right]\hat{\mathbf{z}}\right) \pm \frac{p_\perp \gamma_H}{v_\perp}\left(\mathbf{v}_L \cdot \mathbf{v}_\perp\right)\right] \tag{14.51}$$

where we take into consideration a possible existence of both particle-like and hole-like carriers in the normal state. The factors $\gamma_{\rm O,H}$ are found from eqn (14.42) with help of eqn

(14.50). They are determined by eqn 14.14 and coincide with the result obtained earlier within the Boltzmann kinetic equation. Equation (14.14) reasonably well reproduces the most important feature of the exact expression (14.48) and (14.49) for the lowest level $n = 0$: As a function of $\omega_0\tau$, the parameter γ_{O} first increases linearly, reaches the maximum value of the order unity for $\omega_0\tau \sim 1$ and then decreases as $(\omega_0\tau)^{-1}$ for $\omega_0\tau \gg 1$. The parameter γ_{H} grows monotonously first as $(\omega_0\tau)^2$ and then saturates at $\gamma_{\mathrm{H}} = 1$ for $\omega_0\tau \gg 1$, see Fig. 14.1. The use of τ-approximation allows us to find the distribution function for levels with higher energies, as well. We see that the general trend of the parameters γ_{O} and γ_{H} as functions of $\omega_n\tau$ is the same as for the low-energy level.

The order of magnitude of ω_n is $\Delta_0^2/E_F \sim eH_{c2}/m^*c$. One can say that the behavior of localized excitations in the moving vortex is similar to that of a charge in a magnetic field of the order of H_{c2}. This contrasts to the Bardeen and Stephen (1965) assumption of the effective field being the external field H.

14.4.2 Delocalized excitations

One would expect that, if the density of moving vortices is small enough, the deviation of the distribution function from equilibrium for the excitations with energies $|\epsilon| > \Delta_\infty$ should vanish at large distances, where the excitations are in equilibrium with the crystal lattice or sample boundaries. However, for a finite vortex density, when the distance between vortices is shorter than the magnetic field penetration depth, one has to take into account also the motion of electrons in the magnetic filed with the induction $B \approx H$, where H is the applied filed. Their behavior is determined by the relation between the Larmor radius $r_H = v_F/\omega_c \sim (E_F/\Delta)\,(H_{c2}/H)\,\xi$ and the mean free path ℓ.

The trajectory of a delocalized quasiparticle crosses many vortex unit cells at various distances from vortex axes, i.e., at different impact parameters. Since the distribution function f_1 is constant along the trajectory it should thus be independent of the impact parameter. Let us integrate eqn (14.35) over the unit cell of the vortex lattice. Using eqn (14.38) we obtain

$$\begin{aligned}
&-\frac{1}{2}\int_{S_0} \mathrm{Tr}\left[\mathbf{v}_L\cdot\hat{\nabla}\check{H}\check{g}_-\right]\frac{\partial f^{(0)}}{\partial\epsilon}d^2r - \frac{1}{2}\int_{S_0}\mathrm{Tr}\left[\hat{\nabla}\check{H}\check{g}_-\right]\frac{\partial f_1}{\partial\mathbf{p}}d^2r\\
&+\int_{S_0}\mathbf{v}_F\cdot\nabla\left[g_-f_2 - g_-\left(e\varphi - \frac{e}{c}\mathbf{v}_L\cdot\mathbf{A}\right)\frac{\partial f^{(0)}}{\partial\epsilon}\right]d^2r\\
=&\int_{S_0} J_1 d^2r
\end{aligned}$$

Second line gives zero because the functions under the gradient do not increase with distance; the average gradient should thus vanish. Using the identity eqn (14.34) we find

$$\pi\left(\left[\mathbf{z}\times\mathbf{v}_\perp\right]\mathbf{v}_L\right)\frac{\partial f^{(0)}}{\partial\epsilon} + \pi\left(\left[\mathbf{z}\times\mathbf{v}_\perp\right]\frac{\partial f_1}{\partial\mathbf{p}}\right) + \int_{S_0} J_1 d^2r = 0 \tag{14.52}$$

The integral of the last term is determined by large distances from the vortex core where the Green functions $g_-$and $f_-$are determined by eqn (14.33). Equation (10.74) results in the collision integral

$$J_1 = -\frac{1}{\tau_{\mathrm{imp}}}\left(f_1 - \langle f_1\rangle\right)$$

Therefore,

$$\int_{S_0} J_1 d^2r = -\frac{f_1}{\tau_{\mathrm{imp}}}\frac{\Phi_0}{B}$$

We find from eqn (14.52) that the factors in eqn (14.51) are

$$\gamma'_{\mathrm{O}} = \frac{\omega_c \tau}{\omega_c^2 \tau^2 + 1}; \gamma'_{\mathrm{H}} = \frac{\omega_c^2 \tau^2}{\omega_c^2 \tau^2 + 1} \tag{14.53}$$

where we replace τ_{imp} with τ for brevity. The cyclotron frequency is

$$\omega_c = H|e|v_\perp / cp_\perp \tag{14.54}$$

The modulus of charge appears due to our choice of the z axis.
For high fields, $\omega_c \tau \gg 1$, we obtain

$$f_1 = \mp (\mathbf{p} \cdot \mathbf{v}_L) \frac{\partial f^{(0)}}{\partial \epsilon}$$

This form of the distribution function suggests that the excitations move together with the vortex. Delocalized excitations can not relax at the heat bath, since they can not escape to distances longer than $r_H \ll \ell$. On the contrary, for low fields such that $\omega_c \tau \ll 1$ and $r_H \gg \ell$, we have $f_1 \ll p v_L \left(\partial f^{(0)} / \partial \epsilon\right)$. In this case the delocalized excitations are almost in equilibrium with the heat bath and do not participate in the vortex motion.

The case of low fields corresponds to the situation in electrically neutral superfluid systems, where $r_H = \infty$. Consider it in more detail. The kinetic equations (10.41) and (10.44) at large distances from the vortex give

$$\mathbf{v}_\perp \cdot \nabla (g_- f_2) = -\frac{1}{\tau_{\mathrm{imp}}} (f_1 - \langle f_1 \rangle) \tag{14.55}$$

$$\mathbf{v}_\perp \cdot \nabla f_1 = -\frac{g_-}{\tau_{\mathrm{imp}}} (f_2 - \langle f_2 \rangle) \tag{14.56}$$

We keep the collision integral in the second equation because the distances which we are interested in are of the order and larger than ℓ, thus eqn (14.36) is not sufficient, see page 210 . The functions f_1 and f_2 are independent of the distance along the trajectory for $\xi \ll \rho \ll \ell$. These constants are coupled to each other by the conditions $f_1 = g_- f_2$ for $s \to +\infty$ and $f_1 = -g_- f_2$ for $s \to -\infty$ which follow from the fact that eqns (14.55, 14.56) should not have exponentially increasing solutions. They are the boundary conditions to be imposed at distances s larger than the core size of the vortex, but shorter than ℓ. For delocalized excitations with ϵ above the gap at infinity, these boundary conditions give $f_1 \sim f_2$. As a result both f_1 and f_2 are small. Indeed, since $f_1 \sim f_2$, one can neglect both the term with $\partial f_1 / \partial \mathbf{p}$ and the collision integral in eqn (14.35) compared to the term containing f_2. The distribution functions f_1 and f_2 which come from eqn (14.35) are then, by the order of magnitude, $(v_L \Delta / v_F)(\partial f_0 / \partial \epsilon)$. Hence the distribution function f_1 for $\epsilon > \Delta_\infty$ is by the factor of $\Delta / (E_F \min\{\omega_0 \tau, 1\})$ smaller than its magnitude for $\epsilon < \Delta_\infty$ and should be considered as zero. This agrees with eqn (14.53) in the limit $\omega_c \to 0$.

19.3.4 14.5 Flux flow conductivity

The force from environment on a vortex is determined by the equation (13.10). Remind that the nonequilibrium Green function is, according to eqn (13.15)

$$\hat{g}^{(\mathrm{nst})} = -\frac{i}{2} \frac{\hat{\partial} \left(\hat{g}^R + \hat{g}^A\right)}{\partial t} \frac{\partial f^{(0)}}{\partial \epsilon}$$

$$+ \left(\hat{g}^R - \hat{g}^A\right) f_1 + \left(\hat{g}^R \hat{\sigma}_z - \hat{\sigma}_z \hat{g}^A\right) f_2 \tag{14.57}$$

The main contribution to eqn (14.57) comes from the part containing f_1. The force from environment eqn (13.10) becomes

$$\mathbf{F}_{\text{env}} = -\int d^2r \int \frac{d\epsilon}{4} \int \frac{dS_F}{(2\pi)^3 v_F} \operatorname{Tr}\left[\left(\hat{g}^R - \hat{g}^A\right) \hat{\nabla} \check{H}\right] f_1 \tag{14.58}$$

14.5.1 Localized excitations

Consider first localized excitations in the vortex core with $|\epsilon| < \Delta_\infty$. Using the identity of eqn (14.29) we obtain the contribution from the localized excitations to eqn (14.58)

$$\mathbf{F}_{\text{env}}^{(\text{loc})} = -\pi \int db \int d\epsilon \int \frac{dS_F}{(2\pi)^3 v_F} f_1 \left[\hat{\mathbf{z}} \times \mathbf{v}_\perp\right] \sum_n \frac{\partial \epsilon_n}{\partial b} \delta\left(\epsilon - \epsilon_n\right) \tag{14.59}$$

The integration over db and ds is extended from $-\infty$ to $+\infty$ since the integral converges at distances of the order of ξ which is much shorter than the unit cell size. Equation (14.59) coincides exactly with eqn (14.4) obtained phenomenologically using the concept of semi-classical particles localized in the core.

The Fermi-surface area element $dS_F = dp_\perp dp'$ is constructed out of two tangential momentum increments: $d\mathbf{p}_\perp$ with the magnitude $dp_\perp = p_\perp d\alpha$ in the (xy) plane, and $d\mathbf{p}'$ perpendicular to both $d\mathbf{p}_\perp$ and $\mathbf{v}_F$. One has $(v_\perp / v_F)\, dp' = dp_z$. Therefore, for a Fermi surface isotropic in the (ab) plane,

$$\int \frac{v_\perp p_\perp dS_F}{(2\pi)^3 v_F} = \int \frac{2S\left(p_z\right) dp_z}{(2\pi)^3} \frac{d\alpha}{2\pi} \tag{14.60}$$

where $S\left(p_z\right) = \pi p_\perp^2$ is the area of the cross section of the Fermi surface by the plane $p_z =$ const. The integral in eqn (14.60) gives

$$\frac{2V_F}{(2\pi)^3} = N$$

where V_F is the volume within the Fermi surface and N is the density of electrons (or holes). Equation (14.59) reduces to eqn (14.4) for a spherical Fermi surface.

Performing the average over $d\alpha$ in eqn (14.59) we get with help of eqn (14.51)

$$\begin{aligned}\mathbf{F}_{\text{env}}^{(\text{loc})} =& \pi \sum_n \int \frac{2S\left(p_z\right) dp_z}{(2\pi)^3} \int \frac{db}{2} \frac{\partial \epsilon_n}{\partial b} \frac{\partial f^{(0)}\left(\epsilon_n\right)}{\partial \epsilon} \\ & \times \left[-\gamma_{\text{O}}\left(\epsilon_n\right) \mathbf{v}_L \pm \gamma_{\text{H}}\left(\epsilon_n\right) \left[\hat{\mathbf{z}} \times \mathbf{v}_L\right]\right]\end{aligned} \tag{14.61}$$

which coincides with eqns (14.11, 14.12). The longitudinal force is determined by the sum over the spectrum branches with all the radial quantum numbers n. However, the transverse force contains only the branch with $n = 0$. This is because γ_{H} is an even function of the impact parameter b while the derivative
$\partial \epsilon_n / \partial b$ is even for $n = 0$ and odd for $n \neq 0$. Therefore, the contributions from the branches with $n \neq 0$ drop out. In contrast, the combination $\left(\partial \epsilon_n / \partial b\right) \gamma_{\text{O}}$ in the expression for $\mathbf{F}_{\parallel}^{(\text{loc})}$ is even for all the branches.

Balancing the force from environment by the Lorentz force eqn (14.6) determines the contribution to the transport current from the localized excitations

$$\mathbf{j}_{\text{tr}}^{(\text{loc})} = \sigma_{\text{O}}^{(\text{loc})} \mathbf{E} + \sigma_{\text{H}}^{(\text{loc})} [\mathbf{E} \times \mathbf{B}] / B$$

where the Ohmic and Hall components of conductivity are:

$$\sigma_{\mathrm{O}}^{(\mathrm{loc})} = \frac{2|e|c}{(2\pi)^3 B} \sum_n \left(\int_e + \int_h \right) S\left(p_z\right) dp_z \int \frac{db}{2} \frac{\partial \epsilon_n}{\partial b} \frac{df^{(0)}\left(\epsilon_n\right)}{d\epsilon} \gamma_{\mathrm{O}}\left(\epsilon_n\right) \tag{14.62}$$

$$\sigma_{\mathrm{H}}^{(\mathrm{loc})} = \frac{2ec}{(2\pi)^3 B} \left(\int_e - \int_h \right) S\left(p_z\right) dp_z \int \frac{db}{2} \frac{\partial \epsilon_0}{\partial b} \frac{df^{(0)}\left(\epsilon_0\right)}{d\epsilon} \gamma_{\mathrm{H}}\left(\epsilon_0\right) \tag{14.63}$$

The integrals over the Fermi surface are shown for the electron-like and holelike parts separately. Equations (14.62) and (14.63) coincide exactly with eqns (14.15) and (14.16) obtained by the Boltzmann scheme.

14.5.2 Delocalized excitations

To evaluate the part of the force in eqn (14.58) for excitations with energies $|\epsilon| > \Delta_\infty$ we use the identity eqn (14.34) derived earlier. We obtain

$$\begin{aligned} \mathbf{F}_{\mathrm{env}}^{(\mathrm{del})} =& \pi \sum_n \int \frac{2S\left(p_z\right) dp_z}{(2\pi)^3} \int_{|\epsilon|>\Delta_\infty} \frac{d\epsilon}{2} \frac{\partial f^{(0)}(\epsilon)}{\partial \epsilon} \\ & \times \left[-\gamma_{\mathrm{O}}'\left(\epsilon_n\right) \mathbf{v}_L \pm \gamma_{\mathrm{H}}'\left(\epsilon_n\right) \left[\hat{\mathbf{z}} \times \mathbf{v}_L\right] \right] \end{aligned} \tag{14.64}$$

The force balance eqn (14.7) yields

$$\mathbf{j}_{\mathrm{tr}}^{(\mathrm{del})} = \sigma_{\mathrm{O}}^{(\mathrm{del})} \mathbf{E} + \sigma_{\mathrm{H}}^{(\mathrm{del})} [\mathbf{E} \times \mathbf{B}]/B$$

where

$$\sigma_{\mathrm{O}}^{(\mathrm{del})} = \frac{2|e|c}{(2\pi)^3 B} \left[\int_e S\left(p_z\right) \gamma_{\mathrm{O}}' dp_z + \int_h S\left(p_z\right) \gamma_{\mathrm{O}}' dp_z \right] \left[1 - \tanh \frac{\Delta_\infty}{2T} \right] \tag{14.65}$$

$$\sigma_{\mathrm{H}}^{(\mathrm{del})} = \frac{2ec}{(2\pi)^3 B} \left[\int_e S\left(p_z\right) \gamma_{\mathrm{H}}' dp_z - \int_h S\left(p_z\right) \gamma_{\mathrm{H}}' dp_z \right] \left[1 - \tanh \frac{\Delta_\infty}{2T} \right] \tag{14.66}$$

We use that $\gamma_{\mathrm{O,H}}'$ are independent of ϵ.

19.3.5 14.6 Discussion

14.6.1 Conductivity: Low temperatures

The total conductivities are sums of the contributions from both localized and delocalized excitations: $\sigma = \sigma^{(\mathrm{loc})} + \sigma^{(\mathrm{del})}$. However, the contribution from delocalized excitations is small compared to $\sigma^{(loc)}$ in most of the cases since the

Fig. 14.2. The Hall angle as a function of the purity parameter x.

ratio $\omega_c/\omega_0 \sim H/H_{c2}$ is small in our limit. The exceptions are: (i) the case of very pure samples such that $\omega_c\tau > 1$ even for fields $H \ll H_{c2}$, (ii) the immediate vicinity of T_c.

We start the discussion with the exact solution of the kinetic equation available for low temperatures (14.48), (14.49). Delocalized excitations do not contribute. Equations (14.63), (14.62) give

$$\sigma_{\mathrm{O}} = \frac{(N_e + N_h)\,|e|c}{B}\,\langle\langle\gamma_{\mathrm{O}}\rangle\rangle\,, \sigma_{\mathrm{H}} = \frac{(N_e - N_h)\,ec}{B}\,\langle\langle\gamma_{\mathrm{H}}\rangle\rangle \tag{14.67}$$

where x is determined by eqn (14.47). The electron (hole) densities are $N_{e,h} = 2V_{e,h}/(2\pi)^3$, where $V_{e,h}$ are the volumes confined within the particle- and holelike parts of the Fermi surface, respectively. The factors $\langle\langle\gamma_{\mathrm{O}}\rangle$ and $\langle\langle\gamma_{\mathrm{H}}\rangle$ are shown in Fig. 14.1 as functions of x. The Hall angle defined through $\tan\Theta_{\mathrm{H}} = \sigma_{\mathrm{H}}/\sigma_{\mathrm{O}}$ is plotted in Fig. 14.2.

For a moderately clean limit, $x \ll 1$, the dissipative (Ohmic) component of the conductivity tensor can be written in a form similar to the Bardeen-Stephen model. The upper critical field for clean superconductors at low temperatures has been calculated by Gor'kov (1959 b)

$$H_{c2} = \left(\frac{e^2_{\ln\gamma}}{2}\right)\frac{c\Delta^2_\infty}{|e|v_F^2} \approx 1.04\frac{\Phi_0}{2\pi\xi_0^2}.$$

Here $e_{\ln} \approx 2.718$ is the base of natural logarithm (we denote it $e_{\ln}$ to distinguish from the electronic charge e), $\gamma \approx 1.78$ is the Euler constant, and $\xi_0 = v_F/2\pi T_c$ is the zero temperature coherence length. Equation (3.87) was used to write the upper critical filed in terms of ξ_0. We find with help of this expression

$$\sigma_{\mathrm{O}} = \frac{3\pi x\,(N_e + N_h)\,|e|c}{16B} \approx 0.23\sigma_n\frac{H_{c2}}{B}\ln\frac{\Delta_\infty}{T}. \tag{14.68}$$

Equation (14.68) was first obtained by Larkin and Ovchinnikov (1976). It has an extra logarithmic factor as compared to the Bardeen-Stephen model. The Hall component is small, $\sigma_{\mathrm{H}}/\sigma_{\mathrm{O}} \sim x$.

In the superclean limit $x \gg 1$ the factor $\langle\langle\gamma_{\mathrm{H}}\rangle\rangle = 1$ so that the Hall conductivity is large

$$\sigma_{\mathrm{H}} = (N_e - N_h)\,ec/B$$

while the dissipative component vanishes as x^{-1}. The Hall angle is close to $\pi/2$.

14.6.2 Conductivity: Arbitrary temperatures

14.6.2.1 Superclean limit The superclean limit is reached when $\omega_0\tau \sim \omega_n\tau_n \gg 1$. The parameter $\omega_c\tau$ can be either small or large. In both cases the Hall angle is close to $\pi/2$. Consider first the contribution from localized excitations. In this case $\gamma_{\mathrm{H}} = 1$, and γ_{O} is small. The Hall conductivity from the localized excitations becomes

$$\sigma_{\mathrm{H}}^{(\mathrm{loc})} = \frac{2ec}{(2\pi)^3 B}\left(\int_e - \int_h\right) S\left(p_z\right) dp_z \int \frac{db}{2}\frac{\partial\epsilon_0}{\partial b}\frac{\partial f^{(0)}\left(\epsilon_0\right)}{\partial\epsilon} \tag{14.69}$$

Since $\epsilon_0(b)$ runs from $-\Delta_\infty$ to $+\Delta_\infty$ as b varies from $-\infty$ to $+\infty$ (see Fig. 6.2), we obtain

$$\sigma_{\mathrm{H}}^{(\mathrm{loc})} = \frac{\left(N_e - N_h\right) ec}{B}\tanh\left(\frac{\Delta_\infty}{2T}\right) \tag{14.70}$$

When the purity is so high that also $\omega_c\tau \gg 1$ the delocalized excitations eqn (14.66) give

$$\sigma_{\mathrm{H}}^{(\mathrm{loc})} = \frac{\left(N_e - N_h\right) ec}{B}\left[1 - \tanh\left(\frac{\Delta_\infty}{2T}\right)\right]$$

such that the total conductivity is $\sigma_{\mathrm{H}} = \left(N_e - N_h\right) ec/B$. This is exactly the Hall conductivity of a normal metal in a high magnetic field. The Ohmic conductivity is small.

Despite very strong limitations on the purity of the sample, the superclean regime has been shown to be quite realistic: for example, $\tan\Theta_{\mathrm{H}}$ of the order of unity has been observed by Harris et al. (1994) for low temperatures. In principle, inelastic electron-phonon (and electron-electron) relaxation can also play a role if the impurity content is low enough. The electron-phonon mechanism can become more effective in the superclean regime since, for high-T_c superconductors, the electron-phonon relaxation rate is quite substantial at T_c, though it decreases rapidly for lower temperatures. It is possible that the increase in the Hall angle towards the superclean limit observed by Harris et al. (1994) is due to the increase in τ_{ph} with lowering the temperature.

14.6.2.2 Moderately clean limit This limit corresponds to $\omega_0\tau \ll 1$, and is the most common situation in clean superconductors. Moreover, this regime is always reached as T approaches T_c even for a very pure sample: for temperatures close to T_c, the parameter $\omega_0\tau_0$ decreases with Δ_∞ since $\partial\epsilon_0/\partial b \sim \Delta_\infty^2/v_F$. In
moderately clean limit the conductivities are mainly determined by the localized excitations because $\omega_c\tau \ll \omega_0\tau \ll 1$. One has in the moderately clean limit $\gamma_{\mathrm{H}} = \left(\omega_0\tau_0\right)^2, \gamma_{\mathrm{O}} = \omega_n\tau_n$. The Hall conductivity becomes

$$\sigma_{\mathrm{H}} = \frac{ec}{(2\pi)^2 B}\left(\int_e - \int_h\right) dp_z \int_0^\infty db\frac{\partial f^{(0)}\left(\epsilon_0\right)}{\partial\epsilon}\tau_0^2\left(\frac{\partial\epsilon_0}{\partial b}\right)^3 \tag{14.71}$$

The Ohmic conductivity is

$$\sigma_{\mathrm{O}} = \frac{|e|c}{(2\pi)^2 B}\left(\int_e + \int_h\right) p_\perp dp_z \sum_n \int_0^\infty db\frac{\partial f^{(0)}\left(\epsilon_n\right)}{\partial\epsilon}\tau_n\left(\frac{\partial\epsilon_n}{\partial b}\right)^2 \tag{14.72}$$

The Hall angle is small: $\tan\Theta_{\mathrm{H}} \sim \omega_0\tau$.
For low temperatures, $T \ll T_c$,

$$\sigma_{\mathrm{H}} = \frac{ec}{(2\pi)^2 B}\left(\int_e - \int_h\right) dp_z\tau_0^2\left(\frac{\partial\epsilon_0}{\partial b}\right)^2 \tag{14.73}$$

where the derivative of the energy is taken at $b = 0$. The Ohmic conductivity is

$$\sigma_{\mathrm{O}} = \frac{|e|c}{(2\pi)^2 B}\left(\int_e + \int_h\right) p_\perp dp_z \tau_0 \frac{\partial \epsilon_0}{\partial b} \tag{14.74}$$

since only the level with $n = 0$ is important for $T \ll T_c$. These expressions are generalizations of the results obtained earlier for low temperatures by the exact solution of the kinetic equation assuming the Kramer-Pesch contraction of the vortex core. With $\tau_0 \sim \tau$ and $\partial\epsilon_0/\partial b \sim (\Delta_\infty/\xi_0)\ln(\Delta_\infty/T)$, eqns (14.73) and (14.74) agree with eqns (14.67) in the limit $x \ll 1$.

For temperatures close to T_c, the Ohmic conductivity eqn (14.72) for $B \ll H_{c2}$ is (Kopnin and Lopatin 1995)

$$\sigma_{\mathrm{O}} \sim \sigma_n \frac{H_{c2}}{B}\frac{\Delta_\infty}{T_c} \tag{14.75}$$

Equation (14.75) implies a temperature dependence of the conductivity $\sigma_{\mathrm{O}} \propto (1 - T/T_c)^{3/2}$. The extra factor Δ_∞/T_c in eqn (14.75) as compared to the BardeenStephen model has been indeed observed experimentally by Kambe et al. (1999). The Hall conductivity near T_c is

$$\sigma_{\mathrm{H}}^{(\mathrm{loc})} \sim \frac{Nec}{B}\left(\frac{T_c^2\tau}{E_F}\right)^2\left(\frac{\Delta_\infty}{T_c}\right)^5 \tag{14.76}$$

It is proportional to $(1 - T/T_c)^{5/2}$. This temperature dependence has been confirmed experimentally by Kim et al. (1996).

The temperature dependence of the flux-flow conductivities in clean superconductors differs from that obtained using the TDGL model (see Section 12.5).

This is not unexpected because the TDGL model does not work for clean superconductors. Moreover, eqn (14.75) differs also from the prediction of the microscopic theory for dirty superconductors near T_c (see Section 13.2). However, the microscopic results for dirty superconductors transform into expression (14.75) for clean superconductors at the border where these two limiting cases meet. Equation (13.25) of Section 13.2 can be written as

$$\sigma_{\mathrm{O}} \sim \sigma_n \frac{H_{c2}}{B}\frac{T_c}{\Delta_\infty}\Lambda^2 \tag{14.77}$$

Equation (14.77) is valid as long as $\Delta(T)\tau \ll 1$. To match it with eqn (14.75) we need to take eqn (14.77) in the regime when $T_c\tau \gg 1$. For $T_c\tau \gg 1$ the function $\Lambda^2(T_c\tau) \sim (T_c\tau)^{-2}$ thus eqn (14.77) becomes

$$\sigma_{\mathrm{O}} \sim \sigma_n \frac{H_{c2}}{B}\frac{\Delta_\infty}{T_c}(\Delta_\infty\tau)^{-2}$$

We can see that this expression transforms into eqn (14.75) at the limit of its applicability $\Delta_\infty(T)\tau \sim 1$. Equation (14.75) also transforms into eqn (12.59) obtained in Section 12.8.2 for a clean d-wave superconductor if we put $\Delta_\infty(T)\tau_{\mathrm{imp}} \sim 1$ in eqn (14.75).

The reason for the different dependences of σ_{O} is that, for clean superconductors with $\ell \gg \xi(T)$, the main contribution to conductivity comes from localized excitations only whose number decreases as $T \to T_c$. On the contrary, in dirty superconductors with $\ell \ll \xi$, all the excitations contribute to the conductivities. Deviation of the distribution function from equilibrium increases as T approaches T_c because the diffusion relaxation becomes slow due to an increase in the vortex core size.

We can make an interesting observation concerning the sign of the Hall effect. Comparing eqn (14.73) with the Hall conductivity in the normal state (Abrikosov 1998):

$$\sigma_{\mathrm{H}}^{(\mathrm{del})} = \frac{e^3 B\tau^2}{(2\pi)^2 c^2}\left(\int_e v_\perp^2 dp_z - \int_h v_\perp^2 dp_z\right) \tag{14.78}$$

we see that the sign of the normal-state Hall conductivity can differ from that in the superconducting state. The reason is that the sign is determined by the result of subtraction of electron and hole contributions which are given by integrals over the Fermi surface taken of the functions with different momentum dependences. It is the cyclotron frequency in the normal state, while it is the interlevel spacing in the superconducting state. For example, the hole contribution in the normal state can be larger than the electron contribution, however, electrons can give more than holes in the superconducting state. The sign of the Hall effect in this case will change after the transition from the normal to the superconducting state. The sign reversal depends on the shape of the Fermi surface; it is absent for a simple parabolic spectrum $E_n = p^2/2m^*$ when the flux-flow and the normal-state Hall conductivities have the same sign as was assumed in the earlier models by Bardeen and Stephen (1965) and by Nozières and Vinen (1966). The possibility for the Hall angle anomaly, i.e., for the sign reversal of the Hall angle exists also within the modified TDGL model discussed in Section 12.9.1. It was attributed to the energy dependent density of states at the Fermi surface. In clean superconductors, the origin of the Hall angle sign reversal is also associated with a complicated Fermi surface in the normal state. However, the exact reason for the anomaly is slightly different: In clean superconductors, the energy spectrum of localized excitations in the vortex core plays the most important role. Being dependent on the shape and on the topology of the Fermi surface, the Hall effect anomaly can appear or disappear as the chemical potential E_F varies for different doping levels of the superconducting material. This behavior has been indeed observed in many experiments (see, for example, Nagaoka et al. (1998)).

To conclude the discussion we note that the origin of the Hall effect in clean superconductors differs from the mechanism discussed in Section 12.9. In clean superconductors, it is the dynamics of nonequilibrium excitations which determines the conductivity. It gives a much larger contribution to the Hall conductivity than what we have obtained in Section 12.9. Indeed, the Hall angle in a moderately clean limit is $\tan\Theta_H \sim \omega_0\tau$ while it only is $\tan\Theta_H \sim \lambda^{-1}\,(T_c/E_F)$ for the mechanism associated with the variation of the pairing interaction due to the chemical potential changes induced by a moving vortex. The latter effect does not exist within the quasiclassical approximation. It appears due to violation of the particle-hole symmetry of eqn (13.14); the symmetry breaks down when the density of states assumes an energy dependence beyond the quasiclassical approximation. As a result, eqn (13.14) acquires a small term linear in the time derivative of the order parameter.

We outline briefly the origin of this effect. As we noticed already, it is proportional to the inverse strength of the pairing interaction λ. The $1/\lambda$ contribution to the force on a vortex comes from the regular part of the nonstationary Green function in the expression of the type of eqn (13.14). To calculate it we need to return to the full non-quasiclassical Green functions $\check{G}^{R(A)}$. Equation (13.14) becomes

$$\int d^2r \int \frac{d\epsilon}{4\pi i} \int \frac{d^3p}{(2\pi)^3} f^{(0)}(\epsilon)\,\mathrm{Tr}\left[\left(\check{G}^R_{\epsilon_+,\epsilon_-} - \check{G}^A_{\epsilon_+,\epsilon_-}\right)\hat{\nabla}\check{H}\right] \tag{14.79}$$

We now expand eqn (14.79) in powers of ω up to the first-order terms. The zero-order term represents the static contribution and drops out of the nonequilibrium part of the Green function. The first-order term vanishes within the quasiclassical approximation due to the particle-hole symmetry. As we shall see in a moment, the non-quasiclassical correction in eqn (14.79) from the first-order term is logarithmically divergent due to integration over $d\epsilon$. This divergence is to be cut off at the BCS frequency Ω_{BCS}. Since the characteristic frequencies are

large, one can look for the Green function by expanding the Gor'kov equation (8.20) in powers of the small ratio Δ/ϵ up to the leading term. We obtain

$$F_{corr}^{R(A)} = -\frac{\hat{\omega}\Delta_\omega}{2}\frac{\partial}{\partial\xi_{\mathbf{p}}}\left[G_\epsilon^{R(A)}\bar{G}_\epsilon^{R(A)}\right] \tag{14.80}$$

$F_{\text{corr}}^{\dagger R(A)}$ is obtained by substituting $\hat{\omega}\Delta_\omega$ with $-\hat{\omega}\Delta_\omega^*$. The operator $\hat{\omega}$ is defined by eqn (13.13). It is because of the full derivative with respect to $d\xi_{\mathbf{p}}$ that the quasiclassical contribution to eqn (14.79) vanishes. However, the contribution becomes nonzero if one takes into account an energy dependence of the normalstate density of states in the momentum-space volume element

$$\frac{d^3p}{(2\pi)^3} = \nu\left(\xi_{\mathbf{p}}\right)d\xi_{\mathbf{p}}\frac{d\Omega_{\mathbf{p}}}{4\pi}$$

After an integration over $d\xi_{\mathbf{p}}$ we obtain an expression which has an asymptotics $1/\epsilon$ for large ϵ. It diverges logarithmically and represents a correction to the pairing interaction. Cutting the integral at $\epsilon \sim \Omega_{\mathbf{BCS}}$ we find the correction to the force in the form

$$\mathbf{F}_{\text{env}}^{(corr)} = \frac{i}{2}\left(\frac{1}{\lambda}\frac{d\nu}{d\xi_{\mathbf{p}}}\right)\int d^2r\left[\hat{\nabla}\Delta\frac{\hat{\partial}\Delta^*}{\partial t} - \hat{\nabla}\Delta^*\frac{\hat{\partial}\Delta}{\partial t}\right] \tag{14.81}$$

Here $\hat{\partial}/\partial t = \partial/\partial t \pm 2ie\varphi$ with the upper sign for Δ and the lower sign for Δ^*.
We put $\partial/\partial t = -\mathbf{v}_L \cdot \nabla$ for a moving vortex and calculate the integral in eqn (14.81). We neglect the vector potential for an extreme type-II superconductor with a large Ginzburg-Landau parameter. The scalar potential is proportional to the vortex velocity $\mathbf{v}_L$. As in eqn (12.19), the potential φ contains $v_{L\phi}$, where (ρ, ϕ, z) are the coordinates in the cylindrical frame. Putting $e\varphi = v_{L\phi}\psi$ we observe that the correction to the force is perpendicular to the vortex velocity,

$$\mathbf{F}_\perp^{(\text{corr})} = -\pi\delta N_\Delta w\left[\mathbf{v}_L \times \hat{\mathbf{z}}\right] \tag{14.82}$$

where

$$w = 1 - \Delta_\infty^{-2}\int_0^\infty \psi\frac{d|\Delta|^2}{d\rho}\rho d\rho$$

and δN_Δ is defined by eqn (12.79). Note that $\partial\nu/\partial\xi_{\mathbf{p}} = \partial\nu/\partial E_n = \partial\nu/\partial\mu$ in the normal state.

Equation (14.81) is independent of the impurity scattering because the characteristic energies are large. It is only the potential φ which depends on the real kinetics of a superconductor. Therefore, the function w in eqn (14.82) may vary depending on the parameters of the superconductor. It has been shown by Kopnin (1996) that $w \to 1$ near the critical temperature when $\psi \to 0$. In this limit, eqn (14.82) coincides with eqn (12.78) obtained within the modified TDGL theory and with the result of van Otterlo et al. (1995).

We emphasize once again that nonequilibrium excitations give a much larger contribution to the Hall effect. It is only in a close vicinity of T_c such that $\Delta_\infty^2\tau \sim T_c/\lambda$ when the variations of the pairing interaction become important.

14.6.3 Forces

Let us now discuss the forces acting on a moving vortex. The friction force is determined by the longitudinal components of eqns (14.61) and (14.64). It
is proportional to the mean free path of excitations in the moderately clean regime and vanishes

in the superclean limit. The transverse force deserves a more detailed discussion because it has been a matter of controversy for a long time since first calculated for vortices in helium II by Lifshitz and Pitaevskii (1957) and then by Iordanskii (1964). Consider for simplicity a spherical Fermi surface. The full transverse force from eqns (14.61) and (14.64) is

$$\mathbf{F}_{\perp} = \pi N \left[\int \frac{db}{2} \frac{\partial \epsilon_0}{\partial b} \frac{\partial f^{(0)}(\epsilon_0)}{\partial \epsilon} \langle\langle \gamma_{\mathrm{H}}(\epsilon_0) \rangle\rangle + \left(1 - \tanh \frac{\Delta_\infty}{2T}\right) \gamma'_{\mathrm{H}} \right] [\hat{\mathbf{z}} \times \mathbf{v}_L] \tag{14.83}$$

In the superclean limit, when both $\omega_c \tau$ and $\omega_0 \tau$ are much larger than unity, the factors $\gamma_{\mathrm{H}} = \gamma'_{\mathrm{H}} = 1$, and the transverse force becomes

$$\mathbf{F}_{\perp} = \pi N [\hat{\mathbf{z}} \times \mathbf{v}_L] \tag{14.84}$$

The balance eqn (14.7) against the Lorentz force gives the transport current in the form

$$\mathbf{j}_{\mathrm{tr}} = Ne\mathbf{v}_L \tag{14.85}$$

This equation is consistent with the Helmholtz theorem of conservation of circulation in an ideal fluid: vortices move together with the flow. On the contrary, the transverse force disappears in the limit $\omega_0 \tau, \omega_c \tau \to 0$. This is the overall behavior of the transverse force as a function of the quasiparticle mean free path.

From the historical point of view it is interesting to identify several contribution to the transverse force and consider them one by one. We can present the full transverse force in the form

$$\mathbf{F}_{\perp} = -\pi N_s [\mathbf{v}_L \times \hat{\mathbf{z}}] - \pi N_n [\mathbf{v}_L \times \hat{\mathbf{z}}] + \mathbf{F}_{\mathrm{sf}} \tag{14.86}$$

The force

$$\begin{aligned} \mathbf{F}_{\mathrm{sf}} =& \pi N \int \frac{d\mu}{2} \frac{\partial f^{(0)}}{\partial \epsilon} \left\langle \left\langle \frac{\omega_0}{\omega_0^2 \tau_0^2 + 1} \right\rangle \right\rangle [\mathbf{v}_L \times \hat{\mathbf{z}}] \\ &+ \pi N \left(1 - \tanh \frac{\Delta_\infty}{2T}\right) \frac{1}{\omega_c^2 \tau^2 + 1} [\mathbf{v}_L \times \hat{\mathbf{z}}] \end{aligned} \tag{14.87}$$

is called the spectral flow force (Kopnin et al. 1995).
Let us now write the transport current as $\mathbf{j}_{\mathrm{tr}} = N_s e \mathbf{v}_s$. The force balance $\mathbf{F}_{\mathrm{L}} + \mathbf{F}_{\perp} + \mathbf{F}_{\parallel} = 0$ takes the form

$$\mathbf{F}_{\mathrm{M}} + \mathbf{F}_{\mathrm{I}} + \mathbf{F}_{\mathrm{sf}} + \mathbf{F}_{\parallel} = 0 \tag{14.88}$$

where

$$\mathbf{F}_{\mathrm{M}} = \pi N_s [(\mathbf{v}_s - \mathbf{v}_L) \times \hat{\mathbf{z}}]$$

is the Magnus force. The force

$$\mathbf{F}_{\mathrm{I}} = \pi N_n [(\mathbf{v}_n - \mathbf{v}_L) \times \hat{\mathbf{z}}]$$

is called the Iordanskii force (Iordanskii 1964). The Iordanskii force is the counterpart of the Magnus force for normal excitations. We have included the normal
velocity $\mathbf{v}_n$ to make this similarity more transparent. In our consideration, the normal velocity is always zero $\mathbf{v}_n = 0$ since the equilibrium corresponds to excitations at rest in the reference frame associated with the crystal lattice.

The spectral flow force $\mathbf{F}_{\rm sf}$ is due to the momentum flow from the Fermi sea of normal excitations to the moving vortex via the gapless spectral branch going through the vortex core from negative to positive energies (Volovik 1986, Stone and Gaitan 1987). Due to the time dependent angular momentum of the excitations $\mu(t) = [(\mathbf{r} - \mathbf{v}_L t) \times \mathbf{p}] \cdot \hat{\mathbf{z}}$, there appears a flow (with the velocity $\partial\mu/\partial t$) of spectral levels characterized by the angular momentum μ. Each particle on a level carries a momentum p_F. The momentum transfer to the vortex is effective if the quasiparticle relaxation occurs quickly: the factor $(\omega_0^2\tau_0^2 + 1)^{-1}$ in the first line of eqn (14.87) accounts for the relaxation on localized levels: the relaxation and hence the momentum transfer is complete for $\omega_0\tau \ll 1$, and vanishes in the opposite limit. The first term in eqn (14.87) thus describes a disorder-mediated momentum flow along the anomalous chiral branch $E_0(\mu)$ for energies below the gap. The second line in eqn (14.87) accounts for the spectral flow for energies above the gap. As we have seen on page 293, the delocalized single-particle spectrum also has a chiral branch which is made from the Landau levels for the states with definite angular momenta. The corresponding factor which takes into account the relaxation rate is $(\omega_c^2\tau^2 + 1)^{-1}$.

The spectral flow force vanishes in the limit $\tau \to \infty$ when both $\omega_0\tau \gg 1$ and $\omega_c\tau \gg 1$ such that the transverse force is given by eqn (14.84). On the contrary, the spectral flow force has its maximum value for moderately clean limit, $\omega_0\tau \ll 1$. Equation (14.87) gives in this limit

$$\mathbf{F}_{\rm sf} = \pi N \left[\mathbf{v}_L \times \hat{\mathbf{z}}\right]$$

This completely compensates the first two terms in eqn (14.86), i.e, the Iordanskii force and the part of the Magnus force that contains the vortex velocity. The transverse force vanishes such that the Lorentz force is balanced only by a friction force $\mathbf{F}_{\parallel}$. As a result, the dissipative dynamics is restored.

As we already know, the low field limit $\omega_c\tau \ll 1$ corresponds to a situation when the delocalized excitations are at rest with the heat bath. It is interesting to note that the spectral flow force from above gap states in this case is related to the anomalous contribution to the transverse vortex cross section for scattering of delocalized quasi-particles (Kopnin and Kravtsov 1976 b, Galperin and Sonin 1976). The transverse vortex cross section was calculated by Kopnin and Kravtsov (1976 b)

$$\sigma_\perp = \frac{\pi}{p_\perp}\left[\frac{\epsilon}{\sqrt{\epsilon^2 - \Delta_\infty^2}} - 1\right] \tag{14.89}$$

Inserting eqn (14.89) into the expression for the force exerted on the vortex by scattered excitations,

$$\mathbf{F}_\perp^{(\rm sc)} = \int_{|\epsilon|>\Delta_0} \frac{d\epsilon}{4}\frac{\partial f^{(0)}}{\partial\epsilon}\int \frac{dp_z}{2\pi} p_\perp^3 \sigma_\perp \left[\hat{\mathbf{z}} \times \mathbf{v}_L\right] \tag{14.90}$$

we recover the last two terms in eqn (14.86) that are due to the normal excitations. Indeed, the first term in eqn (14.89) corresponds to the cross section for a vortex in a Bose superfluid (Sonin 1987), $\sigma_\perp = 2\pi/mv_g$, where v_g is the group velocity defined by eqn (1.88). This term gives rise to the Iordanskii force $\mathbf{F}_{\rm I} = -\pi N_n \left[\mathbf{v}_L \times \hat{\mathbf{z}}\right]$. The second term in eqn (14.89) originates from the fact that here, as distinct from the situation in a Bose superfluid, the phase of the single-particle wave function changes by π upon encircling the vortex, while it is the order parameter phase which changes by 2π. It is this singularity, produced by the vortex in the single-particle wave function, which results in the anomalous contribution to the cross section in eqn (14.89). Inserted into eqn (14.90), it exactly reproduces the second term in eqn (14.87) taken for $\omega_c\tau \ll 1$. We see that the spectral flow force is related to a single-particle anomaly associated with the vortex.

19.4 BOLTZMANN KINETIC EQUATION

We derive the canonical Boltzmann kinetic equation for two particular examples. First case is a superconductor with homogeneous in space order-parameter magnitude and current. The second example treats the excitations in the vortex core. The Boltzmann equation is applied to calculate the vortex mass and to study the vortex dynamics in a d-wave superconductor.

19.4.1 15.1 Canonical equations

The microscopic non-stationary theory of superconductivity based on the Green function technique is a powerful method to describe the vortex dynamics both in dirty and in clean superconductors, as well as in other superfluid Fermi systems. However, a practical disadvantage of the method is its mathematically complexity, which tends to hide the physical picture of the phenomenon. For a clean system, where the excitation spectrum is well defined, an alternative way to deal with dynamical processes is based on the quasiclassical Boltzmann kinetic equation. For normal metals, the equivalence of the quasiclassical Green function approach to the Boltzmann equation has been demonstrated by Keldysh (1964). In Chapter 10 we have derived the set of kinetic equations which are suitable for many nonstationary problems in the theory of superconductivity. In the present Chapter we consider two examples for which the generalized kinetic equations can be reduced to the canonical Boltzmann form

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{q}} \cdot \frac{\partial \epsilon_n}{\partial \mathbf{p}} - \frac{\partial \epsilon_n}{\partial \mathbf{q}} \cdot \frac{\partial f}{\partial \mathbf{p}} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \tag{15.1}$$

where $(\partial f/\partial t)_{\text{coll}}$ is the collision integral and $\epsilon_n(\mathbf{q}, \mathbf{p})$ is the quasi-classical excitation spectrum, characterized by the canonically conjugated generalized "coordinate" $\mathbf{q}$ and "momentum" $\mathbf{p}$ of the excitation (n denotes the set of other quantum numbers). The first example is the case of constant in space order parameter and supercurrent. We shall see that the Boltzmann equation is in this case completely similar to its normal-metal counterpart. Another example is the dynamics of excitations in the moving vortex.

19.4.2 15.2 Uniform order parameter

One can easily derive the canonical form of the Boltzmann equation for a situation where the order parameter magnitude, supercurrent, and external fields are almost constant and only slowly varying in space and in time (Betbeder-Matibet
and Nozières 1969, Aronov et al. 1981, 1986). We have already discussed this equation in Section 1.2.3. Let us define the energy of excitations as in eqn (1.54)

$$\epsilon_{\mathbf{p}} = \tilde{\epsilon}_{\mathbf{p}} + \mathbf{p}_s \cdot \mathbf{v}_F$$

where

$$\tilde{\epsilon}_{\mathbf{p}} = \sqrt{\tilde{\xi}_{\mathbf{p}}^2 + |\Delta|^2}, \tilde{\xi}_{\mathbf{p}} = \xi_{\mathbf{p}} + e\Phi \tag{15.2}$$

and

$$\mathbf{p}_s = \frac{1}{2}\left(\nabla\chi - \frac{2e}{c}\mathbf{A}\right)$$

According to the definition eqn (15.2), the excitation energy is always positive. This coincides with the Landau picture of the Fermi liquid. We define the distribution function for the excitations with the energy spectrum eqn (15.2) through our generalized distribution functions $f_1(\epsilon, \mathbf{p})$ and $f_2'(\epsilon, \mathbf{p})$ as

$$n_{\mathbf{p}} = \frac{1}{2}\left[1 - \left(\operatorname{sign}\left(\tilde{\xi}_{\mathbf{p}}\right)\left[f^{(0)}\left(\epsilon_{\mathbf{p}}\right) + f_1\left(\epsilon_{\mathbf{p}}, \mathbf{p}\right)\right] + \frac{\tilde{\epsilon}_{\mathbf{p}}}{\left|\tilde{\xi}_{\mathbf{p}}\right|} f_2'\left(\epsilon_{\mathbf{p}}, \mathbf{p}\right)\right)\right]. \tag{15.3}$$

In equilibrium, it is the Fermi function

$$n_0 = \left(1 + \exp\left[\operatorname{sign}\left(\xi_{\mathbf{p}}\right)\frac{\epsilon_{\mathbf{p}}}{T}\right]\right)^{-1}$$

We also note that for a constant $|\Delta|$ and $\mathbf{p}_s$, the Green functions are

$$g_- = \frac{\epsilon - \mathbf{p}_s \cdot \mathbf{v}_F}{\sqrt{\left(\epsilon - \mathbf{p}_s \cdot \mathbf{v}_F\right)^2 - |\Delta|^2}}\Theta\left[\left(\epsilon - \mathbf{p}_s \cdot \mathbf{v}_F\right)^2 - |\Delta|^2\right]$$

$$f_- = \frac{\Delta}{\sqrt{\left(\epsilon - \mathbf{p}_s \cdot \mathbf{v}_F\right)^2 - |\Delta|^2}}\Theta\left[\left(\epsilon - \mathbf{p}_s \cdot \mathbf{v}_F\right)^2 - |\Delta|^2\right]$$

The radicals are defined as analytical functions of $x = \epsilon - \mathbf{p}_s \cdot \mathbf{v}_F$ on the complex plane of x with the cut between $-|\Delta|$ and $|\Delta|$ (see Fig. 5.1); they change sign when x goes from positive $x > |\Delta|$ to negative $x < -|\Delta|$. If ϵ is replaced with a positive $\epsilon_{\mathbf{p}}$, the Green functions become

$$g_- = \frac{\tilde{\epsilon}_{\mathbf{p}}}{\left|\tilde{\xi}_{\mathbf{p}}\right|}, f_- = \frac{\Delta}{\left|\tilde{\xi}_{\mathbf{p}}\right|}$$

One can check after some algebra that in the linear approximation with respect to the deviation from equilibrium, the two equations (10.52) and (10.53) for the functions $f_1(\epsilon, \mathbf{p})$ and $f_2(\epsilon, \mathbf{p})$ possessing a definite parity in ($\epsilon, \mathbf{p}$) can be written as one equation

$$\frac{\partial n_{\mathbf{p}}}{\partial t} + \mathbf{v}_g \cdot \nabla n_{\mathbf{p}} + \mathbf{f}\frac{\partial n_{\mathbf{p}}}{\partial \mathbf{p}} = I\left\{n_{\mathbf{p}}\right\} \tag{15.4}$$

for the function $n_{\mathbf{p}}$ which has no definite parity in $\tilde{\xi}_{\mathbf{p}}$. Equations (10.52) and (10.53) can be recovered from eqn (15.4) by separating the even and odd parts in $\tilde{\xi}_{\mathbf{p}}$. The group velocity $\mathbf{v}_g$ and the elementary force $\mathbf{f}$ including the Lorentz force are defined in eqn (1.88). Equation (15.4) coincides with eqn (1.87) used in the beginning of the book.

The collision integral in eqn (15.4) has the form

$$I\left\{n_{\mathbf{p}}\right\} = -\frac{1}{2}\left(\frac{\tilde{\xi}_{\mathbf{p}}}{\tilde{\epsilon}_{\mathbf{p}}} J_1\left\{f_1\right\} + \frac{\tilde{\xi}_{\mathbf{p}}^2}{\tilde{\epsilon}_{\mathbf{p}}^2} J_2\left\{f_2\right\}\right)$$

For example, the impurity collision integral becomes

$$\begin{aligned} I^{(\mathrm{imp})}\left\{n_{\mathbf{p}}\right\} = &-\frac{1}{\nu(0)\tau_{\mathrm{imp}}}\int\left(u_{\mathbf{p}}u_{\mathbf{p}'} - v_{\mathbf{p}}v_{\mathbf{p}'}\right)^2 \\ &\times\left(n_{\mathbf{p}} - n_{\mathbf{p}'}\right)\delta\left(\epsilon_{\mathbf{p}} - \epsilon_{\mathbf{p}'}\right)\frac{d^3p'}{(2\pi)^3} \end{aligned} \tag{15.5}$$

The Bogoliubov-de Gennes coherence factors are as in eqn (1.56)

$$u_{\mathbf{p}}^2 = \frac{1}{2}\left(1 + \frac{\tilde{\xi}_{\mathbf{p}}}{\tilde{\epsilon}_{\mathbf{p}}}\right), v_{\mathbf{p}}^2 = \frac{1}{2}\left(1 - \frac{\tilde{\xi}_{\mathbf{p}}}{\tilde{\epsilon}_{\mathbf{p}}}\right)$$

During the transformation we use the relation

$$\frac{\partial \tilde{\epsilon}_{\mathbf{p}}}{\partial \tilde{\xi}_{\mathbf{p}}} = \frac{\partial \epsilon_{\mathbf{p}}}{\partial \xi_{\mathbf{p}}} = \frac{\tilde{\xi}_{\mathbf{p}}}{\tilde{\epsilon}_{\mathbf{p}}}$$

Equation (15.4) has the form of the canonical Boltzmann equation. However, its region of applicability is limited to the situation where the order parameter and the supercurrent are almost uniform in space. One would think that the required scale of spatial variations should be larger than ξ. However, this condition does not guarantee that the obtained results are correct. The point is that the spatial variations also result in distortions of the quasiparticle spectrum. These distortions are not included into eqn (15.2) though they can be important for some of the responses. One example is the distortion of the spectrum provided by eqns (14.31) and (14.32) which are important for calculating the flux flow conductivity.

19.4.3 15.3 Quasiparticles in the vortex core

Another example is the application of the Boltzmann equation to the vortex dynamics in clean superconductors (Stone 1996, Kopnin and Volovik 1997). Equation (14.42) which we derived from kinetic equations (14.35) and (14.36) already looks almost like the canonical equation (15.1). In this Section we present a microscopic verification of (15.1) for a general case (Blatter et al. 1999) when Δ can depend on the quasiparticle momentum as it is the case, for example, in d-wave superconductors. We demonstrate that our kinetic equations for the generalized distribution function derived from the quasiclassical Green function formalism can be further transformed into the simple and physically transparent canonical equation (15.1). We restrict ourselves to the particular example of vortex dynamics; the calculation can be easily generalized to include the dynamics of other topological defects in superfluid Fermi systems.

In Section 14.5, we have derived microscopically the force acting on a moving vortex, eqn (14.59). This equation and eqn (14.4) show that, within the quasiclassical approximation, the force $\mathbf{F}_{\text{env}}$ can be represented as the momentum transfer from the heat bath via the localized quasi-particle excitations to the vortex,

$$\mathbf{F}_{\text{env}} = -\sum_n \int \frac{d^d q d^d p}{(2\pi)^d} f(\mathbf{q}, \mathbf{p}) \frac{\partial \mathbf{p}_n}{\partial t} \tag{15.6}$$

where $\partial \mathbf{p}_n / \partial t = -\partial_{\mathbf{q}} \epsilon_n(\mathbf{q}, \mathbf{p})$ and d is the dimensionality of the problem ($d = 1$ in case of vortices). Our analysis thus provides a microscopic verification of the phenomenological approach used in Section 14.1 for a qualitative description of the vortex dynamics. This approach is based on the concept of semi-classical particles obeying the Boltzmann kinetic equation where the quasiclassical spectrum ϵ_n of excitations in the vortex core plays the role of the Hamiltonian, and the force acting on the vortex results from the elementary force $\partial \mathbf{p}_n / \partial t$ the quasiparticles exert on the vortex core.

The Boltzmann kinetic equation can be successfully applied to many problems of the vortex dynamics. We have seen it already when we calculated the flux flow conductivity in an s-wave superconductor. In this Chapter, we consider two more examples of implementation of this scheme. In Section 15.4 we calculate the vortex mass. Another example is the vortex dynamics in d-wave superconductors to be discussed in Section 15.5.

15.3.1 Transformation into the Boltzmann equation

Let us write the kinetic equations (10.41) and (10.56) again. We have

$$\left(e\mathbf{v}_F\cdot\mathbf{E}g_- + \frac{1}{2}\left[f_-\frac{\hat{\partial}\Delta^*}{\partial t} + f_-^\dagger\frac{\hat{\partial}\Delta}{\partial t}\right]\right)\frac{\partial f^{(0)}}{\partial\epsilon} + \mathbf{v}_F\cdot\nabla\left(f_2 g_-\right) + g_-\frac{\partial f_1}{\partial t}$$
$$-\left(\frac{1}{2}\left[(\hat{\nabla}\Delta)f_-^\dagger + \left(\hat{\nabla}\Delta^*\right)f_-\right] - \frac{e}{c}\left[\mathbf{v}_F\times\mathbf{H}\right]g_-\right)\frac{\partial f_1}{\partial\mathbf{p}}$$
$$+\frac{1}{2}\left(\frac{\partial\Delta}{\partial\mathbf{p}}f_-^\dagger + \frac{\partial\Delta^*}{\partial\mathbf{p}}f_-\right)\nabla f_1 = J_1 \tag{15.7}$$

and

$$g_-\mathbf{v}_F\cdot\nabla f_1 = 0 \tag{15.8}$$

We start with a few preparatory steps before transforming the kinetic equation (15.7) into the Boltzmann equation. For simplicity we consider the limit of low magnetic fields such that $\omega_c\tau \ll 1$. In this case delocalized quasi-particles are practically in equilibrium with the heat bath; their contribution to the force
on the vortex is small, see page 300 . We thus concentrate on the excitations localized in the vortex core.

We again use the coordinate frame associated with the vortex as shown in Fig. 6.1. Remind that the z-axis is chosen parallel to the axis of the vortex, with the positive direction along the vortex circulation $\hat{\mathbf{z}} = \mathrm{sign}(e)\hat{\mathbf{H}}$. The distance along the quasi-particle trajectory as well as the impact parameter are $s = \rho\cos(\phi-\alpha\ )$ and $b = \rho\sin(\phi-\alpha)$, respectively, where ρ and ϕ are the radial distance and the azimuthal angle in the cylindrical frame, and α is the angle between $\mathbf{v}_\perp$ and the x-axis. In this representation, the quasi-classical Green function is specified by the momentum projection on the vortex axis $p_z = p_F\cos\theta$, the momentum direction α in the plane perpendicular to the vortex axis, and the impact parameter b, which is related to the angular momentum $\mu = -bp_\perp$, with $p_\perp$ the momentum projection on the plane perpendicular to the vortex axis. For simplicity, we consider a spherical Fermi surface here. Up to corrections of order $(p_F\xi)^{-1}$ we can assume straight quasi-particle trajectories and thus the angular momentum μ is a conserved quantity even for a non-axisymmetric vortex.

Next, we transform the operators in eqn (15.7). The momentum derivative $\partial\mathbf{p}$ in eqn (15.7) is taken at a constant position vector $\mathbf{r} = (\rho,\phi)$ with respect to variations in the momentum direction, with the magnitude of the momentum being fixed at the Fermi surface. The planar projection can be written as

$$\left(\frac{\partial}{\partial\mathbf{p}_\perp}\right)_{\mathbf{r}} = \hat{\mathbf{v}}_\perp\frac{\partial}{\partial p_\perp} + \frac{[\hat{\mathbf{z}}\times\hat{\mathbf{v}}_\perp]}{p_\perp}\left(\frac{\partial}{\partial\alpha}\right)_{\mathbf{r}}$$

with $\hat{\mathbf{v}}_\perp$ the unit vector in the direction of $\mathbf{v}_\perp$. Changing to variables s and b, the derivative with respect to α becomes

$$\left(\frac{\partial}{\partial\alpha}\right)_{\mathbf{r}} = b\frac{\partial}{\partial s} - s\frac{\partial}{\partial b} + \frac{\partial}{\partial\alpha}$$

and the spatial gradient in the (s,b) frame is

$$\nabla = \hat{\mathbf{v}}_\perp\frac{\partial}{\partial s} + [\hat{\mathbf{z}}\times\hat{\mathbf{v}}_\perp]\frac{\partial}{\partial b} \tag{15.9}$$

In the presence of a vortex, the order parameter has the form $\Delta_{\mathbf{p}}(\rho,\phi) = \Delta_{p_\perp}(\alpha,s,b)e^{i\alpha}$. Here $\Delta_{p_\perp}(\alpha,s,b)$ is the order parameter expressed in the coordinate frame (s,b), where the azimuthal angle is measured from the momentum direction. For a non-axisymmetric vortex

and/or in a d-wave superconductor, $\Delta_{p_\perp}(\alpha, s, b)$ can have an explicit dependence on the angular coordinate α.

To work with various terms in the kinetic equation (15.7) we use transformations very similar to those we made in Section 14.2 to derive identities for the quasiclassical Green functions. Keeping in mind that f_1 is independent of s (see eqn (15.8)), we rewrite the terms in the second and third lines of eqn (15.7) in the form

$$\left(f_-\nabla\Delta^*_{\mathbf{p}} + f^\dagger_-\nabla\Delta_{\mathbf{p}}\right)\cdot\frac{\partial f_1}{\partial\mathbf{p}} = \frac{1}{p_\perp}\left(f_-\frac{\partial\Delta^*}{\partial b} + f^\dagger_-\frac{\partial\Delta}{\partial b}\right)\left(\frac{\partial f_1}{\partial\alpha} - s\frac{\partial f_1}{\partial b}\right) + \left(f_-\frac{\partial\Delta^*}{\partial s} + f^\dagger_-\frac{\partial\Delta}{\partial s}\right)\frac{\partial f_1}{\partial p_\perp} \tag{15.10}$$

and

$$\begin{aligned}&\left(f_-\frac{\partial\Delta^*_{\mathbf{p}}}{\partial\mathbf{p}} + f^\dagger_-\frac{\partial\Delta_{\mathbf{p}}}{\partial\mathbf{p}}\right)\cdot\nabla f_1\\ =&\frac{1}{p_\perp}\left[\left(f_-\frac{\partial\Delta^*}{\partial\alpha} + f^\dagger_-\frac{\partial\Delta}{\partial\alpha}\right) - s\left(f_-\frac{\partial\Delta^*}{\partial b} + f^\dagger_-\frac{\partial\Delta}{\partial b}\right)\right]\frac{\partial f_1}{\partial b}\\ &+\frac{1}{p_\perp}\left[\left(f_-\frac{\partial\Delta^*}{\partial s} + f^\dagger_-\frac{\partial\Delta}{\partial s}\right)b\right]\frac{\partial f_1}{\partial b}\end{aligned} \tag{15.11}$$

Moreover, since $\mathbf{A}$ does not depend on the momentum direction, we can subtract the zero term

$$\frac{e}{c}v_{\perp i}\left(\frac{\partial A_i}{\partial\mathbf{p}}\right)_{\mathbf{r}}\cdot\nabla f_1 = 0$$

from the third line of eqn (15.7). Next, we integrate (15.7) along the quasiclassical trajectory, using $\partial f_1/\partial s = 0$ and one of the Eilenberger equations,

$$i\left(\mathbf{v}_F\cdot\nabla\right)g_- = \Delta f^\dagger_- - \Delta^* f_-. \tag{15.12}$$

After some algebra the second and third lines in the l.h.s. of eqn (15.7) take the form

$$\begin{aligned}&\int\left[\frac{e}{c}\left[\mathbf{v}_F\times\mathbf{H}\right]g_- - \frac{1}{2}\left(f_-\hat{\nabla}\Delta^*_{\mathbf{p}} + f^\dagger_-\hat{\nabla}\Delta_{\mathbf{p}}\right)\right]\frac{\partial f_1}{\partial\mathbf{p}}ds\\ &+\int\left[-\frac{e}{c}v_{\perp i}\frac{\partial A_i}{\partial\mathbf{p}}g_-\nabla f_1 + \frac{1}{2}\left(f_-\frac{\partial\Delta^*_{\mathbf{p}}}{\partial\mathbf{p}} + f^\dagger_-\frac{\partial\Delta_{\mathbf{p}}}{\partial\mathbf{p}}\right)\right]\nabla f_1 ds\\ =&-\frac{1}{p_\perp}\int\left[\frac{1}{2}\left(f_-\frac{\partial\Delta^*}{\partial b} + f^\dagger_-\frac{\partial\Delta}{\partial b}\right) - \frac{e}{c}v_\perp\frac{\partial A_s}{\partial b}g_-\right]\frac{\partial f_1}{\partial\alpha}ds\\ &+\frac{1}{p_\perp}\int\left[\frac{1}{2}\left(f_-\frac{\partial\Delta^*}{\partial\alpha} + f^\dagger_-\frac{\partial\Delta}{\partial\alpha}\right) - \frac{e}{c}v_\perp\frac{\partial A_s}{\partial\alpha}g_-\right]\frac{\partial f_1}{\partial b}ds.\end{aligned} \tag{15.13}$$

The term $\partial A_s/\partial\alpha$ accounts for the explicit α-dependence picked up by the vector potential when expressed in the coordinate frame (s, b). The s derivatives present in eqns (15.10) and (15.11) disappear in eqn (15.13), as can be seen from the identity (14.29). In particular, taking the $\mathbf{v}_\perp$ projection of eqn (14.29) we find,

$$\int \mathrm{Tr}\left[\frac{\partial\check{H}}{\partial s}\check{g}_-\right]ds = 0$$

It is this relation which serves to eliminate the s-derivatives from eqn (15.13).

The first term in the r.h.s. of eqn (15.13) has the form

$$-\frac{1}{2p_\perp}\frac{\partial f_1}{\partial \alpha}\int \mathrm{Tr}\left[\check{g}_-\frac{\partial \check{H}}{\partial b}\right]ds$$

which can also be transformed with help of eqn (14.29). The second term has the form

$$\frac{1}{2p_\perp}\frac{\partial f_1}{\partial b}\int \mathrm{Tr}\left[\check{g}_-\frac{\partial \check{H}}{\partial \alpha}\right]ds$$

It can be transformed using the relation

$$\int ds\,\mathrm{Tr}\left[\frac{\partial \mathcal{H}}{\partial \alpha}\check{g}_-\right] = 2\pi v_\perp \sum_n \delta\left(\epsilon-\epsilon_n\right)\frac{\partial \epsilon_n}{\partial \alpha} \tag{15.14}$$

which can be easily derived in exactly the same way as eqn (14.29). As a result, the r.h.s. of eqn (15.13) takes the simple form

$$-\frac{1}{p_\perp}\sum_n\left[\frac{\partial \epsilon_n}{\partial b}\frac{\partial f_1}{\partial \alpha}-\frac{\partial \epsilon_n}{\partial \alpha}\frac{\partial f_1}{\partial b}\right]\pi v_\perp \delta\left(\epsilon-\epsilon_n\right) \tag{15.15}$$

Next, we concentrate on the first line of eqn (15.7). The driving term $\propto \partial f^{(0)}/\partial\epsilon$ is the source of the non-equilibrium state as produced by time variations of the order parameter together with the electric field. According to eqn (14.38), the first term of the kinetic equation (15.7) can be written in the form

$$-\frac{1}{2}\frac{\partial f^{(0)}}{\partial \epsilon}\int \mathrm{Tr}\left[\left(\mathbf{v}_L\cdot\nabla\right)\check{H}g_-\right]ds$$

which then can be further transformed using eqn (14.29). The second term in the first line of eqn (15.7) vanishes after integration over ds. The next term can be transformed with help of the identity (14.25). We finally obtain

$$\begin{aligned}&\left(\left[\hat{\mathbf{v}}_\perp\times\mathbf{v}_L\right]\cdot\hat{\mathbf{z}}\right)\frac{\partial f^{(0)}}{\partial \epsilon}\frac{\partial \epsilon_n}{\partial b}+\frac{1}{p_\perp}\frac{\partial f_1}{\partial \alpha}\frac{\partial \epsilon_n}{\partial b}\\ &-\frac{1}{p_\perp}\frac{\partial \epsilon_n}{\partial \alpha}\frac{\partial f_1}{\partial b}-\frac{\partial f_1}{\partial t}+\frac{1}{\pi v_\perp}\int_{-\infty}^{\infty}J_n ds=0\end{aligned} \tag{15.16}$$

A similar equation (14.42) has been already derived in Section 14.4 for s-wave superconductors. Note that the term with $\partial f_1/\partial b$ is absent in eqn (14.42) because $\partial\epsilon_n/\partial\alpha=0$ for an axisymmetric vortex in an s-wave superconductor.

Equation (15.16) is nothing but the Boltzmann equation (14.3) used in the beginning of this Chapter. Indeed, consider various terms in eqn (15.16) and compare them with eqn (14.3). Since

$$\frac{\partial \epsilon_n}{\partial b}=-p_\perp\frac{\partial \epsilon_n}{\partial \mu}$$

we have from eqn (15.16)

$$\left(\left[\mathbf{p}_\perp\times\mathbf{v}_L\right]\cdot\hat{\mathbf{z}}\right)\frac{\partial f^{(0)}}{\partial \epsilon}\frac{\partial \epsilon_n}{\partial \mu}+\frac{\partial f_1}{\partial t}$$

$$+\frac{\partial f_1}{\partial \alpha}\frac{\partial \epsilon_n}{\partial \mu} - \frac{\partial \epsilon_n}{\partial \alpha}\frac{\partial f_1}{\partial \mu} = \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \tag{15.17}$$

with $\epsilon_n = \epsilon_n(\mu, \alpha; p_z)$ and $\mu = -bp_\perp$. We denoted the collision integral as

$$\left(\frac{\partial f}{\partial t}\right)_{\text{coll}} = \frac{1}{\pi v_\perp}\int_{-\infty}^{\infty} J_n ds \tag{15.18}$$

The time-derivative term in eqn (14.3) is obtained from eqn (15.17) if we assume that the energy ϵ_n contains a time dependence through the impact parameter $b(t) = -\mu(t)p_\perp^{-1}$, where $\mu(t) = [(\mathbf{r} - \mathbf{v}_L t) \times \mathbf{p}] \cdot \hat{\mathbf{z}}$:

$$\frac{\partial f^{(0)}}{\partial t} = \frac{\partial f^{(0)}}{\partial \epsilon}\frac{\partial \epsilon_n}{\partial \mu}\frac{\partial \mu}{\partial t} = \frac{\partial f^{(0)}}{\partial \epsilon}\frac{\partial \epsilon_n}{\partial \mu}\left([\mathbf{p}_\perp \times \mathbf{v}_L] \cdot \hat{\mathbf{z}}\right) \tag{15.19}$$

and combine it with $\partial f_1/\partial t$ in the total distribution function $f = f^{(0)}(\epsilon_n) + f_1$. Equation (14.3) can be rewritten into the generic form (15.1) with the canonical variables $q = \alpha$ and $p = \mu$.

To summarize the results of this Section, we have started with the exact microscopic description of the non-stationary processes in terms of the Green function technique. Using the quasiclassical approximation, we have been able to reduce the problem of finding the non-equilibrium state of the superconductor with a moving vortex to the problem of solving the canonical Boltzmann equation for the distribution of non-equilibrium excitations localized in the vortex core. The only information needed to find the distribution function is the energy spectrum of the equilibrium excitations in the vortex core. The knowledge of the energy spectrum is also sufficient to calculate the force acting on the moving vortex. The full problem of the vortex dynamics thus reduces to several much easier and more transparent steps which are: finding the equilibrium energy spectrum of the excitations in the vortex core, solving the Boltzmann equation, and, finally, calculating the momentum transfer from the localized excitations.

19.4.4 15.4 Vortex mass

The vortex mass in superfluids and superconductors has been a long standing problem in vortex physics and remains to be an issue of controversies. There are different approaches to its definition. One approach consists in calculating the vortex free energy increase due to the vortex velocity (Duan and Leggett 1992). First used by Suhl (1965) this approach yields the mass of the order of one quasiparticle mass (electron, in case of superconductor) per atomic layer. The increase in the electromagnetic energy gives rise to the so called electromagnetic mass (Coffey and Hao 1991) which, in good metals, is of the same order of magnitude (see Sonin et al. 1998 for an extensive review). Another approach is based on finding the force necessary to support an unsteady vortex motion. Identifying then the contribution to the force proportional to the vortex acceleration, one defines the vortex mass as a coefficient of proportionality. This method was first applied for vortices in superclean superconductors by Kopnin (1978) and since then was used by other authors (see for example, Kopnin and Salomaa 1991, Šimánek 1995). The resulting mass is of the order of the total mass of all electrons within the area occupied by the vortex core. We will refer to this mass as to the dynamic mass. The dynamic mass originates from the inertia of excitations localized in the vortex core and can also be calculated as the momentum carried by localized excitations (Volovik 1997). It is much larger and thus it is much more important than the mass obtained from the energy considerations.

In the present Section we describe how one can apply our kinetic equation approach for calculating the dynamic vortex mass in a general case of a finite relaxation time of nonequilibrium excitations produced by the moving vortex. Using the Boltzmann kinetic equation for quasiparticles localized in the vortex core, we derive the equation for the vortex dynamics which contains the inertia term together with all the forces acting on a moving vortex. We shall see that dynamic mass displays a nontrivial feature: it is a tensor whose components depend on the quasiparticle mean free time. In s-wave superconductors, this tensor is diagonal in the superclean limit. The diagonal mass decreases rapidly as a function of the mean free time, and the off-diagonal components dominate in the moderately clean regime. Our results agree with the previous work (Kopnin 1978, Kopnin and Salomaa 1991, Volovik 1997) in the limit $\tau \to \infty$. For simplicity, we consider the limit $\omega_c \tau \ll 1$ when delocalized excitations are at rest with respect to the heat bath. These results were obtained by Kopnin and Vinokur (1998).

15.4.1 Equation of vortex dynamics

Multiplying eqn (15.17) by $\mathbf{p}_\perp/2$ and summing up over all the quantum numbers, we obtain

$$\mathbf{F}_{\text{env}} = \mathbf{F}_{\text{coll}} - \frac{\partial \mathbf{P}}{\partial t} - \pi N \tanh\left(\frac{\Delta_\infty}{2T}\right)[\mathbf{v}_L \times \hat{\mathbf{z}}] \tag{15.20}$$

where the l.h.s. of eqn (15.20) is the force from the environment on a moving vortex eqn (14.4) derived in Section 14.5.

The first term in the r.h.s. of eqn (15.20) is the force exerted on the vortex by the heat bath via excitations localized in the vortex core:

$$\mathbf{F}_{\text{coll}} = -\frac{1}{2}\sum_n \int \frac{dp_z}{2\pi}\frac{d\alpha d\mu}{2\pi}\mathbf{p}_\perp \left(\frac{\partial f}{\partial t}\right)_{\text{coll}} \tag{15.21}$$

Its component parallel to $\mathbf{v}_L$ gives the friction force. After a little of algebra, the force $\mathbf{F}_{\text{coll}}$ can be written in the form

$$\mathbf{F}_{\text{coll}} = \mathbf{F}_{\text{sf}} + \mathbf{F}_\parallel - \pi N\left(1 - \tanh\frac{\Delta_\infty}{2T}\right)[\mathbf{v}_L \times \hat{\mathbf{z}}]$$

where the friction and spectral flow forces are determined by eqns (14.11) and (14.87), respectively.

The second term in the r.h.s. of (15.20) is the change in the vortex momentum defined as

$$\mathbf{P} = -\frac{1}{2}\sum_n \int \frac{dp_z}{2\pi}\frac{d\alpha d\mu}{2\pi}\mathbf{p}_\perp f_1 \tag{15.22}$$

The force from environment eqn (15.20) takes the form

$$\mathbf{F}_{\text{env}} = \mathbf{F}_{\text{sf}} + \mathbf{F}_\parallel - \frac{\partial \mathbf{P}}{\partial t} - \pi N[\mathbf{v}_L \times \hat{\mathbf{z}}] \tag{15.23}$$

The equation of the vortex dynamics is obtained by the variation of the superconducting free energy plus the external field energy with respect to the vortex displacement. The variation of the superfluid free energy gives the force from the environment, $\mathbf{F}_{\text{env}}$, while the variation of the external field energy produces the external Lorentz force. In the absence of pinning the total energy is translationally invariant. Therefore, the requirement of zero variation of the free energy gives again the condition $\mathbf{F}_L + \mathbf{F}_{\text{env}} = 0$ in the form the force balance. Using our expression for $\mathbf{F}_{\text{env}}$, the force balance can now be written in the form similar to eqn (14.88)

$$\mathbf{F}_{\text{M}} + \mathbf{F}_{\text{I}} + \mathbf{F}_{\text{sf}} + \mathbf{F}_\parallel = \frac{\partial \mathbf{P}}{\partial t} \tag{15.24}$$

where the r.h.s. contains the time derivative of the vortex momentum due to a non-steady vortex motion. For a steady vortex motion, eqn (15.24) reduces to eqn (14.88). The physical meaning of the eqn (15.24) is simple. The l.h.s. of this equation accounts for all the forces acting on a moving straight vortex line: the Magnus force from the superfluid component, the Iordanskii force due to the normal excitations, the spectral flow force and the friction force from the heat bath transferred to the vortex by localized excitations. The r.h.s. of eqn (15.24) comes from the inertia of excitations localized in the vortex core and is identified as the change in the vortex momentum. The definition of eqn (15.22) is similar to that used by Stone (1996) and by Volovik (1997). Note that the delocalized quasiparticles do not contribute to the vortex momentum in the limit $\omega_c\tau \ll 1$ because they are in equilibrium with the heat bath.

15.4.2 Vortex momentum

Having defined the vortex momentum, we calculate the vortex mass. The distribution function of localized particles is given by eqn (14.51). The vortex momentum becomes $P_i = M_{ik}u_k$; it has both longitudinal and transverse components with respect to the vortex velocity. For a vortex with the symmetry not less than the four-fold, the effective mass tensor per unit length is $M_{ik} = M_\parallel \delta_{ik} - M_\perp e_{ikj}\hat{z}_j$ where $M_\parallel = M_{\parallel e} + M_{\parallel h}$ and $M_\perp = M_{\perp e} - M_{\perp h}$ with

$$M_{\parallel e,h} = \frac{1}{4}\sum_n \int_{e,h} \frac{dp_z}{2\pi}\frac{d\mu d\alpha}{2\pi}\frac{df^{(0)}}{d\epsilon}p_\perp^2\gamma_{\mathrm{H}} \tag{15.25}$$

and the same expression for $M_{\perp e,h}$ where γ_{H} is replaced with γ_{O}. The e, h subscripts indicate the integrations over the electron and hole parts of the Fermi
surface, respectively. Only the branch with $n = 0$ gives the contribution to the transverse mass because γ_{O} is odd in μ for $n \neq 0$.

If a vortex moves with a constant velocity, the time derivative of the momentum is zero. However, one can speak of the vortex mass as a proportionality constant between the momentum and velocity. If the vortex acceleration is slow, one can use the solution for a steadily moving vortex to calculate the vortex inertia. For s wave superconductors, γ_{H} and γ_{O} for a steady vortex motion are determined by eqn (14.14). At low temperatures one has

$$M_{\parallel e,h} = \pi N_{e,h}\left\langle\left\langle\frac{\gamma_{\mathrm{H}}}{\omega_0}\right\rangle\right\rangle, M_{\perp e,h} = \pi N_{e,h}\left\langle\left\langle\frac{\gamma_{\mathrm{O}}}{\omega_0}\right\rangle\right\rangle.$$

In the superclean limit $T_c^2\tau/E_F \gg 1$ where $\omega_n\tau \gg 1$ the mass tensor is diagonal $M_{ik} = M_\parallel\delta_{ik}$, and we obtain $M_{\parallel e,h} = \pi N_{e,h}\left\langle\omega_0^{-1}\right\rangle$. The mass is of the order of the mass of all electrons in the area occupied by the vortex core $M \sim \pi\xi_0^2 mN$. The mass decreases with τ. In the moderately clean regime $T_c^2\tau/E_F \ll 1$ where $\omega_n\tau \ll 1$, the diagonal component vanishes as τ^2, and the mass tensor is dominated by the off-diagonal part. If τ_0 is independent of energy

$$M_{\perp e,h} = \pi\tanh\left(\frac{\Delta_0}{2T}\right)N_{e,h}\left\langle\langle\tau_0\rangle\right\rangle$$

We should emphasize an important point that, in contrast to a conventional physical body, the mass of a vortex is not a constant quantity for a given system: it may depend on the frequency ω of the external drive. Indeed, for a nonzero ω we find from eqn (15.17)

$$\gamma_{\mathrm{H}} = \frac{\omega_n^2\tau_n^2}{\omega_n^2\tau_n^2 + (1 - i\omega\tau_n)^2}; \gamma_{\mathrm{O}} = \frac{\omega_n\tau_n(1 - i\omega\tau_n)}{\omega_n^2\tau_n^2 + (1 - i\omega\tau_n)^2}$$

As a result, all the dynamic characteristics of vortices including the conductivity and the effective mass acquire a frequency dispersion. In the limit $\tau \to \infty$, poles in $\gamma_{\mathrm{O,H}}$ appear at a frequency equal to the energy spacing between the quasiparticle states in the vortex core which

gives rise to resonances in absorption of an external electromagnetic field (Kopnin 1978, 1998 b).

19.4.5 15.5 Vortex dynamics in d-wave superconductors

In this Section we consider the vortex dynamics in d-wave superconductors at low temperatures. We specify later which temperatures can be considered as low. In d-wave superconductors, the vortex dynamics is expected to be more intricate due to a peculiar structure of the vortex-core states. The presence of gap nodes introduces the most important difference in the structure of core states compared to an s-wave superconductor. As was shown in Section 6.4, instead of a well-defined quasiclassical Caroli-de Gennes-Matricon interlevel spacing ω_0, the true quantum states in d-wave superconductors have a much smaller separation Ω_0 between quantum levels, $E_0 = -\Omega_0 \left(m + \frac{1}{2}\right)$, which depends on the magnetic field. As a result, there appears another parameter $\Omega_0\tau$ which influences the vortex dynamics. We shall see that a new regime can be reached in superclean superconductors with longitudinal and transverse components of the conductivity tensor independent of the quasiparticle mean free time (Kopnin and Volovik 1997, Kopnin 1998). It is realized when the relaxation rate is smaller than the average distance between the quasiclassical energy levels $\langle\omega_0\rangle \sim T_c^2/E_F$ but larger than the separation between the true quantum-mechanical states Ω_0.

15.5.1 Distribution function

Here we employ the Boltzmann kinetic equation derived in this Chapter. We also use the spectrum of excitations in the vortex core calculated in Section 6.4. In d^-wave superconductors, the gap at large distances from the vortex has the form $\Delta_\infty \left|\sin\left(2\alpha_0\right)\right|$. A particle is classically localized within the vortex core if it has an energy $|\epsilon| < \Delta_\infty$ and moves within the range of angles $\alpha > \alpha_0$ counted from one of the nodes. The angle α_0 is defined such that $|\epsilon| = \Delta_\infty \left|\sin\left(2\alpha_0\right)\right|$. For classically localized quasiparticles, the derivative $\partial\epsilon_n/\partial\mu = -\omega_n(\alpha)$ is the quasiclassical interlevel spacing. We shall consider low temperatures such that the level $n = 0$ is only important. Such temperatures correspond to $T \ll T_c\sqrt{H/H_{c2}}$. Remind that according to eqn (6.100)

$$\omega_0(\alpha) = 8\tilde{\omega}_0\alpha^2 + \omega_c/2 \tag{15.26}$$

where $\tilde{\omega}_0 = \left(\Delta_\infty^2/v_\perp p_\perp\right) L$, and L denotes the logarithmic function of α.
The kinetic equation for the distribution function of localized excitations in a moving vortex has the form of eqn (15.17). We use a τ-approximation eqn (14.8) for the collision integral. We look for a solution in the form $f_1 = f_1\left[\epsilon_0(\alpha,\mu),\alpha\right]$ which does not contain an explicit dependence on μ. Equation (15.17) gives

$$\frac{\partial f_1}{\partial\alpha} - \left(\left[\mathbf{u}\times\mathbf{p}_\perp\right]\hat{\mathbf{z}}\right)\frac{df^{(0)}}{d\epsilon} = U(\alpha)f_1 \tag{15.27}$$

where

$$U(\alpha) = \frac{1}{\omega_0(\alpha)\tau} \tag{15.28}$$

We take the nonequilibrium distribution function again in the form of eqn (14.9). The factors γ_{O} and γ_{H} depend now on α. We assume a Fermi surface with not less than the tetragonal symmetry in the plane perpendicular to the vortex axis. For simplicity, we consider only the particle-like Fermi surface. Taking into account that, for the tetragonal symmetry, the responses do not depend on the direction of the vortex motion with respect to the crystal lattice, one finds two coupled first-order differential equations for $\gamma_{\mathrm{O}}(\alpha)$ and $\gamma_{\mathrm{H}}(\alpha)$:

$$\frac{\partial \gamma_{\mathrm{O}}}{\partial \alpha} - \gamma_{\mathrm{H}} - U(\alpha)\gamma_{\mathrm{O}} + 1 = 0$$
$$\frac{\partial \gamma_{\mathrm{H}}}{\partial \alpha} + \gamma_{\mathrm{O}} - U(\alpha)\gamma_{\mathrm{H}} = 0 \tag{15.29}$$

Equations (15.29) can be easily solved. We have $\gamma_{\mathrm{H}}(\alpha) = \mathrm{Re}\, W; \gamma_{\mathrm{O}}(\alpha) = \mathrm{Im}\, W$ where

$$W = e^{[i\alpha + F(\alpha)]}\left(C - i\int_0^\alpha e^{-[i\alpha' + F(\alpha')]} d\alpha'\right)$$

with

$$F(\alpha) = \int_0^\alpha U(\alpha')\, d\alpha'$$

In the moderately clean limit, $\tilde{\omega}_0\tau \ll 1$, the potential $U(\alpha)$ is always large, and we obtain the local solution as in an s-wave superconductor

$$\gamma_{\mathrm{O}}(\alpha) = \omega_0(\alpha)\tau, \gamma_{\mathrm{H}}(\alpha) = [\omega_0(\alpha)\tau]^2 . \tag{15.30}$$

Consider temperatures $T \ll T_c\sqrt{H/H_{c2}}$ which correspond to the energy range $\epsilon \ll \Delta_\infty\sqrt{H/H_{c2}}$. For these temperatures, excitations are mostly localized in the vortex core as we discussed in Section 6.4.2. Let us find the distribution function in the region of angles α not specifically close to the gap nodes. In the superclean limit $\tilde{\omega}_0\tau \ll 1$, the potential $U(\alpha)$ is small almost everywhere except for vicinities of the gap nodes where it becomes large. For angles larger than $\alpha_1 \sim (\tilde{\omega}_0\tau)^{-1/2}$ from the nodes one can neglect the potential U. As a result, one has for $\delta\alpha < \alpha < \pi/2 - \delta\alpha$ where $\alpha_1 \ll \delta\alpha \ll 1$,

$$\gamma_{\mathrm{O}}(\alpha) = A\cos\alpha + B\sin\alpha$$
$$\gamma_{\mathrm{H}}(\alpha) = 1 - A\sin\alpha + B\cos\alpha \tag{15.31}$$

The constants A and B and the overall behavior of the distribution function are to the highest extent determined by what happens in a close vicinity of the gap nodes. It is this region which is responsible for the whole build-up of nonequilibrium distribution. The constants can be found by matching with the solution in the vicinity of a node, $\alpha \ll 1$, where $\gamma_{\mathrm{O,H}}(\alpha) \propto e^{F(\alpha)}$. This provides the boundary condition

$$\gamma_{\mathrm{O,H}}(+\delta\alpha) = e^{2\lambda_0}\gamma_{\mathrm{O,H}}(-\delta\alpha)$$

across the node. Here $2\lambda_0 = F(+\delta\alpha) - F(-\delta\alpha)$. For energies $\epsilon \ll \Delta_\infty\sqrt{H/H_{c2}}$, the region of angles $\alpha < \alpha_0$ with delocalized trajectories is not important. The range of angles $\alpha \sim \alpha_1$ sets the width of the transition region near a gap node where the distribution function jumps from its value at $\alpha = -\delta\alpha$ to its value at $\alpha = +\delta\alpha$. We obtain using eqn (6.100)

$$\lambda_0 = \tau^{-1}\int_0^\infty \frac{d\alpha}{\omega_0(\alpha)} = \frac{\pi}{4\Omega_0\tau} \tag{15.32}$$

The factor λ_0 contains exactly the integral which determines the true bound states in eqn (6.99). It converges and is determined by angles $\alpha \sim \sqrt{H/H_{c2}}$. The interlevel spacing Ω_0 is as in eqn (6.102)

$$\Omega_0 = \sqrt{\tilde{\omega}_0\omega_c}$$

The solution for $\gamma_{\mathrm{O,H}}$ should be periodically continued to the rest of angles with the period $\pi/2$ since the response function has the same tetragonal symmetry as

Fig. 15.1. The factors $\gamma_{\mathrm{H}}, \gamma_{\mathrm{O}}$ as functions of α. The limit $\lambda_0 \gg 1$ is shown when $\gamma_{\mathrm{O}} \approx \cos\alpha$ while $\gamma_{\mathrm{H}} \approx 1 - \sin\alpha$.

the underlying system: $\gamma(\pi/2-\delta\alpha) = \gamma(-\delta\alpha)$. The behavior of the factors $\gamma_{\mathrm{O,H}}$ as functions of the angle α is illustrated schematically in Fig. 15.1.

Together with the above boundary condition, this gives

$$A = \frac{e^{\lambda_0}\sinh\lambda_0}{2\sinh^2\lambda_0 + 1}; \quad B = \frac{e^{-\lambda_0}\sinh\lambda_0}{2\sinh^2\lambda_0 + 1}. \tag{15.33}$$

We have after averaging over the azimuthal angle α

$$\langle\gamma_{\mathrm{H}}\rangle_\alpha = 1 - \frac{4}{\pi}\frac{\tanh^2\lambda_0}{1+\tanh^2\lambda_0}, \langle\gamma_{\mathrm{O}}\rangle_\alpha = \frac{4}{\pi}\frac{\tanh\lambda_0}{1+\tanh^2\lambda_0}. \tag{15.34}$$

15.5.2 Conductivity

The force on a vortex from the environment where it moves is the momentum transferred from the excitations, eqn (14.4). Only the low energy levels with $n = 0$ are excited. The force takes the form

$$\begin{aligned}\mathbf{F}_{\mathrm{env}} =& \frac{1}{2}\int\frac{dp_z}{2\pi}\frac{d\mu d\alpha}{2\pi}\left[\hat{\mathbf{z}}\times\mathbf{p}_\perp\right]\frac{\partial\epsilon_0}{\partial\mu}f_1 \\ =& -\left[\hat{\mathbf{z}}\times\mathbf{v}_L\right]\int\frac{p_\perp^2}{8\pi}\left\langle\int p_z\right. \\ &+ \mathbf{v}_L\int\frac{p_\perp^2}{8\pi}\frac{dp_z}{d\epsilon}\left\langle\int\gamma_{\mathrm{O}}\frac{df^{(0)}}{d\epsilon}\frac{\partial\epsilon_0}{\partial\mu}d\mu\right\rangle_\alpha \\ & d\mu\rangle_\alpha\end{aligned} \tag{15.35}$$

Since γ_{O} and γ_{H} do not depend on μ, the integration over $d\mu$ in eqn (15.35) can be reduced to the integration over $d\epsilon$. This results in the components of the flux flow conductivity in the form

$$\sigma_{\mathrm{O}} = \frac{|e|c}{B}N\left\langle\langle\gamma_{\mathrm{O}}\rangle\right\rangle, \sigma_{\mathrm{H}} = \frac{ec}{B}N_e\left\langle\langle\gamma_{\mathrm{H}}\rangle\right\rangle. \tag{15.36}$$

where the average over the Fermi surface $\langle\langle\gamma_{\mathrm{H,O}}\rangle\rangle$ is defined by eqn (14.13).

In the moderately clean case, the factors γ_{O} and γ_{H} are given by eqn (15.30). The conductivities are

$$\sigma_{\mathrm{O}} \sim \frac{Nec}{B}\frac{\Delta_\infty^2\tau}{E_F}\ln\left(\frac{T_c}{T}\right) \sim \sigma_n\frac{H_{c2}}{B}\ln\left(\frac{T_c}{T}\right)$$
$$\frac{\sigma_{\mathrm{H}}}{\sigma_{\mathrm{O}}} \sim \frac{\Delta_\infty^2\tau}{E_F}\ln\left(\frac{T_c}{T}\right) \tag{15.37}$$

This is similar to the results for an s-wave superconductor (see Section 14.6).
In the superclean limit $\tilde{\omega}_0\tau \gg 1$ the factors γ_{O} and γ_{H} for low temperatures $T \ll T_c$ are determined by eqn (15.34). The universal regime is reached when $\lambda_0 \gg 1$, i.e., when $T/T_c \ll \sqrt{H/H_{c2}} \ll 1/\left(\tilde{\omega}_0\tau\right)$. One has from eqn (15.34) $\gamma_{\mathrm{H}} = 1 - 2/\pi, \gamma_{\mathrm{O}} = 2/\pi$. The Ohmic and Hall conductivities become

$$\sigma_{\mathrm{O}} = \frac{N|e|c}{B}\frac{2}{\pi}; \sigma_{\mathrm{H}} = \frac{Nec}{B}\left(1 - \frac{2}{\pi}\right) \tag{15.38}$$

These conductivities are universal: they do not depend on the scattering time. The only material characteristics is the number of carriers.

If the mean free path is further increased, the parameter λ_0 decreases. In the limit $\lambda_0 \ll 1$, i.e., for $\sqrt{H/H_{c2}} \gg 1/\left(\tilde{\omega}_0\tau\right)$, we have from eqns (15.34, 15.36)

$$\sigma_{\mathrm{O}} \sim \frac{N|e|c}{B}\frac{1}{\Omega_0\tau}, \sigma_{\mathrm{H}} = \frac{Nec}{B} \tag{15.39}$$

The longitudinal conductivity decreases and finally vanishes while the Hall conductivity reaches its non-dissipative value.

The true interlevel distance Ω_0 introduces a new energy scale which does not exist for superconductors without gap nodes. For conventional superconductors, there is a single characteristic value, the minigap ω_0, which marks a crossover in the relaxation rate $1/\tau$ between two regimes: a dissipative viscous flux flow for $\omega_0\tau \ll 1$, and a dissipation-less vortex motion for $\omega_0\tau \gg 1$. For superconductors with gap nodes, there are two critical values: the average quasiclassical interlevel distance $\langle\omega_0\rangle \sim \tilde{\omega}_0$ and the true interlevel distance Ω_0 which is less than $\langle\omega_0\rangle$. Therefore one has three regimes: (i) Moderately clean regime $\langle\omega_0\rangle\tau \ll 1$ with a high dissipation, (ii) intermediate universal regime $\Omega_0\tau \ll 1$ with a finite dissipation, where the quasiclassical levels are well resolved on average but the true minigap is smaller than the scattering rate; and (iii) the extreme superclean case $\Omega_0\tau \gg 1$ when the true levels are well separated and thus dissipation is small.

20 Collisionless relaxation of the energy gap in superconductors by Volkov, Kogan, 1973

(it is not written, physically why the perturbation relax and why can they appear? these are the main questions)

Abstract

Equations are derived for equal-time Green's functions and describe the dynamics of superconductors during a time that is short in comparison with the electron energy relaxation times τ_{ph} and τ_{ee}.

(which exactly these times are? it means $t < \tau_{ph}, \tau_{ee}$)

The time evolution of small initial perturbations of the order parameter Δ is investigated.

(what physically happens when there are these initial perturbations????)

It is established that following initial perturbations of a definite type (which type?) the energy gap relaxes only as a result of inelastic collisions of the electrons during a time on the order of τ_{ph}, τ_{ee}.

(only! why only?..)

In the general case which???, the order parameter at $t < \tau_{ph}, \tau_{ee}$ oscillates with a frequency $\sim 2\Delta$ and with an amplitude that attenuates asymptotically with time in accord with a power law.

(so in particular case gap decreases and in general case it also oscillates?)

(still don't understand the physics behind this formulas...)

20.0.1 1. Introduction

Much progress has been made recently in the problem of the nonstationary phenomena in superconductors. Equations describing the kinetics of superconductors were obtained. These equations are of simplest form in the case of gapless superconductivity [1–3].

(why for gapless?? and how is it??)

In this case they are closed equations for the order parameter Δ and generalize the Ginzburg-Landau equations.

At the same time, there are many problems in which an important role is played precisely by the presence of the finite gap in the energy spectrum of the superconductor (I thought it is important for all problems...).

One such problem arises in the problem of perturbation relaxation in superconductors.

It is easy to verify that in gapless superconductors the perturbations of Δ decrease exponentially to zero.

(how to verify it??? well, if SC is gapless, then there should be no gap, so it should decrease...)

What remains unclear is the character of the relaxation of Δ in the absence of external field in superconductors with gaps.

(why we need an absence of external field?)

In particular, one can conceive of a situation in which the perturbations of Δ experience natural undamped oscillations [4] (within a time short in comparison with the inelastic relaxation times).

(was it discovered in [4]??? B. I. Ivlev, ZhETF Pis. Red. 15, 441 (1972))

The kinetics of superconductors with finite gaps is described by sufficiently complicated equations, obtained by Éliashberg [5] for equal-time Green's functions.

In this paper we examine, on the basis of equations derived from the Eliashberg equations, the behavior of a small initial perturbation of Δ in the absence of fields, and in a time interval small in comparison with the electron energy-relaxation times

$$\tau_{ph} \approx \hbar\Theta_D^2/T^3, \qquad \tau_{ee} \approx \hbar\varepsilon_F/T^2,$$

which reach 10^{-8} sec in order of magnitude.

(it is interesting to find why these times are such, but now it is better to just believe it)

It will be shown below that for a definite type of perturbation, the characteristic time of variation of the initial perturbation is Δ^{-1}, which is small in comparison with $\tau_{\rm ph}$ and $\tau_{\rm ee}$.

Then the inelastic collisions of the electrons do not influence the evolution of the perturbation during the course of times that are short in comparison with $\tau_{\rm ph}$ and $\tau_{\rm ee}$, and the corresponding collision integrals can be neglected.

This enables us to simplify greatly the initial Eliashberg equations for the equal-time Green's functions and to change over to equations relative to Green's functions with coinciding times (accurate to $\sim \omega_{\mathrm{D}}^{-1}$).

The derived equations can be regarded as "collisionless" kinetic equations for the superconductor.

(what is kinetic here if there are no collisions??)

In Sec. 2 we derive first the Eliashberg equations by a different method, which in essence is a generalization of the Keldysh technique [6] to the case of superconductors. This approach is simplest and most lucid.

20.0.2 2. Derivation of the Fundamental Equations

(find some articles where the same formalism is used! I am not sure now about it)

(write in the theory about Keldysh's method, why is it so??? I don't understand it yet)

Keldysh's method consists of ordering the secondquantization operators in the Green's functions on a double time-dependent contour consisting of two axes, one extending from $-\infty$ to $+\infty$, and the other from $+\infty$ to $-\infty$.

The time on the second axis is assumed to be longer (in the sense of ordering of the operators) than any time on the first axis, while the ordering of the operators on the second axis is in antichronological order, i.e., the operator with the smallest time, closest to $-\infty$, is located to the left.

In accordance with the fact that the time of each of the two ψ operators (quantum fields) in the Green's function can be either on the first or on the second axis of the contour, four functions making up the matrix are possible.

In the case of generalization to include superconductors, it is necessary to introduce also the spin indices of the operators (the Gor'kov-Nambu technique [7]):

$$\psi_{\uparrow}\left(1_i\right) \equiv \psi_1\left(1_i\right), \quad \psi_{\downarrow}^{+}\left(1_i\right) \equiv \psi_2\left(1_i\right),$$

where 1 is the aggregate of the spatial coordinates and time (so it is used only to make formulas more compact, it a special notation), and i is the index of the temporal axis (because we have two time axis, so they can be only 1 and 2).

(writ it in the ordinary notation to compare!)

Thus, the single-particle Green's function of the electrons

$$G_{\alpha\beta}{}^{ik}\left(11'\right) = i^{-1}\left\langle T\psi_{\alpha}\left(1_i\right)\psi_{\beta}{}^{+}\left(1_k{}'\right)\right\rangle \tag{1}$$

(add this formula in ordinary notation to compare them!)

constitutes a matrix both in the spin indices (α, β) and in the temporal indices (i, k) (note that there are no space indices), while the angle brackets denote averaging with the density matrix taken at the instant t_0 at which the Heisenberg operators coincide with the Schrödinger operators.

(I want to see an explicit expression of this averaging)

In addition to the functions (1), we define also the retarded and advanced functions, and also the function introduced by Keldysh [6] :

$$
\begin{aligned}
{G_{\alpha\beta}}^{R}(11') &= i^{-1}\theta(t_1 - t_1')\left\langle\left\{\psi_\alpha(1), {\psi_\beta}^{+}(1')\right\}_{+}\right\rangle, \\
{G_{\alpha\beta}}^{A}(11') &= i\theta(t_1' - t_1)\left\langle\left\{\psi_\alpha(1), {\psi_\beta}^{+}(1')\right\}_{+}\right\rangle, \\
{G_{\alpha\beta}}^{K}(11') &= {G_{\alpha\beta}}^{12} + {G_{\alpha\beta}}^{21} = i^{-1}\left\langle\left[\psi_\alpha(1), {\psi_\beta}^{+}(1')\right]_{-}\right\rangle.
\end{aligned} \tag{2}
$$

(We assuming that we can also plug indices i, k to these expression). (what is the intuition behind these defs? there are some answers that can be given by looking at the Keldysh contour)

From the definitions (1) and (2) we get the following relations [6] (the spin indices have not been written out):

$$
\begin{aligned}
G^{11} + G^{22} &= G^{12} + G^{21}, \\
G^{A} = G^{11} - G^{21} &= G^{12} - G^{22}, \\
G^{R} = G^{11} - G^{12} &= G^{21} - G^{22}.
\end{aligned} \tag{3}
$$

(why? I'll check it later, now I just believe it)

If the system contains no fields acting directly on the electron spins, and if the spin-orbit interaction can be neglected, then

$$
G^{ij*}_{\alpha\beta}(11') = (-1)^{\alpha+\beta} G^{i\bar{j}}_{\bar{\alpha}\bar{\beta}}(11') \tag{4}
$$

where the bar over the index denotes its replacement by the opposite: $\overline{1} = 2$ and $\overline{2} = 1$. (why? maybe it is not hard to prove it? why in SO interaction it is not so?)

From (4) and (3) it follows that

$$
\begin{aligned}
G^{A(R)^*}_{\alpha\beta}(11') &= -(-1)^{\alpha+\beta} G^{A(R)}_{\overline{\alpha\beta}}(11'), \\
G^{K^*}_{\alpha\beta}(11') &= (-1)^{\alpha+\beta} G^{K}_{\overline{\alpha\beta}}(11').
\end{aligned} \tag{5}
$$

(why? I'll prove it later, maybe)

In addition, it is easy to verify that

$$
\begin{aligned}
{G_{\alpha\alpha}}^{K*}(11') &= -{G_{\alpha\alpha}}^{K}(1'1), \\
{G_{\alpha\beta}}^{K}(11') &= {G_{\alpha\beta}}^{K}(1'1), \quad \alpha \neq \beta.
\end{aligned} \tag{6}
$$

(why? I'll prove it later, maybe)

In Dyson's equation for the Green's function

$$
\hat{\sigma}_z \hat{G}_0^{-1}(1) G(11') - \Sigma(12) G(21') = 1 \cdot \delta(1 - 1') \tag{7}
$$

(I don't understand!!!! why do we have a big 2 here??? it was never before!)
(how to write it in the usual notation???)
the operator

$$
\left(G_0^{-1}(1)\right)_{\alpha\beta} = \delta_{\alpha\beta} i\partial/\partial t_1 - (H(1))_{\alpha\beta}
$$

is a matrix in the space of the spin indices, with

$$\hat{H} = \begin{pmatrix} H & 0 \\ 0 & -H^* \end{pmatrix},$$

where $H(1)$ is the single-electron operator of the energy, reckoned from the Fermi energy, with allowance for the external fields.

In addition, $(\sigma_{\mathrm{Z}})_{\mathrm{ik}}$ in (7) is a Pauli matrix, while $(1)^{\mathrm{ik}}_{\alpha\beta} = \delta_{\alpha\beta}\delta_{\mathrm{ik}}$.

In the phonon model, the mass operator is given by a well known sum of diagrams [8] in which, however, each vertex is set in correspondence with the matrix

$$\left(\mathscr{T}^{(0)}(12;3)\right)^{ij;k}_{\alpha\beta} = (\sigma_z)_{ij}\,\delta_{jk}\,(\sigma_z)_{\alpha\beta}\,\delta(1-2)\delta(1-3).$$

(so, below the results of the previous articles are used. the only way to understand them is to read that articles)

In the first-order approximation we have

$$\Sigma^{ij}_{\alpha\beta}(12) = i(-1)^{\alpha+\beta+i+j} G^{ij}_{\alpha\beta}(12) D^{ji}(21) \tag{8}$$

The contribution made to Σ by the direct interaction is also given by the usual sum of diagrams [8], in which each elementary four-corner vertex is set in correspondence with the matrix

$$\begin{aligned}\left(\Gamma^{(1)}(12;34)\right)^{h_1h_2h_4}_{\alpha_1\alpha_2\alpha_3\alpha_4} &= -iv(12)\,(\sigma_z)_{k_1k_2}\,\delta_{k_1k_3}\delta_{k_2k_4}\left\{(\sigma_z)_{\alpha_1\alpha_3}\,\delta(1-3)\,(\sigma_z)_{\alpha_2\alpha_4}\,\delta(2-4)\right.\\ &\left.-(\sigma_z)_{\alpha_1\alpha_4}\,\delta(1-4)\,(\sigma_z)_{\alpha_3\alpha_3}\,\delta(2-3)\right\}.\end{aligned} \tag{9}$$

The mass operators are connected by a relation [6] that follows from (3):

$$\Sigma^{11} + \Sigma^{22} + \Sigma^{12} + \Sigma^{21} = 0$$

The equations for $\mathrm{G^R}, \mathrm{G^A}$, and $\mathrm{G^K}$ follow from (7):

$$G_0^{-1}(1)G^{R(A)}(11') - \Sigma^{R(A)}(12)G^{R(A)}(21') = \delta(1-1') \tag{10}$$

$$G_0^{-1}(1)G^{K}(11') - \Sigma^{R}G^{K} + \Omega G^{A} = 0$$

where

$$\Sigma^{R(A)} = \Sigma^{11} + \Sigma^{12(21)} = -\Sigma^{22} - \Sigma^{21(12)}$$

$$\Omega = \Sigma^{12} + \Sigma^{21} \tag{11}$$

These equations differ from the corresponding equations of Keldysh's paper [6] only in that all the quantities are matrices in the spin indices.

Relations (4) and (5) for the Green's functions correspond to the following relations for the mass operators:

$$\Sigma^{ij}_{\alpha\beta} = (-1)^{\alpha+\beta}\Sigma^{\bar{i}\bar{j}}_{\bar{\alpha}\bar{\beta}} \tag{12}$$

$$\Sigma^{R(A)*}_{\alpha\beta} = -(-1)^{\alpha+\beta}\Sigma^{R(A)}_{\bar{\alpha}\bar{\beta}}, \quad \Omega^*_{\alpha\beta} = -(-1)^{\alpha+\beta}\Omega_{\bar{\alpha}\bar{\beta}}$$

(I have no idea what is the meaning of these relations)

If we write out the spin indices in (10) explicitly and use (5) and (12) then, as can be easily verified, we obtain the Eliashberg equations [5]1). It is necessary to bear in mind here the following correspondence between the notation of our paper and [5] :

$$G_{11}^{R(A)} = -G^{R(A)}, \quad G_{12}^{R(A)} = -F^{R(A)}, \quad G_{11}{}^{K} = -G, \quad G_{12}{}^{K} = -F,$$
$$\Sigma_{11}^{R(A)} = -\Sigma_{1}^{R(A)}, \quad \Sigma_{12}^{R(A)} = \Sigma_{2}^{R(A)}, \quad \Omega_{11} = -\Sigma_{1}, \quad \Omega_{12} = \Sigma_{2}.$$

We write out explicitly only the equations for $\mathrm{G}_{11}^{\mathrm{K}}$ and $\mathrm{G}_{12}^{\mathrm{K}}$, which we shall need later on (together with the conjugate equations):

$$(i\partial/\partial t_1 - H(1))\, G_{11}{}^{K}(11') - \Sigma_{11}{}^{R} G_{11}{}^{K} + \Sigma_{12}{}^{K} G_{12}{}^{K*} + \Omega_{11} G_{11}{}^{A} + \Omega_{12} G_{12}{}^{A*} = 0,$$
$$(i\partial/\partial t_1' + H^*(1'))\, G_{11}{}^{K}(11') + G_{11}{}^{K}\Sigma_{11}{}^{A} + G_{12}{}^{K}\Sigma_{12}{}^{A*} - G_{11}{}^{R}\Omega_{11} + G_{12}{}^{R}\Omega_{12}{}^{*} = 0, \tag{13}$$
$$(i\partial/\partial t_1 - H(1))\, G_{12}{}^{K}(11') - \Sigma_{11}{}^{R} G_{12}{}^{K} - \Sigma_{12}{}^{R} G_{11}{}^{\kappa*} + \Omega_{11} G_{12}{}^{A} - \Omega_{12} G_{11}{}^{A*} = 0,$$
$$(i\partial/\partial t_1' - H(1'))\, G_{12}{}^{K}(11') - G_{12}{}^{K}\Sigma_{11}{}^{A*} + G_{11}{}^{K}\Sigma_{12}{}^{A} - G_{12}{}^{R}\Omega_{11}{}^{*} - G_{11}{}^{R}\Omega_{12} = 0. \tag{14}$$

As is well known, $\left(\Sigma_{12}^{\mathrm{R}} + \Sigma_{12}^{\mathrm{A}}\right)/2$ plays the role of the operator of the energy gap Δ of the superconductor. We represent Σ_{12}^{R} and Σ_{12}^{A} in the form

$$\Sigma_{12}^{R(\boldsymbol{A})}(12) = \Delta(12) + 1/2\left(\Sigma_{12}^{R(\boldsymbol{\Lambda})}(12) - \Sigma_{12}^{\boldsymbol{\Lambda}(R)}(12)\right). \tag{15}$$

We add Eqs. (13) for $\mathrm{G}_{11}^{\mathrm{K}}$ and, in similar fashion, Eqs. (14) for $\mathrm{G}_{12}^{\mathrm{K}}$. The result can be represented in the following form (we neglect renormalization effects described by $\mathrm{Re}\,\Sigma_{11}(\ \mathrm{A})$[5]):

$$(i\partial/\partial t - H(1) + H^*(1'))\, G_{11\kappa}{}^{K}(11') + \Delta(12) G_{12}{}^{K*}(21') + G_{12}{}^{K}(12)\Delta^*(21') = I_{11}(11')$$

$$(i\partial/\partial t - H(1) - H(1'))\, G_{12}{}^{K}(11') - \Delta(12) G_{11}{}^{K*}(21') + G_{11}{}^{K}(12)\Delta(21') = I_{12}(11') \tag{16}$$

Here $t = (t_1 + t_1')/2$, and the form of the "collision integrals " I_{11} and I_{12} can be easily obtained from (13)(15).

(some complicated equations....)

From the expressions for Σ_{12}^{R} and Σ_{12}^{A} (see (11) and (8)) follows the well known results [5]

$$\Delta(12) = 1/4 i^{-1} G_{12}{}^{K}(12)\left[D^A(21) + D^R(21)\right]$$

Here the sum of the phonon functions $D^A + D^R$, regarded as the function $\tau = t_1 - t_2$, is a narrow peak of width $\sim \omega \bar{D}^{-1}$. Therefore the energy gap is

$$\Delta(\mathbf{R}t) = \int d\mathbf{r} d\tau \Delta(12) = \frac{ig^2}{2}\int \frac{d\mathbf{p}}{(2\pi)^3}\int_{-\omega_D}^{\omega_D}\frac{d\varepsilon}{2\pi} G_{12}{}^{K}(\mathbf{R}t;\mathbf{p}\varepsilon) \tag{17}$$

where

$$\mathbf{R} = (\mathbf{r}_1 + \mathbf{r}_2)/2, \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$$
$$G_{12}{}^{K}(\mathbf{R}t;\mathbf{p}\varepsilon) = \int d\mathbf{r} d\tau e^{-i\mathbf{p}\mathbf{r} + i\mathbf{e}\tau} G_{12}{}^{K}(12)$$

In a number of problems, the reciprocal of relaxation times in a superconductor are much smaller than Δ and then the characteristic frequencies of all the quantities as functions of t. In this case I_{11} and I_{12} can be discarded in (16). We can simplify the obtained collisionless kinetic

equations further by using the already mentioned narrowness of $\Delta(12)$ as a function of τ, and by using the fact that to determine $\Delta(\mathrm{Rt})$ we need the function $\mathrm{G}_{12}^{\mathrm{K}}$ at practically equal times ($\tau \lesssim \omega_{\mathrm{D}}^{-1}$). This makes the system of collisionless equations (Eqs. (16) without the right-hand sides) for $\mathrm{G}_{11}^{\mathrm{K}}$ and $\mathrm{G}_{12}^{\mathrm{K}}$ with $\tau \lesssim \omega_D^{-1}$, together with expression (17) for Δ (Rt), a closed system.

We introduce the notation

$$g(\mathbf{p};\mathbf{R}t) = \frac{1}{2\pi}\int_{-\omega_D}^{\omega_D} d\varepsilon {G_{11}}^K(\mathbf{R}t;\mathbf{p}\varepsilon)$$
$$f(\mathbf{p};\mathbf{R}t) = \frac{1}{2\pi}\int_{-\omega_D}^{\omega_D} d\varepsilon {G_{12}}^K(\mathbf{R}t;\mathbf{p}\varepsilon) \tag{18}$$

Taking into account the approximations made above, we obtain from (16)

$$\begin{aligned}&[i\partial/\partial t - H(\mathbf{r}_1 t) + H^*(\mathbf{r}_1{}' t)]\, g(\mathbf{r}_1\mathbf{r}_1{}' t) + \Delta(\mathbf{r}_1 t)\, f^*(\mathbf{r}_1\mathbf{r}_1{}' t) + \Delta^*(\mathbf{r}_1{}' t)\, f(\mathbf{r}_1\mathbf{r}_1{}' t) = 0,\\ &[i\partial/\partial t - H(\mathbf{r}_1 t) - H(\mathbf{r}_1{}' t)]\, f(\mathbf{r}_1\mathbf{r}_1{}' t) - \Delta(\mathbf{r}_1 t)\, g^*(\mathbf{r}_1\mathbf{r}_1{}' t) + \Delta(\mathbf{r}_1{}' t)\, g(\mathbf{r}_1\mathbf{r}_1{}' t) = 0.\end{aligned} \tag{19}$$

We note that these equations are obtained in the BCS model without additional assumptions. In accordance with formula (9) for $\Gamma^{(1)}$ in the self-consistent field approximation we have

$${\Sigma_{\alpha\beta}}^{ij}(12) = -i\,(\sigma_z)_{ij}\left\{(\sigma_z)_{\alpha\beta}\, v(13){G_{\gamma\gamma}}^{ij}(33) - (-1)^{\alpha+\beta}{G_{\alpha\beta}}^{ij}(12)v(21)\right\}.$$

In this approximation $\sum_A^{\mathrm{ij}} = 0$ at $\mathrm{i} \neq \mathrm{j}$ and $\Omega = 0$, and the mass operators $\Sigma_{\alpha\beta}^{\mathrm{R}}$ and $\Sigma_{\alpha\beta}^{\mathrm{A}}$ reduce to $\Sigma_{\alpha\beta}^{11}$ or $-\Sigma_{\alpha\beta}^{22}$.

Next, as usual, we include Σ_{11}^{R} and Σ_{11}^{A} in the chemical potential.

(?????? why???)

And express $\Delta(12) = 1/2\,(\Sigma_{12}^{11} - \Sigma_{12}^{22})$ in the form

$$\Delta(12) = -1/2i{G_{12}}^K(12)v(21) = 1/2ig\delta(12){G_{12}}^K(11) = \Delta(1)\delta(12)$$

When relations (5) are taken into account, we arrive at (19).

It should be noted that the obtained equations (19) describe the behavior of the Green's functions integrated with respect to ϵ (see (18)), and not with respect to $\xi = (p^2 - p_0^2)/2m$ as in Eliashberg's paper [5]. Therefore the first equation of (19) goes over at $\Delta = 0$ into the usual collisionless kinetic equation.

(I'll think about this statement not soon)

20.0.3 3. Collisionless Evolution of Perturbations in a Superconductor

We use Eqs. (19) to investigate the evolution of the initial perturbations in a superconductor during times that are short in comparison with the energy-relaxation times τ_{ph} and τ_{ee}.

(again, physically what happens when there is a perturbation and when relaxation of them happens???)

We confine ourselves to the spatially homogeneous case and assume that all the quantities are independent of the direction of p, and that there are no fields. After applying a Fourier transformation with respect to the difference coordinate $r - r_1 - r_1'$, Eqs. (19) and (17) take the form

$$i\partial g(\xi t)/\partial t + \Delta(t) f^*(\xi t) + \Delta^*(t) f(\xi t) = 0 \tag{20a}$$
$$(i\partial/\partial t - 2\xi) f(\xi t) + 2\Delta(t) g(\xi t) = 0 \tag{20b}$$
$$\Delta(t) = i\frac{\lambda}{2}\int d\xi f(\xi, t) \tag{20c}$$

(comment that this is how we always see the gap, as $\Delta(t) = i\frac{\lambda}{2}\int d\xi f(\xi, t)$. why, by the way???? is it written in Gorkov's papers???)

We took into account here the fact that in accordance with (6) g is pure imaginary. Assume that arbitrary initial perturbations $\delta g_0(\xi)$ and $\delta f_0(\xi)$ have occurred in the superconductor and are described by the deviations of the functions f and g from their stationary values g_0 and f_0. The perturbations can be produced either by electromagnetic radiation or by injecting quasiparticles or pairs into the superconductor. Let us find the evolutions of the perturbations with the aid of the linearized system (20). We see that the problem is quite analogous in its formulation with the Landau problem of evolution of perturbations in a collisionless plasma [9]. The electric self-consistent field in the plasma corresponds in the superconductor to the "field" Δ(t).

The stationary functions f_0 and g_0 can be represented, according to (20), in the form

$$f_0(\xi) = -i\Delta\chi(\xi), g_0(\xi) = -i\xi\chi(\xi), \tag{21}$$

where the form of $\chi = \chi^*$ is determined by the particular conditions for the production of the stationary state. Under thermodynamic equilibrium we have

$$\chi = [\mathrm{th}(\varepsilon/2T)]/\varepsilon,$$

where $\epsilon = (\xi^2 + \Delta^2)^{1/2}$. We linearize (20) with respect to the stationary values in (21):

$$(i\partial/\partial t - 2\xi)\delta f(\xi t) - 2i\xi\chi\delta\Delta(t) + 2\Delta\delta g(\xi t) = 0$$
$$i\partial\delta g/\partial t - 2\Delta\chi\delta\Delta'' + 2\Delta\delta f'(\xi t) = 0$$

The prime and double prime denote here the real and imaginary parts of the function, respectively. We note that, at small perturbations, $\delta\Delta'$ is the change of the gap $|\Delta|$, while $\delta\Delta''/\Delta$ is the change of the phase of the order parameter.

We separate the imaginary and real parts of the equation:

$$\begin{gathered}\partial\delta f''/\partial t + 2\xi\delta f' - 2\xi\chi\delta\Delta'' = 0\\ \partial\delta f'/\partial t - 2\xi\delta f'' - 2\xi\chi\delta\Delta' + 2\Delta\delta g'' = 0\\ \partial\delta g''/\partial t + 2\Delta\chi\delta\Delta'' - 2\Delta\delta f' = 0\end{gathered} \tag{22}$$

For the Laplace transforms of the sought functions $\delta F(s)$ we obtain from (22)
(maybe here, there is a nontrivial mathematics, but in general it is clear what happens)

$$\left(s^2 + 4\varepsilon^2\right)\delta \mathrm{r}'(s) - 4\varepsilon^2\chi\delta\Delta''(s) + 2\xi\chi s\delta\Delta'(s) = \delta\dot{f}_0' + s\delta f_0' - 2\xi\chi\delta\Delta';$$
$$\left(s^2 + 4\varepsilon^2\right)\delta f''(s) - 2\xi\chi s\delta\Delta''(s) + 4\xi^2\chi\delta\Delta'(s) = -\frac{2\xi}{s}\left(\delta\dot{f}_0' + s\delta f_0'\right) + \frac{4\xi^2 v}{s}\delta\Delta_0' + \frac{\delta f_0''}{s}\left(s^2 + 4\varepsilon^2\right) \tag{23}$$

Here $\delta\Delta_0' = [\delta\Delta'(\mathrm{t})]\,\mathrm{t} = 0$, etc. In the derivation of (23) we have expressed δg_0 with the aid of the section equation of (22) in terms of $[\ \mathrm{d}\delta f'/\mathrm{dt}]_{\mathrm{t}=0} \equiv \delta\dot{f}_0'$ and $\delta f_0''$.

We multiply (23) by $\lambda/2$ and integrate with respect to ξ. We confine ourselves to the case of functions χ that are even in ξ. Then, taking (23) into account, we obtain

$$s^2 F(s)\delta\Delta''(s) = \left\langle \frac{\delta\dot{f}_0' + s\delta f_0'}{s^2+4\varepsilon^2} \right\rangle, \tag{24a}$$

$$\left(s^2+4\Delta^2\right) F(s)\left(\delta\Delta'(s) - \frac{\delta\Delta_0'}{s}\right) = \frac{1}{s}\left\langle \frac{2\xi\left(\delta\dot{f}_0' + s\delta f_0'\right)}{s^2+4\varepsilon^2} \right\rangle. \tag{24b}$$

We have introduced here the notation

$$F(s) = \left\langle \frac{\chi}{s^2+4\varepsilon^2} \right\rangle, \quad \langle \ldots \rangle = \frac{\lambda}{2}\int_{-\omega_D}^{\omega D} d\xi \ldots$$

and used the equality $\chi = 1$, which follows from (20c) and (21).
(is it the same as in Eliashberg's work???)

If $\delta\dot{f}_0' = 0$ and $\delta f_0' = 0$ (but $\delta f_0'' \neq 0$) then as seen from (24b), $\delta\Delta'(s) = \delta\Delta_0'/s$, meaning that $\delta\Delta'(\mathrm{t}) = \delta\Delta_0' = \mathrm{const}$ and that the energy gap relaxes only as a result of inelastic collisions during a relaxation time much larger than Δ^{-1}. If $\Delta\dot{f}_0' + s\delta f_0' \neq 0$, then that part of this function which is even in ξ determines the time variation of $\delta\Delta''(\mathrm{t})$, i.e., the change of the phase of the order parameter, while the odd part determines the variation of $\delta\Delta'(\mathrm{t})$, i.e., of the modulus of the gap.

Let us investigate the analytic properties of $F(s)$ in the complex s plane. Near the imaginary axis $s = i\Omega + \delta$, where $\delta \to 0$, we have

$$F(s) = \frac{\lambda}{2}\left\{ \int_{-\omega_D}^{\omega D} \frac{d\xi\chi(\varepsilon)}{4\varepsilon^2 - \Omega^2} - i\frac{\pi}{2}\,\mathrm{sgn}(\Omega\delta)\frac{\chi(\Omega/2)}{\left(\Omega^2-4\Delta^2\right)^{1/2}}\theta\left(\Omega^2-4\Delta^2\right) \right\} \tag{25}$$

At $|\Omega| > 2\Delta$, the imaginary part of $F(s)$ experiences a discontinuity when s crosses the imaginary axis. Therefore $F(s)$ has a branch point at $s = \pm 2\mathrm{i}\Delta$, and the single-valued branch $F(s)$ should be chosen by drawing the cuts as shown in the figure.

The functions in the right-hand side of (24) have the same branch points as $F(s)$. Therefore $\delta\Delta''(s)$ has in the general case branch points at $s = \pm 2i\Delta$, and in addition poles of first and second order at zero, while $\delta\Delta'(s)$ has, in addition to these branch points, also a first-order pole at zero. It is important to emphasize that $\delta\Delta'(s)$ has no poles at the point $s = \pm\mathrm{i}2\Delta$.

Indeed, the right-hand side of (24b) differs from zero if $\delta \dot{f}_0' + s\delta f_0'$ has a part that is odd in ξ. At the same time, from the continuity of this part at the point $\xi = 0$ it follows that it can be represented in the form $\xi\eta(\xi)$, where $\eta(\xi)$ is an even function having no singularities at zero. Then the right-hand part of (24b) can be transformed into

$$\langle\eta\rangle - \left(s^2 + 4\Delta^2\right)\left\langle \frac{\eta}{s^2 + 4\varepsilon^2}\right\rangle.$$

The first term does not depend in general on s, and the second factor at $s^2 + 4\Delta^2$ has the same singularities as $F(s)$.

Since the function $\delta\Delta'(s)$ has no poles at the points $s = \pm i2\Delta$, it follows that the energy gap, even if we neglect the inelastic collisions, has no natural undamped oscillations with frequency 2Δ (cf. [4]).

Let us illustrate the behavior of $\delta\Delta$(t) under the following concrete initial conditions

$$\delta f_0' = 2\alpha_1\xi\chi + \alpha_2\chi, \qquad \delta\dot{f}_0' = 2\beta_1\xi\chi + \beta_2\chi,$$

where α_i and β_i are small constants. It follows then from (24), with allowance for (23), that

$$\delta\Delta''(s) = \frac{\beta_2}{s^2} + \frac{\alpha_2}{s} = \frac{\dot{\Delta}_0''}{s^2} + \frac{\Delta_0''}{s},$$
$$\delta\Delta'(s) = \frac{\Delta_0'}{s} + \frac{\beta_1 + \alpha_1 s}{s\left(s^2 + 4\Delta^2\right)F(s)} - \frac{\beta_1 + \alpha_1 s}{s}$$

Taking the inverse Laplace transform, we find that the imaginary part of the gap, which is proportional to the phases, increases linearly with time:

$$\delta\Delta''(t) = \delta\dot{\Delta}_0''t + \delta\Delta_0'',$$

while the real part is given by

$$\delta\Delta'(t) = \delta\Delta_0' - \beta_1 + \frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} \frac{ds\left(\beta_1 + \alpha_1 s\right)e^{st}}{s\left(s^2 + 4\Delta^2\right)F(s)}. \tag{26}$$

The integral can be represented here in the form of a difference of the integral over the entire closed contour in the figure and the integrals over sections $1, 2, 3,$ and 4 of the second contour. As a result we obtain

(I'll check math later, a good problem in complex analysis)

$$\frac{1}{2\pi i}\int_{-i\infty+\sigma}^{i\infty+\sigma} \frac{ds\left(\beta_1 + \alpha_1 s\right)e^{st}}{s\left(s^2 + 4\Delta^2\right)F(s)} = \frac{\beta_1}{4\Delta^2 F(0)} - \frac{2}{\pi}\int_{2\Delta}^{\infty} d\Omega \frac{\operatorname{Im} F(\Omega)}{|F(\Omega)|^2}\frac{1}{\Omega^2 - 4\Delta^2}\left[\beta_1\frac{\cos\Omega t}{\Omega} - \alpha_1\sin\Omega t\right]. \tag{27}$$

In the calculation we took into account the fact that the imaginary part of $F(s)$ on opposite sides of the cut along the imaginary axis (see the figure) differ in sign, and that $F(s) = F(-s)$.

It is easy to verify that at t = 0 the integral in (26) is equal to β_1 and that $\delta\Delta'(0) = \delta\Delta_0'$, as should be the case. To understand the character of the time variation of the gap perturbation, let us find the asymptotic value of the integral in (27) at $2t\Delta \gg 1$. It is determined by the behavior of the integrand at $\Omega \approx 2\Delta$. It can be shown that $\operatorname{Re} F(s)$ has no singularities as $s \to 2\mathrm{i}\Delta$ (see (25)) and that $\operatorname{Im} F/|F|^2 = (\operatorname{Im} F)^{-1}$ near this point. Substituting Im F from (25) and taking the equilibrium value of χ, we obtain

$$\delta\Delta'(t) = \delta\Delta'_0 - \beta_1 + \frac{\beta_1}{4\Delta^2 F(0)} + \left(\frac{2}{\pi}\right)^{3/2} \frac{1}{\lambda\,\mathrm{th}(\Delta/2T)} \times (2t\Delta)^{-1/2}\left[\beta_1 \cos\left(2\Delta t + \frac{\pi}{4}\right) - 2\alpha_1\Delta \sin\left(2\Delta t + \frac{\pi}{4}\right)\right].$$

We see that the perturbation of the gap, as a function of the time, take the form of oscillations having a frequency $\sim 2\Delta$ and an amplitude that attenuates like $t^{-1/2}$. It is important that with increasing t (but at $\mathrm{t} \ll \tau_{\mathrm{ee}}, \tau_{\mathrm{ph}}$) in the considered collisionless approximation the gap perturbation $\delta\Delta'(\mathrm{t})$ tends.generally speaking neither to zero nor to $\delta\Delta'_0$.

20.0.4 My questions about work by Volkov, Kogan

(?)

21 Collective Modes in Superconductors by Shon

21.1 1. Introduction

The dynamic properties of superconductors have been the subject of an extensive research for many years. Various relaxation processes have been investigated which reveal much about the macroscopic order and the microscopic interaction typical for the superconducting state. Collective oscillations, in particular those of low frequencies, are interesting for the same reasons. In neutral superfluids a large variety of sound-like (propagating) collective oscillations, which have a linear dispersion relation, are well established. However, in superconductors none of these modes has been observed. The main reason is that the particles in the superconductor are charged. Hence, the frequency of density waves is shifted to the plasma frequency, which is little modified by the superconducting transition. Furthermore, the electrons in the superconductor move in a lattice, which in practice is always distorted. Hence, their motion is subject to (Ohmic) damping. It was only after a long lasting search, both theoretically and experimentally, that a sound-like collective mode in superconductors was detected, in the experiments of Carlson and Goldman (1973, 1975). They investigated the order parameter structure factor and found resonant peaks at finite frequencies $\pm\omega_q$, where (ω_4 depends linearly on the wave vector.

Early theoretical descriptions, based on the two-fluid model or the phenomenological time-dependent Ginzburg Landau theory gave neither a proper description of the collective modes nor of the relaxation mechanisms. The discussion of collective modes was incited particularly after the establishment of the microscopic theory by Bardeen, Cooper and Schrieffer (1957). In the original form it failed to account for the Meissner effect in a gauge-invariant way. The remedy of this flaw also leads to the prediction of a soundlike collective density oscillation in a (hypothetical) neutral superconductor (Anderson 1958, Bogoliubov et al. 1959). This mode, which involves oscillations of the phase of the order parameter, is also predicted by the Goldstone theorem. In a (real) charged superconductor, however, the frequency of this density mode is shifted to the plasma frequency and of little physical interest.

In the following years many investigations were concerned with the response of the superconductor to perturbations like electromagnetic response or attenuation of ultrasound. These descriptions involved only the order para-
meter and the thermal quasiparticles, thus precluding many nonequilibrium features. More recently, however, it became clear that the deviations of the quasiparticle distribution in superconductors from equilibrium may be significant. One reason is that the inelastic scattering

times in superconductors are comparatively long. A complete description has to incorporate both degrees of freedom: the quasiparticles and the order parameter. It has to establish a kinetic theory for the quasiparticle distribution function in the superconducting state and to specify the influence of a nonequilibrium distribution on the order parameter. It turns out that in clean superconductors a sufficient theory is of a rather transparent form. The central part is a Boltzmann equation for the distribution function $f(p; r, t)$ (Aronov and Gurevich 1974). It is of the same form as in the normal metal with some rather obvious modifications due to the characteristic superconducting properties. Also in dirty superconductors, where the majority of experiments is performed, one finds a Boltzmann type of equation for the quasiparticle distribution as a function of the energy $f'(E; r, t)$ (Schmid and Schön 1975a). One of the consequences of the coupling between the order parameter and the quasiparticles is that the order parameter relaxation becomes rather slow. Furthermore, the theory accounts for the phenomenon of charge imbalance (Clarke 1972, Tinkham 1972) and its relaxation.

The microscopic description being developed, the collective mode observed by Carlson and Goldman quickly found a theoretical interpretation both in dirty (Schmid and Schön 1975b,c) as well as in clean superconductors (Artemenko and Volkov 1975). The main features are: the collective mode is a high frequency, collisionless oscillation of the supercurrent which is compensated by a counterflow of the normal current. Since the frequency is high a hydrodynamic description is not possible and, in principle, the full energy and/or momentum dependence of the distribution function has to be taken into account. In certain limits, simple approximate descriptions can be developed (see e.g. Pethick and Smith 1979 or Kadin et al. 1980). In the following, we will indicate these approximate pictures. However, the main purpose of this article is to describe quantitatively the collective modes in superconductors on the basis of the full microscopic theory. Because of its experimental relevance we will consider the dirty limit in more detail.

The plan of the chapter is as follows. Section 2 contains a short review of various collective modes. We contrast charged and uncharged systems, the hydrodynamic and collisionless frequency range, superfluids and superconductors. Also we present a qualitative, simple picture of the Carlson Goldman collective mode. In section 3 we present the microscopic theory and evaluate it further in order to describe collective modes at general frequencies. The order parameter structure factor is calculated in section 4. Analytic expressions covering various limits near T_c are presented, and a comparison with experi-
ments is performed. In section 5, we derive further results from the microscopic theory, which are relevant for a complete understanding of the collective modes. We discuss the charge imbalance relaxation, consider neutral superconductors and low temperatures. We review the theoretical literature concerned with the Carlson-Goldman mode and other collective modes. We also discuss the physical manifestation of the collective modes. Section 6 contains some conclusions. In the Appendix we extend the calculation of the order parameter structure factor to cover a larger range of temperatures near the transition temperature.

21.2 2. Review of collective sound-like modes

21.2.1 2.1. Two-fluid hydrodynamics of helium II

Sound-like collective oscillations in superfluids were first investigated in the superfluid phase of helium (He II) in a range of parameters where a hydrodynamic description applies (Landau and Lifshitz 1959, Khalatnikov 1965): In this case, He II can be thought as consisting of two interpenetrating liquids: a superfluid which moves with zero viscosity, and a normal viscous fluid. Both liquids may move relative to one another without friction. The two components are characterized by their densities n_s and n_n, and their velocities v and v_n. The sum $n = n_\mathrm{s} + n_\mathrm{n}$

is the total (number) density and $j = n_{\rm s} v_{\rm s} + n_{\rm n} v_{\rm n}$ is the total (number) current density. The superfluid flow does not carry entropy, and it is a potential flow ($\operatorname{curl} v_{\rm s} = 0$). These properties and the local conservation laws form the basis of the hydrodynamic theory of He II.

From the hydrodynamic description one finds that in bulk He II three different types of propagating collective modes exist: first, second, and fourth sound. (The third sound is a surface wave.) First sound is the density wave familiar from normal systems. In the superfluid both components move in phase with the same velocity $v_{\rm n} = v$. (We ignore thermal expansion.) The phase velocity of this mode $c_1^2 = \hat{i}p/\hat{\partial}(mn)$ is nearly independent of the temperature. In the second sound mode the normal and the superfluid component move in opposite directions such that the total current vanishes and the density remains constant in space. However, the local entropy density is modulated and hence, the second sound is also called entropy wave. Its phase velocity is $c_2^2 = Ts^2 n_{\checkmark}/cn_{\rm n}$ (s is the entropy per unit mass, o the specific heat per unit mass). Fourth sound is found if there exists a strong damping mechanism for the motion of the normal component, e.g. if the He II is immersed in a medium with fine pores. In this case, only the superfluid component performs density oscillations. The phase velocity depends on the superfluid density

$$c_4^2 = \frac{n_n}{n} c_1^2 + \frac{n_n}{n} c_2^2,$$

where the second term in this expression is usually small.
An essential assumption in the derivation of the hydrodynamic equations is that the system is in local thermal equilibrium and that the state of the system can be described completely by the local values of the thermodynamic variables, the superfluid density and velocity. Thus, consistency requires that the frequencies of collective modes found from these equations are small compared to internal relaxation rates $\omega \ll 1/\tau_{\rm eq}$.

21.2.2 2.2. Hydrodynamic description of superfluid Fermi systems

In the hydrodynamic regime the superfluid Fermi systems, superfluid ^{3}He and superconductors, can also be described in terms of a two-fluid picture (Bardeen 1958). The superfluid component represents the Cooper pairs and the normal component the thermally excited quasiparticles. However, in certain respects, the Fermi systems differ significantly from He II.

The superfluid ^{3}He is an anisotropic superfluid with strong interatomic forces. A successful description is based on Landau's theory of Fermi liquids (Landau 1956, 1957, 1958, Leggett 1965, 1966, 1975). First. second, and fourth sound exist (Wölfle 1973), with phase velocities depending on the Landau parameters and the orientation. In addition, the structure of the order parameter leads to a rich variety of further collective modes.*

In superconductors, because of the long-range Coulomb interaction, no hydrodynamic sound-like collective oscillation exists.** First and fourth sound are associated with density oscillations, and hence their frequencies are shifted up to the plasma frequencies depending on the total electron density n or the superfluid density $n_{,}$, respectively. The velocity of the second sound mode remains unchanged. However, second sound is practically always overdamped (Bardeen 1958, Ginzburg 1961, Enz 1974). In a superfluid liquid, the damping is due to friction between the normal component and walls, and may be negligible under suitable conditions. In a superconductor, the quasiparticles forming the normal component always have a finite scattering rate $1, \tau_{cc}$ due to elastic electron-impurity and inelastic electron-phonon scattering. This leads to a damping of the second sound mode given by $\gamma = n_{\rm s}/n_{\rm n}\tau_{\rm sc}$. The collective mode can only be observed if γ is small compared to the resonant frequency $\gamma \ll \omega$. Furthermore, a hydrodynamic description is sufficient only if the frequencies and wave vectors are small*** $\omega \ll 1/\tau_{\rm eq}$ and $q \ll 1/\lambda_{\rm eq} = 1/v_{\rm F}\tau_{\rm eq}$ (Enz 1974).

Since the Fermi velocity is much larger than the velocity of second sound c_2, the second of these conditions is more restrictive. Hence, a window

condition for the frequency of second sound results, which can only be met at temperatures extremely close to the critical temperature T_c (in a range of order $10^{-5}T_c$).

21.2.3 2.3. Collisionless modes

In contrast to the hydrodynamic regime characterized by $\omega \ll 1/\tau_{eq}$, the regime $\omega \gg 1, \tau_{eq}$ is called collisionless. For any fixed frequency one enters this regime when the temperature is lowered, since then the inelastic scattering of quasiparticles and the establishment of local thermal equilibrium become increasingly slower. While in the hydrodynamic region the quasiparticle distribution is always of a thermal form (corresponding to the local values of the thermodynamic variables and the superfluid density and velocity), it may differ significantly from a thermal distribution in the collisionless region. It is therefore essential to take into account the detailed energy and momentum dependence of the distribution function of the quasiparticles.

A familiar example of a collisionless collective mode is zero sound, which exists at low temperatures in uncharged normal Fermi systems, provided that the quasiparticle interaction is repulsive (Landau 1957). The velocity of zero sound c_0 for weak coupling is $c_0 = c_F$, and it increases with coupling strength. The Fermi surface in this mode performs rather complex oscillations.

21.2.4 2.4. The Bogoliubov-Anderson mode

21.2.5 2.4.1. The problem of gauge invariance in superconductors

There has been an extensive discussion of collective oscillations in superconductors in the literature*, starting with the development of the microscopic theory by Bardeen, Cooper and Schrieffer (1957). The reason for this was that the original BCS description gave a correct account of the Meissner effect only when the response to a vector potential A was calculated in a gauge where div $A = 0$. It failed to do so in an arbitrary gauge. (The same problem appears in neutral superfluid Fermi systems where expressions for the current may be obtained which do not satisfy the continuity equation, while the reduction of the moment of inertia - the analogue to the Meissner effect is described correctly.) This insufficiency in the original BCS description arises from the fact that only quasiparticle excitations were included among the excited states. It was pointed out by Bardeen (1957) and demonstrated first in the work of Anderson (1958) that a vector potential with div $A \neq 0$ also couples to a collective density fluctuation mode. Inclusion of this mode in a consistent way leads to a gauge invariant and current conserving theory. This can be done

most systematically by working within a conserving approximation, a procedure eludicated by Baym and Kadanoff (1961) and Baym (1962) and applied to superconductors by Ambegaokar and Kadanoff (1961). An equivalent way of handling the problem was suggested by Nambu (1960) who made use of a generalized Ward identity.

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- A detailed discussion is given by Wölfle (1977, 1978).
  **A systematic discussion is given by Enz (1974).
  ***In section 3 we will express $1/\tau_{eq}$ in terms of microscopic parameters. Relaxation to a local equilibrium requires inelastic scattering processes, hence $\tau_{eq} \geqslant \tau_{sc}$.

12*For a review see Martin (1969).

21.2.6 2.4.2. Uncharged Fermi systems

In all treatments on this subject a low temperature collective mode was found. It can be described as a state in which the phase of the order parameter varies periodically in space and time. Thus, supercurrents are flowing, resulting in a spatial variation of the superfluid density. In an uncharged Fermi system, this mode has a sound-like dispersion relation. The phase velocity is formally equal to the phase velocity of hydrodynamic first sound c_1, although the frequency may be rather large and the mode is not a hydrodynamic mode. The reason is that, in a superconductor, the existence of the energy gap prevents transitions to excited states as long as the frequency is small $\omega < 2\Delta$. The system follows adiabatically the compression and expansion due to the sound wave. Thus, a local equilibrium distribution is maintained although the mechanism is completely different from the relaxation processes occurring in a hydrodynamic first sound situation. The mode was first discovered by Bogoliubov et al. (1959) and is called Bogoliubov-Anderson mode in the literature. It already exhausts the conductivity sum rule (Kadanoff and Martin 1961), thus one can conclude that it is only sound-like collective oscillation of the system at $T = 0$.

The investigations were extended to finite temperatures first by Kadanoff and Martin (1961) and generalized to include strong interactions i.e., the Fermi liquid effects, by Larkin and Migdal (1963) and by Leggett (1965, 1966). The result is that at zero temperature for arbitrary strength of the interaction, the Bogoliubov-Anderson mode has a phase velocity equal to that of first sound. At finite temperatures the existence of the mode depends on the strength of the interaction:
(i) For weak attraction the collective mode is only well defined at $T = 0$. As the temperature is raised, the velocity increases-never reaching ι_F - and the oscillation becomes rapidly damped, disappearing altogether when the average velocity of the quasiparticles becomes of order $v_F/\sqrt{3}$.
(ii) For weak repulsive interaction the mode exists at zero temperature and in the vicinity of the transition temperature. There it merges with the predictions of normal fluid kinetic theory, i.e. it becomes zero sound. The velocity depends strongly on the temperature.
(iii) For strong repulsive interactions the mode exists at all temperatures. The velocity depends only weakly on the temperature. (In this limit also the velocity of zero and first sound do not differ much.) This limit applies to ^{3}He. Indeed the predicted small change in velocity of sound was found experimentally in superfluid ^{3}He (Paulson et al. 1973).

21.2.7 2.4.3. Superconductors

In charged superconductors the frequency of the Bogoliubov Anderson mode is shifted to the plasma frequency, due to the long range Coulomb forces. Since the plasma frequency is barely modified by the superfluid transition we cannot expect experimental evidence of this mode. The plasmon mode, however, is still necessary to maintain gauge invariance. The nature of the BogoliubovAnderson mode in superfluids can also be understood on the basis of Goldstone's theorem. It states that if an infinitesimal uniform change of the ground state of a system produces a new ground state degenerate with it, then an infinitesimal change in the state varying slowly in space will generate an eigenstate of low energy. This means the system must have an excitation branch with no energy gap. In the superfluid Fermi systems the degeneracy is connected with the phase of the order parameter of the condensed phase. In the collective mode the phase varies periodically in space. In a neutral superfluid, the Bogoliubov-Anderson sound mode is the gapless collective excitation required by the Goldstone theorem. In a charged super-conductor, however, the singular nature of the long-range Coulomb forces renders the theorem inconclusive.

21.2.8 2.5 The Carlson-Goldman mode

The first experimental demonstration of the existence of a collective mode with a phonon-like dispersion relation in a superconductor was given by Carlson and Goldman (1973, 1975, 1976). They investigated the frequency and wavevector dependence of the order-parameter structure factor

$$S(q,\omega) = \langle \Delta^*(r,t)\Delta(0,0)\rangle_{q,(\omega)} \tag{2.1}$$

of a short mean free path superconducting aluminium film using a method suggested by Ferrell (1969) and Scalapino (1970).

Above T_c, only one peak in the structure factor centered at $\omega = 0$ was seen (Anderson et al. 1972). Below T_c, an additional symmetric pair of peaks at finite frequencies shows up. It results from a propagating order parameter collective mode. The resonant frequency is almost proportional to the wave vector. It also depends on the temperature. This peak was identified to originate from fluctuations in the phase in the order parameter, whereas the central peak results from fluctuations in the magnitude of the order parameter. When the temperature is lowered, the propagating mode is increasingly damped and the peak is leveled off. Furthermore, there appears a structure at $\omega = 24$.

The experimental values of the structure factor above $\mathrm{T_c}$ are in good agreement with the results from the diffusive time-dependent GinzburgLandau equation (Schmid 1966, Abrahams and Tsuneto 1966). But it is not possible to explain the existence of a propagating mode below T_c by this
equation. Nor is a hydrodynamic description in terms of a two-fluid model sufficient, since the resonant frequency is not small in comparison to internal relaxation rates. We will find below that the mode is characterized by a charge imbalance between the quasiparticles and the condensed phase. Hence, the characteristic relaxation rate is the charge imbalance relaxation rate $1/\tau_{Q*}$ (Clarke 1972, Tinkham 1972). In fact, the mode is a collisionless mode, in the sense that the frequency is much larger than the inelastic electron-phonon collision rate, $\omega \gg 1/\tau_{\mathrm{E}}$, but the elastic electron impurity scattering rate is even larger, $\omega \ll 1/\tau_{\mathrm{imp}}$.

One can find a rather simple physical description of the collective mode in the limit*

$$\Gamma, Dq^2 \ll |\omega| \ll \Delta \ll T_c, \tag{2.2}$$

where Γ measures the pair breaking (which results from inelastic electronphonon scattering, scattering at paramagnetic impurities, magnetic field, supercurrent, ...) D is the diffusion coefficient, and Δ is the energy gap. In this case, one finds from the microscopic theory that the chemical potential of the Cooper pairs Φ obeys the relation

$$\frac{\pi\Delta}{4T}\dot{\Phi} + \frac{1}{2eN_0}\,\mathrm{div}\, j_{\mathrm{s}} = 0 \tag{2.3}$$

We will comment further on this relation in section 5. At present, we merely mention that it can be interpreted as a "continuity equation for the superfluid" (Pethick and Smith 1979) with coefficients determined by the microscopic theory.

Space charges are minimized in this collective mode by a counterflow of supercurrent and normal current. Hence, we are led to consider a situation where

$$\mathrm{div}\,(j_{\mathrm{s}} + j_{\mathrm{n}}) \approx 0. \tag{2.4}$$

We express the supercurrent as

$$j_{\mathrm{s}} = en_{\mathrm{s}}v_{\mathrm{s}}, \tag{2.5}$$

where the superfluid velocity obeys the acceleration equation

$$m\dot{v}_{\mathrm{s}} = -\operatorname{grad}\Phi + eE. \tag{2.6}$$

Since the frequencies considered are still small in comparison to the elastic electron impurity scattering rate, $\omega \ll 1/\tau_{\mathrm{imp}}$, the normal current is proportional to the driving force. Near T_c, in a rough approximation, it is the same as in the
normal state

$$j_{\mathrm{n}} = \sigma E \tag{2.7}$$

These equations allow for longitudinal collective oscillations $\propto \mathrm{e}^{-\mathrm{i}\left(\omega_{\varphi^1} - q\cdot r\right)}$ with dispersion relation

$$\omega_4 = -\frac{1}{2}\mathrm{i}\gamma \pm \sqrt{c_{\mathrm{CG}}^2 q^2 - \gamma^2/4} \tag{2.8}$$

The phase velocity and the damping of this propagating mode are given by

$$c_{\mathrm{CG}}^2 = \frac{n_{\mathrm{s}}}{n}\frac{4T}{\pi\Delta}\frac{v_{\mathrm{F}}^2}{3}, \quad \because= \frac{n_{\mathrm{s}}}{n}\frac{1}{\tau_{\mathrm{imp}}} \tag{2.9}$$

Although normal current and supercurrent are flowing in opposite directions, this mode is not a second sound mode. It is neither a hydrodynamic mode, nor does it involve temperature or entropy fluctuations. It is primarily a longitudinal oscillation of the condensate, the restoring force being given by the gradient of the Cooper pair chemical potential. This is most apparent if we consider a hypothetical uncharged superconductor where a similar but undamped mode is found with $\omega_q = \pm c_{\mathrm{CG}} q$. There the normal current j_{n} remains zero and hence, the mode involves local density fluctuations. In a charged superconductor, the supercurrent oscillation produces an electric field which drives the normal current to perform a counteroscillation. It turns out that the resulting total space charges are small and can be ignored in the continuity equation (2.4). However, the normal current is subject to Ohmic losses, which is the origin of the damping γ. Close to T_{c} where the superfluid density n_{s} is small, the superfluid oscillations can be compensated by small oscillations of the normal current, and hence the damping is small. At lower temperatures the superfluid density and the damping increase. (In order to avoid misinterpretations, we stress the point that the superfluid density n_{s} is merely a coefficient relating superfluid velocity and supercurrent. It depends on the temperature and impurity concentration. It is not related to the fluctuations of the local charge density.)

We note that the phase velocity of the collective mode is larger than that of hydrodynamic fourth sound by a factor $4T/\pi\Lambda$, but is still small in comparison to the Fermi velocity. Near $T_{\mathrm{c}}, n_{\mathrm{s}}$ is given by

$$\frac{n_{\mathrm{s}}}{n} = \frac{7\zeta(3)}{4\pi^2}\frac{\Delta^2}{T^2}\chi\left(\frac{1}{2\pi T_{\mathrm{c}}\tau_{\mathrm{imp}}}\right) \tag{2.10}$$

where Gor'kov's function expresses the dependence on the impurity concentration. It is defined as

$$\chi(x) = \frac{8}{7\zeta(3)}\sum_{v=0}^{\infty}(2v+1)^{-2}(2v+1+x)^{-1} \tag{2.11}$$

In clean superconductors $T_{\mathrm{c}}\tau_{\mathrm{imp}} \gg 1$, when $\chi \approx 1$, we recover the results of Artemenko and Volkov (1975). In the simple description given above, we neglected quasiparticle diffusion

[12*] We use units where $\hbar = k_{\mathrm{B}} = c = 1$.

processes which lead to an additional damping of order Dq^2. Thus, in clean superconductors, the collective mode only exists if the conditions

$$\frac{\Delta^2}{T^2}\frac{1}{\tau_{\rm imp}}, \Gamma \ll |\omega_q| \ll \frac{\Delta}{T}\frac{1}{\tau_{\rm imp}} \tag{2.12}$$

are satisfied, where the upper frequency limit guarantees that $Dq^2 \ll \omega_q$.
In dirty superconductors, where $T_{\rm c}\tau_{\rm imp} \ll 1$, the superfluid density reduces to $n_{\rm s}/n = \pi\Delta^2/2T\tau_{\rm imp}$ and the phase velocity is

$$c_{\rm CG} = \sqrt{2\Delta D}. \tag{2.13}$$

This result (Schmid and Schön 1975b) explains well the experimental findings of Carlson and Goldman (1975) as far as the absolute value and temperature dependence of the phase velocity is concerned. The result (2.9) also gives a qualitative account of the damping, which increases with decreasing temperature. Because of the damping, the collective mode in dirty samples propagates only if

$$\frac{\Delta^2}{T}, \Gamma \ll |\omega| \ll \Delta. \tag{2.14}$$

The virtue of the presentation given above is to provide a physical picture of the propagating mode. However, we still have to justify the relation (2.3), and we have to investigate in more detail the effect of various damping mechanisms, in particular the effect of quasiparticle diffusion. We also will derive in the following the complete order-parameter structure factor of a superconductor near T_c for general frequencies and wave vectors including also fluctuations in the magnitude of the order parameter.

21.3 3. The microscopic theory

21.3.1 3.1. Some remarks

Nonequilibrium phenomena in superconductors can be described on various levels. Under certain conditions a rather simple kinetic theory based on the BCS theory is sufficient. It consists of a Boltzmann equation for the distribution function of the excitations (Aronov and Gurevich 1974) which includes collisions of the excitations, e.g. with impurities and with phonons (Bardeen et al. 1959). Furthermore, the excitations are coupled to the condensate. In order for this description to be valid, variations in space and time have to be sufficiently slow $qv_{\rm F} \ll \Delta$ and $\omega \ll A$. In addition, all scattering processes have to be weak.

In connection with the collective mode larger frequencies and wave-vectors are of interest. This means that we need a more general description, e.g. in terms of microscopic Green functions. We also need the general theory if the interactions with impurities, phonons, or other pair breaking effects are not weak. On the other hand, the characteristic wavelengths of interest are still much larger than the Fermi wavelength. This fact permits significant simplifications. A theory which systematically incorporates this limit from the beginning was developed by Eilenberger (1968) and independently by Larkin and Ovchinnikov (1968). They found that it is sufficient to consider the quasiclassical Green function which is obtained from the full Green function $G(p, E)$ by integration over the magnitude of the momentum. The concept was extended by Eliashberg (1971) for retarded interactions such that inelastic electron-phonon collisions can be included. Later Schmid and Schön (1975a) introduced in this context the quasiparticle distribution. A very compact formulation of the problem based on the Keldysh (1964) technique was given by Larkin and Ovchinnikov (1977, 1981), Hu (1980) and Schmid (1981).

In the following subsection, we will summarize the quasiclassical theory and we will evaluate the equation of motion further in order to describe the collective mode and to calculate the order-parameter structure factor. This means we allow for sufficiently general frequencies and wave vectors. On the other hand, it is sufficient to linearize the theory. Since the propagating collective mode is a collisionless mode, inelastic electron-phonon collisions are relatively weak. They provide, however, an important relaxation and damping mechanism, and we will point out their effect. We will present the explicit calculation only in the dirty limit ($T_{\mathrm{c}}\tau_{\mathrm{imp}} \ll 1$), since most experiments are performed with samples having short mean free paths. Explicit expressions are given for temperatures close to T_c.

21.3.2 3.2. Dirty limit theory

In the dirty limit ($T_{\mathrm{c}}\tau_{\mathrm{imp}} \ll 1$) the quantities of interest can be expressed by the part of the quasiclassical Green function which is isotropic in momentum space (Usadel 1970). This function

$$\overline{\hat{g}}(E;r,t) = \frac{\mathrm{i}}{\pi\,\overline{\mathrm{N}}_0}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\hat{G}(p,E;r,t)$$

obeys the equation of motion*

$$\left[\overline{\hat{\tau}}_3 E - \overline{\hat{1}}_{e\varphi} - \overline{\hat{\Sigma}}_{\mathrm{ph}} - \overline{\hat{\Sigma}}_{\mathrm{sf}} \doteqdot \overline{\hat{g}}\right] = \frac{1}{2}\mathrm{i}D\left[\hat{\partial}_\rho[\hat{g};[\hat{\partial};\overline{\hat{g}}]]\right], \tag{3.1}$$

where $[,]isacommutator.Thefunction\hat{g}$ is normalized,

$$\dot{\hat{g}} \circ \overline{\hat{g}} = \hat{1} \tag{3.2}$$

Apart from the retarded and the advanced Green function, in nonequilibrium situations also a third function $\hat{g}^{\mathrm{K}}$, introduced by Keldysh (1964), is needed, which can be combined with $\hat{g}^{\mathrm{R}}$ and $\hat{g}^{\mathrm{A}}$ to form a matrix in "Keldysh space"

$$\overline{\hat{g}} = \begin{pmatrix} \hat{g}^{\mathrm{R}} & \hat{g}^{\mathrm{K}} \\ 0 & \hat{g}^{\mathrm{A}} \end{pmatrix}. \tag{3.3}$$

The caret indicates that the quantities are still matrices in particle-hole (Nambu) space.
The dot product is defined as

$$A \circ B = \exp\left[\frac{1}{2}\mathrm{i}\left(\partial_E^A\partial_t^B - \partial_t^A\partial_E^B\right)\right]A(E,t)B(E,t), \tag{3.4}$$

where ∂^A acts on A only, ...** In two cases it may be expressed in simple terms: (i) If the external frequencies are small, $\omega \ll \Delta, T$ we can expand the exponential and restrict ourselves to the first few derivatives (see e.g. Schmid 1981). (ii) If we proceed in a linearized theory one of the quantities A or B is a static function and we may use a Fourier transformation to our advantage, e.g.

$$[A(E,t) \circ B(E)]_\omega = A(E,\omega)B(E-\omega/2). \tag{3.5}$$

The gauge-invariant spatial derivative is

$$\overline{\hat{\delta}} = \overline{\hat{1}}\frac{\partial}{\partial r} - i\overline{\hat{\tau}}_3 eA(r,t) \tag{3.6}$$

[12]*For further details see e.g. Schmid (1981). We use the notation introduced in this reference.

and φ and A are the electromagnetic potentials. The scattering at magnetic impurities leads to a self energy

$$\hat{\Sigma}_{\mathrm{ff}} = -\tfrac{\mathrm{i}}{2\tau_{\mathrm{s}}}\overline{\hat{\tau}}_3\overline{\hat{g}\hat{\tau}}_3. \tag{3.7}$$

The electron phonon interaction is described by $\hat{\Sigma}_{ph}$. It plays a twofold role: inelastic collisions are contained in the components $\hat{\Sigma}^{\mathrm{K}}_{\mathrm{ph}}$ and $\hat{\Sigma}^{\mathrm{R}}_{\mathrm{ph}} - \hat{\Sigma}^{\mathrm{A}}_{\mathrm{ph}}$, pair correlations are contained in $\hat{\Sigma}^{R}_{ph} + \hat{\Sigma}^{A}_{ph}$. If we ignore renormalization effects we have

$$\left(\hat{\Sigma}^{\mathrm{R}}_{\mathrm{ph}} + \hat{\Sigma}^{\mathrm{A}}_{\mathrm{ph}}\right)/2 = -\mathrm{i}\Delta\hat{\tau}_\theta; \quad \hat{\tau}_\theta = \mathrm{e}^{-\mathrm{i}\theta\hat{\tau}_3}\hat{\tau}_1, \tag{3.8}$$

where Δ and θ are the magnitude and the phase of the complex order parameter $\Delta = \Delta\mathrm{e}^{-\mathrm{i}\theta}$. The order parameter satisfies the self-consistency relation

$$\Delta(r,t) = \frac{1}{4\mathrm{i}}\lambda\int \mathrm{d}E\left\{\hat{g}^{\mathrm{K}}(E;r,t)\right\}_{12} \tag{3.9}$$

where λ is the electron-phonon coupling constant.
The Keldysh component of the Green's function can be related to a distribution function as follows

$$\hat{g}^{\mathrm{K}} = \hat{g}^{\mathrm{R}}\circ\hat{h} - \hat{h}\circ\hat{g}^{\mathrm{A}}, \quad \hat{h} = \left(1 - 2f^{(\mathrm{L})}\right)\hat{1} - 2f^{(\mathrm{T})}\hat{\tau}_3, \tag{3.10}$$

where in equilibrium $f^{(L)}(E;r,t)$ is equal to the Fermi distribution $f_0(E)$ and $f^{(T)}\,(E;r,t) = 0$. The superscripts refer to two degrees of freedom, the longitudinal and the transverse mode. This terminology refers to the fact that the two modes are associated with changes in the magnitude and the phase of the complex order parameter, respectively. Another distinction between the two modes is that they involve changes either in the energy or in the charge. Thus, they are also called energy mode (L) and charge mode (T), respectively.

The charge density and the current density, finally, are given by

$$\rho(\boldsymbol{r},t) = -2eN_0\left\{\frac{1}{8}\int \mathrm{d}E\,\mathrm{Tr}\left[\hat{g}^{\mathrm{K}}(E;\boldsymbol{r},t)\right] + e\varphi\right\} \tag{3.11}$$

and, omitting the arguments ($E;r,t$) in $\hat{h}$ and $\hat{g}$,

$$j(r,t) = \frac{\sigma}{8e}\int \mathrm{d}E\,\mathrm{Tr}\,\tau_3\left\{[\hat{\partial},\hat{h}] - \hat{g}^{\mathrm{R}}\circ[\hat{\partial},\hat{h}]\circ\hat{g}^{\mathrm{A}} + \frac{1}{2}\left[\hat{g}^{\mathrm{R}};\left[\hat{\partial},\hat{g}^{\mathrm{R}}\right]\right]\circ\hat{h} - \frac{1}{2}\hat{h}\circ\left[\hat{g}^{\mathrm{A}};\left[\hat{\partial},\hat{g}^{\mathrm{A}}\right]\right]\right\} \tag{3.12}$$

Both satisfy the continuity equation $\dot{\rho} + \mathrm{div}\,j = 0$. Furthermore, since in metals the space charges are suppressed almost entirely by Coulomb forces, we have with good accuracy $\rho = 0$ and $\mathrm{div}\,j = 0$. The density of states per spin at the Fermi surface N_0 and the normal state conductivity are related by $\sigma = 2e^2N_0D$.

The equilibrium Green function can be chosen to contain only a $\hat{\tau}_3$ and $\hat{\tau}_\theta$ component

$$\hat{g}_0^{R(A)}(E;r) = \alpha^{R(A)}(E;r)\hat{\tau}_3 + \beta^{R(A)}(E;r)\hat{\tau}_0. \tag{3.13}$$

[12]*In general, spatial variations require an equivalent operation (see Eckern and Schmid, 1981). However, since the wavelengths considered are large in comparison to the Fermi wavelength $1/p_{\mathrm{F}}$, we can neglect this generalization for most purposes.

The retarded (advanced) components satisfy

$$\alpha^2 + \beta^2 = 1$$
$$-\mathrm{i}\left(E \pm \frac{\mathrm{i}}{2\tau_\mathrm{E}}\right)\beta + \Delta\alpha = \alpha\beta\Gamma_0 - \frac{D}{2}\left[\alpha\left(\frac{\partial}{\partial \boldsymbol{r}}\right)^2\beta - \beta\left(\frac{\partial}{\partial \boldsymbol{r}}\right)^2\alpha\right]. \tag{3.14}$$

Here we have omitted the superscript R(A). The inelastic electron phonon collision rate enters with a plus (minus) sign in the equation for the retarded (advanced) quantities. Further pair breaking effects are represented by Γ_0. These are for example, scattering at magnetic impurities, supercurrent flow, or a magnetic field applied parallel to a thin film of thickness d. In these cases one has $\Gamma_0 = 1/\tau_\mathrm{s}, 2m^2Dv_\mathrm{s}^2, \frac{1}{6}e^2DdH^2$, respectively. We can relate α and β to spectral quantities. In particular, the reduced density of states in superconductors is

$$N_1(E) = \frac{1}{2}\left[\alpha^\mathrm{R}(E) - \alpha^\mathrm{A}(E)\right] \tag{3.15}$$

Analytic solutions for α and β can be obtained in different limits. For example, in a homogeneous superconductor with no pair breaking effects other than inelastic electron-phonon scattering, we find

$$\alpha^{\mathrm{R(\ A)}}(E) = \left(E \pm \mathrm{i}/2\tau_\mathrm{E}\right)/\sqrt{\left(E \pm \mathrm{i}/2\tau_\mathrm{E}\right)^2 - \Delta^2}$$
$$\beta^{\mathrm{R(\ A)}}(E) = \mathrm{i}\Delta/\sqrt{\left(E \pm \mathrm{i}/2\tau_\mathrm{E}\right)^2 - \Delta^2} \tag{3.16}$$

where $\mathrm{Im}\sqrt{\ldots} > 0$. In certain cases, it is also sufficient to know the large energy expansions for $E \gg \Delta$

$$\alpha^{\mathrm{R(\ A)}}(E) = \pm\left[1 + \frac{\Delta^2}{2(E \pm \mathrm{i}\Gamma)^2}\right]$$
$$\beta^{\mathrm{R(\ A)}}(E) = \pm\frac{\mathrm{i}\Delta}{E \pm \mathrm{i}\Gamma}\left[1 + \frac{E\Delta^2}{2(E \pm \mathrm{i}\Gamma)^3}\right] \tag{3.17}$$

Here Γ contains all pair breaking mechanisms (i.e. it also includes the effect of electron-phonon collisions); e.g.

$$\Gamma = \frac{1}{\tau_\mathrm{s}} + \frac{1}{2\tau_\mathrm{E}} + \frac{1}{2}D\left(4m^2v_\mathrm{s}^2 - \frac{1}{\Delta}\frac{\partial^2}{\partial \boldsymbol{r}^2}\Delta\right) \tag{3.18}$$

The expansion (3.17) is also valid for strong pair breaking $\Gamma \gg \Delta$. The complete theory is gauge invariant. We can make this explicit by using only the gaugeinvariant quantities

$$\Phi = \frac{1}{2}\hat{\theta} - e\varphi, \quad v_\mathrm{s} = -\frac{1}{2m}\left(\frac{\partial}{\partial r}\theta + 2eA\right) \tag{3.19}$$

Since $\dot{\theta}/2$ is the Cooper pair electrochemical potential, Φ has the meaning of a pair chemical potential. Furthermore, v_s is the superfluid velocity. On the other hand, the gauge where Δ is real ($\theta = 0$) has particular virtues. If one resorts to an expansion of the dot product in terms of time derivatives, this gauge ensures the fastest convergence. In the linearized theory, the real Δ gauge is still a convenient gauge.

21.3.3 3.3. Linearized theory

If the deviations from equilibrium are small, we may proceed in a linearized theory. We split according to

$$f^{\mathrm{L}} = f_0 + f_1^{\mathrm{L}}, \quad f^{\mathrm{T}} = f_1^{\mathrm{T}}$$

$$\hat{g}^{\mathrm{R}} = \hat{g}_0^{\mathrm{R}} + \hat{g}_1^{\mathrm{R}}, \text{ (similarly for A, K)}$$
$$\Delta = \Delta_0 + \Delta_1, v_{\mathrm{s}} = v_{\mathrm{s0}} + v_{\mathrm{s1}}$$

Also Φ is a small quantity. For convenience, we will omit the subscript 0 at $A_0 \equiv \Delta$ in the following. After Fourier transformation with respect to time in order to express the dot product in simple terms we obtain from the K component of eq. (3.1) for general frequencies two (coupled) linearized Boltzmann equations for the two distribution functions $f_1^{\mathrm{L}}(E; r, \omega)$ and $f_1^{\mathrm{T}}(E; \boldsymbol{r}, \omega)$. (Actually, we consider the following combinations of the components of (3.1) : $\{\ldots\}^{\mathrm{K}} - \{\ldots\}^{\mathrm{R}} \circ h + h \circ \{\ldots\}^{\mathrm{A}}$.)

$$\begin{aligned}
&N_1 \dot{f}_1^{\mathrm{L}} - D\frac{\partial}{\partial r}\left[M^{\mathrm{L}}\left(\frac{\partial}{\partial r} f_1^{\mathrm{L}}\right)\right] - 4Dmv_{\mathrm{s0}}\left(\frac{\partial}{\partial r} f_1^{\mathrm{T}}\right) N_2 R_2 \\
&\quad + \dot{d}_1 R_2 \frac{\partial f_0}{\partial E} - 4Dm^2 v_{\mathrm{s0}} \cdot v_{\mathrm{s1}} N_2 R_2 \frac{\partial f_0}{\partial E} - K^{\mathrm{L}}\left(f_1^{\mathrm{L}}\right) = 0 \\
&N_1 \dot{f}_1^{\mathrm{T}} + 2\Delta N_2 f_1^{\mathrm{I}} - D\frac{\partial}{\partial r}\left[M^{\mathrm{T}}\left(\frac{\partial}{\partial r} f_1^{\mathrm{T}} + m\dot{v}_{\mathrm{s1}} \frac{\partial f_0}{\partial E}\right)\right] \\
&\quad - 4Dmv_{\mathrm{s0}}\left(\frac{\partial}{\partial r} f_1^{\mathrm{L}}\right) N_2 R_2 - N_1 \dot{\Phi} \frac{\partial f_0}{\partial E} - K^{\mathrm{T}}\left(f_1^{\mathrm{T}}\right) = 0
\end{aligned} \tag{3.20}$$

Here, we have written for short $f_1^{\mathrm{L}} = -i\omega f_1^{\mathrm{L}}(E; r, \omega), \ldots$, and $N_1 = N_1(E; \omega), \ldots$. We have introduced generalized spectral quantities, depending on frequency shifted equilibrium Green's functions, by

$$\begin{aligned}
N_1(\mathrm{E}; \omega) &= \frac{1}{2}\left[\alpha^{\mathrm{R}}\left(E + \frac{1}{2}\omega\right) - \alpha^{\mathrm{R}}\left(E - \frac{1}{2}\omega\right)\right], \\
N_2(E; \omega) &= \frac{1}{2}\left[\beta^{\mathrm{R}}\left(E + \frac{1}{2}\omega\right) + \beta^{\mathrm{A}}\left(E - \frac{1}{2}\omega\right)\right], \\
R_2(E; \omega) &= (1/2\mathrm{i})\left[\beta^{\mathrm{R}}\left(E + \frac{1}{2}\omega\right) - \beta^{\mathrm{A}}\left(E - \frac{1}{2}\omega\right)\right],
\end{aligned} \tag{3.21}$$

and

$$M^{\mathrm{T}}(E; \omega) = N_1^2(E; \omega) + N_2^2(E; \omega), \quad M^{\mathrm{L}}(E; \omega) = N_1^2(E; \omega) - R_2^2(E; \omega)$$

In the following, we will always write the arguments of the spectral quantities to remind ourselves of this frequency shift, whereas we will drop the arguments of the distribution functions and other quantities where no confusion can occur.

The Boltzmann equations are of the form (3.20) for frequencies satisfying $\omega \ll T$ since we approximated the expression $\frac{1}{2}\mathrm{i}\left\{\tanh\left[\left(E + \frac{1}{2}\omega\right)/2T\right] - \tanh\left[\left(E - \frac{1}{2}\omega\right)/2T\right]\right.$, by $i\omega\left(\hat{c}f_0/\hat{c}E\right)$. The electron-phonon collision integrals are denoted by K^T and K^L. We do not give here their detailed form. Rather, we only mention that they consist of a "scattering out" term $-(1/\tau_{\mathrm{E}})$ $\times N_1(E; \omega) f_1^{\top(L)}$ proportional to the inelastic electron-phonon collision rate;* and a "scattering

in" term involving integrals of f_1. An important property is that K^{T} conserves the charge $\int \mathrm{d}EK^{\mathrm{T}}\left(f_1^{\mathrm{T}}\right)=0$.

Simple expression for the current can be worked out in limiting cases. Near T_{c}, to leading order in Δ/T we find

$$j_0=\frac{\sigma}{e}\frac{\pi\Delta^2}{2T}mv_{\mathrm{s0}}\left[\frac{2}{\pi^2}\psi'\left(\frac{1}{2}+\rho\right)\right] \tag{3.22}$$

where $\rho=\Gamma/2\pi T$ and ψ' is the derivative of the digamma function [e.g. $\psi'\left(\frac{1}{2}\right)=\pi^2/2$]. The linear terms in leading order in Δ/T and $\omega/T, Dq^2/T$ are

$$\begin{aligned} j_1(r,\omega)=&-\frac{\sigma}{e}\int \mathrm{d}E\left\{M^{\mathrm{T}}(E;\omega)\left[\left(\frac{\partial}{\partial r}f_1^{\mathrm{T}}\right)+m\dot{v}_{\mathrm{i1}}\frac{\partial f_0}{\hat{c}E}\right]\right.\\ &\left.+4mv_{\mathrm{s0}}\ \mathrm{N_2(E;\omega)R_2(E;\omega)}f_1^{\mathrm{L}}\right\}\\ &+\mathrm{Lin}\left\{\frac{\sigma}{e}\frac{\pi\Delta^2}{2T}mv_{\mathrm{s}}\left[\frac{2}{\pi^2}\psi'\left(\frac{1}{2}+\rho\right)\right]\right\} \end{aligned} \tag{3.23}$$

By Lin$\{\ldots\}$ we denote the linear part of $\{\ldots\}$, linearized with respect to variations of Δ and v_s. (Also ρ depends on v_s.) In the evaluation of j_0 and of the last term in j_1 we encountered integrals involving the factor $1-2f_0(E)$ in combination with products of retarded quantities only (or advanced only). This fact allowed us to use the expansion (3.17) of α and β, if Δ/T and ω/T are small.

The charge density in the same limit becomes

$$\rho=2eN_0\left\{\int \mathrm{d}EN_1(E;\omega)f_1^{\mathrm{T}}+\Phi\right\} \tag{3.24}$$

The magnitude of the order parameter obeys, near T_c, a generalized Ginzburg-Landau equation. In nonequilibrium situations it depends strongly on the nonequilibrium distribution function. We find

$$\frac{\pi}{8\ \mathrm{T}}\gamma\dot{A}_1-\Psi A=-\mathrm{Lin}\left\{\left[\alpha+\beta\frac{\Delta^2}{T^2}+\gamma\xi^2(0)\left(4m^2v_{\mathrm{s}}^2-\frac{\partial^2}{\partial r^2}\right)\right]A\right\} \tag{3.25}$$

with coefficients depending on the pair breaking strength. For $\Gamma\ll T$ one has $\gamma=1,\alpha=(T-T_{\mathrm{c}})/T_{\mathrm{c}}$, (notice however, that T_{c} is depressed by pair breaking: in the limit considered $T_{\mathrm{c}}=T_{\mathrm{c0}}-\frac{1}{4}\pi\Gamma\big)$, $\beta=7\zeta(3)/8\pi^2$ and the coherence length in the dirty limit is related to D by $\zeta^2(0)=\pi D/8T_c$. For the general strength of the

pair breaking one has (Entin-Wohlman and Orbach 1978)

$$\alpha=\frac{T-T_{\mathrm{c}}}{T_{\mathrm{c}}}\left[1-\rho\psi'\right],\quad \beta=\frac{\psi''+(\rho/3)\psi'''}{(4\pi)^2},$$

and $\gamma=\frac{1}{2}\pi^2\psi'$, where the argument of all $\psi^{(n)}$ is $\frac{1}{2}+\rho$ and $\rho=\Gamma/2\pi T$ contains all pair breaking effect except those written explicitly in (3.25). The distribution function enters in an important way in (3.25) in the gap control function

$$\Psi=-\frac{1}{\Delta}\int \mathrm{d}ER_2(E;\omega)f_1^{\mathrm{L}} \tag{3.26}$$

[12]*Strictly, $1/\tau_{\mathrm{E}}$ depends on the energy and on the frequency ω. However, near T_{c} we may replace it by its normal state value at T_{c} for $E=\omega=0$.

21.4 4. The order-parameter structure factor

21.4.1 4.1. The formal solution

The order parameter structure factor (2.1) in a superconductor is composed of two contributions reflecting fluctuations in the magnitude and in the phase of the order parameter,

$$S(\mathrm{q},\omega) = S^{\mathrm{L}} + S^{\mathrm{T}}. \tag{4.1}$$

It is connected to a pair field susceptibility by the fluctuation dissipation theorem

$$S(q,\omega) = \frac{2T}{\omega}\,\mathrm{Im}[\chi(q,\omega)]. \tag{4.2}$$

We, therefore, calculate the response of the system to an applied perturbation. The contribution from the magnitude fluctuations is obtained from the linearized Ginzburg Landau equation (3.25) where on the right-hand side an additional normalized drive term $1/2N_0$ is added. Then $\chi^{\mathrm{L}}(q,\omega) = \Lambda_1(q,\omega)$.

Similarly, we can proceed using a generalized Ginzburg - Landau equation for the phase, which yields $\chi^{\top}(q,\omega) = -A()_1(q,\omega)$ (Schmid and Schön 1975c). In the real Δ gauge, we correspondingly calculate the response of v_s to a source term, which we add in the continuity equation

$$\dot{\rho} + \mathrm{div}\, j = -2e\Delta. \tag{4.3}$$

Then $\chi^{\mathrm{T}}(q,\omega) = -\mathrm{i}(2m\Delta/q)v_{\mathrm{s1}}(q,\omega)$ for longitudinal oscillations of v_{s1}.
If we insert the expressions for ρ and j_1, (3.24) and (3.23), and make use of the Boltzmann equation (3.10) we find that the continuity equation to leading order in Δ/T takes the form

$$2\Delta \int \mathrm{d}E N_2(E;\omega) f_1^{\mathrm{T}}(E;q,\omega) + \mathrm{i}D\,\mathrm{Lin}\left\{mq\cdot v_{\mathrm{s}}\frac{\pi\Delta^2}{2T}\left[\frac{2}{\pi^2}\psi'\left(\frac{1}{2}+\rho\right)\right]\right\} = \frac{\Delta}{N_0}. \tag{4.4}$$

Furthermore, in Maxwell's equation $\mathrm{div}\,\boldsymbol{E} = 4\pi\rho$ we can neglect the left-hand side as long as the characteristic wavelengths are small in comparison to the Thomas-Fermi screening length $\lambda_{\mathrm{TF}} = (8\pi e^2 N_0)^{-1/2} \approx 10^{-8}$ cm. This means we have approximate charge neutrality, i.e.

$$\int \mathrm{d}E N_1(E;\omega) f_1^{\mathrm{T}} + \Phi = 0 \tag{4.5}$$

Finally, f_1^{L} and f_1^{T} are found as solutions of the Boltzmann equations (3.20).
We consider first situations where no or only a small static supercurrent is flowing in the superconductor, i.e. $v_{\mathrm{s0}} \approx 0$. In this case, the (L)- and (T)-modes decouple. For transparency, we also restrict ourselves in the following to the important case of not too strong pair breaking, i.e. Γ/T is small. We make no assumption on Γ/Δ, and the generalization to strong pair breaking $\Gamma \gtrsim T$ is straightforward (see Entin-Wohlman and Orbach 1978). We are mainly interested in high frequency phenomena where $\omega \gg 1/\tau_{\mathrm{E}}$. In this context we can neglect the electron-phonon collisions, which in turn allows a straightforward (exact) solution of the Boltzmann equation for f_1^{L} and f_1^{T}. On the other hand, near T_{c} we can easily generalize the result to include the effect of electronphonon scattering in an approximate way. To leading order in A/T the collision operator K^T can be replaced by a reduced operator

$$K_{\mathrm{red}}^{\mathrm{T}} = -\frac{1}{\tau_{\mathrm{E}}} N_1(E;\omega)\left[f_1^{\mathrm{T}}(E) + \frac{\partial f_0}{\partial E}\int \mathrm{d}E' N_1(E';\omega)\, f_1^{\mathrm{T}}(E')\right] \tag{4.6}$$

which, like the exact operator, conserves the charge $\int \mathrm{d}E K^{\mathrm{T}}_{\mathrm{red}} = 0$. Also, we can replace K^{L} by the scattering-out term only,

$$K^{\mathrm{L}} \approx -\left(1/\tau_{\mathrm{E}}\right) N_1(E;\omega) f_1^{\mathrm{L}}$$

With these approximations and use of (4.5) the solutions of the Boltzmann equations are

$$\begin{aligned} f_1^{\mathrm{L}} =& \frac{1}{Q^{\mathrm{L}}} \frac{1}{4T\cosh^2(E/2T)} (-\mathrm{i}\omega)\Delta_1 R_2(E;\omega) \\ f_1^{\mathrm{T}} = \frac{1}{\mathrm{Q}^{\mathrm{T}}} \frac{1}{4T\cosh^2(E/2T)} \big[& -\left(-\mathrm{i}\omega + 1/\tau_{\mathrm{E}}\right) \Phi N_1(E;\omega) \\ & -D\omega m q \cdot v_{\mathrm{s1}} M^{\mathrm{T}}(\mathrm{E};\omega)\big] \end{aligned} \tag{4.7}$$

where

$$\begin{aligned} Q^{\mathrm{L}} &= \left(-\mathrm{i}\omega + 1/\tau_{\mathrm{E}}\right) N_1(E;\omega) + Dq^2 M^{\mathrm{L}}(E;\omega) \\ Q^{\mathrm{T}} &= \left(-\mathrm{i}\omega + 1/\tau_{\mathrm{E}}\right) N_1(E;\omega) + 2\Delta N_2(E;\omega) + Dq^2 M^{\mathrm{T}}(E;\omega) \end{aligned} \tag{4.8}$$

(Notice, also in the generalized densities of states the combination $-i\omega + 1/\tau_{\mathrm{E}}$ appears.) We now define the integrals

$$\begin{aligned} X_{ij} &= \int_{+\infty}^{\infty} \mathrm{d}E \frac{1}{\cosh^2 E/2T} \frac{N_i(E;\omega) N_i(E;\omega)}{Q^{\mathrm{T}}} \quad i,j = 1,2 \\ X_{\mathrm{L}} &= \int_{-\infty}^{\infty} \mathrm{d}E \frac{1}{\cosh^2 E/2T} \frac{R_2^2(E;\omega)}{Q^{\mathrm{L}}} \end{aligned} \tag{4.9}$$

In X_{11} we split off the leading term

$$X_{11} = \frac{4T}{-i\omega + 1/\tau_{\mathrm{E}} + Dq^2} + \tilde{X}_{11} \tag{$\prime$}$$

The wave vector dependence thus written explicitly in (4.9') turns out to be most important. Furthermore, we encounter the integral $\int_{-\infty}^{\infty} N_2(E;\omega)\mathrm{d}E = \pi A$.

With these definitions we readily obtain

$$\chi^{\mathrm{L}}(q,\omega) = \frac{4T}{\pi N_0} \left[-\frac{16T_{\mathrm{c}}}{\pi}\alpha - \mathrm{i}\omega + Dq^2 - \frac{2\mathrm{i}\omega}{\pi} X_{\mathrm{L}} \right]^{-1} \tag{4.10}$$

$$\begin{aligned} \chi^{\mathrm{T}}(q,\omega) = \frac{4T}{\pi N_0} \Big\{ & -\mathrm{i}\omega + Dq^2 \\ & + \mathrm{i}\omega \frac{2}{\pi} \left[X_{22} - \frac{(X_{12})^2}{\tilde{X}_{11} - \frac{4TDq^2}{(\mathrm{i}\omega + 1/\tau_{\mathrm{E}} + Dq^2)(-\mathrm{i}\omega + 1/\tau_{\mathrm{E}})}} \right] \Big\}^{-1} \end{aligned}$$

21.4.2 4.2. Analytic expressions

It remains to evaluate the integrals X_{ij}. We find analytic expressions in several limits. In the considered order in Δ/T we may set in the integrals X_{12}, X_{22}, $\widetilde{X}_{11}$ and X_{L} the factor $\cosh^{-2}(E/2T) = 1$. In context with the collective mode an important case is that of negligible pair breaking $\Gamma \ll \omega$. In this limit we find

$$
\begin{aligned}
X_{12} &= -\frac{i}{2}\frac{v\delta}{v^2+\delta^2}(I+rJ)+\frac{\delta^2}{v^2+\delta^2}L \\
X_{22} &= -\frac{\delta}{2}(I-rJ)+\mathrm{i}\frac{\delta^2}{v}L+\frac{\pi}{2} \\
\tilde{X}_{11} &= -\mathrm{X}_{22}+\frac{1}{2}\frac{\delta}{v^2+\delta^2}\left[H+\left(s-2v^2-2\right)I+r^2J\right]+\mathrm{i}\frac{r\delta^2}{v\left(v^2+\delta^2\right)}L+\frac{\pi}{2} \\
X_{\mathrm{L}} &= \frac{\delta}{2}\left(1+\frac{1}{v^2+\delta^2}\right)I+\frac{sr}{2\delta}J+\frac{\mathrm{i}s}{v}L-\frac{\pi}{2}
\end{aligned}
\tag{4.12}
$$

We used the notation $v=\omega/(24)$ and $\delta=D\mathrm{q}^2/(24)$ and introduced

$$
s=\frac{\delta^2}{v^2+\delta^2}-\delta^2, \quad r=\delta^2+v^2-\frac{v^2}{v^2+\delta^2}
$$

The functions, I, J, H are combinations of complete elliptic integrals of the first kind $K(k)$, the second kind $E(k)$, and third kind $\Pi\left(\alpha^2, k\right)$. They are, using the notation of Byrd and Friedman (1971)

$$
\begin{aligned}
I &= \frac{2}{1+v}K(k)\theta(1-v)+\frac{\mathrm{i}}{1+v}K\left(k'\right) \\
J &= \frac{-2}{s(1+v)}\Pi\left(\frac{(1-v)^2}{s},k\right)\theta(1-v)+\frac{\mathrm{i}}{(1+v)\left[(1+v)^2-s\right]}\Pi\left(\frac{4v}{(1+v)^2-s},k'\right) \\
H &= 2(1+v)[K(k)-E(k)]\theta(1-v)+\mathrm{i}(1+v)E\left(k'\right)
\end{aligned}
\tag{4.12'}
$$

with $k=(1-v)/(1+v), k'=\sqrt{1-k^2}$. The real parts of I, J and H vanish for $v>1$. Finally, for $v \leqslant \sqrt{1-\delta^2}$, where $s \geqslant 0$, the functional L is given by

$$
L=\frac{\pi}{4}\frac{v}{\delta^2}\left[\frac{v-\delta^2/\sqrt{s}}{\sqrt{1-(\sqrt{s}+v)^2}}+\frac{v+\delta^2/\sqrt{s}}{\sqrt{1-(\sqrt{s}-v)^2}}\right] \tag{''}
$$

while for $v>\sqrt{1-\delta^2}$, where $s<0$, it is

$$
L=-\mathrm{i}\frac{\pi v}{4\delta\sqrt{-s}}\left[1-\frac{r}{\sqrt{\left(1-s-v^2\right)^2-4sv}}\right]+\frac{\pi v}{2\delta^2}\frac{\left(v+\mathrm{i}\delta^2/\sqrt{-s}\right)}{\sqrt{1-(\mathrm{i}\sqrt{-s}+v)^2}}. \tag{'''}
$$

All square roots are positive real numbers except for the expression $\sqrt{1-(\mathrm{i}\sqrt{-s}+v)^2}$, which has a positive real and negative imaginary part.

Another interesting limit is the case of small wave vectors $Dq^2 \ll \omega$ or $Dq^2 \ll 1/\tau_{\mathrm{E}}$. In this case, we find for general strength of the pair breaking (but still $\Gamma \ll \mathrm{T}$)

$$
\begin{aligned}
X_{12} &= \frac{\pi}{2}\frac{\sqrt{(-\mathrm{i}\omega+2\Gamma)/\left(-\mathrm{i}\omega+1/\tau_{\mathrm{E}}\right)}}{\sqrt{1+(-\mathrm{i}\omega+2\Gamma)\left(-\mathrm{i}\omega+1/\tau_{\mathrm{E}}\right)/(2A)^2}} \\
X_{22} &= \frac{\pi}{2}-\frac{\left(-\mathrm{i}\omega+1/\tau_{\mathrm{E}}\right)}{2\Delta}X_{12} \\
\tilde{X}_{11} &= \frac{-2\Delta}{-\mathrm{i}\omega+1/\tau_{\mathrm{E}}}X_{12} \\
X_{\mathrm{L}} &= \frac{\pi\Delta}{-\mathrm{i}\omega+1/\tau_{\mathrm{E}}}\left\{\sqrt{1+\frac{(-\mathrm{i}\omega+2\Gamma)^2}{(2\Delta)^2}}-\frac{(-\mathrm{i}\omega+2\Gamma)}{2\Delta}\right\}
\end{aligned}
\tag{4.13}
$$

where for $\omega = 0$ the square roots are positive. More precisely, we find the forms (4.13) in both the limits $\Gamma = 1/2\tau_{\mathrm{E}}$ and $\Gamma \gg \Delta$. Furthermore, they differ only by a few percent from the result of a first order expansion in $\Gamma_i\Delta$. Therefore, we use (4.13) as interpolation formulas for all values of Γ.

Although (4.13) is the small Dq^2 limit of the integrals X_{ij}, we stress again that the dominant wave vector dependence is taken into account by splitting X_{11} into two contributions in eq. (4.9') and keeping the Dq^2 written explicitly there.

Finally, we give the results for small wave vectors and weak pair breaking Γ, $Dq^2 \ll \omega \ll \Delta \ll T$, including first order corrections in Dq^2/ω and Γ/ω. In this case

$$
\begin{aligned}
X_{12} &= \frac{\pi}{2}\left(1 + \mathrm{i}\frac{\Gamma_0}{\omega} + \frac{2}{\pi}\mathrm{i}\frac{Dq^2}{\omega}\right) \\
X_{22} &= \frac{\pi}{2} - \mathrm{i}\frac{\omega}{2A}\frac{\pi}{2}\left(1 + \mathrm{i}\frac{\Gamma_0}{\omega} + \mathrm{i}\frac{1}{\omega\tau_{\mathrm{E}}}\right) - \frac{Dq^2}{2\Delta}\left(\ln\frac{8\Delta}{\omega} - 1 + \mathrm{i}\frac{\pi}{2}\right) \\
\tilde{X}_{11} &= -\mathrm{i}\frac{\pi\Delta}{\omega}\left(1 + \mathrm{i}\frac{1/\tau_{\mathrm{E}} - \Gamma_0}{\omega} - \mathrm{i}\frac{4}{\pi}\frac{Dq^2}{\omega}\right) \\
X_{\mathrm{L}} &= -\frac{\pi}{2} + \frac{\pi\Delta}{-\mathrm{i}\omega + 1/\tau_{\mathrm{E}}} - \frac{Dq^2\Delta}{\omega^2}\left(\ln\frac{8\Delta}{\omega} - 4 + \mathrm{i}\frac{\pi}{2}\right)
\end{aligned}
\tag{4.14}
$$

where Γ_0 represents the pair breaking effects other than inelastic electron phonon scattering ($\Gamma = \Gamma_0 + 1/2\tau_{\mathrm{E}}$).

21.4.3 4.3. Results

The order-parameter structure factor $S(q,\omega) = S^{\mathrm{L}}(q,\omega) + S^{\mathrm{T}}(q,\omega)$ is plotted in figures 1 to 4 as a function of the frequency for different wave vectors and temperatures near T_c. We used both the weak pair breaking expressions (4.12) and the small wave-vector expansion (4.13). Several features are quite conspicuous.

First, there is a central peak at small frequencies resulting mainly from the longitudinal structure factor. For vanishing wave vectors q $\to$ 0, this peak reflects the order parameter relaxation with characteristic time τ_1. For instance, for weak pair breaking due to electron-phonon scattering only ($\Gamma = 1/\left(2\tau_{\mathrm{E}}\right) \ll \Delta$), we find (Schmid 1968, Schmid and Schön 1975a)

$$
\frac{1}{\tau_\Delta} = \frac{7\zeta(3)}{\pi^3}\frac{\Delta}{T}\frac{1}{\tau_{\mathrm{E}}}. \tag{4.15}
$$

The generalization to arbitrary pair breaking follows in a straightforward manner from (4.13) (Eckern and Schön 1978). On the other hand, for finite wave-vectors, considered here, the central peak is broader, reflecting the existence of a diffusive mode. From eq. (4.14) we find that to lowest order in Δ/T for $\Gamma \ll Dq^2 \ll \omega \ll \Delta$

$$
\chi^{\mathrm{L}}(q,\omega) = \frac{4T}{\pi N_0}\frac{\omega}{2\Delta}\left[\omega + \mathrm{i}\frac{Dq^2}{\pi}\ln\frac{8\Delta}{\omega}\right]^{-1}. \tag{4.16}
$$

Second, for finite wave vectors, the transverse part of the structure factor at temperatures close to T_{c} shows a peak at finite frequencies. In the limit Γ, $Dq^2 \ll \omega \ll \Delta \ll T$ we can use the expansion (4.14) and find a simple expression*:

$$
\chi^{\mathrm{T}}(q,\omega) = A\left\{\omega^2 + \mathrm{i}\omega\left[\frac{\pi A^2}{2T} + \frac{Dq^2}{\pi}\left(2\ln\frac{8\Delta}{\pi} - 2\right) + \frac{1}{\tau_{\mathrm{E}}} + \Gamma_0\right] - 2\Delta Dq^2\right\}^{-1}, \tag{4.17}
$$

where the prefactor depends strongly on Dq^2

$$A = \frac{4T}{\pi N_0} \frac{1}{Dq^2} \left(i \frac{\pi \Delta^2 \omega}{2T} - 2\Delta D q^2 \right). \tag{4.18}$$

In the form (4.17), we easily see that this peak in the structure factor results from a pole of χ^{T} at

$$\omega_q = -\frac{1}{2} i_\gamma \pm \sqrt{c^2 q^2 - \frac{1}{4}\gamma^2}.$$

which reflects a propagating collective mode. The bare phase velocity of this mode (in the dirty limit) is

$$c_{CG} = \sqrt{2AD}. \tag{4.19}$$

The mode is damped. The mode damping in the dirty limit is given by

$$\gamma = \frac{n_{\mathrm{s}}}{n} \frac{1}{\tau_{\mathrm{imp}}} + \frac{Dq^2}{\pi} \left(2 \ln \frac{8\Delta}{\omega} - 2 \right) + \frac{1}{\tau_{\mathrm{E}}} + \Gamma_0. \tag{4.20}$$

Clearly, the propagating mode only exists at temperatures close to T_{c} as long as the damping $n_{\mathrm{s}}\left((n\tau_{\mathrm{imp}}) = \pi A^2, (2T)\right.$, which is due to ohmic losses of the normal current, is small. On the other hand, the mode is damped due to

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- Various authors, concerned with different limits, have derived this result, extensions, or parts of it before. Schmid and Schön (1975b) and Artemenko and Volkov (1975) considered the case of small Dq^2 and $\Gamma = 0$ in the dirty and clean limits, respectively. Schmid and Schön (1975c) and Dinter (1978a,b) considered general Dq^2 and $\Gamma = 0$. Ovchinnikov (1977) included Dq^2 and Γ_0 to first order and, in addition, allowed for general impurity concentrations. Schön and Ambegaokar (1979) considered small Dq^2, but general Γ as long as $\Gamma/T \ll 1$, and included the effect of an imposed supercurrent. Entin-Wohlman and Orbach (1978) formulated the problem for general Γ/T, but in their final expressions for the damping left out the leading correction in Γ_0/ω [included in eq. (4.17)]. Also Artemenko and Volkov (1979), Kulik et al. (1981) and Cheishvili (1982) discussed the problem for general impurity concentrations.

Fig. 1. The structure factors representing fluctuations of the magnitude and the phase of the order parameter $S^{L} $ and $S^{\bar{T}}$, and the total structure factor, i.e the sum of $S^L + S^T$, are plotted as a function of the frequency ω. The results are those of short mean free path superconductors, the wavevector is fixed at $Dq^2/(2\Delta) = 0.1$ and the reduced temperature is $(T_\mathrm{c} - T)/T_\mathrm{c} = 0.002$. Pair breaking effects are small, $\Gamma, 1/\tau_\mathrm{E} \approx 0$, i.e. the expansion (4.12) is used. The structure factors are shown for positive frequencies only, they are symmetric in ω. They are multiplied by N_0, the density of states at the Fermi surface. The parts with large frequencies are also shown enhanced by a factor 10 in order to resolve better the resonance of S^L at $\omega = 2\Delta$. The central peak of S^L represents a diffusive mode, the peak of $S^\top$ the Carlson-Goldman propagating collective mode.}

diffusion processes of the quasiparticles which is reflected by the second term in %. Hence. the mode propagates only if $Dq^2 \ll \omega$ |. As long as the resonant frequency $|\omega| \approx c_\mathrm{CG} q$ is small in comparison to Δ this condition is always satisfied. Finally. we see that inelastic electron phonon collisions and any other pair breaking mechanism further increase the damping. (See also Cheishvili 1977, 1982 and Galaiko et al. 1978). Summarizing, we conclude that in a short mean free path superconductor the collective mode propagates if the following conditions are satisfied:

$$\Gamma_0, \quad \frac{1}{\tau_\mathrm{E}}, \quad \frac{\Delta^2}{T} \ll |\omega| \ll \Delta. \tag{4.21}$$

The qualitative features which we obtain from the expansion (4.18) are also borne out by the analysis of the more complete results shown in the figures.

In this context it should be added that in superconductors with longer mean free path, the propagating mode is also found (Artemenko and Volkov 1975.

Fig. 2. The (total) order-parameter structure factor is shown as a function of the frequency for different wavevectors: (curve I) $D q^{2 /(2 \Delta)=0.02$, (II) 0.05 , (III) 0.1 , (IV) 0.2 , and (V) 0.5 . The other parameters are the same as in fig. 1. For easier comparison the structure factors are plotted reduced by a factor $Dq^2/(24)$. Notice the following features: (i) the diffusive central peak broadens with increasing Dq^2; (ii) the peak at finite frequency is shifted roughly proportional to q, i.e. it represents a propagating mode; (iii) the damping of the propagating mode becomes stronger with increasing Dq^2; (iv) the resonance at $\omega = 24$ becomes less pronounced as Dq^2 increases.}

1979; Dinter 1978a; Ovchinnikov 1977). The same physical damping mechanisms exist. But, since the superfluid density depends on the impurity concentration the phase velocity and the strength of the damping are different. In clean superconductors the restriction of weak diffusive damping $Dq^2 \ll |\omega|$ results in an upper limit to the frequency $|\omega| \ll \Delta_1 (T\tau_{\text{imp}})$ (Ovchinnikov 1977).

Third, we find that the order parameter structure factor strongly increases at frequencies above $2A$. This is again due to the longitudinal part $S^{\mathrm{L}}(\mathrm{q}, \omega)$. In the limit of very weak pair breaking and very small wave vectors, the structure factor S^{L} vanishes for $\omega \leqslant 24$ and sets in proportional to $\sqrt{(\omega/2\Delta)^2 - 1}$ above $\omega = 24$. Hence, this structure is a resonance. It differs from the structures discussed so far which result from poles of the susceptibility in the complex frequency plane. In the time domain, this resonance corresponds to an oscillation at frequency $\omega = 2\Delta$ which is damped according to a power law in $1/(\Delta t)$. This means the damping is never small. In contrast a pole corresponds to an oscillation damped exponentially. The effect of pair breaking is to smear out this structure (Schön and Ambegaokar 1979) and S^L becomes proportional

Fig. 3. The (total) order-parameter structure factor is shown as a function of the frequency for different (reduced) temperatures: (curve I) $\left(T_{\mathrm{c }}-T\right) / T_{\mathrm{c}}=0.0005$, (II) 0.001 , (III) 0.002 , (IV) 0.005 , (V) 0.01 . The wavevector is fixed at $Dq^2/T_c = 0.05$, the other parameters are the same as in fig. 1. Notice the following features: (i) the peak representing the propagating mode is shifted roughly proportional to $\sqrt{\Delta(T)}$; (ii) the damping of this mode increases with decreasing temperature; (iii) there is a resonance at $\omega = 2\Delta(T)$.}

to $-\mathrm{Im}\sqrt{1+[(-i\omega+2\Pi)/24]^2}$ (see eq. 4.13). Spatial variations have a similar effect (Dinter 1978a,b): From the expansion of (4.12) we find that near $\omega = 2A$ the structure factor S^{L} is proportional to $\sqrt{(\omega/2\Delta)^2-1}\theta(\omega-2\Delta)+Dq^2/2\Delta$.

In a superconductor where a finite supercurrent is flowing in equilibrium already, i.e. $v_{s0} \neq 0$, the fluctuations in the magnitude and phase of the order parameter are coupled. In other words, the longitudinal (energy) mode and the transverse (charge) mode are no longer independent. We can still proceed as before, to calculate the response to an external drive [compare e.g. eq. (4.3)], and obtain the pair field susceptibilities. A technical complication arises from the coupling of the two Boltzmann equations. The problem has been dealt with in the limit of small wave vectors $Dq^2 \ll |-i\omega+1/\tau_E|$ by Schön and Ambegaokar (1979). We merely quote some of the results. The most important effects of the imposed supercurrent are: (i) The gap relaxation time becomes longer and it diverges (mode softening) if the imposed supercurrent approaches the Ginzburg-Landau critical current j_c. For $\Gamma \ll \Delta$ one finds that a spatially

Fig. 4. The (total) order-parameter structure factor is shown as a function of the frequency for different strength of the pair breaking. The parts with large frequencies are also/only shown enhanced by a factor 20 . Curve I represents the situation where the pair breaking is due to inelastic phonon collisions only i.e. $\Gamma=1 /\left(2 \tau_{\mathrm{E }}\right)$, with rate assumed to be $1/(2\Delta\tau_{\mathrm{E}}) = 0.01$. Curves II and III represent the situation with additional stronger pair breaking mechanisms. The expansions (4.13) are used. Hence, the result is correct only for $\omega \gg Dq^2$. (The expansions can also be applied if $1/\tau_E \gg Dq^2$, but the parameters are chosen differently.) The comparison shows the following features: (i) pair breaking effects act as a damping mechanism for the collective mode and reduce the peak frequency; (ii) the resonance at $\omega = 2\Delta$ is washed out by pair breaking effects. Since the expansion (4.13) is used, the damping of the collective mode due to diffusion is not accounted for in fig. 4 . This effect can be seen by comparing curve I of fig. 2 and curve I of fig. 4 , which have roughly the same parameters.}

(q) modulated fluctuation of Δ relaxes with rate

$$1/\tau_A = \frac{1}{2\Delta\tau_{\mathrm{E}}}\left(\frac{14\zeta(3)\Delta^2}{\pi^3 T} + Dq^2 - 16Dm^2 v_{\mathrm{s0}}^2\right), \tag{4.22}$$

where Δ is reduced due to the supercurrent. Hence, at j_{c} homogeneous fluctuations ($q = 0$) of Δ become unstable first. (ii) Both phase velocity and damping of the propagating collective mode are affected. The phase velocity is reduced,

$$c_{\mathrm{CU}}^2 = 2AD\left(1 - \frac{2Dm^2 v_{\mathrm{s0}}^2}{\Delta}\right), \tag{4.23}$$

while the damping is increased due to the pair breaking effect of the imposed supercurrent (see eq. 4.20).

21.4.4 4.4. Comparison with experiment*

In many systems exhibiting a second-order phase transition, the order parameter can be directly coupled to a classical external field. The response of the order parameter to this field determines the characteristic susceptibility and correlation function. In a superconductor however, the order parameter A is off-diagonal in electron number representation. For this reason one cannot couple to Δ with a classical field. Nevertheless, following a suggestion of Ferrell (1969) the pair field susceptibility can be determined by measuring the conductivity of a Josephson junction. The metal of interest which is near its transition temperature forms one side of the junction. The metal on the other side is well below its transition temperature.

The order parameter of the high T_c superconductor provides the off-diagonal coupling to the fluctuating order parameter of the superconductor near T_c. These fluctuations give rise to Josephson currents which appear as an excess conductivity.

The geometry of the junctions used in the experiments is shown in fig. 5. On one side is the superconductor of interest, a film of thickness d, with transition

Fig. 5. The tunneling junction geometry is shown which is used in the experiments to measure the order-parameter structure factor of the superconducting film.

temperature T_c. Since $T \approx T_c$ the penetration depth $\dot{\lambda}(T)$ is large and an applied parallel magnetic field completely penetrates the film. On the other side is a superconductor well below its transition temperature T_c' with penetration depth λ'. The order parameter of this superconductor has a well defined value with negligible fluctuations. The state of the Josephson junction depends on the phase difference across the barrier, which is a function of both the magnetic field and the DC voltage applied to the junction. A field H, in the y-direction, produces a variation of the phase difference in the z-direction characterized by a wave vector

[12]*For a detailed review see also Goldman (1981).

$$q = (2eH/c)\,(\lambda' + d/2)\,, \tag{4.24}$$

A DC voltage V, applied to the junction, produces time variations with frequency

$$\omega = 2\mathrm{eV}. \tag{4.25}$$

Even above T_c an excess current due to pair tunneling is flowing. It turns out to be related to the pair field susceptibility by

$$I_{\mathrm{ex}}(H, V) = \text{ const } \cdot \mathrm{Im}[\chi(q, \omega)]. \tag{4.26}$$

The constant in (4.26) is proportional to $|T|^4$ where T is the tunneling matrix element. Scalapino (1970) derived this result starting from a model Hamiltonian for the pair tunneling. Takayama (1971) and Shenoy and Lee (1974) confirmed this result by microscopic calculations.

Also below T_c, the pair current may serve as a probe for the fluctuations and the pair field susceptibility (see e.g. Dinter 1978a). However, since the expectation value of the fluctuating order parameter is nonzero, also the usual Josephson current (proportional to $|T|^2$) is flowing. This contribution is usually larger than I_{cx}, but can be quenched by applying a magnetic field. Another large contribution to the current results from quasiparticle tunneling. It has to be subtracted numerically from the experimental data. In the experiments of Carlson and Goldman (1975, 1976) short mean free path superconducting aluminium films were investigated. Pair breaking effects were weak. The resulting order-parameter structure factor is shown in fig. 6 (full lines). It is compared with the theoretical result derived above (broken lines) in the limit of vanishing pair breaking (i.e. using 4.12), corrected to include second order contributions in $\Delta/T, \omega/T$, and Dq^2/T given in the Appendix (Schön 1976). Except at very small frequencies we find very good quantitative agreement between both results with no adjustable parameter. We confirm that the central peak results from fluctuations in the magnitude of the order parameter. It is a diffusive peak of the type following from eq. (4.16). Notice that the resolution of the experiments is too rough to detect a homogenous ($q \approx 0$) gap relaxation with rate $1/\tau_1$ (compare eq. 4.15). The peak at finite frequencies results from fluctuations in the phase of the order parameter. The

Fig. 6. The order-parameter structure factor measured in the experiment of Carlson and Goldman (1975) (solid line) is compared with the theoretical result (including corrections in $\Delta / \mathrm{T
, \omega / \mathrm{T}$, Dq^2/T) (broken line). The parameters of the calculated curves are the same as in the experiments. The wavevector is $q = 4 \times 10^4$ cm^{-1}, the diffusion coefficient is $D = 18.6$ s^{-1} cm^2, and the transition temperature is $T_c = 1.7795$ K. Only the absolute value of the structure factor, which is uncertain in the experiments, is scaled to fit at one point, indicated by the arrow.}

frequency where the peak due to the collective mode is found is much larger than the charge imbalance relaxation rate $1/\tau_{Q^*}$ (see below). Hence, the mode is definitely not a hydrodynamic mode, rather it is a collisionless mode.

A magnetic field is applied to create the spatially periodic structure. The data shown in fig. 6 were obtained in a field of 125 G which leads to a wave vector $q = 4 \times 10^4$ cm^{-1}. This field also leads to pair breaking of strength $\Gamma_0 = 1.95 \times 10^9$ s^{-1}, which may account for the difference between experimental and plotted theoretical results at small frequencies. It has no noticeable direct effect on the propagating mode.

Since the magnetic field also lowers the transition temperature, a comparison of the resonance frequencies for different wave vectors should be performed at the same reduced temperature

$$\alpha(T,H) = \frac{[T - T_c(H)]}{T_c(H)} \approx \alpha(T,0) + \frac{\pi e^2 D d^2}{24 T_c(0) c^2} H^2. \tag{4.27}$$

In fig. 7 the dispersion relation $\omega(q)$ at fixed $\alpha(T,H)$ found from the

Fig. 7. The dispersion relation $\omega(q)$ at fixed reduced temperature $-\alpha(T, H)=6.32 \times 10^{-3
$ obtained experimentally (Carlson and Goldman 1975) and from the calculated structure factor is compared.}

Fig. 8. The dependence of the resonant frequency on the reduced temperature $\left(T_{\mathrm{c
}}-T\right) / T_{\mathrm{c}}}$ as measured in the experiment and calculated is compared. The parameters of curve I are the same as in fig. 6. In curve II the wavevector is $q = 4.8 \times 10^4$ cm^{-1} and $T_c = 1.775$ K.}

experiment is shown and compared with the theoretical result. Over the range of wave vectors where the experiment was performed, $\omega(\mathrm{q})$ is almost proportional to q, which means the resonant peak results from a sound-like mode. The phase velocity of this mode, which is obtained from the slope of the curves, depends on the reduced temperature. It is typically of the order $c \approx 10^6$ cm/s. The temperature dependence of the resonant frequency $\omega(q)$ at fixed values of the wave vector is shown in fig. 8. Again the agreement between theoretical and experimental result with no adjustable parameter is excellent.

The effect of the various damping mechanisms on the collective mode, including the pair breaking effect of paramagnetic impurities, was examined in more detail by Aspen and Goldman (1979, 1981). The temperature dependence of γ due to ohmic losses of the normal current (4.20) was confirmed quantitatively. Furthermore, the experiments demonstrate qualitatively the influence of additional pair breaking effects. However, no quantitative systematic agreement, as far as the pair breaking effects are concerned, was obtained. In the same work also the resonance in the structure factor at $\omega \approx 2\Delta$ was investigated in more detail. The smearing of the step seems to be in reasonable agreement with the predictions of the theory.

21.5 5. Extensions and further collective modes

21.5.1 5.1. Space and time dependent conversion

Although there is no unique prescription how to separate the total current into a normal and a supercurrent part, it appears natural to call the contributions to j_1 in (3.23) which contain f_1^{T} and $m\dot{v}_{s_1}$ the normal current. The familiar linear response to an applied homogeneous electric field

$$j(\omega) = [\sigma_1(\omega) + \mathrm{i}\sigma_2(\omega)]\, E(\omega)$$

suggest a slightly different identification, namely $j_{\mathrm{n}}(\omega) = \sigma_1(\omega)E(\omega)$, where

$$\begin{aligned}\sigma_1(\omega) =& \sigma\frac{1}{2\omega}\int_{-\infty}^{\infty}\mathrm{d}E\left[N_1\left(E+\frac{1}{2}\omega\right)\mathrm{N}_1\left(E-\frac{1}{2}\omega\right)+\mathrm{R}_2\left(E+\frac{1}{2}\omega\right)\mathrm{R}_2\left(E-\frac{1}{2}\omega\right)\right]\\ &\times\left(\tanh\frac{E+\frac{1}{2}\omega}{2T}-\tanh\frac{E-\frac{1}{2}\omega}{2T}\right)\end{aligned} \tag{5.1}$$

Here the usual generalized densities of states appear $N_1(E) \equiv N_1(E;\omega=0)$ [compare eqs. (3.15) and (3.21)] and $R_2(E) \equiv R_2(E;\theta)=0)$. Both identifications differ by small terms of order $\omega L^2/T^3$ only. By combining the continuity equation $\dot{\rho} + \operatorname{div} j = 0$ and the Boltzmann equation we find [compare eq. (4.4)]

$$2\Delta\int N_2(E;\omega)f_1^{\mathrm{T}} - \frac{1}{2eN_0}\operatorname{div} j_{\mathrm{s}} = 0 \tag{5.2}$$

We consider situations with temperatures near T_c (which allow us to use the reduced electron phonon collision operator), weak spatial variations ($Dq^2 \ll \omega$ or $Dq^2 \ll 1/\tau_{\mathrm{E}}$) and $v_{\mathrm{s0}} = 0$. We can then easily solve the Boltzmann equation in terms of the pair chemical potential Φ. In the limit considered we have

$$2\Delta\int N_2 f_1^{\mathrm{T}} = 2\Delta\frac{X_{12}}{X_{11}}\int N_1 f_1^{\mathrm{T}} \tag{5.3}$$

The expansion (4.13) is the appropriate limit of X_{12} and in the order considered we may set $X_{1\mathrm{i}} \approx 4T/\left(-i\omega + 1/\tau_{\mathrm{E}}\right)$. Thus, we have

$$-\frac{\pi\Delta}{4T_{\mathrm{c}}}\left[\frac{(-\mathrm{i}\omega+2\Gamma)(-\mathrm{i}\omega+1/\tau_{\mathrm{E}})}{1+(-\mathrm{i}\omega+2\Gamma)(-\mathrm{i}\omega+1/\tau_{\mathrm{E}})/(2\Delta)^2}\right]^{1/2}\Phi-\frac{1}{2eN_0}\operatorname{div} j_{\mathrm{s}}=0 \tag{5.4}$$

In the frequency range $\Gamma, 1/\tau_{\mathrm{E}} \ll \omega \ll \Delta$ this relation reduces to eq. (2.3) which was an essential ingredient of the approximate description of the CarlsonGoldman collective mode in section 2.5. Furthermore, eq. (5.4) describes the conversion of currents. The charge neutrality fixes $\operatorname{div}(j_{\mathrm{n}}+j_{\mathrm{s}})=0$. Near $T_{\mathrm{c}}, j_{\mathrm{n}}$ is roughly equal to the normal state value $j_n \approx \sigma E$. Thus in a stationary experiment with a normal current flowing into a superconductor, the electric field penetrates to a depth

$$\lambda_{Q*}=\sqrt{D\tau_{Q*}} \tag{5.5}$$

The rate $1/\tau_{Q*}$ is the charge imbalance relaxation rate describing the conversion of an injected normal current into supercurrent (Clarke 1972, Tinkham 1972; Schmid and Schön 1975a)

$$\frac{1}{\tau_{Q*}}=\frac{\pi\Delta}{4T}\left(\frac{2\Gamma}{\tau_{\mathrm{E}}}\right)^{1/2}\left(1+\frac{\Gamma}{2\Delta^2\tau_{\mathrm{E}}}\right)^{-1/2} \tag{5.6}$$

Finally, we find from (5.4) that in a spatially homogeneous situation, with no external perturbation, the free decay of a charge imbalance, i.e. a nonequilibrium pair chemical potential, occurs on a time scale τ_{F} (Kadin et al. 1980).

21.5.2 5.2. The "neutral" superconductor

Although of hypothetical nature, the neutral superconductor is an interesting theoretical object since the absence of Coulomb interactions allows for a richer variety of low frequency collective modes. The pair chemical potential and the superfluid velocity are now related directly by

$$\frac{\partial}{\partial r}\phi=-m\dot{v}_{\mathrm{s}} \tag{5.7}$$

Again, a continuity equation is satisfied where now ρ and j are the number density and the corresponding current density. They are given by the same expressions as in the charged case except for an obvious replacement. As before, we find from the continuity equation and the Boltzmann equation

$$2\Delta\int \mathrm{d}E N_2(E;\omega) f_1^{\mathrm{T}}-\frac{1}{2N_0}\operatorname{div} j_{\mathrm{s}}=0 \tag{5.8}$$

but the distribution differs from the result in the charged case. We consider again the case near T_{c}, small wave vectors $Dq^2 \ll \omega$ or $Dq^2 \ll 1/\tau_{\mathrm{E}}$ and $v_{\mathrm{s}0}=0$ and solve the Boltzmann equation for f_1^T in terms of Φ. We now find

$$\frac{\Delta}{2T}X_{12}\frac{(-\mathrm{i}\omega)(-\mathrm{i}\omega+1/\tau_{\mathrm{E}})}{-\mathrm{i}\omega+\frac{\Delta}{2T}\frac{1}{\tau_{\mathrm{E}}}X_{12}}\Phi+\frac{n_{\mathrm{s}}}{n}\frac{v_{\mathrm{F}}^2}{3}m\operatorname{div} v_{\mathrm{s}}=0, \tag{5.9}$$

where we used the free electron connection $2N_0=3n/mv_{\mathrm{F}}^2$. With (5.7) we immediately find that at small frequencies $\omega \ll 1/\tau_{\mathrm{E}}$ a propagating solution exists with frequency ω satisfying

$$\omega^2=c_4^2q^2\left(1-\mathrm{i}\omega\tau_{Q*}\right), \tag{5.10}$$

where $c_4^2=n_{\mathrm{s}}v_{\mathrm{F}}^2/3n$ and τ_{Q*} is given in eq. (5.6). This mode, for ω small in comparison to the charge imbalance relaxation rate $1/\tau_{Q^*}$, is the hydrodynamic fourth sound, familiar from

neutral superfluids. For finite frequencies, it is damped due to the charge conversion processes. In phenomenological treatments this process is described by a second viscosity.

There are in general several such coefficients. However, in a superconductor where the normal current is dissipative, like clamped superfluids, only one coefficient ζ appears (see e.g. Khalatnikov 1969, Putterman 1974, and Schmid 1980). Hence, we can identify

$$\zeta = \tau_{Q*}/2N_0. \tag{5.11}$$

Furthermore, we find from (5.9) that also at large frequencies $\omega \gg \Gamma$ there exist a propagating solution, satisfying

$$\omega^2 + \mathrm{i}\omega\begin{pmatrix} 1 \\ \tau_{\mathrm{E}} + \Gamma_0 \end{pmatrix} - \frac{4T}{\pi A} c_4^2 q^2 = 0. \tag{5.12}$$

This is again the Carlson Goldman mode. The phase velocity $c^2_{(C'_G)} = 4Tc_+^2\pi A$ is the same as in the charged case. Pair breaking effects again lead to damping. However, there is no damping $xn_{\mathrm{s}}/n\tau_{\mathrm{imp}}$ since in the uncharged system the normal component remains at rest and is not subject to ohmic losses*. Although the phase velocity of the Carlson-Goldman mode is larger than that of fourth sound by the factor $\sqrt{4T,(\pi\Lambda)}$ we cannot find simultancously two

propagating solutions at fixed q. The conditions $c_4 q \ll 1/\tau_{Q*} = \pi\Delta/4T\tau_{\mathrm{E}}$ and $1/\tau_{\mathrm{E}} \ll c_{\mathrm{CG}}q$ cannot be met simultaneously.

21.5.3 5.3. Low temperatures

So far we have restricted our attention to temperatures close to T_c. However, we can extend our description to lower temperatures as well, starting from the formal theory presented in section 3. We find that the distribution functions obey the Boltzmann equations (3.20) at all temperatures. The only generalization is that if the frequencies are comparable to T or larger than T we have to retain, here and in other places below, the expression $\frac{1}{2}\left\{\tanh\left[E+\frac{1}{2}\omega\right)/2T\right] - \tanh\left[\left(E-\frac{1}{2}\omega\right)/2T\right]\right\}$ instead of the expansion $-i\omega\left(\partial f_0/\partial E\right)$. At lower temperatures the inelastic electron-phonon collision rate becomes smaller (see e.g. Kaplan et al. 1976) and hence the collisionless regime ($\omega \gg 1/\tau_{\mathrm{E}}$) gains more importance. Whereas near T_{c} we relied on expansions in $A/T, \omega/T$ and Dq^2/T, at low temperatures $T \ll \Delta$ we find simple results if $\omega, Dq^2 \ll A$. If the frequencies and wave-vectors considered satisfy these conditions the charge density is given by eq. (3.24) at all temperatures. Also the current density is given by the expression (3.23) encountered before, with one rather obvious replacement: the last term (if $\omega, Dq \ll \Delta$) becomes, at low temperatures $T \ll \Delta$ for not too strong pair breaking $\Gamma < A$:

$$\mathrm{Lin}\left\{\frac{\sigma}{e}\pi\Delta m v_s\left(1 - \frac{4\Gamma}{3\pi\Delta}\right)\right\} = \mathrm{Lin}\left\{e n_s \boldsymbol{v}_s\right\}. \tag{5.13}$$

Notice, that Γ depends on ι_{s} too. Although we consider here explicitly only the dirty limit, generalizations are possible, as indicated by the second general form of (5.13). For example, the dependence of n_s on the impurity concentrations at low temperatures and weak pair breaking ($\Gamma, T \ll \Delta$) is (Abrahams and Tsuneto 1966)

$$\frac{n_s(x)}{n} = \begin{cases} \frac{1}{x}\left(\frac{\pi}{2} - \frac{\arccos x}{\sqrt{1-x^2}}\right) & \text{for } x \leqslant 1 \\ 1 & \\ x\left(\pi - \frac{\operatorname{arccosh} x}{\sqrt{x^2-1}}\right) & \text{for } x \geqslant 1 \end{cases} \tag{5.14}$$

[12]*In superfluid ^{3}He this mode is overdamped as a consequence of the Fermi-liquid interaction.

where $x = 1, (2\Delta\tau_{\text{imp}})$. Or, in the dirty limit for weak pair breaking and Dq^2, $\omega \ll \text{Max}\{\Delta, T\}$ the temperature dependence of n_s is

$$n_{\text{s}}(T) = \frac{\sigma}{e^2}\pi\Delta m \tanh\frac{\Delta}{2T}. \tag{5.15}$$

From the self-consistency relation one finds at low temperatures the following relation from small frequencies and wave vectors ($\omega, Dq^2 \ll \Delta$), which replaces
the Ginzburg-Landau equation found near T_c

$$\Delta_1(1 - \pi\Gamma/4\Lambda) + \pi D m^2 v_{\text{s0}} \cdot v_{\text{s1}} - \int \mathrm{d}E R_2(E) f_1^{\text{L}}(E) = 0 \tag{5.16}$$

where $R_2(E) \equiv R_2(E; \omega = 0)$.
As far as the transverse (charge) mode is concerned, we again start from the continuity equation which we compare to the Boltzmann equation, integrated over the energy. Hence, for $\omega, Dq^2 \ll \Delta$

$$\begin{aligned}\dot{\Phi} + \frac{1}{2N_0}\,\text{div}\,(n_{\text{s}} v_{\text{s}}) - 2\Delta \int N_2(E) f_1^{\text{T}} \\ + \frac{\mathrm{i}}{2}\Phi \int \left[\tanh\frac{E + \frac{1}{2}\omega}{2T} - \tanh\frac{E - \frac{1}{2}\omega}{2T}\right] N_1(E) = 0\end{aligned} \tag{5.17}$$

In general, this result differs from those obtained near T_{c} since the last term no longer cancels $\dot{\Phi}$. At low temperatures and small frequencies $T, \omega \ll \Delta$ it becomes $(\omega^2/2\Delta)\,\Phi$, i.e. it is small. In this limit also the term $2\Delta \int N_2(E) \int_1^T(E)$, which is most important near T_{c}, is negligible. In fact if the wavevectors are small and inelastic scattering is weak $1/\tau_{\text{E}}, Dq^2 \ll \omega$ it is also $(\omega^2/2\Delta)\,\Phi$ and hence cancels the last term.

From (5.17) we immediately see that in an uncharged superconductor, where $\partial\Phi/\partial\boldsymbol{r} = -m\dot{v}_{\text{s}}$, at low temperatures $T \ll \Delta$ a propagating collective mode exists for not too large wavevectors and frequencies $\omega, Dq^2 \ll \Delta$ with phase velocity

$$c^2 = \frac{n_{\text{s}}}{n}\frac{v_{\text{F}}^2}{3} \tag{5.18}$$

This is the Bogoliubov-Anderson mode. The phase velocity is formally equal to that of hydrodynamic fourth sound, although the frequency may take arbitrary values in comparison to the inelastic scattering rate $1/\tau_{\text{E}}$. In the dirty limit for weak pair breaking $c^2 = \pi\Delta D$. In most treatments concerned with the Bogoliubov Anderson mode, clean superconductors were considered, where at low temperatures $n_{\text{s}}/n \approx 1$. Hence, the phase velocity was found to be formally equal to that of hydrodynamic first sound. For general impurity concentration the phase velocity follows from eqs. (5.14) and (5.18). At higher temperatures both integrals in (5.17) have to be evaluated. If the frequency is large in comparison to the inelastic scattering rate $1/\tau_{\text{f}}$, which of course increases with the temperature, we expect the collective mode to exist at all temperatures. The phase velocity changes with temperature. Near T_{c} we recover the Carlson-Goldman mode of the neutral superconductor discussed in section 5.2.

The Bogoliubov-Anderson mode involves local density fluctuations. Hence it does not exist in a real charged superconductor at most temperatures. A
counterflow of the normal current makes approximate charge neutrality possible. However, we have seen that only very close to T_{c} the resulting Ohmic losses are small enough for the mode not to be overdamped.

21.5.4 5.4. Further theoretical work on the Carlson-Goldman collective mode

From the early data of Carlson and Goldman (1973) a large pair breaking parameter ($\Gamma \approx 10^{11}$ s^{-1}) was erroneously inferred. This would imply that in the temperature range where the experiments were performed, the superconductor is gapless since $\Lambda \leqslant \Gamma$ and the time-dependent GinzburgLandau theory of Gor'kov and Eliashberg (1970) would yield an appropriate description. Motivated by this interpretation Maki and Sato (1974) and Brieskorn et al. (1974) calculated the order-parameter structure factor of a neutral gapless superconductor. Apart from a central diffusive peak centered at $\omega = 0$, they found a resonant peak at finite frequency, resulting from a fourth sound mode. The frequency of this mode follows from the relation (5.10). However, the superfluid density and hence the phase velocity are reduced by the pair breaking. For general Γ/T in the dirty limit c_4 is given by

$$c_4^2 = \frac{\Delta^2 D}{\pi T}\psi'\left(\frac{1}{2}+\rho\right) \tag{5.19}$$

In the extreme gapless case $\Gamma \gg T$ this reduces to $c_4^2 = 2A^2D/\Gamma$ and τ_{Q*} becomes $\tau_{Q*} = \Gamma/\left(2\Delta^2\right)$. Clearly, these results do not apply to the experimental situation, both because of the effect of the charge and because the pair breaking Γ in the experiments was rather weak (Carlson 1975). Nevertheless, the work of Maki and Sato (1974) and Brieskorn et al. (1974) provided the first insight into the physical processes responsible for the peaks in the structure factor.

Later, Bray and Schmidt (1975b) postulated ad hoc a phenomenological description of charged superconductors. They put up a set of rate equations for the normal and superfluid densities and currents, allowing also for conversion between the two components. Their description is similar in spirit to that presented in section 2.5, but in some respects without justification goes beyond it. For example, they postulated that the normal current is driven also by a gradient of a normal chemical potential μ_{n}. It is not clear a priori that in the high frequency regime such a chemical potential is a sensible concept. Moreover, their acceleration equation for the supercurrent differs from eq. (2.6). If they identify certain phenomenological parameters with results of the microscopic theory they reproduce the Carlson Goldman collective mode. However, they stress that the phase velocity is slightly larger than the results quoted above, namely $c^2 = c_{CG}^2 + c_4^2$. The difference results from the not so well motivated -inclusion of $\lambda\mu_{\mathrm{n}}$ ir , which represents a restoring force due to a compression of the normal liquid. In any case, the difference is small (c_4^2 is smaller than $c_{(G,G}^2$ by a factor $\pi\lrcorner/4T$) and hence consistently does not show up
in the results presented above. In the clean limit, Dinter (1978a) quotes such a correction. However, in the dirty limit, by extending our microscopic analysis, we find further corrections, reversing the effect. Also, the damping terms found by Bray and Schmidt do not agree with those derived from the microscopic theory. In gapless superconductors, Bray and Schmidt (1975a) in a model description find some evidence of a soundlike collective mode with phase velocity $c = \sqrt{2}c_4$. The increase in comparison to the fourth sound in neutral gapless superconductors is again due to the compression of the normal fluid, which compensates the superfluid oscillation. However, even within the model the conditions for this mode to exist, are (nearly) impossible to meet.

Another phenomenological approach, based on the established thermodynamic theory of irreversible processes as applied to clamped superfluids, was put forward by Putterman (1977). (Superconductors, where the motion of the normal component is subject to Ohmic losses, correspond for frequencies $\omega\tau_{\mathrm{imp}} \ll 1$ to superfluids where the normal component is clamped.) The conversion of normal and supercurrents at low frequencies is accounted for in this description

by a coefficient of second viscosity $\zeta(0) = \tau_{Q*}/2N_0$ [compare eq. (5.11)], (Landau and Lifshitz 1959, Khalatnikov 1965, Putterman 1974). The underlying physical processes, the inelastic electron phonon-collisions in superconductors, are rather slow. At finite frequencies, one is led to the substitution

$$\zeta(\omega) = \zeta(0)/\left(1 - \mathrm{i}\omega\tau_{\mathrm{E}}\right) \tag{5.20}$$

As a result, at high frequencies $\omega\tau_{\mathrm{E}} \gg 1$ a soundlike collective mode is found with phase-velocity (Putterman 1977, Schmid 1980)

$$c^2 = n_{\mathrm{s}}\zeta(0)/\tau_{\mathrm{E}} = \frac{n_{\mathrm{s}}2T}{mN_0\pi\Delta} \tag{5.21}$$

Also the most important damping proportional to $n_{\hookrightarrow,}\left(n\tau_{\mathrm{imp}}\right)$ is found. Clearly, this mode is the Carlson-Goldman mode. In a neutral superconductor the substitution (5.20) roughly corresponds to the transition from (5.10) to (5.12).

A relatively simple description of many nonequilibrium phenomena in clean superconductors, which resembles somewhat a two-fluid picture, was developed by Pethick and Smith (1979). It is based on the Boltzmann equation description of the excitations in the superconductor of Aronov and Gurevich (1974), and hence is restricted to small wavevectors and frequencies $v_{\mathrm{F}}q, \omega \ll \Delta$, and weak collision processes and pair breaking. Pethick and Smith split (somewhat artificially) changes of the electric charge into a normal and a superfluid charge ρ_{n} and ρ_{s}-not to be confused with the coefficients n_{n} and n_{s} which they relate to the deviation of a normal and a superfluid chemical potential μ_{n} and $\mu_{\mathrm{s}} \equiv \Phi$ from the equilibrium value $\mu_{\mathrm{eq}} : \rho_{\mathrm{s}} = \chi_{\mathrm{s}}\left(\mu_{\mathrm{s}} - \mu_{\mathrm{eq}}\right)$ and $\rho_{\mathrm{n}} = \chi_{\mathrm{n}}\left(\mu_{\mathrm{n}} - \mu_{\mathrm{eq}}\right)$. Near T_{c} the coefficients are found to be $\chi_{\mathrm{s}} = \pi\Delta 2eN_0/4T$ and $\chi_{\mathrm{n}} = 2eN_0$. In this context it is important to realize that in some static or low frequency situations, the distribution function of the excitations can be parametrized with good accuracy by a chemical potential μ_n. The two kinds of charges and the corresponding currents obey two continuity equations, with source terms of opposite signs describing the conversion of charges. At high frequencies $\omega \gg 1/\tau_{\mathrm{E}}$ superfluid and normal charges are separately conserved. If we impose the charge neutrality requirement and assume again the normal current to be given by $j_n = \sigma\boldsymbol{E}$ we recover the approximate macroscopic description presented in section 2.5. Quantitatively, however, the description applies at most for parameters restricted by the conditions (2.2). Damping effects are not well accounted for and the temperature has to be very close to T_{c}.

Another compact formulation describing collective modes and other charge imbalance phenomena was developed by Kadin et al. (1980). They derived, for clean superconductors, from the Boltzmann equation for the excitations in the appropriate limit a single differential equation for the charge imbalance Q^*. We can easily derive the equivalent formulation for dirty superconductors in the present framework. From eq. (5.4) and the charge neutrality it follows that in the limit $Dq^2 \ll \mathrm{Max}\left\{\omega, 1/\tau_{\mathrm{E}}\right\} \ll \Delta$ the charge imbalance

$$Q^* \equiv 2eN_0 \int \mathrm{d}E N_1(E) f_1^{\Upsilon}(E) = -2eN_0\Phi \tag{5.22}$$

obeys the relation

$$\frac{\pi\Delta}{4T}\left(-i\omega + \frac{1}{\tau_{\mathrm{E}}}\right) Q^* + \mathrm{div}\, j_{\mathrm{n}} = 0$$

For simplicity, we ignore pair breaking effects other than the inelastic electron-phonon scattering. Furthermore, near T_c the normal current is approximately

$$j_{\mathrm{n}} \approx \sigma E = \frac{\sigma}{e}\left(\frac{m}{en_{\mathrm{s}}} j_{\mathrm{s}} - \frac{1}{2eN_0}\nabla Q^*\right)$$

Combining these relations and $\operatorname{div} j = 0$ yields (Kadin et al. 1980)

$$\left\{ \frac{4Tn_{\mathrm{s}}}{\pi\Delta}\frac{v_F^2}{3}\nabla^2 - \frac{\partial^2}{\partial t^2} - \left(\frac{n_{\mathrm{s}}}{n}\frac{1}{\tau_{\mathrm{imp}}} + \frac{1}{\tau_{\mathrm{E}}}\right)\frac{\partial}{\partial t} - \frac{n_{\mathrm{s}}}{n}\frac{1}{\tau_{\mathrm{imp}}}\frac{1}{\tau_{\mathrm{E}}}\right\} Q^* = 0 \tag{5.23}$$

In this form one easily recovers the Carlson Goldman mode and further properties mentioned in section (5.1). Kadin et al. (1980) further pointed out
the equivalence of the result (5.23) and the telegraph equation, which describes the propagation of electromagnetic waves down a transmission line with series inductance and resistance, as well as shunt capacitance and leakage conductance.

21.5.5 5.5. Manifestation of the Carlson-Goldman collective mode

The existence of the Carlson Goldman mode as a collective excitation in the superconductor being established. the natural question to ask is, how it can be excited and which physical properties are affected by it. So far the most convincing experimental manifestation of the collective mode is its influence on the excess pair current in Josephson tunnel junctions*, which we discussed in section 4.4.

The collective mode also affects the dielectric function $\varepsilon(q,\omega)$ of the superconductor (Dinter 1978b. Bezuglyi and Bratus' 1981, Bezuglyi et al. 1982). In particular Im[1 &(q. (%)] has a minimum at the resonant frequency. This suggests the possibility of exciting the collective mode in a superconductor by electromagnetic waves incident on its boundary (Bezuglyi et al. 1982). However. the coupling is rather weak: the electric field inside the superconductor at the boundary is smaller by orders of magnitude than that applied from outside. As another example we mention that also the properties of the phonons in the system are affected by the collective mode (Dinter 1978 b). As a result, at fixed wavevector the ionic correlation function shows, besides the resonant peak due to ordinary sound, a second though much smaller peak at the resonant frequency (y_4 of the Carlson-Goldman mode. (Typical values in clean samples are $q \sim 10^2$ cm^{-1} and $\omega_q \sim 10^{10}$ s^{-1}.) So far neither the effect on the dielectric function nor that on the phonons is verified experimentally.

The effect of the Carlson Goldman collective mode is much more likely to be seen near phase slip centers in superconductors. These spatially localized centers exist in quasi-onedimensional superconductors when a current close to the critical current I_c is imposed. In the core of the center the order parameter collapses periodically in time allowing the phase to reset by 2π (Skocpol et al. 1974). During this brakdown the supercurrent $I.\ (\ x,t\ )$ is locally reduced. in fact vanishes periodically in the core ($x = 0$), and hence part of the imposed current I is carried as normal current. Since the conversion of currents is rather slow. the normal current (in particular the time-averaged normal current) is flowing to distances of the order of $i_{1/2\%}$ (see eq. 5.5) away from the core. The frequency of the oscillations is equal to the time-averaged voltage ($\omega_{\mathrm{g}} = 2eV$. This in turn depends on the effective resistance of the phase slip center $\overline{V'} = R_{\mathrm{eff}}\ \left(I - \bar{I}_{\mathrm{c}}\right)$ where

$R_{\mathrm{eff}} = 2\lambda_{Q*}/\sigma A$ and A is the cross-section of the superconductor. One expects qualitatively that the periodic oscillations of normal and supercurrent in the phase slip center also excite the Carlson-Goldman collective mode. However, a quantitative discussion is rather complicated. We mention here the numerical analysis of Kadin et al. (1980). These authors combined in a model treatment the transmission line (eq. 5.23) with an ideal Josephson element to represent the core of the phase slip center. They have to make assumptions on the supercurrent in the core; also the transmission line model ignores all effects of the spatially varying order parameter. Nevertheless they demonstrate rather convincingly the propagation of the collective mode. The mode is strongly damped and only propagates to short distances.

[12]*Compare also Kadin and Goldman (1982).

We can easily estimate the decay length $i.$ for the collective mode. It is approximately

$$i_\gamma \approx 2c_{\mathrm{CG}}/\gamma \approx \left[\left(32/\pi^2\right) DT^2/\Delta^3\right]^{12}$$

in the dirty limit. Since in most situations, except very close to T_c, the inelastic scattering rate $1/\tau_{\mathrm{E}}$ is small compared to Λ^2/T the decay length λ_γ, is shorter than the conversion length $\lambda_{Q*} = [(4/\pi)D\tau_E T/\Delta]^{1/2}$. In clean samples,

$$\lambda_\gamma \approx \left[(64\pi/7\zeta(3))D\tau_{\mathrm{imp}}T^3/\Delta^3\right]^{1/2}$$

is short in comparison to λ_{Q*} as long as $1\tau_E \ll \Delta^2\,(20T^2\tau_{\mathrm{imp}}\,)$. This condition is still realized in most situations, although the opposite limit is conceivable. We conclude that in most cases the waves propagate only to distances which are short in comparison to the distance the time-averaged normal current is flowing. It has been conjectured that these waves may propagate between phase slip centers and synchronize their oscillations, resulting in a voltage locking. This locking has been observed experimentally in closely spaced microbridges (Jillie et al. 1977, Palmer and Mercereau 1977, Lindelof and Bindslev Hansen 1977). However, in view of the different length scales $i_{\ldots} \ll i_{Q*}$, more direct coupling mechanisms may be more important. In fact, a thorough numerical analysis based on the complete theory of (weakly) gapless ($\Gamma \gtrsim \Delta$) superconductors (Baratoff 1982) reveals that neighbouring phase slip centers synchronize even in the gapless superconductor where the propagating mode is overdamped.

In another experiment (Kadin et al. 1981) a phase slip center near a boundary, which can reflect the waves back to the phase slip center, was investigated. The current-voltage characteristic shows several peaks which may be related to resonances of the waves in this arrangement. So far, no quantitative explanation could be given, but the idea is interesting and further work is desirable.

For completeness, we mention the work of Kulik (1979). He considered the interaction of electrons with the Carlson Goldmann collective oscillation. He
quantizes this oscillator according to the usual rules, calling the quantized particles bogolons. The electron-bogolon interaction then is investigated further, in particular its effect on the conversion of charges and on the penetration depth of an electric field into a superconductor. The penetration depth is found to be temperature independent, if the electron-bogolon interaction is the dominant interaction. We doubt whether it makes sense to quantize the -strongly damped-Carlson-Goldman oscillation and whether the bogolons represent a useful particle concept. Also, because the CarlsonGoldman mode frequency must be small $\omega_q \approx c_{\mathrm{CG}}q \lesssim \Delta$, the number of bogolons ought to be smaller than that of excited phonons by a factor of order $(\Delta/T)^{3/2}$. Hence, most likely the electron-phonon scattering is more important than electron-bogolon scattering.

21.5.6 5.6. Amplitude oscillations and further collective modes

It is well known that many superconducting response functions show a marked frequency dependence at the threshold of the creation of quasiparticle pairs at $\omega = 24$. This is an important result of the BCS theory. A number of response functions were calculated, e.g. by Mattis and Bardeen (1958), and further functions for more general parameters by Ovchinnikov (1970). In the present chapter we have found a branch cut in the pair field susceptibility near T_{c} extending to $\omega = 2\Delta$. In the time domain, this implies that the magnitude (amplitude) of the order parameter may oscillate at a frequency 24 . This can be interpreted as virtual decay of Cooper pairs into pairs of quasiparticles due to the quantum-mechanical energy-time uncertainty (Schmid 1968). However, this oscillation is not a collective mode, rather it is a resonance. It is damped according to some power of $1/t$ even in the absence of any collision

process. The oscillation is a quantum-mechanical effect related to the gap in the spectrum and the damping is a consequence of the fact that the excitation spectrum is continuous beyond 24 . In the response functions the resonance is represented by a branch cut extending to $\omega = 2A$. In contrast a collective mode corresponds to a pole in the complex ω plane.

As an illustration, we consider a homogenous superconductor at low temperatures $T \ll A$. A supercurrent is imposed which at time $t = 0$ is switched from j_0 to $j_0 + j_1$ (where j_1 is small). Because of the pair breaking effect of the supercurrent, this change requires an adjustment of the order parameter from Δ to eventually $\Delta + \Delta_1(\infty)$ and of the superfluid velocity from v_{s0} to $v_{s1}(\infty)$. This adjustment takes place in two stages on two very different time scales: Δ^{-1} and the inelastic collision time τ_E. In the first fast stage $\Delta_1(t)$ and $v_{s1}(t)$ perform damped oscillations, reaching after times long compared to Δ^1 but still small compared to τ_E intermediate values $\bar{\Delta}_1$ and $\tilde{v}_{s1}$, which are very close to $\Delta_1(\infty)$ and $v_{s1}(\infty)$, respectively. The difference is proportional to $e^{-d/T}$. Using the response functions derived by Ovchinnikov (1970) we find for 1. $\lesssim t \ll \tau_E$ (Ovchinnikov and Schön 1981)

$$\Delta_1(t) = \tilde{\Delta}_1 \left\{ 1 - \frac{1}{\sqrt{\pi^3 \Delta t}} [(\ln(64\Delta t) + \gamma) \cos(2\Delta t + \pi/4) \right.$$

$$\left. + \pi/2 \sin(2\Delta t + \pi/4)] \right\}$$

$$v_{s1}(t) = \tilde{v}_{s1} \left\{ 1 + \frac{\pi}{2} \frac{1}{(2\Delta t)^2} \cos(2\Delta t) \right\} \tag{5.24}$$

where Euler's constant is $\gamma = 0.577$. After these oscillations are damped out the quasiparticle distribution is still out of equilibrium. It relaxes by inelastic scattering and recombination processes (total rate $1/\tau_E$) which is the second slow stage. Because of the coupling between the distribution function and the order parameter and the superfluid velocity the quasiparticle relaxation results in a further, though smaller ($\propto e^{-\Delta T}$) adjustment of Δ and v_s. This may be detected and used in turn to investigate the relaxation processes (Ovchinnikov et al. 1981).

Littlewood and Varma (1981) observed that in those superconductors where a charge density wave is established, this wave modulates the density of states at the Fermi surface and hence couples to the superconducting order parameter. In the coupled system they find a collective mode with frequency close to, but slightly smaller than 24

$$\omega_g = 2\Delta \left[1 - \frac{2\alpha^2}{\pi^2} \left(1 - \frac{4\Delta^2}{\omega_0^2} \right) \right], \tag{5.25}$$

where α is the dimensionless coupling strength and ω_0 the phonon frequency. The mode called "gap" mode shows up as a very sharp peak in the phonon spectral function $S_{ph}(\omega)$ at the frequency ω_g. In order for it to be observable $\omega_0/24$ must not be too large. In addition $S_{ph}(\omega)$ has a broadened peak at ω_0, and a foot extending down to $\omega = 2\Delta$ reflecting the excitation of quasiparticles. With increasing coupling strength ω_g decreases, the ω_0 peak broadens, the foot becomes larger and the oscillator strength in the gap mode increases. Littlewood and Varma stress that the gap mode involves modulations of the magnitude (amplitude) of the superconducting order parameter. This means that the modulation of a low frequency (ω_0) phonon, coupled suitably to the superconducting order parameter, allows for a long lived collective mode of the order parameter. In contrast, without this coupling we found only a resonance as discussed above, i.e. an oscillation which is strongly damped according to a power law.

The gap mode may be the mode which Sooryakumar and Klein (1980) observed in Raman-scattering experiments on $2H - NbSe_2$ (where $\omega_0/\Delta \approx 5$) as

sharp lines below the superconducting transition temperature $T_c = 7.2$ K. Balseiro and Falicov (1980) explained this peak as a phonon-quasiparticle pair bound state. Beyond the usual BCS interaction they included only the interaction of the ω_0 phonon with the electronic charge. Littlewood and Varma (1981) pointed out that this interaction can only lead to effects close to the plasma frequency of the electron system and since $\omega_{p_1} \gg \omega_0$ has no effect on the phonon self energy near ω_0.

For completeness we mention the possibility of exciton-like modes (see e.g. Martin 1969). They may arise due to electron-electron interactions in angular momentum channels with $l \neq 0$. Crudely they correspond to excited pair states. Their energy may lie in the energy gap of the quasiparticles. However, no conclusive experimental evidence for these modes exists.

21.6 6. Conclusions

In the investigation of the Carlson-Goldman collective mode, similarly as in the case of many relaxation phenomena in superconductors near T_c, a rather good, even quantitative agreement between theory and experiment has been found. This confirms further that the microscopic description summarized in section 3 is correct and complete, at least in the vicinity of the transition temperature. We mention, however, that the analysis of relaxation processes at lower temperatures revealed some unresolved discrepancies between theory and experiment (see e.g. Lemberger and Clarke 1981).

In the literature several descriptions of the collective modes have been presented, which in part appear quite simple. For instance, Artemenko and Volkov (1975) proceed similarly as we do, but start from the clean limit theory of Aronov and Gurevich (1974). It should be kept in mind that this theory is valid only under several restrictions: the system has to be clean, $1/\tau_{\text{imp}} \ll \Delta$, pair breaking effects have to be weak, $\Gamma \ll \Delta$, and only small wavevectors and frequencies are allowed, $\imath_F q, \omega \ll A$. In contrast, the experiments are performed in dirty superconductors, and in connection with the collective modes frequencies of order Δ or larger are interesting, too.

A simple description which resembles a two-fluid picture was deduced by Pethick and Smith (1979) from the clean limit theory. It reproduces the most prominent features of the Carlson-Goldman mode near T_c. It does not include various damping effects-e.g. due to diffusion or inelastic scattering. Since the derivation is rather unsystematic, it is not clear how these effects could be incorporated.

It should be kept in mind, that the Carlson-Goldman collective mode is a rather subtle effect. The currents nearly cancel but still some electric field driving the normal current is left. The damping is proportional to Δ^2/T whereas the approximate descriptions require $\Delta/T \ll 1$. In view of this it appears
fortuitous that any simple macroscopic description of the mode can be developed. We also stress that the concept of a normal chemical potential which has proven a rather accurate parametrization of a nonequilibrium distribution in homogenous, static charge imbalance phenomena (see e.g. Chi and Clarke 1979) no longer does so in high frequency or inhomogenous problems. Descriptions of the collective modes based on this parametrization, e.g. of Kadin et al. (1981), are therefore only approximate. (The transmission line model for example does not account for the damping due to quasiparticle diffusion.) A complete theory has to keep the full functional form of the quasiparticle distribution. It is on this level that the good quantitative agreement with experiments is obtained.

Several of the investigations, based on the microscopic theory (Ovchinnikov 1977, Dinter 1978a,b, Artemenko and Volkov 1979. Kulik et al. 1981) derive the properties of the collective mode directly in a Green function formulation without introducing the concept of the distribution function. In fact, unless ω/A is small the distribution function $f(E;\omega)$ depending on

the generalized densities of states like $N_1(E;\omega)$ which we used in our description, is no longer a classical concept. However, the present formulation has the advantage that it makes contact with the familiar Boltzmann equation description in the low frequency limit. In particular, the inclusion of collision processes appears transparent in this form.

We have restricted the explicit discussion in this article to temperatures close to T_c or to low temperatures. As far as the sound-like collective mode found by Carlson and Goldman is concerned, temperatures very close to T_c are most interesting, but for the quantitative comparison with the experiments (fig. 6,7 and 8) we found it necessary to include corrections in an expansion in Δ/T and ω/T. Since in the high frequency phenomena the inelastic electronphonon collision play only a minor role, the corrections can be determined analytically by straightforward higher order expansion (see the Appendix). Similarly, we could extend the analysis to general temperatures. This requires finding the supercurrent at general temperatures for the space and time dependent problem and to evaluate the integrals $x_{1/}$ (generalized if ω/T is not small). The problem is an integration problem only.

By calculating the order parameter structure factor we have given a complete account of the collective excitations in a superconductor, however, subject to several restrictions. We considered only small (linear) deviations from the equilibrium. The equilibrium state is homogenous in space. We only indicated some of the effects of an imposed supercurrent, which couples amplitude and phase fluctuations of the order parameter. We did not consider coupling to other degrees of freedom in the material, for example, the phonons. These restrictions permit a complete and still rather simple theoretical description, but also exclude many most interesting examples of collective phenomena in superconductors. One which is most intriguing is the existence
of a phase slip center in an otherwise homogenous one-dimensional superconductor. This is clearly a nonlinear phenomenon, involving both magnitude and phase oscillations, which appears if the imposed current is of the order of the critical current. As another example, we mention the Josephson plasma oscillations in a tunnel junction. Here the spatial structure and the Coulomb interaction between charges on the two sides of the junction play a prominent role. This list could be continued, but clearly we could not even try to cover all these collective excitations.

21.7 Appendix. Correction terms in the description of the CarlsonGoldman mode

We have restricted the explicit discussion in the main part of the text to temperatures very close to T_c and to low temperatures. In fact, as far as the sound-like collective mode of Carlson and Goldman is concerned, temperatures near T_c are most interesting. For a quantitative comparison with experiment, deviations from the lowest order expansion in A/T and ω/T may become noticeable. As long as inclastic electron phonon scattering processes can be ignored, i.e. for $\omega \gg 1/\tau_\mathrm{E}$, the generalization to lower temperatures is straightforward, in particular we can easily find the next order corrections in $\Delta/T, \omega/T$, and Dq^2/T.

In this order, the Boltzmann equation and the expression for the charge remain unchanged. Corrections, e.g. from the expansion of $\tanh\left[\left(E+\frac{1}{2}\omega\right)/(2T)\right]-\tanh\left[\left(E-\frac{1}{2}\omega\right)/(2T)\right]$, appear in higher order only. However, the supercurrent acquires additional terms. The corrections to expression (3.23) are

$$j_1' = \frac{\sigma}{e}\frac{\pi\Delta^2}{2T} m v_{\mathrm{s}1}\left[-\frac{1}{2\pi^3}\psi''\left(\frac{1}{2}+\rho\right)\right]\frac{\mathrm{i}\omega - Dq^2/2}{T}, \tag{A.1}$$

where ψ'' is the second derivative of the digamma function, e.g. $\psi''\left(\frac{1}{2}\right) = -14\zeta(3)$. When calculating the pair field susceptibility we again start from the continuity equation, including

a drive term, and make use of the Boltzmann equation. The result is a relation similar to eq. (4.4) but modified in an obvious way to account for div j_1'. Furthermore, the ϕ terms no longer cancel. In the following, we consider only $\Gamma/T \ll 1$.

As long as we can ignore inelastic electron-phonon collisions, the solution of the Boltzmann equation is still of the form (4.7) with $1/\tau_{\mathrm{E}} = 0$. Inclusion of the electron-phonon collision integral operator at temperatures not close to T_c is complicated. The reduced collision operator is sufficient only to lowest order in Δ/T. In general, one has to resort to a numerical solution of the integral equation.

When solving for v_{s1}, the evaluation of $\int N_1 f_1^\top$ and $\int N_2 f_1^\top$ again suggests to consider the integrals X_{ij} defined in eq. (4.9). In their evaluation we now have to retain the factor $\cosh^{-2}(E/2T)$. Also we encounter the integrals

$$X_2 = \int \mathrm{d}E N_2(E;\omega)/\cosh^2(E/2T), \quad X_1 = \int \mathrm{d}E N_1(E;\omega)/\cosh^2(E/2T)$$

Analytic expressions of the integrals X_{ij} for different cases in leading order in Δ/T and ω/T were given in section 4.2. The correction terms are

$$\begin{aligned} X_{12}' &= \frac{\mathrm{i}\omega}{-\mathrm{i}\omega + Dq^2}\frac{7\zeta(3)}{\pi^2}\frac{\Delta}{T} \\ X_{22}' &= 0\left(\frac{\omega\Delta^2}{T^3}\right) \approx 0 \\ X_{11}' &= -\frac{\mathrm{i}\omega\Lambda}{(-\mathrm{i}\omega + Dq^2)^2}\frac{7\zeta(3)}{\pi^2}\frac{\Delta}{T} \end{aligned} \tag{A.2}$$

Furthermore, we have to retain the correction to X_2

$$X_2 = \pi\Delta + i\omega\frac{7\zeta(3)}{\pi^2}\frac{\Delta}{T} \tag{$\prime$}$$

and in the limit considered we have $X_1 = 4T\mathrm{Y}$, where Y is the Yoshida function. Near T_c it is $\mathrm{Y} = 1 - (7\zeta(3)/4\pi^2)\,(\Delta^2/T^2)$. We thus find

$$\begin{aligned} \chi^{\mathrm{I}} = \frac{4T}{\pi N_0}\Bigg\{ &-\mathrm{i}\omega\left[\frac{X_2}{\pi\Delta} - \mathrm{i}\omega\frac{2T}{\pi\Delta^2}(1-\mathrm{Y})\right] + Dq^2\left[1 + \left(\mathrm{i}\omega - Dq^2/2\right)\frac{7\zeta(3)}{\pi^3 T}\right] \\ &+\mathrm{i}\omega\frac{2}{\pi}\left[X_{22} - \frac{\left(X_{12} + \frac{2T}{4}(1-\mathrm{Y})\right)^2}{\tilde{X}_{11} - \frac{4TDq^2}{(-\mathrm{i}\omega+Dq^2)(-\mathrm{i}\omega)}}\right]\Bigg\}^{-1}, \end{aligned} \tag{A.3}$$

where the integrals X_{ij} include the correction (A.2).

Note added in proof

(?????)

In restricted geometries the Coulomb forces are weaker than in the bulk and hence do not shift the frequency of density oscillations to the bulk plasma frequencies. For example in superconducting filaments of cross-section smaller than about 10^{-14} m^2, in a large frequency range at all temperatures below T_{c}, a collective density oscillation exists with a sound-like dispersion relation and weak damping. See

Mooij, J.E., and G. Schön, 1985, Phys. Rev. Lett. 55, 114.

22 Stability of nonequilibrium Fermi distributions with respect to Cooper pairing by Aronov, Gurevich

(current conclusion: I don't understand this article, because it uses Keldysh's formalism, which I don't know yet)

Abstract

The stability of normal and superconducting nonequilibrium Fermi systems relative to Cooper pairing and Cooper pair decay, respectively, is investigated. If the inequality $\hbar/\tau\Delta \ll 1$ is obeyed, where Δ is the superconducting gap thickness and τ is the electron-phonon collision time, the stability condition does not depend explicitly on the method whereby the imbalance is produced, and is expressed only in terms of the superconducting excitation distribution function $m(\epsilon)$. It is shown that if the sign of the difference $1-2m(\epsilon)$ is constant throughout the complete energy region ϵ, the superconducting state will be stable. The stability condition obtained in the paper is used to analyze a number of concrete cases.

22.0.1 1. Introduction

The question of the possibility of Cooper pairing of electrons under strongly nonequilibrium conditions has recently evoked considerable interest from the experimental point of view [1,2] as well as from the point of view of theoretical developments. [3–12]

As Éliashberg has shown, [6] in a number of interesting cases even in nonequilibrium systems the gap parameter Δ is determined from the BCS equation, in which it is only necessary to replace the equilibrium distribution function of the superconducting excitations by the nonequilibrium one:

$$\Delta = -\lambda N(0) \int_{\Delta}^{\omega_D} \Delta \frac{n(-\varepsilon) - n(\varepsilon)}{(\varepsilon^2 - \Delta^2)^{1/2}} d\varepsilon \tag{1.1}$$

where λ is the effective coupling constant of the electron-electron interaction, N(0) is the density of electron states near the Fermi surface, ω_D is the limiting frequency of the Debye phonons, and $n(\epsilon)$ is the (nonequilibrium) distribution function of the superconducting excitations. It is determined by the type of external perturbation creating the nonequilibrium state, and it also depends on Δ as a parameter. If the distribution function is anisotropic, then n(ϵ) should be understood as the corresponding average value over a surface of constant energy. In the so-called symmetric case, which we shall mainly be occupied with in what follows, one has

$$n(-\varepsilon) = 1 - n(\varepsilon). \tag{1.2}$$

Let Eq. (1.1) have a nontrivial solution. For the sake of definiteness, we first assume this solution to be unique. In addition, a solution with $\Delta = 0$ also exists. Thus arises the necessity of comparing these two solutions and ascertaining which of them will actually be realized.

In the equilibrium case this question can be resolved on the basis of simple energy considerations. The solution with $\Delta \neq 0$ would correspond to a smaller energy of the ground state (or, at finite temperatures below the critical temperature-it would correspond to a smaller value of the free energy) than the solution with $\Delta = 0$. In nonequilibrium cases simple energy considerations do not permit us to compare these two states since the energy of the superconductor is not conserved: it interacts with the external system creating the nonequilibrium. Therefore, in order to solve the problem which has been posed, it is natural to investigate the stability of these two states: the state with $\Delta = 0$ with regard to Cooper

pairing, and the state with $\Delta \neq 0$ with regard to the destruction of the pairs. In order to do this it is necessary to calculate the two-particle retarded Green's function of the electrons in a conductor for small total momentum and energy of the electrons. If the Green's function, regarded as a function of the total energy, has a pole in the upper half-plane, this implies instability. The instability of a normal Fermi system with attractive interactions for T = 0 is established in such a manner. [13a] The stability of the normal, nonequilibrium Fermi distribution was first investigated by Batyev [4] in the same way, using Keldysh's diagram technique. [14]

The stability of the state with $\Delta \neq 0$ with respect to small fluctuations of the order parameter Δ is investigated in the present article. The solution of the stability problem is greatly facilitated owing to the fact that here the nonequilibrium distribution function of the excitations can be regarded as fixed. The point is that in order to investigate the stability of the system with regard to pairing, it is sufficient to trace its evolution during a time interval of order $\hbar/\Delta$. This time interval is assumed to be much smaller than the characteristic elec-tron-phonon collision time τ, during which pronounced changes occur in the average values of the quasi-particle distribution functions. We substantially utilize the inequality $\hbar/\tau\Delta \ll 1$ in the construction of the theory. It is found that the superconducting nonequilibrium state is stable in all cases when Eq. (1.1) has a nontrivial solution and the difference $\mathrm{n}(-\epsilon) - \mathrm{n}(\epsilon)$ keeps the same sign for all $\epsilon > 0$.

However, if the sign of the difference $n(-\epsilon) - n(\epsilon)$ is not constant, then the system generally possesses a number of nontrivial properties. Thus, additional modes of collective oscillations, which are not present in the equilibrium superconductor, may exist in it. Furthermore, the system may turn out to be unstable with respect to fluctuations of the order parameter Δ, where the instability manifests itself as an increase in the amplitude of these collective oscillations. An example of such an instability is considered in Sec. 4.

For a system in an external field, the investigation of stability is valid only when the momentum acquired by a Cooper pair during the field's period $1/\Omega$ (or, in the case of a constant field, over the length of the sample) is small in comparison with the coherence length ξ : [1)]

$$eE/\Omega \ll \hbar/\xi = \hbar m\Delta/p_F,$$

where $\mathrm{p_F}$ is the "Fermi momentum," that is, the momentum corresponding to the middle of the energy gap. This criterion turns out to be substantially more stringent than the requirement that $e^2E^2/\mathrm{m}\Omega^2$ be small in comparison with Δ; it characterizes the accuracy of our theory in determining the critical value Δ_c, corresponding to the boundary of the stability region.

In the case of a normal conductor one can show that the influence of the field can be neglected if its work on a pair during one period of the field is small in comparison with the increment of growth of the amplitude of the Cooper pair wave function. The theory can actually be applied to the case when the electron distribution function is noticeably perturbed near the Fermi surface whereas the influence of the field on the dynamics of the electrons is negligible-for example, if the nonequilibrium distribution appeared in connection with the illumination of a thin metal film.

Hitherto the question has only been the dynamical stability of the system with respect to Cooper pairing. However, since the topic of discussion is strongly nonequilibrium systems it is necessary in general to also investigate their stability with respect to perturbations of the quasi-particle distribution functions. This investigation is carried out mainly by standard methods with the help of the kinetic equation. It is only necessary to keep the following point in mind. It turns out to be possible to investigate the dynamical stability of the system for a fixed quasi-particle distribution function. On the other hand, it is impossible to investigate the stability of the distribution function for a fixed spectrum because the magnitude of the gap Δ and together with it the transition probabilities as well depend on the shape of the distribution

function. Therefore, a perturbation of the distribution function must lead to a perturbation of the order parameter, and therefore also leads to a perturbation of the transition probabilities.

We shall use the results of the general theory in order to investigate the stability in a number of specific cases. In particular, one unusual example is consid-ered-the so-called inverse Fermi distribution. We want to explain [2)] the fact (which appears to be paradoxical at first glance) that Eq. (1.1) has a nonvanishing solution Δ for $\lambda > 0$ (repulsion) for the case when the distribution function $n(\epsilon)$ has the following form in the energy range of interest to us: $n(\epsilon) = 1$ for $\epsilon > 0$ and $n(\epsilon) = 0$ for $\epsilon < 0$. In this case the solution with $\Delta = 0$ turns out to be unstable with respect to pairing, and the solution with $\Delta \neq 0$ turns out to be stable.

One might describe the physics of this bound state in the following way. The inverse distribution of Fermi particles is equivalent to an ordinary distribution of Fermi holes with a negative effective mass. But, as is clear from an analysis of the corresponding Schrödinger equation, particles with negative effective mass form bound states in the presence of repulsion. The case of the bielectron [15] is a well-known example of this.

The bound states, which arise in the present case, form a Bose condensate, a gap appears in the spectrum of the single-particle excitations, and the total energy of the system increases upon pairing. One can verify this, in particular, by calculating the energy of the system as a function of Δ. The energy has a maximum at the value of Δ determined from condition (1.1).

Thus, in the present example we verify that energy considerations in general do not permit us to judge the dynamical stability of a nonequilibrium system. It should be noted that it is necessary to consider the inverse Fermi distribution as a purely model example because of the following reason. If it is assumed that, upon a further increase of the energy ϵ within the limits of the same band, the function $n(\epsilon)$ again vanishes, then even though such a scheme is stable with respect to the generation of plasmons and other longitudinal oscillations, [13 b] the characteristic electron-electron collision time τ_{e} in it turns out to be so small that the parameter $\hbar/\tau_{\mathrm{e}}\Delta$ turns out to be of the order of unity in the best case. However, if $n(\epsilon) = 1$ right up to the edge of the forbidden band, then the corresponding system is unstable with respect to the generation of plasmons and other longitudinal oscillations.

We shall use the diagram technique developed by Keldysh. [14] It permits us to determine simultaneously the energy spectrum of a nonequilibrium system and also its kinetic characteristics, that is, in our case we can determine the nonequilibrium distribution function of the superconducting excitations. In particular, an expression for the operator describing the collisions of the excitations with phonons is obtained by such a method. The corresponding expression was previously obtained by Éliashberg. [6]3) However, our expression (2.18) is applicable to any anisotropic, nonequilibrium distribution function, and is not averaged over angles, as was done in [6].

22.0.2 2. Derivation of the Kinetic Equation

Following Keldysh, [14] we define the normal Green's function of the electrons in the following way:

$$\hat{G^{\alpha\beta}}(x,x') \equiv \begin{pmatrix} G_c^{\alpha\beta}(x,x') & G_+^{\alpha\beta}(x,x') \\ G_-^{\alpha\beta}(x,x') & G_c^{\alpha\beta}(x,x') \end{pmatrix} = \begin{pmatrix} G_{11}^{\alpha\beta} & G_{12}^{\alpha\beta} \\ G_{21}^{\alpha\beta} & G_{22}^{\alpha\beta} \end{pmatrix} = \\ = \begin{pmatrix} -i\left\langle T\psi_\alpha(x)\psi_\beta{}^+(x')\right\rangle, & i\left\langle \psi_\beta{}^+(x')\,\psi_\alpha(x)\right\rangle \\ -i\left\langle \psi_\alpha(x)\psi_\beta{}^+(x')\right\rangle, & -i\left\langle \tilde{T}\psi_\alpha(x)\psi_\beta{}^+(x')\right\rangle \end{pmatrix} \tag{2.1}$$

Here α and β are spin indices, T and $\tilde{\mathrm{T}}$ denote T-products and anti-T-products, respectively, and the angular brackets denote averages with respect to the unperturbed Hamiltonian.

In analogous fashion we introduce the matrix functions $\hat{\mathbf{F}}$ and ${}^{+}\hat{\mathbf{F}}$:

$$F^{\alpha\beta}(x,x') = \begin{pmatrix} \langle T\psi_\alpha(x)\psi_\beta(x')\rangle, & -\langle\psi_\beta(x')\psi_\alpha(x)\rangle \\ \langle\psi_\alpha(x)\psi_\beta(x')\rangle, & \left\langle\tilde{T}\psi_\alpha(x)\psi_\beta(x')\right\rangle \end{pmatrix} \equiv \begin{pmatrix} F_c^{\alpha\beta}(x,x') & F_+^{\alpha\beta}(x,x') \\ F_-^{\alpha\beta}(x,x') & F_c^{\alpha\beta}(x,x') \end{pmatrix} \tag{2.2}$$

The function ${}^{+}\hat{F}$ is obtained from $\hat{F}$ by replacing all of the operators ψ in Eq. (2.2) by ψ^{+}.

As is usually done, let us assume that the interaction does not depend on the spins. Then

$$\hat{G}_{a\beta}^{\alpha\beta}(x,x') = \hat{G}(x,x')\,\delta_{\alpha\beta}, \quad \hat{F}^{\alpha\beta}(x,x') = -\hat{F}(x,x')\,I_{\alpha\beta}$$
$$\hat{I} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} i\sigma_y$$

(there is no explanation here, for understanding why we are working this way, one needs to see other articles or books!)

Similarly we find:

$${}^{+}F^{\alpha\beta} = {}^{+}F I_{\alpha\beta}$$

The phonon Green's function is given by

$$\hat{D} = \begin{pmatrix} D_c D_+ \\ D_- \widetilde{D}_c \end{pmatrix} = \begin{pmatrix} -i\langle T\varphi(x)\varphi(x')\rangle, & -i\langle\varphi(x')\varphi(x)\rangle \\ -i\langle\varphi(x)\varphi(x')\rangle, & -i\left\langle\tilde{T}\varphi(x)\varphi(x')\right\rangle \end{pmatrix} \tag{2.3}$$

Later we shall be interested in the case of a spatially homogeneous system, where the Green's functions depend on the difference between the spatial coordinates. Then, assuming that the Green's functions are slowly varying functions of the total time t, and evaluating the Fourier component with respect to the difference of coordinates and times, we obtain the following result in the zero-order approximation (furthermore, we take $\hbar = 1$):

$$\hat{G}_0^{-1} = \left(\frac{i}{2}\frac{\partial}{\partial t} + \varepsilon - \xi_{\mathrm{p}}\right)\hat{\sigma}_z, \tag{2.4}$$

where σ_Z is a matrix which acts on the lower indices (not the spin indices) of the Green's function, $\xi_{\mathrm{p}} = (\mathrm{p}^2/2\ \mathrm{m}) - \mu$, μ denotes the origin with respect to which the energy is measured; later on we choose this to coincide with the middle of the superconducting energy gap.

The Feynman diagram technique can be used for the functions $\hat{G}$ and $\hat{F}$, and consequently the corresponding system of Dyson equations has the form

$$\begin{aligned} \left(\frac{i}{2}\frac{\partial}{\partial t} + \varepsilon - \xi_{\mathrm{p}}\right)\sigma_z\hat{G}(p,t) &= \hat{1} + i\hat{\Sigma}(p,t)\hat{G}(p,t) + i\hat{\Delta}(p,t)^{+}\hat{F}(p,t) \\ \left(\frac{i}{2}\frac{\partial}{\partial t} - \varepsilon - \xi_{\mathrm{p}}\right)\sigma_z{}^{+}\hat{F}(p,t) &= i\hat{\Sigma}^T(-p,t)^{+}\hat{F}(p,t) + i^{+}\hat{\Delta}(p,t)\hat{G}(p,t) \end{aligned} \tag{2.5}$$

(???? why???)

One can isolate the system of equations for $G_c(p,t)$ and ${}^{+}F_c(p,t)$ from the system (2.5) by taking the 11-component in the matrix equations. In this connection we discard the time derivative and also all terms which contain the functions G_+, D_+, and ${}^{+}F_+$. The latter step is permissible if the parameter $1/\tau\Delta$ is small, where τ is the characteristic time for electron-phonon collisions.

(??? why this approach works for noneqSC??? how does it generalizes eq sc approach???)

As an example let us present the expression for the self-energy part Δ_c, which appears in the right-hand side of the equation

$$\Delta_{\mathrm{c}}(\varepsilon, \mathbf{p}) = \int \frac{d^3q}{(2\pi)^3} \int \frac{d\omega}{2\pi} |c_{\mathbf{q}}|^2 D_c(-\mathbf{q}, -\omega) F_{\mathrm{c}}(\mathbf{p}-\mathbf{q}, \varepsilon-\omega), \tag{2.6}$$

where $\mathrm{c_q}$ is the electron-phonon coupling constant (I'll change notation later). Just as in the equilibrium case, the dispersion $\Delta_c(\epsilon, p, t)$ arises for $\epsilon \sim \omega_{\mathrm{D}}$. For our purposes, however, as will be clear from what follows below, energies $\epsilon \ll \omega_D$ are important. In addition, just like in the equilibrium case, frequencies $\omega \approx \Delta\left(0, \mathrm{p_F}\right) \ll \omega_{\mathrm{D}}$ are essential in Eq. (2.6). Taking these inequalities into account, we obtain, just as in the equilibrium case, the following equation for $\Delta_c\left(0, p_F\right)$:

(??? I don't understand how the transformation is done???)

$$\Delta = -\frac{\lambda}{2} \int \frac{d^3p}{(2\pi)^3} \int \frac{d\varepsilon}{2\pi} F_c(\varepsilon, \mathbf{p}) \tag{2.7}$$

where λ is a constant.

For values of $\epsilon \ll \omega_{\mathrm{D}}$ the solution of the system of equations (2.5) for the functions G_c and F_c has the form

(derive!!!)

$$\begin{aligned} G_c(\varepsilon, \mathbf{p}) &= u_{\mathbf{p}}^2 \left(\frac{m_{\mathbf{p}}}{\varepsilon - \varepsilon_{\mathbf{p}} - i\delta} + \frac{1 - m_{\mathbf{p}}}{\varepsilon - \varepsilon_{\mathbf{p}} + i\delta} \right) + v_{\mathbf{p}}^2 \left(\frac{m_{-\mathbf{p}}}{\varepsilon + \varepsilon_{\mathbf{p}} + i\delta} + \frac{1 - m_{-\mathbf{p}}}{\varepsilon + \varepsilon_{\mathbf{p}} - i\delta} \right), \\ {}^{+}F_c(\varepsilon, \mathbf{p}) = F_c(\varepsilon, \mathbf{p}) &= -i u_{\mathbf{p}} v_{\mathbf{p}} \left(\frac{m_{\mathbf{p}}}{\varepsilon - \varepsilon_{\mathbf{p}} - i\delta} + \frac{1 - m_{\mathbf{p}}}{\varepsilon - \varepsilon_{\mathbf{p}} + i\delta} - \frac{m_{-\mathbf{p}}}{\varepsilon + \varepsilon_{\mathbf{p}} + i\delta} - \frac{1 - m_{-\mathbf{p}}}{\varepsilon + \varepsilon_{\mathbf{p}} - i\delta} \right) \end{aligned} \tag{2.8}$$

$$u_{\mathbf{p}}^2 = \frac{1}{2}\left(1 + \frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\right), \quad v_{\mathbf{p}}^2 = \frac{1}{2}\left(1 - \frac{\xi_{\mathbf{p}}}{\varepsilon_{\mathbf{p}}}\right), \quad \varepsilon_{\mathbf{p}} = \left(\Delta^2 + \xi_p^2\right)^{1/2}.$$

Here m_p is an undetermined (for the time being) positive function whose absolute value does not exceed unity. We shall call it the distribution function of the excitations, and we shall write down a kinetic equation for it below. One can verify that it is related to the distribution function of the quasi-electrons by the following equations:

$$\begin{aligned} m_{\mathrm{p}} &= n_{\mathrm{p}} \text{ for } \xi_{\mathrm{p}} > 0, \\ m_{\mathrm{p}} &= 1 - n_{\mathrm{p}} \text{ for } \xi_{\mathrm{p}} < 0. \end{aligned} \tag{2.9}$$

By substituting expressions (2.8) into Eq. (2.7) we obtain Eq. (1.1) of the BCS theory.

(do it!!!)

Using the definitions of the Green's functions, it is not difficult to obtain the following spectral representations for the causal functions:

$$\begin{aligned} G_c(\mathbf{p}, \varepsilon) &= \int_{-\infty}^{\infty} \frac{d\varepsilon'}{2\pi i} \left[\frac{G_+(\varepsilon', \mathbf{p})}{\varepsilon - \varepsilon' - i\delta} - \frac{G_-(\varepsilon', \mathbf{p})}{\varepsilon - \varepsilon' + i\delta} \right], \\ {}^{+}F_c(\mathbf{p}, \varepsilon) &= \int_{-\infty}^{\infty} \frac{d\varepsilon'}{2\pi i} \left[\frac{{}^{+}F_-(\varepsilon', p)}{\varepsilon - \varepsilon' - i\delta} - \frac{{}^{+}F_+(\varepsilon', \mathbf{p})}{\varepsilon - \varepsilon' + i\delta} \right]. \end{aligned} \tag{2.10}$$

One can write down a similar relationship for $F_c(\epsilon, p)$. The relationships for the anti-causal functions are obtained if we interchange the positions of the functions with plus and minus subscripts on the right-hand side of (2.10).

Comparing expressions (2.10) with (2.8), we find

$$G_+(\varepsilon, \mathbf{p}) = 2\pi i \left[u_p^2 m_p \delta(\varepsilon - \varepsilon_p) + v_p^2 (1 - m_{-p}) \delta(\varepsilon + \varepsilon_p)\right],$$
$$G_-(\varepsilon, p) = -2\pi i \left[u_p^2 (1 - m_p) \delta(\varepsilon - \varepsilon_p) + v_p^2 m_{-p} \delta(\varepsilon + \varepsilon_p)\right], \tag{2.11}$$
$$^+F_+(\varepsilon, \mathbf{p}) =^+ F_-(-\varepsilon, -\mathbf{p}) = F_+(\varepsilon, \mathbf{p}) = -2\pi u_p v_p \left[(1 - m_p) \delta(\varepsilon - \varepsilon_p) - m_{-p} \delta(\varepsilon + \varepsilon_p)\right]. \tag{2.12}$$

It is clear from (2.11) that

$$m_{\mathrm{p}} = \int_0^\infty \frac{d\varepsilon}{2\pi i} \left[G_+(p) - G_-(-p)\right]. \tag{2.13}$$

(?? I am lost, what are we doing???)

Thus, in order to derive the kinetic equation for the function m_p, it is necessary to obtain equations for $G_+(p)$ and G$_-$(p). Since these functions are purely imaginary, by taking the real part of the matrix component 12 of the first equation of (2.5) we obtain

$$\frac{\partial}{\partial t} \operatorname{Im} G_+ = \operatorname{Im} \left\{\Sigma_+ G_- - \Sigma_- G_+ + \Delta_+^+ F_- - \Delta_-^+ F_+\right\} \tag{2.14}$$

Here, in connection with the transformation of the right-hand part, it is convenient to use the Lehmann representations for the self-energy parts, which have the form

$$\Sigma_{\mathrm{c}}(\mathbf{p}, \varepsilon) = -\int \frac{d\varepsilon'}{2\pi i} \left[\frac{\Sigma_+(\mathbf{p}, \varepsilon')}{\varepsilon - \varepsilon' - i\delta} - \frac{\Sigma_-(\mathbf{p}, \varepsilon')}{\varepsilon - \varepsilon' + i\delta}\right]. \tag{2.15}$$

This can easily be checked in lowest-order perturbation theory, which is the approximation of interest to us.

As an example, let us write down the perturbation theory expression for one of the self-energy parts, namely Δ_-:

$$\Delta_-(\varepsilon, \mathbf{p}) = \int \frac{d^3 q}{(2\pi)^3} \int \frac{d\omega}{2\pi} |c_{\mathbf{q}}|^2 D_-(\omega, \mathbf{q}) F_-(\varepsilon - \omega, \mathbf{p} - \mathbf{q}). \tag{2.16}$$

Here D$_-(\omega, \mathrm{q})$ is the phonon propagator. In the lowest-order approximation we find

$$D_\pm(\omega, \mathbf{q}) = -2\pi i \left[(N_{\mp\mathbf{q}} + 1) \delta(\omega \pm \omega_{\mathbf{q}}) + N_{\pm\mathbf{q}} \delta(\omega \mp \omega_{\mathbf{q}})\right], \tag{2.17}$$

where N$_{\mathrm{q}}$ is the phonon distribution function.

(??? below, I am fully lost)

A similar relationship is also obtained for $G_-(-p)$. Subtracting one from the other, substituting the explicit expressions for the propagators and the self-energy parts into the right-hand side, and then integrating over ϵ in the semi-infinite limits, we obtain the following expression for the collision integral:

$$\left(\frac{\partial m_{\mathrm{p}}}{\partial t}\right)_{\mathrm{coll}} = 2\pi \sum_{\mathrm{q}} |c_{\mathrm{q}}|^2 \left\{(1 - m_{\mathrm{p}}) m_{\mathrm{p-q}} (u_{\mathrm{p}} u_{\mathrm{p-q}} - v_{\mathrm{p}} v_{\mathrm{p-q}})^2\right.$$

$$\times \left[(1+N_{-\mathrm{q}})\,\delta\left(\varepsilon_{\mathrm{p}}-\varepsilon_{\mathrm{p-q}}+\omega_{\mathrm{q}}\right)+N_{\mathrm{q}}\delta\left(\varepsilon_{\mathrm{p}}-\varepsilon_{\mathrm{p-q}}-\omega_{\mathrm{q}}\right)\right]-m_{\mathrm{p}}\left(1-m_{\mathrm{p-q}}\right)$$
$$\times\left(u_{\mathrm{p}}u_{\mathrm{p-q}}-v_{\mathrm{p}}v_{\mathrm{p-q}}\right)^2\left[N_{-\mathrm{q}}\delta\left(\varepsilon_{\mathrm{p}}-\varepsilon_{\mathrm{p-q}}+\omega_{\mathrm{q}}\right)+(1+N_{\mathrm{q}})\,\delta\left(\varepsilon_{\mathrm{p}}-\varepsilon_{\mathrm{p-q}}-\omega_{\mathrm{q}}\right)\right]$$
$$+\left(1-m_{\mathrm{p}}\right)\left(1-m_{-\mathrm{p+q}}\right)\left(u_{\mathrm{p}}v_{\mathrm{p-q}}+u_{\mathrm{p-q}}v_{\mathrm{p}}\right)^2\Big[(1+N_{-\mathrm{q}})\,\delta\left(\varepsilon_{\mathrm{p}}+\varepsilon_{\mathrm{p-q}}+\omega_{\mathrm{q}}\right)$$
$$+N_{\mathrm{q}}\delta\left(\varepsilon_{\mathrm{p}}+\varepsilon_{\mathrm{p-q}}-\omega_{\mathrm{q}}\right)\Big]-m_{\mathrm{p}}m_{-\mathrm{p+q}}\left(u_{\mathrm{p}}v_{\mathrm{p-q}}+u_{\mathrm{p-q}}v_{\mathrm{p}}\right)^2$$
$$\times\left[N_{\mathrm{q}}\delta\left(\varepsilon_{\mathrm{p}}+\varepsilon_{\mathrm{p-q}}+\omega_{\mathrm{q}}\right)+(1+N_{-\mathrm{q}})\,\delta\left(\varepsilon_{\mathrm{p}}+\varepsilon_{\mathrm{p-q}}-\omega_{\mathrm{q}}\right)\right]\Big\}. \tag{2.18}$$

The first two terms on the right-hand side describe processes which conserve the number of excitations, that is, the scattering of excitations accompanied by the absorption or emission of phonons. The third and fourth terms describe the creation of a pair of quasi-particles from the condensate with the absorption of a phonon and the annilation of a pair with the emission of a phonon. The terms containing δ-functions of the form $\delta\left(\epsilon_p+\epsilon_{p-q}+\omega_p\right)$ vanish identically since in the quasiparticle representation we are using $\epsilon_p>0$. However, instead of quasi-electrons above the gap and quasi-holes below the gap, as we are now doing, one can write down the kinetic equation in another representation used by us earlier, [10,11] which operates by using the quasielectron distribution function $\mathrm{n_p}$ both above and below the gap. 4) In this representation the form of the collision operator remains the same, but since for $\xi_p<0$ the energy of the quasi-electrons in this representation is negative, all of the δ-functions give a contribution to the collision operator.

22.0.3 3. The Two-Particle Green's Function and the Conditions for Stability

In order to investigate stability, it is necessary to determine the two-particle retarded Green's function associated with a small total momentum q and total frequency ω. Here we shall use an effective four-fermion Hamiltonian with an effective coupling constant λ.

In the normal metal the problem reduces to a summation of the series which is graphically shown in Fig. 1 (see [13a]). The vertex part

$$\hat{\Gamma}_{\alpha\beta,\gamma\delta}(q)=\hat{\Gamma}(q)\left(\delta_{\alpha\gamma}\delta_{\beta\delta}-\delta_{\alpha\delta}\delta_{\beta\gamma}\right)$$

satisfies the following matrix equation:

$$\hat{\Gamma}(q)=\hat{\Gamma}_0+i\hat{\Gamma}_0\hat{S}(q)\hat{\Gamma}(q) \tag{3.1}$$

where $\hat{\Gamma}_0=\lambda\hat{\sigma}_Z$. The quantity $\hat{\Gamma}$ has the properties of a boson Green's function and is a 2×2 matrix, and $\widehat{\mathrm{S}}$ is the self-energy part, which is given by

$$\hat{S}=\begin{pmatrix} S_c & S_+ \\ S_- & \hat{S}_c \end{pmatrix}=\begin{pmatrix} \int\frac{d^4p}{(2\pi)^4}G_c(p)G_c(q-p), & \int\frac{d^4p}{(2\pi)^4}G_+(p)G_+(q-p) \\ \int\frac{d^4p}{(2\pi)^4}G_-(p)G_-(q-p), & \int\frac{d^4p}{(2\pi)^4}G_c(p)G_c(q-p) \end{pmatrix}, \tag{3.2}$$

In order to obtain the retarded Green's function $\Gamma_{\mathbf{r}}$ we make a unitary transformation of Eq. (3.1) with the help of the matrix $U=(1+i\sigma_y)/\sqrt{2}$. Finally we have

$$U^{-1}\hat{\Gamma}U=\begin{pmatrix} 0 & \Gamma_a \\ \Gamma_r & \Gamma_d \end{pmatrix},\quad \Gamma_d=\Gamma_++\Gamma_-,\quad \Gamma_{r,a}=\Gamma_c-\Gamma_\pm \tag{3.3}$$

Furthermore, we find

$$U^{-1}\hat{S}U=\begin{pmatrix} S_d & S_r \\ S_a & 0 \end{pmatrix},\quad S_d=S_++S_-,\quad S_{r,a}=S_c+S_\pm \tag{3.4}$$

In deriving these formulas we have used the identities [14]

$$\Gamma_{\mathrm{c}} + \widetilde{\Gamma}_{\mathrm{c}} = \Gamma_{+} + \Gamma_{-} \tag{3.5}$$

$$S_{\mathrm{c}} + S_{\mathrm{c}} + S_{+} + S_{-} = 0 \tag{3.6}$$

Hence

$$\Gamma_r = \lambda + i\lambda S_r \Gamma_r \tag{3.7}$$

Thus, the constructive method of evaluating Γ_{r_1} consists in writing down the graphical equation Fig. 1 directly for the retarded Green's function Γ_r. One can evaluate the self-energy part $\mathrm{S_r}$ appearing in it with the aid of the relationship (3.4).

FIG. 1

FIG. 2

Anomalous electron Green's functions exist in the case of a superconductor. Therefore, in calculating the self-energy parts, the system of equations, which is represented graphically in Fig. 2, arises for the quantities Γ_1, Γ_2 and Γ_3 :

$$\left(\hat{1} - i\lambda \hat{S}_T\right) \begin{pmatrix} I_{1r} \\ I_{2r} \\ I_3 \end{pmatrix} = \begin{pmatrix} \lambda \\ 0 \\ 0 \end{pmatrix} \tag{3.8}$$

Here the elements of the matrix $\hat{\mathrm{S}}_{\mathrm{c}}$ are given by

$$S_{11}(q) = S_{22}(-q) = \int \frac{d^4k}{(2\pi)^4} G_{\mathrm{c}}(k) G_{\mathrm{c}}(q-k),$$

$$S_{12}(q) = S_{21}(q) = \int \frac{d^4k}{(2\pi)^4} F_{\mathrm{c}}(k) F_{\mathrm{c}}(q-k),$$

$$S_{13}(q) = S_{31}(q) = S_{23}(-q) = S_{32}(-q) = -\int \frac{d^4k}{(2\pi)^4} F_{\mathrm{c}}(k) G_{\mathrm{c}}(q-k),$$

$$S_{33}(q) = -\int \frac{d^4k}{(2\pi)^4} \left[G_{\mathrm{c}}(k) G_i(k+q) + F_{\mathrm{c}}(k)^{+} F_{\mathrm{c}}(k+q)\right]. \tag{3.9}$$

By doing the calculation and then analytically continuing the answer, we obtain the following results (the subscript **r** is omitted):

$$S_{11}(q) = \sum_{\mathbf{k}} \left[u^2_{\mathbf{k+q}/2} u^2_{\mathbf{k-q}/2} f_q(\mathbf{k}) + v^2_{\mathbf{k+q}/2} v^2_{\mathbf{k-q}/2} f_{-q}(\mathbf{k}) + 2u^2_{\mathbf{k+q}/2} v^2_{\mathbf{k-q}/2} \varphi_q(\mathbf{k})\right] \tag{3.10}$$

$$S_{12}(q) = -\sum_{\mathbf{k}} u_{\mathbf{k+q}/2} v_{\mathbf{k+q}/2} u_{\mathbf{k-q}/2} v_{\mathbf{k-q}/2} \left[f_q(\mathbf{k}) + f_{-q}(\mathbf{k}) - 2\varphi_q(\mathbf{k})\right]$$

$$S_{13}(q) = -i \sum_{\mathbf{k}} u_{\mathbf{k+q}/2} v_{\mathbf{k-q}/2} \left[u_{\mathbf{k+q}/2} u_{\mathbf{k-q}/2} \left[\varphi_q(\mathbf{k}) - f_q(\mathbf{k})\right]\right.$$
$$\left. - v_{\mathbf{k+q}/2} v_{\mathbf{k-q}/2} \left[\varphi_q(\mathbf{k}) - f_{-q}(\mathbf{k})\right]\right\}$$

$$S_{33}(q) = -S_{12}(q) + \sum_{\mathbf{k}} \left[\left(u^2_{\mathbf{k-q}/2} u^2_{\mathbf{k+q}/2} + v^2_{\mathbf{k-q}/2} v^2_{\mathbf{k+q}/2}\right) \varphi_q(\mathbf{k})\right.$$
$$\left. + u^2_{\mathbf{k+q}/2} v^2_{\mathbf{k-q}/2} \left[f_q(\mathbf{k}) + f_{-q}(\mathbf{k})\right]\right]$$

$$f_q(\mathbf{k}) = i \frac{m_{\mathbf{k+q}/2} + m_{-\mathbf{k+q}/2} - 1}{\omega - \varepsilon_{\mathbf{k+q}/2} - \varepsilon_{\mathbf{k-q}/2} + i\delta} = f_q(-\mathbf{k}) \tag{3.11}$$

$$\varphi_q(\mathbf{k}) = i \frac{m_{\mathbf{k+q}/2} - m_{\mathbf{k-q}/2}}{\omega - \varepsilon_{\mathbf{k+q}/2} + \varepsilon_{\mathbf{k-q}/2} + i\delta} = \varphi_{-q}(\mathbf{k}) \tag{3.12}$$

We obtain the dispersion equation by setting the determinant of the system of equations (3.8) equal to zero:

$$\left\| \hat{1} - i\lambda \hat{S}_r(q) \right\| = 0 \tag{3.13}$$

It follows from Eq. (3.13) that the dispersion equation is invariant under the substitutions $\lambda \to -\lambda$ and $\mathrm{m_k}$
$\to 1 - \mathrm{m_k}$, that is, the spectrum of the collective oscillations and of the two-particle excitations in a system having the usual Fermi distribution and attractive interactions turns out to be the same as in a system with the inverse Fermi distribution and repulsion.

Let us start with an analysis of the stability conditions for the normal metal. For $\Delta = 0$ and $\mathrm{q} = 0$ the dispersion equation (3.13) gives

$$1 = \lambda N(0) \int_{-\omega_D}^{\omega D} d\xi \frac{1 - 2n(\xi)}{\omega - 2\xi + i\delta} \tag{3.14}$$

For the given form of the distribution function $n(\xi)$, this equation has complex roots. If the roots are located in the lower half-plane, then the state is stable. If the function $\mathrm{n}(\xi)$ depends on some external parameter (and in the equilibrium case, it would depend on the temperature), then a root may intersect the real axis for a certain value of this parameter. This is therefore the stability boundary of the normal state. It is determined from the condition that Eq. (3.14) has a real root. In this case, by separating out the imaginary part in Eq. (3.14), we find

$$n(\omega/2) = 0 \tag{3.15a}$$

By choosing the origin with respect to which the energy ξ is measured, and hence the quantity $\omega/2$, to coincide with the point at which $\mathrm{n}(\xi) = 1/2$, we obtain the condition $\omega = 0$ from Eq. (3.15a). By isolating the real part in Eq. (3.14) we obtain the following equation for the determination of the stability boundary:

$$1 = -\frac{1}{2} \lambda N(0) \int_{-\omega_D}^{\omega_D} \frac{d\xi}{\xi} [1 - 2n(\xi)] \tag{3.15b}$$

Here n(ξ) is the average of n_k over the corresponding constant-energy surface. In the case of equilibrium, Eq. (3.15b) goes over into the well-known equation for the determination of the critical temperature T_c. In so far as we are able to judge, Eqs. (3.15) differ somewhat from the corresponding stability conditions derived by Batyev. [4]

Let us proceed to the investigation of a superconductor. When q = 0 Eq. (3.13) takes the form

$$\left[\gamma + \left(4\Delta^2 - \omega^2\right) Z\right]\left[\gamma - \left(4\Delta^2\gamma + \omega^2\right) Z + 2\Delta^2\omega^2 Z^2\right] = 0 \tag{3.16}$$

$$Z(\omega) = \lambda N(0) \int_\Delta^\infty \frac{1 - 2m(\varepsilon)}{\left(\varepsilon^2 - \Delta^2\right)^{1/2}} \frac{d\varepsilon}{\omega^2 - 4\varepsilon^2 + i\omega\delta} \tag{3.17}$$

$$\gamma = 1 + \lambda N(0) \int_\Delta^\infty \frac{d\varepsilon}{\left(\varepsilon^2 - \Delta^2\right)^{1/2}} [1 - 2m(\varepsilon)] \tag{3.18}$$

So that the stationary value, Δ_S, of the parameter Δ is determined from the equation $\gamma = 0$. The relationship $n(-\xi) = 1 - n(\xi)$ was used in the derivation of Eq. (3.16).

When $\gamma = 0$ the left part of Eq. (3.16) breaks up into a product of factors, two of which give the real roots $\omega^2 = 0$ and $\omega = \pm 2\Delta_S$, and the other two factors lead to the equations

$$Z(\omega) = 0, \tag{3.19}$$

$$Z(\omega) = +1/2\Delta_s^2. \tag{3.20}$$

Equation (3.20) doesn't have any solutions. In fact, one can rewrite it in the form

$$\frac{\lambda N(0)}{2} \int_1^\infty dx \frac{1 - 2m(x\Delta)}{\left(x^2 - 1\right)^{1/2} \left(x^2 - \omega^2/4\Delta_2^2 - i\omega\delta\right)} = 1$$

The small factor $\lambda N(0)$ appears to the left; in order of magnitude this factor is equal to $\ln^{-1}(\omega_D/\Delta_0)$, where Δ_0 denotes the equilibrium gap for T = 0. As one can easily verify, the smallness of this factor can not be compensated by the integral, whose order of magnitude does not exceed unity.

Thus, the stability boundary of the superconducting state is determined by Eq. (3.19). In this equation we set $\omega = \omega' - \mathbf{i}\omega''$. The stability boundaries are determined from the condition that ω'' becomes an infinitesimal negative quantity. By isolating the imaginary part in Eq. (3.19), we find that this condition is satisfied provided that one of the following two conditions is fulfilled:

$$\omega' = 0, \tag{3.21a}$$

$$m\left(\omega_i'/2\right) = 1/2. \tag{3.22a}$$

These are also those values of the frequency at which an instability may begin.

We note that Eq. (3.22a) may, in general, have several solutions, in which case we shall distinguish them by the subscript i. The real part of Eq. (3.19) together with conditions (3.21a) or (3.22a) determine those values of the external parameters, characterizing the nonequilibrium nature of the distribution, at which the instability begins. The corresponding equation for case (3.21a) has the form

$$\int_{\Delta_\varepsilon}^\infty \frac{d\varepsilon}{\varepsilon^2} \frac{1 - 2m(\varepsilon)}{\left(\varepsilon^2 - \Delta_\varepsilon^2\right)^{1/2}} = 0 \tag{3.21b}$$

and for case (3.22a)

$$\int_{\Delta_i}^{\infty} \frac{d\varepsilon}{(\varepsilon^2 - \Delta_i^2)^{1/2}} \frac{1 - 2m(\varepsilon)}{\varepsilon^2 - \omega_i'^2/4} = 0 \tag{3.22b}$$

We see that in order for Eqs. (3.21) and (3.22) to have a solution, the function $1 - 2$ m(ξ) must be alternating in sign. [5)]

Thus, if the function m(ϵ) in the important range of variation of ϵ does not pass through the value 1/2, then the superconducting state is stable.

Furthermore, if the function $1-2$ m(ϵ) has only one change of sign, then Eq. (3.22) doesn't have a solution. In this case the stability boundary is determined by Eq. (3.21b). Undamped, collective oscillations with frequencies $\omega^2 < 4\Delta_S^2$ exist in a system in its region of stability; these are determined by the following dispersion equation:

$$\int_{\Delta_\varepsilon}^{\infty} \frac{d\varepsilon}{(\varepsilon^2 - \Delta_\varepsilon^2)^{1/2}} \frac{1 - 2m(\varepsilon)}{\varepsilon^2 - \omega^2/4} = 0 \tag{3.23}$$

Thus, we see that in this case the instability is related to the vanishing of the frequency of one of these undamped, collective modes.

However, if the function $1 - 2$ m(ϵ) changes sign more than once, then a solution of Eq. (3.22) may also exist. And the actual boundary of the stability region is determined by the most stringent condition out of all the conditions which are obtained in this manner. In this case weakly-damped modes with frequencies, whose dispersion equation has the form Z(ω) = 0, may also exist near the boundary of the stability region. The instability may appear as a change in the sign of the attenuation coefficient for one of these modes.

Now let us explain the principle used in selecting the diagrams associated with the derivation of the system of equations (3.8). When $\gamma = 0$ the dispersion equation (3.16) has, in particular, the roots $\omega = \pm 2\Delta_S$. But, as $\omega \to \pm 2\Delta_S$ each of the self-energy parts S_{ik} diverges logarithmically, and the problem reduces to the summation of an infinite sequence of diagrams, that is, it reduces to the solution of the system of equations shown in Fig. 2.

On the other hand, in order to obtain the pole associated with small values of ω and q , that is, the acoustic mode, it is sufficient (to within terms of order λN(0) $\ll 1$) to sum the same sequence of diagrams. In connection with satisfying the self-consistency condition $\gamma = 0$ the principal terms of the series, which do not depend on ω and q , cancel each other, and consequently a pole appears as $\omega \to 0$ and q $\to 0$.

The summation of the sequence of diagrams, permitting us to obtain the roots $\omega = 0$ and $\omega = \pm 2\Delta$, has been carried out in the articles by Migdal and Maleev [17] for the equilibrium superconductor at T = 0.

Now let us investigate the acoustic solution of the dispersion equation, which tends to zero as q $\to 0$[18,19], for values of $q \neq 0$. Carrying out an expansion of the dispersion equation (3.13) correct to terms of order q^2, we obtain the following result for an isotropic distribution:

$$\omega^2 = c^2 q^2, \tag{3.24}$$

At T = 0 and for the equilibrium case, [19] one finds

$$c^2 = v_F^2/3. \tag{3.25}$$

Expression (3.25) is calculated to the zero-order approximation with respect to the small parameter $\lambda N(0)$. The accuracy of our calculation permits us, in principle, to also derive the next approximation in powers of this parameter. There is no attenuation of sound as long as ω^2(q) $< 4\Delta_S^2$. We emphasize that this result also pertains to the inverse Fermi distribution.

22.0.4 4. Examples of the Investigation of Stability

Now let us discuss the stability of the nonequilibrium superconducting states which were previously considered in articles [5,7,8–12]. The nonequilibrium distribution function m(ϵ) in the presence of an ultrahigh-frequency field is found in the article by Ivlev and Éliashberg. [7] The state is stable whenever the corresponding equation for the gap has a solution. Under the same external conditions, i.e., at the same temperature and for the same power of the high-frequency field, the normal state is also stable (in any event, if the temperature is above the superconducting transition point in equilibrium). Thus, one of these two states must be metastable.

A completely analogous situation occurs in the case when the nonequilibrium state of the superconductor can be described with the aid of a quasifield, analyzed by the authors in [9].

A different situation is considered in [8], that of the strongly nonequilibrium state which appears when a conductor is illuminated by light of a definite spectral composition. Here the Fermi step becomes more abrupt than in equilibrium at the given temperature, and the corresponding superconducting state is stable, since the difference $1-2n(\epsilon)$ has a constant sign for $\epsilon > 0$.

FIG. 3 The dependence of Δ_S on the concentration of excitations, x $\left(x_c = 1/3\sqrt{3}\right)$.

FIG. 4. The dependence of the frequencies of the collective modes on the concentration of excitations ($x_c = 1/3\sqrt{3}$).

However, the normal state is unstable in the presence of sufficiently intense illumination. The easiest way to see this is to note that the distribution which appears differs very little from the equilibrium distribution at T = 0. Thus, there is an important distinction between the present case and the situations which we were talking about above.

The superconducting state with an inverse Fermi distribution (with a repulsive interaction between the particles) is also stable, because $n(\epsilon) > 1/2$ in the essential range of positive energies. However, the normal state associated with the inverse Fermi distribution is unstable.

Owen and Scalapino [12] have recently studied the nonequilibrium state of a superconductor having an excess (in comparison with its equilibrium value) concentration of superconducting excitations. 6) They confine their discussion to the stability in this model at T = 0. Introducing ν, the chemical potential of the excitations, we may write the equation of state (1.1) in the following form:

$$v + \left(v^2 - \Delta_s^2\right)^{1/2} = \left(\Delta_0 \Delta_s\right)^{1/2}. \tag{4.1}$$

Here Δ_0 denotes the half-width of the equilibrium gap, that is, the half-width for zero

concentration of the excitations. It is convenient to rewrite expression (4.1) by introducing the dimensionless concentration of excitations:

$$x \equiv \frac{1}{2\Delta_0 N(0)} \sum_{\mathbf{p}} n_{\mathbf{p}} = \frac{1}{\Delta_0} \left(\nu^2 - \Delta_s^2\right)^{1/2} \tag{4.2}$$

Then Eq. (4.1) can be written down in the form [7)]

$$\Delta_0 \left(\Delta_0 - \Delta_0\right)^2 = 4x^2 \Delta_0^3 \tag{4.3}$$

The dependence of Δ_S on x is schematically shown in Fig. 3.

The stability criterion (3.25) gives the following result for the critical value of the quantity ν :

$$v_c = 2\Delta_0/3\sqrt{3} \tag{4.4}$$

which corresponds to the following result for the critical concentration of excitations

$$x_c = 1/3\sqrt{3} \tag{4.5}$$

and for the gap

$$\Delta_{c'} = \Delta_0/3 \tag{4.6}$$

In order to understand which branch of the curve shown in Fig. 3 is unstable, let us consider the dispersion equation (3.19). Its solution has the form

$$\omega_1^2 = 4\Delta_3^2 \left[1 - x^2 \left(\Delta_0/\Delta_3\right)^3\right] \tag{4.7a}$$

$$\omega_2^2 = 4\Delta_3^2 \left\{1 - x^2 \left(\Delta_0/\Delta_3\right)^3 \left[1 - 2x \left(\Delta_0/\Delta_3\right)^{1/3}\right]^2\right\} \tag{4.7b}$$

For $x = x_c$ and $\Delta_s = \Delta_c$ the solution ω_1^2 tends to zero like $(x_c - x)^{1/2}$, and $\omega_2^2 = 32\Delta_c^2/9$. Using the equation of state (4.3), we see that $\omega_1^2 < 0$ for $x < x_c$ and $\Delta_S < \Delta_c$, and therefore this branch of the solution is unstable. The dependences of ω_1^2 and ω_2^2 on the concentration in the stable region are schematically shown in Fig. 4.

Thus, two collective modes exist in the strongly nonequilibrium state in a superconductor, and for $q = 0$ the frequencies of these modes are determined by formulas (4.7). Such modes are not present in equilibrium. The instability which we have been considering-this is an instability of that collective mode which appears in the nonequilibrium state. Since the given state is a state of incomplete equilibrium, it is clear that the criterion that the energy be a minimum should also define, and actually does define in this case, the stability region of the superconducting state.

23 Inelastic Electron Collisions and Nonequilibrium Stationary States in Superconductors by Eliashberg

Abstract

A derivation of the dynamic equations for superconductors is presented, in which inelastic collisions between electrons or between electrons and phonons are taken into account explicitly. It is assumed that the phonons are in equilibrium (play the role of a thermostat). The conditions for which this assumption is valid are discussed. The equations are

written in a form which is analogous to the kinetic equation but their structure is much more complex than for a normal metal. This complication is mainly due to the need for taking into account changes induced in the excitation spectrum density by an alternating external field. For thin films taken as an example, it is shown that under certain conditions the alternating field effect consists mainly of a redistribution of the excitations with respect to energy, the spectrum density remaining constant. In this case the set of equations reduces to an equation for the energy gap, which is identical in form to the BCS equation but has a nonequilibrium distribution function, and to a nonlinear kinetic equation for this function.

23.0.1 1. Choice of Model

ONE of the important problems of the dynamic theory of superconductivity is the investigation of the nonequilibrium stationary states of the superconductor that are produced by continuous application of an alternating field. It is clear that the existence of such states is possible only because of the coupling between the electrons and the thermostat, the role of which is played in final analysis by the cooling medium. The mechanism of this coupling can be quite complicated, but it is realized in any case by the inelastic collisions of the electrons with the phonons, with the walls of the sample, and also with one another. Such collisions, however, are not considered in the simplified BCS model [1], on which the greater part of the dynamictheory investigations is based. This is justified by the relatively low probability of the inelastic electron scattering processes in metals. The corresponding damping of the excitations γ in the normal state is, as is well known, of the order of $\mathrm{T}^3/^{22}_{\mathrm{D}}$ and $\mathrm{T}^2/\mathrm{E_F}$ in electron-phonon and electron-electron scattering, and is comparable with the energy scale characteristic of superconductors, namely the value of the parameter Δ, only in a very small neighborhood of the transition temperature:

$$\left(T_{\mathrm{c}} - \mathrm{T}\right)/T_{\mathrm{c}} \sim \left(T_{\mathrm{c}}/\Theta_D\right)^4, \quad \left(T_{\mathrm{c}}/E_F\right)^2.$$

In the case of bulky superconductors, the perturbation connected with the action of the alternating field is localized near the surface, and the kinetics of the process is determined mainly by the diffusion relaxation mechanism. Of course, such a mechanism by itself does not lead to a stationary regime. Nonetheless, if the time-averaged deviation from equilibrium is small in the stationary state, then the final results do not contain the characteristics of the inelastic processes. On the other hand, it is precisely by virtue of the slowness of the energy relaxation of the electrons that the dynamic properties of the superconductors are
strongly influenced by the deviation from equilibrium in the energy distribution of the electrons; at a small average value, the collision amplitude becomes appreciable. The hydrodynamic approximation is therefore not suitable here, and there is no simple generalization of the Ginzburg-Landau theory for the nonequilibrium case [2–4].

Moreover, the large value of the electron-energy relaxation time gives rise to a noticeable stationary deviation in the energy distribution of the electrons even for relatively weak excitation. A study of the effects that may be connected with this calls for explicit allowance for the inelastic processes. Such an allowance is essential, in particular, for the investigation of the dynamics of superconductors of small dimensions, when the entire volume of the sample is in the nonequilibrium state, and there is no diffusion relaxation.

Before we proceed to derive the equations, we must dwell on the choice of the model. We consider below the simplest case, when the role of the thermostat is played by the lattice, so that the coupling of the electrons with the thermostat is effected by electronphonon interaction. A definite role can be played here also by electron-electron collisions, which we shall also take into consideration.

Let us discuss now the degree to which this assumption can be regarded as justified in different cases.

The simplest situation in this sense occurs in the case of samples of small dimensions, particularly for thin films, which will be discussed here for concreteness. We are interested in films whose thickness is of the same scale as the correlation radius $\xi_0 \sim \hbar \mathrm{v}/\mathrm{T_c}$ (v is the electron velocity on the Fermi surface), i.e., much larger than the wavelength of the thermal phonon at $\mathrm{T} \sim \mathrm{T_c}$. If the ρs (ρ-density, s -speed of sound) in the metal and in the cooling medium are of the same order of magnitude, then the phonon will give up its energy practically in each collision with the wall (see, for example, [5]). On the other hand, the free-path time of a thermal phonon in a bulky sample, connected with
its interaction with the electrons, is of the order of $\hbar \mathrm{v}/\mathrm{sT}$[6], so that the corresponding free path is $\sim \xi_0$ at $\mathrm{T} \sim \mathrm{T_c}$. Thus, at a film thickness $\mathrm{d} < \xi_0$, the phonon distribution will be determined by the collisions with the walls and its deviation from equilibrium will be small. On the other hand, if ρs is much smaller in the medium than in the metal, then to satisfy this condition it is necessary to have correspondingly thinner films. In the case of relatively thick films, when the phonons relax mainly on electrons, their distribution function will align itself to the distribution of the electrons. Since the phonon specific heat is small compared with that of the electrons, the phonons in this case exert no direct influence at all on the stationary distribution of the electronic excitations, which in this method of excitation is determined by the electron-electron collisions and by the condition that determines the equality of the energy flux absorbed by the electrons to the flux carried away through the walls. In this case, generally speaking, the situation does not reduce at all to simple heating: the electron distribution, and with it the phonon distribution, can differ noticeably from the temperature distribution.

All the foregoing pertains to some intermediate temperature region, when $\Delta \sim T$. The situation can change greatly at low temperatures and in the vicinity of $\mathrm{T_c}$. If $\mathrm{T} \ll \Delta$, then at not too intense an irradiation, the number of the excitations is small and they therefore exert a small influence on the phonon system, so that the latter remains at equilibrium also in the case of films thicker than indicated above. The same takes place, albeit for another reason, when $\mathrm{T} \gg \Delta$: if the field frequency is comparable with Δ, then it can be shown [7] that the nonequilibrium excitations are concentrated in the energy region $\epsilon \sim \Delta \ll T$, and the bulk of the excitations remains in equilibrium and plays, together with the phonons, the role of the thermostat.

These last considerations, which pertain to extreme temperatures, remain in force also for a bulky superconductor: the region of low temperatures and the vicinity of T_c are here, too, favorable for the use of the chosen model. At the same time, when the problem is formulated in greater detail, it must be borne in mind that in the case of a bulky sample we deal with a picture that is essentially inhomogeneous in space. The nonequilibrium excitations produced in the region δ of field penetration can then proceed from the surface to a greater or a smaller depth, which depends on the ratio of the transport path time τ, which we shall assume to be determined by the impurities, to the energy relaxation time τ_0. If $\tau \ll \tau_0$, then we have diffusion of the nonequilibrium excitation to a depth $L \sim (l\mathrm{v}\tau_0)^{1/2}$, which, depending on the impurity concentration, on the temperature, and on the field frequency, can be either larger or smaller than δ. Owing to the large value of τ_0, the inequality $L \gg \delta$ can be satisfied also under the condition $l \ll \xi_0$, and we therefore confine ourselves here to a discussion of only this case. The form of the electron distribution will change with increasing distance from the surface. At distances small compared with L, this change is due mainly to the dependence of the diffusion coefficient of the excitations in the superconductor on the energy:
$\mathrm{D}_\epsilon = \left(l\mathrm{v}_\epsilon/3 \sim \sqrt{\epsilon^2 - \Delta^2}\right.$, At distances comparable with L , inelastic processes come into play. If the phonon mean free path connected with their interaction with the electrons is larger than L, as can occur at low temperatures, then the phonons remain in equilibrium, and the electrons

relax over a length L to the tem perature of the phonon thermostat. A similar situation arises, for the reason indicated above, also near T_c. In these cases, the mechanism whereby the heat is carried away does not enter explicitly into the problem.

In the intermediate temperature interval, and also at low temperatures under conditions of sufficiently strong pumping, the number of excitations is large and the equilibrium in their distribution is violated in the entire energy interval. Therefore, as in the foregoing example of a relatively thick film, the phonons have time to align themselves to the electron distribution and drop out of the game. In this case, owing to the electron-electron collisions, thermalization of the excitation gas takes place over distances $\sim_L$, but the corresponding temperature may differ from the initial one, and the heat removal must be taken into account explicitly for its determination.

The foregoing estimates, of course, are schematic and need refinement as applied to concrete problems. Nonetheless, they show that there exists a sufficiently wide circle of problems the investigation of which can be carried out on the basis of the chosen model and does not require explicit allowance for the heat-removal mechanism.

23.0.2 2. Dynamic Equations in the Phonon Model

In order to take the electron-phonon collisions into account in a consistent manner, we derive the dynamic equations on the basis of the phonon model. As in the paper of Gor'kov and the author [3], the starting point is the concept of discrete imaginary frequencies, in which the expansion of the Green's function in powers of the external field has a simple diagram structure. The essential complication that arises when working with the phonon model is that it is necessary to deal here with energy-dependent self-energy parts, which will be determined by the same diagram approximation as in the equilibrium case [8] (Fig.1). $(\mathrm{P} = \{\xi, \mathrm{p}\}, \mathrm{K} = \{\omega, \mathrm{k}\})$:

$$\Sigma_1(P, P-K) = T\sum_{e'}\int \frac{d^3\mathbf{p}'}{(2\pi)^3} D\left(P'-P\right) G\left(P', P'-K\right), \tag{1}$$

$$\Sigma_2(P, P-K) = T\sum_{\varepsilon'}\int \frac{d^3\mathbf{p}'}{(2\pi)^3} D\left(P'-P\right) F\left(P', P'-K\right).$$

The functions G and F are assumed to be exact in the sense that they include, besides the external alternating field, also the self-energies Σ_1 and Σ_2. In contrast, the Green's function of the phonons is assumed to be free:

$$D\left(P'-P\right) = g^2 \frac{{}'_{\ldots} n^2_{n'-\mathbf{p}}}{\omega^2_{\mathbf{p}'-\mathbf{p}} - \varepsilon' - \varepsilon)^2} \tag{2}$$

corresponding to the assumption that the phonons are

FIG. 1

in equilibrium. (The renormalization of the phonon spectrum can be regarded as taken into

account in (2), since it is practically independent of the smearing region near the Fermi surface, which is affected by violation of the equilibrium.)

We shall carry out an analytic continuation of (1) by regarding G and F as sums of diagrams of different order in the alternating field. Since the entire procedure is independent of whether we are dealing with Σ_1 or Σ_2, we shall set the function G in correspondence with the electron line and will not mark the directions of the arrows on the diagrams.

Let us consider a diagram of N -th order in the field. It depends on the frequencies of its extreme electron lines ϵ and $\epsilon-\omega$, and in addition it depends on the frequencies ωl corresponding to the field vertices ($\sum_{l=1}^{N}\omega_l=\omega$). Since in the approximation (1) = (2) a single electron line passes through any diagram G , the field vertices can be numbered in the sequence in which they follow along this line from left to right, regardless of whether this diagram contains the insert Σ or not (Fig. 2). Let us ascertain the analytic structure of G(N) as a function of the variable ϵ at fixed $\omega_l=2\ \mathrm{m}_l\pi\mathrm{Ti}$. We are interested in the final analysis in the continuation of each of the ω_l from the upper halfplane, and we can therefore assume that all the $\mathrm{m}_l>0$. The diagram of Fig. 1a (i.e., the one not containing Σ), is a product of the zeroth-approximation functions $\mathrm{G(P)}=(\xi-\epsilon)^{-1},\xi=\mathrm{v}\,(\mathrm{p}-\mathrm{p}_0)$:

$$G^{(N)}_{\varepsilon,\varepsilon-\omega}\infty G_{\varepsilon}G_{\varepsilon-\omega_1}G_{\varepsilon-\omega_1-\omega_2}\ldots G_{\varepsilon-\omega}$$

and its singularities as functions of ϵ (in this case, poles), located on lines parallel to the real axis: $\mathrm{Im}\,\epsilon=0,\mathrm{Im}\,(\epsilon-\omega_1)=0,\ldots,\mathrm{Im}(\epsilon-\omega)=0$. We shall show now that the same lines are singular also for a diagram $G(N)$ of arbitrary form (for example, Fig 2b). To this end, obviously, it suffices to verify that $\Sigma^{(}\mathrm{N})$ has the same analytic structure as its internal electron line.

To do so, we transform the sum in (1) into a contour intergral, drawing the cuts in the ϵ' plane along the indicated singular lines $\mathrm{G}^{(\mathrm{N})}_{\epsilon',\epsilon'-\omega}$ and along the singular line of the D-function $\mathrm{Im}\,(\epsilon'-\epsilon)=0(\epsilon=(2\mathrm{n}+1)\pi\mathrm{Ti}\ )$, and choosing the integration contour as shown in Fig. 3. We denote the jump of the $\mathrm{G}^{(N)}$ function on its l-th cut by $\delta_l\left(\mathrm{G}^{(\mathrm{N})}\right)$, numbering these cuts in the upward direction. We then obtain the following representation for $\Sigma^{(N)}$:

$$\begin{aligned}\Sigma^{(N)}_{\varepsilon,\varepsilon-\omega}=\int_{-\infty}^{\infty}\frac{dz}{4\pi i}\Big\{\mathrm{cth}\,\frac{z}{2T}\left(D^R-D^A\right)_z G^{(N)}_{z+\varepsilon,z+\varepsilon-\omega}\\ +\mathrm{th}\,\frac{z}{2T}\left[D(z-\varepsilon)\delta_0\left(G^{(N)}\right)+D\left(z-\varepsilon+\omega_1\right)\delta_1\left(G^{(N)}\right)\right.\\ \left.\ldots+D(z-\varepsilon+\omega)\delta_N\left(G^{(N)}\right)\right]\Big\}\end{aligned}\tag{3}$$

Since the integration variable z is real, we see that $\Sigma(N)$ contains the same combinations of ϵ and ω_l as $G^{(N)}$, thus proving the statement made above.

Continuing (3) with respect to ϵ from the regions $\mathrm{Im}(\epsilon-\omega)>0$ and $\mathrm{Im}\,\epsilon<0$, we obtain the contributions of N -th order in the field to Σ^{R} and Σ^{A}, respec-

tively. In these cases, all the combinations of ϵ with ω_l that enter in (3) have imaginary parts of definite sign, and therefore the successive continuation with respect to each of the ω_l does not depend on the order. Shifting then the integration variables in each term of (3) in a simple manner, so as to restore the initial designation of the arguments, and summing over all the N , we arrive at the following formulas:

$$\Sigma_1^{R(A)}(P, P-K) = \int_{-\infty}^{\infty} \frac{d\varepsilon'}{4\pi i} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \left\{ \operatorname{cth} \frac{\varepsilon' - \varepsilon}{2T} \left(D^R - D^\Lambda\right)_{P'-P} \right.$$
$$\left. \times G^{R(\Lambda)}\left(P', P'-K\right) + D^{\Lambda(R)}\left(P'-P\right) G\left(P', P'-K\right) \right\} \tag{4}$$

where

$$G(P, P-K) = \sum_{N=0}^{\infty} \sum_{l=0}^{N} \delta_l \left(G^{(N)}\right) \operatorname{th} \frac{\varepsilon_l}{2T}, \quad \varepsilon_l = \varepsilon - \omega_1 - \ldots - \omega_l \tag{5}$$

Expressions for $\Sigma_2^{\mathrm{R(A)}}$ are obtained from this by replacing $G^{R(A)}$ and G by $F^{R(A)}$ and F. Since all the free lines and the proper spectra forming the diagrams $\mathrm{G}^{\mathrm{R(A)}}$ are retarded (advanced), it follows that $\mathrm{G}^{\mathrm{R(A)}}$ and $F^{R(A)}$ can be expressed in the usual manner in terms of $\Sigma_1^{\mathrm{R(A)}}$ and $\Sigma_2^{\mathrm{R(A)}}$:

$$\left\{ \begin{pmatrix} \xi - \varepsilon - \Sigma_1 + H_1 & -\Sigma_2 \\ \Sigma_2^+ & \xi + \varepsilon - \bar{\Sigma}_1 + \bar{H}_1 \end{pmatrix} \begin{pmatrix} G \\ -F^+ \end{pmatrix} \right\}_{P,P-K}^{R(A)} = \hat{1}(2\pi)^4 \delta(\omega)\delta(\mathbf{k})$$
$$H_1 = -\frac{e}{c}(\mathbf{vA}) + e\varphi, \quad \bar{H}_1 = \frac{e}{c}(\mathbf{vA}) + e\varphi. \tag{6}$$

The function G , introduced by the relation (5), and the analogous function F, play the principal role in the theory of nonequilibrium phenomena. We shall soon show that all the physical quantities, namely the current, Δ, etc., are expressed directly in their terms.

To establish the connection between G and Σ, we rewrite (5) in a somewhat different form, namely, we separate from the contribution of each diagram the terms corresponding to the upper edge of the upper cut and the lower edge of the lower cut (Fig. 3). The separated terms obviously define the retarded and advanced Green's functions, so that we obtain the following definition of the new quantity $\mathrm{G}^{(\mathrm{a})}$:

$$G(P,P-K)=G^R(P,P-K)\,\mathrm{th}\,\frac{\varepsilon-\omega}{2T}-\mathrm{th}\,\frac{\varepsilon}{2T}G^A(P,P-K) \\ +G^{(a)}(P,P-K) \tag{7}$$

It is easy to verify that, in the case of the BCS model, in which Δ enters in the diagrams in analogy to the external field, $\mathrm{G}^{(\mathrm{a})}$ coincides with the anomalous part of the Green's function introduced earlier [3]. All the terms making up $\mathrm{G}^{(\mathrm{a})}$ have a common property in that they have $G^R(P)$ as the incoming free line and $\mathrm{G}^{\mathrm{A}}(\mathrm{P}-\mathrm{K})$ as the outgoing line. The difference from the BCS model consists here in the fact that (5) has a part connected with the self-energy cuts. From (5) and

FIG. 3

(7), taking into account the fact that G contains the self-energies Σ_1 and Σ_2 (and with inverted arrow directions), we readily find that

$$G^{(a)}_{\varepsilon,\varepsilon-\omega}=\Big\{G^R\left(-h+\Sigma_1^{(a)}\right)G^4-F^{\mathrm{q}}\left(-\bar{h}+\bar{\Sigma}_1'^{a}\right)F^{+A} \\ -G^R\Sigma_2^{(a)}F^{+A}-F^R\Sigma_2^{+(a)}\bar{G}^A\Big\}_{\varepsilon,\varepsilon-\omega} \tag{8}$$

where

$$h_{e,\varepsilon-\omega}=\left(\mathrm{th}\,\frac{\varepsilon}{2T}-\mathrm{th}\,\frac{\varepsilon-\omega}{2T}\right)H_1(\omega)$$

and $\Sigma_1^{(a)},\bar{\Sigma}_1^{(a)},\Sigma_2^{(a)},\bar{\Sigma}_2^{+(a)}$ are expressed in terms of the discontinuities on the cuts of the corresponding self-energies with the aid of relations of the type (5) and (7):

$$\Sigma_{\varepsilon,\varepsilon-\omega}=\Sigma^R_{\varepsilon,\varepsilon-\omega}\,\mathrm{th}\,\frac{\varepsilon-\omega}{2T}-\mathrm{th}\,\frac{\varepsilon}{2T}\Sigma^A_{\varepsilon,\varepsilon-\omega}+\Sigma^{(a)}_{\varepsilon,\varepsilon-\omega} \\ =\sum_{N=0}^{\infty}\sum_{l=0}^{N}\delta_l\left(\Sigma^{(N)}\right)\mathrm{th}\,\frac{\varepsilon_l}{2T}. \tag{9}$$

The superior bar denotes an inversion of the arrow directions in the diagrams. In a coordinate representation we have identities of the type

$$\bar{G}^A(x,x')=G^R(x',x),\quad \bar{G}^{(a)}(x,x')=G^{(a)}(x',x);x=\{t,\mathbf{r}\}$$

Finally, we note that in (8) we have used the abbreviated notation [1)] :

$$\{ABC\}_{\mathrm{s,e}-\omega}=\iint_{-\infty}^{\infty}\frac{d\omega_1 d\omega_2}{(2\pi)^2}A_{\mathrm{e},\mathrm{e}-\omega_1}B_{\mathrm{e}-\omega_1,\mathrm{e}-\omega_2}C_{\mathrm{e}-\omega_2,\mathrm{e}-\omega} \tag{10}$$

In exactly the same manner, we can obtain expressions for $\mathrm{F}^{(\mathrm{a})}$, $\mathrm{F}^{+(\mathrm{a})}$, and $\overline{\mathrm{G}}^{(\mathrm{a})}$. For example,

$$F^{(a)}_{\varepsilon,\varepsilon-\omega}=\Big\{G^R\left(-h+\Sigma_1^{(a)}\right)F^A+F^R\left(-\bar{h}+\bar{\Sigma}_1^{(a)}\right)\bar{G}^A \\ +G^R\Sigma_2^{(a)}\bar{G}^A-F^R\Sigma_2^{+(a)}F^A\Big\}_{\varepsilon,\varepsilon-\omega}. \tag{11}$$

It remains for us to express Σ, defined in accordance with (9), in terms of the Green's functions, as was done above for $\Sigma^{R(A)}$ (see Fig. 4). From (3) we obtain, with the aid of the identity $\coth(\mathrm{x}' - \mathrm{x})(\tanh \mathrm{x}' - \tanh \mathrm{x} = 1 - \tanh \mathrm{x} \tanh \mathrm{x}'$,

$$\Sigma_1(P, P-K) = \int_{-\infty}^{\infty} \frac{d\varepsilon'}{4\pi i} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \left\{ \operatorname{cth} \frac{\varepsilon' - \varepsilon}{2T} G(P', P'-K) - \left(G^R - G^A\right)_{P', P'-K} \right\} \left(D^R - D^A\right)_{P'-P} \tag{12}$$

Formulas for Σ_2 and $\Sigma_2^+$are obtained from this by replacing G with F and F^+. We now have a complete system of equations describing, on the basis of the Fröhlich model, the behavior of a superconductor in an alternating field.

FIG. 4

23.0.3 3. Kinetic Equations

Let us eliminate from the equations of the preceding section the anomalous functions $\mathrm{G}^{(\mathrm{a})}, \Sigma^{(\mathrm{a})}$, etc. To this end, we note that $G^{(\mathrm{a})}$ (Eq. (8)) satisfies the following Dyson equation, which is obtained by separating on the left the free line $G^R(P) = (\xi - \epsilon - i\delta)^{-1}$:

$$(\xi - \varepsilon) G^{(a)}(P, P-K) = \left\{ \left(-H_1 + \Sigma_1^R\right) G^{(a)} - \Sigma_2^R F^{+(a)} \right\}_{P, P-K} + \left\{ \left(-h + \Sigma_1^{(a)}\right) G^A - \Sigma_2^{(a)} F^{+A} \right\}_{P, P-K}$$

(the notation is the same as in (8) and (10)). Substituting here in place of the anomalous functions their expressions from (7) and (9) and using Eqs. (6) for $\mathrm{G}^{\mathrm{R(A)}}$ and $\mathrm{F}^{\mathrm{R(A)}}$, we obtain an equation for G(5) :

$$(\xi - \varepsilon) G(P, P-K) = \left\{ \left(-H_1 + {\Sigma_1}^R\right) G - {\Sigma_2}^R F^+ + \Sigma_1 G^A - \Sigma_2 F^{+A} \right\} + (2\pi)^4 \delta(\omega) \delta(\mathbf{k}) \tag{13}$$

In exactly the same manner, we obtain an equation for F:

$$(\xi - \varepsilon) F(P, P-K) = \left\{ \left(-H_1 + {\Sigma_1}^R\right) F + {\Sigma_2}^R \bar{G} + \Sigma_1 F^A + \Sigma_2 \bar{G}^A \right\} \tag{14}$$

and, analogously, for $F^+$and $\bar{G}$. An important property of Eqs. (13) and (14) is the fact that only the Σ of formulas (4) and (12) depend explicitly on the temperature, and we see that

[121)] Depending on the representation, integration is implied here with respect to the intermediate coordinates or with respect to the momenta.

T enters in these formulas as the temperature of the phonon thermostat. The situation here is in full correspondence with that demonstrated by Keldysh [9], whose method could be used in our case. The method chosen by us to obtain the dynamic equations makes it possible to retain the direct connection with the previously described diagram technique [3].

We call attention now to the fact that, as noted a number of times [6,8], the self energies Σ are practically independent of the momentum, i.e., of $\xi = \mathrm{v}\,(\mathrm{p} - \mathrm{p}_0)$. In fact, p enters only in the D function, and since both the external momentum p and the internal momentum p' are close to the Fermi momentum p_0, it follows that $D\,(p - p')$ depends mainly on the angle between them. This enables us to go over in (4) and (12) from integration with respect to p to integration with respect to ξ and the angles

$$\frac{d^3\mathbf{p}}{(2\pi)^3} \approx \frac{mp_0}{2\pi^2}\frac{dO_{\mathbf{p}}}{4\pi}d\xi$$

and to integrate the electron functions in these formulas with respect to ξ. Introducing the notation

$$g_{\varepsilon,\varepsilon-\omega}(\mathbf{p},\mathbf{k}) = \int_{-\infty}^{\infty} d\xi G(P, P-K), f_{\varepsilon,\varepsilon-\omega}(p,k) = \int_{-\infty}^{\infty} d\xi F(P, P-K) \tag{15}$$

we rewrite Σ in the form

$$\Sigma^{R(A)}_{i\varepsilon,\varepsilon-\omega}(\mathbf{p},\mathbf{k}) = \int_{-\infty}^{\infty}\frac{d\varepsilon'}{4\pi i}\int\frac{dO_{\mathbf{p}}}{4\pi}\left\{\mathrm{cth}\,\frac{\varepsilon'-\varepsilon}{2T}\left(D^R - D^A\right)g^{R(A)} + D^{A(R)}g\right\} \tag{/}$$

$$\Sigma_{1\varepsilon,\varepsilon-\omega}(\mathbf{p},\mathbf{k}) = \int_{-\infty}^{\infty}\frac{d\varepsilon'}{4\pi i}\int\frac{dO_{\mathbf{p}}}{4\pi}\left\{\mathrm{cth}\,\frac{\varepsilon'-\varepsilon}{2T}g - \left(g^R - g^A\right)\right\}\left(D^R - D^A\right) \tag{/}$$

and analogously, for $\Sigma_2{}^{2)}; \mathrm{D}^{\mathrm{R(A)}} \equiv \mathrm{D}^{\mathrm{R(A)}}_{\epsilon'-\epsilon}\,(\mathrm{p}' - \mathrm{p})$, $\mathrm{g} \equiv \mathrm{g}_{\epsilon'}, \epsilon' - \omega(\mathbf{p},\mathbf{k})$.

We see now that ξ enters explicitly only in the lefthand sides of (13) and (14), and we can therefore obtain equations directly for g and f. To do so, we write out for G and F equations analogous to (13) and (14), but obtain from (8) and (11) by separating the free righthand line: $\mathrm{G^A(P-K)} = (\xi - \mathrm{vk} - \epsilon + \omega + \mathrm{i}\delta)^{-1}$ in the case of G and $\overline{\mathrm{C}}^{\mathrm{A}}(\mathrm{P-K}) = (\xi - \mathrm{vk} + \epsilon - \omega - \mathrm{i}\delta)^{-1}$ in the case of F :

$$\begin{gathered}(\xi - \mathbf{vk} - \varepsilon + \omega)G(P, P-K) = \left\{G\left(-H_1 + \Sigma_1{}^A\right) - F\Sigma_2{}^{+A}\right.\\ \left.+G^R\Sigma_1 - F^R\Sigma_2{}^+\right\} + (2\pi)^4\delta(\omega)\delta(\mathbf{k})\\ (\xi - \mathbf{vk} + \varepsilon - \omega)F(P, P-K) = \left\{F\left(-\bar{H}_1 + \bar{\Sigma}_1{}^A\right) + G\Sigma_2{}^A\right.\\ \left.+F^R\bar{\Sigma}_1 + G^R\Sigma_2\right\}\end{gathered}$$

Combining these equations with (13) and (14), so as to exclude ξ from the left-hand side, and integrating with respect to ξ, we obtain

$$\begin{gathered}(\omega - \mathbf{vk})g_{\varepsilon,\varepsilon-\omega} = \left\{H_1 g - gH_1 + g\Sigma_1{}^\Lambda - \Sigma_1{}^R g - f\Sigma_2{}^{+\Lambda}\right. \\ \left.+\Sigma_2{}^R f^+ + g^R\Sigma_: - \Sigma_1 g^\Lambda - f^R\Sigma_2{}^+ + \Sigma_2 f^{+\Lambda}\right\}_{\varepsilon,\varepsilon-\omega}\end{gathered} \tag{16}$$

$$\begin{gathered}(2\varepsilon - \omega - \mathbf{vk})f_{\varepsilon,\varepsilon-\omega} = \left\{H_1 f - f\bar{H}_1 + \bar{f}\bar{\Sigma}_1{}^\Lambda - \Sigma_1{}^R f + g\Sigma_2{}^\Lambda\right.\\ \left.-\Sigma_2{}^R\bar{g} + f^R\Sigma_1 - \Sigma_1 f^\Lambda + g^R\Sigma_2 - \Sigma_2\bar{g}^\Lambda\right\}_{\varepsilon,\varepsilon-\omega}\end{gathered} \tag{17}$$

Besides the processes of scattering and decay of excitations, Eqs. (16) and (17) take into account also the renormalization effects. The latter are determined by the quantity $\Sigma_1^{\mathrm{R}} + \Sigma_1^{\mathrm{A}}$

and are practically insensitive to violation of the equilibrium [3)]. We shall disregard them and retain in the equations only the dissipative part Σ_1^{R} :

$$2i\gamma_{\varepsilon,\varepsilon-\omega} \equiv \Sigma_{\mathrm{t}}{}^{\mathrm{R}} - \Sigma_{\mathrm{t}}{}^{\mathrm{A}} = \int_{-\infty}^{\infty} \frac{d\varepsilon'}{4\pi i} \int \frac{dQ_{p'}}{4\pi} \left[\operatorname{cth} \frac{\varepsilon' - \varepsilon}{2T} \left(g^{\mathrm{R}} - g^{\mathrm{A}} \right) - g \right] \left(D^R - D^A \right)$$

With the same accuracy, the role of Δ is assumed by the quantity

$$\Delta = 1/2 \left(\Sigma_2^R + \Sigma_2^A \right) \tag{19}$$

which varies as a function of ϵ in an interval $\sim \omega_D$, so that when $\epsilon \sim \Delta$ and $T \ll \omega_D$ this dependence is immaterial. The first term under the integral sign in (4') for Σ_2 introduces in Δ a contribution $\sim \Delta \left(T/\omega_D \right)^2$. We can therefore use for the determination of Δ the simplified equation

$$\Delta_\omega(k) = \lambda \int_{-\omega_D}^{\omega_D} \frac{d\varepsilon}{4\pi i} \int \frac{dO_P}{4\pi} f_{e,\varepsilon-\omega}(\mathbf{p}, \mathbf{k}) \tag{20}$$

At the same time, for $\Sigma_2^{\mathrm{R}} - \Sigma_2^{\mathrm{A}}$ it is necessary to retain the exact expression:

$$\begin{aligned} 2i\delta_{e,\varepsilon-\omega} \equiv \Sigma_2^R - \Sigma_2{}^\Lambda = \int_{-\infty}^{\infty} \frac{d\varepsilon'}{4\pi i} \int \frac{dO_p}{4\pi} \Big[\operatorname{cth} \frac{\varepsilon' - \varepsilon}{2T} \left(f^R - f^\Lambda \right) \\ - f \Big] \left(D^R - D^\Lambda \right) \end{aligned} \tag{21}$$

After these simplifications, Eq. (16) takes the form

$$(\omega - \mathrm{vk}) g_{\varepsilon,\varepsilon-\omega} = \left\{ H_1 g - g H_1 - f\Delta^* + \Delta f^+ \right\}_{e,e-\omega} + I_{\varepsilon,\varepsilon-\omega} \tag{22}$$

where

$$\begin{aligned} I = \big\{ -i(g\gamma + \gamma g) + i \left(f\delta^+ + \delta f^+ \right) \\ + g^R \Sigma_1 - \Sigma_1 g^\Lambda - f^R \Sigma_2^+ + \Sigma_2 f^{+\Lambda} \big\} \end{aligned} \tag{23}$$

It is easy to verify that the collision integral (23) vanishes when the equilibrium expressions are substituted for $\mathrm{g}_{\epsilon,\epsilon-\omega} = 2\pi\delta(\omega)\mathrm{g}_\epsilon$ and $\mathrm{f}_{\epsilon,\epsilon-\omega} = 2\pi\delta(\omega)\mathrm{f}_\epsilon$:

$$g_\varepsilon = \left(g^R - g^\Lambda \right)_\varepsilon \operatorname{th} \frac{\varepsilon}{2T}, \quad f_\varepsilon = \left(f^R - f^\Lambda \right)_\varepsilon \operatorname{th} \frac{\varepsilon}{2T} \tag{24}$$

i.e., Eq. (16) describes relaxation to the equilibrium.

Even more significant simplifications can be made in Eq. (17) for f. The point is that here the right-hand side contains the variable $\epsilon \sim \Delta, T$, whereas the dissipative parts γ and δ ((18) and (21)) are much smaller in the entire significant temperature interval, with the exception of a very small vicinity of T_c, an estimate for which was given in the beginning of the article. Their role here, as in the expressions for $g^{R(A)}$ and $f^{R(A)}$ consists, roughly speaking, in the fact that they lead to a certain smearing of the singularity in the density of the excitation spectrum near $\epsilon = \Delta$, and are, as a rule, by no means the principal source of such smearing. Therefore such quantities should be neglected in (17) (as also in $g^{R(A)}$ and $f^{R(A)}$, which enter in (23)). As a result we obtain for f the equation

[122)] We have included the factor $\mathrm{mp}_0/2\pi^2$ in the D function (2), where now $\mathrm{g}^2\mathrm{mp}_0/2\pi^2 \equiv \lambda \sim 1$.

[3)] As is well known [6,8] in the equilibrium case $\mathrm{Re}\,\Sigma_{1\epsilon}^{\mathrm{R}} \approx \mathrm{a}\epsilon$, where $\mathrm{a} \sim 1$ has the same value in the normal and in the superconducting states, accurate to within $(\Delta/\omega_{\mathrm{D}})^2$.

$$(2\varepsilon - \omega - \mathbf{vk}) f_{\varepsilon,\varepsilon-\omega}(\mathbf{p},\mathbf{k}) = \left\{ H_1 f - f\bar{H}_1 + g\Delta - \Delta\bar{g} \right\}_{e,\varepsilon-\omega} \tag{25}$$

The system (22)-(25) plays the role of the kinetic equation in the case of the superconducting state. In the variant linearized with respect to the external field, equations of this type have already been used by a number of authors (see, in particular, [10]). It should be noted however, that it was possible to obtain nonlinear equations only because of the procedure of integration with respect to ξ, [4)], whereas the true kinetic equation would correspond to equations for the Green's functions integrated with respect to ϵ. But it is precisely for this reason that the equations given here remain valid also when the concept of the excitation spectrum becomes meaningless. Such a situation will occur, for example, if Δ is a sufficiently rapid function of the coordinates and of the time.

In the next sections we shall obtain for the electronimpurity and electron-electron collision integrals expressions that should be added to (23).

23.0.4 4. Scattering by Impurities

The interaction with impurities is included in the general scheme of the theory, as well as the interaction with the external field [3]. In the imaginary-frequency representation, self-energy parts σ_1, σ_2, and σ_2^+appear in the Green's-function diagrams after averaging over the impurity positions. As a result of analytic continuation, each of them, as in the case of phonons, generates three quantities $\sigma R(A)$ and σ, expressions for which will be derived here under the assumption that the scattering is isotropic:

$$\begin{gathered}
\sigma_1^{R(A)} = \frac{1}{2\pi\tau}\left\langle g^{R(A)} \right\rangle, \quad \sigma_1 = \frac{1}{2\pi\tau}\langle g \rangle, \\
\sigma_2^{R(A)} = \frac{1}{2\pi\tau}\left\langle f^{R(A)} \right\rangle, \quad \sigma_2 = \frac{1}{2\pi\tau}\langle f \rangle, \\
\langle \ldots \rangle = \int \frac{dO_{\mathrm{p}}}{4\pi},
\end{gathered} \tag{26}$$

where τ is the transport free-path time of the electron [5)], and g and f are the functions defined above, calculated with allowance for the impurities. Adding (26) to the corresponding phonon self-energies in (16) and (17), we obtain the integrals for collisions with impurities:

$$I_p^{(imp)} = \int \frac{dO_{p'}}{4\pi} \left\{ \left[g^R g\right]_{pp'} + \left[g g^\Lambda\right]_{pp'} - \left[f^R f^+\right]_{pp'} - \left[f f^{+\Lambda}\right]_{pp'} \right\} \tag{27}$$

$$K_p^{(imp)} = \int \frac{dO_{p'}}{4\pi} \left\{ \left[g^R f\right]_{pp'} + \left[g f^\Lambda\right]_{pp'} + \left[f \bar{g}^\Lambda\right]_{pp'} + \left[f^R \bar{g}\right]_{pp'} \right\} \tag{28}$$

$$[gf]_{pp'} \equiv g_p f_{p'} - g_{p'} f_p$$

which must be included respectively in the right-hand sides of (22) and (25). These expressions possess the property that their averages over the directions of p vanish, as a consequence of the elasticity of the scattering by impurities.

[124)] Equations for the functions integrated with respect to ξ were obtained for the equilibrium case by Eilenberger [[11]].

23.0.5 5. Electron-Electron Interaction

We are interested in the interaction between electrons only to the extent that it leads to collision processes that are of importance for the kinetics. In addition, it makes its own contribution to the kernel of the equation for Δ. However, within the framework of the model equation (20) we can simply assume that λ is defined with allowance for all the interactions. Finally, the electron-electron interaction necessitates, generally speaking, that the electronic system be regarded as a Fermi liquid. We note in this connection that the renormalized quantities that arise in the theory of Fermi liquids are practically insensitive to the transition of the metal into the superconducting state, and it can therefore be assumed that they are accounted for in the employed characteristics of the normal metal, such as the spectrum density, the vertex of the interaction with the external field, etc. On the other hand, we shall not consider effects connected with the possible existence of excitations of the zero-sound type.

In the derivation of the equation for the electronelectron collision integral, we shall start from the fact that in a superconductor, as in a normal metal, only paired collisions are significant: processes in which a large number of excitations take part have a low probability because of degeneracy. Just as in the calculation of the damping of the excitations in a normal Fermi liquid [13], we should pick out in each diagram Σ in succession all the cross sections with three lines (Fig. 4), and take into account the dependence on the discrete imaginary frequencies (and by the same token on the temperature), on the external field, and in the given case on the parameter Δ of the superconducting state only in these cross sections. This

procedure makes it necessary for us to regard both vertices on the diagram of Fig. 4 as the true amplitudes for the scattering of two excitations on the Fermi surface, containing no dependences on the aforementioned parameters which characterize the smearing region.

Since superconductivity is taken into account in the separated lines of the diagram of Fig. 4, Σ_1 and Σ_2 are actually determined by several diagrams of this type, which differ in the disposition of the arrows. Each such diagram corresponds to a different combination of amplitudes. In the imaginary-frequency representation, Σ_1 and Σ_2 can be expressed as follows:

$$\Sigma_1(P, P-K) = T^2 \sum_{\varepsilon_1,\varepsilon_2} \iint \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(2\pi)^6} \left\{ A G_1 G_2 \bar{G}_3 - B F_1 F_2{}^+ G_3 \right\},$$
$$\Sigma_2(P, P-K) = T^2 \sum_{\varepsilon_1,\varepsilon_2} \iint \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(2\pi)^6} \left\{ B G_1 \bar{G}_2 F_3 - A F_1 F_2 F_3{}^+ \right\}, \tag{29}$$

where A and B are two different quadratic forms of the amplitudes, the explicit form of which we do not need. For clarity, we present here expressions for A and B in perturbation theory, when the amplitudes can be replaced by a certain effective potential $\mathrm{V_q}$:

$$A = -2\left|V_{p-p_2}\right|^2 + V_{p-p_2} V_{p-p_1},$$
$$B = -2\left|V_{p_1+p_2}\right|^2 + V_{p-p_1} V_{p-p_2} + V_{p_1+p_2} V_{p-p_i} + V_{p-p_1} V_{p_1;p_2}. \tag{30}$$

We have taken into account the fact that, depending on the method of joining the vertices in Σ, an additional electron loop may or may not arise. The sign of the diagram also changes when a pair of lines $\mathrm{FF}^+$is added. The internal lines correspond to the complete Green's functions including the alternating field, $\mathrm{G_1 = G\,(P_1, P_1 - K_1)}$, etc., and in order to write down the succeeding formulas in symmetrical form, we have determined the internal variables by starting

[125)] If paramagnetic impurities are also present, then σ_1 and σ_2 contain respectively the relaxation times τ_1 and τ_2 [12].

from the "decay" conservation law $\mathrm{P} = \mathrm{P}_1 + \mathrm{P}_2 + \mathrm{P}_3$ and $\mathrm{K} = \mathrm{K}_1 + \mathrm{K}_2 + \mathrm{K}_3$. This is the reason for the appearance of $\overline{\mathrm{G}}$ in (29).

Regarding the diagram $\Sigma\,(\Sigma_1$ or $\Sigma_2)$ of definite order in the external field as a function of the complex variable ϵ at fixed imaginary frequencies of the field vertices, we can verify, just as in the case of the phonons, that it corresponds to cuts on the lines $\mathrm{Im}\,(\epsilon - \Omega_{\mathrm{k}}) = 0$, located between the extreme lowest cut $\mathrm{Im}\,\epsilon = 0$ and the highest cut $\mathrm{Im}(\epsilon - \omega) = 0$. The values of Ω_{k} represent here, too, certain combinations of the frequencies of the field vertices, but the set of these combinations and their total number depend in this case on the distribution of the field vertices over the internal lines of the diagram of Fig. 4. Let us as sume that the G functions of these lines correspond to the sets of cuts $\mathrm{Im}\,(\epsilon_1 - \omega_{1\mathrm{i}}) = 0, \mathrm{Im}\,(\epsilon_2 - \omega_{2\mathrm{k}}) = 0$, $\mathrm{Im}\,(\epsilon_3 - \omega_3 l) = 0$, and let us transform the sum over the frequencies in (29) into a contour integral (a double integral in this case). The transformation is carried out here in two steps: we first replace with an integral the sum over one of the variables, say ϵ_2, at a fixed imaginary value of ϵ_1, and then go over to the integral in the sum over ϵ_1. The external variable ϵ and the field frequencies remain imaginary to the end. The result takes the form

$$\Sigma\infty T^2 \sum_{\varepsilon_1,\varepsilon_2} G_{\varepsilon_1} G_{\varepsilon_2} G_{\varepsilon-\varepsilon_1-\varepsilon_2}$$

$$\begin{aligned}
= \iint_{-\infty}^{\infty} \frac{dz_1 dz_2}{(4\pi i)^2} \Bigg\{ & \sum_{i,k} \delta_i \left(G_{x_1+\omega_{1i}}\right) \delta_k \left(G_{z_2+\omega_{2k}}\right) G_{\varepsilon-\omega_{1i}-\omega_{2k}-z_1-z_2} \operatorname{th}\frac{z_1}{2T} \operatorname{th}\frac{z_2}{2T} \\
& - \sum_{k_1,l} G_{z_1+\varepsilon-\omega_{2k}-\omega_{3l}} \delta_k \left(G_{z_2+\omega_{2k}}\right) \delta_l \left(G_{-z_1-z_2+\omega_{3l}}\right) \operatorname{cth}\frac{z_1}{2T} \operatorname{th}\frac{z_2}{2T} \\
& + \sum_{kl} G_{z_1+\varepsilon-\omega_{2_k}-\omega_{3l}} \delta_k \left(G_{z_2-z_1+\omega_{2k}}\right) \delta_l \left(G_{-z_2+\omega_{3l}}\right) \operatorname{cth}\frac{z_1}{2T} \operatorname{th}\frac{z_2}{2T} \\
& - \sum_{i,l} \delta_i \left(G_{z_1+\omega_{1i}}\right) G_{z_2-z_1+\varepsilon-\omega_{1i}-\omega_{3l}} \delta_l \left(G_{-z_2+\omega_{3l}}\right) \operatorname{th}\frac{z_1}{2T} \operatorname{th}\frac{z_2}{2T} \Bigg\}
\end{aligned} \tag{31}$$

In order not to clutter the formulas, we have omitted the second index of the G functions. The combinations obtained here of ϵ with $\omega_{1\mathrm{i}}, \omega_{2\mathrm{k}}$, and $\omega_3 l$ form the set of cuts of the given diagram Σ. Continuing (31) with respect to ϵ from the regions $\mathrm{Im}(\epsilon - \omega) > 0$ and $\mathrm{Im}\,\epsilon < 0$, and then with respect to all the field frequencies from the upper half-plane we obtain without difficulty Σ^{R} and Σ^{A}. We are interested only in the dissipative part $\Sigma^R - \Sigma^A$, for which we obtain the expression

$$\begin{aligned}
\Sigma^R - \Sigma^A \propto \iint_{-\infty}^{\infty} \frac{d\varepsilon_1 d\varepsilon_2}{(4\pi i)^2} \Big\{ & G_1 G_2 \left({G_3}^R - {G_3}^A\right) + G_1 \left({G_2}^R - {G_2}^A\right) G_3 \\
& + \left({G_1}^R - {G_1}^A\right) G_2 G_3 + \left({G_1}^R - {G_1}^A\right) \left({G_2}^R - {G_2}^A\right) \left({G_3}^R - {G_3}^A\right) \Big\}
\end{aligned} \tag{32}$$

Using a definition of the type (9)

$$\Sigma_{\varepsilon,\varepsilon-\omega} = \sum_{N=0}^{\infty} \sum_{k} \delta_k \left(\Sigma^{(N)}_{\varepsilon,\varepsilon-\omega}\right) \operatorname{th}\frac{\varepsilon - \Omega_k}{2T}$$

we obtain

$$\begin{aligned}
\Sigma\infty \int_{-\infty}^{\infty} \frac{d\varepsilon_1 d\varepsilon_2}{(4\pi i)^2} \Big\{ & G_1 G_2 G_3 + G_1 \left({G_2}^R - {G_2}^A\right) \left({G_3}^R - {G_3}^A\right) \\
& + \left({G_1}^R - {G_1}^A\right) G_2 \left({G_3}^R - {G_3}^A\right) + \left({G_1}^R - {G_1}^A\right) \left({G_2}^R - {G_2}^A\right) G_3 \Big\}
\end{aligned} \tag{33}$$

Then applying these formulas to (29), we obtain expressions for the quantities $\Sigma_1^R - \Sigma_1^A, \Sigma_1$ and $\Sigma_2^R - \Sigma_2^A, \Sigma_2$ of interest to us. Before we present the results, we note that all these quantities are expressed in fact in terms of the Green's functions integrated with respect to ξ, and do not depend on ξ themselves. The point is that owing to the short-range character of the effective interaction, the amplitudes in (30) can be regarded as dependent on the angles only. We can therefore write, for example,

$$\iint \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(2\pi)^6} A G_1 G_2 G_3$$
$$= \left(\frac{mp_0}{2\pi^2}\right)^2 \iint \frac{dO_{\mathbf{p}_1} dO_{\mathbf{p}_2}}{(4\pi)^2} A \iiint d\xi_1 d\xi_2 d\xi_3 \delta\left(\xi_3 - v\left(p_3 - p_0\right)\right) G_1 G_2 G_3$$

The δ function limits here, in the main, the integration with respect to the angle, requiring that $p_3 \equiv \mid p - p_1 - p_2 \mid \approx p_0$, and thus

$$\iint \frac{d^3\mathbf{p}_1 d^3\mathbf{p}_2}{(2\pi)^6} A G_1 G_2 G_3 = \left(\frac{mp_0}{2\pi^2}\right)^2 \frac{1}{p_0 v} \iint \frac{dO_{\mathbf{p}_1} dO_{\mathbf{p}_2}}{(4\pi)^2} \delta\left(\frac{p_3}{p_0} - 1\right) A g_1 g_2 g_3 \tag{34}$$

Taking all these formulas into account, we can now write

$$\begin{aligned}
2i\gamma &= \hat{L}\left[A\left\{g_1 g_2 \bar{g}^3\right\}_1 - B\left\{f_1 f_2{}^+ g_3\right\}_1\right] \\
\Sigma_1 &= \hat{L}\left[A\left\{g_1 g_2 \bar{g}_3\right\}_2 - B\left\{f_1 f_2{}^+ g_3\right\}_2\right] \\
2i\delta &= \hat{L}\left[B\left\{g_1 \bar{g}_2 f_3\right\}_1 - A\left\{f_1 f_2 f_3{}^+\right\}_1\right] \\
\Sigma_2 &= \hat{L}\left[B\left\{g_1 \bar{g}_2 f_3\right\}_2 - A\left\{f_1 f_2 f_3{}^+\right\}_2\right]
\end{aligned}$$

$$\hat{L} = \left(\frac{mp_0}{2\pi^2}\right)^2 \frac{1}{p_0 v} \iint_{-\infty}^{\infty} \frac{d\varepsilon_1 d\varepsilon_2}{(4\pi i)^2} \iint \frac{dO_{p_1} dO_{p_2}}{(4\pi)^2} \delta\left(\frac{p_3}{p_0} - 1\right) \tag{35}$$

where $\{\ldots\}_1$ and $\{\ldots\}_2$ are forms patterned after the curly brackets in (32) and (33), respectively. Substitution of these formulas in (23) leads to an expression for $I^{(e)}$.

23.0.6 6. Charge and Current Densities. Electroneutrality Condition

In the preceding sections we obtained equations for the Green's functions integrated with respect to ξ. It is clear that they can be used to calculate only quantities that are determined completely by the electrons near the Fermi surface. It is therefore necessary to exercise a certain caution when writing down the expressions for the current densities and the electron numbers. If we start from the expressions for j and N in the imaginary-frequency representation, then a direct application of the continuation procedure employed above, with allowance for the definition (5) of the function G, leads to the formulas

$$\mathbf{j}_\omega(k) = -\frac{2e}{m} \int_{-\infty}^{\infty} \frac{d\varepsilon}{4\pi i} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p} G_{e,--\omega}\left(\mathbf{p}_+, \mathbf{p}_-\right) - \frac{e^2}{mc}(NA)_{\omega,k} \tag{36}$$

$$N_\omega(k) = -2 \int_{-\infty}^{\infty} \frac{d\varepsilon}{4\pi i} \int \frac{d^3\mathbf{p}}{(2\pi)^3} G_{e,\varepsilon-\omega}\left(\mathbf{p}_+, \mathbf{p}_-\right); \quad \mathbf{p}_\pm = \mathbf{p} \pm \frac{\mathbf{k}}{2} \tag{37}$$

We shall assume that in the absence of an external field the medium is homogeneous. We stipulate that in the presence of a field the density of the number of electrons remain the same as before. The electroneutrality requirement frees us of the need for taking into account the Coulomb character of the interaction under conditions when the frequencies, gradients, and field

amplitudes are small compared with quantities of atomic scale. To formulate this condition explicitly, we separate that part of the function G which is connected with the contribution made to (36) and (37) by the remote regions, and therefore cannot be calculated by integration with respect to ξ. This part, obviously, should not depend on Δ. It is determined by the regular terms (see (7)) of the function of the normal state, since all of them, with the exception of the free line, vanish upon integration with respect to ξ. Correct allowance for the contribution of these terms to N in first order in the electrochemical potential $\varphi_\omega(\mathrm{k})$[6)] leads to the following expression:

$$N_\omega(k) - N_0(2\pi)^4\delta(\omega)\delta(\mathbf{k})$$
$$= -\frac{mp_0}{\pi^2}\mathbf{r}e\varphi_\omega(k) + \int_{-\infty}^{\infty}\frac{d\varepsilon}{4\pi i}\int\frac{dO_\mathbf{p}}{4\pi}g'_{\varepsilon,\varepsilon-\omega}(\mathbf{p},\mathbf{k})\Bigg], \quad N_0 = \frac{p_0^3}{3\pi^2}$$

where the prime denotes that it is necessary to subtract from g the contribution of the free line $2\pi\mathrm{i}\tanh(\epsilon/2\ \mathrm{T})$. By virtue of the neutrality condition we have

$$e\varphi_\omega(k) = -\int_{-\infty}^{\infty}\frac{d\varepsilon}{4\pi i}\int\frac{dO_\mathbf{p}}{4\pi}g'_{\varepsilon,\varepsilon-\omega}(\mathbf{p},\mathbf{k}). \tag{38}$$

A similar separation of the contribution made to j by the regions far from the Fermi surface leads to a cancellation of the second term in (36) (see also [3]):

$$\mathbf{j}_\omega(k) = -\frac{p_0 e}{\pi^2}\int_{-\infty}^{\infty}\frac{d\varepsilon}{4\pi i}\int\frac{dO_\mathbf{p}}{4\pi}\mathbf{p}g_{e,e-\omega}(\mathbf{p},\mathbf{k}) \tag{39}$$

Using the equations obtained above for g and f, we can easily verify that (38) coincides with the equation $\operatorname{div}\mathbf{j} = 0$.

23.0.7 7. Thin Films in a High-Frequency Field

We apply here the obtained equations to description of a thin film in the field of a wave incident normally onto its surface. We choose a gauge with a vector potential in the plane of the film, and confine ourselves to the case when the film is so thin that $\mathrm{A}_\omega, \Delta$, and the other quantities with them are constant over its cross section. We assume at the same time that the electron mean free path determined by scattering from impurities is small compared with the film thickness, making it possible to disregard the singularities of electron reflection from the walls. This condition also ensures satisfaction of the inequalities $\Delta\tau \ll 1$, $\mathrm{T}\tau \ll 1, \omega\tau \ll 1$ and leads to a weak dependence of g and f on the angle ϑ between p and $\mathbf{A}_\omega$:

$$g = g^{(0)} + g^{(1)}\cos\vartheta, \quad f = f^{(0)} + f^{(1)}\cos\vartheta \tag{40}$$

From (22) and (25), in which in this case it is necessary to put k = 0, we readily obtain equations for the individual components. The equations averaged over the angles do not contain terms corresponding to collisions with impurities, whereas in the equations for $g^{(1)}$ and $f^{(1)}$ these collisions play the principal role, and therefore the inelastic processes can be neglected here. In spite of these simplifications, the equations still retain a rather complicated structure, in accord with the complicated character of the action of the alternating field on the superconducting state.

Let us discuss first the case of the normal state, when $\Delta = 0$ and $g^{R(A)} = \pm\pi\mathrm{i}$. The last equation means that, within the framework of the approximation connected with the integration

[126)] It is obviously necessary to include in φ both the scalar potential of the field and the variation of the chemical potential. Allowance of terms of higher order in φ lead to corrections $\sim \mathrm{e}\varphi/\mathrm{E_F}$. The latter pertains also to terms with the vector potential.

with respect to ξ, the field has no influence on the level density of the normal electrons. We write out, assuming the field to be monochromatic, an equation for the element $g^{(0)}_{\epsilon,\epsilon} \equiv g_\epsilon$, which is diagonal in the energy:

$$0 = \frac{ev}{3c}\left\{A_\omega\left(g^{(1)}_{\varepsilon,\varepsilon+\omega} - g^{(1)}_{\varepsilon-\omega,\varepsilon}\right) + A_{-\omega}\left(g^{(1)}_{\varepsilon,\varepsilon-\omega} - g^{(1)}_{\varepsilon+\omega,\varepsilon}\right)\right\} + I_\varepsilon \tag{41}$$

Here I_ϵ takes the inelastic collisions into account, and its order of magnitude is $I_\epsilon \sim i\tau_0^{-1}\left(g\epsilon - g_\epsilon\right)$, where τ_0 is a certain effective energy relaxation time, and $g_{\epsilon_0} = 2\pi i \tanh(\epsilon/2T)$. If the field frequency is sufficiently high so that $\omega\tau_0 \gg 1$, then the nondiagonal elements $g^{(0)}_{\epsilon,\epsilon-2\omega}$ etc. are small compared with g_ϵ relative to the parameter $(\omega\tau_0)^{-1}$. Therefore $g^{(1)}$ can be expressed only in terms of $g\epsilon$. Since it follows for a normal metal from (27) that $\mathrm{I}^{(\mathrm{imp})} = -(\mathrm{i}/\tau)\mathrm{g}^{(1)}$, we obtain, inasmuch as $\omega\tau \ll 1$,

$$g^{(1)}_{\varepsilon,\varepsilon-\omega} = i\tau\frac{ev}{c}A_\omega\left(g_{\varepsilon-\omega} - g_\varepsilon\right)$$

Substituting in (41), we obtain an equation for g_ϵ :

$$0 = 2i\left(\frac{e}{c}\right)^2\frac{vl}{3}A_\omega A_{-\omega}\left(g_{e-\omega} - 2g_e + g_{e+\omega}\right) + I_e. \tag{42}$$

Introducing the notation

$$g_e = 2\pi i\left(1 - 2n_\varepsilon\right), \tag{43}$$

we can easily verify that the electron-phonon and electron-electron collision integrals obtained above have a canonical form with n_ϵ as the distribution function.

As seen from (42), when $\omega \lesssim \mathrm{T}$ the correction to $\mathrm{n} \in$ linear in the field intensity is of the order of magnitude

$$\delta n_e \sim \frac{\alpha\omega^2\tau_0}{T^2}, \quad \alpha = D\left(\frac{e}{c}\right)^2 A_\omega A_{-\omega}, \quad D = \frac{lv}{3}, \tag{44}$$

which indeed determines the field intensity at which a noticeable deviation from equilibrium takes place in the energy distribution of the electrons [7] : $\alpha \sim \mathrm{T}^2/\omega^2\tau_0$. We note that although (42) contains effects that are nonlinear in the field intensity, they are limited to those corresponding to successive single-quantum transitions. The probability of simultaneous absorption of two quanta has a small factor $(\omega\tau)^{-2}$ compared with cascade absorption.

In the case of the superconducting state at not too low temperatures, when there are enough excitations already in equilibrium ($\mathrm{T} \sim \Delta$), it is natural to expect the field intensity at which an appreciable change in the energy distribution of the excitations takes place to be determined, as before, from (44). The situation is made more complicated, however, by the fact that here the field also influences the density of the spectrum. Formally this is taken into account by the functions $gR(A)$ and $fR(A)$, which enter in the equations, and are altered by the turning on of the field in the case of the superconducting state. The situation here is analogous to that obtaining in the case of a constant magnetic field, although it is made more complicated by retardation effects. Consequently, substantial changes in the spectrum should be expected in fields whose amplitude is comparable with the critical magnetic field of the film, meaning $\alpha \sim \Delta, T$ in the notation (44) [8]. Since τ_0 is quite large (for example, $\tau_0\ \mathrm{T_C} \sim (\mathrm{T_c}/^{\omega}\mathrm{D})^2$), it follows that $\omega^2\tau_0 > \mathrm{T}$ even at not too high frequencies, and the changes in the excitation distribution become significant in fields which are still small compared with $\mathrm{H_c}$. Under such conditions, when calculating the energy distribution of the excitations, we can regard the field in the first approximation only as the source of transition at a constant spectrum density. We shall simplify the equations further as applied to this situation, confining ourselves, as in the example with the normal metal, to single-photon transitions ($\omega\tau_0 \gg 1$).

The diagonal element $g\epsilon$ is determined here, too, from (41), and in Eq. (25) for f $\in$ we should leave out

the field term, since the field enters here alongside with $\epsilon \sim \Delta$, unlike in (41), where it enters in combination with τ_0^{-1}. We thus have simply [9)]

$$f_\varepsilon = (\Delta/\varepsilon) g_\varepsilon. \tag{45}$$

In the equations for $g^{(1)}$ and $f^{(1)}$, which follow from (22) and (25) with allowance for (27) and (28), it is necessary to retain only the diagonal elements g_ϵ and f_ϵ. As to $gR(A)$ and $f^{R(A)}$, their averages over the angles should be replaced by the unperturbed values

$$g_e^{R(A)} = \frac{\varepsilon}{\Delta} f_e^{R(A)} = \pi i \frac{\varepsilon}{\xi_e^{I(A)}},$$

$$\xi_e{}^R = -\left(\xi_e{}^\Lambda\right)^* = \begin{cases} \sqrt{\varepsilon^2 - \Delta^2}\,\mathrm{sign}\,\varepsilon + i\delta, & \varepsilon^2 > \Delta^2, \\ i\sqrt{\Delta^2 - \varepsilon^2}, & \varepsilon^2 < \Delta^2, \end{cases}$$

and the angle parts should be calculated in first order in the field:

$$g_{\varepsilon,\varepsilon-\omega}^{(1)R(A)} = \frac{ev}{c} A_\omega \frac{\pi i}{\left(\xi_\varepsilon + \xi_{\varepsilon-\omega}\right)^{R(\Lambda)} + i/\tau} \left[1 - \frac{\varepsilon(\varepsilon-\omega) + \Delta^2}{\left(\xi_\varepsilon \xi_{\varepsilon-\omega}\right)^{R(\Lambda)}}\right],$$

$$f_{\varepsilon,\varepsilon-\omega}^{(1)R(A)} = -\frac{ev}{c} \frac{\pi i}{\left(\xi_\varepsilon - 1 - \xi_{\varepsilon-\omega}\right)^{R/(\lambda)} + i/\tau} \frac{2\varepsilon - \omega}{\left(\xi_\varepsilon \xi_{\varepsilon-\omega}\right)^{R(\Lambda)}}.$$

Taking all the foregoing into account, we obtain after simple manipulations

$$g_{\varepsilon,\varepsilon-\omega}^{(1)} = i\tau \frac{ev}{c} A_\omega \left[\varepsilon(\varepsilon-\omega) + \Delta^2\right] \left[\frac{g_{\varepsilon-\omega}}{(\varepsilon-\omega)\xi_\varepsilon{}^K} + \frac{g_\varepsilon}{\varepsilon \xi_{\varepsilon-\omega}^A}\right], \quad l < \xi_0 \tag{46}$$

and, substituting (46) in (41), we arrive at an equation for gϵ. We write down the result, introducing the notation

$$g_\varepsilon = 2\pi i \frac{\varepsilon}{\sqrt{\varepsilon^2 - \Delta^2}} \theta\left(\varepsilon^2 - \Delta^2\right)(1 - 2n_\varepsilon) \tag{47}$$

where n_ϵ is defined as an even function of the energy, the equation for which is ($\epsilon \geq \Delta$)

$$0 = 2\alpha\left[U_-\left(n_{\varepsilon-\omega} - n_\varepsilon\right) - U_+\left(n_\varepsilon - n_{\varepsilon+\omega}\right) + V\left(1 - n_\varepsilon - n_{\omega-\varepsilon}\right)\right] + J_\varepsilon$$

$$U_\pm = \frac{\left[\varepsilon(\varepsilon \pm \omega) + \Delta^2\right]\theta(\varepsilon \pm \omega - \Delta)}{\sqrt{\left(\varepsilon^2 - \Delta^2\right)\left[(\varepsilon \pm \omega)^2 - \Delta^2\right]}}$$

$$V = \frac{\left[\varepsilon(\omega - \varepsilon) - \Delta^2\right]\theta(\omega - \varepsilon - \Delta)}{\sqrt{\left[(\omega - \varepsilon)^2 - \Delta^2\right]\left(\varepsilon^2 - \Delta^2\right)}} \tag{48}$$

The expression for $J_\epsilon = I_\epsilon/4\pi$ follows directly from (23), (18), (21), and (35) with allowance for the simplifications listed here:

[127)] Since we are dealing with a thin film in which, as noted in Sec. 1, the phonons are close to equilibrium, the effect under consideration does not reduce to simple heating, although it has the same scale.

[8)] Actually, the superconductivity of the films is destroyed by the current [[14]], but the parameters of the superconducting state nevertheless change in the scale of H_c : the destruction by the current occurs abruptly.

$$J = J^{(ph)} + J^{(e)},$$

$$J_e^{(ph)} = \frac{\pi\lambda}{2\left(sp_0\right)^2}\int_\Delta^\infty d\varepsilon' \int_0^\infty d\omega \frac{\omega^2\varepsilon\varepsilon'}{\left[\left(\varepsilon^2-\Delta^2\right)\left(\varepsilon'^2-\Delta^2\right)\right]^{1/2}}$$

$$\times\left\{\left[n'(1-n)\left(1+N_\omega\right)-n\left(1-n'\right)N_\omega\right]\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\delta\left(\varepsilon'-\varepsilon-\omega\right)\right.$$

$$+\left[n'(1-n)N_\omega-n\left(1-n'\right)\left(1+N_\omega\right)\right]\left(1-\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\delta\left(\varepsilon-\varepsilon'-\omega\right)$$

$$\left.+\left[\left(1-n\right)\left(1-n'\right)N_\omega-nn'\left(1+N_\omega\right)\right]\left(1+\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\delta\left(\varepsilon+\varepsilon'-\omega\right)\right\}$$

$$J_e^{(e)} = \frac{1}{p_0 v}\iint_\Lambda^\infty \int \frac{N_\omega = \left(e^{\omega/T}-1\right)^{-1}}{\left[\left(\varepsilon^2-\Delta^2\right)\left(\varepsilon_1^2-\Delta^2\right)\left(\varepsilon_2^2-\Delta^2\right)\left(\varepsilon_3^2-\Delta^2\right)\right]^{1/2}}$$

$$\times\left\{M_1\left[(1-n)n_1n_2n_3-n\left(1-n_1\right)\left(1-n_2\right)\left(1-n_3\right)\right]\delta\left(\varepsilon-\varepsilon_1-\varepsilon_2-\varepsilon_3\right)\right.$$
$$+3M_2\left[n_1n_2(1-n)\left(1-n_3\right)-nn_3\left(1-n_1\right)\left(1-n_2\right)\right]\delta\left(\varepsilon+\varepsilon_3-\varepsilon_1-\varepsilon_2\right)$$

$$\left.+3M_3\left[n_1(1-n)\ \left(1-n_2\right)\left(1-n_3\right)-\left(1-n_1\right)nn_2n_3\right]\delta\left(\varepsilon+\varepsilon_2+\varepsilon_3-\varepsilon_1\right)\right\}$$

$$M_1 = a\left(\varepsilon\varepsilon_1\varepsilon_2\varepsilon_3-\Delta^4\right)-\frac{b}{3}\left(-\varepsilon^2+\varepsilon_1\varepsilon_2+\varepsilon_1\varepsilon_3+\varepsilon_2\varepsilon_3\right)\Delta^2 \tag{50}$$

$$M_2 = -M_1\left(-\varepsilon_3\right) \quad M_3 = M_1\left(-\varepsilon_2,-\varepsilon_3\right)$$

where a and b are numbers of the order of unity, obtained from (30) by averaging over the angles:

$$a = -2\pi\left(\frac{mp_0}{2\pi^2}\right)^2\iiint\frac{dO_{\mathbf{p}}dO_{\mathbf{p}_1}dO_{\mathbf{p}_2}}{(4\pi)^3}\delta\left(\frac{\left|\mathbf{p}-\mathbf{p}_1-\mathbf{p}_2\right|}{p_0}-1\right)A$$

and b is similarly connected with B. We emphasize that the equations contain the nonequilibrium value of Δ, which should be determined in accordance with (20) and (45) from the BCS equation:

$$1 = \lambda\int_\Delta^{\omega_D}\frac{d\varepsilon}{\sqrt{\varepsilon^2-\Delta^2}}\left(1-2n_\varepsilon\right) \tag{51}$$

with the nonequilibrium distribution function n_ϵ.
In conclusion, we note that the estimates given at the beginning of the section pertained to the temperature region $T \sim \Delta$. Using the results of the direct calculation of the correction to Δ, linear in the field intensity [15,17], we can show that in the vicinity of T_c the condition for the applicability of (48)-(51) is $\omega^2\tau_0 \approx \Delta$, which coincides with the condition given above for $\Delta \sim$ T. The case of low temperatures calls for a special investigation.

24 Diagram technique for nonequilibrium processes by Keldysh

Abstract

[129)] In this case $\Delta = \Delta^*, \mathrm{f} = \mathrm{f}^+, \overline{\mathrm{g}} = -\mathrm{g}$.

A graph technique analogous to the usual Feynman technique in field theory is developed for calculating Green's functions for particles in a statistical system which under the action of an external field deviates to any arbitrary extent from the state of thermodynamic equilibrium. It is found that in order to describe such a system it is necessary to introduce two Green's functions for each type of particles. The equation for one of these functions is a generalization of the usual Boltzmann kinetic equation.

THE use of quantum field theory methods has turned out to be very effective for the formulation and for the solution of problems in statistical physics. A number of prescriptions has been given for the description of thermodynamic characteristics of equilibrium systems [1−6] and of the ground state of a many-body system [7−9]. In the papers of Konstantinov and Perel, [10], Dzyaloshinski [11] and Mills [12] a technique has been developed for the calculation of time dependent quantities (kinetic coefficients) in systems in thermodynamic equilibrium. In carrying out this program the deviation of the system from the Gibbs equilibrium distribution was assumed to be small, i.e., the external field was taken to be sufficiently small. The present paper is devoted to the application of analogous methods to the description of essentially nonequilibrium systems, for example, such systems as the system of electrons in a strong electric field. In the simplest cases in order to describe such systems Boltzmann's kinetic equation or the system of coupled equations proposed by Bogolyubov [13] are used.

We shall obtain below a system of equations for the Green's functions of the particles in the system, and we shall show that one of these equations is essentially analogous to the kinetic equation. The appearance of such an equation-an additional one in comparison with systems in thermodynamic equilibrium-in the problem under consideration is quite natural, since we must have an equation which describes the distribution of particles in the system. In the equilibrium case this is given by the Gibbs canonical distribution. Our diagram technique will be close to Mills' technique for equilibrium systems. This technique has the advantage that it is completely analogous to the usual Feynman technique in field theory, with the only differ-
ence that the number of Green's functions appearing in it is increased. In the course of the discussion there appears automatically a function which describes the distribution of the particles in the system, and the equation for this function plays the role of the kinetic equation.

Following the method of Konstantinov and Perel' we shall define the density matrix ρ for the system by the usual equation

$$i\partial\rho/\partial t = [H_i, \rho]_- \equiv H_i(t)\rho(t) - \rho(t)H_i(t) \tag{1}$$

with the boundary condition

$$\rho(t = -\infty) = \rho_0 = \exp\left\{\Psi_0 - H_0(-\infty))\,/kT\right\} \tag{2}$$

where $\mathrm{H} = \mathrm{H}_0 + \mathrm{H_i}$ is the Hamiltonian of the system

$$H_0(t) = \int \psi_0 + (\mathbf{r}t)\left\{\varepsilon\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right) + e(\mathbf{D}\right\} \\ \times \psi_0(\mathbf{r}t)d\mathbf{r} + H_T \tag{3}$$

The first term in H_0 is the energy of noninteracting electrons in an external electromagnetic field
$\{\mathrm{A(rt)}, \Phi\mathrm{(rt)}\}$, the second is the Hamiltonian of the heat bath in contact with the system under consideration, $\epsilon(p)$ is the dispersion law for the electrons, Ψ_0 is the initial free energy.

The field operators ψ_0 and $\psi_0^+$are defined in the following manner: let $\varphi_\mathrm{p}(\mathrm{rt})$ be the complete system of functions determined by the equation

$$\left\{\varepsilon\left(-i\nabla-\frac{e}{c}\mathbf{A}(\mathbf{r}t)\right)+e\Phi(\mathbf{r}t)-i\frac{\partial}{\partial t}\right\}\varphi_{\mathbf{p}}(\mathbf{r}t)=0 \tag{4}$$

and by the boundary condition at $t\to-\infty$

$$\varphi_{\mathbf{p}}(\mathbf{r}t)\to\exp\left\{i\left(\mathbf{pr}-\varepsilon_{\mathbf{p}}t\right)\right\}, \tag{5}$$

where we consider the external field to be switched off at $t\to-\infty$, i.e., $(A,\Phi)\to 0$. Then the field operator ψ_0(rt) is defined by the relation

$$\psi_0(\mathbf{r}t)=\sum_{\mathbf{p}}a_{\mathbf{p}}\varphi_{\mathbf{p}}(\mathbf{r}t) \tag{6}$$

where a_p are the usual Fermi operators which satisfy the commutation relations

$$\left[a_{\mathbf{p}}^{+}a_{\mathbf{p}'}\right]_{+}\equiv a_{\mathbf{p}}^{+}a_{\mathbf{p}'}+a_{\mathbf{p}'}a_{\mathbf{p}}^{+}=\delta_{\mathbf{pp}'} \tag{7}$$

Utilizing the fact that the functions φ_p(rt) form at any arbitrary instant of time a complete orthogonal system, and formulas (6) and (7), it can be easily shown that for coincident times

$$\left[\psi_0^{+}(\mathbf{r}t),\psi_c\left(\mathbf{r}'t\right)\right]_{+}=\delta\left(\mathbf{r}-\mathbf{r}'\right) \tag{8}$$

and in virtue of this the operators ψ_0(rt) satisfy the free equations of motion

$$i\partial\psi_0/\partial t=[\psi_0,H_0]_{-} \tag{9}$$

i.e., we have from the outset defined the field operators in the interaction representation.

We note that the operator H_0(t) differs in the Schrödinger representation and in the interaction representation in contrast to the usual case, when all the external fields are independent of the time. Having in mind such a definition of the field operators we have from the outset written equation (1) in the interaction representation leaving in it only the operator for the interaction energy H_i which, for the sake of definiteness, we shall in future write in the form

$$H_i(t)=g\int{\psi_0}^{+}(\mathbf{r}t)\psi_0(\mathbf{r}t)\varphi_0(\mathbf{r}t)d\mathbf{r} \tag{10}$$

where g is a dimensionless coupling constant. Such an expression describes the interaction between electrons and phonons in solids, and also can be utilized to describe the Coulomb interaction of charged particles if we write a separate equation for the Coulomb field.

Equation (1) for the density matrix ρ(t) can be formally solved with the aid of the S-matrix

$$S(t,-\infty)=T\exp\left\{-\frac{i}{\hbar}\int_{-\infty}^{t}H_i(\tau)d\tau\right\} \tag{11}$$

which satisfies the equation

$$i\partial S/\partial!=H_i(t)S(t) \tag{12}$$

The symbol T in (11) denotes a time-ordered product defined in the usual manner. Then we have

$$\begin{aligned}\rho(t)&=S(t,-\infty)\rho_0S^{+}(t,-\infty)\\&=S(t,-\infty)\rho_0S(-\infty,t)\end{aligned} \tag{13}$$

The density matrix defined in this manner depends explicitly on the time. The average value of an arbitrary operator $L_0(t)$ at time t has the form

$$\langle L_0(t)\rangle = \mathrm{Tr}\,\{\rho(t)L_0(t)\} \tag{14}$$

where the subscript zero on the operator L_0 shows that this operator is taken in the interaction representation, i.e., its time dependence is determined by the free equation of motion in the external field

$$i\partial L_0/\partial t = [L_0(t), H_0(t)]_- \tag{15}$$

since the density matrix $\rho(t)$ itself was defined by us in the interaction representation. In future, however, we shall have occasion to deal with correlation functions of several field operators taken at different instants of time. In these cases it will be more convenient for us to transfer the entire time dependence to these operators, and to regard the density matrix as being independent of the time, i.e., to go over to the Heisenberg representation. In this case the field operators satisfy the complete equations of motion

$$i\partial\psi/\partial t = [\psi, H]_{\dots} \tag{16}$$

For the time-independent density matrix we can take the value of the matrix determined by expression (13) at a certain fixed instant of time, for example, t = 0, having thus included in it all the changes which the distribution $\rho_0(2)$ had undergone when the external field and the interaction in the system were switched on. In order to convince ourselves that such a procedure is correct we shall, for example, show that in the case of a system in thermodynamic equilibrium it leads to the Gibbs canonical distribution, as it ought. In order to do this we utilize the well-known relation [14,15]

$$\begin{aligned} &S(0,-\infty)H_0S^+(0,-\infty) \\ &\quad = H - i\delta g\frac{dS(0,-\infty)}{dg}S^+(0,-\infty) \end{aligned} \tag{17}$$

where δ is a parameter describing the adiabatic switching-on of the interaction, i.e., the interaction constant g in (10) is considered to be time dependent, in accordance with $g(t) = ge^{\delta t}$. After going to the limit $\delta \to 0$ the second term in (17) turns into a c-number equal to the sum of all the vacuum loops and describing the correction to the free energy of the system [1−3]. Therefore, for the given case of an equilibrium system
$\rho = \rho(0) = S(0,-\infty)\rho_0S^+(0,-\infty) = e^{(\Psi-H)/kT}$,
where H is the total Hamiltonian, while Ψ is the total free energy.

Having defined the density matrix as $\rho(0)$ we must utilize such Heisenberg field operators $\psi(r,t)$, which at t = 0 go over into the free $\psi_0(rt)$. In other words,

$$\psi(\mathbf{r}t) = S(0,t)\psi_0(\mathbf{r}t)S(t,0) \tag{19}$$

where, by definition,

$$\begin{aligned} S\left(t',t\right) &= T\exp\left\{-i\int_t^{t'} H_i(\tau)d\tau\right\} \\ &= S\left(t',-\infty\right)S^+(t,-\infty) \end{aligned} \tag{20}$$

Utilizing these relations and the definition of the Heisenberg density matrix ρ, the average value of the T -product of arbitrary Heisenberg operators $\mathrm{L(t), M\,(t')\ldots}$. can be brought to the form

$$\begin{aligned}\langle TL(t)M\,(t')\ldots\rangle &= \mathrm{Tr}\,\{\rho TL(t)M\,(t')\ldots\}\\ &= \mathrm{Tr}\,\{S(0,-\infty)\rho_0 S(-\infty,0)TL(t)M\,(t')\ldots\}\end{aligned}$$

or, commuting in the argument of Tr the operator $\mathrm{S}(0,-\infty)$ and going over to operators in the interaction representation,

$$\begin{aligned}&\langle TL(t)M\,(t')\ldots\rangle\\ &\quad= \mathrm{Tr}\,\{\rho_0 S(-\infty,0)\,[TS(0,t)L_0(t)S\,(t,t')\,M_0\,(t')\ldots]\\ &\quad\times S(0,-\infty)\} = \mathrm{Tr}\,\{\rho_0 T_c\,[S_c L_0(t)M\,(t')\ldots]\}\end{aligned} \tag{21}$$

where $\mathrm{T_c}$ indicates ordering along the contour c, going from $-\infty$, passing the points $\mathrm{t, t'\ldots}$ and then returning back to $-\infty$; $\mathrm{S_C}$ is the complete S -matrix defined along the whole contour c. Ordering along c means that the points on the return branch of this contour correspond to later times than the points on the original direct branch, and of any two points on the return path the later one is that which is closer to $-\infty$.

However, evaluation of integrals along the contour c is inconvenient. Therefore, we perform in (21) the following identity transformation: we insert in the argument of $\mathrm{T_c}$ the factor $\mathrm{S\,(t_0,+\infty)}\times\ S\,(+\infty,t_0)$ which is identically equal to unity, taking the point t_0 to lie on the contour c later than that one of the points $\mathrm{t, t'\ldots}$ which lies on the extreme right. This is equivalent to cutting the contour C . to the right of the point lying on the extreme right and inserting into it a piece which goes to $+\infty$ and returns. The resultant contour C traverses the whole time axis from $-\infty$ to $+\infty$ and then back again from $+\infty$ to $-\infty$. Formula (21) is thus brought to the final form

$$\langle TL(t)M'\,(t')\ldots\rangle = \mathrm{Tr}\,\{\rho_0 T_C\,(S_C L_0(t)M_0\,(t')\ldots]\} \tag{22}$$

with all the arguments $\mathrm{t, t'\ldots}$ considered as lying on the positive $(-\infty,+\infty)$ branch of the contour C.

But use of the contour C enables us to write in the form of a T-product also the usual correlation function of two operators $\mathrm{L(t)}$ and $\mathrm{M\,(t')}$. Indeed, we shall assign to points lying on the positive (pro-
ceeding from $-\infty$ to $+\infty$) branch of the contour the subscript $+$, and to points lying on the negative branch the subscript - Then evidently we have

$$\langle L(t)M\,(t')\rangle = \mathrm{Tr}\left\{\rho_0 T_C\left[S_C L_0\,(t_-)\,M_0\left(t'_+\right)\right]\right\} \tag{23}$$

since in accordance with the definition of the $\mathrm{T_{C^-}}$ product the point t'_+ always precedes the point t_-. This possibility leads to the fact that the diagram technique obtained below enables us to evaluate directly not only the Green's functions for the particles, but also the usual average values of operators.

The averages of T-products of the type (22) and (23) can be decomposed in the usual manner [2] into a sum of Feynman graphs evaluated in accordance with the following rules.

1. An n-th order graph contains 2 n points along the real time axis. Moreover, each point can lie either on the positive or on the negative branch of the contour C, and this is denoted respectively by the subscripts $+$ or - . An integration with respect to the coordinates of all the internal points is carried out over the whole volume and over time from $-\infty$ to $+\infty$. Each internal point is put in correspondence with a factor g, if its subscript is $+$, and with the factor -g, if its subscript is -.

2. One phonon and two electron lines emerge from (or enter) each point. Each of these lines is set in correspondence with one of four functions depending on the signs of the points which this line joins:
a) if the line goes from the point - to the point +, it is set in correspondence with the function

$$\begin{aligned} G_0^+\left(\mathbf{r}t_+,\mathbf{r}'t'_-\right) &= -i\,\mathrm{Tr}\left\{\rho_0 T_C\left[\psi_0\left(\mathbf{r}t_+\right),\psi_0+\left(\mathbf{r}'t'_-\right)\right]\right\} \\ &= i\left\langle\psi_0^+\left(\mathbf{r}'t_+{}'\right)\psi_0\left(\mathbf{r}t_-\right)\right\rangle_0 \end{aligned} \tag{24}$$

the symbol $\langle\ldots\rangle_0$ denotes averaging using the matrix ρ_0;
b) if the line goes from the point + to the point -, it is set in correspondence with the function

$$\begin{aligned} G_0^-\left(\mathbf{r}t_-,\mathbf{r}'t'_+\right) &= -i\,\mathrm{Tr}\left\{\rho_0 T_c\left[\psi_0\left(\mathbf{r}t_-\right)\psi_0^+\left(\mathbf{r}'t'_+\right)\right]\right\} \\ &= -i\left\langle\psi_0\left(\mathbf{r}t_-\right)\psi_0^+\left(\mathbf{r}'t'_+\right)\right\rangle_0 \end{aligned} \tag{25}$$

c) if the line goes from the point + to the point +, then it is set in correspondence with the usual causal Green's function

$$\begin{aligned} G_0^c\left(\mathbf{r}t_+,\mathbf{r}'t'_+\right) &= -i\,\mathrm{Tr}\left\{\rho_0 T_c\left[\psi_0\left(\mathbf{r}t_+\right)\psi_0{}^+\left(\mathbf{r}'t'_+\right)\right]\right\} \\ &= -i\left\langle T\psi_0\left(\mathbf{r}t_+\right)\psi_0^+ + \left(\mathbf{r}'t'_+\right)\right\rangle_0 \end{aligned} \tag{26}$$

d) and, finally, the line connecting the point with the point - is set in correspondence with the function

$$\begin{aligned} \widetilde{G}_0^c\left(\mathbf{r}t_-,\mathbf{r}'t'_-\right) &= -i\,\mathrm{Tr}\left\{\rho_0 T_C\left[\psi_0\left(\mathbf{r}t_-\right)\psi_0^+\left(\mathbf{r}'t'_-\right)\right]\right\} \\ &= -i\left\langle\widetilde{T}\psi_0\left(\mathbf{r}t_-\right)\psi_0^+\left(\mathbf{r}'t'_-\right)\right\rangle_0 \end{aligned} \tag{27}$$

In the last formula $\widetilde{T}$ denotes the product ordered with respect to the reversed time, i.e.,

$$\widetilde{T}\psi_0(t)\psi_0^+\left(t'\right) = \begin{cases} \psi_0(t)\psi_0^+\left(t'\right), \text{ if } t<t' \\ -\psi_0^+\left(t'\right)\psi_0(t), \text{ if } t>t' \end{cases}$$

We shall henceforth omit everywhere the indices + and - in the time arguments of the Green's functions, since the index ascribed to the Green's function itself uniquely determines the positions of its final and initial points on the branches of the contour C . The functions $\mathrm{G}_0^{\mathrm{c}}$ and $\widetilde{\mathrm{G}}_0^{\mathrm{c}}$ are related to $\mathrm{G}_0^{\pm}$ by the obvious relations

$$\begin{aligned} G_{0^c}\left(\mathbf{r}t,\mathbf{r}'t'\right) &= \theta\left(t-t'\right)G_0^-\left(\mathbf{r}t,\mathbf{r}'t'\right) \\ &\quad +\theta\left(t'-t\right)G_0+\mathbf{r}t,\mathbf{r}'t') \end{aligned} \tag{28}$$

$$\begin{aligned} \widetilde{G}_0^c\left(\mathbf{r}t,\mathbf{r}'t'\right) &= \theta\left(t-t'\right)G_{0'}\left(\mathbf{r}t,\mathbf{r}'t'\right) \\ &+\theta\left(t'-t\right)G_0^-\left(\mathbf{r}t,\mathbf{r}'t'\right) \\ &\theta(x) = 1/2(1+\operatorname{sign}x) \end{aligned} \tag{29}$$

The Green's functions for Bose particles are defined in an analogous manner.

The further rules for calculation are completely analogous to the usual Feynman rules, and, therefore, we shall not state them in detail. We only note that a diagram with 2 n points contains the factor i^{n+2F}, where F is the number of closed loops in it, and that the unconnected diagrams must be omitted since they cancel with the trace of the density matrix which we have for this reason omitted in advance in the denominators of all the formulas (21)-(29).

It should also be noted that the applicability of Wick's theorem to the expansion of the T-products (22), (23) is not entirely obvious. Strictly speaking, when four or more field operators having the same momentum are averaged over the distribution ρ_0, the T-product does not decompose into a sum of products of T-products taken a pair at a time. But for systems of Fermi-particles and of phonons (phonons correspond to a real field) considered by us Wick's theorem is completely rigorous. In other cases the contribution of two or more pairs of operators whose momenta are the same is generally of order 1/N, where N is the number of degrees of freedom, since integration is carried out over all the internal momenta. An exception can occur in the case of systems in which phenomena of the type of Bose-condensation are observed. The technique under consideration is not directly applicable to these systems.

We now define the complete Green's functions:

$$G^c\left(\mathbf{r}t,\mathbf{r}'t'\right) = -i\,\mathrm{Tr}\left\{\rho_0 T_C\left[\psi_0\left(\mathbf{r}t_+\right)\psi_0^+\left(\mathbf{r}'t'_+\right)S_C\right]\right\}, \tag{30}$$

$$\widetilde{G}^c\left(\mathbf{r}t,\mathbf{r}'t'\right) = -i\,\mathrm{Tr}\left\{\rho_0 T_C\left[\psi_0\left(\mathbf{r}t_-\right)\psi_0^+\left(\mathbf{r}'t'_-\right)S_C\right]\right\}, \tag{31}$$

$$G^\pm\left(\mathbf{r}t,\mathbf{r}'t'\right) = -i\,\mathrm{Tr}\left\{\rho_0 T_C\left[\psi_0\left(\mathbf{r}t_\pm\right)\psi_0^+ + \left(\mathbf{r}'t_{\Psi'}\right)S_C\right]\right\}, \tag{32}$$

which are sums of all the connected diagrams with one entering and one emerging line which differ from one another by the location of the initial and the final points on different branches of the contour C. The following relations involving them are preserved

$$G^c\left(\mathbf{r}t,\mathbf{r}'t'\right) = \theta\left(t-t'\right)G^-\left(\mathbf{r}t,\mathbf{r}'t'\right) + \theta\left(t'-t\right)G^+\left(\mathbf{r}t,\mathbf{r}'t'\right) \tag{33}$$

$$\widetilde{G}^c\left(\mathbf{r}t,\mathbf{r}'t'\right) = \theta\left(t-t'\right)G^+\left(\mathbf{r}t,\mathbf{r}'t'\right) + \theta\left(t'-t\right)G^-\left(\mathbf{r}t,\mathbf{r}'t'\right) \tag{34}$$

i.e., only the two functions $G^+$and $G^-$are independent, and even they are related by the obvious expression

$$\lim_{t'\to t}\left\{G^+\left(\mathbf{r}t,\mathbf{r}'t'\right) - G^-\left(\mathbf{r}t,\mathbf{r}'t'\right)\right\} = -i\delta\left(\mathbf{r}-\mathbf{r}'\right) \tag{35}$$

The subsequent technique of calculating the various quantities and the form of the equations are both considerably simplified if, following Mills [12], we go over to the matrix form of writing the equations. We define a two-rowed matrix G in the following manner:

$$G = \begin{pmatrix} G^c & G^- \\ G^+ & \widetilde{G}^c \end{pmatrix} \tag{36}$$

Then, as can be easily seen, summation over the subscript $\pm$ at each point reduces to matrix multiplication of matrices corresponding to lines emerging from that point. But since there are three such lines-two electron lines and one phonon line-it is necessary to connect them by means of a vertex matrix of the third rank

$$\gamma_{ij}^k = \delta_{ij}\left(\sigma_z\right)_{jk} \tag{37}$$

where the subscripts refer to the electron lines, and the superscript to the phonon line; σ_Z is the third Pauli matrix; the fact that for k = 2 it is negative takes into account the factor -1

for points on the return branch of the contour C (in matrix notation the points + correspond to subscripts 1, and the points - correspond to subscripts 2).

An arbitrary matrix element can now be written in the form of a product of blocks of the type

$$G_{li}\gamma_{ij}^{k}G_{jm}D_{kn}$$

in which summation is assumed to be carried out over repeated subscripts. D is the matrix for the Green's functions for Bose-particles defined in
analogy with (36). With this notation we no longer need to distinguish the points denoted by + and - . The technique for the evaluation of any arbitrary graph now becomes a purely Feynman one with the only difference that to each line we now set in correspondence the matrix (36), and to each point the vertex matrix (37).

Summation of the diagrams for the complete Green's function leads to an equation of the type of Dyson's equation:

$$\begin{aligned} G\left(\mathbf{r}t,\mathbf{r}'t'\right) &= G_0\left(\mathbf{r}t,\mathbf{r}'t'\right) \\ &+ ig^2\int G_0\left(\mathbf{r}t,\mathbf{r}_1t_1\right)\Sigma\left(\mathbf{r}_1t_1,\mathbf{r}_2t_2\right)G\left(\mathbf{r}_2t_2,\mathbf{r}'t'\right)d\mathbf{r}_1d\mathbf{r}_2dt_1dt_2 \end{aligned} \tag{38}$$

The self energy matrix Σ is defined by relation (39) in which the complete vertex matrix $\Gamma_{\mathrm{ij}}^{\mathrm{k}}\left(\mathrm{x},\mathrm{x}';\mathrm{y}\right)$ is the sum of all the irreducible diagrams with one incoming (at the point x) and one outgoing (at the point x') electron line and one external phonon line at the point y:

$$\begin{aligned} &\Sigma_{ij}\left(x,x'\right) \\ &= \int \gamma_{ii'k}G_{i,j}\left(x,x_1\right)\Gamma_{j'j'}\left(x_1,x'y\right)D_{k'k}(y,x)d^4x_1d^4y \\ &x = (\mathbf{r},t) \end{aligned} \tag{39}$$

Dyson's equation in integral form (38) appears to be unsatisfactory since it contains the function G_0 which explicitly depends on the initial distribution ρ_0. At the same time from physical considerations it follows that if the system is in contact with a heat bath, then after sufficient time has elapsed it must reach a state which is, generally speaking, not stationary, since the external field depends on the time, but which is independent of the initial conditions. This inconsistency can be easily avoided by going over to the differential form of Dyson's equation, i.e., by multiplying (38) by the operator matrix

$$\hat{G}_0^{-1} = G_0^{-1}(x)\sigma_z \tag{40}$$

where the differential operator G_0^{-1} is defined by

$$G_0{}^{-1}(x) = i\frac{\partial}{\partial t} - \varepsilon\left(-i\nabla - \frac{e}{c}\mathbf{A}(x)\right) - e\Phi(x), \tag{41}$$

x = (rt). Then Eq. (38) is reduced to the form

$$\begin{aligned} &\hat{G}_0{}^{-1}(x)G\left(x,x'\right) \\ &\quad = ig^2\int \Sigma\left(x,x_1\right)G\left(x_1,x'\right)dx_1 + \delta\left(x-x'\right) \end{aligned} \tag{42}$$

In deriving this formula we have utilized the obvious relations

$$G_0^{-1}(x)G_0^{\pm}(x,x') = 0, \quad G_0^{-1}(x)G_0^{c}(x,x') = \delta(x-x'),$$
$$G_0{}^{-1}(x)G_0^{c}(x,x') = -\delta(x-x')$$

Equation (42) contains only complete Green's functions which do not explicitly depend on ρ_0. At the same time its solution is determined only up to an arbitrary solution of the homogeneous equation

$$\hat{G}_0^{-1}(x)h(x,x') = 0 \tag{43}$$

We shall discuss the problem of the unique choice of the solution and of the role of initial conditions later.

Equation (42) represents a system of four in-tegro-differential equations. However, between the four unknown functions $\mathrm{G_{ij}}$ there exist two linear relations (33) and (34) which, naturally, are consistent with the equations. Therefore, by means of a certain linear canonical transformation of the matrices G, γ and D we can reduce the number of independent equations to two. Such a transformation is given by

$$G \to \frac{1-i\sigma_y}{\sqrt{2}} \begin{pmatrix} G^{\mathrm{c}} & G^- \\ G^+ & \widetilde{G}^{\mathrm{c}} \end{pmatrix} \frac{1+i\sigma_y}{\sqrt{2}} = \begin{pmatrix} 0 & G^a \\ G^r & F \end{pmatrix}, \tag{44}$$

where three new Green's functions have been introduced: the advanced one $\mathrm{G^a}$, the retarded one $\mathrm{G^r}$, and the correlation function F which, as we shall see later, is essentially a single-particle density matrix. The matrix σ_y is the second of the three Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The functions $G^{\mathrm{a}}, \mathrm{G^r}$, and F are defined as follows in terms of average values of Heisenberg operators using the Heisenberg density matrix:

$$G^a(x,x') = -i\theta(t-t')\left\langle \left[\psi(x), \psi^+(x')\right]_+ \right\rangle \tag{45}$$
$$G^r(x,x') = i\theta(t'-t)\left\langle \left[\psi(x), \psi^+(x')\right]_+ \right\rangle \tag{46}$$
$$F(x,x') = -i < \left[\psi(x), \psi^+(x')\right]_- \Big\rangle \tag{47}$$

They are related to the G-functions used above by means of the relations

$$\begin{aligned} G^a &= G^c - G^+ = -\widetilde{G}^c + G^- \\ &= -\theta(t-t')\left(G^+ - G^-\right) \end{aligned} \tag{48}$$
$$\begin{aligned} G^r &= G^c - G^- = -\widetilde{G}^c + G^+ \\ &= \theta(t'-t)\left(G^+ - G^-\right) \end{aligned} \tag{49}$$
$$F = G^c + \widetilde{G}^c = G^+ + G^- \tag{50}$$

The D-matrix transforms in a similar fashion. The vertex matrix $\gamma_{\mathrm{ij}}^{\mathrm{k}}$ and the operator $\hat{\mathrm{G}}_0^{-1}$ are also altered:

$$\gamma_{ij}{}^1 = 2^{-1/2}\delta_{ij}, \quad \gamma_{ij}^2 = 2^{-1/2}(\sigma_x)_{ij} \tag{51}$$
$$\hat{G}_0^{-1} = \sigma_x G_0^{-1}(x) \tag{52}$$

The self energy matrix Σ is brought to the form

$$\Sigma = \begin{pmatrix} \Omega & \Sigma^r \\ \Sigma^a & 0 \end{pmatrix} \tag{53}$$

$$\Sigma^r = \Sigma^c + \Sigma^- = -\left(\widetilde{\Sigma}^c + \Sigma^+\right)$$

$$\Sigma^a = \Sigma^c + \Sigma^+ = -\left(\widetilde{\Sigma}^c + \Sigma^-\right)$$

$$\Omega = \Sigma^c + \widetilde{\Sigma^c} = -\left(\Sigma^+ + \Sigma^-\right) \tag{54}$$

The difference between (53), (54) and (44), (48)-(50) is related to the fact that the quantities $\Sigma^{\pm}$in contrast to Σ^{c} and $\tilde{\Sigma}^{\mathrm{c}}$ contain the factor -1, which appears in the vertex (37) for k = 2, an odd number of times.

The general equations (39) and (42) are obviously invariant under the transformation (44); we simply go over to a new representation for the matrices G and D . But the number of independent equations is now obviously equal to two since the functions G^{a} and G^{r} are Hermitian conjugates and the equations for them follow from each other. The physical meaning of the functions G^{r} and F is quite different. Roughly speaking, G^{r} and G^{a} characterize the dynamic properties of the particles in the system under consideration, while F characterizes their static distribution. This can already be seen from the definition of these functions in the absence both of an external field and of an interaction, i.e., for $t \to -\infty$. Then, going over to Fourier transforms with respect to the differences $\mathbf{r} - \mathbf{r}'$ and $\mathrm{t} - \mathrm{t}'$ we obtain

$$G_0(\mathbf{p}\varepsilon) = \int G_0\left(\mathbf{r} - \mathbf{r}', t - t'\right) \exp\left\{-i\left[\mathbf{p}\left(\mathbf{r} - \mathbf{r}'\right)\right.\right.$$
$$\left.\left.-\varepsilon\left(t - t'\right)\right]\right\} d\left(\mathbf{r} - \mathbf{r}'\right) d\left(t - t'\right) \tag{55}$$

and in the limiting case mentioned above

$$G_0^r(\mathbf{p}\varepsilon) = G_0^{a^*}(\mathbf{p}\varepsilon) = 1/\left(\varepsilon - \varepsilon_{\mathbf{p}} - i\delta\right)$$
$$\delta \to +0 \tag{56}$$

$$G_0{}^+(\mathbf{p}\varepsilon) = 2\pi i n_{\mathbf{p}} \delta\left(\varepsilon - \varepsilon_{\mathbf{p}}\right)$$

$$G_0^-(\mathbf{p}\varepsilon) = -2\pi i\left(1 - n_{\mathbf{p}}\right)\delta\left(\varepsilon - \varepsilon_{\mathbf{p}}\right) \tag{57}$$

$$F_0(\mathbf{p}\varepsilon) = 2\pi i\left(2n_{\mathbf{p}} - 1\right)\delta\left(\varepsilon - \varepsilon_{\mathbf{p}}\right) \tag{58}$$

where

$$n_{\mathrm{p}} = \left[\exp\frac{\varepsilon_{\mathbf{p}} - \mu}{kT} + 1\right]^{-1}$$

is the Fermi distribution function, μ is the chemical potential in the equilibrium state. Thus, the functions F and $\mathrm{G}^{\pm}$essentially represent the distribution function for the particles written in different forms.

When the interaction is taken into account the
functions G^{r} and G^{a} also indirectly depend on the distribution of the particles, but the fact that $\mathrm{G}^{\pm}\left(\mathbf{r}\mathrm{t}, \mathbf{r}'\mathrm{t}\right)$ and $\mathrm{F}\left(\mathrm{rt}, \mathrm{r}'\mathrm{t}\right)$ taken at coincident times $\mathrm{t}' = \mathrm{t}$ directly determine the distributions of electrons and of holes follows directly from their definition (32). We shall now show that the equation for G^{+}(or the equation for F equivalent to it) contains within itself the usual Boltzmann kinetic equation under the assumptions under which the latter exists, i.e., when the external field is quasiclassical and the interaction is weak in the sense

$$\hbar/\tau \not\ll \bar{\varepsilon} \tag{59}$$

where τ is the time taken to traverse a mean free path, and $\bar{\epsilon}$ is the average energy of the particles.

The equation for G^+has the form

$$G_0^{-1}(x)G^+(x,x') = ig^2 \int \left\{ \Sigma^+(x,x'')\, G^c(x'',x') \right.$$
$$\left. + \widetilde{\Sigma}^c(x,x'')\, G^+(x'',x') \right\} dr'' \tag{60}$$

The corresponding Hermitian conjugate equation is

$$G_0{}^{-1*}(x')\, G^+(x,x') = ig^2 \int \left\{ \widetilde{G}^c(x,x'')\, \Sigma^+(x'',x') \right.$$
$$\left. + G^+(x,x'')\, \Sigma^c(x'',x') \right\} dx'' \tag{61}$$

We add these two equations, then go over from the variables x, x' to the new variables $(\mathrm{r_S t_S}) = \mathrm{x_S} = (x+x')/2$ and $(r_a t_a) = x_a = (x-x')/2$, and then take a Fourier transform of the type (55) with respect to the difference variable $\mathrm{x_a}$. As a result terms of the type

$$ig^2 \int \Sigma(x_s + x_a', p')\, G(x_s - x_a + x_a', p'') \exp\{i\,[(p'-p)\,x_a$$
$$+ (p''-p')\,x_a]\}\, dx_a dx_a{}' dp' dp'', \quad p = (\mathbf{p}\varepsilon)$$

will appear on the right hand side of the equation.

The quasiclassical nature of the external field means that quantities of the type $p\nabla_{\mathrm{S}}$ are small. At the same time the right hand side of the equation is proportional to the small interaction constant g^2. Therefore, neglecting small terms of the form $\mathrm{g}^2\,(\mathrm{p}\nabla_{\mathrm{S}})$ and introducing a further change of variables $\mathrm{p} \to \mathrm{p} - \mathrm{eA}\,(\mathrm{x_S})/\mathrm{c}, \epsilon \to \epsilon - \mathrm{e}\Phi\,(\mathrm{x_S})$, we obtain

$$\left\{ \frac{\partial}{\partial t_s} + \mathbf{v_p}\nabla_s + \left(e\mathbf{E} + \left[\frac{e\mathbf{v_p}}{c}\mathbf{H} \right] \right) \nabla_{\mathbf{p}} \right\} G^+(x_s p)$$
$$= g^2 \left\{ \Sigma^+(x_s p) \left[G^c(x_s p) + \widetilde{G}^c(x_s p) \right] \right.$$
$$\left. + \left[\Sigma^c(x_s p) + \widetilde{\Sigma}^c(x_s p) \right] G^+(x_s p) \right\} \tag{62}$$

where $\mathbf{v_p} = \nabla_{\mathbf{p}}\epsilon_{\mathrm{p}}$ is the velocity of the particles, E and H are the electric and the magnetic fields:

$* \, [v_p H] = v_p \times H.$

$$\mathbf{E} = -\nabla\Phi - c^{-1}\partial\mathbf{A}/\partial t; \quad \mathbf{H} = \mathrm{rot}\,\mathbf{A} \tag{63}$$

In deriving (62) we have expanded in the left hand side of the equation the operator $\mathrm{G}_0^{-1}(\mathrm{x}) - \mathrm{G}_0^{-1*}(\mathrm{x}')$ in series in terms of the small difference $x_a = (x-x')/2$ up to the first order on the basis of the same quasiclassical condition. The right hand side of (62) can be further simplified by utilizing (50) and (54):

$$\left\{ \frac{\partial}{\partial t} + \mathbf{v}\nabla + e\left(\mathbf{E} + \left[\frac{\mathbf{v}}{c}\mathbf{H} \right] \right) \nabla_{\mathbf{p}} \right\} G^+(xp)$$
$$= g^2 \left\{ \Sigma^+(xp) G^-(xp) - \Sigma^-(xp) G^+(xp) \right\} \tag{64}$$

The right hand side of (64) describes the effect of collisions on the distribution of the particles, with the first term being related to particles entering the state of momentum p from all the other states with momenta p′, while the second term is related to the particles leaving the state. In order to bring this equation to the usual form of the Boltzmann equation we introduce the distribution function $f(prt)$ which, as has been noted previously, coincides with $G^+$for coincident times $t' = t$:

$$f(\mathbf{p}\mathbf{r}t) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} G^+(\mathbf{r}t;\mathbf{p}\varepsilon)d\varepsilon \tag{65}$$

Integration of equation (64) over ϵ then yields

$$\begin{aligned}\left\{\frac{\partial}{\partial t} + \mathbf{v}\boldsymbol{\nabla} + e\left(\mathscr{E} + \left[\frac{\mathbf{v}}{c}\mathbf{H}\right]\right)\boldsymbol{\nabla}\mathbf{p}\right\} f(\mathbf{p}\mathbf{r}t) \\ = \frac{g^2}{2\pi i}\int_{-\infty}^{\infty}\left\{\Sigma^+(\mathbf{r}t;\mathbf{p}\varepsilon)G^-(\mathbf{r}t;\mathbf{p}\varepsilon)\right. \\ \left.-\boldsymbol{\Sigma}^-(\mathbf{r}t;\mathbf{p}\varepsilon)G^+(\mathbf{r}t;\mathbf{p}\varepsilon)\right\} d\varepsilon\end{aligned} \tag{66}$$

In accordance with the assumption that the interaction is small we must evaluate the right hand side of (66) up to the first order in g^2, i.e., we must substitute into (66) and (39) the functions $G_0^{\pm}$, satisfying the free equations

$$G_0{}^{-1}G_0 \pm (\mathbf{r}t;\mathbf{p}\varepsilon) = 0 \tag{67}$$

but normalized with respect to an as yet unknown distribution function f by equations (65) and (35). Symbolically they can be written in the form

$$G_0^+(\mathbf{r}t;\mathbf{p}\varepsilon) = 2\pi i\delta\left(\varepsilon - \varepsilon_{\mathbf{p}} + i\hat{B}\right) f(\mathbf{p}\mathbf{r}t) \tag{68}$$

$$G_0 - (\mathbf{r}t;\mathbf{p}\varepsilon) = -2\pi i\delta\left(\varepsilon - \varepsilon_{\mathbf{p}} + i\hat{B}\right)(1 - f(\mathbf{p}\mathbf{r}t)) \tag{69}$$

where $\hat{B}$ is the operator in the left hand side of (66). The quantity $\delta\left(G_0^{-1}\right)$ in formulas (68), (69), is an integral operator with a kernel satisfying equation (67) and equal to $\delta(r - r')$ for $t' = t$. It can be easily expressed in terms of the functions φ_p (4).

In order to evaluate $\Sigma^{\pm}$we must also define $D_0^{\pm}$. In the majority of problems of solid state physics the phonons can be regarded as being in thermodynamic equilibrium. Then

$$\begin{aligned}D_0 \pm (\mathrm{k}\omega) = -i\pi\left|m_{\mathbf{k}}\right|^2\left\{(1 + N_{\mathbf{k}})\,\delta(\omega \pm \omega_{\mathbf{k}})\right. \\ \left.+N_{\mathbf{k}}\delta(\omega \mp \omega_{\mathbf{k}})\right\}\end{aligned} \tag{70}$$

where k and ω_{k} are the propagation vector and the frequency of the phonon, m_{k} is the matrix element for the interaction with the electron, and

$$N_{\mathbf{k}} = \left[\exp\frac{\omega_{\mathbf{k}}}{kT} - 1\right]^{-1} \tag{71}$$

is the Planck distribution function.
Substituting (68)-(70) into (66) we obtain

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- rot = curl.

$$\hat{B}f(\mathbf{p}\mathbf{r}t) = 2\pi^2 g^2 \int |m_{\mathbf{k}}|^2 \{[(1+N_{\mathbf{k}})\,\delta\,(\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}+\mathbf{k}} + \omega_{\mathbf{k}})$$
$$+N_{\mathbf{k}}\delta\,(\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}+\mathbf{k}} - \omega_{\mathbf{k}})]\, f(\mathbf{p}+\mathbf{k},\mathbf{r}t)(1-f(\mathbf{p}\mathbf{r}t))$$
$$-\,[(1+N_{\mathbf{k}})\,\delta\,(\varepsilon_{\mathbf{p}} - \boldsymbol{\varepsilon}_{\mathbf{p}+\mathbf{k}} - \omega_{\mathbf{k}}) + N_{\mathbf{k}}\delta\,(\varepsilon_{\mathbf{p}} - \varepsilon_{\mathbf{p}+\mathbf{k}} + \omega_{\mathbf{k}})]$$
$$\times\,(1-f(\mathbf{p}+\mathbf{k},\mathbf{r}t))f(\mathbf{p}\mathbf{r}t)\}d\mathbf{k}/(2\pi)^3 \tag{72}$$

i.e., the usual form of Boltzmann's equation. Thus, in deriving it we have utilized two conditions: the quasiclassical nature of the external field

$$\frac{d}{dx}(e\mathbf{A}, e\Phi) \ll \varepsilon\bar{p} \tag{73}$$

and the smallness of the interaction

$$\bar{\varepsilon} \gg g^2\Sigma^{\pm} \sim \hbar/\tau \tag{74}$$

The quasiclassical condition is not obligatory. Proceeding in exactly the same manner, but without expanding (60) and (61) in terms of the external field, we would have obtained the so-called quantum kinetic equation [16,17]. But the condition of smallness of the interaction (74) is necessary in this case also. If it is violated, then the functions $G^{\pm}$do not have a δ-like character (68), (69) as functions of ϵ, and the integration over ϵ in equation (66) cannot be effectively carried out. In such a case we shall not obtain, generally speaking, a closed equation for the distribution function f(prt**t**), i.e., for the G^{+}-function for coincident times $t' = t$. The equation exists only for the complete G-function considered as a function of all its arguments. In the most general case this corresponds to Dyson's equations (39) and (42). If the interaction is not weak, they can be solved if the system has some other parameters, for example density, which enable us to select a set of principal diagrams. However, in this case there will arise the previously mentioned problem of the nonuniqueness of the solution of (42) up to an arbitrary solution of the free equation. We shall now show that such
arbitrariness does not in fact exist, and that there is a unique prescription for the choice of the correct solution.

In order to do this we compare Eqs. (42) and (38) in the representation (44), by writing them separately for each of the functions G^a, G^r and F. In differential form

$$G_0^{-1}G^a = 1 + \Sigma^a G^a, \tag{75}$$
$$G_0^{-1}G^r = 1 + \Sigma^r G^r \tag{76}$$
$$G_0{}^{-1}F = \Omega G^a + \Sigma^r F \tag{77}$$

in integral form, corresponding to equation (38),

$$G^a = G_0^a\,(1 + \Sigma^a G^a) \tag{75a}$$
$$G^r = G_0^r\,(1 + \Sigma^r G^r) \tag{76a}$$
$$F = F_0\,(1 + \Sigma^a G^a) + G_0^r\,(\Omega G^a + \Sigma^r F) \tag{77a}$$

Comparison of (75) - (77) with (75a) - (77a) shows that equations (75), (76) do not contain any arbitrariness: they are solved by the advanced operator G_0^a or the retarded operator G_0^r respectively. For $t' \to t$ their solutions must go over into $\delta\,(r - r')$. A possible lack of uniqueness, i.e., dependence on initial conditions, is associated only with the term proportional to F_0

in equation (77 a), since F_0 depends on the initial distribution, while in the differential form of the equation this term is completely absent. However, it can be easily seen that this term is identically equal to zero. Indeed, we substitute (75) into (77a). Then we have

$$F = F_0 {G_0}^{-1} G^a + {G_0}^r \left(\Omega G^a + \Sigma^r F\right) = {G_0}^r \left(\Omega G^a + \Sigma^r F\right), \quad (78)$$

since $F_0 G_0^{-1} = 0$. Thus, (77) is solved by means of the function G_0^r operating from the left. Equation (77) and its solution only appear to be nonsymmetric with respect to retarded and advanced functions. In place of (77) we can consider the Hermitian conjugate equation equivalent to it

$$ {G_0}^{-1} F = G^r \Omega + F \Sigma^a \quad (79)$$

which is solved by the function G_0^a operating from the right. Thus, solutions of Eqs. (75) - (77) are determined uniquely and independently of the initial distribution at $t = -\infty$.

We should now make clear to what systems the results obtained above are applicable. Evidently, we are dealing with well established solutions, i.e., such solutions which are determined by the effect of the external field at the present and at earlier instants of time, by the interaction with the heat bath, but not by the initial conditions. Generally speaking, these are not stationary solutions if the external field depends on the time. The lack of dependence on the initial conditions means physically
that the initial conditions, even if they had been imposed, were specified in the sufficiently remote past, formally at $t \to -\infty$, while in fact we are dealing with times large in comparison to the relaxation time of the system. However, in discussions of kinetic phenomena, another way of posing the problem is possible: at time t_0 an arbitrary density matrix $\rho(t_0)$ is given and we are dealing with the relaxation of this state towards some well established regime. Such a problem, apparently, can not in principle be described by any diagram technique, since the latter is always based on the concept of the possibility of expressing two-, three-, etc. particle Green's functions in terms of the oneparticle Green's function at least in the form of an infinite series. But an arbitrary specification of the density matrix is equivalent to an arbitrary and independent specification of all these quantities.

The existence of well established solutions independent of initial conditions is evident in the case of systems in contact with a heat bath. In the case of an adiabatic system the problem is less clear and the technique under discussion is applicable with certainty only in the case when the system had been actually in a state of thermodynamic equilibrium and was then taken out of that state by a given external field. The problem of approach to equilibrium from an arbitrary initial state has been discussed in papers by Van Hove [18], Prigogine and Resibois [19], Fujita [20] and others.

In conclusion I wish to express my gratitude to V. L. Gurevich, I. E. Dzyaloshinskiĭ, D. A. Kirzhnits, and E. S. Fradkin for discussions of a number of points dealt with in the present article.

Part VII
Kinetic Equation Method

25 Nonequilibrium Properties of Superconductors (Transport Equation Approach) by Aronov, Galperin, Gurevich, Kozub

25.1 1. Introduction

The purpose of this review article is to formulate a relatively simple quasiclassical approach to the description of nonequilibrium properties of superconductors. We want to present a method which is as close to the Boltzmann equation description for normal metals as possible. The advantage of this approach is that it ensures not only quantitative but also qualitative insight into the physical picture (finally, what is this qualitative insight?).

The electron system of a superconductor has two-component character: it consists of the Cooper pair condensate and normal excitations - quasielectrons and quasiholes. It is the close resemblance in the behaviour of the quasiparticles in superconductors and of the electrons in normal metals that makes one hope that the transport equation can be applicable also for superconductors.

Geilikman (1958) and Kresin (1959) (read these articles!!!) were the first to employ these ideas. These authors investigated the electron contribution to the thermal conductivity of a superconductor. The same approach was used by Privorotskii (1962), who studied ultrasonic absorption (read it!). Afterwards in such a way the solutions of several problems were obtained.

However, the important problem remained to generalize such an approach for the case of large perturbations (what is this parameter of largeness?), when the response is non-linear. Furthermore, it was important to consider the problems connected with moving condensate.

The last but not the least problem was to derive the transport equation from first principles. In the course of such a derivation which will also be presented in this chapter we shall discuss the limits of applicability of the Boltzmann equation.

We should like to stress an essential point of difference between normal metals and superconductors. Quantum effects in superconductors are much more pronounced (which is a very well-known statement, so this paragraph is well-known). The characteristic quantum length for superconductors (the so-called coherence length $\check{\zeta}_0 = \hbar v_F/\Delta$) is much larger than the de Broglic length $\hbar_i p_{\mathrm{F}}$ in normal metals [here $2\pi h$ is the Planck constant, $v_{\mathrm{F}}\,(p_1)$ is the Fermi velocity (momentum) while A is the modulus of the order parameter of a superconductor]. Furthermore, the characteristic energy scale in superconductors is of the order of Δ, which is much smaller than the Fermi energy μ, i.e. the characteristic energy for normal metals.

To understand the role of quantum effects it is natural to start from most general equations, which was done by several authors. A set of equations for the Green functions (with regard to condensate motion) which could be used to analyze transport phenomena was derived by Stephen (1965) (what is there??). This was an essential achievement. However it was difficult to use these equations for explicit calculations of transport phenomena.

Galaiko (1973,1974,1975) (what is there??) proposed a transport equation for the electron density matrix. Rather general problems could be treated with the help of this equation. Still it was much more sophisticated than the usual Boltzmann equation.

A very powerful treatment of transport phenomena was developed on the basis of the set of equations for the Green function integrated over the variable $\check{\zeta}_p = p^2/2m - \mu$ (i.e. over the modulus of the electron momentum p). For small deviations from equilibrium this method was proposed by Eilenberger (1968) [see also the paper by Prange and Kadanoff (1964), who developed the same method for normal metals]. Eliashberg (1971) generalized this method for large deviations from equilibrium, where the corresponding equations are nonlinear. This approach was extended by Larkin and Ovchinnikov (1973,1977) and by Schmid and Schön (1975a). (I need definitely to read it!!!)

Recently Shelankov (1980) proposed a concise form of generalized nonlinear transport equation that transforms directly into the usual Boltzmann equation (discussed in this article) for the case of rare collisions.

Numerous examples of applications of some of those methods can be found in other chapters in this book (Larkin and Ovchinnikov 1985, Galaiko and Kopnin 1985, Eliashberg and Ivlev 1985, Galitskii et al. 1985).

The advantage of these methods is that they remain valid even if the concept of the excitation spectrum is lost, e.g. if the order parameter A varies rapidly as a function of space coordinates r (or time t) or if the momentum is not a good quantum number (e.g. because of frequent collisions with impurities) as in the case of the so-called dirty superconductors (what if momentum is not a good quantum number? I don't know...). However, such widening of the range of applicability of the theory results inevitably in some loss of clarity of the physical picture.

Having all this in mind we have made an attempt to derive a quasi-classical transport equation similar to the Boltzmann equation for the normal state (Gal'perin et al. 1973a. Aronov and Gurevich 1974). This equation is not so general as those mentioned above. However, it describes many problems of interest.

In what follows we shall describe such an approach and some of its applications.

In section 2 the full set of equations describing transport in the electron system of a superconductor is formulated. It consists of the transport equation for the quasiparticles, and the self-consistency equation for the modulus of the order parameter and the equations for electromagnetic potentials; the specific feature of a superconductor is the fact that such a set also includes the continuity equation as an independent one. In that section the conditions of applicability of the approach are also discussed.

In section 3 the classification of perturbations representing different kinds of nonequilibrium phenomena, depending on the features of quasiparticle relaxation, is given.

In section 4 the proposed approach is used for discussion of stationary nonequilibrium phenomena in superconductors in the presence of laser pumping.

In section 5 the main thermodynamic equations-the energy conservation law and the H-theorem-are derived with the help of the transport equation approach. In this way the approach is proved to be consistent with the main thermodynamic principles.

The hydrodynamical approach applicable to the case of strong impurity scattering is presented in section 6. The full system of macroscopic equations is derived and some transport phenomena are discussed. Special attention is given to specific phenomena connected with imbalance of the population of the electron-like and the hole-like branches of the quasiparticle spectrum.

The specific features of the thermoelectric effect in superconductors are discussed in section 7; a critical review of some papers concerning the supercurrent-induced thermoelectric effect is also given there.

In section 8 the acoustical phenomena in superconductors are treated; these are both inho-

mogeneous and time-dependent; the Boltzmann equation approach is a valuable tool to study the response of a superconductor to perturbations of this kind. It is important that this approach makes it possible in a relatively simple way to study corresponding nonlinear problems, such as nonlinear ultrasonic damping and the acousto-electric effect, which are rather difficult to be studied by other methods.

A microscopic derivation of the basic equations with the help of the Keldysh diagram technique is presented in section 9.

25.2 2. The full set of equations describing transport

The form of equations

At finite temperatures the electron system of a superconductor is two-component: it consists of a condensate of Cooper pairs and of normal excitations. The unique electromagnetic behavior of a superconductor is due to the condensate. This is a quantum coherent state described by the order parameter, i.e. by the wave function $\Delta e^{1/}$ (what is in the phase? there is a typo in the book). Its modulus is equal to half of the energy gap in the excitation spectrum.

The kinetic properties (such as thermal conductivity, ultrasonic attenuation ctc.) of superconductors are determined mainly by the normal excitations.

We shall begin with a description of the quasiparticle spectrum in the quasiclassical approximation.

Let us suppose that the following conditions of applicability of the quasiclassical approach hold:

$$\hbar\omega \ll \Delta, \tag{1}$$

$$qd \ll 1. \tag{2}$$

Here ω is the angular frequency of the external perturbation, q is its wave vector, while d is the characteristic quantum length. For the majority of problems d is of the order of the coherence length ξ_0. However, in general, as we shall see below, it may differ for different problems.

One can obtain the quasiparticle spectrum in this approximation from the Bogoliubov-De Gennes equations (see e.g. De Gennes 1968):

$$\begin{aligned} \tilde{\varepsilon}_p &= \varepsilon_p + p_s v, & \varepsilon_p &= \sqrt{\xi_p^2 + \Delta^2}, \\ \tilde{\xi}_p &= \xi_p + \phi + \tfrac{p_s^2}{2m}, & \xi_p &= E_p - \mu = \tfrac{p^2}{2m} - \mu. \end{aligned} \tag{3}$$

(derive this formula with p_s! I haven't seen it yet before) For simplicity, in most of this chapter we consider the case of an isotropic quadratic spectrum: $E(\boldsymbol{p}) = p^2/2m$. The generalization to the anisotropic case is straightforward. Here E_p is the spectrum of electrons in the normal state.* Two gauge-covariant quantities, p_s and ϕ, enter eq. (6). The condensate momentum p_s is given by

$$p_\mathrm{s} = \frac{\hbar}{2}\left(\nabla\chi - \frac{2e}{\hbar c}A\right) \tag{4}$$

while the covariant potential ϕ is

$$\phi = \frac{\hbar}{2}\frac{\partial\chi}{\partial t} + e\varphi \tag{5}$$

where e is the electron charge, c is the light velocity and A and φ are the vector and scalar potentials, correspondingly.

Our purpose is to formulate the transport equation for the distribution function n_p of quasiparticles, i.e. of the normal excitations in a superconductor. This equation has the form

$$\frac{\partial n_p}{\partial t} + \frac{\partial \tilde{\varepsilon}_p}{\partial p}\frac{\partial n_p}{\partial r} - \frac{\partial \tilde{\varepsilon}_p}{\partial r}\frac{\partial n_p}{\partial p} + I\left\{n_p\right\} = 0. \tag{6}$$

Here

$$I = I_{\mathrm{im}} + I_{\mathrm{ph}} + I_{\mathrm{ee}} \tag{7}$$

is the collision operator describing collisions of the quasiparticles with impurities [the first term in eq. (7)], with phonons (second term) and with each other (third term).

The first term has the form

$$\begin{aligned} I_{\mathrm{im}} &= \frac{\pi N_{\mathrm{i}}(2\pi)^4}{m^2}\int (\,\mathrm{d}\rho)_{p'}\left|f_{p,p'}\right|^2\left(u_p u_{p'} - v_p v_{p'}\right)^2\left(n_p - n_{p'}\right)\delta\left(\tilde{\varepsilon}_p - \tilde{\varepsilon}_{p'}\right) \\ (\mathrm{d}p)_{p'} &= \frac{2\,\mathrm{d}^3 p'}{(2\pi\hbar)^3} \end{aligned} \tag{8}$$

(why this formula is such? should it be obvious??? maybe there is a derivation below...)

(intuition here is needed!! can we write explicitly this integral if we substitute functions!!! now it is not clear what is meant here. I'll write this explicit form here)

Here N_{i} is the impurity concentration, $f_{p,p'}$ is the amplitude of scattering of a normal electron from the state p (what is this state? we have a superconductor, actually) to the state p', u_p and v_p are the well-known Bogoliubov-Valatin amplitudes, given by

$$u_p^2 = \frac{1}{2}\left(1 + \tilde{\zeta}_p/\varepsilon_p\right), \quad v_p^2 = \frac{1}{2}\left(1 - \tilde{\zeta}_p/\varepsilon_p\right) \tag{9}$$

Note that it is the quantity ε_p rather than $\tilde{\varepsilon}_p$ that enters these expressions (why???).

The term in the integrand which is proportional to n_p is the so-called "outterm", which describes the collisional events where an excitation with momentum p makes a transition into a state with momentum p'. Another term in eq. (8) proportional to n_p is the so-called "in-term" which describes the reversed transitions.

The collision term I_{ph} has the form

$$I_{\mathrm{ph}} = I^{(\mathrm{s})}\left\{n_p\right\} + I^{(\mathrm{r})}\left\{n_p\right\}$$

where

$$\begin{aligned} I^{(s)}\left\{n_p\right\} = -\frac{2\pi}{\hbar}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\left|c_q\right|^2\left(u_p u_{p-\hbar q} - v_p v_{p-\hbar q}\right)^2 \\ \times\Big\{ n_{p-\hbar q}\left(1-n_p\right)\left[\left(1+N_{-q}\right)\delta\left(\tilde{\varepsilon}_p - \tilde{\varepsilon}_{p-\hbar q} + \hbar\omega q\right) + N_q\delta\left(\tilde{\varepsilon}_p - \tilde{\varepsilon}_{p-\hbar q} - \hbar\omega_q\right)\right] \\ - n_p\left(1-n_{p-\hbar q}\right)\left[N_{-q}\delta\left(\tilde{\varepsilon}_p - \tilde{\varepsilon}_{p-\hbar q} + \hbar\omega_q\right) + \left(1+N_q\right)\delta\left(\tilde{\varepsilon}_p - \tilde{\varepsilon}_{p-\hbar q} - \hbar\omega_q\right)\right]\Big\} \end{aligned} \tag{10}$$

$$\begin{aligned} I^{(r)}\left\{n_p\right\} = -\frac{2\pi}{\hbar}\int \frac{\mathrm{d}^3 q}{(2\pi)^3}\left|c_{\mathrm{q}}\right|^2\left(u_p v_{p-\hbar q} + u_{p-\hbar q} v_p\right)^2\delta\left(\tilde{\varepsilon}_p + \varepsilon_{-p+\hbar q} - \hbar\omega_q\right) \\ \times\left[\left(1-n_p\right)\left(1-n_{-p+\hbar q}\right)N_q - n_p n_{-p+\hbar q}\left(1+N_q\right)\right] \end{aligned} \tag{11}$$

(explain here meaning of the terms and write the explicit form of them! it should be clear, what functions are used???)

Here c_q is the matrix element of the electron phonon interaction (and not an operator), (ω_q is the phonon frequency, and N_q is the phonon distribution function for the phonon wave vector q. This function in general should be determined from the phonon Boltzmann equation (where does this equation come from??)

$$\frac{\partial N_q}{\partial t} + \frac{\partial \omega_q}{\partial q}\frac{\partial N_q}{\partial r} + I_{\text{ph-ph}}\left\{N_q\right\} + I_{\text{ph}-e}\left\{n_p, N_q\right\} = 0 \tag{12}$$

The collision operator $I_{\text{ph-ph}}\left\{N_q\right\}$ describes scattering of phonons by other phonons and lattice defects. The phonon-electron collision operator, $I_{\text{ph-e}}\left\{n_p, N_q\right\}$ can be divided into two parts: $I_{\text{phe}} \equiv I^{(s)}_{\text{phc}} + I^{(r)}_{\text{ph-e}}$, where

$$\begin{aligned} I^{(s)}_{\text{ph-e}} =& \frac{2\pi}{\hbar}\int (\,\mathrm{d}p)_p \left|c_q\right|^2 \left(u_p u_{p+\hbar q} - v_p v_{p+\hbar q}\right)^2 \\ & \times \left[n_p\left(1+n_{p+\hbar q}\right)N_q - n_{p+\hbar q}\left(1-n_p\right)\left(N_q+1\right)\right]\delta\left(\tilde{\varepsilon}_p - \tilde{\varepsilon}_{p+\hbar q} + \hbar\omega_q\right), \end{aligned} \tag{13}$$

$$\begin{aligned} I^{(r)}_{\text{ph-e}} =& \frac{2\pi}{\hbar}\int (\,\mathrm{d}p)_p \left|c_q\right|^2 \left(u_p v_{p-\hbar q} + u_{p-\hbar q} v_p\right)^2 \\ & \times \left[N_q\left(1-n_p\right)n_{\hbar q-p} - n_p n_{\hbar q-p}\left(1+N_q\right)\right]\delta\left(\tilde{\varepsilon}_p + \varepsilon_{\hbar q-p} - \hbar\omega_q\right). \end{aligned} \tag{14}$$

The term $I^{(s)}_{\text{ph-e}}$ describes scattering of phonons by quasiparticles while the term $I^{(r)}_{\text{ph}}$ is due to creation-annihilation processes in the course of electron phonon collisions.

We shall not write the electron-electron collision term explicitly. The explicit expression for this term was derived by Eliashberg (1971) (for the ξ_p-even part of the distribution function) (read it!!!). Let us mention, however, that it describes processes of two types, i.e. processes of the type $2 \to 2$, conserving the number of the quasiparticles (two quasiparticles collide and as a result come into two other quantum states) and processes of the type $3 \leftrightarrows 1$ (as a result of such a collision two quasiparticles are either generated or absorbed).

Containing only the gauge-covariant quantities p_{s} and ϕ, eq. (6) is gauge-covariant itself. The solution of a transport problem in superconductors is in general self-consistent: the non-equilibrium distribution should be sought simultaneously with the excitation spectrum which, in its turn, depends on the excitation distribution, as well as on external fields. Thus, another important equation in our case is the equation for the energy gap in the nonequilibrium state. As we shall see (section 9) (see this proof! because one may ask, why this eq is at all used in noneq case?), in the quasiclassical approximation this equation has the same form as in the equilibrium case:

$$1 = -\frac{\lambda_{\text{e}}}{2}\int (\,\mathrm{d}p)_p \frac{1-2n_p}{\varepsilon_p}, \tag{15}$$

where λ_e is the effective constant of electron-electron coupling. In the vicinity of the transition point to the normal state eq. (15) reduces to the well-known Ginzburg Landau equation, where the term with the space derivatives (of the order of $q\check{\zeta}_0 \ll 1$) is discarded while some additional term due to the nonequilibrium part of the distribution function is present. The latter is sometimes called "the anomalous term" (see Gor'kov and Eliashberg 1968).

Limits of applicability of collision equations

Now let us discuss the limits of applicability of the equations formulated above in more detail. For this purpose, let us evaluate the characteristic length d entering the criterion (2). Let us turn to the pair of the Bogoliubov-De Gennes equations

$$\left[\frac{1}{2m}\left(-\mathrm{i}\hbar\frac{\partial}{\partial r}-\frac{e}{c}\boldsymbol{A}\right)^2+e\varphi-\mu\right]u+\Delta\mathrm{e}^{\mathrm{i}\chi}v=\tilde{\varepsilon}u,$$
$$-\left[\frac{1}{2m}\left(-\mathrm{i}\hbar\frac{\partial}{\partial r}+\frac{e}{c}\boldsymbol{A}\right)^2+e\varphi-\mu\right]v+\Delta\mathrm{e}^{-\mathrm{i}\chi}u=\tilde{\varepsilon}v \tag{16}$$

in the WKB-approximation. In this case one may seek the solution of eq. (16) in the form

$$\begin{pmatrix}u(r)\\ v(r)\end{pmatrix}=\begin{pmatrix}u_p\\ v_p\end{pmatrix}\exp\left(\frac{\mathrm{i}}{\hbar}\int p\ \mathrm{d}r\right) \tag{17}$$

where the variation of the amplitudes u_p and v_p [given by eq. (9)] with the space coordinate r is supposed to be slow.

The general solution of the eq. (16) for a given energy $\tilde{\varepsilon}$ is a superposition of two terms of the type (17), which can be discriminated by the sign of $\tilde{\xi}_p$ (the two branches of the excitation spectrum). The two corresponding absolute values of the momentum, $p^{\pm}$, can be found from

$$\tilde{\varepsilon}=\sqrt{\xi_{\mathrm{p}}^2+\Delta^2}+p_{\mathrm{s}}v \tag{18}$$

In the first approximation in the small parameter $\left|\vec{\xi}_p\right|/\mu$ the solutions are*

$$p_{\pm}=\left[2m\left(\mu\pm\sqrt{\left(\bar{\tilde{\varepsilon}}-\boldsymbol{P}_{\mathrm{s}}v_{\mathrm{F}}\right)^2-\Delta^2}\right)\right]^{1/2} \tag{19}$$

One can see that there are two quantum scales that appear in the theory. The first one, $\hbar/p_{\mathrm{F}}$, is the same as in the normal state. The second one,

$$\frac{\hbar}{|p_+-p_-|}\simeq\frac{\hbar v_{\mathrm{F}}}{\xi_p} \tag{20}$$

describes the large-scale quantum oscillations of the quasiparticle wave lunction. This scale has been introduced by Azbel (1970) as a characteristic length determining the validity of the quasiclassical approach.

An additional limitation is due to electron scattering and has the form

$$1\gg d \tag{21}$$

l being the characteristic electron mean free path. One should keep in mind that the characteristic value of ξ_p [entering eq. (20)] depends on the problem under consideration. Consequently, one should check the validity of these criteria in every case.

If we are interested in the contribution of all quasiparticles and the temperature T is of the order of T_{c}, then d is of the order of $\xi_0(0)$ (the coherence length at $T=0$). If, on the contrary, the quasiparticles with small $\widetilde{\zeta}_p$ are important the situation is more complicated and additional analysis is usually required.

As is the case with condition (2), condition (1) imposed on the perturbation frequency is not always sufficient. At low temperatures a more restrictive criterion exists:

$$\hbar\omega\ll\bar{\xi}_p \tag{22}$$

where $\bar{\xi}_p$ is the characteristic value of $\bar{\xi}_p$.

Another restriction on the perturbation frequency may appear as a result of analysis of the

[12]*For most purposes this approximation is adequate. However, as we shall see later, in the theory of v_s-dependent thermoelectric current this approximation is not sufficient.

equation for the gap if the latter is changed by nonequilibrium processes. If the deviation from equilibrium is small such a criterion, as we shall see, has the form

$$\hbar\omega \ll \Delta^2/T.$$

This criterium was first formulated by Gor'kov and Eliashberg (1968, 1969). In the vicinity of the transition temperature it is much more restrictive than the condition $\hbar\omega \ll \Delta$.

Properties of the collision operators

Now let us discuss properties of the collision operators entering the transport equations for quasiparticles (eq. 6) and phonons (eq. 12). We shall begin with discussion of the impurity collision operator, $I_{\rm im}$ (eq. 8). This operator vanishes when acting upon an arbitrary function of the total excitation energy, $\tilde{\delta}_p$. At the same time, the electron-phonon collision operator $I_{\rm ph}$ [eqs. (10) and (11)] vanishes when acting upon the equilibrium Fermi function $n^{(0)}$,

$$n^{(0)}(\tilde{\varepsilon}) = [\exp(\tilde{\varepsilon}/T) + 1]^{-1} \tag{23}$$

if the phonon distribution is the equilibrium one. It is necessary to emphasize that if $p_s \neq 0$ this function has a current-carrying part which reflects the existence of persistent current in superconductors in the equilibrium state.

The impurity collision operator can be rewritten in the form

$$I_{\rm im}\{n_p\} = \int (\mathrm{d}p)_{p'} W_{pp'} (n_{p'} - n_p) \left(1 + \frac{\widetilde{\xi}_p \widetilde{\xi}_{p'} - \Delta^2}{\varepsilon_p \varepsilon_{p'}}\right) \delta(\tilde{\varepsilon}_p - \tilde{\varepsilon}_{p'}) \tag{24}$$

One can draw an important conclusion from this expression: if $p_s = 0$ the operator $I_{\rm im}$ vanishes when acting upon an arbitrary function of ξ_p. In other words, the impurity scattering smears the quasiparticles over a surface of constant $\widetilde{\xi}_p$ in the momentum space. This smearing results in relaxation of the part of the distribution function which is asymmetric in p. Note that in a situation where $p_{\rm s} = 0$ the operator $I_{\rm im}$ satisfies the relation

$$I_{\rm im} n_p = (|\xi_p|/\varepsilon_p)\, I^{\rm n}_{\rm im} n_p, \tag{24a}$$

where $I^{\rm n}_{\rm im}$ is the impurity-scattering operator for the normal metal.
Let us give as an example an expression for the "out" part of the impurity collision operator in the case $p_{\rm s} = 0$:

$$I^{\rm out}_{\rm im}\, n_p \equiv n_p/\tau^{\rm out} = \left(\left|\widetilde{\zeta}_p\right|/\varepsilon_p\right) n_p/\tau_{\rm im}, \tag{25}$$

where $\tau_{\rm im}$ is the lifetime of the electron with momentum p due to the impurity collisions in the normal state. Thus, if $p_{\rm s} = 0$, the impurity collisions conserve the $\check{\zeta}_p$-parity of the distribution function. This means that the total numbers of quasielectrons ($\tilde{\xi} > 0$) and quasiholes ($\tilde{\xi} < 0$) are conserved separately, and therefore the impurity scattering does not result in relaxation of the inbalance between the two branches, provided $\boldsymbol{p}_{\rm s} = 0$.

One can offer a rather transparent physical interpretation of this fact. Indeed, the coherence factor describing interbranch transitions is proportional to the matrix element squared $\left|\langle \psi^{\rm el}_p \mid \psi^{\rm hole}_{p'} \rangle\right|^2$, where $\left|\psi^{\rm el}_p\right\rangle$ and $\left|\psi^{\rm hole}_p\right\rangle$ are the quasi-electron and quasihole states, correspondingly. If $p = p'$, this amplitude vanishes because $\left|\psi^{\rm el}_p\right\rangle$ and $\left|\psi^{\rm hole}_p\right\rangle$ are the orthogonal solutions of the Bogoliubov-De Gennes equations. If the quasiparticle spectrum is isotropic this property exists for arbitrary directions of p and p' because $\left|\tilde{\zeta}_p\right| = \left|\tilde{\zeta}_{p'}\right|$ for elastic scattering provided $p_{\rm s} = 0$. This is why such an imbalance can relax, e.g., due to electron-phonon collisions (Tinkham 1972).

To make an estimate of characteristic electron -phonon times of relaxation one should know whether the phonon distribution is the equilibrium one or not. For many important cases the non-equilibrium part of the phonon distribution function is small and can be discarded in the course of estimation of the electron-phonon scattering rate. Thus, in this section we shall assume the phonon distribution to be the equilibrium one. If this is the case the electron-phonon imbalance relaxation time τ_{Q} can be estimated as τ_{ph} (provided $T_{\mathrm{c}}-T$ is of the order of T_{c}) or as $\tau_{\mathrm{ph}}(T/\Delta)$ (provided $T_{\mathrm{c}}-T \ll T_{\mathrm{c}}$) (Tink ham 1972, Clarke and Tink ham 1972). Here τ_{ph} is of the order of $h\left(\omega_1^2\right), T^3$, i.e. of the order of the electron-phonon relaxation time in normal metals ((ω_D) is the Debye phonon frequency).

If $p, \neq 0$ or $\Lambda(p)$ is anisotropic the imbalance can relax through the impurity collisions as well. This can be seen straight forwardly from eq. (24) because ε_p and $\varepsilon_{p'}$ rather than $\tilde{\varepsilon}_p$ and $\tilde{\varepsilon}_{p'}$ enter the expression for the coherence factor $\left(u_p u_{p'}-v_p v_{p'}\right)^2$. The relaxation rate is of second order in the quantity ($p_s v, \varepsilon_c$),
ε_{c} being the characteristic quasiparticle energy. The corresponding relaxation time τ_{Q} can be estimated as

$$1/\tau_{\mathrm{Q}} \simeq\left(1/\tau_{\mathrm{im}}\right)\left(p_{\mathrm{F}} v_{\mathrm{s}}/T_{\mathrm{c}}\right)^2, \tag{26}$$

if $T_{\mathrm{c}}-T \ll T_{\mathrm{c}}$ (see Galako and Shumeko 1976, and Shelankov 1977). The same physical considerations are valid if the gap parameter $A(p)$ is anisotropic. If the anisotropy is small so that

$$\frac{\Delta(p)-\langle\Delta\rangle}{\langle\Delta\rangle} \ll 1,$$

where the angular brackets denote the average over the Fermi surface, we have the following estimate

$$1/\tau_{\mathrm{Q}} \simeq\left(1/\tau_{\mathrm{im}}\right) \frac{\langle(\Delta-\langle\Delta\rangle)^2\rangle}{\langle\Delta\rangle^2}. \tag{27}$$

This type of relaxation was also studied by Galako and Shumeko (1976). The imbalance relaxation can be produced also by other mechanisms, such as paramagnetic impurity scattering (Schmid and Schön 1975a). We shall consider here only the nonmagnetic impurity scattering.

Three types of relaxation

Thus, one can discriminate between 3 types of relaxation:
(1) Relaxation of the part of the distribution function asymmetrical in p ("smearing" over a surface of constant $\widetilde{\zeta}_p$). These processes are usually the most rapid. This type of relaxation can be due to impurity collisions, the characteristic time being of the order of τ_{im}. So to describe such a relaxation one can for many cases discard the phonon part in the collision term of the Boltzmann equation for the quasiparticles.
(2) So-called imbalance relaxation, of the part of the distribution function symmetric in p and odd in $\overset{\approx}{\xi}_p$. It may be due to electron - phonon collisions, or (in the presence of p) to impurity collisions, or to other mechanisms discussed above.
(3) Energy relaxation, i.e relaxation of the part of the distribution function depending on the total energy $\tilde{\varepsilon}$. This type of relaxation is due to electronphonon and electron-electron collisions. If $T \simeq T_{\mathrm{c}}$, the characteristic relaxation time due to electron-phonon collisions is of the order of $\tau_{\mathrm{ph}}\left(T_{\mathrm{c}}\right)$. However, at $T \ll T_{\mathrm{c}}$ one should discriminate between the contributions of the two types of inelastic electron-phonon processes mentioned above. The contribution of scattering processes conserving the total number of quasiparticles to the scattering rate is of the order of $\tau_{\mathrm{ph}}^{-1}(T)(T, \Delta)^{12}$, whereas the contribution of creation-annihilation processes is of the order of

$$\tau_{\mathrm{ph}}^{-1}(\Delta)x. \tag{28}$$

Here x is the so-called dimensionless concentration of the quasiparticles, defined as

$$x = \frac{\pi^2\hbar^3}{2mp_{\mathrm{F}}\Delta}\int(\,\mathrm{d}p)_p n_p. \tag{29}$$

The corresponding equilibrium quantity equals (Schrieffer and Ginsberg 1962)

$$x_0 = (\pi T/2\Delta)^{1/2}\exp(-\Delta/T) \tag{30}$$

The rate of electron-electron collisions at $T \simeq T_{\mathrm{c}}$ is of the same order as for normal metals:

$$\tau_{\mathrm{ee}}^{-1} \simeq T_{\mathrm{c}}^2/\hbar\mu, \tag{31}$$

whereas at $T \ll T_{\mathrm{c}}$ two different characteristic electron-electron times exist. The characteristic rate of processes of the type $2 \to 2$ is proportional to the concentration of the quasiparticles and is given by

$$\tau_{\mathrm{ee}}^{-1} \simeq \frac{\Delta^2}{\hbar\mu}x, \tag{32}$$

while for 31 processes (such as recombination or shock ionization) we have

$$\tau_{\mathrm{ee}}^{(\mathrm{r})-1} \simeq \frac{\Delta^2}{\hbar\mu}x^2. \tag{33}$$

Thus, the characteristic time $\tau_{\mathrm{ee}}^{(\mathrm{r})}$ is proportional to x^{-2}.
Before formulating the full set of equations describing transport in superconductors let us note some useful sum rules for the collision operators of the electron and phonon Boltzmann equation (we shall use these rules in section 4). These relations can be obtained from the explicit expressions for the collision operators, eqs. (10), (11), (13) and (14), and have the form

$$\int(\mathrm{d}p)_p I_{\mathrm{ph}}^{(\mathrm{s})}\{n_p, N_q\} = 0 \tag{34}$$

$$\int\frac{\mathrm{d}^3q}{(2\pi)^3}I_{\mathrm{ph-c}}^{(\mathrm{s})}\{n_p, N_q\} = -\frac{1}{2}\int(\,\mathrm{d}p)_p I_{\mathrm{ph}}^{(\mathrm{r})}\{n_p, N_q\} \tag{35}$$

$$\int(\mathrm{d}p)_p\tilde{\varepsilon}_p\left\{I_{\mathrm{ph}}^{(\mathrm{s})} + I_{\mathrm{ph}}^{(\mathrm{r})}\right\} = -\int\frac{\mathrm{d}^3q}{(2\pi)^3}\hbar\omega_{\mathrm{q}}\left(I_{\mathrm{ph-c}}^{(\mathrm{r})} + I_{\mathrm{ph-c}}^{(\mathrm{s})}\right) \tag{36}$$

The first relation is due to the fact that scattering processes conserve the total number of the quasiparticles (as we have mentioned above). The second one describes the conservation of the total number of quasiparticles and high-frequency phonons in the course of creation-recombination events. The last relation represents the energy conservation law in the electron-phonon collisions.

Now we can proceed to the main purpose of the present section, i.e. formulating the full set of equations describing kinetics in superconductors.

The Boltzmann equation and the equation for the modulus of the order parameter include such quantities as the external fields (or, more definitely, potentials A and φ), as well as the phase χ of the order parameter. The fields can be determined from the Maxwell equations

$$\mathrm{curl}\,\boldsymbol{E} = -\frac{1}{c}\frac{\partial H}{\partial t}, \quad \mathrm{curl}\,\boldsymbol{H} = \frac{4\pi}{c}\boldsymbol{j} \tag{37}$$

where j is the total current density.
In normal metals one would need one equation to complete the set. This is the neutrality condition $N = \text{const.}$, where N is the electron concentration. In a superconductor one needs one more equation, which reflects the existence of an extra variable, i.e. the phase χ. For this purpose the continuity equation

$$e\frac{\partial N}{\partial t} + \operatorname{div} j = 0 \tag{38}$$

may be used. It can be obtained by averaging the corresponding operator identity*. The averaging procedure gives the following explicit expressions for N and j (see, e.g. Gal'perin et al. 1973a):

$$N = \int (\mathrm{d}p)_p \left[u_p^2 n_p + v_p^2 \left(1 - n_{-p}\right)\right] \tag{39}$$

$$j = e\int (\ \mathrm{d}p)_p v n_p + eNv_{\mathrm{s}} \tag{40}$$

where $v = \partial E_p/\partial p$.
Equation (38) in normal conductors can be obtained straightforwardly by integration of the Boltzmann equation over $(\mathrm{d}p)_p$ and, thus, is a direct sequence of electron number conservation. In a superconductor the number of excitations is not conserved so the Boltzmann equation itself bears no explicit relation to the particle number conservation. Thus, equation (38) for a superconductor is a subsidiary equation. Combining it with the neutrality condition we obtain the two equations that complete our set

$$N = \ \text{const.}, \tag{41}$$

$$\operatorname{div} j = 0. \tag{42}$$

for most cases it is natural to seek the distribution function n_p as a sum of the Jocal-equilibrium part (eq. 23) and the nonequilibrium term $n_p'' : n_p = n'''\left(\hat{i}_p\right) - n_p''$. Substituting this sum into eq. (4()) and retaining the linear terms in v one gets

$$\begin{aligned} j &= e\left|(dp)_p v n_p'^1 - e\right|^{\hat{i}(dp)_p v(p\cdot v)\ in'^{()}_{i;}} + eNv \\ &= eNv_s + e\int (dp)_p v n_p^{(1)} \end{aligned} \tag{43}$$

The first term in eq. (43) represents the s()-called supercurrent or persistent current the coefficient I. being the so-called superconducting electron density

$$N_{\mathrm{s}} = N_0\left(1 + 2\int_0^\infty \mathrm{d}\xi \frac{\hat{O}n^{(0)}}{\hat{C}\varepsilon}\right) \tag{44}$$

The second term is the normal current. Such a classification is adequate only for approximation linear in τ. Note that it is the "bare" electron velocity t rather than the excitation group velocity ri ; p that enters the expression for the normal current.

Differentiating eq. (4) with respect to time and using eq. (5) we get the following identity for p_s :

[12]*Note that the continuity equation can also be obtained as the imaginary part of the equation for the anomalous Green function.

$$\frac{\mathrm{d}p_{\mathrm{s}}}{\mathrm{d}t} = \nabla\phi + e\boldsymbol{E} \tag{45}$$

which clarifies the physical meaning of the potential to. In some papers isee for example. Stephen 1965. Kemoklidze and Pitaevskii 1967. Betbeder Matibet and Nozires 1969, Galako 1973, 1975) it is represented in another form

$$\frac{\hat{c}p_{\mathrm{s}}}{\hat{c}t} + \frac{1}{m}\left(p_{\mathrm{s}}\nabla\right)p_{\mathrm{s}} = \nabla\left(\phi + p_{\mathrm{s}}^2/2m\right) + eE + \frac{e}{mc}\left[p_{\mathrm{s}} \times H\right], \tag{46}$$

which reflects the idea that the electron chemical potential in a superconductor acquires an additional term

$$\dot{r}\mu = -\left(\phi + \frac{p_{\mathrm{s}}^2}{2m}\right) \tag{47}$$

$\delta\mu$ depends. in particular. on the rate of space and time ariation of the phase of the order parameter %. We shall sometimes refer. for the sake of brevity. to "the transport equation approach" when we mean the approach based on the complete set of equations written in this section.

25.3 3. Different kinds of perturbation and different nonequilibrium phenomena

In considering various nonequilibrium and transport phenomena in superconductors it is useful to discriminate between different kinds of perturbations. We shall base such a classification on the parity of the non-equilibrium distribution in p and $\widetilde{\zeta}_p$. For most of the problems in question this classification is natural.

Thus, we shall consider four kinds of perturbations which create nonequilibrium distributions*

(a) even in ξ_p, odd in p,
(b) odd in $\widetilde{\xi}_p$, even in p,
(c) even in $\widetilde{\xi}_p$, even in p,
(d) odd in $\widetilde{\xi}_p$, odd in p.

(a) Perturbations creating distributions even in $\tilde{\zeta}_p$ and odd in p generate electric current (the typical example is the perturbation proportional to $p_s v$). Such distributions relax through impurity collisions, the response being proportional to τ_{im}.

(b) Perturbations creating distributions even in p and odd in $\widetilde{\zeta}_p$ produce imbalance between the quasielectron and quasihole branches of the excitation spectrum. The imbalance can manifest itself by the potential ϕ. Indeed, making use of the neutrality condition (eq. 41) one obtains

$$\frac{\partial N}{\partial\mu}\phi = -\int (\mathrm{d}p)_p \frac{\tilde{\xi}_p}{\varepsilon_p}\left[n^{(0)}\left(\tilde{\varepsilon}_p\right) - n_p\right] \tag{48}$$

The distribution shifted from equilibrium can relax in this case through phonons, paramagnetic impurities or (in the presence of supercurrent or anisotropy of the gap) through nonmagnetic impurities. The corresponding relaxation time is τ_{Q} introduced in section 2 .

The potential produced by the imbalance is a measurable quantity. For instance, it can be measured in tunnel experiments. The first such experiment was made by Clarke and Tinkham (1972).

Imbalance perturbations are typical for cases where the normal current is inhomogeneous (for instance, if current flows through a $\mathrm{N}-\mathrm{S}$ boundary). The existence of such phenomena

was indicated by Rieger et al. (1971). Afterwards, they were studied theoretically in more detail by Volkov (1974) and by Artemenko and Volkov (1976, 1979).

Note that perturbations of the types (a) and (b) do not lead to changes of the gap, at least in lowest order in T/μ.

(c) Perturbations creating distributions even both in p and in $\tilde{\zeta}_p$ produce no current. Typical examples of such perturbations are "heating" of the excitations by an external microwave field or by an intensive sound wave. The deviations of the distribution function from equilibrium can relax only through inelastic processes (for instance, through scattering by phonons). It is the only type of perturbations that influences the energy gap Δ.

The phenomena of superconductivity stimulation, first considered by Eliashberg (1970) and widely discussed thereafter, are due to perturbations of this kind. Various phenomena of excitation pumping into a superconductor are also described by contributions of this kind to the nonequilibrium distribution function. Some of these phenomena are described in the next section.

(d) Perturbations creating distributions odd both in p and ξ_p are the kind of perturbation that gives rise to thermoelectric effects (see sections 6 and 7).

25.4 4. Stationary nonequilibrium distribution under pumping

The purpose of the present section is to illustrate the role of two different kinds of collisional events described by the collision operator of the Boltzmann equation, i.e. scattering processes and creation- annihilation processes. To this end we shall consider as an example the formation of a nonequilibrium distribution function under pumping of excitations.

There are several different methods to create a nonequilibrium distribution function by pumping, such as (1) laser pumping, (2) tunnel injection, or (3) microwave pumping. Only the case of laser pumping will be considered here. Discussions of other cases and of related phenomena can be found in the review papers by Aronov and Spivak (1978), by Eliashberg and Ivlev (1985) and by Galitskii et al. (1985).

Under laser pumping quasiparticles are created in a wide range of energies $\varepsilon \gg A$. Thereafter they relax down emitting phonons of all frequencies. If the phonon spectrum is linear and $|c_q|^2$, q the steady-state distribution function at high energies goes as some power of the energy. In thin films, according to Vardanyan and Ivlev (1973),

$$n(\varepsilon) \propto (\Delta/\varepsilon)^4. \tag{49}$$

In thick films, because of reabsorption of high frequency phonons (see Aronov et al. 1976)

$$n(\varepsilon) \propto (\Delta/\varepsilon)^5 \tag{50}$$

The distribution function near the gap is practically independent of its form at high energies and has the quasiequilibrium form with nonvanishing chemical potential. The temperature and chemical potential of the quasiparticles are determined by energy and quasiparticle number conservation.

If the film thickness a is smaller than the diffusion length $L = \sqrt{D\tau_\varepsilon}, D$ being the quasiparticle diffusion coefficient while τ_ε is the time of energy relaxation, the spatial distribution of the quasiparticles is homogeneous.

Averaging the Boltzmann equation over the cross-section of the sample, integrating over $(\mathrm{d}p)_p$ and making use of eq. (34) we obtain the condition of quasiparticle number conservation in the following form

[12]*Note that the classification based on $\widetilde{\xi}_p$-parity is effective only if one is interested in the contribution of the lowest nonvanishing order in T/μ (cf. section 7).

$$\int (\mathrm{d}p)_p I^{(r)} \{n_p, N_q\} = \int (\mathrm{d}p)_p \langle Q_p \rangle_{\mathrm{s}} . \tag{51}$$

Here $\langle Q_p \rangle_{\mathrm{s}}$ is the capacity of the source of the quasiparticles at some given energy averaged over the cross-section of the sample.

The high-frequency phonons with $\hbar\omega_{\mathbf{q}} > 2\Delta$ play an especially important role in this effect because the total variation of the number of the quasiparticles is due to the variation of the number of these phonons. Indeed, the disappearance of such a high-frequency phonon is equivalent to the generation of two quasiparticles, as stated by eq. (35).

Let us now integrate the phonon Boltzmann equation (12) over the sample and make use of eq. (35). As a result we obtain the conservation law for the number of quasiparticles in the form

$$\int (\mathrm{d}q) \left(\frac{\partial \omega_q}{\partial q} n \right) N_q \bigg|_{\mathrm{s}} = \frac{1}{2} a \int (\ \mathrm{d}p)_p \langle Q_p \rangle_{\mathrm{s}} - a \int_{\hbar\omega_q > 2\Delta}, \quad (\ \mathrm{d}q) I^{(\mathrm{s})}_{\mathrm{ph-c}}(\mathrm{d}q) \equiv \frac{\mathrm{d}^3 q}{(2\pi)^3}. \tag{52}$$

Here n is the normal to the surface of the sample. The lhs of this equation describes the exchange by high-frequency phonons between the film and the thermal bath; the first term in the rhs represents creation of quasiparticles by laser pumping while the second term describes absorption of phonons by quasiparticles and multiple generation of quasiparticles due to absorption and re-emission of high-frequency phonons.

The energy conservation law can be written in the form

$$R = \int (\mathrm{d}q) \left(\frac{\partial \omega_q}{\partial q} n \right) \hbar\omega_q N_q \bigg|_{\mathrm{s}} = a \int (\ \mathrm{d}p)_p \varepsilon_p \langle Q_p \rangle_{\mathrm{s}} . \tag{53}$$

R is the energy flux density which is absorbed by the superconductor.
The mean free path $l_>$of high-frequency phonons in a superconductor is determined by generation and recombination processes and is of the order of the corresponding mean free path in a normal metal. The mean free path $l_<$of lowfrequency phonons ($\hbar\omega_{\mathrm{q}} < 2\Lambda$) is determined by scattering events and is of the order of

$$l_< \simeq \frac{l_>}{x}, \quad x \ll 1. \tag{54}$$

Here x is the dimensionless concentration of quasiparticles introduced in section 2.

One can discriminate between the following two cases of importance.
(1) A thin film. In this case the mean free paths of all phonons are larger than the effective thickness of the film $\tilde{a} = a, t, t$ being the transparency coefficient of the film boundaries with respect to phonons. All phonons are in equilibrium, the quasiparticles are scattered by the equilibrium phonons, and the quasiparticle lifetime in the sample is determined by recombination. If the scattering time is smaller than the recombination time [according to eq. (28) this means that $x \leqslant (T_0 A)^{1\ 2}$] the quasiparticles have a quasiequilibrium distribution function (Owen and Scalapino 1972):

$$n(\varepsilon) = \exp\left(\frac{v - \varepsilon}{T_0} \right) \tag{55}$$

The chemical potential v is determined by the conservation law for the quasiparticle number and is given by

$$v = A + \frac{T_0}{2} \ln \left[\frac{2\Delta}{\pi T_0} \left(x_0^2 + \frac{\hbar\Omega}{2\Delta} \Gamma \right) \right] . \tag{56}$$

Here

$$I = \frac{8\pi e^2 R_{()}}{c(\hbar\Omega)^2}\tau_A \frac{x}{a} D$$

% is the skin depth. R_r, is the radiation intensity. Ω is the frequency of radiation, τ_1 is the electron phonon scattering time in the normal metal provided the electron energy is equal to $2A$.

As long as the quasiparticle concentration is small the variation of the energy gap relates to the concentration x as

$$\frac{A_0 - \Delta}{A_0} \simeq 2x \ll 1. \tag{57}$$

Here A_0 is the energy gap at $T = 0$ in equilibrium.
(2) A thick film, $l_> < \tilde{a} \ll l_<$, or $x \ll l_> \mid \tilde{a}$. In this case the conditions are essentially different for high-frequency (hw) $_4 - 21$) and low-frequency (hry $_4 - 21$) phonons. The film is transparent for low-frequency phonons. which is not the case for high-frequency phonons because the latter take part in the processes of creation and annihilation of the quasiparticles (Rothwarf and Taylor 1967). In the stationary state these processes establish partial equilibrium among the highfrequency phonons and the quasiparticles. Their distribution functions are given
by

$$\begin{aligned} n(\varepsilon) &= \exp[(v - \varepsilon)/T], \\ N(\omega) &= \begin{cases} \exp[(2v - \varepsilon)/T], & \hbar\omega > 2\Delta, \\ [\exp(\hbar\omega/T_0) - 1]^{-1}, & \hbar\omega < 2\Delta. \end{cases} \end{aligned} \tag{58}$$

Substituting eq. (58) in eq. (52) we obtain the equation for the quasiparticle concentration:

$$\frac{3\Delta^3 x^2}{\pi^3\hbar^3 w^2}\left(t + \frac{2\sqrt{2}a}{\pi l_>}\right) = c\frac{R}{2\Delta}, \quad w \equiv \partial\omega_q/\partial q, \tag{59}$$

where $c \lesssim 1$. The coefficient c measures the fraction of the energy that is transformed into low-frequency phonons and thus does not take part in generation of quasiparticles.

The first term in the lhs describes the escape of high-frequency phonons through the film boundary, the second term in the lhs reflects the variation of the quasiparticle concentration due to absorption of high-frequency phonons and subsequent emission of low-frequency phonons. Therefore the total lifetime of the quasiparticles in the film is given by

$$1/\tau_{\mathrm{e}} = (1/\tau_{\mathrm{r}})\left(\frac{l_>}{\tilde{a}} + \frac{2\sqrt{2}}{\pi}x\right). \tag{60}$$

To determine the temperature let us introduce the characteristic energy of the quasiparticles at which the rate of quasiparticle scattering by low-frequency phonons is equal to the rate of recombination (Aronov and Spivak 1977, 1978). According to eq. (28) this energy is equal to $\varepsilon_1 - \Delta \simeq \Delta x^{2/7}$. Let us suppose that $\varepsilon_1 - \Delta > T$. The last inequality may be fulfilled in a nonequilibrium state only if $x \gg (T/A)^{7,2}$. Let us call the energy interval $A < \varepsilon < \varepsilon_1$ the active region. In the active region fast generation-recombination processes lead to quasi-equilibrium distribution functions with the form (58). These processes redistribute the excess energy (as compared to T) among all the quasiparticles. Quasiparticles reach the active region emitting phonons. The quasiparticle active region is of the order of the width of the active region, $\bar{\varepsilon} - \Lambda \simeq \beta A x^{27}$, where β is of the order of unity. These considerations are sufficient to

determine the temperature T. Let us multiply eq. (52) by 2Δ and subtract it from eq. (55). Then we obtain

$$\int (\mathrm{d}q)(nw)\left(\hbar\omega_q - 2\Delta\right) N_q\Big|_{\mathrm{s}} - a\int_{\hbar\omega_q>2A} (\,\mathrm{d}q) 2\Delta I^{(s)}_{\mathrm{ph-c}}$$
$$- a\int_{\hbar\omega q<2A} (\,\mathrm{d}q)\hbar\omega_q I^{(s)}_{\mathrm{ph-c}} = a\int (\,\mathrm{d}q)_p \left(\varepsilon_p - \Delta\right)\langle Q_p\rangle_s\,. \tag{61}$$

Substituting eq. (58) in eq. (61) we obtain the equation for the temperature in the following form

$$\frac{\tau_{\mathrm{e}}}{\tau_{\mathrm{s}}\left(\tau_0\right)} F\left(T/T_0\right) = \beta\frac{\Delta}{T}x^{2/7}. \tag{62}$$

Here we suppose that the characteristic energy of the quasiparticles which they bring in the active region is $\beta\Delta x^{2/7}$. The function $F\left(T/T_0\right)$ has the form

$$F(t) = t^{7/2}\int_0^\infty \mathrm{d}y y^3 \frac{\mathrm{e}^{-y} - \mathrm{e}^{-ty}}{1-\mathrm{e}^{-ty}}\int_0^\infty \mathrm{d}x \mathrm{e}^{-\mathrm{x}}\frac{2x+y}{\sqrt{x(x+y)}} \tag{63}$$
$$T = T_0, \quad \text{at} \quad \frac{\tau_{\mathrm{s}}}{\tau_{\mathrm{e}}}\frac{\Delta}{T}x^{2,7} \ll 1$$
$$T = \Delta x^{2,7}\left(\frac{\beta l_>}{\tilde{a}}\right)^{2/9}, \quad \text{at} \quad \frac{\tau_{\mathrm{s}}}{\tau_{\mathrm{e}}}\frac{\Delta}{T}x^{2/7} \gg 1$$

and the width of the active region is much greater than the temperature because of the inequality $\left(\tilde{a}/l_>\right)^{2/9} \geqslant 1$.

At larger pumping rates ($1 \geqslant x \geqslant 1 > /\tilde{a}$) the film becomes opaque for low-frequency phonons as well. Their temperature because of quasiparticle phonon scattering becomes equal to the temperature of the quasiparticles. However, the chemical potential of the quasiparticles is by no means equal to zero. Their concentration is determined by eq. (59), where one must retain only the second term in the lhs.

25.5 5. Energy conservation and entropy production

Let us demonstrate that the transport equation approach in the presence of external fields is compatible with the energy conservation law. For simplicity, we shall consider only the elastic scattering of quasiparticles.

For our purpose it is convenient to multiply eq. (6) by the quasiparticle energy $\tilde{\varepsilon}_p$ and integrate it over $(\mathrm{d}p)_p$. The integral of the elastic collision operator vanishes and one can obtain

$$\frac{\partial}{\partial t}\int (\,\mathrm{d}p)\tilde{\varepsilon}_p n_p + \frac{\partial}{\partial x_i}\int (\,\mathrm{d}p)\tilde{\varepsilon}_p\frac{\partial\tilde{\varepsilon}_p}{\partial p_i}n_p = \int (\mathrm{d}p) n_p\frac{\partial\tilde{\varepsilon}_p}{\partial t}, \tag{64}$$
$$\frac{\partial\tilde{\varepsilon}_p}{\partial t} = \frac{\tilde{\xi}_p}{\varepsilon_p}\frac{\partial\tilde{\xi}_p}{\partial t} + v\frac{\partial p_s}{\partial t} + \frac{\Delta}{\varepsilon_p}\frac{\partial\Delta}{\partial t}, \quad \frac{\partial\tilde{\xi}_p}{\partial\tilde{t}} = \frac{\partial}{\partial t}\left(\phi + \frac{p_{\mathrm{s}}^2}{2m}\right). \tag{65}$$

Here we have taken into account that the integral

$$\int (\mathrm{d}p)\frac{\partial}{\partial p_i}\left(A_p\frac{\partial\tilde{\varepsilon}_p}{\partial x_i}\right) \tag{66}$$

where A_p is an arbitrary function of quasimomentum p, vanishes as a consequence of the periodicity of the integrand in p-space, its period being the reciprocal lattice vector.

To get the energy conservation law one should transform the rhs of eq. (64) in the following way

$$\frac{\partial}{\partial t}\left(\phi+\frac{p_{\mathrm{s}}^2}{2m}\right)\int(\mathrm{d}p)\frac{\widetilde{\xi}_p}{\varepsilon_p}n_p+\frac{\partial p_{\mathrm{s}}}{\partial t}\int(\ \mathrm{d}p)vn_p+\frac{1}{2}\frac{\partial\Delta^2}{\partial t}\int(\ \mathrm{d}p)\frac{n_p}{\varepsilon_p}. \tag{67}$$

Using eqs. (3) and (4) we get

$$\int(\mathrm{d}p)n_p\frac{\partial\tilde{\varepsilon}_p}{\partial t}=N\frac{\partial\phi}{\partial t}-\frac{j}{e}\frac{\partial p_{\mathrm{s}}}{\partial t}-\int(\mathrm{d}p)v_p^2\frac{\partial}{\partial t}\left(\phi+\frac{p_{\mathrm{s}}^2}{2m}\right)+\frac{1}{2}\frac{\partial\Delta^2}{\partial t}\int(\ \mathrm{d}p)\frac{n_p}{\varepsilon_p}. \tag{68}$$

To transform the third and fourth terms in the rhs of eq. (5) let us note that

$$\frac{\partial}{\partial t}\int(\ \mathrm{d}p)\tilde{\varepsilon}_pv_p^2=-\int(\mathrm{d}p)v_p^2\frac{\partial}{\partial t}\left(\phi+\ \frac{p_{\mathrm{s}}^2}{2m}\ \right)+\frac{1}{2m}\frac{\partial\Delta^2}{4\partial t}\int(\ \mathrm{d}p){\varepsilon_p}^1. \tag{69}$$

Taking this expression into account and combining it with the self-consistency equation (15), we obtain

$$\int(\mathrm{d}p)n_p\frac{\partial\tilde{\varepsilon}_p}{\partial t}=N\frac{\partial\phi}{\partial t}+\frac{1}{e}j\frac{\partial p_{\mathrm{s}}}{\partial t}+\frac{\partial}{\partial t}\left(\int(\ \mathrm{d}p)\tilde{\varepsilon}_pv_p^2+\frac{\Delta^2}{2\lambda_{\mathrm{e}}}\right). \tag{70}$$

Inserting eq. (70) into eq. (64) we obtain

$$\frac{\partial}{\partial t}(\mathscr{E}-N\phi)+\operatorname{div}\left(\boldsymbol{W}-\frac{1}{e}\phi j\right)=j\boldsymbol{E}. \tag{71}$$

Here the quantity

$$\boldsymbol{W}=\int(\mathrm{d}p)_p\tilde{\varepsilon}_p\frac{c\tilde{\varepsilon}_p}{\partial p}n_p \tag{72}$$

is the electron part of the energy density flux, while

$$\mathscr{E}=\int(\mathrm{d}p)_p\tilde{\varepsilon}_p\left(n_p-v_p^2\right)-\frac{\Lambda^2}{2\lambda_{\mathrm{e}}} \tag{73}$$

is the electron contribution to the energy density. One can show that this quantity is equal to the average of the BCS Hamiltonian in the presence of external fields.

With the help of the Maxwell equation curl $\boldsymbol{H}=(4\pi/c)j$ one can rewrite eq. (71) in a manner which explicitly represents the conservation law for the total (quasiparticle and field) energy

$$\frac{\partial U}{\partial t}+\operatorname{div}u=0. \tag{74}$$

Here

$$U=\mathscr{E}-N\phi+\boldsymbol{H}^2/8\pi \tag{75}$$

is the total energy density while

$$u=W-(\phi/e)j+(c/4\pi)[E\times H] \tag{76}$$

is the total energy flux density.

To take inelastic collisions with phonons into account one should include, into the full set of

equations, also the Boltzmann equation for the phonon distribution function (eq. 12). As a result the explicit expressions for the energy density and the energy flux density should be modified to include the phonon terms.

Now let us obtain the expression for the entropy production rate with the help of the transport equation approach. We shall start with the usual combinatorial expression for the entropy density S of the quasiparticles obeying Fermi statistics:

$$S = \int (\mathrm{d}p)_p \sigma_p \left(n_p\right) \tag{77}$$
$$\sigma_p = -\left[n_p \ln n_p + \left(1 - n_p\right) \ln \left(1 - n_p\right)\right]$$

Differentiating this quantity with respect to time we have

$$\frac{\partial S}{\partial t} = \int (\mathrm{d}p)_p \frac{\partial \sigma_p}{\partial p} \frac{\partial n_p}{\partial t}.$$

Inserting $\partial n_p / \partial t$ from the transport equation (6) we get

$$\frac{\partial S}{\partial t} + \operatorname{div} s = -\int (\mathrm{d}p)_p \frac{\partial \sigma_p}{\partial n_p} I\left\{n_p\right\}, \tag{78}$$

Here

$$s = \int (\mathrm{d}p)_p \frac{\partial \tilde{\varepsilon}_p}{\partial p} \sigma_p \tag{79}$$

is the entropy flux density. The flux is determined in terms of the group velocity $\partial \tilde{\varepsilon}_p / \partial p$, in contrast to the normal current density which is expressed through $v = \partial E_p / \partial p$ (see eq. 40). In the derivation of eq. (78) we have taken into account that integrals of the type (66) with $A_p \equiv \sigma_p$ vanish.

We wish to emphasize that if n_p is an arbitrary function of the total energy $\hat{\varepsilon}_p$ it follows from the explicit expression (79) that the entropy flux density (and therefore the heat flux density) vanishes. In particular, this is the case for the equilibrium state with a finite persistent current.

Indeed, in such a case a_p depends only on the total energy and the integrand in eq. (79) can be written as the full derivative

$$\int (\mathrm{d}p) \frac{\partial \tilde{\varepsilon}_p}{\partial \boldsymbol{p}} \sigma_p = \int (\mathrm{d}p) \frac{\partial \boldsymbol{F}}{\partial \boldsymbol{p}} = 0; \quad F = \int^{\hat{\varepsilon}_p} \sigma_p(\varepsilon) \mathrm{d}\varepsilon$$

To obtain this result one employs no special properties of the quantity σ_p, only its dependence on the $\hat{\varepsilon}_p$. In a similar way one can verify that the quasiparticle velocity averaged over a surface of constant energy $\tilde{\varepsilon}_p = \varepsilon = \text{const.}$, vanishes (for the proof one may consider $\delta\left(\varepsilon - \tilde{\varepsilon}_{p'}\right)$ instead of $\sigma\left(\tilde{\varepsilon}_p\right)$ in the integrand). The absence of the entropy flux in the equilibrium is closely related with this mathematical fact.

Now let us demonstrate the H-theorem for superconductors, i.e. the nonnegative sign of the entropy production rate $(\partial S / \partial t)_{\mathrm{coll}}$, which enters the rhs of eq. (78). For the case of elastic impurity scattering we insert the expression for I_{im} (eq. 8) into eq. (78). Then we interchange the integration variables p and p' and take the half-sum of the expressions in the integrand. Employing the relation $\left|f_{pp'}\right|^2 = \left|f_{p'p}\right|^2$, which is valid for centro-symmetric crystals, we get

$$\left(\frac{\partial S}{\partial t}\right)_{\text{coll}} = \pi N_{\text{i}} \frac{(2\pi)^4}{m^2} \int (\,\mathrm{d}p)_p \int (\,\mathrm{d}p)_{p'} \left|f_{pp'}\right|^2 (n_p - n_{p'}) \times \ln \frac{(1-n_{p'})\, n_p}{n_{p'}\,(1-n_p)} \left(u_p u_{p'} - v_p v_{p'}\right)^2 \delta\left(\tilde{\varepsilon}_p - \tilde{\varepsilon}_{p'}\right) \tag{80}$$

One can see that the integrand is always non-negative. Thus, $(\hat{C}S/\hat{C}t)_{\text{coll}}$ is also non-negative. This statement is the well-known Boltzmann H-theorem for superconductors.

If phonon scattering is taken into account the whole procedure is more cumbersome. In this case one should consider the Boltzmann equations both for normal excitations and for phonons in the superconductor. As a result, one gets the following expression for the total entropy density production of the electron-phonon system:

$$\begin{aligned}\left(\frac{\partial S}{\partial t}\right)_{\text{coll}} =& \pi \int (\mathrm{d}p)_p \int (\,\mathrm{d}p)_{p'} \left|c_{p-p'}\right|^2 \\ &\times \Bigg\{ \left(u_p u_{p'} - v_p v_{p'}\right)^2 \ln \frac{n_p\,(1-n_{p'})\, N_{p-p'}}{(1-n_p)\, n_{p'}\,(1+N_{p-p'})} \\ &\times \delta\left(\tilde{\varepsilon}_p - \tilde{\varepsilon}_{p'} + \hbar\omega_{p-p'}\right) \left[n_p\,(1-n_{p'})\, N_{p-p'} - n_{p'}\,(1-n_p)\,(1+N_{p-p'})\right] \\ &+ \left(u_p v_{p'} + u_{p'} v_p\right)^2 \ln \frac{n_p n_{-p'}\,(1+N_{p+p'})}{(1-n_p)\,(1-n_{-p'})\, N_{p+p'}} \delta\left(\tilde{\varepsilon}_p + \tilde{\varepsilon}_{-p'} - \hbar\omega_{p+p'}\right) \\ &\times \left[n_p n_{-p'}\,(1+N_{p+p'}) - (1-n_p)\,(1-n_{-p'})\, N_{p+p'}\right] \Bigg\}\end{aligned} \tag{81}$$

One can see that there are two types of irreversible processes that contribute to the entropy generation. The first type is simple scattering of quasiparticles by phonons. These processes are represented by the first term in the braces in eq. (81); this term differs from the similar term in the normal state only by the coherence factor $(u_p u_{p'} - l_p v_{p'})^2$ and by the form of energy spectrum $\tilde{\varepsilon}_p$. The second contribution is due to the creation-annihilation processes we have discussed above.

25.6 6. Hydrodynamical approach

In this section we shall obtain a closed set of equations for macroscopic variables describing nonequilibrium behavior of a superconductor in nonuniform external fields. Thereafter these will be referred to as hydrodynamical equations. They are the equations for the full set of macroscopic variables (such as potential ϕ, normal current, temperature gradient, etc.) describing the nonequilibrium state of a superconductor.

One may hope to justify such an approach if inelastic scattering of excitations is relatively rare, i.e. if they are scattered much more often by impurities than by phonons. The second restriction is connected with the characteristic scales of time and space variation of external fields; one should assume these scales to be much larger than the corresponding characteristic electron quantities, τ_{im} and l_{im}, with l_{im} the excitation mean free path due to the impurity collisions.

The procedure of derivation of macroscopic equations is analogous to the well-known Chapman Enskog method (see also Gantsevich et al. 1974, where this method is applied to derive macroscopic equations describing the nonequilibrium behavior of a normal conductor). Let us confine ourselves to the case where the equations are linear in the variable v_{s}.

Assume the quantities $\omega\tau_{\rm im}$ as well as $ql_{\rm im}$ to be small. Then one can iterate the linearized transport equation, as well as the neutrality condition, in powers of these quantities. The linearized transport equation can be written in the form

$$\frac{\partial n_p^{(1)}}{\partial t} + \left(\frac{\xi}{\varepsilon}\dot{\phi} + \dot{p}_{\rm s} v \right) \frac{\partial n^{(0)}}{\partial \varepsilon} + \frac{\xi}{\varepsilon} v \frac{\partial n_p^{(1)}}{\partial r} - \frac{\xi}{T}(v\nabla T)\frac{\partial n^{(0)}}{\partial \varepsilon} + I_{\rm im} n_p^{(1)} + I_{\rm ph} n_p^{(1)} = 0. \tag{82}$$

Let us average eq. (82) over all directions in momentum space. We get

$$\overline{\frac{\partial n_p^{(1)}}{\partial t}} + \frac{\xi}{\varepsilon}\phi\frac{\partial n^{(0)}}{\partial \varepsilon} + \overline{\frac{\xi}{\varepsilon} v \frac{\partial n_p^{(1)}}{\partial r}} + I_{\rm ph} n_p^{(1)} = 0 \tag{83}$$

where the bar denotes the averaged quantity. The averaged impurity collision operator, naturally, vanishes.

Subtracting eq. (83) from eq. (82) and applying an iteration procedure in the small parameters $\omega\tau_{\rm im}$ and $ql_{\rm im}$ to the difference equation one gets

$$n_p^{(1)} = \overline{n_p^{(1)}} - \frac{\xi}{\varepsilon} I_{\rm im}^{-1} v_k \overline{\frac{\partial n_p^{(1)}}{\partial x_k}} - I_{\rm im}^{-1}\left(v_k p_{\rm sk} - v_k \frac{\partial T}{\partial x_k}\frac{\xi}{T} \right)\frac{\partial n^{(0)}}{\partial \varepsilon} \tag{84}$$

One can get the equation for $\overline{\mathrm{n}_p^{(1)}}$ by substituting eq. (84) in eq. (83):

$$\begin{aligned} &\overline{\frac{\partial n_p^{(1)}}{\partial t}} + \frac{\xi}{\varepsilon}\phi\frac{\partial n^{(0)}}{\partial \varepsilon} + I_{\rm ph}\overline{n_p^{(1)}} - \frac{\xi}{\varepsilon} v_{\rm i} I_{\rm im}^{-1} v_{\rm k} \frac{\partial^2 \overline{n_p^{(1)}}}{\partial x_i \partial x_k} - \\ &- \overline{v_i I_{\rm im}^{-1} v_k} \frac{\xi}{\varepsilon}\frac{\partial}{\partial x_i}\left(p_{\rm sk} - \frac{\partial T}{\partial x_{\rm k}}\frac{\xi}{T} \right)\frac{\partial n^{(0)}}{\partial \varepsilon} = 0 \end{aligned} \tag{85}$$

Now let us turn to the calculation of the current density. Inserting eq. (84) into expression (40) for the current we get

$$j_i = eN_{\rm s} v_{\rm si} - \eta_{ik}\frac{\partial T}{\partial x_k} + \frac{1}{e}\sigma_{ik}^{(\rm s)}\dot{\rm p}_{\rm sk} - eD_{ik}\frac{\partial v_{\rm s}}{\partial x_{\rm k}} \tag{86}$$

Here the first term is the supercurrent while the second term is the normal thermoelectric current which we shall consider in more detail in section 7 ; the thermoelectric transport coefficient η_{ik} is given by

$$\eta_{ik} = -\frac{e}{T}\int (\,{\rm d}p)_p v_i I^{-1} v_k \xi_p \frac{\partial n^{(0)}}{\partial \varepsilon_p} \tag{87}$$

The third term is the normal Ohmic current proportional to the (non-steady) electric field $\boldsymbol{E} = e^{-1}\dot{\boldsymbol{p}}_{\rm s}$. The transport coefficient $\sigma_{ik}^{(s)}$ is given by

$$\sigma_{ik}^{(\rm s)} = -e^2 \int (\,{\rm d}p)_p v_i (-{\rm i}\omega + I)^{-1} v_k \frac{\partial n^{(0)}}{\partial \varepsilon_p} \tag{88}$$

This quantity may be called the conductivity of the normal component in the superconducting state.

If only impurity scattering is present then in the immediate vicinity of the critical temperature $T_{\rm c}$ the ratio $\sigma^{(s)}/\sigma^{(n)}$ (where $\sigma^{(n)}$ is the conductivity in the normal state) calculated on the basis of eq. (88) is given by

$$\sigma^{(\rm s)}/\sigma^{(\rm n)} = 1 + (\Delta/2T)\ln\omega\tau_{\rm im}$$

However, this result appears to be incorrect in the general case, for it can be seen that the characteristic energies in the integrand determining the logarithmic term are small. At such small energies one should be cautious and check the applicability of the Boltzmann equation. It follows from eq. (88) that the logarithmic term is determined by the energy ξ_p of the order of $\omega\tau_{\rm im}\Delta$. However, this conclusion in also appears to be incorrect in the general case. It can be seen from the quantum-mechanical approach (see Ovchinnikov 1977 and Artemenko and Volkov 1979) that the characteristic value of ξ_p turns out to be of the order of $(\hbar\omega\Delta)^{1/2}$. If this quantity is much larger than $\omega\tau_{\rm im}\,\Delta$ it cuts off the integral. So in the general case the expression for the ratio $\sigma^{(s)}/\sigma^{(n)}$ has the form

$$\sigma^{(\rm s)}/\sigma^{(\rm n)} = 1 + (\Delta/T)\ln\Lambda; \quad \Lambda = \max\left[\omega\tau_{\rm im}, (\hbar\omega/\Delta)^{1/2}\right]. \tag{89}$$

Such a divergence of the conductivity at very small frequencies could be removed by taking into account other mechanism of momentum relaxation which do not contain the factor ξ_p/ε_p in the expression for the inverse relaxation time, such as electron-phonon scattering, paramagnetic impurities, etc., which can be done on the basis of the Boltzmann equation.

The fourth term in eq. (86) introduces a new macroscopic quantity into the theory, i.e.

$$v_{\rm s} = \int (\mathrm{d}p)_p \,\mathrm{sign}\,\xi_p n_p(1) \tag{90}$$

It has the meaning of the population difference between the two branches with $\xi_p > 0$ and $\tilde{\zeta}_p < 0$ and may be considered as the measure of imbalance between the quasi-electron and quasihole branches (briefly discussed in section 5). The corresponding transport coefficient,

$$D_{ik} = \int (\mathrm{d}p)_p \frac{|\xi_p|}{\varepsilon_p} v_i I^{-1} v_k \frac{\partial n^{(0)}}{\partial \varepsilon_p} \tag{91}$$

which in the isotropic case simply equals $\frac{1}{3}v_{\rm F}^2\tau_{\rm tr}\delta_{ik}2n^{(0)}(\Lambda)$, has the meaning of a diffusion coefficient.

The closed set of hydrodynamical equations can be obtained only in the vicinity of $T_{\rm c}$, i.e. if $T_{\rm c} - T \ll T_{\rm c}$. Taking this restriction into account, multiplying eq. (85) by sign ξ_p and integrating it over p, one obtains

$$\begin{aligned} &\frac{\pi\Delta}{4T}\frac{\partial\phi}{\partial t} + \frac{\phi}{\tau_{\rm Q}}\frac{\partial N}{\partial\mu} - \frac{\partial N}{\partial\mu} D_{ik}\frac{\partial^2\phi}{\partial x_i \partial x_k} \\ &\quad = -\frac{\partial}{\partial x_i}\left(\frac{1}{e^2}\sigma_{ik}^{(\rm n)}\dot{p}_{{\rm s}k} - \frac{1}{e}\eta_{ik}\frac{\partial T}{\partial x_k}\right) \end{aligned} \tag{92}$$

Here we have employed the relation $v_s = \phi\hat{c}N/\hat{c}\mu$, which is a consequence of the neutrality condition. The branch-imbalance relaxation time $\tau_{\rm Q}$ introduced in section 2 can be defined here quantitatively by the relation

$$\phi\frac{\partial N}{\partial\mu}\frac{1}{\tau_{\rm Q}} = \int (\mathrm{d}p)_p \frac{\xi_p}{\varepsilon_p} I n_p^{(1)} \tag{93}$$

One can see from eq. (92) that in homogeneous situation the potential ϕ relaxes in a period of the order of $\tau_{\rm Q}\pi\Delta/4T$. As for the inhomogeneous case, the diffusion should also be taken into account. One can see from eq. (92) that the characteristic length of the spatial variation of the potential ϕ is the diffusion length

$$L_{\rm b} = \sqrt{D\tau_{\rm Q}} \propto \Delta^{-1/2}.$$

It increases near T_c and diverges at $T = T_c$. Note that eq. (92) has the form of the continuity equation for the quantity v_s (or $\phi\partial N/\partial\mu$), the rhs of eq. (92) being the source of an imbalance.

Let us now discuss what becomes of equations (86) and (92) of the hydrodynamical approach in the normal state. In this limit $\Delta \to 0$, and the Einstein relation between D_{ih} and σ_{th} holds. As a result, it can be shown, using eq. (86) and the relation $v_s = \phi\partial N/\partial\mu$, that eq. (92) becomes a trivial consequence of the continuity equation $\operatorname{div} j = 0$.

To solve eq. (92), appropriate boundary conditions should be formulated. Discussions of the boundary conditions may be found in papers by Artemenko and Volkov (1975a, 1976a) and Aronov (1976). For example, for the simplest case of a superconductor-insulator boundary, where no specific boundary mechanism of imbalance relaxation exists, these conditions require the normal current density (and therefore the supercurrent density as well) to vanish on the boundary surface. Artemenko and Volkov (1975a, 1976a) were the first to show that in the presence of a temperature gradient the electric field $E = -e^{-1}\nabla\phi$ appears in a superconductor near an S−N boundary. This thermoelectric field decreases as a function of the distance from the boundary, the characteristic decrease length being L_b. It is necessary to note that other types of perturbations can also produce imbalance in superconductors. For instance, electromagnetic or sound waves can produce photoelectric or acoustoelectric fields of this nature (Aronov 1976). The latter will be discussed in more detail in section 8.

Investigating excess current in Josephson junctions Carlson and Goldman (1975) have discovered a collective mode in superconductors near the transition temperature T_c. Schmid and Schön (1975a,b) and Artemenko and Volkov (1975b) later showed theoretically that a sound-like collective mode can exist in a superconductor near T_c. In such a wave the oscillating quantities are the branch imbalance v_1 (or potential ϕ) and v_1, no spatial charge appearing. The normal current and the supercurrent flow in the opposite directions, the total current being zero. It is the absence of a net current and consequently the
absence of any magnetic field that provides the possibility for these waves to exist in the bulk of a superconductor.

To find the dispersion law for such a collective mode one should apply Fourier transformation to eqs. (86) and (92). Making use of the continuity equation $\operatorname{div} j = 0$ we get (Artemenko and Volkov 1975b)

$$\begin{aligned}\omega^2 =& 4T\bar{\Delta}\frac{N_s}{m}\frac{\partial\mu}{\partial N}\left(q^2 + L_b^{-2}\right) + \mathrm{i}\omega\left(\frac{e^2 N_s}{m\sigma^{(n)}} + \frac{1}{\tau_{ph}}\right)\\ &+ \mathrm{i}\omega D q^2 \frac{4T}{\pi\Delta}\frac{\sigma^{(s)} - \sigma^{(n)}}{\sigma^{(n)}}\end{aligned} \tag{94}$$

It can be readily seen from this equation that if

$$\frac{1}{\tau_{ph}}, \quad \frac{1}{\tau_{im}}\frac{N_s}{N} \ll \omega \ll \frac{1}{\tau_{im}}\frac{N_s}{N}\frac{\sigma^{(n)}}{\sigma^{(s)} - \sigma^{(n)}}, \tag{95}$$

such a collective mode exists, its damping being small. The dispersion law of this mode is given by (Artemenko and Volkov 1975b)

$$\omega^2 = \frac{4T}{\pi\Delta}\frac{N_s}{m}\frac{\partial\mu}{\partial N}q^2, \tag{96}$$

i.e. this mode is of the acoustic type. Near the transition point $N_s \propto \Delta^2 \propto (T_c - T)$, consequently the wave velocity is proportional to $\Delta^{1/2} \propto (T_c - T)^{1/4}$.

To analyse the conditions (95) of small damping one should know the ratio $\left(\sigma^{(s)} - \sigma^{(n)}\right)/\sigma^{(n)}$. As we have seen one cannot determine this quantity correctly using the transport equation approach.

As a conclusion to this section we shall formulate the Onsager relations for the transport coefficient. For this purpose let us calculate the thermal flux density Q. Linearizing eq. (77) for σ_p in $n_p^{(1)}$ and inserting it in eq. (79) we get

$$\mathrm{Q}_i = Ts_i = \int (\mathrm{d}p)_p \xi_p v_i n_p^{(1)} \tag{97}$$

Inserting eq. (84) for $n_p^{(1)}$ we have in linear approximation in the gradients

$$Q_{\mathrm{i}} = \xi_{ik} E_k - \kappa_{ik} \partial T / \partial x_k, \tag{98}$$

where

$$\begin{aligned} \xi_{\mathrm{ik}} &= -e \int (\ \mathrm{d}p)_p \xi_p v_i I^{-1} v_k \partial n^{(0)} / \partial \varepsilon \\ \kappa_{ik} &= -\int (\mathrm{d}p)_p \xi_p v_i I^{-1} v_k \partial n^{(0)} / \partial \varepsilon \end{aligned} \tag{99}$$

One can verify that the matrix elements of the linearized collision operator $l_{pp'}$ obey the relations

$$I_{pp'} \frac{\partial n^{(0)}}{\partial \varepsilon} = I_{-p',-p} \frac{\partial n^{(0)}}{\partial \varepsilon}. \tag{100}$$

These relations are a consequence of the time reversal asymmetry of the mechanical equations. We conclude from eq. (100) that

$$\left(\frac{\partial n^{(0)}}{\partial \varepsilon}\right)^{-1} \left(I^{-1}\right)_{pp'} = \left(\frac{\partial n^{(0)}}{\partial \varepsilon}\right)^{-1} \left(I^{-1}_{-p',-p}\right).$$

Using eqs. (87), (88) and (99) we easily obtain the following relations

$$\sigma_{ik}^{(\mathrm{s})} = \sigma_{ki}^{(\mathrm{s})}, \quad T\eta_{ik} = \zeta_{ki}, \quad \kappa_{ik} = \kappa_{ki}. \tag{101}$$

These are the Onsager relations for a superconductor. They were first established by Luttinger (1964).

25.7 7. Thermoelectric phenomena

Here we shall discuss briefly thermoelectric effects in superconductors. The physical picture of these phenomena is appropriate to point out some specific features of transport in the superconducting state and to illustrate the approach presented above.

It was shown in section 6 that in the presence of a temperature gradient in a superconductor a thermoelectric current of normal excitations proportional to ∇T appears, its nature being similar to the thermoelectric current in the normal state. The corresponding thermoelectric coefficient η in the case of impurity scattering is given by [cf. eq. (87); see Gal'perin et al. 1973, 1974)]

$$\eta = \frac{2\pi^2}{9} \frac{eT}{m} \frac{\hat{C}}{\hat{C}\varepsilon} \left[\tau_{\mathrm{tr}}(\varepsilon) v(\varepsilon) \varepsilon\right]_{\varepsilon=\mu} G(\Delta/T). \tag{102}$$

Here $r(\varepsilon)$ is the density of states while τ_{tr} is the so-called transport relaxation time describing relaxation of the function $n_p^{(1)}$ proportional (for an isotropic excitation spectrum) to the first spherical harmonic. For the case of impurity scattering it is given by

$$\frac{1}{\tau_{\mathrm{tr}}} = \frac{\pi N_i (2\pi)^4}{m^2}$$
$$\int (\,\mathrm{d}p)_p \left|f_{pp'}\right|^2 \left(u_p u_{p'} - v_p v_{p'}\right)^2 \delta\left(\tilde{\varepsilon}_p - \tilde{\varepsilon}_{p'}\right)\left[1 - \cos\left(p, p'\right)\right].$$

The function $G(x)$ is defined by the integral

$$G(x) = \frac{3}{2\pi^2} \int_x^\infty \frac{y^2\,\mathrm{d}y}{\cosh^2(y/2)} \tag{103}$$

If $x \ll 1$, we have

$$G(x) = 1 - 2x^2/\pi^2 \tag{104}$$

Thus, for $T \to T_{\mathrm{c}}$ we get for the transport coefficient η the well-known expression for normal metals. For large values of the argument, $x \gg 1$, we have

$$G(x) = \frac{24}{\pi^2} x^2 e^{-x} \tag{105}$$

so for low temperatures ($T \ll T_{\mathrm{c}}$) the coefficient η is exponentially small.
Ginzburg (1944) was the first to suggest the existence of such a normal thermoelectric current in superconductors. He concluded that in anisotropic or inhomogeneous superconductors the normal thermoelectric current can produce a magnetic field proportional to the temperature gradient. He also pointed out that in the bulk of a homogeneous isotropic superconductor the total current vanishes because the normal current is offset by the supercurrent. This makes it impossible to observe the thermoelectric effect in a simpleconnected isotropic superconductor directly.

However, one can make an independent measurement of the supercurrent which manifests itself by the creation of a phase difference $\delta\chi$ in the order parameter across the sample's boundaries (Gal'perin et al. 1973b, 1974a). The most effective way to observe the phase difference is to make the superconductor a part of bimetallic loop that may also contain weak links (Gal'perin et al. 1973b, 1974b, Garland and Van Harlingen 1974). Such a system may be, for instance, a superconducting quantum interferometer which measures $\delta\chi$ directly. It may also be a usual thermoelectric loop made of two different superconductors; in this case a non-quantized part of magnetic flux proportional to the temperature gradient appears in the cavity of the loop. Thus, the coefficient η in superconductors is measureable. It should give valuable information about the relaxation processes.

A number of experiments was made to observe the thermoelectric magnetic flux (see, for example, Zavaritskii 1974, Selzer and Fairbank 1974, Pegrum et al. 1975, Van Harlingen and Garland 1978). But in most of these experiments (except that of Zavaritskii 1974) the observed temperature-dependent magnetic flux was by several orders of magnitude larger than the theoretically predicted one. We do not intend to discuss in this chapter possible sources of these discrepancies. Some discussion was given by Pegrum and Gunault (1976), and by Kozub (1978); they considered, in particular, an important masking effect. It is due to temperature dependence of the penetration depth which leads to T-dependent redistribution of the magnetic flux between the cavity and the surface layer with a width of the order of the penetration depth.

Several theoretical papers were also published, where the influences of some specific features of quasi particle transport on the thermoelectric effect were studied [phonon drag (Gurevich and Krylov 1975), scattering on magnetic impurities (Kon 1975, Kon and Nika 1976, Gelikman and Reizer 1975)].

One may also expect a specific thermoelectric effect in the situation where a temperature gradient exists in the presence of a supercurrent. Taking symmetry considerations into account

one may conclude that a contribution to the quasiparticle distribution function proportional to $(v_s \nabla T)$ and a corresponding contribution to the current density should exist in this case (Aronov 1974). Noting that $v_{\mathrm{s}} \nabla T$ does not depend on p one might also come to conclusion that this part of the nonequilibrium distribution function relaxes only through collisions with phonons and, as a result, the corresponding contribution to the current density should be proportional to $\tau_{\mathrm{ph}} v_{\mathrm{s}} (v_{\mathrm{s}} \nabla T)$ (Aronov 1974). Unfortunately, such reasoning, employed in several papers (Aronov 1974, Kozub 1978, Gal'perin et al., 1978) seems to be erroneous. Indeed, it can be shown that such an effect should be proportional, apart from τ_{ph}, to the integral

$$\int (\mathrm{d}p) \frac{\partial \tilde{\varepsilon}_p}{\partial p} \delta \left(\varepsilon - \tilde{\varepsilon}_p\right) \tag{106}$$

which, as we have seen in section 6 , vanishes. Thus, we come to the conclusion that in the case of comparatively strong impurity scattering no thermoelectric effect proportional to τ_{ph} exists. This means that all the additional terms in the nonequilibrium distribution function proportional to $v_c \nabla T$ are also proportional to τ_{im}. As a consequence, the corresponding contribution to the thermoelectric current can be estimated as

$$\left(p_{\mathrm{F}} v_{\mathrm{s}} / T\right)^2 \eta \nabla T \tag{107}$$

and may be discarded in standard experimental situations, where $p v_{\mathrm{s}} \ll T$.
Nevertheless, a specific v_{-}-dependent thermoelectric effect does exist in superconductors. It manifests itself by the appearance of the branch imbalance discussed above, and, consequently, of the potential ϕ in the presence of ∇T and v, (Schmid and Schön 1979). The corresponding thermoelectric potential ϕ is proportional to $(v, \nabla T)$; it can be homogeneous if this quantity is constant along the sample. It was shown (Schmid and Schön 1979) that this effect is also connected with the impurity scattering rather than with the phonon scattering. This fact is in accordance with the considerations discussed above.

One may also mention, in conclusion, the effect considered recently by Shelankov (1980) which is analogous to the well-known Nernst effect in normal conductors. If an external magnetic field is parallel to the surface of a superconducting plate supercurrent flows near the surface because of

Meissner-Oxenfeld screening. If a temperature gradient exists along the plate, it produces, in combination with the supercurrent, branch imbalance accompanied by the transverse potential difference $\delta\phi = \phi_1 - \phi_2$, where ϕ_1 and ϕ_2 are the values of the nonequilibrium covariant potential on the surfaces of the plate.

25.8 8. Acoustical properties of superconductors

25.8.1 8.1. Deformational perturbations

In this section we are going to consider another important class of perturbations, i.e. those due to the strain of a superconductor. They appear, for instance, if sound propagates through a superconductor.

The electron dispersion law is affected by the strain. To describe this effect one shoud choose the proper reference frame. For this purpose either Lagrangian or Eulerian coordinates can be used. Consider some point of the elastic continuum. Let us specify its initial position by the vector r; its position after the strain is $r' = r + u, u$ being the displacement vector. The strained state of the elastic medium can be described by the vector u as a function of either the Lagrangian coordinates r, t or the Eulerian coordinates r', t. The Lagrangian reference frame "follows" the strained continuum, and the interaction Hamiltonian in this system has the simplest form. It is the system we shall use hereafter in this section to formulate the

transport equation. The noninertiality and curvilinearity of the system can properly be taken into account by adding extra terms to the interaction Hamiltonian.

The strain field being a slowly-varying function of r and t, we shall use the well-known effective-mass approach. Along the lines of this approach we shall consider an electron as a quasiparticle with the Hamiltonian $E_0(-\mathrm{i}\hbar\hat{c}/\hat{c}r)$ where $E_0(p)$ is the unperturbed energy spectrum (here we, of course, are to reject the previously formulated assumption of isotropy of the electron spectrum). The strain tensor

$$u_{ik} = \left(\frac{1}{2}\right)\left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i}\right) \tag{108}$$

is considered to be small: $|u_{ik}| \ll 1$. In this approximation the one-electron Hamiltonian of a normal metal has the form (see Akhieser 1938, Akhieser et al. 1957)

$$\hat{H} = E_0(\hat{p}) + \lambda_{ik}(\hat{p})u_{ik} - m_0 v\frac{\partial u}{\partial t}, \tag{109}$$

where $\hat{p} = -\mathrm{i}\hbar\hat{d}/\hat{c}r$. Here λ_{ih} is the tensor of deformational potential. The third term describes the well-known Stuart-Tolman effect due to acceleration of the lattice, m_0 being the mass of the free electron.

To obtain the spectrum of the excitations in the superconducting state one should insert the Hamiltonian (108) in the Bogoliubov-De Gennes equation which gives

$$\tilde{\varepsilon}(p,r) = \sqrt{[E_0(p) + \lambda_{ik}u_{ik} + \phi - \mu]^2 + \Delta^2} + p_{\mathrm{s}}v - m_0 v\dot{u}. \tag{110}$$

One can see that the deformational perturbation contributes both to the potential ϕ as the term $\lambda_{ik}u_{ik}$ and to the term $p_{\mathrm{s}}v$ as the item $-m_0 v\dot{u}$. Note that the term $\lambda_{\mathrm{it}}\,(p)u_{ik}$, depending on the electron quasimomentum p, is different for different electrons. The potential ϕ should satisfy the neutrality condition. For instance, for the case of stationary deformation this condition has the form

$$u_{ik}\overline{\lambda_{ik}(p)} + \phi = 0. \tag{111}$$

(Here the bar denotes the average over the Fermi surface.)
To obtain the expression for the acoustic power density $\mathscr{P}$ absorbed per second in a unit volume of a superconductor one should differentiate the expression for the entropy density over time, then use the Boltzmann equation, and finally average the expression over the period of the sound wave. In such a way we get the following expression for $\mathscr{P}$ (Gal'perin 1974):

$$\mathscr{P} = \left\langle \int (\mathrm{d}p)\dot{\tilde{\varepsilon}}(p,r)n_p \right\rangle, \tag{112}$$

where $\tilde{\varepsilon}(p,r)$ is given by eq. (110), angular brackets mean the time average over the period.

25.8.2 8.2. Nonlinear acoustic phenomena in superconductors

Acoustic absorption in superconductors was studied in a number of papers. It was shown that investigation of acoustical properties can give valuable information about the excitation spectrum and other properties of superconductors.

We shall concentrate ourselves on nonlinear acoustic phenomena. On the one hand, the problem of the linear response of a superconductor to acoustic perturbation is relatively simple and has been studied extensively. On the other hand, the problem of nonlinear acoustic phenomena is much more complicated and it is the transport equation approach that permits one to solve it and at the same time to get clear physical understanding at every step of its

solution. This problem is interesting also from the experimental point of view, because nowadays it is possible to observe nonlinear effects in normal metals. (For the theory of nonlinear acoustic phenomena in normal metals see the review article by Gal'perin et al. 1979, where some experimental data are also discussed.)

Therefore, one can hope that experimental observation of the effects of the same nature in superconductors is also possible.

Of a number of possible nonlinear acoustic effects we shall discuss only two here: nonlinear absorption and the acousto-electric effect. In the linear theory the absorption coefficient Γ is independent of sound intensity S. We determine the nonlinear absorption as the physical situation where the coefficient Γ depends on S and it is this dependence that we shall discuss below.

The absorption mechanism depends strongly on the relation between sound wavelength $2\pi/q$ and electron mean free path l. Nonlinear effects in absorption are much more pronounced provided $q\,|\gg 1$, and we shall confine ourselves to this case.

The second, acousto-electric, effect exists irrespective of the value of the quantity ql. This effect has no direct analogue in normal conductors. It amounts to the following. A travelling acoustic wave drags the quasiparticles, which produces a.d.c. normal current, $j^{(\mathrm{a})}$, which is known as the acoustoelectric current. In an isolated sample the normal current is offset by a supercurrent $j^{(s)}$ which is proportional to $\nabla\chi$. The sum $j^{(a)}+j^{(s)}$ being zero one may conclude that a travelling acoustical wave creates a stationary phase difference in a superconductor. The phase difference can be measured in different ways discussed in several papers (Gal'perin et al. 1973a,b, 1974a,b, 1978).

Therefore, the purpose of the theory is to calculate the acousto-electric current $j^{(a)}$.

Let us describe the physics of nonlinear ultrasonic absorption at $ql \gg 1$. To understand this, we shall describe briefly the situation in normal conductors. Let us for simplicity consider interaction with longitudinal acoustic waves. In this case electrons interact with the sound wave only via the deformational potential. Consequently, the sound wave is accompanied by a wave of the electric field acting on the electrons which propagates with the sound velocity w. Such a wave of the effective field materially affects only those electrons whose velocities in the direction of sound propagation, ι_1, are close to the sound velocity, w, because the mean field acting on the rest of the electrons is small. Therefore the interaction of the electrons with the sound wave is determined by the properties of the distribution function in the "resonance" region of velocities and also by the width of this region. In the linear regime the characteristic width of the "resonance region" is of the order of $|v_x - w| \simeq v_{\mathrm{F}}(ql)^{-1}$ and the characteristic interaction time is of the order of τ, τ being the mean free time of the resonance electrons. Consequently, the linear absorption coefficient Γ_0 is independent of the mean free path l. Therefore, this mechanism is known as collisionless damping; it is analogous to the well-known Landau damping of plasma waves (see Landau 1946).

If the sound intensity is sufficiently high the structure of the resonance region is changed. Some of the resonance electrons become "captured" by the
sound wave: their motion in the coordinate system following the travelling wave is finite in the direction of the wave propagation, x. The captured electrons oscillate in the potential wells produced by the acoustic wave. The frequency of the oscillations is of the order

$$\omega_0 \simeq q\tilde{v} = q\sqrt{U_0/m} \tag{113}$$

where $U_0 = \left(\hat{\lambda}_{ik} - \bar{\lambda}_{ik}\right) u_{ik}^{(0)}, \tilde{v} = \sqrt{U_0/m}$ is the characteristic velocity of the oscillatory motion, while $u_{ik}^{(0)}$ is the amplitude value of the tensor u_{ik}. Each scattering event changes the direction of the electron velocity and removes it from the captured group. Therefore, in order that such a group would perform oscillatory motion the inequality

$$\omega_0 \tau \gg 1 \tag{114}$$

should be satisfied. This inequality means that the captured electrons perform many oscillations in the potential wells during the time between collisons, τ.

The formation of a group of captured electrons leads to distortion of the electron distribution in the resonance region of velocities, the width of which is of the order of $\tilde{v}$, and, consequently, to the appearance of nonlinear effects in sound absorption. The distribution function of the untrapped electrons is also changed by the sound wave. However, its variation is relatively large only in the resonance region, i.e. the width of the corresponding region of velocities is also of the order of i. In other words, the number of untrapped resonance electrons is of the order of the number of trapped electrons.

Note that the relative number of resonance electrons is small: it is of the order of $U_0/\mu \ll 1$. Nevertheless, the inequality (114) can be satisfied if the product ql is large enough. Indeed, we have

$$\omega_0 \tau \simeq ql\sqrt{U_0/\mu}.$$

The parameter $\omega_0 \tau$ can be large even if the actual value of the sound intensity is relatively small.

This physical picture is based implicitly on classical consideration. It is valid if (1) the number of quantum levels in each potential well is large or (2) if these levels are smeared by the collisions. It means that our picture is valid if

$$\frac{\hbar^2 q^2}{m} \ll U_0 \quad \text{or} \quad \frac{\hbar^2 q^2}{m} \ll \frac{\hbar}{\tau}. \tag{115}$$

In what follows we shall consider at least one of these conditions to be satisfied.

Now let us turn to our immediate purpose, i.e to a superconductor. Normal excitations in a superconductor behave essentially in the same manner as electrons in normal metals. So one can expect that a similar mechanism of nonlinearity in superconductors exists as a consequence of bunching of normal excitations discussed above.

This situation can be described quantitatively with the help of the transport equation, where one should use as the excitation energy the expression given in the previous section. There are two gauge-covariant potentials, ϕ and p , that enter this expression.

For longitudinal sound we are interested in they both should be determined from the neutrality condition. General results of such an analysis depend on the ratio ω/q in comparison to v_{F}, i.e. on the relative importance of the time and spatial dispersion (see, for instance, Aronov and Gurevich 1976). If the spatial dispersion is more important than the time dispersion, i.e. if $\omega/q < \iota_F$, one may discard the term $p_{\mathrm{s}} v$ from eq. (110). Then the neutrality condition gives

$$\phi = -\lambda_{ik} u_{ik}$$

This is why the quantity

$$\bar{\xi}_p = \xi_p + U = \check{\zeta}_p + \left[\lambda_{ik}(p) - \tilde{\lambda}_{ik}\right] u_{ik} \tag{116}$$

enters the expression

$$\dot{\varepsilon}_p = \sqrt{\xi_p^2 + \Delta^2}$$

for the excitation energy. In this case the effective potential U depends on the quasimomentum p (cf. eq. 5).

To calculate the ultrasonic absorption coefficient, one should obtain the nonequilibrium distribution function n_p by solving the transport equation. Let us divide the distribution function n_p into the sum of two parts $n_p = n^{(0)}\left(\hat{\varepsilon}_p\right) + n_p^{(1)}$, where the first part is averaged over a constant-energy surface. Let us assume that the function $n^{(0)}\left(\tilde{\varepsilon}_p\right)$ to be the equilibrium one (we shall discuss below when this is the case). Then, as can be established on the basis of eq. (15), no essential corrections to the gap are necessitated by the nonequilibrium character of the distribution function. The transport equation for this case has the form:

$$\left(v_x \frac{\tilde{\xi}_p}{\varepsilon_p} - w\right) \frac{\partial n_p^{(1)}}{\partial x} - \frac{\partial U}{\partial x} \frac{\tilde{\xi}_p}{\varepsilon_p} \frac{\partial n_p^{(1)}}{\partial p_x} + \frac{n_p^{(1)}}{\tau} \frac{|\tilde{\xi}|}{\varepsilon_p} = w \frac{\partial U}{\partial x} \frac{\tilde{\xi}_p}{\varepsilon_p} \frac{\partial n^{(0)}}{\partial \tilde{\varepsilon}_p}. \tag{117}$$

Here we use two approximations. First, we have considered $n_p^{(1)}$ to depend on x and t in the combination ($x - wt$) only. A sufficient criterion for this assumption is $\Gamma 1 \ll 1$, which usually holds. Secondly, we have discarded the "in" term of the collision operator. Such a procedure can be justified if one takes into account that the resonance group is small. This is why the probability for an electron to escape from this group is much larger than the probability for a nonresonant electron to join this group.

We shall analyse the eq. 117 with the help of the method of characteristics. One can see that equations to determine the quasiparticle trajectory have the energy integral.

$$\sqrt{\xi_p^2 + \Delta^2} - p_{\mathrm{x}} w = \mathscr{E}. \tag{118}$$

Let us compare this equation with the corresponding one for the normal state, which can be written in the form

$$\tilde{\xi}_p - p_x w = E. \tag{119}$$

One can see that in the normal state the potential $U(x)$ leads to coordinate dependence of the x-component of the quasimomentum, p, i.e. to distortion of the shape of electron trajectory. In the presence of a sound wave it becomes curvilinear rather than a straight line. The absorption coefficient Γ is determined by "group velocity" $v_{gx} = \partial E / \partial p_x$, which in the normal state is $v_x - w$; this quantity should be calculated from eq. (119) and its characteristic value is, as we have already seen, of the order of $\tilde{v}$.

The corresponding group velocity in the superconducting state is equal to $v_x \xi_p / \varepsilon_p - w$. Thus, there are two sources of the influence of the potential U on the group velocity v_{g}. One of them is the distortion of the shape of electron trajectories (just as in the normal state). This mechanism, as we shall see later, is characterized by the same quantity $\omega_0 \tau$ as in the normal state. Thus, the threshold value of U_0, describing the onset of the nonlinear effects, is of the order of $U_0^{(th)} \simeq m/(q\tau)^2$. The second source is specific for the superconducting state. One can see that the group velocity may change even if the trajectory remains a straight line. due to the change of the quantity Ξ_p. i.e. of the modulus of the quasimomentum p. Furthermore, it can even change its sign without a change of the sign of v_x. It is clear that this effect can lead to specific localization of a quasiparticle in the field of an acoustic wave. This effect is somewhat similar to the well-known Andreev reflection of quasiparticles by inhomogeneities of the order parameter.

The influence of such Andreev-like localization on the sound absorption was studied by Galako et al. (1981).

Our immediate purpose is to illustrate the application of the transport equation approach to acoustic phenomena rather than a detailed discussion of different mechanisms of nonlinear acoustic absorption. This is why we shall restrict ourselves to the first mechanism, which is

the simplest. To understand the region of parameters where this mechanism is dominant let us analyse the energy integral (118).

Note that if one may neglect the second term in eq. (118) it leads to the equality $\widetilde{\xi}_p = \xi =$ const. In this case

$$v_x = p_x/m = (2U_0/m)^{1/2} \sqrt{A - \left[\overline{U(x)}/U_0\right]}, \tag{120}$$

where we have introduced the notation

$$\mu - \frac{p_\perp^2}{2m} + \xi \equiv U_0 A$$

(we shall see that the characteristic value of A is of the order of unity).
To estimate the corrections due to the finite value of the term $p_x w$ in comparison with $\sqrt{\xi_p^2 + \Delta^2}$ let us expand the last term in powers of the quantity $\delta\xi_p = \tilde{\xi}_p - \tilde{\zeta}$ up to first order:

$$(\xi\varepsilon)\delta_p \pm \left(2mU_0 w^2\right)^{12} \sqrt{A - \frac{U(x)}{U_0} + \frac{\delta\xi_p}{U_0}} = 0.$$

Here $\varepsilon = \sqrt{\xi^2 + A^2}$. It can be seen from this equation that one can neglect the quantity $\delta\xi_p$ in the second term if

$$\left(2mw^2/U_0\right)^{1/2} \ll \zeta\check{\zeta}/\varepsilon. \tag{121}$$

If this condition is satisfied one can still use eq. (120) for $v_x(x)$, and we have

$$\delta\check{\zeta}_p = \frac{\varepsilon}{\xi} m w v_x(x). \tag{122}$$

To estimate the role of this correction to ξ one should compare the quantity $\delta_{\zeta_p}\mu$ (d d $\xi_{\xi'}$) (ξ') to the velocity $r_{\gamma'}$. Indeed, dividing the transport equation eq. (117) by $\cong i_p$ we obtain an equation similar to that for the normal case. The only difference is that the quantity $w\%_p, \tilde{\zeta}_p$ rather than w enters this equation. The first source of nonlinearity is the change of v_x along the trajectory, the second source is the change of $1\omega_p\tilde{\Xi}_p$ due to the change of $\tilde{\Xi}_p$. Using eq. (122) for $\delta\xi_p$ and comparing the term $\delta\xi_p$ w(d d ξ) (δ) with l', we get the following criterion for the range of applicability of the theory we shall discuss below

$$\frac{mw^2}{T} \min\left(\frac{\Delta}{T}, \frac{\Delta^2}{T^2}\right) \ll 1 \tag{123}$$

[here we have taken into account that $\xi \simeq \max(\sqrt{\Delta T}, T)$]. Combining eq. (121) with the expression for the threshold amplitude $U_0^{(\mathrm{th})}$ we get the second criterion of applicability of our theory:

$$\left(U_0/U_0^{(th)}\right)^{1/2} > \max(1, \omega\tau\varepsilon/\xi). \tag{124}$$

Let us now discuss when it is justified to consider the function $n^{(0)}(\varepsilon)$ as the equilibrium one. There are two kinds of corrections to the function $n^{(0)}\varepsilon$: stationary corrections and corrections with the period of the sound wave. One can verify that only corrections of the first kind may be important. The corresponding upper limit can be obtained by comparison of the energy transfer to a quasiparticle during the energy relaxation time $\tau_{\mathrm{ph}} \simeq \hbar^3\omega_{\mathrm{D}}^2/T^3$ with the characteristic energy scale, $\min(T, T_c - T)$. The resulting estimate has
the form

$$S \ll \frac{NT}{\Gamma \mu \tau_{\mathrm{ph}}} \min\left(T_{\mathrm{c}} - T, T_{\mathrm{c}}\right). \tag{125}$$

This condition is met for most of the interesting cases.
The problem is somewhat more complicated for the case of transverse sound which is not discussed here in detail. For this situation the changes in $n^{(0)}(\varepsilon)$ by intensive acoustic wave were studied by Bulyzhenkov and Ivlev (1976a,b)*. These changes may be of the same nature as in the case of superconductivity stimulation; they are discussed by Eliashberg and Ivlev (1985).

Now let us turn to the calculation of the nonlinear sound absorption coefficient for the case $(\omega_0 \tau \geqslant> 1$. It can be shown (see Gal'perin et al. 1973a) that the absorption coefficient Γ, can be expressed in terms of the nonequilibrium distribution function:

$$\Gamma_{\mathrm{s}} = \left(\rho \omega^2 u^2\right)^{-1} \int (\ \mathrm{d}p) \left\langle \frac{\xi_p}{\varepsilon_p} n_p \frac{\partial U}{\partial x} \right\rangle_\eta \tag{126}$$

where $\langle \cdots \rangle_\eta$ denotes averaging over the wave coordinate $\eta = qx - wt$, and ρ is the crystal density.**

To obtain the nonequilibrium part $n_p^{(1)}$ of the distribution function one may rewrite eq. (117) in terms of the dimensionless quantities

$$\begin{gathered} \eta, \quad \tilde{v}_x \equiv v_x / \left(U_0/m\right)^{1/2}, \quad \tilde{U} \equiv U(x)/U_0 \\ \tilde{v}_x \frac{\hat{i} n_p^{(1)}}{\partial \eta} - \frac{\mathrm{d}\tilde{U}}{\mathrm{d}\eta} \frac{\hat{i} n_p^{(1)}}{\partial \tilde{v}_{\mathrm{x}}} + \frac{\operatorname{sign} \xi_p}{u_0 \tau} u_p^{(1)} = \tilde{U} \frac{w'}{\sqrt{U_0/m}} \frac{\hat{i} n^{(0)}(\varepsilon)}{\partial \varepsilon}. \end{gathered} \tag{127}$$

To solve this equation one should formulate the appropriate boundary conditions. The distribution function of the untrapped quasiparticle should satisfy the periodicity condition in the wave coordinate η while the distribution function of the trapped quasiparticles should satisfy the condition of specular reflection from the walls of the potential wells produced by the sound wave. The general solution was obtained by Gal'perin et al. (1972), and has a rather complicated form. However, in the case of strong nonlinearity ($\omega_0 \tau \gg 1$ it can be expanded in powers of the small parameter $1/\omega_0 \tau$, and the main contribution to the absorption coefficient appears to be of the first order in $1/\omega_0 \tau$. One can

obtain the following result

$$\frac{\Gamma_{\mathrm{S}}}{\Gamma_{\mathrm{S0}}} = 1.1 \overline{\left[\frac{\Lambda_{xx}^2}{q\tau \left(U_0/m\right)^{1/2}} \frac{2}{\mathrm{e}^{d/T} + 1}\right] \cdot \left[\frac{2\Lambda_{xx}^2}{\mathrm{e}^{d/T} + 1}\right]^{-1}}, \tag{128}$$

where Γ_{S0} is the linear absorption coefficient in the superconductor, and $\Lambda_{xx} = \lambda_{xx} - \bar{\lambda}_{xx}$.

25.8.3 8.3. Acousto-electric current and field

For the calculation of the d.c. current generated by a traveling acoustic wave it is natural to seek the solution of the Boltzmann equation (6) in the form

[12]*It can be shown that the nonlinear effects due to changes in $n^{(0)}$ discussed by Bulyzhenkov and Ivlev are suppressed if the nonlinearity discussed above in detail is well pronounced.
**This expression is valid if one can neglect the nonlinear distortion of the acoustic wave itself so that the wave can be represented by single harmonic. This assumption may be justified if $U_0 \ll \mu$ (see Gal'perin and Kozub 1977).

$$n_p = n^{(0)}\left(\varepsilon_p + pv_s\right) + n_p^{(1)} \tag{129}$$

Using the general expression for the current density (see eq. 40) one can obtain, in linear approximation in the small parameter pv_s/Δ,

$$j = j^{(a)} + eN_{\mathrm{s}}v_{\mathrm{s}},$$

where

$$j^{(a)} = \frac{e}{2}\int(\,\mathrm{d}p)_p v\left\langle\left(n_p^{(1)} - n_{-p}^{(1)}\right)\right\rangle_\eta \tag{130}$$

is the current of the dragged normal excitations. To calculate this current, one may neglect the motion of the condensate.

We shall calculate $j^{(a)}$ without solving eq. (6), using the method developed for this purpose by Kagan (1967). Let us multiply eq. (6) by the electron velocity v and integrate it over the surface of constant energy $\varepsilon_p = \varepsilon$ and over the wave period. By definition, for elastic collisions

$$\int(\mathrm{d}p)vIn_p^{(1)}\dot{\delta}\left(\varepsilon - \varepsilon_p\right) = \frac{\left|\breve{\zeta}_p\right|}{\varepsilon_p}\int(\,\mathrm{d}p)I^{(\mathrm{n})}vn_p^{(1)}\delta\left(\varepsilon - \varepsilon_p\right).$$

The only other term of eq. (6) that does not vanish with the accepted accuracy after averaging over the period of the wave is

$$-\int(\mathrm{d}p)v\left\langle\frac{\partial U}{\partial \boldsymbol{r}}\frac{\xi_p}{\varepsilon_p}\frac{\partial n_p^{(1)}}{\partial \boldsymbol{p}}\right\rangle_\eta\delta\left(\varepsilon - \varepsilon_p\right)$$

As a result, we get for $j^{(a)}$ in the time-of-relaxation approximation

$$\begin{aligned} j^{(\mathrm{a})} &= e\int_\Lambda^\infty \mathrm{d}\varepsilon\tau_{\mathrm{tr}}\frac{\varepsilon}{|\xi|}\int(\mathrm{d}p)v\frac{\xi_p}{e_p}\left\langle\frac{\partial U}{\partial r}\frac{\partial n_p^{(1)}}{\partial p}\right\rangle_\eta\delta\left(\varepsilon - \varepsilon_p\right) \\ &= -e\int(\,\mathrm{d}p)\tau_{\mathrm{tr}}(\varepsilon)m^{-1}\left\langle n_p^{(1)}\frac{\partial U}{\partial r}\right\rangle_\eta \operatorname{sign}\xi_p. \end{aligned} \tag{131}$$

Inasmuch as the electrons contributing to the last integral are mainly close to the Fermi surface while the scale of variation of $\tau_{\mathrm{im}}^{(\mathrm{n})}$ is μ, one can take the relaxation time outside the integral. As a result, we get the following relation connecting the acousto-electric current $j^{(a)}$, the absorption coefficient Γ, the sound intensity S and the electron mobility in the normal state, μ_n

$$j^{(a)} = \frac{\mu_{\mathrm{n}}\Gamma_{\mathrm{n}}S}{w}\frac{N_0 - N_{\mathrm{s}}}{N_0} = \mu_{\mathrm{n}}\frac{\Gamma_{\mathrm{s}}S}{w}\mathscr{F}\left(\frac{\Delta}{T}\right), \tag{132}$$

where

$$\begin{aligned} \mathscr{F}\left(\frac{\Delta}{T}\right) &= \left(\int_\Delta^\infty \frac{\varepsilon}{\sqrt{\varepsilon^2 - \Delta^2}}\frac{\partial n^{(0)}}{\partial \varepsilon}\mathrm{d}\varepsilon\right) / \left(\int_\Delta^\infty \mathrm{d}\varepsilon\frac{\partial n^{(0)}}{\partial \varepsilon}\right) \\ &= \frac{1}{4}\left(\mathrm{e}^{\Delta/T} + 1\right)\int_0^\infty \mathrm{d}x\frac{x + (\Delta/T)}{\sqrt{x(x + 2\Delta/T)}}\cosh^{-2}\left(\frac{x}{2} + \frac{\Delta}{2T}\right) \end{aligned} \tag{133}$$

The relation (132) is the generalization of the well-known Weinreich relation (see Weinreich 1957) for superconductors.

Thus, we have calculated the normal acousto-electric current. Inasmuch as the total current in the bulk of a superconductor should vanish we have the following expression for p_{s} :

$$\frac{1}{2}\hbar\nabla\chi = p_{\mathrm{s}} = -\frac{1}{wN_{\mathrm{s}}}\tau\mathscr{F}(\Lambda/T)\Gamma_{\mathrm{s}}S \tag{134}$$

Integrating this equation along the length of the sample and taking into account the law of acoustic intensity spatial variation because of the absorption,

$$\mathrm{d}S\ \mathrm{d}x = -\Gamma S$$

we obtain the following relation for the phase difference across a sample of length L

$$|\delta\chi| = \frac{2\tau\mathscr{F}(\Delta/T)}{\hbar wN_s}\left(S|_{x=0} - S|_{x=L}\right). \tag{135}$$

We think that the simplest experimental arrangement to observe this phase difference is a monometallic superconducting loop, the travelling acoustic wave propagating along a part of the loop. The measured quantity may be the magnetic flux through the loop which equals

$$\phi_{\mathrm{m}} = \phi_0(n + \delta\chi/2\pi), \tag{136}$$

where n is an integer while $\phi_0 = \pi\hbar c/e$ is the magnetic flux quantum.

Now let us turn to the acousto-electric field in superconductors (Aronov 1976) we have mentioned briefly in section 6 . As we have seen from eq. (92), if the normal current is spatially inhomogeneous the potential ϕ becomes inhomogeneous too. Indeed, the rhs of eq. (92) can be written as $-(1/e)\,\mathrm{div}\,j^{(n)}$,

and in the stationary case the quantity $\nabla^2\phi$ is nonvanishing. This means that the presence of an inhomogeneous normal current leads to an electric field $E = -(1/e)\nabla\phi$. One of the sources of such a field can be a travelling acoustic wave which drags quasiparticles of a superconductor. As a result, the d.c. acousto-electric normal current we have discussed above appears. Because of sound attenuation this current is coordinate-dependent. Thus, to obtain the acousto-electric field one should add $-(1/e)\,\mathrm{div}\,j^{(n)}$ to the rhs of eq. (92).

Another important thing is that the acousto-electric field leads to a contribution $\sigma_{ik}^{(\mathrm{n})}E_k$ to the electric current, so the total current density near T_{c} is equal to

$$j_i = eN_{\mathrm{s}}v_{\mathrm{s}i} + j_i^{(\mathrm{a})} + \sigma_{ik}^{(\mathrm{n})}E_k. \tag{137}$$

In a stationary state this current must vanish. As a result, the superflow velocity becomes coordinate dependent, and the equality $eN_{\mathrm{s}}v_{\mathrm{s}i} + j_i^{(a)} = 0$ we have employed above is valid only at distances from the sample boundaries much larger than L_b. One should take this fact into account when analysing the conditions for the observation of acousto-electric effects. Making use of eq. (132) for $j^{(a)}$ and taking into account that $\partial S/\partial x = -\Gamma_{\mathrm{s}}S$ we have at $T_{\mathrm{c}} - T \ll T_{\mathrm{c}}$

$$\left(-\frac{\partial^2}{\partial x^2} + L_{\mathrm{b}}^{-2}\right)\phi = \frac{l}{wN}\Gamma_{\mathrm{s}}\frac{\partial S(x)}{\partial x}. \tag{138}$$

This equation as well as eq. (92) is valid in the vicinity of T_{c}.

Usually the attenuation length Γ_{s}^{-1} is much larger than L_{b}. In such a case one obtains from eq. (138) the following expression for the acousto-electric field $E^{(\mathrm{a})}$

$$E_{\mathrm{x}}^{(\mathrm{a})} = -\frac{1}{e}\frac{\partial\phi}{\partial x} = -\frac{1}{wNe}\Gamma_{\mathrm{s}}S_0\left(\Gamma_{\mathrm{s}}L_{\mathrm{b}}\right)^2\mathrm{e}^{\Gamma_{\mathrm{s}}\mathrm{x}}. \tag{139}$$

Thus, an acoustic wave produces the acousto-electric field $E^{(\mathrm{a})}$ in the bulk of a superconductor.

25.9 9. Derivation of the transport equations for slowly varying external fields

In this section we shall demonstrate the derivation of a basic set of equations describing transport in superconductors. It is convenient to use for this purpose the Keldysh diagram technique (Keldysh 1964). In the course of the derivation we shall follow papers by Gal'perin et al. (1973a) and Aronov and Gurevich (1974).

Let us start from the Dyson equation for the electron Green function. To account for the specific features of the superconducting state one should
introduce, besides the normal matrix Green function

$$\hat{G}^{\alpha\beta} = \begin{pmatrix} G_{\mathrm{c}} & G_{+} \\ G_{-} & \widetilde{G}_{\mathrm{c}} \end{pmatrix} = \begin{pmatrix} -\mathrm{i}\left\langle \mathrm{T}\psi_\alpha(x)\psi_\beta^{+}(x')\right\rangle & \mathrm{i}\left\langle \psi_\beta^{+}(x')\,\psi_\alpha(x)\right\rangle \\ -\mathrm{i}\left\langle \psi_\alpha(x)\psi_\beta^{+}(x')\right\rangle & -\mathrm{i}\left\langle \mathrm{T}\psi_\alpha(x)\psi_\beta^{+}(x')\right\rangle \end{pmatrix}, \tag{140}$$

also the anomalous matrix Green function

$$\hat{F}^{\alpha\beta} = \begin{pmatrix} \left\langle \mathrm{T}\psi_\alpha(x)\psi_\beta(x')\right\rangle & -\left\langle \psi_\beta(x')\,\psi_\alpha(x)\right\rangle \\ \left\langle \psi_\alpha(x)\psi_\beta(x')\right\rangle & \left\langle \widetilde{\mathrm{T}}\psi_\alpha(x)\psi_\beta(x')\right\rangle \end{pmatrix}. \tag{141}$$

In the same way one should also introduce the matrix ${}^{+}F^{x\beta}$, which differs from (141) by replacement of ψ by ψ^{+}. Here α and β are the spin indices, $\psi^{+}$and ψ are the electron creation and annihilation operators, x denotes the combination $(\,r,t\,)$, and T and $\tilde{\mathrm{T}}$ are the time-ordering and antitime-ordering symbols. In this section the angular brackets represent the average over the equilibrium ensemble, external fields being included in the unperturbed Hamiltonian. The interaction is considered to be spin-independent. Therefore

$$\begin{aligned} \hat{G}^{\alpha\beta} &= \delta_{\alpha\beta}\hat{G}, \qquad {}^{+}\hat{\mathrm{F}}^{\alpha\beta} = \hat{\mathrm{I}}_{\alpha\beta}{}^{+}\mathbf{F}, \\ \hat{F}^{\alpha\beta} &= -\hat{I}_{\alpha\beta}\hat{F}, \quad \hat{I}_{\alpha\beta} \equiv \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{142}$$

The Dyson equations for $\hat{G}$ and ${}^{+}\hat{F}$ have the form

$$\begin{aligned} \hat{\sigma}_z G_0^{-1}(x_1)\,\hat{G}(x_1x_2) =& \hat{1}\delta(x_1-x_2) + \mathrm{i}\int \mathrm{d}^4x_3\hat{\Sigma}(x_1x_3)\,\hat{G}(x_3x_2) \\ &+ \mathrm{i}\int \mathrm{d}^4x_3\hat{\Delta}'(x_1x_3)^{+}\,\hat{F}(x_3x_2) \end{aligned} \tag{143}$$

$$\begin{aligned} \hat{\sigma}_z G_0^{*-1+\hat{F}(x_1,x_2)=} & \mathrm{i}\int \mathrm{d}^{+}x_3\hat{\Sigma}^{\mathrm{T}}(x_3x_1)^{+}\,\hat{F}(x_3x_2) \\ &+ \mathrm{i}\int \mathrm{d}^4x_3{}^{+}\hat{\Delta}'(x_1x_3)\,\hat{G}(x_3x_2)\,. \end{aligned} \tag{144}$$

We omit the analogous equation for $\widehat{F}(x_1x_2)$. Here

$$G_0^{-1}(x_1) = \mathrm{i}\hbar\frac{\partial}{\partial t_1} - \left(\frac{1}{2m}\right)\left[\hat{p}_1 - \frac{e}{c}A(x_1)\right]^2 - e\varphi(x_1) + \mu \tag{145}$$

$\hat{p}_1 = -i\hbar\hat{c}/\partial r_1$ being the canonical momentum operator. The self-energy parts $\hat{\Sigma}$ and $\hat{A}'$ are 2×2 matrices in the Keldysh technique. The matrix $\hat{\Sigma}$ is defined according to the rules of the usual Feynman technique in terms of $\hat{G}$ and the phonon Green function matrix $\hat{D}$, which is

defined in the same way as eqs. (140). The electron phonon vertex is represented by the matrix $\Gamma^h_{\prime\prime\prime}(x, x', y)$ with phonon index k and electron indices j, j'; the simple vertex has the matrix structure $\delta_{jj}\left(\hat{\sigma}_z\right)_{jj'k}$. Here

$$\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is the operator which acts on the corresponding matrix indices; we call them the Keldysh indices (not to be confused with the spin indices). The self-energy matrix $\hat{\Delta}'$ differs from $\hat{\Sigma}$ by the replacement of $\hat{G}$ by $\hat{F}$.

We consider the case of slowly varying (in time and space) external fields. To employ this condition it is natural to seek the solutions of Dyson equations (143) and (144) in the form

$$\hat{G}\left(x_1 x_2\right) = \hat{\mathscr{G}} \exp\left[\frac{1}{2}\mathrm{i}\chi\left(\mathrm{x}_1\right) - \frac{1}{2}\mathrm{i}\chi\left(\mathrm{x}_2\right)\right] \tag{146}$$

$$+\,{}^{+}F\left(x_1 x_2\right) = {}^{+}\mathscr{F}\left(x_1 x_2\right) \exp\left[-\frac{1}{2}\mathrm{i}\chi\left(x_1\right) - \frac{1}{2}\mathrm{i}\chi\left(x_2\right)\right] \tag{147}$$

$${}^{+}\hat{\Delta}'\left(x_1 x_2\right) = {}^{+}\Delta\left(x_1 x_2\right) \exp\left[-\frac{1}{2}\mathrm{i}\chi\left(x_1\right) - \frac{1}{2}\mathrm{i}\chi\left(x_2\right)\right] \tag{148}$$

The functions $\hat{\mathscr{G}}$ and $\hat{\mathscr{F}}$ are slowly varying functions of the central coordinate $x = \left(x_1 + x_2\right) 2$ and rapidly varying functions of the difference coordinate $X = x_1 - x_2$, where X represents the set of spatial coordinates ρ and time τ. In accordance with the semiclassical approximation, we neglect the higher-order derivatives of the phase $\chi(x)$.

Performing transformation to the Fourier components in terms of the difference variable X we get from eqs. (143) and (144) the following equation for $\hat{g}(p, x)$:

$$\begin{aligned} &\hat{\sigma}_z\left[\mathscr{G}_0^{-1}\left(p + p_s, \varepsilon - \phi\right) + \frac{1}{2}\mathrm{i}\hbar B\left(p + p_s\right)\right]\hat{\mathscr{G}}(p, x) \\ &\quad = \hat{1} + \mathrm{i}\hat{\Sigma}(p)\hat{\mathscr{G}}(p, x) + \mathrm{i}\hat{\Delta}(p)^{+}\mathscr{F}(p, x) - \frac{1}{2}\{\hat{\Sigma}, \hat{\mathscr{G}}\} - \frac{1}{2}\left\{\hat{\Delta}, {}^{+}\hat{\mathscr{F}}\right\} \end{aligned} \tag{149}$$

Here

$$\{A, B\} = \frac{\hat{i}A\hat{i}B}{\hat{i}x\hat{i}p} - \frac{\hat{i}A\hat{i}B}{\hat{i}p},$$

p denotes the set of variables p and ε,

$$g_0^{-1}(\boldsymbol{p}, \varepsilon) = \varepsilon - \mu - \boldsymbol{p}^2/2m \tag{150}$$

and

$$B\left(p + p_s, x\right) = \frac{\partial}{\partial t} - \frac{\partial\phi}{\partial r}\frac{\partial}{\partial p} + \frac{p + p_s}{m}\frac{\partial}{\partial r} - \frac{\partial}{\partial r}\frac{\left(p + p_s\right)^2}{2m}\frac{\partial}{\partial p}. \tag{151}$$

The corresponding equation for ${}^{+}\mathscr{F}(p, x)$ differs from eq. (149) by the replacement $\mathscr{G} \to {}^{+}\mathscr{F}, p_{\mathrm{s}} \to -p_{\mathrm{s}}, \varepsilon \to -\varepsilon, {}^{+}\hat{\Delta} \to \hat{\Delta}, \hat{\Sigma}(p) \to \hat{\Sigma}^{\mathrm{T}}(-p)$. Note that these equations are gauge-covariant because external fields enter them through the gauge-covariant combinations p_{s} and ϕ only.

In the course of derivation of eq. (149) and the corresponding equation for ${}^{+}\mathscr{F}$ we have used inequalities (1) and (2).

To find the functions $\mathscr{G}_{\mathrm{c}}$ and $\mathscr{F}_{\mathrm{c}}$, i.e. the " 11 " components of the matrices $\mathscr{G}$ and $\mathscr{F}$, it is sufficient to retain in the corresponding equations only those terms which do not themselves contain derivatives with respect to x, and we can discard the terms of the type $\Sigma_{+}\mathscr{G}_{-}$which are proportional to the small parameter $1/4\tau$. Assuming that the electron energy spectrum of the normal metal has already been renormalized by the normal self-energy part Σ_c, we have

$$\mathscr{G}_c(p,x) = u_p^2 A(p) - v_p^2 A(-p) \tag{152}$$

$$+\mathscr{F}_c(p,x) = -iu_p v_p[A(p) + A(-p)] \tag{153}$$

where

$$A(p) = \frac{n_p}{\varepsilon - \tilde{\varepsilon}_p - \mathrm{i}\delta} + \frac{1 - n_p}{\varepsilon - \tilde{\varepsilon}_p + \mathrm{i}\delta}$$

and n_p is a function of the quasimomentum p. We shall see that it has the physical meaning of the quasiparticle distribution function.

The order parameter Δ is represented by the " 11 " matrix element of the anomalous self-energy matrix. One can find the following equation for the element

$$^{+}\Delta(\varepsilon,p,x) = \int (\mathrm{d}p)_{p'} \int \frac{\mathrm{d}\varepsilon'}{2\pi} |c_q|^2 D_c(p'-p,-\varepsilon') + \mathscr{F}_c(\varepsilon-\varepsilon',p',x) \tag{154}$$

where D_c is the phonon causal Green function (in order to obtain this equation one should set the electron-phonon vertex equal to the simple vertex $\Gamma^k_{0jj'} = c_q \delta_{p'-p-hq} (\hat{\sigma}_z)_{j'k}$. One can see that $^{+}\Delta_c(\varepsilon, p_{\mathrm{F}})$ has dispersion, i.e. is essentially dependent on ε at ε of the order of $\hbar\omega_{\mathrm{D}}$. For the usual transport phenomena the energies of the order of T or 1 are important, which are much smaller than the Debye energy $\hbar(\omega_1)$. Taking this into account we may assume that $\Lambda(\varepsilon,p,x)$ is independent of the energy in the energy interval we are interested in. Thus, we can assume $^{+}\Delta_c(\varepsilon,p) \equiv \Delta$ Making use of eqs. (152) and (153) and the criteria (1) and (2), as well as of the spectral representation

$$\mathscr{G}_{\mathrm{c}} = \int^{x} \frac{\mathrm{d}\varepsilon'}{2\pi\mathrm{i}} \left[\frac{\mathscr{G}_{+}}{\varepsilon - \varepsilon' - \mathrm{i}\delta} - \frac{\mathscr{G}_{-}}{\delta - \varepsilon' + \mathrm{i}\delta} \right] \tag{155}$$

and the analogous expression for $^{+}\overline{\mathscr{F}}$, we get the following expressions for "!, and $\mathscr{F}_{+}$

$$\mathscr{G}_{+} = 2\pi\mathrm{i}\left[u_p^2 n_p \delta(\varepsilon - \tilde{\varepsilon}_p) + v_p^2 (1 - n_{-p}) \delta(\tilde{\varepsilon}_{-p} + \varepsilon)\right], \tag{156}$$

$$+\mathscr{F}_{+} = \mathscr{F}_{+} = -2\pi u_p v_p \left[(1 - n_{-p}) \delta(\varepsilon + \tilde{\varepsilon}_{-p}) - n_p \delta(\varepsilon - \tilde{\varepsilon}_p)\right]. \tag{157}$$

The expression for $\mathscr{G}_?$ differs from eq. (156) by the replacement $n_{+.p} \leftrightarrows -n_{\pm p} + 1$ and by the sign; $\mathscr{F}_{-}(p,x) =^{+} \mathscr{F}_{+}(-p,x)$.

$$n_p(x) = \int_0^{\infty} \frac{\mathrm{d}\varepsilon}{2\pi\mathrm{i}} \left[\mathscr{G}_{+}(\varepsilon,p) - \mathscr{G}_{-}(\varepsilon,-p)\right]. \tag{158}$$

The Green functions $\mathscr{G}_{+}$and $\mathscr{G}_{-}$being imaginary, one can find the equation for $n_p(x)$ by adding the Dyson equations for $\mathscr{G}_{+}$an $\mathscr{G}_{-}$to the corresponding Hermitian-conjugate equations. Besides, one can neglect in the rhs of eq. (149) and of the corresponding equation for $\mathscr{G}_{-}$:
(1) the terms containing the space and time derivatives of the normal Green function $\mathscr{G}$, as well as of the normal selfenergies Σ,
(2) the terms proportional to the derivatives of $\Delta_{\pm}$and $^{+}\Delta_{+}$(these are proportional to the small parameter d/l).

The terms proportional to $\partial\Delta/\partial p$ can be also discarded. Thus, the only term with derivatives that should be taken into account on the rhs of the Dyson equation for $\mathscr{G}_+$ is

$$\frac{\hbar\hat{c}A_c\hat{c}+\mathscr{F}}{\partial p}.$$

As a result, the Dyson equation for $\mathscr{G}_+$ acquires the following form

$$\left[\mathscr{G}_0^{-1}\left(p+p_{\mathrm{s}},\varepsilon-\phi\right)+\frac{\mathrm{i}\hbar B\left(p+p_{\mathrm{s}}\right)}{2}\right]\mathscr{G}_+(p,x)+\frac{\hbar}{2}\frac{\partial\Delta_{\mathrm{c}}}{\partial r}\frac{\partial+\mathscr{F}_+}{\partial p}$$
$$=\mathrm{i}\left(\sum_{\mathrm{c}}\mathscr{G}_++\sum_{+}\hat{\mathscr{G}}_{\mathrm{c}}+\Delta_{\mathrm{c}}+\mathscr{F}_++\Delta_++\mathscr{F}_{\mathrm{c}}\right). \tag{159}$$

Taking into account the identities

$$\hat{\mathscr{H}}^+=\hat{\sigma}_1\hat{\mathscr{H}}_1,\quad \hat{j}^+=-\hat{\sigma}_1{}^+\hat{A}_0\hat{\sigma}_1\quad \hat{\sigma}_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}.$$

one can show that the Hermitian-conjugate equation for " 4 , differs from cq. (159) by the replacement $\mathscr{G}_0^{-1}\to-\mathscr{G}_0^{-1}$ in the lhs and $\Sigma\rightleftarrows\mathscr{G},\Delta\rightleftarrows\mathscr{F}$ in the rhs. Combining this equation with eq. (159) and taking into account the Keldysh identity (Keldysh 1964)

$$\Sigma_{\mathrm{c}}+\widetilde{\Sigma}_{\mathrm{c}}+\Sigma_++\Sigma=0$$

and the analogous identity for $\triangle$ one can find

$$\hbar B\left(p+p_{\mathrm{s}}\right)\mathscr{G}_++\mathrm{i}\hbar\frac{\partial\Delta_{\mathrm{c}}}{\partial r}\frac{\partial^+\mathscr{F}^+}{\partial p}=\Sigma_+\mathscr{G}_--\Sigma_-\mathscr{G}_++\Delta_{\mathrm{c}}^+\mathscr{F}_++{}^+\tilde{\Delta}_{\mathrm{c}}\mathscr{F}_+$$
$$+\Delta_+{}^+\mathscr{F}_{\mathrm{c}}+{}^+\Delta_+\mathscr{F}_{\mathrm{c}}. \tag{160}$$

Similarly, we have for $\mathscr{G}_-$

$$-\hbar B\left(p+p_{\mathrm{s}}\right)\mathscr{G}_-+\mathrm{i}\hbar\frac{\partial\Delta_{\mathrm{c}}}{\partial r}\frac{\partial^+\mathscr{F}}{\partial p}=\Sigma_-\mathscr{G}_+$$
$$-\Sigma_+\mathscr{G}_-+\tilde{\Delta}_{\mathrm{c}}{}^+\mathscr{F}_-+{}^+\Delta_{\mathrm{c}}\mathscr{F}_-+\Delta_-{}^+\mathscr{F}_{\mathrm{c}}+{}^+\Delta_-\mathscr{F}_{\mathrm{c}}. \tag{161}$$

Replacing p by $-p$ in eq. (161) and taking into account the identities

$$\Delta_{\mathrm{c}}+\tilde{\Delta}_{\mathrm{c}}+\Delta_++\Delta_-=0,\quad \Delta_+(-p)=\Delta_+(p),\quad \mathscr{F}_\pm(p)=\mathscr{F}_+(-p) \tag{162}$$

and analogous identities for ${}^+\Delta$ and ${}^{+\mathscr{F}}$ (which can be derived from the Dyson equations and the definitions of $\mathscr{F}$ and Δ) we obtain as the sum of eqs. (160) and (161)

$$\hbar\left[B\left(p+p_s\right)\mathscr{G}_+(p,x)-B\left(-p+p_v\right)\mathscr{G}_{(-p,x)}+2\mathrm{i}\frac{\partial\Delta_c}{\partial r}\frac{\partial^+\mathscr{F}_+}{\partial p}\right]$$
$$=\left(\Sigma_+\mathscr{G}_--\Sigma_-\mathscr{G}_+\right)_p+\left(\Sigma_-\mathscr{G}_+-\Sigma_+\mathscr{G}\right)_{-p}$$
$$+\left(\Delta_+{}^+\mathscr{F}_--\Delta_-{}^+\mathscr{F}_++{}^+\Delta_+\mathscr{F}_--{}^+\Delta_-\mathscr{F}_+\right)_p. \tag{163}$$

Substituting the explicit expressions for the Green functions into eqs. (163) we find that the function of the momentum, n_p. obeys the Boltzmann equation (6) and thus has the physical meaning of the distribution function of the quasiparticles.

We have mentioned above that the criteria for the applicability of this theory turn out to be the most stringent near T_c. These criteria appear in the course of analysis of the equations for the anomalous Green functions and the self-consistency condition (154). This condition was derived in the zeroth approximation in the function ${}^{+}\mathscr{F}_c(p)$ [and ${}^{+}\overline{\mathscr{F}}_{\pm}(p)$]. Inserting eqs. (152) and (153) into eq. (154), we can transform the latter to the form of eq. (15).

To determine the range of its applicability let us make estimates of the discarded terms. For this purpose, one should write the equation for ${}^{+}\mathscr{F}_{+}(p)$ taking into account terms proportional to the time derivative and to the derivatives with respect to the central coordinates, as well as terms containing the collision operator. As a result, an additional term appears in the equation for the gap. This term is of the order of

$$\left(\bar{\Xi}_p\right)^{-1}(\hat{c}\Delta/\partial t),$$

where $\widetilde{\xi}_p$ is the characteristic value of $\widetilde{\xi}_p$ for the problem under consideration.

If, for instance, the state is shifted not too far from equilibrium we have $\widetilde{\zeta}_p \simeq T_c$. The criterion of applicability of the equation for the gap is found by comparison of the terms of the type $\left(\hbar/\Delta\tilde{\xi}_p\right)(\partial\Delta/\partial t)$ with the lhs of the equation

$$1 + (\lambda_e/2)\int(\mathrm{d}p)_p\frac{1-2n_p}{\varepsilon_p} = 0$$

The latter is of the order of Δ^2/T_c^2 near equilibrium. Accordingly, for a state not far from equilibrium the neglect of the terms with time derivatives may be justified only if

$$\hbar\omega/\xi_p \simeq \hbar\omega/T_c \ll \Delta^2/T_c^2.$$

This condition was found by Gorkov and Eliashberg (1968, 1969).

25.10 10. Conclusion

We have discussed the transport-equation approach to the description of transport phenomena in superconductors and demonstrated its applications to various problems. One can see that this approach is by many aspects analogous to the classical transport approach successfully employed in normal conductors. Indeed, the transport equation is similar to the corresponding Boltzmann equation in the normal state and the expressions for the transport coefficients are similar as well (see sections 6, 7). So we have formulated a scheme for description of the response of a superconductor to slowly varying (in time and space) external perturbations which is very close to the classical one. Nevertheless, the full set of equations for a superconductor is rather complicated and one should not overestimate the similarity of the two schemes. Indeed, there is the Cooper pair condensate in a superconductor, which is a dynamical system interacting strongly with the normal excitations.

New degrees of freedom call for additional variables: as we have seen, the full set of equations contains extra variables and extra equations as compared to the normal conductors. The condensate motion, as well as the normal excitation distribution, influences the energy gap. Thus, the full set of equattions should be treated self-consistently, which is in general a rather complicated problem. However, even in this case the transport-equation approach seems to be much less involved than more general schemes.

There is another advantage of this approach - it is comparatively easy to interpret its results qualitatively because they are expressed in terms of a quantity with clear physical meaning, i.e. the excitation distribution function.

The most important question concerns the range of applicability of the approach. The corresponding conditions have been formulated in section 2. It should be mentioned, however,

that these conditions are dependent on the character of the problem in question, i.e. on the characteristic quantum time
and space parameters of a system. So it is difficult to give universal criteria for the applicability of the transport-equation approach. One should estimate the characteristic quantum parameters of the system under consideration before applying this approach and one should take care not to break the inequalities of section 2. We have tried to demonstrate this procedure in section 6 (logarithmic term in conductivity). Sometimes, one can see from the final result that the limits of applicability of the transport equation are violated, for instance, the corresponding integrals may diverge. But even in such a case one can usually guess which quantum quantity cuts the integral off. That means that order-of-magnitude estimations can be obtained even in this case: the transport-equation approach has a heuristic value.

We believe that in the future the approach discussed in this review article may prove to be useful for the analysis of many problems connected with transport phenomena in superconductors.

26 Kinetic Equations in Superconducting Thin Films by Chang

26.1 1. Introduction

Within the frame work of the BCS theory (Bardeen et al. 1957), a superconductor is a system of three components: the quasiparticles, the phonons, and the paired electrons. Because of the strong interactions among the three components, disturbing any one component also drives the other two out of equilibrium. In order to understand the nonequilibrium phenomena to be discussed in this chapter, we need a means of describing the response of the three components to an external driving source. We have adopted the kineticequation approach. The excitation spectra of the quasiparticles and the phonons are described by a set of coupled kinetic equations (Bardeen et al. 1959, Ambegaokar and Tewordt 1964, Prange and Kadenoff 1964, Chang 1981). Following the treatment of Eliashberg (1970), the paired electrons are characterized by an energy gap using a modified BCS gap equation that relates the energy gap to the energy spectra of the quasiparticles in a nonequilibrium state.

In general, a superconducting state under driven condition depends upon not only the relaxation times of the sample and the strength of the driving source but also on the specific nature of the driving mechanism. For a given power input, various driving mechanisms excite quasiparticles and phonons differently, i.e., they produce different excitation spectra. The different signatures of the driving sources, as we will see, can be responsible for many interesting phenomena.

In the next section, we will introduce the coupled kinctic equations for the quasiparticle and phonon distributions. We begin by discussing the injection rates, i.e., the excitation rates of quasiparticles and phonons due to four kinds of external sources: quasiparticle injection through a tunnel junction, microwave radiation, light radiation, and phonon radiation. We will show that due to the singularity in the quasiparticle density of states at the gap edge, sharp structures in the quasiparticle and phonon spectra can occur. We also discuss the various relaxation processes and calculate the corresponding collision integrals.

How far away from equilibrium a system can be driven is determined by the relaxation time of the system and the strength of the source. However, in general, the relaxation time depends upon the specific nature of the non-
equilibrium state. Only when a system is driven into a state close to the original equilibrium state, can the relaxation times be expressed in terms of macroscopic dynamic variables and

hence be of interest for consideration. In section 3, we will calculate the relaxation times of quasiparticles and phonons associated with various relaxation processes. These relaxation times are microscopic in nature and depend upon the ambient temperature and the energy of the quasiparticle or phonon states. We will show that the macroscopic relaxation times measured by various techniques are different weighted averages of these microscopic times. We also describe a technique which can measure the microscopic quasiparticle lifetimes. Superconductors driven far from equilibrium are considered in section 4 . We will consider three interesting phenomena. First, the generation of narrow-band phonons. We show that nearly monochromatic phonons of frequency $v \sim 2\Delta/h$ can be generated from a superconductor-insulator-superconductor tunnel junction. Since the energy gap is typically of order 1 meV , the phonon frequency is of order 500 GHz . Next, we consider the enhancement of superconductivity by microwave irradiation and quasiparticle tunnel injection. Finally, we will discuss two simple analytic models: the μ^* and T^* models which have been widely used to describe a superconductor with poor thermal coupling with the ambience. Stability of the driven homogeneous steady state will be discussed briefly using these models.

26.2 2. Basic equations

We will be interested in nonequilibrium states driven in a uniform and steady state manner. The kinetic equations which govern the quasiparticle and phonon distributions, $f(E_k)$ and $n(\Omega_q)$, have the following forms:

$$\frac{\mathrm{d}f(E_k)}{\mathrm{d}t} = I_{\mathrm{qp}}(E_k) + \left.\frac{\partial f(E_k)}{\partial t}\right|_{\mathrm{collision}} \tag{2.1a}$$

and

$$\frac{\mathrm{d}n(\Omega_q)}{\mathrm{d}t} = I_{\mathrm{ph}}(\Omega_q) + \left.\frac{\partial n(\Omega_q)}{\partial t}\right|_{\mathrm{collision}} \tag{2.1b}$$

Here $I_{\text{чр}}$ (E_{h}) and I_{ph} (Ω_{q}) are respectively the energy spectra of the quasiparticle and phonon injection rates. The contributions to the rate of change of the quasiparticle and phonon distributions due to the various interaction mechanisms, are represented by $\left(\hat{\partial} f(E_h), \hat{\partial} t\right)_{\mathrm{coll}}$ and $\left(\hat{\partial} n(\Omega_q)/\hat{c}t\right)_{\mathrm{coll}}$. We calculate each term in eq. (2.1) by Fermi's Golden Rule. This method is the simplest and brings out the physics clearly. Furthermore, it has been shown (Eliashberg 1972, Entin-Wohlman and Orbach 1979a) that as long as the energy gap is well formed, the result of this method is the same as that obtained from a more rigorous Green function approach.

26.2.1 2.1. Injection rates

Here we first discuss the injection rates due to the various driving mechanisms. Collision rates are discussed in section 2.2.

26.2.2 2.1.1. Quasiparticle tunneling injection

Consider a tunnel junction formed by two superconducting thin strips separated by an insulating layer. The tunneling of quasiparticles can be described by the Hamiltonian (Bardeen 1961, 1962, Cohen et al. 1962)

$$H_{\mathrm{T}} = \sum_{k\cdot q,\sigma} T_{kq} C^{+}_{k\sigma} C_{q\sigma} + \text{ Hermitian conjugates.} \tag{2.2}$$

Here we use $k(q)$ for the wavevector of a quasiparticle state in the left (right) film of the junction, and $C^{+}_{k\sigma}\,(C_{k\sigma})$ is the creation (destruction) operator for an electron of wavevector k and spin σ. In terms of the quasiparticle operators γ and γ^{+}(see e.g. Schrieffer 1964):

$$\gamma^{+}_{k\uparrow} = u_k C^{+}_{k\uparrow} - v_k S^{+} C_{-k\downarrow}, \tag{2.3a}$$

$$\gamma_{k\downarrow} = u_k C_{-k\downarrow} + v_k S C^{+}_{k\uparrow}, \tag{2.3b}$$

the tunneling Hamiltonian becomes

$$H_{\mathrm{T}} = \sum_{k,q} T_{kq} \left(u_k \gamma^{+}_{k*} + v_k \gamma_{-k_\downarrow} S^{+}\right) \left(u_q \gamma_{q\uparrow} + v_q S \gamma^{+}_{q\downarrow}\right) + \text{Hermitian conjugates.} \tag{2.4}$$

Here $S^{+}(S)$ is the pair creation (destruction) operator and u and v are the wellknown coherence factors. The four terms in eq. (2.4) give four distinct tunneling processes. It can be easily shown that the Hermitian conjugate represents the reverse processes of these four direct processes. The four processes are shown in fig. 1. Their contribution to the quasiparticle injection can be calculated using the Golden-Rule approximation. We have for the right-hand film, the injection rates due to the four processes:

$$\begin{aligned}
W^{\mathrm{a}}_{q} &= \frac{2\pi}{\hbar\Omega_{\mathrm{R}}} \sum_{k} \left|T_{kq} u_k u_q\right|^2 (f_k - f_q)\, \delta\left(E_q - E_k - eV\right), \\
W^{\mathrm{b}}_{q} &= \frac{2\pi}{\hbar\Omega_{\mathrm{R}}} \sum_{k} \left|T_{kq} v_k v_q\right|^2 (f_k - f_q)\, \delta\left(E_q - E_k + eV\right), \\
W^{\mathrm{c}}_{q} &= \frac{2\pi}{\hbar\Omega_{\mathrm{R}}} \sum_{k} \left|T_{kq} v_k u_q\right|^2 (1 - f_k - f_q)\, \delta\left(E_q + E_k - eV\right), \\
W^{\mathrm{d}}_{q} &= \frac{2\pi}{\hbar\Omega_{\mathrm{R}}} \sum_{k} \left|T_{kq} u_k v_q\right|^2 (1 - f_k - f_q)\, \delta\left(E_q + E_k + eV\right).
\end{aligned} \tag{2.5}$$

Here, the junction voltage is defined to be $V = V_q - V_k$, and Ω_{R} is the volume of the right-hand film under injection.

We first notice that the injection rates depend upon the non-equilibrium quasiparticle distribution. They also depend upon the coherence factors:

$$\mathrm{u}^2_k = \frac{1}{2}\left(1 + \xi_k/E_k\right) \tag{2.6a}$$

and

$$v^2_k = \frac{1}{2}\left(1 - \xi_k/E_k\right), \tag{2.6b}$$

with $\check{\zeta}_h$ being the electron energy measured from the chemical potential and $E_k = \left(\check{\xi}^2_k + A^2\right)^{1/2}$. Since the coherence factors are not symmetric with respect to the chemical potential, the injection rates for quasiparticle states of the same energy E but residing on the two different branches of $\xi_h > 0$ and $\xi_h < 0$ will be different. Therefore the steady-state quasiparticle distribution function will not only depend upon the energy of the state but also upon which branch the state is on. We use $f_{>}\left(E_h\right)$ and $f_{<}\left(E_h\right)$ to represent the distribution of quasiparticles on the $\xi_k > 0$ and $\xi_k < 0$ branches, respectively.

Fig. 1. Four quasiparticle tunneling channels in a superconductor-insulator junction, illustrated by their $T_{q k
$ terms, i.e., those which transfer an electron from k to q. (From Kirtley et al. 1980.)}

To put the injection rates into a more practical form, we carry out the sum over the wavevector k and eliminate T_{kq} in favor of the normal state conductance for the junction G_{nn}. We obtain

$$W^{\mathrm{a}}_{\gtrless}(E_{\mathrm{R}}) = \frac{G_{\mathrm{nn}}}{2e^2 N_{\mathrm{R}}(0)\Omega_{\mathrm{R}}}\rho_{\mathrm{L}}(E_{\mathrm{R}} - eV)\left\{u^2_{\mathrm{L}>}(E_{\mathrm{R}} - eV)u^2_{\mathrm{R}\gtrless}(E_{\mathrm{R}})\left[f_{\mathrm{L}>}(E_{\mathrm{R}} - eV)\right.\right.$$
$$\left.\left. - f_{\mathrm{R}\gtrless}(E_{\mathrm{R}})\right] + u^2_{\mathrm{L}<}(E_{\mathrm{R}} - eV)\, u^2_{\mathrm{R}\gtrless}(E_{\mathrm{R}})\left[f_{\mathrm{L}<}(E_{\mathrm{R}} - eV) - f_{\mathrm{R}\gtrless}(E_{\mathrm{R}})\right]\right\}$$

$W^{\mathrm{b}}_{\gtrless}(E_{\mathrm{R}}) = \mathrm{W}^{\mathrm{a}}_{\gtrless}(E_{\mathrm{R}})$ with the replacements of V by $(-V)$ and u by v.

$$W^{\mathrm{c}}_{\gtrless}(E_{\mathrm{R}}) = \frac{G_{\mathrm{nn}}}{2e^2 N_{\mathrm{R}}(0)\Omega_{\mathrm{R}}}\rho_{\mathrm{L}}(-E_{\mathrm{R}} + eV)\left\{v^2_{\mathrm{L}} > (-E_{\mathrm{R}} + eV)\, u^2_{\mathrm{R}\gtrless}(E_{\mathrm{R}})\right.$$
$$\left[1 - f_{\mathrm{L}} > (-E_{\mathrm{R}} + eV) - f_{\mathrm{R}\gtrless}(E_{\mathrm{R}})\right] + v^2_{\mathrm{L}<}(-E_{\mathrm{R}} + eV)\, u^2_{\mathrm{R}\gtrless}(E_{\mathrm{R}})$$
$$\left.\left[1 - f_{\mathrm{L}<}(E_{\mathrm{R}} + eV) - f_{\mathrm{R}\gtrless}(E_{\mathrm{R}})\right]\right\}$$

$W^{\mathrm{d}}_{\gtrless}(E_{\mathrm{R}}) = W^{\mathrm{c}}_{\gtrless}(E_{\mathrm{R}})$ with replacements of V by $(\ -V\ )$ and interchanged roles of u and v.

Here $G_{\mathrm{nn}} = (4\pi e^2/\hbar)\, T^2 N_{\mathrm{L}}(0) N_{\mathrm{R}}(0)\Omega_{\mathrm{L}}\Omega_{\mathrm{R}}, T$ is the effective tunneling matrix element, Ω is the volume under injection, and $\rho_{\mathrm{L}}(E_{\mathrm{L}}) \equiv \theta(E_{\mathrm{L}} - \Delta_{\mathrm{L}})\,|E_{\mathrm{L}}|\,/\,(E_{\mathrm{L}} - A_{\mathrm{L}})^{1/2}$ is the normalized BCS density of states function for the quasiparticles in the left film. The total quasiparticle injection rate into the right film for the two $\xi_k \gtrless 0$ branches are

$$I_{\mathrm{qp}} \gtrless (E_{\mathrm{R}}) = W^{\mathrm{a}}_{\gtrless}(E_{\mathrm{R}}) + W^{\mathrm{b}}_{\gtrless}(E_{\mathrm{R}}) + W^{\mathrm{c}}_{\gtrless}(E_{\mathrm{R}}) + W^{\mathrm{d}}_{\gtrless}(E_{\mathrm{R}})\,. \tag{2.8a}$$

When the injection rate is not too strong, we can approximate the distribution function in the right-hand side by the equilibrium distribution, i.e. $f_>(E) \sim f_<(E) \sim f_0(E,T)$ with $f_0(E,T) = 1/\left(\exp\left[(E-\mu)/k_{\mathrm{B}}T\right] + 1\right)$. The injection rates for a film in a symmetric junction of $A_{\mathrm{L}} = A_{\mathrm{R}} = A$ and biased at voltages of $V = Ae$ and $3A/e$ are shown in fig. 2 . The quantitative difference in the excitation rates for the $\xi_k > 0$ branch ($I_{\mathrm{qp}} >$), and the $\xi_k < 0$ branch ($I_{\mathrm{qp}} <$), is apparent. For negative voltages, the solid and dotted curves interchange

roles due to the electron-hole symmetry. Furthermore, the injection spectrum due to tunneling from a normal metal can be obtained from eqs. (2.7, 8a) with $\Lambda_{\mathrm{L}} = 0$.

Finally, since, no phonons are injected into the films,

$$I_{\mathrm{ph}}\left(\Omega_q\right) = 0 \tag{2.8b}$$

26.2.3 2.1.2. Microwave irradiation

We consider a film irradiated by microwaves of frequency (1). If the electromagnetic field is described by the vector potential

$$A(r,t) = A_\omega e^{\mathrm{i}(q\cdot r-\omega t)} + \text{ c.c.} \tag{2.9}$$

Fig. 2. The energy spectra of the quasiparticle injection rate in the film of a symmetric superconductor-insulator superconductor tunnel junction. (a) Ambient temperature $T_{a =0.9 T_{c}$ and bias voltage $V = \Delta(T)/e$. (b) $T_{\mathrm{a}} = 0.5T_{\mathrm{c}}$ and $V = 3\Delta(T)/e$.}

The coupling Hamiltonian is of the form (Mattis and Bardeen 1958)

$$\begin{aligned}
H(t) =& -\frac{1}{c}\int j(r)\cdot A(r,t)\mathrm{d}^3r \\
=& \frac{e}{2mc}\sum_{k,\sigma} A_\omega\cdot(2k+q)C^+_{k+q,\sigma}C_{k,\sigma}\mathrm{e}^{-\mathrm{i}\omega t} + \text{ Hermitian conjugate} \\
=& \sum_k \frac{eA_\omega\cdot(2k+q)\mathrm{e}^{-\mathrm{i}\omega t}}{2mc}\left\{L(k+q,k)\left[\gamma^+_{(k+q)\uparrow}\gamma_{k\uparrow} - \gamma^+_{-k\downarrow}\gamma_{-(k+q)\downarrow}\right]\right. \\
& \left. -P(k+q,k)\left[\gamma^+_{(k+q)\uparrow}\gamma^+_{-k\downarrow}S + \gamma_{k\downarrow}\gamma_{-(k+q)\downarrow}S^+\right]\right\} \\
& + \text{Hermitian conjugate,}
\end{aligned} \tag{2.10}$$

where the coherence factors L and P are given by

$$L\left(p',p\right) = u_{p'}u_p + v_{p'}v_p \tag{2.11a}$$

and

$$P\left(p',p\right) = u_{p'}v_p - u_p v_{p'}. \tag{2.11b}$$

Therefore the interaction Hamiltonian provides two processes for exciting the quasiparticles. These are shown in fig. 3. The injection rates due to the two processes can again be calculated from the Golden Rule. After eliminating the matrix element in favor of the more practical quantity R_{N}, the normal-state surface resistance, we obtain

Fig. 3. Basic processes of the electron-microwave interaction. The solid lines represent electrons and the wavy lines represent photons. (a) Scattering process; (b) recombination process or pairbreaking process. The same diagrams can also describe the electron-phonon interaction if the wavy lines are considered representing phonons. (From Chang and Scalapino 1978.)

$$
\begin{aligned}
&I_{\mathrm{qp}}(E) = \tfrac{2R_{\mathrm{N}}H^2}{\mathrm{dN}(0)\omega^2} \\
&\times \Big\{\rho(E-\omega)\left(1+\tfrac{A^2}{E(E-\omega)}\right)[f(E-\omega)(1-f(E))-f(E)(1-f(E-\omega))]\theta(E-\omega-\Delta) \\
&+\rho(E+\omega)\left(1+\tfrac{\Delta^2}{E(E+\omega)}\right)[f(E+\omega)(1-f(E))-f(E)(1-f(E+\omega))]\theta(E+\omega-\Delta) \\
&\qquad +\rho(\omega-E)\left(1-\frac{\Delta^2}{E(\omega-E)}\right)[(1-f(E))(1-f(\omega-E)) \qquad (2.12a) \\
&-f(E)f(\omega-E)]\theta(E-\Delta)\theta(\omega-E-\Delta)\}.
\end{aligned}
$$

Here H is the RF magnetic field. The first two terms in the curly brackets are due to process (a). The last term is contributed by process (b) and is nonzero only when $\hbar\omega > 2\Delta$. We notice that in this case both branches of $\breve{\zeta}_h \gtrless 0$ are excited equally.

Since the coupling of the microwaves to the phonons is usually quite weak, we will assume for this case

$$I_{\mathrm{ph}}\left(\Omega_q\right) = 0. \qquad (2.12b)$$

26.2.4 2.1.3. Heat pumping and light irradiation

For a superconducting thin film attached to a heater at temperature T_{H}, the phonon injection current is given by (Little 1959)

$$I_{\mathrm{ph}}(\Omega) = \frac{\Gamma_s}{2d}\left[n\left(\Omega, T_{\mathrm{H}}\right) - n(\Omega)\right]. \qquad (2.13a)$$

Here Γ is an angularly averaged phonon transmission coefficient between the film and the heater, s is the averaged speed of sound in the film, and d is the film thickness. Since the phonon frequency of interest is of the order 2.1 h . which is much smaller than the Debye frequency v_{D}, Γ and s can be taken to
be constants independent of Ω. Obviously

$$I_{\mathrm{qp}}\left(E_k\right) = 0 \qquad (2.13b)$$

for the case of heat pumping.
As for the case of films under laser irradiation, it has been shown (Chang and Scalapino 1977c) that pseudoinjection terms can be used. They have the forms:

$$I_{\mathrm{qp}}\left(E_k\right) = 0 \qquad (2.14a)$$

and

$$I_{\mathrm{ph}}\left(\Omega_q\right)= \text{ constant }. \tag{2.14b}$$

26.2.5 2.2. Relaxations

We will consider three relaxation processes: (a) the electron-phonon interaction, which is inelastic, (b) the electron-impurity scattering, which is elastic, and (c) the "relaxation" of phonons by escape from the sample film. Usually, these are the most important relaxation mechanisms. Other processes such as electron-magnetic impurities scattering have been discussed by many authors (Schmid and Schön 1975, Entin-Wohlman and Orbach 1979b, Pethick and Smith 1980, Lemberger and Clark 1981), and will not be discussed here.

26.2.6 2.2.1. Electron-phonon collisions

The Hamiltonian for the electron-phonon interaction has the well-known form (see e.g. Schrieffer 1964)

$$H_{\mathrm{ep\ h}}=\sum_{pq\sigma i} g_{q\lambda}\mathrm{C}^{+}_{(p+q),\sigma}\mathrm{C}_{p\sigma}\left(a_{q\lambda}+a^{+}_{-q\lambda}\right) \tag{2.15}$$

In terms of the quasiparticle operators (eq. 2.3) it becomes

$$\begin{aligned} H_{\mathrm{e-ph}}=&\sum_{pq\lambda} g_{q\lambda}\left(a_{q\lambda}+a^{+}_{-q\lambda}\right)\Big\{N(p+q,p)\left[\gamma^{+}_{(p+q)\uparrow}\gamma_{p\uparrow}+\gamma^{+}_{-p\downarrow}\gamma_{-(p+q)\downarrow}\right]\\ &+M(p+q,p)\left[\gamma^{+}_{(p+q)\uparrow}\gamma^{+}_{-p\downarrow}S-\gamma_{p\uparrow}\gamma_{-(p+q)\downarrow}S^{+}\right]\Big\} \end{aligned} \tag{2.16}$$

where

$$N\left(p',p\right)\equiv u_{p'}u_p-\iota_{p'}\iota_p=\frac{1}{2}\left(1-\left(\Delta^2-\xi_p\xi_{p'}\right)/E_pE_{p'}\right) \tag{2.17a}$$

and

$$M\left(p',p\right)\equiv u_{p'}v_p+v_{p'}u_p=\frac{1}{2}\left(1+\left(\Delta^2-\xi_p\xi_{p'}\right)/E_pE_{p'}\right). \tag{2.17b}$$

Therefore the basic processes are the same as for the microwave case (fig. 3) although the matrix elements and the coherence factors involved are different. These processes can change both the quasiparticle and phonon distributions.

Their contribution to the rate of change of the distributions $\hat{c}f\left(E_p\right)/\hat{c}t$ and $\partial n\left(\Omega_q\right)/\partial t$, can be easily calculated using the Golden-Rule approximation. If we further assume strong elastic interactions between the quasiparticle and the impurities, the treatment of Eliashberg in dealing with strong superconductors can be adopted (Eliashberg 1960, Scalapino, 1969) to average out the angular dependence and to eliminate g_q in favor of $x^2F(\Omega)$. We obtain for the quasiparticle relaxation rate

$$\left.\frac{\partial f(\xi)}{\partial t}\right|_{\text{eph}} = \frac{2\pi}{\hbar}\int_{-\infty}^{\infty} \mathrm{d}\xi' \int_0^{\infty} \mathrm{d}\Omega \alpha^2 F(\Omega)$$
$$\times \left\{ \frac{1}{2}\left(1 - \frac{\Delta^2 - \xi\xi'}{EE'}\right)\left[f\left(\xi'\right)\left(1 - f(\xi)\right)(n(\Omega)+1) - \left(1 - f\left(\xi'\right)\right) f(\xi) n(\Omega)\right] \delta\left(\Omega + E - E'\right)\right.$$
$$- \frac{1}{2}\left(1 - \frac{\Delta^2 - \xi\xi'}{EE'}\right)\left[f(\xi)\left(1 - f\left(\xi'\right)\right)\left(n(\Omega)+1\right) - \left(1 - f(\xi)\right) f\left(\xi'\right) n(\Omega)\right] \delta\left(\Omega + E' - E\right)$$
$$\left. + \frac{1}{2}\left(1 + \frac{\Delta^2 - \xi\xi'}{EE'}\right)\left[\left(1 - f(\xi)\right)\left(1 - f\left(\xi'\right)\right) n(\Omega) - f(\xi) f\left(\xi'\right)\left(n(\Omega)+1\right)\right] \delta\left(\Omega - E - E'\right)\right\} \tag{2.18}$$

where the subscript "e-ph" stands for inelastic electron-phonon interactions. The first two terms are contributed by the scattering process (a) with the first (second) term due to the scattering of a quasiparticle from a state of energy $E = \left(\xi^2 + \Delta^2\right)^{1/2}$ to a state of $E' > E\,(E' < E)$ and vice versa. They are proportional to the effective-interaction matrix element squared, α^2; the effective phonon density of states, $F(\Omega)$; the coherence factor $N^2 = \frac{1}{2}\left[1 - \left(\Delta^2 - \xi\xi'\right)/EE'\right]$, and the various quasiparticle and phonon distribution functions. The last term is due to the recombination process and its reverse, the pairbreaking process. We notice that the coherence factor involved is $M^2 = \frac{1}{2}[1 + \left(\Delta^2 - \xi\xi'\right)/EE']$ which is different from the previous case. Furthermore, only phonons of energy $\Omega = E + E' \geqslant 24$ give a nonzero contribution to the last term.

Similarly, the contribution of the electron phonon interaction to the rate of change of the phonon distribution is

$$\left.\frac{\partial n(\Omega)}{\partial t}\right|_{\text{eph}} = \frac{2\pi}{\hbar}\int_{-}^{\prime} \mathrm{d}\xi \int^{\prime} \mathrm{d}\xi' \frac{2N(0)}{N}\alpha^2$$
$$\times \left\{ \frac{1}{2}\left(1 - \frac{\Delta^2 - \xi\xi'}{EE'}\right)\left[f(\xi)\left(1 - f\left(\xi'\right)\right) n(\Omega) - f\left(\xi'\right)\left(1 - f(\xi)\right)(n(\Omega)+1)\right] \delta\left(\Omega + E - E'\right)\right.$$
$$+ \frac{1}{4}\left(1 + \frac{\Delta^2 - \xi\xi'}{EE'}\right)\left[\left(1 - f(\xi)\right)\left(1 - f\left(\xi'\right)\right) n(\Omega)\right.$$
$$\left. - f(\xi) f\left(\xi'\right)\left(n(\Omega)+1\right)\right] \delta\left(\Omega - E - E'\right)\} \tag{2.19}$$

where N and $2N(0)$ are respectively the ion density and the electron density of states on the Fermi surface. The first term is due to the scattering process while the second is due to the pair-breaking process as can be clearly seen by noticing the different coherent factors and the different combinations of the distribution functions involved.

26.2.7 2.2.2. Electron-impurity interactions

We assume that this interaction is elastic and spin independent. It has been shown (Eliashberg 1970a) that the effect of a strong electron-impurity interaction is to average out the angular dependence of states with the same energy. However, for a superconductor with an isotropic gap, because the relevent coherence factor $N\left(p', p\right)$ for elastic scattering vanishes with $\xi_{\mathrm{p}} = -\xi_p'$, quasiparticles on the $\xi_p > 0$ branch cannot be scattered to the $\xi_p' < 0$ branch and vice versa. This means that the effect of elastic scatterings is to average out the population of states residing on the same branch only, and in general the populations of the two branches with $\check{\zeta}_p > 0$ and $\check{\zeta}_p < 0$ are different. Therefore, instead of considering $f(E)$ alone we have to

consider $f_>(E)$ and $f_<(E)$ separately when a population imbalance on the branches exists such as in the case of quasiparticle tunnel injections discussed in section 2.1.1.

For superconductors with an anisotropic energy gap, the coherence factor $N(p', p)$ associated with the elastic scattering is not zero. Therefore, elastic scattering will contribute to the reduction of the branch imbalance, which can be measured by the function $f_-(E) = f_>(E) - f_<(E)$. Tink ham (1972) calculated the contribution to the rate of change of the distribution function $f(E)$ and found a term of the form

$$\left.\frac{\partial f_-(E)}{\partial t}\right|_{\mathrm{e-i}} = -\frac{2f_-(E)}{\tau_{\mathrm{el}}(E)} \tag{2.20a}$$

with the energy-dependent elastic branch-mixing rate $\tau_{\mathrm{el}}^{-1}(E)$ given by

$$\tau_{\mathrm{el}}^{-1}(E) = \tau_{\mathrm{i}}^{-1}\frac{\Delta^2 E}{(E^2-\Delta^2)^{3/2}} \tag{2.20b}$$

where $\tau_i{}^1$ is an energy-independent time rate proportional to the mean squared deviation of the anisotropic gap as well as the impurity concentration, and A is the averaged gap energy in the equilibrium state.

In the presence of a branch imbalance, it is found more convenient to consider, instead of $f_>(E)$ and $f_<(E)$, the functions:

$$f_+(E) \equiv \frac{1}{2}\left[f_>(E) + f_<(E)\right] \tag{2.21a}$$

and

$$f_-(E) \equiv \left[f_>(E) - f_<(E)\right], \tag{2.21b}$$

which are respectively symmetric and antisymmetric with respect to the populations of the two branches. The rate of change of these functions due to electron phonon interactions can be easily obtained from eq. (2.18). We have

$$\begin{aligned}
&\partial f_+(E)|_{eph} = -\frac{2\pi}{h}\int_0^{\prime} \mathrm{d}E'\rho\left(E'\right) \\
&\partial\int_{x^2} F\left(E-E'\right)\Big[\theta(E-E')(1-\frac{\Delta^2}{EE'})\left(f_-(E)\left(1-f_-\left(E'\right)\right)\left(n\left(E-E'\right)\right)\right. \\
&-\left(1-f_+(E)\right)f_+\left(E'\right)n\left(E-E'\right)\Big) + \frac{|\xi\xi'|}{4EE'}f_-(E)f_-\left(E'\right)\Big] \\
&-\left[E\to E' \text{ and } E'\to E\right] \\
&+\alpha^2 F\left(|E+E'|\right)\Big[(1+\frac{\Delta^2}{EE'})\left(f_+(E)f_+\left(E'\right)\left(n\left(E+E'\right)+1\right)\right. \\
&\left.-\left(1-f_+(E)\right)\left(1-f_+\left(E'\right)\right)n\left(E+E'\right)\right) + \frac{|\xi\xi'|}{4EE'}f_-(E)f_-\left(E'\right)\Big]\Big\}
\end{aligned} \tag{2.22a}$$

and

$$\left.\frac{\partial f_{-}(E)}{\partial t}\right|_{e-\text{ ph}} = -\frac{2\pi}{\hbar}\int_0^\infty \mathrm{d}E'\rho\left(E'\right)f_{-}(E)\left\{\alpha^2F\left(E-E'\right)\left(1-\frac{\Delta^2}{EE'}\right)\right.$$
$$\times\left[\theta\left(E-E'\right)\left(1+n\left(E-E'\right)-f_{+}\left(E'\right)\right)+\theta\left(E'-E\right)\left(f_{+}\left(E'\right)+n\left(E'-E\right)\right)\right]$$
$$\left.+\alpha^2F\left(\left|E+E'\right|\right)\left(1+\frac{\Delta^2}{EE'}\right)\left(f_{+}\left(E'\right)+n\left(E+E'\right)\right)\right\}$$
$$+\frac{2\pi}{\hbar}\int_0^\infty \mathrm{d}E'\rho\left(E'\right)f_{-}\left(E'\right)\frac{|\xi\xi'|}{EE'}\left\{\alpha^2F\left(\left|E-E'\right|\right)\right.$$
$$\times\left[\theta\left(E-E'\right)\left(f_{+}(E)+n\left(E-E'\right)\right)+\theta\left(E'-E\right)\left(1+n\left(E'-E\right)-f_{+}(E)\right)\right]$$
$$\left.-\alpha^2F\left(\left|E+E'\right|\right)\left(f_{+}(E)+n\left(E+E'\right)\right)\right\} \tag{2.22b}$$

26.2.8 2.2.3. Phonon escape

Through this mechanism the sample gets rid of the energy injected by the external source. We use a phenomenological term

$$\left.\frac{\hat{c}n(\Omega)}{\hat{c}t}\right|_{\text{cs}} = -\frac{n(\Omega)-n^{-}\left(\Omega,T_{\text{a}}\right)}{\tau_{\text{c}}} \tag{2.23a}$$

to simulate the escape of excess phonons.
Here $n\left(\Omega.T_a\right)$ is the equilibrium phonon distribution at the ambient temperature. T_a. The phonon escape time. $\tau_{\ldots}$. depends strongly upon the genmetry and the acoustic match between the sample and the environment. In fact, the situation here is similar to the case of heat pumping discussed in section 2.1.3, and the phonon escape time is given by

$$\tau_{\text{es}} = 2d/\Gamma s. \tag{2.23b}$$

26.2.9 2.3. Summary on the kinetic equations

From the above discussion it is clear that a superconductor can be disturbed in many different ways. If the "characteristic frequency" associated with the external source (for tunnel junction of quasiparticles the characteristic frequency is eV/h) exceeds 24/h, pair-breaking can occur and the nonequilibrium state would have a quasiparticle density larger than that in equilibrium. However, if the characteristic frequency of the source is small, the primary effect of the external source is to redistribute the quasiparticles by exciting them into higher-energy states (e.g., microwave irradiation with $v < 2\Delta/h$) or to extract quasiparticles out of the film (e.g., tunneling in an asymmetric junction). Furthermore, quasiparticle injection through tunnel junction can create population imbalance on the two quasiparticle branches of $\zeta_k > 0$ and $\zeta_k < 0$.

As for the relaxation back to equilibrium, only the quasiparticle recombination process can reduce the quasiparticle density. When this process takes effect, phonons of energy $\Omega \geqslant 2A/e$ will be created. The effect of the elastic electron impurity scattering is to average out the angular dependence for states of the same energy. However, it is ineffective for the relaxation of the branch imbalance if the gap anisotropy is small. If this is the case, we have to consider the two branches of $\xi_k \gtrless 0$ separately.

Putting all the pieces together, we have three coupled kinetic equations for the distribution functions of the quasiparticles and phonons. Their detailed forms are

$$\frac{\mathrm{d}f_+(E)}{\mathrm{d}t} = I^+_{\mathrm{qp}}(E) + \left.\frac{\partial f_+(E)}{\partial t}\right|_{\mathrm{e-ph}}, \tag{2.24a}$$

$$\frac{\mathrm{d}f_-(E)}{\mathrm{d}t} = I^-_{\mathrm{qp}}(E) + \left.\frac{\partial f_-(E)}{\partial t}\right|_{\mathrm{e-ph}} + \left.\frac{\partial f_-(E)}{\partial t}\right|_{\mathrm{e-i}} \tag{2.24b}$$

$$\frac{\mathrm{d}n(\Omega)}{\mathrm{d}t} = I_{\mathrm{ph}}(\Omega) + \left.\frac{\partial n(\Omega)}{\partial t}\right|_{\mathrm{e-ph}} + \left.\frac{\partial n(\Omega)}{\partial t}\right|_{\mathrm{esc}} \tag{2.24c}$$

with

$$I^+_{\text{чр}}(E) = \frac{1}{2}\left\{I_{\text{чр}} > (E) + I_{\text{чр}} < (E)\right\}$$

and

$$\mathrm{I}_{\mathrm{qp}}(E) = I_{\mathrm{qp}} > (E) - I_{\mathrm{qp}} < (E).$$

The terms $\partial f_\pm(E)/\left.\partial t\right|_{e\mathrm{ph}}$ and $\partial n(\Omega)/\left.\partial t\right|_{e\mathrm{ph}}$ are given by eqs. (2.22) and (2.19).

26.2.10 2.4. Modified gap equation

To solve the kinetic equations, we need to know the energy gap. It has been shown that the energy gap in a nonequilibrium state is governed by the modified BCS gap equation (Eliashberg 1970b), which can be put into the form

$$\ln\left(\Delta/\Delta_0\right) = -\int_1^\infty \mathrm{d}E \frac{2f_+(E)}{\sqrt{(E^2-\Delta^2)}} \tag{2.25}$$

where A_0 is the zero-temperature energy gap.
In summary, the coupled kinetic equations (eq. 2.24) along with the modified gap equation (eq. 2.25) form the basis of our discussion. For a given driving situation, these coupled equations have to be solved simultaneously. Let us first apply these equations to a system driven into a state near equilibrium.

26.3 3. Systems near equilibrium

In a system slightly away from equilibrium, the deviations of the quasiparticle and phonon distributions, $\delta f_{\gtrless}(E)$ and $\delta n(\Omega)$, as well as the energy gap are small compared with their respective equilibrium values. This means that we can use the coupled equations linearized with respect to

$$\delta f_+(E) \equiv f_+(E) - f^\circ(E), \tag{3.1a}$$

$$\delta f_-(E) \equiv f_-(E), \tag{3.1b}$$

$$\delta n(\Omega) \equiv n(\Omega) - n^\circ(\Omega), \tag{3.1c}$$

to describe the system, where $f(E)$ and $n(\Omega)$ are, respectively, the quasiparticle and phonon distributions in the equilibrium state at an ambient temperature T_a. The only terms in the kinetic equations that need to be linearized are those contributed by the electron phonon interaction. In the linear approximation, they have the forms:

$$\left.\frac{\partial f_+(E)}{\partial t}\right|_{e-\mathrm{ph}} = -\int_0^\infty \mathrm{d}E' 4N(0)\rho\left(E'\right) \times \left\{\delta f_+(E)\left[K_1\left(E,E'\right)+K_1\left(-E,-E'\right)+K_1\left(E,-E'\right)\right] +\delta f_+\left(E'\right)\left[K_1\left(E',E\right)+K_1\left(-E',-E\right)-K_1\left(E',-E\right)\right]\right\} - \int_0^\infty \mathrm{d}\Omega N F(\Omega)\delta n(\Omega)\left[K_2(\Omega,E)+K_2(-\Omega,E)-K_2(\Omega,-E)\right] \tag{3.2a}$$

$$\left.\frac{\partial f_-(E)}{\partial t}\right|_{\mathrm{e\cdot ph}} = -\int_0^\infty \mathrm{d}E' 4N(0)\rho\left(E'\right) \times \left\{\delta f_-(E)\left[K_1\left(E,E'\right)+K_1\left(-E,-E'\right)+K_1\left(E,-E'\right)\right] +\delta f_-\left(E'\right)\left[K_3\left(E,E'\right)+K_3\left(-E,-E'\right)+K_3\left(E,-E'\right)\right]\right\} \tag{3.2b}$$

$$\left.\frac{\partial n(\Omega)}{\partial t}\right|_{e-\mathrm{ph}} = -\int_0^\infty \mathrm{d}E 4N(0)\rho(E)\delta n(\Omega)\left[K_2(\Omega,E)+\frac{1}{2}K_2(\Omega,-E)\right] - \int_0^\infty \mathrm{d}E 4N(0)\rho(E)\delta f_+(E)\frac{4N(0)}{NF(\Omega)} \times \left[-\rho(E-\Omega)K_1(E,E-\Omega)+\rho(\Omega+E)K_1(-E,-(E+\Omega)) -\rho(\Omega-E)K_1(E,-(E-\Omega))\right] \tag{3.2c}$$

where the functions K_1, K_2 and K_3 in the integrands are defined to be

$$K_1\left(E,E'\right) \equiv \frac{2\pi}{\hbar}\frac{\alpha^2 F\left(E-E'\right)}{4N(0)}\left(1-\frac{\Delta^2}{EE'}\right)\theta\left(E-E'\right)\left[1+n^\circ\left(E-E'\right)-f^\circ\left(E'\right)\right],$$

$$K_2(\Omega,E) = \frac{2\pi}{\hbar}\frac{\alpha^2 F(\Omega)}{NF(\Omega)}\rho(\Omega+E)\left(1-\frac{\Delta^2}{E(\Omega+E)}\right)\theta(\Omega+E-\Delta)\left[f^\circ(E)-f^\circ(\Omega+E)\right],$$

$$K_3\left(E,E'\right) = \frac{2\pi}{\hbar}\frac{\alpha^2 F(\Omega)}{4N(0)}\left(\rho(E)\rho\left(E'\right)\right)^{-1}\theta\left(E-E'\right)\left[f^\circ(E)+n^\circ\left(E-E'\right)\right].$$

We notice that $\delta f_+(E)$ and $\delta f_-(E)$ are decoupled. Furthermore the phonon equation depends upon δf_+only. This set of coupled equations has been solved, with the assumption $\delta f'(E) = 0$ (Chang and Scalapino 1977c), for the four different driving mechanisms discussed in the previous section. We will not discuss the results for $\delta f_+(E)$ and $\delta n(\Omega)$ here. The more interesting cases of strongly driven systems will be discussed in section 4 . Here we first discuss the derivation and the measurement of the various microscopic relaxation times which are of interest in their own right.

26.3.1 3.1. Microscopic lifetimes: calculation

The quasiparticle and phonon lifetimes in superconductors, due to the electron phonon interaction, have been calculated by Kaplan et al. (1976) using the Green function method. Here we derive them from the linearized kinetic equations (3.2).

26.3.2 3.1.1. Quasiparticle lifetimes

Consider a superconductor with perturbed distributions $\delta f_\pm(E)$ for states of energy E. All other energy states are assumed to be in thermal equilibrium.

The relaxation of the distribution functions due to the electron-phonon interaction is given by

$$\frac{\partial f_{\pm}(E)}{\partial t} = -\delta f_{\pm}(E) \int_0^{\infty} \mathrm{d}E' 4N(0)\rho\left(E'\right)\left[K_1\left(E,E'\right)+K_1\left(-E,-E'\right)+K_1\left(E,-E'\right)\right] \quad (3.3)$$

where the first two terms in the square brackets are due to the scattering process while the last term is due to the recombination process. Therefore, it is natural to define the inelastic quasiparticle relaxation rates for $\delta f_{\pm}(E)$ by

$$\left[\tau_{\mathrm{qp}}^{+}(E)\right]^{-1} = \left[\tau_{\mathrm{qp}}^{-}(E)\right]^{-1} = \left[\tau_{*}^{-1}(E)+\tau_{\mathrm{r}}^{-1}(E)\right], \quad (3.4)$$

with

$$\tau_{\mathrm{s}}^{-1}(E) \equiv \int_0^{\infty} \mathrm{d}E' 4N(0)\rho\left(E'\right)\left[K_1\left(E,E'\right)+K_1\left(-E,-E'\right)\right]$$

and

$$\tau_{\mathrm{r}}^{-1}(E) \equiv \int_0^{\infty} \mathrm{d}E' 4N(0)\rho\left(E'\right) K_1\left(E,-E'\right)$$

This expression for the relaxation rates only differs from that obtained by Kaplan et al. by an overall phonon renormalization factor Z_1^{-1}. The scattering lifetime and the recombination lifetime are plotted in figs. 4 and 5, respectively, where the simple form $x^2F(\Omega) = b\Omega^2$ is assumed. This is quite adequate for all materials except Pb and Hg . The characteristic quasiparticle lifetime τ_0 is given by

$$\tau_0 = \frac{\hbar Z_1}{2\pi b\left(k_{\mathrm{B}}T_{\mathrm{c}}\right)^3}. \quad (3.5)$$

It has been calculated and tabulated, and found to be orders of magnitude different for various materials (Kaplan et al. 1976).

Figure 4 shows the temperature dependence of τ_{s} for quasiparticles of different energies. We see that τ_{s} increases for a quasiparticle of energy $\omega \sim \Delta(T)$ as the temperature decreases, because a quasiparticle at the gap edge cannot emit a phonon and scatter when it is in the lowest quasiparticle energy state. Thus it can only be scattered by thermal phonons whose population decreases as the temperature decreases, therefore τ increases. As for quasiparticles of higher energy, spontaneous phonon emission sets a limit to the scattering lifetime and τ_s saturates at low temperatures. On the other hand, as shown in fig. 5, the recombination lifetime for all quasiparticles increases as the temperature decreases. This is because the recombination process is a binary reaction which requires two quasiparticles. At low temperatures the quasiparticle density decreases exponentially as the temperature is lowered and hence τ_{r} increases.

A comparison of these two lifetimes is shown in fig. 6 where curves for various fixed ratios of τ_s/τ_r are plotted in the quasiparticle-energy-ambienttemperature space for an equilibrium superconductor. For $T \leqslant 0.2T_{\mathrm{c}}$, the scattering process dominates. For $T \gtrsim 0.2T_{\mathrm{c}}$, the scattering process dominates the relaxation of the high-energy quasiparticles, while the recombination process dominates the relaxation of the low-energy quasiparticles. Therefore, a high-energy quasiparticle is most likely to relax by scattering into a low-energy state where it goes through a recombination process to create a pair.

Another quasiparticle lifetime of interest is the lifetime associated with the relaxation of the charge in quasiparticle excitations. Since a quasiparticle of wavevector k is a combination of u_k^2 electron and v_k^2 hole, the net charge of the excitation (in units of $-e$) is

Fig. 4. Temperature and energy dependences of the quasiparticle scattering lifetime. (From Chang and Scalapino 1978.)

$$q_k = u_k^2 - v_k^2 = \xi_k / E_k. \tag{3.6}$$

Therefore in general, an imbalance in the branch population implies a nonzero quasiparticle charge density

$$Q^* \equiv \sum_{k,\sigma} q_k f_k = \int_0^\infty \mathrm{d}E 2N(0) \delta f_-(E) \tag{3.7}$$

The quasiparticle charge relaxation rate that characterizes the relaxation of Q^* is defined as

$$\tau_{Q^*}^{-1} \equiv I_{Q^*}/Q^* \tag{3.8}$$

Here I_{Q^*} is the quasiparticle charge injection rate and is given by

$$I_{Q^*} \equiv \sum_{k,\sigma} q_k I\left(E_k\right) = \int_0^\infty \mathrm{d}E 2N(0) I_{\mathrm{qp}}^-(E) \tag{3.9}$$

Utilizing eqs. (2.24b) and 3.2b) we find

$$\tau_{Q^*}^{-1} = \int_\Delta^{'} \mathrm{d}E 2N(0) f_-(E) \left[\tau_{Q^*}^{-1}(E,T) + \tau_{\mathrm{el}}^{-1}(E)\right] / \int_1^\infty \mathrm{d}E 2N(0) f_-(E) \tag{3.10}$$

with the microscopic inelastic charge relaxation rate given by

$$\begin{aligned}
\tau_{Q^*}^{-1}(E) &\equiv \frac{2\pi}{\hbar} \int_0^\infty \mathrm{d}E' \rho\left(E'\right) \\
&\times \left\{ \left[\alpha^2 F\left(|E'-E|\right) \theta\left(E'-E\right) \left(1 - \frac{\Delta^2}{EE'} - \rho^{-2}\left(E'\right)\right) \left(f^\circ\left(E'\right) + n^\circ\left(E'-E\right)\right) \right] \right. \\
&\left. + \left[E \to -E, E' \to -E'\right] + \left[E \to -E\right] \right\}
\end{aligned} \tag{3.11}$$

Fig. 5. Temperature and energy dependences of the quasiparticle recombination lifetime. (From Chang and Scalapino 1978.)

which is the same expression as that obtained by Chang (1979). The temperature dependence of τ_{Q^*} for five different quasiparticle energies is shown in fig. 7. All curves diverge as T approaches T_c. Otherwise τ_{Q^*} is of the same order as τ_s and τ_{r}.

26.3.3 3.1.2. Phonon lifetimes

Now consider an excess distribution of phonons with energy Ω in a superconductor otherwise at equilibrium. The decay of the excess phonons due to the electron-phonon interaction is governed by [from eq. (3.2c)]

$$\left.\frac{\partial n(\Omega)}{\partial t}\right|_{\mathrm{eph}} = -\int_0^\infty \mathrm{d}E 4N(0)\rho(E)\delta n(\Omega)\left[K_2(\Omega,E)+\frac{1}{2}\,\mathrm{K}_2(\Omega,-E)\right] \tag{3.12}$$

where the first term in the square brackets is due to the scattering process and the second term is due to the pair-breaking process. Therefore, the inelastic phonon relaxation rate is

$$\left[\tau_{\mathrm{ph}}(\Omega)\right]^{-1} = \left[\tau_{\mathrm{s}}^{\mathrm{ph}}(\Omega)\right]^{-1} + \tau_{\mathrm{B}}^{-1}(\Omega), \tag{3.13}$$

where the phonon scattering rate and pair-breaking rate are defined, respectively, as

$$\left[\tau_{\mathrm{s}}^{\mathrm{ph}}(\Omega)\right]^{-1} \equiv \int_0^\infty \mathrm{d}E' 4N(0)\rho\left(E'\right)K_2\left(\Omega,E'\right) \tag{3.14a}$$

and

$$\tau_{\mathrm{B}}^{-1}(\Omega) \equiv \int_0^\infty \mathrm{d}E' 4N(0)\rho\left(E'\right)\frac{1}{2}K_2\left(\Omega,-E'\right) \tag{3.14b}$$

Fig. 6. Locus of points on the energy -temperature plane where $\tau_{\mathrm{s} } / \tau_{\mathrm{r}}=0.5,1,2$, and 4 . The dashed curve represents $\Delta(T)/\Delta_0$, so that all quasiparticle states lie above this curve. (From Kaplan et al. 1976.)}

The phonon scattering and pair-breaking rates are plotted in fig. 8. The most significant feature of the phonon relaxation rate is the dominance of the pair-breaking process over the scattering process for phonons with energy $\Omega \geqslant 24$. The characteristic phonon time

$$\tau_0^{\mathrm{ph}} = \frac{\hbar N}{4\pi^2 N(0)\alpha^2\Delta(0)}, \tag{3.15}$$

with $\Delta(0)$ the zero temperature energy gap, is typically of order 10^{-10} s. Values of τ_0^{ph} for various materials are also calculated and tabulated by Kaplan et al. (1976).

26.3.4 3.2. Microscopic lifetime: measurement

Due to the lack of a high-resolution phonon spectrometer, the phonon distribution in a nonequilibrium superconductor is difficult to measure directly. However, various means have been developed to make indirect measurements. These have been reviewed by Eisenmenger (1981). On the other hand, the quasiparticle distribution can be probed quite accurately by the quasiparticle-tunneling technique (Chang and Scalapino 1976, Willemsen and Gray 1978, Horstman et al. 1981). Furthermore, using a double-junction structure, the tunneling technique has been shown to be capable of measuring the microscopic quasiparticle lifetime due to inelastic collisions as well as elastic collisions (Kaplan et al. 1977, Chang 1977).

Details of the measurement and analysis are given by Kirtley et al. (1980). Here we briefly outline the physics behind the technique, and discuss the result obtained for Al films.

Consider an Al/Al/Pb − Bi junction. The Al/Al junction has a low re-

(From Kirtley et al. 1980.)

sistance and serves as an injector to drive the center Al film out of equilibrium. The driven

Fig. 7. Temperature and energy dependences of the quasiparticle charge-imbalance relaxation time. (From Chang 1979.)

state of this film is probed by the high resistance Al/Pb − Bi detector junction. When the Al/Al junction is biased at zero voltage, four well-known structures in the $I - V$ curve of the detector junction occur at $V_\mathrm{d} = \pm\left(\Delta_\mathrm{Pb} \pm \Delta_\mathrm{A1}\right)/e$. When $V_\mathrm{i} > 2A/e$, four additional structures are observed at $\mathrm{V_d} = \pm\left[\Delta_\mathrm{Pb} \pm \left(\Delta_\mathrm{A1} - |eV_\mathrm{i}|\right)\right]/e$. To bring out the details of these structures, $I_\mathrm{d}(2\omega)$ (which is proportional to $\mathrm{d}^2 I_\mathrm{d}/\mathrm{d}V^2$) versus V, instead of I versus V, is shown in fig. 9. These curves indicate that there are small steps in the detector $I - V$ characteristics at $V_\mathrm{d} = \pm\left(\Delta_\mathrm{Pb} + \Delta_\mathrm{A1} - |eV_\mathrm{i}|\right)/e$ and a small peak and dip at $V_\mathrm{d} = \left(\Delta_\mathrm{Pb} - \Delta_\mathrm{A1} + |eV_\mathrm{i}|\right)/e$. We note that for these structures (occurring at voltages of equal magnitude but opposite sign) the amplitudes are different. This asymmetric effect increases with increasing injection voltage V_i.

It has been shown that these structures result from the sharp structure in the quasiparticle injection rate at energy $E_\mathrm{s} = eV_\mathrm{i} - \Lambda_\mathrm{A1}$ (fig. 2). Effects due to the structure at $(eV_i + A_{A1})$ are too small to be detected. The injected quasiparticles relax and give a steady state distribution which, for the most part, resembles a thermal distribution with an effective temperature $T^* > T_\mathrm{a}$. However, the steady-state distribution also contains a small residue from the sharp structure in the injection spectrum. Figure 10 shows schematically the effects of this

sharp structure on the detector-junction current. The excess quasi-particles enhance the currents contributed by processes (a) and (d) as shown in (c|a) and (c|d). They also block processes (b) and (c), and reduce the currents contributed by them as shown in (c|b) and (c|c). We note that the most effective enhancement and reduction corresponding to each process occur at different detector voltages. This results in the four measured structures in the detector $\mathrm{d}^2 I, \mathrm{dV}^2$ curve, shown in fig. 9 , each corresponding to a different
process. The asymmetry in the detector junction $I - V$ curve is due to the imbalance in the injection rate on the two $\xi \gtrless 0$ branches. We also note that, since the magnitudes of the $I - V$ structures depend only on the magnitude of the sharp structure of the steady-state quasiparticle distribution at $E_\mathrm{s} = eV_\mathrm{i} - \Delta_\mathrm{A1}$ (which in turn is governed by the relaxation times of the quasiparticles of that particular energy), the quasiparticle relaxation time at energy E_s can be obtained from a quantitative analysis of the magnitudes of the $I \cdot V$ structures. Furthermore, by sweeping the injector junction voltage and changing the temperature, the energy and temperature dependences of the relaxation time can be mapped out.

The magnitudes of the current-step heights at $\pm V_\mathrm{d}^\mathrm{s} = \pm\left(\Delta_\mathrm{Pb} + \Delta_\mathrm{A1} - eV_\mathrm{i}\right)/e$ for the injector voltage $V_\mathrm{i} > 2\Delta_\mathrm{A1}/e$ can be readily calculated. We obtained (Chang 1977)

Fig. 8. Phonon decay rates due to scattering and pair-breaking processes. (From Chang and Scalapino 1978.)

$$\Delta I_{\mathrm{d}}\left(\pm V_{\mathrm{d}}^{\mathrm{s}}\right)=\frac{\pi G_{\mathrm{nn}}^{\mathrm{i}} G_{\mathrm{nn}}^{\mathrm{d}}\left(\Delta \Delta_{\mathrm{Pb}}\right)^{1/2}}{8 e^{2} N(0) \Omega} \tau_{\mathrm{qp}} \rho\left(e V_{\mathrm{i}}-\Lambda\right)$$
$$\times\left[1-f^{\circ}\left(\Delta, T^{*}\right)-f^{\circ}\left(e V_{\mathrm{i}}-\Delta, T^{*}\right)\right]\left[1 \pm\left(1-\frac{\Delta^{2}}{\left(e V_{\mathrm{i}}-\Lambda\right)^{2}}\right) \frac{\tau_{\mathrm{el}}}{2 \tau_{\mathrm{qp}}+\tau_{\mathrm{cl}}}\right] \tag{3.16}$$

Here all the unsubscripted functions and parameters such as ρ, Δ, Ω, etc. are those for the center Al film. The inelastic relaxation time τ_{qp} and the elastic relaxation time τ_{e1} are to be evaluated at the energy $E=eV_i-A_{A1}$ and at the temperature T^*. Clearly, the inelastic quasiparticle relaxation time $\tau_{\mathrm{qp}}\left(eV_{\mathrm{i}}-A_{\mathrm{A1}}\right)$ can be obtained from the sum, and the elastic branch mixing time $\tau_{\mathrm{e1}}\left(eV_{\mathrm{i}}-\Delta_{\mathrm{A1}}\right)$ can be obtained from the difference of the current-step magnitudes at $V_{\mathrm{d}}=\pm\left(\Delta_{\mathrm{Pb}}+\Delta_{\mathrm{A1}}-eV_{\mathrm{i}}\right)/e$. Experimental data for a dirty A1 film have

been obtained by Kirtley et al. (1980). The inelastic quasiparticle relaxation time as a function of the quasiparticle energy for four different ambient temperatures is shown in fig. 11 (left). The corresponding curves calculated from eq. (3.4) are shown in fig. 11 (right). The agreement is good. However, the elastic quasiparticle-impurity scattering time obtained using this technique is independent of both energy and temperature within the uncertainty of the measurement. This is different from the theoretical prediction (Tinkham 1972) of a $\left(E^2-\Delta^2\right)^{3/2}/E\Delta^2$ energy dependence, although the magnitude of the predicted τ_{c1} is consistent with the experimental value. At the moment, there is no explanation of this. Data for a clean Al film have been obtained by Moody and Paterson (1981). The same conclusion can be drawn from their data.

Fig. 9. Left-hand figure: second-harmonic current signal and first harmonic voltage modulation versus detector bias voltage, with injection at bath temperature $T_{\mathrm{B }}=1.25 \mathrm{~K}$, and with no injection at a higher bath temperature $T_\beta = 1.38$ K. The cusp in the detector $I - V$ characteristic at $(\Delta_{\mathrm{Pb}} - \Delta_{\mathrm{A1}})/e$ occurs at $V_{\mathrm{d}} = 1.22$mV. The nonequilibrium structure appears at $V_{\mathrm{d}} = (\Delta_{\mathrm{Pb}} + \Delta_{\mathrm{A1}} - |eV_i|)/e = 0.84$mV. The background signal curves for the injection and no-injection conditions match exactly, indicating an effective film temperature under injection $T^* = 1.38$ K. Right-hand figure: same experiment with detector voltage biased above the current step at $(\Delta_{\mathrm{pb}} - \Delta_{\mathrm{A1}} + |eV|,)/e$.}

26.3.5 3.3. Macroscopic lifetimes

26.3.6 3.3.1. The quasiparticle density relaxation time

In a thin superconducting film under a driven situation, excess quasiparticles can be created. The relaxation time of the excess quasiparticle density, τ_{d}, has long been an interesting subject (Rothwarf and Taylor 1967, Gray et al. 1969). Letting δN and I_{qp} be, respectively, the excess quasiparticle density and the injection rate of the quasiparticle, we have

$$\begin{aligned} \delta N &= \sum_{k\pi} \delta f_k \\ &= \int_d^\infty \mathrm{d}E 4N(0)\rho(E)\delta f_+(E) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} I_{\mathrm{qp}}^+ &= \sum_{k\sigma} I_h \\ &= \int_A^\infty \mathrm{d}E 4N(0)\rho(E) I^+(E) \end{aligned} \tag{3.18}$$

The quasiparticle density relaxation time is defined as

$$\tau_{\mathrm{d}} = \delta N / I_{\mathrm{qp}}^+ \tag{3.19}$$

It has been shown that the excess quasiparticle density δN can be extracted from tunneling data (Chang and Scalapino 1976). For example, at low temperature, the tunneling current of a symmetric superconductor -insulatorsuperconductor junction is proportional to the sum of the quasiparticle densities in the two films which form the junction if the junction voltage is biased at a value slightly below 21. Also, to a good approximation, the gap suppression from its equilibrium value is directly proportional to the excess quasiparticle density in a nonequilibrium film.

Fig. 10. Four detector bias situations which give rise to the singularities in the detector currentvoltage curve due to the injector channel c. (From Kirtley et l. 1980.)

Intuitively, one would expect that the relaxation time τ_{d} would be related to the quasiparticle recombination time $\tau_{\mathrm{r}}(E,T)$ and the phonon pair-breaking time $\tau_{\mathrm{B}}(E,T)$ since these processes change the quasiparticle density. Also important will be the phonon escape time τ_{cs} because it governs the relaxation of the energy injected into the system by an external source. To examine how these times are related to the quasiparticle density relaxation time τ_{d}, we first find a relationship between I_{qp} and δN from the kinetic equation (2.23a). Substituting eq. (3.3) into eq. (2.22a) and summing over the quasiparticle states, we have

$$I_{\mathrm{qp}}^{+} = 2\delta N \left\langle \tau_{\mathrm{r}}^{-1} \right\rangle_{\mathrm{qp}} - 2\delta N_{\omega} \left\langle \tau_{\mathrm{B}}^{-1} \right\rangle_{\mathrm{ph}} \tag{3.20}$$

Here

$$\delta N_{\omega} \equiv \int_{2,\Delta}^{\infty} \mathrm{d}\Omega N F(\Omega) \delta n(\Omega) \tag{3.21}$$

is the excess density of phonons with energy $\Omega \geqslant 2\Delta$, while

$$\left\langle \tau_{\mathrm{r}}^{-1} \right\rangle_{\mathrm{qp}} \equiv \int_{\Delta}^{\infty} \mathrm{d}E 4N(0)\rho(E)\delta f^{+}(E)\tau_{\mathrm{r}}^{-1}(E) / \int_{\Delta}^{\infty} \mathrm{d}E 4N(0)\rho(E)\delta f^{+}(E) \tag{3.22a}$$

and

$$\left\langle \tau_{\mathrm{B}}{}^{1} \right\rangle_{\mathrm{ph}} \equiv \int_{21}^{1} \mathrm{d}\Omega N F(\Omega)\delta n(\Omega)\tau_{\mathrm{B}}{}^{1}(\Omega) / \int_{21}^{1} \mathrm{d}\Omega N F(\Omega)\delta n(\Omega) \tag{3.22b}$$

Fig. 11. Left-hand figure: experimental values of the inelastic relaxation time versus the normalized quasiparticle energy for four bath temperatures. Right-hand figure: theoretical calculation of the same quantities. (From Kirtley et al. 1980.)

are, respectively the quasiparticle recombination rate and phonon pairbreaking rate averaged over the excess distribution. $\delta N_{,}$ can be calculated by substituting eq. (3.12) into the phonon kinetic equation (2.22c). We obtain

$$\delta N_\omega = \delta N \left(\left\langle \tau_{\mathrm{sA}}^{-1} - \tau_{\mathrm{sE}}^{-1} - \tau_{\mathrm{r}}^{-1} \right\rangle_{\mathrm{qp}} \right) / \left\langle \left(\tau_{\mathrm{s}}^{\mathrm{ph}} \right)^{-1} + \tau_{\mathrm{B}}^{-1} + \tau_{\mathrm{cs}}^{-1} \right\rangle_{\mathrm{ph}} . \tag{3.23}$$

Here

$$\tau_{\mathrm{sE}}^{-1} \equiv \int \mathrm{d}E' 4N(0)\rho\left(E'\right) K_1\left(e, E'\right) \theta\left(E - E' - 2\Delta\right)$$

$$\tau_{\mathrm{sA}}^{-1} \equiv \int \mathrm{d}E' 4N(0)\rho\left(E'\right) K_1\left(-E, -E'\right) \theta\left(E' - E - 2\Delta\right)$$

are the quasiparticle scattering rates associated with, respectively, the emission and absorption of a phonon of energy $\Omega \geqslant 2.1$. Eliminating is $N_{,}$ from ecls. (3.20 and 3.23), we obtain

$$\tau_{\mathrm{d}}^{-1} = 2 \left\langle \tau_{\mathrm{r}}^{-1} \right\rangle_{\mathrm{qp}} \left\{ 1 - \frac{1 + \left\langle \tau_{\mathrm{sE}}^{-1} - \tau_{\mathrm{sA}}^{-1} \right\rangle_{\mathrm{qp}} / \left\langle \tau_{\mathrm{r}}^{-1} \right\rangle_{\mathrm{qp}}}{1 + \left[\left\langle \left(\tau_{\mathrm{s}}^{\mathrm{ph}} \right)^{-1} + \tau_{\mathrm{es}}^{-1} \right\rangle_{\mathrm{ph}} \right] / \left\langle \tau_{\mathrm{B}}^{-1} \right\rangle_{\mathrm{ph}}} \right\} . \tag{3.24}$$

The above expression for τ_{d}^{-1} can be simplified if the excess quasiparticles have low energies and quasiparticle states of energy $E \geqslant 3\Delta$ are not disturbed. In this case the effect of the quasiparticle scattering process can be neglected and we have

$$\tau_{\mathrm{d}}^{-1} = 2 \left\langle \tau_{\mathrm{r}}^{-1} \right\rangle_{\mathrm{qp}} / \left[1 + \left\langle \tau_{\mathrm{B}}^{-1} \right\rangle_{\mathrm{ph}} / \left\langle \tau_{\mathrm{es}}^{-1} \right\rangle_{\mathrm{ph}} \right] . \tag{3.25}$$

We see that the macroscopic charge density relaxation rate ${\tau_{\mathrm{d}}}^{-1}$ is a direct measure of the microscopic quasiparticle recombination rate averaged over the excess quasiparticle distribution. For the phonon escape time $\tau_{\mathrm{es}} = 0$, we have $\tau_{\mathrm{d}}^{-1} = 2 \left\langle \tau_{\mathrm{r}}^{-1} \right\rangle_{\mathrm{qp}}$, with the factor of 2 due to the fact that each recombination event reduces two quasiparticles. When τ_{es} is finite, the well known phonon trapping factor $\left\langle \tau_{\mathrm{B}}^{-1} \right\rangle_{\mathrm{ph}} / \left\langle \tau_{\mathrm{e}}^{-1} \right\rangle_{\mathrm{ph}}$ first pointed out by Rothwarf and Taylor (1967) reduces τ_{d}^{-1} from the average quasiparticle recombination rate.

26.3.7 3.3.2. Quasiparticle charge relaxation time

The time τ_{Q^*} that governs the quasiparticle charge relaxation is of interest because it is related to many physical quantities that can be directly measured. (See, for example, Clarke

1981.) For example, it has been shown that for a superconductor-insulator superconductor tunnel junction in which one of the films has a charge imbalance Q^* there is a finite quasiparticle tunneling current at zero-bias voltage (Clarke 1972, Tinkham and Clark 1972, Moody and Paterson 1978, Chi and Clarke 1979a). Specifically one has

$$I(V=0) = \frac{G_{n1}\Omega}{2dN(0)e} Q^* \tag{3.26}$$

The quasiparticle injection rate I_{Q^*} can be easily calculated from eq. (3.9) using the $I_{\gtrless}(E)$ obtained in section 2 . Therefore, measuring $I(V=0)$ yields immediately the charge relaxation rate

$$\tau_{Q^*}^{-1} = I_{Q^*}/Q^* = \frac{2dN(0)e}{G_{\rm nn}\Omega} \frac{I_{Q^*}}{I(V=0)} \tag{3.27}$$

which from eq. (3.10) is just the microscopic rate $\tau_{Q^*}^{-1}(E) + \tau_{\rm e1}^{-1}(E)$ averaged over the excess quasiparticle charge distribution $q_k f_k$.

Details of the measurement and the analysis can be found in the articles by Chi and Clarke (1979a) and Clarke (1981).

26.4 4. Systems far from equilibrium

The strength of external sources required to drive a superconductor far from equilibrium has been worked out for different driving situations (Chang and Scalapino 1977b). For example, in the quasiparticle tunnel-injection case, the driving strength is characterized by $G_{\rm nn}/2N(0)e^2\Omega$, while the relaxation time
for the quasiparticle is characterized by τ_0. Therefore the parameter

$$A = \frac{G_{\rm nn}\tau_0}{2N(0)e^2\Omega} \left(\frac{k_{\rm B}T_{\rm c}}{\Delta_0}\right)^3 \tag{4.1}$$

is a good measure of how strongly the system is driven. For Al films of thickness $d \sim 10^3$Å, it is found that the conductance per unit area $G_{\rm nn}d/\Omega$ needs to be greater than $10^4\Omega^{-1}$ cm^{-2} to make $A \sim 0.01$. For Sn , due to the shorter relaxation time τ_0, the required conductance per area would be larger ($\sim 2\times 10^6\Omega^{-1}$ cm^{-2}).

However, the above consideration neglects the phonon-trapping effect discussed in section 3.3.2 where it is shown that, if the thermal coupling between the sample and the environment is weak, the effective quasiparticle relaxation time will be longer than the intrinsic time τ_0. Therefore, in the actual experimental situation, the required strength may be smaller than those given above. Also, in this case of weak thermal coupling, the injected energy will be transferred between the quasiparticle and phonon systems many times before it is finally released from the sample. This will tend to smear out the sharp structures in the injected quasiparticle spectrum and the specific signature of the driving mechanisms may be washed out. We will discuss this case later. First, let us discuss the properties of films with strong thermal coupling where the signatures of the driving sources are significant and contribute to interesting phenomena.

26.4.1 4.1. Narrow band phonon generation

The physics behind narrow-band phonon generation using a nonequilibrium superconducting thin film is easy to understand. Consider a symmetric superconductor-insulator superconductor tunnel junction biased at a voltage slightly above $24/e$. From figs. 1 and 2 it is easy to see that the dominant tunneling process at low temperature is either process (c) or (d), depending upon whether V is positive or negative. Quasiparticles are predominantly injected into states

of energy ranging from A to $eV - A$ for both films in the junction. Energy conservation forbids the excitation of quasiparticles with energy greater than $eV - A$ by process (c) or (d). The excited quasiparticles then relax and recombine into pairs with the emission of phonons of energy ranging from 2Δ to $2(eV - \Delta)$. When the voltage is set at a value just above $2\Delta/e$ the bandwidth, $2(eV - A) - 2A = 2(eV - 2A)$, can be very narrow.

Eisenmenger and Dayem (1967) were the first to suggest the use of tunnel junctions in the generation of high-frequency narrow-band phonons ($v = 24/h \sim 10^{12}$ Hz). The energy spectra of phonons emitted from a symmetric superconductor insulator -superconductor tunnel junction have been investigated experimentally by Dayem and Wiegand (1972) and Dynes et al. (1971, 1972).

Theoretical work to calculate the emitted-phonon energy spectra was first attempted by Dayem and Wiegand (1972). They used a set of linearized kinetic equations. However, in a real situation, it is more likely that, to generate high intensity phonons in narrow energy ranges, the system needs to be driven into nonequilibrium states for which the linear approximation is not valid. Therefore, the full set of nonlinear kinetic equations has to be solved along with the modified gap equation (eqs. 2.23, 24). This work has been carried out by Chang and Scalapino (1977a, 1978) and here we go through their results briefly.

The steady-state quasiparticle and phonon energy spectra are shown in fig. 12. We have neglected the effect associated with the branch imbalance which is unimportant in this case since most of the quasiparticles are injected into states of $E \sim \Delta$. The ambient temperature is set at $T = 0$, the bias voltage is $V = 2A_0/e$, and the phonon trapping factor is $\tau_{\rm es}/\tau_{\rm B}\,(2A_0) = 1, 4,$ and 8 . The parameter A which characterizes the strength of the driving source is set at 0.01 . Here $\tau_{\rm B}\,(2A_0)$ is the pair-breaking time of a phonon with energy $2A_0$ in an equilibrium superconductor at zero temperature. We notice that as expected

the energy gap is suppressed by the injected auasiparticles and the suppression is large for a large phonon-trapping factor. Furthermore, a large phonontrapping factor tends to smear the sharp structure in the quasiparticle spectra at $E = eV - \Delta$ somewhat. As for the phonon spectra, fig. 12 shows that there is an intense narrow band of phonons of energy $\Omega \sim 24$. The width is indeed approximately given by $2(eV - 2\Delta)$ which can be made small by adjusting the voltage V close to the steady state value of $2\Delta/e$. These curves may give the wrong impression that a high phonon-trapping factor generates a high density of narrow-band phonons. The fact that this is not the case can be seen by an examination of the energy spectra of the emitted phonons, which is proportional to $\Omega^2 n(\Omega)/\tau_{\rm es}$ (fig. 13). It can be seen clearly that indeed high phonon-trapping factors tend to smear the sharp structures and broaden the bandwidth of the emitted phonons.

We want to emphasize that the narrow energy bandwidth of the emitted phonons is the result of a nonlinear effect. In fact, for thin films under weakly driven conditions, the scattering process dominates the relaxation of quasiparticles, as can be seen in fig. 6. Since the scattering process necessarily tends

to smear out the quasiparticle energy distribution, the emitted phonons will not have a sharp energy dependence. The reason that we do have narrow-band phonons emitted is because of the large number of quasiparticles injected into the film, so that in the nonequilibrium state a quasiparticle can find a partner to recombine with into a pair with relative ease. Therefore, the recombination process, instead of the scattering process, dominates the relaxation.

Attempts to generate narrow-band phonons using a thin film driven by heat pulses (Narayanamurti and Dynes 1971, Dynes and Narayanamurti 1972) and microwaves (Rutledge et al. 1979) have also been made. However, the results are not as favorable as in the case using tunnel junctions. In the heat pumping case, this is due to the lack of a mechanism to cut off the generation of phonons on the high-energy side. In the microwave case, this is because the coherence factor of the scattering process is larger than that of the recombination

Fig. 12. Steady state quasiparticle (left-hand figure) and phonon (right-hand figure) distributions in thin films under quasiparticle tunnel injection through a symmetric junction. The ambient temperature is $T=0 \mathrm{~K
$ and the voltage is $V = 2\Delta_0/e$. Curves a-c correspond to $\tau_{\mathrm{es}}/\tau_{\mathrm{B}} = 1, 4$ and 8 , respectively. Inset shows the three processes that contribute to the tunneling current. (From Chang and Scalapino 1978.)}

process for the quasiparticle-microwave interactions. This means that before a large quasiparticle density of $E \sim \Delta$ is achieved by strong microwave irradiation of frequency $v \sim 24/\mathrm{h}$, many of them will already be scattered again by microwave photons into states of $E \sim 3A$. The relaxation of these quasiparticles is dominated by the scattering process with the emission of phonons which will not give phonons of narrow bandwidth.

26.4.2 4.2. Enhancement of superconductivity

The energy gap of a superconducting thin film is observed to increase when the film is under microwave irradiation (Kommers and Clarke 1977, Hall et al. 1980, 1981, Horstman et al. 1981), quasiparticle tunnel injection (Gray 1978, Chi and Clarke 1979), and low-frequency phonon injection (Seligson and Clarke 1982).

The enhancement of the energy gap using quasiparticle tunnel injection was first proposed by Parmenter (1961). He noticed that when an asymmetric junction is biased at a voltage smaller than the difference of the gap voltages of the two films, the quasiparticle density in the film with the smaller gap will be reduced and hence superconductivity enhanced. Later, Wyatt et al. (1966) and Dayem and Wiegand (1967) observed enhancement of the critical current in microbridges irradiated by microwaves. Eliashberg (1970), inspired by their results, suggested

Fig. 12. Steady state quasiparticle (left-hand figure) and phonon (right-hand figure) distributions in thin films under quasiparticle tunnel injection through a symmetric junction. The ambient temperature is $T=0 \mathrm{~K
$ and the voltage is $V = 2\Delta_0/e$. Curves a-c correspond to $\tau_{\mathrm{es}}/\tau_{\mathrm{B}} = 1, 4$ and 8 , respectively. Inset shows the three processes that contribute to the tunneling current. (From Chang and Scalapino 1978.)}

that the enhancement was due to the shoveling of quasiparticles from low-energy states to high-energy states. He noticed that it is the occupation of the low-energy states that most severely interferes with the pair correlation of the electrons (see eq. 2.24). Since microwaves of frequency $y < 24/h$ cannot break a pair, it can only excite the low-energy quasiparticles into high-energy states. This results in an enhanced gap and hence an enhanced critical current.

Eliashberg (1970, 1972) and Ivlev and co-workers (1971, 1973) have studied the gap enhancement due to microwave irradiation using the quasiparticle

$$A = \frac{2R_{\mathrm{N}}H^2}{dN(0)\omega^2}\left(\frac{k_{\mathrm{B}}T_{\mathrm{c}}}{\Delta_0}\right)^3 \tau_0.$$

kinetic equation. Chang and Scalapino (1976, 1978) have included nonequilibrium phonon effects by solving the full set of quasiparticle and phonon kinetic equations selfconsistently. They found that the shoveling of quasiparticles into higher energy states by the microwaves can result in a reduction in the quasiparticle density and hence an enhancement of the energy gap. This can be achieved because higher-energy quasiparticles can recombine into pairs faster than the lower-energy ones. To be more specific, we note first that the phonon density of states in the Debye approximation varies as the phonon energy squared. Thus for phonons emitted by recombination of quasiparticles

Fig. 13. Energy spectra of the emitted phonons shown in the previous figure. (From Chang and Scalapino 1978.)

near the gap edge, the final phonon density of states involved is proportional to $(2\Delta)^2$. However, if the two quasiparticles are shoveled into states of energy $\Delta + hv$ before they recombine into a pair, the recombination phonon would have energy $2(A + hv)$ and the final phonon density of states involved would be proportional to $4(4 + hv)^2$. Therefore tunneling excitation of quasiparticles can lead to faster quasiparticle recombination and a net reduction of the quasiparticle density in the steady state.

Numerical solutions of the variations of the gap and the quasiparticle density $N_{\rm qp}$ with microwave coupling for the four different phonon trapping factors $\tau_{\rm es}\,/\tau_{\rm B}\,(2A_0) = 0.2, 2, 8$, and 16 are shown in fig. 14. These curves clearly show that the gap enhancement is related to the quasiparticle-density reduction at small driving amplitudes. When the driving force is strong [e.g. $A \sim 0.2$ in case (d)], nonlinear effects set in and the quasiparticle density can be larger than its equilibrium value. In this case, the gap enhancement is completely due to the redistribution of the quasiparticles.

Theoretical work on the gap enhancement in superconducting films of a symmetric tunnel junction due to quasiparticle tunnel injection has been carried out by Peskovatskii and Seminozenko (1976) and Chang (1978). The reason that the energy gap can be enhanced for both films is similar to the case of microwave irradiation. A junction biased at $V < 2A/e$ can have process (a) and (b) (fig. 1) contribute to the current. The effect of these processes is to shovel quasiparticles from low-energy states to high-energy states as the lowfrequency microwave do (see, for example, fig. 1).

Although the physics behind the gap enhancement is the same for microwave irradiation as for quasiparticle tunnel injection, the experimental situations are quite different. In the tunneling case, the coupling constant is fixed for a given junction, however the bias voltage can be adjusted at will. In the case of microwave irradiation, the coupling amplitude can be changed at will by changing the imput power, but the frequency of the microwaves is difficult to adjust. Experimental studies of the gap enhancement have been carried out by Gray (1978) using a symmetric A1, A1 junction and by Chi and Clarke (1979) using an asymmetric A1/A1 junction.

Fig. 14. The microwave coupling dependences of (a) the steady-state quasiparticle density and (b) the energy gap. (From Chang and Scalapino 1978.) The bath temperature is $T=0.9 T_{c $. The microwave frequency is $v = \Delta_0/2h$. Curves a-d correspond to $\tau_{\rm es}/\tau_{\rm B} = 0.2, 2, 8$ and 16 , respectively. The parameter that represents the coupling strength is}

In fig. 15 the energy gap as a function of the junction bias voltage is shown for coupling constants of $A = 0.05$ and 0.1 and an ambient temperature of $T = 0$. These curves are obtained (Chang 1978) by assuming that there is no branch imbalance. Nonequilibrium phonons are taken into account. The phonon trapping factor is set equal to $\tau_{\rm cs}, \tau_{\rm B}\,(2A_0) = 8$, which is typical for an Al film on sapphire in liquid Helium. It is interesting to note that the energy gap begins to decrease before the bias voltage reaches $2A, h$ when the pair breaking process sets in. This is different from the result of Peskovatskii and Seminozenko (1976) who ignored the nonequilibrium phonon effect and found that the gap increases with increasing voltage. The curves shown in fig. 15 are in good qualitative agreement with the experimental data (Gray 1978). The

magnitude of the maximum gap enhancement calculated is to be slightly less than 4% using these parameters. The experimental value of a maximum enhancement is slightly larger than 1%. Among other things, this could be due to the magnetic field generated by the current or the neglect of the quasiparticle branch imbalance.

Before we leave this subject, I would like to emphasize that this effect can be observed

Fig. 15. Voltage dependence of the gap enhancement due to quasiparticle tunnel injections using a symmetric junction. (From Chang 1978.)

only in samples with good thermal coupling. For a sample with severe phonon trapping, "heating" effects will be important and the energy gap will be suppressed instead of enhanced. A simple heating model has been developed to take the nonequilibrium phonons into account (Mooij 1981). Both the redistribution and the reduction of quasiparticles have been observed in a microwave-driven film which shows gap enhancement (Horstman et al. 1981). What is the maximum amount of the enhancement that can be achieved experimentally, and which mechanism is more responsible for this effect. remain as intriguing questions.

26.4.3 4.3. Analytical models and instability

As mentioned before, only when the thermal coupling between the sample and its environment is strong do the energy distributions of the quasiparticles show the specific structures associated with the external driving source. For weak thermal coupling, the quasiparticle distribution is smooth and can be described by an analytic model. There are two well-known models: the μ^* model by Owen and Scalapino (1972) and the T^* model first proposed by Parker (1975) and later modified by Chang et al. (1979). In the μ^* model it is assumed that the phonons are in thermal equilibrium with the ambience, while the quasiparticles thermalize rapidly to the ambient temperature, but because of a recombination bottleneck these are excess quasiparticles in the steady state. The presence of excess quasiparticles is taken into account by using an effective chemical potential μ^* which is greater than its equilibrium value. In the T^* model, on the other hand, Parker (1975) has assumed that phonons of energy $\Omega < 24$ are in thermal equilibrium with the environment. This is based upon the fact that

these phonons cannot break pairs so that their lifetime associated with the electron-phonon interaction is long compared with the phonon-escape time for the usual situation. This means they have stronger coupling to the environment than to the quasiparticles. Quasiparticles and phonons of energy $\Omega \geqslant 2\Delta$ are assumed to be in equilibrium at a temperature higher than the ambient temperature. Later, Chang et al. (1979) have shown that if the quasiparticles can be described by a Fermi function with an effective temperature T^*, the kinetic equations can be solved exactly to give the phonon distribution

$$n(\Omega) = \frac{\left\{\left[\tau_{\mathrm{s}}^{\mathrm{ph}}\left(\Omega, T^*\right)\right]^{-1} + \tau_{\mathrm{B}}^{-1}\left(\Omega, T^*\right)\right\} n\left(\Omega, T^*\right) + \tau_{\mathrm{es}}^{-1} n\left(\Omega, T_{\mathrm{a}}\right)}{\left[\tau_{\mathrm{s}}^{\mathrm{ph}}\left(\Omega, T^*\right)\right]^{-1} + \tau_{\mathrm{B}}^{-1}\left(\Omega, T^*\right) + \tau_{\mathrm{es}}^{-1}}. \tag{4.2}$$

This distribution function is a weighted average of two Bose functions and has a discontinuity at $\Omega = 24$ because the pair-breaking process can happen to phonons with $\Omega \geqslant 24$ only. The quasiparticle and phonon distributions given by the modified T^* model have been confirmed by numerical solution of the full set of kinetic equations in the limit of high phonon trapping, typically $\tau_{\mathrm{es}}/\tau_{\mathrm{B}}(2A) \geqslant 20$. Experimental measurements of the quasiparticle distribution have been carried out by various groups (e.g., Willemsen and Gray 1978, Jaworski and Parker 1979. Smith et al. 1980). It is found that they can be well approximated by a Fermi function with a T^*, except for the cases in which the quasiparticles are predominantly injected into states near the gap edge (such as the case of tunnel injection with a bias voltage $V \sim 2A/e$). In such cases a Fermi function with two parameters: μ^* and T^*, improves the fitting greatly.

The application of the μ^* model is found good only at low temperatures and for systems with good thermal coupling to the environment. Nevertheless, it has attractive features. It is easy to handle and it predicts an instability for a spatially uniform nonequilibrium state at a sufficiently large driving amplitude (Chang and Scalapino 1974). To understand why this can occur, consider a thin film at $T = 0$. When the driving source is turned on, quasiparticles are created increasing the chemical potential μ^* above the gap edge. The increased quasiparticle density increases μ^* measured from the gap edge and at the same time suppresses the energy gap. If the gap suppression exceeds the increase of μ^* measured from the gap edge, the net result is a decrease of μ^* due to the increased quasiparticle density. Since $\delta\mu^*/\delta N < 0$ implies a flow of quasiparticles from regions of low density to regions of high density, a uniform distribution of quasiparticles cannot be restored by diffusion.

The effect of quasiparticle diffusion has been considered by Scalapino and Huberman (1977) using the μ^* model. Other theoretical attempts to investigate the instability of a homogeneous nonequilibrium steady state have been made (Schmid 1977, Hida 1978, Elesin 1979, Tremblay 1981). Comparison with experimental results shows that more work, both on theory and experiments, needs to be done. This is a fast developing area. The problem of instabilities in nonequilibrium superconductors is intriguing and far from being completely solved.

27 Theory of Superconductivity Stimulation by Eliashberg, Ivlev

27.1 J. Introduction

Among many nonequilibrium phenomena in superconductors the stimulation of superconductivity is one of the most interesting effects. On the one hand, typical kinetic properties of superconductors become apparent here, on the other hand, this effect can be checked experimentally.

By superconductivity stimulation effect we understand the improvement of superconducting properties, i.e., the order parameter, critical current and field enhancement, due to some external influences changing the electron distribution function. These are microwave or acoustic radiation, or tunneling injection of electrons. The order parameter is known to obey the BCS equation

$$\frac{1}{g} = \int_{\Delta}^{\omega_{\mathrm{D}}} \mathrm{d}\varepsilon \frac{1-2n_{\varepsilon}}{\sqrt{\varepsilon^2-\Delta^2}} \tag{1}$$

where g is the electron phonon interaction constant, n_z is the electron distribution function. Due to root singularity in the density of states one may expect the order parameter to be sensitive to details of the distribution function. Parmenter (1961) had already supposed that the distribution function is affected by quasiparticle injection.

Eliashberg (1970) suggested that an electromagnetic field gives rise to superconductivity enhancement. This idea was then developed by Ivlev and Eliashberg (1971) and Ivlev et al. (1970). Being absorbed by quasiparticles, the electromagnetic field changes their distribution function in such a way that at smaller energies there is a deficit of electrons and at larger energies there is an excess of electrons. As is seen from eq. (1) such nonequilibrium state yields an increase in the order parameter. The kinetic theory of nonequilibrium states under microwave radiation has been developed by Eliashberg (1971).

Nonequilibrium stationary states form a large and important class of kinetic phenomena in superconductors. Stability of such states is maintained by an external source functioning continuously. Because of energy exchange with a thermostat, a dynamical equilibrium is established in a sample. To estimate approximately the degree of efficiency of radiation influence on a superconductor we assume that due to an electric field with frequency ω the temperature of the metal changes because of heating. Then the relative temperature change

$$\frac{\partial T}{T} \approx \frac{1}{C}\frac{\tau_{\varepsilon}}{T}\frac{\partial q}{\partial t},$$

where $C \approx mnp_0$ is the heat capacity of electrons, $\partial q/\partial t = \sigma E^2$ is the energy pumping rate, τ_{ε} is the characteristic energy relaxation time. Hence

$$\frac{\delta T}{T} \approx D\left(\frac{e}{c}A\right)^2 \tau_{\varepsilon}\frac{\omega^2}{T^2}$$

where D is the diffusion coefficient; the parameter $\delta T/T$ can play the role of the shift from equilibrium state. It can be rather big due to a long energy relaxation time. Actually, the reverse time of this relaxation is

$$\gamma' = \frac{1}{\tau_{\varepsilon}} \approx \mathrm{g}\frac{T^3}{\omega_D^2}$$

in the case of electron-phonon interaction. Close to the transition temperature of real superconductors the parameter $\gamma \times 10^{-7} - 10^9\ \mathrm{s}^{-1}$, which is much less than the typical scales of the energy gap and transition temperature. That is why this action of the electromagnetic field is more efficient than the usual suppression of the order parameter by the vector potential of the field. If the impurity concentration is large enough, then momentum relaxation associated with elastic scattering on impurities will be more effective than energy relaxation. This means that the nonequilibrium distribution function will be basically isotropic in the momentum space. A non-stationary part of distribution function will be small in the limit $\omega \gg\propto \gamma/\tau_{\varepsilon}$.

Afterwards the theory of stimulated superconductivity has been essentially developed. Aronov and Gurevich (1972) considered stimulation by tunneling injection. Tredwell and

Jacobsen (1975) calculated the acoustic pumping effect. Chang and Scalapino (1977) showed that a contribution to stimulation was made by decreasing the total number of quasiparticles. Aslamasov and Larkin (1978) studied the influence of the gap oscillation effect in a bridge on the stimulation under microwave radiation. Peskovatski and Seminozhenko (1976) calculated the stimulation of superconductivity by tunneling injection. The theory of injection effect on superconductivity was developed by Chang (1978) and by Chi and Clarke (1979).

A detailed survey on experimental achievements in the field of stimulated superconductivity is given in the present volume by Dmitriev, Gubankov, and Nad' (1985). We shall, therefore, not consider these in detail. Since the first experiments by Wyatt et al. (1966) and Dayem and Wiegand (1967) on enhancement of superconductivity in bridges by radiation the mechanism of observed effects has been unclear for a long time.

Due to experimental research of Latyshev and Nad' (1974), Tredwell and Jacobsen (1975), Klapwijk and Mooij (1976), Kommers and Clarke (1977) and
other experimentalists it was possible to establish the adequacy of the experiments and the Eliashberg (1970) theory.

The direct observation of the gap enhancement by Tredwell and Jacobsen (1975) and by Kommers and Clarke (1977) proves most convincingly the validity of this theory. A similar gap enhancement at tunneling injection was observed by Gray (1978). Superconductivity stimulation by electromagnetic field, acoustic wave and tunneling injection can be included in the same group of phenomena.

This paper is made up as follows. In section 2 the equations required are derived. In section 3 the theory of superconductivity stimulated by microwave radiation is given, and the order parameter, critical temperature and current enhancement are considered. The frequency range of existence of the effect and limits of radiation power are considered. In section 4 acoustic pumping is studied. In section 5 the specificity of superconductivity stimulation by tunneling injection is singled out, since electron-hole imbalance may be of great importance. In section 6 the stability of spatially homogeneous nonequilibrium states is discussed.

27.2 2. Derivation of equations

The behaviour of superconductors in electromagnetic and acoustic fields as well as under the effect of tunneling injection is now examined. As well as mentioned above, the difference of electron and hole branches population must be taken into account. Moreover, as often occurs in experiment, the impurity concentration is not small and consequently, the value $\xi_p = v\,(p - p_0)$ is a bad quantum number. Hence we proceed from the Green functions depending on an energy variable ε, and integrated on ξ_p. The kinetic equation for the electron distribution function under pumping conditions was obtained by Eliashberg (1971) with the method of analytical continuation. We will then summarize these equations, branch imbalance taken into account. We use the Keldysh technique (1964) to write equations rather compactly.

We write a Green function before integration as

$$\begin{aligned} -G^{+} =& \frac{n_\varepsilon}{2}\left[\left(\frac{\varepsilon}{\xi_\varepsilon^{\mathrm{R}}}+1\right)\frac{1}{\xi-\xi_\varepsilon^{\mathrm{R}}-\mathrm{i}/2\tau}+\left(\frac{\varepsilon}{\xi_\varepsilon^{\mathrm{A}}}-1\right)\frac{1}{\xi+\xi_\varepsilon^{\mathrm{A}}+\mathrm{i}/2\tau}\right] \\ &+\frac{1-n_{-\varepsilon}}{2}\left[\left(\frac{\varepsilon}{\xi_\varepsilon^{\mathrm{R}}}-1\right)\frac{1}{\xi+\xi_\varepsilon^{\mathrm{R}}+\mathrm{i}/2\tau}+\left(\frac{\varepsilon}{\xi_\varepsilon^{\mathrm{A}}}+1\right)\frac{1}{\xi-\xi_\varepsilon^{\mathrm{A}}-\mathrm{i}/2\tau}\right] \end{aligned} \tag{2}$$

Here the following designations are introduced:

$$\xi_\varepsilon^{\mathrm{R}} = -\left(\xi_\varepsilon^{\mathrm{A}}\right)^{*} = \begin{cases} \sqrt{\varepsilon^2-\Delta^2}\,\operatorname{sign}\varepsilon & \Delta^2<\varepsilon^2 \\ \mathrm{i}\sqrt{\Delta^2-\varepsilon^2} & \varepsilon^2<\Delta^2 \end{cases}$$

τ is the impurities scattering time. A label on the left marks the function according to Gor'kov, and on the right according to Keldysh.

After integration with respect to ξ the Keldysh equations for superconductor functions in matrix form are:

$$
\begin{aligned}
(\omega - vk)^{-} g_{\varepsilon\varepsilon-\omega} &= \frac{e}{c} v \left({}^{-}gA - A^{-}g\right) + \mathrm{i}\left\{\sigma_z{}^{-}\Sigma^{-}g - \sigma_z{}^{-}\Delta^{+}f\right\} + I_{\mathrm{imp}} \\
(2\varepsilon - \omega + vk)^{+} f_{\varepsilon\varepsilon-\omega} &= -\frac{e}{c} v \left({}^{+}fA + A^{+}f\right) + \mathrm{i}\left\{\sigma_z{}^{+}\Sigma^{+}f - \sigma_z{}^{+}A^{-}g\right\} +^{+} K_{\mathrm{imp}} \\
(\omega + vk)^{+} \mathrm{g}_{\varepsilon\varepsilon-\omega} &= \frac{e}{c} v \left({}^{+}gA - A^{+}g\right) + \mathrm{i}\left\{\sigma_z{}^{+}\Sigma^{+}g - \sigma_z + \Delta^{-}f\right\} +^{+} I_{\mathrm{imp}} \\
(2\varepsilon - \omega - vk)^{-} f_{\varepsilon\varepsilon-\omega} &= -\frac{e}{c} v \left({}^{-}fA + A^{-}f\right) + \mathrm{i}\left\{\sigma_z{}^{-}\Sigma^{-}f - \sigma_z{}^{-}A^{+}g\right\} + {}^{-}K_{\mathrm{imp}}
\end{aligned}
\tag{3}
$$

Associated with elastic scattering on impurities the last terms in the equations of system (3) determine the angular parts of Green functions and reduce to zero when averaged over momentum directions. As an example, we will write out two of them:

$$
\begin{aligned}
{}^{-}T_{\mathrm{imp}} &= \frac{1}{2\pi\tau}\left\{\left\langle {}^{-}g\right\rangle \sigma_z{}^{-}g -^{-} g\sigma_z \left\langle {}^{-}g\right\rangle + {}^{-}f\sigma_z \left\langle {}^{+}f\right\rangle - \left\langle {}^{-}f\right\rangle \sigma_z{}^{+}f\right\}, \\
{}^{+}K_{\mathrm{imp}} &= \frac{1}{2\pi\tau}\left\{\left\langle {}^{+}g\right\rangle \sigma_z{}^{+}f -^{+} g\sigma_z \left\langle {}^{+}f\right\rangle + {}^{+}f\sigma_z \left\langle {}^{-}g\right\rangle - \left\langle {}^{+}f\right\rangle \sigma_z{}^{-}g\right\}.
\end{aligned}
\tag{4}
$$

Angle brackets mean averaging over momentum directions.
The second terms in the right-hand sides of equations (3) describe interaction of electrons with phonons as well as inelastic scattering on them, and processes producing Cooper coupling. In expressions (3) and (4) Keldysh matrices are taken for the Green functions and self-energy parts,

$$
g = \begin{pmatrix} g^{\mathrm{c}} & g^{-} \\ g^{+} & \tilde{g}^{\mathrm{c}} \end{pmatrix} \tag{5}
$$

of which the elements both for Green functions and self-energy parts are connected by Keldysh identities (1964).

In the approximation, corresponding to the kinetic equation when

$$
{}^{+}g_{ij}\left(n_{\varepsilon}\right) = {}^{-}g_{ij}\left(1 - n_{-\varepsilon}\right), \quad +_{fij} = -^{-}_{fij}
$$

Green functions diagonal with respect to energy and satisfying equations (3) are

$$
\begin{aligned}
{}^{-}g^{+} &= -\mathrm{i}\pi\left[u_{\varepsilon}\left(1 + \beta_{\varepsilon}\right) + \alpha_{\varepsilon}\right], & {}^{-}g^{\mathrm{c}} &= {}^{-}g^{+} + \mathrm{i}\pi u_{\varepsilon} \\
{}^{-}g^{-} &= -\mathrm{i}\pi\left\{u_{\varepsilon}\left(\beta_{\varepsilon} - 1\right) + \alpha_{\varepsilon}\right\}, & {}^{-}\tilde{g}^{\mathrm{c}} &= {}^{-}g^{-} - i\pi u_{\varepsilon} \\
f^{+} &= -\mathrm{i}\pi \mathrm{v}_{\varepsilon}\left(\beta_{\varepsilon} + 1\right). & f^{\mathrm{c}} &= f^{+} + \mathrm{i}\pi v_{\varepsilon}
\end{aligned}
\tag{6}
$$

Here and below we will use the notations

$$
\begin{aligned}
u_{\varepsilon} &= \frac{\varepsilon}{\Delta} v_{\varepsilon} = \frac{|\varepsilon|}{\sqrt{\varepsilon^2 - \Delta^2}} \theta\left(\varepsilon^2 - \Delta^2\right) \\
\alpha_{\varepsilon} &= n_{\varepsilon} + n_{-\varepsilon} - 1, \quad \beta_{\varepsilon} = n_{\varepsilon} - n_{-\varepsilon}
\end{aligned}
\tag{7}
$$

For self-energy parts associated with electron-phonon interaction we have

$$
\sum_{ij} = g \int \frac{\mathrm{d}\varepsilon'}{2\pi} \int \frac{\mathrm{d}O_{\mathrm{p}}}{4\pi} \gamma^{k}_{i_1 j_1} g_{i_1 j_1} \Gamma^{k_1}_{j_1 j} D_{k_1 k} \tag{8}
$$

A phonon propagator is

$$\begin{aligned}
\mathrm{D}^{\pm} &= 2\pi \mathrm{i}\omega_n \left[N\omega_k f\left(\omega \mp \omega_k\right) + \left(1 + N\omega_k\right) f\left(\omega \pm \omega_k\right)\right] \\
D_{\mathrm{c}(\hat{\mathrm{c}})} &= D^{\pm} \pm \frac{\omega_k}{2}\left[\frac{1}{\omega - \omega_k + \mathrm{if}} - \frac{1}{\omega + \omega_k + \mathrm{if}}\right]
\end{aligned} \tag{9}$$

Performing in eq. (8) integration over momentum directions and using the relation

$$\int \frac{\mathrm{d}O_{\mathrm{p}}}{4\pi} = \int \frac{\omega_k \, \mathrm{d}\omega_k}{2\left(sp_0\right)^2}$$

we write out an expression for ${}^{-}\Sigma^{+}$

$$\begin{aligned}
-\Sigma^{+} = &-\frac{\pi g}{2\left(sp_0\right)^2} \int d\varepsilon' \left(\varepsilon' - \varepsilon\right)^2 \left[u_{\varepsilon_1}\left(\beta_{\varepsilon_1} + 1\right) + \alpha_{\varepsilon_1}\right] \\
&\times \left[\left(1 + N_{\varepsilon_1 - \varepsilon}\right)\theta\left(\varepsilon_1 - \varepsilon\right) + N_{\varepsilon - \varepsilon_1}\theta\left(\varepsilon - \varepsilon_1\right)\right]
\end{aligned} \tag{10}$$

Similarly one can see that

$$\Delta = \frac{1}{2}\mathrm{i}\left(\tilde{\Delta}^{\mathrm{c}} - \Delta^{\mathrm{c}}\right) \tag{11}$$

Due to gauge invariance we assume Δ to be a real value.
It is better to write out equations arising from system (3) for the Green functions diagonal on energy $g_\varepsilon = g^{+} + g^{-}$and $f_\varepsilon = f^{+} + f^{-}$

$$^{\mp}g = 2\mathrm{i}\pi\left(u_\varepsilon \beta_\varepsilon \pm \alpha_\varepsilon\right), \quad {}^{\pm}f = 2\mathrm{i}\pi v_\varepsilon \beta_\varepsilon \tag{12}$$

Such a definition of the distribution function somewhat differs from that in the paper of Larkin and Ovchinnikov (1977). Averaging over momentum directions gives

$$\begin{aligned}
\left\langle(\omega - vk)^{-}g\right\rangle = &\left\langle \frac{e}{c} v\left({}^{-}gA - A^{-}g\right)\right\rangle \\
&+ 2\mathrm{i}\left\{{}^{-}\Sigma^{-} - g^{+} - {}^{-}g^{-}{}^{-}\Sigma^{+} - {}^{-}f^{-}\Delta^{+} - {}^{-}\mathrm{D}^{-} - f^{+}\right\} \\
&+ \left\langle \Delta\left({}^{+}f - {}^{-}f\right)\right\rangle + I^{\mathrm{T}}
\end{aligned} \tag{13}$$

$$\begin{aligned}
\left\langle 2\varepsilon\left({}^{+}f - {}^{-}f\right)\right\rangle = &2\mathrm{i}\left\{{}^{+}f\left({}^{+}\Sigma^{-} - {}^{-}\Sigma^{-}\right) + \left({}^{+}g^{+} - {}^{-}g^{+}\right)\Delta^{-}\right. \\
&\left. - {}^{-}f\left({}^{+}\Sigma^{+} - {}^{-}\Sigma^{+}\right) - \left({}^{+}g - {}^{-}g^{-}\right)A^{+}\right\} - \left\langle vk(+f + f)\right\rangle \\
&- \left\langle \tfrac{e}{c} vA\left({}^{+}f - {}^{-}f\right) + \left({}^{+}f - {}^{-}f\right)\frac{e}{c} vA\right\rangle + K^{\mathrm{T}}
\end{aligned} \tag{14}$$

where I^{T} and K^{T} are associated with tunneling.
In the absence of order parameter phase and in equilibrium $\langle {}^{+}f_\varepsilon - f_\varepsilon\rangle = 0$. Deviation of this value from zero is due to the intrusion of an additional part even in energy to the distribution function. The procedure of obtaining the kinetic equation is to find the function difference from eq. (14) and insert it into eq. (13). In the right-hand side of eq. (13) an effective phonon collision integral is obtained. In addition, such a substitution cancels the London current in a term v kg$_\varepsilon$ corresponding to divergency of the total current. Only its diffusive part is conserved as Gor'kov and Eliashberg (1968) showed by their diagram method.

To proceed solving the system of equations (13), (14) one should find A_ω depending parts of Green functions non-diagonal on energy. This can be done by solving system (3) before averaging over momentum, when impurity collision integrals are given by equations (4). However,

to avoid this cumbersome procedure, by analogy with the technique of Gorkov and Eliashberg (1968) we take into account the fact that in the linear field approximation the off-diagonal Green functions are in the form

$$
\begin{aligned}
{}^{-}G_{\varepsilon\varepsilon-\omega} &= \frac{e}{c} vA\omega \left(-G_{\varepsilon} - G^{\mathrm{A}}_{\varepsilon-\omega} + {}^{-}F_{\varepsilon} + F^{\mathrm{A}}_{\varepsilon-\omega} - {}^{-}G^{\mathrm{R}}_{\varepsilon} - G_{\varepsilon-\omega} + {}^{-}F^{\mathrm{R}}_{\varepsilon} + F_{\varepsilon-\omega}\right), \\
+F_{t,-\omega} &= -\frac{e}{c} vA\omega \left({}^{+}F_{\varepsilon} - G^{\mathrm{A}}_{\varepsilon-\omega} - {}^{+}\mathrm{G}_{\varepsilon} + {}^{+}\mathrm{F}^{\mathrm{A}}_{\varepsilon-\omega} + {}^{+}F^{\mathrm{R}}_{\varepsilon} - G_{\varepsilon-\omega} - {}^{+}\mathrm{G}^{\mathrm{R}}_{\varepsilon} + F_{\varepsilon-\omega}\right),
\end{aligned} \tag{15}
$$

where

$$
\begin{aligned}
G_i =& \frac{1-2n_{\varepsilon}}{2}\left[\left(\frac{\varepsilon}{\xi^{\mathrm{R}}_i}+1\right)\frac{1}{\xi-\xi^{\mathrm{R}}_i-\mathrm{i}/2\tau}+\left(\frac{\varepsilon}{\xi^{\mathrm{A}}_i}-1\right)\frac{1}{\xi+\xi^{\mathrm{A}}_{\varepsilon}+\mathrm{i}/2\tau}\right] \\
&+\frac{1-2n_{-\varepsilon}}{2}\left[\left(\frac{\varepsilon}{\xi^{\mathrm{R}}_{\varepsilon}}-1\right)\frac{1}{\xi+\xi^{\mathrm{R}}_{\varepsilon}+\mathrm{i}/2\tau}+\left(\frac{\varepsilon}{\xi^{\mathrm{A}}_{\varepsilon}}+1\right)\frac{1}{\xi-\xi^{\mathrm{A}}_{\varepsilon}-\mathrm{i}/2\tau}\right] \\
+F_{\varepsilon} =& \frac{2n_{\varepsilon}-1}{2}\left[\frac{\Delta}{\xi^{\mathrm{R}}_{\varepsilon}}\frac{1}{\xi+\xi^{\mathrm{R}}_{\varepsilon}+\mathrm{i}/2\tau}+\frac{\Delta}{\xi^{\mathrm{A}}_{\varepsilon}}\frac{1}{\xi-\xi^{\mathrm{A}}_{\varepsilon}-\mathrm{i}/2\tau}\right] \\
&+\frac{2n_{-\varepsilon}-1}{2}\left[\frac{\Delta}{\xi^{\mathrm{R}}_{\varepsilon}}\frac{1}{\xi-\xi^{\mathrm{R}}_{\varepsilon}-\mathrm{i}/2\tau}+\frac{\Delta}{\xi^{\mathrm{A}}_{\varepsilon}}\frac{1}{\xi+\xi^{\mathrm{A}}_{\varepsilon}+\mathrm{i}/2\tau}\right].
\end{aligned} \tag{16}
$$

At this stage of calculations one should make concrete suppositions on the length of the mean free path of electrons. We will then study superconductors in the dirty limit. In this limiting case we integrate on ξ off-diagonal Green functions (15) and find

$$
\begin{aligned}
&- g_{\varepsilon\varepsilon-\omega} = -\pi\tau\frac{e}{c}vA_{\omega} \\
&\times\left\{(1-2n_{\varepsilon})\left[\frac{\varepsilon(\varepsilon-\omega)+\Delta^2}{\varepsilon\xi_{\varepsilon-\omega}}u_{\varepsilon}+\frac{\varepsilon-\omega}{\xi_{\varepsilon-\omega}}\right]-(1-2n_{\varepsilon})\left[\frac{\varepsilon(\varepsilon-\omega)+\Delta^2}{\varepsilon\xi_{\varepsilon-\omega}}u_{\varepsilon}-\frac{\varepsilon-\omega}{\xi_{\varepsilon-\omega}}\right]\right. \\
&\left.+(1-2n_{\varepsilon-,})\left[\frac{\varepsilon(\varepsilon-\omega)+\Delta^2}{(\varepsilon-\omega)\xi^{R}_{\varepsilon}}u_{\varepsilon-(\omega)}+\frac{\varepsilon}{\xi^{R}_{\varepsilon}}\right]-\left(1-2n_{(\cdot)-\varepsilon}\right)\left[\frac{\varepsilon(\varepsilon-\omega)+\Delta^2}{(\varepsilon-\omega)\xi^{R}_{\varepsilon}}u_{\varepsilon-(1)}-\frac{\varepsilon}{\xi^{R}_{f}}\right]\right\} \\
&\left({}^{+}f-{}^{-}f\right)_{\varepsilon\varepsilon-\omega} = 4\pi\tau\Delta\frac{e}{c}vA\omega\left(\frac{\alpha_{\varepsilon-\omega}}{\xi^{R}_{\varepsilon}}-\frac{\alpha_{\varepsilon}}{\xi^{A}_{\varepsilon-\omega}}\right).
\end{aligned} \tag{17}
$$

Inserting expression (17) into (13) and (14) the equation for the Green function $g_{\varepsilon} = {}^{+}g_{\varepsilon} + {}^{-}g_{\varepsilon}$ can be written. Note that, according to eq. (12),

$$
u_{\varepsilon}\frac{\partial n_{\varepsilon}}{\partial t} = \frac{\partial}{\partial t}\left[g_{\varepsilon} - g_{-\varepsilon} + u_{\varepsilon}\left(g_{\varepsilon}+g_{-\varepsilon}\right)\right]\frac{1}{8\mathrm{i}\pi}. \tag{18}
$$

The kinetic equation is given by (Bulyzhenkov and Ivlev 1978)

$$
u_{\varepsilon}\frac{\partial n_{\varepsilon}}{\partial t} - D\nabla^2 n_{\varepsilon} + 2D\left(\frac{e}{c}A\right)^2\frac{A^2}{\varepsilon^2-\Delta^2}\left(n_{\varepsilon}+n_{\varepsilon}-1\right) + I_{\varepsilon} = a_{\varepsilon} + t_{\varepsilon}. \tag{19}
$$

Here, the vector potential in the left-hand side corresponds to the constant London current, a_{ε} in the right-hand side is the microwave pumping:

$$
\begin{aligned}
a_{\varepsilon} =& D\left(\frac{e}{c}\right)^2 A\omega A_{-\omega} \\
&\times\left\{u_{\varepsilon}u_{\varepsilon-\omega}\left[1+\frac{\Delta^2}{\varepsilon(\varepsilon-\omega)}\right](\beta_{\varepsilon-\omega}-\beta_{\varepsilon})+u_{\varepsilon}u_{\varepsilon+\omega}\left[1+\frac{\Delta^2}{\varepsilon(\varepsilon+\omega)}\right](\beta_{\varepsilon-\omega}-\beta_{\varepsilon})+\alpha_{\varepsilon-\omega}\right. \\
&\left.+\alpha_{\varepsilon+\omega}-u_{\varepsilon}u_{\varepsilon-\omega}\left[1+\frac{\Delta^2}{\varepsilon(\varepsilon-\omega)}\right]\alpha_{\varepsilon}-u_{\varepsilon}u_{\varepsilon+\omega}\left[1+\frac{\Delta^2}{\varepsilon(\varepsilon+\omega)}\right]\alpha_{\varepsilon}\right\}
\end{aligned} \tag{20}
$$

t_ε is the tunnel source of quasiparticles, its expression will be written below. As seen from eq. (19) current is an additional source of branch imbalance
relaxation in a superconductor. This effect was discovered by Gor'kov and Kopnin (1973). If the microwave field is strong enough, it can be the main reason for relaxation and can determine the type of distribution function.

The phonon collision integral is obtained from equations (13), (14) and (18) taking into account eqs. (6)-(9), as shown by Bulyzhenkov and Ivlev (1978):

$$\begin{aligned} I_\varepsilon =& \frac{g}{\omega_{\mathrm{D}}^2}\left\{\int_\varepsilon^\infty \mathrm{d}\varepsilon_1 \left(\varepsilon_1-\varepsilon\right)^2 \left(u_\varepsilon u_{\varepsilon_1}-v_\varepsilon v_{\varepsilon_1}+1\right)\left[n_\varepsilon\left(1-n_{\varepsilon_1}\right)N_{\varepsilon_1-\pi^-}-n_{\iota_1}\left(1-n_\varepsilon\right)\left(1+N_{\varepsilon_1\iota}\right)\right]\right. \\ &+\int_{-\infty}^{\varepsilon} \mathrm{d}\varepsilon_1 \left(\varepsilon-\varepsilon_1\right)^2 \left(u_\varepsilon u_{\varepsilon_1}-v_\varepsilon v_{\varepsilon_1}+1\right)\left[n_\varepsilon\left(1-n_{\varepsilon_1}\right)\left(1+N_{\varepsilon_1}\right)-n_{\varepsilon_1}\left(1-n_\varepsilon\right)N_{l-\varepsilon_1}\right] \\ &+\int_{-\infty}^{-\varepsilon} \mathrm{d}\varepsilon_1 \left(\varepsilon+\varepsilon_1\right)^2 \left(u_\varepsilon u_{\varepsilon_1}+v_\varepsilon v_{\varepsilon_1}-1\right)\left[n_\varepsilon n_{\varepsilon_1}N_{\varepsilon+\varepsilon_1}-\left(1-n_\varepsilon\right)\left(1-n_{\varepsilon_1}\right)\left(1+N_{\varepsilon+\varepsilon_1}\right)\right] \\ &\left.+\int_\varepsilon^\infty \mathrm{d}\varepsilon_1 \left(\varepsilon+\varepsilon_1\right)^2 \left(u_\varepsilon u_{\varepsilon_1}+v_\varepsilon v_{\varepsilon_1}-1\right)\left[n_\varepsilon n_{\iota_1}\left(1+N_{\varepsilon+\varepsilon_1}\right)-\left(1-n_\varepsilon\right)\left(1-n_{\varepsilon_1}\right)N_{\varepsilon+\varepsilon_1}\right]\right\} \end{aligned} \tag{21}$$

This is a collision integral of the same type as that in the quasiparticle description with the variable ξ. Hence eqs. (19)-(21) coincide with the Eliashberg (1971) equations, if $\alpha_\varepsilon = 0$. The collision integral (21) can also be derived from the results in works by Larkin and Ovchinnikov (1977).

The first two terms in (21) conserve the total number of electron excitations like in a normal metal, the last two terms describe non-conservation of the total number of particles. Just these lead to the relaxation of branch imbalance, as has been investigated in detail by Tinkham and Clarke (1972).

Processes corresponding to the first two terms (21) conserve the total number of electronic excitations. The last two terms describe the transition of a pair of excitations into a condensate. This decreases the number of electrons near the Fermi surface. A source producing changes in the chemical potential must be added to the right-hand side of the kinetic equation (19). Tunneling of electrons from another sample into a superconductor is considered as such a source. Note, when obtaining the expression from a tunnel source, the corresponding term in the right-hand side of eq. (19) is formally equivalent to an impurity collision integral of type (4), in which instead of functions averaged over directions there are those pertaining to another metal. The superconductor in question is believed to be at zero voltage, the other metal at voltage V. This metal may be either superconducting or normal. In this metal due to the time dependence of the phases of Green functions,

$$^{\mp}g_{\mathrm{v}}\left(t-t'\right) = {}^{\mp}g\left(t-t'\right)\exp\left[\mp \mathrm{i}V\left(t-t'\right)\right],$$

their Fourier-components are

$$^{\mp}g = {}^{\mp}g(\varepsilon \mp V).$$

Introducing the value w proportional to the probability of tunneling, in accordance with symmetrization (18), the expression for a tunneling source in the kinetic equation (19) is derived from eqs. (13) and (14) (Bulyzhenkov and Ivlev 1978):

$$
\begin{aligned}
t_\varepsilon = w[&u_{\varepsilon-V}\left(\beta_{\varepsilon-V}-\beta_\varepsilon\right)+u_{\varepsilon+1}\left(\beta_\varepsilon-\beta_{\varepsilon+V}\right)+\alpha_{\varepsilon-V}-\alpha_{\varepsilon+V}\\
&-u_\varepsilon\left(u_{\varepsilon-V}+u_{\varepsilon+V}\right)\alpha_\varepsilon+u_\varepsilon u_{\varepsilon-V}\left(\beta_{\varepsilon-V}-\beta_\varepsilon\right)\\
&+u_\varepsilon u_{\varepsilon+V}\left(\beta_{\varepsilon+V}-\beta_\varepsilon\right)+\mathrm{u}_\varepsilon\left(\alpha_{\varepsilon-V}-\alpha_{\varepsilon+V}\right)+\left(\mathrm{u}_{\varepsilon+V}-\mathrm{u}_{\varepsilon-V}\right)\alpha_\varepsilon]\,. \qquad (22)
\end{aligned}
$$

In linear approximation, when $\alpha_\varepsilon = 0$ and $\beta_\varepsilon = -\tanh(\varepsilon/2\tau)$, a tunneling source for the NS contact was obtained by Volkov and Zaitzev (1975). The source (22) in the kinetic equation (19) describes tunneling injection only of normal excitations, whereas the processes of direct particle transition into supercurrent contribute in total tunnel current.

To obtain the tunneling current, one should calculate the right-hand side of the continuity equation by integration of equation (13) with respect to energy. In the right-hand side after integration a non-zero contribution is made only by the tunneling source

$$
\begin{aligned}
&\frac{mp_0e}{\pi^2}\left\{\frac{\pi e}{c}D\nabla(\Delta\beta\Delta A)-D\nabla\int_\Delta^x \mathrm{d}\varepsilon u_\varepsilon\nabla\alpha_\varepsilon\right\}\\
&=2w\int_\Delta^\infty \mathrm{d}\varepsilon\left[u_\varepsilon u_{\varepsilon-V}\left(\beta_{\varepsilon-V}-\beta_\varepsilon\right)+u_\varepsilon u_{\varepsilon+V}\left(\beta_\varepsilon-\beta_{\varepsilon+V}\right)\right.\\
&\quad\left.-\left(\mathrm{u}_{\varepsilon-V}+\mathrm{u}_{\varepsilon+V}\right)\alpha_\varepsilon+\mathrm{u}_\varepsilon\left(\alpha_{\varepsilon-V}+\alpha_{\varepsilon+V}\right)\right]\frac{mp_0e}{\pi^2} \qquad (23)
\end{aligned}
$$

For the equilibrium distribution function, $\alpha_{\varepsilon=0}$ and $\beta_\varepsilon = -\tanh(\varepsilon/2T)$, this expression given the Giaever current [see review by Douglass and Falicov (1964)].

The first term in the left-hand side of eq. (23) is the divergence of the London current, the second term is that of the normal current. As far as the stationary situation is examined, the time derivative of the charge density is absent. The value in the right-hand side of eq. (23) is the total tunnel current. Note that the functions without an argument shift of value V are relevant to the superconductor of interest.

Thus, the kinetic equation (19) with the phonon collision integral (21) and the tunneling source (22) along with the continuity equation (23) allow, in principle, to solve kinetic problems in the situation with shift of a chemical potential. Electroneutrality is performed automatically due to the shift of the initial energy. Responsible for this is an electrostatic potential which in the static case provides a small contribution of the order ϕ/ε_r to all the values except for charge density.

27.3 3. Superconductivity stimulation by electromagnetic field

In this section we observe the effect of superconductivity stimulation by microwaves. As indicated in the Introduction, this effect is due to transfer of quasiparticles by the electromagnetic field into the region of large energies. Consequently, effective cooling occurs in the region of energies with a singularity in the density of states.

We will first assume a superconductor film to be thin enough that phonons interact mainly with the walls of the sample but not with electrons. In this case the phonon system serves as a thermostat for electrons. Therefore, the phonon distribution function may be in equilibrium.

The equations derived in the previous section describe the situation without branch imbalance. If taking $n_\varepsilon = 1 - n_{-\varepsilon}$ and $t_\varepsilon = 0$ in eqs. (19)-(21), we get at $\varepsilon > 0$

$$
\begin{aligned}
I_\varepsilon = 2\alpha u_\varepsilon \Bigg\{ & \left[1 + \frac{\Delta^2}{\varepsilon(\varepsilon-\omega)}\right] u_{\varepsilon-\omega}\left(n_{\varepsilon-\omega} - n_\varepsilon\right) \\
& - \left[1 + \frac{\Delta^2}{\varepsilon(\varepsilon+\omega)}\right] u_{\varepsilon+\omega}\left(n_\varepsilon - n_{\varepsilon+\omega}\right) \\
& - \left[1 - \frac{\Delta^2}{\varepsilon(\omega-\varepsilon)}\right] u_{\omega-\varepsilon}\left(n_\delta + n_{\omega-\varepsilon} - 1\right)\theta(\omega-\Delta-\varepsilon)\Bigg\} \\
\alpha = D\left(\frac{e}{c}\right)^2 A_\omega A_{-\omega} & \qquad (24)
\end{aligned}
$$

In the situation with symmetry between electrons and holes the collision integral is

$$
\begin{aligned}
I_\varepsilon = \frac{2g}{\omega_{\mathrm{D}}^2} \Bigg\{ & \int_1^\varepsilon \mathrm{d}\varepsilon_1 \left(\varepsilon-\varepsilon_1\right)^2 u_\varepsilon u_{\varepsilon_1}\left(1 - \frac{\Delta^2}{\varepsilon\varepsilon_1}\right) \\
& \times \left[n_\varepsilon\left(1-n_{\varepsilon_1}\right)\left(1+N_{\varepsilon-\varepsilon_1}\right) - n_{\varepsilon_1}\left(1-n_\varepsilon\right)N_{\varepsilon-\varepsilon_1}\right] \\
& + \int_\varepsilon^\infty \mathrm{d}\varepsilon_1 \left(\varepsilon_1-\varepsilon\right)^2 u_\varepsilon u_{\varepsilon_1}\left(1 - \frac{\Delta^2}{\varepsilon\varepsilon_1}\right) \\
& \times \left[n_\varepsilon\left(1-n_{\varepsilon_1}\right)\mathrm{N}_{\varepsilon_1-\varepsilon} - n_{\varepsilon_1}\left(1-n_\varepsilon\right)\left(1+N_{\varepsilon_1-\varepsilon}\right)\right] \\
& + \int_A^\infty \mathrm{d}\varepsilon_1 \left(\varepsilon_1+\varepsilon\right)^2 u_\varepsilon u_{\varepsilon_1}\left(1 + \frac{\Delta^2}{\varepsilon\varepsilon_1}\right) \\
& \times \left[n_\varepsilon n_{\varepsilon_1}\left(1+N_{\varepsilon+\varepsilon_1}\right) - \left(1-n_\varepsilon\right)\left(1-n_{\varepsilon_1}\right)N_{\varepsilon+\varepsilon_1}\right]\Bigg\} \qquad (25)
\end{aligned}
$$

Close to the transition temperature when $\Delta \ll T$, if the field frequency is not large, $\omega \ll T$, then the typical scale of correction to the equilibrium part of distribution function is small compared to temperature. In this case, non-local terms will be small and the collision integral (25) can be written in τ approximation, as shown by Eliashberg (1971):

$$
I_\varepsilon = 2\gamma u_\varepsilon \left(n - n_{\mathrm{F}}\right)_\varepsilon, \quad \gamma = 7\zeta(3)\frac{gT^3}{\omega_{\mathrm{D}}^2}. \qquad (26)
$$

The BCS equation for the order parameter is to be added to the equations derived. Close to the critical temperature it differs from the usual GinzburgLandau equation only by a nonequilibrium term

$$
\begin{aligned}
& \frac{T_{\mathrm{c}} - T}{T} - \frac{7\zeta(3)}{8\pi^2}\frac{\Delta^2}{T^2} - \frac{\pi}{T}\alpha\left(1 + b\frac{\omega^2}{\gamma T}\right) + F = 0 \\
& F = -2\int_\Delta^\infty \frac{\mathrm{d}\varepsilon}{\sqrt{\varepsilon^2-\Delta^2}}\left(n - n_{\mathrm{F}}\right)_\varepsilon \qquad (27)
\end{aligned}
$$

Here, the

$$
-\frac{\pi}{T}\alpha\frac{b\omega^2}{\gamma T}(b \approx 1)
$$

term is added to the Ginzburg-Landau equation (22). It cannot be calculated by using τ-approximation for a collision integral since the relevant part of distribution function is localized on energies of the order of the temperature (Ivlev et al. 1973). The value b can be obtained only from a numerical solution of the kinetic equation, in which ail the terms are taken into account in the collision integral.

At high field frequencies the order parameter oscillates weakly and depends on the square of the field averaged in time:

$$\overline{A^2(t)} = 2A_{t,}A$$

We also need the relation between the vector potential and the current density. Later we consider a one-dimensional sample, such as a long superconductor with small cross dimensions compared to the coherence length and I.ondon penetration depth. Due to electroneutrality the current density will be constant along such a sample. Close to the transition temperature the current density has the form

$$j_\omega = \sigma \frac{\mathrm{i}\omega}{c} A_\omega - \frac{\pi\Delta^2}{2Tc}\sigma A_\omega. \tag{28}$$

In the temperature range

$$\omega \gg \Delta^2/T \tag{29}$$

to be examined later one may neglect the London current. In this situation if either current or voltage is given or not, the vector potential does not depend on the order parameter, $j_\omega = \sigma(i\omega/c)A_\omega$. Nonlinear corrections to this ratio are insignificant in comparison to effects of non-linearity in eq. (27).

Thus, the system of equations (24), (26), (27) enables one to calculate nonequilibrium stationary states of a superconductor under electromagnetic radiation.

We first consider the energy dependence of the distribution function in linear approximation on microwave power. It follows from eqs. (24) and (26) with the notation $n'_\varepsilon = n_\mathrm{e} - n_\mathrm{F}$ that

$$n'_\varepsilon = \frac{\alpha}{\gamma}\frac{\omega}{4T}\Big[\frac{\varepsilon(\varepsilon-\omega)+\Delta^2}{\varepsilon\sqrt{(\varepsilon-\omega)^2-\Delta^2}}\theta(\varepsilon-\Delta-\omega) - \frac{\varepsilon(\varepsilon+\omega)+\Delta^2}{\varepsilon\sqrt{(\varepsilon+\omega)^2-\Delta^2}}\theta(\varepsilon-\Delta) + \frac{\varepsilon(\omega-\varepsilon)-\Delta^2}{\varepsilon\sqrt{(\omega-\varepsilon)^2-\Delta^2}}\theta(\varepsilon-\Delta)\theta(\omega-\Delta-\varepsilon)\Big]. \tag{30}$$

Hence it is clear that at $\varepsilon = \Delta + \omega$ the distribution function tends to infinity. If one is to find the form of n_ε, then the next higher orders of expansion in α/γ must be taken into account. This peculiarity is not essential for calculation of the additional term F to the Ginzburg Landau equation (27), however, one should study the details of the distribution function. We consider now the spatially homogeneous problem and point out some properties of the solution of eq. (24):

(1) At $\omega < 24$ there is conservation of number of quasiparticles over the threshold since new particles are not produced by the field:

$$\int_\Delta^\infty \mathrm{d}\varepsilon \frac{\varepsilon}{\sqrt{\varepsilon^2-\Delta^2}} n'_\varepsilon = 0 \tag{31}$$

(2) As the exact distribution function has no singularities, the singularity at $\varepsilon = \Delta + \omega$ in the right-hand side of equation (24) must be compensated. Consequently

$$n(\Delta) = n(\Delta + \omega) \tag{32}$$

(3) In τ-approximation (26) at $\omega < 24$ this property results in

$$n'(\Delta) = 0 \tag{33}$$

The function $n'(\varepsilon)$ is schematically plotted in fig. 1. The dashed line shows the dependence corresponding to (30). Singularities at $\varepsilon = \Delta + n\omega (n = 0, 1, 2, \dots)$ are localized on intervals of energy

$$\varepsilon_1 \approx \Delta(\alpha/\gamma)^2 \tag{34}$$

Naturally, solution (30) is valid almost everywhere if the singularity region (34)

Fig. 1. Energy dependence of the nonequilibrium correction to the distribution function under microwave radiation.

is much smaller than r. Hence we obtain a limit for the field intensity for validity of the linear approximation (30) to calculate the term F in the Ginzburg-Landau equation:

$$\frac{x}{\gamma}\sqrt{\frac{\Delta}{\omega}} \ll 1. \tag{35}$$

Function F has the form

$$\begin{aligned} F =& \frac{\omega^2}{2T}\frac{\alpha}{\gamma}\int_\Delta^x \mathrm{d}\varepsilon \frac{\varepsilon(\varepsilon+\omega)+\Delta^2}{\varepsilon(\varepsilon+\omega)\sqrt{\varepsilon^2-\Delta^2}\sqrt{(\varepsilon+\omega)^2-\Delta^2}} \\ &+ \frac{\alpha\omega}{2T_\gamma}\int_\Delta^{\omega-\Delta} \mathrm{d}\varepsilon \frac{\varepsilon(\varepsilon-\omega)+\Delta^2}{\varepsilon\sqrt{\varepsilon^2-\Delta^2}\sqrt{(\varepsilon-\omega)^2-\Delta^2}}\theta(\omega-2\Delta). \end{aligned} \tag{36}$$

The last term corresponds to production of new quasiparticles by the field. when the frequency is beyond the threshold. Introducing a new function $\phi(p)$: one can rewrite expression (36) as

$$F = \frac{\omega}{2T}\frac{\alpha}{\gamma}\phi\left(\frac{\omega}{\Delta}\right) \quad \phi(p) = \begin{cases} p\ln\frac{8}{ep} & p \ll 1, \\ \pi & 1 \fallingdotseq p. \end{cases} \tag{37}$$

Equation (27) determines the temperature dependence of the order parameter at given frequency and intensity of field. The latter is taken to obey the relation

$$\gamma \ll \omega \ll T. \tag{38}$$

Fig. 2. The dependence of the nonequilibrium term in the Ginzburg-Landau equation on the gap.

Fig. 3. Temperature dependence of the gap under weak microwave radiation. Dashed line shows equilibrium dependence.

The dependence $F(\Delta)$ is shown in fig. 2. At $\Delta = \frac{1}{2}\omega$ the slope is undefined. The temperature dependence is given in fig. 3, where the dashed line shows the

equilibrium function. If the radiation intensity is not too low,

$$\frac{c}{T} \ll \frac{x}{r}, \tag{39}$$

the temperature dependence of the order parameter is determined mainly by the term f and it concides with the equilibrium one only at large A.

The temperature dependence of A obtained has a hysteretic feature. This means, that the jump from one regme to another under changing of temperature has another callse. Now, we will examine the "ideal" picture of superconductivity stimulation. that is, instabilities are supposed not to develop. and the regime remains spatially homogeneous.

From the formula obtained, the distortion of the electron distribution function by radiation yields an increase in the order parameter istimulated superconductivity effect), if the frequency is larger than some critical value (:):

$$\omega_{\mathrm{c}}^2 \ln \frac{xA}{e(1)_{\mathrm{c}}} = 2\pi; \Delta. \tag{0}$$

Hence. as the temperature is increased. T, may be exceeded but the order parameter dres not ger to zero. The maximum temperature in the limit $\omega/T \ll \%/y \ll 1$ is determined by

$$\frac{I(\%)-T}{T_c}=\frac{\omega}{2T}\frac{x}{\gamma}\varphi(2)=\frac{\pi}{2\sqrt{3}}\frac{\omega}{T}\frac{x}{\gamma}. \tag{41}$$

At this temperature 1 - fro. As the temperature is further increased. the order parameter tends to zero.

At high intensities mequalit; (39) does not hold and the perturbation theors on 2 fails. Here radiaton strongly affects the distribution function by spreading it over the range of energes exceeding (%) It is sery important that due to root sirgularity in the density of states there is alady a part of the nonequilibrium distribution function changing at energles which are much smaller than temperature. For this reason the nonequilibrium part of the distribution function can be represented as the sum of two parts,

$$n_2'=n-n_F=n_1(\varepsilon)+n_2(\varepsilon), \tag{42}$$

where the scale of changes in $n_2(2)$ is 7 . but $n_3(2)$ changes at much smaller energies. The collision integral is

$$I\cap n=2;\frac{\varepsilon}{\sqrt{\varepsilon^2-\Delta^2}}n_1(\varepsilon)+I\left\{n_2\right\}. \tag{43}$$

Only the linearized part of the collision integral plays a rote. Such an approximation is valid of $n''=1$. This holds for our case too. From eq. 1251 we
have

$$\begin{aligned}
&I\left\{n_2\right\}=\frac{2g}{\omega_D^2}\\
&\times\left\{\int_1^{\varepsilon}\mathrm{d}\varepsilon_1\left(\varepsilon-\varepsilon_1\right)^2u_\varepsilon u_{\varepsilon_1}\left(1-\frac{\Delta^2}{\varepsilon\varepsilon_1}\right)\left[n_2(\varepsilon)\left(1+N_{\varepsilon-\varepsilon_1}-n_{\varepsilon_1}^{\mathrm{F}}\right)-n_2\left(\varepsilon_1\right)\left(N_{\varepsilon-\varepsilon_1}+n_\varepsilon^{\mathrm{F}}\right)\right]\right.\\
&+\int_\varepsilon^\infty\mathrm{d}\varepsilon_1\left(\varepsilon_1-\varepsilon\right)^2u_\varepsilon u_{\varepsilon_1}\left(1-\frac{\Delta^2}{\varepsilon\varepsilon_1}\right)\left[n_2(\varepsilon)\left(N_{\varepsilon_1-\varepsilon}+n_{\varepsilon_1}^{\mathrm{F}}\right)-n_2\left(\varepsilon_1\right)\left(1-n_\varepsilon^{\mathrm{F}}+N_{\varepsilon_1}-\varepsilon\right)\right]\\
&\left.+\int_\Delta^\infty\mathrm{d}\varepsilon_1\left(\varepsilon_1+\varepsilon\right)^2u_\varepsilon u_{\varepsilon_1}\left(1+\frac{\Delta^2}{\varepsilon\varepsilon_1}\right)\left[n_2(\varepsilon)\left(n_{\varepsilon_1}^{\mathrm{F}}+N_{\varepsilon_1-\varepsilon}\right)+n_2\left(\varepsilon_1\right)\left(n_\varepsilon^{\mathrm{F}}+N_{\varepsilon_1+\varepsilon}\right)\right]\right\}
\end{aligned} \tag{44}$$

Integration of eq. (24) from Δ to ε in the spatially homogeneous case (we assume $\omega<24$) yields

$$\begin{aligned}
&2\gamma\int_\Delta^\varepsilon\mathrm{d}\varepsilon_1u_{\varepsilon_1}n_1\left(\varepsilon_1\right)+\int_\Delta^\varepsilon I\left\{n_2\right\}\mathrm{d}\varepsilon_1\\
&\quad=2\alpha\int_{\varepsilon-\omega}^\varepsilon\mathrm{d}\varepsilon_1u_{\varepsilon_1}u_{\varepsilon_1+\omega}\left(1+\frac{\Delta^2}{\varepsilon_1\left(\varepsilon_1+\omega\right)}\right)\left(n_{\varepsilon_1+\omega}-n_{\varepsilon_1}\right)
\end{aligned}$$

We will consider further the region of energies $\varepsilon\gg\omega$. Since corrections for the distribution function change slowly over the range ω, expansion to first order can be performed in the integral

$$\begin{aligned}
2\gamma\int_\Delta^\varepsilon\mathrm{d}\varepsilon_1u_{\varepsilon_1}n_1+\int_\Delta^\varepsilon T\left\{n_2\right\}\mathrm{d}\varepsilon_1=&2\alpha\omega^2\frac{\varepsilon^2+\Delta^2}{\varepsilon^2-\Delta^2}\left(\frac{\partial n_1}{\partial\varepsilon}+\frac{\partial n_2}{\partial\varepsilon}\right)\\
&+4\alpha\omega^2\frac{\partial n^{\mathrm{F}}}{\partial\varepsilon}\frac{\Delta^2}{\varepsilon^2-\Delta^2}+2\alpha\omega^2\frac{\partial n_{\mathrm{F}}}{\partial\varepsilon}.
\end{aligned} \tag{45}$$

Solution (45) must obey the integral relation

$$2\gamma \int_{\Delta}^{\infty} \mathrm{d}\varepsilon u_{\varepsilon} n_1 + \int_{\Delta}^{\infty} \mathrm{d}\varepsilon_1 I\left\{n_2\right\} = 0 \tag{46}$$

Equation (45) is divided into two parts: one changing considerably and the other changing slowly at energies of the order of temperature. Each is considered separately

$$\begin{gathered} 2\gamma \int_{A}^{\varepsilon} \mathrm{d}\varepsilon_1 u_{\varepsilon_1} n_1 = 2\alpha\omega^2 \frac{\varepsilon^2+\Delta^2}{\varepsilon^2-\Delta^2}\frac{\partial n_1}{\partial \varepsilon} + 4\alpha\omega^2 \frac{\partial n_{\mathrm{F}}}{\partial \varepsilon}\frac{\Delta^2}{\varepsilon^2-\Delta^2} \\ \int_{\Delta}^{\infty} \mathrm{d}\varepsilon u_{\varepsilon} n_1 = 0 \end{gathered} \tag{47}$$

The slowly changing part of the distribution function $n_2(\varepsilon)$ is small on parameter Δ/T compared to n_1. Its contribution to the total number of quasiparticles over the threshold is not small, but it is small in units of F, hence $n_2(\varepsilon)$ is not considered.

By introducing a new variable $y = (1/\Delta)\sqrt{\varepsilon^2-\Delta^2}$, and a new function

$$N(y) = \frac{2T}{\Delta}\int_0^y n_1\left(y_1\right)\mathrm{d}y_1, \tag{48}$$

we obtain from (47) the equation

$$N'' - \frac{y^3}{\left(y^2+2\right)\sqrt{y^2+1}}\beta N = \frac{y}{\left(y^2+2\right)\sqrt{y^2+1}}, \quad \beta = \frac{\gamma\Delta^2}{\alpha\omega^2} \tag{49}$$

to be solved with boundary conditions $N(0) = 0$ and $N(\infty) = 0$; the second condition is the result of conservation of the total number of excitations with energies comparable to temperature.

Basically, this method of solving the initial difference equation (24) consists of making it differential since at large intensity values the characteristic scale of changes in the distribution function is much larger than ω.

Solution (49) can be expressed in the form

$$N(y) = -N_0(y)\int_0^y \mathrm{d}y_1 N_0^{-2}\left(y_1\right)\int_{y_1}^{\infty}\frac{y_2}{\left(y_2^2+2\right)\sqrt{y_2^2+1}}N_0\left(y_2\right)\mathrm{d}y_2, \tag{50}$$

where $N_0(y)$ is the solution of the corresponding homogeneous equation decreasing at infinity. It is best of all to find the form of $N_0(y)$ in the limiting cases of low and high intensities.

First of all we consider low intensities $\beta \gg 1$, i.e., $\alpha/\gamma \ll \Delta^2/\omega^2$. Then in equation (49) small y are of importance:

$$N'' - \frac{1}{2}y^3\beta N = \frac{1}{2}y. \tag{51}$$

The solution of the homogeneous equation is

$$N_0(y) = f\left(\frac{1}{5}\sqrt{2}y^{5/2}\beta^{1/2}\right), \tag{52}$$

where

$$f(\xi) = \int_1^{\infty}\frac{\mathrm{e}^{-\xi t}}{\left(t^2-1\right)^{7/10}}\,\mathrm{d}t$$

Then one can readily find N along with the distribution function at small and large values of the variable compared to $\beta^{-15}, y^{\cdot \ll \beta^{-15}}$ corresponds to small ξ :

$$N(y) = -N_0(0) \int_0^y \mathrm{d}y_1 N_0^{-2}(y_1) \int_0^\infty \frac{1}{2} y_2 N_0(y_2)\, \mathrm{d}y_2;$$

at $y \gg \beta^{-1/5}\xi$ is large and

$$f(\xi) = \xi^{-3/10} \frac{\mathrm{e}^{-\xi}}{2^{7/10}} \int_0^\infty \frac{\mathrm{e}^{-x}}{x^{7/10}}\, \mathrm{d}x$$

Thus, for the distribution function we have

$$n_1(g) = \begin{cases} -\frac{\Delta}{T}\beta^{-2/5} \frac{\pi}{2(40)^{1/5}} \left[\Gamma\left(\frac{1}{5}\right) \sin \frac{2}{5}\pi\right]^{-1} & y \ll \beta^{-1/5}, \\ \frac{\Delta}{\beta T} y^{-3} & \beta^{-1/5} \ll y, \end{cases} \tag{53}$$

The value F can be obtained with logarithmic accuracy when the principal change of the distribution function is localized in the region $y \ll 1$:

$$\begin{aligned} F &= -2 \int_0^\infty \frac{\mathrm{d}y}{\sqrt{y^2+1}} n_1(y) \\ &= 2 \int_0^\infty \mathrm{d}y \left(1 - \frac{1}{\sqrt{y^2+1}}\right) n_1(y) = \int_0^\infty y^2 n_1(y) \mathrm{d}y = \frac{\Delta}{5\beta\ \mathrm{T}} \ln \beta. \end{aligned} \tag{54}$$

We have taken into account that the integral in semi-infinite limits from the distribution function equals zero due to conservation of the total number of quasiparticles.

As was reported, formula (54) is valid for $\beta \gg 1$. Moreover, the given approach holds if the spreading of the distribution function over energy exceeds ω. The last condition is equivalent to

$$\frac{\alpha}{\gamma} \sqrt{\frac{\Delta}{\omega}} > 1,$$

in other words $v^{\omega/A} \ll \beta^{-1/5}$. This is contrary to condition (35) for applicability of perturbation theory. Thus, the condition of validity for eq. (54) is

$$\sqrt{\frac{\omega}{\Delta}} \ll \frac{\alpha}{\gamma} \ll \frac{A^2}{\omega^2}. \tag{55}$$

Note, that at $\alpha_1 \hat{\gamma} \sim_V \overline{(1)/\Lambda}$ on the verge of applicability of perturbation theory formula (54) matches expression (37) with logarithmic accuracy, if $\omega \ll A$.

In the opposite limiting case with sufficiently high intensities, $\beta \ll 1 (\alpha/\gamma \gg \Delta^2/(\omega^2)$, the solution has the following structure. The principal change of nonequilibrium addition to the distribution function occurs on the scale $y \approx 1$, and there is a part with small amplitude but attenuating weakly (over distances $y \approx \beta^{-1/2}$). This is of no importance for calculating F. The solution
of the homogeneous equation $N_0(y)$ is changed just over such long distances. Hence to calculate a localized part of n_1 one takes $N_{0'}$ constant. In this case formula (50) gives

$$n_1(y) = -\frac{\Delta}{2T} \left(\frac{\pi}{2} - \tan^{-1} \sqrt{y^2+1}\right) + n_1' \tag{56}$$

The slowly changing part n_1' has the form

$$n_1' = C\frac{A}{T} f\left(y\beta^{1/2}\right), \tag{57}$$

with f a function of the order of unity. The contribution of function $n_1(y)$ to the total number of quasiparticles is

$$\int_\Delta^\infty \mathrm{d}\varepsilon u_\varepsilon n_1 \approx -C_1\frac{\Delta}{T}\ln\beta + \int_0^\infty n_1'(y)\mathrm{d}y$$
$$\approx -C_1\frac{\Delta}{T}\ln\beta + C\frac{\Delta}{T}\beta^{-1/2}\int_0^\infty f(z)\mathrm{d}z = 0 \quad C_1 \approx 1$$

Now we have an estimation $C \approx \beta^{1/2}\ln\beta \ll 1$, which confirms the abovementioned:
Calculating the value of F by eq. (56), we have

$$F = \frac{\pi}{4}\ln(3+2\sqrt{2})\frac{\Delta}{T} \approx 1.38\frac{\Delta}{T}. \tag{58}$$

Thus, formulae (54) and (58) yield values of F in the limiting cases $\Delta^2/\omega^2 \gg \alpha/\gamma$ and $\Delta^2/\omega^2 \ll \alpha/\gamma$, correspondingly.

To find the temperature dependence of the order parameter one has to use equation (27). It is of importance which is the given value, the amplitude of the current density or that of the electric field. This circumstance depends on experimental conditions. We first consider the situation with the electric field given, i.e., α is constant. The dependence of Δ/T on temperature is shown in fig. 4.

Similarly to the low-power case, the transition from the super conducting to the normal state will occur by a jump with a finite value of the order parameter

$$\Delta \approx \omega(\alpha/\gamma)^{1/2}. \tag{59}$$

At $\Delta \lesssim \omega$ the theory fails. The value of the order parameter (39) corresponds to the highest possible temperature which we also call critical, $T_c(\alpha)$:

$$\frac{T_\mathrm{c}(\alpha) - T_\mathrm{c}}{T_\mathrm{c}} = C\sqrt{p} - \pi p\left(b + \frac{\gamma T}{\omega^2}\right) \quad p = \frac{\alpha}{\gamma}\frac{\omega^2}{T^2}. \tag{60}$$

$C\sqrt{p}(C \approx 1)$ is the maximum value of F. Our theory holds up to $A/T \ll 1$.

Fig. 4. Temperature dependence of the gap under strong microwave radiation.

From (59) it follows that it is valid at $p \ll 1$ or

$$\alpha/\gamma \ll T^2/\omega^2. \tag{61}$$

Let us consider two frequency limits

(1) $\omega \ll \sqrt{\gamma T_c}$.

In this case expression (60) has its maximum at $p = p_c$:

$$\left[\frac{T_c(\alpha) - T_c}{T_c}\right]_{\max} = \frac{c^2}{4\pi}\frac{\omega^2}{\gamma T_c}, \quad p_c = \frac{\alpha_c}{\gamma'}\frac{\omega^2}{T^2} = \left(\frac{c}{2\pi}\frac{\omega^2}{\gamma T}\right)^2 \ll 1.$$

The dependence of the critical temperature on radiation power is shown in fig. 5 as a curve with a maximum. Thus, we have an optimum magnitude of microwave power giving the maximum stimulation of superconductivity.

$$\sqrt{\gamma}T_c \ll \omega \ll T_c. \tag{2}$$

In this case formula (60) also gives formally a curve with a maximum, however. its position is defined by values $p \approx 1$ lying on the edge of applicability of the theory. Apparently, in this case there is a similar picture (fig. 5). Hence the value

$$\left[\frac{T_c(\alpha) - T_c}{T_c}\right]_{\max} \approx 1 \tag{28}$$

We now take a situation with the current density given. Here from eq.

Fig. 5. The dependence of the relative increase of critical temperature on radiation power.

follows

$$\alpha = D\left(\frac{e}{c}\right)^2 \boldsymbol{A}_\omega A_{-\omega} = \frac{\pi^2}{m p_0 \omega^2 \sigma}\frac{j_\omega j_{-\omega}}{1 + \left(\frac{\pi\Delta^2}{2T\omega}\right)^2}. \tag{62}$$

If $\alpha/\gamma \ll T/\omega$, the London current can be neglected. The results coincide with previous ones. At $\alpha/\gamma \geqslant 7/\omega$ according to eq. (62) a formal change in $\alpha \to (2\ \mathrm{T}\omega/\pi\Delta^2)^2\,\alpha$ can be performed. We consider case (2), i.e., frequencies $\omega \gg \sqrt{\gamma T_c}$. Then the value of the order parameter determining the transition into the normal state is

$$\frac{\Delta}{T} \approx \left(\frac{\alpha}{\gamma}\right)^{1/6}\left(\frac{\omega}{T}\right)^{2/3}. \tag{63}$$

The dependence of the transition temperature on power has the form

$$\frac{T_{\mathrm{c}}(\alpha)-T_{\mathrm{c}}}{T_{\mathrm{c}}} \approx \left(\frac{\alpha}{\gamma}\right)^{1/6}\left(\frac{\omega}{T}\right)^{2/3}. \tag{64}$$

The maximum increase in transition temperature on the edge of applicability of the theory is

$$\left[\frac{T_c(x)-T_c}{T_c}\right]_{\max} \approx 1,$$

as in the previous case. Aslamazov and Gavrilov (1980), considering superconductivity stimulation, paid attention to the significance of the depairing term in the Ginzburg-Landau equation.

The dependence of transition temperature on power, when current density is given and the frequency satisfies $\omega \gg \sqrt{\gamma T_{\mathrm{c}}}$, is shown in fig. 6.

To obtain the temperature dependence of the critical current under conditions of superconductivity stimulation, note that near the transition temperature there is a typical value $eA_0/c \ll \Delta$, where the static vector potential A_0 is associated with the direct current:

$$j = -\frac{\pi\sigma}{2T_c}\frac{\Delta^2}{T}A_0 \tag{65}$$

With the Ginzburg-Landau equation (27) the dependence of the current squared the on order parameter can be obtained

$$j^2 = \frac{mp_0}{2}\sigma T_{\mathrm{c}}^3\frac{\Delta^4}{T^4}\left[\frac{T_{\mathrm{c}}-T}{T} - \frac{7\xi(3)}{8\pi^2}\frac{\Delta^2}{T^2} + F\right]. \tag{66}$$

The maximum current is reached at $\Delta = \frac{1}{2}\omega$ if $\omega/T \ll \alpha/\gamma \ll 1$:

$$j_{\max} = \frac{21\sqrt{3}\breve{\zeta}(3)}{64\pi^2}j_0\left(\frac{\omega}{T}\right)^2\left(\frac{T_{\max}-T}{T_{\mathrm{e}}}\right)^{1/2} \approx 0.06 j_0\left(\frac{\omega}{T}\right)^2\left(\frac{T_{\max}-T}{T_{\mathrm{c}}}\right)^{1/2}. \tag{67}$$

The current j_0 is determined in such a way that in the absence of radiation $j_{\mathrm{c}} = j_0\left[(T_{\mathrm{c}}-T)/T_{\mathrm{c}}\right]^{3/2}$. Spreading of root singularity in the density of states associated with constant current, occurs at energies

$$(\varepsilon-\Delta) \approx \Delta^{1/3}\left[D\left(\frac{e}{c}A_0\right)^2\right]^{2/3}.$$

If this region is smaller than ω, then the right-hand side of formula (66) does not depend on A_0, and the maximum current is defined by (67). Since

$$D\left(\frac{e}{c}-A_0\right)^2 \approx (T_{\max}-T),$$

formula (67) is valid not too far from $T_{\max}$ $(T_{\max}-T) \ll \omega$.

Hitherto, we have considered the case of relatively low frequencies, $\omega \ll T$. This corresponds to the microwave range. In the literature the frequency limits of stimulated superconductivity have been discussed by Latyshev and Nad' (1974) and Dmitriev and Khristenko (1978). A lower frequency limit is found from formula (40). We will show that in "the ideal model" neglecting the nonequilibrium of the phonon system, the formation of instabilities and heterogeneity of a sample, the upper frequency limit of the effect is $\omega_{\max} \approx T$.

We consider here the case with frequencies $\omega \gg T$. Close to the transition temperature $\Delta \ll T$, and in order to calculate the distribution function localized at energies of the order of

Fig. 6. The dependence of the relative increase of the critical temperature on power in conditions of fixed high-frequency current density.

the temperature one can put $\Delta = 0$ in the kinetic equation (24). In linear approximation on intensity, as shown by Vardanyan and Ivlev (1973), this gives

$$
\begin{aligned}
n'_\varepsilon \Bigg(& \frac{\varepsilon^3}{3T^3} + 2\int_0^\infty \frac{x^2\,\mathrm{d}x}{\mathrm{e}^x-1} + 2\int_{\varepsilon/T}^\infty \frac{x^2\,\mathrm{d}x}{\mathrm{e}^x+1} + \frac{4\varepsilon}{T}\int_0^{\varepsilon/T} \frac{x\,\mathrm{d}x}{\mathrm{e}^x+1} + \frac{2\varepsilon^2}{T^2}\int_{\varepsilon/T}^\infty \frac{\mathrm{d}x}{\mathrm{e}^x+1} \\
& + \frac{2}{\mathrm{e}^{\varepsilon/T}+1}\int_{\varepsilon/T}^\infty x^2 n'_x \mathrm{d}x + \frac{4\varepsilon/T}{\mathrm{e}^{\varepsilon/T}+1}\int_0^{\varepsilon/T} x n'_x \mathrm{d}x + \frac{2\varepsilon^2/T^2}{\mathrm{e}^{\varepsilon/T}+1}\int_{\varepsilon/T}^\infty n'_x \mathrm{d}x \\
& + \int_{\varepsilon/T}^\infty \frac{x^2\,\mathrm{d}x}{\mathrm{e}^x-1} n'\left(x-\frac{\varepsilon}{T}\right) - \int_0^\infty \frac{x^2\mathrm{e}^x}{\mathrm{e}^x-1}\,\mathrm{d}x n'\left(x+\frac{\varepsilon}{T}\right) - \int_0^{\varepsilon/T} \frac{x^2\,\mathrm{d}x}{\mathrm{e}^x-1} n'\left(\frac{\varepsilon}{T}-x\right) \\
& = 7\xi(3)^\alpha \begin{cases} \tanh\frac{\varepsilon}{2T} - \left[\exp\left(\frac{\omega-\varepsilon}{T}\right)+1\right]^{-1} & \varepsilon < \omega \\ \left[\exp\left(\frac{\varepsilon-\omega}{T}\right)+1\right]^{-1} & \omega < \varepsilon \end{cases}
\end{aligned}
\tag{68}
$$

The value of γ is determined by formula (26). The function n'_t must obey the normalization condition obtained by integration of the kinetic equation (24). This condition has the meaning of a balance of the total number of quasiparticles

$$
\begin{aligned}
\int_0^x n'_r \frac{\mathrm{d}\varepsilon}{T} \Bigg(& 2\int_0^r \frac{x^2\,\mathrm{d}x}{\mathrm{e}^x+1} + \int_T^r \frac{x^2\,\mathrm{d}x}{\mathrm{e}^x-1} \\
& + \frac{4\varepsilon}{T}\int_0^\infty \frac{x\,\mathrm{d}x}{\mathrm{e}^x+1} + \frac{2\varepsilon^2}{T^2}\int_0^\infty \frac{\mathrm{d}x}{\mathrm{e}^x+1}\Bigg) = 7\zeta(3)\frac{\alpha\omega}{\gamma}.
\end{aligned}
\tag{69}
$$

The basic part of the distribution function, localized at $\varepsilon \approx T$ must be found from the solution of the homogeneous equation (68). The source in the righthand side is necessary only for normalization. Neglecting the right-hand side of eq. (68) at $\varepsilon \ll \omega$, we obtain n'_ε at $T \ll \varepsilon$ from the equation

$$
\frac{\varepsilon^3}{3T^3} n'_\varepsilon - \int_0^\infty x^2 n'\left(x+\frac{\varepsilon}{T}\right)\mathrm{d}x = 0 \tag{70}
$$

Hence the solution is

$$
n'_\varepsilon \approx \varepsilon^{-4} \tag{71}
$$

At $\varepsilon \ll T$ equation (68) becomes

$$n'_\varepsilon \int_0^\infty \frac{x^2\,\mathrm{d}x}{\mathrm{e}^x+1} = \frac{\varepsilon}{4T}\int_0^\infty x^2 n'_x \mathrm{d}x + \frac{\varepsilon}{2T}\int_0^\infty \frac{x^2}{\tanh\frac{1}{2}x}\frac{\partial n'}{\partial x}\,\mathrm{d}x \tag{72}$$

giving $n'_c \propto \varepsilon$. With the normalization condition (69) taken into account, the distribution function may finally be represented in the form

$$n'_t = \frac{\alpha}{\gamma}\frac{\omega}{T} f\left(\frac{\varepsilon}{T}\right) \quad f(x) \approx \begin{cases} x & x \ll 1, \\ x^{-4} & 1 \ll x. \end{cases} \tag{73}$$

Calculating value F by means of eq. (27),

$$F = -\frac{2\alpha}{\gamma}\frac{\omega}{T}\int_0^\infty \frac{\mathrm{d}x}{x} f(x) \tag{74}$$

we have $F < 0$. No superconductivity stimulation by radiation with frequency $\omega \gg T$ takes place since under these conditions the basic mechanism is the increase of total number of quasiparticles. Thus one can conclude that the maximum frequency of radiation stimulating superconductivity is $\omega_{\max} \approx T$, if nonequilibrium of the phonon system is neglected.

In real conditions the acoustic contact of a superconducting film with the substrate can be imperfect. It means that before leaving a sample phonons have interacted with electrons. So the distribution function of phonons will differ from the nonequilibrium one. Chang and Scalapino (1971) have calculated just this situation. They proceeded from the kinetic equation (24) with an electron collision integral (25) and a phonon collision integral of the type

$$\begin{aligned} &[N_\omega - N_\omega(T)]\,/\tau_{\mathrm{es}} = \\ &\lambda\left\{2\int_{\Delta t\omega}^\infty \mathrm{d}\varepsilon \frac{\varepsilon(\varepsilon-\omega)-\Delta^2}{\sqrt{\varepsilon^2-\Delta^2}\sqrt{(\varepsilon-\omega)^2-\Delta^2}}\left[n_\varepsilon\left(1-n_{\varepsilon-\omega}\right)\left(1+N_\omega\right)-\left(1-n_\varepsilon\right)n_{\varepsilon-\omega}N_\omega\right]\right. \\ &\quad\left.+\int_1^{\omega-\Delta}\mathrm{d}\varepsilon\frac{\varepsilon(\omega-\varepsilon)+\Delta^2}{\sqrt{\varepsilon^2-\Delta^2}\sqrt{(\omega-\varepsilon)^2-\Delta^2}}\left[n_\varepsilon n_{\omega-\varepsilon}\left(1+N_\omega\right)-\left(1-n_\varepsilon\right)\left(1-n_{\omega-\varepsilon}\right)N_\omega\right]\right\} \end{aligned} \tag{75}$$

The value λ here is proportional to the constant of electron phonon interaction, τ_{es} is the characteristic time of electron interaction with the substrate, and $N_\omega(T)$ is the equilibrium distribution function of substrate phonons with temperature T. At given substrate temperature $T = 0.9T_{\mathrm{c}}$, various frequencies and τ_{es}, in particular, the dependence of the order parameter on radiation power was calculated. Similarly to the case of equilibrium phonons the gap reaches its maximum value at some radiation power. The upper frequency limit of the effect is here less than $\frac{3}{2}\boldsymbol{A}$. Moreover, it was found that the effect reduces when τ_{es} is increased. Due to the τ-approximation for the collision integral to be valid near the transition temperature, the total number of quasiparticles is constant if $\omega < 24$. On deviation from T_{c} down the nonlocal part of the collision integral becomes significant. The total number of quasiparticles may decrease. As shown by Chang and Scalapino (1977), at $T = 0.9T_{\mathrm{c}}$ and a good contact with a thermostat there is an additional mechanism of superconductivity stimulation owing to a decrease of the total number of quasiparticles. This mechanism becomes important not very close to T_{c}. At $T = 0.9T_{\mathrm{c}}$ it is already important. Thus, as seen from these calculations, the stimulation of superconductivity by microwaves is more effective if there is a good acoustic connection between a sample and a thermostat. The influence of sample heating on the jump to the normal state by increasing the power was theoretically calculated by Latyshev and Nad' (1976).

Kozub and Spivak (1981) have studied the influence of electromagnetic radiation on a bulk superconductor placed in a magnetic field parallel to its surface. The value of the energy gap

on the surface is less than that in the volume, therefore, a fraction of the surface quasiparticles appeared to be blocked in this region. The influence of radiation on these particles is much stronger since unblocked particles can readily diffuse into the bulk. Influencing the caught particles, radiation leads to superconductivity stimulation as well. The critical magnetic field is then increased.

27.4 4. Superconductivity stimulation by acoustic waves

As mentioned in the Introduction, acoustic waves of sufficiently large amplitude can serve as a source for pumping energy into the electronic system. The frequency of these waves must be in the range of the typical energy parameters of the superconductor, in particular the gap. The mechanism of acoustic wave attenuation in metal has electron character. Hence the disturbance of the electron system will be considerable.

If the superconducting sample is a thin film on substrate, in which an acoustic wave propagates, then this situation is similar to that with electromagnetic radiation. Analogous effects will also take place in a bulk superconductor through which sound is propagating. If the temperature is close to
the critical one, then $\Delta \ll T$. The basic change of the distribution function will localize at energies of order A. Most of the thermal quasiparticles play the role of a thermostat.

In experiments by Tredwell and Jacobsen (1975) the nonequilibrium state was created in a thin superconducting film by longitudinal waves. They gave. together with interesting experimental data, the kinetic equation for the case of acoustic pumping. This equation is analogous to eq. (24), except for the coherence factor corresponding to the scale potential $\varepsilon(\varepsilon-\omega)-d^2$ and the value α being defined differently. Close to the transition temperature the correction to the distribution function is

$$n'_\varepsilon = \frac{\alpha_1}{\gamma}\frac{\omega}{4T}\Big[\frac{\varepsilon(\varepsilon-\omega)-\Delta^2}{\varepsilon\sqrt{(\varepsilon-\omega)^2-\Delta^2}}\theta(\varepsilon-\Delta-\omega)-\frac{\varepsilon(\varepsilon+\omega)-\Delta^2}{\varepsilon\sqrt{(\varepsilon+\omega)^2-\Delta^2}}\theta(\varepsilon-\Delta) + \frac{\varepsilon(\omega-\varepsilon)+\Delta^2}{\varepsilon\sqrt{(\omega-\varepsilon)^2-\Delta^2}}\theta(\varepsilon-\Delta)\theta(\omega-\Delta-\varepsilon)\Big]. \tag{76}$$

The value x_1 is proportional to the absorbed power of an acoustic wave, and it depends on the dirtiness of a sample. The interaction of a longitudinal sound wave with metal is described by a deformation potential (see, e.g.. Kittel 1963). In the system of coordinates moving in each point along with lattice displacement, the addition to the Hamiltonian, associated with the acoustic wave, is in the form:

$$H' = (\operatorname{div} u)(p\cdot n)(v\cdot n) - \phi$$

where u is the lattice displacement vector, n is the unit vector directed along the wave vector of the sound. ϕ is determined from the condition of electroneutrality, equivalent with good accuracy to the requirement $\overline{H'}=0$ (the overbar means averaging over angles). Hence

$$\phi = \frac{pv}{3}\operatorname{div} u.$$

As is known, attenuation is determined by the Landau mechanism at $k\cdot l \gg 1$, by the diffusion mechanism at $kl \ll 1$. In the first case $\alpha_1 \approx \varepsilon_{\mathrm{F}}^2(ku)^2/kv$, in the second case $x_1 \approx \tau\varepsilon_1^2\,(h(t)^2$ in agreement with the Pippard formula for attenuation of longitudinal sound. We express the energy flux of the acoustic wave $W \approx \rho s(\hat{\partial}u/\hat{\partial}t)^2$ through the α_1/γ value in the dirty case. It is more valid for films

$$W \approx g\left(\frac{T^4}{\varepsilon_{\mathrm{F}}^3\omega_{\mathrm{D}}}\right)\frac{1}{T\tau}\frac{x_1}{\gamma}\rho v s^2 \cdot 10^{-7}\ \mathrm{W/cm^2}, \tag{77}$$

where g is the electron phonon constant, and s the velocity of sound. Substituting typical values of metal parameters $r_{\mathrm{F}} \approx 10^8$ cms.$s \approx 10^6$ cm s,
$\omega_{\mathrm{D}} \approx 10^2$ K, $\varepsilon_{\mathrm{F}} \approx 10^4$ K, $T \approx 1$ K, we get if $T\tau \approx 1$

$$W \approx \frac{\alpha_1}{\gamma}10^{-1}\ \mathrm{W/cm^2} \tag{78}$$

The nonequilibrium state is established when $x_1, \gamma \sim 1$. Hence the desired power level will be 10^{-1} W/cm^2. Experimentally it is very likely to be achieved.

With the function (76), the nonequilibrium term in the Ginzburg Landau equation acquires the form

$$\begin{aligned} F =& \frac{\omega^2}{2T}\frac{\alpha_1}{\gamma}\int_{\Delta}^{\infty}\mathrm{d}\varepsilon\frac{\varepsilon(\varepsilon+\omega)-\Delta^2}{\varepsilon(\varepsilon+\omega)\sqrt{\varepsilon^2-\Delta^2}\sqrt{(\varepsilon+\omega)^2-\Delta^2}} \\ & -\alpha_1\omega \\ & \gamma 2T\int_{\Delta}^{\omega-\Delta}\mathrm{d}\varepsilon\frac{\varepsilon(\omega-\varepsilon)+\Delta^2}{\varepsilon\sqrt{\varepsilon^2-\Delta^2}\sqrt{(\omega-\varepsilon)^2-\Delta^2}}\theta(\omega-2\Delta) \end{aligned} \tag{79}$$

Expression (36) is similar to this. In the limiting cases it gives

$$F = \begin{cases} \frac{\omega^2}{2\Delta T}\frac{\alpha_1}{\gamma} & \omega \ll \Delta, \\ \pi\Delta\alpha_1\left[\left(1-\frac{1}{\sqrt{3}}\right)-\theta(\omega-2\Delta)\right] & \mid(\omega-2\Delta\mid<\Delta, \\ T\gamma\gamma\left[\begin{array}{ll}\pi\Delta & \alpha_1 \\ -\frac{\pi}{2T} & \Delta \ll \omega.\end{array}\right. \end{cases} \tag{80}$$

The Ginzburg Landau equation for the situation with longitudinal sound differs from eq. (27) by the absence of a depairing term depending on a vector potential:

$$\frac{T_c - T}{T} - \frac{7\zeta(3)}{8\pi^2}\Delta^2 T^2 + F = 0 \tag{81}$$

The temperature dependence of the order parameter following from this is shown in fig. 7. We see that only at $A > \frac{1}{2}()$. gap enhancement occurs. If the acoustic intensity is not too low,

$$\frac{\alpha_1}{i} > \frac{\omega}{T}\frac{7\zeta(3)}{16\pi^3}\frac{\sqrt{3}}{\sqrt{3}-1}, \tag{82}$$

then there is superconductivity at temperatures above critical. Tredwell and Jacobsen (1975) used tunnel technique to measure the gap in the presence of pumping by longitudinal sound. The results were compared with given calculations on the basis of formulae of type (79) and (81). Good agreement between theory and experiment was found.

Superconductivity stimulation can be performed by transverse sound as
well. In this case the addition associated with sound to the Hamiltonian is

$$H' = -\frac{\partial u_x}{\partial z}p_x v_z + \frac{e}{c}v_x A_x, \tag{83}$$

where u_v is the lattice displacement, and sound propagates along the =-axis. The first term is the deformation potential. The vector potential arises from the condition that the Maxwell equations be obeyed. In sufficiently pure superconductors close to the transition temperature

Fig. 7. The temperature dependence of the gap at acoustic pumping. Dashed line shows equilibrium dependence.

attenuation will have electromagnetic character, fitting the last term in (83). On the contrary, in a dirty superconductor attenuation is defined by the deformation mechanism fitting the first term in (83). Consequently, at low impurity concentrations the kinetic equation will be of type (24), at high concentrations it will be of type (76) with a scalar coherence factor. Transverse sound will also lead to superconductivity stimulation. Note that besides this effect, strong nonlinear effects can also occur in the absorption and the velocity of sound, induced by changes in the distribution function homogeneous and isotropic with respect to momentum (30), as shown by Bulyzhenkov and Ivlev (1976a, b).

27.5 5. Superconductivity stimulation by tunnel injection

In the situations we have considered above, the redistribution of quasiparticles leading to superconductivity enhancement was provided by absorption of high-frequency electromagnetic and acoustic field quanta. There is also a way,
first proposed by Parmenter (1961), to influence superconductivity by tunnel injection of quasiparticles. In this situation the stationary state appears, as usual, due to energy exchange with a thermostat.

Processes occurring at tunneling can be depicted very roughly as in fig. 8a, where arrows show electron transfer through a tunnel barrier. Electron transfer through a barrier takes place due to the differences in density of states and distribution function at the same energy value.

It should be noted that tunneling can result in a more varied picture of the nonequilibrium state than can radiation by a monochromatic field, since in this case the magnitudes of the energy gaps do not obligatory coincide, and one can vary the voltage V at will, in analogy with the field frequency.

Tunnel experiments are of interest because details of the current voltage characteristics of contacts can give information on the shape of the distribution function instead of integral characteristics as obtained if only the order parameter is known, as shown by Kaplan et al. (1977). The position of the centre of the energy gap in a superconductor is conventionally called the chemical potential of Cooper pairs,

$$\mu_{\mathrm{p}} = -\frac{1}{2e}\frac{\partial \chi}{\partial t},$$

where χ is the phase of the order parameter $\Delta = |\Delta| \mathrm{e}^{\mathrm{i}}$. The correction to the

Fig. 8. Representation of the tunneling processes: (a) asymmetrical scheme; (b) symmetrical scheme.

chemical potential of normal electrons is given by

$$\mu_{\mathrm{n}} = \int_{-\infty}^{\infty} (n_\varepsilon - n_{\mathrm{F}})\, \mathrm{d}\varepsilon \tag{84}$$

The integration in this formula can be extended to $-\infty$, since the function $(n_\varepsilon - n_F)$ in the real situation is well localized near the Fermi surface. An electrostatic potential ϕ fixing the shift of the zero position of electron energy is found from the conditions of electroneutrality

$$\delta\rho = \frac{3ne^2}{m\mathrm{p}_0} (\mu_{\mathrm{n}} - \phi) \tag{85}$$

The value

$$\Phi = \mu_{\mathrm{n}} - \mu_{\mathrm{p}} = \phi - \frac{1}{2e}\frac{\partial\chi}{\partial t} \tag{86}$$

is a gauge-invariant scalar potential determining the difference between the chemical potentials of normal electrons and Cooper pairs. To be more exact, Φ is the electron-hole imbalance. The theory of such phenomena has been developed in detail by Tinkham and Clarke (1972).

A tunnel contact is a convenient object to create and observe branch imbalance. Although it is not the subject of our review, electron-hole imbalance should be taken into account in the consideration of superconductivity stimulation in tunnel contacts. The only exception is the case of a completely symmetric system of tunneling contacts (fig. 8b). Clearly, in the central sample the electron and hole branches will be equally populated, and $\Phi = 0$. This case is closer to the situation with monochromatic radiation, when Φ is negligible in comparison with the parameter $\omega/\varepsilon_{\mathrm{F}}$.

Figure 9 shows a typical scheme to observe nonequilibrium effects on tunneling. A nonequilibrium is created in a superconductor S by injection through a low-resistance contact G (generator) from a superconductor S′ or normal metal N . If the length L of the contact surpasses the diffusion length L_1 then the distribution function will be homogeneous in the mid region far from the ends. In this region it is necessary to place a high-resistance contact with a detecting superconductor $\mathrm{S_D}$ or normal metal $\mathrm{N_D}$ not influencing the state of superconductor S . The gap and the magnitude of electron-hole imbalance can be measured using the current voltage characteristics of the detector.

In the superconductors S and S′ throughout all the stationary state the chemical potential of Cooper pairs remains constant in space. If the parameter ($L_1 - L$) exceeds the imbalance relaxation length, then at $x = L_1$ normal electrons and Cooper pairs will have the same

potentials. The injection voltage V in this case has an ordinary meaning for the superconductors S and S′. They differ only by the equilibrium electron distributions being shifted an amount V.

Fig. 9. Scheme of the experiment on the creation of nonequilibrium states by tunneling.

Note that if ($L_1 - L$) is smaller than the imbalance relaxation length the determination of the voltage V will not be trivial.

As follows from formula (23), the density of tunneling current through a contact is

$$j_{\mathrm{T}} = \frac{2mp_0e}{\pi^2} w \, \mathrm{d} \int_{\Delta}^{\infty} \mathrm{d}\varepsilon \left[u_\varepsilon u'_{\varepsilon-\mathrm{V}} \left(\beta'_{\varepsilon-\mathrm{V}} - \beta_\varepsilon \right) \right.$$
$$\left. + u_\varepsilon u'_{\varepsilon+\mathrm{V}} \left(\beta_\varepsilon - \beta'_{\varepsilon+\mathrm{V}} \right) - \left(u'_{\varepsilon-\mathrm{V}} + u'_{\varepsilon+\mathrm{V}} \right) \alpha_\varepsilon + u_\varepsilon \left(\alpha'_{\varepsilon-\mathrm{V}} + \alpha'_{\varepsilon+\mathrm{V}} \right) \right], \tag{87}$$

where d is the thickness of the superconductor S . The quantities with subscript ε refer to the superconductor S ; and those with shifted arguments $\varepsilon \pm V$ refer to the superconductor S'. The values β_1, α_2 and u_1 are defined by relation (7), in which only for S' the substitution $\Delta \to \Delta'$ should be made. Evidently, $\alpha_a \propto \Phi$.

Passing uniformly through the contact G into the superconductor S , as seen from fig. 9, the tunneling current then flows along the thin film S . This results in a non-zero current divergence in S $(\operatorname{div} j = j_{\mathrm{T}}/d)$. The current consists of London and normal current; according to (27). Even if the distribution function is homogeneous along the film S , the vector potential changes with coordinate:

$$D \frac{mp_0e^2}{2\pi c} \frac{\Delta^2}{T} A = -\frac{j_{\mathrm{T}}}{d} x. \tag{88}$$

The maximum value, at $x = L$, is of the order of magnitude

$$\left[\frac{D}{\gamma} \left(\frac{e}{c} A \right)^2 \right]^{1/2} \propto \frac{j_{\mathrm{T}}}{dmp_0e} \frac{T}{\Delta^2 \gamma} \frac{L}{l_{\mathrm{D}}}. \tag{89}$$

The diffusion length $l_{\mathrm{D}} \propto \sqrt{D/\gamma}$ used in this theory is introduced here. The part of equation (19) even in ε describes the attenuation of the imbalance α_{e} in space. As shown by Schmid and Schön (1975), Artemenko et al. (1978) and Bulyzhenkov and Ivlev (1979), if $(D/\gamma)(eA/c) \geqslant 1$ then this value starts to affect greatly the imbalance relaxation rate. Consequently, the problem can be thought to be spatially homogeneous only in films short enough that $(D/\gamma)(eA/\mathcal{C})^2 \ll 1$. With relation (89) taken into account, we get the limitation on tunnel current density at given length L of the contact and thickness d of the film:

$$\frac{j_{\mathrm{T}}}{dmp_0e} \frac{T}{\Delta^2 \gamma} \frac{L}{l_{\mathrm{D}}} \ll 1. \tag{90}$$

As stated above, the problem on nonequilibrium states at injection can be thought to be spatially homogeneous only under special limitations on contact linear parameters and tunnel current. This long preface for the problem of injection-stimulated superconductivity is due to the complexity of these phenomena compared to pumping by a monochromatic field.

A detailed calculation of tunnel-injection-stimulated superconductivity for the symmetrical situation of the type fig. 8 b has been made by Aronov and Gurevich (1972). For exactly the same case Peskovatski and Seminozhenko (1976) derived the kinetic equation and calculated the change in the order parameter.

The kinetic equation (19), in which no derivatives with respect to time and coordinate are considered and $A_{10} = 0$, is divided into two parts, one even and one odd in the energy:

$$\frac{1}{2}\left(I_\varepsilon + I_{-\varepsilon}\right) = R_\varepsilon, \quad \frac{1}{2}\left(T_\varepsilon - T_{-\varepsilon}\right) = L_\varepsilon \tag{91}$$

R_i and L_i are obtained by the corresponding symmetrizations of the source (22):

$$\begin{aligned} R_\varepsilon &= w[u'_{\varepsilon-V}\left(\beta'_{\varepsilon-V} - \beta_\varepsilon\right) + u'_{\varepsilon+V}\left(\beta_\varepsilon - \beta'_{\varepsilon-V}\right) \\ &\quad +\alpha'_{\varepsilon-V} + \alpha'_{\varepsilon+V} - u_\varepsilon\left(u'_{\varepsilon-V} + u'_{\varepsilon+V}\right)\alpha_\varepsilon] \\ L_\varepsilon &= w[u_\varepsilon u'_{\varepsilon-V}\left(\beta'_{\varepsilon-V} - \beta_\varepsilon\right) + u_\varepsilon u'_{\varepsilon+V}\left(\beta'_{\varepsilon+V} - \beta_\varepsilon\right) \\ &\quad +u_\varepsilon\left(\alpha'_{\varepsilon-V} - \alpha'_{\varepsilon+V}\right) + \left(\mathrm{u}'_{\varepsilon+V} - \mathrm{u}'_{\varepsilon-V}\right)\alpha_\varepsilon]\,. \end{aligned} \tag{92}$$

The distribution function is determined by the sum of the even and odd parts,

$$n_\varepsilon = \frac{1}{2}\left(1 + \alpha_\varepsilon + \beta_\varepsilon\right), \tag{93}$$

where the even part α_1 defines the chemical potential shift and the odd part β_c is responsible for the change in the order parameter. The anomalous term F in the Ginzburg-Landau equation (27) becomes

$$F = -\int_0^\infty \frac{\mathrm{d}\varepsilon}{\varepsilon} u_\varepsilon \beta'_\varepsilon, \tag{94}$$

where $\beta'_\varepsilon = \beta_\varepsilon + \tanh(\varepsilon/2T)$, then in equilibrium $\beta_\varepsilon = -\tanh(\varepsilon/2T)$. As shown by Artemenko and Volkov (1975), close to the transition temperature

$$\alpha_\varepsilon = \frac{1}{u_\varepsilon}\frac{\Phi}{2T}\frac{1}{\cosh(2\varepsilon/2T)}. \tag{95}$$

Hence from eqs. (91) and (92), supposing S' to be in equilibrium, we have

$$\Phi = V\frac{w}{\gamma}\left(\frac{w}{\gamma} + \frac{\pi\Delta}{8T}\right)^{-1}. \tag{96}$$

In the limit of strong injection $w/\gamma \gg A/T$, this yields $\Phi = V$. In the odd part of the collision integral one should keep the part linear in β'_ε and quadratic in α_ε :

$$\frac{1}{2}\left(I_\varepsilon - I_{-\varepsilon}\right) = \frac{1}{2}\left(I_\varepsilon - I_{-\varepsilon}\right)_{\mathrm{lin}} + A_\varepsilon, \tag{97}$$

in which the following notation is introduced:

$$\frac{1}{2}\left(I_\varepsilon - I\right)_{\text{lin}} = \frac{g}{\omega_D^2}$$
$$\times \left\{ \int_\varepsilon^\infty \mathrm{d}\varepsilon_1 \left(\varepsilon_1 - \varepsilon\right)^2 \left(u_\varepsilon u_{\varepsilon_1} - v_\varepsilon v_{\varepsilon_1}\right) \left[\beta'_\varepsilon \left(n^{\text{F}}_{\varepsilon_1} + N_{\varepsilon_1} - \varepsilon\right) - \beta'_{\varepsilon_1} \left(1 - n^{\text{F}}_\varepsilon + N_{\varepsilon_1 - \varepsilon}\right)\right] \right.$$
$$+ \int_\Delta^\varepsilon \mathrm{d}\varepsilon_1 \left(\varepsilon - \varepsilon_1\right)^2 \left(u_\varepsilon u_{\varepsilon_1} - v_\varepsilon v_{\varepsilon_1}\right) \left[\beta'_\varepsilon \left(1 - n^{\text{F}}_{\varepsilon_1} + N_{\varepsilon - \varepsilon_1}\right) - \beta'_{\varepsilon_1} \left(n^{\text{F}}_\varepsilon + N_{\varepsilon - \varepsilon_1}\right)\right]$$
$$\left. + \int_\Delta^\infty \mathrm{d}\varepsilon_1 \left(\varepsilon + \varepsilon_1\right)^2 \left(u_\varepsilon u_{\varepsilon_1} + v_\varepsilon v_{\varepsilon_1}\right) \left[\beta'_\varepsilon \left(n^{\text{F}}_{\varepsilon_1} + N_{\varepsilon + \varepsilon_1}\right) + \beta'_{\varepsilon_1} \left(n^{\text{F}}_\varepsilon + N_{\varepsilon + \varepsilon_1}\right)\right] \right\}$$
$$A_\varepsilon = -\frac{g}{\omega_{\text{D}}^2} \alpha_\varepsilon \left[\int_\Delta^\varepsilon \mathrm{d}\varepsilon_1 \left(\varepsilon^2 + \varepsilon_1^2\right) \alpha_{\varepsilon_1} + 2\varepsilon \int_\varepsilon^\infty \mathrm{d}\varepsilon_1 \varepsilon_1 \alpha_{\varepsilon_1} \right]. \tag{98}$$

Substituting this into eq. (91), we obtain

$$\frac{1}{2}\left(I_\varepsilon - I_{-\varepsilon}\right)_{\text{lin}} = L_\varepsilon + A_\varepsilon. \tag{99}$$

The solution corresponding to L_i is localized at energies $\varepsilon \times \Lambda \ll T$. So to find the solution near the critical temperature the collision integral can be represented in τ-approximation. The solution corresponding to A_ε at energies of the order of the temperature is

$$\beta_\varepsilon'^{(A)} = \frac{\Phi^2}{4T^2} \tanh \frac{\varepsilon}{2T} \left(\cosh \frac{\varepsilon}{2T}\right)^{-2} \quad (\varepsilon \gg \Lambda). \tag{100}$$

As a result, as shown by Ivlev (1978), the potential Φ affects the odd part of the distribution function in the following way:

$$\beta'_\varepsilon = \beta_\varepsilon'^{(A)} + \frac{\omega}{\gamma T} \left(V - \frac{\Phi}{u_\varepsilon^2}\right) \left(u'_{\varepsilon+V} - u'_{\varepsilon-V}\right) \tag{101}$$

with Φ given by expression (96). Here corrections to the β_c are neglected, as it took place in the case of electromagnetic radiation because there is more essential nonlinearity in the value of Φ.

With (101) the anomalous term F (99) becomes

$$F = -\frac{7\zeta(3)}{4\pi^2} \frac{\Phi^2}{T^2} + \frac{w}{\gamma T} \int_0^\infty \frac{\mathrm{d}\varepsilon}{\varepsilon} u_\varepsilon \left(V - \frac{\Phi}{u_\varepsilon^2}\right) \left(u'_{\varepsilon-V} - u'_{\varepsilon+V}\right) \tag{102}$$

Galaiko (1974) has indicated the presence of the term Φ^2, T^2 in the equation for the order parameter.

As follows from eq. (87), the tunnel current is

$$j_{\text{T}} = \frac{4mp_0 l}{\pi^2} wd(V - \Phi) = \frac{4mp_0 e}{\pi^2} wVd \frac{\pi\Delta/8T}{w/\gamma + \pi\Delta/8T} \tag{103}$$

Consequently, inequality (90) takes the form

$$\frac{\Phi}{\Delta} \ll \frac{l_{\text{D}}}{L}, \tag{104}$$

with Φ defined by formula (96). Since $l_{\text{D}} \ll L, \Phi$ is much less than Δ.

The general conclusion can be drawn that the spatially uniform tunneling contact in conditions of branch imbalance is to be considered with care. First of all this refers to the interpretation of experimental data on the basis of homogeneous consideration. The dependence of the order parameter in conditions of imbalance (fig. 8a) is determined by the Ginzburg-Landau equation

$$\frac{T_c - T}{T} - \frac{7\zeta(3)}{8\pi^2}\frac{\Delta^2}{T^2} + F = 0, \tag{105}$$

with the anomalous term (102). Condition (104) should be obeyed.
The picture looks much simpler if a symmetrical scheme is used (fig. 8b), where there is no branch imbalance, though the use of such schemes is less convenient experimentally. As shown by Peskovatski and Seminozhenko (1976) in this case the situation is analogous to pumping by an electromagnetic field, except for a coherence factor. From eqs. (19) and (22) at $\alpha_\varepsilon = 0$ we find the kinetic equation

$$I_\varepsilon = 2wu_\varepsilon \left[u'_{\varepsilon-V}\left(n'_{\varepsilon-V} - n_\varepsilon\right) + u'_{\varepsilon+V}\left(n'_{\varepsilon+V} - n_\varepsilon\right)\right].$$

The phonon collision integral is given by formula (25). Near the transition temperature it has the form (26). Due to this the anomalous term in equation
(105) is given, at small w/γ, as

$$F = \frac{wV}{\gamma T}\int_\Delta^\infty \frac{\mathrm{d}\varepsilon}{\sqrt{\varepsilon^2 - \Delta^2}}\left[\frac{\varepsilon + V}{\sqrt{(\varepsilon+V)^2 - \Delta_1^2}}\theta\left((\varepsilon+V)^2 - \Delta_1^2\right)\right.$$
$$\left. - \frac{|\varepsilon - V|}{\sqrt{(\varepsilon - V)^2 - \Delta_1^2}}\theta\left((\varepsilon - V)^2 - \Delta_1^2\right)\right]. \tag{106}$$

In the limit of small $w/\gamma \ll \Delta/T$ the same formula holds in the asymmetric scheme (fig. 8a) as far as imbalance can be neglected.

From eq. (102) it follows that the relative change of the order parameter is

$$\frac{\delta\Delta}{\Delta} \approx \frac{T^2}{\Delta^2}\frac{4\pi^2}{7\zeta(3)}F. \tag{107}$$

Formulas (106) and (107) permit to calculate the voltage dependence of the change in the order parameter at tunneling. In order to do so we take $\Delta_1 > \Delta$. Then at low voltage $V \ll (\Delta_1 - \Delta)$

$$\frac{\delta\Delta}{\Delta} = \frac{8\pi^2}{7\zeta(3)}\frac{T^2}{\Delta^2}\frac{wV^2}{\gamma T\Delta_1}\int_1^\infty \frac{x^2\,\mathrm{d}x}{\sqrt{x^2-1}\left(x^2 - \Delta^2/\Delta_1^2\right)^{3/2}} \tag{108}$$

At voltages $|V - \Delta_1 + \Delta| \ll \Delta$, with logarithmic accuracy we have

$$\frac{\delta\Delta}{\Delta} = \frac{2\pi^2}{7\zeta(3)}\frac{T^2}{\Delta^2}\frac{w}{\gamma T}\left(\Delta_1 - \Delta\right)\sqrt{\frac{\Delta_1}{\Delta}}\ln\frac{\Delta}{|V - \Delta_1 + \Delta|} \tag{109}$$

At voltages $|V - \Delta_1 - \Delta| \ll \Delta$

$$\frac{\delta\Delta}{\Delta} = \frac{2\pi^3}{7\zeta(3)}\frac{w}{\gamma T}\left(\Delta_1 + \Delta\right)\sqrt{\frac{\Delta_1}{\Delta}}\frac{T^2}{\Delta^2}\theta\left(\Delta_1 + \Delta - V\right) \tag{110}$$

It is seen that for voltages surpassing the value $A_1 + A$ stimulation is absent.
It should be noted that formulae (106)-(110) are valid if

$$\Delta, \Delta_1, V \ll T.$$

Gray (1978) was the first to observe experimentally the tunnel-injectionstimulated superconductivity effect. Superconductivity enhancement was observed up to voltages equal to the sum of the gaps according to formula (110). At high voltages superconductivity suppression

occurred, apparently, due to the phonon system deviating from equilibrium. Chang (1978) calculated injection stimulation with phonon nonequilibrium as well as with electromagnetic radiation.

Detailed experimental and theoretical researches into tunnel injectionstimulated superconductivity were carried out by Chi and Clarke (1979) for aluminium contacts. The theoretical and experimental results agreed well up to voltages $\Delta_1 - \Delta$ in agreement with formula (109). At high voltages the agreement was worse because of phonon non-equilibrium.

27.6 6. Investigation of stability of nonequilibrium states created by microwave radiation $\omega < 24$ near T_c.

Stability of nonequilibrium states occurs owing to the balance of pumping and energy exchange with a thermostat. The local stability of such states means that infinitesimal initial fluctuations attenuate in time. These fluctuations can not be obligatory homogeneous in this case. A superconductor in a nonequilibrium state can, in principle, show instability with respect both to infinitesimal and to finite fluctuations.

In this connection the paper of Eckern et al. (1979) on stability at microwave pumping near the transition temperature should be mentioned. Eckern et al. studied, in particular, the stability with regard to large fluctuations. Using the method of Langevin forces they showed that between T_c and $T_{\mathrm{c\,max}}$ as determined from eq. (41) a temperature T_{k} exists, at which a jump-like transition to the normal state occurs. At temperatures in the range $T_{\mathrm{k}} < T < T_{\mathrm{c\,max}}$ the superconducting state is overheated.

A superconductor is an example of a system where the pumping source and the diffusion coefficient depend on the order parameter determined from the BCS condition as well. All this makes the analysis of stability of superconductor nonequilibrium states a non-trivial problem. It is not the aim of the present chapter to give a detailed survey on the analysis of stability at different ways of pumping. Only microwave radiation with frequency $\omega < 2\Delta$ and temperatures near the critical are studied. Spatially heterogeneous fluctuations are taken to be one-dimensional.

The kinetic equation and the Ginzburg-Landau equation summarized for the non-equilibrium case have the form

$$-\frac{D}{2\gamma}\frac{\partial 2n}{\partial x^2}\frac{\sqrt{\varepsilon^2-\Delta^2}}{\varepsilon} + (n - n_{\mathrm{F}}) = \frac{\alpha}{\gamma}\left[\frac{\varepsilon(\varepsilon-\omega)+\Delta^2}{\varepsilon\sqrt{(\varepsilon-\omega)^2-\Delta^2}}\left(n_{\varepsilon-\omega}-n_\varepsilon\right)\theta(\varepsilon-\Delta-\omega)\right.$$
$$\left.-\frac{\varepsilon(\varepsilon+\omega)+\Delta^2}{\varepsilon_V}\left(n_\varepsilon - n_{\varepsilon+\omega}\right)\theta(\varepsilon-\Delta)\right] \tag{111}$$

$$\Delta\left[\frac{T_{\mathrm{c}}-T}{T} - \frac{7\zeta(3)}{8\pi^2}\frac{\Delta^2}{T^2} + F\right] + \frac{\pi D}{8T}\frac{\partial^2\Delta}{\partial x^2} = 0 \tag{112}$$

F is defined by the ratio (27).

To analyse the local stability of the system (111), (112), time derivatives should be taken into account. We restrict ourselves to investigation of the boundary of the stability region. For this purpose it is sufficient to use static equations. System (111), (112) is linearized with respect to small deviations $\Delta \to \Delta + \Delta_1$ and $n \to n + n_1$, where small values $\Delta_1 \propto n_1 \propto \mathrm{e}^{\mathrm{i}kx}$. Frequencies $\omega \ll \Delta$, and consequently large values of k, are considered to have a rather large first term in the left-hand side of eq. (111). Then the value F is to be varied only on Δ, provided n is constant. It gives for $(\varepsilon - \Delta) \ll \Delta$.

After integration by parts we obtain

$$\delta F = -\Delta_1 \sqrt{\frac{2}{\Delta}} \int_{\Delta}^{\infty} \frac{\mathrm{d}\varepsilon}{\sqrt{\varepsilon - \Delta}} \frac{\partial n'_{\varepsilon}}{\partial \varepsilon} \tag{113}$$

The correction to distribution the function at $\omega \ll \Delta$ and $\Delta < \varepsilon < \Delta + \omega$ is

$$n'_e = -\frac{\alpha}{\gamma} \frac{\omega}{4\ \mathrm{T}} \sqrt{\frac{2A}{\varepsilon + \omega - A}} \cdot \frac{\sqrt{\varepsilon - A}}{\sqrt{\varepsilon - A} + \alpha\sqrt{2A}} \tag{114}$$

This formula takes into account the singularity near threshold shown in fig. 1. Formulas (113) and (114) were derived by Ginzburg and Spivak (1981). In their paper. from formulas analogous to (113), (114) the conclusion about local instability was drawn as well. By substituting eq. (114) into eq. (113) and varying equation (122). the dispersion law of low fluctuations is obtained

$$\sqrt{\frac{2\omega}{\Delta}} \ln\left(\frac{Dk^2}{\gamma}\sqrt{\frac{\omega}{\Delta}}\right) - \frac{\pi D k^2}{8\Delta} = \frac{7\zeta(3)}{\pi^2}\frac{\Delta}{T} \quad \frac{Dk^2}{\gamma}\sqrt{\frac{\omega}{\Delta}} \geqslant 1. \tag{115}$$

Omitting logarithmical dependences, the position of the stability boundary is determined by the relation

$$\frac{A}{T}\sqrt{\frac{A}{(1)}} \propto 1 \tag{116}$$

At these values of the order parameter. as follows from formula (37). the . 1 enhancement is not yet considerable. and temperature does not exceed $T_{\cdot}$ On raising the temperature when A is getting smaller than the value determined by formula (116), instability development will result either in inhomogeneous structure of the superconductor or in the normal state. This question needs further study.

Note, that in calculations of superconductivity stimulation the singularity of the distribution function near the point $\delta = A$ is insignificant. but it is very important in stability analysis. Veglecting this singularity one could draw the wrong conclusion that on microwave radiation the nonequilibrium state of a superconductor is always locally stable up to the maximum temperature.

The validity of formula (115) is basicaliy associated with the presence of root singularity in density of states. Singularity spreading must be small:

$$\varepsilon_0 \ll \Delta \left(\frac{\alpha}{\gamma}\right)^2. \tag{117}$$

Note that formula (115) is valid if the condition $A(\alpha/\gamma)^2 \ll \omega$ is obeyed. The instability considered does not occur in superconductors where the singularity is spread by phonons and inhomogeneities of the sample. Moreover, it is not realized, apparently, at high intensities $(\alpha/\gamma)\sqrt{\Delta/\omega} \gg 1$, when the distribution function is spread strongly.

Hence the conclusion is that in superconductors with a sharp peak in the density of states the experimental observation of superconductivity stimulation becomes complicated due to possible instability. The theory of stimulation of superconductivity reported in previous sections refers to the case of slightly spread singularity in the density of states on the interval of energies not higher than ω.

28 Pair Breaking in Nonequilibrium Superconductors by Beyer, Pethick, Rammer, Smith

28.1 1. Introduction

Nonequilibrium properties of superconductors may be described within two rather different theoretical frameworks. The first one, which characterizes the nonequilibrium state in terms of quasiparticles, is the more intuitive, but it is limited to situations where the variations are small on time scales of order $\hbar/\Delta$ and length scales $\hbar v_{\mathrm{F}}/\Delta$, and where the width of the quasiparticle states may be neglected. Here Δ is the temperature-dependent gap and v_{F} the Fermi velocity. The second one is more general as it is based on the quasiclassical Green function. It can be applied to situations involving rapid spatial and temporal variations, as well as situations when the width of the quasiparticle states is significant, causing the quasiparticle picture to lose its meaning.

The quasiparticle approach to nonequilibrium problems was first used by Bardeen, Rickayzen and Tewordt (1959) in the context of thermal conductivity, and later elaborated by Betbeder Matibet and Nozires (1969), Aronov (1974), Aronov and Gurevich (1974), Pethick and Smith (1978, 1979a), Aronov et al. (1981), and many others. The Green function approach is due to Eilenberger (1968), Eliashberg (1971), Larkin and Ovchinnikov (1968, 1977), and Schmid and Schön (1975).

In the quasiparticle approach the kinetic equation resembles closely the Boltzmann equation used for normal metals, with the superfluid corrections modifying only the dispersion relation and the transition probabilities. The kinetic equation in the Green function approach may be derived using the Keldysh (1964) method. The distribution functions that have been introduced (e.g. Schmid and Schön 1975) differ in some respects from the quasiparticle distribution function, as do the forms of the kinetic equations within the two approaches. This makes a comparison of the two methods less immediate, in particular as regards the important role of pair breaking. Recently, Shelankov (1980) has shown how to define distribution functions in the quasiclassical Green function approach in a way that makes the comparison of the Green function and quasiparticle approaches more direct, but until now most of the calculations within the quasi-classical approach have not employed Shelankov's definitions.

Our purpose in this chapter is to review calculations of the effects of pair breaking in a number of nonequilibrium situations. We shall first discuss the
basic formalism, and compare the Green function and quasiparticle approaches. Then we shall describe calculations carried out with the quasiclassical Green function approach. Until recently, most calculations within this approach had been carried out for temperatures close to T_c, and for weak coupling superconductors. What we wish to show is that the quasiclassical Green function approach can be employed for carrying out realistic calculations of pair-breaking effects away from T_{c}, and for strong coupling superconductors, and that such calculations are not significantly more complicated than those based on the quasiparticle picture. Consequently, a meaningful comparison of theory and experiment is now possible.

Although the kinetic equations for the quasiclassical Green function are more general and contain the quasiparticle Boltzmann equation as a limit, the concept of quasiparticles is so useful that one would like to put precise limits on its applicability. Furthermore, since a number of numerical calculations have been based on the quasiparticle approach, it is desirable to assess their range of validity. A detailed investigation of the relationship between the two approaches, based in part on numerical calculations of charge imbalance, was performed recently by Beyer Nielsen et al. (1982) and here we shall review and extend some of their results and also discuss the case of thermal conductivity, which was treated recently by Beyer Nielsen and Smith (1982). When dealing with pair-breaking phenomena we treat simultaneously the effects

of strong electron phonon coupling, magnetic impurities and supercurrents, and indicate the differences and similarities between these various pair-breaking mechanisms.

The plan of the chapter is as follows. In section 2 we discuss the expressions for the electric and thermal currents, and for the charge imbalance, and compare the results obtained within the two frameworks. Section 3 reviews the effects of pair breaking on charge imbalance, thermal conductivity, and frequency-dependent conductivity, and includes a discussion of strongcoupling effects. Finally, section 4 illustrates the quasiparticle and Green function approaches to the calculation of thermal conductivity and charge imbalance. The physical situation is one in which the relaxation is dominated by impurity scattering in the presence of a superflow. In this last section, we go into more detail in our discussion in order to illustrate some of the concepts introduced in the previous sections of this chapter.

28.2 2. Densities and currents

The relation between the quasiparticle and Green function approaches may be elucidated by comparing the form of the expressions for the densities and currents.

28.2.1 2.1. The quasiparticle method

In the quasiparticle formalism the state of the superconductor is described by a quasiparticle distribution function $f_{p\sigma}$, which gives the local density of quasiparticles of momentum p and spin σ, supplemented by quantities describing the condensate, such as its velocity, v_{s}, and the superconducting gap. The character of the quasiparticle is specified by the dispersion relation relating its energy E_p to the momentum, which in the absence of a supercurrent is

$$E_p = \sqrt{\xi_p^2 + \Delta^2} \tag{2.1}$$

where $\xi_p = \varepsilon_p - \mu, \varepsilon_p$ being the normal state energy and μ the chemical potential. The effective charge carried by a quasiparticle is $q = \breve{\zeta}/E$ (in units of the electronic charge), and the associated current of charge is vq, where $v = \hat{c}E\hat{c}p$ is the group velocity (Pethick and Smith 1979a). This current of (normal) charge differs, however, from the usual normal current, as we shall see below.

The total density of particles in the quasiparticle approach is given by

$$n = \sum_{p\sigma} u_p^2 f_p + v_p^2 \left(1 - f_{-p}\right), \tag{2.2}$$

where f_p is the distribution function for BCS-Bogoliubov quasiparticles, and u_p and v_p are the usual BCS coherence factors

$$u_p^2 = \frac{1}{2}\left(1 + \frac{\xi_p}{E_p}\right) \tag{2.3}$$

and

$$v_p^2 = \frac{1}{2}\left(1 - \frac{\xi_p}{E_p}\right) \tag{2.4}$$

with $\dot{\varsigma}_p = \varepsilon_p - \mu$ as above. In the general case the chemical potential, μ, which may vary in space, is the energy required to add an electron to the superconductor at the point r, without altering the quasiparticle distribution.

The total density n may be separated into two contributions as follows:

$$n = \sum_{p\sigma} v_p^2 + \sum_{p\sigma} \left(u_p^2 - v_p^2\right) f_p. \tag{2.5}$$

The first term is simply the density of electrons at zero temperature, corresponding to a chemical potential μ, provided particle hole symmetry applies. The second term in eq. (2.5) is in accordance with common usage denoted by Q^*. We see that in a certain sense Q^* is the charge associated with the quasiparticles, since each excitation may be regarded as carrying a charge $q_p = u_p^2 - \iota_p^2 = \dot{\zeta}_p, E_p$. It is important to note, however, that q_p depends on the local chemical potential, and therefore when the system is perturbed from
equilibrium, the second term in eq. (2.5) will change because of changes in the coherence factors as well as in the quasiparticle distribution. Note also that when particle-hole symmetry applies f_p in eq. (2.5) may be replaced by the deviation from local equilibrium $\delta f^{le} = f_p - f_p^{le}$. In the absence of a supercurrent, f_p^{le} is the equilibrium (Fermi) distribution function $f^0(E) = [\exp(E/T) + 1]^{-1}$ evaluated using the actual quasiparticle energy in the nonequilibrium state. In the presence of a supercurrent it is given by eq. (2.9). For the second term in eq. (2.5) we therefore have

$$Q^* = \sum_{p\sigma} q_p \delta f_p^{\ell e} = \sum_{p\sigma} q_p \left(f_p - f_p^{\ell e}\right) \tag{2.6}$$

The quantity Q^*, commonly referred to as the charge imbalance, is therefore a measure of the amount by which the quasiparticle distribution is asymmetric about the actual Fermi surface, not the Fermi surface in equilibrium. Physically, it is precisely the quantity measured in tunnel injection experiments of the type pioneered by Clarke (1972) and analysed in detail by Tinkham and Clarke (1972) and by Tinkham (1972).

The charge imbalance Q^* is the quantity of importance in many timeindependent situations. However, another way of writing the expression for the charge density which is particularly useful in time-dependent situations is to express changes in the total charge density as a sum of one contribution coming from changes in the quasiparticle distribution, and another coming from changes in the coherence factors. The two contributions are referred to as the charge of the normal component, and the charge of the superconducting component, respectively, and form the basis for the development of a two-fluid model for the charge density, as described in detail by Pethick and Smith (1978, 1979a).

The total particle number current density j may be written as

$$j = n v_{\mathrm{s}} + \sum_{p\sigma}^{p} f_p \tag{2.7}$$

since the total current carried by a quasiparticle is p/m. The superfluid velocity is denoted by v. When eq. (2.7) is written in terms of δf^{te} , the deviation from local equilibrium, one finds, to first order in v_{s},

$$j = n_{\mathrm{s}} v_{\mathrm{s}} + \sum_{p\sigma}^{p} \delta f_p^{\ell e} \tag{2.8}$$

since $f_p = f_p^{le} + \delta f_p^{le}$, where

$$f_p^{\ell e} = f^0 \left(E_p + p \cdot v_s\right) \tag{2.9}$$

is the distribution function for quasiparticles in equilibrium with the lattice in the presence of superfluid flow. (In this discussion we neglect Fermi liquid effects, which would result in E_p

being a function of v_s.) The superfluid density
n_8 is given in terms of the Yoshida function Y_0 by the usual clean-limit result,

$$\frac{n_{\mathrm{s}}}{n} = 1 - \int_{-\infty}^{\infty} \mathrm{d}\xi \left(-\frac{\partial f^0(E)}{\partial E} \right) = 1 - Y_0 \tag{2.10}$$

The second term in eq. (2.8) is commonly referred to as the normal current j_n, while $j_{\mathrm{s}} = n_{\mathrm{s}} v_{\mathrm{s}}$ is the supercurrent.

The thermal current associated with the quasiparticle excitations is

$$j_{\mathrm{th}} = \sum_{p\sigma} E_p v_p \delta f_p^{le} \tag{2.11}$$

since E_p is the quasiparticle energy measured with respect to the chemical potential. Unlike the normal current j_n the thermal current involves the group velocity v_p of the excitations.

28.2.2 2.2. The Green function method

The basic ingredient of the microscopic approach is the Nambu Green function, which contains information about the nature of the excitation spectrum, the distribution function for excitations and also the properties of the condensate. The Green function is a function of two sets of coordinates, but for many purposes it is convenient to transform to center-of-mass and relative coordinates, and then perform a Fourier transformation, so that the Green function depends on the center-of-mass coordinates and the relative momentum and frequency variables. The Green functions in this mixed representation correspond closely to the Wigner distribution function. The equations for the Green functions are more complicated than those for the distribution function in the quasiparticle approach, because the Green function depends on both momentum and frequency, whereas the quasiparticle distribution function depends only on the momentum. However, in many cases of physical interest sufficient information is contained in the quasiclassical Green function, g, which is obtained by integrating the Green function over the magnitude of the momentum variable and therefore depends on only the frequency and the direction of the momentum, as well, of course, as on the center-of-mass coordinates. For superconducting systems it is convenient to work with a 2×2 matrix representation for Green functions in particle-hole space. The nonequilibrium properties are conveniently described using the formalism developed by Keldysh (1964), who introduced a further 2×2 matrix representation to distinguish the various orderings of these arguments on the chosen integration contour. In the simplest 2×2 matrix representation the Green functions and the self energies are arranged in the form of matrices,

$$G = \begin{pmatrix} G^{\mathrm{R}} & G^{\mathrm{K}} \\ 0 & G^{\mathrm{A}} \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \Sigma^{\mathrm{R}} & \Sigma^{\mathrm{K}} \\ 0 & \Sigma^{\mathrm{A}} \end{pmatrix} \tag{2.12}$$

where

$$\begin{aligned} G^{\mathrm{R}} &= G^{\mathrm{R}}(1,2) = \{G^{>}(1,2) - G^{<}(1,2)\}\, \theta\,(t_1 - t_2)\,, \\ G^{\mathrm{A}} &= -\{G^{>}(1,2) - G^{<}(1,2)\}\, \theta\,(t_2 - t_1)\,, \\ G^{\mathrm{K}} &= G^{>}(1,2) + G^{<}(1,2). \end{aligned} \tag{2.13}$$

Here $G^{>}$and $G^{<}$are 2×2 matrices in Nambu space and are given by

$$G^{>}(1,2) = -i \left\langle \begin{pmatrix} \psi_{\uparrow}(1)\psi_{\uparrow}^{\dagger}(2) & \psi_{\uparrow}(1)\psi_{*}(2) \\ -\psi_{\downarrow}^{\dagger}(1)\psi_{\uparrow}^{\dagger}(2) & -\psi_{\downarrow}^{\dagger}(1)\psi_{\downarrow}(2) \end{pmatrix} \right\rangle \tag{2.14}$$

and

$$G^{<}(1,2) = \mathrm{i}\left\langle \begin{pmatrix} \psi_{\uparrow}^{\dagger}(2)\psi_{\uparrow}(1) & \psi_{v}(2)\psi_{\uparrow}(1) \\ -\psi_{\uparrow}^{\dagger}(2)\psi_{*}^{\dagger}(1) & -\psi_{*}(2)\psi_{*}^{\dagger}(1) \end{pmatrix} \right\rangle. \tag{2.15}$$

In eqs. (2.14) and (2.15) $\psi(1)$ and $\psi^{\dagger}(1)$ denote the electron field operators $\psi\left(r_1, t_1\right)$ and $\psi^{+}\left(r_1, t_1\right)$.

To illustrate the significance of the Keldysh function, G^{K}, we may evaluate eqs. (2.14) and (2.15) for a normal translationally invariant, non-interacting Fermi system with a particle distribution function $f_{p\sigma}$, where p is the particle momentum and E_p the particle energy (measured from the chemical potential). One then gets for the 11 -components of $G^{>}$and $G^{<}$

$$G^{>}(1,2)_{11} = -\mathrm{i}\sum_{p}\left(1 - f_{p\sigma}\right)\exp\left[\mathrm{i}p\cdot\left(r_1 - r_2\right) - \mathrm{i}E_p\left(t_1 - t_2\right)\right] \tag{2.16}$$

and

$$G^{<}(1,2)_{11} = \mathrm{i}\sum_{p} f_{p\sigma}\exp\left[\mathrm{i}p\cdot\left(r_1 - r_2\right) - \mathrm{i}E_p\left(t_1 - t_2\right)\right]. \tag{2.17}$$

From these expressions one sees, by subtraction, that the retarded and advanced functions do not contain the distribution function, but the Keldysh function does, since the sum of (2.16) and (2.17) is given by

$$G^{\mathrm{K}}(1,2)_{11} = -\mathrm{i}\sum_{p}\left(1 - 2f_{p\sigma}\right)\exp\left[\mathrm{i}p\cdot\left(r_1 - r_2\right)_{\mathrm{e}} - \mathrm{i}E_p\left(t_1 - t_2\right)\right]. \tag{2.18}$$

As discussed above, the Green functions may be integrated over the magnitude of the momentum, corresponding to the variable $\xi = p^2/2m - \mu$, with μ the chemical potential. With the center-of-mass coordinates given by $r = \frac{1}{2}\left(r_1 + r_2\right)$ and $t = \frac{1}{2}\left(t_1 + t_2\right)$ the ξ-integrated Green function g is

$$g(\hat{\boldsymbol{p}}, E; rt) = \frac{\mathrm{i}}{\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi G(\boldsymbol{p}, E, rt), \tag{2.19}$$

where p is the momentum variable and E the frequency variable associated with Fourier transforming with respect to the coordinates $r_1 - r_2$ and $t_1 - t_2$, respectively.

The matrix Green function satisfies the Dyson equation

$$\left(G_0^{-1} - \Sigma\right)\otimes G = \delta(1-2), \tag{2.20}$$

where $\otimes$ implies integration over internal variables, $(A\otimes B)(1,2) = \int \mathrm{d}3A(1,3)B(3,2)$. The operator G_0^{-1} is the inverse of the non-interacting Green function. A more convenient equation for G, more closely related to the Boltzmann equation, may be derived by subtracting the conjugate of eq. (2.20) from it:

$$\left[G_0^{-1} - \Sigma\otimes G\right] = 0. \tag{2.21}$$

A kinetic equation may then be derived from the off-diagonal (in Keldysh space) component of (2.21). We shall not describe this procedure in any detail, but refer the reader to the excellent review by Schmid (1981). In the applications discussed in the present chapter the Dyson equation (2.21) simplifies to an ordinary commutator involving the ξ-integrated Green function g instead of G :

$$\left[g_0^{-1} - \Sigma, g\right] = 0. \tag{2.22}$$

This equation must be supplemented by the normalization condition

$$gg = 1, \tag{2.23}$$

obtained by Eilenberger (1968) and Larkin and Ovchinnikov (1968). The diagonal part of equation (2.22) determines the retarded and advanced functions while the off-diagonal part becomes an equation for the distribution function which is associated with g^K. The particular choice of distribution functions made by Schmid and Schön (1975) corresponds to writing first

$$g^{\mathrm{K}} = g^{\mathrm{R}} h - h g^{\mathrm{A}}, \tag{2.24}$$

where g^K, g^R and g^A are 2×2 matrices in Nambu space. A diagonal distribution matrix h is then used to introduce the scalar distribution functions f^L and f^{T},

$$h = \left(1 - 2f^{\mathrm{L}}\right) - 2f^{\mathrm{T}}\tau_3. \tag{2.25}$$

As we mentioned above, the Keldysh component of the Green function contains effects both from changes in the spectral density of excitations, and from changes in their distribution function. In using eq. (2.24) one is attempting to separate out the two contributions to the Keldysh component. However, as we shall show below [see particularly the discussion after eq. (2.30)], eqs. (2.24) and (2.25) do not succeed completely in separating the two contributions to the Keldysh function. An elegant alternative way of making the separation has been described by Shelankov (1980); in the limit when the
quasiparticle picture is valid the distribution function he introduces reduces to the usual quasiparticle distribution function, without any accompanying spectral density factors. In local equilibrium $1 - 2f^{\mathrm{L}} = \tanh(E/2T)$ and $f^{\mathrm{T}} = 0$. In the cases discussed below we may without loss of generality represent g^{R} and g^{A} in the simple form

$$g^{\mathrm{R}} = \alpha\tau_3 + \beta\tau_1, \quad g^{\mathrm{A}} = -\alpha^*\tau_3 + \beta^*\tau_1 \tag{2.26}$$

where $\tau_i (i = 1, 2, 3)$ are the usual Pauli matrices.
The functions α and β depend on the frequency variable E and the direction in momentum space $\hat{\boldsymbol{p}}$. They are written in terms of the real functions N_i, R_i $(i = 1, 2)$ as

$$\alpha = N_1 + \mathrm{i}R_1, \quad \beta = N_2 + \mathrm{i}R_2. \tag{2.27}$$

In the microscopic approach the charge imbalance Q^* introduced in eq. (2.6) takes the form

$$Q^* = -N(0)\frac{1}{4}\int_{-\infty}^{\infty} \mathrm{d}E \left\langle \mathrm{Tr}\, g^K \right\rangle \tag{2.28}$$

where $\langle \ldots \rangle$ denotes the average

$$\langle \ldots \rangle = \int \frac{\mathrm{d}\Omega_{\hat{p}}}{4\pi} \ldots \tag{2.29}$$

and $N(0)$ is the density of states per spin at the Fermi energy in the normal state of the metal.

When (2.24) is inserted into (2.28) one finds

$$Q^* = 2N(0) \int_{-\infty}^{\infty} \mathrm{d}E \left\langle N_1 f^{\mathrm{T}} \right\rangle \tag{2.30}$$

which should be compared with (2.6). In the quasiparticle limit the two distribution functions are thus seen to be related by $N_1 f^{\top} = \delta f^{le}$, since $\xi \mathrm{d}\xi = E\ \mathrm{d}E$, when $E = \sqrt{\xi^2 + \Delta^2}$.

The particle current j is given by

$$j = -\frac{N(0)}{4m} \int_{-\infty}^{\infty} \mathrm{d}E \left\langle p \operatorname{Tr}\left(\tau_3 g^{\mathrm{K}}\right)\right\rangle \tag{2.31}$$

and the heat current, j_{th} , by (Beyer Nielsen and Smith 1983)

$$j_{\mathrm{th}} = -\frac{N(0)}{4m} \int_{-\infty}^{\infty} \mathrm{d}E \left\langle Ep \operatorname{Tr} g^{\mathrm{K}}\right\rangle \tag{2.32}$$

In the static case these may be rewritten as

$$j = \frac{2N(0)p_{\mathrm{F}}}{m} \int_{-,}^{,} dE \left\langle N_1 \hat{p} \left(f^{\mathrm{L}} - \frac{1}{2}\right)\right\rangle \tag{2.33}$$

$$j_{\mathrm{th}} = \frac{2N(0)p_{\mathrm{F}}}{m} \int_{-\infty}^{\infty} \mathrm{d}E \left\langle E\hat{\boldsymbol{p}} N_1 f^{\mathrm{T}}\right\rangle \tag{2.34}$$

where p_{F} denotes the Fermi momentum.
Equations (2.30)-(2.32) are a natural starting point for developing a twofluid model. One writes densities and currents as the sum of two types of contribution-those coming from changes in the distribution function of excitations, which one associates with the normal component, and those coming from changes in spectral densities, which one associates with the superfluid component. Since the quantities $N_1 f^{\mathrm{T}}$ and ($f^{\mathrm{L}} - \frac{1}{2}$) correspond to the distribution function, eqs. (2.30) and (2.34) show that the charge imbalance and the thermal current are naturally related to the normal component, while eq. (2.33) shows that the total current has both normal and superfluid components.

An important point to notice is that deviations in f^{L} and $N_1 f^{\mathrm{T}}$ are naturally related to the quantity δf_p^{le}, the difference between the quasiparticle distribution function, f_p, and an equilibrium distribution evaluated using the actual quasiparticle energy in the nonequilibrium state, $f^0\left(E_p\right)$. In the Green function approach the effects of quasiparticle energy shifts are taken into account implicitly, since the basic variable is the energy, not the momentum. The quasiparticle approach focuses attention on the distribution function for quasiparticles of a given momentum while the Green function approach focuses attention on excitations of a given energy. One consequence of this is that in the limit when the quasiparticle picture is valid, the distribution functions f^{L} and $N_1 f^{\mathrm{T}}$ for fixed E refer to states whose momenta depend on nonequilibrium contributions to the quasiparticle dispersion relation.

It is instructive to compare the expressions (2.30) and (2.33) for the density and current for a clean superconductor in the presence of a superflow characterized by velocity v_{s}. As we shall see in more detail in section 4 , the density-of-states function N_1 is given in the clean limit by

$$N_1 = \frac{|E - p_{\mathrm{F}} v_{\mathrm{s}} \cos\theta|}{\sqrt{\left(E - p_{\mathrm{F}} v_{\mathrm{s}} \cos\theta\right)^2 - \Delta^2}} \Theta\left(|E - p_{\mathrm{F}} v_{\mathrm{s}} \cos\theta| - \Delta\right), \tag{2.35}$$

where θ is the angle between p and the direction of superflow v_{c}. When one inserts eq. (2.35) into eq. (2.33), makes use of the expansion

$$N_1 \simeq N_1^0 - \frac{\partial N_1^0}{\partial E} p_{\mathrm{F}} v_{\mathrm{s}} \cos\theta \tag{2.36}$$

where N_1^0 is given by (2.35) with $\iota_{\mathrm{s}} = 0$, writes $f^{\mathrm{L}} = f^0(E) + \delta f^{\mathrm{L}}$, and integrates by parts, one finds

$$j = n_{\mathrm{s}} v_{\mathrm{s}} + \frac{2N(0)p_{\mathrm{F}}}{m} \int_{-\infty}^{\infty} \mathrm{d}E \left\langle N_1 \delta f^{\mathrm{L}} \hat{\boldsymbol{p}}\right\rangle \tag{2.37}$$

The term $n_s v_s$ in (2.37) originates in the term ($f^0 - \frac{1}{2}$), while the second term in (2.37) comes from the contribution δf^{L}, and corresponds to the normal part of the current j in eq. (2.8).* On the basis of the comparison of (2.37) and (2.8) we conclude that δf^{L}, which may depend on both the direction of momentum and energy, corresponds to δf_p^{le}. Finally let us compare the thermal currents (2.11) and (2.34). It is apparent that we must identify $N_1 f^{\mathrm{T}}$ in the BCS-limit with δf_p. as in the case of charge imbalance. In section 4 we study these distribution functions in more detail in a current-carrying superconductor with impurity scattering as the dominant relaxation mechanism, and solve the Dyson equation for the retarded and advanced Green functions as well as the kinetic equation for f^{T}. In the following section we discuss some examples of pair breaking effects on the calculated currents and densities in both the quasiparticle and Green function frameworks.

28.3 3. Pair-breaking effects

In order to determine the charge imbalance or the currents in a situation of physical interest one must in general solve a kinetic equation for the distribution function as in any other nonequilibrium problem. The kinetic equation depends on the physical quantity of interest as well as on whether one works with BCS-Bogoliubov quasiparticles or with the fully microscopic Green function picture. Naturally these two approaches must in some limit be equivalent. The subject of this section is to indicate when this is so, and to show in some detail how the results obtained within the two approaches differ when pair breaking effects become significant. In the case of charge relaxation such a detailed, quantitative investigation was performed recently by Bever Nielsen et al. (1982), and we shall refer to many of their results in the following. Similar effects in the thermal conductivity were examined recently by Beyer Nielsen and Smith (1982), for situations where electron-phonon scattering and magnetic impurity scattering are the sources of pair breaking. In this section we provide the reader with an overview of what has been achieved as regards the quantitative understanding of the effects of pair breaking on charge relaxation and thermal conduction. We shall not, however, attempt to summarize the work that has gone into solving kinetic equations under a variety of other circumstances such as those related to radiation

surrulated superconductivity or gap relaxation, which are discussed elecwhere in the present volume.

Wher considering par-breaking effects on nonequilibrium properties it is iriportant or sm-phasive that in the quasiparticle approach such effects are included. to lowest order in the appropriate scattering rates. sia the cordlision mitegral in the Boltaman equation. The advantage of the Green function approach is that it allows one to include pair breaking to ail orders by taking mes accosunt the effects of pass breaking on the retarded and advanced Green functuors. We shall see how this works in two important cases of pair hreaking: (a) magnetic impunts and electron phomon scattenng and (b) scattering by ordinary impurities in the presence of a supercurrent.

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- Note that in situations where a perturbation is applied to a current-carrying superconductor the second term in (2.33) may contain terms proportional to v_s. The question of whether such terms are regarded as a contribution to the normal current or the supercurrent depends on ones viewpoint. On the one hand there is dissipation associated with the current, and so it may be considered to be a normal component, but, on the other hand, these currents do enter the expression for the penetration depth, and it may therefore be argued that they should be considered as contributions to the supercurrent.

28.3.1 3.1. Scattering by magnetic impurities and phonons

It is well known that ordinary impurt? scattering has no pair breaking effects. In accordance zith Anderson's theorem. By contrast. scattering by magnetic mpurmes. characteryed by the scattering rate I. mas be expected to influence the density of states significantly when I, becomes comparable with the energy gap 1. Withm the BCS theor; the densit; of states V_{is} ; is gien b; $|u^2 - t^2|^{-1}$, where u and v are the coherence factors given by eqs. (2.3) and 124). When ardten in terms of the quasiparticle energ: Ethe BCS density of states is

$$N_1(E) = \frac{E}{\sqrt{E^2 - 1^2}}\theta(|E| - \Delta) \tag{3.1}$$

since in this limit the real part of x, eq. (2.27), is the usual BCS density of states.

Pair breaking due to scattering by magnetic impurities and that due to electron phonon scatterng both remove the smgularity in eq. 13.11 at the gap codge through in rather different manners. When magnetic impunty scatterng acts on its own, one finds that $N_1(E = \Delta) \sim (\Delta\Gamma_0)^{13}$ for $\Gamma_0 \ll \Delta$, while when electron-phonon scattering acts alone, one has $N_1(E = \Delta) \sim (\Delta\Gamma_p)^{12}$ for $I. = 1.$ where I_s is the rate of scattering of ejectrons by phonom. A further difference between the two pair breaking mechamsms is that in the magnetic impurity case $N_1(E)$ vanishes for $|E| < 1 - \frac{1}{2}(I/\Delta)^{1/3}$ if pair breaking is weak. If " 1 1. while on the case of electron phonon scatterng \, $|E|$, 1 , finte for all value of E. ... ath $\Lambda_1(0) \sim I_2$ dif $I_2 = 1$. The density of state function I_1 (E). which mas be obtaned from the s-integrated retarded (sreen function q^2 ich eqs. 2.26, 2.27), is found in general by solving a quartic equation, which is derlied irom the equation of motoon for a^2 and the normallaturs conduon or a. eq. (2.23). The equation of motom for y^2 is the diagonal part of eq. 12.221. the difference between the Dyson equation and its conjugate. In term: of the
complex renormalization function $Z(E)$ and the gap function $\phi(E)$ of the Eliashberg (1971) theory the equation of motion is

$$ZE\beta - \mathrm{i}\phi\alpha + \mathrm{i}\Gamma_{\mathrm{s}}\alpha\beta = 0 \tag{3.2}$$

and the normalization condition obtained from (2.23) is

$$\alpha^2 + \beta^2 = 1. \tag{3.3}$$

The generalized density of states functions N_1 and $R_i (i = 1, 2)$ are defined by the real and imaginary parts of α and β, according to eq. (2.27). In the BCS limit, where $A = \Phi/Z$ may be chosen to be real, one sees that N_1 is given by eq. (3.1),

$$R_1(E) = -\frac{E}{\sqrt{\Delta^2 - E^2}}\Theta(\Delta - |E|) \tag{3.4}$$

and

$$N_2 = -\Delta R_1/E, \tag{3.5}$$
$$R_2 = \Delta N_1/E. \tag{3.6}$$

When the pair breaking effects of phonons and magnetic impurities are included in eq. (3.2) the density of states functions become finite for all E. In a realistic case the evaluation of N_i and $R_i (i = 1, 2)$ may be quite laborious, since Z and Φ must in general be determined as functions of E by solving the Eliashberg equations. For a strong-coupling material like Pb the resulting density of states has structure at energies around 50 K which may be observed directly in tunneling experiments. The functions N_1 and R_1 serve as input to the kinetic equations

appropriate to the physical situation under consideration. In figs. 1 and 2 we plot N_1 and N_2 for two different cases: (1) a weak-coupling superconductor with magnetic impurities, and (2) a model for Pb . The calculation for the latter case was carried out by Beyer Nielsen and Smith (1982). For the phonon spectra they used Scalapino's (1969) model for the phonon spectrum of Pb , supplemented by a Debye tail at low energies. The densities of states appear remarkably similar to those for the weak-coupling case in the absence of pair breaking, except for some broadening due to the finite quasiparticle lifetime. The most important difference is that the energy is measured in terms of the actual energy gap at the gap edge calculated including strong-coupling effects, not the BCS value of the gap: for the case considered $\Delta(E = \Delta) \simeq 1.21\Delta_{\mathrm{BCS}}$.

The kinetic equation is derived within the Green function approach from the off-diagonal component of eq. (2.22). We shall not go into the details of deriving the collision integral, since rather complete discussions have been given by Schmid (1981) and by Beyer Nielsen et al. (1982). The latter authors showed in detail how strong-coupling effects are included in the formulation,

Fig. 1. The generalized density of states functions $N_{1
$ and N_2 for the case of magnetic scattering alone, for scattering rates $\Gamma_s = 0.02\Delta$ (thin lines), and $\Gamma_s = \Delta$ (thick lines). For $\Gamma_s = \Delta, N_1$ and N_2 are singular at $E = 0$, but for higher scattering rates N_1 and N_2 both remain finite at $E = 0$.}

but did not perform specific calculations for this case. Calculations of thermal conductivity and charge relaxation for the strong-coupling superconductor Pb were performed recently by

Beyer Nielsen and Smith (1982). Below we specify the collision integral appropriate to charge relaxation and thermal conduction in terms of the distribution function $f^{\top}$, which determines the charge imbalance and heat current according to eqs. (2.30) and (2.34). For convenience we omit the effect of ordinary impurity scattering on the thermal conductivity, but this may be included easily. The general form of the kinetic equation is

$$X = I\left(f^{\mathrm{T}}\right) \tag{3.7}$$

where the driving term, X, may be due to tunnel injection or to the combined

Fig. 2. The generalized density of states functions $N_{1 $ and N_2 for the strong-coupling superconductor Pb for the case of scattering by phonons alone at a temperature $T = 0.964T_c$. The phonon spectrum used is described in the text, and the gap Δ, which was obtained by solving the Eliashberg equations, is 0.436 meV . For this model $\mathrm{Re}\, Z(0)/\tau_{\mathrm{in}} \simeq 0.146 k_{\mathrm{B}} T_{\mathrm{c}}$, and the electronphonon scattering rate at the gap edge, $2\left(\mathrm{Im}\, ZE - (R_2/N_1)\,\mathrm{Im}\,\phi\right)_{|E=\Delta}$, equals 0.218Δ at $T = 0.964T_{\mathrm{c}}$.}

influence of a supercurrent and a temperature gradient, as discussed further below. The collision integral $I\left(f^1\right)$ for the case of charge relaxation (Schmid 1981, Beyer Nielsen et al. 1982) is

$$\begin{aligned} I\left(f^{\mathrm{T}}\right) = & -2\left(N_1\,\mathrm{Im}\, ZE + N_2\,\mathrm{Re}\,\Phi\right) f^{\mathrm{T}}(E) \\ & + \int_{-\infty}^{\infty} \mathrm{d}E' B\left(E, E'\right)\left(N_1(E) N_1\left(E'\right) + N_2(E) N_2\left(E'\right)\right) f^{\mathrm{T}}\left(E'\right) \end{aligned} \tag{3.8}$$

where

$$B\left(E,E'\right)=\pi\alpha^2F\left(\left|E-E'\right|\right)\frac{\cosh\left(E'/2T\right)}{\sinh\left(\left|E-E'\right|/2T\right)\cosh(E/2T)}. \tag{3.9}$$

The function $\alpha^2F(E)$ entering eq. (3.9) is the effective phonon density of states, equal to bE^2 in a Debye model, where b is a constant. The collision integral for thermal conduction, calculated by Beyer Nielsen and Smith (1982), is the same as that for charge relaxation, except that α^2F in eq. (3.9) must be replaced by α_1^2F, an effective density of states for transport which is evaluated in the same way as α^2F, but with the inclusion of an additional factor $\hat{p}\cdot\hat{p}$ in the sum over phonon states. Here $\hat{\boldsymbol{p}}$ and $\hat{\boldsymbol{p}}$ are the directions of the initial and final electrons in the process in which an electron is scattered by a phonon. As a convenient measure of the strength of the electron-phonon interaction one may introduce the normal state relaxation time $\tau_{\rm in}$ at $T_{\rm c}$ and at the Fermi energy,

$$\frac{1}{\tau_{\rm in}}=\frac{4\pi}{\operatorname{Re}Z(0)}\int_0^\infty \mathrm{d}E\frac{\alpha^2F(E)}{\sinh\left(E/T_{\rm c}\right)} \tag{3.10}$$

Next we specify the driving term $X_{\rm inj}$ appropriate to tunnel injection of charge. This is

$$X_{\rm inj}\;=\;\text{const.}\;N_1(E)\left(f^0(E-eV)-f^0(E+eV)\right), \tag{3.11}$$

where V is the injection voltage. In the case of thermal conductivity the driving term is

$$X_{\rm th}=\;\text{const.}\;EN_1(E)\left(\partial f^0/\partial E\right), \tag{3.12}$$

as shown in detail in the following section.
Before we discuss the results of solving the kinetic equation (3.7) let us show how to recover the ordinary quasiparticle Boltzmann equation. In this limit one must consistently treat the pair breaking parameters as small. This means that, since N_2 appears explicitly in the collision integral in the combination $2\operatorname{Re}\Phi N_2$, one needs to include in N_2 terms linear in $\Gamma_{\rm s}$ and $\Gamma_{\rm p}$. One finds (Beyer Nielsen et al. 1982)

$$2\operatorname{Re}\Phi N_2=\frac{2|E|}{\sqrt{E^2-\Delta^2}}\left(\frac{E^2}{E^2-\Delta^2}\Gamma_{\rm p}-\operatorname{Im}ZE\right)+\frac{2\Delta^2E^2}{\left(E^2-\Delta^2\right)^2}\Gamma_{\rm s}. \tag{3.13}$$

Combining this term with the remainder of the collision integral (3.8) and introducing the function $N_1f^\top$, which in the limit of weak pair breaking goes over to the distribution function $\delta f'^c$ of the quasiparticle approach. one finds after dividing by $N_1^2=q^{-2}$ the Boltzmann equation

$$\begin{aligned}&\text{const.}\;\left(f^0(E-eV)-f^0(E+eV)\right)q\\&=-2\left(\Gamma_{\rm p}+\frac{\Delta^2}{E^2}\frac{E}{|\xi|}\Gamma_{\rm s}\right)\delta f^{\ell e}+q\int_{-\infty}^{\infty}\mathrm{d}\xi' B\left(E,E'\right)q'\delta f^{\ell e}\left(\xi'\right)\end{aligned} \tag{3.14}$$

Beyer Nielsen et al. (1982) solved the kinetic equation obtained by equating (3.8) and (3.11) for a weak-coupling Debye model and varying amounts of magnetic impurity scattering; the results were compared with

solutions of the Boltzmann equation (3.14). The calculations were performed at arbitrary temperatures, and hence are not restricted by the condition $\Delta\ll T$. The results Beyer Nielsen et al. obtained by solving the Boltzmann equation are consistent with those obtained earlier by Lemberger and Clarke (1981), who also based their analysis on the quasiparticle Boltzmann equation (3.14).

The results of calculations based on the fully microscopic theory and on the Boltzmann equation are compared in fig. 3. Note how close the two sets of results are despite the fact that

the pair-breaking parameter $1/\tau_{\rm s} = \Gamma_{\rm s}/\operatorname{Re} Z(0)$ is
comparable to Δ over much of the temperature range: in the example chosen $1/\tau_{\rm s} = 0.77T_{\rm c}$.

Also shown in the figure is a result due to Schmid and Schön (1975) for the charge relaxation rate ${\tau_{Q^*}}^{-1}$ near T_c. To lowest order in Δ/T, they found the charge relaxation time τ_{Q^*} to be given to a good approximation by

$$\frac{1}{\tau_{Q^*}} = \frac{\pi}{4}\frac{\Delta}{T_{\rm c}}\sqrt{\frac{1}{\tau_{\rm in}}\left(\frac{1}{\tau_{\rm in}} + \frac{2}{\tau_{\rm s}}\right)} \tag{3.15}$$

As one can see from fig. 3, this result is a good approximation only for $\Delta/T \lesssim 0.1$. For energy gaps in the range $0.2 < \Delta/T < 0.8$ the charge relaxation rate is proportional to Δ to a good approximation, but the coefficient differs from that in eq. (3.15). This behaviour is consistent with the experimental data of Lemberger and Clarke (1981). For more discussion of the data and their comparison with theory see Beyer Nielsen et al. (1982).

Turning to the case of thermal conductivity we again consider the effects of pair breaking due to electron-phonon scattering. In pure weak-coupling superconductors, the thermal conductivity should increase with decreasing temperature below T_c, according to weak-coupling calculations. However, the fact that in the strong-coupling superconductor Pb the thermal conductivity is found to fall rapidly just below $T_{\rm c}$ (Watson and Graham 1963) clearly demonstrates the inadequacy of weak-coupling equations for dealing with strong-coupling materials. A kinetic equation for strong-coupling superconductors was derived from the Kubo formula by Ambegaokar and Tewordt (1964), and the thermal conductivity for Pb near $T_{\rm c}$ was calculated in the relaxation time approximation by Ambegaokar and Woo (1965). The latter authors found $\kappa_{\rm s}(T)/\kappa_{\rm n}(T) \simeq 1 - 11\,(1 - T/T_{\rm c})$, where $\kappa_{\rm n}$ and $\kappa_{\rm s}$ are the thermal conductivities in the normal and superconducting states, respectively, whereas experimentally the coefficient in this relation was found to be about -9 , compared with a typical value of -1.6 for weak-coupling superconductors (Gunault 1961). Subsequently calculations for Pb and Hg have been performed by Vashishta and Carbotte (1972), who used essentially the same formalism and approximations. These results show that strong-coupling effects are able to account for $\kappa_{,}\kappa_n$ in Pb behaving very differently from the way it does in weak-coupling superconductors.

Recently Beyer Nielsen and Smith (1982) have derived a kinetic equation for thermal conduction in strong-coupling superconductors on the basis of the quasiclassical Green function approach. Their equations appear to be rather different from those of Ambegaokar and Tewordt (1964), but in fact the two sets of equations may be shown to be equivalent, apart from some terms which are unimportant numerically. Beyer Nielsen and Smith (1982) then solved their kinetic equation for the case of Pb , using functions N_i and R_i calculated from gaps and renormalization factors obtained from solutions of the Eliashberg
equations for a strong-coupling superconductor. The calculations were made at arbitrary temperatures and did not employ the relaxation time approximation. The kinetic equation and the Eliashberg equations were both solved for the same phonon spectrum. which was taken to be that given by a model due to Scalapino (1469), supplemented by a Debye tail at low energics. Close to T_c the theoretical thermal conductivity is given by

$$\kappa_{\rm s}(T)\kappa_{\rm n}(T) \simeq 1 - 7\left(1 - \frac{T}{T_{\rm c}}\right).$$

and $\kappa_s(T)$ displays a maximum well below T_s in agreement with experiment (Mezahov-Deglin 1979).

Strong-coupling effects enter the calculations of transport coefficients in a number of different ways: they change the gap from the BCS value. they give quasiparticle states a finite width, and they modify the collision term, as a consequence of departures of the phonon spectrum from the Debye form. In the calculations for Pb . broadening of the quasiparticle states

is unimportant. and the behaviour of the thermal conductivity is well accounted for by the quasiparticle Boltzmann equation neglecting broddening provided the collision term contains the correct phonon density of states and provided the strong-coupling result for the gap is used (Beyer Nielsen and Smith 14Si). In weah-coupling calculations the phonon spectrum is approximated by a Debye model, but in the calculations of Beter Vielsen and Smith the Debre tail of the phonon spectrum dominates only for $T \leqslant 0.4T_0$ and the structure in the temperature dependence of h, is due mainly to a laree peak in the transorre phonon density of states at an energy of about 50 K .

Calculations of the charge relaxation time for strong-compline superconductors have been carried out by Beyer Vielsen and Smith (19S21. For the case of electron phonon scattering they find that if broddening of the quasiparticle states can be neglected. the charge relaxation rate is given by

$$\tau_{Q^*}^{-1} = (\pi 4)(1T)\tau_{\mathrm{in}}^{-1} \quad \text{close to } T_{\mathrm{c}}$$

provided I is interpreted as the value of $\mathrm{Re}\,\Phi/\ /\ 1$ at the sap edge. * Line the phonon spectrum described abowe they hate carried out calculations for Pb . and find that $\mathrm{e}^{-}\mathrm{F}^{-}$in is about 1.5 times the weak-coupling value at I $T = 1.2$. times it at $dT = 2, 4$ times it at $dT = 3$, and 5 times it at $dT = 5$. As a function of TT_c, the difference between the weak- and strong-coupling calculations is even more pronounced, because of the more rapid rise in . I with decreasing temperature in the strong-coupling calculation.

As a last example of the effects of pair breaking we mention the frequency- sepandent conducuraty of a ding cupercond cator anken may be roce ared by whermptime a mall P} curress on a large Jor coment, as in the angmal experiment by Peters and Meanon (1977). Recently Pitger and Meanon 119401 roprasad and exagudest thus experiment They analy, eas there ereamen-tal data in terms of the theory due to Schmid (1968, 1969) who derived an interpolation formula for the frequency-dependent conductivity in the prevence of electron powon restionerg Pitger and Meanon found that is 2 an necessary to introduce an excess conductivity to account for their experimental results near T_c at relatively high frequencies. This excess conductivity may be explained by the effects of pair breaking, as shown by Beyer Nielsen et al. (1984).

The interpotaton formala dae to S.armed 1964. 19631 applien to tex reagen near T_c, where $\Delta \ll T_c$ and is in addition restricted in validity to the frequency region $\omega\tau_{in} \ll 1$. It was shown by Orchinnikov et al. (1981) that at very low remperatures the frequence: cependence of the conductury a deamermed by the relaxation time at the gap edge. In their recent calculation Beyer Nielsen et al rigit exterded and apposimate reatmente to arbarrary ; above of or. and ΔT subject only to the condition $\omega \ll \Delta$. From numerical solutions to the kinetic equations these authors investigated the validity of the relaxation time spporamator. and anted that ∴ aork quite well sase: the rans 'Ion temperatures where the frequency-dependent conductivity involves two characteristic times, z_{is} and the gap-relaxation time. :- - (r. - (1) 1) 1T.

To illustrate the results we consider the case when T is close to T_c. In the relaxation time approximation Beyer Vieken et al. (1984) find

$$= \frac{\mathrm{i}\sigma_s \pi}{2} \frac{\Delta^2}{T} \int_1 1 - \frac{\Gamma}{3\Gamma_s} \left(1 + \frac{(2 - \mathrm{i}\omega_{sin})^2}{2\,(1 - \mathrm{i}\omega_s)\,(1 - \mathrm{i}\omega_{si})} \right)' \Bigg\}' \tag{13.161}$$

[12]*In agreement with Beyer Nielsen et al. (1982) we use here a definition of τ_Q that includes the renormalization factor ReZ(0). Note that Beyer Nielsen and Smith (1982) used a definition of τ_{Q^*} without Re$Z(0)$.

provided $\omega/T \ll 1$ and $\omega/T \ll \Omega$ $\Omega = \Omega(T)$ where $\Omega = 2D p_s^2 \sigma_n$ is the normal state DC conductivity $p_s = m v_s$ is the momentum of an electron in the condensate $D = \frac{1}{3} t_2^2 \tau_{\text{imp}}$ is the diffusion coefficient in the normal state. and

$$\Gamma_c = \frac{T' = (3)}{6\pi^3} \frac{\Delta^2}{T} = \frac{4}{3\pi} (T_c - T) .$$

Schmid's interpolation formula is

$$== \frac{\mathrm{i}\pi\sigma_s}{2\cos\frac{\Delta^2}{T}\int_1 1 - \frac{\Gamma}{3\Gamma_2}\left(1 - \frac{2}{1-\mathrm{i}\omega_s}\right)'\Big\}} \tag{3.17}$$

which agrees with (3.16) in the limit when $\omega/T \ll 1$. At slightly lower temperatures, down to $0.95\,T_c$, both (3.16) and (3.17) fit the numerical solution quite

well, the former yielding a more accurate description of the collisionless limit $\omega\tau_{\text{in}} \gg 1$. As we mentioned above, calculations of the conductivity have been carried out for a wide range of conditions, but because the results cannot be expressed in particularly simple form we shall not display them here.

Pair breaking effects due to phonons are of importance also to the thermoelectric effect which one observes when a supercurrent passes through a superconductor in the presence of a temperature gradient. This will be treated in the next section.

28.4 4. Impurity scattering in the presence of a supercurrent. Effects of a temperature gradient

In this section we illustrate a number of the points made in the previous sections by discussing in detail the derivation of a kinetic equation appropriate to impurity scattering in the presence of a supercurrent and a temperature gradient. Electron-phonon scattering is neglected in this section in order to keep the discussion simple. Within this model we show how the charge imbalance and the thermal conductivity are calculated from the same kinetic equation. A great virtue of this problem is that the majority of the results can be obtained analytically. In addition, the problem is an important one since impurity scattering is the dominant scattering mechanism in many experiments. Most of the calculations for the charge imbalance have been described by Beyer Nielsen et al. (1982), but here we show in addition how to obtain simple analytic results not only in the clean and dirty limits, but also in between. The results for the dependence of the thermal conductivity on the superfluid velocity have not been published before. The generation of charge imbalance by the simultaneous application of a temperature gradient and a supercurrent was first proposed by Pethick and Smith (1979b) and subsequently observed by Clarke et al. (1979) and Fjordboge et al. (1981). The relaxation by impurities has been considered by Schmid and Schön (1979), Clarke and Tinkham (1980) and Beyer Nielsen et al. (1980).

In the clean limit, a quasiparticle with energy $E > A + p_{\text{F}} l_{\text{s}}$ can scatter into any direction, but for $\Delta - p_{\text{F}} v_{\text{s}} < E < \Delta + p_{\text{F}} v_{\text{s}}$, the momenta of possible states are limited to a cone whose vertex is directed antiparallel to the superfluid velocity. As a consequence, the scattering rates for states in the pocket with energy in the range $\Delta - p_{\text{F}} v_{\text{s}} < E < \Delta + p_{\text{F}} v_{\text{s}}$ are very different from those for states with $E > A + p_{\text{F}} l_{\text{s}}$. The calculations we describe will show how, as the impurity scattering rate is increased, the pocket of states gets smoothed out by depairing due to the finite supercurrent, as was pointed out first by Schön (1981a).

The fundamental equation of motion (2.22) involves the inverse of the noninteracting Green function $\hat{g}_0{}^{-1}$. The matrix $\hat{g}_0{}^{-1}$ is proportional to a unit

matrix in Keldysh space, $\hat{g}_0^{-1} = \hat{1} g_0^{-1}$. In the stationary case, where we consider only impurity scattering in the presence of a supercurrent, g_0^{-1} is a 2×2 matrix in Nambu space,

$$g_0^{-1} = \tau_3 \left(E - p \cdot v_s\right) + \mathrm{i} \frac{p \cdot \nabla}{m} - e\phi \tag{4.1}$$

Here E is the energy (frequency) variable, $p = p_{\mathrm{F}} \hat{p}$ is the momentum, and ϕ the electrostatic potential. The self energy $\hat{\Sigma}^{\mathrm{R}}$ consists of the usual pairing contribution and a part $\hat{\Sigma}_{\mathrm{imp}}$ due to impurity scattering,

$$\hat{\Sigma}^{\mathrm{R}} = -\mathrm{i}\Delta\tau_1 + \hat{\Sigma}_{\mathrm{imp}}. \tag{4.2}$$

We neglect for simplicity the effects of phonon renormalization, although they may readily be kept (Beyer Nielsen et al. 1982). The impurity part of the self energy is calculated with a self-consistent Born approximation. With the further assumption that the scattering in the normal state is pure s-wave, the impurity contribution to the self energy may be written as

$$\hat{\Sigma}_{\mathrm{imp}} = -\frac{\mathrm{i}}{2\tau_{\mathrm{imp}}} \langle \hat{g}(\hat{p}, E, r) \rangle \tag{4.3}$$

where $\langle \ldots \rangle$ denotes the angular average (2.29), and τ_{imp} is the normal state impurity scattering time. The equation of motion (2.22) is solved by the ansatz (2.24-2.26). With this ansatz the normalization condition is simply

$$\alpha^2 + \beta^2 = 1. \tag{4.4}$$

The kinetic equation for $f^{\top}$ is obtained from the component of (2.22) which is off-diagonal in Keldysh space, and proportional to τ_3 in Nambu space:

$$\mathrm{Tr}\left\{\tau_3 \left(\left(g_0^{-1} - \Sigma^{\mathrm{R}}\right) g^{\mathrm{K}} - \Sigma^{\mathrm{K}} g^{\mathrm{A}} - g^{\mathrm{K}} \left(g_0^{-1} - \Sigma^{\mathrm{A}}\right) + g^{\mathrm{R}} \Sigma^{\mathrm{K}}\right)\right\} = 0 \tag{4.5}$$

When g_0^{-1} and Σ are inserted from eqs. (4.1)-(4.5) together with the expressions (2.24) and (2.25) for g^{K} the following kinetic equation results:

$$\begin{aligned} N_1 \frac{p}{m} \cdot \nabla f^{\mathrm{L}} = -\frac{1}{\tau_{\mathrm{imp}}} \{ & \left\{ \left(N_1 \langle N_1 \rangle + N_2 \langle N_2 \rangle\right) f^{\mathrm{T}} \right. \\ & \left. - N_1 \left\langle N_1 f^{\mathrm{T}} \right\rangle - N_2 \left\langle N_2 f^{\mathrm{T}} \right\rangle \right\} - 2AN_2 f^{\mathrm{T}} \end{aligned} \tag{4.6}$$

The integral equation (4.6) is separable and may readily be solved, once $f^{\mathrm{L}} \simeq f^0(E)$ is inserted on the left-hand side. When v_{s} is parallel to ∇T, equation (4.6) may be written as

$$\begin{aligned} c(E) N_1 x = & \left(N_1 \langle N_1 \rangle + N_2 \langle N_2 \rangle\right) f^{\mathrm{T}} \\ & - N_1 \left\langle N_1 f^{\mathrm{T}} \right\rangle - N_2 \left\langle N_2 f^{\mathrm{T}} \right\rangle + \gamma^{-1} N_2 f^{\mathrm{T}}, \end{aligned} \tag{4.7}$$

with $\gamma = \frac{1}{2\tau_{\mathrm{imp}}\,\Delta}$ and $c(E) = v_{\mathrm{F}} \tau_{\mathrm{imp}} \left|\nabla T\right| (E/T) \left(\partial f^0 / \partial E\right)$. We can now find the desired averages $\left\langle N_1 f^{\top} \right\rangle$ and $\left\langle N_1 f^{\top} x \right\rangle$,

$$\left\langle N_1 f^{\mathrm{T}} \right\rangle = c \frac{\left\langle N_1^{-1} \tau x \right\rangle + \gamma \left\langle N_1 x \right\rangle \left\langle N_1^{-2} N_2 \tau \right\rangle}{1 - \left\langle N_1^{-1} \tau \right\rangle} \tag{4.8}$$

and

$$\begin{aligned}\left\langle N_1 f^{\mathrm{T}} \mathrm{x}\right\rangle =& c\left(\left\langle N_1^{-1} \tau x^2\right\rangle+\gamma\left\langle N_1 x\right\rangle\left\langle N_2 N_1^{-2} \tau x\right\rangle\right) \\ &+\left\langle N_1 f^{\mathrm{T}}\right\rangle\left\langle N_1^{-1} \tau x\right\rangle,\end{aligned} \tag{4.9}$$

where the quantity τ is defined by

$$\frac{1}{\tau}=N_1^{-3}\left(N_1\left\langle N_1\right\rangle+N_2\left\langle N_2\right\rangle+\gamma^{-1} N_2\right). \tag{4.10}$$

The results (4.8)-(4.10) are obtained most easily by eliminating $\left\langle N_2 f^{\mathrm{T}}\right\rangle$ by averaging eq. (4.7) over angle. From (4.8) and (4.9) the charge imbalance Q^* and heat current j_{th} are obtained as

$$Q^*=2 N(0) \int_{-\infty}^{\infty} \mathrm{d} E\left\langle N_1 f^{\mathrm{T}}\right\rangle \tag{4.11}$$

and

$$j_{\mathrm{th}}=2 N(0) v_{\mathrm{F}} \int_{-\infty}^{\infty} \mathrm{d} E E\left\langle N_1 f^{\mathrm{T}} x\right\rangle \tag{4.12}$$

Although formally the problem is solved by these results one must know N_1 and N_2 as functions of energy and angle in order to carry out the integrals over these two variables. Let us first discuss some simple limits, when the selfconsistency equations may be solved analytically. The component of (2.22) diagonal in Keldysh space yields one equation which α and β must satisfy,

$$\left(E-p_{\mathrm{F}} v_{\mathrm{s}} \cos \theta\right) \beta-\mathrm{i} \Delta \alpha+\frac{\mathrm{i}}{2 \tau_{\mathrm{imp}}}(\langle\alpha\rangle \beta-\langle\beta\rangle \alpha)=0. \tag{4.13}$$

This must be solved together with the normalization condition (4.4).

28.4.1 4.1. Dirty limit

The effect of strong impurity scattering is to wash out most of the anisotropy associated with the supercurrent. The self-consistency equations may be solved by an ansatz which retains only s and p waves:

$$\begin{aligned}\alpha &= \alpha_0+\alpha_1 x, \\ \beta &= \beta_0+\beta_1 x,\end{aligned} \tag{4.14}$$

where $x=\cos \theta$. One finds readily that, provided $1/\tau_{\mathrm{imp}} \gg \sqrt{E^2-\overline{A^2}}, \alpha_0$ and β_0 satisfy the equations

$$\alpha_0^2+\beta_0^2=1 \tag{4.15}$$

$$E \beta_0-\mathrm{i} \Delta \alpha_0+\mathrm{i} \Delta \frac{\delta^2}{3 \gamma} \alpha_0 \beta_0=0 \tag{4.16}$$

where $\delta=p_{\mathrm{F}} v_{\mathrm{s}} / \Delta$. The equations (4.15) and (4.16) are identical with those for magnetic impurity scattering if one replaces $\Delta \delta^2 / 3 \gamma$ in eq. (4.16) by the rate Γ_{s} for scattering by magnetic impurities. The quantities β_1 and α_1 are obtained by combining the p -wave component of the normalization condition,

$$\alpha_0\alpha_1 + \beta_0\beta_1 = 0, \tag{4.17}$$

with the p-wave component of (4.13). The result is

$$\beta_1 \simeq \frac{1}{\mathrm{i}}\frac{\delta}{\gamma}\alpha_0\beta_0 \tag{4.18}$$

and

$$\alpha_1 \simeq \mathrm{i}\frac{\delta}{\gamma_{\prime}}\beta_0^2. \tag{4.19}$$

One therefore obtains the important relations

$$N_1 = N_1^0 - x\frac{2\delta}{\gamma}N_2^0R_2^0, \tag{4.20}$$

$$N_2 = N_2^0 + x\frac{\delta}{\gamma}\left(N_1^0R_2^0 + N_2^0R_1^0\right) \tag{4.21}$$

which express N_i in terms of the isotropic parts of the density of states functions which are obtained from eqs. (4.15) and (4.16) and the relationships $\alpha_0 = N_1^0 + \mathrm{i}R_1^0$ and $\beta_0 = N_2^0 + \mathrm{i}R_2^0$.

Instead of working directly with the formal solution it is easier to expand N_1f^{T}. Writing

$$N_1f^{\mathrm{T}} = f_0 + f_1x \tag{4.22}$$

one finds from insertion into eq. (4.7)

$$\frac{f_0}{c} = \frac{N_1^0}{N_2^0}\gamma\left\langle N_1x\right\rangle = -\frac{2}{3}\delta N_1^0R_2^0c \tag{4.23}$$

and

$$\frac{f_1}{c} = \frac{1 + 2(\Gamma/\Delta)N_2^0\left(R_2^0\right)^2\left(1 + \left(N_2^0/N_1^0\right)^2\right)}{1 + \left(N_2^0/N_1^0\right)^2 + \gamma^{-1}N_2^0/\left(N_1^0\right)^2}, \tag{4.24}$$

where $\Gamma = \frac{1}{3}\delta^2\Delta/\gamma = 2Dp_{\mathrm{s}}^2$ is the characteristic rate for pair breaking by the

superflow. In the limit $v_{\mathrm{s}} \to 0, N_2^0$ tends to zero for $E > \Delta$, and therefore f_1/c, given by eq. (4.24), tends to unity. The difference between the right-hand side and unity is therefore a measure of how much the heat current is affected by the superflow. We plot $-f_0/c$ and f_1/c for $\Gamma = 10^{-2}\Delta$ in figs. 4 and 5 as functions of E, taking $\gamma = 1$ in (4.24). To find the thermal conductivity, κ, one needs to perform the energy integral in the heat current

$$j_{\mathrm{th}} = \frac{2}{3}N(0)v_{\mathrm{F}}\int_{-\infty}^{\infty}\mathrm{d}EEf_1(E) \tag{4.25}$$

The trace in Nambu space of the Keldysh component of the kinetic equation is

$$N_1\frac{\boldsymbol{p}\cdot\boldsymbol{\nabla}}{m}f^{\mathrm{T}} = -\frac{1}{\tau_{\mathrm{imp}}}\left[\left(N_1\left\langle N_1\right\rangle - R_2\left\langle R_2\right\rangle\right)f^{\mathrm{L}} - N_1\left\langle N_1f^{\mathrm{L}}\right\rangle + N_2\left\langle N_2f^{\mathrm{L}}\right\rangle\right]. \tag{4.26}$$

If, in the spirit of the standard Chapman-Enskog development, one inserts a local equilibrium distribution function ($f^{\mathrm{T}} = 0$) on the left-hand side, the driving term vanishes and f^{L} is then given by a local equilibrium distribution function $f^L = \left(e^{E/T(r)} + 1\right)^{-1}$. Only if one works to higher order in λ/L, where λ is a typical mean free path and L is a typical length scale for spatial variations, does f^{L} deviate from equilibrium.

28.4.2 4.2. Intermediate case

The transition from the clean limit to the dirty limit has been discussed extensively by Schön (1981a,b) and by Beyer Nielsen et al. (1981, 1982). The effect of impurities is found to depend strongly on energy. For the case of weak impurity scattering one finds that the charge relaxation rate for states in the pocket with energy $\Delta - p_{\mathrm{F}} v_{\mathrm{s}} \leqslant E \leqslant \Delta + p_{\mathrm{F}} v_{\mathrm{s}}$ is of order $(1/\tau_{\mathrm{imp}})\left(p_{\mathrm{F}} v_{\mathrm{s}}/\Delta\right)^{1/2}$. Pair breaking in the pocket region can not be treated as small when the charge relaxation rate is comparable to a typical quasiparticle energy, measured with respect to the gap Δ. Thus for energies near Δ the transition between clean and dirty behaviour occurs when

$$\frac{1}{\tau_{\mathrm{imp}}}\left(\frac{p_{\mathrm{F}} v_{\mathrm{s}}}{\Delta}\right)^{1/2} \sim p_{\mathrm{F}} v_{\mathrm{s}}, \qquad \text{or} \qquad \frac{1}{\tau_{\mathrm{imp}}\,\Delta} \sim \left(\frac{p_{\mathrm{F}} v_{\mathrm{s}}}{\Delta}\right)^{1/2}. \tag{4.27}$$

For high energies ($|E - A| \gg \Delta$), the transition between clean and dirty behaviour occurs when

$$\frac{1}{\tau_{\mathrm{imp}}} \simeq E. \tag{4.28}$$

For $|E| - \Lambda - p_{\mathrm{F}} v_{\mathrm{s}} \gg p_{\mathrm{F}} v_{\mathrm{s}}$, Beyer Nielsen et al. (1982) find for the quantity of importance in the expression for the charge imbalance, eq. (4.11),

$$\left\langle N_1 f^{\mathrm{T}} \right\rangle = -\tau_{\mathrm{imp}} v_{\mathrm{s}} \cdot \nabla T \frac{E}{T} \frac{\partial f^0}{\partial E} v_{\mathrm{F}} p_{\mathrm{F}} \frac{\frac{4}{15}\left(E^2 - A^2\right)^2 + 2\left(E^2 - \Delta^2\right)\Gamma_{\mathrm{imp}}^2 + \frac{2}{3}\Gamma_{\mathrm{imp}}^4}{\left(E^2 - \Delta^2 + \Gamma_{\mathrm{imp}}^2\right)^2\left(E^2 - \Delta^2\right)}, \tag{4.29}$$

where $\Gamma_{\mathrm{imp}} = 1/(2\tau_{\mathrm{imp}})$. This result shows that the transition between the clean and dirty limit occurs when $\sqrt{E^2 - A^2} \simeq 1/(2\tau_{\mathrm{imp}})$, which reduces to (4.28) when $|E - \Delta| \gg \Delta$. Close to T_{c} the region when $|E| - \Delta - p_{\mathrm{F}} v_{\mathrm{s}} \leqslant p_{\mathrm{F}} v_{\mathrm{s}}$ contributes little

to the charge imbalance, and to leading order in Δ the charge imbalance (4.11) may be obtained by integrating over E the high energy expression (4.29) with Δ put equal to zero. One finds

$$Q^* = \frac{4}{15} N(0) p_{\mathrm{F}} v_{\mathrm{F}} \tau_{\mathrm{imp}} v_{\mathrm{s}} \cdot \frac{\nabla T}{T} I\left(\Gamma_{\mathrm{imp}}/T\right), \tag{4.30}$$

where

$$\begin{aligned} I\left(\Gamma_{\mathrm{imp}}/T\right) &= \int_{-x}^{\infty} \mathrm{d}E \left(\frac{-\partial f^0}{\partial E}\right)\left(2 + \frac{11\left(\Gamma_{\mathrm{imp}}\right)^2}{E^2 + \Gamma_{\mathrm{imp}}^2} - \frac{8\Gamma_{\mathrm{imp}}^4}{\left(E^2 + \Gamma_{\mathrm{imp}}^2\right)^2}\right) \\ &= 2 + 7y\psi'\left(\frac{1}{2} + y\right) + 4y^2\psi''\left(\frac{1}{2} + y\right) \end{aligned} \tag{4.31}$$

where $y = \Gamma_{\mathrm{imp}}/(2\pi T)$, and ψ' and ψ'' are the tri-gamma function and its derivative. In the clean limit ($\Gamma_{\mathrm{imp}}/T \to 0$), I is 2 and

$$Q^* = \frac{8}{15} N(0) v_{\mathrm{F}} \tau_{\mathrm{imp}} v_{\mathrm{s}} \frac{\nabla T}{T}$$

while in the dirty limit I is 5 and

$$Q^* = \frac{4}{3} N(0) v_{\mathrm{F}} \tau_{\mathrm{imp}} v_{\mathrm{s}} \frac{\nabla T}{T}$$

The transition between these two limiting behaviours is shown in fig. 6, in which eq. (4.31) is plotted as a function of Γ_{imp}/T.

Finally we discuss the lowest energy, E_g, at which there is a finite density of states. This is the lowest energy for which β_0, the angular average of β, has a finite imaginary part. From eq. (4.13) and the normalization condition (4.4) one finds the following quintic equation for β_0 :

$$\frac{2}{3}\gamma\delta^2\beta_0^5 + \left(\delta^2 + 2\delta x_0\gamma^2\right)\beta_0^4 + 4\delta\gamma x_0\beta_0^3 + \left(\gamma^2 + 2\delta x_0\right)\beta_0^2 + 2\gamma\beta_0 + 1 = 0, \tag{4.32}$$

where $\gamma = \Gamma_{\text{imp}}/\Delta, \delta = p_{\text{F}}v_{\text{s}}/\Delta$, and $x_0 = (E-\Delta)/p_{\text{F}}v_{\text{s}}$. In the clean limit one finds

$$E_{\text{g}} \simeq \Delta - p_{\text{F}}v_{\text{s}}\left(1 - \frac{2}{3}\frac{\gamma}{\sqrt{\delta}}\right), \quad \gamma \ll \sqrt{\delta}, \tag{4.33}$$

while in the dirty limit

$$E_{\text{g}} \simeq \Delta - p_{\text{F}}v_{\text{s}}\frac{\sqrt{3}}{2}\left(\frac{\sqrt{\delta}}{\gamma}\right)^{2/3}, \quad \gamma \gg \sqrt{\delta}. \tag{4.34}$$

Note that the transition from the clean to the dirty limit occurs when $\gamma \sim \sqrt{\delta}$, or $1/2\tau_{\text{imp}} \simeq \sqrt{p_{\text{F}}v_{\text{s}}\Delta}$, in agreement with eq. (4.27). A graph of E_{g} versus $\gamma\sqrt{\delta}$ in the intermediate regime is shown in fig. 7.

28.4.3 4.3. Clean limit

In the clean limit quasiparticles with $E > A + p_{\text{F}}v_{\text{s}}$ can scatter in all directions, but for $E < A + p_{\text{F}}v_{\text{s}}$ the solid angle into which quasiparticles can scatter is restricted because the minimum energy of a quasiparticle is $A + p_{\text{F}}v_{\text{s}}x$, where

$x = \hat{p}\cdot\hat{v}_{\text{s}}$. This leads to rather different behaviours of the quasiparticle distribution function depending on whether E is greater or less than $\Delta + p_F v_s$. The charge imbalance for this case was first calculated by Beyer Nielsen et al. (1980), and the result for the thermal conductivity is new. One finds

$$Q^* = 2N(0)v_{\text{F}}\tau_{\text{imp}}\frac{\hat{v}_{\text{s}}\cdot\nabla T}{T}\int_{1-p_{\text{F}},}^{\infty}\mathrm{d}E\left(\frac{\partial f^0}{\hat{c}E}\right)E\frac{\langle q\tau x\rangle}{1-\langle q\tau\rangle} \tag{4.35}$$

and

$$\kappa = 4N(0)v_{\text{F}}^2\tau_{\text{imp}}\frac{1}{T}\int_{1-p_{\text{F}}}^{\infty}\mathrm{d}E\left(\frac{-\partial f^0}{\partial E}\right)E^2\left\{\left\langle q\tau x^2\right\rangle + \frac{\langle q\tau x\rangle^2}{1-\langle q\tau\rangle}\right\}. \tag{4.36}$$

where $q = N_1^{-1}$ and τ is the $\gamma = 0$ limit of eq. (4.10). We now consider the contributions to the integrands for $E > A + p_{\text{F}}v_{\text{S}}$ and $E < \Delta + p_{\text{F}}v_{\text{S}}$.

(a) $E > \Delta + p_{\text{F}}v_{\text{s}}$. For $E - \Delta > p_{\text{F}}v_{\text{s}}$ one may expand quantities in powers of v_{s}, and one finds

$$\begin{aligned}
\langle q\tau\rangle &= 1 - \frac{1}{3}\frac{\delta^2}{\left(e^2-1\right)^2} + \cdots \\
\langle q\tau x\rangle &= -\frac{4}{45}\frac{\delta^3 e}{\left(e^2-1\right)^3} + \cdots \\
\left\langle q\tau x^2\right\rangle &\simeq \frac{1}{3} - \frac{7}{45}\frac{\delta^2}{\left(e^2-1\right)^2} + \cdots
\end{aligned} \tag{4.37}$$

(b) $\Delta - p_{\mathrm{F}} v_{\mathrm{s}} < E < \Delta + p_{\mathrm{F}} v_{\mathrm{s}}$. In the pocket one finds

$$q\tau = \begin{array}{c} 2\sqrt{\varepsilon - (1+x)} \\ \sqrt{\varepsilon}\frac{4}{3}\varepsilon - (1+x) \end{array}, \tag{4.38}$$

where $\varepsilon = 1 + (E-1)p_{\mathrm{F}} l'$ s is a dimensionless energy parameter which varies from 0 to 2 . For the angular averages one finds

$$\begin{aligned} \langle q\tau \rangle &= 2\left(1 - \frac{\pi}{3\sqrt{3}}\right) \simeq 0.7908. \\ \langle q\tau x \rangle &= -2\left(1 - \frac{\pi}{3\sqrt{3}}\right) + 2\varepsilon\left(1 - \frac{4\pi}{9\sqrt{3}}\right), \\ \left\langle q\tau x^2 \right\rangle &= 2\left(1 - \frac{\pi}{3\sqrt{3}}\right) - 4\varepsilon\left(1 - \frac{4\pi}{9\sqrt{3}}\right) + \varepsilon^2\left(\frac{12}{5} - \frac{32\pi}{27\sqrt{3}}\right). \end{aligned} \tag{4.39}$$

In fig. 8 we plot the function $-\langle q\tau x\rangle(1 - \langle q\tau\rangle)$, which determines the charge

imbalance according to (4.35), as a function of ε. The corresponding plot for the function $\langle q\tau x^2\rangle + \langle q\tau x\rangle^2/(1 - \langle q\tau\rangle)$ in eq. (4.36) is shown in fig. 9.

In order to calculate the physically observable quantities, Q^* and κ, as functions of the temperature, the energy integrals (4.35) and (4.36) must be performed. For the case of charge imbalance, this was done by Beyer Nielsen et al. (198()). In the case of thermal conductivity one finds in this clean limit a

change linear in v_{s} from the usual $v_{\mathrm{s}} = 0$ result for the impurity-limited thermal conductivity. Because of the above mentioned smearing effects on the pocket caused by the impurities, such an effect is probably not experimentally observable.

28.5 5. Conclusion

We have reviewed the quasiclassical approach to non-equilibrium phenomena in superconductors and discussed solutions to kinetic equations that are appropriate to a variety of physical circumstances when pair breaking effects are significant. The results are compared with those based on solving a conventional kinetic equation for Bogoliubov quasiparticles. In many situations the latter approach yields surprisingly accurate answers, but there are cases, such as the thermoelectric effect associated with a supercurrent and a temperature gradient, when pair breaking effects under normal experimental conditions require use of the quasiclassical approach. From the point of view of getting reliable numerical solutions to the kinetic equations and obtaining the associated relaxation rates the quasiclassical approach is not significantly more difficult than the conventional Boltzmann equation method, although the latter is conceptually simpler. The theoretical results reviewed in this chapter have been derived without making any assumptions on the magnitude of the energy gap relative to the temperature. They are therefore applicable at all temperatures and make possible a detailed comparison of experiment with theory.

Fig. 3. Plot of the quantity $\left[\left(\pi \tau_{Q^{*} } / 4 \tau_{\mathrm{in}}\right)\left(1+2 \tau_{\mathrm{in}} / \tau_{\mathrm{s}}\right)^{1 / 2}\right]^{-1}$ as a function of Δ/T. The dashed line is the Schmid and Schön expression Δ/T_c for this quantity. (See eq. 3.15.) The full curves show results calculated from the Boltzmann equation, and from the equations with broadening included ("exact"), as described in the text. The curves are for $\tau_{\mathrm{in}}/\operatorname{Re} Z(0) = 130/T_{\mathrm{c}}$, appropriate for Sn , and for (i) the case of no magnetic scattering ($\tau_s \to \infty$) and (ii) a case with magnetic scattering strong compared with phonon scattering ($\tau_{\mathrm{in}}/\tau_{\mathrm{s}} = 100$). For intermediate values of the magnetic scattering rate the curves vary smoothly between the two cases exhibited. The injection voltage is given by $eV = 30\Delta$.}

Fig. 4. The function $-f_{0 / c$, divided by $2\delta/3$, (eq. 4.23), as a function of E/Δ for $\Gamma = 10^{-2}\Delta$. The quantity f_0 is proportional to the charge imbalance on a shell of energy **E**.}

Fig. 5. The function f_{1 / c (eq. 4.24) as a function of E/Δ for $\Gamma = 10^{-2}\Delta$ and $\gamma = 1$. The quantity f_1 is proportional to the heat current on a shell of energy E. (See eq. 4.25 .)}

Fig. 6. The quantity I (eq. 4.31) as a function of $\Gamma_{1 m n T$.}

Fig. 7. The minimum energy $E_{\mathrm{g }}$ at which the density of states is finite, as a function of $\gamma/\sqrt{\delta}$ (full line). The dashed lines show the asymptotic expressions (4.33) and (4.34) for low and high values of $\gamma/\sqrt{\delta}$, respectively.}

Fig. 8. The angle-averaged deviation function $-\langle q\tau x\rangle/(1-\langle q\tau\rangle)$, produced by a supercurrent and a temperature gradient, in the pocket region $E-p_{F v_{s} \leqslant E \leqslant \Delta+p_{F} v_{b}$ for a clean superconductor. This quantity is proportional to the charge imbalance on a shell of energy E. (See eq. 4.35.)}

Fig. 9. The first moment of the distribution function, $\left\langle q \tau x^{2 \right\rangle+\langle q \tau x\rangle^{2} /(1-\langle q \tau\rangle)$, produced by a supercurrent and a temperature gradient, in the pocket region, $\Delta - p_F v_s \leqslant E\Delta + p_F v_s$, for a clean superconductor. This quantity is related to the heat current on a shell of energy E. (See eq. 4.36.)}

29 The Kinetic Theory of Superconductors with Excess Quasiparticles by Galitskii, Elesin, Kopaev (!!!!!!)

29.1 Introduction

The influence of an electromagnetic field and tunnel injection on superconductors leads to the creation of quasiparticles so that their number is greater than in thermal equilibrium. This situation takes place if the field frequence ω or the voltage across the junction V exceeds the gap 24*. Thus, a nonequilibrium state of the superconductor arises, which is accompanied by a number of interesting phenomena observed experimentally, namely: first-order transitions of the type superconductor-normal metal (or Δ_1-superconductor- Δ_3 superconductor), instabilities and hysteresis effects, the formation of spatially inhomogeneous states of a new type, etc. Quite a large number of papers and reviews deal with the theoretical investigation of superconductors with excess quasiparticles (Tremblay 1981, Elesin and Kopaev 1981). The shape of the energy distribution of quasiparticles created by an electromagnetic field or tunnel injection is the most important parameter of the theory. It appears that the order parameter, the stability of the state and the response to an external magnetic field may be very sensitive to the quasiparticle energy distribution. Therefore, a correct determination of the shape of the quasiparticle distribution function (QDF) is the fundamental task of the theory.

The general problem of simultaneously describing the dynamic and kinetic properties of quasiparticles was solved by Keldysh (1964). The Keldysh technique has made it possible to derive systems of equations for an exciton and a photon dielectric (semiconductor in a strong electromagnetic field, Elesin 1970a) and for superconductors (Elesin et al. 1973, Aronov and Gurevich 1973). Gor'kov and Eliashberg (1968) developed another method for describing nonequilibrium systems; using this method, Eliashberg (1970, 1971a) derived a system of equations for a superconductor. This system includes the equation for the order parameter $A(r,t)$, and kinetic equations for the QDF $n(\xi,r,t)$ and phonons $N(\varepsilon,r,t)$ (Chang and Scalapino 1977).

The purpose of this review is to describe the present state of the theory of these phenomena and to show how the theoretical results agree with experimental data obtained for superconductors with excess quasiparticles. Most attention is given to a consistent knetic theory, which makes it possible to

describe unambiguously a wide range of phenomena using solutions of the kinetic equations.

The review claims in no way to provide a complete bibliography, because it relies mainly on papers dealing with the kinetic theory and also on papers, which have appeared in recent years. Earlier papers are considered in detail in the reviews by Tremblay (1981), Elesin and Kopaev (1981), Langenberg (1975), and Aronov and Spivak (1978).

The present review consists of two parts. Section 1 deals with the singleband approximation, which assumes that the field frequency is less than the energy gap E_{g} between the conduction band and the valence band; section 2 considers the opposite case $\omega > E_{\mathrm{g}}$, where the electromagnetic field induces transitions between the bands (the two-band approximation).

The basic equations for the gap $\Delta, \mathrm{QDF} n(\xi,r,t)$ and phonons $N(\varepsilon,r,t)$ are presented in section 1.1. The electromagnetic field appears on the right-hand side of the kinetic equation for quasiparticles and acts as a source of excess quasi-particles. A similar role is played by tunnel injection.

Irrespective of the nature of the sources, it is convenient to divide them into two classes (Elesin 1974): "wide sources" create quasiparticles with energies in the range $\Delta\langle\varepsilon\langle\bar{\varepsilon},\bar{\varepsilon}\rangle\rangle$ (these are realized under optical pumping $\omega \geqslant \Delta$, tunnel injection $V \gg \Delta$, ionization by charged

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- We assume $\hbar = c = 1$.

particles, etc.); "narrow sources" create quasiparticles in a small energy range $\varepsilon - 2\Delta \ll \Delta$ (these are realized at threshold values of the frequency and voltage).

Section 1.2 is devoted to the most interesting class of states - those of superconductors with wide quasiparticle sources.

Experimental investigations in this field began after the work of Testardi (1971), in which he reported the measurement of the resistance of superconducting films irradiated with laser pulses. Testardi came to the conclusion that the superconductor normal metal transition observed was caused by excess quasiparticles, created by the electromagnetic field, rather than by "simple heating". Later on, Sai-Halasz et al. (1974), Hu et al. (1974) and Golovashkin et al. (1975) discovered that the resistance of a superconductor irradiated with light appeared gradually, starting at a certain critical power β_c, and reaching normal values in a fairly wide range of β (but not in a jump-like manner as in the equilibrium case). Those authors have assumed that a new type of nonequilibrium intermediate state may be possible, in which the superconducting and normal phases co-exist.

Further investigations of superconductors with optical (Golovashkin et al. 1978, 1979a,b, 1980, Mitsen 1981) and especially tunnel injection (Iguchi 1977, Fush et al. 1977, Mitsen 1979, Pan and Rudenko 1981, Mitsen et al. 1981) have confirmed the existence of an inhomogeneous state in nonequilibrium superconductors, and also have revealed other phenomena such as first-order phase transitions (Iguchi 1977, Fush et al. 1977), broadening of current-voltage characteristics, hysteresis effects, etc.

It should be noted that all these phenomena are observed in thin films (3003000Å). Such a small thickness is necessary in order to ensure that nonequilibrium phonons are removed before they come into equilibrium with the lattice and quasiparticles. Otherwise, the external action leads just to heating of the superconductor.

The first theoretical model was suggested by Owen and Scalapino (1972); this model was further developed in several papers (Baru and Sukhanov 1975, Scalapino and Huberman 1977, Hida 1978a,b) and was called the μ^*-model. It is assumed in the μ^*-model that the QDF can be represented in the form

$$n_{\mu^*} = \left[\exp\left(\frac{\varepsilon - \mu^*}{T}\right) + 1\right]^{-1} \tag{1}$$

where μ^* is the positive chemical potential and T is the lattice temperature. The μ^*-model predicted the inverse population of quasiparticles ($n > \frac{1}{2}$), firstorder transition at a certain critical concentration, diffusive instability, anomalous paramagnetism, and other effects. Comparison of the μ^*-model with experiment has revealed a number of essential difficulties (Golovashkin et al. 1979b, Tremblay 1981). Some time later Parker (1975) suggested a thermal model (the so-called T^*-model), in which the QDF was used in the form

$$n_{T^*} = \left[\exp\frac{\varepsilon}{T^*} + 1\right]^{-1} \tag{2}$$

The Parker function differs from the function of thermal (equilibrium) quasiparticles

$$n_{\mathrm{T}} = \left[\exp\frac{\varepsilon}{T} + 1\right]^{-1} \tag{3}$$

in that the lattice temperature is replaced by a certain effective temperature T^* determined from balance relations. The T^*-model describes well the dependence of A on the pumping power β in a certain range of β values, but it does not predict first-order transitions and instabilities, and is, therefore, insufficient to describe phenomena near the phase transition region, for example, the inhomogeneous state.

A third model was suggested by Elesin (1974, 1976, 1979a) and was developed further in papers by Elesin (1977b), Elesin et al. (1977a, b, 1979). Elesin and Levchenko (1978), Eckern

et al. (1979) and Imai (1980). It is based on the analysis and solution of the kinetic equations for quasiparticles. It was shown in those papers that the QDF differed significantly from the corresponding functions in the μ^* - and T^*-models, the specific features of $n(\xi)$ being of general character and having a clear physical meaning.

Specific features of the QDF are caused by the fact that quasiparticle relaxation is a coherent process. The energy distribution of excess quasi-
particles decays slower than that of thermal quasiparticles, so $n(\xi)$ satisfies the conditions (Elesin 1976)

$$n(\xi = 0) < n_T(\xi = 0), \tag{4}$$

$$n(\xi) < \frac{1}{2}. \tag{5}$$

Here n_{T} is the thermal QDF taken with the same Δ and temperature $T(\Delta)$ according to the BCS theory.

This means in terms of the quasiequilibrium distribution (1) that μ^* is negative and T^* exceeds T_{c}, i.e. the quasiparticle distribution is as if it were "overheated" (Elesin and Kopaev 1981).

The function $n(\xi)$ is less than $\frac{1}{2}$ and leads to an ambiguous dependence of the gap on the pumping power (Elesin 1976), characteristic of coherent instability (Elesin 1977a), which in turn gives rise to a laminated inhomogeneous state. This laminated state represents alternating superconducting and normal regions.

The model described [called the E-model by Eckern et al. (1979) and Tremblay (1981)] predicts a first-order transition into the normal or inhomogeneous state provided the optical pumping power (or voltage across the junction) exceeds a certain critical value.

The E-model makes it possible to predict and describe a wide range of phenomena, in particular the unsteady intermediate state under optical pumping, and the steady state under tunnel injection. In section 1.3.5 conclusions deduced from the μ^*-, T^*- and E-models are compared with experimental results.

The kinetic theory of superconductors with a narrow quasiparticle source is briefly outlined in section 1.3, following mainly the papers by Elesin (1974, 1979b), Aronov and Spivak (1976), Schön and Tremblay (1979), and Gray and Willemsen (1978). In this case specific features of the QDF are determined by the effect of the order parameter on the source type, since the rate of threshold quasiparticle production is proportional to the difference $\omega - 24$. As in the E -model, the order parameter is an inhomogeneous function and the state appears to be unstable against the so-called "threshold instability" predicted by Elesin (1974) and leading to a laminated inhomogeneous state. The difference is, however, that in the latter case the state is laminated into phases with different but finite values of the order parameter.

Experimental investigations of the threshold tunnel injection were performed by Dynes and Narayanamurti (1972) and by Gray and Willemsen (1978) who proved the existence of a two-gap inhomogeneous state.

The second part of this review presents the theory for the case of the twoband model. Both the bands should be taken into account here because an electromagnetic field with a frequency higher than the energy gap E_{g} gives rise to interband transitions. Besides interband transitions discussed in sections 1.1 and 1.3, the electromagnetic field leads to two additional effects.

First, nonequilibrium electrons in the conduction band and holes in the valence band are created due to absorption of the field. This shifts the Fermi level in metals and doped semiconductors. In pure semiconductors and insulators the electrons occupy (under the action of the field) the bottom of the conduction band.

Second, interband transitions may result in a dynamic effect-a change in the electron spectrum. In fact, as was shown by Galitskii et al. (1969), a gap $2\hat{i}$, proportional to the product of the interband transition dipole moment and the field amplitude, arises in the spectrum of electrons in the valence and conduction bands.

Electrons and holes form field-coupled pairs, which are similar in many respects to Cooper pairs and electron-hole pairs of an exciton insulator.

The appearance of a gap in the electron spectrum, and, hence singularities in the density of states, significantly affects the state of a superconductor.

Depending on the concentration of equilibrium electrons in the conduction band, the field may either suppress superconductivity (Elesin and Kopaev 1972) or increase the critical temperature (Elesin and Kopaev 1974). The net result depends on the relative position of the Fermi level and the gap 2λ created by the field.

The electromagnetic field may also lead to a change in the quasi-particle energy distribution. In particular, the function may become asymmetric or even inverse ($n > \frac{1}{2}$) (Elesin 1975).

In fact, if the field is sufficiently strong so that the gap 2% exceeds the Debye frequency (ω_{D}. the quasiparticles cannot recombine through the gap and are accumulated at energies larger than E_{F}. The inverse distribution may, in turn, create conditions for the appearance of a superconducting state with repulsion between the electrons (Kirzhnits and Kopaev 1973, Galitskii et al. 1973) and also for abnormal paramagnetism (Galitskii et al. 1973).

29.2 1.1. The dynamic and kinetic equations in Single-band approximation

The treatment of nonequilibrium processes in superconductors requires a system of equations which simultaneously describes both dynamic and kinetic properties. In the weak-coupling approximation this system is reduced to the order parameter equation and kinetic equations for the distribution functions of quasiparticles and phonons (Elesin 1977a).

29.2.1 1.1.1. The order parameter equation

In the spatially homogeneous case the order parameter equation is of the form (Eliashberg, 1970, 1971, Elesin et al. 1973)

$$\Delta = -g \int_0^{\omega_{\mathrm{D}}} \frac{\mathrm{d}\xi}{\varepsilon}(1-2n(\xi))\Delta, \quad n(\xi) = \int \frac{n_p \, \mathrm{d}\Omega}{4\pi}, \quad \varepsilon = \sqrt{\xi^2+\Delta^2} \tag{6}$$

where g is the electron phonon coupling constant $(g<0), n(\check{\zeta})$ is the QDF satisfying the kinetic equation (9), and ω_{D} is the Debye frequency; hereafter by Δ we mean the modulus of the order parameter.

The spatial variation of $\Delta(r)$ in a nonequilibrium superconductor is described by an equation similar to the Ginzburg-Landau equation:

$$\xi_0^2 \left(\frac{\partial}{\partial r} - 2\mathrm{i}e\mathscr{A}\right)^2 \Delta(r) = \frac{\partial U}{\partial \Delta} \tag{7}$$

$$U(\Delta) = \int_0^{\Delta} \Delta' \mathrm{d}\Delta' \left[\frac{1}{|g|} - \int_0^{\omega_{\mathrm{D}}} \mathrm{d}\xi \frac{1-2h(\xi,r)}{\left(\xi^2+(\Delta')^2\right)^{1/2}}\right] \tag{8}$$

which is obtained (Elesin 1977a) under the assumption that Δ is small and varies slowly within the coherence length $\check{\zeta}_0$: here $\check{\zeta}_0$ is a functional of the QDF of the normal metal $n_0(\xi)$:

$$\xi_0^2 = \frac{p_0^2}{3m}\int_0^\infty \frac{\mathrm{d}\xi}{\xi}\left(\frac{\mathrm{d}}{\mathrm{d}\xi}\frac{2n_0-1}{\xi}\right)$$

The description of unsteady processes requires, in general, a time equation for A. But in most cases the analysis can be restricted to equation (7), because the time needed for the establishment of a steady state in the system of Cooper pairs is much less than that for a system of quasiparticles.

The vector potential $\mathscr{A}$ includes also the potential of a electromagnetic field of a frequency ω. The inclusion of $\mathscr{A}$ into the dynamic equation (7) for Δ makes it possible to describe the direct influence of the field on the order parameter. This influence was studied in detail by Gorkov and Eliashberg (1969) [see also the review by Dmitriev and Khristenko (1978)], and it was shown that the effect is significant at low frequencies satisfying the condition ($\omega^2 < \omega_\mathrm{c}^2 = A\tau_\mathrm{ph}$ (τ_ph is the phonon energy relaxation time of quasiparticles). Let us note that since the values of τ_ph are large ($\tau_\mathrm{ph}\Delta \gg 1$) the above condition is met at frequencies much lower than $2A$.

At frequencies $\omega > 2\Delta > \omega_\mathrm{c}$ the main contribution is due to the second channel, where the electromagnetic field acts as a source of quasiparticles. Therefore, in further consideration we can neglect the contribution of the electromagnetic field in eq. (7) and take it into account only in the kinetic equation (9).

29.2.2 1.1.2. The kinetic equation for the quasiparticle distribution function

The kinetic equation for the QDF is of the form

$$\frac{\partial n(p,r,t)}{\partial t} + \frac{\partial n}{\partial r}\frac{\partial \varepsilon}{\partial p} - \frac{\partial \mathrm{n}}{\partial p}\frac{\partial \varepsilon}{\partial r} = I_\mathrm{imp} + I_\mathrm{ph} + I_\mathrm{ee} + Q, \tag{9}$$

where ε is the quasiparticle energy, Q is the quasiparticle source, and $I_\mathrm{ph}, I_\mathrm{ce}, I_\mathrm{imp}$ are the collision integrals of quasiparticles with phonons, quasiparticles and impurities, respectively; the last integral can be written in the form

$$I_\mathrm{imp} = -\frac{\xi}{\varepsilon\tau_\mathrm{imp}}\{n(p,r,t) - n(\xi,r,t)\}, \tag{10}$$

where τ_imp is the relaxation time at the impurity.
The equation for the QDF $h(\xi,r,t)$ integrated over angles is obtained from eq. (9) in an ordinary way, if we take into account that $\tau_\mathrm{imp}/\tau_\mathrm{ph}$ is small:

$$\frac{\partial}{\partial t}n(\xi,r,t) + \frac{|\xi|}{\varepsilon}L^2\left\{\frac{\partial}{\partial r_\mathrm{i}}\left[-\frac{\partial h}{\partial r_\mathrm{i}} + \frac{\partial\varepsilon}{\partial r_\mathrm{i}}\frac{\partial h}{\partial\varepsilon}\right] + \frac{\partial\varepsilon}{\partial r_\mathrm{i}}\frac{\partial}{\partial\varepsilon}\left[\frac{\partial n}{\partial r_\mathrm{i}} - \frac{\partial\varepsilon}{\partial r_\mathrm{i}}\frac{\partial n}{\partial\varepsilon}\right]\right\}$$
$$= I_\mathrm{ph} + I_\mathrm{ee} + Q. \tag{11}$$

The phonon collision integral (Eliashberg 1971) is

$$I_{\mathrm{ph}} = S^{+} - S^{-} - S^{\mathrm{R}} \tag{12}$$

$$S^{+} = \frac{1}{\tau_{\mathrm{ph}}\Delta_0^{k+2}} \int_{\xi}^{\omega_{\mathrm{D}}+\xi} \mathrm{d}\xi' \left(\varepsilon' - \varepsilon\right)^{k+1} v1 - \eta\frac{\Delta^2}{\varepsilon\varepsilon'}\Big)$$
$$\times \left[(1-n)n'\left(1+N_{n'-\varepsilon}\right) - h\left(1-h'\right)N_{\varepsilon'-\varepsilon}\right] \tag{13}$$

$$S^{-} = \frac{1}{\tau_{\mathrm{ph}}\Delta_0^{k+2}} \int_{0}^{\zeta} \mathrm{d}\xi' \left(\varepsilon' - \varepsilon\right)^{k+1} \left(1 - \eta\frac{\Delta^2}{\varepsilon\varepsilon'}\right)$$
$$\times \left[n\left(1-n'\right)\left(1+N_{\varepsilon-\varepsilon'}\right) - (1-n)n'N_{\varepsilon-k'}\right] \tag{14}$$

$$S^{\mathrm{R}} = \frac{1}{\tau_{\mathrm{ph}}\Delta_0^{k+2}} \int_{0}^{\omega^{-\xi}} \mathrm{d}\xi'' \left(\varepsilon + \varepsilon'\right)^{k+1} \left(1 + \eta\Delta^2\right)$$
$$\times \left[nn'\left(1+N_{\varepsilon+\varepsilon'}\right) - (1-n)\left(1-k'\right)N_{\varepsilon+\varepsilon'}\right] \tag{15}$$

$$\frac{1}{\tau_{\mathrm{ph}}} = \frac{\eta|g|\Delta_0}{2}\left(\frac{\Delta_0}{\omega_{\mathrm{D}}}\right)^{k+1}, \quad n\left(\xi'\right) \equiv n', \quad \varepsilon\left(\xi'\right) \equiv \varepsilon'. \tag{16}$$

Here N_r is the phonon distribution function, k is a factor describing the dependence of the matrix element of the electron phomon interaction on q $\left(M_{\mathrm{q}}^2 \sim q^{\mathrm{k}}\right)$, η is a parameter taking the values $+1$ for a superconductor (Eliashberg 1971, 1970) and -1 for an exciton and photon insulator (Elesin

1970a). The first two terms in eq. (12) describe the energy relaxation of quasiparticles (S^{+}, "arrival"; S^{-}, "leaving"), the third term S^{R} describes quasiparticle recombination with the emission (absorption) of phonons. The coherence factors ($1 \pm \eta\Delta^2/\varepsilon\varepsilon'$) appear due to coherence of the interaction between quasiparticles and phonons and play very important roles in the phenomena discussed below.

In the one-band model $n(\zeta)$ is an even function in the case of electromagnetic pumping and symmetric tunnel injection. The collision integral (12) has been written with due account for this fact. In the general case the coherence factors ($1 \mp \eta\left(\Delta^2/\varepsilon\varepsilon'\right)$) should be replaced by

$$\left(1 \pm \eta\frac{\xi\xi'}{\varepsilon\varepsilon'} \mp \eta\frac{A^2}{\varepsilon\varepsilon'}\right).$$

In the two-band model of a superconductor in a strong electromagnetic field of frequency $\omega > E_{\mathrm{g}}$ the kinetic equation (12) is modified as follows.

The quasiparticle energy ε is replaced by $\tilde{\varepsilon} = \sqrt{\varepsilon^2 + \lambda^2}$, and a term is introduced which describes recombination of electrons and holes through the forbidden band (Elesin 1970a).

The quasiparticle collision integral for superconductors I_{ee} was found by Eliashberg (1970) and can be represented in the form

$$
\begin{aligned}
I_{\mathrm{ee}} &= S_{\mathrm{ee}}^{+} - S_{\mathrm{ee}}^{-} \qquad (17)\\
S_{\mathrm{ee}}^{+} =& \frac{1}{\tau_{\mathrm{ec}}\Delta_0^2}\iiint_0^1 \mathrm{d}\xi_1\ \mathrm{d}\xi_2\ \mathrm{d}\xi_3\\
&\times \{M_1(1-n)n_1n_2n_3\delta\left(\varepsilon-\varepsilon_1-\varepsilon_2-\varepsilon_3\right)\\
&\quad + 3M_2(1-n)n_1n_2\left(1-n_3\right)\delta\left(\varepsilon-\varepsilon_1-\varepsilon_2+\varepsilon_3\right)\\
&\quad +3M_3(1-n)n_1\left(1-n_2\right)\left(1-n_3\right)\delta\left(\varepsilon-\varepsilon_1+\varepsilon_2+\varepsilon_3\right)\},\\
S_{\mathrm{ee}}^{-} =& \frac{1}{\tau_{\mathrm{ec}}\Delta_0^2}\iiint_0^\infty \mathrm{d}\xi_1\ \mathrm{d}\xi_2\ \mathrm{d}\xi_3\\
&\times \{M_1 n\left(1-n_1\right)\left(1-n_2\right)\left(1-n_3\right)\delta\left(\varepsilon-\varepsilon_1-\varepsilon_2-\varepsilon_3\right)\\
&\quad + 3M_2 n\left(1-n_1\right)\left(1-n_2\right)n_3\delta\left(\varepsilon-\varepsilon_1-\varepsilon_2+\varepsilon_3\right)\\
&\quad +3M_3 n\left(1-n_1\right)n_2n_3\delta\left(\varepsilon-\varepsilon_1+\varepsilon_2+\varepsilon_3\right)\},\\
M_1 =& \left(1-\frac{\Delta^4}{\varepsilon\varepsilon_1\varepsilon_2\varepsilon_3}\right)-\frac{b\Delta^2}{3a\varepsilon}\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}+\frac{1}{\varepsilon_3}-\frac{\varepsilon^2}{\varepsilon_1\varepsilon_2\varepsilon_3}\right),\\
M_2 =& M_1\left(-\varepsilon_3\right). \quad M_3 = M_1\left(-\varepsilon_1,-\varepsilon_2\right), \quad \frac{1}{\tau_{\mathrm{ec}}} = \frac{a\Delta_0^2}{E_F},
\end{aligned}
$$

where a and b are dimensionless quantities of the order of unity dependent on the electron-electron interaction matrix element; M_i are the coherence factors, which describe the coherence of quasiparticle interaction.

It is useful to derive the law of conservation of the number of quasiparticles $\bar{n}$; this law can easily be found by integrating eq. (11) with respect to ξ :

$$
\begin{aligned}
&\frac{\partial\bar{n}}{\partial t} + L^2\frac{\partial}{\partial r_{\mathrm{i}}}\int_\varepsilon \frac{\xi}{\varepsilon}\left(\frac{\partial n}{\partial r_{\mathrm{i}}}-\frac{\partial\varepsilon}{\partial r_{\mathrm{i}}}\frac{\partial n}{\partial\varepsilon}\right)\frac{\mathrm{d}\xi}{\Delta_0}\\
&\quad + \frac{1}{\tau_{\mathrm{ph}}}\int_0^{\omega_{\mathrm{D}}}\frac{\mathrm{d}\xi}{\Delta_0^{k+3}}\int_0^{\omega_{\mathrm{D}}-\xi}\mathrm{d}\xi'\left(\varepsilon+\varepsilon'\right)^{k+1}\left(1+n\frac{\Delta^2}{\varepsilon\varepsilon'}\right)\varphi_{\mathrm{ph}}\left(\xi,\xi'\right)\\
&\quad - \int_0^\infty I_{\mathrm{ee}}\frac{\mathrm{d}\xi}{\Delta_0} = \bar{Q} \qquad (18)\\
&\varphi_{\mathrm{ph}}\left(\xi,\xi'\right) = n(\xi)n\left(\xi'\right) - N_{\varepsilon+\varepsilon'}\left(1-n(\xi)-n\left(\xi'\right)\right)\\
&\bar{n} = \frac{1}{\Delta_0}\int_0^\infty \mathrm{d}\xi n(\xi,r,t), \quad \bar{Q} = \frac{1}{\Delta_0}\int_0^\infty \mathrm{d}\xi Q \qquad (19)
\end{aligned}
$$

29.2.3 1.1.3. Sources of nonequilibrium quasiparticles

The nonequilibrium states considered in this review, are formed due to the production of quasiparticles by external sources (electromagnetic and ultrasonic fields with $\omega > 2\Delta$, tunnel injection with $V > 2\Delta$, ionizing particles, etc.). Although the results obtained below are valid, to a certain extent, for all those sources, we shall consider in detail only two of them: electromagnetic pumping Q_ω and symmetric tunnel injection Q_{V}. According to Eliashberg (1970a), Q_ω can be represented in the form

$$Q_\omega = Q_\omega^{(1)} + Q_\omega^{(2)} + Q_\omega^{(3)}, \tag{20}$$

$$Q_\omega^{(1)} = \alpha_\omega \rho^\omega(\omega - \varepsilon)\left(1 - n_\varepsilon - n_{\omega-\varepsilon}\right),$$

$$Q_\omega^{(2)} = Q_{-\omega}^{(3)} = \alpha_\omega \rho^\omega(\varepsilon - \omega)\left(n_{\varepsilon-\omega} - n_\varepsilon\right),$$

$$\rho^{(\omega)}(\omega - \varepsilon) = \frac{(\omega - \varepsilon) - \left(\Delta^2/\varepsilon\right)^{(1+k\omega)/2}}{\sqrt{(\omega - \varepsilon)^2 - \Delta^2}} - 0(\omega - -\ddot{\Delta}), \quad 0(x) = \begin{cases} 1, & x \geqslant 0 \\ 0, & x < 0 \end{cases} \tag{21}$$

where

$$\alpha_\omega = \frac{E^2 e^2 L^2}{2m\omega^2 \pi \left(1 + \omega^2 \tau_{\text{imp}}^2\right) \tau_{\text{ph}}} \tag{22}$$

$$L^2 = \frac{v_{\text{F}}^2 \tau_{\text{imp}} \tau_{\text{ph}}}{3} \tag{23}$$

$k_\omega = +1, \alpha_{(1)}$ is the electromagnetic injection parameter, L is the quasiparticle diffusion length, and v_{F} is the velocity of electrons at the Fermi surface.

Note that expression (22) corresponds to absorption at free electrons. In certain superconductors other mechanisms, for instance interband absorption, may give the main contribution. Since the film thickness d usually exceeds the reciprocal absorption coefficient, its explicit form is insignificant and it can be taken as $1/d$.

The source which really creates quasiparticles is described by the quantity $Q_\omega^{(1)}$, while $Q_{(2,3)}^{(2,3)}$ describes the energy redistribution of quasiparticles present in the system, since $\int_0^x \mathrm{d}\xi \left(Q^{(2)} + Q^{(3)}\right) = 0$.

In the case of tunnel injection through the symmetric SIS junction eq. (19) may be used for Q_{V}, in which, however, one should put $\omega = V, k_{\text{V}} = -1, \alpha_\omega = \alpha_{\text{V}}$ (Kirichenko et al. 1976):

$$\alpha_v = \frac{2\sigma_N \tau_{\text{ph}}}{de^2 v(0)}, \tag{24}$$

where σ_{N} is the junction conductivity in the normal state, d is the film thickness, and $r(0)$ is the density of electron states. An essential difference between Q_{V} and Q_c is that the coherence factor $\lambda^2\varepsilon$ does not appear in Q_{V}.

As was already noted in the Introduction, the sources can be divided, irrespective of their nature, into two classes: wide sources and narrow sources. This classification appeared to be quite convenient for understanding the physics of the phenomena and analytically solving the equations in the limiting cases of wide and narrow sources. The fact is that in these cases one can separate the processes through which the order parameter affects the QDF formation.

Indeed, in the case of a wide source Δ disappears from the expression for the source $Q^{(1)}$, that is, $Q^{(1)} = \alpha_\omega \theta(\omega - \varepsilon)$, and it affects only the relaxation processes. Since the "density of source action" is small (Elesin 1974), the kinetic equation may be assumed homogeneous (one may take $Q = 0$), so the relationship between n and Q can be found from the condition (18) that the number of quasiparticles is conserved.

In the case of a narrow source the order parameter has a decisive effect on the source. Note that $Q^{(2)}$ and $Q^{(3)}$ do not play an essential role in the phenomena considered if (1) $> 2A$. One should bear in mind that the concept of a narrow source has a meaning only if the injection intensity is sufficiently low, so that the change in the parameter $\delta A^{\text{is small, } \delta A \ll A_0}$. On the contrary, in the case of intensive injection where $\Delta_0 - \delta\Delta \ll \Delta_0$, a source with frequency $\omega - 2A_0 \ll 1_0$ becomes wide because the energies of quasiparticles produced are $\omega \sim 2A_0 \ll 1$. In the case of tunnel injection the equation for an external circuit should be taken into account:

$$\mathscr{E} = V + JR, \quad \bar{J} = \int \mathrm{d}^2 r J(V, d)$$

$$J(V, \Delta) = J_1 + J_2$$
$$J_1 = \alpha_V \int_{\Delta}^{V-\Delta} \mathrm{d}\varepsilon \rho(\varepsilon) Q_V^{(1)}, \quad J_2 = 2\alpha_V \int_{\Delta}^{\infty} \mathrm{d}\varepsilon \rho(\varepsilon) Q_V^{(3)} \tag{25}$$

where $\mathscr{E}$ is the emf, R is the external resistance, J is the current flowing through the SiS junction, and $\rho(\varepsilon) = \varepsilon/\sqrt{\varepsilon^2 - \Delta^2}$.

29.2.4 1.1.4. The kinetic equation for phonons

The phonon distribution function $N(\varepsilon)$ may differ from the equilibrium one

$$N_T(\varepsilon) = \left[\exp \frac{\varepsilon}{T} - 1\right]^{-1} \tag{26}$$

and it can be found from the following kinetic equation (Chang and Scalapino 1977):

$$\frac{\partial N}{\partial t} + \frac{N - N_T}{\tau_{\mathrm{es}}} = \frac{1}{\tau_{\mathrm{B}}} \int_{\Delta}^{\varepsilon-\Delta} \frac{\mathrm{d}\varepsilon'}{\Delta_0} \rho(\varepsilon') \rho(\varepsilon - \varepsilon')$$
$$\times \left[n_{\varepsilon'} n_{\varepsilon-\varepsilon'} - N(1 - n_{\varepsilon'} - n_{\varepsilon-\varepsilon'})\right] \left(1 + \frac{\Delta^2}{\varepsilon'(\varepsilon - \varepsilon')}\right) + \frac{2}{\tau_\beta} \int_{\Delta}^{\infty} \frac{\mathrm{d}\varepsilon'}{\Delta_0} \rho(\varepsilon) \rho(\varepsilon + \varepsilon')$$
$$\times \left[n_{\varepsilon+\varepsilon'}(1 - n_{\varepsilon'}) - N(n_{\varepsilon'} - n_{\varepsilon+\varepsilon'})\right] \left(1 - \frac{\Delta^2}{\varepsilon'(\varepsilon + \varepsilon')}\right), \tag{27}$$

$$\tau_{\mathrm{es}} = \frac{4d}{s}, \quad \frac{1}{\tau_\beta} = \frac{\pi |g| \Delta_0 \omega_{\mathrm{D}}}{2E_{\mathrm{F}}}, \tag{28}$$

where s is the acoustic velocity. The second term in eq. (27) takes into account the fact that phonons can penetrate from the film into the substrate; the third and fourth terms account for the production of phonons under recombination and scattering of quasiparticles, respectively.

29.2.5 1.1.5. The kinetics of quasiparticles in superconductors

The energy distribution of excess quasiparticles is formed due to the source and relaxation processes. The effect of the source $Q^{(1)}$ lies in the break of a Cooper pair and production of two quasiparticles. A wide source creates quasiparticles with high energies $\varepsilon \ll \Delta$. The quasiparticle created gives its energy to the electron gas the lattice.

In the first case the quasiparticle energy is consumed to increase the energy of quasiparticles already produced and to create new quasiparticles via impact ionization through the gap 24 . This is the way multiple production of quasi-
particles occurs, the process being characterized by the multiplication coefficient r_{cc}.

In the second case the energy is consumed for the emission of nonequilibrium phonons. Eventually, the quasiparticles reach the bottom of the band and recombine with emission of a phonon ($\Delta \ll \omega_{\mathrm{D}}$), forming a Cooper pair again. Nonequilibrium phonons may either leave the film within a time τ_{es} or undergo reabsorption within a time τ_β, producing a pair of quasiparticles (or increasing their energy) and thereby leading to multiplication with the coefficient r_{ph}. The relative probability for phonons to leave and to be reabsorbed is characterized by the parameter $\gamma = \tau_{\mathrm{es}} / \tau_\beta$, which is limited to the right by phonon-phonon interaction.

The kinetics of quasiparticles in superconductors is similar, in certain respects, to the kinetics of electrons in semiconductors in which electron-hole pairs are produced (Ryvkin 1963). Indeed, the parameter 24 acts here as the energy gap and quasiparticles as electrons and holes. However, nonequilibrium processes in superconductors have essential distinctive features.
(1) The gap 24 in superconductors is very sensitive to the concentration and energy distribution of quasiparticles and may even become zero.
(2) Since the ratio $2\Delta/\omega_{\mathrm{D}}$ is small, scattering and recombination are singlephonon processes and occur at approximately the same rate (Elesin 1974). This means that recombination and scattering times are approximately equal, so the QDF must differ from the quasiequilibrium functions (occurring if the recombination time is large in comparison with the scattering time) and, in particular, from the functions in the μ^*-model (Owen and Scalapino 1972) (except for the case of low T and β).
(3) The order parameter significantly affects the relaxation and production of quasiparticles.

29.3 1.2. Superconductors with a wide quasiparticle source. Optical pumping and tunnel injection at high voltages in Single-band approximation

29.3.1 1.2.1. The distribution function of excess quasiparticles

Consider first a spatially homogeneous case.
At present, there are many papers in which the QDF $n(\xi)$ has been determined analytically (Vardanyan and Ivlev 1973, Elesin 1974, 1976, 1977a,b, Eckern et al. 1979) and numerically (Elesin et al. 1977a,b, Imai 1980) for a wide range of the parameters. The main properties and specific features of $n(\xi)$ are of general character and have a clear physical meaning.

Let us first discuss the properties of $n(\xi)$, which satisfy the kinetic equation with phonon relaxation. The contribution of electron-electron collisions is usually small (Eliashberg 1970, Vardanyan and Ivlev 1973) and will be considered below.

As was established by Elesin $(1974, 1976), n(\xi)$ is a monotonously decaying function localized mainly in the energy range $\xi \sim \Delta_0$, which does not exceed $\frac{1}{2}$. Moreover, the QDF is not only less than $\frac{1}{2}$, it satisfies the more strict "overheating" condition (4) (Elesin et al. 1979).

Condition (4) is very important and permits the classification of QDF. In particular, the "overcooled" QDF satisfying a condition inverse to eq. (4) lead to essentially different consequences (the so-called NE-model) (Eckern et al. 1979, Elesin 1976, 1977a). The thermal function serves as if it were a limiting boundary separating the functions $n(\zeta)$ in the E- and NE-models (Elesin 1977b).

The "overheating" character of the QDF is closely related to the coherence factors ($1 \pm \eta\left(\Delta^2/\varepsilon\varepsilon'\right)$) in the integral of collisions between quasiparticles and phonons (12). According to Elesin (1976), a coherence correction n_1 can be separated in the QDF, so $n(\xi)$ is represented in the form

$$n(\xi) = n_0(\varepsilon) + n_1(\xi) \tag{29}$$

where $n_0(\varepsilon)$ statisfies eq. (11) with the coherence factors equal to zero ($\eta = 0$), and the function n_1 is proportional to the coherence factors $\eta\Delta^2/\varepsilon\varepsilon'$.

To elucidate the specific features of $n(\xi)$, it is convenient to consider the simplest model, assuming in eq. (12) $k = -1, T = 0, \gamma = 0$. This model implies an exact solution (Elesin 1976, 1977a) for $\Delta/\Delta_0 \ll 1$:

$$n_0(\varepsilon) \equiv n_0(\xi) = \frac{1}{2}\left(1 - \frac{\xi}{\sqrt{\xi^2 + 4\Delta_0^2\bar{n}^2}}\right), \tag{30}$$

$$n_1 = -\frac{\Delta^2}{\varepsilon}\left(\frac{1}{4\bar{n}\Delta_0}\ln\frac{2\bar{n}\Delta_0}{\Delta}\right), \quad \eta = 1. \tag{31}$$

Substitution of $n_0(\xi)$ into the equation (6) for the gap yields a critical concentration of quasiparticles $\bar{n}_c$, at which the gap vanishes ($\Lambda = 0$).

$$\bar{n}_c = \frac{1}{2}. \tag{32}$$

Using then the normalization conditions (18), we find the critical power

$$\beta_c = n_c^2 = \frac{1}{4}$$

It can be seen from eqs. (30) and (31) that $n(\xi) < \frac{1}{2}$ and satisfies the overheating condition (4) for $\eta = 1$ due to the coherence correction.

A decrease in $n(\xi)$ near small $\xi \sim \Delta$ is caused by an increase in the quasiparticle recombination rate proportional to $(1 + \Delta^2/\varepsilon\varepsilon')$, which is especially pronounced as $\zeta \to 0$. Note that if η were equal to -1 , the distribution would be "overcooled" (Elesin 1976, Eckern et al. 1979) (the NE-model).

In the general case of finite temperatures, the actual matrix elements are $M_q^2 \sim q, k = 1$, and if the effect of phonon reabsorption is taken into account, the specific features of $n(\xi)$ remain unchanged.

The main properties of $n(\xi)$ were formulated by Elesin (1976, 1977*b*). Let us first recall those properties of the function $n_0(\varepsilon)$ which are close to those of $n_T(\varepsilon)$.
(1) Expansion of the integrals of $n_0(\varepsilon)$ includes only even powers of Δ^2 [as in the case of $n_T(\varepsilon)$].
(2) The value of $n_0(\varepsilon)$ is less than $\frac{1}{2}, n_0(0) = \frac{1}{2}$ (as for n_T).
(3) The asymptotic behaviour of $n_0(\varepsilon)$ for large ε has a power form (Vardanyan and Ivlev 1973, Elesin 1976),

$$n_0(\varepsilon) \sim \varepsilon^{-(k+3)}, \quad \varepsilon \to \infty \tag{33}$$

unlike the exponential decay for $n_T = e^{-\varepsilon/T}$.
(4) The functions $n_0(\varepsilon)$ for $\Delta = 0$ pass into "critical" functions $n_0(\breve{\zeta})$ satisfying the kinetic equation with $\Lambda = 0$. These functions are characterized by a single parameter ${a_1}^{1/2}$,

$$n_0(\xi) \equiv n_0\left(\xi/a_1^{1/2}\right), \quad a_i = \int_0^\infty \xi^i n_0(\xi)\mathrm{d}\xi \tag{34}$$

which plays the role of effective temperature. Substitution of the function $n_0(\xi)$ into the gap equation makes it possible to find a critical value a_{1c} at which the gap becomes zero:

$$\frac{1}{|g|} = \int_0^{\omega_D/{a_{1c}}^{1/2}} \frac{1 - n_0(x)}{x}\,\mathrm{d}x.$$

In particular, for $\gamma = 0, T = 0$ we have (Elesin et al. 1977)

$${a_{lc}}^{1/2} = 0,58\Delta_0 \simeq T_c, \quad a_{0c} = 0.4\Delta_0 \tag{35}$$

Once $n_0(x)$ is known, one can find the critical concentration of quasiparticles

$$\bar{n}_{\mathrm{c}} = a_0/\Delta_0 = 0.4 \tag{36}$$

or in the dimensionless form

$$N_{\mathrm{c}} = v(0)A_0\bar{n}_{\mathrm{c}}. \tag{37}$$

The quantity $\bar{n}_c$ practically coincides with the concentration of thermal quasiparticles $\bar{n}_f = T_{\mathrm{c}} \ln 2 = 0.4$, indicating once more that the properties of n_0 and n_T are close for $\xi \leqslant \Delta_0$.

Substituting $n_0(\xi)$ into the normalization condition (18), we find the critical power $\beta_{\mathrm{c}} = \bar{Q}$, which corresponds to the vanishing of the gap (see section 1.2.4).

The coherence correction n_1 for small Λ/Δ_0 can be found in the general form and expressed as

$$n_1 = \ddot{-}\eta\frac{\Lambda^2}{\pi\varepsilon}\varphi, \tag{38}$$

$$\varphi = \frac{\pi}{a_2}\int_0^\infty \xi \mathrm{d}\xi \varphi_{\mathrm{ph}}(\xi, 0) \tag{39}$$

where $\varphi_{\mathrm{ph}}(\xi, 0)$ is the function (19), into which the "critical" distribution functions of quasiparticles $n_0(\xi)$ and phonons $N_0(\xi)$ are substituted.

Relations (38), (39) and (19) imply the following properties of n_1 :
(1) The sign of the "coherence" correction n_1 is determined by the sign of $\eta = (\pm 1)$ in the coherence factors. In superconductors $\eta = 1, n_1 < 0$; in an exciton insulator $\eta = -1, n_1 > 0$.
(2) The correction n_1 is proportional to $\varphi(\xi, \xi')$ (see eq. 19), which is a measure of the "nonequilibrium of the system". The function $\varphi(\xi, \xi')$ represents the recombination collision integral, which vanishes in equilibrium. The maximum of φ is reached at $T = 0, \gamma = 0$ and is equal to (Elesin et al. 1977a)

$$\varphi_{\mathrm{m}} = \frac{\pi a_1}{2a_2} = \frac{0.71}{\Delta_0} \tag{40}$$

Obviously, as T and the film thickness (i.e. γ) increase, the number of nonequilibrium phonons N, grows and φ correspondingly decreases, tending to zero but always remaining a positive quantity. From the physical viewpoint this is related to the fact that $\varphi(\xi, \xi')$ is proportional to the difference between the quasiparticle recombination rate (the first term) and the production rate due to reabsorption of thermal and nonequilibrium phonons. For $\beta > 0$ the number of recombining particles always exceeds the number of particles produced due to reabsorption of non-equilibrium phonons, so $\varphi > 0$.
(3) Subtracting from $n(0)$ (eq. 29) the function $n_7(0)$ and taking into account that $n_0(0) = n_T(0)^*$, we obtain the "overheating" condition (4)

$$n(\xi = 0) - n_T(\xi = 0) = n_1(\xi = 0) < 0 \tag{41}$$

It can be seen from eq. (41) that the difference between the QDF $n(\xi = 0)$ and the equilibrium one $n_T(\xi = 0)$ is caused by the coherence correction n_1. It is obvious therefore that as γ and T grow, when $\varphi \to 0$ and $n_1 \to 0, n(\xi)$ tends to $n_0(\varepsilon)$.

The above properties and the form of $n(\xi')$ were confirmed by computeraided numerical solution of the complete system of equations for A, n and N (eqs. 6, 11, 12, 36) (Elesin et al. 1977a,b, 1979, Imai 1980).

Elesin et al. (1979) calculated the functions $n(\xi)$ for a wide range of the parameters β, Δ, T and γ. It was assumed in solving the equations that the source acts in the energy range

$9A_0 < \omega < 10A_0$ and it was shown that its shape does not influence the results obtained (if $\omega \gg \Delta_0$).

Fig. 1. Dependence of the quasiparticle distribution function on energy, for $T = 0$ and $\gamma = 0.15$ and the following values of the order parameter $\Delta / \Delta_{0 $ (and correspondingly β): (a) 0.95 ; (b) 0.8 ; (c) 0.6 ; (d) 0.4 ; (e) 0.5 ; (f) 0.01 . (Elesin et al. 1979.)}

Figure 1 shows a typical form of the QDF $n\left(\xi \Lambda_0\right)$ for various β, with $T = 0$, $\gamma = 0.15$.

In order to illustrate that the "overheating" condition (4) is really satisfied for the distribution function at $\xi = 0$, fig. 2 plots the values of $n(\xi = 0)$ as a function of Δ and, for comparison, the values of the thermal QDF $n_{\mathrm{T}}(\xi = 0)$. As can be seen from fig. 2, $n(\xi = 0)$ is less than $n_{\mathrm{T}}(\xi = 0)$ for any γ and T, in accordance with condition (4).

Numerical solution of the system $(6, 11, 12, 26)$ has been performed by Imai (1980), confirming the results of Elesin et al. (1979). Imai (1980) performed his calculations, using the general formula for a quasiparticle source Q. for frequencies (1)$\Delta_0 = 2, 1$ and 8, 20, reabsorption parameters $\gamma = 0, 1, 10$, and different temperatures. The QDF for (1) $\Delta_0 = 8, 20$ (wide source) differ little from each other (see fig. 3) and from those obtained by Elesin et al. (1979). This con-

firms the conclusion, substantiated by Elesin (1974), that the shape of a wide source is insignificant. The reason is that $n(\xi)$ is formed mainly for $\xi \leqslant \Delta_0$ due to rapid relaxation of quasiparticles with $\xi \gg \Delta_0$.

Figure 3 clearly shows the effect of the reabsorption coefficient γ on $n(\zeta)$. The function n with $\gamma = 1$ decays faster with increasing ξ than n with $\gamma = 0$.

It would be of interest to approximate the monotonous function n by the quasiequilibrium QDF

[12]*This equality can easily be demonstrated for the simple case $N_\varepsilon = 0$. Then, according to Elesin (1976), $n_0(s) = \frac{1}{2}\left(1 - 2\Delta a_1/a_2\right)$, the ratio $2a_1/a_2$ is equal to $1/2T_{\mathrm{c}}$ (Elesin et al. 1977a), so we arrive at the equality in question.

Fig. 2. Dependence of $2n(\xi = 0)$ on the order parameter: (a) according to the BCS theory; $(b - d)$ nonequilibrium distribution function with (b) $\gamma=0, T / \Delta_{0 =0$, (c) $\gamma = 0.15, T/\Delta_0 = 0$, (d) $\gamma = 0.15$, $T/\Delta_0 = 0.44$. (Elesin et al. 1979.)}

$$n_{\mathrm{F}}(\xi) = \left[\exp\left(\frac{\varepsilon - \mu^*}{T^*}\right) + 1\right]^{-1} \tag{42}$$

considering μ^* and T^* as fitting parameters*. It should be noted first of all that for any values of μ^* and T^* the functions n and n_{F} cannot be fitted to each other exactly in the whole energy range (the discrepancy is especially large for $\gamma \to 0, T \to 0$).

If approximation is performed with respect to the minimum of the deviation $(n - n_{\mathrm{F}})$ within the entire energy range, μ^* and T^* can be found to $5 - 10\%$ for

Fig. 3. Dependence of the QDF $n(\varepsilon)$ on $\varepsilon / \Delta_{0 $ for $T/T_{\mathrm{c}} = 0.5, \Delta/\Delta_0 = 0.5$: (a) $\omega/\Delta_0 = 8.20, \gamma = 0$; (b) $\omega/\Delta_0 = 8, \gamma = 1$. (Imai 1980.)}

[12]*The fact that this is only an approximation is clearly seen from the following example. At $T = 0, \gamma = 0, n(\xi)$ is determined from eq. (12) when neither of the terms S^+(or S), S^{R} vanishes separately (as it must be in equilibrium).

any set of the parameters β, γ and T (Elesin et al. 1979). The dependences of μ^* and T^* on β for various T and γ are shown in fig. 4.

As can be seen from the figure, both parameters are different from zero, the effective chemical potential always being negative, $\mu^* < 0$, and the effective temperature T^* can exceed T_c. Both parameters are functions of $T_{,\gamma}$ and β, and in the interval $\beta_c < \beta < \beta_H$ they are two-valued functions of β. Negative values of μ^* mean, obviously, that the value of the function n is less than $\frac{1}{2}$. For small A the chemical potential can be related to the coherence correction:

$$\mu^*/4T_c = n_1(\xi = 0), \tag{43}$$

which makes quite clear why μ^* has a negative value in superconductors.
It is interesting to note that the "effective temperature" T^* exceeds T_c (the ratio T, T, may be as high as 1.55), but at the same time the order parameter is nonzero ($1.A_0 \geqslant 0.3$). This means that T^* cannot be considered as a temperature and also clearly indicates (together with $\mu^* < 0$) that quasiparticles are "overheated".

The conclusions that μ^* is negative and $T^* > T_c$ were confirmed by Imai (1980) who obtained (by approximating n for two points $\varepsilon = A$ and $\varepsilon = 3A$ with

Fig. 4. Dependence of the parameters $T^{*} $ and μ^* on the source power β. The values of the parameters γ and T : (a) $\gamma = 0, T = 0$; (b) $\gamma = 0.02, T = 0$; (c) $\gamma = 0.02, T/\Lambda_0 = 0.44$; (d) $\gamma = 0.15, T = 0$; (e) $\gamma = 0, T/\Delta_0 = 0.44$; (f) $\gamma = 0.15, T/\Delta_0 = 0.44$. (Elesin et al. 1979.)}

$\omega/\Delta_0 = 20, \gamma = 0$) the following values:

$$\mu^* = -0.91\Delta, T^*/T_c = 1.4.^*$$

Let us compare the QDF in the E-model with the QDF in the μ^*- and T^*-models. [The μ^* - and T^*-models have been analysed in detail by Tremblay (1981).] It should be stressed that although in the E-model the quasiparticle distribution function can be parametrized in an approximate manner, using μ^* and T^*, the fact that both these parameters must change simultaneously means that the predictions of this theory differ significantly from those of the μ^* - and T^*-models. This has to be clear from the above considerations and the discussion which follows.

Equation (43) shows that if the coherence correction is neglected, the chemical potential μ^* can be put equal to zero, to give the function $n_0(\varepsilon)$ close to that in the T^*-model. Hence, the T^*-model does not allow for the coherence of interaction. Obviously, the T^*-model becomes a good approximation for $;\gg 1, T \to T_c$. A similar conclusion was made by Chang et al. (1979). As far as the μ^*-model is concerned, the function (1) possesses the properties $n_{\mu^*}(0) > n_1(0), n_{\mu^*}(0) > \frac{1}{2}, \mu^* > (T \to 0)$, which differ significantly from those for $n(\xi)$ in the E-model, the discrepancy being related to the fact that the QDF (1) satisfies the kinetic equation under the condition (Vardanyan and Ivlev 1973, Elesin 1976)

$$\bar{n} < \tilde{n} = \frac{1}{28}\left(T/\Delta_n\right)^{7/2} \ll 1. \tag{44}$$

This means that the μ^*-model is inapplicable for $T = 0$ [see the paper by Elesin et al. (1976b) where the solution has been found for $T = 0$ and $\ddot{n} \to 0$] and also for $\bar{n} > \tilde{n}$. And it is this region, into which the non-trivial predictions of the μ^*-model fall.

As was noted by Elesin (1976, 1977b, 1979a) and Eckern et al. (1979), a model alternative to the E-model is, in principle, possible, if one takes $\eta = -1$ in the coherence factors (the NE-model, where N stands for negative). In this model the function satisfies the condition

$$\frac{1}{2} > n_{\mathrm{NE}}(\xi = 0) > n_{\mathrm{T}}(\xi = 0) \tag{45}$$

that is, the distribution is "overcooled" (as in the μ^*-model), but n_{NF} does not exceed $\frac{1}{2}$. The NE-model leads to predictions significantly different from those of the μ^*-model, though in the former model the chemical potential is positive by virtue of eq. (43).

Let us consider the effect of electron-electron collisions. It is usually accepted (Eliashberg 1970, Vardanyan and Ivlev 1973) that electron electron collisions have only a small effect on the form of the QDF for $\xi < A_0$ because

the ratio C of the constants of electron-electron and electron-phonon interactions,

$$C = \frac{\tau_{\mathrm{ph}}}{\tau_{\mathrm{ee}}} = \frac{a\omega_{\mathrm{D}}^2}{\pi|g|E_{\mathrm{F}}\Delta_0}, \tag{46}$$

is small.

However, corrections to $n(\xi)$ due to electron-electron (e-e) interaction may, in general, be significant even for small C since the nonequilibrium state is very sensitive to the form of $n(\xi)$. Moreover, in certain superconductors $C > 1$, for instance, in aluminium [see the Tables in the articles by Aronov and Spivak (1978) and Elesin and Kopaev (1981)].

The effect of e e interaction on the form of $n(\xi)$ has been studied by Aronov and Spivak (1979) and Elesin (1981).

[12]*"Overheating" of quasiparticles has been observed experimentally by Willemsen and Gray (1978) (see section 1.2.11).

Elesin (1981) studied the behaviour of $n(\check{\zeta})$ near the phase transition region ($A/A_0 \ll 1$) with due account for e-e collisions, and showed that this behaviour remains qualitatively unchanged, at least for not too large G.

In fact, it follows from eq. (17) that the function $n(\xi)$ has the properties

$$n_0(\xi) < \frac{1}{2}, \quad n_0(0) = \frac{1}{2}, \quad (\text{ any } C)$$

and

$$\xi \to \infty, \quad n_0(\xi) \sim \xi^{-4}.$$

If the electron-phonon and electron-electron interactions are taken into account, the coherence correction can be represented in the form

$$n_1 = -\frac{\Delta^2 a_1}{2\varepsilon\pi\tilde{a}_2}\left(\varphi_{\text{ph}} - G\varphi_{\text{ee}}\right) \tag{47}$$

where

$$\varphi_{\text{ee}} = \iint_0 \mathrm{d}x_1\ \mathrm{d}x_2 \frac{x_1^2 + x_2^2 + x_1 x_2}{x_1\left(x_1 + x_2\right) x_2} \varphi_{\text{ee}}\left(x_1, x_2\right) \tag{48}$$

$$\begin{aligned}\varphi_{\text{ec}}\left(x_1, x_2\right) &= n_0\left(x_1 + x_2\right)\left[1 - n_0\left(x_1\right)\right]\left[1 - n_0\left(x_2\right)\right] \\ &- n_0\left(x_1\right) n_0\left(x_2\right)\left[1 - n_0\left(x_1 + x_2\right)\right]\end{aligned} \tag{49}$$

$$\tilde{a}_2 = a_2 + C S_{\text{ee}}^{+}\left(n_0, 0, 0\right) \tag{50}$$

Let us note first of all that the contribution to $n(\xi)$ from the coherence factors due to electron-electron interaction appears to be opposite to that due to electron-phonon interaction. This means that inclusion of electron electron collisions leads to a positive correction to the chemical potential.

The nonequilibrium function φ_{ee} is proportional to the difference of recombination and production rates, so the equilibrium QDF n_{T} makes φ_{ec} vanish, as well as φ_{ph} .

Yet, there is an essential difference between φ_{ph} and φ_{ee}. This lies in the fact that in thin films (and for $T \to 0$), when the number of phonons is small, quasiparticle production processes due to phonons are suppressed and φ_{ph} reaches a value of the order of unity. A second term describing quasiparticle production is always present in φ_{ee}, so this function is small and, probably, always less than φ_{ph}.

Hence, for not very large values $C < 1$ the main contribution is due to φ_{ph}, so we obtain the earlier results, but with a certain renormalization (Elesin 1981).

If C is sufficiently large (which is unlikely to occur in practice), the situation appears to be more complicated because the problem of finding $n_0(\zeta)$ for large C and calculating φ_{ph} and φ_{ee} is difficult to solve. Although estimates performed with due account for the fact that φ_{ce} has to decrease with increasing C indicate that the phonon contribution should dominate, it is not excluded that n_1 may change sign (which would lead to the NE-model). Only particular calculations may give the final answer.

Aronov and Spivak (1979) investigated the effect of electron-electron collisions on $n(\xi)$; this function was sought in the form

$$n = \exp\left(\frac{\mu^* - \varepsilon}{T^*}\right) \tag{51}$$

and μ^* and T^* were found from the conservation laws for the number of quasiparticles and for the energy (for details see Aronov and Spivak 1978). These authors conclude that the

chemical potential may become negative in a certain pumping power range. if electron-electron collisions are taken into account. This contradicts, in general, the result obtained by Elesin (1981) (see eq. 47), because the contribution of the coherence factors to the function n_1 due to electron-electron interaction leads to a positive correction to μ^* as well. Direct comparison of the results is, however, difficult because the function (51) is applicable only for very low pumping intensities.

29.3.2 1.2.2. The order parameter

The order parameter is quite sensitive to the energy distribution of quasiparticles, especially for $\breve{\xi} = 0$. "Overheating" of quasiparticles leads, as Elesin (1976) was the first to show, to an ambiguous dependence of the order parameter A on the pumping power β, starting with a certain critical value β_c. Following the paper by Elesin (1976), let us derive an equation for A, assuming that AA_0 is small. Using the properties of $n_0(\delta)$ and relation (34), we obtain from eq. (6) to within Δ^2

$$\Delta\left(\delta + \alpha\Delta^2 + 2\int \frac{n_1}{\varepsilon}\,\mathrm{d}\xi\right) = 0 \tag{52}$$

$$\delta = \frac{a_1 - a_{1c}}{a_{1c}} \tag{53}$$

where α is a numerical coefficient to be defined below. Note that the expansion in eq. (52) appears to be a good approximation in a wide range of Δ (Elesin 1977b). Substituting n_1 into eq. (52), we obtain the equation for the order parameter and the expression for the energy (8) of a non-equilibrium superconductor (Elesin 1976, 1977b):

$$\Delta\left(\delta + \alpha\Delta^2 - \varphi\Delta\right) = 0 \tag{54}$$

$$U(\Delta,\delta) = \Delta^2\left(\frac{\delta}{2} - \frac{\varphi\Delta}{3} + \frac{\alpha\Delta^2}{4}\right) \tag{55}$$

This result was confirmed in subsequent papers (see, for example, Eckern et al. 1979). Equations (54) and (55) differ from the Ginzburg-Landau equation by the presence of a term, linear in Δ, with the coefficient of "non-equilibrium" φ (and also by the form of coefficients δ and α). The solution of eq. (54) is

$$\Delta_1 = 0, \quad \Delta_{2,3} = \frac{\varphi}{2\alpha}\left(1 \mp \sqrt{1 - \frac{4\alpha\delta}{\varphi^2}}\right), \tag{56}$$

i.e. for $\varphi \neq 0$ two more solutions exist besides the trivial solution $\Delta_1 = 0$: one solution, Δ_3, decays with increasing β; the other, Λ_2, grows with β and appears for $\delta > 0$ or $\beta > \beta_c$. Both solutions merge at $\delta = \delta_H = \varphi^2/4\alpha$ so the "two-value" interval of $\Delta(\beta)$ is equal to $\varphi^2/4\alpha$. The coefficients $\delta = c\,(\beta - \beta_c)\,, \beta_c, \alpha$ and φ depend, in general, on T,;and k. They have been determined by Elesin et al. (1977a,b, 1979) and have been presented by Elesin and Kopaev (1981). In particular, for $\gamma = 0, T = 0, k = 1$ we have $\varphi = 1, \alpha = 1.9, C = 0.37, \beta_c = 0.83$; and for $\gamma = 0, T \to T_c, k = 1$ one obtains (Elesin 1976)

$$\varphi = \beta\zeta_3, \quad \delta = \frac{T - T_c}{T_c} + 2\beta\zeta_2 \tag{57}$$

where ζ_i is a coefficient of the order of unity.
It is also useful to consider the dependence of Δ on temperature for $\beta \neq 0$ (Elesin 1976). It can easily be seen that for

$$T > T_{\mathrm{c}}\,(1 - 2\beta\zeta_2)$$

the dependence $\Delta(T)$ becomes two-valued. Numerical calculations (Elesin et al. 1977a,b, 1979) made it possible to find $A(\beta)$ in a wide range of β, γ and T. The dependences $\Delta(\beta)$ for $k = 1$ are shown in fig. 5. It can be seen from this figure that for all T and γ there is a branch of the solution $A(\beta)$, increasing with β, which appears at a power exceeding the critical one. As T and γ grow, the two-value region of $\Delta(\beta, T)$ get narrower, but the slope of $\mathrm{d}\Delta\mathrm{d}\beta$ remains positive.

Fig. 5. Dependence of the order parameter on the source power β for different values of the parameters γ and T : (a) 0,0 ; (b) $0.02, 0$; (c) $0, 0.44$; (d) $0.02, 0.44$; (e) $0.15, 0$; (f) $0.15, 0.44$; (g) $1.0, 0$.

(Elesin et al. 1979.)

Even more detailed calculations of $\Delta(\beta)$ have been performed by Imai (1980). For the same values of the parameters they agree completely with those of Elesin et al. (1977b, 1979). Figure 6 presents the dependences of Δ/Δ_0 on β/β_{c} for $T/T_{\mathrm{c}} = 0.5$ and different values of ω/Δ_0 and γ. The cases $\omega/\Delta_0 = 8$ or 20, $\gamma = 0$ differ little and coincide with the corresponding cases in fig. 4. A decrease in (1) A_0 leads to a considerable increase in Δ on the upper branch. This is illustrated by the curve with $\omega/\Delta_0 = 2.1$. In particular, the value of $\Delta\,(\beta_{\mathrm{c}})\,/\Delta_0$ reaches 0.72 . The curves $\Delta(\beta)$ with $\gamma = 1$ in fig. 6, and also with $\gamma = 1, 10$, show the effect of phonon reabsorption on the dependence $\Delta(\beta)$. It can be seen that the ambiguity region remains considerable even for large $\gamma = 10$, the decay being slower for smaller values of $(\omega), A_0$. These results agree with conclusions of Elesin (1977b) that $\mathrm{d}A\ \mathrm{d}\beta > 0$ for whatever large γ. The conclusions are based on the fact that the nonequilibrium function φ, which determines the derivative $\mathrm{d}\Delta/\mathrm{d}\beta$ (or $\mathrm{d}\Delta/\mathrm{d}\delta$), is positive

$$\left.\frac{\mathrm{d}\Delta}{\mathrm{d}\delta}\right|_{A=0} = \frac{1}{\varphi}. \tag{58}$$

Relation (58) implies several important consequences. If $\varphi < 0$, the anomalous solution disappears and A vanishes for $\beta = \beta_c$ with a finite negative derivative $\mathrm{d}\Delta/\mathrm{d}\beta$ (or $\mathrm{d}\Delta/\mathrm{d}\delta$). In the limiting case $\varphi = 0$ (equilibrium or

disregard of coherence) a second-order transition with infinite derivative occurs.

Such a strong dependence of Δ on n_1 can be explained by the fact that n_1 appears in the gap

Fig. 6. Dependence of $\Delta / \Delta_{0
$ on β/β_c for $\gamma = 0, 1, 10$. Solid curve: $\omega/\Delta_0 = 2.1$; dashed curve: $\omega/\Delta_0 = 8.$}

equation together with the function $1-2n_0(\varepsilon)$, which is small near $\xi = 0$. And it is this interval of small ξ, where $n_0(0) = \frac{1}{2}$, which contributes mainly to the integral (6).

Figure 7 presents the dependences for three cases:
(1) $\eta = 1, \varphi > 0$ (this case corresponds to the E-model, first-order transition);
(2) $\eta = 0, \varphi = 0$ (second-order transition);
(3) $\eta = -1, \varphi < 0$ (the NE-model, second-order transition with a finite derivative).

Now compare the results with those of the μ^* - and T^*-models. Obviously, case (2) corresponds to the T^*-model, in which $\Delta(\beta)$ is single-valued.

In the μ^*-model Δ is a two-valued function for any β (Tremblay 1981) (for $T \to 0$), while in the E -model the second solution appears only for $\beta > \beta_c$. Note that the prediction of the μ^*-model differs also from that of the NE-model, in which $\Delta(\beta)$ is single-valued (case 3). The above consideration referred to a spatially homogeneous state. Inhomogeneous states are described by the general system of equations (7, 11). In a number of cases they can be reduced

to one order-parameter equation. To this end, one has to find a solution of the kinetic equation (11), substitute it into the gap equation (7) and integrate with respect to ζ.

This calculation procedure was performed in the works of Elesin (1977b) and Eckern et al. (1979). In the general case, an inhomogeneous nonequilibrium state of a superconductor is characterized by two lengths: the coherence length ξ_0 and the quasiparticle diffusion length L.

The equation for Δ in the limiting case $\xi_0 \gg L$ and also for any ξ_0 and L under the assumption that Δ varies slowly was derived by Elesin (1977a) [for details see the review by Elesin and Kopaev (1981)]. Eckern et al. (1979) derived a similar equation for any L and $\check{\zeta}_0$ (as $T \to T_c$) without assuming slow variation of A. This can be done by passing, in the kinetic equation (11), to a new representation, in which quasiparticle diffusion occurs under the condition $\varepsilon(r) = \text{const}$ [i.e. n depends on r only through $A(r)$]. The importance of this representation was noted by Smith (1977).

Using the method suggested by Eckern et al. (1979), Elesin and Kopaev (1981) derived a closed equation for Δ

$$\hat{\Pi}\,(\xi_0, L)\,\Delta = \frac{\partial U(\Delta, \delta)}{\partial \Delta}, \tag{59}$$

Fig. 7. Dependence of $\Delta / \Delta_{0 $ on β/β_{c} for (1) the T*-model, (2) the E -model and (3) the NE-model. (Elesin 1977b.)}

where the differential operator is given by $\hat{\Pi}\left(\xi_0, L\right)$,

$$\hat{\Pi}\left(\xi_0, L\right) \Delta = \left[\xi_0^2 \nabla^2 + L^2 \nabla^2 \left(\delta + \alpha \Delta^2\right) - L^2 \xi_0^2 \nabla^4\right] \Lambda, \tag{60}$$

and U is the energy of a superconductor (eq. 55).
For $\xi_0 \gg L$ eq. (59) coincides with the equation obtained by Elesin (1977a); for $T \to T_{\mathrm{c}}$ it coincides with the equation derived by Eckern et al. (1979).

For $\xi_0 \gg L$ eq. (59) has a structure similar to that of the Ginzburg-Landau equation, an essential difference being the presence of a third-power term proportional to the nonequilibrium function φ.

Time derivatives of Δ (needed for stability analysis) can be of two types. First, there is a term $\tau_\Delta(\partial\Delta/\partial t)$, which takes into account the variation of Δ with the order parameter relaxation time τ_1 (Eckern et al. 1979). This term is significant only for the analysis of the stability of the normal state. Second, there is a derivative which allows for a change in Δ due to variation of the quasiparticle distribution function. In the general case, this term is nonlinear in $\delta/\partial t$. But to analyse the stability, we are interested in small decrements $\tilde{\gamma}$ (near the stability limit), i.e. only in the term linear in $\hat{c}/\hat{c}t$. In this approximation the following equation for $\hat{c}\Delta$, $\hat{c}t$ can be obtained (Elesin 1977b, Eckern et al. 1979, Elesin and Kopaev 1981):

$$-\left(\tau_A + D\right) \partial\Delta/\partial t + \hat{\Pi}\Delta = \partial U/\partial\Delta, \tag{61}$$

$$D = \Lambda^2 \int_0^\infty \frac{1-2n}{\varepsilon^3}\, \mathrm{d}\xi. \tag{62}$$

For $\gamma = 0, T = 0, D = \pi\Delta a_1/a_2$, eq. (61) coincides with that in the article by Elesin et al. (1980a); for $\gamma = 0, T \to T_{\mathrm{c}}, D = \pi\Delta/4T_{\mathrm{c}}$ it coincides with the equation derived by Eckern et al. (1979).

29.3.3 1.2.3. The number of superconducting electrons and magnetic properties

The magnetic properties of superconductors are closely related to the number of superconducting electrons N_{S}, because the linear response to a weakly varying magnetic field is

(Abrikosov and K halatnikov 1958)

$$J = -\frac{e^2 N_0}{mc} N_{\mathrm{S}}, \tag{63}$$

where N_{S} is given by

$$N_{\mathrm{S}} = 1 + 2\int_0^\infty \frac{\mathrm{d}n}{\mathrm{d}\varepsilon}\,\mathrm{d}\xi. \tag{64}$$

In equilibrium superconductors N_{S} is positive, in particular $N_{\mathrm{S}} \sim \Delta^2$ for $T \to T_{\mathrm{c}}$, which corresponds to anomalous diamagnetism.

The sign of $N_{\mathrm{S}'}$ determines also the stability of a superconductor against inhomogeneous perturbations (see section 1.2.6), the case $N_s > 0$ corresponding to stability.

The sign of N_{S} has long been a debatable problem, but now it can be considered as solved, to a great extent: in all cases studied $N_{\mathrm{S}} > 0$ for superconductors.

Let us prove first of all, following the papers by Elesin (1977a,b), that the sign of N_{S} is related to the sign of the coherence correction to the ODF n_1. Substituting $n(\xi)$ into N_{S} and using the general properties of $n_0(\varepsilon)$ (see section 1.2.1), we obtain the expression for N_{S} to within Δ^2

$$N_{\mathrm{S}} = 1 + 2\int_0^\infty \frac{\mathrm{d}n_0(\xi)}{\mathrm{d}\xi}\,\mathrm{d}\xi + 2\int_0^\infty \frac{\mathrm{d}n_1}{\mathrm{d}\varepsilon}\,\mathrm{d}\xi = 2\int_0^\infty \frac{\mathrm{d}n_1}{\mathrm{d}\varepsilon}\,\mathrm{d}\xi. \tag{65}$$

Since $n_0(0) = \frac{1}{2}$, the first two terms in eq. (65) cancel, and N_{S} is determined only by the coherence correction n_1.

Since the derivative $\mathrm{d}n_1/\mathrm{d}\varepsilon = -n_1/\varepsilon$ (see eq. 38), the expression for N_{S} can be rewritten in the form

$$N_{\mathrm{s}} = \eta\varphi\Delta, \tag{66}$$

which implies that in the nonequilibrium case ($\varphi > 0$) the sign of N_{S} is determined by the sign of the coherence factor η. If $\eta = 1$ (superconductor), $N_{\mathrm{s}} > 0$, that is, nonequilibrium superconductors with excess quasiparticles remain diamagnetic, and $N_{\mathrm{S}} \sim \Delta$. The positive sign of $N_{\mathrm{S}'}$ was confirmed in subsequent papers (Eckern et al. 1979, Elesin et al. 1979, Imai 1980) both analytically and numerically for the whole interval of Δ values and for a wide range of the other parameters.

Obviously, in the NE-model ($\eta = -1$) the sign of N_{s} becomes negative. This possibility was discussed by Elesin (1977b, 1979a) and by Eckern et al. (1979).

In the μ^*-model N_{S} is negative (Chang and Scalapino 1974, Baru and Sukhanov 1975), i.e. the μ^*-model predicts anomalous paramagnetism.* From the mathematical point of view this is related to a more rapid, in comparison with n_{T}, decay of the QDF with energy in the μ^*-model (the property of "supercooling").

The study of the effect of magnetic field and superconducting current on the state of superconductors with excess quasiparticles is an important problem. The fact is that a current is usually passed through a film to measure the voltage across the film in the resistive state, as, for example, is the case with optical pumping or injection. In a number of cases the currents needed are almost equal to the critical ones. Therefore, Elesin (1980) considered the effect of magnetic field H and current state on the character of the phase transition in a superconductor with a wide source of quasiparticles. If the film is thin $\Big(d < \tilde{\lambda}_3, \zeta_0; \tilde{\lambda}$ is the London length) and the magnetic field is parallel to it, the

[12]*It should be noted that the possibility of anomalous paramagnetism in nonequilibrium superconductors ($N_{\mathrm{S}} < 0$) with inverse quasiparticle distribution was first pointed out by Galitskii et al. (1973) (see section 2.5).

dependence $\Delta(\beta)$ retains its ambiguous character and the critical value β_c decreases with increasing magnetic field:

$$\beta_{\rm c}(H) = \beta_{\rm c} - \tilde{h}^2, \quad \tilde{h}^2 = \frac{1}{24}\left(\frac{H}{\tilde{H}_{\rm c}}\right)^2 \left(\frac{d}{\tilde{\lambda}^2}\right)^2, \quad \tilde{H}_{\rm c} = 4\pi v(0)\Delta_0\zeta, \tag{67}$$

where $\hat{H}_{\rm c}$ is the "critical magnetic field", ζ is of the order of unity.
By varying the superconductor free energy (55), which allows for the contribution due to superconducting current and magnetic field, one can find the expression for the order parameter of a thin film:

$$\Delta = \frac{\varphi}{2\alpha}\left[1 \mp \sqrt{1 - \frac{4\alpha\delta(H)}{\varphi^2}}\right], \quad \delta(H) = \delta + \hat{h}. \tag{68}$$

It can be seen that the dependence $\Delta(\beta)$ has two branches, which appear for $\beta > \beta_{\rm c}(H)$, given by eq. (67).

Thus, the dependence $\Delta(\beta)$ retains its ambiguous character in a magnetic field. Note that as the magnetic field H increases at fixed β, the ordinary solution Δ_3 decreases, while a new solution Δ_2 grows. The diminishing of the critical power $\beta_{\rm c}(H)$ is of interest for experimental investigations because it permits the reduction of overheating of irradiated films.

It can also be shown (Elesin 1977a) that the value $\delta_0(H)$, at which the Gibbs energies of the normal and superconducting phases become equal to each other, is

$$\delta_0(H) = \delta_0 - \hat{h}. \tag{69}$$

In other words, the dependence $\Delta(\beta)$ shifts as a whole with increasing H. It should be noted that the results obtained here (and subsequent results) may find quantitative application for small δ.

Now let us consider the current state. If the current J is specified, expressing then the velocity in terms of Δ and J, we arrive at the following equation for Δ :

$$\Delta\left(\delta + \alpha\Delta^2 - \varphi A + \frac{\tilde{J}^2}{\Delta^2}\right) = 0, \quad \hat{J}^2 = \frac{J^2}{J_{h,}^2}. \tag{70}$$

In this case the solution, which grows with β, does exist for all δ since the main contribution for small Δ is due to the term $\widetilde{J}^2/\Delta^2$, which is present in the equation also for $\beta = 0$.

Let us find the quantity $\delta_0(J)$. Using the expression for the energy and eq. (70), we easily obtain

$$\delta_0(J) = \delta_0 - \frac{9}{4}\alpha\tilde{J}^2. \tag{71}$$

Thus, the power needed for the first-order transition or for the transition of a superconductor into the inhomogeneous state decreases with increasing current.

The decrease in the power becomes appreciable, if the current and magnetic field attain the values

$$J \simeq \frac{2\sqrt{2}}{9\alpha} J_{k\rho}, \quad H \simeq \tilde{H}_{\rm c}\left(24\beta_{\rm c}\right)^{1/2}\frac{\tilde{\lambda}}{d} \tag{72}$$

The critical current of a film vanishes at $\delta = \varphi/4\alpha$.
As was noted by Elesin (1976) the QDF determines the sign of the superconducting current in an external alternating electric field E_0 of frequency Ω. It can be shown (Elesin et al. 1973)

that the expression for this current (if scattering from impurities is taken into account) is, for $\Omega \ll \Delta$:

$$J_{\mathrm{S}} = \mathrm{i}\sigma_{\mathrm{S}} E_0, \quad \sigma_S = \frac{\pi A}{\Omega}\sigma_{\mathrm{N}}(1 - 2n(\xi = 0)) \tag{73}$$

If the functions satisfy the condition (5), $n(0) < \frac{1}{2}$ (the E-, NE- and T^*-models), the conductivity is positive, typical of equilibrium superconductors. In the μ^* model σ_{S} changes its sign. Note that the change in the sign can be verified experimentally.

Moreover, if σ_{s} were negative, this would lead to a specific electrical instability (Elesin and Kopaev 1981), which arises as soon as $n(0) > \frac{1}{2}$. Indeed, the fluctuation of the charge would give rise to an electric field, which in turn would produce a current enhancing the charge fluctuation. The development of this instability would lead to the destruction of the superconducting state or to the formation of a certain inhomogeneous state.

29.3.4 1.2.4. The critical power of optical pumping

As was shown by Elesin (1974) (see section 1.2.2), the order parameter vanishes at a certain critical power β_c. This is the boundary from where a second solution for Δ appears. The critical power can be found by substituting the "critical" QDF $n_0(\xi)$ into the normalization condition (18):

$$\begin{aligned}\beta_{\mathrm{e}}(T, \gamma') = & \frac{1}{\Lambda_0^4}\int_0^{(\omega_1)} \mathrm{d}\xi \int_0^{(\omega_1)^-} \mathrm{d}\xi' \left(\xi + \xi'\right)^2 \\ & \times \left[n_0 n_0' - N_{\xi+\xi'}\left(1 - n_0 - n_0'\right)\right] + C\int_0^{\infty} \frac{\mathrm{d}\xi}{\Delta_0} I_{\mathrm{ee}}(\xi)\end{aligned} \tag{74}$$

This expression for β_{c} includes two terms having different physical meaning.
The last term, with a negative sign, describes the impact ionization of quasiparticles by high-energy quasiparticles, that is, it describes the multiplication of quasiparticles. Large ξ contribute to this term (see Vardanyan and Ivlev 1973), where the QDF $n_0(\xi) \sim \beta/\xi^4$ is proportional to the source power β. Transferring the last term to the left-hand side, we obtain instead of β_{c}

$$\beta_{\mathrm{c}} \to \beta_{\mathrm{c}}\left(1 + r_{\mathrm{ee}}\right) \tag{75}$$

where r_{ee} is the multiplication coefficient. For $C \ll 1, r_{\mathrm{ee}}$ has been found by Vardanyan and Ivlev (1973).

The addend $N\left(1 - n - n'\right)$ in the first term describes the production of quasiparticles due to equilibrium N_T and non-equilibrium $N = N_0(\xi) + N_v(\xi)$ phonons, where $N_0(\xi)$ stands for phonons produced under recombination and scattering of low-energy particles ($\xi \lesssim \Delta_0$) and N_{V} denotes phonons produced under scattering of high-energy quasiparticles. The contribution of these quasiparticles depends directly on the intensity β and describes the multiplication of quasiparticles due to phonon absorption (the multiplication coefficient is r_{ph}). In the remaining integral those energies $\xi \leqslant \Delta_0$ are significant, at which $n_0(\xi)$ is close to the thermal one (see section 1.2.1), so the following substitution is possible (Elesin 1977a):

$$\varphi_{\mathrm{ph}}\left(\check{\zeta}, \xi', T, \eta'\right) = \frac{\varphi_{\mathrm{ph}}\left(\xi, \xi', T, \gamma = 0\right)}{1 + \gamma'\left(\xi + \xi'\right)/\Lambda_0} \tag{76}$$

The factor $\left(1 + \gamma\left(\xi + \xi'\right)/\Delta_0\right)$ accounts for an increase in the quasiparticle lifetime due to reabsorption processes. Obviously, it can be taken out of the integral sign in the form $(1 + \eta_i)$ where η_i is a quantity of the order of unity. Summarizing the results for $\beta_c(T, \gamma)$, we obtain

$$\beta_{\mathrm{c}}(T,\gamma) = \beta_{\mathrm{c}}(T)\left(1+r_{\mathrm{ec}}+r_{\mathrm{ph}}\right)^{-1}\left(1+\gamma\eta_{\gamma}\right)^{-1} \tag{77}$$

$$\beta_{\mathrm{c}}(T) = \frac{1}{\Delta_0^4}\int_0^{\infty}\mathrm{d}\xi\int_0^{\infty}\mathrm{d}\xi'\left(\xi+\xi'\right)^2 \times\left[n_0(\xi)n_0\left(\xi'\right)-N_{\mathrm{T}}\left(\xi+\xi'\right)\left(1-n_0(\xi)-n_0\left(\xi'\right)\right)\right] \tag{78}$$

The coefficient r_{ph} is insensitive to ; and the dependence $\beta_{\mathrm{c}}\left(\gamma'\right)$ is mainly determined by the factor $(1+\gamma\eta_i)$.

The temperature dependence of the critical power is given by a universal function $\beta_c(T)$. At $T=0$ it reaches a maximum of 0.84 (Elesin et al. 1977a). As T grows, β_{c} decreases due to the production of thermal quasiparticles and for $T\to T_{\mathrm{c}}$ it is described by the expression $\beta_{\mathrm{c}}(T)=A\left(T_{\mathrm{c}}-T\right)$ (Elesin 1976). The dependence $\beta_{\mathrm{c}}(T)$ in the entire temperature range was found by Elesin et al. (1979) and is shown in fig. 8. The relationship between the measured power $W_{\mathrm{c}}\,(\mathrm{W/cm^2})$ and dimensionless power β_{c} is given by the expression (Golovashkin et al. 1980)*

$$W_{\mathrm{c}} = \beta_{\mathrm{c}}(T)\frac{N_{\mathrm{c}}\hbar\omega}{4\tau_{\mathrm{ph}}\left(1+\tau_{\mathrm{ee}}+\tau_{\mathrm{ph}}\right)\left(1+\gamma\eta_{\gamma}\right)k_1^2k(\omega)} \tag{79}$$

Fig. 8. Theoretical dependence of the dimensionless pumping β_{c} on T/T_{c}. (Golovashkin et al. 1980.)}

where $k(\omega), k_1^2$ are the coefficients of absorption and reflection of electromagnetic field. If one takes into account that usually in superconductors the thickness $d\gg k^{-1}(\omega), d^{-1}$ can be substituted for $k(\omega)$.

Comparison of W_{c} with experimental data is performed in section 1.2.11.

29.3.5 1.2.5. Current-voltage characteristics of tunnel junctions

The variation of Δ due to quasiparticle injection in tunnel junctions with a sufficiently low resistance may lead to a radical change in their currentvoltage characteristics.

The effect of weak injection on the energy gap has been considered theoretically in several papers (see, for example, Kirichenko and Seminozhenko 1977).

Strong injection in SiS junctions at high voltages ($V\gg A$, wide source) has been studied by Elesin and Levchenko (1978) and Elesin et al. (1980a, b).

It was shown by Elesin et al. (1977b) that the order parameter vanishes at a critical voltage V_c, which satisfies the condition

$$\alpha_V\int_0^{\gamma_c}\frac{\mathrm{d}\xi}{\Delta_0}\left(1-2n_0\right)=\beta_c(T,\gamma) \tag{80}$$

[12]*Note that there is a misprint in the corresponding equation in the article by Elesin and Kopaev (1981); instead of $r_{\mathrm{e}}r_{\mathrm{f}}$ their formula should read $r_{\mathrm{ee}}+r_{\mathrm{ph}}$.

In particular, for $T=0, \alpha_V \ll 1$, the critical voltage

$$\frac{V_c}{\Delta_0}=\frac{\beta_c(0)}{\alpha_v} \gg 1 \tag{81}$$

is large in comparison with Δ, so under these conditions tunnel injection is equivalent to the effect of a wide source. Therefore, for $V>V_{\mathrm{c}}$ the dependence $\Delta(V)$ becomes many-valued. The multiplicity of $\Delta(V)$ leads, naturally, to multiplicity of the current voltage characteristics $J(V)$. Depending on the values of the parameters $\alpha_V, \hat{\gamma}$ and T, three types of $I-V$ characteristics (shown in fig. [9)] may exist. They may be classified according to relationships between

the characteristic values of the currents J_c, J_0 and J_{th} (Elesin et al. 1980a.b):

$$J_{\mathrm{c}}=J\left(V_{\mathrm{c}}, \Delta=0\right), \quad J_0=J\left(V_0, \Delta=0\right), \quad J_{\mathrm{th}}=J\left(V_{\mathrm{H}}, \Delta_{\mathrm{H}}\right), \tag{82}$$

where V_0 is the voltage at which the energies of the normal and superconducting phases become equal. Current-voltage characteristics of the first type correspond to the condition $J_0>J_c>J_{\mathrm{th}}$, those of the second type to $J_0>J_{\mathrm{th}}>I_{\mathrm{c}}$, and those of the third type to $J_{\mathrm{th}}>J_0>J_{\mathrm{c}}$. As was shown by Elesin et al. (1980a) (see below, section 1.2.10), current voltage characteristics of types I and II lead to transition into the inhomogeneous state (with $d_3 \neq 0$ and $J_1=0$), while characteristics of type III lead to a first-order transition into the normal state.* The conditions for characteristics of each type to occur have been derived by Elesin et al. (1980b). Note that growth of T and γ may cause characteristics of type III to become those of type II.

The temperature dependence of the critical voltage $V_c(T)$ and the corresponding critical dimensionless current $J_c(T)$

$$\begin{aligned} J_c(T) &= \alpha_1 \int_0^1 \frac{d\xi}{d_0}\left(1-2n_0\right)+2\alpha_1 \int_0^1 \frac{d}{d_0}\left[n_0(\xi)-n_0\left(\xi+I_c\right)\right] \\ &= \beta_c(T)+2\alpha_V \frac{T_c}{A_0} \ln\left[1+\exp\left(-\frac{V_c}{T_c}\right)\right] \end{aligned} \tag{83}$$

differs slightly from $\beta_c(T)$:

$$V_c(T)=\begin{cases} \beta_c(0)/\alpha_V, & T \to 0, \\ 2\sqrt{\frac{T_c \beta_c(T)}{\alpha_V \Delta_0}} \sim \sqrt{1-T/T_c}, & T \to T_c, \end{cases} \tag{84}$$

Equations (83). (84) imply that I_c and J_c should decrease with increasing T. that is, the dependence is qualitatively similar to that of the power under optical pumping (cf. section 1.2.4).

29.3.6 1.2.6. Stability analysis. Coherent instability under optical pumping

The stability of nonequilibrium superconductors has been studied with respect to two classes of perturbations: fast perturbations (at times much less than the phonon relaxation time τ_{ph}) and slow perturbations (at times of the order of τ_{ph}).

The stability with respect to fast inhomogeneous perturbations was studied in several articles (Kemoklidze and Pitaerskii 1967. Aronor and Gurevich 1973. Chang and Scalapino 1974. Baru and Sukhanov 1975. Elesin 1977. Eckern et al. 1979. Imai 1980). The criteria obtained were identical, though

[12]*Transition into the inhomogeneous state for type-III characteristics is possible, if the injection current is reduced after the transition into the normal state (see fig. 9).

different in form. and in the limit $yl. = 1$ ig is the perturbation wave sector) they reduce to the following expression for the decrement $\tilde{\%}$ (Elesin and Kopaev 1981, Eckern et al. 1979)

$$\tilde{\gamma} = -q^2 L^2 \frac{N_{\mathrm{s}}}{D}. \tag{85}$$

The sign of ; 1s determined by the signs of the number of superconducting electrons. 1 isee sectoon 1.2 .3) and of the quantit: 1) lisee eq. 621 . which are the functionals of steady-state QDF.

In the E-model X is prositive (see 1.2 .3) and J) - (1) bi wrtue of the properts (5) (n-in - is) the decrement is negatioe. - 11. Thus. the s.stem is stable against fast perturbatuns. and instabilittes are stabilized due to quasi-particle diffusion. This result as abotained b; Elesin 199-69 and confirmed by Flesm 11977a.b). and alor in papers by Feckern et al. 119'y) and Imal 11geron by somew hat different methods. I mas has performed computer-aded calculatoms in a wide range of the parameters.

In the NE-model $N - ()$ isce section 1.2.3) and D remains positive $\left(n - \frac{1}{2} \cdot 1\right.$. so the decrement ; is posstrie. In this case anomalous diffuston of quasiparticles a ould lead io the erhancement of inhomogenerous perturbations. that 13. to the so-called diffusion instability (see below, section 1.2.7).

The μ^*-model, in which $1, -1$). also predicts diffusion instability. I Chang and Scalapino 1974. Baru and Sukhano: 1975). It should be noted. however. that in the μ^*-model D mas becrme negative $\left(n - \frac{i}{2}\right)$ (for example. for $T \to (1)$ and $\mu^* > \sqrt{\frac{4}{3}} A$).

In the T^*-model $N_{\mathrm{S}} > 0, D > 0, \tilde{\gamma} < 0$, and the superconductor is stable.
The stability; against sion perturbaturs has been studied b; Elesin 119-al. Eckern et al. (1979) and Imai (198()).

A tope of instabilit: pecific of superconductors atth a aide quasiparticle source was predicted b: Fiesm 19'GTai. aho called it coherent mstability. since it is related to the coherence of the interactum betseen quasiparticle and phomons. The instabilit: arroes in the state with $1 = 1$. corresponding to the increasing solutions lef. sectoon 1.2.21. To study coherent instabilit: one can proceed from the hinctic equation iflesin $19^{\top -}$ a. imal 1 yxon. ar the kinetic equation modified for $T \to T$. If.ckern et al. 1979). or equation iffi, for 1 1Elesin and Kopaes 19811. Let usconsider. following the paper by Elesin and Kopaev (1981), a homogeneous perturbation

$$\Delta(t) = \Delta + \dot{\Delta} \exp(\dot{r}t). \tag{86}$$

Substituting eq. $|xf_1|$ into eq. $|6|$ 1. ae obtain for the dimensoniev decrement

$$\tilde{i} = -\frac{1}{D} \frac{\hat{c}^2 U}{Di1^2}, \tag{87}$$

or using expression (55) for $U(\Delta, \delta)$

$$\tilde{\gamma} = \frac{2}{\Delta\delta}, \quad \Delta\delta \equiv \frac{\hat{c}\Delta}{\hat{c}\delta}. \tag{88}$$

Equation (88) shows that the increasing solution Δ_2 is unstable, while the solution Δ_3 is stable (the stability of the solution $\Delta_1 = 0$ is discussed below). It is a simple matter to demonstrate that the system passes into the normal state for $\Delta(t = 0) < \Delta_2(\delta)$ and into the superconducting state for $\Delta(t = 0) > \Delta_2(\delta)$.

The physical meaning of instability is as follows. Suppose that the gap in the state with A_2 on the increasing branch slightly decreases due to a fluctuation. Then the probability of quasiparticle recombination ($1 + A^2/\varepsilon\varepsilon'$) will decrease and, hence, the number of quasiparticles (near $\xi \sim \Delta$) will increase. The equation for Δ implies that this will lead to a further diminishing of the gap, and so on.

It is interesting to note that when the system passes into the normal state (at a fixed β) an additional number of quasiparticles δn are produced due to a decrease in the re-combination rate.

It would be useful to relate the stability of a system to peculiarities of the energy $U(\Delta)$ by virtue of eq. (87). The dependence $U(\Delta)$ is shown in fig. 10 .

It can be seen that $U(\Delta)$ has three extrema Δ_1, Δ_2 and Δ_3, the values Δ_1 and Δ_3 corresponding to the minimum of $U(\Delta)$ and Δ_2 to the maximum. It is natural, therefore, that the solution Δ_2 is unstable.

Different energies $U\left(A_1\right)$ and $U\left(A_3\right)$ may correspond to the stable solution

A_1 and A_3. Thus, one of these states (with a higher energy) will be metastable and the other will be really stable. At a certain pumping power $\delta=\delta_0$ the energies of the states become equal and co-existence of the superconducting and normal phases, i.e. an inhomogeneous state, is possible. As was shown by Eckern et al. (1979), the time needed for a system to pass from the metastable to the stable state for $\delta \neq \delta_0$ is very long, if strong fluctuations and the nuclei of the stable phase are absent. Therefore, depending on the external conditions, a superconductor may undergo a first-order transition into the normal state either at $\delta=\delta_0$ or at $\delta=\delta_{\mathrm{H}}$ and $\delta=0$, the transition in the latter case being accompanied with a hysteresis.

Using eq. (61), it is not difficult to obtain the decrement for inhomogenous perturbations:

$$\tilde{\gamma}=\frac{2}{\Delta \delta}-q^2 L^2 \tag{89}$$

The instability decrement is maximal at $q=0$, but it remains positive also for inhomogeneous perturbations for $q^2 L^2<^2 /\Delta \delta$. This means that lamination into the superconducting and normal phases, i.e. transition into an inhomogeneous state, is possible under appropriate conditions. Figure 10 shows descriptively the lamination process. Suppose that an inhomogeneous fluctuation $\Delta(x)$ arises in a superconductor, which is in the state with Δ_2. Due to coherent instability, the order parameter in the state with $\Delta(x)<\Delta_2$ will drop down to $\Delta_1=0$, while in the state with $\Delta(x)>\Delta_2$ it will increase to Δ_3.

Equation (61) permits the analysis of stability of the normal state $\Delta_1=0$. The corresponding decrement $\tilde{\gamma}$ is (Elesin et al. 1981, Eckern et al. 1979)

$$\tilde{\gamma}=-\delta-q^2 \xi_0^2-\delta q^2 L^2-L^2 \xi_0^2 q^4 \tag{90}$$

that is, the normal state $A_1=0$ is stable for $\delta>0$. The diffusion of quasiparticles and the order parameter leads to an additional stabilization. This result was obtained earlier by Elesin et al. (1976a) for the case of $\check{s}_0 \gg L$ and any $\check{s}_0 q$. Equation (61) permits also the study of stability of inhomogeneous solutions

$$\Delta(r, t)=\Delta(r)+\tilde{\Delta} \exp (\tilde{\gamma} t), \delta(t)=\delta+\tilde{\delta} \exp (\tilde{\gamma} t) \tag{91}$$

In this case we obtain for $\tilde{\Delta}(r)$ the equation

$$\left(\hat{\Pi}+\frac{\partial^2 U}{\partial \Delta^2}\right) \tilde{\Delta}(r)+\left(\frac{\partial^2 U}{\partial \Delta \partial \delta}\right) \tilde{\delta}=-\tilde{\gamma} D \tag{92}$$

the decrement $\tilde{\gamma}$ being an eigenvalue of this equation.

The conditions of coherent instability may be satisfied under tunnel injection at high voltages $V \gg \Delta_0$ (Elesin and Levchenko 1978). The possibility of fixing the current is an essential difference from the case of optical pumping (Elesin and Levchenko 1978). In the regime of a given current (corresponding
to large R) eq. (25) should be added and a change in the voltage,

$$V(t)=V+\tilde{V} \exp (\tilde{\gamma} t), \tag{93}$$

should be taken into account. We then obtain for $\hat{i}$ (Elesin and Kopaev 1981)

$$\tilde{\gamma} = \frac{1}{\Delta_{\mathrm{V}}} \frac{1 + R\,\mathrm{d}J/\mathrm{d}V}{1 - RJ_{\mathrm{V}}} - q^2 L^2, \quad J_{\mathrm{V}} = \left(\frac{\partial J}{\partial V}\right)_{\Delta} \tag{94}$$

$$\frac{\mathrm{d}J}{\mathrm{d}V} = J_{\mathrm{V}} + J_{\Delta}\Delta_{\mathrm{V}}, \quad J_{\Delta} = \left(\frac{\partial J}{\partial \Delta}\right)_{\mathrm{V}}, \quad \Delta_{\mathrm{V}} = \frac{\partial \Delta}{\partial V}. \tag{95}$$

Consider the current-voltage characteristic of type I (see fig. 9). If the current exceeds J_{th} , we are first at the descending portion of the current-voltage characteristic corresponding to $A = A_2$. The homogeneous solution is stable against homogeneous perturbations for a fixed current. Indeed, since on the descending portion of the current-voltage characteristic $\mathrm{d}J/\mathrm{d}V < 0$ (but $J_V > 0$ and $1 + RJ_{\mathrm{V}} > 0$), $\tilde{\gamma}$ is negative for R satisfying the condition $|\mathrm{d}J/\mathrm{d}V|R > 1$. This is how, for example, the state with $\Delta = \Delta_2$ is approached in a superconductor.

This state, however, appears to be unstable against inhomogeneous perturbations which do not alter the total current, i.e. satisfy the condition

$$\int \mathrm{d}^2 r \tilde{\Delta}(r) = 0 \tag{96}$$

In this case the decrement $\tilde{\gamma}$ becomes positive for perturbations which vary in space sufficiently slow, $\tilde{\gamma} = \Delta_{\mathrm{V}}^{-1} - q^2 L^2$, and the sample is laminated into the normal $\Delta_1 = 0$ and superconducting Δ_3 phases (see above) along the current direction, the regions with smaller Δ and higher currents alternating with those of larger Δ and lower currents.

A similar lamination occurs for current voltage characteristics of types II and III.

29.3.7 1.2.7. Diffusion instability

Diffusion instability caused by possible anomalous diffusion of quasiparticles has been predicted by Chang and Scalapino (1974) and by Baru and Sukhanov (1975). The diffusion of quasiparticles is assumed to proceed from the region with large order parameter land correspondingly low quasiparticle concentration $\bar{n}$) to the region with small Δ (and large $\bar{n}$).

Instability develops in the following way. Suppose that Δ decreased in a certain region of space. Then, the gradient of Δ leads to a flow of quasiparticles, which further diminishes A, and so on. The conditions under which diffusion instability may occur. (85), are determined by the sign of the quantities N_{S} and D (eq. 64), which is a functional of $n(\xi)$. As was already noted, the QDF in the μ^* - and NE-models satisfy the condition $n(0) > n_T(0)$ ("supercooled" QDF) and yield $N_{\mathrm{S}} < 0, D > 0$ to meet the conditions for the occurrence of diffusion instability. However, in superconductors with optical and tunnel pumping the quasiparticle distribution is "overheated" and $N_{\mathrm{S}} > 0$, so diffusion instability cannot take place.*

At the same time, in an exciton insulator where the signs of the coherence factors are opposite to those in superconductors (see section 1.1.2) the NEmodel is applicable. The possibility of diffusion instability occurring in an exciton insulator was pointed out by Elesin (1979a) who derived the conditions for diffusion instability to take place and calculated the wave vector with which the spatial variation of $\Delta(r)$ is modulated.

29.3.8 1.2.8. Inhomogeneous states in the E-model

The development of coherent instability leads to the formation of an inhomogeneous laminated state (Elesin 1977a) described by the steady-state equation (59). The structure of the inhomogeneous state in the E-model was studied by Elesin (1977a), Elesin et al. (1980a, b) and by Eckern et al. (1979).

It is a simple matter to understand the situation in the one-dimensional case when quasi-particle diffusion is neglected ($\xi_0 \gg L$). In this case eq. (59) takes the form (Elesin 1977a)

$$\frac{\mathrm{d}^2\Delta}{\mathrm{d}x^2} = \frac{\partial U}{\partial \Delta}, \quad x \equiv \frac{x'}{\xi_0}, \tag{97}$$

$$\left.\frac{\mathrm{d}\Delta}{\mathrm{d}x}\right|_{x=+1/2} = 0, \tag{98}$$

where l is the film length.
The first integral of this equation is

$$\frac{1}{2}\left(\frac{\mathrm{d}\Delta}{\mathrm{d}x}\right)^2 = C - U(\Delta), \quad C = U\left(\Delta^{(2)}\right) = U\left(\Delta^{(3)}\right) \tag{99}$$

where C is a constant and $A^{(i)}(i = 1, 2, 3, 4)$ are the roots of the equation

$$C - U\left(\Lambda^{(i)}\right) = 0.$$

Analysis of the trajectories leads to the following possible spatially inhomogeneous solutions for Δ (Elesin 1977a; see also Eckern et al. 1979, and Tremblay 1981):
(1) oscillating from $\Delta^{(2)}$ to $\Delta^{(3)}$;
(2) narrow layers with $\Delta \neq 0$ or $\Delta = 0$ (soliton-type solutions):
(3) monotonous solutions.

In order to find out which of these solutions are stable, one has to analyse their stability with the help of eq. (92). This analysis shows (Elesin and Kopaev 1981, Eckern et al. 1979, Tremblay 1980) that only the monotonous solutions are stable, with stability for $\delta \neq \delta_0$ and under the additional conditions

$$\frac{\mathrm{d}\bar{J}}{\mathrm{dV}} < 0, \quad R > \left|\frac{\mathrm{d}\bar{J}}{\mathrm{dV}}\right| \tag{100}$$

which are satisfied for a given tunnel injection current.
Among the monotonous solutions there is a specific one (connecting the singular points $A_{10} = 0$ and $\Delta_{30} \neq 0$), which is called the laminated solution. This solution occurs for a certain value of $\delta_0, C = 0$, at which the energies of the phases are equal

$$U\left(\Delta_{10}\right) = U\left(\Delta_{30}\right), \quad \delta_0 = {}^{2/9}\,\mathrm{d}, \quad \Delta_{30} = {}^{2/3}\alpha, \tag{101}$$

$$\Delta(x) = \frac{\Delta_{30}}{1 + \exp\left[-\sqrt{\delta_0}\,(x + x_0)\right]}, \tag{102}$$

where x_0 is the position of the junction boundary. It is noteworthy that the boundary conditions do not fix the position x_0.

The monotonous solutions for $\delta \neq \delta_0$ were found by Elesin et al. (1980b). In particular, for long films ($l \gg \xi_0$) and $\delta > \delta_0$ we have

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- Yet, it is not forbidden, in principle. For example, the possibility is not excluded for diffusion instability to occur in superconductors with strong electron electron interaction (see section 1.2.1). The assumption has been made (Eckern et al. 1979) that the case $N_\mathrm{S} < 0$ is possible under phonon pumping.

$$\Delta(x) = \Delta_3 - \frac{4by}{y^2 - 2by + (\Delta_2 - \Delta_1)^2}, \tag{103}$$

$$y = (\Delta_2 - \Delta_1) \exp \sqrt{\frac{\alpha b}{2}} \left(x + \frac{1}{2} l \right) \tag{104}$$

$$b = (\Delta_3 - \Delta_2)(\Delta_3 - \Delta_1), \quad b = \Delta_1 + \Delta_2 - 2\Delta_3.$$

The order parameter $\Delta(x)$ increases monotonously from its minimal value $\Delta_{\min} = \Delta\left(-\frac{1}{2} l_2\right)$ to the maximal one $\Delta_{\max} = \Delta\left(\frac{1}{2} l\right)$, and for $\delta \to \delta_0$

$$\Delta_{\min} = \frac{2}{3} \sqrt{\frac{\delta - \delta_0}{\delta_0}} \tag{105}$$

and $\Delta_{\max} = \Delta_3$ increases slowly to $2/3\alpha$.
Let us now take into account the diffusion of quasiparticles, considering, for simplicity, the inverse limiting case $L \gg \xi_0$. It was shown (Elesin 1977a, Eckern et al. 1979, Elesin and Kopaev 1981), for the one-dimensional case, that the inclusion of quasiparticle diffusion does not qualitatively alter the structure of the inhomogeneous state. The width of the intermediate region $\check{\zeta}_0$ is replaced by L, and the parameter δ_0 changes slightly (by $5°_0$). According to Eckern et al. (1979) and to Elesin and Kopaev (1981), the pumping value at which the
phase energies become equal is

$$\delta_0(L) = \frac{15}{64\alpha} = \frac{2.5}{9\alpha} \tag{/}$$

In the two-dimensional and three-dimensional cases the problem is much more complicated and cannot be solved completely. Although assumptions have been made that the qualitative results must be the same (see Volkov and Kogan 1968), it is quite possible that geometrical factors may affect the structure of the inhomogeneous state.

The effect of spatial inhomogeneities of the original films and also of boundary conditions on the structure and stability of inhomogeneous states requires further investigation. In particular, the boundary condition corresponding to a superconductor-insulator interface has been used in eq. (98). In particle, however, this choice of the boundary condition is not unique. For example, under optical pumping current metal contacts are placed at the ends of the sample. A superconductor-metal interface requires, of course, another boundary condition.
1.2.9. Inhomogeneous unsteady state in superconductors with optical pumping In superconductors with optical pumping the frequencey ω is fixed ($R = 0!$). Therefore, according to eq. (100) a steady-state inhomogeneous solution exists for a single value of the pumping power δ_0. For $\delta \neq \delta_0$ the phase boundary will move at a velocity proportional to the difference $\delta - \delta_0$ (Elesin 1977a). In particular, for $L \ll \xi_0$ the phase boundary velocity is

$$v_c = \frac{(\delta - \delta_0)}{\delta_0} \frac{3\xi_0 \delta_0^{1/2}}{\tau\Delta} \tag{106}$$

In fact, we seek a solution of eq. (61) in the form of a travelling wave $\Delta(x - v_c t)$. The equation for $\Delta(x - v_c t)$ will differ from eq. (97) by the term

$$\frac{v_c \tau \Delta}{\xi_0} \frac{\partial \Delta}{\partial x}$$

which has the meaning of a friction force. Multiplying the modified equation by $\mathrm{d}\Delta/\mathrm{d}x$ and integrating it along the "motion trajectory", we find the velocity given by eq. (106). For $\delta > \delta_0$ a stationary wave is propagated, which causes the sample to pass from the superconducting

into the normal phase. On the contrary, for $\delta > \delta_0$ the wave transforms the normal phase into the superconducting phase.

In the opposite limiting case $L > \breve{\zeta}_0$, the phase boundary travels at a velocity (Elesin 1977a)

$$v_{\mathrm{c}} = \frac{L}{\tau_{\mathrm{ph}}} \frac{\delta - \delta_0(L)}{\delta_0(L)} \zeta \tag{107}$$

where ζ is of the order of unity. The motion of the boundary is similar, to a certain extent, to the motion of combustion wave boundaries in gaseous mixtures (or to a liquid-vapour interface). Indeed, a transition into the normal phase, accompanied by the production of excess quasiparticles (an analog of reaction heat), takes place at the interface, since the state with $\Delta < \Delta_2$ (for $\delta > \delta_0$) is unstable. The excess quasiparticles diffuse into the neighbouring region to reduce the gap there to $\Delta < \Delta_2$. In this region the transition into the normal phase occurs again and so on, that is, a wave of "combustion" of the superconducting phase propagates at a velocity $\sim L/\tau_{\mathrm{ph}}$. Obviously, after some time the sample gets filled with a single phase, so the inhomogeneous state under optical pumping can be observed in an unsteady regime.

In most of the experiments known to us the inhomogeneous state is observed (for example, as an increase in the resistance) in a pulsed, i.e. unsteady regime, the pulse duration τ_{u} being of the order of filling time. Therefore, Elesin (1977a) suggested to interprete a gradual transition, observed as a change in the resistance (and other phenomena) within the framework of the unsteady inhomogeneous state model. According to this model, the resistance appears after the normal phase (spanning the film over its width b) arises, the size of this phase growing at a velocity v_{c} (107). The value of the total film resistance by the end of the pulse, R_1, can be written as

$$\frac{R_{\mathrm{J}}}{R_{\mathrm{N}}} = \frac{v_{\mathrm{c}} \tau_{\mathrm{u}}}{l} = \frac{\tau_{\mathrm{u}} L \left(\delta - \delta_0(L)\right)}{\tau_{\mathrm{ph}} l \delta_0} \delta = \frac{\tau_{\mathrm{u}} L}{\tau_{\mathrm{ph}} l} \frac{\beta - \hat{\beta}_0}{2 \delta_0 \beta_{\mathrm{c}}} \zeta, \tag{108}$$

where R_{N} is the film resistance in the normal state, l is the film length, $\widetilde{\beta}_0 = (1 + \delta_0)^2 \beta_c$. It can be seen from eq. (108) that the resistance appears at a power exceeding β (which almost coincides with β_c, since $\delta_0 \ll 1$) and grows linearly with β. The time needed to span the width of the film gives rise to a time delay in the appearance of the resistance relative to the pumping pulse:

$$\tau_{\mathrm{delay}} = v_{\mathrm{c}}/b \tag{109}$$

This time delay must decrease with increasing power and temperature.
The model suggested permits a wide range of experimental facts to be explained, though it encounters certain difficulties. Comparison with experiment has been performed in a number of papers and reviews (see, for example, Tremblay 1981) and will be discussed below (section 1.2.11).

While making comparisons, one has to remember that the model does not allow for certain experimental complications. Some of them were already mentioned in section 1.2.8.

Special attention should be drawn to a possible peculiar "pinning" of phase boundaries at film inhomogeneities. Indeed, if the film contains regions, with a size of the order of L, with a higher value of the order parameter Δ_0 (and hence, with a lesser τ_{ph}), the motion of the phase boundary may cease until the power increases to $\bar{\beta}_0(r)$, the critical power in the inhomogeneity region.

It should be noted that the unsteady inhomogeneous state may be observed under electromagnetic pumping with $\omega > 2A$, and also under tunnel injection with a constant voltage.

29.3.9 1.2.10. The inhomogeneous state of superconductors with tunnel injection

Steady-state inhomogeneous states in the E-model are possible for a given tunnel injection current (Elesin and Levchenko 1978). The current determines the phase boundary.

In order to prove that the stability conditions (100) are satisfied, one has to find the current-voltage characteristic of the transition in the inhomogeneous state $\bar{J} = \bar{J}(V)$. This was done in the papers by Elesin et al. (1980a, b), in which the following expressions were obtained for the dimensionless current $\bar{J}$ and the derivative $\mathrm{d}\bar{J}/\mathrm{d}V$ for $V > V_0$:

$$\bar{J} = \alpha_V \left(V - 2\bar{A}^2/V\right) \tag{110}$$

$$\bar{\Delta}^2 = \Delta_3^2 - \frac{1}{l\sqrt{2\alpha\bar{b}}}\left[1 - 2\sqrt{b}\left(\Delta^{(1)} + \Delta^{(2)} + \Delta^{(3)}\right)\ln\left|\frac{2\sqrt{b}+b}{2\sqrt{b}-b}\right|\right] \tag{111}$$

$$\frac{\mathrm{d}\bar{J}}{\mathrm{d}V} = \alpha_V\left(1 - \frac{2^{2/3}\Delta_3}{\sqrt{\alpha}V_0 l\,(V - V_0)}\right). \tag{112}$$

The derivative $\mathrm{d}J\ \mathrm{d}\ V$ is positive everywhere (in particular for $V < V_\mathrm{c}$), except for the region $V - V_0 \ll V_0$ where it becomes negative and tends to $-\infty$. The current-voltage characteristic $J(V)$ of the SiS junction in the inhomogeneous state is shown in fig. 9 (section 1.2.5).

The stability conditions (100) imply that such a characteristic is unstable. In fact.

$$\text{if } J > J_\mathrm{th}\ (V < V_\mathrm{c})\,,\ \text{then} \quad \mathrm{d}J,\ \mathrm{d}V > 0$$

and the state is unstable for $V \leqslant V_\mathrm{c}$.
Consider a current-voltage characteristic of the type I. For $J > J_\mathrm{th}$ the voltage of the system decreases jumpwise (J is fixed!) to $V \geqslant V_0$, where $\mathrm{d}\bar{J}/\ \mathrm{d}V|_{\mathrm{v}\geqslant\mathrm{v}_0} < 0$, so the system is stable, according to eq. (100).

For long films stability is attained at $V = V_0$, so the inhomogeneous distribution is described by the laminated solution (102). The sample is laminated into phases $A_1 = 0$ and $\Delta_{30} \neq 0$.

As the current increases, the volume of the normal phase grows fand that of the superconducting phase diminishes) for a fixed voltage $I - I_0$ (the vertical portion of the $I - V$ characteristic).

For $J = J_0$ the system passes completely into the normal state. The corresponding $I - V$ characteristic is shown in fig. 9 by a solid line.

The inhomogeneous state described is possible for $I - V$ characteristics of types I and II only. For I-V characteristics of type III the system passes into
the normal state before the junction voltage reaches the value V_0, since $J_\mathrm{H} > J_0$. Therefore, for type-III characteristics the sample goes over jumpwise, at $J = J_\mathrm{H}$, from the homogeneous superconducting into the homogeneous normal state. However, as the current decreases the system passes, at $J = J_0$ and $V = V_0$, into the inhomogeneous state of the type described above.

A qualitatively similar behaviour may be expected in the two-dimensional case where current pinching occurs and the parameters of the system are of the order of Δ. Consider, for example, a solution in the form of a superconducting radially symmetric pinch of the radius ρ_0. If $\rho_0 \ll l$, the boundary conditions are satisfied automatically. The order parameter A varies from a certain value $\Delta = \Delta_\mathrm{max}$ at the centre of the pinch to $\Delta = 0$ at the periphery.

As in the case of superconductors with an S -shaped $I - V$ characteristic (Volkov and Kogan 1968), the superconducting pinch is stable at the descending portion of the $I - V$ characteristic in the given current regime, that is under conditions (100). It is a simple matter to verify

that the derivative $\mathrm{d}J/\mathrm{d}V$ becomes negative and tends to $-\infty$ as $V \to V_0$. Thus, the IV characteristic has a vertical portion at $V = V_0$.

29.3.10 1.2.11. Comparison of theory and experiment. Conclusion

The main predictions which follow from the models considered in this review, are listed in table 1. As can be seen from the table, the T^* - and BCS models are very close to each other, as far as certain properties are concerned. These models seem to be intermediate between the E- and NE-models, which appear to give opposite results with respect to properties 1, 2, and 4-8.

The properties 3,7 and 10 are common for the E- and NE-models.
All ten properties can be verified experimentally, to a certain accuracy. It seems useful to recall the temporal and spatial scales characteristic of experiments. The relaxation times of quasiparticles τ_{ph} near $\xi \sim A_0$ are of the order of $10^{-8} - 10^{-10}$ s. The length of typical light pulses is $10^{-7} - 10^{-5}$ s. This is enough for quasiparticles and phonons to relax to their steady-state value. Tunnel experiments are usually performed under steady-state conditions.

The typical lengths: the coherence length at $T = 0$ is $\xi_0 = (10^{-6} - 10^{-4})$ cm; the quasiparticle diffusion length $L = \sqrt{D\tau_{\mathrm{ph}}} = (10^{-4} - 10^{-3})$ cm; the approximate dimensions of the samples: width 10^{-1} cm, length 1 cm , thickness 10^{-5} cm.

Determination of the form of the QDF and $A(\beta)$ (properties 2-4). The number of works dealing with direct experimental determination of the QDF is rather scanty (Balkashin et al. 1977, Jaworsky and Parker 1979, Kaplan et al. 1977, Willemsen and Gray 1978). The last work is the most comprehensive; it is devoted to the determination of $n(\xi)$ in thin aluminium films with tunnel injection.

Willemsen and Gray processed their data with the help of the function (42).

Table 1
Main predictions of models considered.

Property		
	E-model	T*-model
(1) Kinetic equation for $n(\zeta)$	Satisfied for $\eta = 1$	Satisfied for η
(2) $n(\zeta)$	$n(0) < n_{\mathrm{T}}(0)$ overheating	Equilibrium w
(3) $n(\xi)$	$n < 1/2$	$n < 1/2$
(4) Form of $\Delta(\beta)$	two-valued	singlevalued
(5) Type of phase transition	1st-order	2nd-order
(6) Magnetic properties	$N_{\mathrm{S}} > 0$ diamagn.	$N_{\mathrm{S}} > 0$ diama
(7) Sign of supercon. current	$\sigma_{\mathrm{S}} > 0$	$\sigma_{\mathrm{S}} > 0$
(8) Instability	coherent	absent
(9) Inhomogeneous state	laminated: unsteady (ω = const.), steady (I = const.)	absent
(10) Critical power	$\beta_c(0) \neq 0, \beta_c$ drops with T	$\beta_c(0) \neq 0, \beta_c$

They showed that at a high junction voltage (corresponding to the wide-source criterion) the parameters μ^* and T^* aro nonzero and depend on V' (see fig. 4), the chemical potential μ^* being negative and T^* exceeding T_c. The ratio T^*/T_{c} is higher than unity and reaches 1.5, while $A_1 A_0 > 0.3$. These data as well as the dependence of μ^* and T^* on the voltage agree well (both qualitatively and quantitatively) with the predictions of the E-model (Elesin et al. 1979) (see section 1.2.1), indicative of "overheating" of quasiparticles (property 2). The
dependence $A(V)$ coincides with that shown in fig. 5. Note that the dependence of Δ on pumping power up to $V = V_{\mathrm{c}}$ agrees quantitatively with the corresponding dependence predicted by

the T^*-model. Therefore, the results of a number of papers exhibiting good agreement with the T^*-model can also be considered as confirming the E-model. (More careful measurements are needed to distinguish the E-model from the T^*-model.) One has to remember that the results presented usually refer to Δ values below a certain Δ_c, and then the transition into the normal or inhomogeneous state occurs (properties 4, 5). [Willemsen and Gray (1978) report $\Delta_{c'}\Delta_0 = 0.3$, in the paper by Lambert et al. (1977) $\Delta_c/\Delta_0 = 0.3$, etc.]

Lambert et al. (1977) determined $n(\check{\zeta} = 0)$ by ultrasonic absorption measurements. The corresponding dependence is presented in the paper by Elesin and Kopaev (1981) and illustrates good agreement with the theory. It is also shown that $n(0) < \frac{1}{2}$ (property 3).

Fuchet et al. (1977) showed that $n < \frac{1}{2}$ and $\Delta_c/\Delta_0 = 0.6$.
Thus, experiment confirms properties (2-4) of the E-model.
Inhomogeneous state under optical pumping. A rather detailed comparison between theory and experiment was performed by Tremblay (1981) who concluded that the E-model permits a consistent and also the most complete explanation of some experimental results.

At the same time, certain problems arose, first of all the problem of stationarity. The fact is that the E-model predicts a change in the film resistance with time (see section 1.2.8). Yet, some experiments revealed stationarity, as reported, for example, in the paper by Sai-Halasz et al. (1974). The reflection coefficient does not change during the laser pulse $\tau_{\mathrm{u}} = 10^{-8}$ s.

Such stationarity was also observed by Golovashkin et al. (1975) for short laser pulses, while for long pulses ($\tau_{\mathrm{u}} > 10^{-6}$ s) those authors found that the film resistance increased with time, in accordance with the E-model, and suggested an explanation of the stationarity observed.

According to Mitsen (1981), the stationarity is caused by the small duration of the laser pulse. The distance travelled by the normal phase during the pulse is very small, so the increase in the resistance cannot be detected.

Below we shall describe the results reported by Golovashkin et al. (1975. 1978, 1979a,b, 1980), which are interpreted by those authors within the framework of the E-model, and also of certain more recent works concerning optical pumping.

Many important results have been obtained by Golovashkin et al. (1975, 1978. 1979a,b, 1980) who studied the behaviour of lead, niobium. Nb_3Sn and V_3Si films irradiated by pulses of semiconductor, argon and neodymium lasers. At powers W exceeding a certain critical value $W_c(T)$ (dependent on temperature) a gradually increasing resistance appeared that gave rise to a measurable voltage V. Figure 11 shows the typical time dependence of V across

the lead film for different temperatures, and also the trace of the laser pulse with the length 10μ s; $W = 30$ W/cm^2. It can be seen from the figure that the signal across the sample appears at a temperature much lower than $T_6 = 7.2$ K. As the temperature increases, the signal amplitude grows to reach the value V_N across the film in the normal state.

During the laser pulse, the voltage across the sample is not constant, but increases with time reaching a maximum by the end of the pulse. Golovalshkin et al. have estimated, using the dependence $V(t)$, the velocity of the normal phase boundary to obtain the value $v_{\mathrm{c}} = 2.5 \times 10^4$ cm/s $(W/W_{\mathrm{c}} = 1.5)$, which agrees within an order of magnitude with eq. (107).

Using the value of the velocity v_{c}, one can find a minimal pulse length τ_0 at which "stationarity" is still observed

$$\tau_0 = \frac{0,1\bar{d}}{v_c} = 10^{-6} \text{ s},$$

where $\bar{d} = 0.1$ cm is the characteristic size of the normal region. It is clear from this point of view that for $\tau_{\mathrm{u}} = 10^{-8}$ s "stationarity" had to be observed by SaiHalasz et al. (1974). The signal delay $\delta\tau$ with respect to the beginning of the laser pulse is an essential effect. It follows from fig. 11 that $\delta\tau$ drops linearly with increasing T (or W), in accordance with the prediction

of the E-model (see properties $1, 2, 4$). Substitution of $b = 10^{-1}$ cm and $v_{\mathrm{c}} = 2.5 \times 10^5$ cm/s into eq. (109) yields $\delta\tau = 4\mu$ s, which agrees with experiment.

The growth of the resistance with time and the signal delay effect confirm properties 8,9 of the E-model.

Figure 12 shows the typical voltage-power dependence $V(\tau_{\mathrm{u}})$. It can be seen that the film resistance at the end of the pulse increases with power, in accordance with eq. (108). The dependence $V(W)$ (fig. 11) is of threshold character. The film passes into the resistive state for $W > W_c$.

A detailed comparison of the dependences of W_{c} on T and film thickness d with the existing models was performed by Golovashkin et al. (1979a,b, 1980) (see fig. 13a,b). They observed not only qualitative, but also good quantitative agreement with the E -model (property 10). The μ^*-model leads to qualitatively different dependences (Golovashkin et al. 1979a, b, 1980).

Inhomogeneous states with similar properties have been discovered by Janik et al. (1975) and by Buisson et al. (1978).

Buisson et al. (1978) observed the occurrence of a resistive state in Al films ($d = 500$ and 800Å) irradiated with pulses of a dye laser with a duration of 6 ns and energy (in the pulse) 0.5μ J. Dynes et al. (1977) irradiated aluminium films of $d = 200$Å by an argon laser with a maximal power of 0.5 W , and measured the change in the transmission coefficient at microwave frequencies (100 GHz) under optical pumping. They assumed that an intermediate state in the form of

superconducting and normal phases occurred in the film, the normal phase volume increasing with W.

Smith (1980) studied inhomogeneous states in a uniform geometry [for details see the review by Tremblay (1981)].

We have not come across publications concerning direct experimental verification of diamagnetism (property 6) and the sign of the superconducting current (property 7). But we are neither acquainted with communications on these anomalies, which are easy to observe. As far as the effect of current is concerned, several papers reported a decrease in the critical power W_{c} with increasing J, in accordance with the predictions of the E-model (see section 1.2.3).

In summary, one may conclude that the E-model explains the main experimental facts. The T^* - and μ^*-models have been discussed in detail by Tremblay (1981). We note only that the T^*-model agrees qualitatively with the E-model for powers W below the critical value W_{c}.

The experimental data presented above are at variance with the μ^*-model for properties $2, 3, 8, 9, 10$ and, probably, 6 and 7 . It should be noted that the μ^* model is difficult to compare with experiment. The fact is that the μ^*-model does not provide an expression for experimentally determined W_c, but gives the value n_c at which instability occurs, and this value n_c does not correspond to the moment when the resistance appears (Tremblay 1981).

Tunnel injection at high voltages. According to the E-model, the injection of quasiparticles in SiS junctions at $V >\Rightarrow$ must lead to a multi-valued dependence of Δ and I on V, and also to a first-order transition into the normal or inhomogeneous state (see section 1.2.10).

The E-model predicts these phenomena, in principle, for any α_V. It should be noted, however, that they are difficult to observe for very small α_{v} and, correspondingly, large V. Indeed, in this case the S-shaped portion of the IV characteristic is rather difficult to detect against the high-voltage background, and moreover intensive heat release ($\sim 1/\alpha_{\mathrm{V}}$) masks the effect. The values $\alpha_{\mathrm{v}} \gtrsim 0.1$ are the most suitable for observation.

Tunnel injection for $V \gg A$ was studied in a number of works (see the reviews by Tremblay 1981 and by Elesin and Kopaev 1981), but we shall concentrate here on the most typical ones.

Junctions with double injection (e.g. Al-I-Al-I-Al) of the size $10^{-1} \times 10^{-1} \times 10^{-5}$ cm are usually employed in experiment.

One of the junctions with "high" resistance ($10^{-2} - 10^{-4}$Ohmcm^2) acts as a detector and

controls the properties of the middle sample perturbed by the injector junction with a low resistance ($10^{-4} - 10^{-6}$Ohmcm^2). Most tunnel experiments are carried out under steady-state conditions. Since the total contact resistance is small, the given current regime is practically always realized in these experiments.

First-order transitions into the normal state were observed by Fush et al. (1977) in tin films for $T < T_\lambda$, and also by Mitsen (1979) ($T > T_\lambda$) when the current exceeded a certain critical value. Their results are not at variance with the E-model, which predicts the first-order transition for type-III IV characteristics (see section 1.2.10), and also with the μ^*-model. However, Fush et al. (1977) believe that the attainment of the film critical current could be the cause of the transition.

The inhomogeneous state was observed by Iguchi (1977) at a temperature $T > T_\lambda$ (as a gradual change in the resistance).

A transition into the inhomogeneous state very close to that predicted by the E-model was observed by Mitsen (1979) and by Pan and Rudenko (1981).

Mitsen (1979) studied the junction Sn – I – Sn. Its current-voltage characteristic is shown in fig. 14. Curves 1 and 2 refer to different values of α_{V}. At a certain injection current J_{th}, to which the voltage V_{H} corresponds, a negative voltage jump to $V = V_0$, followed by a vertical portion of the $I - V$ characteristic, is observed across the junction. (The currents J_{th} are less than the critical ones.) As the current decreases, hysteresis is observed. The results obtained are in good qualitative agreement with the E-model (see section 1.2.10) and correspond to $I-V$ characteristics of types I or II. In particular, $V_{\mathrm{H}}, J_0, J_{\mathrm{th}}$ grow with the decreasing α_{v}. Mitsen has estimated, using experimental data, the critical concentration of quasiparticles from the relation

$$\frac{N_{\mathrm{c}} v'}{\tau_{\mathrm{ph}}} = \frac{J_{\mathrm{th}}}{e}$$

to obtain $N_{\mathrm{c}} = 0.4 \times 10^{19}$ cm^{-3}, which is in good agreement with the theoretical estimate (see eq. 37) for tin

$$N_{\mathrm{c}} = v(0) a_0 = v(0) \Delta_0 \cdot 0.4 = 0.31 \times 10^{19} \text{ cm}^{-3}.$$

Note that Mitsen's 1979 paper contains a misprint; the theoretical formula used therein overestimates N_{c} by a factor of 4 .

According to Mitsen (1979) heating, which occurs in such experiments, does not affect the results.

Pan and Rudenko (1981) investigated the $I - V$ characteristic of the junction Nb – I – Nb (see fig. 15). For $J > J_{\mathrm{th}}$, a voltage jump across the junction was observed, and a voltage appeared across the Nb. The voltage increased linearly with current, indicative of a transition into the normal state at $J = J_{\mathrm{th}}$. As the current was decreased, a vertical portion (i.e. the inhomogeneous state) appeared (see fig. 15a).

This behaviour corresponds to the third-order transition in the E-model. Pan and Rudenko (1981) measured the temperature dependence $J_{\mathrm{th}}(T)$ (see fig. 15b) and showed that it agrees qualitatively with $\beta_c(T)$, that is, with the prediction of the E-model. Using a sophisticated technique, they eliminated the effect of overheating on the dependence $J_{\mathrm{th}}(T)$. Otherwise, experiment reveals a nonmonotonous dependece $J_{\mathrm{th}}(T)$ similar to that observed by Iguchi (1977). In this respect, the work of Iguchi can also be interpreted within the framework of the E-model, especially if allowance is made for the fact that Iguchi (1977) observed the inhomogeneous state at temperatures up to T_c, whereas according to the μ^*-model diffusion instability disappears at $T > 0.6T_{\mathrm{c}}$.

There are works, besides the aforementioned ones, which neither fit the Emodel. For example, Iguchi et al. (1981) state that they have observed an inhomogeneous state with four

gaps.

Thus, one may conclude that at present the E-model is the most satisfactory, both from the theoretical and from the experimental point of view. This model represents the kinetic theory of superconductors with excess quasiparticles.

At the same time, one has to remember that the E-model does not take into account experimental complications associated with inhomogeneities of films and radiation, with heating, and so on. Therefore, only detailed quantitative comparison between theory and experiment, with due account for specific experimental conditions, can decide between one model or another.

29.3.11 1.3. Superconductors with a narrow quasiparticle source. Threshold electromagnetic pumping and tunnel injection in Single-band approximation

If the frequency of the electromagnetic field (or the voltage V) is close to the threshold, the state of a superconductor appears to be essentially nonequilibrium. The QDF differs significantly from the quasiequilibrium one (especially for $T = 0, \hat{i} = 0$) and grows with ξ. The dependence of the gap on

frequency (voltage) becomes three-valued. The state is unstable against "threshold instability", which may lead to a first-order transition of the superconductor from the state with Δ_0 into the state with lesser gap A_3 or into the inhomogeneous state. The latter represents an alternation of phases with Δ_0 and A_3, different from zero. In other words, a superconductor with threshold injection exhibits a behaviour very close to that described by the E-model for a wide source. Therefore, let us consider the basic phenomena, using the analogy just mentioned. The theory of a narrow source was developed in papers by Elesin (1974, 1979b), Aronov and Spivak (1976), Schön and Tremblay (1979), Elesin et al. (1980a, b), Gray and Willemsen (1980), and Hida (1978a, b), and inhomogeneous states with two values of the gap were observed experimentally by Dynes et al. (1977) and by Gray and Willemsen (1978).

29.3.12 1.3.1. The quasiparticle distribution function

Let us confine ourselves to the case $\gamma = 0 (T = 0)$. The kinetic equation for a narrow source can be simplified significantly if one takes into account that scattering processes are strongly suppressed as compared to recombination, due to the factors $(\varepsilon - \varepsilon')^2$ and coherence factors $(1 - \Delta^2/\varepsilon\varepsilon')$. At $T = 0$ when the terms $Q^{(2,3)}$ describing the energy redistribution of quasiparticles can be neglected in the expression for the source, we arrive at the following equation (Elesin 1974, Aronov and Spivak 1976):

$$n_z \bar{n} = \alpha_\omega (\hat{\delta})^{K_{(\omega)}} \frac{1 - n_z - n_{1-z}}{\sqrt{1-z}} \Theta(z)\Theta(1-z) \tag{113}$$

where

$$\bar{\delta} = \frac{\omega - 2\Delta}{2\Lambda}, \quad z = (\varepsilon - \Delta)/\Delta\bar{\delta}$$

In the case of tunnel injection, $\alpha_V \bar{\delta}^{-1/2}$ must be substituted for $\alpha_\omega \bar{\delta}^{-1/2}$ because the coherence factor is absent (see eq. 21).

Equation (113) shows that the shape of the source has a major effect on the QDF. Aronov and Spivak (1976) have obtained an exact solution of eq. (113):

$$n_z = A \frac{\sqrt{z}\Theta(z)\Theta(1-z)}{\sqrt{z(1-z)} + A(\sqrt{z} + \sqrt{1-z})} \tag{114}$$

where the coefficient A satisfies the normalization condition. It can easily be seen that $n(z)$ is different from zero in the interval $0 < z < 1$ and takes the values $n(0) = 0$ and $n(1) = 1$, i.e., in the case of a narrow source the distribution function differs significantly from the equilibrium one, has a positive derivative, and can reach unity. The coefficients $A_{(1)}, A_{\vee}$ appear to be essentially different. For example, for small $\alpha_{(1)} A_{(1)}$ is also small, while A_V may become large as $\widehat{\delta} \to 0$. Therefore, the QDF for electromagnetic pumping is nonzero in a

narrow interval near $z = 1$, and for tunnel injection n_z is of the order of unity almost in the whole interval $0 < z < 1$. An exact solution for n_z has been found for $T \to T_{\mathrm{c}}$ as well, and it can be represented in the form (Elesin et al. 1980a)

$$n = n_{\mathrm{T}} + n_2 + \hat{n}$$

where n_2 is a smooth function due to the action of $Q^{(2)}$, and the "narrow" part $\hat{n}$ is given by

$$\hat{n} = n_z(T = 0)\psi_{\mathrm{T}}, \quad \psi_{\mathrm{T}} = 1 - n_{\mathrm{T}}(z) - n_{\mathrm{T}}(1 - z) \tag{115}$$

The correction $\hat{n}$ possesses the property (114). If reabsorption is taken into account, the QDF becomes smoother and tends to the equilibrium one in the limit $\gamma \to \infty$. Figure 16, taken from a paper by Imai (1980), shows the QDF for $\gamma = 1, 10$.

29.3.13 1.3.2. The order parameter

A narrow source produces quasiparticles with energies in a narrow interval $\omega - 2\Delta$, so the quasiparticles are localized near $\varepsilon \sim A$. The local character of $n(\xi)$ makes it possible to find, by virtue of eq. (6), a universal relationship
between A and $\bar{n}$, the quasiparticle concentration (see, for example, Elesin 1977b):

$$\Delta = \Delta_0(1 - 2\bar{n}), \quad \bar{n} \ll 1. \tag{116}$$

In order to obtain a closed system for Δ and $\bar{n}$ in the case of small α_ω, it is sufficient to use the normalization condition (18). Taking into account the local character of $n(\xi)$, we arrive at the relation (Elesin 1974)

$$\bar{n}^2 = \alpha_{()} \frac{\omega - 2A}{2\Delta_0} \Theta(\omega - 2\Delta) \quad (\text{ at } T = 0) \tag{117}$$

which expresses the equality of the quasiparticle recombination rate (the lefthand side) and the rate of production of quasiparticles due to the electromagnetic field (since the $\alpha_{()}$are small, the $n(\zeta)$ in Q are put equal to zero). It is important to note that the production rate is proportional to the interval $\omega - 2\Delta$ and depends significantly on A. In other words, in the case of a narrow source the order parameter controls the quasiparticle production rate. This leads, as was shown by Elesin (1974), to a characteristic threshold instability (see section 1.3.3) and a multi-valued dependence of Δ on ω (Elesin 1979b, Schön and Tremblay 1979).

Indeed, eliminating $\bar{n}$ from eqs. (115) and (116) we obtain an equation for A whose solution in the interval $-\alpha_\omega < \delta < 0$ is of the form

$$\Delta_5 = \Delta_0, \quad \Delta_{4,3} = \Delta_0 \left[1 - 2\alpha_\omega \left(1 \mp \sqrt{1 + \delta/\alpha_\omega}\right)\right]. \tag{118}$$

Three solutions, Δ_5, Δ_4 and Δ_3, exist in this interval.
The existence of three solutions is related to the fact that Δ influences the quasiparticle production rate. The solution $A_5 = A_0$ corresponds to the case where the source is switched off ($\omega < 2\Delta_0$). If, on the contrary, $\Delta = \Delta_{4.3} < \Delta_0, \Delta$ decreases and the source produces

quasiparticles; then solutions increasing with ω and solutions, Δ_4, decaying with ω are possible. Qualitatively similar results are also obtained under tunnel injection ($\omega \to V$):

$$\Delta_5 = \Delta_0, \quad \Delta_4 = V/2, \quad \Delta_3 = \Delta_0 \left(1 - 2\sqrt{\frac{1}{2}\pi\alpha_V}\right) \tag{119}$$

But now the solution Δ_3 is constant and Δ_4 depends "universally" on the voltage (see fig. 17a). This relation is different from eq. (118) because the coherence factors are absent in the source for tunnel injection.

Note that multiplicity of $\Delta(\omega)$ occurs for any α_ω. The conclusion on the existence of three solutions $\Delta(V)$ (Elesin 1979b) was simultaneously made also by Schön and Tremblay (1979). They solved graphically the equation for Δ for $T \to T_c$. The case of finite temperatures was also considered by Elesin et al. (1980c), where the contribution due to the terms $Q_v^{(2)}$ and $Q_v^{(3)}$ was taken into account.

The multiplicity of $\Delta(\omega)$ and $\Delta(V)$ must lead to a multi-valued dependence of the absorbed electromagnetic power and of current through the tunnel junc-

tion $J(V)$ [for details see Elesin and Kopaev (1981)]. Indeed, it was shown by Elesin (1979b) and by Schön and Tremblay (1979) that the IV characteristic is S-shaped near $V = 2A$.

Figure 17 shows the dependence $\Delta(V)$ and the $I - V$ characteristics in the entire voltage range for different α_V, as calculated analytically (Elesin 1979b, Schön and Tremblay 1979) and numerically (Elesin et al. 1980a, b). The second portion of the S-shaped $I - V$ characteristic is due to the coherent mechanism (see section 1.2). Qualitative concepts on IV characteristics with a negative slope at $V = 24$ have been used by Gray and Willemsen (1978).

29.3.14 1.3.3. Threshold instability

Elesin (1974) predicted a type of instability, for the case of a narrow source ($\omega \approx 24$), that was called "threshold instability". This occurs as follows. Suppose that at an initial moment $\Delta = \Delta_0, \bar{n} = 0$ and $\omega \leqslant 2\Delta_0$. A fluctuation of $\bar{n}$ leads to a decrease in Δ (see eq. 116) and, hence, to an increase in the width of the source action, which is proportional to $\omega - 24$ (see eq. 117). This, in turn, increases n, and so on. A similar instability can arise in superconductors with tunnel injection (Elesin 1979b, Schön and Tremblay 1979) for $V \leqslant 24$, and was observed experimentally (Dynes et al. 1977, Gray and Willemsen 1978).

To find the decrement, we shall use the equation for Δ, the deviation of Δ from A_0, which coincides with the quasi-particle concentration $\bar{n}$ (see eq. 116) (Elesin 1979, Elesin and Kopaev 1981) ($L \gg \zeta_0$ at $T = 0$)

$$\bar{\Delta} = \frac{\Delta_0 - \Delta}{2\Delta_0} = \bar{n}, \quad \frac{\partial \bar{n}}{\partial t} - L^2 \frac{\partial^2 \bar{n}}{\partial r^2} = \frac{\partial U(\bar{n}, \delta)}{\partial \bar{n}}$$
$$U(\bar{n}, \delta) = -\int_0^{\bar{n}} \alpha \bar{n}' \left(2\bar{n}'^2 - \bar{Q}_\omega\right), \quad \bar{Q}_\omega = \alpha_\omega(\delta + 2\bar{n})\Theta(\delta + 2\bar{n}) \tag{120}$$

Here $U(\bar{n}(\Delta))$ is the energy of the system, similar to that in the E-model, which has minima at Δ_5 and Δ_3 and a maximum at Δ_4. The corresponding decrement is (Elesin 1979b)

$$\tilde{\gamma} = \frac{1}{2\Delta} \frac{\partial^2 U}{\partial \Delta^2} - q^2 L^2 = \alpha_\omega \left(\frac{\partial \Delta}{2\Delta \partial \delta}\right)^{-1} - q^2 L^2 \tag{121}$$

This relation implies that the increasing solution Λ_4 (corresponding to the maximum of $U(\bar{n}(\Delta))$ is unstable, and quasiparticle diffusion stabilizes the perturbations. Thus, the situation is completely analogous, in qualitative respect, to that in the E-model (see sections 1.2.6 and

1.2.8) if we make the substitutions $A_1 \to A_5, A_2 \to A_4, A_3 \to A_3$, so we can use the above results.

In particular, one is to observe first-order transitions $A_5 = A_0 \to A_3$ at $\delta = \delta_0$, hysteresis ($\Delta_3 \to \Delta_0$ at $\delta = -\alpha_\omega$) or the inhomogeneous state.

29.3.15 1.3.4. Inhomogeneous states

Threshold instability leads to an inhomogeneous state, which is described by the steady-state equation (119).

The steady-state inhomogeneous solution is laminated, it separates the homogeneous regions with $\Delta_5 = \Delta_0$ and $\Delta_3 \neq 0$ and is realized at a certain frequency ($_0$ or voltage V', [which corresponds to the equality $U(A_3) = U(A_5)$]. For an electromagnetic field the laminated solution is of the form (Elesin 1979b)

$$\bar{\Delta}(x) = \begin{cases} 3\left[x - x_0 - 4\sqrt{2/\alpha(1)}\right]^2, & x \leqslant x_0 \\ \bar{\Delta}_{30} \tanh^2\left[\sqrt{\alpha\omega/2}\,(x - x_0) + C_2\right], & x > x_0 \end{cases} \tag{122}$$

where $\tanh C_2 = \frac{1}{2}, x_0$ is the coordinate of the transition boundary, and

$$\nearrow_{30} = \frac{3\alpha_\omega}{2}, \quad \frac{\omega_0}{2A_0} = 1 - \frac{3\alpha_\omega}{4}. \tag{123}$$

Qualitatively similar results are also obtained for tunnel injection. In particular, the quantities V_0 and A_{30} are given by (Elesin 1979b)

$$\frac{A_0 - A_{30}}{\Delta_0} = \sqrt{\pi\alpha_{\mathrm{V}}}, \quad \frac{V_0}{2A_0} = 1 - \sqrt{\frac{\pi\alpha v}{4}}. \tag{124}$$

Analogous results have been obtained by Gray and Willemsen (1978), Hida (1978), and Schön and Tremblay (1979).

The $I-V$ characteristics of an SiS junction resemble those considered in section 1.2. Indeed, for currents J exceeding J_{th} ($\mathrm{d}J/\mathrm{d}V < 0$) threshold instability arises, which leads to lamination into pinches with Δ_5 and Δ_3, and is accompanied, by virtue of eqs. (100) and (125), with a jumpwise voltage decrease to V_0. A laminated solution with $\Delta_{50} = \Delta_0$ and Δ_{30} is formed. The position of the phase boundary x_0 is determined by the total current.

As the current grows further, the volume of the phase with Δ_{30} increases at a constant voltage V_0 (vertical portion of the $I - V$ characteristic), until the sample is completely filled with the phase A_{30}.

29.3.16 1.3.5. Comparison with experiment. Conclusion

The model considered agrees well with experimental data concerning the observation of the "two-gap state" under threshold tunnel injection (Dynes et al. 1977, Gray and Willemsen 1978).

It explains the form of the $I - V$ characteristics for a "detector" and a "generator" (see fig. 18), in particular, the vertical portion at $V = V_0$, jumps and hysteresis of the generator $I - V$ characteristic, and the form of the $I - V$ characteristic of the "detector" (lamination of the phase with A_{30}). The model describes quantitatively the relationship between the difference $\Delta_0 - \Delta_{30}$ observed and the injection coefficient α_V and V_0 (see eq. 124); it also explains the non-threshold character of the effect, and so on.

It is believed at present that experiment is described well, at least qualitatively, by the theory considered above (Tremblay 1981, Elesin and Kopaev 1981).

29.4 2. Effect of electromagnetic field on superconductors in the two-band approximation

If the field frequency ω exceeds the energy gap, the valence band should be taken into account besides the conduction band. The theory of such an effect was developed by Galitskii et al. (1969, 1973), Elesin and Kopaev (1972, 1974), Elesin (1975), and Kirzhnits and Kopaev (1973).

29.4.1 2.1. The electron spectrum of a crystal in a strong electromagnetic field

Consider a crystal in the field of a strong electromagnetic wave with the vector potential

$$\mathscr{A} = \mathscr{A}_0 \cos(\omega t - kr), \quad k\mathscr{A}_0 = 0. \tag{125}$$

Suppose that the frequency ω exceeds the energy gap between the valence and conduction bands, which have a maximum and a minimum, respectively, at the centre of the Brillouin zone.

In this case the field induces resonant electron transitions between the valence band and the conduction band, which satisfy the condition $E_{\mathrm{c}}(p) - E_{\mathrm{v}}(p+k) - \omega = 0$. If the photon wave vector k is neglected in comparison with the electron quasimomentum p, the resonant transitions will occur with conservation of the quasimomentum (direct transitions).

Assuming the field to be sufficiently strong [see below for the corresponding condition, eq. (127)], we can include the resonance portion of the interaction with electromagnetic field into the zero-order Hamiltonian, whose eigenstates can be determined exactly (Galitskii et al. 1969):

$$H_0(t) = \sum_p E_p \left(a_p^+ a_p + b_{-p}^+ b_{-p}\right) + \lambda_p \left(a_p^+ b_{-p}^+ \mathrm{e}^{-\mathrm{i}\omega t} + b_{-p} a_p \mathrm{e}^{\mathrm{i}\omega t}\right) \tag{126}$$

Here $a_p^+$and $b_{-p}^+$are the operators of production of an electron in the conduction band and a hole in the valence band, respectively,

$$\lambda_p = \frac{1}{2} e A_0 v_{\mathrm{cv}}(p), \quad v_{\mathrm{cv}}(p) = \frac{1}{m} \int \psi_{\mathrm{c}p}^* \hat{p} \psi_{\mathrm{v}p} \, \mathrm{d}^3 r,$$

v_{cv} is the matrix element of the interband transition rate, and ψ_{cp} and ψ_{vp} are the Bloch functions of electrons in the conduction and valence bands.

The use of the Hamiltonian (126) as the zero-order approximation is valid if the frequency of transitions between the bands due to the field is much higher than the collision frequency $1/\tau$

$$\lambda \gg 1/\tau, \tag{127}$$

where by τ we mean the smallest among the electron-phonon collision time, electron electron collision time and annihilation time τ_A. The Hamiltonian $H_0(t)$ depends on time. Using the unitary transformation (Galitskii et al. 1969)

$$U(t) = \exp\left\{-\frac{1}{2} i\omega t \sum_p \left(a_p^+ a_p + b_{-p}^+ b_{-p}\right)\right\} \tag{128}$$

it is convenient to go over to a representation in which the Hamiltonian is time independent:

$$
\begin{aligned}
H_0 &= U^+(t)H_0(t)U(t) - \mathrm{i}U^+\frac{\partial U}{\partial t} \\
&= \sum_p \left[\xi_p\left(a_p^+a_p + b_p^+b_p\right) + \lambda(p)\left(a_p^+b_p^+ + b_pa_p\right)\right], \qquad (129)\\
\xi_p &= \frac{1}{2m}\left(p^2 - p_0^2\right), \quad \frac{p_0^2}{2m} = \mu_0 = \frac{\omega - E_{\mathrm{g}}}{2}.
\end{aligned}
$$

The canonical transformation

$$
a_p = u_p\alpha_p + v_{-p}\beta_{-p}^+, \quad b_{-p} = u_p\beta_{-p} - v_{-p}\alpha_p^+ \qquad (130)
$$

reduces the Hamiltonian (129) to the diagonal form

$$
H_0 = \sum_p \varepsilon_p\left(\alpha_p^+\alpha_p + \beta_{-p}^+\beta_{-p}\right). \qquad (131)
$$

The ordinary anticommutation relations for Fermi particles hold true for the operators of new quasiparticles, and the functions u_p, ι_p and the quasiparticle energy are given by

$$
\left.\begin{matrix} u_p^2 \\ v_p^2 \end{matrix}\right\} = \frac{1}{2}\left(1 + \frac{\xi}{\varepsilon_p}\right), \qquad (132)
$$

$$
\varepsilon_p = \sqrt{\xi_p^2 + \lambda_p^2}. \qquad (133)
$$

The operators $\alpha_p\left(\alpha_p^+\right)$and $\beta_p\left(\beta_p^+\right)$are the parameters of annihilation (production) of quasiparticles of species α and β, which describe the superposition of electrons and holes.

The quasiparticle spectrum has a gap at $p = p_0$. Thus, an electromagnetic field of a frequency exceeding the energy gap leads to an essential dynamic effect, namely the occurrence of a gap in the energy spectrum of electrons of the conduction and valence band.

This result was first derived by Galitskii et al. (1969) when they considered the interaction of a strong resonance field with pure semiconductors (i.e. semiconductors in which the concentration of conduction band electrons tends to zero for $T \to 0$). Obviously, this result is also valid for doped semiconductors and metals.

29.4.2 2.2. The quasiparticle distribution function

Consider now the kinetics of quasiparticles, described by the quasiparticle distribution functions

$$
n_\alpha = \left\langle\alpha_p^+\alpha_p\right\rangle, \quad n_\beta = \left\langle\beta_p^+\beta_p\right\rangle \qquad (134,135)
$$

The equations for these functions have been derived by Galitskii et al. (1969) using the Bogolyubov method and by Elesin (1970a) using the Keldysh technique. These equations differ from eqs. (12) and (13 15) by the sign of the coherence factors and by the presence of a collision term accounting for annihilation between electrons and holes

$$
I_{\mathrm{A}} = \frac{1}{\tau_{\mathrm{A}}}\left[v_p^4\left(1 - k_p\right)^2 - u_p^4n_p^2\right], \quad \frac{1}{\tau_{\mathrm{A}}} = \frac{4}{3}e^2\omega\left|v_{\mathrm{cv}}\right|^2. \qquad (136)
$$

First we consider the simplest case where the concentration of equilibrium electrons is small as compared with that of electrons produced by the field ($E_{\mathrm{F}} \to 0$). If annihilation (136) is disregarded (usually $\tau_{\mathrm{ph}} \ll \tau_{\mathrm{A}}$), the QDF for this case is (Galitskii et al. 1969)

$$n_\alpha(p) = n_\beta(p) = n_{\mathrm{T}} = \left[\exp\frac{\varepsilon}{T} + 1\right]^{-1}. \tag{137}$$

At zero temperature $n_p = 0$ and the so-called saturation state is reached. In this state the energy distribution of electrons and holes

$$f_p = \left\langle a_p^+ a_p \right\rangle = \left\langle b_p^+ b_p \right\rangle = v_p^2 \tag{138}$$

has the form of the Fermi step smeared out near the boundary energy μ_0 by the value i. (it is assumed that $i \ll \mu_0$). In other words, the electrons (holes) occupy the portion of the conduction (valence) band from $p = 0$ to $p = p_0$. This is easy to explain.

Indeed, the law of conservation of momentum implies that electrons with momentum p_0 must pass from the valence band into the conduction band.

When interacting with the lattice, the electrons emit phonons and occupy the states with lesser momenta up to p_0.

In the saturation state the absorption coefficient of the electromagnetic field vanishes (Galitskii et al. 1969). A nonzero value of the absorption coefficient is obtained if annihilation (136) is taken into account. An allowance for annihilation, by virtue of the condition $\tau_{\mathrm{A}} > \tau_{\mathrm{ph}}$, changes n_p in a small region near $\xi = 0$, this change being dependent on the parameter (Elesin 1975)

$$\eta_{\mathrm{A}} = \frac{\tau_{\mathrm{ph}}}{\tau_{\mathrm{A}}}\left(\frac{\mu_0}{\lambda}\right)^2 \tag{139}$$

and on the ratio $2\lambda/\omega_{\mathrm{D}}$.

Solutions of the kinetic equation for quasiparticles have been studied in detail for both steady-state and unsteady cases (Elesin 1970a, 1975, Galitskii and Elesin 1973, Galitskii et al. 1978).

For example, in the case $\lambda \ll \omega_{\mathrm{D}}, \eta_{\mathrm{A}} \ll 1, n(\xi)$ is no longer an even function and can be represented in the form

$$n(\xi) = \begin{cases} v_p^2, & -a_0\,(\omega_{\mathrm{D}} < \xi < \infty \\ 0, & \xi < a_0\omega_{\mathrm{D}}, \quad a_0 = \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi n(\xi)'}{\omega_{\mathrm{D}}} \end{cases} \tag{140}$$

where a_0 is proportional to the annihilation rate.

If $2\lambda > \omega_{\mathrm{D}}, \eta_{\mathrm{A}} \ll 1$, the solution of the kinetic equation will be (Elesin 1975)

$$n(\varepsilon) = \left[\exp\frac{\varepsilon - \mu^*}{T} + 1\right]^{-1}, \quad n(\varepsilon) = \begin{cases} 1, & \varepsilon < \mu^* \\ 0, & \varepsilon > \mu^*, \end{cases} \quad T = 0, \tag{141}$$

where $\mu^*(0)/\mu_0 = \tau/3, T$ is the lattice temperature.

An inverse population exists in the conduction (valence) band. This form of the distribution is related to the fact that the gap $2\lambda > \omega_{\mathrm{D}}$ forbids electron transitions (the rate of single band recombination is zero) into the state with $\xi < 0$ from the state with $\xi > 0$, where the electrons are accumulated due to indirect virtual transitions in the strong field (cf. the μ^*-model).

If we take into account the presence in the conduction band of equilibrium electrons with concentration n_d, we have to use the kinetic equation for each QDF $n_\alpha = n_\beta$; the QDFs are now related by

$$n_{\mathrm{d}} = \int \frac{\mathrm{d}^3 p}{(2\pi)^3}\left(n_\alpha(p) - n_\beta(p)\right) \tag{142}$$

29.4.3 2.3. The decay and spectrum of interacting quasiparticles

The results obtained above are valid if $\lambda\tau > 1$. The point is, which τ actually appears in this condition.

This problem was considered in a number of papers, using the efficient methods developed in superconductivity theory (Abrikosov et al. 1962) and the approach suggested by Keldysh (1964).

The spectrum and decay of quasiparticle excitations were found by Elesin (1969, 1970a) when sufficiently strong electron-phonon interaction was taken into account. It was shown that the gap in the excitation spectrum does remain, whatever strong the electron-phonon interaction be, and the decay is small.

A similar result has been obtained for electron-electron interaction.
The most significant effect on the spectrum was found to be due to scattering from charged (nonmagnetic!) impurities considered by Elesin (1970b). The impurity lamination was shown to lead to the destruction of an electron hole pair and to vanishing of the gap in the excitation spectrum at a certain time of scattering from an impurity λ_{imp} :

$$\tau_{\text{imp}} = 1/\lambda. \tag{143}$$

The reason is that the field of a charged impurity acts on the pair components with oppositely directed forces, due to the different signs of the electron and hole charges. The situation resembles that for a superconductor with magnetic impurities.

In pure crystals the decay of excitations is mainly determined by annihilation processes (interband recombination of electrons), the contribution being of two types. First, there is a contribution $\sim 1/\tau_A$, which is related to the probability of the direct escape of an electron from a state near the gap into the valence band. It is usually small because $\tau_{\text{A}} \geqslant \tau_{\text{ph}}$.

Second, there is a contribution (Elesin 1970a)

$$\gamma_{\text{A}} = \frac{1}{\tau_{\text{A}}}\frac{\mu_0}{\lambda} \gg \frac{1}{\tau_{\text{A}}}, \tag{144}$$

which is essentially nonequilibrium.
Expression (144) is obtained if the QDF (140) is substituted into the decay formula (Elesin 1970a)

$$\gamma_{\text{A}} = \frac{\pi\zeta_0}{2\omega_{\text{D}}^2}\int_{-\mu_0}^{\infty} \mathrm{d}\xi\xi^2 n(\xi) \tag{145}$$

where ζ_0 is the constant of electron-phonon interaction.
The result has a simple physical meaning. The electrons, which find themselves, due to the field, in the conduction band at $p = p_0$ in the energy range i. escape rapidly (as compared with interband recombination, since $\tau_{\text{ph}} \ll \tau_{\text{A}}$) into the energy range μ_0, from which they recombine at a rate of $1, \tau_{\text{A}}$. Since the resultant rate of electron escape from the energy range μ_0 is limited by recombination, the phonon time disappears from eq. (144), but a factor μ_0/λ.
appears. It takes into account that recombination occurs from the whole energy range μ_0.

Thus, the gap does exist if the following condition is satisfied:

$$\lambda > \frac{1}{\tau_{\text{A}}}\frac{\mu_0}{\lambda}, \tag{146}$$

which can be rewritten in the form

$$Ik(\omega) > \frac{n_0}{\tau_{\text{A}}}, \tag{147}$$

where I is the field intensity, $k(\omega)$ is the absorption coefficient. Condition (147) means that the rate of electron production must exceed the rate of electron annihilation. This is the condition needed for electrons to occupy the bottom of the conduction band up to the energy μ_0 (the saturation state) and this condition is relatively easy to realize ($I \sim 10^5$ W/cm^2) (Elesin 1970a).

29.4.4 2.4. The effect of a strong electromagnetic field on the properties of superconductors

The effect on superconductors of an electromagnetic field giving rise to interband transitions has been studied in several papers (Buimistrov 1968, Kumar and Sinha 1968, Blazhin and Selivanenko 1970, Aronov and Gurevich 1972, Elesin and Kopaev 1972, 1974).

Buimistrov (1968), Kumar and Sinha (1968) and Blazhin and Selivanenko (1970) considered the effect of a field on T_c due to renormalization of the matrix element of electron-phonon interaction caused by the mixing of wave functions.

The paper by Aronov and Gurevich (1972) considers the effect of a resonance field, which causes electrons above the gap ($\xi > 0$) to pass into an empty band. In this case a decrease in the number of thermal electrons near $\xi = 0$ and, hence, an increase in T_c was expected.

Aronov and Gurevich (1972) assumed that the QDF is of the form (1) with $\mu^* < 0$ and came to the conclusion that T_c must tend to infinity. Unfortunately, this conclusion has not been confirmed by the corresponding calculations of $n(\xi)$ and is, therefore, open to doubt.

The effect of simultaneous dynamic and kinetic action of the field was studied by Elesin and Kopaev (1972, 1974) for the cases of absence and presence of electrons in the conduction band, respectively.

According to (Elesin and Kopaev 1972), the kinetic action lies in the formation of a Fermi sphere of nonequilibrium electrons in the conduction band.

In this state superconductivity may arise due to an ordinary phonon mechanism and to the formation of a superconducting gap at the Fermi surface μ_0.

The dynamic action means that a field gap 2λ (see section 2.1) is formed at the Fermi surface. As is shown by Elesin and Kopaev (1972), the field suppresses superconductivity if λ exceeds the energy of the Cooper pair A_0 :

$$\lambda \geqslant \Delta_0, \quad \Delta_0 = 2\omega_\mathrm{D} \exp\left(-\frac{1}{|g|}\right). \tag{148}$$

If $\lambda < \Delta_0$, the superconducting gap decreases,

$$\Delta = \sqrt{\Delta_0^2 - \lambda^2} \tag{\prime}$$

Consider this problem for $T = 0$ and $1/\tau_A = 0$, following Elesin and Kopaev (1972). The initial Hamiltonian of the system represents the Hamiltonian H_0 (129). The system is described by Green's functions $G_\parallel$and $F_\parallel{}^+$(i is the band number) and also by the function G_{12}, which corresponds to electron-hole pairing. Solving the system, we obtain

$$G_{11}(p,\omega) = \frac{\omega + \xi}{\tilde{D}}, \quad F_{11}^+ = \frac{-i\Lambda^*}{\tilde{D}}, \quad G_{21} = \frac{i\lambda}{\tilde{D}} \tag{149}$$

$$\tilde{D} = \omega^2 - \tilde{\varepsilon}_p^2, \quad \tilde{\varepsilon}_p = \sqrt{\xi^2 + \Delta^2 + \lambda^2} \tag{150}$$

$$1 = |g| \int \frac{\mathrm{d}\xi}{\sqrt{\xi^2 + \lambda^2 + \Delta^2}} \tag{151}$$

The equation for the gap (151) yields the solutions (148) and (148'). Interestingly, the dispersion relation for quasiparticles (150) includes the gap $\sqrt{\xi^2+\lambda^2} = \Delta_0$, which remains constant for $0 \leqslant \lambda < \Delta_0$.

The field suppresses the Meissner effect, reducing the number of superconducting electrons (Elesin and Kopaev 1974):

$$N_{\mathrm{S}} = 1 - (\lambda/\Delta_0)^2\,, \quad 0 \leqslant \lambda \leqslant \Delta_0. \tag{152}$$

At the same time, T_{c} almost does not change. This can be explained by the fact that the gap makes the production of thermal excitations difficult, which must lead to an effective increase in T_{c}. Indeed, the ratio $T_{\mathrm{c}}/\Delta(0)$ is

$$\frac{T_c}{\Delta(0)} = 0.57\frac{1-0.3\lambda^2/\Delta_0^2}{1-\lambda^2/\Delta_0^2}, \tag{153}$$

i.e. it coincides with that in the BCS theory for $\lambda = 0$ and exceeds 0.57 if $\lambda \neq 0$.

For crystals with equilibrium electrons in the conduction band the effect of the field is different.

As was shown by Elesin and Kopaev (1974), in this case T_c may rise significantly because the field gap does not coincide with the Fermi level. Therefore, the field gap does not suppress the superconducting gap. On the
contrary, renormalization of the electron spectrum due to λ leads to an increase in the density of states at the Fermi level, which corresponds to a rise in T_{c}. Elesin and Kopaev (1974) obtained the following expression for T_{c} :

$$T_{\mathrm{c}} = 2.28\frac{\tilde{n}^2}{\lambda}\exp\left(-\frac{\tilde{n}}{\lambda}g_0\right), \quad g_0 = \ln\frac{0.57\lambda}{T_{\mathrm{c}0}}, \quad \tilde{n} = \frac{Nd}{3n_0}\mu_0. \tag{154}$$

It is assumed here that $i \gg T_{\mathrm{c}0}, T_{\mathrm{c}} \ll 4\hat{n}^2/\lambda$. Note that the density of states at the Fermi level is proportional to the ratio $\lambda/\tilde{n} \gg 1$, so the factor $\tilde{n}/\lambda$ in the exponential in eq. (154) accounts for an increase in the density of states at the Fermi level; the pre-exponential factor $\tilde{n}^2/\lambda$ is related to a decrease in the energy region of interelectron attraction due to the formation of the gap 2λ.

The maximum value of T_{c} is

$$T_{\mathrm{cm}} = 2.28\lambda_{\mathrm{m}}\left(\ln\frac{\lambda_{\mathrm{m}}0.57}{T_{\mathrm{c}0}}e\right)^{-1}, \quad \frac{\lambda_{\mathrm{m}}}{\tilde{n}} + 1 = \ln\frac{0.57\lambda_{\mathrm{m}}}{T_{\mathrm{c}0}}. \tag{155}$$

Note that for a field intensity $\sim 10^5$ W/cm^2 the value of λ_{m} is of the order of 10^{13} cm^{-1} and corresponds to $T_{\mathrm{cm}} \sim 100$ K.

29.4.5 2.5. Superconductivity in systems with repulsion and inverse quasiparticle distribution

An inverse distribution of quasiparticles may give rise to superconductivity of a new type-superconductivity with repulsion between electrons. Indeed, if in eq. (6) we change the sign both in $(1-2n)$ and in g, the equation will have a non-trivial solution. This was noted by Kirzhnits and Kopaev (1973).

In order to understand the nature of a bound state thus formed, it is convenient to use the concept of quasiholes (Elesin et al. 1973), for which the distribution is not inverse, the potential energy of interaction is positive (repulsion), and the kinetic energy is negative due to the negative sign of the hole mass. Obviously, since the potential and kinetic energies have opposite signs, the bound state may, in principle, occur.

The main difficulty for this state to occur lies in the initial creation and upkeep of the inverse quasiparticle population.

To avoid this difficulty, Kirzhnits and Kopaev (1973) suggested a "semiconductor model", in which the dielectric gap is assumed to make the inverse distribution possible. In this case the gap arises near the band extrema, rather than at the Fermi surface.

Another possibility was suggested by Galitskii et al. (1973) where a "selfsustained" superconducting state with $2A > (\theta_1)$ and $n\left(\xi_1^2\right) > \frac{1}{2}$ was shown to exists in a metal within a certain energy range near the Fermi level in the case of repulsion between electrons.

The inverse population near the Fermi surface is created due to a resonance electromagnetic field at a frequency of $\omega > E_g$. As was shown above (see section 2.1), the gap 2λ arises in this case near the Fermi surface. If $2\hat{\lambda} > \omega_{\mathrm{D}}$, the QDF becomes inverse* (see section, 2.2). In this case the equation

$$1 = g_{\mathrm{ee}} \int_0^{\omega} \frac{\mathrm{d}\xi(2n-1)}{\tilde{\varepsilon}} \tag{156}$$

has the solution

$$\Delta = \begin{cases} \sqrt{\Delta_0^2 - \lambda^2}, & 0 \leqslant \lambda \leqslant \Delta_0 \\ 0 & , \quad \lambda > \Delta_0 \end{cases} \tag{157}$$

$$\Delta_0 = \frac{2\mu^2}{\tilde{\omega}} \exp\left(-\frac{1}{g_{\mathrm{ee}}}\right)\left(1 - \exp\left(-\frac{2}{g_{\mathrm{ee}}}\right)\right)^{-1}.$$

This solution implies that λ should be chosen so as to be larger than $\omega_{\mathrm{D}}/2$ but less than Δ_0. Obviously, the condition

$$2\Delta_0 > \omega_{\mathrm{D}} \tag{158}$$

must also be satisfied, which is quite real due to large values of the constants g_{ee}.

Since the gap in the quasiparticle spectrum is $\sqrt{\lambda^2 + \Delta^2} = \Delta_0$, we can let λ adiabatically tend to zero (i.e. switch off the field), retaining, nevertheless, condition (158).

This means that in the nonequilibrium state and in the presence of a source transferring quasiparticles from the state below the gap into the state above it (in order to compensate for a certain recombination of quasiparticles due to multiphonon and electron-electron processes) the solution with the gap A_0 is self-sustained-the gap creates an inverse quasiparticle distribution, which in turn gives rise to a gap.

Conditions needed for such a state to be stable and the power of an additional source were considered in detail by Galitskii et al. (1973), Elesin (1970a), and Elesin and Kopaev (1981). The properties of inverse superconductivity were also studied in those papers at greater length. Here we shall only touch upon the effect of anomalous paramagnetism of such superconductors, first discovered by Galitskii et al. (1973). This effect is closely related to the sign of N_{S} (see section 1.2.2).

Indeed, the current which arises due to a slowly varying magnetic field is expressed by eq. (63) where N_{s} is given by eq. (64). Using the expression for $n(\xi)$, we find that $N_{\mathrm{s}} < 0$ and is equal to

$$N_{\mathrm{S}} = -\left[\frac{2\mu}{\sqrt{\mu^2 - \Delta^2}} - 1\right] \simeq -1, \quad \mu \gg \Delta. \tag{159}$$

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- Note that this state does arise without an additional source, which transfers quasiparticles from the state below the gap into the state above it (Elesin 1975).

Obviously, the negative sign of N_{S} is associated with "supercooling" of the inverse distribution (and with the fact that $n(0) > \frac{1}{2}$).

Fig. 9. Current voltage characteristics of junctions for high voltages: (a) type I; (b) type II; (c) type III. The dashed line corresponds to a homogeneous junction, the solid line to an inhomogeneous junction. (Elesin et al. 1980a,b.)

Fig. 10. (a) The dependence $\Delta(\beta)$; (b) dependence of energy $U(\Delta)$ on Δ; (c) dependence of Δ on the coordinate (laminated solution). (Elesin 1977a.)

Fig. 11. Voltage across a Pb film under illumination with laser light for different temperatures (top: oscillogram of the laser pulse); $V_{\mathrm{N }}$ is the voltage across the sample in the normal state. (Mitsen 1981.)}

Fig. 12. Dependence on pumping power of the reduced voltage across a Pb sample under laser illumination; $\tau_{\mathrm{u
}=1 \mu \mathrm{~s}, T=6.75 \mathrm{~K}$. (Mitsen 1981.)}

Fig. 13. (a) Dependence of the critical power on reduced temperature for Pb films of different thickness. Solid lines represent theoretical dependences $W_{1
\left(T T_{1}\right)$ according to eq. (79): (1300∧ : (A) 500Å : (x) 1000Å. (b) Dependence of the critical power on the thickness of a Pb film for $T/T_{\mathrm{c}} = 0.7$. Solid line: theoretical dependence $W_{\mathrm{c}}(d)$ according to eq. (79). (Golovashkin et al. 1980.)}

Fig. 14. Current-voltage characteristics of $\mathrm{Sn -\mathrm{I}-\mathrm{Sn}$ junctions: (1) $\eta_T = 2.8 \times 10^{-3}\Omega, S = 0.16$ mm^2, $T = 1.7$ K, $d = 1200$Å; (2) $\eta_{\mathrm{T}} = 5.5 \times 10^{-3}\Omega, d = 2600$Å, $T = 1.7$ K. (Mitsen 1979.)}

Fig. 15. (a) Current-voltage characteristic for $\mathrm{Nb -\mathbf{I ~ P b}$ junction. (Pan and Rudenko 1981.) (b) Dependence of J_{H} on the local temperature $T_1(+)$ and on the bath temperature $\mathrm{T_b}(\overline{\hat{s}})$. Solid line is the temperature dependence of $\mathrm{J_H}(T)/J_{\mathrm{H}}(0) = \beta_{\mathrm{c}}(T)/\beta_{\mathrm{c}}(0)$ taken from the article by Pan and Rudenko (1981).}

Fig. 16. The dependence $n(\varepsilon)$ for $\omega / \Delta_{0 =2.1, T / T_{\mathrm{c}}}=0.5, \Delta / \Delta_{0}=0.5$, and $\gamma = 0, 1, 10$. (Imai 1980.)}

Fig. 17. (a) Dependence of the order parameter Δ on the junction voltage for different values of the injection parameter $\alpha_{V : \alpha_{1}<\alpha_{2}<\alpha_{3}<\alpha_{4}<\alpha_{5}$. (b) Dependence of the tunnel junction current $J(V)$ on the voltage for different α_{v}. (Elesin 1979b.)}

Fig. 18. Current-voltage characteristic of (a) generator and (b) detector; $T=1.169 \mathrm{~K
$. (Gray and Willemsen 1978.)}

30 Quasiparticle relaxation dynamics in superconductors with different gap structures: theory and experiments on $YBa_2Cu_3O_{7-\delta}$ by by Kabanov, Demsar, Podobnik, Mihailovic

Abstract

Photoexcited quasiparticle relaxation dynamics are investigated in a $YBa_2Cu_3O_{7-\delta}$ superconductor as a function of doping δ and temperature T using ultrafast time-resolved optical spectroscopy. A model calculation is presented which describes the temperature dependence of the photoinduced quasiparticle population n_{pe}, photoinduced transmission $\Delta\mathcal{T}/\mathcal{T}$ and relaxation time τ for three different superconducting gaps: (i) a temperature-dependent collective gap such that $\mathbf{\Delta}(T) \to 0$ as $T \to T_c$, (ii) a temperature-*independent* gap, which might arise for the case of a superconductor with pre-formed pairs and (iii) an anisotropic (e.g. d-wave) gap with nodes. Comparison of the theory with data of photoinduced transmission $|\Delta\mathcal{T}/\mathcal{T}|$, reflection $|\Delta\mathcal{R}/\mathcal{R}|$ and quasiparticle recombination time τ in $YBa_2Cu_3O_{7-\delta}$ over a very wide range of doping ($0.1 < \delta < 0.48$) is found to give good quantitative agreement with a temperature-*dependent* BCS-like isotropic gap near optimum doping ($\delta < 0.1$) and a temperature-*independent* isotropic gap in underdoped $YBa_2Cu_3O_{7-\delta}$ ($0.15 < \delta < 0.48$). A pure d-wave gap was found to be inconsistent with the data.

30.1 Introduction

Ultrafast optical time resolved experiments performed on high-T_c superconductors in recent years have shown that a substantial transient change of the optical transmission or reflection can be induced in these materials by ultrashort laser pulse photoexcitation (PE). What makes these studies particularly interesting is the fact that in optimally doped superconductors where T_c is a maximum, the amplitude of the observed photoinduced transmission signal, $|\Delta\mathcal{T}/\mathcal{T}|$, (or reflection $|\Delta\mathcal{R}/\mathcal{R}|$) appears to increase dramatically below T_c[1, 2, 3, 2]. Furthermore, recently it was found that in underdoped $YBa_2Cu_3O_{7-\delta}$ ($\delta > 0.15$), the increase in amplitude could not be correlated with T_c, but rather with the so-called "pseudogap" temperature T^* which increases with increasing δ[5]. These observations together suggest rather strongly that the photoinduced effects are related to the opening of a gap (or pseudogap) in the density of states close to the Fermi energy. For optimally doped cuprate superconductors - on which most experiments were done so far - there has been substantial agreement in the literature regarding the experimental data. However, various groups have proposed very different explanations for the observed effects[1, 3, 2, 6]and their origin have remained controversial[7].

In this paper we present a calculation of the photoinduced optical response in a superconductor for weak excitation and compare our results with systematic experimental data on $YBa_2Cu_3O_{7-\delta}$ as a function of δ and T. We derive expressions for the temperature dependence of the photoexcited quasiparticle density n_{pe} and photoinduced transmission amplitude $|\Delta\mathcal{T}/\mathcal{T}|$ for different possible gaps (or pseudogaps) which might be applicable to high-T_c cuprates. The simplest case considered is that of a gap, whose temperature dependence is mean-field like and is approximated using a BCS function. The second case considered is for a temperature-independent "pseudogap" which is relevant for a superconductor where condensation of pre-formed pairs into a superconducting state occurs at T_c but pairing takes place above this temperature. In this case the gap is not due to a collective effect, but essentially represents the pair binding energy E_B and is - to first approximation - temperature independent. Finally we discuss the case of an anisotropic gap with nodes, such as one might expect for a d-wave

superconductor. In section V we present a calculation of the temperature dependence of the quasiparticle relaxation rate τ. Comparisons of the theoretical predictions with experimental data are made in each case and a detailed discussion is given in section VI.

30.2 Experimental details

The time-resolved experiments were performed using 100 fs, 800 nm pulses from a Ti:sapphire laser using single high frequency modulation of the pump at 200 kHz. The $YBa_2Cu_3O_{7-\delta}$ thin film samples were grown on MgO or $SrTiO_3$ substrates and annealed in oxygen to obtain different O concentrations δ. The T_c was measured by measuring the AC susceptibility in each case. The transition widths, defined as the temperature where χ drops to 90% of full diamagnetism, were typically 1-2 K for the $\delta < 0.15$ and 4-7 K for $\delta > 0.15$. The oxygen concentration was determined from T_c, which brings in some uncertainty near optimum doping, since the optimum T_c for YBCO is 90 K on $SrTiO_3$ substrates and 89 K on MgO substrates, which is 2-3 K lower than in single crystals. The typical film thicknesses were 100-120 nm and all experiments were performed in transmission through the sample. The typical energy of the laser pulse is 0.2 nJ, and the laser spot size typically 100μm in diameter. In estimating the carrier density, the absorption length was assumed to be $l \simeq 0.1$ μm. The CW temperature rise of the superconductor film due to laser heating was calibrated using a 30μm YBCO superconducting microbridge from the same batch and on the same substrate as in the optical experiments by measuring the change in resistivity with a 4-probe contact method. This temperature offset was then taken into account in the plots shown in Figures 1-6. The error in the sample temperature is thus reduced to less than ± 2 K.

The photoinduced transmission $\Delta\mathcal{T}/\mathcal{T}$ through two different films with $T_c = 90$ and 53 K respectively is shown in Fig. 1 at different temperatures above and below T_c. There are two decay components in each case, a fast component with $\tau \sim 0.3 - 2$ps, and a longer-lived component with $\tau_L > 10$ ns, which appears as a nearly flat background already discussed elsewhere[2]. The latter will not be analysed here, except that it will be taken into account in the fitting procedure. The time-evolution of the traces was fitted (shown by the solid lines in Fig.1) using a model with a *single* exponential decay and a Gaussian temporal profile pump pulse, from which the amplitude of the photoinduced transmission, $|\Delta\mathcal{T}/\mathcal{T}|$, and relaxation time of the fast component, τ, was determined. The temperature dependence of $|\Delta\mathcal{T}/\mathcal{T}|$ and τ derived from the fits are plotted in Figs. 2, 4 and 5 and will be discussed together with the theory in the following sections.

30.3 Initial photoexcited carrier relaxation

The initial phase of PE carrier relaxation after absorption of a pump laser photon proceeds very rapidly. After the laser pulse excites an electron-hole pair, these PE carriers thermalize among themselves with a characteristic time of $\tau_{e-e} \sim \frac{\hbar E_F}{2\pi E^2}$, where E is the carrier energy measured from the Fermi energy E_F. In this thermalization process, quasiparticle avalanche multiplication due to electron-electron collisions takes place as long as τ_{e-e} is less than the electron-phonon (*e-ph*) relaxation time τ_{e-ph}. Electron-phonon relaxation becomes important when the quasiparticle (QP) energy is reduced to $E = \sqrt{\frac{\hbar E_F}{2\pi\tau_{e-ph}}} \simeq 30 - 50$meV (assuming $E_F \simeq 0.1 - 0.2$eV). τ_{e-ph} has been determined experimentally for the case of a $YBa_2Cu_3O_{7-\delta}$[8] as well as $Bi_2Sr_2CaCu_2O_{8+x}$ and $Bi_2Sr_2Ca_2Cu_3O_{10+x}$[9] from relaxation time fits in time resolved experiments using intense laser pulses. Using Allen's formula [10] the *e-ph* relaxation time for YBCO has been found to be $\tau_{e-ph} = \frac{T_e}{3\lambda\langle\omega^2\rangle} \simeq 100$fs for initial carrier temperatures $T_e = E_I/C_e$ in the range 3000 K[8] and $\tau_{e-ph} \simeq 60$fs for $T_e \simeq 410$ K [9]. Here E_I is the energy density per unit volume deposited by the laser pulse, C_e is the electronic specific heat, λ is a

constant which characterizes the electron-phonon interaction and $\langle\omega^2\rangle$ is a mean-square phonon frequency. λ has been determined from these experiments to be in the range $0.9 < \lambda < 1$ both in $YBa_2Cu_3O_{7-\delta}$ and $Bi_2Sr_2CaCu_2O_{8+x}$ superconductors. In experiments which we are considering here[1, 2, 5], the laser intensities are significantly smaller, and the photoexcited electronic temperature T_e is only few K in excess of the lattice temperature, i.e. $T_e \simeq T$, which gives an *e-ph* relaxation time $\tau_{e-ph} \simeq 10$fs. Under these near-equilibrium conditions where the carrier temperature is near the lattice temperature, this value of τ_{e-ph} can be compared to the relaxation time determined from infrared reflectivity measurements. For energies of the order of $30-50$ meV, $\hbar/\tau \sim 3000$ cm $^{-1}$ ($\tau \sim 16$ fs)[11] in good agreement with the estimated $\tau_{e-ph} \simeq 10$fs. In the absence of a gap in the low-energy density of states, this is the timescale of the electron -phonon thermalization.

In the presence of the gap the situation is strongly modified and a bottleneck in the relaxation occurs after $t \geq 10$ fs. As a result QPs accumulate near the gap, forming a non-equilibrium distribution. Each photon thus creates $30 \sim 40$ QPs given by $E_{pump}/2\mathbf{\Delta}$,where $E_{pump} = 1.5$eV is the laser pump photon energy and $2\mathbf{\Delta}$ is the energy gap. Because the final relaxation step across the gap is strongly suppressed [12, 13] the QPs together with high frequency phonons (with $\hbar\omega > 2\mathbf{\Delta}$) form a near-steady state distribution. The QP recombination dynamics of this system is governed by the emission and reabsorption of high frequency phonons. Phonons with $\hbar\omega < 2\mathbf{\Delta}$ do not participate in the direct relaxation of the QPs, since in this case the QP final states would lie in the gap. (A quantitative justification for the assumption that phonons play a dominant role in the QP relaxation will be given in section V.) The basis for these pump-probe experiments is that this near-equilibrium QP population can be very effectively probed using excited state absorption using a suitable second optical probe pulse, giving direct information on QP dynamics and the nature of the gap itself.

Since the probe laser photon energy E_{probe} is typically *well above* the plasma frequency in high-T_c superconductors, we make the approximation that the transition probability for the probe light is given by the Fermi golden rule, with photoinduced quasiparticles as initial states and with final unoccupied states well above the Fermi energy in a band which lies approximately at $E_0 \sim E_{probe}$. The amplitude of the photoinduced absorption (PA) $|\Delta\mathcal{A}/\mathcal{A}|$ is then proportional to the photoinduced transmission $|\Delta\mathcal{T}/\mathcal{T}|$ (and in the linear approximation also to $|\Delta\mathcal{R}/\mathcal{R}|$) which is in turn proportional to the number of photoexcited quasiparticles n_{pe}. The probe signal is thus weighted by the O-Cu charge-transfer dipole matrix element and the joint DOS, so $|\Delta\mathcal{T}/\mathcal{T}| \propto -n_{pe}\rho_f\,|M_{ij}|^2$ where n_{pe} is the photoexcited carrier density, ρ_f is the final density of unoccupied states and $M_{ij} = \langle\mathbf{p}.\mathbf{A}\rangle$ is the dipole matrix element. Although recent experiments on $YBa_2Cu_3O_{7-\delta}$ with different E_{probe} show the existence of a resonance for $E_{probe} \sim 1.5$ eV[2], we assume here that the adiabatic approximation can be applied, so the photoinduced carriers do not cause any change in ρ_f or M_{ij}. In this case the effect of the 1.5 eV resonance is to significantly enhance the sensitivity of the probe.

30.4 Temperature dependence of the quasiparticle density

30.4.1 Isotropic gap.

In discussing the theoretical explanation for the observed effects, it is important to make the distinction between experiments *(i)* in which the photoexcited charge carrier density is substantially less than the normal state carrier density, $n_{pe} \ll n_c$ so the laser makes only a weak perturbation on the superconductor[1, 2, 5] and *(ii)* those where $n_{pe} \sim n_c$ and photoexcitation is sufficiently strong to close the superconducting gap[2, 3]. Whereas for the latter case *(ii)*, a theoretical description was proposed by Mazin[6], so far there has been no theoretical calculation for the photoinduced effects in a superconductor for the case of weak photoexcita-

tion *(i)*. The number of photogenerated quasiparticles in low-excitation density experiments is $n_{pe} \lesssim 3 \times 10^{-3}$/unit cell[1, 2, 5]. On the other hand, the typical quasiparticle concentration in a high-T_c superconductor is $n_0 = 2N(0)\mathbf{\Delta} \simeq 0.2 - 0.4$/unit cell, where $N(0)$ is the density of states at E_F. The number of photoexcited QPs is small compared to the normal state density $n_{pe}/n_0 \lesssim 10^{-2}$, so the weak photoexcitation approximation is clearly justified and the photoexcited quasiparticles make only a small perturbation of the distribution functions. Assuming that the energy gap is more or less isotropic (no nodes), we can approximate nonequilibrium phonon (n_{ω_q}) and quasiparticle (f_ϵ) distribution functions as follows [14]:

$$n_{\omega_q} = \begin{array}{ll} \frac{1}{\exp(\frac{\hbar\omega_q}{k_B T})-1} & \hbar\omega_q < 2\mathbf{\Delta} \\ \frac{1}{\exp(\frac{\hbar\omega_q}{k_B T'})-1} & \hbar\omega_q > 2\mathbf{\Delta} \end{array} \tag{30.1}$$

$$f_\epsilon = \frac{1}{\exp(\frac{\epsilon}{k_B T'}) + 1} \tag{30.2}$$

where T is the lattice temperature and T' is the temperature of quasiparticles and high frequency phonons with $\hbar\omega_q > 2\mathbf{\Delta}$.

We can calculate n_{pe} using Eqs.(1), (2) and by considering the conservation of energy. Assuming $\mathbf{\Delta}$ is temperature *independent* and large in comparison to T, the conservation of energy has the following form:

$$(n_{T'} - n_T)\,\mathbf{\Delta} + \left(n_{T'}^2 - n_T^2\right) \frac{\nu\mathbf{\Delta}}{2\hbar\Omega_c N(0)^2 k_B T'} = \mathcal{E}_I \tag{30.3}$$

here $n_{T'}, n_T$ is the number of thermally excited quasiparticles per unit cell at T' and T respectively, $\mathcal{E}_I$ is the energy density per unit cell deposited by the incident laser pulse, Ω_c is phonon frequency cut off and ν is the effective number of phonon modes per unit cell participating in the relaxation. Taking into account that $n_{pe} = (n_{T'} - n_T) \ll n_T$ and assuming that $n_T = 2N(0)k_B T \exp(-\mathbf{\Delta}/k_B T)$, the number of photogenerated quasiparticles at temperature T is given by:

$$n_{pe} = \frac{\mathcal{E}_I/\mathbf{\Delta}}{1 + \frac{2\nu}{N(0)\hbar\Omega_c}\exp(-\mathbf{\Delta}/k_B T)}. \tag{30.4}$$

For the case of a temperature-dependent gap $\mathbf{\Delta}(T)$, Eq.(3) will be slightly modified:

$$\left((n_{T'} - n_T) + \frac{\nu}{\hbar\Omega_c N(0)^2 \pi\mathbf{\Delta}(T)}\left(n_{T'}^2 - n_T^2\right)\right)(\mathbf{\Delta}(T) + k_B T/2) = \mathcal{E}_I \tag{30.5}$$

We approximate $n_T \simeq 2N(0)\sqrt{\pi\mathbf{\Delta}(T)k_B T/2}\exp(-\mathbf{\Delta}(T)/T)$ [13] with $k_B T \ll \mathbf{\Delta}$, resulting in a slightly modified expression for n_{pe}[15]:

$$n_{pe} = \frac{\mathcal{E}_I/(\mathbf{\Delta}(T) + k_B T/2)}{1 + \frac{2\nu}{N(0)\hbar\Omega_c}\sqrt{\frac{2k_B T}{\pi\mathbf{\Delta}(T)}}\exp(-\mathbf{\Delta}(T)/k_B T)}. \tag{30.6}$$

We note that in Eqs. (4) and (6), the explicit form of n_{pe} (and hence also $|\Delta\mathcal{A}/\mathcal{A}|$ or $|\Delta\mathcal{T}/\mathcal{T}|$) depends only on the ratio $k_B T/\mathbf{\Delta}$, showing that the intensity of the photoresponse is a universal function of $k_B T/\mathbf{\Delta}$ as long as the particular functional form of the temperature dependence $\mathbf{\Delta}(T)$ is the same. Another important feature of the expressions (4) and (6) is that at $T = 0$, $n_{pe} \propto 1/\mathbf{\Delta}(0)$, which allows a determination of $\mathbf{\Delta}(0)$, as long as the experimental parameters required to calculate $\mathcal{E}_I$ are recorded sufficiently precisely.

Since the decay of each phonon involves the creation of two quasiparticles, the number of high frequency phonons is proportional to the square of the number of quasiparticles $n^2_{T'}$ at temperature T'. (Taking into account conservation of energy[14], at higher pulse energies when n_{pe} becomes large, we expect a crossover of the photoresponse from a linear dependence on laser power, $n_{pe} \propto I$ to a square root dependence $n_{pe} \propto \sqrt{I}$ at very high intensities.)

To enable comparison of the temperature dependences of the photoinduced transmission given by Eqs. (4) and (6) with experiments, we have plotted them as a function of temperature in Figures 2b) and c) respectively. The values for the constants for $YBa_2Cu_3O_{7-\delta}$ are: $\nu = 10 - 20$, $N(0) = 2.5 - 5$ eV^{-1}spin^{-1}cell^{-1}, $\Omega_c = 0.1$ eV. At low temperatures n_{pe} is essentially T-independent in both cases, falling off at higher temperatures. However, from the figure it is clear that the high-temperature behaviour is quite different in the two cases. In the case of the T-dependent gap (Fig.2c)), the decrease in $|\Delta\mathcal{T}/\mathcal{T}|$ is much more pronounced and starts quite close to T_c at $T/T_c \approx 0.8$ dropping to zero *at* T_c. In the temperature-independent gap case (Fig. 2b)), the photoinduced transmission starts to drop at much lower temperatures near $T/T^* \approx 0.4$, dropping exponentially at high temperatures. Also, a notable prediction of Eq. (6) for the T-dependent gap is a small peak near $T/T_c \approx 0.7$, which is not present in the case of a temperature-independent gap formula (4).

In Fig. 2a) we have plotted the data for the photoinduced transmission amplitude for $YBa_2Cu_3O_{7-\delta}$ as a function of temperature for three different δ. Both the magnitude and the temperature dependence are seen to be strongly doping-dependent. For comparison with theory, superimposed on the theoretical curves in Figure 2 b) and c) we have plotted the normalised photoinduced reflectivity and transmission data for $YBa_2Cu_3O_{7-\delta}$ with different δ. The data for a number of underdoped samples[5] in the range $0.15 < \delta < 0.48$ have been scaled onto a common temperature scale as suggested by the formula (4) and plotted as a function of T/T^* in Figure 2b). T^* is defined as the point where the amplitude of signal $|\Delta\mathcal{T}/\mathcal{T}|$ drops to some fixed value (for example to 5% of the maximum value). The data can be seen to fit the theoretical curve remarkably well.

For near-optimally doped $YBa_2Cu_3O_{7-\delta}$ ($\delta < 0.1$) the data are scaled by T_c and are plotted as a function of reduced temperature T/T_c. However, this time the data are superimposed on the prediction for the T-dependent BCS-like gap Eq. (6) in Figure 2c). In this case also, the fit is seen to be good, including the small maximum at $T/T_c \approx 0.7$. This maximum is particulary well observed in the data by Han et al.[1] and by Mihailovic et al.[5], although it is not evident in the data of Stevens et al.[2].

The experimental data thus appear to scale onto *one* of the *two* theoretical curves, the near-optimally doped $YBa_2Cu_3O_{7-\delta}$ data ($\delta < 0.1$) agreeing very well with Eq. (6) derived for a temperature-dependent gap, while the underdoped data ($\delta > 0.15$) fit expression (4) for a T-independent gap. The fact that the underdoped sample data scale onto one universal curve in Figure 2b), while the near-optimally doped data scale onto the curve in Fig. 2c), is a consequence of the scaling properties of Eqs.(4) and (6) and is a rather remarkable confirmation of the theoretical model by the experiment. From the fits of the underdoped sample data, we obtain gap values of $2\Delta = (5 \pm 1)k_BT^*$ for underdoped samples and $2\Delta = (9 \pm 1)k_BT_c$ for the near-optimally doped samples, the latter in good agreement with other optical experiments[11, 16].

Further confirmation of the model comes from the predicted linear intensity dependence in Eqs (4) and (6). In Figure 3 we plot the T-dependence of $|\Delta\mathcal{T}/\mathcal{T}|$ for near optimally doped sample ($T_c = 89$K) for three different pump intensities. The data are seen to scale linearly with intensity as shown by the insert where $\mathcal{E}_I$ is changed over one order of magnitude. (Unfortunately, measurements over a larger range of intensities were not possible because of steady-state laser heating.)

30.4.2 Anisotropic gap with nodes

If the gap is anisotropic and contains nodes on the Fermi surface - such as for d-wave or strongly anisotropic s-wave pairing - there is no clear gap in the spectrum. Nevertheless the quasiparticle DOS still has a strong energy dependence and a separation of low and high energy quasiparticles still exists, albeit is less pronounced than in the isotropic case. For an anisotropic gap with nodes we can approximate the quasiparticle DOS as a function of energy ε as:

$$N(\varepsilon) = N(0)\left(\frac{\varepsilon}{\mathbf{\Delta}_a}\right)^{\eta} \quad (\varepsilon \ll \mathbf{\Delta}_a), \tag{30.7}$$

where $\mathbf{\Delta}_a$ is a characteristic energy scale separating the low and high-energy QPs, and the exponent η depends on the topology of the nodes on the Fermi surface. For a two dimensional Fermi surface (with nodes) $\eta = 1$, while for a 3-dimensional case $\eta = 2$.

Using equation (7) we can estimate the average energy accumulated by quasiparticles after each laser pulse:

$$\Delta E = 2FN(0)(k_B T'^{(\eta+2)} - k_B T^{(\eta+2)})/\mathbf{\Delta}_a^{\eta} = \mathcal{E}_I \tag{30.8}$$

$$F = \int_0^{\infty} \frac{x^{\eta+1}}{\exp(x)+1} dx \tag{30.9}$$

$F = 1.80$ for $\eta = 1$, and $F = 5.68$ for $\eta = 2$, and T' and T are the QP temperature and lattice temperatures respectively. At low temperatures, the phonon contribution to the total energy is exponentially small (Eq.(8)) and can be neglected. In this case, the high and low energy phonons are separated by an energy scale of the order of $\mathbf{\Delta}_a$. (Note that in general $\mathbf{\Delta}_a$ scales with the gap, but is not equal to the amplitude of the gap.)

Solving Eq.(8) with respect to the QP temperature T' we obtain:

$$T' = \left[\frac{\mathcal{E}_I \mathbf{\Delta}_a}{2FN(0)} + k_B T^{(\eta+2)}\right]^{1/(\eta+2)}. \tag{30.10}$$

The number of thermally excited quasiparticles is determined by the equation:

$$n_T = 2GN(0)k_B T^{(\eta+1)}/\mathbf{\Delta}_a^{\eta}, \tag{30.11}$$

where G is given by the Eq.(9), except that $\eta + 1$ is replaced by η, so $G = 0.82$ for $\eta = 1$ and $G = 1.80$ for $\eta = 2$. Combining the two equations (10) and (11), we derive the following equation for the number of photogenerated quasiparticles as a function of temperature:

$$n_{pe} = \frac{2GN(0)k_B T^{(\eta+1)}}{\mathbf{\Delta}_a^{\eta}}\left[\left(1 + \frac{\mathcal{E}_I \mathbf{\Delta}_a^{\eta}}{2FN(0)k_B T^{(\eta+2)}}\right)^{\frac{\eta+1}{\eta+2}} - 1\right], \tag{30.12}$$

The expression for the photoinduced carrier density Eq.(12) is plotted in Figure 4 with $\eta = 2$ (solid line) and the same values of $N(0)$ and $\mathcal{E}_I$ as before. Instead of an exponential fall-off with increasing temperature, the photoexcited quasiparticle number decreases rapidly well below T_c according to a power law. In the 2-dimensional case with $\eta = 1$, (and the same parameters), the temperature dependence is shown by the dashed line in Figure 4 and differs only at the lowest temperatures. Using a T-dependent gap (e.g. of BCS form) in Eq. (12), the curve for n_{pe} is virtually indistinguishable from the case where $\mathbf{\Delta}_a^{\eta}$ is temperature-independent. The reason for this is that the effect of the T-dependent gap is only important as $T \to T_c$, but there n_{pe} is already small.

At low temperatures we can neglect the 1 in the round brackets and obtain:

$$n_{pe} = \frac{2GN(0)}{\mathbf{\Delta}_a^{\eta}}\left[\left(\frac{\mathcal{E}_I \mathbf{\Delta}_a^{\eta}}{2FN(0)}\right)^{\frac{\eta+1}{\eta+2}} - k_B T^{(\eta+1)}\right]. \tag{30.13}$$

A feature of the d-wave model - which is particularly important for comparison with experiment - is the peculiar sub-linear behaviour of n_{pe} with laser pump intensity, $n_{pe} \sim \mathcal{E}_I^{\frac{\eta+1}{\eta+2}}$ at low temperatures. In the high temperature limit on the other hand we can expand the term in brackets to obtain a crossover to linear behaviour, with a temperature-dependent slope:

$$n_{pe} = 2G\frac{\mathcal{E}_I}{k_B T}\frac{\eta+1}{\eta+2} \tag{30.14}$$

It should be pointed out that in the high temperature limit, the phonon contribution should become increasingly important and the decrease of the number of photogenerated QP will eventually become exponential, just as in the case of an isotropic gap.

The data for the temperature dependence of the induced transmission for underdoped and optimally doped $YBa_2Cu_3O_{7-\delta}$ are also plotted on Figure 4, superimposed with the predicted response for the d-wave case. Clearly the T-dependence data cannot be described using a *pure* d-wave gap irrespective of dimensionality or the form of temperature dependence of $\mathbf{\Delta}(T)$ (Eq.(12)). The linear intensity dependence shown in Fig. 3 is consistent with this observation, and is also not consistent with a d-wave gap, which predicts a sub-linear response. We note that although the data apparently exclude the possibility of a *pure* d-wave gap in underdoped and near-optimally doped YBCO, it does not entirely rule out a mixed symmetry gap with some d-wave admixture.

30.5 The quasiparticle relaxation rates.

As discussed in the section III, the relaxation rate of the photoinduced QPs is dominated by the energy transfer from high-frequency phonons with $\hbar\omega > 2\mathbf{\Delta}$ to phonons with $\hbar\omega < 2\mathbf{\Delta}$. To describe the relaxation of nonequilibrium quasiparticles in this case we consider the kinetic equation for phonons, taking into account phonon-phonon scattering[17]:

$$\frac{\partial n_\omega}{\partial t} = I_{ph-ph}\{n_\omega\} \tag{30.15}$$

where phonon-phonon scattering integral has the form:

$$\begin{aligned} I_{ph-ph}\{n_\omega\} &= 2\pi \sum_{q_1,q_2} |w_{q_1,q_2}|^2 \{\frac{1}{2}\left[(n_\omega+1)\, n_{\omega_1} n_{\omega_2} - n_\omega\,(n_{\omega_1}+1)\,(n_{\omega_2}+1)\right]\delta\left(\omega-\omega_1-\omega_2\right) \\ &\quad + \left[(n_\omega+1)\,(n_{\omega_1}+1)\, n_{\omega_2} - n_\omega n_{\omega_1}\,(n_{\omega_2}+1)\right]\delta\left(\omega_2-\omega-\omega_1\right)\}, \end{aligned} \tag{30.16}$$

w_{q_1,q_2} is the anharmonic coupling constant and n_ω is the phonon distribution function. (Conservation of the phonon momentum is not relevant because of umklapp scattering.) We neglect the *electron-phonon* scattering integral, because at normal temperatures the relative contribution of quasiparticles to the relaxation of phonons is small due to the relatively small number of quasiparticles compared to the phonons $N(0)\Omega_c \ll \nu$. Electron-phonon collisions can play an important role only at temperatures $T' \ll T_c$ but since - as shown in the previous section - n_{pe} is nearly constant at low temperature, T' cannot be small. The distribution function Eq.(1) reduces to 0 the parts of the collision integral Eq.(16) which describe the scattering of high and low frequency phonons separately. Only the part of the collision integral that does not conserve the number of phonons with $\omega < 2\mathbf{\Delta}$ differs from zero.

To estimate the relaxation time, we multiply Eq. (15) by ω_q and sum over q, satisfying the condition $0 < \omega_q < 2\mathbf{\Delta}(T)$:

$$\frac{\partial E_<}{\partial t} = \sum_{q(\omega_q<2\mathbf{\Delta})} \hbar\omega_q I_{ph-ph}\{n_\omega\}. \tag{30.17}$$

If we suppose that the coupling constant is momentum independent $w_{q_1,q_2} = w$, then:

$$\begin{aligned}\sum_{q(\omega_q<2\mathbf{\Delta})} \omega_q I_{ph-ph}\{n_\omega\} &= 2\pi w^2 \int_0^{2\mathbf{\Delta}} \omega\rho(\omega)\, d\omega \int \{\frac{1}{2} \\ &\left[(n_\omega+1)\, n_{\omega'} n_{\omega-\omega'} - n_\omega (n_{\omega'}+1)(n_{\omega-\omega'}+1)\right] \rho\left(\omega'\right)\rho\left(\omega-\omega'\right) + \\ &\left[(n_\omega+1)(n_{\omega'}+1)\, n_{\omega+\omega'} - n_\omega n_{\omega'}\left(n_{\omega+\omega'}+1\right)\right] \rho(\omega')\rho(\omega'+\omega)\}d\omega' \end{aligned} \tag{30.18}$$

here $\rho(\omega) = 3\nu\omega^2/\Omega_c^3$ is the phonon density of states in the Debye approximation. We restrict ourselves to a discussion of the relatively high temperature limit where the phonon distribution function can be replaced by $k_B T/\hbar\omega$. The energy accumulated by the phonons with $\hbar\omega < 2\mathbf{\Delta}$ is:

$$E_< = \int_0^{2\mathbf{\Delta}} n_\omega \rho(\omega)\, \hbar\omega d\omega = \frac{8\nu k_B T \mathbf{\Delta}^3}{(\hbar\Omega_c)^3} \tag{30.19}$$

In the integral Eq.(18) the first term is equal to 0, because it has only phonon distribution functions with $\hbar\omega < 2\mathbf{\Delta}$. The first non-zero contribution describes the decay of the phonon with $\hbar\omega > 2\mathbf{\Delta}$ to two phonons with $\hbar\omega < 2\mathbf{\Delta}$:

$$I_1 \simeq \frac{13}{12}\pi w^2 \left(\frac{3\nu}{(\hbar\Omega_c)^3}\right)^3 k_B^2 T(T'-T)(2\mathbf{\Delta})^7 \tag{30.20}$$

The second contribution describes the inelastic scattering of high frequency phonon with creation of one low frequency phonon with $\omega < 2\mathbf{\Delta}$:

$$I_2 \simeq \frac{8}{3}\pi w^2 \left(\frac{3\nu}{(\hbar\Omega_c)^2}\right)^3 k_B^2 T'(T'-T)\mathbf{\Delta}^4 \tag{30.21}$$

Eq.(21) shows that the inelastic scattering of high frequency phonons is dominant in the energy relaxation of the energy if $\mathbf{\Delta} \ll \hbar\Omega_c$ and we can neglect I_1.

Taking into account Eqs.(17), (19-21) we obtain expressions describing the relaxation of the equilibrated QP-phonon temperature:

$$\frac{\partial T}{\partial t} = \frac{1}{\tau_{ph}}(T'-T) \tag{30.22}$$

$$\frac{1}{\tau_{ph}} = \frac{9\pi\nu^2 w^2 k_B T' \mathbf{\Delta}(T)}{(\hbar\Omega_c)^3} \tag{30.23}$$

Note that Eq.(22) is rather similar to that derived by Allen[10] for the temperature relaxation in the electron-phonon system in the normal metals, except that Eq.(22) describes the phonon energy relaxation modified by the gap in the quasiparticle spectrum.

The phonon-phonon relaxation time (Eq.23) can be expressed in terms of an experimental parameter, namely the Raman phonon linewidth Γ_ω. Using the Fermi golden rule we can calculate Γ_ω for $k_B T \ll \hbar\omega$:

$$\Gamma_\omega = 2\pi w^2 \sum_{q,\nu,\nu'} \left(n_{\omega_{q,\nu}}+1\right)\left(n_{\omega_{-q,\nu'}}+1\right)\delta(\omega_{q,\nu}+\omega_{-q,\nu'}-\omega) \simeq \frac{3\pi w^2\nu^2\omega^2}{4\hbar\Omega_c^3}. \tag{30.24}$$

The phonon relaxation rate can thus be expressed in terms of Γ_ω in the following form:

$$\frac{1}{\tau_{ph}} = \frac{12\Gamma_\omega k_B T' \mathbf{\Delta}(T)}{\hbar\omega^2} \tag{30.25}$$

For a temperature dependent gap $\mathbf{\Delta}$, the relaxation time is expected to show a divergence $\tau_{ph} \propto 1/\mathbf{\Delta}(T)$ as is indeed observed in optimally doped cuprates[1, 5, 9, 18]. A similar divergence of the relaxation time has been calculated previously by Schmidt and Schön[19] and Tinkham[17], albeit for somewhat differently created nonequilibrium situations.

The typical relaxation timescale τ_{ph} given by the formula Eq.(25) is very close to the experimentally observed values. From the data on the Raman linewidth of the A_{1g}-symmetry apical O(4) phonon mode in YBCO - which has been shown to be particularly anharmonic - $\Gamma_\omega \approx 13$ cm^{-1} and $\omega \simeq 400$ cm^{-1}[21]. At $T^{'} \simeq T = T_c/2$, and using $\mathbf{\Delta}_{T_c/2} \approx 200$ cm^{-1} we obtain $\tau_{ph} =$ 0.8 ps.

At low temperatures the quasiparticle temperature $T^{'}$ is much higher than the lattice temperature T and the formula is expected to fail. However, from the experiments (Figure 2) we see from the temperature dependence of $|\Delta\mathcal{T}/\mathcal{T}|$ that the number of photogenerated quasiparticles is nearly constant at low T. $T^{'}$ can therefore be estimated from the equation:

$$n_{T'} - n_T = \mathcal{E}_I/\mathbf{\Delta}(0) \tag{30.26}$$

Taking into account that $n_T \simeq 2N(0)\mathbf{\Delta}(0)\exp(-\mathbf{\Delta}(T)/k_BT)$ we obtain:

$$k_BT^{'} \simeq \mathbf{\Delta}(T)/\ln\left\{1/(\mathcal{E}_I/2N(0)\mathbf{\Delta}(0)^2 + \exp(-\mathbf{\Delta}(T)/k_BT))\right\} \tag{30.27}$$

At low T, the nonequilibrium temperature $T^{'} \simeq \mathbf{\Delta}(T)/\ln\{(2N(0)\mathbf{\Delta}(0)^2/\mathcal{E}_I\} \simeq T_c/2$, while in the high temperature limit, the exponent in the logarithm becomes large and $T^{'} \simeq T$. Combining Eq.(25) and Eq.(27) we obtain an expression for the QP relaxation time as a function of lattice temperature T and photoexcitation energy $\mathcal{E}_I$ which is valid for all temperatures $0 < T < T_c$:

$$\frac{1}{\tau_{ph}} = \frac{12\Gamma_\omega\mathbf{\Delta}(T)^2}{\hbar\omega^2\ln\{1/(\mathcal{E}_I/2N(0)\mathbf{\Delta}(0)^2 + \exp(-\mathbf{\Delta}(T)/k_BT))\}}. \tag{30.28}$$

To enable comparison of this formula with experiments, we again use Raman data on high-frequency optical phonon linewidths, a value of gap frequency of 200 cm^{-1} ($2\mathbf{\Delta} \simeq 9k_BT_c$) from the fits in Figure 2c), the same values of $N(0) = 2.2 - 5$ eV^{-1}cell^{-1}spin^{-1} as in section III, and a BCS function for $\mathbf{\Delta}(T)$. $\mathcal{E}_I$ was again calculated using an incident laser energy of 0.2 nJ incident on a 100 nm thick film with a spot size of $\sim 100\mu$m diameter. We obtain a temperature dependence of the relaxation time τ_{ph} as shown in Figure 5. A comparison of the theory with the data for optimally doped $YBa_2Cu_3O_{7-\delta}$ samples[1, 5] in Figure 5 shows good agreement with the theory[22] with *no adjustable parameters*, especially near T_c. Some experimental data in the literature[1, 23] show an upturn in relaxation time at low temperatures, which is also weakly present in the data in Fig. 5. Although Eq.(28) is derived for temperatures close to T_c, an upturn at low temperature can still be reproduced by this formula if we reduce the energy per pulse $\mathcal{E}_I$. (In fact in the limit of $\mathcal{E}_I \to 0$, Eq. (28) gives $\tau_{ph} \propto 1/T$ at low T.) Indeed by setting $\mathcal{E}_I/N(0)$ in Eq. (28) as an adjustable parameter, rather than inserting fixed values as we have done, the low-temperature upturn in the data can be reproduced more accurately.

To end this section we briefly make a quantitative comparison of the relaxation rates for a gapped and gapless material. Comparing the relaxation time for the case of a gapped material with relaxation time in a normal metal described by Allen's formula[10] we obtain:

$$\frac{\tau_{ph}}{\tau_{e-ph}} \sim \frac{\lambda\hbar^3\omega^4}{4\pi(k_BT)^2\mathbf{\Delta}\Gamma_\omega} \tag{30.29}$$

Inserting experimental values for ω and Γ_ω we find the anharmonic phonon decay time in a superconductor to be approximately two orders of magnitude *slower* than electron-phonon relaxation in a gapless material as $\tau_{ph}/\tau_{e-ph} \sim 60 - 200$ for $T \simeq T_c$ as expected.

The calculation was performed assuming that the rate limiting step for QP relaxation is anharmonic decay of high-energy phonons, while direct electron phonon relaxation was assumed to be small. This is now justified quantitatively by a calculation of the direct electron-phonon relaxation rate. The contribution of phonon-electron scattering to the relaxation of phonon system can be estimated by deriving the corresponding relaxation rate τ_1^{-1} using the usual electron-phonon collision integral (see Appendix):

$$\frac{1}{\tau_1} = \frac{\pi N(0)\lambda \mathbf{\Delta}(T)\Omega_c}{2\nu} \tag{30.30}$$

We see that the electron-phonon relaxation rate, when compared to the phonon relaxation rate is reduced by the factor $N(0)\hbar\Omega_c/\nu \ll 1$:

$$\frac{\tau_{ph}}{\tau_1} = \frac{\pi\hbar^2 N(0)\lambda\omega^2\Omega_c}{24\nu\Gamma_\omega k_B T'} < 1 \tag{30.31}$$

and can therefore be neglected. However if the number of phonon modes involved in the relaxation is significantly smaller, the ratio τ_{ph}/τ_1 may approach unity. In this case the electron-phonon collisions can also contribute to the phonon relaxation. However the temperature dependence of the relaxation rate $1/\tau_1$ is very similar to the anharmonic relaxation τ_{ph}, both being proportional to $\mathbf{\Delta}(T)$ and so the two contributions to the relaxation may be difficult to separate experimentally. The data shown in Figure 5 are fit to τ_{ph} without any fitting parameters, so it appears to be reasonable to assume that anharmonic relaxation described by Eqs. (25) and (28) is dominant in $YBa_2Cu_3O_{7-\delta}$ near optimum doping.

30.6 Discussion

Near optimum doping in $YBa_2Cu_3O_{7-\delta}$ ($\delta < 0.1$) we find that the theoretical model gives a good quantitative fit to the temperature dependence of the photoinduced transmission (Figure 2c)), provided we use a temperature-dependent gap (Eq. (6)). The relatively square shape of the temperature dependence of $|\Delta\mathcal{T}/\mathcal{T}|$, the weak maximum near $T/T_c = 0.7$ and the rapid drop in the signal amplitude above this temperature is a consequence of the particular temperature dependence of the gap with the property that $\mathbf{\Delta}(T) \to 0$ as $T \to T_c$. In underdoped YBCO on the other hand (Fig. 2b)), the temperature dependence is much weaker and can only be reproduced by the calculation using a temperature-independent gap (Eq. (4)). It is *not possible* to reproduce the temperature dependence of the optimally doped data using a temperature-independent gap and vice versa, it is not possible to describe the underdoped data with a T-dependent BCS-like gap.[24]

A similar conclusion regarding the evolution of the gap with doping is reached on the basis of the comparison of the calculated temperature dependence of the relaxation time τ with the data. The divergence at T_c shown in Figure 5 for samples near optimum doping together with the temperature dependence of $|\Delta\mathcal{T}/\mathcal{T}|$ lead to the inescapable conclusion that there is a gap in $YBa_2Cu_3O_{7-\delta}$ with ($\delta \sim 0.1$) which closes rapidly at T_c. On the other hand, the notable *absence* of this divergence of τ for underdoped $YBa_2Cu_3O_{7-\delta}$ ($\delta > 0.15$)[5] is consistent with the existence of a T-independent gap deduced from the temperature dependence of $|\Delta\mathcal{T}/\mathcal{T}|$.

Plotting the temperature T^* and T_c as a function of δ in Figure 6, we find that T^* drops with increasing doping in a characteristic fashion, following the so-called "pseudogap" behaviour reported by other experimental techniques[25, 26, 27]. Near optimum doping, the "pseudogap" becomes observable close to T_c. Such a situation occurs in BCS superconductors where gap formation is a collective effect and Cooper pairs form simultaneously with the formation of a phase coherent collective state at T_c. However, from Figure 2c) and Figure 3 there is some evidence of remaining QP excitations above T_c even near $\delta \sim 0.1$. The T-independent "pseudogap" thus appears to be still present near optimum doping, and the BCS case is not quite realized.

The underdoped state is distinctly different, with a T-independent normal state pseudogap appearing at $T^* \gg T_c$. The changes in the photoinduced transmission $|\Delta\mathcal{T}/\mathcal{T}|$ at T^* in this case are simply a result of an increase population of pairs with decreasing temperature. Since the observed changes in $|\Delta\mathcal{T}/\mathcal{T}|$ are associated only with pairing and not phase coherence (which occurs at T_c), we see no change in $|\Delta\mathcal{T}/\mathcal{T}|$ at T_c in agreement with predictions of pre-formed pairing models of the underdoped state[28, 29]. The pseudogap in this case signifies the pair binding energy, and its value is determined by the microscopic pairing mechanism. It is to first approximation temperature independent and decreases with an inverse law with increasing carrier concentration, which is consistent with increased semi-classical (Debye-Hückel) screening between charge carriers [28].

Since the temperature dependence of $|\Delta\mathcal{T}/\mathcal{T}|$ is actually exponential at high temperatures (Eq.4), it is more appropriate to discuss the doping dependence of the magnitude of the energy gap Δ rather than of T^*. This avoids the somewhat arbitrary criterion for determining T^* of an asymptotic function. Such a plot of Δ versus doping δ is shown in Figure 7. As predicted on the basis of screening arguments above, $\mathbf{\Delta}$ appears to follow an inverse law $\mathbf{\Delta} \propto 1/x$ where $x = 0.6 - \delta$ is proportional to the carrier concentration.

The doping dependence of $\mathbf{\Delta}$ can be independently confirmed from the same data set by plotting the low-temperature value of $|\Delta\mathcal{T}/\mathcal{T}|$ as a function of δ. As can be seen from the formulae (4,6), the value of $|\Delta\mathcal{T}/\mathcal{T}|$ at $T = 0$ is proportional to $\mathcal{E}_I/\mathbf{\Delta}$, from which the zero-temperature gap $\mathbf{\Delta}_{T=0}$ can be determined. A scaled plot of $1/\left|\Delta\mathcal{T}/\mathcal{T}\right|_{T\to 0}$ as a function of doping δ in Figure 7 independently confirms the inverse relation $\mathbf{\Delta} \propto 1/(0.6-\delta)$. We note that the observation of an inverse law is in agreement with recent experiments on $La_{2-x}Sr_xCuO_4$[30], [31] suggesting that it may be a universal feature of the cuprates.

A somewhat surprising feature of the data is that they are described so well by an isotropic gap over the whole range of doping. A d-wave gap is apparently not consistent with either the temperature dependence of $|\Delta\mathcal{T}/\mathcal{T}|$ or the linear intensity dependence of $|\Delta\mathcal{T}/\mathcal{T}|$. It is also not consistent with the observation of a single exponential decay of $\Delta\mathcal{T}/\mathcal{T}$ with time, since in the d-wave case we would expect a distribution of τs, and certainly no single divergence of τ at T_c should be observed in the d-wave case. A possible explanation for the absence of a d-wave signature is that in YBCO the gap in the bulk of the sample is more or less isotropic in contrast to the surface, where it may be more d-like[32]. Another possibility for the apparent discrepancy between the present optical data and, for example, Raman experiments[33] are the different weights that the matrix elements have in the probing process. In the present experiments the dipole transition matrix elements average over the whole Fermi surface (FS) weighted by areas of large electron momentum. Anisotropy of the FS is not emphasized in this case. This is different than in Raman measurements, where the electronic scattering intensity is zero for an isotropic parabolic band, and Raman intensity comes only from the anisotropic regions of the FS.

We conclude by noting that the comparison of the theoretically predicted temperature dependence of the photoinduced absorption signal amplitude *and* relaxation time τ with the experimental results on underdoped and optimally doped $YBa_2Cu_3O_{7-\delta}$ gives good *quantitative* agreement. The evolution of the gap from a temperature-independent Δ to a temperature-dependent collective gap $\Delta(T)$ is a particularly striking feature of the data as is the apparent inverse dependence of the gap magnitude with doping. Although the model calculation presented in this paper was performed with the superconducting cuprates in mind, it is quite general and can, for example, also be applied to a case of photoexcitation experiments on CDW systems or narrow gap semiconductors. It could also be extended for other non-equilibrium situations as might occur in non-equilibrium superconducting high-T_c devices for example.

30.7 Appendix

To estimate the electron-phonon relaxation rate we take into account the electron-phonon collision integral[17]:

$$I_{ph-e}\{n_\omega\} = -4\pi \sum_k |M_{kk'}|^2 \left[f_k(1-f_{k'})n_q - f_{k'}(1-f_k)\left(n_q+1\right)\right]\delta(\epsilon_k - \epsilon_{k'} + \omega_q)$$

$M_{kk'}$ is electron-phonon coupling constant and n_ω, f_k are phonon and electron distribution functions.

Note that we neglect coherence factors in the electron-phonon collision integral. This is a reasonable approximation for the BCS case if the characteristic energy of quasiparticles is large in comparison with the gap $T > \mathbf{\Delta}$. For underdoped case also, the normal state gap is temperature independent and hence coherence factors are equal to 0. It is easy to show that the collision integrals satisfy the conservation energy law [10]:

$$2\sum_k \epsilon_k(\frac{\partial f_k}{\partial t}) + \sum_q \omega_q(\frac{\partial n_\omega}{\partial t}) = 0.$$

To calculate electron-phonon collision integral we define "electron-phonon spectral function" $\alpha^2 F$:

$$\alpha^2 F(\epsilon, \epsilon^{'}, \Omega) = \frac{2}{N(0)} \sum_{k,k'} \left|M_{k,k'}\right|^2 \delta(\omega_q - \Omega)\delta(\epsilon_k - \epsilon)\delta(\epsilon_{k'} - \epsilon^{'})$$

Following Allen[10], we suppose that $\alpha^2 F(\epsilon, \epsilon^{'}, \Omega) \simeq \alpha^2 F(\epsilon_F, \epsilon_F, \Omega) = \alpha^2 F(\Omega)$. For the sake of simplicity we make the Debye approximation for the electron-phonon spectral function $\alpha^2 F(\Omega) = \frac{\lambda \Omega^2}{\Omega_c^2}$.

$$\begin{aligned}\sum_{q(\omega_q<2\mathbf{\Delta})} \omega_q I_{e-ph}\{n_\omega\} &= -2\pi N(0)\int_0^{2\mathbf{\Delta}} \Omega\alpha^2 F(\Omega)d\Omega \int_{\mathbf{\Delta}}^{\infty} \{(f(\epsilon) - f(\epsilon+\Omega))\, n_\Omega - \\ &\quad f(\epsilon+\Omega)\,(1-f(\epsilon))\}d\epsilon v A1 \end{aligned} \tag{30.32}$$

The integral Eq.(A1) can be easily evaluated:

$$I_3 \simeq \frac{4\pi k_B N(0)\lambda \mathbf{\Delta}^4}{\hbar^2 \Omega_c^2}(T^{'} - T) \tag{A2}$$

Formula (A2) leads to the Eq.(30) for electron-phonon relaxation rate.

Figure 1. The normalized photoinduced transmission $\Delta\mathcal{T}/\mathcal{T}$ as a function of time delay for two different samples with a) $T_c = 53$K and b) $T_c = 90$K, each at three different temperatures. The time-evolution of the traces was fitted (solid lines) using a model with a *single* exponential decay and a Gaussian temporal profile pump pulse.

Figure 2. a) The photoinduced transmission amplitude $|\Delta\mathcal{T}/\mathcal{T}|$ in $YBa_2Cu_3O_{7-\delta}$ as a function of temperature for three different δ : for $\delta = 0.48$ (open squares), $\delta = 0.18$ (solid triangles) and $\delta \simeq 0.1$ (open circles). b) The solid line is a plot of the photoinduced transmission amplitude as a function of temperature for a temperature-independent gap using Expression (4) as a function of T/T^*. The data points are for underdoped samples with $\delta = 0.18$ with $T_c = 77$ K (solid triangles), $\delta = 0.32$ with $T_c = 62$ K (open triangles), $\delta = 0.44$ with $T_c = 53$ K (solid circles) and $\delta = 0.48$ with $T_c = 48$ K (open squares) respectively. c) The solid line is the calculated photoinduced transmission amplitude from Eq. (6) for a temperature-dependent gap as a function of normalized temperature T/T_c and data for the near-optimally doped samples $\delta \sim 0.1$ ($T_c = 90$K) from Mihailovic et al.[5] (open circles), Stevens et al. [2] (open squares) and Han et al. [1] (solid triangles). Note: the original datapoints of Ref.[1] were shifted by 7K in temperature. The solid lines in a) are fits with Eq. (4) for $\delta = 0.48$ and $\delta = 0.18$, and Eq. (6) for $\delta \simeq 0.1$.

Figure 3. The T-dependence of $|\Delta\mathcal{T}/\mathcal{T}|$ for near optimally doped sample ($T_c = 89$K) for three different laser intensities. The data has been normalized with respect to the energy density per unit volume deposited by the laser pulse, $\mathcal{E}_I$. The insert shows low temperature $|\Delta\mathcal{T}/\mathcal{T}|$ vs. $\mathcal{E}_I$, where $|\Delta\mathcal{T}/\mathcal{T}|$ scales linearly with $\mathcal{E}_I$ and $\mathcal{E}_0 = 8 \times 10^{-5}$ eV cell^{-1}spin^{-1}.

Figure 4. A plot of the temperature dependence of the photoinduced transmission amplitude (i.e. PE carrier density) for a pure d-wave gap (Eq. (12)). The solid line is for a 3D case, while the dashed line is for a 2D case using the same parameters. (The ordinate scale is the same for both curves.) The data (symbols) exhibit qualitatively different behaviour than predicted by the d-wave model, irrespective of doping δ. The data were scaled in the same way as in Fig.2. The open squares represent data for $\delta > 0.15$ while the solid squares are for $\delta < 0.1$.

Figure 5. The relaxation time τ as a function of temperature for a temperature-dependent gap (Eq. (28)). The data are for the near-optimally doped samples $\delta \sim 0.1$ ($T_c = 90$K) from Han et al.[1] (full circles) and Mihailovic et al.[5] (open circles). Note: the data points of Han et al[1] were shifted by 7K in temperature.

Figure 6. T_c (open squares) and T^* (solid squares) as a function of δ. T^* is defined as the point where the amplitude of signal $|\Delta\mathcal{T}/\mathcal{T}|$ drops to 5% of maximum of the T-dependent signal (the T-independent contribution, which is assumed to be due to Allen relaxation is subtracted).

Figure 7. The dependence of the energy gap Δ on doping δ. The open squares correspond to values of Δ determined from fits to the data using Eqs.(4) and (6), while the solid circles are obtained by plotting the scaled magnitude of the inverse of the photoinduced transmission at low temperature, $1/\left|\Delta\mathcal{T}/\mathcal{T}\right|_{T\to 0}$. In the latter case, the data were taken in a single experimental run, with carefully controlled laser operating conditions to ensure that Δ can be compared quantitatively. The dashed line is a guide to the eye showing $\Delta \propto 1/x$ bahaviour.

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Part VIII
TDGL: Properties

31 Time Variation of the Ginzburg-Landau Order Parameter by Gordon, Montgomery, Noer, Pickett, Tobon (1966)

Abstract

We use a nonequilibrium form of the Green's-function formulation of the BCS theory of superconductivity to investigate the circumstances under which differential equations in space and time, i.e., "time-dependent Ginzburg-Landau equations," give a valid description of the space and time variation of the order parameter Ψ in superconductors. We find that if the variations are sufficiently slow, time-dependent GinzburgLandau equations exist near absolute zero and near the transition temperature. In the former case the equation has wave-like character, and in the latter case it is of diffusion type, with the restriction that either the characteristic frequency of the time variation of Ψ is greater than the gap frequency or the ratio of the Fermi velocity to the product of the characteristic wavelength and frequency of the space-time variation of Ψ is greater than unity. Under all other circumstances and at general temperatures, there are no differential equations to describe the variations of Ψ. We discuss also the influence of slowly varying timedependent fields and derive the dependence of charge and current densities on the variations of Ψ. Local electrodynamics are assumed. The necessary modifications for the case of dilute superconducting alloys are described. Applications are made to the questions of collective modes, nucleation, and London's theory near absolute zero.

31.1 I. INTRODUCTION

THE phenomenological theory of Ginzburg and Landau [1] (GL) describes the equilibrium properties of a superconductor in a spatially varying magnetic field. The GL equations are differential equations which relate the spatial variation of the order parameter Ψ to the vector potential and the current.

Gor'kov [2,3] was able to show that the GL equations are a consequence of the microscopic theory of superconductivity [4,5] when the temperature T is close to the transition temperature T_c. The order parameter of the GL theory is identified with the energy-gap function of the microscopic BCS theory. Subsequently, the ideas of Gor'kov were extended to lower temperatures by others [6−8] and corresponding GL equations were derived from the microscopic theory. A common feature of all this work is the requirement that the order parameter and vector potential vary slowly over distances of the order of the coherence length and that the electrodynamics be local (London limit).

In this paper, we investigate the question of whether or not it is possible to extend the Gor'kov formulation

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[1] V. L. Ginzburg and L. D. Landau, Zh. Eksperim. i Teor. Fiz. 20, 1064 (1950).

[2] L. P. Gor'kov, Zh. Eksperim. i Teor. Fiz. 36, 1918 (1959) [English transl.: Soviet Phys.-JETP 9, 1364 (1959).

to the nonequilibrium situation in order to discuss the time variation of the order parameter by means of time-dependent GL equations. These would be differential equations in space and time relating the variations of the order parameter to space and time-varying fields and the charge and current densities. In the case of a pure superconductor, we find time-dependent GL equations only when T is near absolute zero and when T is near T_c. In the former case, the equation for the order parameter is wave-like and in the latter case it is of diffusion-type.

In a recent note, Stephen and Suhl [9] addressed themselves to the same question and used a formulation which is in many respects similar to the one with which we begin our discussion. We differ with them in our methods of analysis and we defer relevant remarks to the concluding Sec. VII of this paper.

We mention two limitations to our development here. Firstly, we take the thermal excitations (normal fluid) in the superconductor to be always at rest and in equilibrium with the local value of the energy-gap parameter and the external fields. This presupposes that the characteristic time for interaction between the heat bath (lattice vibrations) and the thermal excitations is short compared to the characteristic time for normal to superfluid conversion. Secondly, in anticipation of the case of the charged superconductor where density fluctuations are suppressed due to the longrange nature of the Coulomb interaction, we do not, even in the neutral case, solve simultaneously for the motion of the order parameter and the motion of the density. That is, we take the variation of the order parameter to be the sole driving term. More detailed remarks on these points will be made at appropriate places in the text.

We devote the greater part of the paper to the neutral pure superconductor. In the charged case, we restrict

ourselves to the London limit and give only the leading terms in the external fields. We discuss briefly the effect of impurities on some of the results. In Sec. II, we develop the formalism for treating the nonequilibrium case and apply it to the situation that the order parameter has sufficiently small time and space variations. In Sec. III, we use the formalism to extend the Gor'kov expansion procedure [2] when the order parameter varies slowly to the case of time as well as space variations and we derive time-dependent GL equations. In Sec. IV, we show that the expansion procedure of Sec. III is valid only at $T = 0$ and we derive the correct timedependent GL equation near T_c by analyzing the wave number and frequency-dependent response to space and time variations of the order parameter. Section V is devoted to the charge and current expressions and the influence of external electromagnetic fields. The modifications required for dilute alloys are discussed in Sec. VI. In the concluding Sec. VII, we discuss the results and their applications.

31.2 II. NONEQUILIBRIUM EQUATIONS

Our general purpose is to derive equations for the order parameter $\Psi(\mathbf{r}, t)$. This quantity is proportional to the local self-consistent pair field or energy-gap function $\Delta(\mathbf{r}, t)$ and we shall restrict ourselves to cases where the space and time variations of Δ are small and occur only over distances greater than the coherence length ξ_0 and over times greater than $\hbar/kT_c$. We

[3] A. A. Abrikosov, L. P. Gor'kov, and I. E. Dzyaloshinski, Methods of Quantum Field Theory in Statistical Mechanics, translated by R. A. Silverman (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963), Chap. 7.

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begin by deriving integral equations relating the energy gaps at different space-time points. The procedure is an extension to the nonequilibrium, time-dependent case of previously used methods [6–8] which were based on Gor'kov's original derivation [2] of the equilibrium GL equations for temperatures near T_c. We employ Green's function methods [3,10] throughout and restrict ourvelves to the BCS model of superconductivity.

In the equilibrium case, one gets the energy gap from a calculation of the quasiparticle self-energy [11] by means of equilibrium thermal Green's functions. In the present case, when time-varying fields and interactions are present, we construct nonequilibrium one-particle Green's functions as follows. We imagine that in the distant past ($t = -\infty$), the system is in thermodynamic equilibrium with Hamiltonian H and chemical potential μ. We let time-dependent interactions occur which we describe by the Hamiltonian $H'(t)$ which includes external time-dependent fields and the time-dependent energy gap through the self-consistent pair potential as described below. We work in the interaction representation for $H + H'$ so that states develop in time according to the time-development operator U :

$$U(t, t') = T \exp\left[-i \int_{t'}^{t} H'(t_1)\, dt_1\right] \tag{2.1}$$

where the symbol T is the usual time-ordering operator. The single-particle Green's function is then defined as

$$\begin{aligned} G(1,2) = &- i \left\langle\left\langle T\psi(1)\psi^\dagger(2)\right\rangle\right\rangle \\ = &- i \operatorname{Tr} e^{-\beta(H-\mu N)} T\left[U(-\infty, t_1)\, \psi(1) U(t_1, t_2)\right. \\ &\left.\times \psi^\dagger(2) U(t_2, -\infty)\right] \left[\operatorname{Tr} e^{-\beta(H-\mu N)}\right]^{-1}, \end{aligned} \tag{2.2}$$

$F(1,2) = \langle\langle T\psi(1)\psi(2)\rangle\rangle$.

In Eq. (2.3), we have defined the "anomalous" Green's function. [5] The electron field operators $\psi, \psi^\dagger$ are in the interaction representation. The space-time point $\mathbf{r}_1, t_1$ is denoted by the symbol 1 . Time-dependent properties of the system are determined by the Green's functions G and F which are thermal averages over the equilibrium grand canonical ensemble at $t = -\infty$. For example, the order parameter at $\mathbf{r}_1, t_1$ is proportional to $F(1,1)$.

In the equilibrium case one can proceed to find the energy gap by adopting a simple form for that part of the Hamiltonian density which gives the superconducting interaction, which, following Gor'kov, [5] we take to be local in space and time:

$$\begin{aligned} H(1) &= g\psi^\dagger(1)\psi^\dagger(1)\psi(1)\psi(1) \\ &= -\Delta(1)\psi^\dagger(1)\psi^\dagger(1) - \Delta^*(1)\psi(1)\psi(1) \end{aligned}$$

where g is the coupling constant ($g < 0$ for an attractive interaction) and Δ is the (constant) self-consistent pair potential or energy gap which is to be evaluated in a self-consistent manner by using the above form of the Hamiltonian to generate the equations of motion for the equilibrium G and F and then solving for Δ from the solution for $F : \Delta(1) = -igF(1,1) =$ const. The equations of motion can be written as integral equations and Δ is then found in the form

$$\Delta(1) = ig \int d^3 r_2 \int_{-\infty}^{+\infty} dt_2 G_0(2,1) G(2,1) \Delta(2) \tag{2.4}$$

where G_0 denotes the normal-state Green's function ($g = 0$).

[1210] L. P. Kadanoff and G. Baym, Quantum Statistical Mechanics (W. A. Benjamin, Inc., New York, 1962).
[11] Reference 3, Sec. 34.3.

In the nonequilibrium case, the pair potential varies with time but the Green's functions defined in Eqs. (2.2) and (2.3) satisfy formally the same equations of motion as in the equilibrium situation. However, because the boundary conditions are different, the corresponding integral equations are different. In fact, the integral equations are manifestly causal. That is, the gap function at time t_1, for example, is determined only by values of the gap function at earlier times. In this way, one can write down causal integral equations for G and F in the nonequilibrium case. From the latter, one obtains the integral equation for the gap function which corresponds to Eq. (2.4).

$$\Delta(1) = ig \int d^3 r_2 \int_{-\infty}^{t_1} dt_2 [G_0 < (2,1) G^{<}(2,1) \\ - G_0 > (2,1) G > (2,1)] \, \Delta(2) \tag{2.5}$$

where

$$G^{>}(2,1) = -i \left\langle\left\langle \psi(2)\psi^{\dagger}(1) \right\rangle\right\rangle, \quad G^{<}(2,1) = i \left\langle\left\langle \psi^{\dagger}(1)\psi(2) \right\rangle\right\rangle.$$

The two terms of Eq. (2.5) describe separately the electron and hole contributions to $\Delta(1)$. If Δ is taken as constant in time, Eqs. (2.4) and (2.5) are equivalent.

The boundary conditions of the nonequilibrium case are nicely kept track of in a scheme devised by Kadanoff and Baym. [12] Following them, we go over to new Green's functions defined in the imaginary time domain in the usual manner of the Matsubara technique by replacing t by $-i\tau$.[13] In the present case, however, τ is defined between τ_0 and $\tau_0 + \beta$. The new Green's functions are given by

$$\begin{aligned} \mathcal{G}(1,2) &= -\operatorname{Tr} e^{-\beta(H-\mu N)} T \left[\mathfrak{u}\left(\tau_0+\beta, \tau_1\right) \psi(1) \mathfrak{u}\left(\tau_1, \tau_2\right) \right. \\ &\qquad \left. \times \psi^{\dagger}(2) \mathfrak{u}\left(\tau_2, \tau_0\right) \right] / \operatorname{Tr} e^{-\beta(H-\mu N)}, \\ \mathfrak{F}(1,2) &= \operatorname{Tr} e^{-\beta(H-\mu N)} T \left[\mathfrak{u}\left(\tau_0+\beta, \tau_1\right) \psi(1) \mathfrak{u}\left(\tau_1, \tau_2\right) \right. \\ &\qquad \left. \times \psi(2) \mathfrak{u}\left(\tau_2, \tau_0\right) \right] / \operatorname{Tr} e^{-\beta(H-\mu N)}, \end{aligned} \tag{2.7}$$

where the time-ordering operator T orders decreasing values of $\tau - \tau_0$ from left to right, the time development operator $\mathcal{U}$ is given by

$$\mathfrak{u}\left(\tau, \tau'\right) = T \exp\left[-\int_{\tau'}^{\tau} H'\left(\tau_1\right) d\tau_1 \right] \\ \left(\tau_0 + \beta > \tau_1, \tau_2 > \tau_0\right) \tag{2.8}$$

and the τ dependence of operators ($H', \psi, \psi^{\dagger}$) in the new interaction representation is given by

$$O(\tau) = e^{(H-\mu N)\tau} O e^{-(H-\mu N)\tau}. \tag{2.9}$$

As in the equilibrium case, analytic continuation of the new Green's functions gives the actual Green's functions. In the present case, the rule is to let all $\tau \to it$ and $\tau_0 \to -i\infty$. For example, $\mathcal{G}(\tau)$ continued in this manner is just the causal $G(t)$ defined in Eq. (2.2). In performing the continuation [in Eq. (2.6), say], it is necessary to write

$$\mathfrak{u}\left(\tau_0+\beta, \tau_1\right) = \mathfrak{u}\left(\tau_0+\beta, \ \tau_0\right) \mathfrak{u}\left(\tau_0, \tau_1\right) \to \\ \mathfrak{u}(-i\infty+\beta, -i\infty) U\left(-\infty, t_1\right),$$

where the last factor is the ordinary time-development operator of Eq. (2.1) obtained from $\mathcal{U}\left(\tau_1 \to it_1, \tau_2 \to it_2\right) = U\left(t_1, t_1\right)$. The other factor, $\mathfrak{U}(-i\infty+\beta, -i\infty)$, is from its definition

equal to unity if one takes into account an "adiabatic switching" parameter $\eta \to 0^+$by means of which we insure $H'(t=-\infty)=0$. That is, we include a factor $\exp \eta|t|$ in the definition of $H'(t)$.

Now, $\mathcal{G}(1,2)$ satisfies periodicity properties similar to those of the thermal Green's functions in the usual (equilibrium) Matsubara technique. [13] In the present case, [12] we have

$$\mathcal{G}\left(\tau_0+\beta,\tau'\right)=-\mathcal{G}\left(\tau_0,\tau'\right).$$

It is then possible to expand $\mathcal{G}$ in a Fourier series whose coefficients, as in the usual technique, are just the corresponding Green's functions $\mathcal{G}\left(\omega_n\right)$ having discrete frequencies $i\omega_n=\pi i(2n+1)/\beta$. If one calculates quantities like $\mathcal{G}\left(\omega_n\right)$, then analytic continuation of $z=i\omega_n$ to the real axis in the usual manner gives $G(\omega)$, the Fourier transform of the original $G(t)$ given in Eq. (2.2).

It is easily verified, that the functions $\mathcal{G},\mathfrak{F}$ satisfy the usual equations of motion (Gor'kov equations [5]) with time-dependent external fields, if they are present, included. Thus, in matrix form, the equations of motion are

$$\begin{aligned}&\begin{pmatrix} L_1^- & \Delta(1) \\ -\Delta^\dagger(1) & L_1^+ \end{pmatrix}\begin{pmatrix} \mathcal{G}(1,2) & \mathcal{F}(1,2) \\ \mathfrak{F}^\dagger(1,2) & -\mathcal{G}(2,1) \end{pmatrix}\\ &=\delta(1,2)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\end{aligned} \tag{2.10}$$

where $L_1{}^\pm=\pm\partial/\partial\tau_1+(\boldsymbol{\nabla}\pm ie\mathbf{A}/c)^2/2m+\mu,\mu$ is the chemical potential and we restrict ourselves here to the case where there is only a transverse vector potential acting. The general case will be discussed in Sec. V. The energy gap $\Delta(1)$ may be obtained from the anomalous Green's function $\mathfrak{F}$ by the relation

$$\Delta(1)=-g\mathfrak{F}\left(1,1^+\right). \tag{2.11}$$

It should be pointed out that the energy gap function of Eqs. (2.10) and (2.11) is not in fact the actual gap function. The latter is given by the expression

$$-ge^{-2i\mu t_1}\mathfrak{F}\left(1,1^+\right).$$

This can be verified by comparison of the analytic continuation of $\mathfrak{F}$ [Eq. (2.7)] and the actual F of Eq. (2.3).

We solve the equations of motion by iteration on the slow variation of the energy-gap function in the following manner (we exclude here the external field to which we shall return in Sec. V): Imagine a superconductor with a constant gap parameter (not necessarily the equilibrium one) Δ_l. We call this the local gap. The corresponding local Green's functions $\mathcal{G}_l,\mathfrak{F}_l$ solve the equations of motion, Eq. (2.10), with Δ replaced by Δ_l. It is easily verified that the solution of the equations of motion for a superconductor whose gap function varies in an arbitrary manner is given by the following integral equations

[1212] Reference 10, Chap. 8.

[13] Reference 3, Chap. 3.

$$\mathcal{G}(1,2) = \mathcal{G}_l(1,2) - \int d3\mathcal{G}_l(1,3)\delta\Delta(3)\mathfrak{F}^\dagger(3,2) - \int d3\mathfrak{F}_l(1,3)\delta\Delta^\dagger(3)\mathfrak{G}(3,2) \tag{2.12}$$

$$\mathfrak{F}^\dagger(1,2) = \mathfrak{F}_l^\dagger(1,2) + \int d3\mathfrak{G}_l(3,1)\delta\Delta^\dagger(3)\mathfrak{G}(3,2) - \int d3\mathfrak{F}_l^\dagger(1,3)\delta\Delta(3)\mathfrak{F}^\dagger(3,2) \tag{2.13}$$

where $\delta\Delta(3) = \Delta(3) - \Delta_l$ is the deviation of the actual gap function at the space-time point 3 from the constant local value Δ_l.

We assume the varations $\delta\Delta$ to be small and iterate the integral equations. For example, the energy gap at point 1 requires knowledge of $\mathfrak{F}(1,1)$ which is given, to second order in $\delta\Delta$, by the following expression:

$$\begin{aligned}\mathfrak{F}^\dagger(1,1) &= \mathfrak{F}_l^\dagger(1,1) + \int d3\mathfrak{G}_l(3,1)\delta\Delta^\dagger(3)\mathfrak{g}_l(3,1) - \int d3\mathfrak{F}_l^\dagger(1,3)\delta\Delta(3)\mathfrak{F}_l^\dagger(3,1) \\ &- \iint d3d4\Big[\mathfrak{g}_l(3,1)\delta\Delta^\dagger(3)\mathfrak{g}_l(3,4)\delta\Delta(4)\mathfrak{F}_l^\dagger(4,1) + \mathfrak{g}_l(3,1)\delta\Delta^\dagger(3)\mathfrak{F}_l(3,4)\delta\Delta^\dagger(4)\mathfrak{g}_l(4,1) \\ &+ \mathfrak{F}_l^\dagger(1,3)\delta\Delta(3)\mathfrak{g}_l(4,3)\delta\Delta^\dagger(4)\mathfrak{g}_l(4,1) - \mathfrak{F}_l^\dagger(1,3)\delta\Delta(3)\mathfrak{F}_l^\dagger(3,4)\delta\Delta(4)\mathfrak{F}_l^\dagger(4,1)\Big].\end{aligned} \tag{2.14}$$

In the causal perturbation theory which we are using, as remarked above, all interactions $H'(\tau)$ [including the pair potential or energy gap $\Delta(\tau)$] include an adiabatic switching parameter $\eta \to 0^+$so we have to include, for example, the factor $\exp(-i\eta\tau_3)$ in $\delta\Delta(3)$. We recall also that the time integrations all run between τ_0 and $\tau_0 + \beta$ and that $\tau_0 \to -i\infty$ at the end.

The next step is to set $\Delta_l = \Delta(1)$. Then we may evaluate the right-hand side of Eq. (2.14), for example, in terms of $\Delta(1)$ by either Taylor series expansion of Δ (3) about $\Delta_l = \Delta(1)$ (Sec. III) or by Fourier analysis (Sec. IV). As we shall see, the procedures will be valid only if the variations in Δ are sufficiently slow.

31.3 III. TAYLOR SERIES METHOD

The original derivation, valid near T_c, of the GL equation by Gor'kov [2] rests on the fact that the energy gap is slowly varying in space compared to the Green's function. We may extend his method by taking advantage of the fact that the Green's functions $\mathcal{G}_l$ and $\mathfrak{F}_l$ which appear in the kernels of the integral equations for $\mathcal{G}$ and $\mathfrak{F}$ [for an example, see Eq. (2.14)] fall off rapidly compared to what we assume are the allowed variations in Δ. The spatial variation of $\mathcal{G}$ and $\mathfrak{F}$ is exponential fall off with a characteristic length given in order of magnitude at all temperatures below the critical temperature T_c by the coherence length $\xi_0 \approx v_F/kT_c$ (v_F = Fermi velocity) while the variations in Δ and electromagnetic fields occur over distances of the order of the penetration depth λ. In what follows, we assume $\lambda \gg \xi_0$ so that we may expand the spatial variation of Δ in Taylor series since it will change only slightly over ξ_0, the distance of importance in the integrands.

In the Taylor series technique, the time variation of Δ is treated in the same manner as the space variation: We assume that the variation of Δ is slow compared to the times of importance in the integrands. These times are again determined by the behavior of $\mathcal{G}_l$ and $\mathfrak{F}_l$ which fall off exponentially in times of the order of $\hbar/kT_c$ as long as Δ_l is not different in order of magnitude from the equilibrium gap.

We now have a well-defined method for deriving a differential equation satisfied by the energy gap. In the integrands of the integral equation for the gap
[Eq. (2.14)], we expand $\Delta(3)$ in a space-time Taylor series about the local value $\Delta_l = \Delta(1)$:

$$\delta\Delta(3) = \Delta(3) - \Delta(1) = \partial_i \Delta(1) {x_{31}}^i + \frac{1}{2}\partial_i\partial_j\Delta(1){x_{31}}^i{x_{31}}^j + \left[\partial\Delta(1)/\partial\tau_1\right]\tau_{31} + \frac{1}{2}\left[\partial^2\Delta(1)/\partial{\tau_1}^2\right]{\tau_{31}}^2 \tag{3.1}$$

where ∂_i means $\partial/\partial{x_1}^i$ and repeated indices are summed from 1 to 3 . We insert this expression into the integral equation for the gap and we keep only terms involving two derivatives. We obtain in this way the timedependent GL equation

$$\Delta^\dagger(1) = -g\mathfrak{F}_l^\dagger(1,1) + \mathscr{L}\left(\Delta, \Delta^\dagger\right)$$

where $\mathscr{L}\left(\Delta, \Delta^\dagger\right)$ is a nonlinear operator on $\Delta(1), \Delta^\dagger(1)$ which contains space and time derivatives with coefficients determined by the integrals of the integral equation, Eq. (2.14), after the series substitution of Eq. (3.1) for Δ (3). There are no cross terms in involving space and time derivatives since we kept only two derivatives and terms with one space derivative vanish by space inversion symmetry.

Before deriving the time-dependent GL equation at an arbitrary temperature, we discuss the situation when $T \approx T_c$. In this case, the method is very similar to the original Gor'kov derivation [2] of the time-independent GL equation. Near T_c, Δ itself is small everywhere and we modify the integral equation for Δ by taking $\Delta_l = 0$. Thus $\mathfrak{F}_l = 0$ and $\mathcal{G}_l$ is simply $\mathcal{G}_n$, the normal-state Green's function. Then the term in Eq. (2.14) for $\Delta^\dagger(1)$ which gives rise to derivatives in the GL equation is simply
$\Delta^\dagger(1) = -g\int d3 g_n(3,1)\Delta^\dagger(3)g_n(3,1)\exp\left(-i\eta\tau_3\right)$,
where $\eta \to 0^+$is the adiabatic switching parameter. We now expand Δ (3) in Taylor series about the point 1 according to Eq. (3.1). The space parts give rise to the usual terms of the time-independent GL equation. We content ourselves here with the discussion of the timederivative terms. We may note that our procedure is only valid when q_0, the characteristic frequency of the time variation of Δ is greater than Δ itself. Otherwise, it is not valid to drop Δ everywhere on the right-hand side of Eq. (2.14).

The time-derivative terms arise from the following part of Eq. (3.2):

$$-g\int d^3r_3 \int_{\tau_0}^{\tau_0+\beta} d\tau_3 \mathfrak{G}_n^2(3,1)\left[\Delta^{\dagger\prime}\tau_{31} + \tfrac{1}{2}\Delta^{\dagger\prime\prime}\tau_{31}^2\right] \times \exp\left(-i\eta\tau_3\right) \tag{3.3}$$

where $\Delta^{\dagger'}$ denotes $\partial\Delta^\dagger(1)/\partial\tau_1$. In performing the τ_3 integral we find that the convergence factor containing η causes contributions $\exp\left(-i\eta\tau_0\right)$ to vanish when we let $\tau_0 \to -i\infty$. The result of the τ_3 integration is simply

$$-g\sum_{\mathrm{p}} \tanh(\beta\xi/2)\left[\Delta^{\dagger\prime}(2\xi + i\eta)^{-2} + \Delta^{\dagger\prime\prime}(2\xi + i\eta)^{-3}\right]$$

where we have first transformed the Green's function to momentum space and $\xi = p^2/2m - \mu$. Making the substitution $\sum_p \to N_0\int d\xi$, where N_0 is the density of states at the Fermi surface in the normal metal, we find the time-derivative terms to be

$$-gN_0\left(K_2\Delta^{\dagger\prime} + K_3\Delta^{\dagger\prime\prime}\right)$$

where

$$K_n = \int d\xi \tanh(\beta\xi/2)(2\zeta + i\eta)^{-n}$$

We assume particle-hole symmetry and evaluate the K_n by residues. The result is

$$K_n = 2(\beta/2\pi i)^{n-1}\left(1 - 2^{-n}\right)\zeta(n) \tag{3.4}$$

where ζ is the Riemann zeta function. After the substitution $\tau \to it$, we may add the time-derivative terms to the space derivatives of the usual GL equation. The result is

$$\begin{aligned}\{\nabla^2/4m - (1/c)\partial/\partial t + (3/8\epsilon_F)\,\partial^2/\partial t^2 + (1/6)\,[1 - T/T_c \\ -7\zeta(3)|\Delta|^2/8\pi^2\,(kT_c)^2]\}\,\Delta^\dagger = 0,\end{aligned} \tag{3.5}$$

where ϵ_F = Fermi energy, $b = 7\zeta(3)\epsilon_F/6\pi^2\,(kT_c)^2$, $c = (8kT_c/\pi)\,b$. The first time-derivative term is larger than the second by a factor $[2\pi^3/7\zeta(3)]\,(kT_c/q_0) \gg 1$ so that we may neglect the latter. We therefore have a differential equation of the diffusion type to describe the space-time variation of the order parameter near T_c when $q_0 > \Delta$. The equation describes how the selfconsistent superconducting interaction (pair potential) drives the order parameter towards its equilibrium value.

Actually, Eq. (3.5) as written is not for the actual gap parameter as discussed in the previous section below Eq. (2.11). The time derivatives in the equation must be altered to include the time-dependent phase involving the chemical potential. We defer this question to the end of this section.

We now return to the case of arbitrary temperature: Δ is not small. We must use the full integral equation of Eq. (2.14). We treat first the space derivatives by inserting the space part of the Taylor series [Eq. (3.1)] into the integral equation. It is convenient to make the Fourier expansion in space and time for the Green's
functions which appear on the right-hand side of the resulting expression for $\mathfrak{F}^\dagger(1,1)$. Thus, for example,

$$\mathfrak{F}_l(1,3) = \beta^{-1}\sum_\omega \exp\left(-i\omega\tau_{13}\right)\sum_{\mathbf{p}} \mathfrak{F}_l(\omega,\mathbf{p})\exp\left(i\mathbf{p}\cdot\mathbf{r}_{13}\right),$$

where ω labels the discrete Matsubara frequency [13]$\pi(2n+1)/\beta$ and $\tau_{13} = \tau_1 - \tau_3, \mathbf{r}_{13} = \mathbf{r}_1 - \mathbf{r}_3$. Then the factors ${\mathbf{r}_{31}}^i$ which appears on the right-hand side from the expansion of $\delta\Delta(3)$ can be replaced by the derivatives with respect to the i th component of $\mathbf{p}$ which in turn are transferred to the appropriate ${\mathfrak{F}_l}^\dagger(\omega,\mathbf{p})$, $\mathscr{F}_l(\omega,\mathbf{p})$ or $\mathscr{G}_l(\omega,\mathbf{p})$ by partial integration in p space. The result of these steps is

$$\begin{aligned}\Delta^\dagger(1) = &- g\mathfrak{F}^\dagger(1,1) = -g\beta^{-1}\sum_{\mathrm{p},\omega}\{\mathfrak{F}^\dagger + \left(\partial\mathcal{G}^-/\partial p_i\right) \\ &\times\left(\partial\mathcal{G}/\partial p_j\right)\left[\frac{1}{2}\partial_i\partial_j\Delta^\dagger(1)\right] - \left(\partial\mathfrak{F}^\dagger/\partial p_i\right)\left(\partial\mathfrak{F}^\dagger/\partial p_j\right) \\ &\times\left[\frac{1}{2}\partial_i\partial_j\Delta(1)\right] - \left(\partial\mathcal{G}^-/\partial p_i\right)\mathcal{G}\left(\partial\mathfrak{F}/\partial p_j\right) \\ &\times\left[2\partial_i\Delta^\dagger(1)\partial_j\Delta(1)\right] - \left(\partial\mathcal{G}^-/\partial p_i\right)\mathfrak{F}\left(\partial\mathcal{G}/\partial p_j\right) \\ &\times\left[\partial_i\Delta^\dagger(1)\partial_j\Delta^\dagger(1)\right] + \left(\partial\mathfrak{F}^\dagger/\partial p_i\right)\mathfrak{F}^\dagger\left(\partial\mathfrak{F}^\dagger/\partial p_j\right) \\ &\times\left[\partial_i\Delta(1)\partial_j\Delta(1)\right]\}\end{aligned} \tag{3.6}$$

where, on the right-hand side, we have dropped the subscripts and arguments of all the Green's functions. The notation $\mathcal{G}^-$means $\mathcal{G}_l(-\omega,-\mathbf{p})$. No linear derivative appears because of space-inversion symmetry.

To evaluate the coefficients of the space derivatives of Δ on the right-hand side of Eq. (3.6), we use the fact that $\mathcal{G}_l$ and $\mathfrak{F}_l$ solve the Gor'kov equations [5] with the local gap $\Delta_l = \Delta(1)$. Thus,

$$\mathcal{G}_l = -(i\omega + \xi)/\left(\omega^2 + E^2\right), \quad \mathfrak{F}_l = \Delta(1)/\left(\omega^2 + E^2\right) \tag{3.7}$$

where $E^2 = \xi^2 + |\Delta(1)|^2$. It it here that we make the assumption [14] regarding the behavior of the thermal excitations to which we referred in the Introduction. The distribution of thermally excited quasiparticles as determined from $\mathcal{G}_l$, for example, is given by the Fermi distribution for energy E (which contains the local gap) and temperature T. That is, we assume the gap in the spectrum to be determined by the local value of the varying pair potential and the normal fluid to be in equilibrium at temperature T with this spectrum. Thus, the interaction of the normal fluid with the phonon heat bath is taken to be faster than the rate at which the pair potential itself varies.

The rest of the calculation is straightforward. We introduce Eq. (3.7) into Eq. (3.6). Derivatives with respect to p_i give rise to terms proportional to p_i. The integrands are peaked near $p = p_F$, the Fermi momentum, so we may replace $p_i p_j \partial_i \partial_j$ by $\frac{1}{3}{p_F}^2 \partial_i \partial_i$ and $\sum_p$ by $N_0 \int d\xi$. The final result is

$$\begin{aligned}
&\left(1 + \frac{1}{2} g N_0 I_1\right) \Delta^\dagger(1) \\
&= -\left(g\pi N_0 \epsilon_F/6m\right) \Big\{ J_3 \nabla^2 \Delta^\dagger(1) - \frac{1}{2} J_5 \left(\Delta \nabla^2 \Delta^\dagger + \Delta^\dagger \nabla^2 \Delta\right) \\
&\quad \times \Delta^\dagger(1) - \frac{1}{2} J_5 \left(2\Delta^\dagger \nabla \Delta + 3\Delta \nabla \Delta^\dagger\right) \cdot \nabla \Delta^\dagger(1) \\
&\quad + (5/8) J_7 \left(\Delta^\dagger \nabla \Delta^\dagger + \Delta \nabla \Delta^\dagger\right)^2 \Delta^\dagger(1) \Big\}
\end{aligned} \tag{3.8}$$

where

$$I_1 = \int d\xi \tanh(\beta E/2) E^{-1} \tag{3.9}$$

$$J_n = \beta^{-1} \sum_\omega \left[\omega^2 + |\Delta(1)|^2\right]^{-n/2} \tag{3.10}$$

I_1 is formally divergent but may be evaluated by using the usual cutoff of the ξ integral or the fact that $I_1 = -2/gN_0$ when Δ is the equilibrium value of the gap at the temperature T. Note that the coefficients of the derivatives depend on $\Delta(1)$ itself. The result, Eq. (3.8), has been found previously by Werthamer. [6]

We treat the time-derivative terms by a method parallel to that used above near T_c. We consider the case $q_0 \ll \Delta$ as is consistent with the Taylor series method. We insert the time part of the Taylor series [Eq. (3.1)] into the integral equation of Eq. (2.14). In the terms on the right-hand side, we use the Fourier coefficients

$$\begin{aligned}
&\mathcal{G}_l(\mathbf{p}, \tau) = \frac{1}{2} \left\{ (1 + \xi/E) e^{-E\tau} \left[n_E - \theta(\tau)\right] \right. \\
&\left. -(1 - \xi/E) e^{E\tau} \left[n_E - \theta(-\tau)\right] \right\} \\
&\mathfrak{F}_l^\dagger(\mathbf{p}, \tau) = -\left[\Delta^\dagger(1)/2E\right] \left\{ e^{-E\tau} \left[n_E - \theta(\tau)\right] \right. \\
&\left. + e^{E\tau} \left[n_E - \theta(-\tau)\right] \right\}
\end{aligned} \tag{3.11}$$

[12][14] One of us (EA) wishes to acknowledge an informative discussion with L. P. Kadanoff on this point.

where $n_E = (\exp \beta E + 1)^{-1}$. We perform the time integrations as before, taking advantage of the convergence factor η. The result is the time-derivative terms of the time-dependent GL equation:

$$\begin{aligned}
&\left(1 + \frac{1}{2} g N_0 I_1\right) \Delta^\dagger(\mathbf{r}, t) \\
&= -\left(g\pi N_0 \epsilon_F / 6m\right) \Big\{ J_3 \nabla^2 \Delta^\dagger - \frac{1}{2} J_5 \left(\Delta \nabla^2 \Delta^\dagger + \Delta^\dagger \nabla^2 \Delta\right) \Delta^\dagger \\
&- \frac{1}{2} J_5 \left(2\Delta^\dagger \nabla \Delta + 3\Delta \nabla \Delta^\dagger\right) \cdot \nabla \Delta^\dagger + (5J_7/8) \\
&\times \left(\Delta^\dagger \nabla \Delta + \Delta \nabla \Delta^\dagger\right)^2 \Delta^\dagger - \left(3/2\pi {v_F}^2\right) \Big[I_3 \Delta^{\dagger\prime} \\
&- \frac{1}{2} I_5 \left(\Delta \Delta^{\dagger\prime}\right) \quad + \Delta \Delta'' \Big) \Delta^\dagger - \frac{1}{2} I_5 \left(2\Delta^\dagger \Delta' + 3\Delta \Delta^{\dagger\prime}\right) \Delta^{\dagger\prime} \\
&\qquad (5I_7/8) \left(\Delta \Delta' + \Delta \Delta^\dagger\right)^2 \Delta^\dagger \Big] \Big\}
\end{aligned} \tag{3.12}$$

where

$$I_n = \int d\xi \tanh(\beta E/2) E^{-n} \tag{3.13}$$

and, as in Eq. (3.5), we have not corrected the partial time derivatives (denoted by primes) to include the time dependence arising from the chemical potential.

We may simplify the GL equation in the limiting case $T \to 0$. For $T = 0$, we find

$$\begin{aligned}
J_{2n+1} = I_{2n+1}/2\pi &= \left[(n-1)! 2^{n-1}\right]^2 / \left[\pi (2n-1)! |\Delta|^{2n}\right] \\
1 + \frac{1}{2} g N_0 I_1 &= \frac{1}{2} g N_0 \ln |\Delta_0/\Delta|^2
\end{aligned} \tag{3.14}$$

where Δ_0 is the equilibrium gap at $T = 0$. The relation between the GL order parameter $\Psi(\mathbf{r}, t)$ and the energygap function $\Delta(\mathbf{r}, t)$ is given, at $T = 0$, by $\Psi = (N/2)^{1/2} \Delta/\Delta_0$ where N is the total electron density. Then the time-dependent GL equation at $T = 0$ is simply

$$\begin{aligned}
&N_0 \ln |\Psi/\Psi_0|^2 \\
&= \left(\Psi_0^4 / 6m \Delta_0^2 |\Psi|^6\right) \Big\{ 2\Psi^2 \left[\Psi^\dagger \nabla^2 \Psi^\dagger - \left(\nabla \Psi^\dagger\right)^2\right] - \left(\Psi^\dagger\right)^2 \\
&\times \left[\Psi \nabla^2 \Psi - (\nabla \Psi)^2\right] - \left(3/v_F^2\right) \Big(2\Psi^2 \left[\Psi^\dagger \Psi^{\dagger\prime} - \left(\Psi^{\dagger\prime}\right)^2\right] \\
&- \left(\Psi^\dagger\right)^2 \left[\Psi \Psi'' - (\Psi')^2\right] \Big) \Big\},
\end{aligned} \tag{3.15}$$

where $\Psi_0 = (N/2)^{1/2}$ and v_F is the Fermi velocity. We have written the equation in a form suitable for the inclusion of external fields as discussed in Sec. V. The prescription for altering the derivatives in Eq. (3.15) to include fields is given later in Sec. VII, Eq. (7.1). In the absence of fields, Eq. (3.15) may be written in a more compact form which displays explicitly the longitudinal density oscillation for a neutral superconductor which we discuss again in Sec. VII:

$$\begin{aligned}
N_0 \ln \left(\Psi/\Psi_0\right)^2 &= \left(\Psi_0^4 / 6m \Delta_0^2 |\Psi|^2\right) \\
&\times \left[\nabla^2 - \left(3/v_F^2\right) \partial^2/\partial t^2\right] \ln \left[\left(\Psi^\dagger\right)^2 / \Psi\right]
\end{aligned} \tag{3.16}$$

Unfortunately, the time-dependent GL equation we have derived in this section is valid only at absolute zero. The reason is that at finite temperatures, there is the possibility of local conversion of the thermally excited normal excitations to superfluid. Mathematically, the difficulty is due to the existence of a singularity in the product of Green's functions which form the kernel of the integral equation [Eq. (2.14)] for the gap function. Let q, q_0 be the characteristic wave number and frequency of the space-time variations of the order parameter. In the Taylor series method of this section, we calculate the time (space) variation by setting $q\,(q_0)$ equal to zero and then taking the limit as $q_0(q)$ goes to zero. In the actual situation, we have either $q/q_0\xi \lessgtr 1$. Then if $q/q_0 < 1$, we are calculating the time derivatives correctly and the space derivatives incorrectly and the reverse in the other case. In the superconductor, the relevant ratio is vq/q_0 where v is the group velocity of the thermally excited quasiparticle. If the ratio is greater than one, local conversion, or diffusion, of normal fluid represented by excitations of wavenumber q may occur since the Crerenkov condition $\Delta E = \mathbf{v}\cdot\mathbf{q} = q_0$ can be satisfied. At absolute zero, there are no thermal excitations and for $q_0 < \Delta_0$ there can be no conversion from pair dissociation either. In this case, the ratio vq/q_0 is irrelevant and the $T = 0$GL equation we have derived in this section is correct (in the neutral case). If the Crerenkov condition is satisfied, then the results derived in this section for the space derivatives are valid at all temperatures, while the time-derivative terms are incorrect. In fact, as we shall show in the next section where we analyze the frequency and wave-number-dependent response, a differential equation describing the time variation cannot be written in the Cerenkov region except near T_c where we again find a diffusion equation identical to the one derived above for the case $q_0 > \Delta, T \approx T_c$ [Eq. (3.5)]. The remarks we have just made apply to the space derivatives at finite temperatures in the case $vq/q_0 < 1$. In the special case $q_0 > \Delta, T \approx T_c$, however, the ratio vq/q_0 is again irrelevant and the diffusion-type equation of Eq. (3.5) is valid.

We turn now to the question of the time-dependent phase of the order parameter. As remarked in connection with Eq. (3.5), we have only derived time-dependent equations for the gap function as defined in Eq. (2.11). It does not include the phase factor involving the chemical potential. We may reintroduce the latter simply by the replacements

$$\Psi' \to (\partial/\partial t + 2i\mu)\Psi, \quad \Psi^{\dagger\prime} \to (\partial/\partial t - 2i\mu)\Psi^{\dagger}. \tag{3.17}$$

However, this reveals another difficulty with the Taylor series method. As remarked above, in the calculation of the time-derivative terms, the spatial variation is taken to be zero. However, when there is a spatial term involving, for example, $\nabla^2\Psi$ in the GL equation $T = 0$, then we expect the kinetic energy of the moving pairs to be reflected in the chemical potential which should be replaced by $\mu - \nabla^2/8m$. In this case, the time-derivative operators become $[\partial/\partial t + 2i\,(\mu - \nabla^2/\ 8m)]\Psi$. The new chemical potential which we have written here is that for an isolated specimen. In the case that there is a local-pair kinetic energy so that the chemical potential is μ far away from the point in question then the over-all constancy of the chemical potential is maintained by a local density change in the case of a neutral superconductor. In the charged case, the density is almost constant and a small electrostatic potential is built up. The addition of $-\nabla^2/8m$ to the chemical potential is essential for the Galilean invariance of the $T = 0$GL equation. With the extra term present, $\Psi(\mathbf{r} - \mathbf{q}t/m, t)\exp\left[i\left(\mathbf{q}\cdot\mathbf{r} - q^2t/4m\right)\right]$ is a solution when $\Psi(\mathbf{r}, t)$ is. This fact is connected with the satisfying result that $\Psi(\mathbf{r}, t)\exp(i\mathbf{q}\cdot\mathbf{r})$ satisfies the GL equation at $T = 0$ if $\Psi(\mathbf{r}, t)$ does. All the terms with q vanish identically. This is consistent with the original Gor'kov equations [5] at $T = 0$ which have the same property and reflects the fact that for the currentcarrying case with q small, the whole Fermi surface shifts slightly but the superconducting correlations are unchanged (rigidity of the wave function in the London sense, cf. Sec. V). The modification of the time derivatives due to the kinetic energy of the moving pairs which we have been discussing here appears naturally in the correct calculation of the behavior of the order parameter which we present in the next section.

31.4 IV. FREQUENCY AND WAVE-NUMBER RESPONSE TO GAP VARIATIONS

In this section, we analyze in detail the conditions under which a differential GL equation exists. The method is simply to retain finite values of q, q_0 in the kernels of the integral equation for the gap parameter, Eq. (2.14) and to investigate whether expansions in powers of q, q_0 are valid. For simplicity, we limit the discussion to the linear terms [first two terms on the right-hand side of Eq. (2.14)] of the integral equation. It is convenient to write the equations for $\Delta^\dagger(1)$ and $\Delta(1)$ together in matrix form:

$$\Lambda(1) = -gF_l(1,1) - g\int d3K(3,1)\delta\Lambda(3) \tag{4,1}$$

where

$$\Lambda = \begin{pmatrix}\Delta^\dagger\\ \Delta\end{pmatrix}, \quad F_l = \begin{pmatrix}\mathfrak{F}_l^\dagger\\ \mathfrak{F}_l\end{pmatrix}, \quad K = \begin{pmatrix}\mathfrak{G}_l^2 & -\mathfrak{F}_l^{\dagger 2}\\ -\mathfrak{F}_l^2 & \mathcal{G}_l^2\end{pmatrix}.$$

We take for $\Lambda(3)$ the variation $\Lambda(\mathbf{q}, q_0)\exp i(\mathbf{q}\cdot\mathbf{r}_3 - \Omega\tau_3)$ where $i\Omega$ is a complex frequency which we afterward continue to the real axis $i\Omega \to q_0 + i\eta$ in such a way as to recover the adiabatic switching parameter η thereby insuring causality. The first term on the right-hand of Eq. (4.1) is proportional to the integral I_1 as in the previous section [see Eqs. (3.6)-(3.9)]. With these substitutions, the equation for $\Lambda(1)$ becomes

$$\begin{aligned}\left(1 + \tfrac{1}{2}gN_0I_1\right)\Lambda(1) = -g\Lambda(\mathbf{q}, q_0)\exp i(\mathbf{q}\cdot\mathbf{r}_1 - q_0t_1)\\ \times\left[L(\mathbf{q}, q_0) - L(0,0)\right]\end{aligned} \tag{4.2}$$

where
$L(\mathbf{q}, q_0) = \left[\int d3K(3,1)\exp i(\mathbf{q}\cdot\mathbf{r}_{31} - \mathbf{\Omega}_{31})\right]_{\Omega\to q_0+i\eta}$.

The matrix components of L are calculated as in the previous section. There, the Green's functions in momentum space have been given in Eq. (3.11) and a typical τ integral performed in Eq. (3.3). The result is

$$\begin{aligned}&L_{11} = L_{22} = (N_0/8q)\int_{-q}^{+q}dx\int d\xi\left[(1 - n_+ - n_-)\right.\\ &\times(1 + \xi_+\xi_-/E_+E_-)D_+ - (n_+ - n_-)\\ &\left.\times(1 - \xi_+\xi_-/E_+E_-)D_-\right]\\ &L_{12}/\left[\Delta^\dagger(1)\right]^2\\ &= L_{21}/[\Delta(1)]^2 = -(N_0/8q)\int_{-q}^{+q}dx\int d\xi\\ &\times\left[(1 - n_+ - n_-)D_+ - (n_+ - n_-)D_-\right]/E_+E_-\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}\mathscr{D}_\pm = \left[(E_+ \pm E_- + q_0 + i\eta)^{-1}\right.\\ \left.+ (E_+ \pm E_- - q_0 - i\eta)^{-1}\right]\end{aligned} \tag{4.5}$$

In the above, $n_\pm = (\exp\beta E_\pm + 1)^{-1}$ and the subscripts $\pm$ indicate that the momentum argument is $\mathbf{p} \pm \mathbf{q}/2$. Further, $x = \mathbf{p}\cdot\mathbf{q}/p$. We have dropped terms in the ξ integrals which vanish due to particle-hole symmetry. We shall refer to the second term (containing $n_+ - n_-$)

of each component of L as the "irregular part" arising from thermally excited quasiparticles. The first terms are called regular parts.

It is in Eqs. (4.4) and (4.5) that the kinetic-energy correction to the chemical potential which we mentioned at the end of Sec. III makes itself evident. That is, ${E_\pm}^2 = {\xi_\pm}^2 + |\Delta(1)|^2$, where $\xi_\pm = p^2/2m \pm px/2m - (\mu - q^2/\,8m)$ and μ is the chemical potential. To handle the term with q^2, imagine that the local value of the chemical potential happens to be $\mu_l = \mu + q^2/8m$. Then $q^2/8m.$ drops out of the energy, but the local gap parameter
oscillates with μ_l. That is, μ_l replaces μ in (3.17). We may then proceed with μ_l and after expanding in powers of q, q_0, we restore the actual case by the replacement $\mu_l \to \mu - \nabla^2/8m$. All we have done is to isolate that part of the local "mechano-chemical" potential which is due to the kinetic energy of superfluid motion in order to handle the space derivatives of Ψ in a proper manner.

We now investigate the conditions under which it is possible to expand the kernel $L\left(\mathbf{q}, q_0\right)$ in powers of q, q_0. If the expansion exists, then the coefficient of $\mathbf{q}^n$ is the coefficient of $(-i\boldsymbol{\nabla})^n$ in the GL equation and the coefficient of ${q_0}^n$ is the coefficient of $(i\partial/\partial t)^n$.

Case A. $T \approx T_c, q_0 > \Delta$
This case corresponds to the first calculation of Sec. III which resulted in Eq. (3.5). In the present method, we drop Δ everywhere in the kernel and find, from Eq. (4.4),

$$\begin{aligned} L_{11} = L_{22} = (N_0/2q) \int dx \int d\xi \tanh(\beta\xi/2) \\ \times \left[(2\xi + v_F x + q_0 + i\eta)^{-1} + (2\xi + v_F x - q_0 - i\eta)^{-1}\right] \\ L_{12} = L_{21} = 0 \end{aligned}$$

After subtracting $L(0,0)$ we can see that the expansion of the kernel in q, q_0 is regular in the ratio $v_F q/q_0$. We recover the Taylor series result of the previous section. Thus:

$$L\left(\mathbf{q}, q_0\right) - L(0,0) = N_0\left[-K_2 q_0 + K_3 q_0^2 + K_3 v_F^2 q^2/3\right]$$

where the K_n integrals are defined in Eq. (3.4). Transforming back to coordinate space we find

$$\begin{aligned} \left(1 + \frac{1}{2} g N_0 I_1\right) \Delta^\dagger = -g N_0 b \left[\nabla^2/4m + (3)8\epsilon_F\right) \\ \times(\partial/\partial t - 2i\mu)^2 - (1/c)(\partial/\partial t - 2i\mu)\Big] \Delta^\dagger \end{aligned} \tag{4.6}$$

which, if we expand I_1 near T_c, becomes the diffusion equation given in Eq. (3.5) where the constants b, c are defined.

Case B. $T \approx T_c, v_F q > q_0$
This is the Čerenkov region near T_c. We shall find, as remarked at the end of Sec. III, that the diffusion equation again obtains. We limit ourselves to the expansion of the kernel in powers of q_0 since in the Cerenkov region, as discussed earlier, the spatial variation is described correctly by the Taylor series method of the previous section. In each matrix component of L, Eqs. (4.4, 5), the second term (containing $n_+ - n_-$) describes the effect of thermally excited quasiparticles. The energy denominators $D_-$of these "irregular" terms, $E_+ - E_- \pm q_0$, go to zero when q and q_0 do thereby causing irregularities in the expansion of the kernel. However, since the leading term of $n_+ - n_-$is itself proportional to q, in the case that $vq > q_0$ we may hope to expand these irregular denom-
inators in the first power of q_0/vq and retrieve a first time-derivative GL equation. It turns out to be possible (because of the temperature dependence of the group velocity v of the low-lying excitations) only when $T \approx T_c$. In all other cases at finite temperatures, these irregular terms

are impossible to expand and a timedependent GL equation cannot be derived. At absolute zero, of course, the irregular terms are zero.

We illustrate these points by examining the imaginary part of L. The regular terms have no imaginary part if $q_0 < \Delta$ as may be seen from the structure of the denominators in $\mathfrak{D}_+$[Eq. (4.5)]. The case $q_0 > \Delta$ has already been treated so we only have to consider the irregular part of L. The terms in question are

$$\begin{aligned}\operatorname{Im} L_{11}^{\text{irr}} = &-(\pi N_0/4q)\int dx \int d\xi\,(n_+ - n_-)\\ &\times(1-\xi_+\xi_-/E_+E_-)\,\delta\,(E_+ - E_- - q_0)\end{aligned} \tag{4.7}$$

For L_{12}, the term in square brackets is replaced by $\left[\Delta^\dagger(1)\right]^2/E_+E_-$.

We note immediately that for small q and the limit $(dE/dp)q/q_0 = vq/q_0 \to 0$ the δ function in Eq. (4.7) can never be satisfied and there is no contribution. We therefore consider the Ccrenkov region. The terms we are discussing will be seen to give rise to the first time derivative of the diffusion equation when $\Delta(T)/v_F q$ is small, that is for $T \approx T_c$.

The δ function, considered as a function of x, can only be satisfied for $\xi > -\frac{1}{2}(v_F q - q_0 S) = \xi_1$ where $S^2 = 1 + 4|\Delta|^2/(v_F{}^2 q^2 - q_0{}^2)$. We have assumed that $v_F q > [2q_0\Delta(T)]^{1/2}$, thereby restricting ourselves to the neighborhood of T_c. We perform the x integral and find

$$\begin{aligned}\operatorname{Im} L_{11}^{\text{irr}} = &(\pi N_0/2v_F q)\int_{\xi_1}^{\infty} d\xi\,\{(\xi/E) - (E+q_0)\\ &\times\left[(E+q_0)^2 - |\Delta|^2\right]^{-1/2}\Big\}\,[n\,(E+q_0) - n(E)]\end{aligned}$$

For L_{12}, replace the curly bracket by $-\left[\Delta^\dagger(1)\right]^2 \times E^{-1}\left[(E+q_0)^2 - |\Delta'|^2\right]^{-1/2}$. An analysis of the integrand shows that for the case $v_F q \gg \Delta \gg q_0\,(T \approx T_c)$ we may drop Δ and q_0 in the first square bracket which becomes $-2\theta(-\xi)$ and take $\xi_1 = \frac{1}{2}v_F q$. For $\beta\Delta \ll 1$, the second square bracket may be expanded to the first order in q_0. In this way, we get the results

$$\begin{aligned}\operatorname{Im} L_{11}^{\text{irr}} &= (\pi N_0/8)\,(q_0/kT) = iN_0K_2q_0\\ \operatorname{Im} L_{12}^{\text{irr}} &= \operatorname{Im} L_{21}^{\text{irr}} = O\left(\Delta^2\right) = 0\end{aligned}$$

We may expand L_{11} in powers of q in a straightforward manner. In the present case, $v_F q \gg \Delta \gg q_0, \beta\Delta \ll 1$, and the result is the same as in the preceding case A of this section. We then find, as before,

$$L\,(\mathbf{q}, q_0) - L(0,0) = N_0\left[-K_2q_0 + K_3v_F^2q^2/3\right]$$

and the time-dependent diffusion-type GL equation is
valid near T_c in the Čerenkov region as well as when $q_0 > \Delta$.

Before leaving the case $T \approx T_c$, we should convince ourselves that the real part of the kernel L_{11} makes no contribution to the time dependence. An analysis of the real part shows that it is at most of order $q_0{}^2/v_F q\Delta$ and we may safely neglect it as being of higher order than the imaginary term of first order in q_0 which we are keeping.

The conclusions of this section are as follows:
Case A. When $q_0 \gg \Delta$, the result of the Taylor series method in the case $\Delta \to 0$ as given at the beginning of Sec. III is valid.

Case B. When $q_0 < v_F q$ (Čerenkov region), the spatial derivatives may be obtained from the regular part of the kernel L by expansion in powers of q, or equivalently by the Taylor series method. However, a time-dependent GL equation only exists for $T \approx T_c$ or $T = 0$ (see

below). In the former case, the irregular part of the kernel which arises from thermally excited quasiparticles contributes a first time derivative and the equation is of diffusion type and is the same as in case A. For general temperatures, it is impossible to expand the irregular part.

Case C. When $\Delta > q_0 > v_F q$, the time derivatives are obtained from the regular part of the kernel and are the same as the result of the Taylor series method. However, above absolute zero, the irregular term cannot be expanded [it behaves like $(v_F q/q_0)^2$], there is no differential equation, even near T_c. For $T \approx 0$, see below.

Case D. When $T \approx 0 (kT \ll \Delta)$, there are no thermal excitations and the irregular part of the kernel is zero. Then everything is given by the expansion of the regular part and the results of the Taylor series method at $T = 0$ are recovered.

31.5 V. CHARGE, CURRENT, AND EXTERNAL FIELDS

We now turn to the expressions for the charge and current densities which are to be associated with the space-time variations of the order parameter.

For clarity, we first consider the case of a neutral superconductor. We can obtain the density and current responses from the integral equation for the Green's function $\mathcal{G}$ given by Eq. (2.12). For slow variations of the gap, we can replace $\mathcal{G}^\dagger$ and $\mathcal{G}$ by their zeroth-order local values $\mathfrak{F}_l^\dagger$ and $\mathfrak{g}_l$ as before. We then have, for the deviation of $\mathcal{G}$ from its local value when the gap is constant, $\Delta = \Delta_l$ (we limit ourselves to terms linear in $\delta\Delta$):

$$\delta\mathcal{G}(1,2) = -\int d3\,[\mathfrak{F}_l(1,3)\mathfrak{G}_l(3,2)\,\delta\Delta^\dagger(3) + \mathcal{G}_l(1,3)\mathfrak{F}_l^\dagger(3,2)\delta\Delta(3)\Big]$$

From $\delta\mathcal{G}$, we may obtain the charge and current responses [15]

$$\delta\rho(1) = \rho(1) - \rho_l = 2\delta\mathcal{G}\left(1,1^+\right), \tag{5.1}$$

$$\mathbf{j}(1) = (i/2m)\left[(\nabla_2 - \nabla_1)\,2\delta\mathcal{G}(1,2)\right]_{2\to 1^+}. \tag{5.2}$$

The subsequent calculations are quite similar to Secs. III and IV. From the Taylor series method, valid at $T = 0$, we find

$$\delta\rho = (iN_0/4)\,I_3\left(\Delta^\dagger\Delta' - \Delta\Delta^{\dagger\prime}\right) \tag{5.3}$$

$$\mathbf{j} = -\left(i\pi N_0 v_F^2/6\right) J_3\left(\Delta^\dagger\nabla\Delta - \Delta\nabla\Delta^\dagger\right) \tag{5.4}$$

where I_n and J_n have been defined in Eqs. (3.13) and (3.10). Note that at $T = 0$, Eq. (3.14) applies for I_n and J_n.

The continuity equation at $T = 0$ is satisfied by these expressions as may be verified by using the GL equation at $T = 0$ given in Eq. (3.15) which relates the space and time derivatives of Δ. It is to be noted that the density involves a first time derivative since the GL equation contains Δ''. If the spatially varying chemical potential is introduced into the time derivatives of Eqs. (3.15) and (5.3) as discussed at the end of Sec. III, then the continuity equation still obtains provided terms of order ∇^4 are neglected in the GL equation, Eq. (3.15).

When $T \neq 0$, we restrict ourselves to the Crenkov region. Then, as before, the spatial derivatives have been treated correctly and the expression we have given for j remains valid [Eq. (5.4)]. For the density, we use the method of Sec. IV and we have to consider the irregular terms in the density kernel which arise from thermal excitations. These are proportional to

$$(q_0/q)\int dx\int d\xi\,(n_+-n_-)$$
$$\times\left[E_+E_-\left(E_+-E_--q_0-i\eta\right)\right]^{-1} \tag{5.5}$$

The notation is that of Eq. (4.4). Equation (4.7) is the corresponding irregular term of the gap kernel. The leading term of the real part of the expression (5.5) is just

$$2q_0\int d\xi E^{-2}\partial n/\partial E = q_0\left(2\pi J_3 - I_3\right)$$

If we combine this with the contribution of the regular term found from the Taylor series in Eq. (5.3), the factor I_3 in the latter equation becomes $2\pi J_3$ and we may write

$$\delta\rho = \left(i\pi N_0 J_3/2\right)\left(\Delta^\dagger\Delta' - \Delta\Delta^{\dagger\prime}\right), \tag{5.6}$$
$$\mathbf{j} = -\left(i\pi N_0 v_F^2/6\right) J_3\left(\Delta^\dagger\nabla\Delta - \Delta\nabla\Delta^\dagger\right). \tag{5.7}$$

There are two remarks to be made. First of all, we are neglecting the imaginary part of the expression (5.5) for the irregular part of the density kernel. This

neglected part corresponds to a dissipative contribution to $\delta\rho$ but it cannot be expressed in differential form and, unlike the situation for the analogous real part of the gap kernel ${L_{11}}^{\text{irr}}$ [cf. Eq. (4.4)], we have not succeeded in demonstrating that it is smaller than the term we keep. We may note, however, that this dissipative term is proportional to ${q_0}^2$ as may easily be seen from the expression (5.5). Secondly, with the approximations used (first order in q, q_0) and the neglect of the dissipative term, the final expressions, Eqs. (5.6) and (5.7), for $\delta\rho$ and j do not satisfy the continuity equation according to the time-dependent GL equation in the Čerenkov region near T_c given by Eq. (4.6). Of course, the superfluid density $\rho_s \propto |\Delta|^2 J_3$ is not conserved under Cerenkov conditions as discussed at the end of Sec. III. However, the original equations are consistent with continuity. Indeed, if we write, as in Sec. IV [Eq. (4.1)] and in this section,

$$\Lambda(1) = -gF_l(1,1) - g\int d3K_\Lambda(3,1)\delta\Lambda(3)$$
$$\rho(1) = \rho_l + \int d3K_\rho(3,1)\delta\Lambda(3)$$
$$j(1) = (i/2m)\left[(\nabla_2-\nabla_1)\int d3K_j(1,3,2)\delta\Lambda(3)\right]_{2\to1}$$

then we can prove $q_0\rho\left(q,q_0\right) = \mathbf{q}\cdot\mathbf{j}\left(q,q_0\right)$.

We turn now to the case of a charged system in the presence of external fields. As we have earlier remarked, we restrict ourselves to the local limit. Then we may take electromagnetic fields to be everywhere slowly varying in space and time. That is, they will be essentially constant in a region ξ_0 (coherence length) and over times of the order of $1/kT_c$. Then we may follow the procedure used by Gor'kov in the static case. [2,3] If $A_\mu = (\mathbf{A}/c, -\phi)$, $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$, and $x_\mu = (\mathbf{r}, t)$, it may be shown that

$$G\left(x,x'\right) = \exp\left[ieA_\mu(x)\left(x-x'\right)_\mu\right]G^0\left(x,x'\right)$$
$$F\left(x,x'\right) = \exp\left[ieA_\mu(x)\left(x+x'\right)_\mu\right]F^0\left(x,x'\right) \tag{5.8}$$

[1215] Reference 3, Sec. 37.1; L. P. Kadanoff and P. Martin, Phys. Rev. 124, 670 (1961).

where G^0, F^0 satisfy the Gor'kov equations without A_μ. This form for the Green's functions evidently implies that we should replace $\partial\Delta/\partial x_\mu$ by $(\partial/\partial x_\mu - 2ieA_\mu)\,\Delta$ wherever it appears in all of our previous expressions.

In addition, we have to modify the charge expression: The prescription of Eq. (5.8) does not account for the static local density shift due to the scalar potential ϕ (Thomas-Fermi term), $\delta\rho = -2N_0e^2\phi$. Furthermore, the chemical potential appearing in the phase of the order parameter is now a local one μ_l which reflects the local density change ($\mu_l = \mu + \delta\rho/2N_0e$). Then for the charge and current, we have

$$\delta\rho = (i\pi N_0 e J_3/2)\left[\Delta^\dagger(\partial/\partial t + 2i\tilde{\mu})\Delta - \text{ c.c. }\right] - 2N_0e^2\phi \tag{5.9}$$

$$\mathbf{j} = \left(-i\pi N_0 v_F^2 e J_3/6\right)\left[\Delta^\dagger\nabla'\Delta - \text{ c.c. }\right] \tag{5.10}$$

where [16]

$$\tilde{\mu} = \mu_l + e\phi - \boldsymbol{\nabla}'^2/8m \tag{5.11}$$

$$\boldsymbol{\nabla}' = \boldsymbol{\nabla} - 2ie\mathbf{A}/c \tag{5.12}$$

These same results may be derived by a perturbation procedure and similar expressions have previously been written down by others. [7,9,17] From Eq. (5.11) we see that if there is a local spatial variation of Δ, then it may be balanced by a scalar potential ϕ to make the mechano-electrochemical potential constant throughout an isolated specimen. Similar observations have been made before by several people. [18] In the presence of the long-range Coulomb interaction, we impose the condition $\delta\rho = 0$, otherwise the motion of the order parameter couples to the plasmon mode. This puts a restriction on the motion of the order parameter through Eq. (5.9). We shall refer to this again in Sec. VII.

If we follow the prescription above for the GL equation, Eq. (3.15), at absolute zero we find that the vector potential disappears completely (recall, we have the London gauge $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$). This does not occur for the impure case to be discussed in the next section. The result is expected and reflects the London rigidity of the superfluid wave function against a magnetic field. Under these circumstances, the expression for the current reduces simply to the London equation $\mathbf{j} = -e^2\mathbf{A}N\ /\ mc$. The application of the prescriptions of this section to the time-dependent GL equation at $T = 0$ is made in Sec. VII.

31.6 VI. MODIFICATIONS FOR DILUTE ALLOYS

In this section we outline the procedure necessary to discuss the case of dilute superconducting alloys and we apply it to the linear terms of the GL equation. The general procedure for dealing with the case of a dirty superconductor is well known [19] : It is convenient to work always in the imaginary frequency domain rather than in the time domain of the Matsubara technique. In the pure case, the corresponding $\mathfrak{F}(\omega, p)$ and $\mathcal{G}(\omega, p)$ are given in Eq. (3.7). In the impure case, the technique for averaging over the positions of randomly placed impurities is as follows: One replaces ω and Δ by $\tilde{\omega} = \eta\omega$ and $\tilde{\Delta} = \eta\Delta$ where $\eta = 1 + 1/2\tau\mathcal{E}$. Here $1/2\tau$ is the decay rate from impurity scattering in the normal state and $\mathscr{E} = \left[\omega^2 + |\Delta|^2\right]^{1/2}$. In addition, one must include the vertex corrections due to the impurity scattering. Our method is similar to that used by Maki and Tsuneto. [20] Finally, we observe that

[1216] In Eq. (5.11) we have included the kinetic-energy correction to the chemical potential which we discussed at the end of Sec. III.

[17] P. W. Anderson, N. R. Werthamer, and J. M. Luttinger, Phys. Rev. 138, A1157 (1965). Our charge

just as in the theory of conductivity, [19] the variations in the energy gap couple with variations in density via the impurity scattering. In the pure case we may write our starting equations [Eqs. (2.12) and (2.13)] in matrix form

$$G = G_l - G_l \delta\Lambda G \tag{6.1}$$

where

$$G = \begin{pmatrix} \mathcal{G} & \mathfrak{F} \\ \mathfrak{F}^\dagger & -\mathcal{G}^- \end{pmatrix}, \quad \delta\Lambda = \begin{pmatrix} 0 & \delta\Delta \\ \delta\Delta^\dagger & 0 \end{pmatrix}$$

where the Green's function $\mathcal{G}^-$is defined by $\mathcal{G}^-(1,2) = g(2,1)$. In the presence of impurity scattering, we have to modify these expressions as described above. G goes over to the Green's function for the alloy and the vertex $\delta\Lambda$ has to be corrected for impurity scattering and coupling to the density. Call the corrected vertex $\delta\tilde{\Lambda}$. Then Eq. (6.1) remains the same but with $\delta\Lambda$ replaced by $\delta\tilde{\Lambda}$ and Green's functions for the alloy are used in the matrix G. The corrected vertex is given by the matrix

$$\delta\tilde{\Lambda} = \begin{pmatrix} \pi & \lambda \\ \lambda^\dagger & \pi^\dagger \end{pmatrix},$$

where λ is the corrected gap vertex and π is the density vertex. These quantities satisfy an integral equation similar to that of Maki and Tsuneto [20] which we may write as follows

$$\delta\tilde{\Lambda} = \delta\chi + n|v|^2 K \delta\tilde{\Lambda},$$

where $|v|^2$ represents symbolically two scatterings from the same impurity and n is the number of impurities. We have rearranged the notation so that $\delta\tilde{\Lambda}, \delta\chi$ are four-rowed column vectors and K is a 4×4 matrix:

$$\delta\chi = \begin{bmatrix} \delta\Delta \\ \delta\Delta^\dagger \\ 0 \\ 0 \end{bmatrix} \quad \delta\tilde{\Lambda} = \begin{bmatrix} \lambda \\ \lambda^\dagger \\ \pi \\ \pi^\dagger \end{bmatrix},$$

$$K = \begin{pmatrix} \mathcal{G}_l^- \mathcal{G} & -\mathfrak{F}_l \mathfrak{F} & -\mathcal{G}_l \mathfrak{F} & \mathfrak{F}_l \mathcal{G}^- \\ -\mathfrak{F}_l^\dagger \mathfrak{F}^\dagger & \mathcal{G}_l^- \mathcal{G} & -\mathfrak{F}_l^\dagger \mathcal{G} & \mathcal{G}_l \mathcal{G}^\dagger \\ \mathcal{G}_l \mathfrak{F}^\dagger & \mathfrak{F}_l^\dagger & \mathcal{G}_l \mathcal{G} & \mathfrak{F}_l \mathfrak{F}^\dagger \\ -\mathfrak{F}_l^\dagger \mathcal{G}^- & -\mathcal{G}_l - \mathcal{G} & \mathfrak{F}_l^\dagger \mathfrak{F} & \mathcal{G}_l -^- \mathcal{G}^- \end{pmatrix}. \tag{6.2}$$

Note that $\delta\chi$ gives the uncorrected vertex part and corresponds to $\delta\Lambda$ in the pure case. There is no bare density change; π and π^+represent the density change generated from $\delta\Delta$ and $\delta\Delta^\dagger$ in the presence of impurity scattering.

We now have coupled integral equations for G and $\delta\tilde{\Lambda}$. We solve them by iteration. As remarked above, we are going to limit ourselves to linear terms in $\delta\chi$. It is a straightforward matter to write the equations to the second order in $\delta\chi$. In fact, the static case has already been treated by Tewordt. [21] Here, we replace G by G_l everywhere in the kernels of the integral equa-

tions. It is then easy to solve the integral equation for $\delta\tilde{\Lambda}$, a quantity labeled by incoming 4 -momentum $p_+ = p + q/2$, outgoing 4-momentum $p_- = p - q/2$ and 4 momentum transfer q :

expression for $q = 0$ (no kinetic-energy term) is equivalent to that of this reference if in the latter one takes full Thomas-Fermi screening. That is, one assumes the total electrochemical potential to be constant all over.

[18] See, for example, Ref. 17 and M. Stephen, Phys. Rev. 139, A197 (1965).

[19] Reference 3, Sec. 39.

[20] K. Maki and T. Tsuneto, Progr. Theoret. Phys. (Kyoto) 28, 163 (1962).

[1221] L. Tewordt, Phys. Rev. 137, A1745 (1965).

$\delta\tilde{\Lambda}(p,q) = \chi(q) + n\sum_{p'} |v_{\mathrm{p,p'}}|^2 K^l\left(p_+{}', p_-{}'\right)\delta\tilde{\Lambda}\left(p', q\right)$,
where K^l is given by Eq. (6.2) with $\mathscr{G}_l$ and $\mathfrak{F}_l$ everywhere. Since we are dealing with elastic impurity scattering, the frequency part of the 4 -momentum p' in Eq. (6.3) is the same as that of p. We neglect the small dependence of $\delta\tilde{\Lambda}$ on $\mathbf{p}$[19] and perform the integral over $\mathbf{p}'$. We then find

$$\delta\tilde{\Lambda} = (1-L)^{-1}\delta\chi$$

where

$$L = n\sum_{p'} |v_{p'p'}|^2 K^l\left(p', q\right) \approx \left(8\pi^2\tau\right)^{-1}\int d\xi' d\Omega' K^l\left(p_+{}', p_-{}'\right) \tag{6.4}$$

In Eq. (6.4), $\delta\Omega'$ is the element of solid angle for the vector $\mathbf{p}'$. The calculations of the matrix components of L is a straightforward matter and is carried out in Appendix A where the result is displayed. By matrix manipulation we can find $\delta\tilde{\Lambda} = (1-L)^{-1}\delta\chi$. The result is

$$\mathcal{E}_+\mathcal{E}_-(1-2I)\delta\tilde{\Lambda} = \begin{pmatrix} \left[\mathcal{E}_+\mathcal{E}_- - \left(\mathcal{E}_+\mathcal{E}_- - \omega_+\omega_-\right)I\right]\delta\Delta - \Delta^2 I\delta\Delta^\dagger \\ -\left(\Delta^\dagger\right)^2 I\delta\Delta + \left[\mathcal{E}_+\mathcal{E}_- - \left(\mathcal{E}_+\mathcal{E}_- - \omega_+\omega_-\right)I\right]\delta\Delta^\dagger \\ -i\left[\Delta^\dagger\omega_+\delta\Delta + \Delta\omega_-\delta\Delta^\dagger\right]I \\ -i\left[\Delta^\dagger\omega - \delta\Delta + \omega_+\Delta\delta\Delta^\dagger\right]I \end{pmatrix}$$

where $\omega_\pm = \omega \pm \frac{1}{2}\Omega, \mathcal{E}_\pm = \left[\omega_\pm^2 + |\Delta|^2\right]^{1/2}$. Here Ω is the Matsubara frequency $2\pi n/\beta$ associated with the frequency component of the 4 -momentum q. In the end, $i\Omega \to q_0 + i\eta$ as in Sec. IV [cf. Eq. (4.3)]. The integral I is defined by (see Appendix A)

$$I = (1/4\tau q)\int_{-q}^{+q} dx\left(\mathcal{E}_+ + \mathcal{E}_- + 1/\tau + iv_F x\right)^{-1}$$

We use the result for $\delta\tilde{\Lambda}$ in the 2,1 and 1,2 components of the matrix equation for G [Eq. (6.1)] to get the linear term of the equation for the energy gap which is analogous to Eqs. (4.1)-(4.3) of the pure case:

$$\Delta^\dagger(1) = -(g/\beta)\left\{\sum_{\mathbf{p},\omega}\mathfrak{F}_l^{a\dagger}(\mathbf{p},\omega) - \sum_{\alpha=1}^{4}\sum_{\mathbf{p},\omega}\left[K_{2,\alpha}{}^l\left(p_+, p_-\right) - K_{2,\alpha}{}^l(p,p)\right] \times \delta\tilde{\Lambda}_\alpha(\mathbf{q},\Omega)\Big|_{i\Omega\to q_0+i\eta}\exp i\left(\mathbf{q}\cdot\mathbf{r}_1 - q_0 t_1\right)\right\} \tag{6.5}$$

where the a on $\mathfrak{F}_l{}^{a\dagger}$ inducates that we use the Green's function in the alloy.

We observe that the first term on the right-hand side of Eq. (6.5) is similar to the ordinary gap in the alloy. It is well known that nonmagnetic impurity scattering has no effect; therefore, its value is once again $-\frac{1}{2}gN_0I_1X\Delta^\dagger(1)$ [cf. Eq. (3.8)]. What remains is the evaluation of the integrals

$$(N_0 \mid 2\beta q) \sum_{\omega} \int_{-q}^{+q} dx \int d\xi \left[K_{2,\alpha}^{l}(p_+, p_-) \right.$$
$$\left. - K_{2,\alpha}^{l}(p,p) \right] \delta\tilde{\Lambda}_{\alpha}(\mathbf{q}, \Omega)$$

where $x = \mathbf{p} \cdot \mathbf{q}/p$. We perform the ξ integral first. This has already been done in the evaluation of the matrix L in Appendix A. The result is

$$\begin{aligned}
&\left(1 + \frac{1}{2} g N_0 I_1\right) \Delta^{\dagger}(1) \\
&\quad = -\left(g\pi N_0/2\beta q\right) \sum_{\omega} \left[4 v_F q \tau I/(1-2I) - \left(v_F q/\mathcal{E}\right) \delta_{q,0} \delta_{\Omega,0}\right] \\
&\quad \times \left[\left(1 + \omega_+ \omega_- / \mathcal{E}_+ \mathcal{E}_-\right) \delta\Delta^{\dagger} - \left(\Delta^{\dagger 2}/\mathcal{E}_+ \mathcal{E}_-\right) \delta\Delta\right],
\end{aligned} \tag{6.6}$$

where, on the right-hand side, after performing the frequency sum and continuing $i\Omega \to q_0 + i\eta$, we interpret $\boldsymbol{q}, q_0$ as $-i\boldsymbol{\nabla}$ and $i\partial/\partial t$ just as in Sec. IV. We remark that the effect of impurity scattering appears only in the quantity

$$g_{\tau}\left(q, \mathcal{E}_+ + \mathcal{E}_-\right) = 4 v_F q \tau I/(1-2I) \tag{6.7}$$

and that the pure case is obtained immediately with g_{∞}.
We now investigate the terms on the right-hand side of Eq. (6.6) which can appear in the GL equation. The various limits now involve τ and are quite complicated. We do not give a complete discussion here. We begin with the space-derivative terms, proportional to q^2. For these, when $v_F q > q_0$, we may set $\Omega = 0$. Then we have the following simplifications

$$\mathcal{E}_{\pm} = \mathcal{E}, \quad \omega_{\pm} = \omega$$
$$I = (1/2 v_F q \tau) \tan^{-1}\left[v_F q \tau/(2\mathcal{E}\tau + 1)\right]$$

To obtain the term in $(v_F q)^2$, we expand the q-dependent quantity $g_{\tau}(q, 2\mathcal{E})$ of Eq. (6.7):

$$g_{\tau}(q, 2\mathcal{E}) \approx (v_F q/\mathcal{E}) \left[1 - \left(v_F^2 q^2/12\mathcal{E}\right)(\mathcal{E} + 1/2\tau)^{-1}\right]$$

The linear term in q is cancelled by the second term in the square bracket of Eq. (6.6). We drop it and find for the term proportional to q^2

$$\begin{aligned}
&-\left(g\pi N_0 \epsilon_F/6m\right) \left[\widetilde{J}_3 \nabla^2 \Delta^{\dagger}\right. \\
&\quad \left. -\frac{1}{2} \widetilde{J}_5 \left(|\Delta|^2 \nabla^2 \Delta^{\dagger} + \Delta^{\dagger 2} \nabla^2 \Delta\right)\right]
\end{aligned} \tag{6.8}$$

where

$$\widetilde{J}_n = (1/\beta) \sum_{\omega} \mathcal{E}^{1-n} (\mathcal{E} + 1/2\tau)^{-1}$$

Comparison with Eq. (3.8) shows that the space derivative terms which are linear in the gap variation have precisely the same form as in the pure case but with the coefficients J_n replaced by $\widetilde{J}_n$. The contribution of the second-order terms in $\delta\Delta$ have a similar structure and may be extracted from Ref. 21.

To discuss the time-derivative terms it is necessary to evaluate the sum on frequencies and perform the continuation $i\Omega + q_0 + i\eta$. This is a delicate matter with which we deal in Appendix

B. In the pure case, we recover the results of Sec. IV. We may anticipate the results for the dirty case at $T = 0$ by the following argument. In the pure case, we found that the second time-derivative terms are found at $T = 0$ by setting $q = 0$ first since the irregular terms arising from thermal excitations are not present. If we do this for the alloy, we expect that the impurity scattering does not alter the time-derivative terms since the scattering is elastic This conjecture is easily verified. The relevant quantity depending on q, τ is, from Eqs. (6.6) and (6.7),

$$(1/2q)g_\tau\left(q, \mathcal{E}_+ + \mathcal{E}_-\right) \underset{q\to 0}{\longrightarrow} v_F/\left(\mathcal{E}_+ + \mathcal{E}_-\right)$$

which is independent of τ as is the rest of the right-hand side of Eq. (6.6). Therefore, for the second timederivative terms at $T = 0$ we have the same expression as in the pure case. Actually, this result holds even if $q \to 0$ if the scattering is strong enough. The discussion is in Appendix B. The same conclusion (pure result) obtains for the diffusion, or first time derivative, term when $q_0 > \Delta$ near T_c.

Finally, we look for the first time derivative near T_c. We must investigate the imaginary part of the righthand side of Eq. (6.6) which is linear in q_0. We assume the real part is small as we were able to prove in the pure case in Sec. IV. The details of the analysis of the imaginary part are presented in Appendix B. The result is that in the extremely dirty limit, the same diffusion term is obtained near T_c as in the pure case but the Cerenkov condition is relaxed since the momentum is not sharp in the alloy.

We conclude that the effect of alloying on the linear GL terms proportional to $\delta\Delta, \delta\Delta^\dagger$ appears only in the space derivatives as given in Eq. (6.8). The time derivative terms given in Secs. III and IV for the pure case are unaffected.

31.7 VII. REMARKS AND CONCLUSIONS

31.8 A. Comparison with Previous Work

The most prominent feature of superconductivity as well as superfluidity in liquid helium is that quantummechanical effects are exhibited on a macroscopic scale, From the analysis of flux quantization, [22] for example, we know that the order parameter Δ plays the role of a macroscopic wave function. It is therefore reasonable to expect the time-dependent generalization of the GL equation to have the same form as the Schrödinger equation, similar to the Pitaevskii equation [23] for the condensate wave function in the Bose system. In fact, Ginzburg proposed just such an equation back in 1951.[24] Our analysis has lead us to a more complicated result even at $T = 0$ [see Eq. (3.16)] where one would expect a purely wave-like behavior for Δ. That the derivative

$\partial^2/\partial t^2$ appears together with $\frac{1}{3}{v_F}^2\partial^2/\partial x^2$ in Eq. (3.16) is of course due to the fact that superconductivity involves electrons near the Fermi surface and is not surprising if we remember that the system possesses a phonon-like collective mode with the dispersion relation ${q_0}^2 = \frac{1}{3}{v_F}^2 q^2$ (see below). Nevertheless, the essential features of a quantum-mechanical wave are retained in the expression for the phase of Δ, that is for the chemical potential with which Δ oscillates in time. This is clear from Eqs. (5.9) and (5.11), for example. Consequently, as we shall see below, the London acceleration equation results, not from the time-dependent GL equation, but from the charge equation. In considering this problem, one must bear in mind that the chemical potential and hence the charge density are not directly related to the magnitude of Δ but rather to its phase.

[1222] J. Bardeen, Phys. Rev. Letters 7, 162 (1961).

[23] E. P. Gross, Nuovo Cimento 20, 454 (1961); L. P. Pitaevskii, Zh. Eksperim. i Teor. Fiz. 40, 646 (1961) [English transl.: Soviet Phys.- JETP 13, 451 (1961)].

[24] V. L. Ginzburg, Zh. Eksperim. i Teor. Fiz. 21, 979 (1951).

A more recent discussion of the time variation of the GL order parameter has been made by Stephen and Suhl [9] in a brief note. They discussed the problem near T_c from a point of view similar to that of the present work. They omitted all dissipative terms and therefore failed to obtain the diffusion equation.

The spatial part of the GL equation, generalized to all temperatures, in the local limit has previously been worked out by several people. [6–8,21] In particular, Tsuzuki [7] and Werthamer [8] give the result in a form which is equivalent to ours.

Finally, we would like to mention the recent work of Stephen [16] who has discussed transport equations for superconductors by the Green's function technique. In his paper, local equilibrium is assumed to have been established. It is the development of this local equilibrium by interelectronic interactions which is described by the diffusion equation of our work.

31.9 B. Collective Modes

The time-dependent GL equation near $T = 0$ which is of the wave type [Eqs. (3.14)-(3.15)] leads to the propagation of collective modes in the following manner: We first linearize the equation about equilibrium. Note that $I_1 \approx (-2/gN_0) - 2\pi\Delta_e J_3{}^e \operatorname{Re}\chi$ where χ is the deviation of the gap from its (real) equilibrium value Δ_e and $J_3{}^e$ is given by Eq. (3.10) with $\Delta = \Delta_e$. The result of the linearization is

$$\begin{aligned}\Delta_e{}^2 J_3{}^e \operatorname{Re}\chi = \left(v_F{}^2/12\right)\left(J_3{}^e\nabla^2\chi^\dagger - \Delta_e{}^2 J_5{}^e\nabla^2 \operatorname{Re}\chi\right)\\ - (1/8\pi)\left(I_3{}^e\chi^{\dagger\prime\prime} - \Delta_e{}^2 I_5{}^e \operatorname{Re}\chi''\right)\end{aligned}$$

where $I_n{}^e$ is given by Eq. (3.13) with $\Delta = \Delta_e$. We equate separately the real and imaginary parts and assume the variation $\exp(iq\cdot r - iq_0 t)$. We find the dispersion relations

$$\begin{aligned}(\operatorname{Im}\Delta)q_0{}^2 &= (2\pi J_3{}^e/3I_3{}^e)\, v_F{}^2 q^2\\ (\operatorname{Re}\Delta)q_0{}^2 &= \left[8\pi/\left(I_3{}^e - \Delta_e{}^2 I_5{}^e\right)\right]\\ &\quad\times\left[\left(J_3{}^e - \Delta_e{}^2 J_5{}^e\right) v_F{}^2 q^2/12 + \Delta_e{}^2 J_3{}^e\right]\end{aligned}$$

The first dispersion relation, for oscillations of the imaginary part of Δ, corresponds to the Anderson mode [25] of the neutral superconductor. At $T = 0$, we have $2\pi J_3 = I_3$ and the mode propagates with the velocity $v_F/\sqrt{3}$ as may be seen immediately from Eq. (3.16). That this mode couples to the density is easily seen from Eq. (5.6). Of course, in the charged case, when the long-range Coulomb interaction is present, this longitudinal density fluctuation occurs at the plasma frequency. This comes about because the motion of Δ is restricted through Eq. (5.9), Poisson's equation and the requirement $\delta\rho = 0$. For a detailed discussion from the Green's function point of view, see Ambegaokar and Kadanoff. [26]

We note from Eq. (5.6) that the oscillations of the real part of Δ which are described by the second dispersion relation give no density fluctuations. Unfortunately, the second dispersion relation is inconsistent with our approximations. At $T = 0$, the threshold is $q_0 = 8\pi J_3/(I_3 - \Delta_e{}^2 I_5)$ and is greater than Δ_e itself, i.e., the condition for slow variation is not satisfied. At higher temperatures, we cannot produce a differential equation in the regime $\Delta > q_0 > v_F q$.

31.10 C. Nucleation

We have used the time-dependent GL equation near T_c to investigate the question of nucleation of superconducting regions due to interelectronic interactions. In reduced units, the important terms of Eq. (3.5) or Eq. (4.6) are the diffusion equation

$$\partial\Psi/\partial t = \nabla^2\Psi + \left(1 - |\Psi|^2\right)\Psi,$$

where $\Psi = 1$ in equilibrium, the unit of length is $v_F/\Delta_e\sqrt{6}$, the unit of time is $\pi^3 kT_c/8{\Delta_e}^2$ and Δ_e is, as before, the equilibrium value of the gap (we have $\hbar = 1$, as usual). An interesting example is that of the growth of the order parameter Ψ starting from an initial distribution representing a superconducting bubble of width 1 and height $\ll 1$. The growth of Ψ offers no surprises. The bubble spreads faster than it grows. Before $t = 2$ it has spread to a width ≈ 10 while the height at the center has doubled. Of course, if the curvature of the initial Ψ is great enough, it will initially decrease in the maximum region as diffusion takes place to regions of small Ψ.

31.11 D. London Acceleration Equation

An important problem related to the time variation of the order parameter is that of determining under what conditions one can derive London's acceleration equation for the superfluid. [27] We have throughout made the assumption that the thermal excitations are at rest; we therefore limit the present discussion to $T = 0$.

We begin by writing the time-dependent GL equation of Eq. (3.15) and we include the external fields according to the prescriptions of Sec. V [cf. in particular, Eqs. (5.8)-(5.12)]. Then the only change in Eq. (3.15) is to replace the space and time derivatives of ψ by

$$\begin{aligned} \boldsymbol{\nabla}'\Psi &= (\boldsymbol{\nabla} - 2ie\mathbf{A}/c)\Psi \\ D\Psi &= \left[\partial/\partial t + 2i\left(\mu_l + e\phi - \nabla'^2/8m\right)\right]\Psi. \end{aligned} \tag{7.1}$$

As we remarked near the end of Sec. V, the terms involving A in the spatial derivative part cancel when the London gauge is used. However, we may proceed in a general gauge. It is convenient to separate the real and imaginary parts of Ψ by writing

$$\Psi = Re^{iS}$$

where $R(\mathbf{r}, t)$ and $S(\mathbf{r}, t)$ are real. The GL equation for Ψ reduces to

$$\begin{aligned} \left(4m\Delta_0^2 R^2/R_0^4\right)\ln\left(R/R_0\right) &= \frac{1}{3}\left\{\left[\nabla^2 R/R - (\nabla R/R)^2\right]\right. \\ &\quad \left. - (3/v_{F^2})\left[R''/R - (R'/R)^2\right]\right\}, \end{aligned} \tag{7.2}$$

$$\nabla^2 S - \left(3/v_F^2\right)S'' = 2e\left[(1/c)\nabla\cdot\mathrm{A} + \left(3/v_F^2\right)\phi'\right] \tag{7.3}$$

In writing these equations, we have, as before, restricted ourselves to second derivatives and dropped cross terms in space and time derivatives. We have as well not written terms of higher order than first in the fields when a derivative is also involved (to do better is straightforward but tedious). Equation (7.3) has been written before by Ambegaokar and Kadanoff. [26] It is shown there how Eq. (7.3) arises in a gauge-invariant perturbation treatment of the external fields.

[1225] P. W. Anderson, Phys. Rev. 112, 1900 (1958).
[26] V. Ambegaokar and L. P. Kadanoff, Nuovo Cimento 22, 914 (1961).
[27] F. London, Superfluids (John Wiley & Sons, Inc., New York, 1950), Vol. 1, p. 54.

We can also give the charge and current expressions in terms of R and S by making the appropriate transformations of Eqs. (5.9) and (5.10). The results are

$$\mathbf{j} = Ne\mathbf{v}, \tag{7.4}$$

$$\rho = -eN_0 \left[S' + 2\left(\mu_l + e\phi\right) - \nabla^2 R/4mR + mv^2\right] - 2N_0 e^2 \phi, \tag{7.5}$$

where $\mathbf{v} = (\boldsymbol{\nabla} S - 2e\mathbf{A}/c)/2m$ is the superfluid velocity.
We now show that the London acceleration equation follows from Eq. (7.5) if we can neglect the spatial variation of R. We note that the external field does not appear in Eq. (7.2) for R. We may then take $R = R_0$, the equilibrium value, and from the gradient of Eq. (7.5) we derive

$$\partial \mathbf{v}/\partial t + \mathbf{v} \cdot \boldsymbol{\nabla} \mathbf{v} = (e/m)(\mathbf{E} + \mathbf{v} \times \mathbf{H}/c) - (1/m)\boldsymbol{\nabla}\left[\mu_l + (\rho/2eN_0 + e\phi)\right], \tag{7.6}$$

which is just the acceleration equation.
We wish to make several comments on these results: First, we remark that the derivation of the London equation depended on the neglect of the $\nabla^2 R$ term in Eq. (7.5) which is justified if the external fields do not appear in Eq. (7.2) for R. At $T = 0$, they only can enter in the time-derivative part of the GL equation. As
indicated below Eqs. (7.2) and (7.3), we have not succeeded in formulating precise criteria for dropping the field-dependent terms. However, in case all the timederivative terms are small then everything follows. The criterion for this is

$$q_0/\Delta_0 \ll 1/\kappa \approx \xi_0/\lambda \approx v_F q/\Delta_0\sqrt{3} \tag{7.7}$$

where λ is the penetration depth and κ is the Abrikosov parameter. Under this condition then, $|\Psi| = R$ changes quasistatically. Also, from Eq. (7.3), we find that the superfluid behaves as an incompressible ideal fluid:

$$\nabla^2 S - 2e\nabla \cdot \mathbf{A}/c = \nabla \cdot \mathbf{v} = 0$$

Thus, we conclude that at $T = 0$ and in the local limit, London's theory is valid if condition (7.7) is satisfied. As remarked earlier, it is interesting to note that the hydrodynamical equation arises not from the GL equation itself but rather from the charge equation.

At finite temperature, Eqs. (7.4) and (7.5) are still valid provided we assume that the normal fluid is at rest and that the other approximations of Sec. V which led to Eqs. (5.9) and (5.10) for the charge and current are valid. We can still drop the time-derivative terms in the GL equation when the criterion (7.7) is satisfied. However, the equation for R which corresponds to Eq. (7.2) now contains the phase S and the vector potential A explicitly (recall that A only drops out of the spatial derivative part of the GL equation at $T = 0$). Therefore, another condition is required for neglecting the spatial derivative of R on the right-hand side of Eq. (7.5) to obtain the acceleration equation. We do not pursue this matter here because of our restrictive assumption concerning the motion of the normal component. In the impure case also the situation is much more complicated since the equation for R will contain $\mathbf{A}$, even at $T = 0$.

Note added in proof. Using a slightly different method, M. P. Kemoklidze and L. P. Pitaevskii, Zh. Eksperim. i Teor. Fiz. 50, 243 (1966) [English transl.: Soviet Phys.-JETP 23, 160 (1966)] have derived results similar to ours for the neutral pure superconductor at absolute zero.

31.12 APPENDIX A

In this Appendix, we evaluate the integrals of Eq. (6.4) in order to determine the elements of the matrix L
which determines the vertex corrections in the impure case. The integrand is the matrix K^l defined in Eq. (6.2) with $\mathcal{G}_l$ and $\mathfrak{F}_l$ everywhere. These Green's functions are
$\mathcal{G}_l(\mathbf{p},\omega) = -(\xi + i\tilde{\omega})/\left(\xi^2+\boldsymbol{\delta}^2\right), \quad \mathcal{G}_l^-(\mathbf{p},\omega) = -(\xi - i\tilde{\omega})/\left(\xi^2+\boldsymbol{\delta}^2\right), \quad \mathfrak{F}_l(\mathbf{p},\omega) = \tilde{\Delta}/\left(\xi^2+\boldsymbol{\delta}^2\right), \quad \mathfrak{F}_l^\dagger(\mathbf{p},\omega) = \tilde{\Delta}^\dagger/\left(\xi^2+\boldsymbol{\delta}^2\right),$
where $\xi = p^2/2m - \mu, \omega$ is the Matsubara frequency $(2n+1)\pi/\beta, \boldsymbol{\delta}^2 = \tilde{\omega}^2 + |\tilde{\Delta}|^2$ and $\Delta = \Delta_l = \Delta(1)$. Quantities with tildes are the original quantities multiplied by $\eta_\omega = 1 + 1/2\tau\mathcal{E}$. Thus, $\tilde{\omega} = \eta_\omega\omega$.

As an example, we compute L_{12}.

$$L_{12} = \left(1/8\pi^2\tau\right)\int d\xi' d\Omega' K_{12}{}^l\left(p_+{}', p_-{}'\right) = \left(1/8\pi^2\tau\right)\int d\xi' d\Omega' \mathcal{G}_l{}^-\left(\mathbf{p}_+{}', \omega_+\right)\mathcal{G}_l\left(\mathbf{p}_-{}', \omega_-\right)$$
$$= (1/4\pi\tau q)\int dx \int d\xi\left(\xi_+ - i\tilde{\omega}_+\right)\left(\xi_- + i\tilde{\omega}_-\right)\left[\left(\xi_+^2 +\right.\right.$$

In the above, $\xi_\pm = \xi \pm v_F x, \omega_\pm = \omega \pm \frac{1}{2}\Omega$ where Ω is the Matsubara frequency $2n\pi/\beta$ associated with the frequency component of the 4 -momentum q. It is convenient to perform the ξ integral first. The result is
where

$$L_{12} = \left(1 + \omega_+\omega_-/\mathcal{E}_+\mathcal{E}_-\right) I$$

$$I = (1/4\tau q)\int_{-q}^{+q} dx\left(\mathcal{E}_+ + \mathcal{E}_- + 1/\tau + iv_F x\right)^{-1}$$

The other elements of L are evaluated in a similar manner. The complete matrix is given by

$$L = \frac{I}{\mathcal{E}_+\mathcal{E}_-}\begin{pmatrix} \mathcal{E}_+\mathcal{E}_- + \omega_+\omega_- & -(\Delta)^2 & i\omega_+\Delta & i\Delta\omega_- \\ -\left(\Delta^\dagger\right)^2 & \mathcal{E}_+\mathcal{E}_- + \omega_+\omega_- & i\Delta^\dagger\omega_- & i\omega_+\Delta^\dagger \\ -i\omega_+\Delta^\dagger & -i\Delta\omega_- & \mathcal{E}_+\mathcal{E}_- - \omega_+\omega_- & \Delta\Delta^\dagger \\ -i\Delta^\dagger\omega_- & -i\omega_+\Delta & \Delta^\dagger\Delta & \mathcal{E}_+\mathcal{E}_- - \omega_+\omega_- \end{pmatrix}$$

31.13 APPENDIX B

We outline in this Appendix the analysis of Eq. (6.6). We rewrite the required frequency sum by shifting the origin of ω by $\frac{1}{2}\Omega$ and defining $U = i\mathcal{E} = \left[-\omega^2 - |\Delta|^2\right]^{1/2}$. We then have, in an otherwise obvious notation, the frequency sum

$$-\left(gnN_0/2v_F q\beta\right)\sum_\omega g_\tau\left(q, U+U_-\right)\left[\left(1 - \omega\omega_-/UU_-\right)\delta\Delta^\dagger + \left(UU_-\right)^{-1}\left(\Delta^\dagger\right)^2\delta\Delta\right], \quad \text{(B} \tag{B1}$$

where

$$g_\tau(q, Z) = 4v_F q\tau I/(1-2I) \tag{B2}$$

$$I = (i/4q\tau)\int_{-q}^{+q} dx\left(Z + i/\tau - v_F x\right)^{-1} \tag{B3}$$

We express the sum as an integral in the standard manner by considering $i\omega$ to be the complex variable z :

$$
\begin{aligned}
&-\left(g N_0 / 8 i v_F q\right) \oint d z \tanh(\beta z / 2) g_\tau\left(q, U+U_-\right) \\
&\quad \times\left[\left(1+z z_- / U U_-\right) \delta \Delta^\dagger+\left(U U_-\right)^{-1}\left(\Delta^\dagger\right)^2 \delta \Delta\right]
\end{aligned}
\tag{B4}
$$

where $z_- = z - i\Omega$ and U is now given in terms of z by $U = \left[z^2 - |\Delta|^2\right]^{1/2}$. The contour encircles the poles of
$\tanh \beta z/2$ in the counterclockwise sense and no other singularities of the integrand are included.

In addition to the poles, the integrand has four branch points coming from the square roots U and U_-. They are located at $z = \pm\Delta, \pm\Delta + i\Omega$. We choose cuts extending from these points to $\pm\infty$. There are no other singularities arising from $g_\tau(q, Z)$. We deform the contour so that it encircles the cuts as shown in Fig. 1. It is not difficult to show that the integrals along 1 and 8 are equal. The same is true for the pairs $(2,7), (3,6)$, and $(4,5)$. Thus, the integral is just twice the contribution from the lower cuts. Now we perform the continua-

Fig. 1. Contours for the integration of Eq. (B4). The arrows indicated how the upper cuts approach the lower ones in the continuation $i\Omega \to q_0 + i\eta$. The upper branch points go to $\pm|\Delta| + q_0 + i\eta$. The numbers indicate portions of the integration that are referred to in the text.

tion $i\Omega \to q_0 + i\eta$. The upper cuts approach the lower ones and take up the position shown in Fig. 1. The upper branch points (for U_-) are shifted and then U_- has different phases along the lower cuts (integrals $3, 4, 7, 8$) than it had before continuation. In the pure case, this leads to the appearance of poles in the integrand of the integral I which appears in the definition of $g_\infty(q, Z)$ [cf. Eq. (B2)]. These poles insure energy
and momentum conservation in the diffusion process, they only appear (for $q_0 < \Delta, T \approx T_c$)

when $v_F q > q_0$, and they give rise to the first time-derivative terms. In the alloy, it is more complicated since the momentum is no longer a good quantum number. We remark, however, that no poles appear between the cuts but that the phases of U_-must be kept careful track of. The result of the analysis is

$$
\begin{aligned}
-\left(gN_0/4iv_F q\right)\{-2i\int_{\Delta}^{\Delta+q\mathbf{0}} dy\tanh(\beta y/2)\left[\operatorname{Re} g_\tau\left(q,Z_1\right)\left(yy_-\delta\Delta^\dagger+\Delta^{\dagger 2}\delta\Delta\right)/ww_- - \operatorname{Im} g_\tau\left(q,Z_1\right)\delta\Delta^\dagger\right] \\
+\int_{\Delta}^{\infty} dy\left[\tanh\left(\beta y_+/2\right)-\tanh(\beta y/2)\right] g_\tau\left(q,Z_2\right)\left[\left(1-yy_+/ww_+\right)\delta\Delta^\dagger-\Delta^{\dagger 2}\delta\Delta/ww_+\right] \\
-\int_{\Delta}^{\infty} dy\left[\tanh\left(\beta y_+/2\right) g_\tau^*\left(q,Z_3\right)-\tanh(\beta y/2) g_\tau\left(q,Z_3\right)\right]\left[\left(1+yy_+/ww_+\right)\delta\Delta^\dagger+\Delta^{\dagger 2}\delta\Delta/\right.
\end{aligned}
\tag{B5}
$$

where

$$
\begin{aligned}
w &= \left[y^2-\Delta^2\right]^{1/2}, & w_- &= \left[\Delta^2-(y-q_0)^2\right]^{1/2} \\
w_+ &= \left[(y+q_0)^2-\Delta^2\right]^{1/2}, & y_\pm &= y\pm q_0
\end{aligned}
$$

and

$$
Z_1 = w+iw_-, \quad Z_2 = w_+ - w, \quad Z_3 = w_+ + w.
$$

We may understand the role of the three integrals by passing to the pure limit and comparing with Sec. IV. All three integrals contribute to the second timederivative terms; they combine in a simple manner when $q \to 0$. However, at $T = 0$, where we expect this procedure to be valid, the second integral is zero and it is sufficient to consider only the first and third.

The diffusion term near T_c comes from the second and third integrals which contain $[\tanh \beta y_+/2 - \tanh \beta y/2]$, the factor which describes thermal excitations.

For the space derivatives, we set $q_0 \to 0$ at once and only the third integral survives in the pure case. This remains true in the alloy but we have already analyzed this case in the main text.

To analyze the second time-derivative terms we look at absolute zero. Then only the first and third integrals survive and are given by

$$
\begin{aligned}
&-\left(gN_0/2v_F q\right)\left\{\int_{\Delta}^{\Delta+q_0} dy\left[\operatorname{Im} g_\tau\left(q,Z_1\right)\delta\Delta^\dagger-\operatorname{Re} g_\tau\left(q,Z_1\right)\right.\right. \\
&\times\left(yy_-\delta\Delta^\dagger+\Delta^{\dagger 2}\delta\Delta\right)/ww_-\Big]+\int_{\Delta}^{\infty} dy\operatorname{Im} g_\tau\left(q,Z_3\right) \\
&\times\left[\left(1+yy_+/ww_+\right)\delta\Delta^\dagger+\Delta^{\dagger 2}\delta\Delta/ww_+\right]\Big\}
\end{aligned}
$$

To evaluate the functions g_τ we work in the very dirty limit $v_F q\tau = ql \ll 1$. Then from Eq. (B2) we may write

$$
\begin{aligned}
g_\tau(q,Z) &\approx \left(2iv_F q/Z\right)\left[1+\frac{1}{3}v_F^2 q^2\tau^2(Z\tau+i)^{-2}\right] \\
&\times\left[1-i\left(v_F^2 q^2\tau/3Z\right)(Z\tau+i)^{-2}\right]^{-1}
\end{aligned}
\tag{B6}
$$

In the first integral, $|Z_1| = |w+iw_-| \approx (q_0\Delta)^{1/2} \ll v_F q \ll 1/\tau$. We furthermore assume that the impurity scattering is strong enough that $v_F q\tau \ll (q_0\Delta)^{1/2}/v_F q \ll 1$. Then $g_\tau\left(q,Z_1\right)$ ls simply $2iv_F/q/Z_1$ which is the same result as for the pure case when $q \to 0$. The other

integral contains $g_\tau(q, Z_3)$ where $(2q_0\Delta)^{1/2} \leq Z_3 = w_+ + w \leq \infty$. Then in the same extremely dirty limit

$$\operatorname{Im} g_\tau(q, Z_3) \approx (2v_F q/Z_3)\left[1 + \frac{1}{3}{v_F}^2 q^2\tau^2 \times \left({Z_3}^2\tau^2 + 1\right)^{-1}\right] \approx 2v_F q/Z_3$$

which once again is the pure result when $q \to 0$. The conclusion is that if $(v_F q)^2 \tau (q_0\Delta)^{-1/2} \ll 1$ then the second time-derivative terms are the same as in the pure case no matter how q goes to zero.

We conclude this Appendix with a discussion of the diffusion term near T_c. We need only the imaginary part of the whole expression (B4). The first integral does not contribute. The contributions of the second and third integrals may be combined by a change of variable from y to $Z_2 = ({y_+}^2 - \Delta^2)^{1/2} - (y^2 - \Delta^2)^{1/2}$ in the second integral and to $Z_3 = ({y_+}^2 - \Delta^2)^{1/2} + (y^2 - \Delta^2)^{1/2}$ in the third. We work near T_c so we neglect second order in Δ where feasible. The result is

$$\begin{aligned} &-(gN_0 i/4v_F q)\int_{q_0}^{\infty} dZ\,\{\tanh[\beta(ZT + q_0)/4] \\ &-\tanh[\beta(ZT - q_0)/4]\}\operatorname{Reg} g_\tau(Z)\delta\Delta^\dagger/T \\ &T^2 = 1 + 4|\Delta|^2/\left(Z^2 - q_0^2\right) \end{aligned}$$

Because of the temperature-dependent statistical factor, the integrand is exponentially small if Z is as large as $1/\tau$. It behaves as $\exp(-\xi_0/l) \ll 1$ where ξ_0 is the coherence length. Then, from Eq. (B5) we have

$$\operatorname{Reg} g_\tau(q, Z) \approx 2v_F q\left(v_F^2 q^2\tau/3\right)\left[Z^2 + \left(v_F^2 q^2\tau/3\right)^2\right]^{-1}$$

Furthermore, we may expand $\tanh[\beta(ZT \pm q_0)/4]$ about $q_0 = 0$, keep only the first-order terms and drop $q_0\frac{1}{2}\pi\delta(Z)$. The final result is the same as in the pure case: everywhere else. The result is

$$\begin{aligned} &-(gN_0 i\beta q_0/4)\int_0^{\infty} dZ\,\operatorname{sech}^2\left[\beta\left(Z^2 + 4|\Delta|^2\right)^{1/2}/4\right] \\ &\times\left(\frac{1}{3}Z\tau v_F^2 q^2\right)\left[\left(Z^2 + 4|\Delta|^2\right)^{1/2}\left(Z^2 + \frac{1}{9}v_F^4 q^4\tau^2\right)\right]^{-1}\delta\Delta^\dagger \end{aligned}$$

We consider the case ${v_F}^2 q^2\tau/3 \gg 2\Delta$. Then we may drop $|\Delta|^2$ everywhere and pass to the limit of strong scattering so that the τ-dependent term may be replaced by

$$-(gN_0 i\pi\beta q_0/8)\,\delta\Delta^\dagger.$$

The approximations we have used to obtain this result are summarized as follows:

$$\begin{aligned} &ql \ll 1, \\ &T \approx T_c, \\ &l/\xi_0 \approx kT_c\tau \ll 1, \\ &\Delta \ll v_F^2 q^2\tau \ll kT_c. \end{aligned}$$

32 A Time Dependent Ginzburg-Landau Equation and its Application to the Problem of Resistivity in the Mixed State by A. Schmid, 1966

Abstract

A time dependent modification of the Ginzburg-Landau equation is given which is based on the assumption that the functional derivative of the Ginzburg-Landau free energy expression with respect to the wave function is a generalized force in the sense of irreversible thermodynamics acting on the wave function. This equation implies an energy theorem, according to which the energy can be dissipated by i) production of Joule heat; ii) irreversible variation of the wave function. The theory is a limiting case of the BCS theory, and hence, it contains no adjustable parameters. The application of the modified equation to the problem of resistivity in the mixed state reveals satisfactory agreement between experiment and theory for reduced temperatures higher than 0.6 .

32.0.1 I. Introduction

In the theory of superconductivity, the Ginzburg-Landau equation [1] has proven to be very successful. The most interesting consequences of this equation have been found by Abrikosov [2] who investigated the behavior of superconductors of the second type - mostly metals of short mean free path in the normal state - in a magnetic field. He found that the magnetic field will penetrate in the superconductor in quantized units of flux (fluxoids, flux lines), if the field is greater than a critical field H_{c1}, and that a second order phase transition takes place at a field $H_{c2} > H_{c1}$. This particular state of a superconductor is called the mixed state. Ginzburd and Landau proposed their equation on a phenomenological basis; it was later shown by Gorkov [3] that this equation can be derived from the BCS theory [4] in the case of thermodynamic equilibrium and for temperatures close to the transition temperature.

Recently Kim, Hempstead and Strnad [5] found in experiments that the voltage-current ratio doesn't vanish for a type II superconductor in a magnetic field, that Ohm's law applies, and that the resistivity depends on the magnetic field. Since fluxoids experience a Lorentz force exerted by the transport current, they thought of a frictional force due to the uniform motion of the fluxoids (flux flow) which counterbalances this driving force. Subsequently, Stephen and Bardeen [6] treated this problem theoretically on the basis of a local model that is a generalization of the London theory. Their calculations agree well with the experimental results, at least in the case of low temperatures. As far as the energy dissipation in this resistive state of type II superconductors is concerned, Stephen and Bardeen consider only losses due to Joule heat of the normal electrons. According to Tinkham [7], an additional dissipative mechanism due to the finite relaxation time of the superelectrons should be taken into account.

Since the Ginzburg-Landau equation deals only with thermodynamic equilibrium (where each physical quantity is constant in time), it cannot be applied to the above mentioned problem. Yet, one may ask whether there exists a generalization of the Ginzburg-Landau equation where the effective wave function (proportional to the gap) is allowed to change in time such that the problem of flux flow would be included in the solutions.

Generalizations of the Ginzburg-Landau equation, which include time dependent terms, have already been proposed by Stephen and Suhl [8], by Anderson et al. [9], and by Jakeman and Pike [10].

The same problem will be discussed here, since the author feels that there may be different approaches to that problem. In particular, the aim of this paper is to present the simplest type of time dependent equation which allows the superconductor to relax in its equilibrium state if

no external driving forces are present. The effects of impurity scattering will be included in this equation. Keeping in mind this aim, we will derive a set of equations in section II, using the Gorkov formulation [11] of the BCS theory. Of course, we will have to limit ourselves to the case of small and slowly varying gap. This set allows to determine all field quantities; and it will be shown in section III that these equations imply a theorem on energy conservation. In section IV the consequences of these equations will be discussed for flux flow in type II superconductors in the cases where the magnetic induction is small (H_{c1}) or close to the upper critical field (H_{c2}); and satisfactory agreement will be found with experiment.

32.0.2 II. A Time Dependent Ginaburg-Landau Equation

In the field free case, the simplest equation which allows the Ginzburg-Landau wave function ψ to relax to its equilibrium value is given by

$$\gamma\hbar\frac{\partial\psi}{\partial t} = \frac{1}{2m}(\hbar\nabla)^2\psi + \alpha\psi - \beta|\psi|^2\psi. \tag{II.1}$$

This type of equation has already been considered by Landat and Khalatnikov [12] in connection with superfluid helium. Eq. (II.1) implies that a deviation of ψ from its equilibrium value $\sqrt{\alpha/\beta}$ has to vanish proportional to $\exp(-t/\tau_{\mathrm{R}})$, where
the relaxation time τ_{R} is given by

$$\tau_{\mathbb{R}} = \frac{\hbar\gamma}{2\alpha} \tag{II.2}$$

provided that γ is real and positive *.
In order to determine γ which appears in eq. (II.1), we start from the Gorkov equation [11] without external fields and consider the approximation

$$\begin{aligned}\Delta^*(x) = -ig\int dx_1 G_0(x_1,x)\,\Delta^*(x_1)\,G_0(x_1,x) + \\ + ig\int dx_1 dx_2 dx_3 G_0(x_1,x)\,\Delta^*(x_1)\,G_0(x_1,x_2)\times \\ \times\,\Delta(x_2)\,G_0(x_3,x_2)\,\Delta^*(x_3)\,G_0(x_3,x)\end{aligned} \tag{II.3}$$

which is sufficient for a small gap $\Delta(x)$[3].$G_0(x,x')$ is the Green's function of the metal in the normal state, and x stands for $\boldsymbol{r}, t$.

Since the superconductor is supposed to be not in thermodynamic equilibrium, we have to be careful about the integration with respect to time. We follow the procedure given by Kadanoff and Baym [13]. This implies that, at the time $-\infty$, the equilibrium state of the metal is the normal state. This seems to be contradictory at first sight, because the metal is superconducting in the temperature range of interest. However, we imagine that we are able to switch the electronelectron interaction responsible for superconductivity such that the interaction constant is zero at $t = -\infty$, and then, assumes slowly its usual value. Since this switching process will be completed long before the times of interest, it doesn't affect the results and hence, should be considered merely a mathematical trick **. This procedure amounts to the replacement of the coupling constant g by $e^{\delta t}g$ (δ small and positive), and of $\Delta(x)\,(= gF^+(x,x))$ by $e^{\delta t}\Delta(x)$.

We now expand $\Delta(x_1)$ in the first integral of the r.h.s. of eq. (II.3) as follows

$$\Delta\left(\boldsymbol{r}_1 t_1\right)=\Delta(\boldsymbol{r} t)+\frac{\partial \Delta(\boldsymbol{r} t)}{\partial q_j}\left(q_{1 j}-q_j\right)+\frac{1}{2} \frac{\partial^2 \Delta(\boldsymbol{r} t)}{\partial q_j \partial q_k}\left(q_{1 j}-q_j\right)\left(q_{1 k}-q_k\right)+$$
$$+\frac{\partial \Delta(\boldsymbol{r} t)}{\partial t}\left(t_1-t\right) ; \quad\left(q_1, q_2, q_3\right)=\boldsymbol{r} \tag{II.4}$$

In the second integral we take the $\Delta\left(x_j\right)$ just equal to $\Delta(x)$. This approximation is the same used by Gorkov [3], with the exception of the last term in eq. (II.4), which takes the time dependence of $\Delta(x)$ approximately into account. All the integrals which do not depend on the time integration variable (except by the factors G_0), turn out to be identical with Gorkov's result [3] in the case of no time dependence. The additional term,

$$-i g \frac{\partial \Delta^*(\boldsymbol{r} t)}{\partial t} \int d x_1 G_0\left(x_1, x\right) e^{\delta t_1}\left(t_1-t\right) G_0\left(x_1, x\right) \tag{II.5}$$

is the same for superconductors with and without impurities. We assume the latter to be the case, where the Fourier transform of G_0 with respect to the space

coordinate is given by

$$G_0^{>}(\boldsymbol{k} ; t_1, t)=-\frac{i}{\hbar} \frac{e^{-\frac{i \varepsilon_k}{\hbar}\left(t_1-t\right)}}{e^{-\beta \varepsilon_k}+1} ; \quad G_0^{<}=\frac{i}{\hbar} \frac{e^{-\frac{i \varepsilon_k}{\hbar}\left(t_1-t\right)}}{e^{+\beta \varepsilon_k}+1},$$

where $\beta=1 / k T$ and $\varepsilon_k=\hbar^2 k^2 / 2 m-\varepsilon_{\mathbf{F}}$ ($\varepsilon_{\mathbf{F}}$ is the Fermi energy). Following the procedure of ref. [13], we find the expression (II.5) to be equal to

$$\begin{aligned}
& -i g \frac{\partial A^*}{\partial t} \int \frac{d^3 \boldsymbol{k}}{(2 \pi)^3} \int_{-\infty}^i d t_1\left(\left(G_0^{<}(\boldsymbol{k} ; t_1, t)\right)^2-\left(G_0^{>}(\boldsymbol{k} ; t_1, t)\right)^2\right) e^{\delta t_1}\left(t_1-t\right) \\
& =\frac{\partial A^*}{\partial t}-i g N(0) \\
& =-\frac{\partial \Delta^*}{\partial t} \frac{\pi \hbar g N(0)}{8 k T_{\mathrm{c}}} ; *
\end{aligned} \tag{II.6}$$

$N(0)$ being the density of states at the Fermi surface.
Following Gorkov, we put

$$\psi(\boldsymbol{r} t)=\frac{7 \zeta(3) n_0 \chi(\varrho)}{16 \pi^2 k^2 T_{\mathrm{c}}^2} \cdot \Delta(\boldsymbol{r} t) ; \quad \zeta(3)=1.202 \tag{II.7}$$

where $\chi(\varrho)$ is defined by eq. (15) of ref. [3b] and obtain eq. (II.1) as a substitute of eq. (II.2). The coefficients of eq. (II.1) are given by

$$\alpha=\frac{1}{\eta} \frac{T_{\mathrm{c}}-T}{T_{\mathrm{c}}} ; \beta=\frac{1}{\eta} \frac{2}{n_0 \cdot \chi(\varrho)} ; \quad \gamma=\frac{1}{\eta} \cdot \frac{\pi}{8 k T_{\mathrm{c}}} ; \quad \eta=\frac{7 \zeta(3) \varepsilon_{\mathrm{F}}}{12 \pi^2 k^2 T_{\mathrm{c}}} \chi(\varrho) ; \tag{II.8}$$

where n_0 is the density of electrons. Using these values, we find the relaxation time τ_{R}, eq. (II.2), equal to

$$\tau_{\mathrm{R}}=\frac{\pi}{16} \frac{\hbar}{k T_{\mathrm{c}}}\left(1-T / T_{\mathrm{c}}\right)^{-1} \approx 0.7 \frac{\hbar}{\Delta(T=0)}\left(1-T / T_{\mathrm{c}}\right)^{-1}$$

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- Actually, it is only necessary to have $\operatorname{Re} \gamma>0$; we will find, however, γ to be real. γ cannot be purely imaginary, since then a stationary solution $|\psi|=$ const. can be found arbitrarely far off from the equilibrium value.
  ⋆⋆ Stephen and Surll [7] expand the Gorkov equation in a power series of $\Delta(x)-\Delta_0$, where Δ_0 is the zero field equilibrium value of the gap and avoid thereby this switching process. On the other hand, for a type II superconductor near $H_{c2}, \Delta(x)-\Delta_0$ might be quite large whereas $\Delta(x)$ is small.

which is very close to the relaxation time considered by Tinkham [7].
In the case of external fields, we include the effects of a vectorpotential $\boldsymbol{A}(\boldsymbol{r}t)$, $\operatorname{curl}\boldsymbol{A}=\boldsymbol{H}$, by replacing $\hbar\nabla$ in eq. (II.1) by $\hbar\nabla+(2ie/c)\boldsymbol{A}(\boldsymbol{r}t)^{\star\star}$. In order to have a gauge invariant equation, we have to substitute $\hbar\partial/\partial t$ by a term of the form $\hbar\partial/\partial t-2ie\tilde{\Phi}(\boldsymbol{r}t)$. A gauge transformation consists in the simultaneous substitutions

$$\psi\to\psi e^{(2ie/\hbar c)p};\quad A\to A+\nabla\varphi;\quad \tilde{\Phi}\to\tilde{\Phi}-\frac{1}{c}-\frac{\partial p}{\partial t}.$$

This form of a gauge transformation indicates a relationship between the effective potential $\tilde{\Phi}$ and the electric potential Φ. We propose the relation

$$\tilde{\Phi}=\Phi-\mu/e \tag{II.9}$$

where μ is the local chemical potential of the superconductor ***. We have implicitly assumed that a local thermodynamic equilibrium can be defined.

Considering the modifications discussed above, we obtain the final form of eq. (IT.1) :

$$\gamma\left(\hbar\frac{\partial}{\partial t}-2ie\tilde{\Phi}\right)\psi=\frac{1}{2m}\left(\hbar\nabla+\frac{2ie}{c}\boldsymbol{A}\right)^2\psi+\alpha\psi-\beta|\psi|^2\psi \tag{A}$$

In order to obtain equations for the quantities $\boldsymbol{A}$ and Φ, we have to determine expressions for the sources of the electromagnetic field, the current density $\boldsymbol{j}$ and the charge density ϱ.

The current density is given by

$$\boldsymbol{j}(\boldsymbol{r}t)=\left[\frac{\hbar e}{m}\left(\nabla_{\boldsymbol{r}}-\nabla_{\boldsymbol{r}'}\right)G\left(x,x'\right)+\frac{2ie}{mc}\boldsymbol{A}(x)G\left(x,x'\right)\right]_{\boldsymbol{r}=\boldsymbol{r}';t=t'-0}$$

where $G\left(x,x'\right)$ is the Green's function in the superconducting state. Considering the Gorkov equation, we find $G\left(x,x'\right)$ in sufficient approximation equal to

$$\begin{aligned}G\left(x,x'\right)&=G_0\left(x,x'\right)-\\&-\int dx_1dx_2G_0\left(x,x_1\right)\Delta\left(x_1\right)G_0\left(x_2,x_1\right)A^*\left(x_2\right)G_0\left(x_2,x'\right)\end{aligned} \tag{II.10}$$

The first term of the r.h.s. of eq. (II.10) is the Green's function of the metal in the normal state and, consequently, the current density $\boldsymbol{j}_{\mathrm{n}}$ resulting from this term is the same as the current density in the normal state. By Ohm's law, we have

$$\boldsymbol{j}_{\mathrm{n}}=\sigma\tilde{\boldsymbol{E}}$$

where the effective field strength $\tilde{\boldsymbol{E}}$ is given by

$$\tilde{\boldsymbol{E}}(\boldsymbol{r}t)=-\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A}(\boldsymbol{r}t)-\nabla\tilde{\Phi}(\boldsymbol{r}t) \tag{TI.11}$$

In the second term, we expand

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- Calculating the coefficient of the next order term $\partial^2\Delta/\partial t^2$, we are led to the assumption that the expansion parameter of this series is of the order $\hbar/kT_{\mathrm{c}}$.
  ** See the discussion given by Maki [14].
  *** The chemical potential μ is the energy necessary to add an electron in thermodynamic equilibrium (for instance, at constant temperature), whereby the electromagnetic fields have to be kept fixed. It is a rather unphysical concept, which is shown by the fact that μ appears only in the combination (II.9). In the case $\mathbf{A}=\mathbf{0},\mu-e\boldsymbol{\Phi}$ is the electrochemical potential.

$$\Delta\left(\boldsymbol{r}_{\mathrm{n}} t_{\mathrm{n}}\right)=\Delta(\boldsymbol{r} t)+\left(\boldsymbol{r}_{\mathrm{n}}-\boldsymbol{r}\right) \nabla \Delta(\boldsymbol{r} t); \tag{II.12}$$

omitting terms as $\frac{\partial^2 A}{\partial q_j \partial q_k}$ and $\frac{\partial A}{\partial t}$, which, according to the treatment above, should be considered of the same order of magnitude. In this way, we get the second contribution to the current density equal to the expression given by Ginzburg and Landau*. We insert these results in Maxwell's second equation and get a second basic equation:

$$\begin{aligned}
&\frac{1}{c^2} \frac{\partial^2}{\partial t^2} \boldsymbol{A}+\frac{1}{c} \frac{\partial}{\partial t} \nabla \Phi+\operatorname{curl} \operatorname{curl} \boldsymbol{A}=\frac{4 \pi}{c}\left(\boldsymbol{j}_{\mathrm{n}}+\boldsymbol{j}_{\mathrm{s}}\right) \\
&\boldsymbol{J}_{\mathrm{n}}=-\sigma\left[\frac{1}{c} \frac{\partial}{\partial t} \boldsymbol{A}+\nabla \tilde{\Phi}\right] \\
&\boldsymbol{j}_{\mathrm{s}}=\frac{i e \hbar}{m}\left(\psi^{*} \nabla \psi-\psi \nabla \psi^{*}\right)-\frac{4 e^2}{m c} \boldsymbol{A}|\psi|^2
\end{aligned} \tag{B}$$

We now derive an expression for the charge density $\varrho(\boldsymbol{r} t)$ of the electrons. Subtracting the positively charged background, we have

$$\varrho=2 i e G\left(x, x^{\prime}\right)_{r=r^{\prime} ; t=t^{\prime}-0}+e n_0$$

where $G\left(x, x^{\prime}\right)$ is given by eq. (II.10). Corresponding to the two parts of G, we put $\varrho=\varrho_{\mathrm{n}}+\varrho_{\mathrm{s}}{}^{\star}$.

The charge density ϱ_n in the normal state is proportional to the change of the chemical potential and, therefore, by eq. (II.9), proportional to $\Phi-\tilde{\Phi}$. We put

$$\varrho_{\mathrm{n}}=-\frac{1}{4 \pi} \lambda_{\mathrm{D}}^{-2}(\Phi-\tilde{\Phi})$$

where λ_{D} is the Debye screening length, $\lambda_{\mathrm{D}}=\left[8 \pi e^2 N(0)\right]^{-1/2}$.
In the expression of ϱ_s, which is proportional to the second term of the r.h.s. of eq. (II.10) when $x=x^{\prime}$, we expand $\Delta\left(\boldsymbol{r}_{\mathrm{n}} t_{\mathrm{n}}\right)$ according to eq. (II.3) and obtain in the case of no external fields

$$\varrho_{\mathrm{S}}=c_1|\nabla \Delta(\boldsymbol{r} t)|^2+c_2\left[\Delta^{*} \nabla^2 \Delta(\boldsymbol{r} t)+\Delta \nabla^2 \Delta^{*}(\boldsymbol{r} t)\right]+i c_3\left[\Delta^{*} \frac{\partial \Delta}{\partial t}-\Delta \frac{\partial \Delta^{*}}{\partial t}\right].$$

As far as the time dependence in this problem is concerned, the calculation of the coefficients c_j follows the prescription given above. The coefficients c_1 and c_2 depend on the presence of impurities and can be calculated using Gorkov's method [3b]. For simplicity, we consider only the case of isotropic scattering where $c_2=\frac{1}{2} c_1$; then

$$c_1|\nabla \Delta|^2+c_2\left[\Delta^{*} \nabla^2 \Delta+\Delta \nabla^2 \Delta^{*}\right]=\frac{1}{2} c_1 \nabla^2|\Delta|^2.$$

We put the expressions for the charge densities in Maxwell's third equation and obtain:

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- One may wonder why the magnitude of the normal current is unaffected by the presence of superconductivity, since one expects the normal conductivity to decrease by increasing magnitude of the gap. One can obtain such corrections by expanding $\Delta(rt)$, eq. (II.12), to higher orders in the derivatives.

$$
\begin{aligned}
-\frac{1}{c}\frac{\partial}{\partial t}\nabla \boldsymbol{A} - \nabla^2\Phi &= 4\pi\left(\varrho_{\mathrm{n}} + \varrho_{\mathrm{s}}\right); \\
\varrho_{\mathrm{n}} &= -\frac{1}{4\pi}\lambda_{\mathrm{D}}^{-2}(\Phi - \tilde{\Phi}); \\
\varrho_{\mathrm{s}} &= \frac{e}{2\varepsilon_{\mathrm{F}}}\cdot\frac{\hbar^2}{2m}\nabla^2|\psi|^2 + \frac{3}{\varepsilon_{\mathrm{F}}\chi(\bar{\varrho})}\times \\
&\times\left[\frac{i\hbar}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) + 2e\tilde{\Phi}|\psi|^2\right].
\end{aligned}
\tag{C}
$$

It may be interesting to note that even in the equilibrium case, where $\frac{\partial\psi}{\partial t} = \tilde{\Phi} = 0$, the charge density is not necessarily zero **.

The set of eqs. (A), (B), and (C) corresponds to 6 independent equations. In principle it is, therefore, possible to determine the 7 unknown quantities $|\psi(\boldsymbol{r},t)|$,

$\arg\psi(\boldsymbol{r},t), A_i(\boldsymbol{r},t), \Phi(\boldsymbol{r},t)$, and $\tilde{\Phi}(\boldsymbol{r},t)$, if one takes into account the irrelevance of the gauge of the wave function and of the electromagnetic field.

Eq. (C) has some features in common with the "additional Ginzburg-Landau equation" proposed by Anderson et al. [9]. However, this equation does not give a direct physical insight, mainly because it contains both Φ and $\tilde{\Phi}$, of which only $\tilde{\Phi}$ is of physical importance. We derive a more appropriate equation, starting from the continuity equation

$$
\operatorname{div}\boldsymbol{j} = \operatorname{div}\left(\boldsymbol{j}_{\mathrm{n}} + \boldsymbol{j}_{\mathrm{s}}\right) = -\frac{\partial}{\partial t}\boldsymbol{o}.
$$

Considering the definitions of $\boldsymbol{j}_{\mathrm{n}}$ and $\boldsymbol{j}_{\mathrm{s}}$, eq. (B), and transforming div $\boldsymbol{j}_{\mathrm{s}}$ by means of eq. (A), we obtain

$$
\begin{aligned}
1 - \frac{\partial}{\partial t}\nabla\boldsymbol{A} + \nabla^2\tilde{\Phi} &= \frac{1}{\sigma}\operatorname{div}\boldsymbol{j}_{\mathrm{s}} - \frac{1}{\sigma}\frac{\partial}{\partial t}\varrho; \\
\operatorname{div}\boldsymbol{j}_{\mathrm{s}} &= \frac{4\gamma e}{\hbar}\left[\frac{i\hbar}{2}\left(\psi^*\frac{\partial}{\partial t}\psi - \psi^{-\frac{\partial}{\partial t}}\psi^*\right) + 2e\tilde{\Phi}|\psi|^2\right].
\end{aligned}
\tag{II.13}
$$

In many cases, $\frac{\partial}{\partial t}\varrho$ is a quantity which can be neglected (we will meet only this situation in the following calculations). Then, eq. (II.13) has a structure similar to eq. (B). In correspondence to the London penetration depth λ_{L} of the magnetic field, a penetration depth $\lambda_{\tilde{\Phi}}$ of the effective potential $\tilde{\Phi}$ can be defined which is given by

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- We emphasize that ϱ_n is the charge density in a corresponding normal state and, therefore, the meaning of ϱ_n is quite different from the concept of the density of normal electrons used in the two fluid model. Consequently, ϱ_{s} is not the density of super electrons; for instance, we will find $\varrho_{\mathrm{s}} = 0$ in the zero field equilibrium case.

  ** We may calculate the charge density using eq. (C), by noting that $\nabla^2|\psi|^2$, and hence Φ will vary over distances approximately equal to the coherence length ξ (about 10^{-6} to 10^{-7} cm). Therefore, $\nabla^2\Phi \leqslant \lambda_{\bar{D}}^{-2}\Phi\left(\lambda_{\mathrm{D}} \approx 10^{-7}\text{ cm}\right)$, and in a first step we have

$$
\Phi = \lambda_{\mathrm{D}}^2\cdot\frac{2\pi e}{\varepsilon_{\mathrm{F}}}\cdot\frac{\hbar^2}{2m}\nabla^2|\psi|^2
$$

Using this result, we get

$$
\varrho = -\frac{1}{4\pi}\nabla^2\Phi = -\lambda_{\mathrm{D}}^2\frac{e}{2\varepsilon_{\mathrm{F}}}\cdot\frac{\hbar^2}{2m}\nabla^4|\psi|^2
$$

ϱ may be estimated to be less than 10^4 esu $\approx 10^{-5}$Coul/cm^3, and the variations of Φ should be less than 10^{-6} esu $\approx 10^{-3}$ volts.

$$\lambda_{\tilde{\Phi}} = \sqrt{\frac{\hbar\sigma\beta}{8e^2\gamma\alpha}}$$

For superconductors of short mean free path in the normal state, $\lambda_{\tilde{\Phi}}$ is equal to

$$\lambda_{\tilde{\Phi}} = 0.42\xi$$

where $\xi = \frac{\lambda_{\mathrm{L}}}{x}$ is the coherence length *.

32.0.3 III. The Energy Theorem

In this section the balance of the free energy density of a superconductor in an electromagnetic field will be discussed on the basis of eqs. (A), (B) and (C). The free energy is the appropriate thermodynamic potential since the process discussed in this paper are at constant temperature.

The free energy density of the electromagnetic field F_{em} is given by

$$F_{\mathrm{em}} = \frac{1}{8\pi}\left[\boldsymbol{H}^2 + \boldsymbol{E}^2\right] = \frac{1}{8\pi}\left[(\mathrm{curl}\,\boldsymbol{A})^2 + \left(\frac{1}{c}\frac{\partial}{\partial t}\boldsymbol{A} + \nabla\Phi\right)^2\right]. \tag{III.1}$$

The free energy density of the superconductor may be considered as consisting of two parts:
(i) the free energy density F_n of the metal in a normal state corresponding to the same values of its variables (temperature and density);
(ii) the free energy difference F_{sn} between superconducting and normal state.
F_{sn} is given by the expression of Ginzburg and Landau [1]

$$F_{\mathrm{sn}} = \frac{1}{2m}\left|\left(\hbar\nabla + \frac{2ie}{c}\boldsymbol{A}\right)\psi\right|^2 - \alpha|\psi|^2 + \frac{\beta}{2}|\psi|^4 \tag{III.2}$$

since this relation is correct even in the case where the Ginzburg-Landau equation (II.1) is not fulfilled. We only have to remember that eq. (II.1) can be derived by minimizing F_{sn} , which means that F_{sn} displays the correct functional dependence on its variables in the neighborhood of the equilibrium states.

In the normal state, the only parameter which can be varied at constant temperature, is the density - ϱ/e. Using the thermodynamic relation $\frac{\partial F_{\mathrm{n}}}{\partial(-\varrho/e)} = \mu^*$, we find

$$F_{\mathrm{n}} = -\frac{1}{e}\int \mu d\varrho' \tag{III.3}$$

where some constant part has been discarded.
We calculate the change of the total energy density in time

$$\frac{dF}{dt} = \frac{dF_{\mathrm{em}}}{dt} + \frac{dF_{\mathrm{n}}}{dt} + \frac{dF_{\mathrm{sn}}}{dt} \tag{III.4}$$

According to standard electrodynamies, the first term on the r.h.s. of eq. (III.4) can be written as follows

$$\frac{dF_{\mathrm{em}}}{dt} = -\,\mathrm{div}\,\boldsymbol{S} - \boldsymbol{j}\cdot\boldsymbol{E}; \tag{III.5}$$
$$\boldsymbol{S} = \frac{c}{4\pi}\boldsymbol{E}\times\boldsymbol{H};$$

[12] $\star\lambda_{\mathrm{L}}$ and $\varkappa$ are defined in ref. [1].

where $\boldsymbol{S}$ is the Pointing vector.
Using $\frac{\partial \varrho}{\partial t} = -\operatorname{div} \boldsymbol{j}$, we find the second term equal to

$$\frac{dF_{\mathrm{n}}}{dt} = \frac{1}{e} \operatorname{div} \mu \boldsymbol{j} - \frac{1}{e} \boldsymbol{j} \nabla \mu \tag{III.6}$$

In calculating the last term, we obtain in a first stage

$$\begin{aligned} \frac{dF_{\mathrm{sn}}}{dt} = -j_{\mathrm{s}} \frac{1}{c} \frac{\partial}{\partial t} \boldsymbol{A} + 2 \operatorname{Re} \frac{\partial \psi}{\partial t} \left[-\frac{1}{2m} \left(\hbar \nabla - \frac{2ie}{c} \boldsymbol{A} \right)^2 - \alpha + \beta |\psi|^2 \right] \psi^* + \\ + 2 \operatorname{Re} \frac{\hbar}{2m} \operatorname{div} \frac{\partial \psi}{\partial t} \left[\hbar \nabla - \frac{2ie}{c} \boldsymbol{A} \right] \psi^* \end{aligned} \tag{III.7}$$

where the definition of $\boldsymbol{j}_{\mathbf{s}}$, eq. (B), has been used. Considering eqs. (II.13) and (B), we find that

$$\begin{aligned} -\boldsymbol{j}_{\mathrm{s}} \nabla \tilde{\Phi} + 2 \operatorname{Re} \left(\frac{1}{\hbar} 2ie \tilde{\Phi} \psi \right) \gamma \left[\hbar \frac{\partial}{\partial t} + 2ie \tilde{\Phi} \right] \psi^* + \\ + 2 \operatorname{Re} \frac{1}{2m} \operatorname{div}(-2ie \tilde{\Phi} \psi) \left[\hbar \nabla - \frac{2ie}{c} \boldsymbol{A} \right] \psi^* \\ = - \boldsymbol{j}_{\mathrm{s}} \nabla \tilde{\Phi} - \tilde{\Phi} \nabla \boldsymbol{j}_{\mathrm{s}} + \nabla \left(\boldsymbol{j}_{\mathrm{s}} \tilde{\Phi} \right) = 0 \end{aligned}$$

The left hand expression will be added to r.h.s. of eq. (III.7). Using eq. (A), we put eq. (III.7) in the following form

$$\begin{aligned} \frac{dF_{\mathrm{sn}}}{dt} = -\boldsymbol{j}_{\mathrm{s}} \left[\frac{1}{c} \frac{\partial}{\partial t} \boldsymbol{A} + \nabla \tilde{\Phi} \right] - \frac{2\gamma}{\hbar} \left| \left(\hbar \frac{\partial}{\partial t} - 2ie \tilde{\Phi} \right) \psi \right|^2 + \\ + \operatorname{div} \operatorname{Re} \frac{1}{m} \left| \left[\hbar \nabla - \frac{2ie}{c} \boldsymbol{A} \right) \psi^* \right] \left[\left(\hbar \frac{\partial}{\partial t} - 2ie \tilde{\Phi} \right) \tilde{\psi} \right] \end{aligned} \tag{III.8}$$

Adding up the results (III.5, III. 6 and III.8), we finally obtain

$$\begin{aligned} \frac{dF}{dt} &= -W - \operatorname{div} \boldsymbol{J}_{\mathrm{E}} \\ W &= \sigma \left[\frac{1}{c} \frac{\partial}{\partial t} \boldsymbol{A} + \nabla \tilde{\Phi} \right]^2 + \frac{2\gamma}{\hbar} \left| \left(\hbar \frac{\partial}{\partial t} - 2ie \tilde{\Phi} \right) \psi \right|^2 \\ \boldsymbol{J}_{\mathbf{E}} &= \boldsymbol{S} - \mu \boldsymbol{j}/e - \operatorname{Re} \frac{\mathbf{1}}{m} \left[\left(\hbar \nabla - \frac{2ie}{c} \boldsymbol{A} \right) \psi^* \right] \left[\left(\hbar \frac{\partial}{\partial t} - 2ie \tilde{\Phi} \right) \psi \right] \end{aligned} \tag{D}$$

We interprete this relation in the following way. The increase of the free energy density is always less than the energy accumulated by the energy current $\boldsymbol{J}_{\mathbf{E}}$. The quantity W obviously represents a loss in reversible energy and, therefore, is the power dissipated in irreversible processes. The first part of W corresponds to the production of Joule heat of the normal electrons, whereas the second part is due to the finite relaxation time ($\gamma \neq 0$) of the wave function ψ. In terms of irreversible thermodynamics, this last part can be considered as the product of the time variation of a generalized coordinate, namely $\frac{\partial \psi}{\partial t} - \frac{2ie}{\hbar} \tilde{\Phi}_{\psi}$, times the force acting on it, which is given by the r.h.s. of eq. (A)*.

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- Since the derivative is taken at fixed electromagnetic field, μ is the chemical, not the electrochemical potential.

32.0.4 IV. The Resistivity of Type II Superconductors

According to Abrikosov [2], there are two limiting cases for type II superconductors in the mixed state:
(i) the vicinity of the point of second order phase transition, where the magnetic field is close to H_{c2}, and $|\psi|^2$ is small;
(ii) the magnetic field is so small that only few fluxoids are in the superconductor.

In these two cases, we will determine that solution of eqs. (A), (B) and (C) which corresponds to the experimental situation in the investigations of Kim, Hempstead and Strnad [5]. The usual way of thinking of this situation is that a fixed transport current causes the fluxoids to move, which in turn gives rise to a voltage drop across the superconductor. On the other hand, one can equally well consider the voltage drop fixed, which maintains a transport current and forces the fluxoids to move. The latter picture is appropriate in case (i), where the superconductor is close to the normal state; and the first one is good in case (ii), where one is not far from ideal superconductivity.

According to the idea of a steady flow of fluxoids [5,6], we assume, in both cases, the time dependence of all quantities $X(\boldsymbol{r},t)$ to be of the form $X(\boldsymbol{r}-\boldsymbol{v}t)$, where $\boldsymbol{v}$ is the drift velocity of the fluxoids. This implies that

$$\frac{\partial}{\partial t}X(\boldsymbol{r},t) = -\boldsymbol{v}\cdot\nabla X(\boldsymbol{r},t). \tag{IV.1}$$

Since the demagnetisation factors of the samples used in experiment are fairly high, we take the mean magnetic field in the superconductor, the induction B, as variable. The cases (i) and (ii) correspond to $B \to H_{c2}$ and $B \to 0$, respectively.

- The contribution to W which is connected with the time variation of the modulus of ψ, namely $2\gamma\hbar\left(\frac{\partial}{\partial t}|\psi|\right)^2$, is similar to the expression proposed by Tinkham [7] in order to account for relaxation effects in superconductors.

The ratio $R = \varrho_{\rm i}/\varrho_{\rm n}$, where $\varrho_{\rm f}$ and $\varrho_{\rm n}$ are the resistivities of the considered mixed state and of the normal state, respectively, will be a function of B such that $R\left(H_{\mathbf{c2}}\right) = 1$ and $R(0) = 0$. The restriction to the limiting cases (i) and (ii) means that we will determine dR/dB when $B = H_{c2}$ and $B = 0$. We note that $\varrho_{\rm r} = \langle\tilde{H}\rangle/\langle j\rangle(\langle\cdots\rangle$ means the average value) has to be calculated for vanishing small $\langle\tilde{H}\rangle$, which allows to simplify the calculations considerably.

Conveniently, we rewrite eqs. (A), (B) and (C) using reduced units. We put

$$\boldsymbol{r} = \lambda_{\rm L}\boldsymbol{r}'; \quad \boldsymbol{A} = \sqrt{2}\lambda_{\rm L}H_{\rm cb}\boldsymbol{A}'; \quad \psi = \sqrt{\frac{\alpha}{\beta}}\psi';$$
$$t = \frac{\hbar\gamma}{\alpha}t'; \quad \Phi(\tilde{\Phi}) = \frac{\alpha}{2e\gamma}\Phi'\left(\tilde{\Phi}'\right);$$

where $\lambda_{\rm L}\left(=\sqrt{mc^2\beta/16\pi e^2\alpha}\right)$ is the zero field penetration depth, $H_{\rm cb}\,(=4\pi\alpha^2/\beta)$ the thermodynamic critical field. The substitutions in the first line are the usual ones; in the second line the scaling factors $\frac{\hbar\gamma}{\alpha}\,(=2\tau_{\rm R})$ and $\frac{\alpha}{2e\gamma}$ are about $10^{-11}\frac{T_c}{T_c-T}$ sec, and $10^{-7}\frac{T_c-T}{T_c}$ esu. Using these substitutions and dropping the primes on the newly introduced quantities, we obtain

$$\left(\frac{1}{\varkappa}\nabla + i\boldsymbol{A}\right)^2 \psi + \psi - \left(\frac{\partial}{\partial t} - i\tilde{\Phi}\right)\psi - |\psi|^2\psi = 0 \tag{$\prime$}$$

$$\nu_1\left(\frac{\partial^2}{\partial t^2}\boldsymbol{A} + \frac{1}{\varkappa}\frac{\partial}{\partial t}\nabla\Phi\right) + \operatorname{curl}\operatorname{curl}\boldsymbol{A} =$$

$$= -\Sigma\left(\frac{\partial}{\partial t}\boldsymbol{A} + \frac{1}{\varkappa}\nabla\tilde{\Phi}\right) + \frac{i}{2\varkappa}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) - \boldsymbol{A}|\psi|^2 \tag{$\prime$}$$

$$\frac{1}{\varkappa}\frac{\partial}{\partial t}\nabla\boldsymbol{A} + \frac{1}{\varkappa^2}\nabla^2\Phi = \nu_2(\Phi - \tilde{\Phi}) - \nu_3\nabla^2|\psi|^2 -$$

$$-\nu_4\left[\frac{i}{2}\left(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}\right) + \tilde{\Phi}|\psi|^2\right] \tag{$\prime$}$$

where $\varkappa$ is the Ginzburg-Landau constant, and

$$\begin{aligned}
&\nu_1 = \tfrac{14\zeta(3)}{\pi^4}\left(\tfrac{\varkappa\lambda_{\mathrm{D}}}{\lambda_{\mathrm{L}}}\right)^2; \quad \nu_2 = \left(\tfrac{\lambda_{\mathrm{L}}}{\varkappa\lambda_{\mathrm{D}}}\right)^2; \\
&\nu_3 = \tfrac{1}{6}\tfrac{e^2 p_{\mathrm{P}}}{\hbar k T_{\mathrm{e}}}\tfrac{\chi(\varrho)}{x^2}; \qquad \nu_4 = \tfrac{14\zeta(3)}{3}\tfrac{e^2 p_{\mathrm{F}}}{\hbar k T_{\mathrm{c}}}\tfrac{\varepsilon_{\mathrm{F}}}{kT_{\mathrm{c}}}\cdot\chi(\varrho); \\
&\Sigma = \tfrac{m\beta}{4e^2\hbar\gamma}\cdot\sigma.
\end{aligned} \tag{IV.2}$$

Concerning the order of magnitude of these coefficients, we observe that $v_1\,(v_2; \nu_3; \nu_4)$ is roughly about $10^{-4}\,(10^4; 10^5; 10^{10})$.Σ decreases with decreasing σ; in the limit of small σ (mean free path shorter than the coherence length ξ_0), Σ assumes a constant value

$$\Sigma_0 = \frac{14\zeta(3)}{\pi^4} = 0.173 \tag{IV.3}$$

In these reduced units, the energy balance takes the form (the unit of energy is $H_{\mathrm{cb}}^2/4\pi$)

$$\begin{aligned}
&\frac{1}{2}\frac{dF}{dt} = -W - \operatorname{div}\boldsymbol{J}_{\mathrm{E}} \\
&W = \Sigma\left(\frac{\partial}{\partial t}\boldsymbol{A} + \frac{1}{\varkappa}\nabla\tilde{\Phi}\right)^2 + \left(\left(\frac{\partial}{\partial t} - i\tilde{\Phi}\right)\psi\right|^2 \\
&\boldsymbol{J}_{\mathrm{E}} = \boldsymbol{E}\times\boldsymbol{H} - \frac{1}{\varkappa e}\mu\boldsymbol{j} - \operatorname{Re}\frac{1}{\varkappa}\left[\left(\frac{1}{\varkappa}\nabla - i\boldsymbol{A}\right)\psi^*\right]\left[\left(\frac{\partial}{\partial t} - i\tilde{\Phi}\right)\psi\right].
\end{aligned} \tag{$\prime$}$$

We are interested in the approximation linear in the driving field $\langle\tilde{E}\rangle$. In this approximation, eq. (II.13) allows to determine $\tilde{\Phi}$ directly provided the equilibrium values of $\psi, \boldsymbol{A}$, and Φ are known. Since we may safely assume the drift velocity to be proportional to $\langle\tilde{E}\rangle$, a quantity $\frac{\partial X}{\partial t} = -\boldsymbol{v}\cdot\nabla X$ can be replaced by - $\boldsymbol{v}\cdot\nabla X_{\mathrm{e}}$, where X_{e} is the equilibrium value of X. Therefore, we write eq. (II.13) in the following form *

$$\begin{aligned}
&-\frac{\Sigma}{x^2}\nabla^2\tilde{\Phi} + \tilde{\Phi}\,|\psi_{\mathrm{e}}|^2 = \\
&= \frac{i}{2}v\cdot\left(\psi_{\mathrm{e}}^*\nabla\psi_{\mathrm{e}} - \psi_{\mathrm{e}}\nabla\psi_{\mathrm{e}}^*\right) - v\cdot\nabla\left(\frac{\Sigma}{\varkappa}\nabla\cdot\boldsymbol{A}_{\mathrm{e}}\right) - v\cdot\nabla\left(\frac{v_1}{x^2}\nabla^2\Phi_{\mathrm{e}}\right);
\end{aligned} \tag{IV.4}$$

where, according to eq. (C$'$),

$$\frac{1}{\varkappa^2}\nabla^2\Phi_{\mathrm{e}} = v_2\Phi_{\mathrm{e}} - v_3\nabla^2\,|\psi_{\mathrm{e}}|^2\,. \tag{IV.5}$$

(i) Vicinity of phase transition. If $B \to H_{c2}, |\psi|^2 \to 0$. Therefore, a zero order solution of eqs. (B') and (IV.4) is

$$\begin{aligned} \boldsymbol{A}^{(0)} &= \left(0, H^{(0)}x, 0\right) \\ \tilde{\Phi}^{(0)} &= -\varkappa \tilde{E}_x^{(0)} x \end{aligned} \tag{IV.6}$$

which corresponds to an effective electric field $\tilde{\boldsymbol{E}}^{(0)} = \left(\tilde{E}_x^{(0)}, v_x H^{(0)}, 0\right)$ perpendicular to the magnetic field $\boldsymbol{H}^{(0)} = \left(0, 0, H^{(0)}\right)$. Thereby, we have neglected the magnetic field due to the transport current, which is supposed to be compensated.

We insert the quantities (IV.6) in (A'), perform the time derivative according to eq. (IV.1), and neglect the cubic term. Since there is no dependence on z, the general solution of this equation is

$$\psi^{(0)} = \sum_k C_k e^{iky} \varphi_k(x) \tag{IV.7}$$

where

$$\frac{1}{\varkappa^2}\frac{\partial^2}{\partial x^2}\varphi_k + \left(\frac{ik}{\varkappa} + iH^{(0)}x\right)^2 \varphi_k + \varphi_k + v_x \frac{\partial}{\partial x}\varphi_k + ikv_y\varphi_k - i\varkappa \tilde{E}_x^{(0)} x \varphi_k = 0$$

Putting $\varphi_k = u_k \exp\left(-\frac{1}{2}\varkappa^2 v_x x\right)$, we find a harmonic oscillator type equation for $u_k(x)$ [15]. In order to have φ_k bounded, we have to put $v_y = -\tilde{E}_x^{(0)}/H^{(0)}$. Furthermore, we have to require that $H^{(0)}$ is as large as possible. This implies $v_x = 0$; and, therefore, $\tilde{\boldsymbol{E}}(0) = \left(\tilde{E_x^{(0)}}, 0, 0\right)$. These results may be put in the following form

$$\tilde{E}(0) = -v \times H^{(0)} \tag{IV.8a}$$

which, in ordinary units, corresponds to *

$$c\tilde{\boldsymbol{E}}^{(0)} = -\boldsymbol{v} \times \boldsymbol{H}^{(0)} \tag{IV.8b}$$

In addition, we have

$$\begin{aligned} H^{(0)} &= \varkappa \left[1 - \left(\varkappa E_x^{(0)}/2H^{(0)}\right)^2\right] \\ \varphi_k &= \exp\left(-\frac{1}{2}\varkappa H^{(0)}\left[x + \frac{1}{\varkappa H^{(0)}}\left(k + i\varkappa \tilde{E}_x^{(0)}/2H^{(0)}\right)\right]^2\right) \end{aligned} \tag{IV.9}$$

In the next step, we determine $\boldsymbol{j}_\delta^{(1)}$, which is obtained by inserting $\boldsymbol{A}^{(0)}$ and $\psi^{(0)}$ in the corresponding expression:

$$\begin{aligned} j_{sx}^{(1)} &= \frac{\varkappa}{2H^{(0)}}\tilde{E}_x^{(0)}\left|\psi^{(0)}\right|^2 && -\frac{1}{2\varkappa}\frac{\partial}{\partial y}\left|\psi^{(0)}\right|^2 \\ j_{sy}^{(1)} &= && +\frac{1}{2\varkappa}\frac{\partial}{\partial x}\left|\psi^{(0)}\right|^2 \end{aligned} \tag{IV.10}$$

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- One may doubt whether eq. (IV.1) can be applied to quantities as $A, \Phi, \tilde{\Phi}$ and ψ, which have no direct physical meaning. However, if one uses the relation (IV.1) for all these quantities simultaneously, possible mistakes will cancel (this implies for instance that the electric field is now defined by $\boldsymbol{E} = (\boldsymbol{v}\cdot\nabla)\boldsymbol{A} - \frac{1}{\varkappa}\nabla\Phi$). One may convince oneself that, indeed, eq. (IV.4) is gauge invariant.

The mean value $\left\langle \boldsymbol{j}_{\mathrm{s}}^{(1)} \right\rangle$ has only a component in x direction equal to

$$\frac{x}{2\tilde{H}^{(0)}} \langle |\psi^{(0)}\Big|^2 \Big\rangle \tilde{E}_x^{(0)}.$$

The ratio R is given by

$$R = \frac{\varrho_{\mathrm{f}}}{\varrho_n} = \frac{\Sigma\langle \tilde{E}\rangle}{\Sigma\langle \tilde{E}\rangle + \langle j_{\mathrm{s}}\rangle} = \left(1 + \langle j_{\mathrm{s}}\rangle / \Sigma\langle \tilde{E}\rangle\right)^{-1}.$$

For the purpose of calculating $\frac{dR}{dB}\big|_{B=H_{c2}}$, we can neglect terms of higher order than $\left|\psi^{(0)}\right|^2$ in $\langle j_{\mathrm{s}}\rangle \mid \Sigma\langle \tilde{E}\rangle$ and, therefore, we are allowed to replace $\langle j_{\mathrm{s}}\rangle$ by $\left\langle j_{\mathrm{s}}^{(1)}\right\rangle$, and $\langle \tilde{E}\rangle$ by $\left\langle \tilde{E^{(0)}}\right\rangle$. Hence

$$R = 1 - \frac{x}{2\Sigma\ddot{H}(0)} \langle |\psi^{(0)}\Big|^2 \Big\rangle. \tag{IV.11}$$

In the linear approximation, $\langle |\psi^{(0)}|^2 \rangle$ may be replaced by its equilibrium value, which, according to Abrikosov [2], is given by**

$$\langle |\psi^{(0)}|^2 \rangle = \frac{2\varkappa^2}{1 + 1.16\,(2\varkappa^2 - 1)} \left(1 - \frac{B}{H_{c2}}\right). \tag{IV.12}$$

Considering eqs. (IV.11, IV.12), we find the slope of $R(B), B = H_{c2}$, equal to

$$\frac{dR(B)}{dB}\bigg|_{B=H_{c2}} = \frac{x^2}{\Sigma\left[1 + 1.16\,(2\varkappa^2 - 1)\right]} \frac{1}{H_{c2}(\bar{T})}. \tag{IV.13a}$$

This equation holds equally in ordinary units. For superconductors of short mean free path ($l < \xi_0$), the value of $\sum$ is given by eq. (IV.3); then we have

$$\frac{dR(B)}{d\bar{B}}\bigg|_{B=H_{c2}} = \frac{2.5}{H_{c2}(T)} \tag{IV.13b}$$

We compare relation (IV.13) with experimental results. For different reduced temperatures T/T_{c}, a plot $R = R(B)$ of experimental data is given in Fig. 4 of ref. [5a], and in Fig. 2a of ref. [5b]. According to relation (IV.13), the slope of R changes discontinuously at $B = H_{c2}$, whereas the experimental curves have a rounded knee in the transition region. Our assumption is that this rounded knee results from experimental circumstances. Therefore, we will neglect the rounded part of the experimental curve and take $\frac{dR}{dB}\big|_{B=H_{c2}}$ equal to the steepest slope of $R(B)$, just before it reaches the value 1 . Using this prescription, we find
a) sample $Nb_{0.5}Ta_{0.5}$

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- More generally, this equation reads

$$c\langle \tilde{\boldsymbol{E}}\rangle = -\boldsymbol{v} \times \langle \boldsymbol{H}\rangle.$$

Sometimes, $\langle \tilde{\boldsymbol{E}}\rangle$ is referred to as the electric field induced by moving fluxoids [5]. However, this argument has not always been found to be convincing [16]. An alternative possibility toexplain this equation is to state that, if not subject to extra forces, the superconductive wave function moves with a drift velocity such that the mean electric field in the moving frame of reference vanishes.
** Concerning the value 1.16 see ref. [17].

T/T_c	0.7	0.6	0.5	0.3
$H_{c2}\frac{dR}{dB}\Big\vert_{B=H_{c2}}$ 2.2	2.0	2.0	2.5	

b) sample $Nb_{0.1}Ta_{0.9}$

T/T_c	0.92	0.79	0.64	0.44	0.35
$H_{c2}\frac{dR}{dB}\Big\vert_{B=H_{c2}}$ 2.8	3.0	3.0	4.4	6.3	

The data confirm relation (IV.13) in so much, as $H_{c2}\frac{dR}{dB}\big|_{B=H_{c2}}$ does not depend on temperature, if the temperature is not too low, say $T/T_c \geqq 0.6$. We find $H_{c2}\frac{dR}{dB}\big|_{B=H_{c2}}$ equal to 2.1 ± 0.1, and 2.9 ± 0.1 for superconductor a) and b), respectively. In both cases, the mean free path of the superconductors is short enough such that the theoretical value 2.5 should be appropriate for $H_{c2}\frac{dR}{dB}\big|_{B=H_{ea}}$. Within 20%, this value agrees with the experimental result. This should be considered a good agreement on account of the uncertainties in defining the final slope of the experimental curves. As far as the low temperature behavior is concerned, there is no doubt that in this case the equations (A), (B) and (C) cannot be considered as accurate enough.
(ii) Small fields. In this case, the calculation follows in principle the procedure of Stephen and Bardeen [6]. We will calculate the energy dissipated by a moving fluxoid. This energy has to be equal to the energy input via transport current, since the total free energy of the superconductor is constant in a state of steady flow.

Considering eq. (D′), the total dissipated power (per unit volume) is given by $D = \frac{1}{V}\int d^3\boldsymbol{r} W(\boldsymbol{r},t)$, where V is the volume of the superconductor. Reasoning in the same way as done in deriving eq. (IV.4), we find that the leading part of D, which is proportional to the square of the drift velocity $\boldsymbol{v} = (0, v, 0)$ is obtained by replacing A and ψ by their equilibrium values. Therefore,

$$D = \frac{1}{V}\int d^3\boldsymbol{r}\Sigma\left(v\frac{\partial}{\partial y}A_e - \frac{1}{\varkappa}\nabla\tilde{\Phi}\right)^2 + \frac{1}{V}\int d^3\boldsymbol{r}\left|\left(-v\frac{\partial}{\partial y} - i\tilde{\Phi}\right)\psi e\right|^2; \tag{IV.14}$$

where $\tilde{\Phi}$ is defined by eq. (IV.4).

In equilibrium, the wave function of a single fiuxoid is, in cylindrical coordinates, of the form

$$\psi_e = f(r)e^{i\varphi}$$

We do not know $f(r)$ exactly; however, the approximation

$$f(r) = \frac{\varkappa r}{\sqrt{\varkappa^2 r^2 + 2}}$$

is reasonable and allows simple calculations. When $x \gg 1$, this ansatz yields the energy per unit length of the fluxoid $\varepsilon = 2\pi[\ln\varkappa + 0.52]/\varkappa^2$, which agrees sufficiently with Abrikosov's value $\varepsilon = 2\pi[\ln x + 0.08]/x^2$.

We put

$$\boldsymbol{Q} = \boldsymbol{A}_e + \frac{1}{\varkappa}\nabla\varphi = Q(r)[-\sin\varphi, \cos\varphi, 0]; \tag{IV.1.5a}$$

and in equilibrium, eq. (B′) becomes

$$\frac{d}{dr}\frac{1}{r}\frac{d}{dr}rQ(r) = f^2 Q(r) \tag{IV.16}$$

The appropriate solution of this equation is

$$Q(r) = \frac{\sqrt{x^2r^2/2+1}}{xr}\frac{K_1\left(\frac{1}{x}\sqrt{x^2r^2+2}\right)}{K_1(\sqrt{2}/x)} \tag{IV.15b}$$

where K_n is the modified Bessel function of second kind and order n.
Concerning eq. (IV.4), which determines $\tilde{\Phi}$, we can discard the last term of the r.h.s. of this equation. Since $\varkappa^2\nu_2 \gg 1$, it follows from eq. (IV.5) that $\Phi_\mathrm{e} = \frac{\nu_3}{\nu_2}\nabla^2 f^2$. Therefore, $v\frac{\partial}{\partial y}\frac{\nu_1}{x^2}\nabla^2\Phi_\mathrm{e}$ is equal to $v\frac{\partial}{\partial y}\frac{\nu_1\nu_3}{x^2\nu_2}\nabla^4 f^2$ and hence, it is smaller than the first term by roughly the factor $\frac{\nu_1\nu_3}{\nu_2} \approx 10^{-3}$. This result has an interesting consequence. It actually means that in sufficient approximation the total current $\boldsymbol{j}_\mathrm{n} + \boldsymbol{j}_\mathrm{s}$ is free of divergences.

Since div $\boldsymbol{A}_\mathrm{e} = 0$, eq. (IV.4) becomes

$$-\frac{\Sigma}{x^2}\nabla^2\tilde{\Phi} + f^2\tilde{\Phi} = -vf^2\frac{\partial}{\partial y}\varphi = -vf^2\frac{\cos\varphi}{r} \tag{IV.17}$$

$\tilde{\Phi}$ is obviously proportional to $\cos\varphi$. Putting

$$\tilde{\Phi} + v\frac{\partial}{\partial y}\varphi = P(r)\cos\varphi \tag{IV.18a}$$

we obtain the differential equation of $P(r)$ quite similar to eq. (IV.16). From there we get the solution

$$P(r) = v\frac{\sqrt{x^2r^2/2+1}}{xr}\frac{K_1\left(\sqrt{x^2r^2+2/\Sigma}\right)}{K_1(\sqrt{2/\Sigma})} \tag{IV.18b}$$

Corresponding to eqs. (IV.15, IV.18), the magnetic and electric fields vanish proportional to e^{-r} as $r \gg 1$ and, therefore, fluxoids which are sufficiently separated will not influence each other. If there are N fluxoids per unit area, the resulting field is the sum of the fields of the individual fluxoids. Calculating the mean effec-
tive electric field in this way, we obtain

$$\left\langle \tilde{\boldsymbol{H}}_x \right\rangle = N\int dxdy\left[v\frac{\partial}{\partial y}\left(A_\mathrm{e}\right)_x - \frac{1}{x}\frac{\partial}{\partial x}\tilde{\Phi}\right] = -\frac{2\pi}{x}Nv \tag{IV.19}$$

and $\left\langle \tilde{E}_y \right\rangle = 0$. Furthermore, the mean magnetic field B is $-\frac{2\pi}{\varkappa}N$ such that - (see eq. (IV.8) -

$$\left\langle \tilde{E}_x \right\rangle = -vB \tag{IV.20}$$

We found the divergence of the total current to be zero. Since the internal currents of the superconductor are free of divergences, we conclude, that the transport current $\boldsymbol{j}_T$ is constant in space, equal to $\langle \boldsymbol{j} \rangle$. We assume $\boldsymbol{j}_T$ parallel to the mean electric field. Therefore, the energy input per unit volume via the transport current is given by

$$\left\langle \boldsymbol{J}_\mathrm{T}\cdot\tilde{\boldsymbol{E}} \right\rangle = j_\mathrm{T}\left\langle \tilde{\boldsymbol{E}}_x \right\rangle. \tag{IV.21}$$

The expression (IV.21) has to be equal to the dissipated power. Considering eq. (IV.14), we put $D = N\eta v^2 = N\eta_1 v^2 + N\eta_2 v^2$, and using the results (IV.15, IV.18), we obtain

$$\eta_1 = \frac{\pi}{2}\left(\frac{\Sigma}{\varkappa^2} - \frac{K_1^2(\sqrt{2}/\varkappa) - K_0^2(\sqrt{2}/\varkappa)}{K_1^2(\sqrt{2}/\varkappa)} + \frac{K_1^2(\sqrt{2}/\Sigma) - K_0^2(\sqrt{2}/\Sigma)}{K_1^2(\sqrt{2/\Sigma})}\right); \tag{IV.22a}$$
$$\eta_2 = \frac{\pi}{2}\left(\frac{1}{2} + \frac{K_2(\sqrt{2/\Sigma})K_0(\sqrt{2/\Sigma}) - K_1^2(\sqrt{2/\Sigma})}{K_1^2(\sqrt{2/\Sigma})}\right).$$

In the limit of show mean free path, we have $\sqrt{2/\Sigma} = 3.4$, and hence,

$$\eta_1 = 0.11\pi; \quad \eta_2 = 0.43\pi \tag{IV.22b}$$

We are now able to calculate the ratio R. Using eqs. (IV.21, IV.22), we obtain

$$R = \Sigma\frac{\langle\tilde{E}\rangle}{\langle\tilde{j}\rangle} = \Sigma\frac{\langle\tilde{E}\rangle^2}{N\eta v^2} = \frac{2\pi\Sigma}{\eta}B$$

where B has to be a positive quantity. From there, we find the slope of R at $B = 0$ equal to

$$\left.\frac{dR}{dB}\right|_{B=0} = \frac{2\pi}{\eta}\cdot\frac{1}{H_{c2}(T)} \tag{IV.23a}$$

which reduces to

$$\left.\frac{dR}{dB}\right|_{B=0} = \frac{0.64}{H_{c2}(T)} \tag{IV.23b}$$

in the above mentioned limit.
Kim, Hempstead and Strnad [5a] have represented their experimental results by the following formula

$$\left.\frac{dR}{d\bar{B}}\right|_{B=0} = \frac{1}{H_{c2}(T=\bar{0})}\cdot f\left(T/T_c\right)$$

where $f\left(T/T_c\right)$ is a universal function practically independent of T and such that $f(0) = 1$. For $T/T_c \approx 0.6$, experimental and theoretical results coincide; however, the general temperature dependence is different. As far as the lack of agreement at lower temperatures is concerned (at $T = 0$, the theoretical value is about 40%
too low), it may be explained very well by the restricted validity of the eqs. (A-C). In the high temperature range, however, we do not think the experimental formula to be very reliable. The initial slope of R cannot remain finite as $T \to T_c$, since the B values for which R has to reach unity ($B = H_{c2}$) will approach zero. (Note that the behavior of the sample $Nb_{0.1}Ta_{0.9}$ of ref. [5b] seems to indicate an appreciable variation of the initial slope and thus does not confirm the assumption $f \approx 1$.)

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33 Time-dependent Ginzburg-Landau theory for a weak-coupling superconductor by Stoof (1992)

(warning: low cited article!!!! is it a good one???)

Abstract

Using a functional approach to the Keldysh formalism we develop a time-dependent GinzburgLandau equation for the long-wavelength and low-frequency dynamics of a weak-coupling superconductor, which is also valid in situations that deviate significantly from thermal equilibrium. The theory takes the interactions between quasiparticles and Cooper pairs into account exactly and is explicitly Galilean-invariant at zero temperature. However, the temporal and spatial dependence of the order parameter is assumed to be such that the effects of Landau damping can be neglected. We also consider the screening properties of the superconductor and show that the inclusion of the polarizability of the electron gas is necessary for a complete description.

33.0.1 I. INTRODUCTION

Even before the advent of the microscopic BCS theory, the phenomenological Ginzburg-Landau [1] theory proved to be a powerful tool in the study of the equilibrium properties of a superconductor in slowly varying electromagnetic fields. [2] Therefore, it came as no surprise that soon after the formulation of the microscopic theory, [3] Gorkov was able to relate both approaches by identifying the energy gap $\Delta(x)$ with the order parameter of the phase transition. [4]

However, for the description of nonequilibrium properties we need an extension of the original GinzburgLandau theory, because the time dependence of the order parameter becomes of the utmost importance. Examples are the nucleation and spinodal growth of the superconducting phase, [5] the dynamics of Josephson junctions, and various phase-slip phenomena associated with the motion of vortices. In particular, the decay of the supercurrent in quasi-one-dimensional superconductors due to both temperature and quantum fluctuations seems to be poorly understood at present [6] and may offer the possibility to verify experimentally the predictions of the timedependent Ginzburg-Landau theory to be derived in this paper.

Theoretically, the above-mentioned phenomena are usually studied at temperatures near the critical temperature T_c, where Δ/k_BT can be considered as a small quantity and the form of the time-dependent GinzburgLandau theory is well established. [7–9] Below T_c the situation is less clear. Abrahams and Tsuneto [8] have made an extensive study of this region and find that a timedependent Ginzburg-Landau equation can only be derived at zero temperature, since at nonzero temperatures the collective (Bogoliubov-Anderson) mode is Landau damped. Nevertheless, neglecting this process, a timedependent Ginzburg-Landau theory can be formulated at all temperatures, which is generally used to describe the physics of phase-slip phenomena. [10]

This procedure has, however, an important drawback: since Δ/k_BT is not small well below T_c, Abrahams and Tsuneto arrive at their results by expanding the equations of motion for $\Delta(\mathbf{x},t)$ around the value of the gap at some point ($\mathbf{x}',t'$), which requires "zeroth-order" Green's functions corresponding to this local gap $\Delta\left(\mathbf{x}',t'\right)$. Because these are not known one has to resort to a physically motivated approximation. Abrahams and Tsuneto assume that the dynamics of the quasiparticles is fast compared to the dynamics of the Cooper pairs and hence that the quasiparticles are at all times in equilibrium with respect to $\Delta\left(\mathbf{x}',t'\right)$. Unfortunately, this is not the case in conventional superconductors as pointed out recently by Ao et al. [11] They therefore propose a sudden approximation, in which the quasiparticles do not react to a change in the condensate wave function and only provide for a "static" background.

Although the static picture is certainly an improvement over the adiabatic one, there still are some problems associated with it. First, Ao et al. include quasiparticle relaxation due to electron-electron, electron-phonon, and electron-impurity scattering in their discussion but neglect the interaction between quasiparticles and condensate, which might induce a more rapid

temporal behavior. Second, their time-dependent Ginzburg-Landau theory is not Galilean-invariant at zero temperature, which contradicts BCS theory. From a formal point of view this is somewhat disturbing, although its effect is negligible in the limit of small currents for which the static picture was developed.

In an attempt to overcome these problems we present in Sec. II a functional approach to the Keldysh formalism, [12] which is used in Secs. III and IV to derive an effective long-wavelength and low-frequency action for the energy gap and thus the desired Ginzburg-Landau theory. In Sec. III we first consider the neutral case and in Sec. IV we subsequently show how the electromagnetic screening properties of the charged superconductor can be included in the description. Finally in Sec. V we summarize some conclusions of this work.

33.0.2 II. NONEQUILIBRIUM THEORY

A way in which we can conveniently study nonequilibrium processes and take the interaction between quasi-
particles and condensate into account is by means of a functional formulation of the Keldysh theory. Such a formulation was recently developed to discuss the nucleation of Bose-Einstein condensation in a dilute atomic gas [13] and gives the opportunity to follow the time evolution of the system of interest, whenever an initial density matrix $\rho\left(t_0\right)$ is specified. In the following we take a $\rho\left(t_0\right)$ corresponding to an essentially arbitrary distribution $N(E)$ of quasiparticles with energy E and gap Δ_0, the latter being obtained self-consistently from the appropriate BCS gap equation for the quasiparticle distribution $N(E)$. Subsequently, we determine the equations of motion for $\Delta(\mathbf{x}, t)$ at times $t \gg t_0+O\left(\hbar / k_B T_c\right)$ when the transients have died out and the Ginzburg-Landau equation acquires a particular useful form.

Following closely the treatment of the Bose gas, we start the discussion of the Keldysh formalism by writing the generation functional of all Green's functions for a
system of fermions with spin states $|\alpha\rangle$ as a functional integral

$$\begin{aligned} Z\left[J, J^*\right] = & \int d\left[\psi^*\right] d[\psi] \exp\left(\frac{i}{\hbar} S\left[\psi^*, \psi\right]\right) \\ & \times \exp\left[i \sum_\alpha \int_C d t \int d \mathbf{x}\left[\psi_\alpha^*(\mathbf{x}, t) J_\alpha(\mathbf{x}, t)\right.\right. \\ & \left.\left.+J_\alpha^*(\mathbf{x}, t) \psi_\alpha(\mathbf{x}, t)\right]\right) \end{aligned} \tag{1}$$

over the Grassmann variables $\psi_\alpha(\mathbf{x}, t)$ and $\psi_\alpha^*(\mathbf{x}, t)$ defined on the Keldysh contour C, which consists of a chronological branch from t_0 to infinity and an antichronological branch from infinity to t_0.[14] Here, the sources $J_\alpha(\mathbf{x}, t)$ and $J_\alpha^*(\mathbf{x}, t)$ are anticommuting c-number fields and the action $S\left[\psi^*, \psi\right]$ is given by

$$\begin{aligned} S\left[\psi^*, \psi\right] = & \sum_\alpha \int_C d t \int d \mathbf{x} \psi_\alpha^*(\mathbf{x}, t)\left[i \hbar \frac{\partial}{\partial t}+\frac{\hbar^2 \nabla^2}{2 m}+\mu\right] \psi_\alpha(\mathbf{x}, t) \\ & -\frac{1}{2} \sum_{\alpha, \alpha^{\prime}} \int_C d t \int d \mathbf{x} \int d \mathbf{x}^{\prime} \psi_\alpha^*(\mathbf{x}, t) \psi_{\alpha^{\prime}}^*\left(\mathbf{x}^{\prime}, t\right) V\left(\mathbf{x}-\mathbf{x}^{\prime}\right) \psi_{\alpha^{\prime}}\left(\mathbf{x}^{\prime}, t\right) \psi_\alpha(\mathbf{x}, t) \end{aligned} \tag{2}$$

for particles with an effective mass m and a spinindependent interaction $V\left(x-x^{\prime}\right)$. Notice that the chemical potential μ is time dependent in principle, but for a weak-coupling superconductor is well approximated by a constant. Moreover, in the case of singlet pairing the effectively attractive interaction $V\left(\mathbf{x}-\mathbf{x}^{\prime}\right)$ can be replaced by the Gorkov potential $-g \delta\left(\mathbf{x}-\mathbf{x}^{\prime}\right)$ and the second term on the right-hand side of Eq. (2) becomes

$$g\int_C dt\int d\mathbf{x}\psi_\uparrow^*(\mathbf{x},t)\psi_\downarrow^*(\mathbf{x},t)\psi_\downarrow(\mathbf{x},t)\psi_\uparrow(\mathbf{x},t)$$

denoting the spin-up and spin-down states by $|\uparrow\rangle$ and $|\downarrow\rangle$, respectively.

To find the effective action and thus the desired Ginzburg-Landau theory for the order parameter $\langle\Delta(\mathbf{x},t)\rangle = -g\langle\psi_\downarrow(\mathbf{x},t)\psi_\uparrow(\mathbf{x},t)\rangle$ of the phase transition we perform a Hubbard-Stratonovich transformation by multiplying the generating functional $Z[J,J^*]$ by

$$1 \equiv \mathcal{N}\int d[\Delta^*]\,d[\Delta]\exp\left(-\tfrac{1}{g}\int_C dt\int d\mathbf{x}\Big|\,\Delta(\mathbf{x},t) + g\psi_\downarrow(\mathbf{x},t)\psi_\uparrow(\mathbf{x},t)\Big|^2\right) \tag{3}$$

and evaluating the Gaussian integral over the fermionic fields. The latter is most easily accomplished in Nambu space, which implies the introduction of the vector quantities:

$$\psi(\mathbf{x},t) \equiv \begin{bmatrix}\psi_\downarrow(\mathbf{x},t)\\ \psi_\uparrow^*(\mathbf{x},t)\end{bmatrix}, \quad J(\mathbf{x},t) \equiv \begin{bmatrix}J_\downarrow(\mathbf{x},t)\\ -J_\uparrow^*(\mathbf{x},t)\end{bmatrix} \tag{4}$$

and their hermitian conjugates. In this manner we find that [9]

$$Z[J,J^*] = \mathcal{N}\int d[\Delta^*]\,d[\Delta]\exp\left[\frac{i}{\hbar}S[\Delta^*,\Delta]\right]\exp\left(-i\int_C dt\int d\mathbf{x}\int_C dt'\int d\mathbf{x}' J^\dagger(\mathbf{x},t)G(\mathbf{x},t;\mathbf{x}',t')\,J(\mathbf{x}',t')\right) \tag{5}$$

with the effective action

$$S[\Delta^*,\Delta] = -i\hbar\,\mathrm{Tr}\left[\ln G^{-1}\right] - \frac{1}{g}\int_C dt\int d\mathbf{x}|\Delta(\mathbf{x},t)|^2 \tag{6}$$

expressed in terms of the exact one-particle Green's function

$$\begin{aligned} iG(\mathbf{x},t;\mathbf{x}',t') &\equiv \left\langle T\left[\psi(\mathbf{x},t)\psi^\dagger(\mathbf{x}',t')\right]\right\rangle \\ &= \begin{pmatrix} \left\langle T\left[\psi_\downarrow(\mathbf{x},t)\psi_\downarrow^\dagger(\mathbf{x}',t')\right]\right\rangle & \langle T\left[\psi_\downarrow(\mathbf{x},t)\psi_\uparrow(\mathbf{x}',t')\right]\rangle \\ \left\langle T\left[\psi_\uparrow^\dagger(\mathbf{x},t)\psi_\downarrow^\dagger(\mathbf{x}',t')\right]\right\rangle & \left\langle T\left[\psi_\uparrow^\dagger(\mathbf{x},t)\psi_\uparrow(\mathbf{x}',t')\right]\right\rangle \end{pmatrix}, \end{aligned} \tag{7}$$

where the average is calculated using the initial density matrix $\rho(t_0)$ and T represents the time-ordering operator on the

Keldysh contour. Introducing also a δ function on the contour by means of $\int_C dt'\delta(t,t') = 1$, this Green's function obeys the equation of motion,

$$\begin{bmatrix} i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\nabla^2}{2m} + \mu & \Delta(\mathbf{x},t) \\ \Delta^*(\mathbf{x},t) & -\left(-i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2\nabla^2}{2m} + \mu\right)\end{bmatrix} G(\mathbf{x},t;\mathbf{x}',t') = \hbar\delta(\mathbf{x}-\mathbf{x}')\,\delta(t,t') \tag{8}$$

defining also its inverse G^{-1}. The latter plays an important role in the following section where we evaluate the effective action of Eq. (6) and derive a time-dependent Ginzburg-Landau equation for the order-parameter $\langle\Delta(\mathbf{x},t)\rangle$ by means of a gradient expansion.

33.0.3 III. TIME-DEPENDENT GINZBURG-LANDAU THEORY

We now turn to the actual calculation of the action $S[\Delta^*,\Delta]$ and the time-dependent Ginzburg-Landau equation, which follows from the principle of least action, i.e., $\delta S[\Delta^*,\Delta]/\delta\Delta^*(\mathbf{x},t) = 0$. As mentioned above we take for $\rho(t_0)$ the density matrix of an

ideal gas of quasiparticles with an energy distribution $N(E)$ and the dispersion relation $E(\mathbf{k}) = \left[\epsilon(\mathbf{k})^2 + |\Delta_0|^2\right]^{1/2}$, where $\epsilon(\mathbf{k}) = (\hbar^2/2m)\,\mathbf{k}^2 - \mu$ and the energy gap Δ_0 is found from the usual BCS gap equation

$$\int \frac{d\mathbf{k}}{(2\pi)^3} \frac{[1 - 2N(\mathbf{k})]}{2E(\mathbf{k})} = \frac{1}{g} \tag{9}$$

We also take the limit $t_0 \to -\infty$, since here we are not interested in the fast [of $O\left(\hbar/k_B T_c\right)$] transients, which are especially important for a discussion of the nucleation of the superconducting phase.

We proceed as follows. First, we expand the exact action $S\left[\Delta^*, \Delta\right]$ around its minimum $S\left[\Delta_0^*, \Delta_0\right]$ by writing $\Delta(\mathbf{x}, t) = \Delta_0 + \Delta'(\mathbf{x}, t)$ and using $\Delta'(\mathbf{x}, t)/\left|\Delta_0\right|$ as a small parameter. Furthermore, we only consider fluctuations $\Delta'(\mathbf{x}, t)$ of long wavelength and small frequency. Explicit calculation at zero temperature shows that this expansior converges quite rapidly and we only have to include quadratic fluctuations to obtain an accurate description of the action in this region. Second, we deduce from this the form of an approximate action $S_{\text{neu}}\left[\Delta^*, \Delta\right]$ by requiring that mass is a conserved quantity and that the expansion of this action around $S_{\text{neu}}\left[\Delta_0^*, \Delta_0\right] = S\left[\Delta_0^*, \Delta_0\right]$ reproduces the previous result.

Performing the first step we introduce the "zerothorder" normal and anomalous Green's functions associated with the initial density matrix $\rho\left(t_0\right)$ by means of the Dyson equation $\boldsymbol{G}^{-1} = \boldsymbol{G}_0^{-1} - \boldsymbol{\Sigma}$ and the self-energy matrix

$$\hbar\Sigma\left(\mathbf{x}, t; \mathbf{x}', t'\right) = \begin{bmatrix} 0 & -\Delta'(\mathbf{x}, t) \\ -\Delta'^*(\mathbf{x}, t) & 0 \end{bmatrix} \delta\left(\mathbf{x} - \mathbf{x}'\right) \delta\left(t, t'\right) \tag{10}$$

Substituting this in Eq. (6) gives

$$\begin{aligned} S\left[\Delta^*, \Delta\right] = & - i\hbar \operatorname{Tr}\left[\ln G_0^{-1}\right] + i\hbar \sum_{n=1}^{\infty} \frac{1}{n} \operatorname{Tr}\left[\left(G_0\Sigma\right)^n\right] \\ & - \frac{1}{g} \int_C dt \int d\mathbf{x} \left|\Delta_0 + \Delta'(\mathbf{x}, t)\right|^2 \end{aligned}$$

The terms linear in $\Delta'^*(\mathbf{x}, t)$ and $\Delta'(\mathbf{x}, t)$ cancel, because Eq. (9) is equivalent to $\Delta_0 = -g\left\langle T\left[\psi_\downarrow(\mathbf{x}, t)\psi_\uparrow(\mathbf{x}, t)\right]\right\rangle$, where the field operators evolve in time according to the interaction picture based on G_0^{-1}. Hence, we have

$$\begin{aligned} S\left[\Delta^*, \Delta\right] = & S\left[\Delta_0^*, \Delta_0\right] + i\hbar \sum_{n=2}^{\infty} \frac{1}{n} \operatorname{Tr}\left[\left(G_0\Sigma\right)^n\right] \\ & - \frac{1}{g} \int_C dt \int d\mathbf{x} \left|\Delta'(\mathbf{x}, t)\right|^2 \end{aligned} \tag{11}$$

Before we can evaluate the time integrations in this expression we must realize that both $\Delta'(\mathbf{x}, t)$ and $G_0\left(\mathbf{x}, t; \mathbf{x}', t'\right)$ are functions on the Keldysh contour C. Therefore, the latter can be decomposed into the analytical functions $\boldsymbol{G}_0^{>}$and $\boldsymbol{G}_0^{<}$by

$$\begin{aligned} G_0\left(\mathbf{x}, t; \mathbf{x}'t'\right) = & G_0^{>}\left(\mathbf{x}, t; \mathbf{x}'t'\right) \Theta\left(t, t'\right) \\ & + G_0^{<}\left(\mathbf{x}, t; \mathbf{x}'t'\right) \Theta\left(t', t\right) \end{aligned} \tag{12}$$

using the Heaviside function on the contour $\Theta\left(t, t'\right)$.[14,15] Substituting this decomposition into Eq. (11) and defining the retarded and advanced Green's functions as

$$G_0^{(\pm)}(\mathbf{x},t;\mathbf{x}'t') \equiv \pm\,\Theta\left(\pm\left(t-t'\right)\right) \times \left[G_0^{>}(\mathbf{x},t;\mathbf{x}'t') - G_0^{<}(\mathbf{x},t;\mathbf{x}'t')\right] \tag{13}$$

we arrive at a real time formalism that explicitly reflects the principle of causality, because essentially only retarded quantities are present. In addition, it is now possible to apply a Fourier transformation on both space and time and to find the effective action for the long-wavelength and low-frequency fluctuations by means of a Taylor expansion of the various coefficients.

As an example we take the coefficient of the diagonal $|\Delta'(\mathbf{k},k_0)|^2$ term. The discussion of the nondiagonal terms proportional to $\Delta'^*(\mathbf{k},k_0)\,\Delta'^*(-\mathbf{k},-k_0)$ and $\Delta'(\mathbf{k},k_0)\,\Delta'(-\mathbf{k}',-k_0)$ is similar and is not repeated in the following. Physically, three processes are involved in the calculation of the quadratic fluctuations. In terms of the usual coherence factors $u(\mathbf{p})$ and $v(\mathbf{p})$ obeying $|u(\mathbf{p})|^2 = \frac{1}{2}[1+\epsilon(\mathbf{p})/E(\mathbf{p})]$, $\quad |v(\mathbf{p})|^2 = \frac{1}{2}[1-\epsilon(\mathbf{p})/E(\mathbf{p})]$, and $2u(\mathbf{p})v(\mathbf{p}) = -\Delta_0/E(\mathbf{p})$, we find for the contribution due to the production of two quasiparticles:

$$-\frac{1}{2}\int \frac{d\mathbf{k}'}{(2\pi)^3}\frac{1}{\hbar k_0^+ - E\left(\mathbf{k}/2+\mathbf{k}'\right) - E\left(\mathbf{k}'/2-\mathbf{k}'\right)}\left[1 - N\left(\frac{\mathbf{k}}{2}+\mathbf{k}'\right) - N\left(\frac{\mathbf{k}}{2}-\mathbf{k}'\right)\right]\left|u\left(\frac{\mathbf{k}}{2}+\mathbf{k}'\right)\right|^2\left|u\left(\frac{\mathbf{k}}{2}-\right.\right.$$

whereas the contribution due to the creation of two quasiholes becomes

$$\frac{1}{2}\int \frac{d\mathbf{k}'}{(2\pi)^3}\frac{1}{\hbar k_0^+ + E\left(\mathbf{k}/2+\mathbf{k}'\right) + E\left(\mathbf{k}/2-\mathbf{k}'\right)}\left[1 - N\left(\frac{\mathbf{k}}{2}+\mathbf{k}'\right) - N\left(\frac{\mathbf{k}}{2}-\mathbf{k}'\right)\right]\left|v\left(\frac{\mathbf{k}}{2}+\mathbf{k}'\right)\right|^2\left|v\left(\frac{\mathbf{k}}{2}-\mathbf{k}\right.\right.$$

Finally, the possibility of Landau damping associated with the breakup of a Cooper pair into a quasiparticle and a quasihole leads to

$$\int \frac{d\mathbf{k}'}{(2\pi)^3}\frac{1}{\hbar k_0^+ - E\left(\mathbf{k}/2+\mathbf{k}'\right) + E\left(\mathbf{k}/2-\mathbf{k}'\right)}\left[N\left[\frac{\mathbf{k}}{2}+\mathbf{k}'\right] - N\left[\frac{\mathbf{k}}{2}-\mathbf{k}'\right]\right]\left|u\left[\frac{\mathbf{k}}{2}+\mathbf{k}'\right]\right|^2\left|v\left(\frac{\mathbf{k}}{2}-\mathbf{k}'\right)\right|^2.$$

It is now important to realize that the first two terms have a well-defined Taylor expansion around $(\mathbf{k},k_0) = (0,0)$, because the dispersion relation of the quasiparticles and quasiholes has a gap. Unfortunately, this is not true for the last term, which in principle prevents the formulation of the time-dependent Ginzburg-Landau equation as a simple partial differential equation at nonzero temperatures in agreement with the conclusions of Abrahams and Tsuneto. [8] However, for low frequencies, Landau damping can be neglected and the last term is very accurately approximated by its static value, which is analytic in $\mathbf{k}$ and has a well-behaved long-wavelength expansion at all temperatures. In this manner the evaluation of the effective action for the quadratic fluctuations becomes straightforward although somewhat tedious for the gradient terms.

Making use of particle-hole symmetry and introducing the functions $I_n(|\Delta_0|)$ by means of

$$I_n(|\Delta_0|) \equiv \frac{|\Delta_0|^{2n}}{B\left(\frac{1}{2},n\right)}\int_{-\infty}^{\infty}\frac{d\epsilon}{E^{2n+1}}[1-2N(E)] \tag{14}$$

where the Beta function $B\left(\frac{1}{2},n\right)$ fixes the normalization such that in the zero-temperature limit $I_n \to 1$, we find for the potential terms in the effective Lagrangian density,
$-\frac{N(0)}{4}\frac{I_1}{|\Delta_0|^2}\left[\left(\Delta_0^*\right)^2\left(\Delta'\right)^2 + 2\left|\Delta_0\right|^2\left|\Delta'\right|^2 + \left(\Delta_0\right)^2\left(\Delta'^*\right)^2\right]$,
denoting the density of states for one spin projection at the Fermi surface by $N(0)$. Hence, the

most simple form of the Lagrangian density $\mathcal{L}_{\text{pot}}\ (\Delta^*, \Delta)$ that reproduces this result if expanded around Δ_0 and also agrees with the Ginzburg-Landau theory near the critical temperature, is

$$\mathcal{L}_{\text{pot}}\left(\Delta^*, \Delta\right) = \frac{N(0)}{2} I_1 |\Delta|^2 \left(1 - \frac{|\Delta|^2}{2\left|\Delta_0\right|^2}\right) \tag{15}$$

For the time-derivative terms we find the off-diagonal part

$$\frac{N(0)}{24} I_2 \frac{\hbar^2}{\left|\Delta_0\right|^2} \left[\frac{\left(\Delta_0\right)^2}{\left|\Delta_0\right|^2} \Delta'^* \frac{\partial^2 \Delta'^*}{\partial t^2} + \Delta' \frac{\partial^2 \Delta'}{\partial t^2} \frac{\left(\Delta_0^*\right)^2}{\left|\Delta_0\right|^2}\right]$$

and the diagonal part

$$-\frac{N(0)}{4}\left[I_1 - \frac{I_2}{3}\right] \frac{\hbar^2}{\left|\Delta_0\right|^2} \left(\Delta'^* \frac{\partial^2 \Delta'}{\partial t^2}\right).$$

Clearly, this result cannot be represented by a Lagrangian density $\mathcal{L}_{\text{time}}\ (\Delta^*, \Delta)$ proportional to $|\partial\Delta/\partial t|^2$ as is usually assumed. We need additional terms of the form $|\Delta\partial\Delta/\partial t|^2$ and $\left(\Delta^*\partial\Delta/\partial t\right)^2 + \left(\Delta\partial\Delta^*/\partial t\right)^2$ that are also of second order in the time derivatives but of fourth order in the order parameter $\Delta(\mathbf{x}, t)$. Together with the requirement that we should reproduce the correct theory near the critical temperature we obtain

$$\begin{aligned}\mathcal{L}_{\text{time}}\ (\Delta^*, \Delta) =& \frac{N(0)}{4}\left[I_1 + I_2\left[1 - \frac{|\Delta|^2}{\left|\Delta_0\right|^2}\right]\right] \\ & \times \frac{\hbar^2}{\left|\Delta_0\right|^2}\left|\frac{\partial\Delta}{\partial t}\right|^2 - \frac{N(0)}{24} I_2 \frac{\hbar^2}{\left|\Delta_0\right|^4}\left(\frac{\partial|\Delta|^2}{\partial t}\right)^2\end{aligned} \tag{16}$$

Finally, we have to determine the gradient part $\mathcal{L}_{\text{grad}}\ (\Delta^*, \Delta)$. In terms of the Yosida function $Y\left(|\Delta_0|\right) = \int_{-\infty}^{\infty} d\epsilon(-dN/dE)$, its first derivative

$$Z\left(|\Delta_0|\right) \equiv -\left|\Delta_0\right|^2 \frac{dY}{d\left|\Delta_0\right|^2} = \frac{\left|\Delta_0\right|^2}{2} \int_{-\infty}^{\infty} d\epsilon \left(\frac{d^2 N}{dE^2}\right) \frac{1}{E} \tag{17}$$

and the Fermi velocity v_F we find for the off-diagonal part of the effective Lagrangian,

$$\begin{aligned}-\frac{N(0)}{24}(1 - Y - Z) & \frac{\hbar^2 v_F^2}{3\left|\Delta_0\right|^2} \\ & \times \left[\frac{\left(\Delta_0\right)^2}{\left|\Delta_0\right|^2} \Delta'^* \frac{\partial^2 \Delta'^*}{\partial \mathbf{x}^2} + \Delta' \frac{\partial^2 \Delta'}{\partial \mathbf{x}^2} \frac{\left(\Delta_0^*\right)^2}{\left|\Delta_0\right|^2}\right]\end{aligned}$$

and for the diagonal part,

$$\frac{N(0)}{6}\left(1 - Y - \frac{Z}{2}\right) \frac{\hbar^2 v_F^2}{3\left|\Delta_0\right|^2}\left(\Delta'^* \frac{\partial^2 \Delta'}{\partial \mathbf{x}^2}\right)$$

In a similar way as for the time-derivative terms we deduce from this result that

$$\begin{aligned}\mathcal{L}_{\text{grad}}\left(\Delta^*, \Delta\right) =& -\frac{1}{2}\frac{\rho_s}{|\Delta|^2}\left|\frac{\hbar}{2m}\frac{\partial\Delta}{\partial \mathbf{x}}\right|^2 \\ & + \frac{N(0)}{24}(1 - Y - Z)\frac{\hbar^2 v_F^2}{3\left|\Delta_0\right|^4}\left(\frac{\partial|\Delta|^2}{\partial \mathbf{x}}\right)^2\end{aligned} \tag{18}$$

having also terms which are second order in the derivatives but fourth order in the order parameter. The superfluid mass density is given by

$$\rho_S(|\Delta|) = 2mn\frac{|\Delta|^2}{|\Delta_0|^2}\left[(1-Y)\left[1-\frac{|\Delta|^2}{2\,|\Delta_0|^2}\right]\right.$$
$$\left.-\frac{Z}{2}\left[1-\frac{|\Delta|^2}{|\Delta_0|^2}\right]\right] \tag{19}$$

which reduces to the expected expression $\rho_S = mn(1-Y)$ if $|\Delta| = |\Delta_0|$. [16]

Furthermore, putting $|\Delta_0| = |\Delta|$ we exactly have the result obtained by Werthamer. [17] However, in the spirit of the above calculation we should not identify $|\Delta_0|$ with $|\Delta|$, because this implies local equilibrium of the quasiparticles with the condensate and we are back to the treatment of Abrahams and Tsuneto. Notice that in the situation of a superflow, the momentum dependence of the gap found from Eq. (18) is

$$|\Delta_0(\mathbf{k})| = |\Delta_0|\left[1-\frac{Z}{4}\frac{\hbar^2 v_F^2}{3\,|\Delta_0|^2}\mathbf{k}^2 + O\left(\mathbf{k}^4\right)\right] \tag{20}$$

which can also be derived directly from the BCS gap equation. In particular, at zero temperature this implies that below the critical current, $|\Delta_0(\mathbf{k})|$ is independent of momentum, which is a consequence of Galilean invariance.

This concludes the derivation of the time-dependent Ginzburg-Landau theory for a neutral BCS-type superfluid. In summary, the complete nonequilibrium longwavelength ($\hbar v_F k \ll |\Delta_0|$) and low-frequency ($\hbar k_0 < \hbar v_F k$) action of the order parameter is

$$S_{\text{neu}}\left[\Delta^*,\Delta\right] = \int dt \int d\text{x}\left[\mathcal{L}_{\text{time}}\left(\Delta^*,\Delta\right) + \mathcal{L}_{\text{grad}}\left(\Delta^*,\Delta\right)\right.$$
$$\left.+\mathcal{L}_{\text{pot}}\left(\Delta^*,\Delta\right)\right]$$
$$\equiv \int dt \int d\text{x}\mathcal{L}_{\text{neu}}\left(\Delta^*,\Delta\right) \tag{21}$$

where the various contributions are given in Eqs. (15), (16), (18), and (19). Substituting $\Delta(\mathbf{x},t) = \Delta_0 \exp[i\theta(\mathbf{x},t)]$ the Lagrangian density for the phase fluctuations becomes

$$\mathcal{L}_{\text{neu}}(\theta) = \frac{N(0)}{4}\hbar^2 I_1\left[\left(\frac{\partial\theta}{\partial t}\right)^2 - v_F^2\frac{1-Y}{3I_1}\left[\frac{\partial\theta}{\partial \text{x}}\right]^2\right], \tag{22}$$

showing explicitly that the Goldstone mode has the velocity $v_F\sqrt{(1-Y)/3I_1}$. Hence, at zero temperature, the velocity is equal to $v_F/\sqrt{3}$ and in agreement with the calculations of Bogoliubov [18] and Anderson. [19] In the next section we discuss how these results are modified in the case of a superconductor where the effect of screening is of great importance for the dynamics of the system. [20]

33.0.4 IV. SCREENING PROPERTIES

We introduce electromagnetism in the functional approach by performing the minimal coupling substitution
$i\hbar\partial/\partial t \to i\hbar\partial/\partial t + e\phi$ and $-i\hbar\nabla \to -i\hbar\nabla + e\mathbf{A}$ in $S\left[\psi^*,\psi\right]$ and adding the action for the free electromagnetic field, which reduces in the Coulomb gauge to

$$S_{\mathrm{em}}[\phi, \mathbf{A}] = \int_C dt \int d\mathbf{x} \frac{\epsilon_0}{2} (- \phi \nabla^2 \phi - \mathbf{A} \cdot \frac{\partial^2 \mathbf{A}}{\partial t^2} + c^2 \mathbf{A} \cdot \nabla^2 \mathbf{A}] \tag{23}$$

Applying the same Hubbard-Stratonovich transformation as before and subsequently integrating over the fermionic fields $\psi_\alpha(\mathrm{x}, t)$, we find that the order parameter $\Delta(x, t)$ is gauge-invariantly coupled to the electromagnetic field by means of the covariant derivatives $i\hbar\partial/\partial t + 2e\phi$ and $-i\hbar\nabla + 2e\mathbf{A}$, as is expected of a field with charge $-2e$, and in addition that the photon propagator is dressed due to the polarizability of the electron gas. Hence, the effective action for the charged case acquires the form

$$S_{\mathrm{ch}} [\Delta^*, \Delta, \phi, \mathbf{A}] = \int dt \int d\mathbf{x} [\mathcal{L}_{\mathrm{neu}} (\Delta^*, \Delta) + \mathcal{L}_{\mathrm{em}}(\phi, \mathbf{A}) + \mathcal{L}_{\mathrm{pol}}(\phi, \mathbf{A}) + \mathcal{L}_{\mathrm{int}} (\Delta^*, \Delta, \phi, \mathbf{A})] \tag{24}$$

The interaction Lagrangian density is found from the minimal coupling prescription and is equal to

$$\mathcal{L}_{\mathrm{int}} (\Delta^*, \Delta, \phi, \mathbf{A}) = -\delta\rho\phi + \mathbf{J} \cdot \mathbf{A} \tag{25}$$

where the charge-density fluctuation $\delta\rho$ and current density $\mathbf{J}$ are given by

$$\delta\rho \equiv \rho + en = -eN(0) \left[I_1 + I_2 \left(1 - \frac{|\Delta|^2}{|\Delta_0|^2} \right) \right] \frac{1}{|\Delta_0|^2} \times \left[\frac{i\hbar}{2} \left[\Delta^* \frac{\partial \Delta}{\partial t} - \Delta \frac{\partial \Delta^*}{\partial t} \right] + 2e|\Delta|^2 \phi \right] \tag{26}$$

and

$$\mathbf{J} = -\frac{e\rho_S}{2m^2|\Delta|^2} \left[\frac{-i\hbar}{2} \left[\Delta^* \frac{\partial \Delta}{\partial \mathbf{x}} - \Delta \frac{\partial \Delta^*}{\partial \mathbf{x}} \right] + 2e|\Delta|^2 \mathbf{A} \right] \tag{27}$$

respectively. Because of gauge invariance, the electric charge is locally conserved and the stationary condition $\delta S_{\mathrm{ch}}/\delta\theta(\mathrm{x}, t) = 0$ can be written as the continuity equation

$$\frac{\partial \delta\rho(\mathbf{x}, t)}{\delta t} + \nabla \cdot \mathbf{J}(\mathbf{x}, t) = 0 \tag{28}$$

where the density fluctuations are constrained by the Euler-Lagrange equation $\delta S_{\mathrm{ch}}/\delta|\Delta(\mathbf{x}, t)| = 0$, leading to $\delta\rho(\mathbf{x}, t) = 0$ in lowest order. However, small deviations from charge neutrality arise in higher orders, which can easily be incorporated if necessary for a particular application of the Ginzburg-Landau equation proposed in this paper.

Finally, to obtain the equations for the electromagnetic potentials also, we have to consider $S_{\mathrm{pol}} [\phi, \mathbf{A}]$. It should be noted that this contribution was usually neglected in previous attempts to derive a time-dependent GinzburgLandau equation, although it turns out to be essential for a correct description of the screening properties of a superconductor. Introducing the four-vector $A = (\phi/c, \mathbf{A})$ we have

$$S_{\text{pol}}[\phi, \mathbf{A}] = \int dt \int d\mathbf{x} \int dt' \int d\mathbf{x}' \frac{1}{2} A^{\mu}(\mathbf{x}, t) \times \Pi_{\mu v}^{(+)}(\mathbf{x}, t; \mathbf{x}', t') \times A^{v}(\mathbf{x}', t') \tag{29}$$

with $\Pi_{\mu\nu}^{(+)}$the retarded polarization tensor of the freeelectron gas. Again, due to gauge invariance, $\Pi_{\mu\nu}^{(+)}$is a transverse tensor and can be written in momentum space as

$$\Pi_{\mu v}^{(+)}(k) = \left[g_{\mu v} - \frac{k_{\mu} k_{v}}{k^2} \right] \pi(k) \tag{30}$$

using $k = (k_0/c, \mathbf{k})$ and the Lorentz metric $g_{\mu v}$ with signature $(-1, 1, 1, 1)$. Thus, the evaluation of $S_{\text{pol}}[\phi, \mathbf{A}]$ is particularly convenient in the Coulomb gauge. In this gauge and neglecting the small contribution from Landau diamagnetism, we find that

$$S_{\text{pol}}[\phi, \mathbf{A}] = \int dt \int d\mathbf{x} \int dt' \int d\mathbf{x}' \frac{1}{2c^2} \phi(\mathbf{x}, t) \times \prod_{00}^{(+)}(\mathbf{x}, t; \mathbf{x}', t') \times \phi(\mathbf{x}', t') \tag{31}$$

and $-\Pi_{00}^{(+)}/c^2$ the density-density response function $\chi_{\rho\rho}$.
In the long-wavelength limit we have the well-known result [9]

$$\prod_{00}^{(+)}(\mathbf{k}, k_0) \underset{\frac{kv_F}{k_0} \to 0}{\sim} -\frac{ne^2}{m} \frac{\mathbf{k}^2 c^2}{k_0^2} \tag{32}$$

which formally leads to
$S_{\text{pol}}[\phi, \mathbf{A}] = -\int dt \int d\mathbf{x} \frac{1}{2} \phi(\mathbf{x}, t) \frac{ne^2}{m} \frac{\partial^2/\partial \mathbf{x}^2}{\partial^2/\partial t^2} \phi(\mathbf{x}, t)$.

Considering again the Bogoliubov-Anderson mode we find that in the limit of infinite wavelength, a solution to the equations of motion is $\delta\rho = 0, \mathbf{J} = \mathbf{A} = \mathbf{0}$, $\phi(t) = \phi_0 \exp(-i\omega_p t - i\pi/2)$, $\theta(t) = 2e\phi_0/\hbar\omega_p \exp(-i\omega_p t)$, and $\omega_p = (ne^2/m\epsilon_0)^{1/2}$ the plasma frequency. Thus, the dispersion of the collective mode is pushed up to the plasma frequency as is expected of a density fluctuation in a system with long-range Coulomb interactions. [19,20]

However, for the purposes of this paper and in agreement with the neglect of Landau damping in the discussion of the neutral superfluid, we are more interested in the region $\hbar k_0 < \hbar k v_F$ where it is appropriate to use the static limit of $\prod_{00}^{(+)}$. In this limit we obtain

$$\Pi_{00}^{(+)}(\mathbf{k}, k_0) \underset{\frac{k_0}{kv_F} \to 0}{\sim} \frac{\epsilon_0 c^2}{\lambda_{\text{TF}}^2}, \tag{34}$$

with λ_{TF} the Thomas-Fermi screening length $(2\epsilon_0\epsilon_F/3ne^2)^{1/2}$. Substituting this into Eq. (31) finally gives

$$S_{\text{pol}}[\phi, \mathbf{A}] = \int dt \int d\mathbf{x} \frac{\epsilon_0}{2} \phi(\mathbf{x}, t) \frac{1}{\lambda_{\text{TF}}^2} \phi(\mathbf{x}, t) \tag{35}$$

The Euler-Lagrange equations for the electromagnetic field are then

$$\nabla^2 - \frac{1}{\lambda_{\mathrm{TF}}^2}\Bigg)\phi(\mathbf{x},t) = -\frac{\delta\rho(\mathbf{x},t)}{\epsilon_0} \tag{36a}$$

$$\left[\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right]\mathbf{A}(\mathbf{x},t) = -\mu_0\mathbf{J}_T(\mathbf{x},t) \tag{36b}$$

where $\mathbf{J}_T(\mathbf{x},t)$ denotes the transverse part of the current density. Together with Eqs. (26) and (27), we see that the screening of the electric field is due to both the normal as well as the superconducting part of the system, whereas the screening of the magnetic field (the Meissner effect) is only due to the superconducting part. Therefore, the typical length scale associated with the former is λ_{TF}, while the length scale associated with the latter is equal to the London penetration depth $\lambda_L = \left(m^2/\mu_0\rho_S e^2\right)^{1/2}$ for a spatially slowly varying order parameter. Clearly, this intuitively reasonable result is only obtained if the polarizability of the electron gas is taken into account and $S_{\mathrm{pol}}\left[\phi,\mathbf{A}\right]$ is included in the complete action. Without this contribution, the electric potential is only coupled to the superfluid density fluctuations and only the superconducting part of the system can lead to screening of the electric field.

33.0.5 V. CONCLUSIONS

In summary, we have shown that it is possible to derive a time-dependent Ginzburg-Landau equation for the order parameter of a superconductor in a highly nonequilibrium situation and in the limit that $\hbar k_0 < \hbar v_F k \ll |\Delta_0|$. We have also shown how the screening properties of the superconductor can be accounted for. In particular, the different screening lengths for the longitudinal and transverse parts of the electromagnetic field are included in a natural way.

The most important restriction on the applicability of the above theory is caused by the neglect of Landau damping. However, it seems likely that the effects of Landau damping cannot be simply incorporated in the
framework of a Ginzburg-Landau theory and requires a more advanced approach, which takes the quasiparticles into account explicitly. [21] Nevertheless, it is of interest to investigate the predictions of the theory presented in this paper, because for a certain application the neglect of Landau damping might turn out to be justified and can in any case be checked in a self-consistent fashion. It will, in particular, be interesting to see if in this way a better agreement with experimental data on the resistance of quasi-one-dimensional superconductors can be obtained.

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Part IX
TDGL: Phenomena

34 Time-Dependent Superconductivity in SNS Bridges: An Example of TDGL Theory by De Lozanne, Beasley

34.1 1. Introduction

Superconducting microbridges have played a major role in the recent advances toward a fundamental understanding of time-dependent, nonequilibrium superconductivity. The complex temporal and spatial variations of the superconducting wavefunction that are associated with the phase slippage process in such bridges have provided important and challenging theoretical problems. Phase slippage necessarily must arise when a finite DC voltage difference exists across a superconductor; consequently the current voltage characteristics of the various microbridge structures have been the focus of much interest. Historically interest in these questions arose out of the failure of the traditional time-dependent Ginzburg Landau (TDGL) theory to account for the I-V curves of filamentary superconductors and constriction-type superconducting microbridges. A comprehensive review of these developments along with others can be found in the chapter by Kadin and Goldman (1985) in the present volume.

There is, however, a class of superconducting microbridges for which the traditional TDGL theory appears to work. These are so-called planar superconductor normal-superconductor (SNS) bridges. An understanding of these bridges is thus of interest as a contrast to the more conventional constriction bridges and as a testing ground for the applications of TDGL theory. They also appear to have considerable technological promise as a practical alternative to tunneling Josephson junctions in many applications. In this chapter we review the current understanding of the dynamical behavior of such bridges and discuss the theoretical implications in the broader framework of time-dependent, nonequilibrium superconductivity.

We begin in the next section with a description of these bridges and the physical basis upon which they have been modeled. In section 3 the TDGL equations are discussed along with the limits of their validity. The equations are cast into a simple form for the case of a narrow, gapless microbridge. Some numerical results illustrating the predictions of this theory are also given in this section. Comparisons with the experimental results available to date are presented in section 4.

34.2 2. SNS Microbridges

34.2.1 2.1. Bridge structure

Figure 1 illustrates various types of SNS Josephson junction structures. Most of the quantitative experimental work on SNS microbridges has been done using bridges with a planar (fig. 1a, 1c) or step-edge (fig. 1b, 1d) geometry. SNS junctions with a sandwich geometry (fig. 1e) were studied earlier but are less satisfactory because of the inherently small resistances and voltages involved. Also they are generally wide compared to the Josephson penetration depth λ_J, and therefore the phase slippage process is accomplished by the more complicated phenomenon of vortex flow. Bridges on the other hand have much larger resistances, hence higher voltages, and can be narrow enough to be onedimensional, which provides considerable theoretical simplification. The fabrication and general properties of these bridges have been

described elsewhere (Komarovskikh et al. 1975, Warlaumont et al., 1979, van Dover et al. 1980, 1981, de Lozanne et al. 1983, Dilorio et al. 1983, Liengme et al. 1983). Similar structures have also been made with semimetal (Gudkov et al. 1979. Ivanov et al. 1981) and degenerate semiconductor (Ruby and van Duzer 1981, Alfeev et al. 1979) weak links. These structures can be thought of as SNS devices with a lower electron density in N. Thus given the appropriate parameters they presumably can also be modeled by the TDGL theory.

A common denominator of all SNS bridges that have actually been fabricated is that the banks are SN bilayers themselves (1a-1d). This is in contrast to the more idealized structure usually assumed theoretically in which the banks are homogeneous superconductors (fig. 1f). The fact that the normal metal extends into the banks turns out to have some advantage in the practical performance of the devices, reduced heating, for example. Moreover, it ensures continuity of the order parameter across the actual bridge/bank interface since it is all one material.

34.2.2 2.2. Model

One may view the Josephson coupling in planar SNS microbridges as arising in two essentially independent steps (see fig. 2) (van Dover et al. 1981). First, superconductivity is induced into N in the SN bilayer banks by virtue of the proximity effect (fig. 2a). This first step can be treated theoretically (at least near T_c) by simply solving the Ginzburg-Landau equations in S and N with SN boundary conditions similar to those of de Gennes. Second, once the order parameter in N is known, in establishing the Josephson coupling one may essentially forget about the presence of the superconductor and treat the structure as a homogeneous bridge with rigid boundary conditions, i.e., with the order parameter constant in the banks independent of what happens in the bridge. These rigid boundary conditions are enforced by the presence of the S

layer, and in most practical cases by the variable thickness geometry of the bridge itself*. With this picture it follows as mentioned above that there is no discontinuity of the order parameter at the bank bridge boundary because the same material (N) is on both sides of this boundary (fig. 2b). For purposes of comparison, the behavior generally assumed in a sandwich SNS junction and an ideal planar SNS bridge are shown in fig. 3. Finally, given rigid boundary conditions and the value of the order parameter in N . in principle the behavior of the bridge can be calculated using the appropriate equations ($\mathrm{G-L}$ or

Usadel equations for the static case, the TDGL cquations for the dynamic case).

Strong support for this model in the case of the static properties of planar SNS bridges has been given by the semi-quantitative agreement of its predictions with the measurements of the temperature dependence of the $I_{\mathrm{c}}R_{\mathrm{N}}$ product for Nb/Au and Pb/Cu planar microbridges (van Dover et al. 1980, 1981, Liengme et al. 1983). On the basis of this success the same ideas have been extended to dynamic situations using the traditional TDGL equations. and again satisfactory agreement is found with the experimental results, at least at the present stage of development. These results are the topic of the remainder of this review.

34.3 3. TDGL theory and results

34.3.1 3.1. The TDGL equations ,

Originally, the TDGL equations were shown by Gor'kov and Eliashberg (1968) to be rigorously valid at all temperatures in a superconductor rendered gapless by paramagnetic impurities. Kramer and Watts-Tobin (1978) obtained a more general version that is valid for a

[12]*For a complete review of variable thickness bridges and their relation to rigid boundary conditions, see the article by Likharev (1979).

Fig. 1. Schematic representation of the various types of SNS structures: (a) planar microbridge, no constriction in N (Warlaumont et al. 1979, Liengme et al. 1983); (b) step-edge microbridge, no constriction in N (de Lozanne et al. 1983); (c) planar microbridge, with constriction in N (van Dover et al. 1980, Komarovskikh et al. 1975); (d) step-edge microbridge, with constriction in N (Dilorio et al. 1983); (e) planar SNS sandwich; (f) structure commonly assumed in theoretical models.

dirty superconductor near T_c, taking into account inelastic electron-phonon scattering. Their result reduces to that of Gor'kov and Eliashberg in the limit of zero gap. Finally, Schön and Ambegaokar (1979) derived a TDGL equation which includes various pairbreaking mechanisms on equal footing: namely electron-phonon scattering, paramagnetic impurities, supercurrent depairing, magnetic fields and spatial variation of the order parameter. Again, this latest result reduces to the previous two in the appropriate limits. The generalized pair-breaking parameter of Schön and Ambegaokar has the following form:

$$\Gamma = \frac{1}{2\tau_E} + \frac{1}{\tau_S} + \frac{D}{2}Q^2 - \frac{D}{2}\frac{\nabla^2\Delta_0}{\Delta_0}, \tag{1}$$

where τ_E is the inelastic electron-phonon scattering time, τ_S is the magnetic spin flip time, D is the diffusion coefficient ($D = v_f^2\tau_{imp}/3$), Δ_0 is the temperature-dependent equilibrium pair potential and

$$Q = \nabla\theta + 2eA$$

is proportional to the superfluid momentum.
The general TDGL equation (Schön and Ambegaokar 1979) is given by*

Fig. 2. Model used to analyze the behavior of planar constriction SNS microbridges (fig. 1c). (a) Behavior of the order parameter as one moves down from the superconductor into the normal metal. The sharp change of the order parameter at the SN interface is due to the different material parameters in each metal. (b) Behavior of the order parameter as a function of distance along the bridge. Note that there is no discontinuity of the order parameter at $X = 0$ and $X = L$ because there is no change of material parameters at those points. (From van Dover et al. 1980, 1981.)

$$\frac{\pi}{8kT}\left[\frac{\eta}{2|\Delta|\tau_{\mathrm{E}}}\left(\frac{\partial}{\partial t}+2\mathrm{i}e\Phi\right)+\left(2|\Delta|\tau_{\mathrm{E}}\zeta-\frac{\eta}{2|\Delta|\tau_{\mathrm{E}}}\right)\frac{|\dot{\Delta}|}{|\Delta|}\right]\Delta$$
$$=\left[\alpha-\beta|\Delta|^2+\xi^2(0)(\nabla-2\mathrm{i}eA)^2\right]\Delta \tag{2}$$

where

$$\begin{aligned}
&\eta=(2\gamma\tau)^{1/2}\left(1+\gamma/2\Delta_0^2\tau\right)^{-1/2}, && \alpha=\left(T_{\mathrm{c}}-T\right)/T_{\mathrm{c}},\\
&\zeta=\left[1+(\gamma/\Delta_0)^2\right]^{1/2}-\gamma/\Delta_0+1/2\Delta_0\tau, && \beta=7\zeta(3),8\pi^2k^2T_{\mathrm{c}}^2,\\
&\gamma=-\mathrm{i}\omega/2+\Gamma, && \xi^2(0)=\tfrac{\pi\hbar}{8kT_{\mathrm{c}}}D,\\
&1/\tau=-\mathrm{i}\omega+1/\tau_{\mathrm{E}},
\end{aligned} \tag{3}$$

and the current is determined by

$$j=\frac{\pi\sigma_0}{4ekT}\,\mathrm{Im}\left\{\Delta^*(\nabla-2\mathrm{i}e\boldsymbol{A})\Delta\right\}-\sigma_0(\nabla\Phi-\dot{A}), \tag{4}$$

where Im stands for the imaginary part. It is assumed in the derivation of this TDGL equation that the quasiparticle relaxation processes are fast enough to justify neglecting the effects of diffusion. This is another way of saying that the equations are local. This is valid as long as

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- Minor typographical errors in the original publication (Schön and Ambegaokar 1979) are corrected here.

Fig. 3. Comparison of the spatial dependence of the order parameter along the direction of current flow for the SNS structures shown in figs. 1e and 1f. (a) Sandwich structure. Current flow is in the vertical direction. The order parameter in S is pulled down near the boundaries due to the large area of contact with N ; the discontinuity at the interfaces is due to the different material parameters in S and N. (b) Commonly assumed theoretical structure (Likharev 1979, Likharev and Yakobson 1975), see fig. 1f. Here the current flow is in the horizontal direction. The normal state material parameters are assumed to be the same in S and N , hence there is no discontinuity at the $\mathrm{S / \mathrm{N}$ interfaces. Furthermore, the small area of contact between S and N (due to the variable thickness geometry) produces negligible depression of the order parameter in S near the interfaces.}

$$|\Delta|^2/k_{\mathrm{B}}T < 1/\tau_{\mathrm{E}}. \tag{5}$$

Furthermore, it is assumed that the problem is linear (small departures from equilibrium) and that the frequency is low:

$$\omega \ll 1/\tau_{\mathrm{E}}. \tag{6}$$

This general TDGL equation reduces to the result of Kramer and WattsTobin (1978) when electron-phonon interactions are the only scattering mechanism, so that $\gamma = 1/2\tau_{\mathrm{E}}$. The equation reduces further to the original TDGL equation,

$$\frac{\pi}{8kT}\left[\frac{\partial}{\partial t} + 2\mathrm{i}e\Phi\right]\Delta = \left[\alpha - \beta|\Delta|^2 + \xi^2(0)(\nabla - 2\mathrm{i}eA)^2\right]\Delta \tag{7}$$

obtained by Gor'kov and Eliashberg (1968), in the limit of zero gap. In fact, in this limit, all of these three equations agree even in the numerical coefficients, which have customarily been thought to be different for Gor'kov and Eliashberg's equation. This can be seen easily when the latter equation is written in the notation of Hu and Thompson (1972). The confusion has stemmed from the use of different normalization factors as explained in greater detail below.

For the case of narrow microbridges (width $< \lambda_{\mathrm{J}}, \xi$) magnetic self-field effects are negligible ($A =$ constant $\equiv 0$) so that the current distribution and the order parameter are constant over

the cross-section of the bridge; consequently the TDGL equations become one-dimensional. To normalize these equations it is customary to measure lengths in terms of $\xi(T)$ and Δ in terms of $\Delta_0 = \sqrt{|\alpha|/\beta}$. (Note that this definition applies both below and above T_c. Below T_c, Δ_0 is the equilibrium order parameter. Above T_c, it is simply a measure of the magnitude of Δ at which the nonlinear term becomes important.) Then, given the normal state conductivity σ_0, currents, voltages and times are measured in terms of

$$J_0 = \frac{\pi\sigma_0\Delta_0^2}{4ekT\xi} \tag{8}$$

$$V_0 = \frac{\xi J_0}{\sigma_0} \tag{9}$$

and

$$\tau_J = \frac{\hbar}{2eV_0} \tag{10}$$

With these normalizations the ordinary TDGL equations become

$$u(\dot{\psi} + \mathrm{i}\mu\psi) = -\left(1 + |\psi|^2\right)\psi + \psi'' \tag{11}$$

$$j = \mathrm{Im}\left(\psi^*\psi'\right) - \mu' \tag{12}$$

Here the prime and the dot represent partial derivatives with respect to normalized length and time respectively. Following the usual custom we have introduced the parameter

$$u = \frac{\tau_A}{\tau_j} = \frac{\pi^2}{16}\frac{\Delta_0^2}{kT}\frac{1}{\alpha} \tag{13}$$

which is the relaxation time of the order parameter in normalized units. Note also that τ_J can be physically interpreted as the relaxation time governing current transfer between the normal and superconducting channels. Finally, note that u depends on the value of Δ_0, which is not the same under all conditions. For an ordinary superconductor near $T_\mathrm{c}, \Delta_0(T) = \Delta_\mathrm{g}(T)$ where Δ_g is the energy gap. In this case $u = 5.79$. When paramagnetic impurities are present, Δ_0 is different (Kulik and Yanson 1972) and one gets $u = 12$, reflecting the different normalization.

34.3.2 3.2. Applicability of the TDGL equations to planar SNS bridges

There are two issues to be considered in assessing the applicability of the above TDGL equations to planar SNS microbridges. The first is whether the equations themselves apply in a normal metal and the second deals with the boundary conditions at the ends of the bridge.

As regards the first, one would argue that the equations should apply in the normal metal because they have been shown to be valid when a wide variety of pair-breaking mechanisms are present and the material is dirty (Schön and Ambegaokar 1979). One may therefore conclude that they should be valid in a dirty normal metal, which is by nature gapless. Note, for example, that the Aslamazov Larkin (1968) result for the fluctuation conductivity above T_c can be derived from TDGL theory (Tinkham 1980). On the other hand, the usual $\mathrm{G-L}$ requirement that $T \approx T_\mathrm{c}$ is certainly not valid for a normal bridge with $T_\mathrm{cN} = 0$. In the derivation of the GL equation from the microscopic theory this requirement is used to ensure that the order parameter and its spatial variations are small. The former is certainly true far above T_{cN}. Thus it seems that these equations should be at least qualitatively correct for a

dirty normal metal into which superconductivity has been induced by the proximity effect. but where now the material parameters should be treated phenomenologically. In particular, u is no longer known and must be treated as an adjustable parameter. The length $\check{\xi}$ (and hence the parameter α) can presumably be
estimated reliably from the familiar expressions for in the theory of the proximity effect (van Dover et al. 1980, 1981).

Linfortunately there is an additional complication in comparing the results attained from SNS microbridges with the existing theoretical calculations based on egs. (11) and (12). Although some calculations have been carried out treating u as an adjustable parameter in all cases known to us it has been assumed that $|\psi| = 1$ (i.e.. $\psi = e^{15\cdot\omega}$) at the end of the bridge. This is true only for an SSS variable thickness bridge. where in fact the theory is not adequate over any accessible temperature region for other reasons. For an S.NS bridge A. (I) $\neq A_{,}$, but rather is given by the proximity coupling in the S.S. bilayer (fig. 2). If the basic TDGL equations are normalized with respect to A (()) rather than Δ_0, only the first TDGL equation is changed and becomes

$$u(\psi + i\mu\psi) = \left(1 + \eta|\psi|^2\right)\psi + \psi'' \tag{14}$$

where $\eta = \left\{A(())A_{(i}\right)^2$ gauges the strength of the nonlinear term. Now. also. $J_{(j)}$. $V_{(0.)} =$, and u are still determined by eqs. (8). (9). (10) and (13), respectively. but where $A_{.}(f)$ is substituted for A_f. Hence in comparing experimental results with these calculations. there is an implicit assumption that $\eta = 1$, whereas for $T \gg T_{\mathrm{cN}}$ one might expect $\eta \gg 1$. The really quite good agreement between theory and experiment found empirically despite this problem may reflect the fact that for rigid boundary conditions as $l \to ()$ the nonlinear term becomes negligible compared to the gradient term in any event (Aslamazov and Larkin 1968).

34.3.3 3.3. Results of TDGL

As discussed above most of the work with the TDGL equations has typically used the form given by eqs. (11) and (12) with the boundary conditions $\omega(x = 1), t) = 1$ and $\psi(x - L, t) = e^{\omega}$. These equations are usually solved numerically by standard methods. both with analog computers (Fjordboge and Lindelof 1978) and with digital computers (van Dover et al. 198(). 1981. Likharev and Yakobson 1975. Kramer and Baratoff 1977). In this section we present numerical solutions of eqs. (11) and (12) that give some insight into the behavior of the equations and their dependence on the relevant parameters ide Lozanne 1982).

Two important parameters determine the solution of eqs. (11) and (12), namely the normalized relaxation time. u, and the normalized length of the bridge. $l \equiv L \div (T)$. As the relaxation time of the order parameter $(u)g(v)es$ to tero one would expect the results to approach those of the Resistively Shunted Junction (RSJ) model for a microbridge (i.e.. with no capacitance). in which the only relaxation mechanism is the $\mathrm{L} - \mathrm{R}$ time constant of the Josephson circuit which is given by τ_{j} in our notation. A number of numerical "experiments" demonstrate this. The simplest one is to consider a junction biased with $I_h < I_{.}$.

letting the equations achieve their time-independent solution. Then the current is suddenly turned off and the phase of the order parameter measured as a function of time. The result, shown in fig. 4a, indicates that the phase has an exponential time dependence, $\phi = \phi_0 \mathrm{e}^{-t/t_0}$, where t_0 is a constant which increases with increasing relaxation time (u). This "experiment" can be easily visualized with the pendulum analogy of a Josephson junction. At $t = 0$ the pendulum is dropped from an initial angle ϕ_0 with the vertical and it moves down through a fluid whose "viscosity" is determined in part by . As gets smaller the pendulum relaxes faster. As $u \to 0$ there is still some viscosity left, because the junction is now modeled by the RSJ equations with zero capacitance, i.e., the overdamped case. Figure 4 b shows the behavior of

t_0 as a function of u. As $u \to 0$ we see that $t_0 \to 6\tau_{\mathrm{j}}$. [In the RSJ model $t_0 \to 1\tau_{\mathrm{j}}$; the different numerical factor is due to the difference in $I_{\mathrm{c}}R : (I_{\mathrm{c}}R)_{\mathrm{RSJ}} = 6\,(I_{\mathrm{c}}R)_{\mathrm{TDGL}}$].

Figure 5 shows the time averaged voltage as a function of bias current for a number of values of u (van Dover et al. 1981). The results of the RSJ model are shown for comparison. At these low voltages the TDGL $I - V$'s are similar in shape to those of the RSJ model. In fact, over this voltage range the TDGL $I \cdot V$ can be fitted to the RSJ form by introducing an effective resistance R_{eff} ,

$$V^2 = R_{\mathrm{eff}}^2 \left(I^2 - I_{\mathrm{c}}^2\right). \tag{15}$$

At higher voltages the $I - V$'s are fundamentally different from the RSJ model,

as shown below and originally noted by Likharev and Yakobson (1975). The behavior of R_{eff} as a function of u is shown in fig. 6a. Again, we see that as $u \to 0$ the effective resistance approaches that of the RSJ model ($R_{\mathrm{eff}} \to R_{\mathrm{N}}$, the device resistance). The dependence of R_{eff} on the length of the bridge is comparatively weaker, as shown in fig. 6b. Analytic expressions have also been obtained by Likharev and Yakobson (1975).

34.4 4. Comparison with experiments

The measured $I - V$ curves of several planar SNS microbridges (those in fig. 1c) follow the predictions of the TDGL equations with $u = 12$ as is illustrated in fig. 7. This value of u should be regarded as a best fit, although it was originally thought to be the theoretically predicted value. Here the current and voltage axes of the TDGL results have been normalized to match the measured critical current and normal resistance, respectively [this effectively determines the normalizations given in eqs. (8.9)]. SNS bridges with other geometries (figs. 1a,b,d) show similar features (de Lozanne et al. 1983, Komarovskikh et al. 1975. Liengme et al. 1983, de Lozanne 1982), although direct comparison between theory and experiment has not been made.

A further test of the theory, and probably a much more demanding one, is to compare it with experiments in which the current bias has a high frequency

AC component, as is the case for a microbridge irradiated with microwaves. Here the microwave frequency should be on the order of the characteristic Josephson frequency ($2el_cR/\hbar$) or higher, in order to probe the dynamics of the order parameter when it is out of equilibrium. Such a test has been carried out using a well-characterized planar SNS microbridge, whose $I \cdot V$'s have been reported earlier (van Dover et al. 1981). The reduced length of this bridge ($l = 3.5$) has been determined from SEM pictures together with the measured resistivity of the normal metal (which determines its coherence length). and independently from measurements of $I_cR(T)$ (van Dover et al. 1981). Thus the parameters of the TDGL theory are previously determined to be $l = 3.5, u = 12$. The comparison with the experimental $I - V$ is shown in fig. 8a; the current axis is scaled to match the measured critical current, and the voltage axis is scaled so that it matches the measured voltage at a bias of 0.7 mA . The agreement is quite good. Also shown is the RSJ $\boldsymbol{I} - \boldsymbol{V}$, scaled in a similar way, which gives a much poorer fit. When microwaves are applied to this microbridge sharp constant-voltage steps appear (van Dover et al. 1981). The only modification needed in the theoretical models is to add an AC term to the DC current bias. The amplitude of the AC bias is adjusted so that the theoretical critical current

matches the experimental one with microwaves. The current and voltage axes cannot be scaled because they were already defined by the $I - V$ without microwaves. Figure 8 b shows that the TDGL equations give a very good fit to most of the IV, much better than that of the RSJ model (Russer 1972).

Warlaumont et al. (1979) have shown that the hysteresis observed in the low voltage part of the $I - V$ curve of SNS microbridges can be related to the relaxation time

of the order parameter in the context of the TDGL theory. They found experimentally that as the temperature is lowered hysteresis sets in when $\Gamma = ul^2 \geqslant 2$, where $l \equiv L/\xi_{\mathrm{N}}(T_{\mathrm{cs}})$. Thermal effects were ruled out as a possibility for this hysteresis. Numerical simulations by Kramer and Baratoff (1977) give hysteresis for $\Gamma \geqslant 38$ when $l = 1$. The same result has been obtained by Fjordbge and Lindelof (1978) using analog simulations. This discrepancy between the experimental and numerical values of Γ could be due to the fact that $l > 1$ in the experiments. Yet another possibility is that the formulas that determine Γ are strictly valid near T_{c}, which in this case is zero, so that the experiment is far above T_{c}.

34.5 5. Conclusions

From a semi-phenomenological standpoint the use of the TDGL equations to explain the behavior of SNS microbridges has been rather successful. Both the $I - V$ curve and, more importantly, the $I - V$ with microwave steps show close agreement between experiment and theory. A fully rigorous comparison, i.e., one without any adjustable parameters, is hindered by the fact that the bridge material (N) is far above its T_{c} and therefore outside of the GL region. This results in phenomenological parameters which show up in the normalization factors.

To make the theory apply far above T_{c} might require a full microscopic treatment. Work along these lines has been carried out by Golub (1980), yet the Green's function formalism used in that paper is not yet amenable to modeling of actual devices. In fact, the only result that could be compared with experiments is the effective resistance, R_{eff} . So far the formalism has proved too complex to yield $I - V$ curves or to reduce to a TDGL-like equation. More work is needed in this area to obtain results that can be meaningfully compared with experiments.

The fact that SNS microbridges seems to be modeled well by the TDGL equations and the technological promise that they hold will hopefully provide interest for more fundamental theoretical work in this field.

Fig. 4. (a) Relaxation of the phase as a function of reduced time showing an exponential dependence $\phi=\phi_{0 \mathrm{e}^{-t / t_{0}}}$. (b) Dependence of t_0 on the relaxation time of the order parameter τ_Δ (in reduced units of τ_j).}

Fig. 5. Effect of the relaxation time of the order parameter on the $I - V$ near $I_{\mathrm{c }}$. As the relaxation time is decreased the TDGL curves approach the RSJ result (dashed curve).}

Fig. 6. Behavior of the low voltage effective resistance $R_{\text {err }}$ as a function of the relaxation time (a) of the order parameter and (b) of the reduced length.}

Fig. 7. Comparison of the experimental $I - V$ curves of a planar SNS microbridge (van Dover et al. 1980, 1981) near its critical temperature ($T_{\mathrm{cSN }}$) with the results of the TDGL model.}

Fig. 8. Comparison of the experimental $I - V$ curves of a planar SNS microbridge (van Dover et al. 1980, 1981) with the results of the TDGL and RSJ models in the case of (a) DC bias and (b) $\mathrm{DC +\mathrm{AC}$ bias. The data were taken at $T = 8.05$ K, while $T_{\mathrm{cSN}} = 8.85$ K.}

35 The Behavior of a Superconductor in a Variable Field by Gor'kov, Eliashberg

Abstract

The dynamical equations previously found for the model of a gapless superconductor are used for the calculation of some nonlinear high-frequency effects.

The high-frequency region is primarily considered here, the parameter Δ mainly being determined by the time average of the square of the field intensity.

Actually, the results do not depend on the choice of model. In each individual case, the boundary of the high frequency region should be specified.

For type I superconductors, the behavior of Δ near the boundary is calculated for the presence of a large-amplitude high-frequency field. The case of the simultaneous effect of a constant and a varying field is also considered. Special interest attaches to the effect of a high frequency field on the hysteresis pattern calculated for a stationary Ginzburg field. Formulas are derived which describe the destruction of the superconductivity of films of various thicknesses in a variable field.

For type II superconductors, the destruction of surface superconductivity by a high-frequency field is calculated. In particular, the dependence of the field strength H_{c3} on the variable field amplitude is found, and equations are obtained for the description of the dynamics of the surface layer.

(so, in a nutshell, we shine a laser on a SC, what will be??? this is what should be answered in this article)

35.0.1 1. Introduction

The properties of superconductors in a nonstationary electromagnetic field have thus far been inadequately studied.

On the one hand, experimental difficulties connected with heating in a strong variable field do not permit us to outline a clear picture of the phenomena, and on the other hand, theoretical results based on modern microscopic representations are virtually nonexistent.

(why inadequately??)

A general approach to the problem of nonstationary properties of a superconductor was proposed previously by the authors in [1], but it was noted at the same time that the general arrangement of the problem, independent of the experimental conditions, is possible in the region in which there are sufficient normal carriers to guarantee heat elimination, and the superconductivity is a relatively small effect. This is, above all, the region near the critical temperature or, for example, the case considered in detail in [1] of gapless superconductivity in alloys with paramagnetic impurities.

For the latter problem, the generalization of the Ginzburg-Landau equation in a varying field was written in [1] :

$$\dot{\Delta} + \frac{\tau_s}{2}\left\{\pi^2\left(T^2 - T_c^2\right) + \frac{|\Delta|^2}{2}\right\}\Delta + D\left(-i\nabla - \frac{2e}{c}\mathbf{A}\right)^2\Delta = 0, \tag{1}$$

$$\mathbf{j} = \sigma\mathbf{E} - \frac{2\sigma\tau_s}{c}|\Delta|^2(\mathbf{A} - \nabla\varphi).$$

Equations (1) are valid in the region of frequencies comparable with the characteristic scale frequency $\omega_0 = 2\tau_S\left|\Delta_0\right|^2$ (Δ_0 is the equilibrium value of the gap). These equations, obtained

in [1] for the special case of paramagnetic alloys, nevertheless allows us to understand the qualitative picture of the phenomena taking place in superconductors in a variable field. In this connection, we note that the parameter κ of the Ginzburg-Landau theory in this system

$$\varkappa^2 = c^2/48\pi\sigma D \tag{2}$$

can be either small or large, depending on which initial pure metal was chosen for the preparation of the alloy. In other words, we shall regard below the system (1) as a reasonable model, which allows us to study the properties of both Pippard ($\kappa \ll 1$) and London ($\kappa \gg 1$) superconductors.

We transform to the dimensionless variables used in the Ginzburg-Landau theory: [2]

$$H' = -\frac{H}{\sqrt{2}H_{cm}}, \quad A' = \frac{A}{\sqrt{2}H_{cm}\delta_L}, \quad \Delta' = \frac{\Delta}{\Delta_0}, \tag{3}$$

and to the nondimensional time parameter

$$t' = \omega_0 t. \tag{\prime}$$

(Here, we shall omit the primes for the nondimensional quantities. The transition to the ordinary variables is everywhere indicated in the corresponding places.) In what follows, we shall limit ourselves to the consideration of one-dimensional situations (films, half-space), where one can assume Δ to be real. In terms of the new variables for all quantities, which depend only on a single coordinate, the set of equations for Δ and the field A has the form

$$\begin{gathered} 12\Delta + \left(\dot{\Delta}^2 - 1\right)\Delta - \frac{1}{x^2}\frac{\partial^2\Delta}{\partial z^2} + A^2\Delta = 0, \\ \frac{\partial^2\Delta}{\partial z^2} = \dot{A} + \Delta^2 A. \end{gathered} \tag{\prime}$$

35.0.2 2. Pippard Superconductors ($\kappa \ll 1$)

For slow changes in the field, up to the point at which the amplitudes of the magnetic field is no longer smaller than the critical value, the change in Δ is small. If the fields exceed critical, then a first-order transition takes place in thick samples, as a result of which a moving separation boundary appears between the normal and superconducting phases. The kinetics of such a transition was considered by I. Lifshitz. [3] We shall therefore not linger over the details of the situation. We only note that the criteria of slowness of change in the field depends on the degree of perfection of the sample and of its surface, inasmuch as precisely these circumstances determine the formation of the boundary. We shall therefore study below the case of high frequencies ($\omega \gg 1$) where the formation and motion of the separation boundary are impossible.

For proof of the latter assertion, we recall that, as is well known, the structure of the boundary separating the two phases is characterized by two lengths: the London penetration depth and the scale of change in Δ (in dimensionless units, these are 1 and $1/\kappa$, respectively). Thus, for $\kappa \ll 1$, the location of the boundary is fixed with an accuracy to within a length ~ 1. At the same time, it is seen from the second of Eqs. (1') that, during the period of change of the field, the boundary cannot go out from the surface a distance greater than $\omega^{-1/2} \ll 1$.

If $\omega \gg 1$, then $\Delta(\mathrm{t}) = \Delta + \Delta_1(\mathrm{t})$, where the time average $\overline{\Delta_1(t)} = 0$ and the value of Δ_1 is small (see below). Therefore, the equation for the static part of the gap has the usual form of Ginzburg-Landau theory:

$$\left(\Delta^2-1\right)\Delta-\frac{1}{v^2}\frac{\partial^2\Delta}{\partial\tau^2}+\overline{A^2\Delta}=0, \tag{4}$$

where, however, the variable part of A, A_1 (before time averaging) is determined from the usual equations of the normal skin effect:

$$\partial^2 A_1/\partial z^2=\dot{A}_1. \tag{/}$$

Assuming that the skin depth is less than all the characteristic dimensions of the problem, we find for A_1, in the case of a monochromatic field,

$$A_1=\frac{h_1}{\sqrt{2}}\delta_s\exp\left(-\frac{z}{\delta_s}\right)\cos\left(\omega t-\frac{z}{\delta_s}-\frac{\pi}{4}\right),$$
$$\overline{A_1^2}=\frac{h_1^2\delta_s^2}{4}\exp\left(-\frac{2z}{\delta_s}\right),$$

where h_1 is the amplitude of the alternating field, $\delta_S=\sqrt{2/\omega}$.

If a constant field h_0 also figures in the problem, then its penetration depth is Δ^{-1}.

35.0.3 A. Boundary of a Bulk Superconductor

Let us consider the superconducting half-space $z>0$ and assume that the field is sufficiently small so that one can determine the change of Δ by perturbation theory: $\Delta=1+\Delta_1$ $(\Delta_1\ll 1)$. Then

$$2\Delta_1-\frac{1}{x^2}\frac{\partial^2\Delta_1}{\partial z^2}+\frac{{h_1}^2}{4}{\delta_s}^2\exp\left(-\frac{2z}{\delta_s}\right)+{h_0}^2e^{-2z}=0.$$

The solution of this equation with the boundary condition $(\partial\Delta/\partial\mathbf{z})_{\mathbf{z}=0}=0$ is

$$\Delta_1(z)=\frac{{h_1}^2{\delta_s}^4x^2}{8\left(2-x^2{\delta_s}^2\right)}\left[\exp\left(-\frac{2z}{\delta_s}\right)-\frac{\sqrt{2}}{x\delta_s}\exp(-x\sqrt{2}z)\right] \tag{5}$$
$$+\frac{{h_0}^2x^2}{2\left(2-x^2\right)}\left[e^{-2z}-\frac{\sqrt{2}}{x}\exp(-x\sqrt{2}z)\right]$$

or, for $\kappa\ll 1$ and $\mathrm{z}=0$,

$$\Delta(0)=1-\left(\frac{x{h_1}^2{\delta_s}^3}{8\sqrt{2}}+\frac{x{h_0}^2}{2\sqrt{2}}\right). \tag{5'}$$

Solving the first of Eqs. (1') for the variable part $\Delta_1(\mathrm{t})$, we find that the component $\Delta_{1,2\omega}$ is equal to ($\kappa\ll 1$)

$$\Delta_{1,2\omega}(z)=\frac{h_1^2\delta_s^4x^2}{16}\left[\exp\left(-\frac{2z}{\delta_s}\right)-\frac{e^{i\pi/4}}{2\sqrt{3}x}e^{-kz}\right],$$

where $\mathrm{k}=4\sqrt{3}\left(\kappa/\delta_\mathrm{S}\right)\mathrm{e}^{-\mathrm{i}\pi/4}$. In other words, in the absence of a constant component, $\Delta_{12\omega}$ has a diffusion "tail" at comparatively small distances $\mathrm{z}\sim\delta_\mathrm{S}/\kappa$ and its value

$$\Delta_{1,2\omega}(z)\approx-\frac{{h_1}^2{\delta_s}^4\chi}{32\sqrt{3}}e^{i\pi/4}e^{-kz}$$

is smaller by the factor $\delta_S \ll 1$ than the constant component Δ_1(z) of (5'). Therefore, we see that in the region where $h_1^2\delta_s^3\kappa \leq 1$, the stationary part of Δ is actually large in comparison with $\Delta_1(t)$, the rapidly oscillating part of the gap, which justifies our assumption in the derivation of the averaged equations (4).

Equations (5) and (5') are naturally valid only so long as $h_1^2\delta_S^2\kappa \ll 1$ and $h_0^2\kappa \ll 1$. Continuation of the solution into the region of stronger fields when $\kappa \lesssim 1$ generally requires numerical calculations similar to those which were carried out by Ginzburg [4] in the determination of the supercooling field. In the region $\kappa^{\ll 1}$, however, an analytical approach is possible, which was not noted in [1]. Having it in mind to study the qualitative picture, we now make use of it. The essential fact is that the field arises in the superconductor at a distance from the surface $\sim_{\delta_0}$ for the variable and $1/\Delta(0)$ for the constant component of the magnetic field. If these distances are small in comparison with $1/\kappa$, then there is no field at great distances from the surface of the sample and the solution for Δ(z) is [2]

$$\Delta = \mathrm{th}[\varkappa(z + C)/\sqrt{2}]. \tag{6}$$

The problem consists of connecting the solution in the region of change of the field with the solution (6). Taking it into account that in the region of the field itself, the function Δ and its derivative

$$\frac{\partial\Delta}{\partial z} = x^2\left\{\left(\Delta^2(0) - 1\right)\Delta(0)z + \Delta(0)\int_0 \bar{A}^2 dz\right\} \tag{7}$$

are slowly changing, we can "join" (7) and (6) together at z $\ll 1/\kappa$. This is possible if

$$h_1{}^2\delta_s{}^3 > 1, \quad h_0{}^2/\Delta^2(0) \gg 1. \tag{8}$$

The conditions (8) overlap for $\kappa \ll 1$ with the region of applicability of (5) and (5'). Thus the "joining" conditions are

$$\Delta(0) = \mathrm{th}(\varkappa C/\sqrt{2})$$
$$\varkappa\Delta(0)\int_0^\infty \overline{A^2}dz = \frac{1}{\sqrt{2}\,\mathrm{ch}^2(\varkappa C/\sqrt{2})} = \frac{1}{\sqrt{2}}\left(1 - \Delta(0)^2\right). \tag{9}$$

Comparing the first of the conditions (9), as $C \to \infty$,

$$\Delta(0) = 1 - 2e^{-x\sqrt{2}C}$$

with the expression (5′), we see that C is logarithmically large for small $\kappa h_1^2\delta_S^3$ and $h_0^2\kappa$.

We now undertake a detailed study of the general case of (8). The second of Eqs. (9), in which C is contained, can be written in the form

$$h_0^2 = \frac{\sqrt{2}}{x}\Delta(0)^2\left(1 - \Delta(0)^2\right) - \frac{h_1^2\delta_s^3\Delta(0)^3}{4}. \tag{10}$$

If there is no constant field ($h_0 = 0$), it then follows from (10) that

$$\Delta(0) = -\frac{xh_1^2\delta_s^3}{8\sqrt{2}} + \left(1 + \frac{x^2h_1^4\delta_s^6}{128}\right)^{1/2}. \tag{11}$$

For small fields (11) coincides with (5'); in the limit of large field amplitudes,

$$\Delta(0) \approx 64/\varkappa h_1{}^2\delta_s{}^3 \tag{\prime}$$

asymptotically approaches zero. Therefore, in the superconductor for $\Delta(x)$ in the limit of extremely strong fields, we obtain essentially the dependence

$$\Delta(z) = \mathrm{th}(xz/\sqrt{2}).$$

Thus, for the infinite half-space, the superconductivity is not destroyed in a high-frequency field. There is only one stationary regime. Once again, we recall that we assume a small Joule heating of the boundary, for which it is necessary to remove heat from the surface of the same.

Before concerning ourselves with the case of the complete Eq. (10), let us first make more precise the results obtained in [4], which pertain to the superheating field. Let $\mathrm{h}_1 = 0$; then $\mathrm{h}_0^2 = \sqrt{2}\kappa^{-1}\Delta(0)^2\,(1-\Delta(0)^2)$. This is a curve with a maximum, i.e., $\Delta(0)$ as a function of h_0 is not well defined. The location of the maximum determines the superheating field $\mathrm{H_S}$ for $\kappa \ll 1$:

$$H_{\mathrm{s}} = \frac{H_{\mathrm{cm}}}{2^{1/4}\sqrt{x}} \approx \frac{0.84}{\sqrt{x}} H_{\mathrm{cm}},$$

which differs somewhat from the result obtained in [4] by numerical calculation. Another circumstance is much more interesting. In Fig. 5 of [4] is given the curve $\psi(0)$ (i.e., $\Delta(0)$) of the field h_0. When inverted as a plot of h_0 vs. $\Delta(0)$, this curve has a single maximum, while the important fact is that this curve must reach the point (in our variables) $\Delta(0) = 0,\ \mathrm{h}_0 = 1/\sqrt{2}$, which corresponding to a first order phase transition (formation of a boundary between the normal and superconducting phases). The region $\Delta(0) \to 0$ cannot be considered by our method, inasmuch as, along with the condition $h_0^2/\Delta^2(0) \gg 1$, the condition of the localization of the field near the boundary $1 \ll \Delta(0)/\kappa$ must be satisfied. However, upon satisfaction of the latter condition as $\kappa \to 0$ Eq. (10) ($h_1 = 0$) is still valid in the region $h_0 \ll 1, \Delta(0) \ll \sqrt{\kappa}$, i.e., curve (10) enters into the region of fields smaller than critical.

In Fig. 1 a graph is constructed of the dependence of $2\kappa h_0^2$ on $\Delta^2(0)$, from which it is seen that in the region indicated by the dashed line, $\Delta(0)$ has four values as a function of h_0. In the usual scale of critical fields, this region is shown schematically in Fig. 2. It is evident that the portion of the curve AB will also be metastable.

FIG. 1

FIG. 2

It is physically probable that this portion corresponds to a certain surface current which screens the external field.

Our discussion refers to the limiting case $\kappa \to 0$. Now, assuming h_0 to be given in Eq. (10), we shall consider the destruction of the superconductivity by an alternating field in the presence of a constant field less than critical ($h_0 < 1/\sqrt{2}$). The dependence of $\Delta(0)$ on h_1, in contrast with (11) becomes non-single valued. The maximum of h_1^2 corresponds to the value

$$\Delta \mathrm{cr}^2 = \frac{3\sqrt{2}\varkappa h_0^2}{1 + \left[1 + 6\sqrt{2}\varkappa h_0^2\right]^{1/2}} \approx \frac{3}{\sqrt{2}} x h_0^2,$$

which generally corresponds to much larger amplitudes of the variable field

$$\frac{{h_1}^2 {\delta_s}^3}{4} = \frac{2{h_0}^2}{\Delta_{\mathrm{cr}^3}} \sim \frac{1}{h_0 \mathrm{x}^{3/2}}.$$

The value of $\Delta_{\mathbf{cr}}^2$ still satisfies our criterion of the localizability of the field

$$1 < \Delta_{\mathrm{cr}}/x \sim h_0/\sqrt{x}.$$

The existence of ambiguity of $\Delta(0)$ as a function of h_1 means that in the region $\Delta^2(0)$ there exists still another branch, similar to the case in a constant field discussed above, which is stable in the sense that $\Delta(0)$ falls off with increase in amplitude of the alternating fields. We did not succeed in analyzing this branch analytically. However, it can be established that in the presence of a constant magnetic field, an increase in the amplitude of the alternating field can produce a "collapse" of the regime, described by Eqs. (5), (5') and (11). This confirmation generally refers only to sufficiently small $\kappa(\sqrt{\kappa} \ll 1)$.

35.0.4 B. Superconducting Plates

As before, let $\kappa \ll 1$. We consider the case of a film with thickness $2d \ll 1/\kappa$. Then, as is known, we can neglect the dependence of Δ on the coordinate, as a result of which, we can omit in (1) terms with second derivative in z, and the set (1) is considerably simplified. In [1] the destruction of superconductivity by an alternating field was studied in detail for the case in which the dimensions of the layer are small in comparison with the penetration depth ($d \ll 1$). Such films in a magnetic field experience a second-order phase transition. In particular, it was discovered in [1] that in the case of a high-frequency field, the destruction of the superconductivity is determined by the mean-square field. Here we would wish only to add several observations, which touch on the properties of such a film in a slowly changing, strong field.

It was shown in [1] that generally, in a slow field, the change of $\Delta(\mathrm{t})$ follows the field change adiabatically. However, if we want to obtain a steady solution (one not depending on the initial conditions), in which a periodic behavior of Δ takes place in a periodically changing magnetic field then, in accord with [1], the condition $1 - \mathrm{h}^2 > 0$ is always necessary (notation from [1]). In other words, the regime, even in a suitably slow field, in which the plate undergoes periodic transition from the superconducting state to the normal, cannot always be established and must "collapse" for sufficiently large amplitude.

The new regime is determined by the value of the random fluctuations which arise during the time when the field is less than critical and the normal phase of the film is made absolutely unstable. Such a regime will certainly be aperiodic. The discovery of a collapse of the periodic regime and the study of the new regime is of considerable interest to us. We have based our arguments above on the formulas of [1], but it is clear that a similar "turbulent" regime will

exist for all films with 2 d $< \sqrt{5}$, for which the phase transition in the magnetic field is of second order. [2] We are dealing here with variable fields with amplitude of the order of the value of the critical field H_{cm}.

We now proceed to the case of high frequencies $\omega \gg 1$ ($\delta_S \ll 1$) and $1 \ll$ d $\ll 1/\kappa$. We shall assume that a constant magnetic field h_0 is applied to the film from both sides. So far as the variable field with amplitude h_1 is concerned, let it be applied only from one side. Inasmuch as $\delta \ll 1$, the variable skin field penetrates only very slightly into the layer. Averaging the field (4) over the z coordinate (from $-d$ to $+d$), we obtain by a direct solution of the Ginzburg-Landau equations [2] the relation

$$\frac{[\Delta^2 - (1-\alpha_1)] \cdot 2\Delta^2 \mathrm{ch}^2 \Delta d}{1 - (\mathrm{sh}\, 2\Delta d)/2\Delta d} = h_0^2, \tag{12}$$

where

$$\alpha_1 = \frac{1}{2d} \int_{-d}^{+d} \overline{A_1^2} dz = \frac{h_1^2}{16d} \delta_s^3$$

By the substitution of variables

$$\Delta' = \frac{\Delta}{\sqrt{1-\alpha_1}}, \quad d' = d\sqrt{1-\alpha_1}, \quad h_0' = \frac{h_0}{1-\alpha_1} \tag{13}$$

we reduce it to an equation similar to that studied for films in [2,4].

If $h_0^2 = 0$, then $\Delta^2 = 1 - \alpha_1$ falls off monotonically to zero with increase in the power of the variable field. We see that (comparable with [1]) the superconductivity for films with $d \ll 1/\kappa$ exists only if the amplitude of the field does not exceed some limit

$$h_1{}^2 < 4\sqrt{2} d \omega^{3/2} \tag{14}$$

Let a certain level of power of the variable field be given. Then the phase transition of the film in the constant magnetic field is significantly changed. For d$' < \sqrt{5/2}$, this transition is of second order. In other words, according to (13), in the presence of radiation a second order phase transition takes place in a constant field for larger thicknesses:

$$\alpha_{\mathrm{cr}} = \sqrt{5}/2\sqrt{1-\alpha_1}. \tag{15}$$

The dependence of the critical field on the thickness is determined now by the relation

$$h_{\mathrm{c2}} = \sqrt{3} d^{-1} \sqrt{1-\alpha_1}. \tag{16}$$

For d $>$ d$_{\mathrm{cr}}$, from (15), the phase transition is a first order transition. Equation (16) has the meaning of a supercooling field. The superheating field is determined by the maximum of the curve (12). For d$' \gg 1$, the field h$'$ was determined in [4] : h$_S' = 0.6\sqrt{\mathrm{d}'}$. Therefore, from (13),

$$h_{\mathrm{s}} = 0.6\sqrt{d}\,(1-\alpha_1)^{5/4}. \tag{17}$$

The relations (16) and (17) determine the region of hysteresis (the region of ambiguity of the dependence of Δ on h_0). It is evident that this region contracts. However, in the presence of an alternating field, we cannot establish the criterion which corresponds to the thermodynamic criterion determining that field h$_{\mathrm{ocr}}$ in which the first order phase transition will be actually established. Similar experiments on the displacements of the critical field can easily be conducted.

In turn, by fixing the constant field h_0 in the limits between (16) and (17), we conclude, inasmuch as h_0' in this range has associated with it two values of Δ, that the ambiguity of Δ exists also with respect to the dependence on the power of the variable field for fixed h_0. It is seen from (16) that this ambiguity, i.e., the possibility of collapse of the superconducting steady state, arises for

$$\alpha_1 = 1 - \frac{h_0^2 d^2}{3} \quad (d < d_{\mathrm{cr}}) .$$

The new regime obviously corresponds to the skin effect in the normal phase.

We have shown above that in the semi-infinite sample, the superconductivity does not vanish in a rapidly alternating field up to very high powers. On the contrary, according to (14), there is always a limiting value of the power in a thin film, above which the superconducting state cannot exist. We shall show that for thin films $\kappa d \sim 1$, a transition takes place from the first situation to the second.

For this case, we consider a thick film ($\kappa d \sim 1$) in a rapidly changing field ($\delta_S \ll 1$), applied from one side ($z = 0$). Then, as also for the case of an infinite halfspace, there is no field in the greater part of the film thickness. However, the gap is not equal to unity on the second wall (at $z = 2d$). We shall denote its value $\Delta(2d) \equiv \Delta_1$; then the first integral of Eq. (4) without a field can be written in the form

$$\frac{d\Delta}{dz} = x \left[\left(\Delta_1^2 - \Delta^2\right) \left(1 - \frac{\Delta_1^2 + \Delta^2}{2}\right) \right]^{1/2} ,$$

where we used the fact that $(\partial\Delta/\partial z)_{z=2d} = 0$. Assuming as in Sec. A that

$$\int_0^\infty \overline{A^2} dz = \alpha \gg \delta_s$$

and joining the results, we obtain two conditions

$$\int_{\Delta_0}^{\Delta_1} d\Delta \left[\left(\Delta_1^2 - \Delta^2\right) \left(1 - \frac{\Delta_1^2 + \Delta^2}{2}\right) \right]^{-1/2} = 2xd$$

$$\left[\left(\Delta_1^2 - \Delta_i^2\right) \left(1 - \frac{\Delta_1^2 + \Delta_0^2}{2}\right) \right]^{1/2} = \Delta_0 x \alpha$$

(where $\Delta_0 = \Delta(0), \alpha = \mathrm{h}_1^2 \delta_{\mathrm{S}}^3/8$). Introducing the variable $\Delta/\Delta_1 = \cos \mathrm{z}, 0 < \mathrm{x} < \pi/2$, we transform both conditions to the much more convenient form

$$\int_0^u \frac{dx}{\sqrt{k^2 + \sin^2 x}} = \frac{2xd}{\sqrt{k^2+2}}, \tag{18}$$

$$\sqrt{k^2 + \sin^2 u} = \alpha x \sqrt{k^2+2} \operatorname{ctg} u,$$

while

$$u = \arccos\left(\Delta_0/\Delta_1\right), \quad \Delta_1 = \sqrt{2}/\left(k^2+2\right). \tag{20}$$

Equation (19) can be solved relative to u :

$$\cos^2 u = 1 + 1/2 \left\{ k^2 + \alpha x - \left[\left(k^2 + \alpha x\right)^2 + 4\alpha x \right]^{1/2} \right\}$$

After this, the problem of the determination of Δ_0 and Δ reduces to the numerical solution of Eq. (18). However, we shall not deal with this, bearing in mind only to show that there

exists some critical thickness, above which no increase in the power can destroy the superconductivity over the entire volume of the film. For this purpose, we note that the destruction of superconductivity means k $\to \infty$ ($\Delta_1 \to 0$). It then follows from (18) that u = 2κ d, while

$$xd < \pi/4 \tag{21}$$

Here the threshold power that destroys superconductivity is determined by

$$1 = \alpha_{\mathbf{cr}} x \operatorname{ctg} 2xd \tag{22}$$

For κd $\ll 1$, we again get (13). For κd ~ 1, we see that for destruction of the superconductivity in the film by the alternating field alone, high powers are necessary ($\kappa \ll 1$):

$$h_1^2 \varkappa \omega^{-3/2} \sim 1$$

Summing up what has been said about type I superconductors, we note only that although the majority of the formulas were obtained by us in the limiting case $\kappa \ll 1$, the phenomena described above undoubtedly are also valid for $\kappa \sim 1$. In this case, the situation deals with entirely realistic powers of the alternating field.

35.0.5 3. London Superconductors (Alloys with $\kappa \gg 1$)

The case of low frequencies of the alternating field for alloys is even more complicated in comparison with type I superconductors. If the amplitude of the variable field exceeds the critical value of the field H_{C_1}, for which vortices are formed, then the penetration of the field into the superconductor is connected with the motion of the quantized vortices. This effect has always had a hysteresis character, both because of the presence of the surface barrier and because of the imperfection of the sample. Therefore, it seems to us that one should consider the general picture of a similar nature rather on a phenomenological basis rather than on the basis of a microtheory constructed for the homogeneous alloy. In this connection, we limit ourselves to the remark that the "motion" of the boundary of the field, for the reasons given, takes place with a velocity v that is less than follows from the second of Eqs. (1'): $v < \sqrt{\omega}$, and in what follows, we shall study the case of high frequencies $\delta_S \ll 1/\kappa$, where the motion of the vortices is practically non-existent. [1)]

In weak fields we have (5') for the correction to the gap as before ($\kappa \gg 1$, but $\kappa\delta_S \ll 1$). For strong fields, we can apply the method of "joining" the two solutions described by Eqs. (7), (9) of the previous section. The joining together of the two solutions (7) and (6) takes place at distances $z \sim \delta \ll 1/\kappa$. Therefore, the criterion of applicability of the method is

$$h_1^2 \delta_s^3 \gg 1 \tag{23}$$

and Eq. (5') ($\kappa\delta_0 \ll 1$) can be obtained from (11) as the limiting case.

Thus, in this limiting high-frequency case, all the results of the previous section are valid. These were obtained for the half-space and film (if $h_0 = 0$). Once again, however, we note the

[121)] In experiments in a variable field, the amplitude of which is greater than H_{cl}, it is naturally always possible to have the existence of a finite vortex density in the sample which is oriented in a different way from these, so that the mean field connected with them was equal to zero. The formation of vortices is always possible, since this is one of the mechanisms of Joule loss. The regime in the variable field in the present of vortices, which are formed in fluctuating fashion, will necessarily be aperiodic (it is possible that a similar picture was observed in [[5,6]]; however, theoretical investigation of it is too complicated, and we shall in what follows digress from the possibility of a similar mechanism of collapse of the established regime.

fact that for films with d $> 1/\kappa$ for type II superconductors, there is the possibility of a new solution with the formation of vortices inside the film.

We now proceed to the inverse situation, where $\delta_s \gg 1/\kappa$.

$$\Delta^2(z) = 1 - \overline{A^2}(z) \tag{24}$$

The applicability of the quasiclassical solution (24) is limited to the vicinity of x_0, where $\Delta\left(z_0\right) = 0$. If the field is large enough,

$$h_1^2 \delta_s^2 / 4 \gg 1 \tag{25}$$

and such a point exists, one must connect the solution (24) with the solution of the equation ($x = z - z_0$)

$$\frac{1}{x^2}\Delta'' + \frac{4x}{\delta_s}\Delta - \Delta^3 = 0 \tag{26}$$

with the conditions for the latter

$$\partial\Delta/\partial x \to 0, \quad x \to -\infty, \quad \Delta\infty x^{1/2}, \quad x \to +\infty.$$

By the substitution

$$x = \left(\delta_s/4x^2\right)^{1/3}\xi, \quad \Delta = \Delta\left(4/\delta_s x\right)^{1/3}$$

we reduce (26) to the form

$$\bar{\Delta}'' + \xi\bar{\Delta} - \Delta^3 = 0 \tag{\prime}$$

Equation (26') is the transcendental equation of Painleve (see [7]), relative to which it is shown that this solution is always a single-valued function of the complex variable ξ. Thus, even in this case, the superconductivity in a strong variable field for the half-space changes monotonically with increase in the power. However, in contrast with the type I superconductors (see [6]), to the left of the point z_0 there arises a region of normal phase. The very point z_0 moves to the right with increase in the field. Without solving (26) and (26'), we can see that, to the left of z_0, Δ decays according to the law

$$\Delta\infty \exp\left\{-\frac{4}{3}\frac{x}{\sqrt{\delta_s}}\left|z - z_0\right|^{3/2}\right\}. \tag{27}$$

The two latter problems, considered below in this section, pertain to the surface superconducting layer in fields $H_{C2} < H < H_{C_3}$, the existence of which was shown first by Saint-James and de Gennes [8]. We shall first show that a sufficiently large varying field destroys the surface layer. According to [8], we ought to study the existence of an infinitely small localized solution of Eq. (1) for Δ in a constant field h_0 and an alternating field $h_1(t)$. Let $\delta_S \gg 1/\kappa$ and the solution $\Delta(z, y)$ is sought in the form $\Delta(x, y) = \Delta(z)e^{iky}$. In correspondence with [8], we get for $\omega \gg 1$ Eq. (4) in the form

$$-\frac{1}{x^2}\frac{\partial^2\Delta_0}{\partial z^2} + \left[{h_0}^2\left(z - z_0\right)^2 + \frac{h_1^2}{4}\delta_s^2 \exp\left(-\frac{2z}{\delta_s}\right)\right]\Delta_0 = \Delta_0 \tag{28}$$

The region of change of $\Delta_0(z)$ is of the order of $z \sim z_0 \sim 1/\kappa$ (see below). Therefore the exponent above can be replaced by unity. By the change of variables

$$z' = z\left(1 - \frac{h_1^2\delta_s^2}{4}\right)^{1/2}, \quad h'^2 = h_0^2\left(1 - \frac{h_1^2\delta_s^2}{4} -\right)^{-3/2} \tag{29}$$

we transform (28) to the form studied in [8]. This equation has its first solution for h′ = 1.7κ. Then

$$h_{c3} = 1.7x\left(1 - \frac{h_1^2\delta_s^2}{4}\right)^{3/4}$$

As soon as h_{c_3} becomes equal to $\kappa\,(H = H_{c_2})$, the superconducting state becomes absolutely unstable in the bulk of the sample. Thus the condition for the vanishing of the surface layer is

$$1,7\left(1 - 1/4h_1^2\delta_s^2\right)^{3/4} = 1 \tag{30}$$

In conclusion, we consider a problem of unquestioned methodological interest-the derivation of equations which control the superconductivity in the surface layer for a varying field. We shall consider the case of small subcriticalities:

$$H(t) - H_{c3} \ll H_{c3} \tag{31}$$

and low frequencies. Thus, let $h(t) = h_{c_3} + h_1(t)$. Then, assuming that $\Delta_{=c}(t)\Delta_0(z)$, where $\Delta_0(z)$ is the solution of (28) for $h_1 = 0$, and multiplying the first of Eqs. (1) by this expression, integrating over z , we get

$$\begin{aligned} 12c\overline{\dot{\Delta}_0}^2 + c^4\overline{\Delta_0{}^4} - c^2\overline{\Delta_0{}^2} - \frac{c^2}{\lambda^2}\left(\overline{\Delta_0\frac{\partial^2\Delta_0}{\partial z^2}}\right) \\ + \left(A' + h_{c3}\left(z - z_0\right)\right)^2\Delta_0{}^2c^2 = 0 \end{aligned} \tag{32}$$

Here integration over z is indicated by the bar, A′ is the field composed of the external field h_1 and the field of the superconducting currents. By virtue of the smallness of h_1 we can expand (31) in Eq. (32) in terms of A' :

$$12c\dot{c}\overline{\Delta_0^2} + c^4\overline{\Delta_0^4} + 2h_{c3}\overline{A'(z)\left(z - z_0\right)\Delta_0^2(z)}c^2 = 0 \tag{33}$$

The field A' satisfies the equation

$$\partial^2 A'/\partial z^2 = \dot{A}' + c^2\Delta_0^2(z)\left\{h_{c3}\left(z - z_0\right) + A'(z)\right\} \tag{34}$$

Equation (34) must be solved by the method of successive approximations, taking it into account that c is small. For this purpose, we introduce the notation

$$A'(z) = A_1'(z) + A_S(z)$$

where

$$\partial^2 A_S/\partial z^2 = c^2h_{c3}\Delta_0^2(z)\left(z - z_0\right)$$

Then, following [9], we integrate the last term in (33) by parts:

$$2\overline{A'(z)h_{c3}\left(z - z_0\right)\Delta_0^2(z)}c^2 = -2\overline{h'(z)h_S(z)} \tag{35}$$

where h′ = ∂A$_1'$/∂z + ∂A$_S$/∂z and we have used the property noted in [8] that the total current in the superconducting layer is equal to zero:

$$h_{S,z} = 0 = h_{c3}c^2\int_0^\infty \Delta_0^2(z)\left(z - z_0\right)dz = 0 \tag{36}$$

For A$_1'$, we have, according to (34), the equation

$$\frac{\partial^2 A_1'}{\partial z^2} = \dot{A}_1' + \dot{A}_S + c^2 \Delta_0^2(z) \left[A_S + A_1'\right] \tag{$\prime$}$$

In Eq. (33), the term with the time derivative is substantial in the region $\omega \sim c^2$. We shall be guided by this scale of frequencies. It is then seen from (36) that the screening currents are small and the field h' in the first approximation can be set equal to $h' = h_1(t) + h_S$. Thus (35) is transformed to the form

$$-2h_1(t)\overline{h_S(z)} - 2\overline{h_S^2(z)} \tag{37}$$

where

$$h_S(z) = -c^2 h_S \int_z^\infty \Delta_0^2(z)\,(z - z_0)\,dz$$

The calculation of all the integrals entering into (33) and (37) is generally possible only numerically. However, in the limit $\kappa \gg 1$, the calculation can be greatly simplified. In particular, it is possible to omit the term $\overline{\mathrm{h}_\mathrm{S}^2}$ in (37). Actually, this term makes a contribution to the coefficient before c^4 in (33). Let $\overline{\Delta_0^2} = 1$. Then, inasmuch as the characteristic length dimension is $1/\kappa$, while the field is $\mathrm{h_{c3}} \sim \kappa$, we have

$$\overline{h_S(z)}^{\,\overline{\Delta_0^4} \sim x,} \quad h_S(z) \sim c^2/\varkappa \underset{\sim}{\sim} c^2/h_{c3}, \quad \overline{h_S^2(z)} \sim c^4/\varkappa.$$

Final formulas are conveniently obtained by using the variational method of determination of $\Delta_0(z)$ proposed by Kittel (see [9]). Choosing Δ_0(z) in the form $\mathrm{e}^{-\mathrm{rz}^2}$, we find

$$r = \frac{x h_{\mathrm{c3}}}{2}\sqrt{1 - \frac{2}{\pi}}, \quad z_0 = \frac{1}{\sqrt{2\pi r}}, \quad h_{c3} \approx 1.66$$

Using the latter expression, after simple calculations, we get

$$12\dot{c} + \frac{c^3}{\sqrt{2}} + \frac{h_1(t)}{h_{\mathrm{c3}}} c = 0 \tag{38}$$

The general solution of an equation of the type (38) is given in [1]. By means of it one can, from Eqs. (34'), find the corrections, for example, to the impedance in a metal, arising from the surface layer.

Part X
Higgs Mode in Nonequilibrium Superconductivity

36 Higgs mode in the quench dynamics of a confined ultracold Fermi gas in the BCS regime by Hannibal, Kettmann, Croitoru, Vagov, Axt, Kuhn

Abstract

The Higgs amplitude mode of the order parameter of an ultracold confined Fermi gas in the BCS regime after a quench of the coupling constant is analyzed theoretically. Characteristic features are a damped oscillation which at a certain transition time changes into a rather irregular dynamics. We compare the numerical solution of the full set of nonlinear equations of motion for the normal and anomalous Bogoliubov quasiparticle excitations with a linearized approximation. In doing so the transition time as well as the difference between resonant systems, i.e., systems where the Fermi energy is close to a subband minimum, and off-resonant systems can be well understood and traced back to the system and geometry parameters.

36.1 Introduction

Ultracold Fermi gases have been subject of many experimental and theoretical studies during recent years (see e.g. [Bloch *et al.*(2008), Giorgini *et al.*(2008), Yefsah et al.(2013)Yefsah, Sommer, Ku, Cheuk, Ji, Bakr, and Zwierlein, Tey et al.(2013)Tey, Sidorenkov, Guajardo, Grimm, Ku, Zwierlein, Hou, Pitaevskii, and Stringari, Sidorenkov et al.(2013)Sidorenkov, Tey, Grimm, Hou, Pitaevskii, and Stringari]). They provide a unique system to study key concepts of condensed matter theory. This is because in these systems many parameters such as the particle density, the Fermi energy, the confinement potential, or the interaction strength between the Fermions, which in a solid state system are typically fixed quantities, can be externally controlled in a wide range [Chin et al.(2010)]. In particular, magnetic-field Feshbach resonances provide the means for controlling the interaction strength between fermions by varying an external magnetic field. The tunability of the s-wave scattering length, which is the dominant interaction channel, makes ultracold Fermi gases ideal for exploring different regimes of interacting many-body systems in a single system. This includes the limiting regimes of weakly attracting fermions, which condense into Cooper pairs forming a Bardeen-Cooper-Schrieffer (BCS) phase below a certain temperature T_C, and repulsive dimers formed by two fermions, which can undergo a Bose-Einstein condensation (BEC). These two limiting regimes are separated by the strongly interacting BCS-BEC crossover regime where the scattering length diverges and the system exhibits unitary properties [Zwerger(2011)].

In addition to the variable interaction strength, ultracold atomic gases offer a unique opportunity to explore the influence of a confinement on the pairing correlations, because dimensionality and confinement can be precisely controlled by tuning external parameters [Bloch *et al.*(2008), Martikainen and Törmä(2005), Dyke et al.(2011)Dyke, Kuhnle, Whitlock, Hu, Mark, Hoinka, Lingham, Hannaford, and Vale, Shanenko *et al.*(2012)]. Varying the confinement, which is often well approximated by a harmonic confinement potential, allows one to access new degrees of freedom. Restricting the Fermi gases to quasi-low dimensionality may, e.g., offer the possibility for an experimental evidence of unconventional phases, like the

Fulde-Ferrell-Larkin-Ovchinnikov (FFLO) state [Fulde and Ferrell(1964), Larkin and Ovchinnikov(1972), Yonezawa et al.(2008)Yonezawa, Kusaba, Maeno, Auban-Senzier, Pasquier, Bechgaard, and Jérome, Tomadin et al.(2008)Tomadin, Polini, Tosi, and Fazio, Devreese et al.(2011)Devreese, Wouters, and Tempere, Croitoru et al.(2012)Croitoru, Houzet, and Buzdin]. Moreover it may help to study and get experimental insight into shape resonances, theoretically predicted for quasi-low dimensional conventional superconductors [Shanenko et al.(2006)Shanenko, Croitoru, Zgirski, Peeters, and Arutyunov]. In Ref. [Dyke et al.(2011)Dyke, Kuhnle, Whitlock, Hu, Mark, Hoinka, Lingham, Hannaford, and Vale] the first quantitative measurements of the transition from 2D to quasi-2D and 3D in a weakly interacting Fermi gas has been reported. At low atom numbers, the shell structure associated with the filling of individual transverse oscillator states has been observed. On the theoretical side the ground state properties of a ^{6}Li gas confined in a cigar-shaped laser trap have been investigated predicting size-dependent resonances of the superfluid gap [Shanenko *et al.*(2012)], similar to the case of superconducting nanowires [Shanenko et al.(2006)Shanenko, Croitoru, Zgirski, Peeters, and Arutyunov], yielding an atypical BCS-BEC crossover.

An important effort is now devoted to the exploration of the out-of-equilibrium behavior of trapped ultracold atomic Fermi gases and, in particular, to the determination of their dynamical properties. The dynamics has been studied in the normal as well as in the condensed phase, observing second sound [Sidorenkov et al.(2013)Sidorenkov, Tey, Grimm, Hou, Pitaevskii, and Stringari] and soliton trains [Yefsah et al.(2013)Yefsah, Sommer, Ku, Cheuk, Ji, Bakr, and Zwierlein], and showing a low-frequency oscillation of the cloud after a change of the system confinement or optical excitation [Guajardo et al.(2013)Guajardo, Tey, Sidorenkov, and Grimm, Riedl et al.(2008)Riedl, Guajardo, Kohstall, Altmeyer, Wright, Denschlag, Grimm, Bruun, and Smith, Bartenstein et al.(2004)Bartenstein, Altmeyer, Riedl, Jochim, Chin, Denschlag, and Grimm, Pantel et al.(2012)Pantel, Davesne, Chiacchiera, and Urban, Lepers et al.(2010)Lepers, Davesne, Chiacchiera, and Urban]. Furthermore, state-of-the-art technology allows one to change the coupling constant on such short time scales that it is possible to explore the regime where the many-body system is governed by a unitary evolution with nonequilibrium initial conditions. In ultracold atomic Fermi gases the dynamics may be initiated by readjusting the pairing interaction through switching an external magnetic field in the region of a Feshbach resonance (i.e., a quantum quench) or by a rapid change of the confinement potential of the trap [Riedl et al.(2008)Riedl, Guajardo, Kohstall, Altmeyer, Wright, Denschlag, Grimm, Bruun, and Smith]. Due to the small energies in the trapping potential the dynamics in the Fermi gases take place on a millisecond timescale. Therefore, in contrast to metallic superconductors, where sub-picosecond excitations are required to achieve non-adiabatic dynamics [Papenkort et al.(2007)], in atomic gases the non-adiabatic regime can be reached already by excitations in the sub-millisecond range.

Spontaneous symmetry breaking gives rise to collective modes of the order parameter which are classified into the Higgs amplitude mode, and the Goldstone mode, the latter corresponding to a phase oscillation of the gap [Nambu(1960), Varma(2002), Volovik and Zubkov(2014)]. The Higgs mode has been subject of intensive theoretical and experimental [Sooryakumar and Klein(1980), Littlewood and Varma(1981), Matsunaga *et al.*(2014)] research efforts in the past. On the theoretical side the non-adiabatic temporal response of the order parameter to (quasi-)instantaneous perturbations has been studied. Different regimes of an oscillatory temporal behavior of the pairing potential were theoretically predicted in homogeneous fermionic condensates [Barankov *et al.*(2004), Barankov and Levitov(2006), Yuzbashyan *et al.*(2006), Dzero *et al.*(2007)]. It was shown that the amplitude of the order parameter oscillates without damping when the coupling constant is increased above a certain critical value [Barankov and Levitov(2006)]. On the other hand, the gap vanishes when the coupling constant is decreased

below another critical value. In between these two limiting regimes the amplitude exhibits damped dephased oscillations and the system goes to a stationary steady state with a finite gap [Barankov and Levitov(2006)]. In extended systems the approach to a stationary state occurs in an oscillatory way with an inverse square root decay in time of the amplitude of the oscillations [Yuzbashyan *et al.*(2006)]. A similar evolution was predicted for conventional bulk superconductors [Papenkort et al.(2007), Papenkort et al.(2008)] where the non-adiabatic regime is reached by excitation with short, intense terahertz pulses. An experimental realization was reported in [Matsunaga *et al.*(2013)]. In contrast, in finite length superconducting nanowires a breakdown of the damped oscillation and a subsequently rather irregular dynamics has been predicted [Zachmann et al.(2013)].

In this paper we present a theoretical analysis of the short-time BCS dynamics of a ^{6}Li gas confined in a cigar-shaped laser trap. The excitation is modeled by a sudden change of the interaction strength which can be achieved through a Feshbach resonance by an abrupt change of the external magnetic field [Chin et al.(2010)]. Applying the well-known BCS theory in mean field approximation to ultracold Fermi gases we show that the change of the coupling strength induces a collective oscillation of the Bogoliubov quasiparticles close to the Fermi level. This results in a damped amplitude oscillation of the BCS gap, which corresponds to the Higgs mode [13]. Like in the case of confined BCS superconductors this oscillation breaks down after a certain time revealing rather chaotic dynamics afterwards. We explain these dynamics in terms of coupled harmonic oscillators which can be derived by linearizing the quasiparticle dynamics obtained from the Heisenberg equation of motion.

In doing so we first derive the quasiparticle equations of motion from the inhomogeneous Bogoliubov-de Gennes Hamiltonian (Sec. 36.2) which, because of using the standard contact-type interaction, requires a proper regularization of the gap equation. Starting from the ground state calculated according to Ref. [Shanenko *et al.*(2012)] we then calculate the dynamics of the superfluid gap after an instantaneous change of the coupling constant. The numerical results as well as their explanation follow in Sec. 36.3, where we first discuss a rather small system and then proceed to a larger, experimentally more easily accessible system. Finally, in Sec. 36.4 we summarize our results and give some concluding remarks.

36.2 Theoretical approach

Our approach aims at describing the dynamics of the superfluid order parameter $\Delta(\mathbf{r}, t)$ of an ultracold ^{6}Li Fermi gas, confined in a cigar-shaped, axial symmetric harmonic trapping potential

$$V_{\text{conf}}(x, y, z) = \frac{1}{2}m\omega_\perp^2(x^2 + y^2) + \frac{1}{2}m\omega_\parallel^2 z^2. \tag{36.1}$$

Here, m is the mass of the ^{6}Li atoms and $\omega_\perp$ ($\omega_\parallel$) is the confinement frequency in the x-y-plane (z-direction), respectively. Choosing $\omega_\perp \gg \omega_\parallel$ yields an elongated cigar-shaped trap where the oscillator length $l_\alpha = \sqrt{\hbar/(m\omega_\alpha)}$ provides a measure of the system length. The eigenvalues

$$\varepsilon_m = \hbar\omega_\perp(m_x + m_y + 1) + \hbar\omega_\parallel(m_z + \frac{1}{2}) - E_F \tag{36.2}$$

are measured with respect to the Fermi energy E_F. The index m refers to the combination of quantum numbers m_x, m_y, and m_z. For this geometry the one-particle states form one-dimensional subbands, labeled by (m_x, m_y) [cf. Fig. **??**], while the states within each subband

[13] In the case of a nonlinear system given here the Higgs and the Goldstone mode are in general coupled. However, due to the considered weak excitations and thus weak dynamical coupling the system behaves approximately linear and the Higgs and the Goldstone mode decouple.

are labeled by m_z. Each subband has a constant one-particle density of states and thus the overall density of states exhibits finite jumps whenever a new subband appears.

We consider the gas to be composed of two spin states, $\uparrow$ and $\downarrow$, and start from the inhomogeneous BCS Hamiltonian at $T = 0\,\mathrm{K}$. Within the Anderson approximation we then derive equations of motion for the corresponding Bogoliubov quasiparticle excitations.

36.2.1 Hamiltonian

The usual inhomogeneous BCS Hamiltonian for an effective BCS-type contact interaction reads [De Gennes(1989)]

$$\begin{aligned} H_{BCS} = &\int \left[\Psi_\uparrow^\dagger(\mathbf{r}) H \Psi_\uparrow(\mathbf{r}) + \Psi_\downarrow^\dagger(\mathbf{r}) H \Psi_\downarrow(\mathbf{r})\right] d^3r \\ &- g \int \Psi_\uparrow^\dagger(\mathbf{r}) \Psi_\downarrow^\dagger(\mathbf{r}) \Psi_\downarrow(\mathbf{r}) \Psi_\uparrow(\mathbf{r})\, d^3r, \end{aligned} \tag{36.3}$$

where $\Psi_\uparrow(\mathbf{r})$ and $\Psi_\downarrow(\mathbf{r})$ are the field operators for up and down spin, respectively, $H = p^2/2m + V_{\mathrm{conf}} - E_F$ is the one-particle Hamiltonian and g is the coupling constant of the contact interaction $V(\mathbf{r}) = -g\,\delta(\mathbf{r})$. In the limit of low temperatures the main contribution to the interaction between two fermionic atoms in different internal spin states is given by scattering processes at low momentum. The description of those can be replaced by the widely known pseudopotential only depending on the scattering length a [Bloch *et al.*(2008)], which yields $g = -\frac{4\pi\hbar^2 a}{m}$.

A BCS-like mean field expansion in terms of anomalous expectation values and a particle-hole transformation, leaving spin-up operators unchanged, $\Psi_\uparrow^\dagger = \Phi_\uparrow^\dagger$, while interchanging spin-down ones, $\Psi_\downarrow^\dagger = \Phi_\downarrow$, leads to the Bogoliubov-de Gennes (BdG) Hamiltonian [Datta and Bagwell(1999)]

$$\begin{aligned} H_{BdG} = &\int \Phi_\uparrow^\dagger(\mathbf{r}) H \Phi_\uparrow(\mathbf{r})\, d^3r - \int \Phi_\downarrow^\dagger(\mathbf{r}) H \Phi_\downarrow(\mathbf{r})\, d^3r \\ &+ \int \left(\Delta(\mathbf{r}) \Phi_\uparrow^\dagger(\mathbf{r}) \Phi_\downarrow(\mathbf{r}) + \Delta^*(\mathbf{r}) \Phi_\downarrow^\dagger(\mathbf{r}) \Phi_\uparrow(\mathbf{r})\right) d^3r, \end{aligned} \tag{36.4}$$

where

$$\Delta(\mathbf{r}) = -g\,\langle \Psi_\downarrow(\mathbf{r}) \Psi_\uparrow(\mathbf{r})\rangle = -g\left\langle \Phi_\downarrow^\dagger(\mathbf{r}) \Phi_\uparrow(\mathbf{r})\right\rangle \tag{36.5}$$

is the BCS order parameter. From Eq. (36.4) it becomes apparent that the corresponding eigenvalue equation can be written as the one-particle Bogoliubov-de Gennes equation

$$\begin{pmatrix} H & \Delta(\mathbf{r}) \\ \Delta^*(\mathbf{r}) & -H^* \end{pmatrix} \begin{pmatrix} u_M(\mathbf{r}) \\ v_M(\mathbf{r}) \end{pmatrix} = E_M \begin{pmatrix} u_M(\mathbf{r}) \\ v_M(\mathbf{r}) \end{pmatrix}. \tag{36.6}$$

H_{BdG} can thus be diagonalized by Bogoliubov's transformation, using the eigenfunctions $u_M(\mathbf{r})$ and $v_M(\mathbf{r})$. This introduces non-interacting quasiparticles with energy E_M obeying fermionic commutation relations with the corresponding creation operator

$$\gamma_M^\dagger = \int \left[u_M(\mathbf{r}) \Phi_\uparrow^\dagger(\mathbf{r}) + v_M(\mathbf{r}) \Phi_\downarrow^\dagger(\mathbf{r})\right] d^3r. \tag{36.7}$$

The spectrum of the BdG equation is symmetric with respect to the Fermi energy and thus the eigenstates of the BdG equation occur in pairs. Labeling $M \to m, \alpha$ for states $E_M > 0$ and $M \to m, \beta$ for $E_M < 0$, respectively, one finds the relations $u_{m,\beta} = -v_{m,\alpha}^*$ and $v_{m,\beta} = u_{m,\alpha}^*$ for the eigenstates. Therefore, all quantities can be expressed solely using the α

state wave functions [Datta and Bagwell(1999)]. In the following we omit the α, β index of the eigenfunctions while they are still necessary for the quasiparticle operators. Hereafter –in the case of the eigenfunctions– the index m refers to the α states. For our further calculations it is convenient to transform to the excitation picture ($\alpha \to a$, $\beta \to b$) with $\gamma_{m\alpha} = \gamma_{ma}$ and $\gamma_{m\beta} = \gamma^\dagger_{mb}$, where all quasiparticle excitations vanish in the ground state. We can rewrite the order parameter in the basis given by the eigenfunctions, where it reads:

$$\begin{aligned}\Delta(\mathbf{r},t) = -g \sum_{m,n} & v^*_m(\mathbf{r}) u_n(\mathbf{r}) \left\langle \gamma^\dagger_{ma} \gamma_{na} \right\rangle \\ & + u_m(\mathbf{r}) u_n(\mathbf{r}) \left\langle \gamma_{mb} \gamma_{na} \right\rangle \\ & - v^*_m(\mathbf{r}) v^*_n(\mathbf{r}) \left\langle \gamma^\dagger_{ma} \gamma^\dagger_{nb} \right\rangle \\ & + u_m(\mathbf{r}) v^*_n(\mathbf{r}) \left(\left\langle \gamma^\dagger_{nb} \gamma_{mb} \right\rangle - \delta_{mn} \right). \end{aligned} \tag{36.8}$$

This yields the well-known result for the ground state order parameter

$$\Delta_{GS}(\mathbf{r}) = g \sum_m u_m(\mathbf{r}) v^*_m(\mathbf{r}), \tag{36.9}$$

which has to be solved self-consistently with Eq. (36.6) [De Gennes(1989)]. Focusing on the underlying physics, we exploit the Anderson approximation (A.A.) [Anderson(1959)], choosing the BdG wave functions $u_m(\mathbf{r})$ and $v_m(\mathbf{r})$ proportional to the one-particle wave functions of the confinement potential $\varphi_m(\mathbf{r})$, i.e.,

$$u_m(\mathbf{r}) = u_m \varphi_m(\mathbf{r}) \quad \text{and} \quad v_m(\mathbf{r}) = v_m \varphi_m(\mathbf{r}). \tag{36.10}$$

Here the amplitudes of the BdG wave functions u_m, v_m are obtained from the BdG Eq. (36.6) and read

$$u_m = \sqrt{\frac{1}{2}\left(1 + \frac{\varepsilon_m}{E_m}\right)} \qquad v_m = \sqrt{\frac{1}{2}\left(1 - \frac{\varepsilon_m}{E_m}\right)} \tag{36.11}$$

with the quasiparticle energies given by

$$E_m = \sqrt{\varepsilon_m^2 + \Delta_m^2} \tag{36.12}$$

and the one-particle energies ε_m given by Eq. (36.2). Applying the Anderson approximation to Eq. (36.6) additionally yields $\Delta_{mn} = \langle m | \Delta(\mathbf{r}) | n \rangle = \Delta_m \delta_{mn}$, where $|m\rangle$ are the one-particle eigenfunctions. The Anderson approximation has been tested in several nanostructured geometries and no qualitative deviations have been found [Chen et al.(2009)Chen, Croitoru, Shanenko, and Peeters]. It is applied to all our calculations.

From Eqs. (36.6) and (36.9) we obtain the well-known BCS-like self-consistency equation, also referred to as gap equation. The ground state order parameter in the state $|m\rangle$ is given by

$$\Delta_m = -\sum_{m'} V_{mm'} \frac{\Delta_{m'}}{2E_{m'}}, \tag{36.13}$$

Here $V_{mm'}$ is the interaction matrix element

$$V_{mm'} = -g \int |\varphi_m(\mathbf{r})|^2 \, |\varphi_{m'}(\mathbf{r})|^2 \, d^3r, \tag{36.14}$$

which exhibits maxima for states at the subband minimum (i.e., states with low m_z).

The contact interaction used here leads to a well-know ultraviolet divergence in the summation over all states, i.e., in Eq. (36.13), which can be regularized by applying a scattering length regularization [Bloch *et al.*(2008)]. This has been established for the homogeneous gap equation and subsequently extended to the inhomogeneous gap equation (36.13), where Ref. [Bruun et al.(1999)] gives a careful derivation for confined systems. However, Refs. [Bruun and Heiselberg(2002), Shanenko *et al.*(2012)] state that a much simpler regularization is sufficient since the results are not sensitive to the details of the method used. The corresponding regularized gap equation reads

$$\Delta_m = -\frac{1}{2}\sum_{m'} V_{mm'}\Delta_{m'}\left(\frac{1}{E_{m'}} - \frac{1}{\varepsilon_{m'}+E_{\mathrm{F}}}\right), \tag{36.15}$$

which can be rewritten as a multiplication by a factor

$$\chi_{m'} = \left(1 - \frac{E_{m'}}{\varepsilon_{m'}+E_{\mathrm{F}}}\right). \tag{36.16}$$

Thus, each quasiparticle state m' is weighted by a factor $\chi_{m'}$. In order to extend this in a consistent manner to the nonequilibrium case, in which Δ deviates from its ground state value Δ_{GS} and is determined by the full Eq. (36.8), the same procedure has to be applied to the nonequilibrium version of Eq. (36.15) as will be discussed below [see Eq. (36.26)] [14].

Figure 36.1: (Color online) One-particle and quasiparticle properties of a resonant system (red) and an off-resonant system (blue). (a) Schematic one-particle states for two differently sized systems (i.e., different $\omega_\perp$) (b) quasiparticle energy vs. one-particle energy, solid lines: subband $(0,0)$, dashed lines: subbands $(1,0)$, $(0,1)$ (higher subbands are not visible), subbands of resonant system are shifted up by 3 peV for better visibility, and (c) quasiparticle density of states.

Figure **??** shows the dependence of the quasiparticle energies E_m on the one-particle energies ε_m. For all subbands crossing the Fermi energy finite minima of the quasiparticle energy evolve at $\varepsilon = 0$. Subbands with the minimum close to the Fermi energy (i.e., $\varepsilon_{m_x,m_y,m_z=0} \approx$

[14] This regularization introduces an additional dependency on the quasiparticle energy E_m and thus the ground state gap Δ_m in the dynamical equations. Since –in the dynamical case– the time-dependent gap $\Delta_m(t)$ differs from the ground state value this may seem unusual. However, we have checked that the method is not sensitive on the detailed parameters used, e.g., on using one constant value for all Δ_m in order to omit the newly introduced dependence. Thus the proposed regularization scheme is a suitable extension since the physical behavior of the system remains unchanged.

0) exhibit a larger Δ_m due to the larger interaction matrix elements $V_{mm'}$. This leads to a rather shallow parabolic-like minimum in these subbands and, thus, to a maximum in the corresponding density of states (see Fig. **??**). Subbands showing this feature will be called resonant in the following and systems in which such a subband exists are referred to as resonant systems. The behavior can be compared to the well-known parabolic-like dispersion relation in the homogeneous BCS theory with $\Delta = \text{const}$. A similar behavior has also been obtained for nanostructured superconducting systems [Shanenko et al.(2010)Shanenko, Croitoru, Vagov, and Peeters]. Overall the density of states combined with the interaction matrix $V_{mm'}$ leads to the quantum size oscillations of the order parameter upon changing the lateral size of the system (i.e., $\omega_\perp$) found in Ref. [Shanenko *et al.*(2012)].

36.2.2 Dynamics

In this paper we consider an excitation of the Fermi gas by a quantum quench, i.e., a sudden change of the coupling constant $\tilde{g} \to g$. Since in the region of a Feshbach resonance the coupling constant strongly depends on an external magnetic field $\mathbf{B}$, this can be achieved experimentally by rapidly switching $\mathbf{B}$ from an initial value $\mathbf{B}_i$ to a final value $\mathbf{B}_f$. We assume that this switching process occurs on a time scale much faster than the characteristic time scale of the order parameter dynamics, such that the excitation can be taken to be instantaneous. This assumption is realistic since fast linear magnetic ramps with rates of 240 G/ms are experimentally available [Chin et al.(2006)Chin, Miller, Liu, Stan, Setiawan, Sanner, Xu, and Ketterle] while the timescale of the gap dynamics is of the order of $\tau_\Delta \approx \hbar/\Delta_{GS} \sim 1\,\text{ms}$ [Barankov *et al.*(2004)]. The excitations considered in this paper require a shift in the magnetic field of a few gauss, which indeed can be assumed to be instantaneous on the typical ms time scale in ultracold Fermi gases. In this case during the switching of the magnetic field the state of the system remains unchanged.

As usual, the dynamics of a quantum mechanical system can be described in different basis systems, which from a mathematical point of view are all equivalent. In our case, to calculate the dynamics of the order parameter after a quench from the *initial* system ($\tilde{u}_m(\mathbf{r})$, $\tilde{v}_m(\mathbf{r})$, $\tilde{g}$) to the *final* system ($u_m(\mathbf{r})$, $v_m(\mathbf{r})$, g) we choose a time-independent basis rather than remaining in the diagonal basis. For convenience we take the basis corresponding to the eigenstates of the system after the switching, i.e., to the coupling constant g. All our calculations are thus carried out in the basis $u_m(\mathbf{r})$, $v_m(\mathbf{r})$.

The initial state, which is characterized by the ground state order parameter $\tilde{\Delta}_{GS}(\mathbf{r})$, corresponding to the coupling constant $\tilde{g}$ and the basis functions $\tilde{u}_m(\mathbf{r})$, $\tilde{v}_m(\mathbf{r})$, has to be expressed in terms of the basis $u_m(\mathbf{r})$, $v_m(\mathbf{r})$, which in particular gives rise to non-vanishing quasiparticle excitations in this basis. Since the confinement potential is unchanged, also the corresponding one-particle wave functions $\varphi_m(\mathbf{r})$ remain unchanged. According to Eq. (36.10), in the present case only the BdG amplitudes u_m and v_m change while the spatial shapes of $u_m(\mathbf{r})$ and $v_m(\mathbf{r})$ remain unchanged. Therefore, all orthogonality relations are preserved and only diagonal quasiparticles are populated. For the initial values of the normal and anomalous expectation values, respectively, one finds

$$\left\langle \gamma^\dagger_{ma}\gamma_{ma} \right\rangle\Big|_{t=0} = \left(v_m\tilde{u}_m - u_m\tilde{v}_m\right)^2 \tag{36.17}$$

$$\left\langle \gamma^\dagger_{ma}\gamma^\dagger_{mb} \right\rangle\Big|_{t=0} = \left(v_m\tilde{u}_m - u_m\tilde{v}_m\right)\left(v_m\tilde{v}_m + u_m\tilde{u}_m\right). \tag{36.18}$$

In addition

$$\left\langle \gamma^\dagger_{ma}\gamma_{ma} \right\rangle = \left\langle \gamma^\dagger_{mb}\gamma_{mb} \right\rangle, \left\langle \gamma_{mb}\gamma_{ma} \right\rangle = \left\langle \gamma^\dagger_{ma}\gamma^\dagger_{mb} \right\rangle^* \tag{36.19}$$

holds for all times $t \geq 0$.

Since the instantaneous order parameter $\Delta(t)$ deviates from the ground state value of the final system Δ_{GS}, the Hamiltonian in the basis $u_m(\mathbf{r})$, $v_m(\mathbf{r})$ becomes non-diagonal depending on the difference $(\Delta(t) - \Delta_{GS})$. It thus becomes explicitly time dependent according to

$$\begin{aligned} H_{BdG} = &\sum_m E_{ma}\gamma^\dagger_{ma}\gamma_{ma} - E_{mb}\left(1 - \gamma^\dagger_{mb}\gamma_{mb}\right) \\ &+ \sum_{m,n}\left[(\Delta - \Delta_{GS})_{u^*_m v_n} + (\Delta^* - \Delta^*_{GS})_{v^*_m u_n}\right]\gamma^\dagger_{ma}\gamma_{na} \\ &+ \sum_{m,n}\left[(\Delta - \Delta_{GS})_{u^*_m u^*_n} - (\Delta^* - \Delta^*_{GS})_{v^*_m v^*_n}\right]\gamma^\dagger_{ma}\gamma^\dagger_{nb} \\ &- \sum_{m,n}\left[(\Delta - \Delta_{GS})_{v_m v_n} - (\Delta^* - \Delta^*_{GS})_{u_m u_n}\right]\gamma_{mb}\gamma_{na} \\ &- \sum_{m,n}\left[(\Delta - \Delta_{GS})_{v_m u^*_n} + (\Delta^* - \Delta^*_{GS})_{u_m v^*_n}\right]\left(1 - \gamma^\dagger_{mb}\gamma_{nb}\right). \end{aligned} \tag{36.20}$$

with

$$\begin{aligned} (\Delta - \Delta_{GS})_{u^*_m v_n} &= \int u^*_m(\mathbf{r})\left[\Delta(\mathbf{r},t) - \Delta_{GS}(\mathbf{r})\right] v_n(\mathbf{r}) d^3r \\ &\underbrace{=}_{\text{A.A.}} (\Delta - \Delta_{GS})_m\, u_m v_m \delta_{mn}. \end{aligned} \tag{36.21}$$

Here, the Anderson approximation has been applied to the dynamical equations as proposed in Ref. [Zachmann et al.(2013)], yielding $(\Delta - \Delta_{GS})_{mn} \approx (\Delta - \Delta_{GS})_m\, \delta_{mn}$. The time evolution of the system is thus described by the time-dependent quasiparticle expectation values. The corresponding equations of motion can be obtained via Heisenberg's equation of motion.

For the considered instantaneous change of the coupling constant only diagonal expectation values are excited. The required equations of motion read

$$i\hbar\frac{d}{dt}\left\langle\gamma^\dagger_{ma}\gamma_{ma}\right\rangle = a_m\left\langle\gamma^\dagger_{ma}\gamma^\dagger_{mb}\right\rangle^* - a^*_m\left\langle\gamma^\dagger_{ma}\gamma^\dagger_{mb}\right\rangle \tag{36.22}$$

$$\begin{aligned} i\hbar\frac{d}{dt}\left\langle\gamma^\dagger_{ma}\gamma^\dagger_{mb}\right\rangle = &-2\,E^{(\mathrm{ren})}_m\left\langle\gamma^\dagger_{ma}\gamma^\dagger_{mb}\right\rangle \\ &+ a_m\left(1 - 2\left\langle\gamma^\dagger_{ma}\gamma_{ma}\right\rangle\right), \end{aligned} \tag{36.23}$$

where

$$E^{(\mathrm{ren})}_m = E_m + 2\,u_m v_m \mathrm{Re}\left[(\Delta - \Delta_{\mathrm{GS}})_{\mathrm{m}}\right], \tag{36.24}$$

$$a_m = v^2_m(\Delta - \Delta_{GS})_m - u^2_m(\Delta - \Delta_{GS})^*_m, \tag{36.25}$$

and

$$\begin{aligned} (\Delta - \Delta_{GS})_m = -\sum_k \Bigg[&2v_k u_k\left\langle\gamma^\dagger_{ka}\gamma_{ka}\right\rangle \\ &+ u^2_k\left\langle\gamma_{kb}\gamma_{ka}\right\rangle - v^2_k\left\langle\gamma^\dagger_{ka}\gamma^\dagger_{kb}\right\rangle\Bigg] V_{mk}\chi_k. \end{aligned} \tag{36.26}$$

In Eq. (36.26) again the regularization factor χ_k has been introduced. Equations (36.22)-(36.26) represent a finite set of coupled ordinary differential equations that we solve numerically.

It is interesting to note that the evolution of the anomalous expectation values [Eq. (36.23)] corresponds to a set of harmonic oscillators with energies approximately given by $2E_m$ [first term

in Eq. (36.23)] while Eq. (36.22) as well as the second terms of Eqs. (36.23) and (36.24) contain nonlinear couplings to all other oscillators via the factor $(\Delta - \Delta_{GS})_m$, which vanishes when the order parameter agrees with its ground state value. We will come back to this separation into linear and nonlinear terms in Sec. 36.3.2.

36.3 Results

In the following the temporal evolution of the amplitude of the spatially averaged gap

$$\bar{\Delta} = \frac{1}{V} \int d^3r \Delta(\mathbf{r}) \tag{36.27}$$

will be shown and analyzed for different system parameters. Here, the normalization volume V is set to $V = l_x l_y l_z$ with l_α being the oscillator length in α direction [15].

In order to concentrate on the physics we will start our analysis by investigating a very small system: All the main features occurring in the dynamics of larger, experimentally accessible confinements arise in small systems, too, but with a strongly reduced degree of numerical complexity. Thus, the dynamics of the superfluid gap will at first be explained on the basis of small systems. The results for a larger system will be shown afterwards.

36.3.1 Full model, small system

An exemplary result for the gap dynamics after a quantum quench, obtained by changing the scattering length from $\tilde{a} = -140\,\text{nm}$ to $a = -135\,\text{nm}$ for a system with the confinement frequencies $f_\perp = \omega_\perp/2\pi = 11.2\,\text{kHz}$ and $f_\parallel = \omega_\parallel/2\pi = 240\,\text{Hz}$, is shown in Fig. 36.2. The Fermi energy has been set to $E_F = 100\,\hbar\omega_\parallel$ yielding $1/(k_F a) \approx -1$ according to Ref. [Shanenko *et al.*(2012)]. As can be seen the amplitude of the gap shows an initial drop corresponding to the decreased coupling and thus decreased ground state gap. Afterwards a smoothly damped oscillation around the new ground state value of the gap occurs, which after a certain transition time t_c turns into an irregular, rather chaotic oscillation. Here t_c is defined as the time of the first deviation [16] from a smooth oscillation.

The inset of Fig. 36.2 suggests that this irregular oscillation after t_c is the result of a superposition of several frequencies. Here a segment of the Fourier spectrum of the gap dynamics is shown. The spectrum is composed of a series of sharp peaks in the range of 8.9 peV to about 30 peV, which can each be assigned to a corresponding quasiparticle state, i.e., $\hbar\omega_m \approx 2E_m$. The main peaks at the lower end of this series correspond to the frequency of the initial damped oscillation and to the dominant frequencies of the irregular dynamics afterwards. Their values are given by the quasiparticle energies closest to the Fermi level. These lie in the vicinity of a quasiparticle subband minimum. The corresponding frequencies are thus given by $\hbar\omega_m \approx 2\Delta_{(m_x,m_y,m_{\min})}$, where $m_{\min}$ is the z quantum number referring to the state with minimal quasiparticle energy, i.e., the state at the Fermi energy. The other peaks belong to higher quasiparticle states and decrease continuously with increasing energy.

While the qualitative picture of the gap dynamics is the same for all investigated systems, the quantitative values of the features mentioned above crucially depend on the system parameters. On the one hand the ground state gap and thus the mean value of the oscillation and its frequency contributions strongly depend on the perpendicular confinement $f_\perp$ (due to the size-dependent superfluid resonances [Shanenko *et al.*(2012)]) and on the scattering length a. The

[15] We want to remark that this definition is only for illustrative purposes. The normalization only scales the gap dynamics but does not enter the dynamical equations.

[16] In the numerical analysis the transition time t_c is assumed to be reached if the distance between two adjacent maxima shows a deviation of more then 50% compared to the averaged distance of all preceding maxima.

transition time, on the other hand, increases with decreasing parallel confinement $f_\parallel$ –i.e., with increasing system length– as can be seen in Fig. 36.3. Here the gap dynamics is shown for the same perpendicular confinement and excitation as in Fig. 36.2 but for increasing system length, i.e., decreasing $f_\parallel$ (from bottom to top). Figure 36.3 shows that the transition time t_c moves to larger times as the length of the system increases. In addition a revival of the oscillation can be seen for the two largest systems with $f_\parallel = 96\,\mathrm{Hz}$ and $f_\parallel = 80\,\mathrm{Hz}$, which for smaller systems would occur after the breakdown.

A quantitative analysis of the transition time for different perpendicular confinements $f_\perp$ over a wide range of parallel confinements is shown in Fig. 36.4. Here the transition time is plotted against the inverse parallel confinement frequency $\frac{1}{f_\parallel} \sim l_z^2$. One can see that t_c is independent of the size of the gas in the x-y-plane, since the values for every $f_\perp$ lie on the same curve. It is only influenced by the confinement in z-direction, where a linear dependence on $\frac{1}{f_\parallel}$ can be observed. This is in full agreement with the behavior found for superconducting quantum wires [Zachmann et al.(2013)]. Even a beating-like pattern was found for thin quantum wires which corresponds to the revivals seen in Fig. 36.3.

To investigate the smooth regime of the gap dynamics Fig. 36.5 shows calculations for a rather large system length and two different perpendicular confinements. For this case of large lengths in Ref. [Zachmann et al.(2013)] it was found that thick quantum wires exhibit a damping of the gap oscillation given by a power law $\sim t^{-\alpha}$ with $\alpha = 3/4$ for resonant systems and $\alpha = 1/2$ for off-resonant ones. However, thin quantum wires were found to differ from this power law showing a more irregular oscillation but still a rather fast decay of the gap oscillation when resonant subbands are present.

Figure 36.5 shows that this situation applies to ultracold Fermi gases as well. Here a calculation of the gap dynamics is shown for a resonant system (upper, red curve), which is again characterized by the same perpendicular confinement as in Figs. 36.2 and 36.3, as well as for a system far away from resonance (lower, blue curve; $f_\perp = 12.9\,\mathrm{kHz}$). The excitation is the same as before and the parallel confinement frequency is chosen as $f_\parallel = 48\,\mathrm{Hz}$, which corresponds to a rather long cloud. It can be seen that both systems show an initial decay of the gap oscillation until a minimal amplitude is reached. In the resonant case this initial decay is rather strong and fast. Here, the amplitude of the oscillation falls close to zero. Afterwards it exhibits revivals until the smooth oscillation breaks down. In contrast, the off-resonant system shows an only moderate, comparatively slow decay of the oscillation, which after a short time exhibits a nearly constant amplitude. Thus, on the one hand the decay of the oscillation is much stronger in the resonant than in the off-resonant case. On the other hand revivals and a beating like pattern occur for the resonant case before the breakdown while systems far away from resonance exhibit a nearly constant oscillation amplitude. The inset of Fig. 36.5 shows that these different temporal evolutions correspond to different Fourier spectra. Here the Fourier transforms of the resonant (positive y-axis) and the off-resonant (negative y-axis) system are shown. The resonant spectrum is composed of several strong components at the lower end and weaker peaks towards higher energies. In contrast, the off-resonant spectrum –it is shifted due to the smaller gap– contains only one dominant frequency part at low energies while the rest of the spectrum is strongly suppressed. Thus, resonant systems on the one hand correspond to spectra with several strong modes and weaker high-frequency parts. Off-resonant systems on the other hand are strongly dominated by one single mode with only minor contributions from other energies.

The features described above –the damped oscillation, the breakdown, and the irregular dynamics– will be explained in the following section. As mentioned before, they also occur in BCS-superconductors in a very similar way, where cigar-shaped Fermi gases correspond to thin, short quantum wires. The following explanations therefore apply to both ultracold Fermi gases and confined BCS-superconductors thus showing the close relation between these systems.

36.3.2 Linearized dynamics, small system

To analyze the mechanisms underlying the gap dynamics and its features we introduce a linearized set of equations of motion. This can be derived by neglecting all terms of second and higher order in the quasiparticle excitations (this is valid due to the weak excitation investigated in this paper, which leads to $\left|\langle\gamma^\dagger_{ma}\gamma_{mb}\rangle\right| \ll \left|\langle\gamma^\dagger_{ma}\gamma^\dagger_{mb}\rangle\right| \ll 1$). In doing so, Eq. (36.22) can be neglected since by inserting Eqs. (36.25) and (36.26) into this equation only terms of at least second order in the normal excitations or products of anomalous and normal excitations contribute, which are to be neglected in the linearized case.

Equation (36.23) reduces to a closed set of equations for the anomalous excitations which, by performing the derivative in time of one equation and inserting the other, can be separated into its real and imaginary parts leading to

$$\begin{aligned}\frac{d^2}{dt^2}\langle\gamma^\dagger_{ma}\gamma^\dagger_{mb}\rangle + \omega_m^2\langle\gamma^\dagger_{ma}\gamma^\dagger_{mb}\rangle &= \\ \sum_{k\neq m}\left[A_{km}\mathrm{Re}(\langle\gamma^\dagger_{ka}\gamma^\dagger_{kb}\rangle) + iA_{mk}\ \mathrm{Im}\ (\langle\gamma^\dagger_{ka}\gamma^\dagger_{kb}\rangle)\right]\chi_k,\end{aligned} \tag{36.28}$$

where again all terms nonlinear in the quasiparticle expectation values have been neglected. This equation describes a set of linearly coupled harmonic oscillators with the uncoupled frequencies

$$\omega_m = \sqrt{\left(\frac{2E_m}{\hbar}\right)^2 - A_{mm}\chi_m} \tag{36.29}$$

and

$$A_{km} = \frac{2}{\hbar^2}V_{km}\left(E_k + \frac{\varepsilon_m\varepsilon_k}{E_k}\right) - \frac{1}{\hbar^2}\sum_l \frac{\varepsilon_k}{E_k}\frac{\varepsilon_l}{E_l}V_{ml}V_{lk}\chi_l. \tag{36.30}$$

The coupling strength of the oscillators A_{km} with $k \neq m$ therein is weak due to the –in this case– mostly small matrix elements V_{km}. The shift of the eigenfrequencies of the coupled system [Eq. (36.28)] with respect to the uncoupled frequencies [Eq. (36.29)] should therefore be small, as should be the shift of the uncoupled frequencies compared to the bare ones $2E_m/\hbar$.

Figure 36.6 shows the dynamics of the BCS gap obtained from Eq. (36.28) compared to the full dynamics [17]. The parameters correspond to the system shown in Fig. 36.2 and are exemplary for all investigated systems. The linearized equations clearly reproduce the full dynamics and all its features in very good agreement. The inset shows that the positions as well as the strengths of most of the frequencies in the Fourier spectra are well described by this approximation. The spectrum corresponding to the full equations of motion (upper, red curve) and the one corresponding to the linearized equation (lower, black curve) show only slight differences. Only one low lying weak frequency component close to zero, which is present in the linearized version (outside the range shown in the inset of Fig. 36.6) as well as weak side peaks that occur adjacent to every main peak in the full dynamics are not fully reproduced. The latter can be attributed to the nonlinear couplings. Their influence on the dynamics, however, is obviously negligible.

The main features of the gap dynamics can thus be explained on the basis of Eq. (36.28), which can be solved analytically. The analytic solution can be expressed in terms of a linear superposition of simple (co)sine-oscillators. The corresponding frequencies are determined by

[17] In case of possible divergences of the linearized solution one can artificially lower the coupling strength in the dynamical calculations in order to prevent them. This means that while the ground state is still calculated with the coupling g, in the dynamical calculations (i.e., in Eqs. (36.28),(36.29) and (36.30)) the coupling has to be slightly lowered by a small amount, i.e., $g \to g' \lesssim g$. It should be mentioned that the full (nonlinear) equations never lead to divergences.

the uncoupled oscillator frequencies ω_m and the coupling A_{km}. They are thus completely fixed by the the system parameters and the BCS gap, but they are independent of the initial conditions. In the considered case of weak coupling the coupled spectrum is only slightly shifted compared to the uncoupled frequencies. The eigenfrequencies of the coupled dynamics are thus approximately given by twice the quasiparticle energies, as observed above.

The amplitudes of the different eigenmodes of the coupled system are determined by the initial values of the dynamics and thus depend on the details of the excitation. In general, one observes that each of the quasiparticle oscillators $\langle\gamma^\dagger_{ma}\gamma^\dagger_{mb}\rangle$ carries strong contributions from oscillators in the vicinity of its uncoupled frequency ω_m and in areas of a high density of states. These areas are located close to the minimum of a quasiparticle subband (see section 36.2), i.e., near $\hbar\omega = 2\Delta_{(m_x,m_y,m_{\min})}$. In addition one observes that the contributions from each oscillator to the lowest energies are mostly in phase while these to higher energies are more and more out of phase. A sum of all quasiparticle oscillators, which according to Eq. (36.8) yields the BCS gap, thus leads to dynamics dominated by low-lying frequencies at a quasiparticle band minimum. These can be understood as cumulative peaks created by collective oscillations of all quasiparticles in the system.

All spectra obtained by the full equations indeed show such a spectrum with rather dense, strong frequency contributions near $\hbar\omega = 2\Delta_{(m_x,m_y,m_{\min})}$ and relatively widespread suppressed frequency contributions at higher energies.

The explanations given above have shown that the gap dynamics can be understood as a linear superposition of quasiparticle oscillations which themselves are given by a sum of simple oscillators. To finally explain the main features of the gap dynamics in the time domain –the damping, the transition and the irregular oscillation– one can therefore use an even simpler picture of a set of independent cosine-oscillators with the frequencies ω_m [18].

Due to the abrupt excitation all these oscillators start in phase at maximum deflection. Starting to oscillate they will soon dephase. A sum of all oscillators (i.e., the gap) thus performs a damped oscillation (cf. Ref. [Yuzbashyan *et al.*(2006)]). Here systems with several strong frequency contributions –i.e., resonant systems– show a rather fast and persistent damping since a large part of the cumulative amplitude is able to dephase. Off-resonant systems in contrast exhibit only a slight decay of the oscillation since the main part of the cumulative oscillation is carried by one single mode.

Proceeding in time the damped oscillation continues until all oscillators are completely dephased and the amplitude of the oscillation is minimal. Then the oscillators start to rephase, the amplitude grows and the oscillation reappears (see Fig. 36.3; for off-resonant systems this effect is strongly suppressed since one single frequency dominates the spectrum). As soon as the first adjacent pair of oscillators goes back in phase again this beating-like pattern is interrupted. At this moment a spike in the cumulative oscillation indicates the breakdown of the regular oscillation and thus the transition time t_c. Afterwards all other frequencies rephase successively and create a rapid sequence of spikes which leaves a picture of an irregular oscillation.

The preceding argumentation suggests that the time of this breakdown should be inversely proportional to the maximum spacing of adjacent quasiparticle energies. Strictly speaking the transition times are found to be determined by adjacent quasiparticles from the same subband [Zachmann et al.(2013)], i.e.,

$$t_c \approx \frac{2\pi\hbar}{2\delta E_{\max}} \approx \frac{\pi\hbar}{(\varepsilon_{m_x,m_y,m_z+1} - \varepsilon_{m_x,m_y,m_z})} = \frac{1}{2f_\parallel}. \tag{36.31}$$

It thus increases $\sim 1/f_\parallel$ with decreasing parallel confinement frequency since the energy spacing of the atomic spectrum then decreases. This is in full agreement with the relation found before.

[18] The main features can actually be reproduced by plotting a weighted sum of such cosine functions.

In fact, Eq. (36.31) gives exactly the linear curve shown in Fig. 36.4. This indicates that the breakdown of the smooth initial oscillation of the BCS gap indeed is due to adjacent frequencies rephasing in time.

36.3.3 Full model, large system

As a final example the gap dynamics of a rather large system with $f_\parallel = 50\,\mathrm{Hz}$ and $f_\perp \approx 1.03\,\mathrm{kHz}$ again after a sudden change of the scattering length from $-140\,\mathrm{nm}$ to $-135\,\mathrm{nm}$ is shown in Fig. 36.7. This corresponds to a gas with $l_\parallel \approx 5.4\,\mu\mathrm{m}$ and $l_\perp \approx 1.2\,\mu\mathrm{m}$ and is thus on an experimentally accessible length-scale [Bartenstein et al.(2004)Bartenstein, Altmeyer, Riedl, Jochim, Chin, Denschlag, and Grimm]. The Fermi energy is chosen as $E_F = 250\hbar\omega_\parallel$ which corresponds to $N = 12258$ atoms in the trap.

Figure 36.7 shows that the qualitative behavior of the gap dynamics is the same as for the smaller systems: A slowly decaying oscillation of the gap occurs. The transition time $t_c = 10\,\mathrm{ms}$ calculated from Eq. (36.31) is in good agreement with a small bump in the curve at $t \approx 9.7\,\mathrm{ms}$, the first deviation from a smooth oscillation. Afterwards more and more deviations occur and the gap dynamics becomes successively irregular.

When looking at the damping of the gap oscillation one can see that although the system is resonant –one subband is close to the Fermi energy– the strength of the damping is situated somewhere between the resonant and the off-resonant case of Fig. 36.5. This is due to the weak perpendicular confinement and therefore large number of states contributing to the condensate: Compared to the overall number of relevant states the resonant ones give only a small contribution to the gap. For larger systems the resonances are thus less pronounced [Shanenko *et al.*(2012)].

The Fourier transform in the inset of Fig. 36.7 shows a familiar picture, too, with strong contributions at the lower end of the spectrum and successively decaying peaks towards higher energies. The difference with respect to the spectra shown before is on the one hand the high density of peaks for the large system. This is due to the weaker confinement and thus higher density of bare and quasiparticle states. On the other hand the gap in the Fourier spectrum and thus the main frequency of the oscillation is comparatively small. This is because of the larger ratio of the perpendicular to the parallel length $l_\perp/l_\parallel$. With the Fermi energy fixed at $E_F = 250\hbar\omega_\parallel$ this leads to a comparatively low particle density of the trapped gas and thus a weaker condensate and a smaller gap.

36.4 Conclusion

In conclusion, we have calculated the Higgs amplitude dynamics in the BCS phase of an ultracold ^{6}Li gas confined in a cigar-shaped trap. The dynamics is induced by a quantum quench resulting from a sudden change of an external magnetic field. We have shown that the amplitude of the spatially averaged gap performs a damped oscillation breaking down after a certain time t_c, which is determined by the parallel confinement frequency $f_\parallel$, i.e., by the length of the cloud. Afterwards a rather irregular oscillation involving many different frequencies occurs.

We have investigated the influence of the confinement on the gap dynamics and the impact of the size-dependent superfluid resonances on its qualitative behavior. It turned out that in the case of a resonant system, i.e., a system where the Fermi energy is close to a subband minimum, the dynamics of the order parameter exhibits a strong damping and, for sufficiently long systems, a revival before eventually the irregular regime is reached. In contrast, in an off-resonant system the damping is much less pronounced and the oscillation before the transition to the irregular regime is mainly determined by a single frequency.

By analyzing the linearized version of the equations of motion for the quasiparticle excitations we were able to interpret the observed features of the dynamics. It turned out that for the excitations studied in this paper the linearized equations well reproduce the dynamical behavior of the gap, except for some slight details resulting from the nonlinearities in the full equations of motion. From the linearized model it becomes evident that the system approximately behaves like a set of weakly coupled harmonic oscillators. The frequencies as well as the couplings of these oscillators are completely determined by the system parameters after the excitation while the amplitudes of the different eigenmodes depend on the details of the excitation. The analysis revealed in particular that the transition time to the irregular dynamics is directly related to the energy separation of the one-particle energies while the differences between resonant and non-resonant systems is caused by the different densities of states and coupling efficiencies close to the subband minima.

Figure 36.2: (Color online) Dynamics of the spatially averaged gap after a sudden change of the scattering length from $-140\,\mathrm{nm}$ to $-135\,\mathrm{nm}$; inset: Fourier transform of the gap dynamics. The confinement frequencies are $f_\perp = 11.2\,\mathrm{kHz}$ and $f_\parallel = 240\,\mathrm{Hz}$.

Figure 36.3: (Color online) Dynamics of the spatially averaged gap showing the change of the transition time (i.e., the time of the first deviation from a smooth oscillation) for a decreasing parallel confinement frequency (from bottom to top). The perpendicular confinement frequency is $f_{\perp} = 11.2\,\mathrm{kHz}$. The arrows mark the transition time t_c.

Figure 36.4: (Color online) Transition times of three different perpendicular confinements in dependence of the inverse parallel confinement frequency $\frac{1}{f_\parallel} \sim l_z^2$ compared to the prediction from the linearized theory (dashed curve).

Figure 36.5: (Color online) Dynamics of the averaged gap for a resonant ($f_\perp = 11.2\,\mathrm{kHz}$; red curve) and an off-resonant system ($f_\perp = 12.9\,\mathrm{kHz}$; blue curve); in both cases $f_\parallel = 48\,\mathrm{Hz}$. Inset: Fourier spectra.

Figure 36.6: (Color online) Gap dynamics derived from the linearized equation of motion (black) compared to the full dynamics (red); inset: Fourier transforms. The confinement frequencies are $f_\perp = 11.2\,\mathrm{kHz}$ and $f_\parallel = 240\,\mathrm{Hz}$.

Figure 36.7: (Color online) Gap dynamics for a large system after a sudden change of the scattering length from -140 nm to -135 nm; inset: Fourier transform of the gap dynamics. The confinement frequencies are $f_{\perp} = 1.03$ kHz and $f_{\parallel} = 50$ Hz.

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37 Higgs mode in a strongly interacting fermionic superfluid by Behrle, Harrison et al.

37.0.1 Main theory

Higgs and Goldstone modes are possible collective modes of an order parameter upon spontaneously breaking a continuous symmetry. Whereas the low-energy Goldstone (phase) mode is always stable, additional symmetries are required to prevent the Higgs (amplitude) mode from rapidly decaying into low-energy excitations. In high-energy physics, where the Higgs boson[1] has been found after a decades-long search, the stability is ensured by Lorentz invariance. In the realm of condensed–matter physics, particle–hole symmetry can play this role[2] and a Higgs mode has been observed in weakly-interacting superconductors[3, 4, 5]. However, whether the Higgs mode is also stable for strongly-correlated superconductors in which particle–hole symmetry is not precisely fulfilled or whether this mode becomes overdamped has been subject of numerous discussions[6, 7, 8, 9, 10, 11]. Experimental evidence is still lacking, in particular owing to the difficulty to excite the Higgs mode directly. Here, we observe the Higgs mode in a strongly-interacting superfluid Fermi gas. By inducing a periodic modulation of the amplitude of the superconducting order parameter Δ, we observe an excitation resonance at frequency $2\Delta/h$. For strong coupling, the peak width broadens and eventually the mode disappears when the Cooper pairs turn into tightly bound dimers signalling the eventual instability of the Higgs mode.

Spontaneous symmetry breaking occurs when an equilibrium state exhibits a lower symmetry than the corresponding Hamiltonian describing the system. The system then spontaneously picks one of the energetically degenerate choices of the order parameter and due to the specific

energy landscape this process is accompanied by new collective modes. The typical picture, which exemplifies spontaneous symmetry breaking, uses a Mexican-hat shaped energy potential (see Figure 1a) that suggests the emergence of two distinct collective modes: the gapless "Goldstone mode", which is associated with long-wavelength phase fluctuations of the order parameter, and an orthogonal gapped mode, the "Higgs mode", which describes amplitude modulations of the order parameter. While Goldstone modes, such as phonons, appear necessarily when continuous symmetries are broken, stable Higgs modes are scarce, since decay channels might be present. The best-known example of a Higgs mode appears in the Standard Model of particle physics where this mode gives elementary particles their mass [1].

In the non-relativistic low-energy regime usually encountered in condensed-matter physics, the existence of a stable Higgs mode cannot be taken for granted[6]. However, under certain conditions, other symmetries, such as particle-hole symmetry, can play the role of Lorentz invariance and induce a stable Higgs mode. A notable example of a low-energy particle-hole symmetric theory hosting a stable Higgs mode is the famous Bardeen-Cooper-Schrieffer (BCS) Hamiltonian describing weakly interacting superconductors [2, 12]. Evidence for the Higgs mode has been found in conventional BCS superconductors [3, 4, 5]. However, experimental detections have been solely indirect as the Higgs mode does not couple directly to electromagnetic fields owing to the gauge invariance required for its existence. The far-reaching importance of the Higgs mode is further illustrated by its observation in a variety of specially tuned systems such as antiferromagnets [13], liquid ^{3}He [14], ultracold bosonic atoms near the superfluid/Mott-insulator transition [15, 16], spinor Bose gases [17], and Bose gases strongly coupled to optical fields [18]. In contrast, weakly-interacting Bose-Einstein condensates do not exhibit a stable Higgs mode [6, 10, 11].

In recent years, research has focused on advanced materials exhibiting superconductivity far beyond the conventional BCS description, such as cuprates, pnictides, and the unitary Fermi gas. Many of these materials are characterized by strong fermionic correlations. Even though in this context the existence of a Higgs mode has been the topic of theoretical debates [7, 8, 9, 10, 11], experimental evidence for the Higgs mode in systems exhibiting strong correlations between fermions is still absent.

Here, we spectroscopically excite the Higgs mode in a superfluid Fermi gas in the crossover between a weakly-interacting BCS superfluid and a Bose-Einstein condensate (BEC) of strongly-coupled dimers (Fig. 1). We induce a periodic modulation of the amplitude of the superconducting order parameter Δ and find an excitation resonance near twice the superconducting gap value. On the BCS side, the spectroscopic feature agrees with the theoretical expectation of the Higgs mode. On the BEC side of the crossover, we find strong broadening beyond the predictions of BCS theory and, eventually, the disappearance of the mode as predicted for a weakly-interacting BEC [6, 10, 11].

Our measurements are conducted in an ultracold quantum gas of $\sim 4 \times 10^6$ ^{6}Li atoms prepared in a balanced mixture of the two lowest hyperfine states $|1\rangle$ and $|2\rangle$ of the electronic ground state. The gas is trapped in a harmonic potential with frequencies of $(\omega_x, \omega_y, \omega_z) = 2\pi \times (91, 151, 235)\,\mathrm{Hz}$ and is subjected to a homogeneous magnetic field, which is varied in the range of $740 - 1000\,\mathrm{G}$ in order to tune the s-wave scattering length a near the Feshbach resonance located at $834\,\mathrm{G}$. This results in an adjustment of the interaction parameter of the gas in the range of $-0.8 \lesssim 1/(k_F a) \lesssim 1$, i.e. across the whole BCS/BEC crossover. The Fermi energy in the center of the gas is $E_F \simeq h \times (34 \pm 3)\,\mathrm{kHz}$ at each of the considered interaction strengths and sets the Fermi wave vector $k_F = \sqrt{8\pi^2 m E_F/h^2}$, where m denotes the mass of the atom and h is Planck's constant.

Excitation of the Higgs mode requires a scheme which couples to the amplitude of the order parameter rather than creating phase fluctuations or strong single-particle excitations. Previous theoretical proposals[19, 8, 20] for exciting the Higgs mode in ultracold Fermi gases have focused

Figure 37.1: Principle of the Higgs mode excitation. (a) Mexican hat potential of the free energy as a function of real and imaginary part of the complex order parameter Δ. The equilibrium state order parameter takes spontaneously one of the values at the energy minima. (b) We employ radiofrequency dressing of the paired superfluid by off-resonant coupling to an unoccupied state $|3\rangle$. This results in a periodic modulation of both the occupation of the state $|3\rangle$ (c) and the superconducting gap (d). Shown are numerical simulations for a coupling constant $1/(k_F a) = -0.6704$, $\hbar\Omega_R = 0.0353 E_F$ and $\hbar\delta = -0.3247 E_F$. (e) By adjusting the modulation frequency, we achieve an excitation of the Higgs mode in the Mexican hat.

on a modulation of the interaction parameter $1/(k_F a)$, however, experimentally only single-particle excitations have been observed from such a modulation[21]. We have developed a novel excitation scheme employing a radiofrequency (rf) field dressing the state $|2\rangle$ with the initially unoccupied hyperfine state $|3\rangle$ thereby modulating the pairing between the $|1\rangle$ and $|2\rangle$ states, see Figure 1b and c. Previous experiments investigating ultracold gases with rf spectroscopy[22, 23, 24, 25] have focused on studying single-particle excitations. To this end, there, the duration of the rf pulse τ had been chosen shorter than the inverse of the Rabi frequency Ω_R, such that the spectra could be interpreted in the weak-excitation limit using Fermi's golden rule. In contrast, here, we employ an rf drive far red-detuned from single-particle resonances in the interacting many-body system and in the long-pulse limit $\Omega_R \tau \gg 1$, in order to couple to the amplitude of the order parameter. To illustrate this, consider first an isolated two-level system of the $|2\rangle$ and the $|3\rangle$ state coupled by a Rabi frequency Ω_R with detuning δ from the resonance. The occupation probability of the atoms in the $|2\rangle$ state is $p_{|2\rangle} = 1 - \Omega_R^2/{\Omega'_R}^2 \sin^2(\Omega'_R t/2)$, i.e. the continuous rf dressing leads to a time-periodic modulation of the occupation of the $|2\rangle$-state with the effective Rabi frequency $\Omega'_R = \sqrt{\Omega_R^2 + \delta^2}$.

In the many-body problem of the BCS/BEC crossover, the situation is complicated by the dispersion of the (quasi-) particles and the presence of interactions. In particular, a continuum of excitations typically occurs above the energy of the lowest single-particle excitation to state $|3\rangle$ (see Figure 2a). Deep in the BCS regime, the continuum of excitations is related to the different momentum states and the excitation scheme can be approximated by coupling each occupied momentum state of the BCS quasi-particles in level $|2\rangle$ to the corresponding momentum state in state $|3\rangle$ since the rf dressing transfers negligible momentum. The effective Rabi frequency $\Omega'_{R,k} = \sqrt{\Omega_R^2 + \delta_k^2}$ and therefore the excitation probability becomes momentum dependent by the many-body detuning $\hbar\delta_k = \hbar\delta - E_k - \xi_k$, where ξ_k is the single-particle dispersion, and $E_k = \sqrt{\xi_k^2 + |\Delta|^2}$ is the quasi-particle dispersion and Δ is the s-wave superconducting order parameter. A red-detuned rf drive, as employed here, avoids resonant coupling to the single-particle excitations, however, still modulates off-resonantly the occupation of the excited states as shown in Figure 2b.

The off-resonant periodic modulation of the occupation of the state $|2\rangle$ with controllable

Figure 37.2: Illustration of the excitation scheme for one modulation frequency. (a) The radiofrequency field is red-detuned from the single-particle excitation of the interacting system. It creates an off-resonant excitation to the state $|3\rangle$ with a varying detuning for different momenta. (b) Time-evolution of the momentum-resolved occupation of the $|3\rangle$ state with momentum $\boldsymbol{k}$ for a fixed value of $\frac{1}{k_F a} = -0.63$, a Rabi frequency $\hbar\Omega_R = 0.038E_F$, and a detuning $\hbar\delta = -0.34E_F$. Blue: $|\boldsymbol{k}|/k_F = 0$, red: $|\boldsymbol{k}|/k_F = 0.8$, green: $|\boldsymbol{k}|/k_F = 1.1$. (c) Spectral weight of the momentum-resolved gap $A_{\mathbf{k}}(\omega)$ (see Methods). The circles indicate the Higgs mode, the stars mark the response to the modulation frequency and the crosses indicate the quasiparticle excitations at $2E_k$. The position of the star at $k = 0$ approximately represents the effective modulation frequency for the chosen parameters. (d) Fourier spectra (momentum integrated) of the occupation of the $|3\rangle$ state (red) and $|\Delta|$ (blue). The dashed line is the expected location of the Higgs mode at $2|\Delta|$. Subplots b, c, and d are for the same driving and detuning parameters.

Figure 37.3: Excitation spectra of the Higgs mode. (a) Excitation spectra of the Higgs mode for different interaction strengths, $1/(k_F a)$, as labeled in the figure. The different levels of background condensate fraction are due to the different $1/(k_F a)$. The solid lines shows the Gaussian fit to the high frequency side of the spectra. The error bars show the standard deviation of approximately four measurements. (b) Time-of-flight image of the condensate with the thermal background subtracted at $1/(k_F a) \approx -0.43$. Rings indicate momentum intervals of $0.02\,k_F$. (c) Momentum-resolved analysis of the Higgs excitation inside the condensate by averaging the optical density in the color-coded rings in (b) for different modulation frequency. The resonance occurs at the same modulation frequency for all momenta.

frequency $\Omega'_{R,k}$ induces a modulation of the amplitude of the order parameter $|\Delta|$ (Fig. 1e, for details see Methods) and hence couples directly to the Higgs mode. To illustrate this mechanism, we numerically solve the minimal set of coupled equations of motion describing the evolution of the order parameter in the presence of an rf coupling to state $|3\rangle$ (see Methods and Supplementary Figure). We see that the Fourier spectrum of $|\Delta|$ for one modulation frequency displays – aside from a response corresponding to the modulation frequencies $\Omega'_{R,k}$ – a sharp peak at the gap value $2|\Delta|$, see Figure 2d. By the momentum-resolved representation (Figure 2c) we identify this peak to be dominated by the Higgs excitation with a momentum-independent dispersion. The amplitude of the latter is maximum when the averaged effective drive frequency $\hbar\bar{\Omega}_{mod} \approx 2|\Delta|$ in accordance with its expected frequency. The Higgs mode is a collective mode of the system and even for the harmonically trapped gas exhibits a unique frequency. Numerical studies in the BCS limit have shown that in harmonically trapped systems, the Higgs mode should occur at the frequency of twice the superconducting gap evaluated at the maximum density of the gas [8, 26, 27, 28, 29] and hence we use this as our reference for the value of the gap in order to compare with theory and other experiments.

In the experiment, we search for the Higgs mode by measuring the energy absorption spectrum of the fermionic superfluid in the $|1,2\rangle$ states for different interactions. Using Ω_R and δ as adjustable parameters, we dress the $|2\rangle$ state by the $|3\rangle$ state with adjustable modulation frequency given by the effective Rabi frequency. We choose a drive frequency in the single–

particle excitation gap. For our experiments we measure the modulation frequency Ω_{mod} and amplitude α of the time-dependent population of the $|3\rangle$ state (for calibration, see Methods and Supplementary Figure). We then use a constant excitation amplitude $p_{|3\rangle} \simeq 0.5\%$ and apply the modulation for a fixed period of 30 ms. After the excitation we conduct a rapid magnetic field sweep onto the molecular side of the Feshbach resonance and convert Cooper pairs into dimers and measure the condensate fraction of the molecular condensate in time-of-flight imaging. The change in the condensate fraction provides us with a sensitive measure of the excitation of modes in the quantum gas. In Figure 3a we plot the measured spectra as a function of the modulation frequency for different values of $1/(k_F a)$. On the BCS-side of the Feshbach resonance up to unitarity, $1/(k_F a) < 0$, we observe clear resonances for which the condensate fraction reduces, signaling the excitation of a well defined mode. For $1/(k_F a) > 0$, the energy absorption peak is gradually washed out and broadens significantly. Far on the BEC side, for $1/(k_F a) \simeq 1$, we cannot observe a resonance and conclude that the Higgs mode is absent. The resonances generally exhibit an asymmetric line shape, which we fit with a Gaussian to the high-frequency side in order to extract the peak position and width. A contribution to the asymmetric peak shape stems from the momentum-dependence of the effective Rabi frequencies $\Omega'_{R,k}$. As indicated in Figures 2a and b, the effective detuning (and hence the modulation frequency) varies with increasing momentum k. Therefore, a resonant excitation at the Higgs mode frequency can be achieved for high momenta k even though for low momenta the modulation frequency is below the resonant excitation.

In order to demonstrate the collective mode nature of our resonance, we perform a number of checks. Firstly, we verify that the excitation resonance frequency (to within 4%) and shape is independent of the modulation strength in the range of $0.001 < \alpha < 0.2$ and modulation duration between $\tau = 0.5$ ms and $\tau = 30$ ms (for the definition of α, see Methods and Supplementary Figure). Secondly, we have confirmed that the observed resonance peak is not caused by single-particle excitations by measuring the excitation probability to the $|3\rangle$ state vs. modulation frequency Ω_{mod} and finding a featureless broad spectrum. This and the following checks have been performed with a modulation amplitude of $\alpha = 0.2$ and a modulation time of 0.5 ms, which is much shorter than the trap period of ~ 5 ms. Hence the measurement is insensitive to thermalization effects and/or density redistribution within the cloud. Thirdly, we check the momentum-dependence of the resonance. After the modulation, we perform a time-of-flight expansion for a period of 15 ms, which is approximately a quarter period of the residual harmonic potential during ballistic expansion. This procedure maps the initial momentum states to positions in the absorption image [30]. We analyse the detected condensate density in momentum intervals of $0.02\,k_F$ and find that the excitation resonance is at the same frequency for all momentum intervals, see Figure 3b and 3c. Finally, we have searched for possible quasiparticle excitations resulting from our interaction modulation by employing standard rf spectroscopy [24, 25, 31] after the interaction modulation. The spectra only show a very weak and broad background independent of the modulation frequency. This behaviour is not unexpected since the contribution of quasi-particle excitations is smeared out in the presence of a trap as we confirmed numerically using the local-density approximation.

In Figure 4a, we plot the position of the fitted peak of the energy absorption spectra vs. the interaction parameter $1/(k_F a)$ evaluated at the center of the sample. It has been suggested[8] that the Higgs mode frequency is close to twice the superconducting gap in the BCS/BEC crossover and can therefore be used as an approximative measure of the gap. In the crossover regime, the exact value of $|\Delta|$ is yet unknown and both experiments and numerical calculations are challenging. We compare our data to gap measurement using different methods[31, 32] and several numerical calculations[33, 34, 35, 36, 37, 38]. As compared to the previous experimental results, our extracted value is somewhat larger. We note that previous gap measurements rely on fitting the onset of a spectral feature whereas our method is based on fitting a Gaussian

Figure 37.4: Observation of the Higgs mode. (a) Measured peak positions of the energy absorption spectra (black circles). For comparison we show numerical simulations of the gap parameter multiplied by 2: BCS mean field theory (blue dashed line), ref. [33] (red), ref. [34] (green), ref. [35] (orange), ref. [36] (purple), ref. [37] (brown), ref. [38] (pink). The grey symbols show the experimental data of ref. [32]. (b) Measured fullwidth at half maximum (FWHM) of the absorption peaks (black circles). For comparison, the BCS mean field theory gap is also shown (blue dashed line). The error bars in a and b represent the standard errors.

to a slightly skewed spectral feature and both methodologies could be susceptible to small systematic uncertainties. An upper bound is provided by the theoretical result of mean-field theory (dashed line), which is known to overestimate the superconducting gap.

In Figure 4b, we plot the full-width-at-half-maximum of the Gaussian fits to the energy absorption peaks. Utilizing the BCS model for the momentum-dependent excitation discussed above, we estimate the width of our excitation resonance to be of order Δ, see Supplementary Figure. Hence, we cannot directly interpret the linewidth of our spectra as the decay rate of the Higgs mode but only as lower limit of the lifetime. On the BCS side of the resonance we find good agreement with our model and towards the BEC side the measured width far exceeds the prediction, indicating that the Higgs mode becomes strongly broadened, for example, due to the violation of particle-hole symmetry resulting in a decay into Goldstone modes[39, 6, 10, 11]. Extending, in the future, our novel experimental scheme with a better momentum resolution will provide a route to finally explore the decay mechanisms of the Higgs mode, the understanding of which is a cornerstone in both high-energy particle physics and condensed-matter physics.

37.0.2 Methods

Preparation

Using standard techniques of laser cooling and sympathetic cooling in a mixture with Sodium atoms in a magnetic trap, we prepare $\sim 5 \times 10^7$ cold fermionic Lithium atoms in a crossed-beam optical dipole trap of wavelength 1070 nm in an equal mixture of the two lowest hyperfine states $|1\rangle$ and $|2\rangle$. Using subsequent evaporative cooling in a homogeneous magnetic field of 795 G, in immediate vicinity of the Feshbach resonance at 834 G, we produce a condensate in the BCS/BEC crossover regime with a temperature of $T/T_F = 0.07 \pm 0.02$. After preparation of the fermionic superfluid, the magnetic offset field is adiabatically adjusted in the range between 740 G and 1000 G in order to control the interaction parameter $1/(k_F a)$ in the range of $-0.8 < 1/(k_F a) < 1$, i.e. across the whole BCS/BEC crossover region.

Calibration of spectroscopy and analysis

We experimentally calibrate the modulation frequency and amplitude to take into account energy shifts owing to interaction effects of the initial and final states and the efficiency of the rf antenna setup. To this end, we drive Rabi oscillations with set values of detuning δ and power and measure the population $p_{|3\rangle}$ as a function of time during the rf drive $p_{|3\rangle} = \alpha \sin^2(\Omega_{\text{mod}} t/2)$. This provides us with a direct measurement of the modulation frequency and amplitude. In order to model the data, we assume a Lorentzian line shape $\alpha = \frac{\Omega_R^2}{\Omega_R^2 + (\delta - \delta_0)^2}$, however, we allow for a frequency shift $\delta_0(k_F a)$ by which the detuning δ is corrected as compared to the Zeeman-energy resonance of the free atom. The fit parameter δ_0 absorbs the effects of interactions in the final state of the spectroscopy, the condensation energy of the initial state, and the averaging of different momentum states and densities in the trap. Experimentally, the calibration is performed at a value of $\alpha \simeq 4\%$ for which we obtain agreement with the Lorentzian model to a few percent.

We check for unpaired atoms in the $|2\rangle$ state for red and blue detuned radio-frequency with respect to δ_0 as a result of the modulation. This has been achieved by rapidly ramping the field to 450 G with approximately 4 G/μs, which allows for the detection of free atoms rather than paired atoms. In the case of a red-detuned radio-frequency modulation no enhancement of the signal of unpaired atoms could be observed over the whole range of modulation frequencies. However, blue detuned radio-frequency modulation increases significantly the number of unpaired atoms due to single particle excitations to the continuum, see Supplementary Figure 1.

Additionally, we vary the driving strength α and observe that the resonance position of the peak with respect to the modulation frequency does not vary, see Supplementary Figure 2.

Theoretical modelling

The experimental system can be described taking three different fermionic levels into account. Initially the system is prepared in a balanced mixture of states $|1\rangle$ and $|2\rangle$. Since we are mainly interested in the excitation mechanism and for this mainly the presence of a difference in the interaction strength between states ($|1\rangle$ and $|2\rangle$) and ($|1\rangle$ and $|3\rangle$) is needed, we take here only the interaction between these two states into account and decouple this term within the s-wave BCS channel. Using the rotating wave approximation for the coupling between the states $|2\rangle$ and $|3\rangle$, we obtain the Hamiltonian

$$H = H_{BCS} + \sum_{\boldsymbol{k}} (\varepsilon_{\boldsymbol{k}} - \hbar\delta) n_{\boldsymbol{k},3} + \frac{\hbar\Omega_R}{2} \sum_{\boldsymbol{k}} \left(c^\dagger_{\boldsymbol{k},3} c_{\boldsymbol{k},2} + c^\dagger_{\boldsymbol{k},2} c_{\boldsymbol{k},3} \right)$$

with

$$H_{BCS} = \sum_{\boldsymbol{k}} \varepsilon_{\boldsymbol{k}} (n_{\boldsymbol{k},1} + n_{\boldsymbol{k},2}) + \sum_{\boldsymbol{k}} \left\{ \Delta^* c_{-\boldsymbol{k},2} c_{\boldsymbol{k},1} + \Delta c^\dagger_{\boldsymbol{k},1} c^\dagger_{-\boldsymbol{k},2} \right\}. \tag{37.1}$$

Here $\Delta = \frac{g}{V} \sum_{\boldsymbol{k}} \langle c_{-\boldsymbol{k},2} c_{\boldsymbol{k},1} \rangle$, Ω_R is the Rabi-frequency, g the interaction strength, V the volume, and the momentum independent detuning is $\hbar\delta = \hbar\omega_{rf} - (\varepsilon^0_3 - \varepsilon^0_2)$, where ε^0_n is the bare energy for the state $n = 2, 3$ and $\varepsilon_k = \hbar^2 k^2/(2m)$ is the single-particle dispersion. We determine g from the scattering length using the expression provided in Ref. [23].

In order to determine the time evolution of the order parameter, we derive a closed set of equations for the expectation values

$$\begin{aligned}
\hbar \frac{\partial}{\partial t} \langle c_{-\boldsymbol{k},2} c_{\boldsymbol{k},1} \rangle &= i\{-2\epsilon_{\boldsymbol{k}} \langle c_{-\boldsymbol{k},2} c_{\boldsymbol{k},1} \rangle - \frac{\hbar\Omega_R}{2} \langle c_{-\boldsymbol{k},3} c_{\boldsymbol{k},1} \rangle + \Delta (n_{\boldsymbol{k},1} + n_{-\boldsymbol{k},2} - 1)\} \\
\hbar \frac{\partial}{\partial t} \langle c_{-\boldsymbol{k},3} c_{\boldsymbol{k},1} \rangle &= i\{-\frac{\hbar\Omega_R}{2} \langle c_{-\boldsymbol{k},2} c_{\boldsymbol{k},1} \rangle - (2\epsilon_{\boldsymbol{k}} - \hbar\delta) \langle c_{-\boldsymbol{k},3} c_{\boldsymbol{k},1} \rangle + \Delta \langle c^\dagger_{-\boldsymbol{k},2} c_{-\boldsymbol{k},3} \rangle\} \\
\hbar \frac{\partial}{\partial t} \langle c^\dagger_{-\boldsymbol{k},2} c_{-\boldsymbol{k},3} \rangle &= i\{\Delta^* \langle c_{-\boldsymbol{k},3} c_{\boldsymbol{k},1} \rangle + \hbar\delta \langle c^\dagger_{-\boldsymbol{k},2} c_{-\boldsymbol{k},3} \rangle - \frac{\hbar\Omega_R}{2} (n_{-\boldsymbol{k},2} - n_{-\boldsymbol{k},3})\} \\
\hbar \frac{\partial}{\partial t} n_{\boldsymbol{k},1} &= -2 \, \mathrm{Im}(\Delta^* \, \langle c_{-\boldsymbol{k},2} c_{\boldsymbol{k},1} \rangle) \\
\hbar \frac{\partial}{\partial t} n_{-\boldsymbol{k},2} &= -2 \, \mathrm{Im}(\Delta^* \, \langle c_{-\boldsymbol{k},2} c_{\boldsymbol{k},1} \rangle) + \hbar \, \Omega_R \, \mathrm{Im}(\langle c^\dagger_{-\boldsymbol{k},2} c_{-\boldsymbol{k},3} \rangle) \\
\hbar \frac{\partial}{\partial t} n_{-\boldsymbol{k},3} &= -\hbar \, \Omega_R \, \mathrm{Im}(\langle c^\dagger_{-\boldsymbol{k},2} c_{-\boldsymbol{k},3} \rangle),
\end{aligned}$$

where the number densities are defined as $n_{\boldsymbol{k},m} = \langle c^\dagger_{\boldsymbol{k},m} c_{\boldsymbol{k},m} \rangle$ with $m = 1, 2, 3$. We solve these equations numerically discretizing both time t and momentum k and using the self-consistency condition $\Delta = \frac{g}{V} \sum_{\boldsymbol{k}} \langle c_{-\boldsymbol{k},2} c_{\boldsymbol{k},1} \rangle$ at each time step ensuring both the convergence for the time-step dt and the momentum spacing. Typical values taken are $dk/k_F = 5 \times 10^{-4}$, $dt = 5 \times 10^{-4} \hbar/E_F$ and the cutoff for the momentum sum is $E_c = 100 \, E_F$.

The momentum resolved spectral weight of the gap shown in Figure 2c is computed as

$$A_{\boldsymbol{k}}(\omega) = V/g \left| \mathscr{F} \left\{ \left| \Delta_{\boldsymbol{k}}(t) \right| - \frac{1}{T} \int_0^T dt \left| \Delta_{\boldsymbol{k}}(t) \right| \right\} \right| \tag{37.2}$$

with the momentum-dependent order parameter $\Delta_{\boldsymbol{k}} = g/V \langle c_{-\boldsymbol{k},2} c_{\boldsymbol{k},1} \rangle$. We use $T = 400\hbar/E_F$ for the calculation.

Time evolution of the population of state $|3\rangle$

We compare the theoretical evolution of the population of atoms in state $|3\rangle$ (see Figure 1c) during the application of the rf dressing to the experimental results. Supplementary Figure 3 shows the population of the atoms in state $|3\rangle$ normalized to the initial atom number in state $|1\rangle$. The simulation and experiment were performed with the same effective modulation frequency, Ω_{mod}, and maximum atom transfer. Both curves show damped oscillations of the population of state $|3\rangle$ with time. The initial time behaviour up to approximately three oscillations agrees well between theory and experiment which means that the dominant damping mechanism is due to a dephasing of the different momentum components. Afterwards the experimental results show a stronger damping which we attribute to other damping mechanisms such as for example the presence of collisions in the experiment which are not considered in the theoretical description.

Time evolution of the condensate fraction

We show the evolution of the condensate fraction during the application of the rf dressing in Extended Data Fig. 4. After different durations of the application, the drive time, the rapid mapping to the condensate fraction has been performed and the condensate fraction has been taken. The drive amplitude was chosen to be 0.05%. As a response, an oscillation of the condensate fraction close to the expected Rabi frequency can be observered over several oscillation periods with an amplitude of the order of 0.05 %.

Comparison of the experimental and theoretical spectra

In Supplementary Figure 5, we show a comparison of the experimentally measured spectra with the theoretical simulation. In order to gain insight into the structure expected from the excitation scheme, we theoretically extract the weight of the Higgs excitation for different effective modulation frequencies and plot these in the lower panel of Supplementary Figure 5 for $1/(k_{\mathrm{F}}a) = -0.63$. To evaluate the area under the Higgs, for each momentum, we integrate around the Higgs excitation peak (shown in Figure 2c) and then sum over all momenta along the Higgs excitation line. Let us note that this procedure leads to the artefact that at high modulation frequencies still a non-vanishing contribution to the weight is found, which, however, can be attributed to the excitation of quasi-particles in a homogeneous system and would vanish in a trapped system as considered experimentally. More importantly, we see that even though the Higgs mode has a very sharp frequency (as shown in Fig. 2c) and therefore a long life-time, the resulting spectra show a much broader peak. The width of the peak is due to the excitation procedure. In particular, a resonant excitation of the Higgs mode is already possible if the effective modulation frequency lies below the sharp frequency of the Higgs mode, since then already some of the Rabi frequencies of the higher momentum components (compare stars in Fig. 2c) can resonantly excite the Higgs mode. Thus, the broadening of the spectral feature is mainly due to the particular excitation scheme and not a measurement of the life-time of the Higgs mode. Let us conclude by pointing out that the full width at half maximum in both the theoretical and the experimental spectra is approximately $|\Delta|$.

Local density approximation for the quasi-particle excitations

To study the effect of the harmonic trapping on the quasi-particle excitations we performed a calculation of the system's dynamics within the local-density approximation (LDA). Within LDA we treat points of different density as effectively homogeneous systems with rescaled interaction $1/[k_F(\mathbf{r})a]$, Fermi energy $E_F(\mathbf{r})$ and chemical potential consistent with the system's density profile. We assume the latter to be the profile for non-interacting fermions as typically the density profiles only changes slightly for the considered interactions. The time evolution of

the superconducting order parameter of the homogeneous system is performed locally for each point in the trap and rescaled to give $\frac{\Delta(\mathbf{r},t)}{E_F(\mathbf{r}=0)}$. We then take the density-weighted average of its Fourier transform. It is important to note that the Higgs - due to its collective nature - cannot be treated in this formalism, so that we remove the Higgs peak in each Fourier transform by hand before we take the trap average. Integrating the resulting spectrum gives the "background excitation weight" (cf. Supplementary Figure 6). In contrast to the peaked quasi-particle structure of a homogeneous system, we find the trap averaged "background excitation weight" to be significantly broadened resulting in a featureless, broad background.

38 Higgs Mode in Superconductors by Shimano, Tsuji

Abstract

When a continuous symmetry of a physical system is spontaneously broken, two types of collective modes typically emerge: the amplitude and phase modes of the order-parameter fluctuation. For superconductors, the amplitude mode is recently referred to as the "Higgs mode" as it is a condensed-matter analogue of a Higgs boson in particle physics. Higgs mode is a scalar excitation of the order parameter, distinct from charge or spin fluctuations, and thus does not couple to electromagnetic fields linearly. This is why the Higgs mode in superconductors has evaded experimental observations over a half century after the initial theoretical prediction, except for a charge-density-wave coexisting system. With the advance of nonlinear and time-resolved terahertz spectroscopy techniques, however, it has become possible to study the Higgs mode through the nonlinear light-Higgs coupling. In this review, we overview recent progresses on the study of the Higgs mode in superconductors.

38.1 Introduction and a Brief Historical Overview

Spontaneous breaking of a continuous symmetry is a fundamental concept of phase transition phenomena in various physical systems ranging from condensed matter to high-energy physics [1]. For instance, a ferromagnetic transition is characterized by the spontaneous breaking of spin rotational symmetry. Likewise, superconductivity is characterized by the spontaneous breaking of U(1) rotational symmetry with respect to the phase of a macroscopic wavefunction [2].

When a continuous symmetry is spontaneously broken, there emerge two types of collective modes in general: fluctuations of the phase and amplitude of the order parameter as schematically shown in **Figure** 38.1***a***. The phase mode (also termed Nambu-Goldstone (NG) mode) is primarily a gapless (massless) mode [3, 4, 5, 6, 7, 8] as required by the symmetry. An acoustic phonon is such an example that is associated with spontaneous breaking of translational symmetry in a crystal lattice. The amplitude mode, on the other hand, is generally a gapped (massive) mode since its excitation costs an energy due to the curvature of the potential (**Figure** 38.1***a***). The amplitude mode (especially for superconductors) is often called the Higgs mode according to the close analogy with the Higgs boson in particle physics. It may sound strange that one talks about the Higgs particle in condensed matter systems, but in fact the origin of the idea of Higgs physics can be found in the course of the study of superconductivity [3, 4, 5, 6, 7, 8, 9]. Namely, the standard model in particle physics can in some sense be viewed as a relativistic version of the Ginzburg-Landau theory [2], i.e., a low-energy effective theory of superconductors. The emergence of the Higgs mode is a universal and fundamental phenomenon in systems with spontaneous symmetry breaking.

In the case of superconductors where the order parameter couples to gauge fields, the Higgs mode occupies a special status since it is the lowest collective excitation mode (**Figure** 38.1***b***).

Figure 38.1: (*a*) A schematic picture of the Higgs (red) and Nambu-Goldstone (blue) modes represented on the Mexican-hat free-energy potential as a function of the complex order parameter Ψ. (*b*) A schematic excitation spectrum of an *s*-wave superconductor. Due to the Anderson-Higgs mechanism, the Nambu-Goldstone mode acquires an energy gap in the order of the plasma frequency ω_p, while the Higgs mode remains in low energy with an energy gap 2Δ, above which the quasiparticle excitation continuum overlaps.

The massless phase mode is absorbed into the longitudinal component of electromagnetic fields, and is lifted up to high energy in the scale of the plasma frequency due to the Anderson-Higgs mechanism [4, 5, 6, 9, 10, 11, 12, 13]. As a result, the amplitude mode becomes stable against the decay to the phase mode in superconductors. The existence of the amplitude mode in superconductors was suggested by Anderson [5, 14] soon after the development of a microscopic theory of superconductivity by Bardeen, Cooper, and Schrieffer (BCS) [15]. In an *s*-wave (BCS-type) superconductor, the Higgs-mode gap energy $\omega_{\rm H}$ coincides with the superconducting gap energy 2Δ [5, 16, 17, 2, 12] (see also [20, 21]). This is eventually consistent with Nambu's conjectural sum rule [21, 22]: $\omega_{\rm H}^2 + \omega_{\rm NG}^2 = 4\Delta^2$ with the NG mode gap energy $\omega_{\rm NG} = 0$. Since the Higgs mode lies at the lower bound of the quasiparticle continuum near zero momentum, the decay of the Higgs mode to single-particle excitations is suppressed and becomes a much slower power law [16]. For nonzero momentum, the Higgs mode can decay by transferring its energy to quasiparticle excitations. The energy dispersion and the damping rate of the Higgs mode has been derived within the random phase approximation [2, 12].

Provided that the theory of the Higgs particle originates from that of superconductivity and that the Higgs particle has been discovered in LHC experiments in 2012 [23, 24], it would be rather surprising that an experimental observation of the Higgs mode in superconductors, a home ground of Higgs physics, has been elusive for a long time. There are, however, good reasons for that: The Higgs mode does not have any electric charge, electric dipole, magnetic moment, and other quantum numbers. In other words, the Higgs mode is a scalar excitation (which is distinct from, e.g., charge fluctuations). Therefore it does not couple to external probes such as electromagnetic fields in the linear-response regime. Another reason is that the energy scale of the Higgs mode lies in that of the superconducting gap, which is in the order of millielectron volts in typical metallic superconductors. To excite the Higgs mode, one needs an intense terahertz (THz) light source (1 THz $\sim$ 4 meV $\sim$ 300 μm), which has become available only in the last decade.

One exception (and the first case) for the observation of the Higgs mode in superconductors in the early stage was the Raman scattering experiment for a superconductor 2H-$NbSe_2$ [3, 26]. It is exceptional in the sense that superconductivity and charge density wave (CDW) coexist

in a single material. The Raman peak observed near the superconducting gap energy 2Δ in 2H-$NbSe_2$ was first interpreted as a single-particle excitation across the gap [3, 26]. Soon later, Littlewood and Varma have theoretically elucidated that the peak corresponds to the scalar excitation of the amplitude mode (i.e., the Higgs mode) [2, 12] (see also Reference [27]). A renewed interest in the Raman signal has recently revealed the importance of the coexisting CDW order for the Higgs mode to be visible in the Raman response [28, 29, 30]. Indeed, the Raman peak of the Higgs mode has been shown to be absent in a superconducting NbS_2, which is a material similar to $NbSe_2$ but has no CDW order [28].

It has been a long-standing issue as to whether the Higgs mode can be observed in superconductors without CDW order. On the theoretical side, there have been various proposals for the excitation of the Higgs mode, including a quench dynamics [16, Barankov and Levitov(2004), 32, Barankov and Levitov(2006), 34, 35, 36, 37] as well as a laser excitation [38, 39, 40, 41, 42, 43, 44]. It is only after the development of ultrafast pump-probe spectroscopy techniques in the low-energy terahertz-frequency region that a clear observation of the Higgs-mode oscillation has been reported in a *pure s*-wave superconductor NbN [4] (with 'pure' meaning no other long-range order such as CDW). Subsequently, it has been experimentally demonstrated through the THz pump-probe and third harmonic generation (THG) measurements [46] that the Higgs mode can couple to electromagnetic fields in a *nonlinear* way. In the THG experiment, a resonant enhancement of THG was discovered [46] at the condition $2\omega = 2\Delta$ with ω the frequency of the incident THz light, indicating the nonlinear light-Higgs coupling in a two-photon process [42].

Meanwhile, it has been theoretically pointed out that not only the Higgs mode but also charge density fluctuation (CDF) can contribute to the THG signal [47]. It was shown that the CDF gives a much larger contribution than the Higgs mode in a clean-limit superconductor within the BCS mean-field approximation. While the relative magnitude of the Higgs-mode and CDF contributions to the THG has been under debate [47, 48, 49, 50], theoretical progresses have recently been made on the theory of the light-Higgs coupling, showing that both the effects of phonon retardation [48] and nonmagnetic impurity scattering [51, 52, 53, 54] drastically enhance the Higgs-mode contribution to nonlinear optical responses, far exceeding the CDF contribution.

The experiments have been extended to high-T_c cuprate superconductors, showing the presence of the Higgs-mode contribution to the pump-probe signal in d-wave superconductors [55]. Recently the THG has also been identified in various high-T_c cuprates [56]. The Higgs mode can be made visible in a linear optical response if dc supercurrent is flowing in a superconductor [57]. The infrared activated Higgs mode in the presence of supercurrents has recently been observed in a superconductor NbN [58].

In this review article, we overview these recent progresses in the study of the Higgs mode in superconductors (where the Anderson-Higgs mechanism is taking place), with an emphasis on the experimental aspects. In a broader context, collective amplitude modes are not limited to superconductors but are ubiquitous in condensed matter systems that may not be coupled to gauge fields (hence without the Anderson-Higgs mechanism). Those include squashing modes in superfluid ^{3}He [59], amplitude modes in bosonic and fermionic condensates of ultracold-atom systems [16, 61], amplitude modes in quantum antiferromagnets [13], and so on. A comprehensive review including these topics can be found in Reference [6].

38.2 Nonlinear Light-Higgs Coupling

In this section, we review from a theoretical point of view how the Higgs mode in superconductors couples to electromagnetic fields in a *nonlinear* way. It is important to understand the mechanism of the nonlinear light-Higgs coupling for observing the Higgs mode in experi-

ments. In fact, the Higgs mode does not have a linear response against electromagnetic fields in usual situations, which has been an obstacle for experiments to detect the Higgs mode in superconductors for a long time.

38.2.1 A phenomenological view

Let us first take a phenomenological point of view based on the Ginzburg-Landau (GL) theory [2], which provides us with a quick look at the nonlinear light-Higgs coupling. For early developments based on the time-dependent GL theory, we refer to References [64, 65, 66, 67, 68]. In the GL theory, the free energy density f is assumed to be a function of a complex order-parameter field $\psi(\boldsymbol{r})$,

$$f[\psi] \quad = f_0 + a|\psi(\boldsymbol{r})|^2 + \frac{b}{2}|\psi(\boldsymbol{r})|^4 + \frac{1}{2m^*}|(-i\nabla - e^*\boldsymbol{A})\psi(\boldsymbol{r})|^2, \tag{38.1}$$

where $a = a_0(T - T_c)$, a_0 and b are some constants, m^* and e^* are the effective mass and the effective charge of the Cooper pair condensate, and $\boldsymbol{A}$ is the vector potential that represents the external light field ($\boldsymbol{E}(t) = -\partial\boldsymbol{A}(t)/\partial t$). The amplitude of the order parameter corresponds to the superfluid density (i.e., $|\psi|^2 = n_s$), while the phase corresponds to that of the condensate.

The free energy density (38.1) is invariant under the global U(1) phase rotation $\psi(\boldsymbol{r}) \to e^{i\varphi}\psi(\boldsymbol{r})$ and more generally under the gauge transformation $\psi(\boldsymbol{r}) \to e^{ie^*\chi(\boldsymbol{r})}\psi(\boldsymbol{r})$, $\boldsymbol{A}(\boldsymbol{r}) \to \boldsymbol{A}(\boldsymbol{r}) + \nabla\chi(\boldsymbol{r})$ for an arbitrary scalar field $\chi(\boldsymbol{r})$. In addition, the GL free energy density (38.1) is invariant against the particle-hole transformation $\psi(\boldsymbol{r}) \to \psi^\dagger(\boldsymbol{r})$. The presence of the particle-hole symmetry (including the time derivative terms in the action) is crucial [27, 6, 69] in decoupling the Higgs amplitude mode from the phase mode (Nambu-Goldstone mode) in electrically neutral systems such as superfluid ultracold atoms. Without the particle-hole symmetry, the Higgs mode quickly decays into the phase mode and becomes short-lived. In superconductors (which consist of electrically charged electrons), on the other hand, the phase mode acquires an energy gap in the order of the plasma frequency due to the Anderson-Higgs mechanism. This prevents the Higgs mode from decaying into the phase mode. The particle-hole symmetry itself is realized as an approximate symmetry in the Bogoliubov-de Gennes Hamiltonian, which describes microscopic low-energy physics of superconductors around the Fermi energy at the mean-field level.

When the temperature goes below the critical temperature (i.e., $a < 0$), the global U(1) symmetry is spontaneously broken, and the system turns into a superconducting state. The order-parameter fluctuation from the ground state ($\psi(\boldsymbol{r}) = \psi_0$) can be decomposed into amplitude $H(\boldsymbol{r})$ and phase $\theta(\boldsymbol{r})$ components,

$$\psi(\boldsymbol{r}) \quad = [\psi_0 + H(\boldsymbol{r})]e^{i\theta(\boldsymbol{r})}. \tag{38.2}$$

Then, the GL potential can be expressed up to the second order of the fluctuation as

$$f \quad = -2aH^2 + \frac{1}{2m^*}(\nabla H)^2 + \frac{e^{*2}}{2m^*}\left(\boldsymbol{A} - \frac{1}{e^*}\nabla\theta\right)^2(\psi_0 + H)^2 + \cdots. \tag{38.3}$$

The first term on the right hand side of Equation 38.3 indicates that the Higgs mode has an energy gap (**Figure** 38.1***b***) proportional to $(-a)^{1/2} \propto (T_c - T)^{1/2}$. The microscopic BCS mean-field theory predicts that the Higgs gap is actually identical to the superconducting gap 2Δ [5, 16, 17, 2, 12]. The phase field $\theta(\boldsymbol{r})$ does not have such a mass term, so that at first sight one would expect to have a massless Nambu-Goldstone mode. However, $\theta(\boldsymbol{r})$ always appears in the form of $(\boldsymbol{A} - \nabla\theta/e^*)$ in Equation 38.3, which allows one to eliminate $\theta(\boldsymbol{r})$ from the GL

Figure 38.2: (a) Diagrammatic representation of the nonlinear light-Higgs coupling $\boldsymbol{A}^2 H$ in Equation 38.4. (b) Third harmonic generation mediated by the Higgs mode. (c) Pump-probe spectroscopy mediated by the Higgs mode.

potential by taking a unitary gauge $\boldsymbol{A}' = \boldsymbol{A} - \nabla\theta/e^*$. As a result, we obtain (denoting $\boldsymbol{A}'$ as $\boldsymbol{A}$)

$$f \;= -2aH^2 + \frac{1}{2m^*}(\nabla H)^2 + \frac{e^{*2}\psi_0^2}{2m^*}\boldsymbol{A}^2 + \frac{e^{*2}\psi_0}{m^*}\boldsymbol{A}^2 H + \cdots . \tag{38.4}$$

One can see that the phase mode is absorbed into the longitudinal component of the electromagnetic field. At the same time, there appears a mass term for photons (the third term on the right hand side of Equation 38.4) with the mass proportional to $\sqrt{e^{*2}\psi_0^2/m^*}$ (Anderson-Higgs mechanism). Due to this, electromagnetic waves cannot propagate freely inside superconductors but decay exponentially with a finite penetration length (Meissner effect).

From Equation 38.4, one immediately sees that there is no linear coupling between the Higgs and electromagnetic field. This is consistent with the fact that the Higgs mode does not have an electric charge, magnetic moment, and other quantum numbers, which has been a main obstacle in observing the Higgs mode by an external probe for a long time. On the other hand, there is a nonlinear coupling term $\boldsymbol{A}^2 H$ in Equation 38.4, which is responsible for the Higgs mode to contribute in various nonlinear processes. In **Figure** 38.2$\boldsymbol{a}$, we illustrate the diagrammatic representation of the nonlinear light-Higgs coupling, where two photons are coming in with frequencies ω_1 and ω_2 and one Higgs is emitted with frequency $\omega_1 + \omega_2$. This is analogous to the elementary process of the Higgs particle decaying into W bosons that had played a role in the discovery of the Higgs particle in LHC experiments [23, 24].

Due to the nonlinear interaction between the Higgs mode and electromagnetic fields, one can induce a nonlinear current given by

$$\boldsymbol{j} \;= -\frac{\partial F}{\partial \boldsymbol{A}} = -\frac{ie^*}{2m^*}[\psi^\dagger\nabla\psi - (\nabla\psi^\dagger)\psi] - \frac{e^{*2}}{m^*}\boldsymbol{A}\psi^\dagger\psi. \tag{38.5}$$

Again, we expand ψ around the ground state ψ_0 in Equation 38.5 and collect leading terms in H, which results in

$$\boldsymbol{j} \;= -\frac{2e^{*2}\psi_0}{m^*}\boldsymbol{A}H. \tag{38.6}$$

This is nothing but the leading part of the London equation $\boldsymbol{j} = -(e^{*2}n_s/m^*)\boldsymbol{A}$ (let us recall that the amplitude of the order parameter corresponds to the superfluid density, $|\psi|^2 = n_s$).

When a monochromatic laser with frequency ω drives a superconductor, the amplitude of the order parameter oscillates with frequency 2ω due the nonlinear coupling $\boldsymbol{A}^2 H$ (**Figure** 38.2$\boldsymbol{a}$). Together with the oscillation of $\boldsymbol{A}$ with frequency ω in Equation 38.6, the induced nonlinear current shows an oscillation with frequency 3ω. As a result, one obtains the third harmonic generation mediated by the Higgs mode (**Figure** 38.2$\boldsymbol{b}$). Since the Higgs mode has an energy 2Δ at low momentum, one can resonantly excite the Higgs mode by a THz laser with a resonance condition $2\omega = 2\Delta$. Given that the nonlinear current is proportional to the Higgs-mode

Figure 38.3: (a) Momentum distribution of a superconducting state (above) and its pseudospin representation due to Anderson (below). (b) Pseudospin precession induced by a laser field. Figures are taken from Reference [46].

amplitude (Equation 38.6), the third harmonic generation can be resonantly enhanced through the excitation of the Higgs mode. The Higgs-mode resonance in third harmonic generation has been experimentally observed for a conventional superconductor [46].

Another example of nonlinear processes to which the Higgs mode can contribute is a pump-probe spectroscopy where pump and probe light is simultaneously applied (**Figure** 38.2$\boldsymbol{c}$). In this case, photons with different frequencies ω_{pump} and ω_{probe} are injected, and those with the same respective frequencies are emitted. There are three possibilities of the frequency carried by the Higgs mode: $\omega_{\text{pump}} \pm \omega_{\text{probe}}$ and $\omega_{\text{pump}} - \omega_{\text{pump}} = \omega_{\text{probe}} - \omega_{\text{probe}} = 0$. If one uses a THz pump and optical probe, i.e., $\omega_{\text{pump}} \lesssim 2\Delta$ and $\omega_{\text{probe}} \gg 2\Delta$, the first two channels do not contribute since the excitation of the Higgs mode is far off-resonant. The remaining zero-frequency excitation channel contributes to the pump-probe spectroscopy. This process has been elaborated in the experimental study of the Higgs mode in d-wave superconductors [55].

38.2.2 A microscopic view

In the previous subsection, we reviewed the phenomenology for the nonlinear light-Higgs coupling using the Ginzburg-Landau theory. However, strictly speaking, the application of the Ginzburg-Landau theory to nonequilibrium problems is not microscopically justified in the case of gapped superconductors [70, 71, 72]. The reason is that in a usual situation one cannot neglect the effect of quasiparticle excitations whose relaxation time is longer than the time scale of the order-parameter variation. Moreover, the Higgs mode and quasiparticle excitations are energetically degenerate at low momentum (**Figure** 38.1$\boldsymbol{b}$), so that it is inevitable to excite quasiparticles at the same time when one excites the Higgs mode. This motivates us to take a microscopic approach for further understanding.

Let us adopt the time-dependent BCS or Bogoliubov-de Gennes equation, which is efficiently represented by the Anderson pseudospin $\boldsymbol{\sigma}_{\boldsymbol{k}} = \frac{1}{2}\langle \Psi_{\boldsymbol{k}}^\dagger \boldsymbol{\tau} \Psi_{\boldsymbol{k}} \rangle$ [5]. Here $\Psi_{\boldsymbol{k}}^\dagger = (c_{\boldsymbol{k}\uparrow}^\dagger, c_{-\boldsymbol{k}\downarrow})$ is the Nambu spinor, $\boldsymbol{\tau} = (\tau_x, \tau_y, \tau_z)$ are Pauli matrices, $c_{\boldsymbol{k}\sigma}^\dagger$ ($c_{\boldsymbol{k}\sigma}$) is the creation (annihilation) operator of electrons with momentum $\boldsymbol{k}$ and spin σ, and $\langle \cdots \rangle$ denotes the statistical average. Physically, the x and y components of the Anderson pseudospin correspond to the real and imaginary parts of the Cooper pair density, respectively, and the z component corresponds to the momentum occupation distribution of electrons (**Figure** 38.3$\boldsymbol{a}$). With this notation, the BCS mean-field Hamiltonian is written as

$$H_{\text{BCS}} = 2\sum_{\boldsymbol{k}} \boldsymbol{b}_{\boldsymbol{k}}(t) \cdot \boldsymbol{\sigma}_{\boldsymbol{k}}, \tag{38.7}$$

$$\boldsymbol{b}_{\boldsymbol{k}}(t) = \left(-\Delta'(t), -\Delta''(t), \frac{1}{2}(\epsilon_{\boldsymbol{k}+\boldsymbol{A}(t)} + \epsilon_{\boldsymbol{k}-\boldsymbol{A}(t)}) \right). \tag{38.8}$$

Figure 38.4: (*a*) A numerical result for the time evolution of the superconducting gap $\Delta(t)$ driven by an electric field pulse $E(t)$ with the frequency $\omega = 2\pi/5 \sim 1.26$. Here we take a two dimensional square lattice with a bandwidth 8 at half filling. The interaction is $V = 4$, and the initial temperature is $T = 0.02$. The polarization of the electric field is parallel to x axis. The left (right) vertical axis represents the value of $\Delta(t)$ ($E(t)$) as indicated by the arrows. (*b*) The amplitude of the 2ω oscillation of the superconducting gap as a function of $2\omega/2\Delta$. (*b*) is reproduced from Reference [42].

Here $\boldsymbol{b}_{\boldsymbol{k}}(t)$ is a pseudomagnetic field acting on pseudospins, Δ' and Δ'' are the real and imaginary parts of the superconducting gap function, respectively, and $\epsilon_{\boldsymbol{k}}$ is the band dispersion of the system. The equation of motion for the pseudospins is given by the Bloch-type equation,

$$\frac{\partial}{\partial t}\boldsymbol{\sigma}_{\boldsymbol{k}}(t) = 2\boldsymbol{b}_{\boldsymbol{k}}(t) \times \boldsymbol{\sigma}_{\boldsymbol{k}}(t), \tag{38.9}$$

supplemented by the mean-field condition $\Delta'(t) + i\Delta''(t) = \frac{V}{N}\sum_{\boldsymbol{k}}[\sigma^x_{\boldsymbol{k}}(t) + i\sigma^y_{\boldsymbol{k}}(t)]$ with V the attractive interaction and N the number of k points. The time evolution of the superconducting state is thus translated into the precession dynamics of Anderson pseudospins (**Figure** 38.3***b***).

The coupling to the electromagnetic field appears in the z component of the pseudomagnetic field, $b^z_{\boldsymbol{k}} = \frac{1}{2}(\epsilon_{\boldsymbol{k}+\boldsymbol{A}(t)} + \epsilon_{\boldsymbol{k}-\boldsymbol{A}(t)})$. If we expand $b^z_{\boldsymbol{k}}$ in terms of $\boldsymbol{A}(t)$, we obtain $b^z_{\boldsymbol{k}} = \epsilon_{\boldsymbol{k}} + \frac{1}{2}\sum_{ij}\frac{\partial^2\epsilon_{\boldsymbol{k}}}{\partial k_i \partial k_j}A_iA_j + O(A^4)$. The linear coupling term vanishes, which is consistent with the phenomenology that we have reviewed in the previous subsection. The leading nonlinear coupling term is quadratic in $\boldsymbol{A}(t)$. In **Figure** 38.4***a***, we show a numerical result for the time evolution of the gap function $\Delta(t)$ when the system is driven by a multi-cycle electric-field pulse. One can see that the 2ω oscillation of the gap is generated during the pulse irradiation, after which a free gap oscillation continues with the frequency 2Δ and the amplitude slowly damping as $t^{-1/2}$ [16].

For a monochromatic wave $\boldsymbol{A}(t) = \boldsymbol{A}_0 \sin\omega t$, one can solve the equation of motion semi-analytically by linearizing with respect to the quadratic coupling $A_i(t)A_j(t)$ around the equilibrium solution. The result shows that the amplitude of the gap function varies as [42]

$$\delta\Delta(t) \propto |2\omega - 2\Delta|^{-\frac{1}{2}}\cos(2\omega - \phi) \tag{38.10}$$

for $\omega \sim \Delta$, where ϕ is a phase shift. The induced oscillation frequency is 2ω, reflecting the nonlinear coupling between the Higgs mode and electromagnetic fields, $\boldsymbol{A}^2 H$. The oscillation amplitude diverges when the condition $2\omega = 2\Delta$ is fulfilled (**Figure** 38.4***b***). This is much the same as the spin resonance phenomenon: The forced collective precession of the pseudospins with frequency 2ω due to the nonlinear coupling resonates with the Higgs mode whose energy

coincides with 2Δ. The power of the divergence $\frac{1}{2}$ in Equation 38.10 is smaller than that of a resonance with an infinitely long-lived mode ($\sim |2\omega - \omega_*|^{-1}$). The reason is that the precession gradually dephases due to the momentum-dependent frequency $\omega_{\boldsymbol{k}} = 2\sqrt{\epsilon_{\boldsymbol{k}}^2 + \Delta^2}$ for each pseudospin so that the average of the pseudospins $\delta\Delta(t) = \frac{V}{N}\sum_{\boldsymbol{k}} \delta\sigma_{\boldsymbol{k}}^x(t)$ shows a power-law damping of oscillations. The phase shift ϕ exhibits a $\frac{\pi}{2}$ jump as one goes across the resonance by changing the drive frequency ω.

In the case of the square lattice and the polarization of the electric field parallel to the diagonal direction in the xy plane, for example, the current is given by $\boldsymbol{j}(t) \propto \boldsymbol{A}(t)\delta\Delta(t)$ [42, 46], which corresponds to the phenomenological relation $\boldsymbol{j}(t) \propto \boldsymbol{A}(t)H(t)$ (Equation 38.6). As expected, the Higgs mode induces a resonant enhancement of the third harmonic generation. Later, it has been pointed out [47] that the current relation is valid only in rather special situations such as the square lattice and the polarization parallel to the diagonal direction. In general, the BCS mean-field treatment (in the clean limit) suggests that the Higgs-mode contribution to the third harmonic generation is subdominant as compared to that of individual quasiparticle excitations. Since the energy scales of the Higgs mode and the quasiparticle pair excitations are in the same order ($\sim 2\Delta$, see **Figure** 38.1***b***), the competition between the two contributions always matters. This point will be further examined in the next subsection.

38.2.3 Impurity and phonon assisting

As we have seen in the previous subsection, the Higgs-mode contribution to the third harmonic generation is generally subdominant in the BCS clean limit. However, there is a growing understanding from recent studies [48, 49, 51, 52, 53, 54] that if one takes into account effects beyond the BCS mean-field theory in the clean limit (such as impurity scattering and phonon retardation) the light-Higgs coupling strength drastically changes.

To understand the impact of these effects, let us consider the linear optical response. The current induced by a laser field is decomposed into the paramagnetic and diamagnetic components $\boldsymbol{j} = \boldsymbol{j}_{\text{para}} + \boldsymbol{j}_{\text{dia}}$ with [73]

$$\boldsymbol{j}_{\text{para}} = \sum_{\boldsymbol{k}\sigma} \frac{\partial \epsilon_{\boldsymbol{k}}}{\partial \boldsymbol{k}} \langle c_{\boldsymbol{k}\sigma}^\dagger c_{\boldsymbol{k}\sigma} \rangle, \tag{38.11}$$

$$\boldsymbol{j}_{\text{dia}} = -\sum_{\boldsymbol{k}\sigma i} \frac{\partial^2 \epsilon_{\boldsymbol{k}}}{\partial \boldsymbol{k} \partial k_i} A_i \langle c_{\boldsymbol{k}\sigma}^\dagger c_{\boldsymbol{k}\sigma} \rangle. \tag{38.12}$$

As we have seen in the pseudospin picture in the previous subsection, only the diamagnetic coupling contributes to the optical response in the mean-field level with no disorder, resulting in $\operatorname{Re}\sigma(\omega) \propto \delta(\omega)$ and $\operatorname{Im}\sigma(\omega) \propto 1/\omega$ for the superconducting state. For finite frequency ($\omega \neq 0$), the real part of the optical conductivity vanishes in the BCS clean limit. This does not necessarily mean that the real part of the optical conductivity is suppressed in real experimental situations. For example, in superconductors with a disorder the real part of the optical conductivity is nonzero and not even small for $\omega \neq 0$. In fact, a superconductor NbN used in the experiment of the third harmonic generation turns out to be close to the dirty regime ($\gamma \gg 2\Delta$, where γ is the impurity scattering rate) as confirmed from the measurement of the optical conductivity [4].

In much the same way as the optical conductivity, the nonlinear response intensity is strongly affected by the impurity effect. The γ dependence of the relative order of magnitudes of the third-order current is summarized in Table 2, where ϵ_F is the Fermi energy and the unit of $\boldsymbol{j}^{(3)}$ is taken such that the diamagnetic-coupling contribution of the quasiparticles is set to be of order 1 (which does not significantly depend on γ). In the BCS clean limit ($\gamma \to 0$), we have only the diamagnetic coupling, for which the Higgs-mode contribution is subleading in the

Table 2: Relative order of magnitudes of the third-order current $\boldsymbol{j}^{(3)}$ in general situations [53].

mode	channel	clean $\to$ dirty
Higgs	dia ($\boldsymbol{A}^2$)	$(\Delta/\epsilon_F)^2$
	para ($\boldsymbol{p}\cdot\boldsymbol{A}$)	$(\epsilon_F\gamma/\Delta^2)^2 \to (\epsilon_F/\gamma)^2$
Quasiparticles	dia ($\boldsymbol{A}^2$)	1
	para ($\boldsymbol{p}\cdot\boldsymbol{A}$)	$(\epsilon_F\gamma/\Delta^2)^2 \to (\epsilon_F/\gamma)^2$

order of $(\Delta/\epsilon_F)^2$ compared to the quasiparticles. However, in the presence of impurities ($\gamma \neq 0$) the paramagnetic coupling generally emerges to contribute to the third harmonic generation. Its relative order of magnitude changes as $(\epsilon_F\gamma/\Delta^2)^2 \to (\epsilon_F/\gamma)^2$ from the clean to dirty limit. The maximum strength of the paramagnetic coupling is achieved around $\gamma \sim \Delta$, where the relative order reaches $(\epsilon_F/\Delta)^2$ for both the Higgs mode and quasiparticles. Thus, the impurity scattering drastically enhances the light-Higgs coupling: in the clean limit the Higgs-mode contribution is subleading to quasiparticles, whereas in the dirty regime it becomes comparable to or even larger than the quasiparticle contribution. The precise ratio between the Higgs and quasiparticle contributions may depend on details of the system. Recent studies [52, 53, 54] suggest that in the dirty regime the Higgs-mode contribution to THG is an order of magnitude larger than the quasiparticle contribution. An enhancement of the paramagnetic coupling also occurs for strongly coupled superconductors due to the phonon retardation effect [48].

38.2.4 Further developments

So far, we have reviewed the phenomenological and mean-field treatments of the nonlinear light-Higgs coupling for conventional s-wave superconductors with or without disorder. For strongly coupled superconductors with an electron-phonon coupling constant $\lambda \gtrsim 1$ (which is the case for NbN superconductors [74, 75, 76]), one has to take account of strong correlation effects. One useful approach for this is the nonequilibrium dynamical mean-field theory [77], which takes into account local dynamical correlations by mapping a lattice model into a local impurity model embedded in an effective mean field. With this, the Higgs mode in strongly coupled superconductors modeled by the Holstein model has been analyzed [42, 48, 78, 79]. A closely related nonequilibrium Keldysh method has been employed to study the Higgs mode in a time-resolved photoemission spectroscopy [43, 80] and time-resolved optical conductivity [81]. Another approach is to use the gauge-invariant kinetic equation [82, 83, 84, 85], where the shift of the center-of-mass momentum of Cooper pairs ("drive effect") caused by a laser pulse has been emphasized.

For multi-band superconductors with multiple gaps such as MgB_2, there are not only Higgs modes corresponding to amplitude oscillations of multiple order parameters but also the so-called Leggett mode [86], which corresponds to a collective oscillation of the relative phase between different order parameters. Based on the mean-field theory, possible laser excitations of the Higgs and Leggett modes in multiband superconductors have been studied [53, 87, 88, 89, 90].

For non-s-wave superconductors, one can expect even richer dynamics of the order parameters. For example, d-wave superconductors allow for various different symmetries of amplitude modes [9]. The quench dynamics of the d-wave [92] and $p+ip$-wave [93] Higgs modes have been explored. For recent experimental progresses on pump-probe spectroscopies and third harmonic generation in d-wave superconductors, we refer to Section 38.3.3. It has also been proposed [94] that by measuring Higgs modes for various quench symmetries one can obtain

information on the symmetry of the gap function ("Higgs spectroscopy").

38.3 Experiments

In this section, we overview recent progresses on experimental observations of the Higgs mode in superconductors. Those include Raman experiments in CDW-coexisting superconductors 2H-$NbSe_2$ and TaS_2 (Section 38.3.1), THz spectroscopies and third harmonic generation in a pure s-wave superconductor NbN (Section 38.3.2) and in high-T_c cuprates (Section 38.3.3), and THz transmittance experiments in NbN under supercurrent injection (Section 38.3.4).

38.3.1 Higgs mode in a superconductor with CDW

The pioneering work on the observation of the Higgs mode in superconductors dates back to 1980 when a Raman experiment was performed in 2H-$NbSe_2$, a transition metal dichalcogenide in which superconductivity coexists with CDW [3, 26]. A new peak was observed below T_c, distinct from the amplitude mode associated with the CDW order. Although it was initially recognized as a pair breaking peak, soon after it was identified as the collective amplitude mode of the superconducting order (i.e., the Higgs mode) [2, 12] (see also [27, 6]).

A renewed Raman experiment has been conducted recently, revealing the transfer of the oscillator strength from the amplitude mode of CDW (amplitudon) to the Higgs mode [28]. Subsequently the Raman experiment under hydrostatic pressure has been performed in 2H-$NbSe_2$ [30] (**Figure** 38.5). With applying the hydrostatic pressure the CDW order is suppressed and concomitantly the Raman peak identified as the Higgs mode was shown to disappear with leaving only the Cooper pair breaking peak. A more clear separation between the in-gap Higgs mode and the pair breaking peak was recently reported in a similar CDW-coexisting superconductor, TaS_2 [95]. These results indicate that the existence of CDW plays an important role on the visibility of the Higgs mode in the Raman spectrum. A theoretical study has demonstrated that, due to the coupling of the superconducting order to the coexisting CDW order, the Higgs mode energy is pushed down below the superconducting gap 2Δ [29]. Accordingly, the Higgs mode becomes stable due to the disappearance of the decay channel into the quasiparticle continuum.

38.3.2 Higgs mode in a pure s-wave superconductor

Nonadiabatic quench with a single-cycle THz pump pulse

One way to excite the Higgs mode is a nonadiabatic quench of the superconducting state, which can be induced by, for example, a sudden change of the pairing interaction that generates the order-parameter dynamics,

$$\frac{|\Delta(t)|}{\Delta_\infty} \simeq 1 + a\frac{\cos(2\Delta_\infty t + \phi)}{\sqrt{\Delta_\infty t}}, \tag{38.13}$$

at long time. Here $2\Delta_\infty$ is the long-time asymptotic superconducting gap, a is some constant, and ϕ is a phase shift. The behavior (Equation 38.13) has been theoretically derived on the basis of the time-dependent BCS mean-field theory [16, Barankov and Levitov(2006), 34, 35]. A similar dynamics has also been investigated after a laser-pulse excitation [38, 39, 40, 41, 43, 44].

Despite intensive theoretical studies on the Higgs mode in superconductors, the observation of the Higgs mode has long been elusive except for the case of 2H-$NbSe_2$, until when the ultrafast THz-pump and THz-probe spectroscopy was performed in a conventional s-wave superconductor $Nb_{1-x}Ti_xN$ [4]. In this experiment, quasiparticles are instantaneously injected

Figure 38.5: Raman spectra in E_{2g} symmetry of 2H-$NbSe_2$ measured at various temperatures and pressures. Under the ambient pressure ($P < P_c$) and in the superconducting phase ($T < T_c$) (green), a sharp peak identified as the Higgs mode appears below the superconducting gap $2\Delta \sim 20$ cm^{-1} marked by the gray vertical line. In the CDW phase ($T > T_c$) (blue), only the CDW amplitude mode is observed around 40 cm^{-1}. Under the pressure ($P > P_c$) where CDW collapses, only a pair breaking peak is discerned at $T < T_c$ (brown), and no peak is identified at $T > T_c$ (red). Inset: Raman spectra in the superconducting state without CDW in 2H-$NbSe_2$ (above 4 GPa) and non-CDW-coexisting NbS_2 (0 GPa), both of which show only the pair breaking peak. Frequency is normalized by 2Δ. Figure is taken from Reference [30].

Figure 38.6: Higgs-mode oscillation after the nonadiabatic excitation of quasiparticles in NbN induced by a monocycle THz pump. (a) The temporal evolution of the change of the probe electric field, δE_{probe}, as a function of the pump-probe delay time t_{pp} at various pump intensities. The solid curves represent the results fitted by a damped oscillation with a power-law decay. (b) The oscillation frequency f obtained from the fits and the asymptotic gap energy $2\Delta_\infty$ as a function of the pump intensity. Figures are taken from Reference [4].

at the superconducting-gap edge to quench the superconducting order parameter nonadiabatically. A strong single-cycle THz pump pulse with the center frequency around 1 THz ($\sim$ 4 meV) was generated from a $LiNbO_3$ crystal by using the tilted-pulse-front method [96, 97, 98], in order to excite high density quasiparticles just above the gap of $Nb_{1-x}Ti_xN$ thin films with $2\Delta = 0.72$-1.3 THz ($\sim$ 3.0-5.4 meV) at 4 K. The details of the THz pump-THz probe scheme were given in Ref. [99]. The subsequent dynamics of the superconducting order parameter was probed by a weaker probe THz pulse in transmission geometry. **Figure** 38.6 shows pump-probe delay dependence of the transmitted probe THz electric field at a fixed point of the probe pulse waveform which reflects the order parameter dynamics. A clear oscillation is identified after the pump with the oscillation frequency given by the asymptotic value of the superconducting gap $2\Delta_\infty$ caused by the quasiparticle injection. The decay of the oscillation is well fitted by a power-law decay predicted for the Higgs-mode oscillation. With increasing the pump intensity, the oscillation period shows a softening, as the asymptotic value of $2\Delta_\infty$ becomes smaller with higher quasiparticle densities. By recording the probe THz waveform at each pump-probe delay, one can extract the dynamics of the complex optical conductivity $\sigma(\omega) = \sigma_1(\omega) + i\sigma_2(\omega)$ (not shown). The spectral-weight oscillation with frequency $2\Delta_\infty$ was clearly identified both in the real and imaginary parts [100], the latter of which reflects the oscillation of the superfluid density (i.e., the amplitude of the superconducting order parameter).

The intense single-cycle THz pump pulse was crucial to excite the quasiparticles instantaneously at the gap edge which acts as a nonadiabatic quench of the order parameter, thereby inducing a free oscillation of the Higgs mode. On the contrary, when a near-infrared optical pulse was used as a pump, a relatively slow rise time ($\sim$ 20 ps) was observed for the suppression of the order parameter, though dependent on the excitation fluence [99, 101]. In this case, the photoexcited hot electrons (holes) emit a large amount of high frequency phonons

($\omega > 2\Delta$) (HFP) which subsequently induces the Cooper pair-breaking until quasiparticles and HFP reach the quasiequilibrium. This process was well described by the Rothwarf-Taylor phonon-bottleneck model [101], and usually takes a time longer than $(2\Delta)^{-1}$, which breaks the nonadiabatic excitation condition needed for the quench experiment. It should be noted here that even if one uses the near-infrared or visible optical excitation, the instantaneous quasiparticle excitation can occur if the stimulated Raman process is efficient [102].

The decay of the Higgs mode is also an important issue. Since the Higgs-mode energy 2Δ coincides with the onset of quasiparticle continuum, the Higgs mode inevitably decays into the quasiparticle continuum even without any collisions [16, 17]. However, this does not mean that the Higgs mode is an overdamped mode. In fact, the decay due to the collisionless energy transfer is expected to be proportional to $t^{-1/2}$ in the BCS superconductor, ensuring a much longer lifetime compared with the oscillation period. The energy dispersion of the Higgs mode for finite wavenumber q has been calculated as

$$\omega_H \simeq 2\Delta + \frac{v_F^2}{12\Delta}q^2 - i\frac{\pi^2 v_F}{24}q \tag{38.14}$$

where v_F is the Fermi velocity [12]. From Equation 38.14 one can estimate that as the wavenumber q increases and the mode energy enters into the quasiparticle continuum, the lifetime of Higgs mode is shortened and turns into an overdamped mode. For the case of NbN, $\Delta \sim 0.6$ THz and $v_F \sim 2 \times 10^6$ m/s [76] and the imaginary part exceeds the real part in Equation 38.14 at $q > \Delta/v_F \sim 0.3$ $(\mu\text{m})^{-1}$. In the single-cycle THz-pump experiment in a thin film of NbN [4], the value of in-plane wavenumber q estimated from the spot size of the pump (~ 1 mm) is at most ~ 1 $(\text{mm})^{-1}$, which is far smaller than Δ/v_F. In such a small q limit, the Higgs mode is considered as a well-defined collective mode [12].

Multicycle THz driving with subgap frequency and third harmonic generation

To excite the Higgs mode in an on-resonance condition, one can use a narrowband multicycle THz pump pulse with the photon energy tuned below the superconducting gap 2Δ. With this, the amplitude of the superconducting order parameter was shown to oscillate with twice the frequency of the incident pump-pulse frequency ω during the pulse irradiation (**Figure** 38.7***a***-***d***) [46]. The observed 2ω modulation of the order parameter was attributed to the forced oscillation of the Higgs mode caused by electromagnetic fields. In fact, the time-dependent Ginzburg-Landau theory shows that there is a coupling term between the Higgs (H) and field ($\boldsymbol{A}$) as expressed by $\boldsymbol{A}^2 H$ (Section 38.2.1), which results in the 2ω-order parameter modulation. The nonlinear electromagnetic response of superconductors has been studied in the frequency range far below 2Δ and near T_c where the Ginzburg-Landau theory is applicable [64, 70, 103, 104, 105]. In the theory developed by Gor'kov and Eliashberg (GE), a closed set of equations for the order parameter Δ and the field $\boldsymbol{A}$ was derived, which predicts the 2ω modulation of Δ and the THG from the induced supercurrent $\boldsymbol{j} \propto \boldsymbol{A}^2\Delta$ [70]. The GE theory was experimentally demonstrated through the observation of THG of a microwave field in a paramagnetically doped superconductor [104].

The THz pump experiments have shed a new light on the investigation of the nonlinear electromagnetic response of superconductors at the gap frequency region. Subsequent to the observation of the 2ω modulation of the order parameter, namely the forced oscillation of the Higgs modes, the THG was observed under the irradiation of a subgap multicycle THz pump pulse in NbN thin films as shown in **Figure** 38.7***e***, ***f***. Importantly, the THG was shown to be resonantly enhanced when a condition $2\omega = 2\Delta$ is satisfied [46]. The temperature dependence of THG intensity for various frequencies in a NbN film is shown in **Figure** 38.8: the temperature dependence of the gap 2Δ and the incident frequencies (**Figure** 38.8***a***), the raw data of THG intensities (**Figure** 38.8***b***), those normalized by the internal field inside the superconducting

Figure 38.7: Forced Higgs oscillation and third harmonic generation from a superconducting NbN film under the multicycle THz pump. (a) Schematic setup for the multicycle THz pump and THz probe spectroscopy. BPF: a bandpass filter, WGP: a wire grid polarizer. (b) Temperature dependence of the superconducting gap energy. Horizontal line indicates the center frequency of the pump pulse, $\omega = 0.6$ THz. (c) Waveform of the pump THz electric field E_{pump} and the squared one, $|E_{\text{pump}}|^2$. (d) The change in the transmitted probe THz electric field δE_{probe} as a function of the pump-probe delay time t_{pp} at the tempareture range $\omega > 2\Delta(T)$ (top panel) and $\omega < 2\Delta(T)$ (bottom panel). Increase of δE_{probe} corresponds to the reduction of the order parameter. (e) Power spectra of the transmitted pump THz pulse above and below $T_c = 15$ K. (f) THG intensity as a function of the pump THz field strength. Figures are taken from Reference [46].

Figure 38.8: Temperature dependence of THG signal from NbN. (a) Temperature dependence of the supercondcuting gap $2\Delta(T)$. Twice the THz pump frequencies 2ω are shown by horizontal lines. (b) Measured THG intensities at each frequencies as a function of temperature. (c) THG intensities normalized by the temperature dependent internal field inside the superconducting film and the transmittance of the THG signal, which corresponds to the susceptibility of THG, $|\chi^{(3)}|^2$. (d) Further normalized value of $|\chi^{(3)}|^2$ by the temperature dependent gap $\Delta(T)^4$ as a function of normalized frequency $2\omega/2\Delta(T)$. Relative intensities between different frequencies are arbitrary. Data are taken from Reference [100].

film which is proportional to the third-order susceptibility $|\chi^{(3)}|^2$ (**Figure** 38.8***c***), and the THG intensity further normalized by the proportional factor of the THG intensity $\Delta(T)^4$ versus the frequency normalized to the gap, $2\omega/2\Delta(T)$ (**Figure** 38.8***d***). These results clearly show the two-photon resonances of the Higgs mode with the field.

As previously described in Section 38.2.3, however, it was theoretically pointed out that there is a contribution also from the quasiparticle excitation (or called charge density fluctuations) in THG, which also exhibits the $2\omega = 2\Delta$ resonance [47]. Within the BCS mean-field approximation in the clean limit, the quasiparticle term was shown to exceed the Higgs mode contribution in THG. At the same time, it was predicted that the THG from the quasiparticle term should exhibit polarization dependence with respect to the crystal axis while the Higgs term is totally isotropic in a square lattice [47]. In the experiments in NbN, the THG intensity was shown to be independent from the incident polarization angle with respect to the crystal orientation and notably the emitted THG does not have orthogonal components with respect to the incident polarization of the fundamental (ω) wave [49]. Such a totally isotropic nature of THG cannot be well accounted for by the CDF term even if one considers the crystal symmetry of NbN [50] and indicates the Higgs mode as the origin of THG. Recently, the importance of $\boldsymbol{p} \cdot \boldsymbol{A}$ term on the coupling between the Higgs mode and the gauge field was elucidated. In particular, the $\boldsymbol{p} \cdot \boldsymbol{A}$ term contribution is largely enhanced when one consider the phonon retardation effect beyond the BCS approximation [48] and more prominently when one considers the nonmagnetic impurity scattering effect [52, 53, 54], dominating the quasiparticle term by an order of magnitude as described in Section 38.2.3. This paramagnetic term was also shown to dominate the diamagnetic term, although its ratio depends on the ratio between the coherence length and the mean free path. The NbN superconductor is in the dirty regime as manifested by a clear superconducting gap structure observed in the conductivity spectrum, and the paramagnetic term should significantly contribute to THG.

38.3.3 Higgs mode in high-T_c cuprates

Higgs modes can exist not only in s-wave superconductors but also in unconventional superconductors. For example, in d-wave superconductors such as high-T_c cuprates one can expect various types of collective modes, including not only the totally isotropic oscillation of the gap function in relative momentum space (A_{1g} mode) but also those that oscillate anisotropically (e.g., A_{2g}, B_{1g}, B_{2g} modes) [9]. These are reminiscent of Bardasis-Schrieffer mode for the case of s-wave superconductors [106]. Possible collective modes have been classified based on the point group symmetry of the lattice [9], and the relaxation behavior of the A_{1g} mode induced by a quench has been discussed [92]. A theoretical proposal has been made to observe the Higgs mode in a d-wave superconductor through a time- and angle-resolved photoemission spectroscopy [80].

The observation of the Higgs mode in d-wave superconductors was recently made in THz-pump and optical-probe experiments in high-T_c cuprates, $Bi_2Sr_2CaCu_2O_{8+x}$ (Bi2212) [55]. An oscillatory signal of the optical reflectivity that follows the squared THz electric field was observed, which is markedly enhanced below T_c as shown in **Figure** 38.9. This signal was interpreted as the THz-pump induced optical Kerr effect, namely the third-order nonlinear effect induced by the intense THz pump pulse. In the Bi2212 system, both A_{1g} and B_{1g} symmetry components with respect to the polarization-angle dependence (which should be distinguished from the relative-momentum symmetry classification mentioned above) were observed in the THz-Kerr signal. The doping dependence shows that the A_{1g} component is dominant in all the measured samples from under to near-optimal region. From the comparison with the BCS calculation of the nonlinear susceptibility, the A_{1g} component was assigned to be the d-wave Higgs-mode contribution.

Figure 38.9: Forced oscillation of Higgs mode in an optimally doped (OP90) $Bi_2Sr_2CaCu_2O_{8+x}$ measured by THz pump and optical reflection probe experiment. Temperature dependences of reflectivity change $\Delta R/R$ as a function of pump-probe delay time in (*a*) A_{1g} and (*b*) B_{1g} components. Red dashed lines indicate T_c. (*c*) The A_{1g} and B_{1g} components against the pump-probe delay time at 10 K. Fitting curves are also shown by solid lines. (*d*) Temperature dependences of the A_{1g} decaying component (blue) (incoherent quasiparticle excitation), the A_{1g} oscillatory component (red) (forced Higgs oscillation), and the B_{1g} oscillatory component (green) (likely the charge density fluctuation). Figures are taken from Reference [55].

The observed oscillatory signal corresponds to the 2ω modulation of the order parameter as observed in an s-wave NbN superconductor under the subgap multicycle THz pump irradiation. Accordingly, like in the case of s-wave systems, one can expect the THG signal from high-T_c cuprates. Recently, THG was indeed observed ubiquitously in $La_{1.84}Sr_{0.16}CuO_4$, $DyBa_2Cu_3O_{7-x}$, $YBa_2Cu_3O_{7-x}$, and overdoped $Bi_2Sr_2CaCu_2O_{8+x}$ [56] by using 0.7 THz coherent and intense light source at the TELBE beamline at HZDR.

38.3.4 Higgs mode in the presence of supercurrents

So far, we have seen the Higgs mode excitation by the nonadiabatic quench with instantaneous quasiparticle injection or through the quadratic coupling with the electromagnetic wave. Recently, it has been theoretically shown that when there is a condensate flow (i.e., supercurrent), the Higgs mode linearly interacts with electromagnetic waves with the electric field polarized along the direction of the supercurrent flow [57]. Namely, the Higgs mode should be observed in the optical conductivity spectrum at the superconducting gap $\omega = 2\Delta$ under the supercurrent injection. This effect is explained by the momentum term in the action, $S \propto \int \boldsymbol{Q}^2(t)|\Delta(t)|^2 dtdr$, where $\boldsymbol{Q}(t) = \boldsymbol{Q}_0 + \boldsymbol{Q}_\Omega(t)$ is the gauge-invariant momentum of the condensate, $\boldsymbol{Q}_0$ is the dc supercurrent part, and $\boldsymbol{Q}_\Omega(t) = \mathrm{Re}[\boldsymbol{Q}_\Omega \exp(i\Omega t)]$ is the time-dependent part driven by the ac probe field, respectively. $\Delta(t) = \Delta_0 + \delta\Delta(t)$ is the time-dependent superconducting order parameter. The action S includes the integral of $\delta\Delta_{2\Omega}\boldsymbol{Q}^2_{-\Omega}$ and $\delta\Delta_\Omega \boldsymbol{Q}_0 \boldsymbol{Q}_{-\Omega}$ where $\delta\Delta_{\Omega(2\Omega)}$ is the Fourier component of the oscillating order parameter. The first term corresponds to the quadratic coupling of the Higgs mode to the gauge field as described Section 38.2. The second term indicates that the Higgs mode linearly couples with the gauge field under the supercurrent with the condensate momentum $\boldsymbol{Q}_0$ parallel to the probe electric field.

Recently, in accordance with the theoretical prediction, the Higgs mode was observed in the optical conductivity in an s-wave superconductor NbN thin film under the supercurrent injection by using THz time-domain spectroscopy [58]. **Figure** 38.10***a*** shows the schematic experimental setup. **Figures** 38.10***b*** and ***c*** show the change of the real part of the optical conductivity induced by the supercurrent injection with the probe polarization (***b***) parallel and (***c***) orthogonal to the current direction. A clear peak is observed at the gap edge $\omega = 2\Delta$ in the parallel geometry, showing a good agreement with the spectra calculated on the basis of the theory developed by Moor et al. [57] as represented in **Figure** 38.10***d***. This result indicates that, without resorting to the sophisticated nonlinear THz spectroscopy, the Higgs mode can be observed in the linear response function when the condensate has a finite momentum, expanding the feasibility for the detection of the Higgs mode in a wider range of materials.

The visibility of the Higgs mode in the optical conductivity has also been addressed in a two-dimensional system near a quantum critical point [7, 110] (see also [111, 112]). Experimentally, an extra spectral weight in the optical conductivity below the superconducting gap was observed in a strongly disordered superconducting film of NbN, which was interpreted in terms of the Higgs mode with its mass pushed below the pair breaking gap due to the strong disorder [5]. However, the assignment of the extra spectral weight below the gap remains under debates. It can also be described by the Nambu-Goldstone mode which acquires the electric dipole in strongly disordered superconductors [114, 115, 116, 117]. The extra optical conductivity was also accounted for by the disorder induced broadening of the quasiparticle density of states below the gap [118].

38.4 Future Perspective

Having seen the Higgs mode in s-wave and d-wave superconductors, it is fascinating to extend the observation of the Higgs mode to a variety of conventional/unconventional super-

Figure 38.10: (*a*) Schematic view of the terahertz transmittance experiments under supercurrent injection. The change of the optical conductivity induced by supercurrent injection taken with the THz probe electric field (*b*) parallel or (*c*) perpendicular to the current direction. The optical conductivity spectra measured without the current is also plotted in (*b*). The dashed line represents a fit by the Mattis-Bardeen model [107, 108], and the superconducting gap estimated from the fit is indicated by the vertical arrow. (*d*) Theoretically expected changes of optical conductivity induced by the supercurrent injection with respect to the normal state conductivity σ_N. Figures are taken from Reference [58].

conductors.

The study of the Higgs mode in multiband superconductors would be important, as it would give deeper insights into the interband couplings. For instance, the application of the nonlinear THz spectroscopy to iron-based superconductors is highly intriguing, as it may provide information on the interband interactions and pairing symmetry [87, 119, 120]. The Leggett mode, namely the collective mode associated with the relative phase of two condensates [86] is also expected to be present in multiband superconductors. The Leggett mode has been observed in a multiband superconductor MgB_2 by Raman spectroscopy [121]. The nonlinear coupling between the Leggett mode and THz light has also been discussed [53, 87, 88, 89, 90]. Recently, a THz-pump THz probe study in MgB_2 has been reported, and the results were interpreted in terms of the Leggett mode [122]. However, the dominant contribution of the Higgs mode over the Leggett mode in the nonlinear THz responses was theoretically pointed out in the case of dirty-limit superconductors [53]. Therefore, further experimental verification of the Higgs and Leggett modes is left as a future problem.

The ultimate fate of the behavior of the Higgs mode in a strongly correlated regime and, what is more, in a BCS-BEC crossover regime is an intriguing problem. In particular, the decay profile of the Higgs mode has been predicted to change from the BCS to BEC regime [36, 37, 8, 124, 125]. Experimentally, such a study has been realized in a cold-atom system, showing a broadening of the Higgs mode in the BEC regime [61]. Further investigation on the Higgs mode in the BCS-BEC crossover regime will be a future challenge in solid-state systems.

The Higgs mode in spin-triplet p-wave superconductors (whose candidate materials include Sr_2RuO_4) is another issue to be explored in the future. Like in the case of superfluid ^{3}He [22, 59, 126], multiple Higgs modes are expected to appear due to the spontaneous symmetry breaking in the spin channel, and its comparison with high-energy physics would be interesting.

As yet another new paradigm, the observation of the Higgs mode would pave a new pathway for the study of nonequilibrium phenomena, in particular for the photoinduced superconductivity [127, 128, Hu *et al.*(2014), 130]. Being a fingerprint of the order parameter with a picosecond time resolution, the observation of the Higgs mode in photoinduced states should offer a direct evidence for nonequilibrium superconductivity. In strongly correlated electron systems exemplified by the unconventional superconductors, elucidation of the interplay between competing and/or coexisting orders with superconductivity is an important issue to understand the emergent phases. The time-domain study of collective modes associated with those orders is expected to provide new insights for their interplay.

Part XI
Models of Pump-Probe Spectroscopy of Nonequilibrium Superconductivity

39 Coherent dynamics and pump-probe spectra of BCS superconductors by Papenkort, Axt, Kuhn (2007)

Abstract

The density matrix formalism is employed to calculate pump-probe spectra of BCS superconductors in the coherent regime. Two dynamical regimes, one adiabatic and one nonadiabatic, can be clearly distinguished. In the adiabatic regime, the modulus of the BCS order parameter can be identified with half the gap in the probe spectra. In the nonadiabatic regime, the order parameter oscillates in real time, whereas the gap observed in the spectra reflects its temporal average. The transition between these regimes is accompanied by a qualitative change of the intensity dependence of the dynamical gap renormalization. Furthermore, a hole-burning effect occurs if the spectral shape of the pump pulse is sufficiently sharp. A probe pulse preceding the pump pulse leads to spectral oscillations, and both the gap before and after the pump pulse are visible in the probe spectra.

39.0.1 I. INTRODUCTION

The nonequilibrium dynamics of superconductors has been of active interest for a long time. In recent years, in particular, the ultrafast response to optical excitations has been studied experimentally. [1–4] Most profoundly understood from a theoretical viewpoint are BCS superconductors. Traditional approaches for the nonequilibrium dynamics of the latter systems are based either on the time-dependent Ginzburg-Landau theory [5,6] (TDGL) or the Boltzmann equation [6,7] (BE). The TDGL is a phenomenological description where the dynamical problem is reduced to tracing a single variable, the order parameter $\Delta(t)$. This procedure is applicable when a local thermal equilibrium compatible with the instantaneous value of $\Delta(t)$ is established on a time scale much shorter than the typical variation of $\Delta(t)$. This requirement restricts the validity of the TDGL to only a few limiting cases. The BE approach goes beyond the TDGL by keeping all quasiparticle distributions as dynamical variables, but coherences are left out. It usually also involves a gradient expansion for the spatial and momentum dependences of the distribution functions and is, thus, applicable only when the latter variations are slow.

Recently, BCS-type systems have been analyzed on short time scales for situations far away from thermal equilibrium. Here, a nonadiabatic regime is reached where it is no longer sufficient to follow only the dynamics of quasiparticle distribution functions. In particular, it is necessary to fully account for the time evolution of anomalous expectation values, i.e., coherences between different quasiparticle states. Indeed, a number of recent papers have been devoted to the derivation of explicit solutions to initial-value problems within the framework of the mean-field approximation. [8–11] The initialvalue problems are meant to model a system immediately after a sudden change of parameters, e.g., a fermionic alkali gas where the strength of the pairing interaction can be abruptly readjusted by switching external magnetic fields.

In this paper, we explore the nonadiabatic regime for conventional metallic superconductors driven by ultrashort laser pulses. As the action of the laser is not restricted to a single point in time, we cannot apply the solution to the initialvalue problem; we rather use the density matrix formalism to

model the coherent collisionless response of the system. Such a description has been worked out many years ago for the linear response. [12] Here, we extend this description to the non-linear regime. This approach allows us to fully account for the dynamics of all quasiparticle occupations and coherences within the limits of the mean-field theory.

Driving a system into the regime of nonadiabatic coherent dynamics by ultrafast laser excitations has been routinely done for many systems. This holds not only for systems with long decoherence times like, e.g., atomic systems, but also for systems such as semiconductors with decoherence times on the subpicosecond time scale. [13] For BCS superconductors, however, such scenarios are widely unexplored, even though typical quasiparticle lifetimes are of the order of nanoseconds. [14,15] Resonant studies of BCS systems in the coherent regime require sources generating pulses much shorter than these times with carrier frequencies of the order of the superconductor gap, that is, in the terahertz scale. Only recently, ultrashort terahertz sources have become available [16,17] and, thus, paved the way toward the nonadiabatic regime.

Compared to other materials like semiconductors, such studies for BCS-type systems are interesting because the BCS model has a number of distinct features such as a ground state exhibiting a coherent superposition of states with different numbers of electrons. In view of these unique properties, also the coherent dynamics within the BCS model should be characteristically different from other systems. It is the goal of the present paper to provide a model study revealing the special dynamical behavior of a BCS superconductor driven by ultrafast pulses in the coherent regime. In a first part, we study the reaction of the superconductor to a single laser pulse. We show that by reducing the pulse length, a transition takes place from an adiabatic to a nonadiabatic regime, where the order parameter oscillates and a description only in terms of quasiparticle occupations becomes insufficient. In addition, we calculate pump-probe spectra which are directly observable quantities. Interestingly, the oscillations of the order parameter that occur in the nonadiabatic regime are not reflected by corresponding shifts of the pump-probe spectra. From the pump-probe spectra, the dynamical gap renormalization can be extracted as a function of the pulse intensity. We find that the transition to the nonadiabatic regime is accompanied by a change in the intensity dependence of the shift. The intensity dependence also deviates characteristically from corresponding results for the dynamical Stark shift in atomic or semiconductor systems. [18−21]

39.0.2 II. THEORY

39.0.3 A. Mean-field approximation of the BCS model

We begin with the BCS-Hamiltonian [22,23]

$$H_{\mathrm{BCS}} = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} c^{\dagger}_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} - W_0 \sum_{\mathbf{k},\mathbf{k}' \in \mathcal{W}} c^{\dagger}_{\mathbf{k}'\uparrow} c^{\dagger}_{-\mathbf{k}'\downarrow} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}, \tag{1}$$

where $c^{\dagger}_{\mathbf{k},\sigma}$ and $c_{\mathbf{k},\sigma}$ are the creation and annihilation operators for electrons with momentum $\hbar\mathbf{k}$ and spin $\sigma, \varepsilon_{\mathbf{k}} = \hbar^2\mathbf{k}^2/2m - E_F, m$ is the effective mass, and E_F is the Fermi energy. The second sum describes the attractive interaction induced by electron-phonon coupling; as usual, the retardation is neglected so that the interaction becomes instantaneous. It is restricted to electrons whose energies deviate from E_F by no more than $\hbar\omega_D$ (ω_D is the Debye frequency), so $\mathcal{W}$ is the set of all k vectors with $|\varepsilon_{\mathbf{k}}| \leq \hbar\omega_D$. We have also adopted the standard BCS assumption that pairing only occurs between electrons with opposite momentum and spin orientation, because in a translationally invariant system these are the dominant terms for the formation of the BCS ground state. [24,25] W_0 is a positive number that determines the strength of this interaction.

We now apply a mean-field (MF) approximation to the interaction part of the Hamiltonian: [26] $ab \approx \langle a\rangle b + a\langle b\rangle - \langle a\rangle\langle b\rangle$, with $a = c^\dagger_{\mathbf{k}'\uparrow}c^\dagger_{-\mathbf{k}'\downarrow}$ and $b = c_{-\mathbf{k}\downarrow}c_{\mathbf{k}\uparrow}$. Neglecting a constant, this yields

$$H_{\mathrm{MF}} = \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}} c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} - \Delta \sum_{\mathbf{k}\in\mathcal{W}} c^\dagger_{\mathbf{k}\uparrow\uparrow} c^\dagger_{-\mathbf{k}\downarrow} - \Delta^* \sum_{\mathbf{k}\in\mathcal{W}} c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}, \tag{2}$$

where $\Delta = W_0 \Sigma_{\mathbf{k}\in\mathcal{W}} \langle c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow}\rangle$ is the (complex) mean-field parameter.

We can diagonalize H_{MF} for a fixed Δ with the aid of a Bogoliubov transformation. [23,27] The order parameter Δ can change with time, so we will call the value for which the transformation is done Δ_1 in order to distinguish it more easily from the current value of the order parameter. This transformation is usually done with $\Delta_1 = \Delta_0$, where Δ_0 is the order parameter in the BCS ground state, but we will later on need to diagonalize H_{MF} with an arbitrary complex order parameter. The transformation is defined by two constants $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$, which are allowed to be complex in order to account for a complex Δ_1, and by

$$\begin{aligned} \alpha_{\mathbf{k}} &= u_{\mathbf{k}} c_{\mathbf{k}\uparrow} - v_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow}, \quad \alpha^\dagger_{\mathbf{k}} = u^*_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow} - v^*_{\mathbf{k}} c_{-\mathbf{k}\downarrow}, \\ \beta_{\mathbf{k}} &= u_{\mathbf{k}} c_{-\mathbf{k}\downarrow} + v_{\mathbf{k}} c^\dagger_{\mathbf{k}\uparrow}, \quad \beta^\dagger_{\mathbf{k}} = u^*_{\mathbf{k}} c^\dagger_{-\mathbf{k}\downarrow} + v^*_{\mathbf{k}} c^*_{\mathbf{k}\uparrow}. \end{aligned} \tag{3}$$

This transformation has the important property that as long as the condition $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$ holds, the new quasiparticle operators $\alpha_{\mathbf{k}}, \alpha^\dagger_{\mathbf{k}}, \beta_{\mathbf{k}}$, and $\beta^\dagger_{\mathbf{k}}$ satisfy the fermionic commutation relations just like $c_{\mathbf{k}\uparrow}, c^\dagger_{\mathbf{k}\uparrow}, c_{\mathbf{k}\downarrow}$, and $c^\dagger_{\mathbf{k}\downarrow}$. The following
choice of $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ satisfies this condition and, furthermore, diagonalizes H_{MF} :

$$\begin{aligned} u_{\mathbf{k}} = 0, \quad v_{\mathbf{k}} = 1 \quad & \text{if } \varepsilon_{\mathbf{k}} < -\hbar\omega_D \\ u_{\mathbf{k}} = 1, \quad v_{\mathbf{k}} = 0 \quad & \text{if } \varepsilon_{\mathbf{k}} > \hbar\omega_D \\ \left\{ \begin{array}{c} u_{\mathbf{k}} = \sqrt{\frac{1}{2}\left(1 + \frac{\varepsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right)} \\ v_{\mathbf{k}} = \frac{\Delta_1}{|\Delta_1|}\sqrt{\frac{1}{2}\left(1 - \frac{\varepsilon_{\mathbf{k}}}{E_{\mathbf{k}}}\right)} \end{array} \right\} \quad & \text{if } |\varepsilon_{\mathbf{k}}| \le \hbar\omega_D, \end{aligned} \tag{4}$$

with

$$E_{\mathbf{k}} = \sqrt{\varepsilon^2_{\mathbf{k}} + |\Delta_1|^2} \tag{5}$$

This choice is not mandatory; in particular, a phase parameter remains free, which we have used to ensure that $u_{\mathbf{k}}$ is always real, whereas $v_{\mathbf{k}}$ is complex whenever Δ_1 is complex. For $\Delta_1 = \Delta_0$, the vacuum state defined by $\alpha_{\mathbf{k}}|0\rangle = \beta_{\mathbf{k}}|0\rangle = 0$ is the BCS ground state.

For $\Delta_1 = \Delta$, this transformation turns H_{MF} into the diagonal form (again, an additive constant is being ignored)

$$H_{\mathrm{MF}} = \sum_{\mathbf{k}\notin\mathcal{W}} |\varepsilon_{\mathbf{k}}| \left(\alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}}\right) + \sum_{\mathbf{k}\in\mathcal{W}} E_{\mathbf{k}} \left(\alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}}\right). \tag{6}$$

However, Δ changes with time, and we have to take this into account by one of the following means: First, we can take our quasiparticle basis to be time dependent and include in our equations of motions additional terms which arise through such a time-dependent transformation. [28] Second, we can keep the transformation parameter Δ_1 at a fixed value and take into account that H_{MF} is no longer diagonal. Both options are equivalent; here, we will stick to the second one. For an arbitrary value of Δ, H_{MF} reads

$$H_{\mathrm{MF}} = \sum_{\mathbf{k}\notin\mathcal{W}} |\varepsilon_{\mathbf{k}}| \left(\alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}}\right) + \sum_{\mathbf{k}\in\mathcal{W}} \Big[R_{\mathbf{k}} \left(\alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}} + \beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}}\right) + C^*_{\mathbf{k}}\alpha^\dagger_{\mathbf{k}}\beta^\dagger_{\mathbf{k}} + C_{\mathbf{k}}\beta_{\mathbf{k}}\alpha_{\mathbf{k}}\Big] \tag{7a}$$

with

$$R_{\mathbf{k}} = \frac{\varepsilon^2_{\mathbf{k}} + \mathrm{Re}\left(\Delta^*\Delta_1\right)}{E_{\mathbf{k}}} \tag{7b}$$

and

$$C_{\mathbf{k}} = \Delta^*_1 \left\{ \frac{\varepsilon_{\mathbf{k}}}{E_{\mathbf{k}}} \left[1 - \mathrm{Re}\left(\frac{\Delta}{\Delta_1}\right)\right] + i\,\mathrm{Im}\left(\frac{\Delta}{\Delta_1}\right)\right\} \tag{7c}$$

which, of course, is reduced to the above diagonal form [Eq. (6)] if Δ_1 is equal to Δ.

39.0.4 B. Interaction with an electromagnetic field

The superconductor is exposed to a laser field with wave vector $\mathbf{q}_0 = q_0\mathbf{e}_{\mathbf{x}}$ and angular frequency $\omega_0 = q_0 c$, which in the Coulomb gauge is described by a transverse vector potential

$$\mathbf{A}_{\mathbf{q}}(t) = \mathbf{A}_0 \exp\left[-\left(\frac{2\sqrt{\ln 2}t}{\tau}\right)^2\right] \left(\delta_{\mathbf{q},\mathbf{q}_0} e^{-i\omega_0 t} + \delta_{\mathbf{q},-\mathbf{q}_0} e^{i\omega_0 t}\right) \tag{8}$$

with amplitude $\mathbf{A}_0 = A_0\mathbf{e}_{\mathbf{y}}$ and full width at half maximum (FWHM) τ.
This vector potential affects the electrons through the Hamiltonian

$$H_{\mathrm{em}} = \sum_{\mathbf{k},\mathbf{q},\sigma} \frac{1}{2m} \left[e\hbar(2\mathbf{k}+\mathbf{q})\cdot\mathbf{A}_{\mathbf{q}} + e^2 \sum_{\mathbf{q}_i} \mathbf{A}_{\mathbf{q}-\mathbf{q}_i}\cdot\mathbf{A}_{\mathbf{q}_i}\right] c^\dagger_{\mathbf{k}+\mathbf{q},\sigma} c_{\mathbf{k},\sigma} \tag{9}$$

Using Eq. (3) again, this becomes

$$H_{\mathrm{em}} = H^{(1)}_{\mathrm{em}} + H^{(2)}_{\mathrm{em}} + \text{ complex number} \tag{10a}$$

$$H^{(1)}_{\mathrm{em}} = \frac{e\hbar}{2m} \sum_{\mathbf{k},\mathbf{q}} (2\mathbf{k}+\mathbf{q})\cdot\mathbf{A}_{\mathbf{q}} \Big[\left(u^*_{\mathbf{k}}u_{\mathbf{k}+\mathbf{q}} + v^*_{\mathbf{k}}v_{\mathbf{k}+\mathbf{q}}\right)\alpha^\dagger_{\mathbf{k}+\mathbf{q}}\alpha_{\mathbf{k}} - \left(u_{\mathbf{k}}u^*_{\mathbf{k}+\mathbf{q}} + v_{\mathbf{k}}v^*_{\mathbf{k}+\mathbf{q}}\right)\beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}+\mathbf{q}} + \left(u_{\mathbf{k}+\mathbf{q}}v_{\mathbf{k}} - u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{q}}\right)\alpha^\dagger_{\mathbf{k}+\mathbf{q}}\beta^\dagger_{\mathbf{k}} + \left(u^*_{\mathbf{k}}v^*_{\mathbf{k}+\mathbf{q}} - u^*_{\mathbf{k}+\mathbf{q}}v^*_{\mathbf{k}}\right)\beta_{\mathbf{k}+\mathbf{q}}\alpha_{\mathbf{k}}\Big] \tag{10b}$$

$$H^{(2)}_{\mathrm{em}} = \frac{e^2}{2m} \sum_{\mathbf{k},\mathbf{q}} \sum_{\mathbf{q}_i} \mathbf{A}_{\mathbf{q}-\mathbf{q}_i}\cdot\mathbf{A}_{\mathbf{q}i} \Big[\left(u^*_{\mathbf{k}}u_{\mathbf{k}+\mathbf{q}} - v^*_{\mathbf{k}}v_{\mathbf{k}+\mathbf{q}}\right)\alpha^\dagger_{\mathbf{k}+\mathbf{q}}\alpha_{\mathbf{k}} + \left(u_{\mathbf{k}}u^*_{\mathbf{k}+\mathbf{q}} - v_{\mathbf{k}}v^*_{\mathbf{k}+\mathbf{q}}\right)\beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}+\mathbf{q}} + \left(u_{\mathbf{k}+\mathbf{q}}v_{\mathbf{k}} + u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{q}}\right)\alpha^\dagger_{\mathbf{k}+\mathbf{q}}\beta^\dagger_{\mathbf{k}} + \left(u^*_{\mathbf{k}}v^*_{\mathbf{k}+\mathbf{q}} + u^*_{\mathbf{k}+\mathbf{q}}v^*_{\mathbf{k}}\right)\beta_{\mathbf{k}+\mathbf{q}}\alpha_{\mathbf{k}}\Big] \tag{10c}$$

We are interested in pump-probe spectra, so we need to calculate the electric current density $\mathbf{j}\left(\mathbf{r}_0, t\right)$. The spatial Fourier transform of its Fock space operator is

$$\hat{\mathbf{j}}_{\mathbf{q}_0} = \hat{\mathbf{j}}^{(1)}_{\mathbf{q}_0} + \hat{\mathbf{j}}^{(2)}_{\mathbf{q}_0} \tag{11a}$$

$$\hat{\mathbf{j}}^{(1)}_{\mathbf{q}_0} = \frac{-e\hbar}{2mV} \sum_{\mathbf{k},\sigma} (2\mathbf{k} + \mathbf{q_0})\, c^{\dagger}_{\mathbf{k},\sigma} c_{\mathbf{k}+\mathbf{q}_0,\sigma} \tag{11b}$$

$$\hat{\mathbf{j}}^{(2)}_{\mathbf{q}_0} = -\frac{e^2}{mV} \sum_{\mathbf{k},\mathbf{q},\sigma} \mathbf{A}_{\mathbf{q}_0-\mathbf{q}} c^{\dagger}_{\mathbf{k},\sigma} c_{\mathbf{k}+\mathbf{q},\sigma} \tag{11c}$$

where V is the normalization volume. When calculating the linear spectrum of our probe pulse, we neglect $\mathbf{j}^{(2)}_{\mathbf{q}_0}$ as it only entails an offset in the imaginary part $\omega \operatorname{Im} \sigma$ of the spectrum. [24]

Finally, we transform $\hat{\mathbf{j}}^{(1)}_{\mathbf{q}_0}$ according to Eq. (3):

$$\begin{aligned}\hat{\mathbf{j}}^{(1)}_{\mathbf{q}_0} =& \frac{-e\hbar}{2mV} \sum_{\mathbf{k}} (2\mathbf{k} + \mathbf{q_0}) \Big[\left(u_{\mathbf{k}} v_{\mathbf{k}+\mathbf{q}_0} - u_{\mathbf{k}+\mathbf{q}_0} v_{\mathbf{k}}\right) \alpha^{\dagger}_{\mathbf{k}} \beta^{\dagger}_{\mathbf{k}+\mathbf{q}_0} \\ &+ \left(u^*_{\mathbf{k}+\mathbf{q}_0} v^*_{\mathbf{k}} - u^*_{\mathbf{k}} v^*_{\mathbf{k}+\mathbf{q}_0}\right) \beta_{\mathbf{k}} \alpha_{\mathbf{k}+\mathbf{q}_0} + \left(u_{\mathbf{k}} u^*_{\mathbf{k}+\mathbf{q}_0}\right. \\ &\left. + v^*_{\mathbf{k}} v^*_{\mathbf{k}+\mathbf{q}_0}\right) \alpha^{\dagger}_{\mathbf{k}} \alpha_{\mathbf{k}+\mathbf{q}_0} - \left(u^*_{\mathbf{k}} u_{\mathbf{k}+\mathbf{q}_0} + v^*_{\mathbf{k}} v_{\mathbf{k}+\mathbf{q}_0}\right) \beta^{\dagger}_{\mathbf{k}+\mathbf{q}_0} \beta_{\mathbf{k}} \Big]\end{aligned} \tag{12}$$

39.0.5 C. Calculating the superconductor's response

To determine $\mathbf{j}^{(1)}_{\mathbf{q}_0}$, it obviously suffices to know the following expectation values:

$$\begin{aligned} &\left\langle \alpha^{\dagger}_{\mathbf{k}} \beta^{\dagger}_{\mathbf{k}+\mathbf{q}} \right\rangle, \quad \left\langle \alpha^{\dagger}_{\mathbf{k}} \alpha_{\mathbf{k}+\mathbf{q}} \right\rangle, \\ &\left\langle \beta_{\mathbf{k}} \alpha_{\mathbf{k}+\mathbf{q}} \right\rangle, \quad \left\langle \beta^{\dagger}_{\mathbf{k}+\mathbf{q}} \beta_{\mathbf{k}} \right\rangle. \end{aligned} \tag{13}$$

Starting from Heisenberg's equation of motion with the Hamiltonian $H = H_{\mathrm{MF}} + H_{\mathrm{em}}$ [Eqs. (7a) and (10)], we obtain a set of differential equations for these variables. As an example, the equation of motion for the first variable is shown in the Appendix [Eq. (A1)]. The others have a similar structure. We observe that the equations contain no other expectation values than those in Eq. (13). This means that we have found a closed set of differential equations; that we do not get an infinite hierarchy of equations is a virtue of the meanfield approximation.

There is still an infinite number of $\mathbf{k}$'s and $\mathbf{q}$'s. However by looking closely at the equations of motion, we can see that there is no interaction with the electromagnetic field for every term with $\mathbf{k}, \mathbf{k}+\mathbf{q} \notin \mathcal{W}$, as long as we start from the BCS ground state; so we will impose the restriction $\mathbf{k}, \mathbf{k}+\mathbf{q} \in \mathcal{W}$. [34]

As we can see from Eq. (A1), only expectation values with indices $(\mathbf{k}, \mathbf{k}+n\mathbf{q}_0)$ with an integer n have to be considered. We can further reduce the number of tuples ($\mathbf{k}, \mathbf{k}+n\mathbf{q}_0$) we have to calculate because $H^{(1)}_{\mathrm{em}}$ couples only tuples with $n_1 - n_2 \equiv \Delta n = \pm 1$ and $H^{(2)}_{\mathrm{em}}$ couples only those with $\Delta n \in \{-2, 0, 2\}$. New excitations can only arise for $n = \pm 1$ through $H^{(1)}_{\mathrm{em}}$ and for $n = -2, 0,$ and 2 through $H^{(2)}_{\mathrm{em}}$. Now $H^{(1)}_{\mathrm{em}}$ is proportional to $\mathbf{A}_0$ and $H^{(2)}_{\mathrm{em}}$ is proportional to ${\mathbf{A}_0}^2$. This means that if $\mathbf{A}_0$ is small enough, we may ignore all tuples with $n > N$. We have found that it suffices to set $N = 4$, which we have done in all the calculations presented herein.

For simplicity, we have computed these equations in two and not in three dimensions. Two-dimensional calculations may be of interest on their own, but they should also provide us with a good approximation for the three-dimensional superconductor for the following reason: As we can see from Eqs. (8) and (10), the laser field couples only indices ($\mathbf{k}, \mathbf{k}+n\mathbf{q}_0$) and ($\mathbf{k}', \mathbf{k}'+n'\mathbf{q}_0$) with $\mathbf{k}-\mathbf{k}' = j\mathbf{q}_0$ with an integer j. So, leaving the second index aside for a moment, we have

one-dimensional subspaces of the k space, wherein the matrix elements are strongly coupled, but two different subspaces are completely independent of each other except for the weak indirect coupling through the mean-field parameter Δ, which is, of course, dependent on all of these subspaces and vice versa affects all of them. As all of these subspaces are similar to each other, it seems like a good approximation to choose a very coarse discretization in the directions perpendicular to $\mathbf{q}_0$ or even to reduce the number of dimensions. So we have made our calculations in two dimensions using the rather coarse discretization of 32 points in the direction perpendicular to $\mathbf{q}_0$. Using 128 discretization points instead produces very similar results indeed, and even using only one dimension leads to the same qualitative behavior, as will be seen later.

FIG. 1. (Color online) Time dependence of the order parameter $|\Delta|$. All pump pulses are centered at $t = 0$ and have the same energy. Note that we have plotted $2|\Delta|$ as this is the value that corresponds to the gap energy. The inset shows the results of the onedimensional calculation.

39.0.6 III. RESULTS

For our numerical computation we use the following parameters which are motivated by the experimental values for lead: $^{29}2\Delta_0 = 2.7$meV (Δ_0 is the order parameter for the BCS ground state and, therefore, half the energy gap), $\hbar\omega_D = 8.3$meV, $E_F = 9.47$eV, and $m = 1.9m_0$ with the free electron mass m_0. The pump and probe pulse frequencies are $\omega_0 = 4.50 \times 10^{12}$ s^{-1} and $\omega_p = 3.80 \times 10^{12}$ s^{-1}, so $\hbar\omega_0 = 2.96$meV is greater and $\hbar\omega_p = 2.50$meV is less than the energy gap $2\Delta_0$.

The calculations start from the BCS ground state at zero temperature. Pump and probe pulses are Gaussian in time, but are cut off at a thousandth of their maximum amplitude. The pulses do not overlap.

39.0.7 A. Effects of the pump pulse

Figure 1 shows twice the modulus of the order parameter $|\Delta|$ as a function of time for different widths of the pump pulse. As a consequence of the action of the pump pulse, $|\Delta|$ decreases. We observe an adiabatic and a nonadiabatic regime: If the pulse is sufficiently wide ($\tau = 10$ps), $|\Delta|$ remains constant after the pump pulse has ended. However, for short pump pulses, $|\Delta|$ continues to oscillate even when the pump pulse has been switched off long ago. The amplitude of this oscillation increases with decreasing pulse width. The decrease of $|\Delta|$ is due to the fact that the laser pulse populates the expectation values $\left\langle \alpha_{\mathbf{k}}^{\dagger}\alpha_{\mathbf{k}} \right\rangle$ and $\left\langle \beta_{\mathbf{k}}^{\dagger}\beta_{\mathbf{k}} \right\rangle$, just as an increase in temperature does.

If we change into the instantaneous Bogoliubov space $[\Delta_1 = \Delta(t)]$ for any given time t after the pump pulse has expired, the time derivative of these expectation values is zero as H_{MF} is diagonal. In the adiabatic regime, the expectation value $\left\langle \alpha_{\mathbf{k}}^{\dagger}\beta_{\mathbf{k}}^{\dagger} \right\rangle$ now is almost zero, so $|\Delta|$ stays constant when the pump pulse has subsided. This means that the new state can be described by a quasiparticle distribution without coherences, although the quasiparticles are not the same as in the ground state.

FIG. 2. (Color online) Occupations $\langle \alpha_{\mathbf{k}}^{+}\alpha_{\mathbf{k}} \rangle$ with $\Delta_1 = \Delta(t)$ immediately after the pump pulses. The inset shows how a change in $|\Delta|$ affects the k positions of excitations with a fixed energy.

The short pulse ($\tau = 0.4$ps) lies beyond the adiabatic regime. Setting $\Delta_1 = \Delta(t)$ again, the expectation value $\left\langle \alpha_{\mathbf{k}}^{\dagger} \beta_{\mathbf{k}}^{\dagger} \right\rangle$ is not zero, so a description that takes into account only the quasiparticle distribution is no longer possible. Its time derivative is not zero; as $\left\langle \alpha_{\mathbf{k}}^{\dagger} \beta_{\mathbf{k}}^{\dagger} \right\rangle$ changes, it affects Δ, and then H_{MF} is no longer diagonal, so $\left\langle \alpha_{\mathbf{k}}^{\dagger} \alpha_{\mathbf{k}} \right\rangle$ and $\left\langle \beta_{\mathbf{k}}^{\dagger} \beta_{\mathbf{k}} \right\rangle$ will also change. This gives rise to the oscillation we observe for the short pulses.

As soon as the pump pulse is switched off, we can regard the time evolution as an initial-value problem as it is discussed in Ref. 11. There it is shown that one possible solution for the time dependence of the modulus of the order parameter is an oscillation with frequency $\omega = 2\Delta_\infty/\hbar$ decaying with $1/\sqrt{\Delta_\infty t/\hbar}$, where Δ_∞ is the value $|\Delta(t)|$ asymptotically reaches. Our results are in very good agreement with this prediction.

In Fig. 2, we have plotted the expectation value $\left\langle \alpha_{\mathbf{k}}^{\dagger} \alpha_{\mathbf{k}} \right\rangle$ for a fixed time immediately after the pump pulse. $\left\langle \beta_{\mathbf{k}}^{\dagger} \beta_{\mathbf{k}} \right\rangle$ looks exactly the same. The longest pulse with $\tau = 10$ps creates a population with two pronounced maxima. The positions of these maxima correspond approximately to half the pump pulse energy, $\frac{1}{2}\hbar\omega_0$, because every excitation consists of two Bogoliubovian quasiparticles. The maxima are broadened both by the energy uncertainty of the pump pulse and by the fact that $|\Delta|$ changes while the pump pulse is active. In the inset of Fig. 2, we have sketched how a change in $|\Delta|$ affects the quasiparticle energies and, thereby, shifts the positions of the excitation maxima. The short pulses' energy spectrum is too broad to create such sharp maxima, instead we get a broad population centered at the Fermi wave number k_F. There are also two secondary maxima (e.g., at $k \simeq 2.17284 \times 10^{10}$ m^{-1}) which are best visible for the longest pump pulse. These are created by two-photon processes, as can be seen from the fact that they are rather small and that their k values belong to the full energy of the pump pulse.

Performing the same calculation in one dimension on one of the one-dimensional subspaces with $k_y = 1.120 \times 10^{10}$ m^{-1}, which is roughly $\frac{1}{2}k_F$, yields quasiparticle distributions which are very similar to those presented above. However, there are some differences in the time evolution of the order

parameter, as can be seen in the inset of Fig. 1. In particular, the shift in the energy gap $2|\Delta|$ is slightly different. Nonetheless, the qualitative behavior is the same. The probe spectra, which will be discussed in the next section, again are virtually independent of whether the number of dimensions is 1 or 2 , although, of course, they reflect the changes in the values of $|\Delta|$.

Figure 3 shows the intensity dependence of the shift in $|\Delta|$. The data for this figure have been calculated in only one dimension; as discussed previously, this should be a good approximation for higher dimensional systems. For the longer pulses, the shift starts linearly but then flattens. The flattening occurs because the pump pulses create rather sharp maxima in the quasiparticle distribution, so due to Pauli blocking it gets increasingly difficult to create new quasiparticles with higher intensities. A linear shift is also observed in semiconductors and atoms, e.g., in the dynamical Stark effect. [18–21] For the shortest pump pulse, the shift is quadratic with a small linear term. This is because shorter pulses need higher intensities to yield a noticeable shift and, with these intensities, the quadratic term of the shift dominates.

39.0.8 B. Probe spectra

In Fig. 4, the probe spectra are shown. The pump pulse precedes the probe pulse, and the pulses do not overlap. The probe pulse has a FWHM of 0.25 ps and is calculated in linear approximation. A Hann window is applied to $j^{(1)}(t)$ prior to Fourier transforming it, [30] and the spectrum itself is calculated via $\sigma(\omega) = j^{(1)}(\omega)/E(\omega) = j^{(1)}(\omega)/[i\omega A(\omega)]$.

FIG. 3. (Color online) Intensity dependence of the shift in $|\Delta|$. Δ_∞ is the asymptotic value of $|\Delta|$ for long times.

The first thing we notice about our probe spectra is that both real and imaginary parts have a sharp edge at the same frequency, and the real part is zero below this frequency. This is the energy gap and we expect it to end at $2|\Delta|$, which is correct as long as $|\Delta|$ does not oscillate. If it does oscillate, the edge is located at the average value of $2|\Delta|$; the spectrum does not oscillate itself, i.e., it is independent of the exact timing of the probe pulse, even though the probe pulse width is shorter than one period of the oscillation. So this oscillation cannot be perceived by means of a simple probe spectrum.

The real part of the spectra for the longer pulses additionally exhibits a small dip (cf. inset of Fig. 4). Its corresponding energy is two times the quasiparticle energy of the popu-

lation maxima. This means that the sharp population maxima which the pump pulse has created now inhibit the absorption through Pauli blocking, leading to this hole-burning effect.

Figure 5 shows the spectra for a negative time delay $\delta t = -25$ps, which means that the probe pulse precedes the pump pulse. We observe a spectral oscillation whose period $\delta\omega = 2\pi/(-\delta t)$ is inversely proportional to the delay δt. Such spectral oscillations are well known in semiconductors where the same relation between $\delta\omega$ and δt applies. [31–33] In contrast to typical semiconductor results, both the energy gap before and after the pump pulse, indicated by dashed lines, can be seen in the spectra as sharp features. Physically, under these excitation conditions, the pump pulse modifies the dynamics of the current induced by the preceding probe pulse. The main oscillation frequency of the current, which is the gap

frequency, changes rather abruptly from the equilibrium value to the reduced value determined by the pump pulse.

39.0.9 IV. CONCLUSION

We have calculated the coherent dynamics of a BCS superconductor using the density matrix formalism. Short pump pulses put the system into a coherent superposition state which can no longer be described by its quasiparticle distribution alone. In this nonadiabatic regime, the modulus of the order parameter performs a damped oscillation. In addition, the intensity dependence of the shift in $|\Delta|$ is very different from the one in the adiabatic regime.

The reaction to the pump pulse may be measured with the aid of pump-probe spectroscopy. One, thereby, gains information about maxima in the quasiparticle distribution and about the modulus of the order parameter, but its oscillation remains invisible. If the probe pulse precedes the pump pulse, spectral oscillations occur and both the gap before and after the pump pulse are visible in the spectra as sharp features.

39.0.10 APPENDIX: EQUATIONS OF MOTION

We show the equation of motion for only one of the four expectation values in Eq. (13); the remaining ones are similar. We have used the fact that $\mathbf{A_q}$ is of a special form [see Eq. (8)] and that all $u_{\mathbf{k}}$ are real. $R_{\mathbf{k}}$ and $C_{\mathbf{k}}$ are the same as in Eq. (7a). As discussed previously, we have limited ourselves to the subset of momentum space called $\mathcal{W}$, where the attractive interaction occurs. The term independent of $\mathbf{A_q}(t)$ is, therefore, valid for $\mathbf{k}, \mathbf{k}+\mathbf{q} \in \mathcal{W}$ only.

$$
\begin{aligned}
i\hbar\frac{d}{dt}\left\langle \alpha^\dagger_{\mathbf{k}}\beta^\dagger_{\mathbf{k+q}}\right\rangle = &-\left(R_{\mathbf{k}}+R_{\mathbf{k+q}}\right)\left\langle \alpha^\dagger_{\mathbf{k}}\beta^\dagger_{\mathbf{k+q}}\right\rangle + C^*_{\mathbf{k+q}}\left\langle \alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k+q}}\right\rangle + C^*_{\mathbf{k}}\left(\left\langle \beta^\dagger_{\mathbf{k+q}}\beta_{\mathbf{k}}\right\rangle - \delta_{\mathbf{q},0}\right) + \frac{e\hbar}{2m}\sum_{\mathbf{q}'=\pm\mathbf{q}_0} 2\mathbf{k}\cdot \mathbf{A} \\
&\times\Big[-\left(u_{\mathbf{k}}u_{\mathbf{k+q}'}+v^*_{\mathbf{k}}v_{\mathbf{k+q}'}\right)\left\langle \alpha^\dagger_{\mathbf{k+q}'}\beta^\dagger_{\mathbf{k+q}}\right\rangle + \left(u_{\mathbf{k+q-q}'}u_{\mathbf{k+q}}+v_{\mathbf{k+q-q}'}v^*_{\mathbf{k+q}}\right)\left\langle \alpha^\dagger_{\mathbf{k}}\beta^\dagger_{\mathbf{k+q-q}'}\right\rangle \\
&+\left(u_{\mathbf{k+q-q}'}v^*_{\mathbf{k+q}}-u_{\mathbf{k+q}}v^*_{\mathbf{k+q-q}'}\right)\left\langle \alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k+q-q}'}\right\rangle + \left(u_{\mathbf{k}}v^*_{\mathbf{k+q}'}-u_{\mathbf{k+q}'}v^*_{\mathbf{k}}\right)\left(\left\langle \beta^\dagger_{\mathbf{k+q}}\beta_{\mathbf{k+q}'}\right\rangle - \delta_{\mathbf{q},} \right. \\
&+\frac{e^2}{2m}\sum_{\mathbf{q}'=0,\pm 2\mathbf{q}_0}\left(\sum_{\mathbf{q}_i=\pm\mathbf{q}_0}\mathbf{A}_{\mathbf{q}'-\mathbf{q}_i}(t)\cdot\mathbf{A}_{\mathbf{q}_i}(t)\right)\Big[-\left(u_{\mathbf{k}}u_{\mathbf{k+q}'}-v^*_{\mathbf{k}}v_{\mathbf{k+q}'}\right)\left\langle \alpha^\dagger_{\mathbf{k+q}'}\beta^\dagger_{\mathbf{k+q}}\right\rangle \\
&-\left(u_{\mathbf{k+q}}u_{\mathbf{k+q-q}'}-v^*_{\mathbf{k+q}}v_{\mathbf{k+q-q}'}\right)\left\langle \alpha^\dagger_{\mathbf{k}}\beta^\dagger_{\mathbf{k+q-q}'}\right\rangle + \left(u_{\mathbf{k+q}}v^*_{\mathbf{k+q-q}'}+u_{\mathbf{k+q-q}'}v^*_{\mathbf{k+q}}\right)\left\langle \alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k+q-}} \right. \\
&+\left(u_{\mathbf{k}}v^*_{\mathbf{k+q}'}+u_{\mathbf{k+q}'}v^*_{\mathbf{k}}\right)\left(\left\langle \beta^\dagger_{\mathbf{k+q}}\beta_{\mathbf{k+q}'}\right\rangle - \delta_{\mathbf{q},\mathbf{q}'}\right)\Big]. \qquad \text{(A1)}
\end{aligned}
$$

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FIG. 4. (Color online) Real and imaginary parts of the probe spectra. The insets magnify the dashed boxes.

FIG. 5. Real part of probe spectra with negative time delay $\delta t = -25$ps between pump and probe pulses. The dashed lines show the energy gap before and after the pump pulse.

40 Kinetics of a superconductor excited with a femtosecond optical pulse by Kabanov, Demsar, Mihailovic

Abstract

Superconducting state dynamics following excitation of a superconductor with a femtosecond optical pulse is studied in terms of a phenomenological Rothwarf and Taylor model. Analytical solutions for various limiting cases are obtained. The model is found to account for the intensity and temperature dependence of both photoinduced quasiparticle density, as well as pair-breaking and superconducting state recovery dynamics in conventional as well as cuprate superconductors.

In recent years numerous studies of non-equilibrium carrier dynamics in superconductors (SC) have been performed utilizing femtosecond real-time techniques [1-18]. Research focused on the identification of relaxation processes and direct measurements of the relaxation times. One of the open issues at the moment is whether cuprates are in the so called phonon bottleneck regime as conventional SCs[19], or in the weak bottleneck regime, where relaxation is governed by the biparticle recombination kinetics [9, 17, 18]. The theoretical model that has been most commonly used to interpret the dynamics is a phenomenological Rothwarf-Taylor (RT) model which describes the evolution of quasiparticle (QP) and high frequency phonon (HFP) populations via a set of two non-linear differential equations[20], which were shown recently[21] to follow from the general set of kinetic equations for a SC[22]. While the RT model has been known for almost 40 years, no rigorous attempt to solve it has been made thus far, and neither has a comparison to the experimental data been made.

In this Letter we present a detailed study of the evolution of the SC state following excitation by ultrashort laser pulse using the RT model. We have obtained analytical solutions of the model in the limit of a strong and a weak bottleneck, which are in excellent agreement with numerical simulations. The solutions enable comparison of the model to the experimental results. We show that RT model can account for most of the experimental observations in conventional as well as cuprate SC, and that both conventional and cuprates SCs are in the strong bottleneck regime, where SC state recovery is governed by the HFP decay dynamics.

Rothwarf and Taylor have pointed out that the phonon channel should be considered when studying the SC relaxation[20]. When two QPs with energies $\geq \Delta$, where Δ is the SC gap, recombine a HFP ($\omega > 2\Delta$) is created. Since HFP can subsequently break a Cooper pair creating two QPs the SC recovery is governed by the decay of the HFP population. The dynamics of QP and HFP populations is determined by[20]:

$$\begin{aligned} dn/dt &= I_0 + \eta N - Rn^2 \\ dN/dt &= J_0 - \eta N/2 + Rn^2/2 - \gamma(N - N_T) \end{aligned} \tag{40.1}$$

Here n and N are concentrations of QPs and HFPs, respectively, η is the probability for pair-breaking by HFP absorption, and R the bare QP recombination rate with the creation of a HFP. N_T is the concentration of HFP in thermal equilibrium at temperature T, and γ their decay rate. I_0 and J_0 represent the external sources of QPs and HFPs, respectively [12]. Physically γ is governed by the fastest of the two processes: anharmonic decay of HFP ($\omega < 2\Delta$ phonons do not have sufficient energy to break Cooper-pairs) [4] and the diffusion of HFP into the substrate [19]. While the main T-dependence in γ appears near T_c[4], some T-dependence is expected at low-T as well. However, as shown in Refs.[23, 24] this T-dependence is very weak in the temperature range of interest and we can consider γ to be T-independent.

Since η/R has the dimensionality of concentration we introduce dimensionless QP and HFP concentrations, $q \equiv Rn/\eta$ and $p \equiv RN/\eta$, while $\theta = \eta t$ and $\tilde{\gamma} \equiv \gamma/\eta$ are the dimensionless

time and HFP decay rate, respectively. Since a femtosecond optical pulse is usually shorter than characteristic timescales of SC dynamics, I_0 and J_0 could be approximated by δ-functions leading to the initial concentrations of QPs $q_0 = Rn_0/\eta$ and HFP $p_0 = RN_0/\eta$ [12]. As a result Eqs. (40.1) reduce to

$$dq/d\theta = p - q^2 \tag{40.2}$$
$$dp/d\theta = -p/2 + q^2/2 - \tilde{\gamma}(p - p_T). \tag{40.3}$$

with the initial conditions $p(0) = p_0$, $q(0) = q_0$. Here $p_T = RN_T/\eta$ is the dimensionless concentration of HFPs in thermal equilibrium. Thermal equilibrium concentrations of HFPs and QPs (q_T) satisfy the detailed balance equation $p_T = q_T^2$.

We investigated various limiting cases. We refer to the situation when the photoinduced density is small $(p_0 - p_T), (q_0 - q_T) \ll 1$ as a weak perturbation regime, while the opposite case is referred to as a strong perturbation. In case $\tilde{\gamma} \ll 1$ we have a strong bottleneck (quasi-equilibrium between QPs and HFPs is established, and SC state recovery is governed entirely by $\tilde{\gamma}$) while in the opposite case ($\tilde{\gamma} \gg 1$) SC is in a weak bottleneck regime.

It is useful to estimate the values of thermal QP and HFP concentrations and compare them with η/R. If $D(\omega) = 9\nu\omega^2/\omega_D^3$ is the phonon density of states with ν the number of atoms per unit cell and ω_D the Debye energy, it follows that $N_T = \frac{36\nu\Delta^2 T}{\omega_D^3}\exp(-2\Delta/k_B T)$. On the other hand the QP density per unit cell at temperature T is given by [4]:

$$n_T \simeq N(0)\sqrt{2\pi\Delta k_B T}\exp(-\Delta/T). \tag{40.4}$$

Here $N(0)$ is the electronic density of states per unit cell. From $\eta/R = n_T^2/N_T$ follows that $\eta/R = \frac{N(0)^2\pi\omega_D^3}{18\nu\Delta}$. Regardless of the value of η/R, it follows that the high perturbation limit is reached when the photoexcitation density is close to the density required for complete depletion of the SC state. To show this, we estimate the temperature $\widetilde{T}$ where $q_{\tilde{T}} \approx 1$ ($n_{\tilde{T}} \approx \eta/R$). It follows that $\widetilde{T} \approx \Delta/\ln(\frac{\Delta^2\nu E_F}{\omega_D^3})$; thus n_T is comparable to η/R only in the close vicinity of T_c.

Strong bottleneck regime. Let us consider the solution of Eqs.(40.2,40.3) for the case when $\tilde{\gamma} \ll 1$. In this case two distinct regimes can be defined: (I) prebottleneck dynamics, describing the processes on timescale much shorter than $\tilde{\gamma}^{-1}$, and (II) the SC state recovery at $\theta \geq 1/\tilde{\gamma}$.

(I) Prebottleneck dynamics describes the short timescale evolution of concentrations $p(\theta)$ and $q(\theta)$ preceding the relaxation. For such short times we can neglect the last term in Eq.(40.3). It leads to the conservation law $q(\theta) + 2p(\theta) = q_0 + 2p_0$ and exact solutions for QP and HFP populations, respectively, in the pre-bottleneck regime are obtained[12]:

$$q(\theta) = \left[-\frac{1}{4} - \frac{\xi^{-1}}{2} + \frac{\xi^{-1}}{1 - K\exp(-\theta/\xi)}\right] \tag{40.5}$$
$$p(\theta) = \frac{1}{2}\left[\frac{1}{8} + \frac{\xi^{-1}}{2} + \frac{\xi^{-2}}{2} - \frac{\xi^{-1}}{1 - K\exp(-\theta/\xi)}\right] \tag{40.6}$$

Here $\xi^{-1} = \sqrt{1/4 + 4p_0 + 2q_0}$ and $K = \frac{(4q_0+1)-2\xi^{-1}}{4q_0+1)+2\xi^{-1}}$. At times $\theta > 1$ concentrations $q(\theta)$ and $p(\theta)$ reach the quasi stationary solution

$$q_s = \frac{1}{4}(\sqrt{1 + 16p_0 + 8q_0} - 1) \quad ; \qquad p_s = q_s^2. \tag{40.7}$$

The prebottleneck dynamics depends on the initial conditions, and has two distinct regimes characterized by the parameter K. The regime $0 < K \leq 1$ corresponds to the situation when $q_0 > q_s$, and $q(\theta)$ rapidly decreases during the formation of the bottleneck - Fig. 1(a). On the

other hand $-1 \leq K < 0$ represents the situation when $p_0 > p_s$, and the QP density increases up to times $\theta \approx \tilde{\gamma}^{-1}$ - Fig. 1(b). This situation is realized in MgB_2[12].

(II) Superconducting state recovery dynamics. At large times ($\theta \geq 1/\tilde{\gamma}$) right hand side in Eqs.(40.2,40.3) is determined by the HFP decay term. To describe the recovery we note that at large times the difference $q^2 - p \approx \tilde{\gamma}(p - p_T)$ while $p(\theta)$ and $q(\theta)$ are slowly decreasing. We introduce a function $s(\theta) = q(\theta)^2 - p(\theta)$. Cleary $s(\theta) \ll p, q$ and is slowly decaying ($ds/d\theta \ll s$). By neglecting the derivative $ds/d\theta$ it follows from the definition of $s(\theta)$ that $2q\frac{dq}{d\theta} = \frac{dp}{d\theta}$, which simplifies Eqs.(40.2,40.3) to

$$(1 + 2\tilde{\gamma} + 4q(\theta))\frac{dq(\theta)}{d\theta} = -2\tilde{\gamma}(q(\theta)^2 - q_T^2) \quad . \tag{40.8}$$

After integration, we obtain an analytical solution:

$$-2\tilde{\gamma}\theta = (2 + \frac{1 + 2\tilde{\gamma}}{2q_T}) \ln [\frac{q - q_T}{q_s - q_T}] + (2 - \frac{1 + 2\tilde{\gamma}}{2q_T}) \ln [\frac{q + q_T}{q_s + q_T}] \tag{40.9}$$

Combining solutions for prebottleneck and recovery dynamics we obtain an approximate solution for $q(\theta)$ valid over the entire time range

$$q(\theta) = q_1(\theta) + q_2(\theta) - q_s, \tag{40.10}$$

Here $q_1(\theta)$ and $q_2(\theta)$ are determined by Eqs.(40.5) and (40.9) respectively. To illustrate that Eq.(40.10) describes the solutions of Eqs.(40.2,40.3) we compare the numerical solution with approximation given by Eq.(40.10). Fig.1 shows excellent agreement for all the limiting cases.

Figure 40.1: Analytical solutions of RT equations (open symbols) compared to numerical solutions (solid lines) in several limiting cases at $T = 0$ K. Panels (a) and (b) represent the strong bottleneck limit ($\tilde{\gamma} = 0.1$) for different initial conditions (see text). Analytical solution for $q(\theta)$ is given by Eq.(40.10). Panel (c) presents the weak bottleneck case ($\tilde{\gamma} = 10$). The inset to panel (a) presents the inverse of the photoinduced QP density as a function of θ for various temperatures (q_T). At low-T ($q_T \longrightarrow 0$) the time dependence is linear, mimicking the bimolecular kinetics, while at higher T it becomes nearly exponential (dashed curve presents an exponential decay fit). Indeed, similar behavior was recently reported [18].

Eq.(40.9) can be simplified in the low and high-T limits:

i) Low temperature limit ($q_T \ll 1$).
In this case $2 \ll \frac{1+2\tilde{\gamma}}{2q_T}$ and Eq.(40.9) reduces to:

$$q(\theta) = q_T + \frac{2q_T(q_s - q_T)\exp(-\theta/\tau_T)}{(q_s + q_T) - (q_s - q_T)\exp(-\theta/\tau_T)} \tag{40.11}$$

The relaxation rate $\tau_T^{-1} = 4\tilde{\gamma}q_T/(1+2\tilde{\gamma})$ is strongly reduced by the additional T-dependent factor $q_T \propto n_T$. The behavior is consistent with the T-dependence of the relaxation time τ in most of the SCs: cuprate[9, 8, 7] and conventional [12, 14]. The low-T divergence of τ was originally attributed to the bi-particle prebottleneck dynamics[9] where τ is divergent due to strong reduction of the thermal QP density. Here we show that at low-T τ diverges also in the case of a well established bottleneck.

In the limit of $\theta/\tau_T \ll 1$ Eq.(40.11) further reduces to $q(\theta) = q_s/(1+\frac{2q_s\tilde{\gamma}\theta}{1+2\tilde{\gamma}})$, showing identical dynamics to that of the biparticle recombination - see inset to Fig. 1(a). If one defines the relaxation rate as the slope $\tau^{-1} = |dq(\theta)/d\theta|/(q_s - q_T)$ at $\theta \to 0$, as proposed in Ref.[17], the SC relaxation rate is intensity dependent

$$\tau^{-1} = \frac{2(q_s + q_T)\tilde{\gamma}}{(1+2\tilde{\gamma})}. \tag{40.12}$$

The intensity dependence is observed only if $q_s \gg q_T$; in this case τ^{-1} is proportional to the excitation intensity. It follows from Eq.(40.12), that the observation of pronounced intensity dependence of τ is constrained only to the very low-T (determined by the Δ/kT_c ratio) and is more likely to be observed in cuprates than in conventional SCs. Indeed, the intensity dependent recovery dynamics has been recently observed at low-T in BiSCO[18], in agreement with the solution of RT equations in the strong bottleneck regime - inset to Fig. 1(a).

ii) High temperature limit ($q_T \gtrsim 1$).
In this case $2 \gg \frac{1+2\tilde{\gamma}}{2q_T}$ and Eq.(40.9) reduces to:

$$q(\theta) = \sqrt{q_T^2 + (q_s^2 - q_T^2)\exp(-\theta/\tau_T)} \tag{40.13}$$

Here the relaxation rate $\tau_T^{-1} = \tilde{\gamma}$ is weakly T-dependent (due to intrinsic T-dependence of $\tilde{\gamma}$), and does not depend on q_s or p_s, implying that the relaxation rate is intensity independent. The same is true also for τ^{-1}.

Weak bottleneck ($\tilde{\gamma} \gg 1$). In this case at short times $\theta \approx \tilde{\gamma}^{-1} \ll 1$ the last term in Eq.(40.3) is dominant, and the solution has the form

$$p(\theta) = p_T + (p_0 - p_T)\exp(-\tilde{\gamma}\theta)\ . \tag{40.14}$$

The HFP density reaches its thermodynamic value on a time scale determined by $\tilde{\gamma}^{-1}$. On such a short time scale QP density has not been changed, therefore we can substitute p with p_T in Eq.(40.2). Taking into account that $q_T^2 = p_T$ we obtain the solution:

$$q(\theta) = \frac{q_T[q_0 + q_T + (q_0 - q_T)\exp(-2q_T\theta)]}{q_0 + q_T - (q_0 - q_T)\exp(-2q_T\theta)}. \tag{40.15}$$

Eqs.(40.14,40.15) present the solution of Eqs.(40.2,40.3) for all timescales. As seen in Fig. 1(c) the difference between the analytical and numerical solutions is negligible. Similar to the case of a strong bottleneck the relaxation rate determined as the slope of the signal at $\theta = 0$ is found to be intensity dependent if $q_0 \gg q_T$ with $\tau^{-1} = (q_0 + q_T)$. The main difference between the strong and weak bottleneck cases is that the absolute value of τ^{-1} is in the strong bottleneck case reduced by the HFP decay time.

Photoinduced QP density. Earlier [4] we have discussed the T and excitation intensity dependence of the photoinduced signal amplitude Q (proportional to the photoinduced QP density) assuming the bottleneck condition and the energy conservation law. These results were recently confirmed using a realistic phonon density of states[25]. However, solution of Eqs.(40.2,40.3) allows unambiguous comparison of $Q(T)$ with the expected T-dependence in a SC - without an additional assumption of the energy conservation as in Refs.[4, 25].

In the strong bottleneck regime initial dynamics leads to the stationary densities of QPs and HFPs determined by Eqs.(40.7). The initial concentrations can be written as $q_0 = q_T + \Delta q$ and $p_0 = p_T + \Delta p$, where Δq, Δp are photoinduced initial concentrations of QPs and HFPs, respectively. Assuming weak perturbation limit Δp, $\Delta q \ll 1$ the amplitude of the signal is given by:

$$Q(T) \propto q_s - q_T \approx \frac{2\Delta p + \Delta q}{4\sqrt{1 + 16p_T + 8q_T}} \tag{40.16}$$

Taking into account that $p_T = q_T^2$ and $q_T = Rn_T/\eta$, and normalizing Q to its low-T value, $A = Q/Q_{T\to 0K}$, we obtain $n_T \propto A^{-1} - 1$. This relation is expected to be general and valid for any SC irrespective on the gap symmetry allowing direct estimation of the T-dependence of QP density in thermal equilibrium.

Importantly, in the weak bottleneck case Q should be proportional to the pump intensity, and T-independent at low-T since $q(\theta = 0) = q_0$, while increasing when $T \to T_c$ due to closing of the SC gap. This T-dependence has not been observed thus far.

In Fig. 2 we plot the T-dependences of n_T (extracted from the raw data via $n_T \propto A^{-1} - 1$) and the SC recovery time τ for MgB_2 and the cuprate superconductor $Tl_2Ba_2CuO_{6+\delta}$. The two datasets capture the main features observed on other conventional [10, 14] or cuprate [4, 5, 9, 6, 7, 8] superconductors. In both Q gradually decreases upon increasing T towards T_c [4, 5, 9, 6, 7, 8, 10, 14], while the low-T relaxation time increases when $T \longrightarrow 0$. The only exception thus far was the near optimally doped YBCO, where τ is roughly constant at low-T [4, 6]. The main difference between cuprates and conventional SC is a 1-2 orders of magnitude difference in τ, which can be understood in terms of an order of magnitude difference in the value of the SC energy gap[16].

While at low-T and weak perturbations both strong and weak bottleneck scenarios suggest slowing down of the relaxation upon cooling as well as intensity dependent relaxation, the fact that the amplitude gradually decreases upon T-increase suggests that all SC studied thus far are in the strong bottleneck regime (in the weak bottleneck scenario Q should increase upon increasing T). In order to test the agreement of the model with the data, we fit n_T with Eq.(40.4), where BCS form for $\Delta(T)$ was used. We find good agreement between the extracted values for $2\Delta/T_c$ from the fit with the values from other experimental techniques[16, 26].

We fit the T-dependence of the relaxation time τ with Eq.(40.12). Assuming $\tilde{\gamma}$ to be T-independent and expressing q_T with Eq.(40.4), Eq.(40.12) is rewritten to $\tau(T) = 1/(\Phi + A(\Delta T)^{1/2}\exp(-\Delta/T))$. Here $\Phi = B(q_s - q_T)$, while A and B are fitting parameters. Since the T-dependence of $q_s - q_T$ is measured directly - see Eq.(40.16), we are left with 3 fitting parameters: A, B and Δ. The T-dependence of τ is governed mainly by $\exp(-\Delta/T)$ term, while A and B account for the magnitude of q_T and $(q_s - q_T)$ respectively. At the intermediate temperatures $q_s - q_T \ll q_T$, and τ is governed by the T-dependence of q_T showing $\exp(\Delta/T)$ behavior. At low enough T, however, $q_s \gg q_T$ and the relaxation time saturates. The values of Δ extracted from relaxation time data are somewhat lower than the values extracted from the fit to n_T. This can be partially attributed to the fact that the T-dependence of $\tilde{\gamma}$ was neglected. More importantly, improper (or missing) treatment of the laser heating can also cause an underestimate of Δ obtained from the fit. Namely, due to laser excitation the equilibrium T of the probed spot can be substantially higher than that of the cold finger[27], especially at low-T. Therefore, the T-dependence of τ is not as steep as if the temperature scale was corrected for

Figure 40.2: Comparison of the experimental data on T-dependence of n_T and τ on a) MgB_2 [12, 13] and b) $Tl_2Ba_2CuO_{6+\delta}$ [5] with the RT solution. n_T is extracted from the raw data (shown in insets) using $n_T \propto A^{-1} - 1$ and fit with Eq.(40.4), while τ was fit with Eq.(40.12). The extracted values of $2\Delta/T_c$ ratio are also shown.

heating, giving rise to an underestimate of Δ. Note that heating is particularly pronounced in experiments using high repetition rate laser systems like in Ref.[5]. Finally, we should note that the value of Δ extracted is also somewhat dependent on the T-dependence of Δ, i.e. when fitting we used a BCS form. However, the accuracy of the available experimental data is not sufficient enough to speculate about the deviation of the temperature dependence from the BCS functional form.

In conclusion, we studied the evolution of a SC state following perturbation with ultrashort optical pulses. We derived analytical solutions of RT model for the strong and weak bottleneck regimes. Analytical solutions account for both pair-breaking dynamics[12] as well as SC state recovery, and are in excellent agreement with numerical results. They enable comparison to experimentally measured quantities like the SC state recovery time and the photoinduced QP density. Comparison with published data suggest that both cuprate and conventional SCs follow the RT scenario, both being in the strong bottleneck regime.

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41 Signatures of nonadiabatic BCS state dynamics in pump-probe conductivity by Krull, Manske, Uhrig, Schnyder

Abstract

We theoretically study the pump-probe response of nonequilibrium BCS superconductors coupled to optical phonons. For ultrashort pump pulses a nonadiabatic regime emerges, which is characterized by oscillations of the superconducting order parameter as well as by the generation of coherent phonons. Using the density-matrix formalism, we compute the pump-probe response in the nonadiabatic regime of the coupled Bogoliubov quasiparticle-phonon system and determine the signatures of the order parameter and the phonon oscillations in the pump-probe conductivity. We find that the nonadiabatic dynamics of the BCS superconductor reflects itself in oscillations of the pump-probe response as functions of delay time δt between pump and probe pulses. We argue that from the

analysis of this oscillatory behavior both frequency and decay time of the algebraically decaying order-parameter oscillations can be inferred. Similarly, the coherent phonons are evidenced in the pump-probe conductivity by oscillations with the frequency of the phonons. Remarkably, we find that the oscillatory response in the pump-probe conductivity is resonantly enhanced when the frequency of the order-parameter oscillations is tuned to the phonon energy.

41.1 Introduction

The nonequilibrium response of superconductors has been the subject of considerable interest for a long time [Kopnin(2001), Orenstein(2012)]. In the past few years, fueled by recent advances in femtosecond terahertz (THz) laser technology [Fülöp et al.(2012)Fülöp, Pálfalvi, Klingebiel, Almási, Krausz, Karsch, and Hebling], many experimental [Matsunaga et al.(2013)Matsunaga, Hamada, Makise, Uzawa, Terai, Wang, and Shimano, Matsunaga and Shimano(2012), Mansart et al.(2013)Mansart, Lorenzana, Mann, Odeh, Scarongella, Chergui, and Carbone, Beck et al.(2013)Beck, Rousseau, Klammer, Leiderer, Mittendorff, Winnerl, Helm, Gol'tsman, and Demsar, Dal Conte et al.(2012)Dal Conte, Giannetti, Coslovich, Cilento, Bossini, Abebaw, Banfi, Ferrini, Eisaki, Greven et al., Beck et al.(2011)Beck, Klammer, Lang, Leiderer, Kabanov, Gol'tsman, and Demsar, Rettig et al.(2012)Rettig, Cortés, Thirupathaiah, Gegenwart, Jeevan, Wolf, Fink, and Bovensiepen, Pashkin et al.(2010)Pashkin, Porer, Beyer, Kim, Dubroka, Bernhard, Yao, Dagan, Hackl, Erb et al., Mansart et al.(2009)Mansart, Boschetto, Savoia, Rullier-Albenque, Forget, Colson, Rousse, and Marsi, Saichu et al.(2009)Saichu, Mahns, Goos, Binder, May, Singer, Schulz, Rusydi, Unterhinninghofen, Manske et al., Perfetti et al.(2007)Perfetti, Loukakos, Lisowski, Bovensiepen, Eisaki, and Wolf, Kaindl et al.(2005)Kaindl, Carnahan, Chemla, Oh, and Eckstein, Averitt et al.(2001)Averitt, Rodriguez, Lobad, Siders, Trugman, and Taylor, Kaindl et al.(2000)Kaindl, Woerner, Elsaesser, Smith, Ryan, Farnan, McCurry, and Walmsley] and theoretical [Volkov and Kogan(1974), Amin et al.(2004)Amin, Bezuglyi, Kijko, and Omelyanchouk, Barankov *et al.*(2004)Barankov, Levitov, and Spivak, Yuzbashyan et al.(2005a)Yuzbashyan, Altshuler, Kuznetsov, and Enolskii, Yuzbashyan et al.(2006)Yuzbashyan, Tsyplyatyev, and Altshuler, Nicol and Carbotte(2003), Kabanov et al.(2005)Kabanov, Demsar, and Mihailovic, Unterhinninghofen et al.(2008)Unterhinninghofen, Manske, and Knorr, Papenkort et al.(2007)Papenkort, Axt, and Kuhn, Papenkort et al.(2008)Papenkort, Kuhn, and Axt, Papenkort et al.(2009)Papenkort, Kuhn, and Axt, Schnyder et al.(2011)Schnyder, Manske, and Avella, Akbari et al.(2013)Akbari, Schnyder, Manske, and Eremin, Zachmann et al.(2013)Zachmann, Croitoru, Vagov, Axt, Papenkort, and Kuhn] studies have focused on the ultrafast time response of the BCS state subjected to nonadiabatic perturbations. For example, Matsunaga *et al.*[Matsunaga and Shimano(2012), Matsunaga et al.(2013)Matsunaga, Hamada, Makise, Uzawa, Terai, Wang, and Shimano] have employed THz pump-THz probe spectroscopy to investigate the nonadiabatic dynamics of superconducting thin films after the injection of Bogoliubov quasiparticles with energies just above the gap edge. Transient oscillations with the frequency of the amplitude mode of the superconducting order parameter have been observed in these measurements. Besides pump-probe experiments, also tunneling spectroscopy has recently been used to study the ultrafast response of superconducting tunnel junctions [Ponomarev et al.(2013)Ponomarev, Van, Kuzmichev, Kulbachinskii, Mikheev, Sudakova, and Tchesnokov].

On the theory side, intensive efforts have been devoted to the analysis of persistent and damped oscillations in the Bogoliubov quasiparticle occupations, which are induced by either an interaction quench [Volkov and Kogan(1974), Amin et al.(2004)Amin, Bezuglyi, Kijko, and Omelyanchouk, Barankov *et al.*(2004)Barankov, Levitov, and Spivak, Yuzbashyan

et al.(2005a)Yuzbashyan, Altshuler, Kuznetsov, and Enolskii, Yuzbashyan et al.(2006)Yuzbashyan, Tsyplyatyev, and Altshuler] or by ultrafast photoexcitations [Akbari et al.(2013)Akbari, Schnyder, Manske, and Eremin, Zachmann et al.(2013)Zachmann, Croitoru, Vagov, Axt, Papenkort, and Kuhn, Schnyder et al.(2011)Schnyder, Manske, and Avella, Papenkort et al.(2007)Papenkort, Axt, and Kuhn, Papenkort et al.(2008)Papenkort, Kuhn, and Axt, Papenkort et al.(2009)Papenkort, Kuhn, and Axt]. In particular, it has been shown that excitation pulses with pulse duration $\tau_{\rm p}$, see Eq. (41.2), shorter than the inverse BCS gap energy $|\Delta|$, i.e., $\tau_{\rm p} \ll h/(2|\Delta|)$, generate coherent oscillations in the order parameter amplitude and in the quasiparticle occupations with frequency $\omega_{\Delta_\infty} \approx 2|\Delta|/\hbar$ [Volkov and Kogan(1974), Amin et al.(2004)Amin, Bezuglyi, Kijko, and Omelyanchouk, Barankov *et al.*(2004)Barankov, Levitov, and Spivak, Yuzbashyan et al.(2005a)Yuzbashyan, Altshuler, Kuznetsov, and Enolskii, Yuzbashyan et al.(2006)Yuzbashyan, Tsyplyatyev, and Altshuler]. Moreover, in the presence of an optical-phonon branch with energy of the order of $\hbar\omega_{\rm ph} \approx 2|\Delta|$, coherent quasipersistent phonon oscillations can be generated [Schnyder et al.(2011)Schnyder, Manske, and Avella]. While the creation of these coherent oscillations is well understood theoretically, it has remained an open question whether and how this nonadiabatic BCS state dynamics could be observed in experimental pump-probe spectra [Papenkort et al.(2008)Papenkort, Kuhn, and Axt].

In this paper, we numerically simulate the pump-probe response of nonequilibrium BCS superconductors coupled to optical-phonon modes and determine the signatures of the coherent Cooper pair and phonon dynamics in the pump-probe conductivity. Our aim is to provide systematic calculations of the pump-probe response, such that experimental measurements can more easily be interpreted and compared to the theoretical predictions. Using density-matrix theory [Rossi and Kuhn(2002)], we derive equations of motion for the coupled Bogoliubov quasiparticle-phonon system treating the phonons at a fully quantum kinetic level. The pump-pulse-induced dynamics of this model is investigated in the collisionless regime, i.e., at time scales shorter than the quasiparticle relaxation time, for which BCS mean-field theory is applicable [Volkov and Kogan(1974), Yuzbashyan et al.(2005b)Yuzbashyan, Altshuler, Kuznetsov, and Enolskii]. The pump-probe experiments of Matsunaga *et al.* [Matsunaga and Shimano(2012), Matsunaga et al.(2013)Matsunaga, Hamada, Makise, Uzawa, Terai, Wang, and Shimano] have shown that in $Nb_{1-x}Ti_xN$ thin films this collisionless regime lasts up to 10 ps, considerably longer than the period of the order parameter oscillations, which is of the order of 1 ps.

We study different hierarchies of the involved time scales, focusing for the most part on the case where the pump-pulse length $\tau_{\rm p}$ is much shorter than both the phonon period $\tau_{\rm ph} = (2\pi)/\omega_{\rm ph}$ and the dynamical time scale of the superconductor $\tau_\Delta \approx h/(2|\Delta|)$. In this regime, both the phononic and quasiparticle subsystems evolve in a nonadiabatic fashion, leading to order-parameter oscillations and the creation of coherent phonons. We present a detailed analysis of the generic features in the pump-probe response resulting from these oscillatory behaviors. In particular, it is shown that the coherent Cooper pair dynamics as well as the finite density of coherent phonons produce oscillations in the pump-probe conductivity as a function of the delay time δt between pump and probe pulses (see Figs. 41.1-41.4). Interestingly, the pump-probe signal is resonantly enhanced and exhibits strong nondecaying oscillations when the frequency of the order-parameter oscillation is tuned to the phonon energy. This condition can be achieved by adjusting the integrated pump laser intensity (see Fig. 41.5).

The remainder of this paper is organized as follows. In Sect. 41.2 we introduce the microscopic Hamiltonian of the model and specify the parameters for our numerical simulations. The equations of motion are derived in Sect. 41.3 using density-matrix theory. Our numerical results are presented and analyzed in Sects. 41.4 and 41.5. Finally, we summarize and conclude our findings in Sect. 41.6. Some technical details and additional plots are presented in

Appendixes 41.7 and 41.8, respectively.

41.2 Microscopic Hamiltonian

We consider a single-band s-wave superconductor coupled to an optical-phonon mode described by the Hamiltonian $H = H_{\rm sc} + H_{\rm ph} + H_{\text{el-ph}}$, where $H_{\rm sc}$ represents the mean-field BCS Hamiltonian

$$H_{\rm sc} = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k},\sigma} - \sum_{\mathbf{k}\in W} \left(\Delta c^\dagger_{\mathbf{k},\uparrow} c^\dagger_{-\mathbf{k}\downarrow} + \Delta^* c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow} \right), \tag{41.1a}$$

the free-phonon part $H_{\rm ph}$ is given by

$$H_{\rm ph} = \sum_{\mathbf{p}} \hbar\omega_{\rm ph} \left(b^\dagger_{\mathbf{p}} b_{\mathbf{p}} + \frac{1}{2} \right), \tag{41.1b}$$

and $H_{\text{el-ph}}$ denotes the interaction between electrons and phonons

$$H_{\text{el-ph}} = \frac{1}{\sqrt{N}} \sum_{\mathbf{p},\mathbf{k},\sigma} g_{\rm ph} \left(b^\dagger_{-\mathbf{p}} + b_{\mathbf{p}} \right) c^\dagger_{\mathbf{k}+\mathbf{p},\sigma} c_{\mathbf{k},\sigma}, \tag{41.1c}$$

where N is the number of lattice sites. In Eq. (41.1a) $c_{\mathbf{k},\sigma}$ ($c^\dagger_{\mathbf{k},\sigma}$) represents the electron annihilation (creation) operator with momentum $\mathbf{k}$ and spin σ, $\epsilon_{\mathbf{k}} = \frac{\hbar^2\mathbf{k}^2}{2m} - E_{\rm F}$ is the electron dispersion relation, m denotes the effective electron mass, and $E_{\rm F}$ stands for the Fermi energy. The second sum in Eq. (41.1a) is taken over the set W of all $\mathbf{k}$ vectors with $|\epsilon_{\mathbf{k}}| \leq \hbar\omega_{\rm c}$, $\omega_{\rm c}$ being the frequency cutoff. The superconducting order parameter $\Delta(t)$ is assumed to be of s-wave symmetry, with $\Delta(t) = \frac{W_0}{N} \sum_{\mathbf{k}\in W} \langle c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow} \rangle$, and W_0 an attractive interaction constant.

We emphasize that we are generally looking for signatures of the dynamics after the pump pulses and, in particular, in the dynamics of certain low-lying phonons. The phonon mode (41.1b) considered here does not generate the superconductivity. The pairing interaction is assumed to be mediated by other bosons at higher energies, i.e., for instance by spin fluctuations or by phonons at high energies of the order of the Debye energy $\hbar\omega_{\rm D} \approx 30$meV. This energy scale is much larger than the energy scale of the Bogoliubov quasiparticles so that these other bosons influence the low-energy dynamics only indirectly via virtual processes, and hence, they do not need to be treated explicitly. The corresponding electron-boson couplings are assumed to be integrated out and enter in the mean-field treatment of superconductivity via W_0.

For our numerical calculations we have to fix the parameters. Motivated by the numbers for Pb [Poole et al.(2007)Poole, Farach, Creswick, and Prozorov] we fix the parameters as follows: gap in the initial state $\Delta(t_{\rm i}) = 1.35$ meV, Fermi energy $E_{\rm F} = 9479$ meV, energy cutoff $\hbar\omega_{\rm c} = 8.3$ meV, and effective electron mass $m = 1.9m_0$, with m_0 being the free electron mass. The operator $b^\dagger_{\mathbf{p}}$ ($b_{\mathbf{p}}$) in Eq. (41.1b) creates (annihilates) phonons with wave vector $\mathbf{p}$ and frequency $\omega_{\rm ph}$, where $\omega_{\rm ph}$ is assumed to be constant for simplicity. Similarly, the coupling between electrons and phonons is taken to be of Holstein form [Eq. (41.1c)] with a momentum-independent interaction constant $g_{\rm ph}$.

In the following, we study the nonequilibrium response of Hamiltonian (41.1) to a short intense pump pulse which injects a nonthermal distribution of Bogoliubov quasiparticles into the system. The considered pump pulse is of Gaussian shape with photon frequency $\omega_{\rm p}$, photon wave vector $\mathbf{q}_{\rm p} = |\mathbf{q}_{\rm p}|\hat{\mathbf{e}}_x$, full width at half maximum (FWHM) $\tau_{\rm p}$, and amplitude $\mathbf{A}_{\rm p} = |\mathbf{A}_{\rm p}|\hat{\mathbf{e}}_y$. Working within the Coulomb gauge, the pump pulse is expressed in terms of the transverse vector potential [foo(a)],

$$\mathbf{A}_{\mathbf{q}}(t) = \mathbf{A}_{\rm p} e^{-\left(\frac{2\sqrt{\ln 2}t}{\tau_{\rm p}}\right)^2} \left(\delta_{\mathbf{q},\mathbf{q}_{\rm p}} e^{-i\omega_{\rm p}t} + \delta_{\mathbf{q},-\mathbf{q}_{\rm p}} e^{i\omega_{\rm p}t} \right). \tag{41.2}$$

Thereby, the coupling of the pump pulse to the superconductor is described by $H_{\mathrm{em}} = H_{\mathrm{em}}^{(1)} + H_{\mathrm{em}}^{(2)}$, where

$$H_{\mathrm{em}}^{(1)} = \frac{e\hbar}{2m} \sum_{\mathbf{k},\mathbf{q},\sigma} (2\mathbf{k} + \mathbf{q}) \mathbf{A}_{\mathbf{q}}(t) c^{\dagger}_{\mathbf{k}+\mathbf{q},\sigma} c_{\mathbf{k},\sigma}, \tag{41.3a}$$

$$H_{\mathrm{em}}^{(2)} = \frac{e^2}{2m} \sum_{\mathbf{k},\mathbf{q},\sigma} \left(\sum_{\mathbf{q}'} \mathbf{A}_{\mathbf{q}-\mathbf{q}'}(t) \mathbf{A}_{\mathbf{q}'}(t) \right) c^{\dagger}_{\mathbf{k}+\mathbf{q},\sigma} c_{\mathbf{k},\sigma}. \tag{41.3b}$$

We stress that the wave vector $\mathbf{q}_{\mathrm{p}}$ must be kept finite in order to desribe the effect of the pump pulse correctly. If it were set to zero, all linear couplings of the electromagnetic field to the fermions are gone. This can be most clearly seen if we consider the metallic case without superconductivity. The single-band model does not allow any direct excitation process $\Delta\mathbf{q} = 0$ but only indirect ones. Thus, neglecting the finiteness of the wave vector prevents any excitation in the linear regime. Moreover, certain effects such as the lowering of the order parameter $\Delta(t)$ and the Pauli blocking would be ill described if we set $\mathbf{q}_{\mathrm{p}} = 0$ [Papenkort et al.(2007)Papenkort, Axt, and Kuhn].

The absorption spectrum of the nonequilibrium state is measured by a probe pulse, which follows the pump pulse after a certain delay time δt. The probe pulse has the same shape as the pump pulse, Eq. (41.2), but much weaker intensity. We consider both negative and positive pump-probe delay times δt, depending on whether the probe pulse precedes the pump pulse ($\delta t < 0$) or follows after it ($\delta t > 0$). For the numerical computations we assume that the pump pulse is centered in time at $t = 0$ ps and has photon energy $\hbar\omega_{\mathrm{p}} = 3$ meV, which is slightly larger than twice the gap energy $2|\Delta(t_{\mathrm{i}})|$ of the superconductor in the initial state at $t = t_{\mathrm{i}}$. The probe pulse is taken to be very short in time with FWHM $\tau_{\mathrm{pr}} = 0.25$ ps and center energy $\hbar\omega_{\mathrm{pr}} = 2.5$ meV (see Fig. 41.10 in Appendix 41.7).

Thus, the probe pulse contains a broad range of frequencies which cover almost the entire energy range of excited quasiparticles induced by the pump pulse similar to recent experiments [Dal Conte et al.(2012)Dal Conte, Giannetti, Coslovich, Cilento, Bossini, Abebaw, Banfi, Ferrini, Eisaki, Greven et al., Matsunaga and Shimano(2012), Matsunaga et al.(2013)Matsunaga, Hamada, Makise, Uzawa, Terai, Wang, and Shimano]. Nonlinear couplings between the superconductor and the probe pulse are neglected because probing is done with much weaker intensity in the linear response regime.

41.3 Equations of motion

In order to simulate the pump-probe conductivity, we need to determine the temporal evolution of the electric current density $\mathbf{j}_{\mathbf{q}_{\mathrm{pr}}}(\delta t, t)$, where $\mathbf{q}_{\mathrm{pr}} = |\mathbf{q}_{\mathrm{pr}}|\,\hat{\mathbf{e}}_x$ is the wave vector of the probe pulse and

$$\begin{aligned} \mathbf{j}_{\mathbf{q}_{\mathrm{pr}}}(\delta t, t) \quad &= \frac{-e\hbar}{2mV} \sum_{\mathbf{k},\sigma} (2\mathbf{k} + \mathbf{q}_{\mathrm{pr}}) \left\langle c^{\dagger}_{\mathbf{k},\sigma} c_{\mathbf{k}+\mathbf{q}_{\mathrm{pr}},\sigma} \right\rangle (\delta t, t) \\ &\quad - \frac{e^2}{mV} \sum_{\mathbf{k},\mathbf{q},\sigma} \mathbf{A}_{\mathbf{q}_{\mathrm{pr}}-\mathbf{q}} \left\langle c^{\dagger}_{\mathbf{k},\sigma} c_{\mathbf{k}+\mathbf{q},\sigma} \right\rangle (\delta t, t). \end{aligned} \tag{41.4}$$

Formally, the current depends on two times, namely the delay time δt between pump and probe pulse and the actual time t at which it is measured or computed, respectively. For the numerical calculations we neglect the second term in Eq. (41.4) since it only results in a constant offset of the imaginary part of the conductivity spectra. Then, the pump-probe conductivity $\sigma(\delta t, \omega)$

is obtained from Eq. (41.4) via

$$\sigma(\delta t,\omega) = \frac{j(\delta t,\omega)}{i\omega A(\delta t,\omega)}, \tag{41.5a}$$

where

$$j(\delta t,\omega) = \int_{-\infty}^{\infty} dt\; \hat{\mathbf{e}}_y \cdot \mathbf{j}_{\mathbf{q}_{\rm pr}}(\delta t,t)\, e^{i\omega t} \tag{41.5b}$$

and

$$A(\delta t,\omega) = \int_{-\infty}^{\infty} dt\; \hat{\mathbf{e}}_y \cdot \mathbf{A}_{\mathbf{q}_{\rm pr}}(\delta t,t)\, e^{i\omega t} \tag{41.5c}$$

denote the Fourier transformed y components of the current density $\mathbf{j}_{\mathbf{q}_{\rm pr}}(\delta t,t)$ and the vector potential $\mathbf{A}_{\mathbf{q}_{\rm pr}}(\delta t,t)$ of the probe pulse, respectively. In the literature, there are also other ways discussed to determine the conductivity; see, for instance, Ref. [Eckstein et al.(2010)Eckstein, Kollar, and Werner]. However, we think that in the present context of a pump and a probe pulse the above procedure suggests itself and is closest to what is experimentally done. Hence, the pump-probe conductivity is fully determined by the time evolution of the expectation values $\langle c^\dagger_{\mathbf{k},\sigma} c_{\mathbf{k}+\mathbf{q}_{\rm pr},\sigma}\rangle(\delta t,t)$, which we numerically compute by integrating the corresponding equations of motion. To keep the notation light, we in the following omit the dependence on the delay time δt because it is externally fixed by the timing of the pump and the probe pulse. Below we discuss how the dependence on the time t is computed.

41.3.1 Density-Matrix Formalism

Using the density-matrix formalism, we derive equations of motion for the quasiparticle densities and the mean phonon amplitudes $\langle b_{\mathbf{p}}\rangle$ and $\langle b^\dagger_{-\mathbf{p}}\rangle$ in this section. For this purpose, it is advantageous to perform a Bogoliubov transformation of the electron operators which diagonalizes the Hamiltonian $H_{\rm sc}$ in the initial state. That is, we introduce new fermionic operators $\alpha_{\mathbf{k}}$ and $\beta_{\mathbf{k}}$, with

$$\alpha_{\mathbf{k}} = u_{\mathbf{k}} c_{\mathbf{k},\uparrow} - v_{\mathbf{k}} c^\dagger_{-\mathbf{k},\downarrow}, \qquad \beta_{\mathbf{k}} = v_{\mathbf{k}} c^\dagger_{\mathbf{k},\uparrow} + u_{\mathbf{k}} c_{-\mathbf{k},\downarrow}, \tag{41.6}$$

where $v_{\mathbf{k}} = \Delta(t_i)/|\Delta(t_i)|\sqrt{(1-\epsilon_{\mathbf{k}}/E_{\mathbf{k}})/2}$, $u_{\mathbf{k}} = \sqrt{(1+\epsilon_{\mathbf{k}}/E_{\mathbf{k}})/2}$, and $E_{\mathbf{k}} = \sqrt{\epsilon^2_{\mathbf{k}} + |\Delta(t_i)|^2}$. We emphasize that the coefficients $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ do not depend on time, i.e., the temporal evolution of the quasiparticle densities is computed with respect to a fixed time-independent Bogoliubov-de Gennes basis in which the initial state is diagonal. All physical observables, such as the electric current density $\mathbf{j}_{\mathbf{q}_{\rm pr}}(t)$, the order parameter amplitude $|\Delta(t)|$, and the lattice displacement $U(\mathbf{r},t)$ can now be expressed in terms of the new Bogoliubov quasiparticle densities $\langle\alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}'}\rangle$, $\langle\beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}'}\rangle$, $\langle\alpha^\dagger_{\mathbf{k}}\beta^\dagger_{\mathbf{k}'}\rangle$, and $\langle\alpha_{\mathbf{k}}\beta_{\mathbf{k}'}\rangle$. For example, for the current density $\mathbf{j}_{\mathbf{q}_{\rm pr}}(t)$ we find

$$\begin{aligned}\mathbf{j}_{\mathbf{q}_{\rm pr}}(t) = \frac{-e\hbar}{2mV}\sum_{\mathbf{k}}(2\mathbf{k}+\mathbf{q}_{\rm pr}) \\ \times\Big[(u_{\mathbf{k}}v_{\mathbf{k}+\mathbf{q}_{\rm pr}} - v_{\mathbf{k}}u_{\mathbf{k}+\mathbf{q}_{\rm pr}})\big(\langle\alpha^\dagger_{\mathbf{k}}\beta^\dagger_{\mathbf{k}+\mathbf{q}_{\rm pr}}\rangle + \langle\alpha_{\mathbf{k}+\mathbf{q}_{\rm pr}}\beta_{\mathbf{k}}\rangle\big) \\ +(u_{\mathbf{k}}u_{\mathbf{k}+\mathbf{q}_{\rm pr}} + v_{\mathbf{k}}v_{\mathbf{k}+\mathbf{q}_{\rm pr}})\big(\langle\alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}+\mathbf{q}_{\rm pr}}\rangle - \langle\beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}+\mathbf{q}_{\rm pr}}\rangle\big)\Big].\end{aligned} \tag{41.7}$$

Due to the interaction term $H_{\text{el-ph}}$ in Eq. (41.1), the kinetic equations for the single-particle density matrices $\langle\alpha^\dagger_{\mathbf{k}}\alpha_{\mathbf{k}'}\rangle$, $\langle\beta^\dagger_{\mathbf{k}}\beta_{\mathbf{k}'}\rangle$, etc., are not closed: Instead, $H_{\text{el-ph}}$ leads to an

infinite hierarchy of equations of higher-order density matrices [Rossi and Kuhn(2002)]. In order to study the generation of coherent phonons, it is sufficient to break this hierarchy at first order of the electron-phonon coupling strength $g_{\rm ph}$. That is, phonon-assisted quantities, such as $\langle \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}+\mathbf{q}} b_{\mathbf{p}} \rangle$, are factorized as follows [Schnyder et al.(2011)Schnyder, Manske, and Avella, Herbst et al.(2003)Herbst, Glanemann, Axt, and Kuhn]:

$$\langle \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}+\mathbf{q}} b_{\mathbf{p}} \rangle = \langle \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}+\mathbf{q}} \rangle \langle b_{\mathbf{p}} \rangle. \tag{41.8}$$

Finite expectation values of the mean phonon amplitudes $\langle b_{\mathbf{p}} \rangle$ and $\langle b^\dagger_{-\mathbf{p}} \rangle$ correspond to a non-vanishing lattice displacement,

$$U(\mathbf{r}, t) = \sum_{\mathbf{p}} \sqrt{\frac{\hbar}{2M\omega_{\rm ph} N}} \left(\langle b_{\mathbf{p}} \rangle + \langle b^\dagger_{-\mathbf{p}} \rangle \right) e^{+i\mathbf{p}\mathbf{r}}, \tag{41.9}$$

where M denotes the reduced mass of the lattice ions. We remark that breaking the hierarchy at first order in $g_{\rm ph}$ amounts to neglecting all correlations among quasiparticles and phonons. In particular, relaxation processes due to quasiparticle-phonon and phonon-phonon scattering are not taken into account. Since we focus on time scales shorter than the coherent-phonon and quasiparticle life-times, we neglect all of these higher-order processes, which are expected to give rise to an exponential damping of the coherent phonon and order parameter oscillations. Recent pump-probe experiments on $\mathrm{Nb}_{1-x}\mathrm{Ti}_x\mathrm{N}$ films [Matsunaga and Shimano(2012), Matsunaga et al.(2013)Matsunaga, Hamada, Makise, Uzawa, Terai, Wang, and Shimano] have shown that these relaxation processes occur on time scales of tens of picoseconds, which is much larger than the period of the order parameter oscillations.

By factorizing higher-order density matrices according to Eq. (41.8), a closed set of differential equations for the quasiparticle density matrices and the mean phonon amplitudes can be derived using Heisenberg's equation of motion. A derivation of these differential equations and some other technical details are given in Appendix 41.7. In Sects. 41.4 and 41.5, we numerically solve this set of differential equations to obtain the temporal evolution of the order parameter amplitude $|\Delta(t)|$, the lattice displacement $U(\mathbf{r}, t)$, and the current density $\mathbf{j}_{\mathbf{q}_{\rm pr}}(t)$. From the Fourier transform of the latter quantity, the pump-probe conductivity $\sigma(\delta t, \omega)$ is readily obtained using Eq. (41.5). In order to study different time-scale regimes, we adjust in the following the pump-pulse width $\tau_{\rm p}$, the phonon energy $\hbar\omega_{\rm ph}$, and the integrated pump-pulse intensity $|\mathbf{A}_{\rm p}|^2 \tau_{\rm p}$. For each regime we determine the signatures of the coherent Cooper pair and phonon dynamics in the pump-probe response.

41.4 Pump-probe response in the absence of phonons

Before studying the evolution of the coupled Bogoliubov quasiparticle-phonon system in Sect. 41.5, it is instructive to first consider the pump-probe response of a BCS superconductor in the absence of phonons. Hence, we first solve the set of equations of motion (41.13) for $g_{\rm ph} = 0$ and compute $\sigma(\delta t, \omega)$ for positive and negative pump-probe delay times δt.

41.4.1 Positive pump-probe delay time

We start by discussing the case where the probe pulse follows the pump pulse after a positive delay time δt. Both the nonadiabatic and the adiabatic regimes are considered, corresponding to $\tau_{\rm p} \ll \tau_\Delta$ and $\tau_{\rm p} \gg \tau_\Delta$, respectively; see Figs. 41.1 and 41.2.

Nonadiabatic regime, $\tau_{\mathbf{p}} \ll \tau_\Delta$

In Fig. 41.1(a) we plot the real part of the pump-probe signal, $\mathrm{Re}[\sigma(\delta t, \omega)]$, versus delay time δt and frequency ω for the regime $\tau_{\mathrm{p}} \ll \tau_\Delta$, where the Bogoliubov quasiparticle densities build up coherently. The imaginary part of $\sigma(\delta t, \omega)$, which shows similar features as the real part, is presented in Fig. 41.11 of Appendix 41.8. In the nonadiabatic regime, the ultrafast photoexcitations lead first to a monotonic growth and then to rapid oscillations in the quasiparticle occupations. Correspondingly, as $t \to \infty$, the order parameter amplitude $|\Delta(t)|$ first decreases monotonically and then approaches the asymptotic value $\Delta_\infty < |\Delta(t_i)|$ in an oscillatory fashion with oscillation frequency [Volkov and Kogan(1974), Amin et al.(2004)Amin, Bezuglyi, Kijko, and Omelyanchouk, Barankov *et al.*(2004)Barankov, Levitov, and Spivak, Yuzbashyan et al.(2005a)Yuzbashyan, Altshuler, Kuznetsov, and Enolskii, Yuzbashyan et al.(2006)Yuzbashyan, Tsyplyatyev, and Altshuler],

$$\omega_{\Delta_\infty} = 2\Delta_\infty/\hbar; \tag{41.10}$$

see Figs. 41.1(d) and 41.1(e). In the collisionless limit, i.e., in the absence of relaxation processes, the oscillations in the quasiparticle densities are undamped, whereas the order parameter oscillations show an algebraic $1/\sqrt{t}$ decay, due to destructive interference among quasiparticle densities with different momenta [Fig. 41.1(d)].

Remarkably, we find that this coherent oscillatory dynamics reveals itself in the pump-probe signal through algebraically decaying oscillations as a function of delay time δt; see Figs. 41.1(a)-(c). These oscillations are most prominent at the frequency ω_{Δ_∞} corresponding to twice the asymptotic gap energy, i.e., at $\hbar\omega_{\Delta_\infty} = 2\Delta_\infty = 2.4690$ meV, where $\sigma(\delta t, \omega)$ exhibits a sharp edge as a function of ω; see Figs. 41.1(a) and 41.1(b). The delay-time dependence of $\mathrm{Re}[\sigma(\delta t, \omega_0)]$ for fixed ω_0 is approximately given by

$$\mathrm{Re}[\sigma(\delta t, \omega_0)] = A + B\frac{\cos(\omega_{\Delta_\infty}\delta t + \Phi)}{\sqrt{\delta t}} + C\delta t, \tag{41.11}$$

as shown by the excellent fits to the numerical data in Figs. 41.1(b) and 41.1(c). Here, Φ is an overall phase and A, B, and C are fit parameters that depend on ω_0. Hence, as it is with the order parameter oscillations, oscillations in the pump-probe signal are characterized by an amplitude decaying as $1/\sqrt{\delta t}$ and a frequency $\omega_{\Delta_\infty} = 2\Delta_\infty/\hbar$ that is determined by the asymptotic gap value Δ_∞. We note that the linear increase in the pump-probe signal of Fig. 41.1(b) can be attributed to slow oscillations that are related to the finite size of the system.

We conclude that the nonadiabatic BCS state dynamics is clearly visible in $\sigma(\delta t, \omega)$ in terms of oscillations as a function of delay time δt. This is one of the key findings of the present work. The frequency dependence of the pump-probe signal, on the other hand, does not show any sign of the order parameter oscillations [Fig. 41.1(a)] [Papenkort et al.(2007)Papenkort, Axt, and Kuhn]. This result is in qualitative agreement with recent pump-probe experiments on $Nb_{1-x}Ti_xN$ thin films [Matsunaga et al.(2013)Matsunaga, Hamada, Makise, Uzawa, Terai, Wang, and Shimano].

Adiabatic regime, $\tau_{\mathbf{p}} \gg \tau_\Delta$

It is interesting to contrast the pump-probe response induced by ultrashort pump pulses [Fig. 41.1(a)] with the one in the adiabatic regime, where the pump pulse duration τ_{p} is much longer than the dynamical time scale of the superconductor τ_Δ. The absorption spectrum $\mathrm{Re}[\sigma(\delta t, \omega)]$ as a function of delay time δt and frequency ω for a pump pulse with $\tau_{\mathrm{p}} \gg \tau_\Delta$ is shown in Fig. 41.2(a). In this adiabatic regime, the pump pulse excites only the normal

quasiparticle densities, $\langle \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}'} \rangle$ and $\langle \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}'} \rangle$, whereas the anomalous ones, $\langle \alpha^\dagger_{\mathbf{k}} \beta^\dagger_{\mathbf{k}'} \rangle$ and $\langle \alpha_{\mathbf{k}} \beta_{\mathbf{k}'} \rangle$, remain unoccupied. Hence, the order parameter $\Delta(t)$ does not oscillate, instead it decreases monotonically towards the asymptotic value Δ_∞ [Fig. 41.2(c)]. Correspondingly, the pump-probe signal $\sigma(\delta t, \omega)$ does not exhibit any oscillations, neither as a function of delay time nor of frequency [Fig. 41.2(a)]. As in Fig. 41.2(a), $\mathrm{Re}[\sigma(\delta t, \omega)]$ has a sharp edge at the frequency ω_{Δ_∞} corresponding to twice the energy of the asymptotic gap value Δ_∞. However, otherwise, it is almost featureless, except for a small dip just below the pump frequency ω_p; see Fig. 41.2(b). This reduced absorption in the vicinity of ω_p is due to Pauli blocking which leads to a saturation in the narrowly peaked quasiparticle distributions [Papenkort et al.(2007)Papenkort, Axt, and Kuhn, foo()].

41.4.2 Negative pump-probe delay time

Let us now turn to the situation where the probe pulse precedes the pump pulse with $\delta t < 0$. We focus our analysis on the nonadiabatic case, $\tau_\mathrm{p} \ll \tau_\Delta$, since in the opposite regime, $\tau_\mathrm{p} \gg \tau_\Delta$, the pump-probe signal does not show any interesting characteristics as a function of delay time δt.

Nonadiabatic regime, $\tau_\mathbf{p} \ll \tau_\Delta$

In Fig. 41.3(a) we present the real part of the pump-probe response, $\mathrm{Re}[\sigma(\delta t, \omega)]$, versus negative delay time δt and frequency ω for an ultrashort pump pulse with $\tau_\mathrm{p} = 0.5$ ps $\ll \tau_\Delta$. Both the energy gap before and after the pump pulse are clearly visible in the frequency dependence of $\sigma(\delta t, \omega)$. That is, the pump-probe signal displays two sharp edges as a function of frequency, one at twice the gap energy in the initial state $\hbar\omega_{\Delta(t_i)} = 2\Delta(t_i) = 2.7$ meV and one at twice the asymptotic gap value $\hbar\omega_{\Delta_\infty} = 2\Delta_\infty = 1.677$ meV [green and blue traces in Fig. 41.3(a)]. Between those two edges, $\sigma(\delta t, \omega)$ shows spectral oscillations in ω with a frequency $\delta\omega$ that is inversely proportional to the delay time, i.e., $\delta\omega = (2\pi)/|\delta t|$.[Papenkort et al.(2007)Papenkort, Axt, and Kuhn]

Interestingly, we find that the pump-probe response $\sigma(\delta t, \omega)$ also exhibits a rich oscillatory behavior in the delay-time dependence, with multiple frequencies that depend on the absorption energy $\hbar\omega$; see Figs. 41.3(a) and 41.3(b). This is revealed most clearly in Fig. 41.3(b), which shows the Fourier transformed pump-probe signal in two-dimensional frequency space, i.e., $\mathrm{Re}(\sigma)$ as a function of Fourier frequency $\omega_{\delta t}$ of the delay time and absorption frequency ω. We observe that for the absorption energy $\hbar\omega$ within the interval $0 < \hbar\omega < \hbar\omega_{\Delta_\infty}$, $\mathrm{Re}[\sigma(\delta t, \omega)]$ oscillates in δt with the frequency $\omega_{\Delta(t_i)}$. The oscillatory behavior of $\mathrm{Re}[\sigma(\delta t, \omega)]$ in the interval $\hbar\omega_{\Delta_\infty} < \hbar\omega < \hbar\omega_{\Delta(t_i)}$ is even more intriguing, as it shows signatures of how the gap decreases while the pump pulse acts on the sample [cf. Fig. 41.3(c)]. In other words, it is found that for these absorption energies $\mathrm{Re}[\sigma(\delta t, \omega)]$ oscillates in δt with three frequencies that are approximately given by $\omega_{\Delta(t_i)}$, $\omega_{\Delta(t_i)} + \omega$, and $\omega_{\Delta(t_i)} - \omega$.

41.5 Pump-probe response in the presence of an optical phonon mode

In this section we analyze the pump-probe response of a superconductor coupled to an optical phonon. To that end, we numerically solve the set of kinetic equations (41.13) for finite electron-phonon coupling g_ph. The correlation expansion used to derive the equations of motion of the coupled Bogoliubov quasiparticle-phonon system is valid as long as g_ph is smaller than the superconducting energy scales. Hence, for the numerical calculations we take $g_\mathrm{ph} = 0.1$ meV $\ll \Delta(t_i)$, in which case the influence of the phonon subsystem on the superconductor becomes negligibly small. As in the previous section, we focus on ultrashort

pump pulses with $\tau_\mathrm{p} \ll \tau_\mathrm{ph}, \tau_\Delta$ that drive both the quasiparticle and the phonon subsystems in a nonadiabatic fashion.

41.5.1 Positive pump-probe delay time

First, we examine the results for positive pump-probe delay time, where the probe pulse follows the pump pulse after the delay time $\delta t > 0$.

Nonadiabatic regime, $\tau_\mathrm{p} \ll \tau_\mathrm{ph} \approx \tau_\Delta$

The most interesting case corresponds to the situation where an ultrashort pump pulse drives quasiparticle oscillations with a frequency that is close to resonance with the phonon energy, i.e., $\tau_\mathrm{p} \ll \tau_\mathrm{ph} \approx \tau_\Delta$, see Figs. 41.4, 41.5, and 41.12. Figure 41.4(a) displays the real part of the pump-probe signal as a function of delay time and frequency for a pump pulse with width $\tau_\mathrm{p} = 0.5$ ps and amplitude $|\mathbf{A}_\mathrm{p}| = 10 \cdot 10^{-8}$ Js/(Cm). For this choice of pump-pulse intensity, the order parameter oscillations are close to resonance with the phonon frequency, i.e., $|2\Delta_\infty/\hbar - \omega_\mathrm{ph}| \ll \omega_\mathrm{ph}$ [Fig. 41.4(f)]. This leads to a beating behavior in the lattice displacement $U(\mathbf{r}, t)$ [Schnyder et al.(2011)Schnyder, Manske, and Avella]; see Fig. 41.4(e). Correspondingly, we observe two distinct edges in the frequency dependence of $\sigma(\delta t, \omega)$, one at twice the asymptotic gap value $\hbar\omega_{\Delta_\infty} = 2\Delta_\infty = 2.2491$ meV and one at the phonon energy $\hbar\omega_\mathrm{ph} = 2$ meV [blue and red traces in Fig. 41.4(a)]. The maxima of these two edges show an oscillatory behavior as a function of delay time [Figs. 41.4(b) and 41.4(c)], reflecting the nonadiabatic dynamics of both the phonon mode and the Bogoliubov quasiparticles.

The delay-time evolution of $\mathrm{Re}[\sigma(\delta t, \omega)]$ at ω_{Δ_∞} is well described by Eq. (41.11), as evidenced by the good fit in Fig. 41.4(b). That is, $\mathrm{Re}[\sigma(\delta t, \omega_{\Delta_\infty})]$ oscillates with frequency ω_{Δ_∞} and an amplitude decaying as $1/\sqrt{\delta t}$. The oscillations at ω_ph, on the other hand, exhibit a beating phenomenon, approximately given by

$$\mathrm{Re}[\sigma(\delta t, \omega_\mathrm{ph})] = A + B\cos(\omega_+ \delta t + \Phi_1)\cos(\omega_- \delta t + \Phi_2)\,, \tag{41.12}$$

where $\omega_\pm = (\omega_{\Delta_\infty} \pm \omega_\mathrm{ph})/2$, and A, B, Φ_1, and Φ_2 are fit parameters. As demonstrated by the gray lines in Fig. 41.4(c), Eq. (41.12) fits our numerical results well. Just as the lattice displacement $U(\mathbf{r}, t)$, $\mathrm{Re}[\sigma(\delta t, \omega_\mathrm{ph})]$ exhibits quantum beats, i.e., it oscillates with frequency ω_+ and an amplitude that is modulated with frequency ω_-. Note that the deviations between the fit function (41.12) and the numerical results of Fig. 41.4(c) for $\delta t \lesssim 7$ ps are due to a transient oscillatory behavior.

By adjusting the pump-pulse intensity, we can bring the order parameter oscillations into exact resonance with the phonon mode [Schnyder et al.(2011)Schnyder, Manske, and Avella]. This is illustrated in Fig. 41.5, where we plot $\mathrm{Re}[\sigma(\delta t, \omega)]$, $|\Delta(t)|$, and $U(\mathbf{r}, t)$ for a pump pulse with $\tau_\mathrm{p} = 0.5$ ps and $|\mathbf{A}_\mathrm{p}| = 11.64 \cdot 10^{-8}$ Js/(Cm). In this resonant case, the lattice displacement oscillates with frequency $\omega_\mathrm{ph} = \omega_{\Delta_\infty} = 2$ meV$/\hbar$ and an amplitude that grows like $\sqrt{t}$; see Fig. 41.5(e). Concomitantly, the frequency dependence of the pump-probe signal $\sigma(\delta t, \omega)$ shows just one sharp edge at $\hbar\omega_{\Delta_\infty} = \hbar\omega_\mathrm{ph} = 2$ meV, whose maximum oscillates as a function of δt [blue trace in Fig. 41.5(a)]. Remarkably, these oscillations are undamped and their amplitude is considerably larger than in the off-resonant case [compare Fig. 41.4(b) with Fig. 41.5(b)]. In fact, the δt dependence of $\mathrm{Re}[\sigma(\delta t, \omega)]$ at ω_{Δ_∞} is very well captured by formula (41.12) with $\omega_+ = \omega_{\Delta_\infty}$ and $\omega_- = 0$, as demonstrated by the fits in Fig. 41.5(b). At absorption energies $\hbar\omega$ different from $\hbar\omega_{\Delta_\infty}$, however, $\mathrm{Re}[\sigma(\delta t, \omega)]$ shows algebraically decaying oscillations in δt with frequency ω_{Δ_∞} and an amplitude decreasing as $1/\sqrt{\delta t}$. This is exemplified in Fig. 41.5(c), which reveals that the delay-time evolution of the pump-probe signal for $\omega \neq \omega_{\Delta_\infty}$ is well described by Eq. (41.11) [gray lines in Fig. 41.5(c)].

Adiabatic regime, $\tau_{\mathbf{p}} \gg \tau_{\Delta}$, $\tau_{\mathbf{ph}}$

We contrast the results for the nonadiabatic regime [Figs. 41.4 and 41.5] with those for the adiabatic case, shown in Fig. 41.6(a), where we present the pump-probe signal for a superconductor photoexcited by a long pump pulse with pulse duration $\tau_{\rm p} \gg \tau_{\Delta}, \tau_{\rm ph}$. We observe that in this regime the lattice displacement does not oscillate with the phonon frequency, but exhibits large transient oscillations with frequency $\omega_{\rm p}$ that occur within the interval $\approx [-\tau_{\rm p}, +\tau_{\rm p}]$ during which the pump laser acts on the system [Figs. 41.6(c) and 41.6(d)]. The probe signal $\mathrm{Re}[\sigma(\delta t, \omega)]$ is almost featureless as a function of delay time δt [Figs. 41.6(a)]. It only displays a sharp edge in the frequency dependence at $\hbar\omega_{\Delta_\infty} = 2\Delta_\infty$ and a small dip just below the pump pulse frequency $\omega_{\rm p}$ which arises due to Pauli blocking [cf. Fig. 41.2(a)]. However, since no coherent phonons with frequency $\omega_{\rm ph}$ are being created, $\mathrm{Re}[\sigma(\delta t, \omega)]$ exhibits no feature at the phonon energy $\hbar\omega_{\rm ph}$ [red trace in Fig. 41.6(a)]. Also the transient oscillations of Fig. 41.6(c) are not visible in the pump-probe response as a function of frequency or delay time.

In closing, we show in Fig. 41.7 the real part of the pump-probe signal $\mathrm{Re}[\sigma(\delta t, \omega)]$ for the special case, where the central energy of the pump pulse is in resonance with the phonon energy, i.e., $\hbar\omega_{\rm p} = \hbar\omega_{\rm ph}$. Such a resonant pumping of the phonon leads to undamped coherent phonon oscillations, which persist even after the pump pulse has passed [Fig. 41.7(c)] [Schnyder et al.(2011)Schnyder, Manske, and Avella]. This resonant response of the phonon system in turn gives rise to an enhanced oscillatory behavior in the delay-time dependence of the absorption spectrum $\mathrm{Re}[\sigma(\delta t, \omega)]$ at the phonon frequency $\omega_{\rm ph}$ [red trace in Fig 41.7(a)].

41.5.2 Negative pump-probe delay time

Finally, we discuss the pump-probe conductivity with negative delay times $\delta t < 0$ for a superconductor coupled to optical phonons. As in Sect. 41.4.2 we focus on the nonadiabatic regime $\tau_{\rm p} \ll \tau_{\Delta}, \tau_{\rm ph}$, in which order parameter oscillations as well as coherent phonons are generated [Figs. 41.8(c) and 41.8(d)].

Nonadiabatic regime, $\tau_{\mathbf{p}} \ll \tau_{\Delta}, \tau_{\mathbf{ph}}$

In Fig. 41.8(a), the real part of the pump-probe signal is plotted versus negative delay time δt and frequency ω for a short pump pulse with $\tau_{\rm p} = 0.5$ ps. Similar to Fig. 41.3(a), we observe two sharp edges in the frequency dependence of $\mathrm{Re}[\sigma(\delta t, \omega)]$, one at twice the energy of the asymptotic gap value $\hbar\omega_{\Delta_\infty} = 1.677$ meV and a smaller one at twice the gap energy of the unperturbed superconductor $\hbar\omega_{\Delta(t_i)} = 2\Delta(t_i) = 2.7$ meV [blue and green traces in Fig. 41.8(a)]. In addition, there appears a third feature at the phonon frequency $\hbar\omega_{\rm ph} \approx 1.19\hbar\omega_{\Delta_\infty}$, indicated by the red trace in Fig. 41.8(a). As a function of delay time the pump-probe signal shows an intricate oscillatory behavior, which reflects the nonadiabatic dynamics of both the Bogoliubov quasiparticle and phonon subsystems. The spectral distribution of these oscillations in δt is depicted in Fig. 41.8(b). It is quite similar to the one in the absence of phonons (cf. Sect. 41.4.2). For absorption frequencies ω within the interval $\omega_{\Delta_\infty} < \omega < \omega_{\Delta(t_i)}$, $\mathrm{Re}[\sigma(\delta t, \omega)]$ oscillates in general with three different frequencies, approximately given by $\omega_{\Delta(t_i)}$, $\omega_{\Delta(t_i)} + \omega$, and $\omega_{\Delta(t_i)} - \omega$. Besides these, the Fourier spectrum of $\mathrm{Re}[\sigma(\delta t, \omega)]$ at the absorption energy $\omega = \omega_{\rm ph}$ also shows a peak at the phonon energy $\hbar\omega_{\rm ph} = 2$ meV [white arrow in Fig. 41.8(b)].

41.6 Summary and Conclusions

In this paper we have used the density matrix formalism to simulate the pump-probe response of nonequilibrium superconductors coupled to optical phonons. Considering different

hierarchies of time scales, we have performed systematic calculations of the pump-probe conductivity, which allows for a direct comparison of pump-probe experiments to the theoretical predictions. For sufficiently short pump pulses the superconductor can be driven into a nonadiabatic regime, which is characterized by rapid oscillations in the Bogoliubov quasiparticle densities. In turn, these sudden changes in the quasiparticle occupations lead to order parameter oscillations and the creation of coherent phonons. We have shown that the pump-probe absorption spectrum as a function of positive and negative delay times δt shows clear signatures of the coherent nonadiabatic dynamics of both the quasiparticle and the phonon subsystems.

In particular, for positive delay times, the absorption spectrum at twice the frequency of the asymptotic gap value exhibits algebraically decaying oscillations in δt with the same frequency as the order parameter oscillations (Fig. 41.1). The coherent dynamics of the phonons reveals itself in terms of a beating phenomenon as a function of δt in the absorption spectrum at the phonon frequency (Fig. 41.4). Interestingly, this oscillatory response in the probe spectra can be strongly amplified by bringing the frequency of the order parameter oscillations into resonance with the phonon energy (Fig. 41.5). For negative delay times, the pump-probe signal shows an even richer oscillatory response in the delay-time dependence, with multiple frequencies that change as a function of absorption energy (Figs. 41.3 and 41.8). This intricate behavior contains information on how the superconducting condensate is depleted while the pump pulse acts on the sample.

Our theoretical findings are qualitatively consistent with recent pump-probe experiments by Matsunaga *et al.* [Matsunaga et al.(2013)Matsunaga, Hamada, Makise, Uzawa, Terai, Wang, and Shimano], which have observed order parameter oscillations in the pump-probe conductivity of $Nb_{1-x}Ti_xN$ thin films. In agreement with our theoretical result, these oscillations are algebraically damped and their frequency corresponds to twice the asymptotic gap energy. These experiments have demonstrated that up to five cycles of order parameter oscillations can be experimentally observed, and that the coherence of the quasiparticles is maintained over a period of up to 10 ps. The theoretical study of relaxation processes due to quasiparticle-quasiparticle and quasiparticle-phonon scattering, which in the case of $Nb_{1-x}Ti_xN$ films become dominant at time scales larger than 10 ps, remains as an important direction for future research. Pump-probe spectroscopy has the unique capability to resolve coherent oscillations as well as relaxation processes in the time domain, thereby yielding important information on the intrinsic time and energy scales of the superconductor.[Matsunaga and Shimano(2012), Matsunaga et al.(2013)Matsunaga, Hamada, Makise, Uzawa, Terai, Wang, and Shimano, Mansart et al.(2013)Mansart, Lorenzana, Mann, Odeh, Scarongella, Chergui, and Carbone] We hope that the findings of this paper will stimulate further time-resolved measurements of superconductors in the nonadiabatic regime.

41.7 Numerical simulation of the pump-probe response

In this appendix, we present some technical details about the numerical implementation of the pump-probe response simulation. The closed set of equations of motion describing the time evolution of a superconductor coupled to a phonon branch is presented in Sec. 41.7.1. The discretization of the Brillouin zone is discussed in Sect. 41.7.2 of this appendix. Some further details about the implementation of the probe pulse can be found in Sect. 41.7.3.

41.7.1 Equations of motion

The equation of motion for the normal quasiparticle density $\langle \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}+\mathbf{q}} \rangle$ is given by

$$
\begin{aligned}
i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \left\langle \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}'} \right\rangle &= (R_{\mathbf{k}'} - R_{\mathbf{k}}) \left\langle \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}'} \right\rangle + C_{\mathbf{k}'} \left\langle \alpha^\dagger_{\mathbf{k}} \beta^\dagger_{\mathbf{k}'} \right\rangle + C^*_{\mathbf{k}} \left\langle \alpha_{\mathbf{k}'} \beta_{\mathbf{k}} \right\rangle \\
&+ \frac{e\hbar}{2m} \sum_{\mathbf{q}'=\pm\mathbf{q}_0} 2\mathbf{k}\mathbf{A}_{\mathbf{q}'} \left(-L^+_{\mathbf{k},\mathbf{q}'} \left\langle \alpha^\dagger_{\mathbf{k}+\mathbf{q}'} \alpha_{\mathbf{k}'} \right\rangle + L^+_{\mathbf{k}',-\mathbf{q}'} \left\langle \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}'-\mathbf{q}'} \right\rangle - M^-_{\mathbf{k}',-\mathbf{q}'} \left\langle \alpha^\dagger_{\mathbf{k}} \beta^\dagger_{\mathbf{k}'-\mathbf{q}'} \right\rangle - M^-_{\mathbf{k},\mathbf{q}'} \left\langle \alpha_{\mathbf{k}'} \beta_{\mathbf{k}+\mathbf{q}'} \right\rangle \right) \\
&+ \frac{e^2}{2m} \sum_{\mathbf{q}'=0,\pm 2\mathbf{q}_0} \left(\sum_{\mathbf{q}_i=\pm\mathbf{q}_0} A_{\mathbf{q}'-\mathbf{q}_i} A_{\mathbf{q}_i} \right) \left(-L^-_{\mathbf{k},\mathbf{q}'} \left\langle \alpha^\dagger_{\mathbf{k}+\mathbf{q}'} \alpha_{\mathbf{k}'} \right\rangle + L^-_{\mathbf{k}',\mathbf{q}'} \left\langle \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}'-\mathbf{q}'} \right\rangle + M^+_{\mathbf{k}',-\mathbf{q}'} \left\langle \alpha^\dagger_{\mathbf{k}} \beta^\dagger_{\mathbf{k}'-\mathbf{q}'} \right\rangle + M^- \right. \\
&+ \frac{g_{\mathrm{ph}}}{\sqrt{N}} \sum_{\mathbf{p}} \left(\left\langle b^\dagger_{-\mathbf{p}} \right\rangle + \langle b_{\mathbf{p}} \rangle \right) \left(-L^-_{\mathbf{k},\mathbf{p}} \left\langle \alpha^\dagger_{\mathbf{k}+\mathbf{p}} \alpha_{\mathbf{k}'} \right\rangle + L^-_{\mathbf{k}',-\mathbf{p}} \left\langle \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}'-\mathbf{p}} \right\rangle + M^+_{\mathbf{k},\mathbf{p}} \left\langle \alpha_{\mathbf{k}'} \beta_{\mathbf{k}+\mathbf{p}} \right\rangle + M^+_{\mathbf{k}',-\mathbf{p}} \left\langle \alpha^\dagger_{\mathbf{k}} \beta^\dagger_{\mathbf{k}'} \right.
\end{aligned}
$$

where

$$
R_{\mathbf{k}} = \frac{\epsilon^2_{\mathbf{k}} + \mathrm{Re}(\Delta^* \Delta_0)}{E_{\mathbf{k}}}, \qquad C_{\mathbf{k}} = \Delta^*_0 \left(\frac{\epsilon_{\mathbf{k}}}{E_{\mathbf{k}}} \left(1 - \mathrm{Re}\left(\frac{\Delta}{\Delta_0} \right) \right) - i\,\mathrm{Im}\left(\frac{\Delta}{\Delta_0} \right) \right), \tag{41.13b}
$$

and with the shorthand notation $L^\pm_{\mathbf{k},\mathbf{q}} = u_{\mathbf{k}} u_{\mathbf{k}+\mathbf{q}} \pm v_{\mathbf{k}} v_{\mathbf{k}+\mathbf{q}}$ and $M^\pm_{\mathbf{k},\mathbf{q}} = v_{\mathbf{k}} u_{\mathbf{k}+\mathbf{q}} \pm v_{\mathbf{k}+\mathbf{q}} u_{\mathbf{k}}$. The equations of motion for the other quasiparticle densities are of similar form[Schnyder et al.(2011)Schnyder, Manske, and Avella] and are not presented for brevity. The equation of motion for the mean phonon amplitude $\langle b_{\mathbf{p}} \rangle$ is given by

$$
i\hbar \frac{\mathrm{d}}{\mathrm{d}t} \langle b_{\mathbf{p}} \rangle = \hbar\omega_{\mathrm{ph}} \langle b_{\mathbf{p}} \rangle + \frac{g_{\mathrm{ph}}}{\sqrt{N}} \sum_{\mathbf{k}} \left(L^-_{\mathbf{k},-\mathbf{p}} \left(\left\langle \alpha^\dagger_{\mathbf{k}-\mathbf{p}} \alpha_{\mathbf{k}} \right\rangle + \left\langle \beta^\dagger_{\mathbf{k}} \beta_{\mathbf{k}-\mathbf{p}} \right\rangle \right) + M^+_{\mathbf{k},-\mathbf{p}} \left(\left\langle \alpha^\dagger_{\mathbf{k}-\mathbf{p}} \beta^\dagger_{\mathbf{k}} \right\rangle - \left\langle \alpha_{\mathbf{k}} \beta_{\mathbf{k}-\mathbf{p}} \right\rangle \right) \right). \tag{41.13c}
$$

Equations (41.13a)-(41.13c) together with the equations of motion for the other three quasiparticle densities form a closed set of differential equations, which we solve using a standard Runge Kutta algorithm.

41.7.2 Discretization

To solve the closed set of differential equations for the expectation values, e.g. $\left\langle \alpha^\dagger_{\mathbf{k}} \alpha_{\mathbf{k}+\mathbf{q}} \right\rangle$, numerically, we have to restrict the numbers of considered points in momentum space. The first restriction is that we only take expectation values with indices $\mathbf{k}$ and $\mathbf{k}+\mathbf{q} \in W$ into account. Furthermore, only expectation values with indices ($\mathbf{k}$, $\mathbf{k}+n\mathbf{q}_0$) with an integer n have to be consider, as we can see from Eq. (41.13a). The external electromagnetic field breaks translational invariance and may add or subtract momentum $\mathbf{q}_0$. For small amplitudes $|\mathbf{A}_{\mathrm{p}}|$ the off-diagonal elements decrease rapidly as increases because the contribution at ($\mathbf{k}$, $\mathbf{k}+n\mathbf{q}_0$) $= O(|\mathbf{A}_{\mathrm{p}}|^{|n|})$. Thus, we set all entries with $n > 4$ to zero. With this choice of the $\mathbf{k}$, $\mathbf{k}+\mathbf{q}$ values, we are able to solve the equations of motion. To reduce the numerical effort we perform a quasi-one-dimensional calculation. The discretization mesh studied is depicted in Fig. 41.9. It has been shown that this quasi-one-dimensional simulation yields a good approximation for two- and three-dimensional systems [Papenkort et al.(2007)Papenkort, Axt, and Kuhn].

41.7.3 Implementation of pump and probe pulses

We have considered both long and short pump pulses, with widths $\tau_{\mathrm{p}} \ll \tau_\Delta$ and $\tau_{\mathrm{p}} \gg \tau_\Delta$, respectively. The width of the probe pulse, however, is always chosen to be short, with

$\tau_{\rm pr} = 0.25$ ps, such that the nonequilibrium superconductor is probed with a broad range of frequencies. Both the pump and the probe pulses are cut off at a thousandth of their maximum amplitude. Hence, for $|\delta t| \gtrsim 2(\tau_{\rm p} + \tau_{\rm pr})$ there is no overlap between pump and probe pulses. Figure 41.10 depicts the temporal evolution of the pump and probe pulses.

For the pump pulse, which has a high intensity, we take into account both linear and nonlinear couplings to the superconductor [cf. Eq. (41.13a)]. The probe pulse, on the other hand, has weak intensity and is therefore only treated within linear approximation, i.e., terms of second order and higher in the probe field $A_{\rm pr}(t)$ are neglected. As a result, the equations of motion simplify.

To compute the effects of the probe pulse we use further following approximations. First, all off-diagonal terms, such as $\left\langle \alpha^\dagger_{\mathbf{k}} \beta^\dagger_{\mathbf{k}+\mathbf{q}_{\rm pr}} \right\rangle$, are zero before the probe pulse is switched on. Second, we use the same momentum grid for pump and probe pulse. Strictly speaking, due to the different wave vectors of the pump and probe pulses, both pulses act on different momentum grids. However, due to the small values of the photon wave vector this approximation is valid. For the diagonal elements we approximate

$$\left\langle \alpha^\dagger_{\mathbf{k}+\mathbf{q}_{\rm pr}} \beta^\dagger_{\mathbf{k}+\mathbf{q}_{\rm pr}} \right\rangle \approx \left\langle \alpha^\dagger_{\mathbf{k}+\mathbf{q}_{\rm p}} \beta^\dagger_{\mathbf{k}+\mathbf{q}_{\rm p}} \right\rangle \text{ etc.} \tag{41.14}$$

and restrict the off-diagonal elements to

$$\left\langle \alpha^\dagger_{\mathbf{k}+m\mathbf{q}_{\rm pr}} \beta^\dagger_{\mathbf{k}+n\mathbf{q}_{\rm pr}} \right\rangle = 0 \quad \text{if } |m-n| > 1. \tag{41.15}$$

As mentioned above the probe pulse is computed only in linear order in $A_{\rm pr}(t)$. So we neglect the parts in the equation which are proportional to $\frac{e^2}{2m}$. Additionally, all contributions in linear order of $A_{\rm pr}(t)$ and the ones relevant to $g_{\rm ph}$ simplify due to the approximation Eq. (41.15), because not every addend in Eq. (41.13a) has to be taken into account due to the fact that the corresponding expectation value is set to zero.

41.8 Imaginary part of the pump-probe response

For brevity, we have only presented the results for the real part of the pump-probe response in the main text. Here, we show the results for the imaginary part, where we can observe the same signatures.

41.8.1 Pump-probe response in the absence of phonons

In Fig. 41.11(a) we plot the imaginary part of the conductivity versus delay time δt and frequency ω in the nonadiabatic regime with positive delay time. The corresponding real part is depicted in Fig. 41.1. As discussed in the main text, we detect algebraically decaying oscillations as a function of the delay time. For all frequencies except $\omega = \omega_{\Delta_\infty}$, the delay-time dependence can be described by Eq. (41.11). As an example, the conductivity at $\omega = 0.81\omega_{\Delta_\infty}$ is shown in Fig. 41.11(c). Here a perfect match of the fit and the data is shown. The oscillation at $\omega = \omega_{\Delta_\infty}$, where we observe a sharp edge in $\sigma(\delta, \omega)$ as a function of ω, is governed by two frequencies, one is given by ω_{Δ_∞} and the other by $2\omega_{\Delta_\infty}$. This is due to the fact that the amplitude of the oscillation with frequency ω_{Δ_∞} is very small so that we can observe higher harmonics of this oscillation. To describe the delay-time dependence of the conductivity at this frequency, we have to modify Eq. (41.11) in the following way:

$$\begin{aligned} &\mathrm{Im}[\sigma(\delta t, \omega_0)] = \\ &A + \frac{B_1 \cos(\omega_{\Delta_\infty}\delta t + \Phi_1) + B_2 \cos(2\omega_{\Delta_\infty}\delta t + \Phi_2)}{\sqrt{\delta t}} + C\delta t. \end{aligned} \tag{41.16}$$

The perfect match of this fit shows that this is a good approximation; see Fig. 41.11(b).

41.8.2 Pump-probe response in the presence of an optical phonon mode

In Fig. 41.12 the imaginary part of the conductivity versus delay time δt and frequency ω is shown in the nonadiabatic regime with positive delay time near resonance. The corresponding real part is depicted in Fig. 41.4. Also in the imaginary part we detect two edges in the frequency dependence of $\mathrm{Im}[\sigma(\delta t, \omega)]$, one at $\hbar\omega_{\Delta_\infty} = 2.2491$ meV and one at the phonon energy $\hbar\omega_{\mathrm{ph}} = 2$ meV [blue and red traces in Fig. 41.12(a)]. The delay-time evolution of $\mathrm{Im}[\sigma(\delta t, \omega)]$ at ω_{Δ_∞} is again well described by Eq. (41.11) (not shown) . The delay-time evolution of $\mathrm{Im}[\sigma(\delta t, \omega)]$ at ω_{Δ_∞} is well described by Eq. (41.12) and shows the beating just as it occurs in the lattice displacement (not shown). In summary, the imaginary part of the conductivity shows the same signatures as its real part.

Re[σ] [arb. units]
(a)
δt [ps]
ω/ω_{Δ∞}
(d)
|Δ| [meV]
t [ps]
(e)
FT-Amp.
ħω [meV]

(b)
Re[σ(ω=ω_{Δ∞})] [arb. units]
(c)

Figure 41.2: (Color online) (a) Real part of the pump-probe conductivity, $\mathrm{Re}[\sigma(\delta t, \omega)]$, versus ω and $\delta t > 0$ for the adiabatic regime [$\tau_\mathrm{p} = 20$ ps, $|\mathbf{A}_\mathrm{p}| = 0.5 \cdot 10^{-8}$ Js/(Cm)] in the absence of phonons. Panel (b) shows a detail of the pump-probe response for frequencies just below the pump frequency $\omega_\mathrm{p} = 1.21\omega_{\Delta_\infty}$. The small dip near $\omega \approx 1.15\omega_{\Delta_\infty}$ is due to Pauli blocking. (c) Time dependence of $|\Delta(t)|$ for the same parameters as in panel (a).

Figure 41.3: (Color online) (a) Absorption spectrum $\mathrm{Re}[\sigma(\delta t, \omega)]$ versus ω and $\delta t < 0$ for the nonadiabatic regime [$\tau_\mathrm{p} = 0.5$ ps, $|\mathbf{A}_\mathrm{p}| = 13880 \cdot 10^{-8}$ Js/(Cm)] in the absence of phonons. (b) Fourier transform of the data in panel (a), i.e., $\mathrm{Re}(\sigma)$ as a function of Fourier frequency $\omega_{\delta t}$ and absorption frequency ω. (c), (d) Time dependence of $|\Delta(t)|$ and spectral distribution of

Re(σ) [arb. units]
(a)
δt [ps]
ω/ω_Δ
(d)
|Δ| [meV]
t [ps]
(e)
U(0,t) [arb. units]
t [ps]
(f)
FT-Amp.
ω [meV]

(b)
Re[σ(ω=ω_Δ∞)] [arb. units]
(c)
units]

Re(σ) [arb. units]
(a)
δt [ps]
ω/ω_Δ
(d)
|Δ| [meV]
t [ps]
(e)
U(0,t) [arb. units]
t [ps]
(f)
FT-Amp.
ω [meV]

(b)
Re[σ(ω=ω_Δ∞=ω_ph)] [arb. units]
(c)
rb. units]

Figure 41.6: (Color online) (a) Absorption spectrum $\mathrm{Re}[\sigma(\delta t, \omega)]$ versus ω and $\delta t > 0$ for the adiabatic regime [$\tau_\mathrm{p} = 20$ ps, $|\mathbf{A}_\mathrm{p}| = 0.5 \cdot 10^{-8}$ Js/(Cm)] in the presence of an optical phonon mode with energy $\hbar\omega = 2$ meV and coupling strength $g_\mathrm{ph} = 0.1$ meV. (b), (c) Time dependence of the order parameter amplitude $|\Delta(t)|$ and the lattice displacement $U(0,t)$, respectively, for the same parameters as in panel (a). (d) Spectral distribution of the coherent phonon oscillations.

Figure 41.7: (Color online) (a) Real part of the pump-probe signal, $\mathrm{Re}[\sigma(\delta t, \omega)]$, versus ω and $\delta t > 0$ for the adiabatic regime [$\tau_\mathrm{p} = 20$ ps, $|\mathbf{A}_\mathrm{p}| = 0.5 \cdot 10^{-8}$ Js/(Cm)] in the presence of an optical phonon mode that is in resonance with pump pulse energy, i.e., $\hbar\omega_\mathrm{ph} = \hbar\omega_\mathrm{p} = 3$ meV. The electron-phonon coupling strength is taken to be $g_\mathrm{ph} = 0.1$ meV. (b), (c) Time dependence of the order parameter amplitude $|\Delta(t)|$ and the lattice displacement $U(0,t)$, respectively, for the same parameters as in panel (a). (d) Spectral distribution of the coherent phonon oscillations.

Figure 41.8: (Color online) (a) Real part of the pump-probe signal, $\mathrm{Re}[\sigma(\delta t, \omega)]$, versus ω and $\delta t < 0$ for the nonadiabatic regime [$\tau_\mathrm{p} = 0.5$ ps, $|\mathbf{A}_\mathrm{p}| = 13 \cdot 10^{-8}$ Js/(Cm)] in the presence of a phonon mode with energy $\hbar\omega_\mathrm{ph} = 2$ meV $\approx 1.19\,\hbar\omega_{\Delta_\infty}$ and coupling strength $g_\mathrm{ph} = 0.1$ meV. (b) Fourier transform of the data in panel (a), i.e., $\mathrm{Re}(\sigma)$ as a function of Fourier frequency

Figure 41.9: (Color online) Sketch of the two-dimensional momentum space. The red line represents the Fermi surface $k = k_\mathrm{F}$ and the solid black lines are the boundary given by the radial component $k = k_\mathrm{F} \pm k_\mathrm{c}$ (k_c, radial component of the cutoff vector). The chosen k_y values are indicated by the dotted line. In the inset the discretization of the k_x values is shown. Only k_x values with a distance of q_0 couple to each other.

Figure 41.10: (Color online) Temporal evolution of the pump field $A_{\mathbf{q}_\mathrm{p}}(t)$ (solid lines) and and probe field $A_{\mathbf{q}_\mathrm{pr}}(\mathrm{t})$ (dashed lines) with pulse widths $\tau_\mathrm{p} = 0.5$ ps and $\tau_\mathrm{pr} = 0.25$ ps, respectively. The black solid (dashed) line represents the Gaussian envelope of the pump (probe) pulse. The blue and red traces show the real and imaginary parts of the pulse fields, respectively. The pump pulse is centered at $t = 0$ ps and the probe pulse at $t = \delta t = 2$ ps.

Figure 41.11: (Color online) (a) Imaginary part of the pump-probe response, $\mathrm{Im}[\sigma(\delta t, \omega)]$, versus ω and $\delta t > 0$ for the nonadiabatic regime [$\tau_c = 0.5$ ps, $|\mathbf{A}_0| = 8 \cdot 10^{-8}$ Js/(Cm)] in the

Figure 41.12: (Color online) Imaginary part of the pump-probe signal, $\mathrm{Im}[\sigma(\delta t, \omega)]$, versus ω and $\delta t > 0$ for the nonadiabatic regime [$\tau_\mathrm{p} = 0.5$ ps, $|\mathbf{A}_\mathrm{p}| = 10 \cdot 10^{-8}$ Js/(Cm)] in the presence of an optical phonon mode with energy $\hbar\omega_\mathrm{ph} = 2$ meV and coupling strength $g_\mathrm{ph} = 0.1$ meV.

Part XII
Phenomena in Wires and Links

42 Theory of the Resistive State in Narrow Superconducting Channels by Galaiko, Kopnin

42.1 1. Introduction

The resistive state is the state in which the superconductor has a finite resistance to a d.c. current. Despite the apparent peculiarity of such a state (since superconductivity is usually understood as zero resistance to d.c. current) it can be observed rather frequently and indicates that the lack of resistance itself is not a fundamental property of superconductivity.

Resistive states can be observed in a wide variety of phenomena. One of the well-known examples is the mixed state of type-II superconductors where an electric field is generated by vortex motion due to the transport current through the sample (see the reviews by Gorkov and Kopnin 1975 and Larkin and Ovchinnikov 1985). One can note the growing interest to a related phenomenon where a finite resistance appears when the transport current passes along the magnetic field direction so that the vortex motion is much more complicated (see for example Brandt 1980 and Clem 1980). Resistive states can be observed also in type-I superconductors. One can mention the intermediate state of superconducting cylinders in the current-carrying state and also the two-dimensional mixed state at the inner surface of a hollow cylinder when the superconductivity is destroyed by the current (Landau and Sharvin 1969). These problems are reviewed by Dolgopolov and Landau (1985) in this volume. A finite resistance is observed also in wide and thick superconducting strips with dimensions larger than the characteristic superconducting lengths: the coherence length $\xi(T)$ and the magnetic field penetration depth $\hat{\lambda}(T)$. The electric field is generated in this case by the motion of flux tubes which are created near the sample surface.

Finally, the resistive state is observed in long narrow superconducting channels (including long superconducting microbridges) and whiskers with lateral dimensions less than the coherence length and the magnetic field penetration depth. It is this type of resistive states which is considered in the present review.

This choice is not fortuitous. We shall see later that this problem has a fundamental character, and its explanation requires some new (as compared with the classical understanding of superconductivity) ideas and the very recent achievements of the time-dependent microscopic theory. On the other hand, this type of resistive state is well studied experimentally and presents
more interest for theoreticians, due to its effective one-dimensionality (all the quantities depend only on the coordinate along the channel) which reduces the mathematical complexity considerably and makes the physical picture more transparent. The major physical effects involved have a close similarity with the processes taking place in Josephson junctions and weak links (see the review by Aslamazov and Volkov 1985, this volume) and with the processes of the electric field penetration into a superconductor near the normalsuperconducting interface (Artemenko and Volkov 1979).

The present review describes only the theory of the resistive state of narrow superconducting channels. The experimental results are given with the only purpose to outline qualitatively the main features of the phenomenon and do not claim any completeness. Moreover, we do not intend to survey all the available literature on this problem, which is very extensive, but would like to concentrate on the basic ideas and results of the theory which help to understand this

interesting and unusual phenomenon.

The review has the following structure. In the second part of the present section we give briefly the main experimental results characterizing the major properties of the resistive state of narrow superconducting channels. In section 2, the general results are discussed which follow from the fact that superconductivity and electric field coexist in the sample. Here the concept of "phase slip centers" (PSCs) is introduced, and the so-called "quantization rule" is derived, which is a generalization of the well-known Josephson relation. Section 3 deals with the Langer and Ambegaokar fluctuation theory of PSC describing the thermodynamical activation of PSCs in the immediate vicinity of the critical temperature. Some other qualitative models of the resistive state are also described in this section. In sections 4 and 5 we discuss the results of solutions of time-dependent microscopic superconductivity equations by means of numerical and analytical methods, respectively. The theory of the voltage steps in the current voltage characteristics of superconducting channels with a finite length is considered in section 6. Finally, in section 7 we discuss the overall state of the theory and its correspondence to the experimental data.

42.1.1 1.1. Basic experimental results

By narrow superconducting channels we mean samples of which the lateral dimensions are less than (or in practice, of the order of) the characteristic superconducting lengths: the coherence length $\xi(T)$ and the magnetic field penetration depth $i(T)$. Such conditions are met by narrow strips and whiskers. For example, the lateral dimensions for tin must be on the order of some tenths of μm. The length of the samples must be sufficient (tens or hundreds of μm). The purity of the samples influences, comparatively slightly, the main features of their resistive state, and the current voltage characteristics of narrow evaporated strips and whiskers are qualitatively rather similar.

Fig. 1. The schematic current-voltage characteristics of narrow superconducting channels for: (a) channels with a finite length and (b) for long channels. The dashed line is Ohm's law.

Experiments are usually done at temperatures close to the critical one since in this case one can reduce the Joule heating which is quite large in the resistive state.

The major characteristic of the resistive state is its current-voltage characteristic (CVC). Qualitatively, for not very long channels, it has the form represented in fig. la (Meyer and von Minnigerode 1972, Skocpol et al. 1974, Smith 1980, Galako et al. 1976).

One can compare the experimental CVC with the characteristic obtained from the Ginzburg Landau theory, shown in fig. 2. The Ginzburg-Landau theory predicts that the superconductivity should disappear and the sample should go into the normal state when the current exceeds some value known as the Ginzburg Landau critical current j_c (for a review of Ginzburg Landau theory see. e.g.. De Gennes 1966). The principal difference between the CVC in fig. 1 and that in fig. 2 is that a broad region in currents and voltages exists, in practice. where superconductivity and a d.c. electric field are present simultaneously. This state is what we call the resistive state of the superconductor.

Fig. 2. Current-voltage characteristic of a narrow channel in the Ginzburg-Landau theory. The dashed line is Ohm's law.

Note that it is absent in the classical Ginzburg-Landau theory.
The important feature of CVCs of channels with a finite length is the voltage steps observed for both whiskers (Meyer and von Minnigerode 1972) and narrow strips (Skocpol et al. 1974, Smith 1980). Samples in the resistive state have also essential nonstationary properties. First of all, the a.c. Josephson effect can be observed in these samples under external microwave irradiation (Skocpol et al. 1974). The generation of lower frequencies is also observed (Churilov et al. 1969, Galaiko et al. 1973), the generation being localized at the ends of the sample.

We can see that the behavior of narrow superconducting channels in the resistive state differs significantly from the classical picture of the currentcarrying superconductor. Moreover, small lateral dimensions make impossible the explanation of resistivity in the usual terms of vortex motion or normal phase regions through the sample. We can state therefore that the resistive state in narrow superconducting channels is a new type of superconducting state.

42.2 2. The general picture of the resistive state. Phase slip centers

The theory has to explain first of all the most striking point, namely, how superconductivity can exist in a sample in which a d.c. electric field is present. The Cooper pairs are charged and should therefore be accelerated in presence of an electric field until they reach the critical velocity and the superconductivity disappears. As we have seen, however, this is not the case in practice. The point is that the motion of Cooper pairs in a superconductor and, generally, the behavior of a superconductor in an electromagnetic field is governed not by the vector, $\boldsymbol{A}$, and scalar, ϕ,-potentials of the electromagnetic field but by the so-called gauge invariant potentials (see, e.g., Larkin and Ovchinnikov 1985).

$$\boldsymbol{Q} = A - \frac{\hbar c}{2e}\nabla\chi, \quad \Phi = \phi + \frac{\hbar}{2e}\frac{\partial\chi}{\partial t}. \tag{1}$$

Here χ is the phase of the complex order parameter $\Delta\exp(\mathrm{i}\chi)$. In contrast to the potentials A and ϕ the gauge invariant potentials Q and Φ have real physical meaning. For example, the potential Q is related to the velocity of Cooper pairs:

$$V_{\mathrm{s}} = -\frac{e}{mc}Q.$$

The meaning of the potential Φ will be discussed below. With the potentials (1) one can write that

$$\boldsymbol{E} = -\frac{1}{c}\frac{\partial\boldsymbol{Q}}{\partial t} - \nabla\Phi,$$

and hence

$$\frac{\partial v_s}{\partial t} = \frac{e}{m}(\boldsymbol{E} + \nabla\Phi). \tag{2}$$

Equation (2) describes the above-mentioned process of Cooper pair acceleration. The last term in the right-hand side of eq. (2) plays a very important role. Note the gauge invariant potential Φ can be considered as the difference between the chemical potentials of quasiparticles $\mu_{\mathrm{e}} = -e\phi$ and Cooper pairs (per particle) $\mu_{\mathrm{p}} = \frac{1}{2}\hbar(\hat{c}\chi/\hat{c}t)$. In equilibrium $\mu_{\mathrm{e}} = \mu_{\mathrm{p}}$ and $\Phi = 0$. To restrict the velocity of Cooper pairs from growing to infinity, the electric field $\boldsymbol{E}$ has to be averaged out by the term $\nabla\Phi$. It means that the resistive state of a superconductor should be essentially nonequilibrium with $\Phi \neq 0$. The potential Φ in a superconductor, however, cannot be too large either, otherwise the conductivity also would be destroyed. We suppose that the sample is sufficiently long. Then the potential difference $\delta\phi$ between two points which are far enough apart can take up rather large values. In order to keep the potential Φ limited, the difference $\delta\phi$ has to be compensated by a corresponding difference of the pair chemical potentials $\delta\mu_{\mathrm{p}}$ or, which is the same, by a corresponding growth rate of the phase difference between the two points.

The resulting picture can, in principle, be visualized in two different ways. The first, static, model suggests that a periodic structure develops along the sample, the order parameter being equal to zero in points of maximum $|\Phi|$. The neighboring superconducting regions have chemical potentials μ_{p} which differ from each other by the potential difference $\delta\phi$ between these regions. Therefore, within each superconducting region $\boldsymbol{E} = -\nabla\Phi$, $\left(\partial\boldsymbol{Q}/\partial t\right) = 0, \mu_{\mathrm{p}} = \mathrm{const}$. The finite magnitude of Φ results from the compensation of the voltage drop $\delta\phi$ by the difference $\delta\mu_{\mathrm{p}}$ of the pair chemical potentials between the neighboring superconducting regions. In the points where the order parameter is zero the macroscopic phase coherence is broken, and μ_{p} and Φ experience jumps. This is illustrated schematically in fig. 3 where A is the modulus of the order parameter and x is the coordinate along the sample. This static model was proposed by Fink (1973a,b), Fink and Poulsen (1974a,b, 1975) and by Galako (1974, 1975, 1976a,b, 1977), Galako and Shumeko (1976), Galako et al. (1976) and Bezuglyi et al. (1977).

Such a picture ignores, however, a very important point, namely, that the neighboring superconducting regions have different pair chemical potentials $\mu_{\mathrm{p}} = \frac{1}{2}\hbar(\partial\chi/\partial t)$. This implies that the phase difference between the neighboring regions will grow with time and the order parameters in them will strongly interact with each other. Indeed, let us write the complex order parameter

Fig. 3. The schematic behavior of the order parameter Δ and the potentials $\mu_{\mathrm{e}}, \mu_{\mathrm{p}}$ and Φ in the static model of the resistive state.

$\psi = \Delta \mathrm{e}^{\mathrm{i}x}$ in the form

$$\psi = \Delta^{(1)}(x)\exp\left(2\mathrm{i}\mu_{\mathrm{p}}^{(1)}t/\hbar\right) + \Delta^{(2)}(x)\exp\left(2\mathrm{i}\mu_{\mathrm{p}}^{(2)}t/\hbar\right),$$

where $\Delta^{(1,2)} \exp\left(2i\mu_{\rm p}^{(1,2)} t/\hbar\right)$ are the order parameters in the two neighboring superconducting regions. It is clear that the modulus of the order parameter $\Delta = |\psi|$ will strongly oscillate in regions of overlaps of $\Delta^{(1)}(x)$ and $\Delta^{(2)}(x)$ and the static model will not be valid here. The question arises, whether the static model is applicable at all. This problem is discussed in more detail in section 5; for now we note only that, as we shall see later, the oscillations, fortunately, take place only in quite narrow regions near points of broken phase coherence; practically the order parameter oscillates only slightly over the whole interval between the two neighboring points.

In the present review we choose another approach which suggests the nonstationary behavior from the very beginning. As has been said above, the potential difference $\delta\phi$ between two points x_1 and x_2 far enough apart has to be compensated by the growth rate of the phase difference between the two points in order to maintain a finite magnitude of the gauge invariant potential Φ, i.e.,

$$\delta\phi + \frac{\hbar}{2e}\frac{\partial}{\partial t}(\delta\chi) \simeq 0. \tag{3}$$

This can be conveniently illustrated in the space where the real and imaginary parts of the order parameter $\psi = \psi_1 + \mathrm{i}\psi_2$ are plotted on two coordinate axes and the third axis is the x-coordinate along the sample (Langer and Ambegaokar 1967, Rieger et al. 1972) (see fig. 4). In the uniform state the modulus $A = (\psi_1^2 + \psi_2^2)^{1,2}$ is constant and the loops of the helix in fig. 4 have the same radius. The phase difference $\delta\chi$ grows with time and the loops of the helix become more dense. This can not, however, continue for infinitely long

Fig. 4. A representation of the complex order parameter $\psi=\psi_{1 +\mathrm{i} \psi_{2}$ in the space $\{\psi_1, \psi_2, x\}$. The only possibility for the helix to loose a loop is to turn its radius to zero at some point.}

since the phase difference $\delta\chi$ determines the velocity of the condensate

$$v_{\rm s} \sim \frac{\hbar\delta\chi}{m\,(x_2 - x_1)}$$

which, therefore, will grow, thus resulting again in destruction of superconductivity. For the phase difference to be restricted some mechanism of reduction is required. It can be seen

from fig. 4 that the helix could lose one or more loops if its radius A would become zero in some point between x_1 and x_2.

The points where the order parameter turns to zero and its phase experiences jumps by multiples of 2π are known as phase slip centers (PSCs).

To maintain the superconductivity in the sample the phase slip process has to repeat in time, the average period t_0 between phase slips being related to the average voltage V across the points x_1 and x_2 by eq. 3 . Since the phase difference changes by 2π when one loop is thrown off, we get after averaging eq. (3) over time

$$2eV = \frac{2\pi\hbar}{t_0} \tag{4}$$

i.e., the usual Josephson relation. Later we shall give a more formal derivation of such an equation, on the basis of the topological properties of PSCs.

Two different mechanisms of PSC formation are possible. First, PSCs can be formed as a result of thermodynamic fluctuations in the system. The probability of such an event is proportional to $\exp(-\delta F/T)$ where δF is the freeenergy barrier between two uniform states before and after the phase slippage. It is clear that this process is more probable near the critical temperature where the barrier is small. This mechanism was proposed by Langer and Ambegaokar (1967); they suggested also the general picture of phase slippage described above.

When the temperature is lowered from the critical temperature, the probability of fluctuation formation of PSC decreases rapidly and this process is replaced by the intrinsic, and therefore more fundamental, process. One can visualize the process of PSC excitation due to a sufficiently large d.c. current as being similar to the excitation of string vibrations due to the motion of a bow with a constant velocity. From a more formal point of view, this means that the processes of PCS formation, i.e., the order parameter oscillations in some points of the channel, are the result of nonlinearities of limiting-cycle type, which are characteristic for the system.

If the superconducting channel has structure defects the formation of PSCs will take place most probably in "weak points" where the order parameter is suppressed by external causes. In sufficiently long and uniform samples, however, PSCs should be formed periodically in time and in coordinate along the channel because of the space and time homogeneity.

The process of PSC formations can be conveniently represented in twodimensional space-time $\{x; ct\}$ (Ivlev and Kopnin 1978). In fig. 5 the dots designate PSCs, i.e., the points in space-time where the modulus of the order

parameter is zero, $\Delta = 0$. We assume that these points form a periodic structure in the space $\{x; ct\}$. It can be easily seen that the process of throwing off n loops when A goes to zero in fig. 4 is equivalent to the condition that the phase of the order parameter should change by $2\pi n$ after going along a closed contour around a PSC in the space time $\{x; ct\}$ in fig. 5, n being an integer. Therefore, the phase slip centers can be considered as vortex-like topological singularities in the space-time $\{x; ct\}$.

For these topological singularities, one can derive (Ivlev and Kopnin 1978) a "quantization rule" for the electric field "flux" in the space-time $\{x; ct\}$, which is analogous to the quantization of the magnetic flux in the usual space. We introduce the two-dimensional vectors

$$\rho = \{x, ct\}, \quad q = \{Q_x, -\Phi\}, \quad a = \{A_x, -\phi\},$$

which are related by

$$q = a - \frac{\hbar c}{2e}\frac{\partial \chi}{\partial \rho} \tag{5}$$

Fig. 5. The space-time ${x ; c t\
$. The solid dots indicate PSCs (see text). The contour ℓ encloses a unit cell of the PSC structure in the space-time $\{x; ct\}$.}

After integration of the vector $\boldsymbol{q}$ along a closed contour around a PSC one gets, with eq. (5),

$$\oint_{\ell} q \, \mathrm{d}p = \oint a \, \mathrm{d}\rho - \frac{\hbar c}{2e} \oint \frac{\partial \chi}{\partial \rho} \, \mathrm{d}\rho. \tag{6}$$

With the obvious formula

$$E = -\left(\frac{1}{c}\right)\left(\frac{\hat{c}A}{\hat{\partial} t}\right) - \left(\frac{\hat{c}\phi}{\partial x}\right) = (\operatorname{curl} a)_3,$$

the first integral in the right-hand side can be transformed according to the Stokes theorem into an integral of (curl $a)_3$ over the area enclosed by the contour ℓ. The second integral in the right-hand side of eq. (6) gives the phase change $2\pi n$ after going around a PSC. So we have

$$\oint_{t} q \, \mathrm{d}\rho = \int_{S} E \, \mathrm{d}xc \, \mathrm{d}t - \frac{2\pi\hbar c}{2e} n.$$

If we consider a separate PSC the integral in the left-hand side, along the infinitely remote contour, vanishes. In case of a periodic array of PSCs, this integral vanishes also if one chooses the boundary of a unit cell of the PSC lattice in the space-time $\{x; ct\}$ as the contour of integration. So we get

$$\int_{S_0} E \, \mathrm{d}S = \phi_0 n, \tag{7}$$

where $\phi_0 = \pi\hbar c/e$ is the "flux quantum" which is formally equal to the quantum of magnetic flux in the ordinary space. The integration in eq. (7) extends over
the unit cell area S_0 in the space $\{x; ct\}$: $\mathrm{d}S = \mathrm{d}xc \, \mathrm{d}t$. The "quantization rule" (7) is the generalization of the Josephson relation to the case of a spatially distributed a.c. potential. When E is time-independent, eq. (7) immediately gives the usual Josephson relation (4). The quantization rule (7) is very useful since it gives the space- and time-averaged electric field (the one which is measured experimentally) in terms of the periods of the PSC structure in time, t_0, and in the coordinate, L,

$$\langle E \rangle = \frac{\phi_0 n}{c t_0 L}. \tag{8}$$

42.3 3. The phenomenological theories of the resistive state

42.3.1 3.1. Fluctuation-induced phase slip centers

In this section we discuss briefly basic qualitative theories suggested for the resistive state.

When the current through a sample is very small (the exciting force is small) the thermodynamic fluctuations in the system are the main mechanism for the PSC formation. Their probability is proportional to $\exp(-\delta F/T)$ and they are significant only in the close vicinity of T_c where the activation energy is low,

$$\delta F \sim (F_{\mathrm{n}} - F_{\mathrm{s}})\, \xi(T) S \propto (T_{\mathrm{c}} - T)^{3/2}$$

Experimental evidence is now available which verifies that thermodynamic fluctuations are responsible for the resistive state in the temperature range very close to the critical one: $1 - T/T_{\mathrm{c}} \leqslant 10^{-4}$ (Webb and Warburton 1968, Lukens et al. 1970. Newbower et al. 1972). The theory of this phenomenon was considered first by Little (1967). A more comprehensive theory was proposed by Langer and Ambegaokar (1967) and was developed further by McCumber and Halperin (1970) and by Tucker and Halperin (1971). We describe here the main results of the Langer and Ambegaokar (LA) theory.

As it was said above, the rate of phase "throw-off"'s or the frequency of PSC formation is related to the voltage across the region of the superconducting channel, where the PSC is formed, by eq. (8) or by eq. (4). To find the resistivity of the superconducting channel, one has to express the frequency of the fluctuation formation of PSCs in terms of the current through the channel. The LA theory treats this problem in the following way.

Let us consider a channel with length L_0 and suppose for definiteness that the order parameter satisfies cyclic boundary conditions at its ends, i.e., that the helix in fig. 4 has an integer number of loops. The wave function of the superconducting electrons, which is proportional to the complex order para-
meter, has the form

$$\psi_k = \Delta_k \exp(ikx) \tag{9}$$

in the uniform current-carrying state. Here $k = 2\pi m/L_0$. The corresponding current density

$$j = \frac{2e}{i\hbar}\left(\psi^* \nabla \psi - \psi \nabla \psi^*\right)$$

is equal to $j_k = (4e/h) k \Delta_k^2$. The free energy

$$F = \int \mathrm{d}^3 r \left[|\nabla \psi|^2 - \alpha |\psi|^2 + \frac{1}{2}\beta |\psi|^4 \right] \tag{10}$$

[where $\alpha = \alpha_0 \left(T_{\mathrm{c}} - T\right)$] in the current-carrying state (9) is

$$F\left\{\psi_k\right\} = S L_0 \left[\left(k^2 - \alpha\right) \Delta_k^2 + \frac{1}{2}\beta \Delta_k^4 \right], \tag{11}$$

where S is the cross-sectional area of the channel. One can easily calculate the free-energy difference between the state (9) with m loops and the state with ($m - 1$) loops:

$$\delta F_1 = \frac{\partial F}{\partial k} \frac{2\pi}{L_0} = \frac{S \pi k}{e} j.$$

Hence the transition frequency between the states $k \leftrightarrows k-(2\pi/L_0)$ is

$$\hat{\gamma} = \Omega(T) \exp\left(-\frac{\delta F_0}{T} \pm \frac{\dot{\delta} F_1}{2T}\right). \tag{12}$$

The prefactor $\Omega(T)$ in the LA theory has the form

$$\Omega(T) = \frac{2\pi N_{\mathrm{e}}}{\tau},$$

where $N_{\mathrm{e}} = SL_0 n$ is the number of electrons in the sample (n is the electron density) and τ is a characteristic time of microscopic processes remaining undetermined in the LA theory. δF_0 is the free-energy barrier between the states ψ_k and $\psi_{k(2\pi L_0)}$. After calculating the probability flux of the system from the state ψ_k into the state $\psi_{k-(2\pi/L_0)}$ and equating it to $2eV/\hbar$ in accordance with eq. (4), we find

$$\frac{2eV}{\hbar} = \frac{4\pi n S L_0}{\tau} \sinh\left(\frac{\delta F_1}{2T}\right) \exp\left(-\frac{\delta F_0}{T}\right).$$

Langer and Ambegaokar assumed that the time τ entering this expression is the mean free time characterizing the conductivity of the normal metal
$\sigma_{\mathrm{n}} = ne^2\tau/m$. Then the resistivity of the sample in the limit $j \to 0$ is

$$\frac{\rho_{\mathrm{s}}}{\rho_{\mathrm{n}}} = \frac{\pi^2 \hbar^2 n^2 S^2}{mT} \exp\left(-\frac{\delta F_0(j \to 0)}{T}\right) \tag{13}$$

For a complete calculation of the resistivity in eq. (13) one has to find the free-energy barrier δF_0 between two states which have different numbers of loops. The free-energy barrier is the saddle point of the functional (10) in the space of functions $\psi(x)$. This implies that the function $\psi_0(x)$, corresponding to the "pass" solution, must satisfy the condition $\delta F/\delta\psi = 0$, i.e., must be a solution of the Ginzburg-Landau equation

$$\nabla^2\psi + \psi - |\psi|^2\psi = 0. \tag{14}$$

Equation (14) is written in the dimensionless units of Ginzburg Landau theory, where x is measured in $\xi(T)$ and ψ is measured in $\psi_{\mathrm{GL}} = (\alpha/\beta)^{1/2}$. The current density in these units has the form

$$j = (1/2\mathrm{i})\left(\psi^* \nabla\psi - \psi\nabla\psi^*\right) \tag{15}$$

It is convenient to separate modulus and phase of the order parameter $\psi = \Delta \exp(\mathrm{i}\chi)$ and introduce the gauge invariant variables $\Delta = |\psi|$ and $Q = -\nabla\chi$ (here $A = 0$). Equations (14) and (15) become

$$\frac{\partial^2\Delta}{\partial x^2} + \left(1 - \Delta^2 - Q^2\right)\Delta = 0 \tag{16}$$

$$j = -\Delta^2 Q, \quad \operatorname{div} j = 0 \tag{17}$$

Equations (16), (17) have the first integral

$$\left(\frac{\partial\Delta}{\partial x}\right)^2 + \Delta^2 - \frac{1}{2}\Delta^4 + \frac{j^2}{\Delta^2} = \text{ const.} \tag{18}$$

The solution $\Delta(x)$ can be easily obtained if one uses the analogy with particle motion in the potential

$$U(\Delta) = \Delta^2 + \frac{j^2}{\Delta^2} - \frac{1}{2}\Delta^4. \tag{19}$$

In this analogy Δ corresponds to the coordinate of the particle and x to time. The potential $U(\Delta)$ is shown in fig. 6. Bounded solutions for Δ are possible only when the constant in the right-hand side in eq. (18) is less than or equal to $U_{\max}(j)$ (see fig. 6). The solution $\Delta(x)$ represents in this case a function, periodic in x, which can be written in the implicit form

$$x = \int_{\Delta_0}^{\Delta} \frac{\mathrm{d}\Delta}{[C - U(\Delta)]^{1/2}} \tag{20}$$

Fig. 6. The schematic form of the "potential" $U(\Delta)$ for eq. (18).

where the zero of x is chosen in the point $A = A_2$. We need a solution which transforms into the uniform solution $\Delta = \Delta_2$ with the given current j as $x \to \pm\infty$. One can obtain such a solution taking $C = U_{\max}$:

$$x = \int_{1,2}^{\Delta} \frac{\mathrm{d}\Delta}{[U_{\max} - U(\Delta)]^{1/2}}. \tag{21}$$

This function is schematically shown in fig. 7 . The order parameter A_0 in the uniform current-carrying state is related to the current j by

$$j = \Delta_0^2\sqrt{1 - \Delta_0^2}. \tag{22}$$

Equation (22) has solutions only if $j \leqslant j_c = 2/3\sqrt{3}$. The value j_c is known as the Ginzburg-Landau critical current. When $j > j_c$ a uniform superconducting state is impossible.

The solution (21), shown in fig. 7 , represents the function corresponding to the maximum of the free-energy functional $F_{\{}\psi\}$ in the space of functions ψ,

along the line which connects the two states before and after the formation of a PSC and passes through the states with minimum possible free energies.

We are interested here in small currents, $j \to 0$. In this case eq. (21) takes the form

$$\Delta(x) = \tanh\frac{|x|}{\sqrt{2}} \tag{23}$$

Fig. 7. The LA solution (21) for eq. 18.

This solution passes through zero at $x = 0$ and corresponds directly to a PSC. With eq. (23) and eq. (10) it is easy to calculate the free-energy barrier

$$\delta F_0(j=0) = \frac{8\sqrt{2}}{3}\frac{\alpha_0^{3/2} S}{2\beta}\left(T_\mathrm{c} - T\right)^{3/2} \tag{24}$$

Equations (13) and (24) are the main result of the LA theory.
It should be noted, however, that the LA estimation of the prefactor $\Omega(T)$ in eq. (12) gives rise to very serious doubts. Its exact calculation requires direct knowledge of the dynamics of the system's transition through the potential barrier. This problem was considered by McCumber and Halperin (1970) on the basis of the time-dependent Ginzburg-Landau equations supplemented by the Langevin random forces describing fluctuations. According to the results of McCumber and Halperin the prefactor $\Omega(T)$ in eq. (12) appears to be

$$\Omega(T) = \frac{N(T)}{\tau(T)},$$

where $\tau(T) = \pi\hbar/8\,(T_\mathrm{c} - T)$ is the characteristic relaxation time in the Ginzburg-Landau theory for the modulus of the order parameter, and $\mathrm{N}(T)$ is the effective number of statistically independent subsystems within the length of the sample, which is proportional to $L_0/\check{\zeta}(T)$. At small currents they get for the prefactor

$$\Omega(T) = \frac{\sqrt{3}L_0}{2\pi^{3/2}\xi(T)\tau(T)}\left(\frac{\delta F_0}{T}\right)^{1/2}$$

The estimation of this expression shows that the prefactor in the theory of McCumber and Halperin is about 10 orders of magnitude smaller than that in the LA theory.

42.3.2 3.2. Spontaneous formation of phase slip centers

As was said previously, the fluctuation mechanism of PSC formation plays a role only in a very narrow interval around of T_c. If one goes beyond this special interval the probability of fluctuations becomes extremely small. Moreover, if the current through a sample is sufficiently small there will be no mechanism of formation of PSC and the voltage across the channel will be
zero, i.e., the sample will be in a purely superconducting state. When the current exceeds a limiting value (dependent on the sample length) the conditions for spontaneous formation of PSC are satisfied (the system falls in a region of attraction to a limiting cycle) and one PSC appears in the sample. This is reflected as a voltage step in the CVC (see fig. 1a). When the current is increased further, two, three and more PSCs may appear in the sample, accompanied by corresponding voltage steps in the CVC. This happens in samples of finite length. In an

infinitely long sample, the total number of PSCs is always large and the density increases smoothly with the current; the corresponding CVC has the form of a smooth curve (fig. 1b).

It is clear from the discussion above that the nonlinearity of a system must play the main role in the resistive state. This inevitably leads to the conclusion that it is very complicated to obtain the exact solution of the problem, if this is possible at all. For this reason we first outline some qualitative theories of the resistive state.

The qualitative picture of PSC formation is the following: the velocity of the superconducting condensate increases under the action of an electric field. Then at some time and place in the sample the velocity reaches such a value (not necessarily equal to the critical velocity) that the superconducting state becomes unstable and the order parameter falls to zero at this place. When Δ turns to zero the phase coherence is destroyed. Then the formation of the superconducting condensate starts again in the vicinity of this point since the sample is at a temperature below T_c. The superconductivity is restored, however, with a different phase, so that the phase difference between the superconducting regions to the right and to the left of a PSC is changed by 2π compared to the phase difference in the initial state. This process repeats in the time determined by eq. (4).

Rieger et al. (1972) have proposed a numerical model for PSC formation. They defined the moment of PSC formation as the moment when the free energy of a sample section, which increases due to the acceleration of Cooper pairs by an electric field, becomes higher than the free energy of the state which this section would have if the phase difference between the ends were 2π less. This condition is, of course, quite artificial and the model proposed by Rieger et al. is useful only as a first step in solving the problem of the resistive state.

We postpone the consideration of the PSC dynamics to later sections and discuss now a qualitative picture of formation of the voltage across the PSC: we shall follow here the model suggested by Skocpol. Beasley and Tinkham (1974) (SBT-model).

The total current density can be represented as a sum of the supercurrent j_s and the normal current $j_n = \sigma E$:

$$j = \sigma E + j_s = -\sigma \left(\frac{\partial \Phi}{\partial x} + \frac{1}{c} \frac{\partial Q}{\partial t} \right) + j_s. \tag{25}$$

All the quantities in the right-hand side of eq. (25) depend on time, varying periodically with the period $t_0 = 2\pi/\omega_J$, where ω_J is the Josephson frequency. We are interested in the time-averaged voltage which is measured experimentally. After averaging eq. (25) over time the term $\partial Q/\partial t$ vanishes due to periodicity and we have

$$j = -\sigma \frac{\partial \bar{\Phi}}{\partial x} + \bar{j}_s \tag{26}$$

It was said in section 2 that the gauge invariant potential Φ can be considered as the difference between the chemical potentials of Cooper pairs, $\mu_p = (\hbar/2)\partial\chi/\partial t$, and quasiparticles, $\mu_e = -e\phi$, i.e. $\Phi = (\mu_p = \mu_e)/e$. In an equilibrium superconductor, it always holds that $\mu_e = \mu_p$ and $\Phi = 0$.

As we have seen Φ has to be nonzero in the resistive state. This type of nonequilibrium results from the possibility, characteristic for a superconductor, of imbalance between the occupation numbers of electron-like and hole-like branches in the energy spectrum (the so-called "branch imbalance"). We shall not consider the physical nature of this effect in detail since it goes beyond the scope of this review. Numerous original literature is available on this question (see, e.g., Tinkham and Clarke 1972, Tinkham 1972, Schmid and Schön 1975, Artemenko et al. 1978, also the review by Artemenko and Volkov 1979). The important point here is that this nonequilibrium, just as any other nonequilibrium, has a characteristic relaxation time, τ_Q

(Tinkham and Clarke 1972, Tinkham 1972). Due to the quasiparticle diffusion, the nonequilibrium created in some place of a superconductor is transferred to other places and decays with a characteristic diffusion length $l_E = \sqrt{D\tau_Q}$, where D is the diffusion coefficient. The length l_E is the relaxation length of branch imbalance, i.e of Φ, in case this nonequilibrium does not depend on time (i.e., is maintained by external sources). This length is known as the electric field penetration depth. since $\boldsymbol{E} = -\nabla\Phi$ in the static case. The relaxation process of the difference $\mu_{\mathrm{p}} - \mu_{\mathrm{e}}$ occurs by means of interaction of excitations with the condensate through phonons (see, e.g., Artemenko and Volkov 1979) and that is why the relaxation time, τ_Q, contains the long electron-phonon collision time $\tau_{\mathrm{ph}} \sim T_{\mathrm{D}}^2\ \hbar/T^3$ where T_{D} is the Debye temperature. As a result, the electric field penetration depth appears to be quite large. Calculations of the length l_E near the critical temperature (Schmid and Schön 1975, Artemenko et al. 1978) yield $l_{\mathrm{E}} = (4DT\tau_{\mathrm{ph}}/\pi\Delta)^{1,2}$; this considerably exceeds the coherence length $\check{\zeta}(T)$.

Now we can write down the equation for $\bar{\Phi}$ in the form

$$\hat{c}^{\hat{2}}\bar{\Phi} = \frac{1}{l_E^2}\bar{\Phi} \tag{27}$$

The simplest approximation is to consider l_E independent of x. The solution of eq. (27) is then

$$\bar{\Phi} = C\sinh\left(\frac{x - x_1}{l_{\mathrm{E}}}\right)$$

where C is a constant and x is the point where $\bar{\Phi} = 0$. We assume for simplicity that the ends of the superconducting channel are connected with the bulk superconducting "banks" which are in equilibrium so that $x = x_1$ corresponds to the end of the channel. From eq. (27) we get

$$j - j_{\mathrm{s}}(x) = -\frac{\sigma}{l_{\mathrm{E}}}C\cosh\left(\frac{x - x_1}{l_{\mathrm{E}}}\right)$$

Let us suppose that the sample has a finite length and there is only one PSC in it, at the point $x = x_0$. The potential drop from x to the PSC is

$$\bar{\Phi}_1 = C\sinh\left(\frac{x_0 - x_1}{l_{\mathrm{E}}}\right)$$

Adding here the potential drop $\bar{\Phi}_2$ from the point x_0 to the point x_2 where $\bar{\Phi} = 0$ (the other end of the channel) we get

$$\bar{V} = \bar{\Phi}_1 + \bar{\Phi}_2 = \frac{l_{\mathrm{E}}}{\sigma}\left[\tanh\left(\frac{x_0 - x_1}{l_{\mathrm{E}}}\right) + \tanh\left(\frac{x_2 - x_0}{l_{\mathrm{E}}}\right)\right]\left(\mathrm{j} - \bar{\mathrm{j}}_{\mathrm{s}}\left(\mathrm{x}_0\right)\right). \tag{28}$$

(a)

(b)

If $|x_0 - x_1|, |v_2 - x_0| \gg l_{\mathrm{E}}$ then $\bar{v} = \frac{2l_{\mathrm{E}}}{\sigma}\left(j - \bar{j}_{\mathrm{s}}\left(x_0\right)\right)$, and the differential resistivity for one PSC is

$$\rho_{\mathrm{d}} = 2l_{\mathrm{E}}/\sigma$$

The voltage $\bar{V}$ is the difference between the time-averaged chemical potentials of Cooper pairs to the right and to the left of the PSC. The schematic behavior of the potentials and of the order parameter in the SBT-model is shown in fig. 8. The fine experiment by Dolan and Jackel (1977) verified perfectly the spatial distribution of the potentials shown in fig. 8a.

Fig. 8. The behavior of (a) the potentials $\mu_{\mathrm{p }}$ and μ_{e} and (b) the order parameter Δ in the SBT model. The bars indicate time-averaged values.}

If a superconducting channel has a uniform structure the first PSC appears in the middle and we get from eq. (28)

$$\bar{\Phi} = -\left(j - j_{\mathrm{s}}\left(x_0\right)\right) \frac{l_{\mathrm{E}}}{\sigma} \sinh\left[\left(x - x_1\right)/l_{\mathrm{E}}\right] / \cosh\left(L_0/2l_{\mathrm{E}}\right)$$

where $L_0 = (x_2 - x_1)$ is the length of the channel. The supercurrent is

$$j_{\mathrm{s}}(x) = j + \sigma \frac{\partial \bar{\Phi}}{\partial x} = j - \left[j - \bar{j}_{\mathrm{s}}\left(x_0\right)\right] \cosh\left(\frac{x - x_1}{l_{\mathrm{E}}}\right) / \cosh\left(\frac{L_0}{2l_{\mathrm{E}}}\right)$$

The supercurrent reaches its maximum at the ends of the channel (connected with the superconducting banks) where it is equal to

$$j_{\mathrm{s\,max}} = j\left[1 - \operatorname{sech}\left(\frac{L_0}{2l_{\mathrm{E}}}\right)\right] + \bar{j}_{\mathrm{s}}\left(x_0\right) \operatorname{sech}\left(\frac{L_0}{2l_{\mathrm{F}}}\right)$$

The supercurrent cannot exceed the Ginzburg Landau critical current j_c. On the other hand, when the total current is increased the condition $j_{s\,\max} \leqslant j_c$ will not be valid, starting with some value of the total current. In this case the solution with one PSC only becomes impossible. This means that one more PSC should appear in the sample, accompanied by a voltage step in the CVC. When the current is increased further there would appear a third and more PSCs, each formation of a new PSC being accompanied by a corresponding voltage step. We see that the simple SBT-model can also give a qualitative explanation for the voltage steps in the CVC (Tinkham 1979). This question will be discussed in more detail in section 6 on the basis of the microscopic equations.

The simple physical considerations discussed above give a key to the understanding of the nature of the resistive state. They must be compatible, however, with the known dynamic properties of superconductors which follow from the microscopic theory. This problem is treated in other sections of the present review.

42.4 4. The microscopic theory: results of numerical solutions of timedependent superconductivity equations

42.4.1 4.1. Time-dependent superconductivity equations

The formation of PSCs is an essentially dynamic process and should be described by time-dependent superconductivity equations. Unfortunately, a system of time-dependent equations for superconductors is in general very complicated. It contains, in addition to equations for the superconducting variables Δ, Q and Φ, generalized kinetic equations for "distribution functions" of excitations (see, e.g., Larkin and Ovchinnikov 1985). To get a closed set of equations for $\Delta, \boldsymbol{Q}$ and Φ is possible only in some special cases. We write down here dynamic equations which can be derived from the microscopic theory if all variations in the quantities are sufficiently slow in space and time. We require that $Dk^2, \omega \ll \tau_{\rm ph}^{-1}$ where D is the diffusion coefficient, $\tau_{\rm ph}$ is the electron-phonon relaxation time, and ω and k are the characteristic frequency and wavevector of the problem, respectively.* In spite of these restrictions the equations which we shall use here have a sufficiently general form and are valid in a quite reasonable temperature range, from the experimental point of view. These equations have been obtained in the most general case by Kramer and Watts-Tobin (1978a) and by Watts-Tobin et al. (1981) (see also Golub 1976 and Schön and Ambegaokar 1979). They are as follows

$$-\frac{\pi}{8T}\left(4\tau_{\rm ph}^2 A^2+1\right)^{1/2}\frac{\partial\Delta}{it}+\frac{\pi D}{8T}\left(\nabla^2\Lambda-\frac{4e^2}{c^2}Q^2\Lambda\right) \tag{29}$$
$$+\frac{T_c-T}{T_c}\Delta-\frac{7\zeta(3)\Delta^3}{8\pi^2T^2}=0$$

$$\left(4\tau_{\rm ph}^2\Delta^2+1\right)^{-1/2}\Delta^2\Phi+\frac{D}{c}\,\mathrm{div}\left(\Delta^2 Q\right)=0 \tag{30}$$

$$j=-\sigma\left(\frac{1}{c}\frac{\partial Q}{\partial t}+\nabla\Phi\right)-\frac{\pi\sigma\Lambda^2 Q}{2cT} \tag{31}$$

One can easily get from eq. (31) and the electroneutrality condition $\mathrm{div}\, j=0$, that

$$D\nabla^2\Phi+\frac{D}{c}\,\mathrm{div}\,\frac{\partial\boldsymbol{Q}}{\partial t}=\frac{\pi}{2T}\frac{\Lambda^2\Phi}{\left(4\tau_{\rm ph}^2\Delta^2+1\right)^{1/2}} \tag{32}$$

Equation (32) describes the branch imbalance relaxation mentioned above.

When Δ is constant, it determines the penetration depth

$$l_{\rm E}=\left(\frac{2DT}{\pi\Delta^2}\right)^{1/2}\left(4\tau_{\rm ph}^2\Delta^2+1\right)^{1/4}$$

for a d.c. electric field.

In the gapless case $\tau_{\rm ph}\Delta \ll 1$, eqs. (29)-(32) transform into the timedependent Ginzburg-Landau equations (TDGL) (Abrahams and Tsuneto 1966, Schmid 1966). In the opposite case $\Delta\tau_{\rm ph} \gg 1$, eq. (29) coincides with the dynamic equation for a superconductor with a gap, which has been obtained first by Gor'kov and Eliashberg (1969), and eq. (32) defines the electric field penetration depth

$$l_{\rm E}=\left(D\tau_{\rm ph}\frac{4T}{\pi\Delta}\right)^{1/2} \tag{33}$$

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- From now on we put $\hbar=1$.

which coincides with the results of Schmid and Schön (1975) and Artemenko et al. (1978). We can see from these equations that the characteristic frequency scale is of the order of

$$\omega_0 = \frac{\pi \Delta_{\mathrm{GL}}^2}{2T}, \tag{34}$$

where

$$\Lambda_{\mathrm{GL}} = \left[\frac{8\pi^2 T\left(T_{\mathrm{c}} - T\right)}{7\zeta(3)}\right]^{1/2} \tag{35}$$

is the equilibrium order parameter. The characteristic scale for spatial variations of Δ is the coherence length

$$\xi(T) = \left[\frac{\pi D}{8\left(T_c - T\right)}\right]^{1/2} \tag{36}$$

The ratio $l_{\mathrm{E}}/\xi(T)$ is of the order of $\left(\Delta^2 \tau_{\mathrm{ph}}^2 + 1\right)^{1/4}$ and is always larger than unity. Equations (29) (32), are, therefore, valid in the temperature range $1 - T_1 T_{\mathrm{c}} \ll \left(\tau_{\mathrm{ph}} T_{\mathrm{c}}\right)^{-1}$. We shall see later, however, that the Josephson frequency of oscillations in the resistive state is larger than the ω_0 from eq. (34) so that the actual temperature range of validity of eqs. (29)-(32) will be somewhat narrower.

It will be convenient to use dimensionless variables. We shall measure distances in $\xi(T)$, the order parameter in Δ_{GL}, frequencies in ω_0, and currents in units of $\sigma\omega/2e\xi$. In the new units the potentials are $\Phi = \varphi + (\partial\chi/\partial t)$, $Q = A_x - (\partial\chi/\partial x)$ and the equations take the form

$$-u\left(\frac{\Delta^2}{\Gamma^2} + 1\right)^{1,2} \frac{\lambda A}{\partial t} + \frac{\hat{\imath}^2 \Delta}{\partial x^2} + \left(1 - \Delta^2 - Q^2\right)\Delta = 0, \tag{37}$$

$$u\Delta^2\left(\frac{\Delta^2}{\Gamma^2} + 1\right)^{-1/2} \Phi + \frac{\partial\left(\Delta^2 Q\right)}{\partial x} = 0 \tag{38}$$

$$j = -\frac{\partial Q}{\partial t} - \frac{\partial \Phi}{\partial x} - \Delta^2 Q \tag{39}$$

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 Q}{\partial x \partial t} = u\Delta^2\left(\frac{\Delta^2}{\Gamma^2} + 1\right)^{-1/2} \Phi. \tag{40}$$

Here Γ is the pair-breaking term

$$\Gamma = \left(2\tau_{\mathrm{ph}} A_{\mathrm{GL}}\right)^{-1} = \frac{\pi}{8\sqrt{u}\left(\tau_{\mathrm{ph}} T_{\mathrm{c}}\right)\left(1 - T/T_{\mathrm{c}}\right)^{1/2}}. \tag{41}$$

The coefficient $u = \pi^4, 14\zeta(3) = 5.79$. The electric field penetration depth in the new units is, from eq. (40),

$$l_{\mathrm{E}} = \left(\frac{\Delta^2}{\Gamma^2} + 1\right)^{1/4} / u^{1/2}\Delta,$$

and for $\Gamma \ll 1$ it will be

$$l_{\mathrm{E}} = \left(u\Gamma\Delta\right)^{-1/2}. \tag{42}$$

The equations (29)-(33) or (37)-(40) form the basis for our analysis of the resistive state.

42.4.2 4.2. Numerical results

We have said previously that phase slip processes have essential nonlinear nature so it is very difficult to obtain a mathematically exact analytical solution even for the comparatively simple time-dependent equations (29)(32), and such a solution is not yet found. Numerical methods of solving these equations can be very helpful here.

The first results in this line have been obtained by Kramer and Baratoff (1977). They investigated the TDGL equations which are valid in case of gapless superconductivity. In terms of eqs. (37)-(40) this corresponds to the limit $\Gamma \gg 1$, when the lack of a gap in the energy spectrum is due to strong electron-phonon interaction:

$$-u\frac{\partial\Delta}{\partial t}+\frac{\partial^2\Delta}{\partial x^2}+\left(1-\Delta^2-Q^2\right)\Delta=0 \tag{43}$$

$$u\Delta^2\Phi+\frac{\partial\left(\Delta^2 Q\right)}{\partial x}=0 \tag{44}$$

$$j=-\frac{\partial Q}{\partial t}-\frac{\partial\Phi}{\partial x}-\Delta^2 Q \tag{45}$$

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2 Q}{\partial x\partial t}=u\Delta^2\Phi. \tag{46}$$

The system of equations (43)-(46) can be represented in a complex form if one introduces the complex order parameter $\psi=\Delta\exp(\mathrm{i}\chi)$. We can ignore the magnetic field produced by a current since the lateral dimensions of the sample are small, and put $A=0$. Then we have

$$\begin{aligned}-u\left(\frac{\partial}{\partial t}+\mathrm{i}\phi\right)\psi+\frac{\partial^2\psi}{\partial x^2}+\left(1-|\psi|^2\right)\psi=0,\\ j=-\frac{\partial\phi}{\partial x}+\frac{1}{2\mathrm{i}}\left(\psi^*\frac{\partial\psi}{\partial x}-\psi\frac{\partial\psi^*}{\partial x}\right).\end{aligned} \tag{48}$$

Similar equations appear for gapless superconductors with large concentrations of magnetic impurities (Gor'kov and Eliashberg 1968), one has to put $u=12$ in the latter case. The TDGL equations (47), (48) differ from the usual Ginzburg-Landau equations (14), (15) by the term with the time derivative in the gauge invariant form in eq. (47) and by the additional term with the normal current in eq. (48).

Kramer and Baratoff (1977) solved eqs. (47), (48) for two values of the parameter $u: u=5.79$ and $u=12$; they obtained the following results:

(1) When the current is less than some value j_1 ($j_1=0.326$ for $u=5.79$ and $j_1=0.284$ for $u=12$) perturbations with respect to the purely superconducting state decay, and the superconductor returns to a uniform current-carrying state with the order parameter from eq. (22). Note that in this case $j_1<j_c$, where $j_c=(2/3\sqrt{3})=0.385$ is the Ginzburg-Landau critical current.

(2) When the current is larger than a threshold value j_2 (where $j_2=0.335$ for $u=5.79$ and $j_2=0.291$ for $u=12$) the superconducting current-carrying state transforms into an expanding domain of the normal phase and finally disappears. Thus, the superconducting state appears to be unstable for $j>j_2$. The threshold current j_2 coincides with the stability limit of the normal-superconducting interface studied by Likharev (1974).

(3) In the interval between j_1 and j_2 there exists a solution which corresponds to phase slippage. This solution behaves as follows. The channel remains superconducting over its length, but at some point local oscillations of the order parameter Δ take place. When Δ turns to zero the

phase χ experiences a jump of 2π. When the current approaches j_1 the oscillation period tends to infinity and the solution transforms into the Langer and Ambegaokar solution (20) asymptotically as $t \to \pm x$. The amplitude of oscillations decreases with increasing current. The Kramer and Baratoff solution for $u = 5.79$ and $j = j_1$ is shown in fig. 9.

These results were the first direct verification of the existence of a solution with phase slippage and have given evidence for the existence of a limiting

Fig. 9. The behavior of Δ and $\mu_{\mathrm{e }}$ at different times according to numerical computations by Kramer and Baratoff (1977) for the TDGL equations. The parameters are $u = 5.79$ and $j = j_1 = 0.326$.}

cycle with oscillations of the required type for a system described by eqs. (47), (48) or eqs. (43)-(45).

Equations (43)-(45) were also investigated more recently by Ivlev et al. (1980a,b). They use the gapless equations for modelling the processes in superconductors with a gap in the energy spectrum. The main idea was as follows. When there is a gap (i.e. when $\Gamma \ll 1$) the electric field penetration depth (42) is large: $1_E \sim \Gamma^{-1/2} \gg 1$. This situation can be simulated in terms of eqs. (44) and (46) by giving a small value to the parameter u : since l_{E} in eq. (46) is proportional to $u^{-1/2}$ this will ensure $l_{\mathrm{E}} \sim u^{-1/2} \gg 1$. Therefore, by means of the simplest TDGL equations (43)-(46), it turns out that it is possible to describe the most important feature (with respect to the resistive state) of real superconductors (i.e., superconductors with a gap), namely, the large value of the electric field penetration depth $l_E/\xi \gg 1$.

Ivlev et al. (1980a,b) solved eqs. (43)-(46) numerically with $u = 0.01$. The total length of the channel was divided into equal segments with length L and the equations were solved on one segment $0 \leqslant x \leqslant L$ with $L = 40$. The boundary conditions $\hat{c}\Delta^2/\hat{r}x = 0$ and $\Phi = \phi + (\partial\chi/\partial t) = 0$ were imposed, which follow from the periodicity of the structure with period L and from the symmetry of the problem with respect to the points $x = 0, L$ taking account for the even parity of A and j_ς and the odd parity of Φ. The length L represents, therefore, the distance between the neighboring PSCs which form a periodic structure along the channel. The solution describing the oscillations in A is shown in fig. 10.

These results have one important feature: the oscillations of the order parameter are localized in a rather narrow region around a PSC while the variables A, Q and Φ are practically independent of time for the greatest part of the distance between two PSCs. The reasons why this is so will be discussed in the next section.

The numerical investigations were then continued by Kramer and WattsTobin (1978a,b) and by Watts-Tobin et al. (1981) directly for eqs. (37)-(40)

with different values of the parameter Γ. It was found that the range of currents where the solution with phase slippage exists is broadened with decreasing Γ, the current j_2 being

Fig. 10. The coordinate dependence of Δ at different times according to numerical results of Ivlev et al. (180a,b) for TDGL equations. The parameters are $u = 0.01$ and $j = 0.4$. The curves 1,2 , and 3 are for $t = 0, t = 0.053$, and $t = 2.188$, respectively. The period of oscillation $t_{0 =2.52$.}

proportional to Γ^{-1} at small Γ. The oscillations of the order parameter occur qualitatively in the same way as in the case of the simple TDGL equations. The results of Watts-Tobin et al. (1981) for Δ, with $L = 6, j = 0.4$ and different values of Γ, are represented in fig. 11. Watts-Tobin et al. mentioned the fact that, with decreasing in Γ, the curves $\Delta(x, t)$ approach the curve which corresponds to the steady-state dependence of the order parameter near the boundary with a normal phase, i.e., the curve, satisfying the boundary condition $\Delta(x = 0) = 0$. The explanation for this will be given in section 5.2.

The described numerical results are of great importance. They prove first of all the very fact of the existence of solutions which correspond to phase slippage and verify thereby the validity of the general understanding of the nature of the resistive state. The numerical methods, however, cannot give complete information on the resistive state properties and how they depend on the various sample parameters and the real experimental conditions. Therefore, analytical solutions of time-dependent equations are needed since only they can answer these questions. The recent progress in the microscopic theory of the resistive state along this line will be discussed in the following sections.

42.5 5. The microscopic theory: results of analytical solutions of timedependent superconductivity equations

42.5.1 5.1. The stability analysis of the current-carrying normal state

In this section we investigate the stability of the current-carrying normal state with respect to infinitely small fluctuations of the order parameter. When the

order parameter is infinitely small we have always the gapless situation $\Delta\tau_{\rm ph} < 1$, so that a superconductor will be described by eqs. (47) and (48). The linearized eqs. (47) and (48) have the form

$$u\frac{\partial\psi}{\partial t} - \mathrm{i}ujx\psi - \frac{\partial^2\psi}{\partial x^2} - \psi = 0. \tag{49}$$

The solution of this equation can be written in the form (Ivlev et al. 1980b)

Fig. 11. The coordinate dependence of Δ for different values of Γ according to Watts-Tobin et al. (1981). The full lines correspond to minima of $\Delta(x = 0)$ and the dashed lines to maxima of $\Delta(x = 0)$.

$$\psi(x,t) = \left(\frac{u}{4\pi t}\right)^{1/2} \exp\left(\frac{t}{u} - \frac{\mathrm{j}^2\mathrm{t}^3}{12u}\right) \int_{-\alpha}^{\infty} \mathrm{d}y\psi(y,0) \exp\left\{ \frac{\mathrm{i}jt}{2}\ (x+y) - \frac{u(x+y)^2}{4t} \right\}. \tag{50}$$

The term t^3 in the exponent accounts for the acceleration of the Cooper pairs by an electric field, and results in the destruction of superconductivity. This leads to the conclusion that, according to Gor'kov (1970) and Kulik (1970), the normal state is stable with respect to infinitely small fluctuations.

This conclusion, however, does not apply to fluctuations with a finite magnitude. The point is that the behavior of a superconductor depends significantly on how an electric field is created in it. In a circuit with a given voltage the normal state is stable at any electric field, independent of how small it is. In a circuit with a given current, however, expulsion of the electric field from the superconductor may take place either totally or partially, the result being either the pure superconducting or the resistive state. This can occur if the order parameter reaches a value of the order of unity in some place of a superconducting channel. This can be spoken of as a nucleus of the superconducting phase in the normal state. This nucleus will either grow or decrease, depending on the current, and the normal state will be either unstable or stable. Interesting numerical results concerning the stability of both the normal and the superconducting state have been obtained by WattsTobin et al. (1981).

Suppose that a fluctuation of Δ appears with amplitude.$_1$ and size larger than ξ. Then, as can be seen from eq. (50), it will evolve according to

$$\Delta \simeq \Delta_0 \exp\left(\frac{t}{u} - \frac{j^2 t^3}{3}\right)$$

passing through a maximum $A_{\max} = A_0 \exp(2, 3uj)$. Thus, if the amplitude of an initial fluctuation is larger than or of the order of

$$\Delta_{\mathrm{c}} = \exp\left(-\frac{2}{3uj}\right) \tag{51}$$

it can reach a value of the order of unity during its further evolution, after which the system can go either into a uniform current-carrying superconducting state locally stable at $j < j_{\mathrm{c}}$, or into the resistive state.

From eq. (50) one can draw an important conclusion about the static model of the resistive state discussed in section 2 . In the static model the order parameter is equal to zero at some points in the sample and $\mu_{\mathrm{p}} = \frac{1}{2}(\hat{c}\chi, \hat{c}t)$ experiences jumps at these points (see fig. 3). Such a solution for eqs. (47) and (48) has been obtained by Fink (1973a, b) and Fink and Poulsen (1974a, b, 1975). We examine now, using eq. (50), what will happen to this static solution if one does not impose the special condition $\psi(0,t) = 0$. Near the point $x = 0$ and at the initial instant of time, $\Delta = \alpha|x| \ll 1$. We find from eq. (44) that $Q = \frac{1}{3}u|\Phi(x = \pm 0)||x|$. Therefore, the order parameter near a static PSC should have the form

$$\psi_{\mathrm{st}}(x,t) = \alpha|x| \exp\left\{\mathrm{i}|\Phi(x = \pm 0)|\left(t - \frac{x^2 u}{6}\right)\operatorname{sgn} x\right\} \tag{52}$$

If we now compare expression (52) with the solution (50) of the timedependent equation (49) when $\Delta \ll 1$, we can see that ψ_{st} from eq. (52) is not a solution of eq. (49), actually, and that the true solution of eq. (49), with the initial condition $\Delta = \alpha|x|$, gives $|\psi(0,t)| = (t\alpha/u)^{1/2}$, which is nonzero at any time $t \neq 0$. This means that a PSC cannot exist as a static object since there is no way to fulfil the condition $\psi(0,t) = 0$, and also that the solution of Fink and Poulsen is unstable near a PSC. This conclusion is supported also by the numerical results of Ivlev et al. (1980a,b). Indeed, curve 1 in fig. 10 corresponds to the solution of Fink and Poulsen. In this case, however, there was no condition $\Delta = 0$ imposed at $x = 0, L$ and the curve does not pass through zero at $x = 0, L$ at subsequent moments of time but oscillates from zero to unity.

42.5.2 5.2. The structure of the phase slip centers

We consider below the time-dependent equations (37)-(42) in the most interesting case, from both the experimental and the theoretical point of view, when there is a gap in the energy spectrum of the superconductor, $\Gamma \ll 1$, which corresponds to temperatures $(T_{\mathrm{c}}\tau_{\mathrm{ph}})^{-2} \ll 1 - T/T_{\mathrm{c}}$. We remind ourselves that the electric field penetration depth is large in this case: $l_E \sim \Gamma^{-1/2} \gg 1$. The following analysis will be based on the results of Ivlev et al. (1980a,b) and Ivlev and Kopnin (1981a).

According to the quantization rule (7) the time- and space-averaged electric field is related to the periods of the PSC structure. In the dimensionless units of section 4.1 this relation has the form

$$\langle E \rangle = \frac{2\pi n}{L t_0}. \tag{53}$$

The integer n indicates how many times 2π the phase looses after formation of each PSC. We choose $n = 1$ as is the case for the ordinary vortices in type-II superconductors.

When a uniform sample (without any defects) is considered, it is clear that the only mechanism which determines the distance between PSCs is the relaxation of the potential Φ, i.e.,

the relaxation of the imbalance between the chemical potentials of the Cooper pairs, $\mu_{\rm p}$. and quasiparticles, $\mu_{\rm c}$ (the imbalance being created during phase slippages). This means that the distance should be of the order of the electric field penetration depth $|_1$. If the current through the sample is not too large, i.e., $j \sim \sigma E \sim j_{\rm c}(j \sim 1$ in our units) the period of oscillations t_0 should be of the order of $L^{-1} \sim \Gamma^{1/2}$ according to eq. (53).

When $\Gamma \ll 1$ one can write eqs. (38) and (39) in the form

$$j = -\frac{\partial Q}{\partial t} + \frac{1}{u\Gamma}\frac{\partial}{\partial x}\left[\frac{1}{\Delta}\frac{\partial}{\partial x}\left(\Delta^2 Q\right)\right] - \Delta^2 Q. \tag{54}$$

Estimation of the first term in the right-hand side of eq. (54) gives $Qt_0^{-1} \sim Q_{\sim}\Gamma^{-1/2}$, where $Q_{\sim}$ is the oscillating part of $Q : Q = Q + Q_{\sim}$, and the bar indicates averaging over time. The second and the third term are of the order Q at distances of order $I_E \sim \Gamma^{-1/2}$. It follows that the oscillating part of Q should be small, $Q_{\sim} \sim \Gamma^{1/2}\bar{Q}$ so that $Q \simeq \bar{Q}$. The oscillating part $Q_{\sim}$ will be of the same order as $\bar{Q}$ only at shorter distances from a PSC where the first and the second term in eq. (54) will be of the same order, i.e., at distances from a PSC of the order of $x_1 = \Gamma^{-1/4} \ll L$. In ordinary units we have $x_1 = [\xi(T)l_E]^{1/2}$. It is important that the oscillations of the order parameter remain small at such distances. This can be seen from the estimation of the relevant terms in eq. (37), which gives

$$\frac{\Delta}{\Gamma}\frac{\partial \Delta}{\partial t} \sim \Delta$$

at distances $x \gtrsim 1$. Hence, the oscillating part of the order parameter $\Delta_{\sim} \sim \Gamma t_0 \sim \Gamma^{3/2}$. This swing of oscillations in Δ is not sufficient to drive the order parameter to zero in the region where $\Delta \sim 1$. This means that oscillations in Δ could result in Δ going to zero at some moments in time only in the region where $\bar{A}$ is small: $\bar{\Delta} \ll 1$. This region of very small Δ determines the characteristic size of a PSC, i.e., of that region of the sample where the order parameter oscillates strongly. We designate the size of this region as x_2. Since in this region $\Delta \sim x_2(\partial\Delta/\partial x)$ and $(\partial\Delta/\partial x) \sim 1$, the size x_2 is of order Δ, therefore $x_2 \ll 1$.

We see that all the variables A, Q, and Φ oscillate only slightly at distances far enough from a PSC: $x \gg x_1$. Note that x_1 is still much shorter than the distance L between two neighboring PSCs. At distances $x_1 \ll x \ll L$ where all variables are practically static one can use eqs. (37)-(39) in the form

$$\frac{\partial^2 \Delta}{\partial x^2} + \left(1 - \Delta^2 - \frac{j_{\rm s}^2}{\Delta^4}\right)\Delta = 0 \tag{55}$$

$$j = -\frac{\hat{c}\Phi}{\partial x} + J_{\rm s} \tag{56}$$

$$u\Gamma\Delta\Phi = \frac{\hat{\partial} j_{\rm s}}{\partial x} \tag{57}$$

where one has averaged over time. In these equations the supercurrent $j_{\rm s} = -\Delta^2 Q$ is introduced. The second-derivative term $\lambda^2\Delta/\lambda x^2$ can be omitted in eq. (55) at distances $x \gg 1$; so

$$j_{\rm s} = \Delta^2\sqrt{1 - \Delta^2}. \tag{58}$$

Equations (56)-(58) can be integrated easily:

$$\Phi^2 = \frac{2}{u\Gamma}\int_{j}^{s_{s0}} \frac{(j - j_s)\,\mathrm{d}j_s}{\Delta\,(j_s)} = \frac{2}{u\Gamma}\int_{\Delta_0}^{\Delta} \frac{\left(j - \Delta^2\sqrt{1-\Delta^2}\right)(3\Delta^2 - 2)\,\mathrm{d}\Delta}{\sqrt{1-\Delta^2}} \tag{59}$$

$$\frac{L}{2} - x = \frac{1}{u\Gamma}\int_{j_s}^{j_{s0}} \frac{\mathrm{d}j_s}{\Delta\,(j_s)\,\Phi\,(j_s)} = \frac{1}{u\Gamma}\int_{\Delta_0}^{1} \frac{(3\Delta^2 - 2)\,\mathrm{d}\Delta}{\sqrt{1-\Delta^2}\Phi(\Delta)} \tag{60}$$

Here Δ_0 and j_{s0} are related by eq. (58) and represent the values of Δ and j_s midway between two PSCs. The integration limits are chosen on the basis of the fact that the potential Φ is equal to zero at the point $x = \frac{1}{2}L$ due to the symmetry of the structure with respect to the midpoint between two PSCs and due to the odd parity of Φ.

Thus, j_{s0} is the maximum value of the supercurrent which is achieved in the midpoint between two PSCs. The integration in eqs. (59), (60) should be carried out on the branch of the function $\Delta\,(j_s)$ which is thermodynamically stable, i.e., corresponds to $0 \leqslant j_s \leqslant j_c$ and $1 \geqslant \Delta \geqslant \sqrt{\frac{2}{3}}$ (see fig. 12).

Fig. 12. The function $j_{s (1)$. The thermodynamically stable branch is shown by the solid curve and the unstable one is shown by the dashed curve.}

Now we consider the region closely around a PSC: $x \ll L \sim \Gamma^{-1/2}$. Estimating the derivative $\hat{c}j_s/\partial x$ according to eq. (38) we have $\hat{\lambda}j_s/\hat{\lambda}x \sim I\Phi$, and

$$(\partial j_s/\partial x) \sim \Gamma^{1/2} \tag{61}$$

since $\Phi = \pi {t_0}^1 \sim \Gamma^{-1/2}$. Due to eq. (61) one can consider j_s independent of x at distances $x \ll \Gamma^{-1/2}$. After averaging eq. (37) over time one can write it in the region of static Δ, i.e., at $x \gg x_2$, in the form

$$\frac{\partial^2\Delta}{\partial x^2} + \left(1 - \Delta^2 - \frac{\overline{j_s^2}}{\Delta^4}\right)\Delta = 0 \tag{62}$$

With the condition (61), the solution of eq. (62) can be obtained similarly to the LA solution (20) (see section 3.1) where the potential will be

$$U(A) = \Delta^2 - \frac{1}{2}\Delta^4 + \frac{\overline{j_s^2}}{\Delta^2}$$

(see figs. 6, 7). This solution has to match at $x \geqslant 1$ with solution (60) which has a very small derivative $\hat{C}A_1\hat{C}x \sim \Gamma^{1/2}$. Extending solution (20) to $x \gg 1$ with zero derivative corresponds to $C = U_{\max}$. Thus, the proper solution has the form

$$x = \int_{\Delta_2}^{\Delta} \frac{\mathrm{d}\Delta}{\sqrt{U_{\max} - U(\Delta)}} \tag{63}$$

Here $x = 0$ corresponds to the minimal value $\Delta = A_2$.
As has been shown previously, the order parameter has to reduce to very small values when approaching a PSC within the static region of A. This corresponds to $\Delta_2 \ll 1$. A small value of Δ_2 is possible only when $j_{\mathrm{s}}^2 \ll 1$. It is clear from the potential $U(A)$ that $A_2 = \sqrt{2\overline{j_{\mathrm{c}}^2}}$. In this case eq. (63) reduces approximately to

$$\Delta = \tanh(|x|/\sqrt{2}) \tag{64}$$

which holds at $\Delta \gg \sqrt{\overline{j_{\mathrm{s}}^2}}$.
The value Δ_2 is, by order of magnitude, the average Δ in the oscillation region. Since $\Delta \sim j_s$ in this region the value of Δ can be easily estimated from eq. (37). Since $Q \sim j_s/\Delta^2 \sim \Delta^{-1}$ we have

$$\frac{\Delta}{\Gamma}\frac{\partial\Delta}{\partial t} \sim \frac{1}{\Lambda}$$

and, hence, $A \sim (\Gamma t_0)^{1/3} \sim \Gamma^{1/2}$. For the size of the oscillation region we find $x_2 \sim \Gamma^{1/2}$. In ordinary units this gives

$$x_2 = \xi(T)\sqrt{\Gamma}.$$

It is interesting to follow the behavior of the potential Φ in the dynamic region. The time-averaged potential Φ does not practically vary at distances small compared with l_E. This is seen from eq. (38) averaged over time:

$$\frac{r^2\bar{\Phi}}{r^2} = u\Gamma \cdot 1\bar{\Phi}$$

(if $x \geqslant x_2$). Thus the change in Φ is small at distances $x \ll \Gamma^{-12}$ and Φ remains finite when $x \to 0$. Its value $\Phi(x = \pm 0) = \pm\Phi_0$ is given by eq. (59) where one should put $\Delta = 1$ at $x \ll l_E$. On the other hand, the instantaneous values of Φ $(x = 0; t)$ must be zero at any time except for the moments of phase slip, $t = 0$, $t_0, \ldots$, when they go to infinity. This follows from the symmetry of the PSC
structure with respect to the point $x = 0$ and the odd parity of Φ. From eq. (39) one easily concludes that the growth of Φ from zero to values of the order of $\Phi_0 \sim t_0^{-1}$ occurs within the dynamic region $x \sim x_2$.

Thus we obtain the following results:
(1) A phase slip region has a size of the order of $\xi(T)\Gamma^{1/2}$. Here all the variables oscillate strongly. At the moments when $\Delta = 0$ the phase χ experiences jumps of 2π and $\Phi(x = 0)$ goes to infinity. The amplitude of oscillations in Δ is of order $\Delta_{\mathrm{GL}}\sqrt{\Gamma}$.
(2) As one goes away from a PSC the oscillations in Δ decay rapidly and Δ becomes practically independent of time at distances $x \gg x_2 = \zeta(T)\Gamma^{1/2}$. Here $\Phi = \bar{\Phi}$, where $\bar{\Phi}$ is equal to $\bar{\Phi}\,(x \ll l_E) = \Phi_0$ from eq. (59). The behavior of $\Delta(x)$ is given by eq. (64). The order parameter takes its equilibrium value $\Delta = A_{\mathrm{GL}}$ (in dimensionless units $\Delta = 1$) at $x \gtrsim \xi$. The supercurrent oscillates but remains still small so that the whole current is practically due to the normal excitations.
(3) At distances $(\xi l_E)^{1/2} \ll x \sim L$ oscillations of all the quantities are negligible. Here relaxation of $\Phi = (\mu_{\mathrm{p}} - \mu_{\mathrm{e}})/e$ takes place. This kind of inequilibrium is created in a PSC region as a result of phase slips. As the potential Φ and the normal current $j_{\mathrm{n}} = -\sigma(\hat{C}\Phi/\hat{C}x)$ decrease,

the supercurrent j_s increases and Δ diminishes. The supercurrent j_s reaches its maximum at the midpoint between two PSCs.

The behavior of the order parameter Φ and the currents j_s and j_n is shown in fig. 13. The behavior of the order parameter near PSCs is in good agreement with the results of numerical computations by Watts-Tobin et al. (1981).

At the end of this subsection we outline the temperature range of validity of our results. The Josephson frequency of oscillations ω_j is of the order of $(\Delta^2/T)\,\Gamma^{-1/2}$. Since our equations are valid if $\Gamma \ll 1$ and $\omega_J \ll \tau_{ph}^{-1}$, one has to use temperatures in the interval

$$(\tau_{ph}T_c)^{-2} \ll 1 - T/T_c \ll (\tau_{ph}T_c)^{-6/5}\,.$$

42.5.3 5.3. The current-voltage characteristic

Having averaged the expression for the electric field $E = -iQ, it - i\Phi/ix$ over time, one obtains $\bar{E} = -\partial\bar{\Phi}/\partial x$. Hence the time- and space-averaged electric field, which will also be designated by E, has the form

$$E = 2\Phi_0/L, \tag{65}$$

where Φ_0 and L are given by eqs. (59) and (60), with the boundary conditions found above: $\Delta = 1$ at $x \ll l_E$. We have for the structure period L from eq. (60)

$$L = \frac{2}{u\Gamma}\int_{\Delta_0}^{1} \frac{(3\Delta^2 - 2)\,\mathrm{d}\Delta}{\sqrt{(1-\Delta^2)}\Phi(\Delta)}, \tag{66}$$

where $\Phi(\Delta)$ is taken from eq. (59). For the current-voltage characteristic

Fig. 13. The behavior of (a) Δ and j_{s}, and (b) Φ and $\bar{\Phi}$ in the (solid lines) static and (dashed lines) dynamic regions. The values which Δ and j_s can take during oscillations are shaded. (Ivlev and Kopnin 1981.)

(CVC) we finally obtain

$$E\int_{\Delta_0}^{1} \frac{(3\Delta^2 - 2)\,\mathrm{d}\Delta}{(1-\Delta^2)^{1/2}\,f(\Delta;\Delta_0)} = 2f(1;\Delta_0) \tag{67}$$

where

$$f^2\left(\Delta;\Delta_0\right)=\int_{\Delta_0}^{\Delta}\frac{\left(j-x^2\sqrt{1-x^2}\right)\left(3x^2-2\right)\mathrm{d}x}{\left(1-x^2\right)^{1/2}} \tag{68}$$

Therefore only a solution of the equations for the potential in the static region is needed to calculate the CVC. At this point, the results of the dynamic theory of the resistive state (Ivlev and Kopnin 1978, 1981a, Ivlev et al. 1980a,b) bexome istry closes io that of the statuc model developed by Csalark (1974. 1975, 1976a.b, 19-7 (salare et al. (1976). and (salaikes and Shumeikes (1976). One can see from a comparares of fig. 3. zhich illustrates the static model. anth fig 13 that the potentials $\mathcal{I}$ in both cater behave ser; simularly coer aimset the whole distance: between Ps C. except for a narroz regron rear the PSC. In the static model, the formation of a wollage across a PSC, has the came possual nature an in the dynamic theor? and occurs through reiaxatoms of the branch imbalance $\mathcal{L}-1\mu_0-\mu_0)$ e over distances larger than the coher ence length () new thould note. However, that the results of the statue model of Csalaiks and co-zosters taken as they: samd have one shordermung. The thing is that Csalaiks and cro-mreks integrate "yuators for she protential of the pe of eq. in and 1571 using the thermodynamically unstable branch 11/1 ahan corresponde to $0-1-$ fisee fig. 12). Thes soluton cannot exast in a real phyucal outem Theresure, the results of these works have to be corrected accordingly.

Further consideratoons derend on the ratus of the channel iergeth. L. . to the distance. L. methosen PSC. If I, $\sim L$, there is room only for a finite rumber of PSCA in the channel and the behaver)? of the C.VC is determened by the creation, of new Pses ath increasing current. Than problem and be discussed in sation - ffere we cronsider the opposite case. "Lo. In the case the number of P'st. an the channel as large and one can consider theor density to merease amouthly with increasing current. Flquatums 167 and 1641 represent a set of carves deermired by the froe parameter 1_0 for /h. I . These curves are shown in fig. 14. Addumai physical consideratums are necessar; for a proper chooe of the parameter 1. Here one can use the princhple of minimum entropy production iminimum dissupatoral. According to thas principle. a regme with the munumum prossible E should be realized in the system a nen the current is fines. This ersure: the minimum disapation ijE. Thes regme corresponds to the upper curve in fig. 14. characteriad hy 1, = 0.8 wor is $\therefore 0\to 6.1$. This choice is equisalent to the conditon that ite supercarren thould have the maximum possible :alue. We thall assume that 1 , in aha: follows.

In order to return to ordinar: Lones in ey: iffl and ifil one should reprece E by $\left(2\sigma E3\sqrt{3}j_c\right)$ and j by $\left(2j3\sqrt{3}j_c\right)$. As a result we get

$$\frac{\sigma E}{h}\int^{1}\frac{\left(3x^2-2\right)\mathrm{d}x}{\left|t-x^2\right|\pi|\sqrt{2}|}=3\sqrt{3}f\left(1:\sqrt{\frac{2}{3}}\right). \tag{69}$$

The function $f\left(x;\sqrt{\frac{2}{3}}\right)$ is defined by eq. (68).

I wis acparate regors can be distungulshed in the CTr figl Now fig. 141 (1) The initial part, $j-j_c\ll j_c$. The structure period is logarithmically large in

this case:

$$L=2l_E\left(\frac{\Delta_{\mathrm{GL}}}{\Delta_0}\right)^{1/2}\ln\left(\frac{Cj_{\mathrm{c}}}{j-j_{\mathrm{c}}}\right),$$

where $C\sim 1$, and the CVC has the form

$$j-j_{\mathrm{c}}\simeq j_{\mathrm{c}}\exp\left(-0.91\left(j_{\mathrm{c}}/\sigma E\right)\right).$$

Fig. 14. The CVC calculated according to eqs. (67), (68) with different values of $\Delta_{0 $. The parts of CVCs with excess currents $j_{\text{exc}} = aj_{\text{c}}$ are seen: curve (1) $\Delta_0 = \sqrt{\frac{2}{3}}, a = 0.68$; (2) $\Delta_0 = 0.9, a = 0.61$; (3) $A_0 = 0.975, a = 0.35$; (4) Ohm's law.}

It has zero slope as $E \to 0$, resulting from the infinite length of the channel.
(2) The region of large currents: $j \gg j_{\text{c}}$. Here the CVC proceeds parallel to Ohm's law: $j = \sigma E + j_{\text{exc}}$ with the excess current $j_{\text{exc}} = aj_{\text{c}}$, where

$$a = \frac{3^{3/2}}{4}\left[\int_{\sqrt{\frac{2}{3}}}^{1} \frac{x^2\left(3x^2-2\right)\mathrm{d}x}{p\left(x:\sqrt{\frac{2}{3}}\right)}\right] / p\left(1;\sqrt{\frac{2}{3}}\right)$$

and

$$p^2\left(x;x_0\right) = \int_{x_0}^{x} \frac{3y^2-2}{\sqrt{1-y^2}}\,\mathrm{d}y.$$

The CVC of eq. (69) has been obtained by Ivlev et al. (1980b). They calculated also the CVC for the more general case when the supercurrent j_s influences the branch-imbalance relaxation. As we have said previously only a
solution of the static equations for the potential distribution is needed to calculate the CVC. In the static region the equation for the potential Φ looks like eq. (32) with the difference that its right-hand side contains a factor describing the effect of the supercurrent (Bulyzhenkov and Ivlev 1979):

$$D\nabla^2\Phi = \frac{\pi\Delta}{4T\tau_{\text{ph}}} q(z)\Phi$$

where

$$q(z) = 1 + \frac{2z}{\pi}\int_1^{\infty} \frac{\sqrt{x^2-1}\,\mathrm{d}x}{x\left[\mathrm{x}\sqrt{x^2-1}+z\right]} = \begin{cases} 1 & \text{if} \quad z \ll 1 \\ \sqrt{z} & \text{if} \quad 1 \ll z \ll T^2/\Delta_{\text{GL}}^2 \end{cases}$$

and $z = \left(4De^2, c^2\right) Q^2\tau_{\text{ph}}$. In the units of eqs. (34)-(36) this equation can be written in a form similar to eq. (57):

$$\frac{\partial j_s}{\partial x} = u\Gamma\Delta q(z)\Phi, \tag{70}$$

where now $z = 8\left(T_{\mathrm{c}} - T\right)\tau_{\mathrm{ph}} Q^2/\pi$. Equations (55), (56) and (70) can be integrated easily. As a result we have $(Q^2 = 1 - \Delta^2)$:

$$\Phi^2 = \frac{2}{u\Gamma}\int_{\Delta_0}^{\Delta} \frac{\left(j - \Delta^2\sqrt{1-\Delta^2}\right)\left(3\Delta^2 - 2\right)}{\sqrt{1-\Delta^2}q(z)}\mathrm{d}\Delta,$$
$$\frac{1}{2}L - x = \frac{1}{u\Gamma}\int_{\Delta_0}^{\Delta} \frac{\left(3\Delta^2 - 2\right)\mathrm{d}\Delta}{\sqrt{(1-\Delta^2)}\Phi(\Delta)q(z)}$$

instead of eqs. (59), (60). The CVC determined by eq. (65) looks like the one considered above and its behavior is qualitatively similar to that in fig. 14.

These results are valid in a wider temperature range:

$$\left(\tau_{\mathrm{ph}} T_{\mathrm{c}}\right)^{-2} \ll 1 - T/T_{\mathrm{c}} \ll \left(\tau_{\mathrm{ph}} T_{\mathrm{c}}\right)^{-1/2}$$

It can be seen from the data in table 1 that this temperature range corresponds

Table 1
Values of the parameter $\tau_{\mathrm{ph} } k_{\mathrm{B}} T_{\mathrm{c}} / \hbar$ for various superconductors.

Material	$\frac{T_{\mathrm{c}}}{(\mathrm{K})}$	$\frac{\tau_{\mathrm{ph}}}{(\mathrm{s})}$	$\tau_{\mathrm{ph}} k_{\mathrm{B}} T_{\mathrm{c}}/\hbar$
Pb	7.2	2×10^{-11}	2×10^{1}
In	3.4	10^{-10}	4×10^{1}
Sn	3.8	3×10^{-10}	10^{2}
Al	1.2	10^{-8}	10^{3}

to temperatures at which actual experiments are carried out. Therefore, these CVCs allow not only qualitative but even quantitative comparison with experiment.

42.5.4 5.4. The upper boundary of the resistive state

The resistive state of narrow superconducting channels occupies a rather broad region of currents, experimentally (see fig. 1a,b). The lower boundary of this region, as we have seen, is of the order of the critical Ginzburg-Landau current j_{c} and has the temperature dependence $(T_{\mathrm{c}} - T)^{3/2}$. More difficult is the question of the resistive state's upper boundary, i.e., the lowest current for which deviations from Ohm's law can be observed. Strictly speaking, there should always be deviations from Ohm's law due to fluctuation-induced creation of Cooper pairs in the normal state, the Cooper pairs then being accelerated and destroyed by the electric field (Gorkov 1970, Kulik 1970). These fluctuation corrections should be, however, extremely small for currents as large as those for which deviations are observed, experimentally.

The first attempt to obtain the boundary at which the normal state of a channel transforms into the resistive one was made by Galaiko (1974, 1976, 1977) in the framework of the static model of the resistive state. He studied the nucleation of the superconducting phase on the background of the normal state of a current-carrying channel. According to his results the normal state becomes unstable below some critical current

$$j_2 = \frac{4\sigma\left(T_{\mathrm{c}} - T\right)}{e}\left(\frac{2\pi T_{\mathrm{c}}}{7\zeta(3)D}\right)^{1/2}. \tag{71}$$

This result, however, has the same shortcoming as the static model itself; it does not account for the inevitable oscillations of the order parameter in the overlapping region of the nuclei of

the superconducting phase. As a result of these oscillations, the static equation for the modulus of the order parameter which has been used in obtaining eq. (71) would fail, so eq. (71) seems not to solve the problem of the upper boundary of the resistive state.

Since the normal current-carrying state is stable with respect to infinitely small fluctuations, the resistive state should appear and vanish through a transition of first-order type. This means that, when the current is increased above some critical current j_2, the dynamic resistive state described in section 5.2 should vanish in a jump-like way and the sample should become normal. We estimate here the upper boundary of the dynamic state (Ivlev et al. 1980b). According to eqs. (59) and (66), the structure period decreases with increasing current and is of the order of $L \sim \xi(T)\,(j_c/j\Gamma)^{1/2}$. When the current j is of the order of j_c/Γ the distance between PSCs would be of the order of $\check{\zeta}(T)$.

The order parameter has to be small at PSCs and of the order of unity in the midpoints between PSCs. It is clear that such a solution cannot exist if the distance between PSCs is less than $\check{\zeta}(T)$. We conclude that the resistive state would vanish when the current is larger than

$$j_2 \sim \frac{j_\mathrm{c}}{\Gamma} \propto \frac{\sigma T^{1/2}\tau_\mathrm{ph}}{eD^{1/2}}\,(T_\mathrm{c}-T)^2 \tag{72}$$

This estimation of the critical current j_2 is in good agreement with the results of numerical computations by Watts-Tobin et al. (1981).

Joule heating of the sample plays an important role in the destruction of the resistive state. This heating can produce hot-spots in the sample where superconductivity is locally destroyed, these hot spots developing then into purely normal state (Volkov and Kogan 1974). Moreover, Joule heating can cause the destruction of superconductivity merely due to overheating the sample. These effects depend, however, on the conditions of sample cooling and the structural homogeneity of the channel, and we will not consider them here. We note, however, that, even with ideal cooling which keeps the temperature of the crystal lattice of the sample equal to that of the external medium, the electron system will be overheated with respect to phonons. This would result in an effective decrease in T_c, and the superconductivity can vanish completely if the current is large enough. An estimation of such a current can be made if in eq. (29) one takes into account the correction to the distribution function produced by the overheating of electrons. This correction is proportional to $\tau_\mathrm{ph}De^2E^2\,(\partial^2 f_\varepsilon/\partial\varepsilon^2)$, where f_α is the equilibrium distribution function. As a result, eq. (29) will also contain a term

$$\frac{7\zeta(3)\tau_\mathrm{ph}De^2E^2\Delta}{2\pi^2T^2}.$$

The critical field E_2 is found from equating this term to $(T_\mathrm{c}-T)\,\Delta/T$ which gives

$$j_2 = \frac{\sigma\pi}{e}\left[\frac{2T_\mathrm{c}\,(T_\mathrm{c}-T)}{7\zeta(3)\tau_\mathrm{ph}D}\right]^{1/2} \tag{73}$$

This result has been obtained by Galaiko (1976b, 1977) and is valid at temperatures $1-T/T_\mathrm{c} \gg (\tau_\mathrm{ph}T_\mathrm{c})^{-1}$. At higher temperatures, $1-T/T_\mathrm{c} \ll (\tau_\mathrm{ph}T_\mathrm{c})^{-1}$, the role of heating decreases, and the current j_2 will be determined by eq. (72).

It should be noted that the described results concerning j_2 are mainly qualitative, and therefore the problem of the upper critical current still remains unsolved.

42.6 6. Voltage steps in the current-voltage characteristics of superconducting channels with finite length

Voltage steps can be observed in the CVC of a not very long superconducting channel. They are related to the creation of new PSCs in a channel when the current through the sample is

increased. The qualitative picture of voltage steps based on the SBT model (Skocpol et al. 1974, Tinkham 1979) was described in section 3.2. We consider here a more thorough approach based on the microscopic theory. The problem of voltage steps was solved first by Bezuglyi et al. (1977), we outline their results below.

Since the dynamic region of oscillations is very narrow and is localized near PSCs one can use here, just as in the case of the long channels in section 5.3, the static equations only. We assume that a superconducting channel with the length L_0 is connected with the current source by means of normal contacts (see fig. 15). When the current is less than $j^{(1)}$, where $j^{(1)}$ corresponds to the creation of the first PSC, the voltage across the channel is determined by the penetration of the electric field into the superconducting channel from the normal contacts (see, e.g., Artemenko et al. 1978, Ivlev et al. 1980d.e). This state is represented in fig. 15a. The order parameter is sharply depressed from the equilibrium value $\Delta_{\rm GL}$ to zero at distances of order $\xi(T)$ near the SN interface. Going further away from the interface, Φ and $j_{\rm n}$ decrease. $j_{\rm s}$ increases and Δ diminishes at distances of order of l_E.

When the current is increased above $j^{(1)}$ the first PSC appears in the middle of the channel (fig. 15b). This state remains in the interval $j^{(1)} < j < j^{(2)}$ and then the second PSC appears, and so on. The conditions near the SN-boundary (Ivlev et al. 1980d,e) at distances larger than ξ but small compared to I_E and those at the boundary of the static region near a PSC (see section 5.2) are the same: $j_{\rm s} = 0$ and $\Delta = 1$, hence, we can use eq. (59) directly to determine the voltage across the channel.

Suppose that there are n PSCs in the channel, then the potential formed across one PSC is

$$\Phi_n^2(A) = \frac{2}{u\Gamma} \int_{\Lambda_{0\rm n}}^{A} \frac{\left(j - \Delta^2\sqrt{1-\Delta^2}\right)(3\Delta^2 - 2)}{\sqrt{1-\Delta^2}}\, {\rm d}\Delta \tag{74}$$

Here A_{0n} is the value of Δ at the midpoint between the neighboring PSCs (or at the midpoint between the SN-boundary and the nearest PSC). The parameter $\Delta_{0\rm n}$ is related now to the channel length L_0 by

$$\frac{L_0}{2(n+1)} = \frac{1}{u\Gamma} \int_{\Delta_{0\rm n}}^{1} \frac{(3\Delta^2 - 2)\,{\rm d}\Delta}{\sqrt{1-\Delta^2}\Phi(\Delta)} \tag{75}$$

The total voltage drop across the channel is

$$V = 2(n+1)\Phi_n(\Delta = 1). \tag{76}$$

Equations (74)-(76) describe the CVC of a channel completely, but their explicit form is very complicated. We restrict ourselves to the case of weak currents, $j - j_{\rm c} \ll j_{\rm c}$ and $n \sim 1$, and suppose that $L_0 \gg l_E$. We get from eqs. (74) and (75)

$$\frac{L_0}{2(n+1)} = \frac{1}{(u\Gamma A_{0\rm n})^{1/2}} \ln \frac{C j_{\rm c}}{j - j_{\rm s0}^{(n)}}. \tag{77}$$

Here $C \sim 1$, and $j_{\rm s0}^{(n)} = \Delta_{0\rm n}^2\sqrt{1-\Delta_{0\rm n}^2}$ is the supercurrent in the midpoints between PSCs.

If one increases the total current j keeping the number n of PSCs in the channel constant then the current $j_{\rm so}^{(n)}$ will also increase according to eq. (77). When $j_{so}^{(n)}$ reaches the critical current j_c the solution with n PSCs becomes impossible and the $(n+1)$ th PSC appears. The current $j^{(n)}$ at which the nth PSC forms, i.e., when the nth step in the CVC appears, is determined by eq. (77):

$$j^{(n)} - j_{\rm c} = j_{\rm c} \exp\left\{-\frac{\sqrt{u\Gamma}L_0}{2n}\left(\frac{2}{3}\right)^{1/4}\right\}; \quad n = 1, 2, \ldots \tag{78}$$

Fig. 15. The behavior of A, the currents j_{s} and j_{n}, and $\bar{\Phi}$ in a superconducting channel with normal contacts. (a) $j < j^{(1)}$ and PSC is absent. (b) $j^{(1)} < j < j^{(2)}$, there is one PSC in the middle.}

The magnitude of a step does not practically depend on n and is equal to the voltage $2\Phi_0$ across one PSC, where

$$\Phi_0^2 = \frac{2}{u\Gamma}\int_{\sqrt{\frac{2}{3}}}^{1} \frac{\left(j_{\mathrm{c}} - \Delta^2\sqrt{(1-\Delta^2)}\right)(3\Delta^2 - 2)}{\sqrt{(1-\Delta^2)}}\mathrm{d}\Delta \simeq \frac{0.150}{u\Gamma} \tag{79}$$

if n is of order of unity. The current dependence of the voltage Φ_n across one PSC in the interval between the nth and the $(n+1)$ th step has the form

$$\Phi_n^2 = \Phi_0^2 + \frac{2}{u\Gamma}(j - j_{\mathrm{c}})\int_{\sqrt{\frac{2}{3}}}^{1} \frac{3\Delta^2 - 2}{\sqrt{(1-\Delta^2)}}\mathrm{d}\Delta. \tag{80}$$

Hence, the CVC in this interval has a constant slope with differential resistance

$$\left(\frac{\partial V}{\partial j}\right)_n = \frac{2(n+1)}{u\Gamma\Phi_0}p^2\left(1;\sqrt{\frac{2}{3}}\right) \simeq 2.06\frac{n+1}{\sqrt{u\Gamma}}. \tag{81}$$

The CVC is shown in fig. 16.

In ordinary units we get for the differential resistance

$$\left(\frac{\partial V}{\partial j}\right)_n = 2.06(n+1)\frac{l_E}{\sigma},$$

where l_{E} is given by eq. (33). One can note that the numerical value $2.06 l_{\mathrm{E}}/\sigma$ of the differential resistance produced by one PSC agrees surprisingly well with the phenomenological result of the SBT-model (Skocpol et al. 1974), which gives

Fig. 16. The schematic CVC of a channel with finite length according to Bezuglyi et al. (1977).

$$\left(\frac{\partial V}{\partial j}\right)_n = 2l_E/\sigma$$

(see section 3.2).
The initial part of the CVC from $j = 0$ to $j = j^{(1)}$ corresponds to the superconducting state of the channel and is determined by the electric field penetration from normal contacts into the superconductor. This part of the CVC is described in the general case by the results obtained by Ivlev et al. (1980 d, e). When the channel is connected with superconducting banks the voltage will not be observed in the initial part of the CVC and one should replace ($n + 1$) by n in eqns. (75)-(81).

42.7 7. Conclusions

We can generally state that the main features of the resistive state in narrow superconducting channels find their explanation in the recent theory. This concerns first of all the over-all behavior of a CVC and the voltage steps in it. The a.c. Josephson effect observed experimentally by Skocpol et al. (1974) is without doubt related to PSC oscillations with the Josephson frequency (Ivlev and Kopnin 1982). As to the generation of lower frequencies (Churilov et al. 1969, Galaiko et al. 1973, 1976), there is as yet no satisfactory explanation for this effect. It seems to be due to the motion of the PSC structure as a whole (Galaiko et al. 1976) but final conclusions are still difficult to draw. Also there are insufficient data for the verification of the temperature dependence of the upper critical current j_2. Experimental data obtained by Galako et al. (1976) suggest a linear dependence of $j_2\gamma\,(T_c - T)$ which does not agree with either eq. (72) or eq. (73).

Despite these and some other questions still open, the general picture of the resistive state as a structure of PSCs is established quite reliably. Experiments verify not only the overall shape of the CVC, but also such fine features as the spatial distribution of the potentials μ_e and μ_p (Dolan and Jackel 1977) and the temperature and magnetic field dependence of the differential resistance of a PSC (Kadin et al. 1978).

The concept of phase slippage in the resistive states of different types, of superconductors turns out to be very useful. The vortex motion in type-II superconductors and in superfluid helium and the motion of flux tubes in typeI superconductors are also mechanisms of phase slippage. These are the processes for which the term "phase slippage" has been introduced (Anderson 1966). We face, however, a qualitatively new type of phase slippage in narrow superconducting channels. The point is that phase slippage in bulk superconductors is carried out by means of moving defects of a superconducting structure, which exist in the system even

when dissipation is absent. In narrow superconducting channels, however, PSCs exist only in the presence of dissipation. A system with PSCs is essentially dissipative and methods based on thermodynamic considerations are not suitable here.

The concept of phase slippage also proves to be very useful for the description of the two-dimensional mixed state and the intermediate state of a superconducting wire in a current-carrying state (see the review by Dolgopolov and Landau (1985) in this volume). Ivlev and Kopnin (1980) and Ivlev et al. (1980c) have shown that the coexistence of electric field and superconductivity in these cases is also due to a phase slippage mechanism like PSC in the resistive state in narrow superconducting channels.

43 Nonequilibrium Phenomena in Superconducting Weak Links by Aslamazov, Volkov

43.1 1. Introduction

Soon after the pioneering work of Josephson was published (1962), a microscopic theory for the Josephson phenomenon was constructed for tunneling junctions; the stationary case was considered by Ambegaokar and Baratoff (1963), while a nonstationary theory was developed by Werthamer (1966), and by Larkin and Ovchinnikov (1966) with the help of the Gorkov microscopic equations. Besides in tunnel structures, the Josephson phenomenon was discovered and investigated in a number of other superconducting weakly connected systems (see Likharev's review, 1979, and Barone and Paterno 1981), including the so-called S-c-S junctions with a direct Ohmic contact between two superconductors. The weak link in such systems is due to the geometrical constriction (c) at the point of contact. When a constriction of that kind (a weak link) connects the superconducting films, the so-called bridges are produced (fig. 1a), while in the case of bulk superconductors, one obtains point superconducting contacts (fig. 1b). The $S-c-S$ systems possess a number of practical advantages.

A microscopic theory of the S-c-S systems was long to come into being; some progress was achieved only recently, owing to the development of a general theory of nonequilibrium processes in superconductors (Eliashberg 1971, Gor'kov and Eliashberg 1968, Larkin and Ovchinnikov 1975, 1977,

Fig. 1. Schematic picture of (a) a variable-thickness bridge and (b) a point of contact.

Schmid and Schön 1975). This is due to the fact that nonequilibrium effects are not essential in tunnel junctions. The voltage V shifts the Fermi levels in the electrodes with respect to each other by the value eV, while the transitions of electrons through the isolating layer may be described by the tunnel Hamiltonian in the first approximation with regard to transparency. The distribution functions of electrons remain the equilibrium ones (of the Fermi type).

The situation differs in bridges and contacts. When a current in the bridge (or at the contact) exceeds the critical value, then the energy gap (more precisely, the order parameter)

oscillates in time with Josephson frequency $\omega_j = 2eV/\hbar$; an electric field is created in the bridge and, as a result, the distribution function of quasiparticles may considerably differ from the equilibrium one. The nonequilibrium processes which arise result in substantial differences in the behaviour of bridges and contacts as compared to tunnel junctions.

In this chapter the nonequilibrium phenomena in S-c-S systems will be considered and the behaviour of these systems, governed by the above mentioned phenomena, will be analysed.

43.1.1 1.1. Josephson effect in systems with direct Ohmic contact

Let us start with a simple theory, developed by Aslamazov and Larkin (1969) on the basis of the Ginzburg-Landau equations. The theory does not take into account nonequilibrium phenomena, but it enables us to visualize the Josephson effect in systems with a direct contact. In the case when the characteristic dimension of the constriction L is small [as compared to the size of the Cooper pair $\xi(T) = \xi(0)\left(1 - T/T_c\right)^{-1/2}$], the order parameter $\hat{\Delta}$ changes rapidly in space and in the Ginzburg-Landau equation the gradient term is of first-rate importance,

$$\nabla^2 \hat{A} = 0. \tag{1.1}$$

The boundary condition for this equation is $\hat{C}\hat{A}/\hat{C}n = 0$ (on the superconductor surface). Far away from the constriction (when the distance is large as compared to $\breve{\zeta}$) the current density is assumed to be small with regard to the critical one, and therefore the value of $\hat{\Delta}$ is equal to its value in the bulk superconductor.

We shall search for the solution of eq. (1.1) in the form

$$\Delta(r) = \Delta_0 \left\{ f(r) \mathrm{e}^{1x_1} + [1 - f(r)] \mathrm{e}^{1x_2} \right\}, \tag{1.2}$$

where the function $f(r)$ tends to 1 when going from the point of contact towards one of the superconductors and tends to zero when directed towards the other one. It is clear that in this case A_0 is the absolute value of the order parameter far away from the contact, while χ_1 and χ_2 stand for the values of its phase on both sides of the contact.

Substituting the expression (1.2) into eq. (1.1), we see that the function $f(r)$ also satisfies the Laplace equation which may be solved when the geometry of the contact is known; thus the dependence of the order parameter on the coordinates can be found. In particular, for a narrow (one-dimensional) contact of length $2L$ we obtain:

$$\begin{aligned} \hat{\Delta}(x) &= \Delta_0 \left(\cos\chi_0 + \mathrm{i}\frac{x}{L}\sin\chi_0 \right) \\ |\hat{A}(x)| = \Delta(x) &= \Delta_0 \left(\cos^2\chi_0 + \frac{x^2}{L^2}\sin^2\chi_0 \right)^{1/2} \end{aligned} \tag{1.3}$$

where the phase difference $\chi_2 - \chi_1 = 2\chi_0$.

In order to describe the electrodynamical properties of the system, let us write the equation for the current. Assuming the mean free path l of electrons in the contact to be much less than dimensions of the contact, we obtain the expression

$$j = -\sigma\nabla\varphi - \frac{\pi e v D}{4T_c}\mathrm{i}\left(\hat{\Delta}\nabla\hat{\Delta}^* - \hat{\Delta}^*\nabla\hat{\Delta} \right), \tag{1.4}$$

where σ is the conductivity of the metal in the normal state, φ is the scalar potential, v is the density of states and $D = V_{\mathrm{F}}1/3$ is the diffusion coefficient. The second term corresponds to the Ginzburg-Landau equation for the density of superconducting current, written for simplicity in the "dirty" limit, $1 \ll \xi(0)$.

The potential φ, as well as the order parameter $\hat{A}$, can be found from the Laplace equation and, therefore, may be expressed in terms of the function $f(r)$. Thus, substituting the expression (1.2) into eq. (1.4), we get

$$j = -\nabla f(r)\left[\sigma V - \frac{\pi e v D \Delta_0^2}{2T_c}\sin 2\chi_0\right] \tag{1.5}$$

where $V = \varphi_2 - \varphi_1$ is the contact voltage. Integrating the expression over the contact cross-section, we obtain the total current,

$$I = \frac{V}{R} + I_{\mathrm{c}}\sin\left(\frac{2e}{\hbar}\int^{\mathrm{t}} V\ \mathrm{d}t'\right), \quad I_{\mathrm{c}} = \frac{\pi A_0^2}{4eRT} \tag{1.6}$$

where R is the resistance of the contact in the normal state, while the phase difference of the order parameter is expressed via the voltage with the use of the Josephson relationship

$$h\frac{\partial \chi_0}{\partial t} \equiv h\dot{\chi}_0 = eV. \tag{1.7}$$

When the total current through the contact does not exceed I_c, then, as seen from eq. (1.6), it has the solution $V = 0$. This corresponds to a constant superconducting current passing through the contact (a stationary Josephson
effect). The value I_{c} as determined by eq. (1.6) is the critical current of a contact. The same expression for the critical current was obtained, on the basis of the microscopic equations of Eilenberger (1968), by Kulik and Omelyanchuk (1975), and Kulik et al. (1977), who also considered the nonlocal case $l \gg L$ and the case of arbitrary temperatures.

For $I > I_{\mathrm{c}}$ an alternating voltage appears at the contact (the nonstationary Josephson effect). This voltage can be found by solving eq. (1.6):

$$V(t) = \frac{R\left(I^2 - I_{\mathrm{c}}^2\right)}{I - I_{\mathrm{c}} + 2I_{\mathrm{c}}\sin^2\left(eR\sqrt{I^2 - I_{\mathrm{c}}^2 t}\right)} \tag{1.8}$$

By averaging the expression (1.8) over time, one obtains for the currentvoltage characteristic (CVC) curve of the contact

$$\bar{V} = R\sqrt{I^2 - I_{\mathrm{c}}^2} \tag{1.9}$$

which is a hyperbola. The mean voltage, as should be expected, is related to the frequency of electromagnetic oscillations by the Josephson relation: $\hbar\omega_{\mathrm{J}} = 2e\bar{V}$.

Let us now study how the CVC is affected by the external microwave field (AC-field). In this case the total current through the contact oscillates with the frequency ω of the external field:

$$I(t) = I_0 + I_1\sin(\omega t + \delta) \tag{1.10}$$

where I_1 is proportional to the amplitude of the external field and is usually small compared to I_0. From eq. (1.6) one can obtain an expression of the first order with respect to I_1 relating the mean current I_0 and the mean voltage $\bar{V}$ on the contact

$$RI_0 = \sqrt{\bar{V}^2 + V_{\mathrm{c}}^2} - \frac{I_1 R V_{\mathrm{c}}}{2\sqrt{\bar{V}^2 + V_{\mathrm{c}}^2}}\sin\left(\omega t - \frac{2e\bar{V}t}{\hbar} - \delta\right) \tag{1.11}$$

where $V_{\mathrm{c}} = RI_{\mathrm{c}}$, while the bar denotes averaging in time. As seen from eq. (1.11), at resonance voltage, $\bar{V} = \hbar\omega/2e$, a "step" appears in the CVC of the contact; the mean current at a fixed voltage varies continuously depending on the phase shift δ between the oscillations

of the external field and the Josephson-type oscillations in the contact. It should be noted that perturbation theory can no longer be used for finding the shape of the CVC near the "step".

The theory developed in this section clarifies the main properties of $\mathrm{S}-\mathrm{c}-\mathrm{S}$ systems and corresponds to the experimental data at small voltages (compared to the absolute value of the order parameter) and temperatures close enough to T_{c}. However, the theory in question is incapable of explaining such important features as the shoulder on the CVC for contacts with decreasing temperature, the stimulation of the contact's superconductivity by an external

SHF field, and a certain excess current (a permanent addition to the normal current) at high voltages. This is due to the fact that the theory does not take into account the nonequilibrium distribution of electrons on the energy levels. In the theory, the energy relaxation time was assumed to be small as compared to the other characteristic parameters of the system.

Some progress in solving the problem has been achieved by means of the time-dependent Ginzburg-Landau equation derived by Gorkov and Eliashberg (1968) for superconductors with paramagnetic impurities. Thus, Likharev and Yakobson (1975) used this equation to obtain the expression for the excess current, while Jensen and Lindelof (1976) explained the appearance of the shoulder on the CVC curve. However, the time-dependent Ginzburg Landau equation is (strictly speaking) applicable only to gapless superconductors. A consistent theory of these phenomena, taking into account the nonequilibrium electron distribution on the energy levels in the contact, is developed in sections 2 and 3.

43.1.2 1.2. Methods for studying nonequilibrium phenomena in weak links

The microscopic equations for the Green functions $\widehat{G}$ are taken as the basis for analysing the nonequilibrium phenomena (see ch. 4). In the general case, these complicated equations can not be solved. However, one can obtain approximate solutions in two limiting cases (depending on the relative value of the contact length $2L$ as compared to the characteristic parameter η). When the bridge length $2L$ satisfies the inequality

$$2L \gg \sqrt{D/\Delta} \equiv \eta(T) \sim \xi(T)\left(1-T/T_{\mathrm{c}}\right)^{1/4} \tag{1.12}$$

then the gradient terms in the equations for $\hat{G}$ turn out to be small and the so called local approximation may be used. This means that the quasiparticles' spectrum $\varepsilon = V\overline{\zeta^2+\Delta^2}$ (here $\check{\zeta} = (p-p_F)\,V_F$) and the density of states are of the same form as they are in a homogeneous superconductor, except that the gap Δ depends on coordinates and time: $\Delta = \Delta(x,t)$. The density of states $N(\varepsilon)$ given below differs from zero for $\varepsilon > \Delta(x,t)$ and equals

$$N(\varepsilon) = \frac{\varepsilon}{\sqrt{\varepsilon^2-\Delta^2(x,t)}}, \quad \varepsilon > \Delta(x,t). \tag{1.13}$$

In this case, one can introduce the energy distribution function of the quasiparticles $F(\varepsilon)$ for which a simple kinetic equation is available. The order parameter can be found from the Ginzburg Landau equation, in which the nonequilibrium phenomena may be described by means of an additional term. It then turns out that $\mathrm{S}-\mathrm{c}-\mathrm{S}$ systems have a specific electron energy diffusion mechanism which is stronger than the direct acceleration of electrons by the field, as first considered by Eliashberg (1970) for uniform systems. In the center of the bridge, the absolute value of the order parameter decreases as compared
to its value A_0 in the banks (due to the high density of the superconducting current). Therefore, the electrons with energies $\varepsilon < \Delta_0$ are confined in the potential well formed by Δ and they carry on a bounded motion. In the case of alternating current (for example, in an external AC field or under Josephson oscillations), oscillations of the order parameter in the contact appear,

whereas its value in the banks is not changed (the current density in the banks is small). As a result, the electrons reflect from the walls of the "shaking" potential well and their energy increases. However, for energy values $\varepsilon > A_0$ the electrons can freely diffuse from the contact to the banks in which equilibrium is maintained. Thus, the mechanism described results in a decrease of the quasiparticles' number in the contact; this corresponds to effective cooling and stimulation of superconductivity in the contact.

In the other limiting case,

$$2L \ll \eta(T), \tag{1.14}$$

the gradient terms in the equations of microscopic theory play the dominant role. Therefore, the excitation spectrum and the density of states $N(\varepsilon)$ are considerably changed. Thus, in the energy interval

$$\tilde{\Delta} \equiv \Delta_0 \left|\cos \chi_0\right| < \varepsilon < \Delta_0 \tag{1.15}$$

the density of states $N(\varepsilon)$ equals (see Appendix A2.1)

$$N(\varepsilon) = \frac{\varepsilon}{\sqrt{\varepsilon^2 - \tilde{\Delta}^2}} \cosh\left(a\frac{x}{L}\right) \cos\left(\frac{\pi}{2}\frac{x}{L}\right) \tag{1.16}$$

Here

$$a = \ln\left|\frac{\Delta_0 \sin\chi_0 + \sqrt{\varepsilon^2 - \tilde{\Delta}^2}}{\sqrt{\Delta_0^2 - \varepsilon^2}}\right|$$

When comparing eqs. (1.3) and (1.16), one can see that the function $N(\varepsilon)$ also differs from zero in the "forbidden" region when $\varepsilon < A(x,t)$. It becomes zero only when the energy ε becomes less than the order parameter in the center of the bridge $\tilde{A}$. Thus, the energy gap in the bridge coincides with the value of the order parameter in the center of the bridge $\tilde{A}(t)$. The physical interpretation of the processes occurring in the short bridge becomes less illustrative. However, this shortcoming is compensated by the fact that the complicated equations for the Green functions are considerably simplified and admit an exact solution over a wide range of currents and voltages. This case will be considered in section 3.

To simulate a bridge or contact take as a model a bridge with variable thickness; this model was widely used (see Likharev's review, 1979). According to the model (see fig. 1a) a bridge is a filament of length $2L$ and diameter d (where $d \ll L$), connecting two bulk superconductors (superconducting banks). The smallness of the filament diameter as compared to its length $2L$ allows us
to assume that all the functions in the bridge depend on one coordinate x along the filament. As we shall see, the final results do not depend on the geometrical characteristics of the model and are determined only by the normal resistance of the bridge R. One can show that the same results will be valid for bridges of differing shapes (for example for contacts in the shape of a hyperboloid of revolution, etc., fig. 1b). It should also be noted that everywhere in this chapter, except in section 3.5, we shall consider the "dirty" case, i.e. we shall assume the free path $l = V_{\mathrm{F}}\tau_{\mathrm{ir}}$ to be less than the correlation length $\xi(0)$ and the dimensions of the bridges: $l \ll \{\xi(0), d, L\}$.

43.2 2. Stimulation of superconductivity in bridges of length $2L > \eta(T)$

One of the most striking manifestations of the nonequilibrium electron energy distribution in bridges is the increase of the critical current in the presence of an external microwave field. This

phenomenon may be observed both in long superconducting bridges whose properties resemble those of uniform film structures and in short bridges and point superconducting contacts (see ch. 3).

A theory explaining the stimulation of superconductivity in films in the presence of AC fields was constructed by Eliashberg (1970) and Ivlev and Eliashberg (1971) and is described in chapter 6. In the present chapter we deal with nonuniform systems, for which the dependence of the superconducting order parameter on the coordinates is essential. We investigate the case of bridges which are not too short: with the length of the bridge $2L$ larger than the characteristic parameter $\eta(T)$, since for $2L < \eta$ the critical current decreases in the presence of an AC field (section 3.1).

One more manifestation of the superconductivity stimulation with a nonequilibrium electron distribution is the bending of the bridge's CVC for decreasing temperature. The reason for the nonequilibrium state in this case is not the external AC field, but the intrinsic voltage of the system which appears on the contact for currents exceeding the critical value. This phenomenon, as well as stimulation by microwave fields, takes place when the bridge length satisfies the inequality $2L > \eta(T)$.

The results described in this section were obtained by Aslamazov and Larkin (1976, 1978).

43.2.1 2.1. General expressions for the order parameter and electron distribution function in contacts

The stimulation of superconductivity manifests itself in the most pronounced manner near the critical temperature T_c of a superconductor. Therefore, to describe the phenomenon, we shall use the Ginzburg Landau equation, in
which the nonequilibrium of the electron energy distribution is taken into account. With sufficiently smooth and slow changes of the order parameter,

$$\hbar\omega < \hbar D/L^2 < \Delta \tag{2.1}$$

this equation may be written as

$$\frac{\pi D}{8T}\frac{\partial^2\Delta}{\partial r^2} - \frac{TI_s^2}{2\pi e^2 v^2 DS^2\Delta^3} + \frac{T_c - T}{T_c}\Delta - \frac{7\zeta(3)\Delta^3}{8\pi^2 T^2} + \Phi(\Delta) = 0 \tag{2.2}$$

$$\Phi(\Delta) = \Delta\int_\Delta^\infty \left[F(\varepsilon) - \tanh\frac{\varepsilon}{2T}\right]\left(\varepsilon^2 - \Delta^2\right)^{-1/2}\,\mathrm{d}\varepsilon \tag{2.3}$$

The electron energy distribution function $n(\varepsilon)$ is expressed in terms of $F(\varepsilon)$ using

$$n(\varepsilon) = \frac{1}{2}[1 - F(\varepsilon)] \tag{2.4}$$

and is assumed to be weakly dependent on time and coordinates (the energy relaxation time of electrons in superconductors, τ, is large as compared to both the period of electromagnetic field, $\sim \omega^{-1}$, and the time $\sim L^2 D$ of the spatial electron diffusion along the contact). The second term in eq. (2.2) describes the variations of the order parameter's phase %, connected with the value of the superconducting current I_s by the formula

$$I_s = \pi e v D S A^2 \nabla\chi/2T \tag{2.5}$$

where S is the cross-sectional area of the contact. The use of the equilibrium expression (2.5) for the superconducting current is justified in the case when the function F differs slightly from the equilibrium function $\tanh(\varepsilon 2T)$ (see Appendix of ch. 13. For higher microwave radiation

power or a higher voltage on the contact, the deviation $F(\varepsilon) - \tanh(\varepsilon/2T)$ may be on the order of the equilibrium value $\tanh(\varepsilon/2T)$. In this case the nonequilibrium expression for the superconducting current should be used, as derived in the Appendix of ch. 13. This, however, affects only the numerical coefficients in the final results.

Before averaging the function $F(\varepsilon, r, t)$ over time and coordinates let us write out, in the case $2L > \eta$, the kinetic equation obtained by Larkin and Ovchinnikov (1977):

$$D\frac{\partial^2 F}{\partial \boldsymbol{r}^2} - \frac{\partial F}{\partial t}\frac{\partial}{\partial \varepsilon}\left(\varepsilon^2 - \Delta^2\right)^{1/2} + \frac{\partial F}{\partial \varepsilon}\frac{\partial}{\partial t}\left(\varepsilon^2 - \Delta^2\right)^{1/2} = I_{\mathrm{ph}}(F), \tag{2.6}$$

where in the right-hand side a collision integral is written. The direct effect of the electromagnetic field on the distribution function is not taken into account in this equation, since it is small for contacts which are not too long. Using this equation one can obtain the averaged-out equation (over time and
coordinates) for the function $F(\varepsilon)$:

$$\frac{1}{\tau_\varepsilon}\left[F(\varepsilon) - \tanh\frac{\varepsilon}{2T}\right]\left\langle\overline{\varepsilon\left(\varepsilon^2 - A^2\right)^{-1/2}}\right\rangle = \frac{\partial}{\partial \varepsilon}\left(D_\varepsilon \frac{\partial F}{\partial \varepsilon}\right) \tag{2.7}$$

$$D = -\left\langle \begin{matrix}\hat{c}\theta \\ it\end{matrix} \left(D_{\hat{c}r^2}^{\partial^2}\right)^{-1} \frac{\partial(l}{it}\right\rangle \cdot \theta = \left(\varepsilon^2 - A^2\right)^{12} - \left\langle\left(\varepsilon^2 - A^2\right)^{12}\right., \tag{2.8}$$

where the left-hand side of eq. (2.7) describes the relaxation of the quasiparticles while the right one gives the energy diffusion; D, is the energy diffusion coefficient, the brackets $/\cdots$, denote averaging over the area of the contact where : $-A$, and the bar denotes averaging over time. The boundary condition for finding the inverse operator in eq. (2.8) requires the derivative normal to the surface $\varepsilon = A(r)$ to be equal to zero. Equations (2.7) and (2.8) are valid when condition (2.1) is fulfilled. These equations are derived in Appendix 1.

43.2.2 2.2. Stimulation of the critical current of the bridge for low-power microwave radiation

In order to find the maximum value of a superconducting current passing through the contact in the presence of an AC field by means of eq. (2.5). one must know the nonequilibrium distribution function and the order parameter in the contact, which may be obtained from eqs. (2.2) and (2.7). For low-power radiation, the nonequilibrium term $\Phi(\Delta)$ in eq. (2.2) is small and therefore, when calculating the energy diffusion coefficient D_ε, one may use the dependence of the order parameter $A(r.t)$ determined from the equilibrium Ginzburg Landau equation. Then, by solving the kinetic equation (2.7) one can find the nonequilibrium distribution function and, using eq. (2.3), find the nonequilibrium term $\Phi(\Delta)$ and the correction to the critical current.

The time dependence of the order parameter is determined by the alternating voltage induced on the contact by the AC field: the phase difference of the order parameter in the banks is related to the voltage V' by the Josephson relation (1.7). Therefore both superconducting and normal currents pass through the contact. However. unless the value of the superconducting current exceeds the critical value, the DC component of the normal current and hence the mean voltage on the contact equals zero. The given parameter is the radiation power. which is proportional to the mean square of the voltage on the contact, and may be measured in dimensionless units:

$$P = \frac{e^2}{h}\frac{\tau_{\frac{l}{V^2}}}{T} \tag{2.9}$$

Fig. 2. The order parameter $\Delta(x)$ and distribution function of quasiparticles $n(\varepsilon)$ in the bridge. Curve (a) corresponds to low radiation power; hatching shows the regions of rarefication and condensation (double hatching) of quasiparticles as compared to the equilibrium distribution (the sloping straight line). The vertical straight line "b" relates to high radiation power, when equal energy distribution is established for the quasiparticles confined inside the bridge.

First, let us consider the most interesting case of a long ($L \gg \xi$) and narrow contact (fig. 2) when the order parameter $.A = A(x)$. over almost the entire area

inside the contact, is close to its value for an infinitely long contact and only near the edges in the domain $L - x \lesssim \xi$ a "linking" takes place with the equilibrium value Δ_0 at the banks (the current density at the banks is small). If, in addition, the superconducting current I_s is close to its critical value for an infinite contact,

$$I_{\mathrm{c}}^0 = 9.1 e\hbar^{-1/2}\nu S D^{1/2}\left(T_{\mathrm{c}} - T\right)^{3/2}$$

then the order parameter is close to the value $\Delta_\infty = \left(\frac{2}{3}\right)^{1/2}\Delta_0$. The spatial variations of the order parameter, $\Delta^2(x) - \Delta^2(0)$ practically over the entire contact area, as seen from eq. (2.2) are of the order

$$\Delta_\alpha^2 - \Delta^2(0) \sim \Delta_\infty^2\left(\xi^2/L^2\right)$$

and therefore are relatively small. The scale of these variations can be characterized by means of the parameter $\delta = A_\infty - \Delta(0)$.

Thus the electrons with energies $\xi < \Delta_0$ prove to be confined in the contact and exercise a finite motion reflecting from the walls of the potential well at the points where $A = \varepsilon$. In an external AC field, the order parameter in the contact depends on time, $\Delta = \Delta(x,t)$, and electrons reflect from the walls of the "shaking" potential well. As a result, electron energy diffusion occurs as described by eq. (2.8). When the electron energy is not too close to the value of the order parameter in the contact center, $\varepsilon - \Delta_\infty \gg \delta$, then the bottom of the well may be considered flat; for the energy diffusion coefficient D, we then obtain from eq. (2.8):

$$D_1 = \frac{1}{4D}\frac{1}{\varepsilon^2 - A^2}\left\langle\left[\int_0^1\left(\left\langle\frac{\hat{i}A^2}{\hat{i}t}\right\rangle - \frac{\hat{i}A^2}{\hat{i}t}\right)\mathrm{d}x\right]^2\right\rangle. \tag{2.10}$$

As seen from this formula, the energy diffusion is due to the fact that the amplitude of oscillations Δ depends on the coordinates (because, if not, then $\langle\partial\Delta^2/\partial t\rangle = \partial\Delta^2/\partial t$) and the bottom of the well does not oscillate as a whole. In order to find the time dependence of the order parameter, let us note that

$$\frac{\partial \Delta^2}{\partial t} = \frac{\partial \Delta^2}{\partial \chi_0}\dot{\chi}_0 = \frac{eV}{\hbar}\frac{\partial \Delta^2}{\partial \chi_0}. \tag{2.11}$$

To calculate the derivative $\partial\Delta^2/\partial\chi_0$, we shall use the system obtained from eqs. (2.2) and (2.5) by substituting the equilibrium function $F(\varepsilon) = \tanh(\varepsilon/2T)$ into it:

$$\frac{\pi D}{8T}\frac{\partial^2 \Delta}{\partial x^2} - \frac{T I_{\mathrm{s}}^2}{2\pi e^2 v^2 D S^2 A^3} + \frac{T_{\mathrm{c}} - T}{T_{\mathrm{c}}}\Delta - \frac{7\zeta(3)}{8\pi^2}\frac{\Delta^3}{T^2} = 0, \tag{2.12}$$

$$\chi_0 = \frac{2TI_{\mathrm{s}}}{\pi e v S D}\int_0^L \Delta^{-2}(x)\mathrm{d}x, \quad \Delta(\pm L) = \Delta_0. \tag{2.13}$$

The value of the current I_{s} in the system will be considered constant and equal to the critical current of the contact without irradiation (the superconducting current oscillates near to the maximum and therefore the amplitude is small).

The set of equations (2.12) and (2.13) can be easily solved; using the derivative $\hat{c}A^2/\hat{c}t$ obtained by means of eqs. (2.11) and (2.8) we can find the energy diffusion coefficient

$$D_\varepsilon = 0.44 e^2 v^2 S^2 D \frac{\dot{\chi}_0^2 \Delta_\infty^8}{T^2 I_s^2}\frac{1}{\varepsilon^2 - \Delta_\infty^2} = 0.23 P \frac{T^2 \Delta_0^2}{\tau_\varepsilon\left(\varepsilon^2 - \Delta_\gamma^2\right)}, \quad \varepsilon - \Delta_\infty \gg \delta. \tag{2.14}$$

As seen from eq. (2.14), the energy diffusion is most intensive near the bottom of the well, and sharply decreases with increasing ε.

In the other limiting case, $\varepsilon - \Delta_\alpha \ll \delta$, for finding D_ε the region close to the center of the contact is essential; there the order parameter changes according to the parabolic law; thus the general equation (2.8) shows that the coefficient D_b ceases to depend on energy and, up to a numerical coefficient, is determined by eq. (2.14) at $\varepsilon - \Delta_\infty \sim \delta$.

The expression obtained for the energy diffusion coefficient shows how to solve the kinetic equation (2.7) and find the nonequilibrium distribution function. Hence we shall use the results obtained for the uniform case (Ch. 6). Although the cause of energy diffusion in the uniform case is the direct acceleration of electrons by the electric field, the dependence of D. on energy has the same form as that in eq. (2.14). Therefore, at energies close to the value of the gap A, an intensive energy diffusion, as shown in chapter 4, leads to
a diminishing number of quasi-particles, as in the uniform case

$$F(\varepsilon) - \tanh(\varepsilon/2T) = 0,1\tau^{1/10}P^{2/5}, \quad \varepsilon - \Delta_\infty \ll \varepsilon_{\mathrm{p}}, \tag{2.15}$$

$$\tau = \left(T_{\mathrm{c}} - T\right)/T_{\mathrm{c}}, \quad \varepsilon_{\mathrm{p}} = T\tau^{1/10}P^{2/5}. \tag{2.16}$$

At higher energies, the correction may be found by substituting the equilibrium function $F(\varepsilon) = \tanh(\varepsilon/2T)$ into the right-hand side of the kinetic equation (2.7):

$$F(\varepsilon) - \tanh\frac{\varepsilon}{2T} = -0.23P\frac{\mathrm{T}\Delta_0^2}{\left(\varepsilon^2 - A_x^2\right)^{3\ 2}}, \quad \varepsilon - \Delta_\infty \gg \varepsilon_{\mathrm{p}} \tag{2.17}$$

The negative sign in the right-hand side of eq. (2.17) indicates an excess of quasiparticles in a given energy interval, as compared to the equilibrium distribution [the distribution function $n(\varepsilon)$ is related to $F(\varepsilon)$ by eq. (2.4)]. Thus (Tinkham. 1979) the quasiparticles are rarefied at energies close to the gap. whereas, on the contrary, they condense in the region of higher energies (fig. 2).

Equations (2.15) and (2.17) describe the nonequilibrium distribution function for a long bridge at energies not too close to the value of the order parameter in the contact: $\varepsilon - d, \gg \delta$.

since the bottom of the potential well was assumed flat. Hence, for sufficiently long contacts, the inequality $\varepsilon_{\mathrm{p}} \gg \delta$ is fulfilled, starting from very small radiation powers, so that for all significant energy values the nonequilibrium distribution function is given by eus. (2.15) and (2.17).

Let us also note that for energies $\varepsilon > \Delta_0$ we must take into account the spatial diffusion of electrons from the contact to the banks. Let us assume that the diffusion is intensive enough and that the distribution function of electrons with energies $\varepsilon > \Delta_0$ in the contact is the same as in the banks (i.e the equilibrium one). Deviations from equilibrium in bridges of constant thickness when the spatial diffusion is hampered, will be discussed in section 2.3 .

The next step is to find the nonequilibrium term $\Phi(1)$ in the GinzburgLandau equation, using the known distribution function. As seen from eq. (2.3), the sign of $\Phi(\Delta)$ and hence the sign of the effect [stimulation or suppression of superconductivity-see eq. (2.2)] is determined by the rival contributions of energies close to the gap, where there is a lack of quasiparticles $[F(\varepsilon) - \tanh(\varepsilon/2T) > 0$, effective cooling], and high energies with excess quasiparticles $[F(\varepsilon) - \tanh(\varepsilon : 2T) < 0$, overheating]. Hence the above contributions almost entirely compensate each other; however, the residue (where the radiation power is not too small) is positive and linearly increases with increasing radiation power (stimulation!).

In order to verify this, let us write out the nonequilibrium term $\Phi(\Delta)$ as follows

$$\Phi\left(\Delta_\infty\right) = \int_{\Delta_\infty}^{\Delta_0} \frac{\Delta_\infty}{\left(\varepsilon^2 - \Delta_\infty^2\right)^{1/2}} \left[F(\varepsilon) - \tanh\frac{\varepsilon}{2T}\right] \mathrm{d}\varepsilon$$

$$= \int_{\Delta_\infty}^{\Delta_0} \left\langle \frac{\varepsilon}{\left(\varepsilon^2 - \Delta^2\right)^{1/2}} \right\rangle \left[F(\varepsilon) - \tanh\frac{\varepsilon}{2T}\right] \mathrm{d}\varepsilon + \Phi^{\mathrm{res}}\left(\Delta_\infty\right) \tag{2.18}$$

$$\Phi^{\mathrm{res}}\left(\Delta_{,}\right) = \int_{\Delta^1}^{\Delta_0} \left[\frac{\Delta_x}{\left(\varepsilon^2 - \Delta_\infty^2\right)^{1/2}} - \left\langle \frac{\varepsilon}{\left(\varepsilon^2 - \Delta^2\right)^{1,2}} \right\rangle\right] \left[F(\varepsilon) - \tanh\frac{\varepsilon}{2T}\right] \mathrm{d}\varepsilon \tag{2.19}$$

It follows from the kinetic equation (2.7) that the integrand in the first term of eq. (2.18) is the total derivative of the quasi-particles' flux $D_\varepsilon \partial F/\partial\varepsilon$. The integral is small, since for $\varepsilon = \Delta_\infty$ the derivative $\hat{\partial} F/\hat{C}\varepsilon$ vanishes (the flux is zero at the bottom of the well), while for $\varepsilon = \Delta_0$ the value of D_ε is already small (eq. 2.14). This corresponds to the above-mentioned almost complete cancellation of the contributions from the energy regions with rarefication and condensation of quasiparticles. In order to calculate the residue $\Phi^{\mathrm{res}}\left(\Delta_\gamma\right)$, we shall use eq. (2.19). In this integral it is reasonable to divide the interval of integration with respect to energy into two regions: $\Delta_\infty < \varepsilon < \Delta_\infty + \varepsilon_{\mathrm{p}}$ and $\Delta_\infty + \varepsilon_{\mathrm{p}} < \varepsilon < \Delta_0$, in which the integrals may be computed with logarithmic precision [the value of ε_{p} is determined by expression (2.16)]. For energies $\Delta_\infty + \varepsilon_{\mathrm{p}} < \varepsilon < \Delta_0$ we shall use eq. (2.16) for $F(\varepsilon)$ and consider that

$$\left\langle \left(\varepsilon^2 - \Delta^2\right)^{-1/2} \right\rangle = \left(\varepsilon^2 - \Delta_\infty^2\right)^{-1/2}$$

(assuming that $\varepsilon_{\mathrm{p}} \gg \delta$). As a result, we obtain:

$$\begin{aligned}\Phi_1^{\mathrm{res}}\left(\Delta_x\right) &= \int_{\Delta_y + \varepsilon_p}^{\Delta_0} \frac{\Delta_\infty - \varepsilon}{\left(\varepsilon^2 - \Delta_\infty^2\right)^{1/2}} \left[F(\varepsilon) - \tanh\frac{\varepsilon}{2T}\right] \mathrm{d}\varepsilon \\ &= 0.1 PT \ln\left(\Delta_0/\varepsilon_p\right)\end{aligned} \tag{2.20}$$

The positive sign of the nonequilibrium corresponds to stimulation of superconductivity in the contact. The contribution to the nonequilibrium term $\Phi_2^{\mathrm{res}}\left(\Delta_\infty\right)$ from the low energy

region $\Delta_\infty < \varepsilon < \Delta_\infty + \varepsilon_\mathrm{p}$ is due to the difference between $\left\langle (\varepsilon^2 - \Delta^2)^{-1/2} \right\rangle$ and $(\varepsilon^2 - \Delta_\infty^2)^{-1/2}$, and this is small for sufficiently long contacts. When calculating $\Phi_2^{\mathrm{rec}}(\Lambda)$ with logarithmic accuracy, one can take the lower limit of integration equal to $\Delta_\infty + \delta$, using eq. (2.15) for the distribution function. The dependence $\Delta(x)$ is found from eq. (2.12) and permits the calculation of $\left\langle (\varepsilon^2 - \Delta^2)^{-1/2} \right\rangle$. As a result, we obtain

$$\begin{aligned}\Phi_2^{\mathrm{res}}(\Delta_\infty) &= 0,1\tau^{1/10} P^{2/5} \Delta_\infty \int_{\Delta_x+\delta}^{\Delta_x+\varepsilon_\mathrm{p}} \left[\left(\varepsilon^2 - \Delta_\infty^2\right)^{-1/2} - \left\langle \left(\varepsilon^2 - \Delta^2\right)^{-1/2} \right\rangle \right] \mathrm{d}\varepsilon \\ &= -0.08 \frac{(D\Delta_0)^{1/2}}{L} \tau^{-3/20} P^{2/5} \ln \frac{\varepsilon_\mathrm{p}}{\delta} \end{aligned} \tag{2.21}$$

This contribution to the nonequilibrium term shows suppression of superconductivity, but its value decreases with increasing length L of the contact. Let us note the surprising fact that after the cancellation of the main
contributions from the low- and high-energy regions (the first integral in 2.18), the positive term Φ_1^{res} in the residue $\Phi^{\mathrm{res}} = \Phi_1^{\mathrm{res}} + \Phi_2^{\mathrm{res}}$ comes precisely from the high-energy region where the quasiparticles are dense.

After the nonequilibrium term $\Phi(\Delta)$ is determined, we shall find the critical current in the AC field using the Ginzburg-Landau equation. In the case of a long contact, one can neglect the first (gradient) term and, as a result, for the superconductivity current will be obtained

$$I_\mathrm{s}^2 = 2\pi e^2 v^2 D S^2 \left(\Delta^3/T\right) \left[\tau\Delta - \frac{7\zeta(3)}{8\pi^2} \frac{\Delta^3}{T^2} + \Phi(\Delta) \right] \tag{2.22}$$

The critical current I_c is found by maximizing the expression. At low irradiation powers $\Phi(\Delta)$ is small. Therefore, to find the correction to the critical current, the value of $\Delta = \Delta_\propto$ may be substituted into the expression; this value corresponds to the critical current of the contact without irradiation; for $\Phi(\Delta_\propto) = \Phi_1^{\mathrm{res}}(\Delta_\propto) + \Phi_2^{\mathrm{res}}(\Delta_\infty)$ one can use eqs. (2.20) and (2.21). As a result, we obtain

$$\frac{I_\mathrm{c} - I_{cx}^0}{I_{cx}^0} = 0.06\tau^{-3/2} P \ln\left(\frac{\Delta_0}{\varepsilon_\mathrm{p}}\right) - 0,14 \frac{\eta}{L} \tau^{-23/20} P^{2/5} \ln\left(\frac{\varepsilon_\mathrm{p}}{\delta}\right) \tag{2.23}$$

This expression has a minimum for powers $P_1 \sim (\eta/L)^{5/3} \tau^{7/12}$. Therefore, at very low irradiation power (the lesser it is, the higher is L, and the closer is T to a T_c) a decrease of the critical contact current should result. However, starting from powers of the order of P_1, the critical current linearly increases with increasing irradiation power. A linear stimulation of the critical current was observed in many experiments, and in several cases one managed to reveal a minimum of the critical current at a low irradiation power. The experimental situation is discussed in detail in ch. 1.

Expression (2.23) obtained for the dependence of the critical current of the contact on the irradiation power relates to the case of a long contact, $2L > \xi$. For short contacts, $\eta < 2L < \xi$ [the first condition implies the validity of the original expressions (2.2), (2.4) and (2.7)] the situation is somewhat different. The power interval in which the critical current decreases increases in this case. However, starting from the power $P_2 \sim \tau^{3/2}$, the nonequilibrium term in the Ginzburg-Landau equation (2.2) again becomes greater than the first term describing the spatial changes of Δ. As a result, the order parameter ceases to depend on coordinates as it does in the case of a long contact, and stimulation of the critical current occurs. This case is considered in the next section.

We shall also give here a comparison of the values of the critical current stimulation phenomena when stimulated by an AC field with a direct acceleration by an electric field (Eliashberg

mechanism) and for the stimulation due to the "shaking" of the potential well. Using the results of the uniform case (ch. 6), we conclude that the stimulation due to low-power electric fields is also linear, the corresponding coefficient in $(\Delta/T)(\eta/L)^2$, however, being less than that in eq. (2.23). Thus in the region where the results obtained are valid, $\eta < 2L < (D/\omega)^{1/2}$, the stimulation in the bridge is mainly due to the "shaking" of the potential well. The right-hand inequality has a simple physical meaning: during a period of the electromagnetic oscillation $\sim \omega^{-1}$ an electron diffusing along the contact experiences many reflections from the "shaking" walls of the well (the diffusion time is L^2/D). This condition is violated by increasing the contact length L, and then the relative contribution of "shaking" decreases (to find it, one must take into account the frequency dependence D_ε). Then, the direct acceleration of electrons by the field may become more important.

43.2.3 2.3. Superconductivity stimulation at high irradiation powers

With increasing irradiation power the nonequilibrium term in the GinzburgLandau equation (2.2) increases as well. For its estimation, we shall use eq. (2.20), obtained by means of perturbation theory which implies that the term in question becomes predominant in eq. (2.2) for $P > \tau^{3/2}$. Hence, for both long and short contacts, the order parameter for nearly all the contact, except the edges, only slightly depends on coordinates, while its value in the contact $\tilde{\Delta}$ is related to the superconductivity current I_s by eq. (2.22), in the right-hand side of which only the summand containing the nonequilibrium ferm $\Phi(\Lambda)$ is essential. Thereby, the deviation of the distribution function from that of equilibrium may be still small when the power $P < \tau^{1/2}$. Thus, in the power interval $\tau^{3/2} < P < \tau^{1/2}$ the nonequilibrium term in the Ginzburg-Landau equation is already large, while the distribution function is still close to the equilibrium one. In this case the critical current I may be found accordingly (in a self-consistent way). Assuming the current to be a given parameter we shall find D_i using eq. (2.14). Then by solving the kinetic equation (2.7) in the framework of perturbation theory (by substituting into the right-hand side of $F(\varepsilon) = \tanh(\varepsilon/2T)$), we shall find the correction to the equilibrium distribution function and, using it (and eq. 2.20) find the nonequilibrium term $\Phi_1(\tilde{A})$ (the contribution Φ_2 is small for sufficiently long contacts as it is for low irradiation power). As a result, we obtain

$$\Phi_1(\tilde{\Delta}) = 0.03\mathrm{e}^2 v^2 S^2 D P \frac{\tilde{\Delta}^6}{T^2 I_\mathrm{s}^2} \ln \frac{\Delta_0 - \tilde{\Delta}}{\tilde{\varepsilon}_\mathrm{p}} \tag{2.24}$$

By substituting this expression into eq. (2.22) (in which only the nonequilibrium term should be taken into account) we obtain a self-consistent equation for finding the current I_s. The critical current may be found by maximizing I_s, which is realized at a $\tilde{A}$ close to A_0 and where I_s is equal to

$$I_\mathrm{c} = 0.53 I_{\mathrm{c}\infty}^0 \tau^{-3/8} P^{1/4} \ln^{1/4}\left(\tau^{1/2} P\right). \tag{2.25}$$

Thus, in an intermediate power region the critical current depends nonlinearly on the irradiation power ($\tau^{3/2} < P < \tau^{1/2}$).

In the high-power region, $P > \tau^{1/2}$, the critical current reaches saturation. Thereby, the distribution function has a limit shape (fig. 2) and ceases to depend on the irradiation power: for all the electrons located inside the contact, a homogeneous equipartition energy distribution is set up, whereas the distribution function of electrons capable of diffusing into the banks remains equilibrium:

$$F(\varepsilon) - \tanh\frac{\varepsilon}{2T} = \begin{cases} \frac{\Delta_0 - \varepsilon}{2T}, & \varepsilon < \Delta_0 \\ 0, & \varepsilon > \Delta_0 \end{cases} \tag{2.26}$$

Equation (2.26) is formally obtained from the kinetic equation (2.7), in which the left-hand term (the relaxation term) for high irradiation power may be put equal to zero. Then, for the quasiparticles' flow we get $D_\varepsilon(\partial F/\partial\varepsilon) = \text{const.}$; since this equals zero at the bottom of the potential well, then for all energies $(\partial F/\partial\varepsilon) = 0$, i.e. $F(\varepsilon) = \text{const.}$ This constant may be found using the "linking" condition with the equilibrium distribution function at $\varepsilon = A_0$. The result obtained may be easily understood from the physical viewpoint in the case when we are coming from a low-power region: the region of rarefication near the bottom of the well increases with increasing power and finally it stabilizes for all the electrons "located" in the well.

For a known distribution function one can find the nonequilibrium term using eq. (2.3):

$$\Phi(\Delta) = \frac{\Delta\Delta_0}{2T}\left[\ln\frac{1+(1-\Delta^2/\Delta_0^2)^{1/2}}{\Delta/\Delta_0} - \left(1-\frac{\Delta^2}{\Delta_0^2}\right)^{1/2}\right]. \tag{2.27}$$

By substituting this expression into the expression (2.27) for a superconducting current (in which again the nonequilibrium term is essential), we see that the maximum current corresponds to $\tilde{\Delta} = 0,7\Delta_0$ and is equal to

$$I_c^* = 0,43evSD^{1/2}\Delta_0^{5/2}/\hbar^{3/2}T. \tag{2.28}$$

This is the maximum critical current of the contact in the microwave field. Note that eq. (2.28) is, in fact, found with accuracy up to a numeric coefficient, since in our derivation of eq. (2.22) the equilibrium expression was used for a superconducting current (see section 2.1).

Let us now compare the expression (2.28) for the maximum critical current with that of a critical current without irradiation, I_c^0. The current I_c^* does not depend on the bridge length and decreases when the temperature approaches T_c according to the law τ^{54}. The value of I_c^0 also decreases when nearing T_c, however, the law by which it decreases depends on the bridge dimension. For a short bridge ($2L < \xi$) the critical current is $I_\mathrm{c}^0 \sim \tau$, while for a long one ($2L > \breve{\zeta}$)
the critical current will be $I_\mathrm{c}^0, \sim \tau^{32}$. Therefore, the relative correction to the critical current far from T_c (when $2L > \xi$) increases with increasing temperature according to the law $\tau^{-1/4}$, then reaches its maximum, $\sim [L/\xi(0)]^{1/2}$, at the temperature when $2L \sim \xi$, and in the interval $\xi > 2L > \eta$ decreases according to the law $\tau^{1/4}$. For $2L < \eta$ the relative correction to the critical current is already small. The temperature dependence of the stimulation effect of the critical current is shown in fig. 3. Thus, the effect of superconductivity stimulation is most pronounced in the temperature interval when $2L \sim \breve{\xi}$. The picture described here is in a good agreement with experimental data for contacts which are not too long (see ch. 5).

When the maximum superconductivity current was found, the direct suppression of superconductivity by the AC field was not taken into account. This is valid until the power of the AC field is not too high, and the AC component of the superconducting current is snall as compared to the value of the maximum current (I_s is generally speaking less than I_c^* since the oscillations of the superconducting current occur near the maximum). The amplitude of the alternating current I may be estimated using eq. (2.5):

$$\tilde{I}_C(t) \sim \mathrm{evSD}\left(\Delta^2 T\right)\omega^{-1}\hat{i}(\nabla\%)\hat{i}t.$$

where $\nabla\chi \sim \chi_0/L$ and the value of $\dot{\chi}_0$ is related to the voltage on the contact V in accordance to eq. (2.9). Comparing the values $I_\mathrm{s}(t)$ and I_s^* we see that suppression leads to a considerable decrease of the superconductivity current at the voltages $eV \sim \hbar\omega(L/\eta)$. The corresponding dimensionless power
$P \sim (\hbar\omega^2\tau_\varepsilon/T)\,(L/\eta)^2$ proves to be proportional to the large value of the energy relaxation time τ_ε and therefore is high as compared to the power $P \sim \tau^{1/2}$ at which the critical current reaches saturation.

Fig. 3. Temperature dependence of the ratio of the maximum critical bridge current $I_{\mathrm{c }}^{*}$ in an AC field to that in the absence of radiation (I_c^0) ($2L$ is the length of the bridge; $\xi_0 = \zeta(0)$ is the coherence length at zero temperature).}

Let us also discuss the questions relating to the spatial diffusion of electrons from the contacts to the banks. This diffusion was assumed to lead to an equilibrium for electrons with energies $\varepsilon > A_0$. This condition is fulfilled in variable-thickness bridges, when the thickness of the film is larger in the banks, while the character of electron diffusion is spatial. However, in bridges of constant thickness, because of the two-dimensional character of the diffusion, the distribution function slowly (according to a logarithmic law) approaches equilibrium when going from the contact to the banks. In this case the contact is a source of nonequilibrium quasiparticles which propagate far towards the banks (to the distance $\sim \sqrt{D\tau_\varepsilon}$).

The "shaking" of the order parameter in such contact results in an insufficiency of quasiparticles with energies $\varepsilon > \Delta_0$ in the region of thickness $\sim \omega$ near the value Δ_0. Therefore, in constant-thickness bridges stronger stimulation may be observed than in those with a variable thickness. This phenomenon was experimentally observed by Latyshev and Nad' (1979) (see ch. 1).

43.2.4 2.4. Superconductivity stimulation in bridges by direct current

When the current through a contact exceeds the critical value, the mean voltage $\bar{V}$ appears on the contact, and the superconducting current has an alternating component with the frequency $\omega = 2e\bar{V}, \hbar$ (the nonstationary Josephson effect). In this case the electron energy distribution function differs from the equilibrium one; the cause of the nonequilibrium is both direct acceleration of electrons by the electric field and oscillation of the order parameter in the contact, produced by the alternating constituent of the superconducting current.

In this section we shall again consider contacts of length $2L$ satisfying condition (2.1). In this case the energy diffusion of electrons is mainly due to the oscillations of the order parameter in the contact, as it was in the case of superconductivity stimulation by an external AC field. However, the amplitude of the oscillations of the order parameter's modulus in the contact in this case can no longer be considered small (see section 1.1).

The condition (2.1) for the contact length provides for sufficiently smooth and slow variations of the order parameter, so that in order to describe the nonequilibrium phenomena one can again use the set of equations (2.2), (2.5) and (2.7). As a result, the energy electron diffusion caused by the oscillations of the order parameter in the contact for a current $I > I_c$ leads to a

decreasing number of quasiparticles (with the energies $\varepsilon < \Delta_0$) trapped in the contact. This corresponds to effective cooling of the contact, and the superconducting current through the contact strongly increases even for low voltage values on the contact.

In order to verify this, we shall first calculate the value of the nonequilibrium term $\Phi(\Lambda)$ for Δ close to Λ_0 and show that $\Phi(\Delta) > 0$. To do this, it suffices to find the deviation of the distribution function from equilibrium, $F(\varepsilon) - \tanh(\varepsilon/2T)$ for ε close to Δ_0. Beginning from low voltages, the distribution function assumes the limiting nonequilibrium form: for $\varepsilon < \Delta_0$ the distribution is homogeneous with respect to energies and $F(\varepsilon)$ weakly depends on energy; at the same time for $\varepsilon > \Delta_0$ the function $F(\varepsilon)$ is close to the equilibrium one, since the electrons with such energies easily diffuse from the contact and mix with the equilibrium electrons in the banks (the current density there is small).

Thus, the calculation may be performed in a self-consistent manner. Using the limiting expression (2.26) for $F(\varepsilon)$ when ε is close to Δ_0 we shall find the value of the nonequilibrium term by the expression (2.3) for Δ close to Δ_0; then solving the Ginzburg Landau equation with the known term $\Phi(\Delta)$, we determine the dependence of the order parameter on time and coordinates. Next, we use eq. (2.8) to calculate the energy diffusion coefficient D_2. With its help we can solve the kinetic equation (2.7) and confirm the validity of the assumption on the form of the distribution function.

By substituting the limiting expression (2.26) for $F(8)$ into eq. (2.3) we obtain for the nonequilibrium term, when Δ is close to Δ_0

$$\Phi(\Delta) = \frac{\sqrt{2}}{3}\frac{\Delta_0^2}{T}\left(1 - \frac{\Delta}{\Delta_0}\right)^{3/2} \tag{2.29}$$

Comparison of the orders of magnitude of the terms in the Ginzburg Landau equation (2.2). where $\Phi(\Delta)$ is given by eq. (2.29), shows that considerable variations of the order parameter (by the value of $\tilde{A} - \Lambda_0$) occur only near the edges of the contact at a distance $\Delta x \sim \eta\left(1 - \tilde{\Delta}/\Delta_0\right)^{-1/4}$, while the value of the order parameter in the contact $\bar{A}$ is determined by the value of the superconducting current,

$$\hbar T I_s^2/2\pi e^2 v^2 S^2 \Delta_0 = \tilde{\Delta}^2 \Phi(\tilde{\Delta}) \tag{2.30}$$

The oscillations of $\tilde{A}$ are caused by those of the superconducting current. which, in turn, are due to the oscillations of the phase difference 2% of the order parameter in the banks. The derivative %, is related to the alternating voltage V by the Josephson relation. Therefore using eqs. (2.30), (2.5) and (1.7) we can find the derivative

$$\partial\tilde{\Delta}/\partial t = -2.3eV\frac{\eta}{L}\Delta_0\left(1 - \frac{\tilde{\Delta}}{\Delta_0}\right)^{1/4} \tag{2.31}$$

so that for $\tilde{\Delta}$ close to Δ_0, the voltage $V = RI$. since mainly normal current flows through the contact (I is a given value of the total current through the contact).

To find D_ε one should also know the dependence of the order parameter on coordinates. Let us assume that near the edges the value Δ^2 linearly depends on coordinates, while it is constant elsewhere. Then, using eq. (2.8) and taking into account eq. (2.31) we get

$$D_\varepsilon = 0.1\frac{\eta}{L}e^2 IR\bar{V}\Delta_0\left(1 - \frac{\varepsilon}{\Delta_0}\right)^{-1/4}. \tag{2.32}$$

We see that for ε close to Δ_0 the energy diffusion coefficient strongly increases and this causes the stabilization of equally populated levels (2.26) in some region of energies close to Δ_0 even at low voltages $\bar{V}$.

The limiting nonequilibrium distribution function can be found from the kinetic equation (2.7) by setting its left-hand side (relaxation term) equal to zero:

$$F(\varepsilon) = C \int_0^{\varepsilon} D_{\varepsilon}^{-1} \, \mathrm{d}\varepsilon \tag{2.33}$$

where the constant C is found from the linkage condition at $\varepsilon = \Delta_0$ with the equilibrium value ($\Delta_0/2T$). By substituting expression (2.32) for $D_{;}$ into this formula, we have

$$F(\varepsilon) = \frac{\Delta_0}{2T} \left[1 - \left(1 - \frac{\varepsilon}{\Delta_0} \right)^{5/4} \right] \tag{2.34}$$

where the second summand is written with accuracy up to a numerical coefficient, since for a precise determination using eq. (2.33) one must know D_{z} at all the energy values.

The expression (2.34) for $F(\varepsilon)$ thus derived shows that deviations of the distribution function from the constant value $(\Delta_0/2T)$ are, for ε close to A_0, smaller than $(\Delta_0 - \varepsilon)$. Therefore, we can state that the nonequilibrium additional term $F(\varepsilon) - \tanh(\varepsilon/2T) \approx (\Delta_0 - \varepsilon)/2T$ is positive, this corresponding to an insufficiency of quasiparticles and superconductivity stimulation in the contact. The nonequilibrium term in the Ginzburg-Landau equation is then also positive and is given by (2.29).

Let us now estimate the voltages at which eq. (2.29) gives the correct value of the nonequilibrium term for values of Δ not close to Δ_0 as well. For this it is necessary for the limiting expression (2.26) for $F(\varepsilon)$ to be also applicable in the region $\varepsilon \lesssim \Delta_0$. Let us find, with the help of perturbation theory, the correction F^1 to the limiting distribution function. By substituting the limiting expression (2.26) into the left-hand side of the kinetic equation (2.7) and using the expression (2.32) for $D(\varepsilon)$ we obtain up to a numerical coefficient

$$F^1 = -\frac{\hbar L}{e^2 \eta} \frac{1}{\tau_\varepsilon \bar{V}} \frac{\Delta_0^2}{TIR} \left(1 - \frac{\varepsilon}{\Delta_0} \right)^{5/4} . \tag{2.35}$$

For ε close to Δ_0 this correction is always small. However, for it to be small as compared to the limiting value $\Delta_0/2T$ at $\varepsilon \lesssim A_0$ the voltage $\bar{V}$ must satisfy the condition

$$\bar{V} > \left(\hbar/e^2 \tau_\varepsilon \right) (L/\eta) \left(\Delta_0/IR \right) . \tag{2.36}$$

When this condition is fulfilled, formula (2.29) gives the correct order of magnitude for the nonequilibrium term for values of Δ not close to Δ_0 as well.

Let us discuss the shape of the CVC of the contact. It considerably differs from hyperbolic for $2L > \eta$ when the nonequilibrium effects are taken into account. In fact, the nonequilibrium effects lead to a considerable increase of the maximum superconducting current I_{s}^* which can flow through the contact. The current is determined by maximizing the right-hand side of eq. (2.30), the maximum being necessarily reached at some value $\tilde{\Delta} < \Delta_0$ (the right-hand side vanishes for $\tilde{\Delta} = 0$ and $\tilde{\Delta} = \Delta_0$, while in the intermediate region it is positive). Using eq. (2.29) for I_{s}^* we obtain (up to a numerical coefficient):

$$I_{\mathrm{s}}^* = 0.4 evSD^{1/2} {A_0}^{5/2} / \hbar^{3/2} T. \tag{2.37}$$

This value for $2L > \eta$ is always higher than the critical current of the contact I_{c} (it is L/η times higher for short contacts, $\eta < 2L < \xi$, and $\tau^{-1/4}$ times more for long ones, $2L > \xi$).

Thus already at low voltages, determined by expression (2.36), the current through the contact considerably increases and becomes close to I_{s}^*. When the voltage is increased further, the current grows slowly, since its change is caused only by the increase of the normal component. A plateau and a typical shoulder* (fig. 4) appear on the CVC. The voltage corresponding to the

Fig. 4. The current-voltage characteristic of the bridge with length $2L > \eta(T)$ (nonequilibrium effects are taken into account).

inflexion point can be obtained by substituting the current value (2.37) into expression (2.36). For short contacts, $\eta < 2L < \xi$, this value does not depend on the contact's characteristics:

$$V^* \sim \hbar T / e\tau_\varepsilon \Delta_0$$

The dynamic picture of the Josephson effects in the plateau region is the following. During the greater part of the period the superconducting current close to I_s^* flows through the contact accompanied by a relatively weak normal current. The value of the superconducting current is determined by the phase difference of the order parameter within the banks. Near the maximum value the superconducting current I_s quadratically depends on the phase difference, and for the total current I through the contact one has the equation

$$I = \frac{\hbar}{eR}\dot{\chi}_0 + I_s^* - I_s^* \frac{4\eta^2}{L^2}\left(\chi_0 - \chi_0^{\max}\right)^2 \tag{2.38}$$

where $\chi_0^{\max}$ is the value of the phase difference corresponding to the maximum superconductivity current. The first term corresponds to the normal current through the contact, determined by Ohm's law, the derivative $\hat{c}^2 I_s / \partial \chi_0^2$ in the last summand is found with accuracy up to a numerical coefficient [using eqs. (2.5) and (2.30)].

The solution of equation (2.38) is of the form

$$\chi_0 - \chi_0^{\max} = \frac{L}{2\eta}\left(\frac{I - I_s^*}{I_s^*}\right)^{1/2} \tan\left(\frac{2eR}{\hbar}\frac{\eta}{L}\left[I_s^*\left(I - I_s^*\right)\right]^{1,2} t\right) \tag{2.39}$$

As seen from eq. (2.39), for I close to I_s^* the phase χ in fact hardly changes for the greater part of the period and it is mainly the superconducting current which passes through the contact; this justifies the use of eq. (2.38) in the expansion of the superconducting current in terms of the phase difference values.

The rare pulses of normal current result in the appearance of a voltage on the contact. However, the shape of the pulses can not be obtained from eq. (2.38) [one needs to know the precise form of the dependence $I_s\left(\chi_0\right)$], while the mean voltage is expressed in terms of the oscillation frequency by means of the Josephson relation:

$$\bar{V} = \hbar\omega / 2e = (\eta / L) R \sqrt{I_s^*\left(I - I_s^*\right)} \tag{2.40}$$

[18]*The term "a shoulder" is used for describing the shape of the $I(V)$ characteristic and the term "a foot" is used in the case of a $V(I)$ characteristic. Here we use the term "a shoulder".

Thus, in the high-voltage range a new term is added to the current I_s^*; it depends quadratically on the voltage.

The law describing how the current reaches the plateau from the low voltage range may be obtained using eq. (2.35) which gives a correction to the limiting distribution function. After using this and eq. (2.3) to calculate the correction of the non-equilibrium term $\Phi(\Delta)$ and then using formula (2.30), we obtain the
correction to the maximum current

$$I - I_s^* \approx -\left(\Delta_0 L/e^2\eta R\right)\left(\hbar/\tau_\varepsilon \bar{V}\right). \tag{2.41}$$

This implies that with increasing voltage the current approaches the value of I_s^* according to a hyperbolic law.

The current-voltage characteristic of the bridge at $2L > \eta$ is shown in fig. 4. The temperature dependence of the inflexion point and the shape of the CVC in the plateau region are in good agreement with the experimental data of Octavio et al. (1978) Gubankov et al. (1977) and Divin and Nad' (1978) (see ch. 1).

43.3 3. Short bridges, $2L < \eta(T)$

In this case we shall use the microscopic equations for the 4×4 matrix Green function (see ch. 4)*. The components of the matrix $\hat{G}$ are 2×2 matrices of the retarded, G^{R} (advanced, G^{A}) Green functions determining the system's spectrum, as well as the matrix $\hat{G}$ determining the kinetic properties of the system. The matrix $\hat{G}$ may be expressed in terms of the quasiparticles' distribution functions $F^{(L)}$ and $F^{(T)}$ as follows (Larkin and Ovchinnikov 1975, 1977)

$$\hat{G} = \hat{G}^{\mathrm{R}}\hat{F} - \hat{F}\hat{G}^{\mathrm{A}}, \quad \hat{F} = F^{(\mathrm{L})}\hat{1} + F^{(\mathrm{T})}\hat{\sigma}_2. \tag{3.1}$$

Here $\hat{\sigma}_z$ is the Pauli matrix. The function $F^{(L)}$ characterizes the symmetrical part of the populations of the excitation spectrum branches. It determines the gap in the spectrum and in the equilibrium case is equal to $F_{\mathrm{eq}}^{(\mathrm{L})} = \tanh(\varepsilon, 2T)$. The function $F^{(T)}$ describes the energy distribution of the population imbalance of the electron-like and hole-like branches of the quasiparticles' spectrum. It determines the electric potential φ [in the general case the gauge invariant potential $\phi = e\varphi + (\hbar/2)(\hat{C}\chi/\hat{C}t)$] and equals zero in the equilibrium case.**

For the functions $F^{L/I'}$ the corresponding kinetic equations may be obtained (see ch. 4). However, in the present case of short bridges it is easier to use directly the original equations, rather than to apply the presentation of $\hat{G}$ in the form (3.1). Then, providing the condition (1.14) is fulfilled, in the equation for the anisotropic part of $\hat{G}$ determining the current, one can neglect all the terms except the gradient. Then we obtain:

$$\frac{\partial \hat{G}_1^{\mathrm{R(\ A)}}}{\partial x} = 0, \quad \frac{\partial \hat{G}_1}{\partial x} = 0. \tag{3.2}$$

The terms neglected are small with respect to the parameters L^2A/D and $L^2/D\tau_\varepsilon$, where τ_ε is the energy relaxation time. This implies that the anisotropic components $\hat{G}_1^{\mathrm{R(A)}}$ and $\hat{G}_1$ do not depend on x :

$$\hat{G}_1^{\mathrm{R(\ A)}} = \hat{C}^{\mathrm{R(\ A)}}, \quad \hat{G}_1 = \hat{C} \tag{3.3}$$

In order to determine the dependence on time of the matrices $\hat{C}^{\mathrm{R(A)}}$ and $\hat{C}$, we shall use the exact normalization relation (see Appendix 2.1), which should be written at the points $x = \pm L$

[18]*This section is based on the results of Artemenko, Volkov and Zaitsev (1979).

**For more detail see the reviews by Artemenko and Volkov (1979), Pethik and Smith (1979), and Schmid (1981).

where all the functions except $\hat{C}^{\mathrm{R(A)}}$ and $\hat{C}$ are known. Thus we shall express the matrix $\hat{C}$ in terms of the functions in the banks. Similarly, the current I is determined by the values of the order parameter in the banks.

The normalization relation describes, in particular, the energy relaxation which will prove to be significant for quasiparticles with energies (1.15). The total current I is expressed through the matrix $\hat{C}$ by means of the formula

$$I(t) = -\frac{1}{LR}\int_0^\infty \mathrm{d}\varepsilon\, \mathrm{Tr}\left(\hat{\sigma}_z\hat{C}\right) \equiv -\frac{1}{LR}\int_0^\infty \mathrm{d}\varepsilon C_z(\varepsilon, t) \tag{3.4}$$

One cannot find the matrix $\hat{C}$ for arbitrary voltages V (and hence, for the frequencies ω), since to do this one has to solve a complicated integral equation. It is, however, possible in the limiting cases when V is small as compared to Δ_0 and for high voltages V. For small V one can use the equilibrium expansion in V/Δ_0 i.e. one can consider the function G to be slightly changed for the period of time of the order of Δ_0^{-1}. At higher voltages $V \gg A_0$ one can carry out the expansion of all the functions in a power series in Δ_0/V and then find the sum of the principal terms of the series.

43.3.1 3.1. Small voltages, $V \ll \Delta_0$; temperatures close to T_c

Now let us consider the case of small voltages V [respectively, the case of the frequencies $\omega = (2e/\hbar)V], V \ll \Delta_0$. We shall restrict the temperatures to those close to the critical one T_c since only then the characteristic frequency of the Josephson oscillations $\omega_c = (2e/\hbar)V_c$ (where $V_c = I_c R = \pi\Delta_0^2/4T$) will be less than Δ_0. For a given current $I \gtrsim I_c$ the harmonics in the spectrum of the Josephson oscillations have approximately the same value. However, to realize bias voltage conditions is very difficult since the resistance of the bridge is rather low. At the same time, the temperature T should not be too close to T_c since we shall assume the smearing of the density of states due to the inelastic processes to be small. This implies the restriction $\Delta_0\tau_\varepsilon \gg 1$.

The Green function determining the current, $\hat{G}_1$, is represented as a sum of two parts: the regular, $\hat{C}^{(\mathrm{r})}$, and anomalous, $\hat{\mathrm{C}}^{(\mathrm{a})}$, parts. The first of them differs from zero for currents $I < I_c$ (i.e. in the equilibrium case) as well. When
substituting $C_{=}^{(r)}$ into eq. (3.4) we obtain the ordinary Josephson current* (see eq. 1.6).

The form of the anomalous function $C_z^{(a)}$ essentially depends on energy. When the quasiparticles' energy exceeds the energy gap in the banks ($\varepsilon > A_0$), then the form of $C_{=}^{(a)}$ at $\varepsilon \sim T$ is the same as that in the normal state at a given current. Integrating $C_{=}^{(4)}$ over the energy region $\varepsilon \sim T$ gives the ordinary Ohmic current V/R.

When the quasiparticles' energy is within the range determined by (1.15), the form of the function $C_{=}^{(a)}$ depends on the energy relaxation time, while its value is anomalously large. The high value of $C_z^{(a)}$ is caused by the slow relaxation of the quasiparticles confined inside the region $|x| < L$ (rather than inside the region determined by the inequality $\varepsilon \geqslant \Delta(x,t)$ in the previous section, since in this case the quasiparticles could move in the "forbidden" region $\tilde{\Delta} < \varepsilon < \Delta(x,t)$ as well!). Hence, the relaxation of these quasiparticles occurs either because of inelastic processes, or due to their expulsion into the energy region $\varepsilon > \Delta_0$. The current. Integrating $C_{:}^{(a)}$ over the energy region $\varepsilon \sim T$ gives the ordinary Ohmic "anomalous") is equal to

$$I^{(a)} = I_c P\{\chi_0\}, P\{\chi_0\} = \frac{\sin(2\chi_0)}{2} \int_0^1 \frac{k^3\, dk}{\sqrt{1-k^2}\sqrt{1-k^2\sin^2\chi_0(t)}}$$
$$\times \int_{-\infty} d\theta \frac{d\%_0(\theta)}{d\theta} \frac{\sin(2\%_0(\theta)}{\sqrt{1-k^2\sin^2\chi_0(\theta)}} \exp\left(\frac{\theta - t}{\tau_c}\right) \tag{3.5}$$

The total current through the bridge equals

$$I = I_c \sin(2\chi_0) + \frac{V}{R} + I_c P\{\chi_0\} \equiv I_c \widetilde{P}\{\chi_0\} + \frac{\hbar}{eR}\frac{\partial \chi_0}{\partial t}. \tag{3.6}$$

The anomalous current $I_c P\{\chi_0\}$ in eq. (3.6) gives the deviation from a simple resistive model (see formula eq. 1.6). This deviation does not depend on the bridge length L [certainly, provided the condition (1.14) is fulfilled] and is determined by the ratio between the inverse time of the energy relaxation $\nu_i \equiv \tau_\varepsilon^{-1}$ and the characteristic frequency of the phase difference's variation $2\%_0(t)$. Let us consider the limiting cases
(a) $\lambda \equiv V_c \tau_\varepsilon = (\pi \Delta_0^2 \tau_\varepsilon / 4T) \ll 1$.

This means, that at $V \lesssim V_c$ the oscillations in the bridge proceed slowly in the time scale τ_ε. At voltages satisfying the condition $V \ll v' = V_c$ i. the functions depending on $\%(\theta)$ in (3.5) can be taken out from under the integral sign [by

substituting $\chi_0(\theta)$ for $\chi_0(t)$] and after integration over k one obtains

$$P\{\chi_0\} = \frac{\partial \chi_0}{\partial t} \tau_\varepsilon \cot\chi_0 \left(2[\chi_0] - \sin(2\chi_0)\right)$$
$$= \frac{\partial \chi_0}{\partial t} \tau_\varepsilon \sum_{n=0}^{\infty} b_n \cos(2n\chi_0) \tag{3.7}$$

Here $[\chi_0]$ is a function with period π, which is equal to χ_0 at $|\chi_0| < \pi/2$, $b = 21\text{n}2 - 1 \simeq 0.39$, and $b_1 = 41\text{n}2 - 3 \simeq -0.23$.
The first term of the series in eq. (3.7) determines the changes of the contact resistivity, the rest gives a contribution to the interference current.

The form of the CVC is found using the formula

$$V = \pi \left[\int_{-\pi/2}^{\pi/2} d\chi_0 \left(\frac{\partial \chi_0}{\partial t}\right)^{-1} \right]^{-1}. \tag{3.8}$$

The integral can be calculated by substituting $\partial\chi_0/\partial$t from (3.6) and (3.7) into (3.8) and integrating over the complex plane. Then we obtain:

$$V = R\left(I^2 - I_c^2\right)^{1/2} \left[1 + \lambda u\left(I/I_c\right)\right]^{-1}, \tag{3.9}$$

where

$$u(a) = \frac{\sqrt{a^2-1}}{a} \ln \frac{2\left(a + \sqrt{a^2-1}\right)}{a} + \frac{1}{a} \operatorname{arctg} \frac{1}{\sqrt{a^2-1}} - 1$$

The difference between the CVC and the resistive model, determined by the factors in the brackets, is small because i is small. For $I \gg I_c$ the asymptotic resistance of the bridge,

[18]*The derivation of the equations required can be found in the article by Artemenko et al. (1979a) and in Appendix 2.

$R_{as} = R\{1 + \lambda(21n2 - 1)\}^{-1}$ decreases slightly due to the contribution of the quasiparticles with energies (1.15).

At voltages $V \gg \nu_r$, the functions in eq. (3.5) containing $\chi_0(\theta)$ oscillate rapidly in time. Therefore, one can integrate eq. (3.5) over time θ considering the exponent as constant and subtracting from the expression obtained its timeaveraged value*. Then for $P\{\chi_0\}$ we obtain:

$$P\{\chi_0\} = \sin(2\chi_0)\left\{\int_0^1 \mathrm{d}k \frac{k}{\sqrt{1-k^2}} \frac{\left\langle\sqrt{1-k^2}\sin^2\chi_0\right\rangle_t}{\sqrt{1-k^2\sin^2\chi_0}} - 1\right\}. \tag{3.10}$$

The averaging over time is carried out using the function ($\hat{c}\chi_0/\hat{c}t$), which can be expressed from eq. (3.6):

$$\left\langle\sqrt{1-k^2\sin^2\chi_0}\right\rangle_t = \frac{V}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\left(1-k^2\sin^2\chi_0\right)^{1/2}\ \mathrm{d}\chi_0}{\mathrm{d}\chi_0/\mathrm{d}t}$$

$$= \frac{V}{\pi R}\int_{-\pi/2}^{\pi/2} \frac{\left(1-k^2\sin^2\chi_0\right)^{1/2}\ \mathrm{d}\chi_0}{\mathrm{I} - I_c\tilde{P}\{\chi_0\}}. \tag{3.11}$$

In the limiting case $\lambda \ll 1$ considered, the inequality $V \gg \nu_\varepsilon$ means that $I \gg I_c$. Then, neglecting the term $I_c\widetilde{P}\{\chi_0\}$ in eq. (3.11), we obtain

$$\left\langle\sqrt{1-k^2\sin^2\chi_0}\right\rangle_t = \frac{2}{\pi}E(k),$$

where $E(k)$ is the complete elliptic integral. By substituting this expression into eq. (3.10), we find for the current

$$I = \frac{\hbar}{eR}\frac{\partial\chi_0}{\partial t} + I_c\frac{2}{\pi}\sin(2\chi_0)\int_0^1 \mathrm{d}k \frac{kE(k)}{\sqrt{1-k^2}\sqrt{1-k^2\sin^2\chi_0}} \tag{3.12}$$

Thus, at high oscillating frequencies, the anomalous current $I^{(a)}$ becomes nondissipative and does not depend on τ_ε. Here, the dependence $\widetilde{P}\{\chi_0\}$ differs from sinusoidal (see fig. 5). This implies that upon irradiation of the bridge

Fig. 5. The curve $P\left(\chi_{0 \right)$ obtained by numerical calculation of the integral in (3.12). The dots denote the function $\sin(2\chi_0)$.}

[18]*The last circumstance was not taken into account by Schmid et al. (1980), which resulted in their conclusion concerning rather strong stimulation of superconductivity in short bridges.

with frequency ω subharmonic current steps appear on the CVC of the bridge at voltages $2eV = (n/m)\hbar\omega$ (n and m are integers).
(b) $\lambda = \tau_\varepsilon V_c \gg 1$.

This means that during τ_ε a lot of oscillations occur in the bridge. Then practically for all V the form of the function $P\{\chi_0\}$ is determined by (3.10). By substituting (3.11) into (3.10), we obtain for $\widetilde{P}\{\chi_0\}$ the integral equation

$$\tilde{P}\{\chi_0\} = \int_{-\pi/2}^{\pi/2} \mathrm{d}\chi_1 \frac{Q(\chi_0, \chi_1)}{I - I_c \tilde{P}\{\chi_1\}} \tag{3.13}$$

where

$$Q(\chi_0, \chi_1) = \frac{V}{2\pi R} \sin(2\chi_0) \int_0^1 dk \frac{k}{\sqrt{1-k^2}} \left[\frac{1 - k^2 \sin^2 \chi_1}{1 - k^2 \sin^2 \chi_0}\right]^{1/2}$$

and the voltage V is determined by eq. (3.8). Thus, for $\lambda \gg 1$ the very form of $\tilde{P}\{\chi_0\}$ depends on the current value I.

When $I \gtrsim I_c$, then the function $\tilde{P}\{\chi_0\}$ can apparently be found only numerically. However, we can make some conclusions about the form of the CVC. Let the voltage V range within the interval $v_r \ll V \ll V_c$. Then the main contribution to the integral (3.8) is determined by a region near the maximum of the function $\widetilde{P}\{\chi_0\}$, which can be expanded in this region as follows:

$$\tilde{P}(\chi_0) = \tilde{P}\{\chi_m\} - \frac{1}{2}\left|\tilde{P}_{//}\right| (\chi_0 - \chi_m)^2 .$$

Then, from (3.13) we can find

$$\widetilde{P}\{\chi_0\} = \sin(2\chi_m) \int_0^1 \frac{k\ \mathrm{d}k}{\sqrt{1-k^2}} \left[\frac{1 - k^2 \sin^2 \chi_m}{1 - k^2 \sin^2 \chi_0}\right]^{1/2} \tag{3.14}$$

The angle χ_m is determined from the equation $\partial \tilde{P}\{\chi_0\}, \hat{c}\chi_0 = 0$. Using eq. (3.14), we can find $2\chi_m = 1.92$. From eq. (3.8), we obtain the formula for the CVC

$$V = R\left[2\left|\widetilde{P}_{\chi\chi}\right| (I - 0.94 I_c)\right]^{1/2} \tag{3.15}$$

Hence, at small voltages the current I is below the critical one, i.e there is a hysteresis on the CVC.

Let us also study the effect of a weak AC signal with frequency $\omega \gg v_\varepsilon$ on the critical current I_c. In this case $\chi_0(t) = \chi_c + \chi_-(t)$, where χ_c is a constant part and $\chi_\sim = \chi_1 \sin(\omega t)$ is a small variable part. The current is found from the expressions (3.6) and (3.10):

$$\begin{aligned} I =& \frac{\hbar}{eR}\frac{\partial \chi_-}{\partial t} + I_c \sin(2\chi_0(t)) \\ &+ I_c \sin(2\chi_0) \int_0^1 \mathrm{d}y \frac{\overline{q(\chi_0, y)} - q(\chi_0, y)}{q(\chi_0, y)} \end{aligned} \tag{3.16}$$

where

$$q(\chi_0, y) = \left[\left(1 + y^2\right) + \left(1 - y^2\right)\cos(2\chi_0)\right]^{1/2}$$

while the averaging is performed over the period $2\pi, \omega$. Let us expand this expression in powers of χ_1 up to second order, average it over time and put $\chi_0 = \pi/2$. Then we obtain

$$I_{\max} = I_{\mathrm{c}} \left(1 - \frac{1}{4}\pi\chi_1^2\right)$$

Hence, in this case, a weak AC signal leads to the suppression of I_{c}, as was the case for the resistive model. It is of interest to note, however, that the anomalous current $I^{(\mathrm{a})}$ under the influence of radiation leads to the stimulation of the superconducting current:

$$I_{\mathrm{c}}\overline{P\left\{\chi_0\right\}} = I_{\mathrm{c}} \left(2 - \frac{1}{4}\pi\right) \chi_1^2.$$

However, the "Josephson" suppression $I_c\overline{\sin\left(2\chi_0\right)} = I_c\left(1 - 2\chi_1^2\right)$ proves to be stronger. In longer bridges ($2L > \eta$), the stimulation of I_{c} proves to be proportional to the parameter $I_{\mathrm{c}}(2L/\eta)$ and is large as compared to I_{c}. Here we can neglect the suppression of the maximum current by the superconducting current for the radiation powers considered.

The question concerning the effect of strong radiation on the current I of a short bridge was not considered.

43.3.2 3.2. The form of the distribution functions

Let us discuss the form of the distribution functions $F^{(L)}$ and $F^{(T)}$ (see eq. 3.1); this will enable us to clarify the physical background of the processes taking place*. For energies $\varepsilon > A_0$, the function $F^{(T)}$ describing the distribution of the potential Φ is determined (see Appendix 2.2) by the formula:

$$F^{(\mathrm{T})} = \frac{\partial\chi_0}{\partial t}\frac{\sqrt{\varepsilon^2 - \widetilde{\Delta}^2}}{2T\Delta_0\sin\chi_0}\tanh\left[\frac{x}{L}\operatorname{arcsinh}\frac{\Delta_0\sin\chi_0}{\sqrt{\varepsilon^2 - \Delta_0^2}}\right]\cosh^{-2}(\varepsilon/2T).$$

At higher energies ($\varepsilon \gg \Delta_0$) the function $F^{(T)}$ has the same form as in the normal state,

$$F^{(T)} \simeq \frac{\partial\chi_0}{\partial t}\left(\frac{x}{2L}\right)\cosh^{-2}(\varepsilon/2T);$$

the Ohmic current V/R is proportional to the integral over energies of the gradient of this function $\hat{\partial}F^{(\mathrm{T})}/\hat{\partial}x$. The function $F^{(\mathrm{L})}$ determining the order parameter Δ at $\varepsilon > \Delta_0$ approximately coincides with that in the equilibrium state: $F^{(\mathrm{L})} \simeq \tanh(\varepsilon/2T)$.

For energies in the interval determined by (1.15), the function $F^{(L)}$ in the bridge does not depend on x and equals

$$\begin{aligned} F^{(L)} - \tanh(\varepsilon/2T) =& \frac{(\Delta_0^2 - \varepsilon^2)}{4T\left|\sin\chi_0\right|}\int_{-\infty}^{t} \mathrm{d}t_1 \frac{\partial\chi_0\left(t_1\right)}{\partial t_1} \\ &\times \frac{\sin\left[2\chi_0\left(t_1\right)\right]\exp\left[\left(t_1 - t\right)/\tau_\varepsilon\right]\operatorname{sgn}\varepsilon}{\left[\Delta_0^2\sin^2\chi_0(t) - (\Delta_0^2 - \varepsilon^2)\sin^2\chi_0\left(t_1\right)\right]^{1/2}} \end{aligned} \tag{3.17}$$

It follows from (3.17) that the deviation of the distribution function $F^{(L)}$ from the equilibrium $\tanh(\varepsilon/2T)$ tends to zero at $\varepsilon \to \Delta_0$, i.e. the function $F^{(L)}$ is continuous at $\varepsilon = \Delta_0$. Let us determine the explicit form of the function $F^{(\mathrm{L})}$ at $V\tau_\varepsilon \ll 1$. By performing the integration in (3.17) in the same way as it was done in section (3.1) we find

[18]*It should be noted, that the function $F^{(L)}$ coincides with the function $F(\varepsilon)$ of section 2.

$$F^{(\mathrm{L})} - \tanh\left(\frac{\varepsilon}{2T}\right) = \tau_\varepsilon \frac{\partial \chi_0}{\partial t} \frac{\Delta_0^2 - \varepsilon^2}{2T\varepsilon} \cot \chi_0$$
$$= \frac{\Delta_0^2 - \varepsilon^2}{T\varepsilon} - \frac{\tilde{\Delta}^2 (\tau_\varepsilon \mathrm{V_c})}{\Delta_0^2 \sin(2\chi_0)} \left[\frac{I}{I_\mathrm{c}} - \sin(2\chi_0)\right]. \tag{3.18}$$

We have substituted here the expression for $\partial\chi_0/\partial t$ from eqs. (3.6) and (3.7) and neglected the small summand containing λ. Since

$$F^{(\mathrm{L})} - \tanh\frac{\varepsilon}{2T} = -2\delta n$$

where δn is the variation of the quasiparticles' population number, equation (3.18) implies that a decrease of the energy gap in the center of the bridge $\tilde{\Delta}$ leads to the rarefication of the quasiparticle gas ($\delta n < 0$), while an increase of the gap $\tilde{\Delta}$ leads to the compression of the quasiparticle gas ($\delta n > 0$). Indeed, when $I > 0$, the phase χ_0 increases with time, $\chi_0 \sim It$, while the gap in the center of the bridge $\tilde{\Delta} = \Delta_0 |\cos\chi_0|$ decreases for $0 < \chi_0 < \pi/2$ [then $\sin(2\chi_0) > 0$] and increases for $-\frac{1}{2}\pi < \chi_0 < 0$ [then $\sin(2\chi_0) < 0$]. Thus, there is much in common in the processes occurring in short bridges ($2L < \eta$) and in longer ones ($2L > \eta$). The difference is that the distribution function in short bridges ceases to depend on τ_ε at voltages which are high as compared to V_c, and the anomalous current becomes of the order of magnitude of the Josephson current. In bridges with length $2L > \eta$, the distribution function reaches the limiting form (it does not depend on τ_ε) at voltages higher than ($T/\Delta)\, v_\varepsilon$, while the superconducting current increases as much as (L/η) times.

43.3.3 3.3. High voltages, $V > \Delta_0$; arbitrary temperatures

As we mentioned in the Introduction, at $V \gg \Delta_0$ the function G_1 can be found by taking the sum of a power series in Δ_0/V. The formulas for current in this case resemble the corresponding expressions in the case of tunneling junctions (Larkin and Ovchinnikov 1966, Werthamer 1966).

The oscillating components of the current has the form

$$I_\mathrm{osc} = I_1 \sin(2Vt) + I_2(V)\cos(2Vt), \tag{3.19}$$

where

$$I_1 \sim (\Delta_0/V)\,, \quad I_2 \sim (\Delta_0/V)\ln(V/\Delta_0) \quad \text{at} \quad \Delta \sim T.$$

Thus, the amplitudes of the Josephson current components in the bridge decrease with an increase of the voltage V.

As for the direct current I_DC through the bridge (contact), here differences appear with respect to the tunneling junctions. The expression for I_{DC} can be obtained by summing the series in formula (A.2.4). Also using the formula (3.4) after a number of rather tedious calculations we obtain:

$$I_\mathrm{DC} = \frac{1}{2R}\int \mathrm{d}\varepsilon N_\mathrm{s}(\varepsilon + v) N_\mathrm{s}(\varepsilon - v)$$
$$\times \left[D(\varepsilon + v) + D(\varepsilon - v) - 1\right]\left\{\tanh\left(\frac{\varepsilon + v}{2T}\right) - \tanh\left(\frac{\varepsilon - v}{2T}\right)\right\} \tag{3.20}$$

Where $v = V/2$, and

$$D(\varepsilon) = \left[\operatorname{arcsinh} f^{\mathrm{R}}(\varepsilon) - \operatorname{arcsinh} f^{\mathrm{A}}(\varepsilon)\right]\left[f^{\mathrm{R}}(\varepsilon) - f^{\mathrm{A}}(\varepsilon)\right]^{-1},$$
$$N_{\mathrm{s}}(\varepsilon) = \varepsilon\left(\varepsilon^2 - \Delta^2\right)^{-1/2} = \frac{1}{2}\left[g^{\mathrm{R}}(\varepsilon) - g^{\mathrm{A}}(\varepsilon)\right].$$

The expression (3.20) resembles that for the quasiparticle current through the tunnel junction. The integrand is the product of the density of states in the banks N_{s}, shifted with respect to each other along the energy scale by the value V, and the difference of the distribution functions, shifted by V as well. The difference from the tunneling junction consists in the appearance of the factor in the square brackets, which can be given the name of the "tunneling probability" $P(\varepsilon)$. The coefficient $P(\varepsilon)$ has a singularity at $|\varepsilon \pm v| < \Delta$, since here $f^{\mathrm{R}} = f^{\mathrm{A}}$, while the numerator in D differs from zero when tracing the points of branching from above (in f^{R}) and from below (in f^{A}). The product $N_{\mathrm{s}}(\varepsilon + v)N_{\mathrm{s}}(\varepsilon - v)P(\varepsilon)$ differs from zero at all energies, i.e. all the states contribute to the current, including those under the gap. This is clear from the physical point of view, since in the presence of a channel with direct conductivity, quasiparticles with energy $\varepsilon < A$ also contribute to the current. Such quasiparticles, moving in the bridge, transform themselves in the banks into Cooper pairs [in the pure case this process corresponds to the Andreev reflection (Andreev 1964)].

The contribution to conductivity of the states with the energy $|\varepsilon \pm v| \lesssim \Delta$ results in an excess current. Let us calculate the integral in (3.30), taking into account the fact that

$$\left(f^{\mathrm{R}} - f^{\mathrm{A}}\right) D(\varepsilon) = \ln\left(\frac{\varepsilon + \Delta}{\varepsilon - \Delta}\right)\theta(|\varepsilon| - \Delta) + \ln\left(\frac{\Delta + \varepsilon}{\Delta - \varepsilon}\right)\theta(\Delta - |\varepsilon|),$$
$$\left(g^{\mathrm{R}} - g^{\mathrm{A}}\right)\left(f^{\mathrm{R}} - f^{\mathrm{A}}\right)^1 = \frac{\varepsilon}{\Delta}\theta(|\varepsilon| - \Delta) + \frac{\Delta}{\varepsilon}\theta(\Delta - |\varepsilon|).$$
$$\theta(x) = 1, \quad x > 0 \quad \text{and} \quad \theta(x) = 0, \quad x < 0.$$

We obtain:

$$I_{\mathrm{DC}} = \frac{V}{R} + I_{\mathrm{exc}} \tanh\left(\frac{eV}{2T}\right), \quad I_{\mathrm{exc}} = \frac{\Delta_0}{eR}\left(\frac{1}{4}\pi^2 - 1\right) \tag{3.21}$$

The second term in (3.20) describes the excess current, which does not depend on voltage at $V > (\Delta_0, T)$. The value of I_{exc} is considerably higher than I_{c} near T_{c}, and $I_{\mathrm{exc}} \sim I_{\mathrm{c}}$ at $T \ll T_{\mathrm{c}}$. The formula (3.21) correlates well with experimental data (see ch. 1).

The excess current appears also in the contact $\mathrm{S} - \mathrm{c} - \mathrm{N}$ (c is the constriction) where its value is two times smaller (Artemenko et al. 1979b). In the case of a $\mathrm{S}_1 - \mathrm{c} - \mathrm{S}_2$ contact the gap Δ_0 in (3.21) should be replaced by $(\Delta_1 + \Delta_2)/2$ (Artemenko et al. 1981).

The expression (3.21) and the analysis carried out in the previous subsection, makes it possible to represent the form of the CVC of the short bridge or that of the contact at $V < A_0$ and $V > A_0$. Fig 6 shows the form of the CVC of the short bridge at $\Delta/T \ll 1$ and $\tau_\varepsilon V_{\mathrm{c}} \ll 1$. In this case, according to (3.9) at low voltages ($V < \Delta_0$) the CVC practically does not differ from the hyperbola obtained for the resistive model. At voltages $V > A_0$, the excess current appears and the form of the CVC is determined by (3.21).

43.3.4 3.4. Bridges with low impurity content (pure case)

A stationary Josephson phenomenon in pure contacts was considered by Kulik and Omelyanchuk (1977). They have assumed the following model of the contact: in a thin, impenetrable screen there exists an orifice with diameter d much smaller than the correlation length $\check{\zeta}(0)$ and the mean free path l. A theory of the nonstationary Josephson effect within the framework of this model was constructed by Zaitsev (1980)*. The conclusions of the theory in the pure case qualitatively agree with those obtained for "dirty" contacts; in

Fig. 6. Form of the current-voltage characteristic of a bridge of length $2L < \eta(T)$ in the high- and low-voltage regions; $\left(e \tau_{\varepsilon V_{\mathrm{c}}} \hbar^{-1}\right)<1,\left(\Delta_{0} / T\right)<1$). The dots show the CVC corresponding to the resistive model. The straight line shows Ohm's law.}

particular, at $V \ll \Delta$ and $\Delta \ll T$ the current is determined by eq. (3.6), in which the functional $P\{\chi_0\}$ has the form

$$P\{\chi_0\} = \frac{\sin(2\chi_0)}{2|\cos\chi_0|}\int_{-\infty}^{t} \mathrm{d}\theta \frac{\hat{\tau}\chi_0}{\partial\theta}\frac{\sin[2\chi_0(\theta)]}{|\cos\chi_0(\theta)|}\exp\left(\frac{\theta - t}{\tau_i}\right)$$

When $\lambda = \tau_\varepsilon V_{\mathrm{c}} \ll 1$, then the current equals

$$I = \frac{1}{R}\frac{\partial\chi_0}{\partial t}\left[1 + \lambda - \lambda\cos(2\chi_0)\right] + I_{\mathrm{c}}\sin(2\chi_0)\,.$$

If $\lambda \gg 1$, then at $V \gg \nu_\varepsilon$

$$I = \frac{\hbar}{eR}\frac{\partial\chi_0}{\partial t} + I_{\mathrm{c}} b \frac{\sin(2\chi_0)}{|\cos\chi_0|}$$

where the coefficient b is connected with the current I and voltage V by the following parametric relation

$$I = 2I_{\mathrm{c}} s b(s), \quad V = 2V_{\mathrm{c}}\sqrt{s^2-1}\,b(s), \tag{3.22}$$
$$b(s) = \frac{\sqrt{s^2-1}}{\pi}\ln\frac{s+1}{s-1}.$$

The formula (3.22) determines the CVC of the pure contact. Its form is shown in fig. 7. The hysteresis in the CVC of a pure contact for $\lambda \gg 1$ is more pronounced than it is in a "dirty" contact.

At higher voltages ($V \gg \Delta$) there is an excess current on the CVC and its value equals $I_{\mathrm{exc}} = \frac{8}{3}A/eR$.

Thus, in a pure contact the value of I_{exc} exceeds that in a "dirty" one.

[18]*In this work a theory of the pure S – c – N contact is also given. An excess current in a pure S -c-N contact was also calculated by Dorozhkin (1980) and Klapwijk et al. (1982) with the use of simple physical ideas.

Fig. 7. The shape of the CVC of a short bridge in the low-voltage range in the case of $e \tau_{\varepsilon V_{c} \hbar^{-1}>1$. The curve corresponds to numerical calculation using eq. (3.22) for the pure case; the straight dashed line shows Ohm's law.}

43.4 4. Discussion

The analysis carried out has shown that in S-c-S systems, in contrast to tunneling junctions, nonequilibrium processes are of importance. This is mainly due to the contribution to conductivity of quasiparticles confined in the bridge or contact, i.e. those quasiparticles whose energy ranges within the interval $\tilde{A} < \varepsilon < A_0$. Such quasiparticles exercise a bounded motion in the contact, in contrast to the electrons with energies $\varepsilon > \Delta_0$, which diffuse into the banks, where equilibrium is maintained. Therefore, under nonequilibrium conditions the distribution function of the confined electrons relaxes slowly due to inelastic electron-phonon collisions only. Hence, the contribution to the current by such nonequilibrium particles is large and depends on the energy relaxation time τ_ε. Phenomena which are the result of nonequilibrium processes qualitatively differ from each other, depending upon the relative magnitudes of the bridge length $2L$ and the characteristic parameter $\eta(T)$.

For short bridges ($2L < \eta$), the confined particles contribute to the current I which is described, at small voltages ($eV < A_0$), by the functional $P\{\chi_0\}$ [see eqs. (3.5) and (3.6)]. If the characteristic frequency of the Josephson oscillations $2eV_c/\hbar = 2eRI_c/\hbar$ is low as compared to the inverse energy relaxation time ${\tau_r}^1$, then the additional current $I^{(a)}$ caused by the confined quasiparticles is dissipative. Despite the fact that the current $I^{(a)}$ is proportional to τ_{e}, it only results in a small increase of the bridge conductivity in this case.

When the characteristic frequency of the Josephson oscillations $2eV_c/\hbar$ is high as compared to τ_ε^{-1}, then the anomalous current $I^{(a)}$ becomes nondissipative; it does not depend on the energy relaxation time and its value is of the order of the critical current I_{c}. Here the dependence of the nondissipative current I_5 on the phase difference $2\%_0$ differs from the sinusoidal one (it is sinusoidal in the stationary case). A small hysteresis appears on the CVC at small voltages.

The results mentioned above were obtained by Artemenko et al. (1979a). Schmid et al. (1980) made the same qualitative conclusions about the nature of the current $I^{(a)}$: dissipative for small V_{c} and nondissipative for higher V_{c}. However, the expression they obtained for the distribution function of the confined quasiparticles at higher V_c (as compared to $h/2e\tau_n$), differs from the one given in section 3.2, and this resulted in their conclusion about stimulation and appearance of a shoulder in the CVC in short bridges of length $2L < \eta$.

An analysis of the shape of the CVC for small voltages ($V \ll \tau_\varepsilon$) was performed by Golub (1976), who also obtained a weak increase of conductivity. However, in

this work he assumed the gradient terms to be small, so that the additional contribution to the bridge's conductivity contained a small parameter $[L/\xi(T)]^2$. His assumption can be justified only for longer bridges $2L > \eta(T)$. Besides, Golub did not take into account the anomalous current component $I^{(a)}$ which even for $2L \approx \eta(T)$ gives a greater contribution than the one he obtained.

The expression for the current I in short bridges ($2L < \eta$) at higher voltages ($eV > A_0$) resembles that for the current in tunneling junctions. The current components I_{osc} which oscillate in time decrease with an increase of V : $I_{\text{osc}} \sim (\Lambda_0/V)$ as in the case of tunneling junctions; but the situation for DC current differs from the case of tunneling junctions. Besides the Ohmic component V, R, the current I_{DC} in a short bridge contains the so-called excess current I_{exc} tanh(eV/2 T), where I_{exc} does not depend on V. Both the excess current and the anomalous current at small voltages are determined by the contribution of quasiparticles with energy $\varepsilon \lesssim \Delta_0$. In contrast to tunneling junctions, where the states under the gap $\varepsilon \leqslant \Delta_0$ do not contribute to the current I_{DC}, in the case of a bridge, all the states contribute to the current I_{DC}. This is due to the fact that, for a bridge, all the quasiparticles move under the action of the electric field, including those with energy $\varepsilon \leqslant \Delta_0$. The charge in the banks is transferred by Cooper pairs, while the quasiparticle current is converted into the pair current by the mechanism of Andreev reflection of quasiparticles (strictly speaking this is true only in the pure limit). In tunneling junctions, such a process is of the second order with respect to transparency (the electron must go through the barrier and return back in the form of a hole); therefore the current I_{exc} will appear in tunneling junctions in higher order approximations with respect to transparency. In the bridge the number of states with energy $\varepsilon \lesssim \Delta_0$ is proportional to Δ_0, so the excess current I_{exc} here is proportional to the order parameter value. This agrees with the experimental data (see ch. 1).

When the length of the bridge $2L$ exceeds the characteristic parameter $\eta(T)$. the nonequilibrium effects may lead to stimulation of superconductivity in the contact. The stimulation occurs in an external AC field or when the value of the current in the contact exceeds some critical value (stimulation by DC current). The cause of stimulation is strong energy diffusion due to the oscillations of the order parameter (gap) in the contact, which lead to a decrease of the number of the quasiparticles in the contact and respectively to the effective decrease of temperature. As a result, a specific shoulder appears on the current voltage characteristics of the contact, while the critical current in the AC field considerably increases.

We emphasize once more that the stimulation phenomenon occur only when the length of the bridge is sufficiently large ($2L > \eta(T)$). When the critical temperature T_{c} is approached, the parameter $\eta = (D, 1_0)^{1\,2}$ increases and this
condition is necessarily violated. Near T_{c} for $2L < \eta$ the CVC is close to a hyperbola (except for the high-voltage range, where excess current is present), while the critical current in the presence of an AC field decreases (see section 3.1). This agrees with the experimental data described in ch. 1 : the shoulder on the CVC and the stimulation of the critical current are observed only at a sufficient distance from T_{c} when $2L > \eta(T)$, and, as shown by Gubankov et al. (1977) and Latyshev and Nad' (1979), the two phenomena are correlated and manifest themselves simultaneously.

Let us also discuss some of the problems which, to our mind, are of interest for further study. The question of the agreement between the experimental and theoretical data, relating to the initial part of the bridge's CVC (when $2L > \eta$) is not quite clear. In order to find the current-voltage characteristic of the bridge in the low-voltage region, one can no longer use the limiting expression (2.26) for the distribution function. Here the voltage range where perturbation theory is valid is very small; in order to find the initial part of the CVC, one has to solve the set of equations (2.2), (2.5) and (2.7) simultaneously, which can not be done analytically.

The development of the theory for the high-frequency case ($\omega > D/L^2$) is also of interest; then a transition to the uniform nonequilibrium situation, considered by Eliashberg (1970) and Ivlev (1971) should occur.

The theory developed does not take into account the suppression of superconductivity by the AC current. This suppression is not significant in the range of AC radiation powers where the nonequilibrium phenomena are manifested to their full extent (due to a great energy relaxation time), but this may lead to current oscillation on further increase of the radiation power.

As for short bridges of length $2L < \eta$, the voltage region $eV = A_0, 2A_0$ etc. remains unstudied; here singularities may appear on the CVC. Besides, the effect of a high amplitude alternating signal on the bridge (e.g. its effect on the critical current) has not been studied.

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43.5 Appendix 1

The complete distribution function may be written as follows:

$$F = F_{00} + F_{01} + F_{10} + F_{11}, \tag{A1.1}$$

where

$$\begin{aligned} &F_{00} = F(\varepsilon) = \langle \bar{F} \rangle, \quad F_{01} = \overline{F - \langle F \rangle}, \\ &F_{10} = \langle F - \bar{F} \rangle, \qquad F_{11} = F - \bar{F} - \langle F \rangle + \langle \bar{F} \rangle, \end{aligned}$$

and the brackets $\langle \cdots \rangle$ mean averaging over the contact area in which $\Delta < \varepsilon$, while the overbar stands for averaging in time.

By analogy, one can write

$$\begin{aligned} &\left(\varepsilon^2 - \Delta^2\right)^{1/2} = \left(\varepsilon^2 - \Delta^2\right)_{00}^{1/2} + \left(\varepsilon^2 - \Delta^2\right)_{10}^{1/2} \\ &+ \left(\varepsilon^2 - \Delta^2\right)_{01}^{1/2} + \left(\varepsilon^2 - \Delta^2\right)_{11}^{1/2} \end{aligned} \tag{A1.2}$$

where the indices mean the appropriate averaging.

By substituting the expressions (A1.1) and (A1.2) into equation (2.6) and using the momentum presentation, we obtain:

$$\begin{aligned} &- Dk^2 \left(F_{01} + F_{11}\right) + \left(\frac{\partial}{\partial t} F_{10} + \frac{\partial}{\partial t} F_{11}\right) \frac{\partial}{\partial \varepsilon} \left(\varepsilon^2 - \Delta^2\right)^{1/2} \\ &- \frac{\partial F}{\partial \varepsilon} \left[\frac{\partial}{\partial t} \left(\varepsilon^2 - \Delta^2\right)_{10}^{1/2} + \frac{\partial}{\partial t} \left(\varepsilon^2 - \Delta^2\right)_{11}^{1/2}\right] = -I_{\text{ph}}. \end{aligned} \tag{A1.3}$$

By averaging this equation in time, we obtain

$$\begin{aligned} &-Dk^2 F_{01} + \overline{\left(\frac{\partial}{\partial t} F_{10} + \frac{\partial}{\partial t} F_{11}\right) \left[\frac{\partial}{\partial \varepsilon} \left(\varepsilon^2 - \Delta^2\right)_{10}^{1/2} + \frac{\partial}{\partial \varepsilon} \left(\varepsilon^2 - \Delta^2\right)_{11}^{1/2}\right]} \\ &- \overline{\left(\frac{\partial}{\partial \varepsilon} F_{10} + \frac{\partial}{\partial \varepsilon} F_{11}\right) \left[\frac{\partial}{\partial t} \left(\varepsilon^2 - \Delta^2\right)_{10}^{1/2} + \frac{\partial}{\partial t} \left(\varepsilon^2 - \Delta^2\right)_{11}^{1/2}\right]} = -I_{\text{ph}}. \end{aligned} \tag{A1.4}$$

In deriving eq. (A1.4) we used the condition $\bar{F}_{11} = 0$, and also

$$\overline{\left(\frac{\partial}{\partial t}F_{10}+\frac{\partial}{\partial t}F_{11}\right)\frac{\partial}{\partial\varepsilon}\left[(\varepsilon^2-\Delta^2)_{00}^{1/2}+(\varepsilon^2-\Delta^2)_{01}^{1/2}\right]}=0,$$
$$\overline{\frac{\partial}{\partial t}\left[(\varepsilon^2-\Delta^2)_{10}^{1/2}+(\varepsilon^2-\Delta^2)_{11}^{1/2}\right]\frac{\partial}{\partial\varepsilon}\left(F_{00}+F_{01}\right)}=0, \tag{A1.5}$$

the means of the total derivatives with respect to time.
By averaging eq. (A1.4) over coordinates and using the condition $\langle F_{11}\rangle=0$. we obtain

$$\frac{\partial}{\partial t}F_{10}\frac{\partial}{\partial\varepsilon}\left(\varepsilon^2-\Delta^2\right)_{10}^{1/2}-\frac{\partial}{\partial\varepsilon}F_{10}\frac{\partial}{\partial t}\left(\varepsilon^2-\Delta^2\right)_{10}^{1/2}$$
$$+\left\langle\frac{\partial}{\partial t}F_{11}\frac{\partial}{\partial\varepsilon}\left(\varepsilon^2-\Delta^2\right)_{11}^{1/2}\right\rangle-\left\langle\frac{\partial}{\partial\varepsilon}F_{11}\frac{\partial}{\partial t}\left(\varepsilon^2-\Delta^2\right)_{11}^{1/2}\right\rangle=-I_{\mathrm{ph}}. \tag{A1.6}$$

Using integration by parts, eq. (A1.6) may be written in the form

$$\frac{\partial}{\partial\varepsilon}\left[\frac{\partial}{\partial t}F_{10}\left(\varepsilon^2-\Delta^2\right)_{10}^{1/2}\right]-\frac{\partial}{\partial\varepsilon}\left\langle F_{11}\frac{\partial}{\partial t}\left(\varepsilon^2-\Delta^2\right)_{11}^{1/2}\right\rangle=-I_{\mathrm{ph}}. \tag{A1.7}$$

Let us show now that the first term in eq. (A1.7) equals zero (we find $\hat{\partial}F_{10}/\partial t$), and also find the expression for F_{11}. To effect this we have to subtract from (A1.3) the expression (A1.4) averaged in time and use the fact that the variable part of Δ is small. Then we obtain

$$-Dk^2F_{11}+\left(\frac{\partial}{\partial t}F_{10}+\frac{\partial}{\partial t}F_{11}\right)\frac{\partial}{\partial\varepsilon}\left(\varepsilon^2-\Delta^2\right)_{00}^{1/2}$$
$$-\frac{\partial}{\partial\varepsilon}F_{00}\left[\frac{\partial}{\partial t}\left(\varepsilon^2-\Delta^2\right)_{10}^{1/2}+\frac{\partial}{\partial t}\left(\varepsilon^2-\Delta^2\right)_{11}^{1/2}\right]=0 \tag{A1.8}$$

By averaging eq. (A1.8) over the coordinates, we obtain for $\partial F_{10}/\partial t$:

$$\frac{\partial}{\partial t}F_{10}=\frac{\partial}{\partial\varepsilon}F_{00}\left[\frac{\partial}{\partial t}\left(\varepsilon^2-\Delta^2\right)_{10}^{1/2}\right]\left[\frac{\partial}{\partial\varepsilon}\left(\varepsilon^2-\Delta^2\right)_{00}^{1/2}\right]^{-1}. \tag{A1.9}$$

Using expression (A1.9), one can transform the first term of eq. (A1.7.) as follows:

$$\frac{\partial}{\partial\varepsilon}\left[\overline{\frac{\partial}{\partial t}F_{10}\left(\varepsilon^2-\Delta^2\right)_{10}^{1/2}}\right]$$
$$=\frac{\partial}{\partial\varepsilon}\left\{\frac{1}{2}\frac{\partial}{\partial\varepsilon}F_{00}\left[\frac{\partial}{\partial\varepsilon}\left(\varepsilon^2-\Delta^2\right)_{00}^{1/2}\right]^{-1}\frac{\partial}{\partial t}\overline{\left[\left(\varepsilon^2-\Delta^2\right)_{10}^{1/2}\right]^2}\right\} \tag{A1.10}$$

The last expression equals zero, being the mean value of the derivative.
To find F_{11}, we subtract the same equation, averaged over coordinates, from (A1.8):

$$-Dk^2F_{11}+\frac{\partial}{\partial t}F_{11}\frac{\partial}{\partial\varepsilon}\left(\varepsilon^2-\Delta^2\right)_{00}^{1/2}-\frac{\partial}{\partial\varepsilon}F_{00}\frac{\partial}{\partial t}\left(\varepsilon^2-\Delta^2\right)_{11}^{1/2}=0 \tag{A1.11}$$

The second term in eq. (A 1.11) for the case of low frequencies $\omega\ll Dk^2$ is small compared to the first one. Therefore, we obtain:

$$F_{11}=-\left(Dk^2\right)^{-1}\frac{\partial}{\partial\varepsilon}F_{00}\frac{\partial}{\partial t}\left(\varepsilon^2-\Delta^2\right)_{11}^{1/2} \tag{A1.12}$$

By substituting eq. (A1.12) into eq. (A1.7), we obtain for the averaged function F_{00} :

$$-\frac{\partial}{\partial\varepsilon}\left\langle\frac{\partial}{\partial t}\left(\varepsilon^2-A^2\right)_{11}^{1/2}\left(Dk^2\right)^{-1}\frac{\partial}{\partial t}\left(\varepsilon^2-A^2\right)_{11}^{1/2}\right\rangle\frac{\partial}{i\varepsilon}F_{00}=I_{\mathrm{ph}} \tag{A1.13}$$

Near T_{c} at $\Delta<T$ the collision integral I_{ph} may be written in the form

$$I_{\mathrm{ph}}(F)=\tau_\varepsilon^{-1}\left[F(\varepsilon)-\tanh\frac{\varepsilon}{2T}\right]\overline{\left\langle\varepsilon\left(\varepsilon^2-\Delta^2\right)^{-1/2}\right\rangle} \tag{A1.14}$$

Finally, to find the dependence of the averaged distribution function $F(\varepsilon)=F_{00}$, we obtain eq. (2.7), where the energy diffusion coefficient D_i, is given by eq. (2.8).

43.6 Appendix 2

43.7 A2.1. Retarded (advanced) Green functions

As mentioned before, in short contacts ($2L<\eta$) the anisotropic parts of the Green functions $\hat{C}^{\mathrm{R}(A)}$ do not depend on x [see eqs. (3.2) and (3.3)]. They are related to the isotropic parts $\hat{G}^{\mathrm{R(A)}}$ as follows (see ch. 4)

$$\hat{C}=-l\hat{G}\frac{\partial\hat{G}}{\partial x}. \tag{A2.1}$$

To shorten the expression, the indices $R(A)$ are omitted here. The solution of eq. (A2.1) has the form

$$\hat{G}(x)=\hat{G}(0)\exp\left(-\hat{C}\frac{x}{l}\right). \tag{A2.2}$$

The relations of normalization (see ch. 4) imply, that the matrices $\hat{G}(0)$ and $\hat{C}$ anticommutate, while $(\hat{G}(0))^2=\hat{1}$. Let us put in (A2.2) $x=\pm L$, then add and substract the results obtained. This will give:

$$\hat{G}_+=\hat{G}(0)\cosh(\hat{C}L/l),\hat{G}_-=-\hat{G}(0)\sinh(\hat{C}L/l). \tag{A2.3}$$

Here $\hat{G}_\pm=\frac{1}{2}[\hat{G}(L)\pm\hat{G}(-L)]$. Let us multiply the matrices $\hat{G}_\pm$. Taking into account the anticommutation of the matrices $\hat{G}(0)$ and $\hat{C}$, we find

$$\hat{C}=-\frac{1}{2L}\operatorname{arcsinh}\left(2\hat{G}_+\hat{G}_-\right)=-\frac{1}{2L}\sum_{k=0}^{\infty}\alpha_{2k+1}\left(2\hat{G}_+\hat{G}_-\right)^{2k+1}. \tag{A2.4}$$

The matrices $\widehat{G}_\pm$may be easily calculated, if one notes that

$$\hat{G}(\pm L)=\hat{S}(\pm L)\hat{g}\hat{S}^+(\pm L), \tag{A2.5}$$

where, for example,

$$\hat{g}^R=g^R\dot{\sigma}_z+f^R\dot{\hat{\sigma}}_i,\quad g^R=\frac{\varepsilon}{\Delta_0}f^R=\varepsilon,\zeta^R,\quad \zeta^R=\sqrt{(\varepsilon+\mathrm{i}0)^2-A_0^2}.$$

i.e. $\hat{g}^{\mathrm{R}}$ is the equilibrium retarded Green function with the real order parameter. The matrix $\hat{S}$ makes it possible to take into account the phase difference in the bridge

$$\hat{S}(\pm L)=\cos\left(\chi_0/2\right)\pm\mathrm{i}\hat{\sigma}_z\sin\left(\chi_0/2\right). \tag{A2.6}$$

Considering the phase $\%_0$ as slowly changing with time and using eq. (A2.6), we find

$$\hat{G}_+ = g\hat{\sigma}_z + \mathrm{i}\hat{\sigma}_y f\cos\chi_0, \quad \hat{G}_- = \mathrm{i}\hat{\sigma}_x f\sin\chi_0 \tag{A2.7}$$

where $\hat{\sigma}_{x,y}$ are the Pauli matrices. Using the condition that $\left(\hat{G}_+\hat{G}_-\right)^2$ is a unit matrix, one can easily calculate the sum in eq. (A2.4), obtaining

$$\hat{C}^{\mathrm{R(A)}} = C_z^{\mathrm{R(A)}}\hat{\sigma}_z + C_y^{\mathrm{R(A)}}\hat{\sigma}_y, \quad C_z = -\mathrm{i}\frac{l}{L}\frac{\Delta_0\cos\chi_0}{\tilde{\zeta}}A = -\frac{\mathrm{i}\Delta_0\cos\chi_0}{\varepsilon}C_y \tag{A2.8}$$

where

$$A = \frac{1}{2}\operatorname{arcsinh}\left(\frac{\xi}{\zeta}f\sin\chi_0\right) = \operatorname{arcsinh}\left(f\sin\chi_0\right)$$
$$\tilde{\zeta}^{\mathrm{R}} = \sqrt{(\varepsilon+\mathrm{i}0)^2 - \tilde{\Delta}^2}; \quad \tilde{\Delta} = \Delta_0\left|\cos\chi_0\right|$$

The formula for $\hat{C}^{\mathrm{R(A)}}$ was first obtained by Kulik and Omelyanchuk (1975), Mitsay (1976) and also, using another method (presented here) by Artemenko et al. (1979a).

Let us now determine the isotropic parts of $\hat{G}^{\mathrm{R(A)}}(x)$, through which the reduced density of states $N(\varepsilon)$ is expressed. Let us represent the exponent in eq. (A2.2) in the form of a series. Working in the same way as we did in finding' $\hat{C}^{\mathrm{R(A)}}$, we obtain

$$\hat{G}(x) = \hat{G}(0)\left[\cosh\left(A - \frac{x}{L}\right) + \frac{-\varepsilon\hat{\sigma}_y + i\hat{\sigma}_z\Delta\cos\chi_0}{\tilde{\zeta}}\sinh\left(A\frac{x}{L}\right)\right]. \tag{A2.9}$$

The matrix $G(0)$ [as it follows from (A2.9)] is related to $G_+$by

$$\hat{G}(0) = \hat{G}_+\cosh A = \frac{\zeta}{\zeta}\hat{G}_+ \tag{A2.10}$$

Using eqs. (A2.9) and (A2.10), we determine the density of states

$$N(\varepsilon) = \frac{1}{2}\operatorname{Tr}\hat{\sigma}_z\left(\hat{G}^{\mathrm{R}} - \hat{G}^{\mathrm{A}}\right) = \frac{\varepsilon}{2\zeta^{\mathrm{R}}}\cosh\left(A^{\mathrm{R}}\frac{x}{L}\right) - \frac{\varepsilon}{2\zeta^{\tilde{\mathrm{A}}}}\cosh\left(A^{\mathrm{A}}\frac{x}{L}\right) \tag{A2.11}$$

We are interested in energies $|\varepsilon| > \tilde{\Delta}$ (since at $|\varepsilon| < \tilde{\Delta}, N(\varepsilon) = 0$); then

$$\tilde{\zeta}^{\mathrm{R}} = -\tilde{\zeta}^{\mathrm{A}} = \sqrt{\varepsilon^2 - \tilde{\Delta}^2}\operatorname{sgn}\varepsilon.$$

If the energy exceeds the gap in the banks $|\varepsilon| > \Delta_0$, then $A^{\mathrm{R}} = -A^{\mathrm{A}} \equiv a\operatorname{sgn}\varepsilon$, where a is determined in (1.16). Therefore, we have

$$N(\varepsilon) = \frac{|\varepsilon|}{\sqrt{\varepsilon^2 - \tilde{\Delta}^2}}\cosh\left(a\frac{x}{L}\right) \tag{A2.12}$$

When energy ranges within the interval (1.15), then for finding A the branching points should be followed in A^{R} from above, while in A^{A} they should be followed from below. Then we obtain

$$A^{\mathrm{R}} = -\left(A^{\mathrm{A}}\right)^* = a - \mathrm{i}\frac{\pi}{2}\operatorname{sgn}\left(\sin\chi_0\right).$$

This taken into account, we find from eq. (A2.11) the expression (1.16) for $N(\varepsilon)$.

43.8 A2.2. The Green function $\hat{G}_1$. Distribution functions

For finding the $\hat{G}_1 = \hat{C}$ matrix independent of x (see eq. 3.3), we use the normalization relationship

$$\hat{G}^{\mathrm{R}}\hat{C} + \hat{C}\hat{G}^{\mathrm{A}} = -\left(\hat{C}^{\mathrm{R}}\hat{G} + \hat{G}\hat{C}^{\mathrm{A}}\right) \tag{A2.13}$$

Expression (A2.13) should be written down at the points $x = \pm L$, where all the functions, except $\hat{C}$, are known. There are several ways of determining $\hat{C}$. It is convenient to present C in the form

$$\hat{C} = \hat{C}^{\mathrm{R}}\hat{F} - \hat{F}\hat{C}^{\mathrm{A}}, \tag{A2.14}$$

where matrix $\hat{F}$ also does not depend on x and has the form $\hat{F} = F_x\hat{\sigma}_x + F_1\hat{1}$. Let us substitute eq. (A2.14) into eq. (A2.13) and take into account the anticommutation of G^{R} and C^{R}. Then we obtain:

$$\hat{C}^{\mathrm{R}}(\hat{B} - \hat{G}) + (\hat{B} - \hat{G})\hat{C}^{\mathrm{A}} = 0, \tag{A2.15}$$

where $\hat{B} = \hat{G}^{\mathrm{R}}\hat{F} - \hat{F}\hat{G}^{\mathrm{A}}$. The determinant of eq. (A2.15) composed of the elements of the matrix $\hat{C}^{R(A)}$ is not equal to zero. This implies that

$$\hat{G}^{\mathrm{R}}\hat{F} - \hat{F}\hat{G}^{\mathrm{A}} = \hat{G} = \hat{G}^{\mathrm{R}}\hat{F}_0 - \hat{F}_0\hat{G}^{\mathrm{A}}. \tag{A2.16}$$

Here we introduced the matrix $\hat{F}_0$, which is connected at $x = \pm L$ with the equilibrium distribution function in the banks, $\hat{F}_0 = \hat{S}(t)F_{\mathrm{eq}}\,(t - t')\,S^{+}\,(t')$. The Fourier image $F_{\mathrm{eq}}\,(t - t')$ equals $F_{\mathrm{eq}}\,(\varepsilon) = \tanh(\varepsilon/2T)$.

Let us add and substract the equations (A2.16) at the points $x = \pm L$. Then we obtain:

$$\hat{G}^{\mathrm{R}}_{\pm}\hat{F}^{(\mathrm{a})} - \hat{F}^{(\mathrm{a})}G^{\mathrm{A}}_{\pm} = \hat{G}^{\mathrm{R}}_{\pm}\hat{F}_{-} - \hat{F}_{-}\hat{G}^{\mathrm{A}}_{\mp}, \tag{A2.17}$$

where

$$\hat{F}^{(a)} = \hat{F} - \hat{F}^{(r)}, \quad \hat{F}^{(r)} = \hat{F}_{+}, \quad \hat{F}_{+} = \frac{1}{2}\left[\hat{F}_0(L) \pm \hat{F}_0(-L)\right].$$

The regular function $\hat{F}^{(r)}$ determines the superconducting current, while the anomalous one $\hat{F}^{(a)}$ stands for the quasiparticle current. The functions $\hat{F}_{\pm}$are easily found at $V \ll \Delta_0$ with the use of (A2.6).

$$\hat{F}_{+} = \hat{1}\tanh(\varepsilon/2T), \quad \hat{F}_{-} = \mathrm{i}\hat{\sigma}_2\frac{1}{4T}\frac{\partial\chi_0}{\partial t}\cosh^{-2}\left(\frac{\varepsilon}{2T}\right). \tag{A2.18}$$

Let $|\varepsilon| > \Delta_0$. Consider the version of equation (A2.17) with the minus signs in the left-hand side. Calculating the trace we find:

$$F^{(\mathrm{a})}_x = -\frac{i\varepsilon}{4T\Delta_0\sin\chi_0}\frac{\partial\chi_0}{\partial t}\cosh^{-2}\left(\frac{\varepsilon}{2T}\right) \tag{A2.19}$$

Multiplicating eq. (A2.17) by σ_γ and calculating the trace gives $F^{(a)}_1 = 0$. For the function $C^{(a)}_z$ determining the current (3.4), we obtain from (A2.14) and (A2.8): $C^{(\mathrm{a})}_z = -\mathrm{i}\left(C^{\mathrm{R}}_v + C^{\mathrm{A}}_v\right)F^{(\mathrm{a})}_x$.

Integrating in (3.4) over the region $|\varepsilon| > \Delta$ we obtain the Ohmic current $V'R$.
Let energy range within the interval (1.15). We consider the version of eq. (A2.17) with the plus signs in the left-hand side. Multiplying this equation by $\hat{\sigma}_z$ and calculating the trace, then multiplying by $\hat{\sigma}_1$ and calculating the trace once more, we see that the left-hand side contains

the coefficients of $F_1^{(a)}$ and $F^{(\mathrm{a})}$ which are factors of the type ($f^{\mathrm{R}} - f^{\mathrm{A}}$), which equal zero within the energy interval considered, when no energy relaxation is taken into account. To obtain the final value of $F_1^{(a)}$ and $F_x^{(a)}$, we take into account the damping in the spectrum: then $f^{\mathrm{R}} - f^{\mathrm{A}} = iv_\varepsilon(\hat{\partial} f, \hat{\partial}\varepsilon)$. Besides, one should perform the quasiclassical expansion in the left-hand side. For example:

$$\hat{G}_-^{\mathrm{R}}\hat{F}^{(\mathrm{a})} = \hat{G}_-^{\mathrm{R}}(\varepsilon)\hat{F}^{(\mathrm{a})}(\varepsilon) + \frac{1}{2\mathrm{i}}\left\{\frac{\partial\hat{G}^{\mathrm{R}}}{\partial t}\frac{\partial\hat{F}^{(\mathrm{a})}}{\partial\varepsilon} - \frac{\partial\hat{G}^{\mathrm{R}}}{\partial\varepsilon}\frac{\partial\hat{F}^{(\mathrm{a})}}{\partial t}\right\}$$

Then, after simple transformations, we obtain the equation for the function $\widetilde{F}_1^{(a)} = F_1^{(a)}\exp(v_t t)$:

$$\begin{aligned}\frac{\partial\tilde{F}_1^{(\mathrm{a})}}{\partial t} &+ \frac{\partial(\ln\sin\chi_0)}{\partial t}\left[\frac{\partial\ln\sqrt{\Delta_0^2-\varepsilon^2}}{\partial\varepsilon}\right]^{-1}\frac{\partial\tilde{F}_1^{(\mathrm{a})}}{\partial\varepsilon}\\ &= -\frac{1}{2T}\frac{\partial\chi_0}{\partial t}\frac{\cot\chi_0}{\cosh^2(\varepsilon/2T)}\cdot\left[\frac{\partial\ln\sqrt{\Delta_0^2-\varepsilon^2}}{\partial\varepsilon}\right]^{-1}\end{aligned} \tag{A2.20}$$

From (A2.20) we find the expression for $\tilde{F}_1^{(a)}$ and hence also for $F_1^{(a)}$:

$$F_1^{(a)} = \frac{\Delta_0}{4T}\cosh -2\left(\frac{\varepsilon}{2T}\right)\int_{-\infty}^{t}\mathrm{d}\theta\frac{\partial\chi_0(\theta)}{\partial\theta}\frac{\sin[2\chi_0(\theta)]\exp[\theta-t)\tau]}{|\kappa|\sqrt{\kappa^2-\sin^2\chi_0(\theta)}} \tag{2}$$

where $\kappa = [\Delta_0\sin\chi_0(t)]/\sqrt{\Delta_0^2-\varepsilon^2}$. The function $F_x^{(a)}$ is small as compared with $F_1^{(a)}$ with regard to the parameter $\max((1,1,1)1_0$. Using the expressions (A2.21). (A2.14) and (A2.8) we find from eq. (3.4) the anomalous current $I^{(1)}$ (see eq. 3.5).

To determine the distribution functions we consider eq. (A2.16) and present the matrix. $\hat{F}$ in the form of eq. (3.1). Let us substitute the expression for $\hat{G}^{R(A)}(x)$ from (A2.9) into (A2.16) and equalize the coefficients of the matrices
$\dot{\sigma}_1(k=1,x,y,z)$. Then we obtain:
(a) $|\varepsilon| > \Lambda_0$. Then

$$F^{(T)} = \frac{1}{4T}\frac{i\%}{\hat{C}t}\frac{\sqrt{\varepsilon^2-\Delta_0^2}}{\Delta_0\sin\%}\sqrt{\frac{\varepsilon^2-\tilde{\Delta}^2}{\varepsilon^2-\Delta_0^2}}\frac{\tanh(a(x/L))}{\cosh^2(\varepsilon/2T)}.$$

The function $F^{(L)} = \tanh(\varepsilon/2T)$, i.e. it remains the equilibrium one.
(b) $\tilde{A} < |\varepsilon| < A_0$. Then

$$F^{(L)} - \tanh(\varepsilon/2T) = F_1^{(a)}.$$

ard F; is determuned by expression (A2.21). The function F at the energers under consideration is small.

Part XIII
Main Experiments on Nonequilibrium Superconductivity

44 Dynamical Effects in Nonequilibrium Superconductors: Some Experimental Perspectives by Kadin, Goldman

44.1 1. Introduction

Dynamical processes on a microscopic level play an essential part in all major phenomena in condensed matter physics. What is more remarkable in the case of superconductivity is that these microscopic processes are strongly coupled to the macroscopic order parameter. It is the interaction between dynamics on these two levels that gives rise to the great richness of phenomena in nonequilibrium superconductivity.

In order to obtain a clear picture of the nature of the nonequilibrium state, it is important to consider the equilibrium state and how it is maintained. For many purposes, a superconductor can be considered a coupled system of three components: the condensate, the quasiparticles, and the phonons (see fig. 1). The ground state condensate of the Cooper pairs carries all of the "super" properties of superconductivity, and is characterized by a complex order parameter, or equivalently as will be shown below, by the magnitude of the energy gap A and the condensate chemical potential $\varepsilon_{..}$At a temperature of absolute zero, the condensate constitutes the entire system, but for higher T, excitations above the condensate appear as quasiparticles. Since the quasiparticles are charged (either electron-like or hole-like), charge as well as energy can be transferred between the condensate and the quasiparticles. The quasiparticle system is characterized by a distribution function $f(6)$. which plays a central role in nonequilibrium superconductivity. Finally, the quasiparticles are coupled to the phonons, and these two systems can exchange energy, but not, of course. charge. We recognize that the existence of the Cooper pairs

Fig. 1. Schematic picture of a superconductor. comprised of the condensate the quasiparticles and the phonons. Different types of probes and perturbations couple directly to the different subsystems of the superconductor.

requires the exchange of virtual phonons, but for our purposes, the condensate and the phonons can be considered as not directly interacting.

Thermal equilibrium is maintained by allowing the phonons in the superconductor to exchange energy with a thermal bath at temperature T (in practice, usually via phonon coupling to some other system or material). The phonons in turn interact with the quasiparticles via inelastic electron-phonon scattering, maintaining a thermal distribution of quasiparticles. The superconducting condensate carries no entropy, and thus is effectively at $T = 0$. However, it is coupled to the quasiparticles by the BCS gap equation and by overall charge neutrality, so that in equilibrium a thermal value $\Delta(T)$ results.

Although we will identify a generalized two-fluid picture from the above separation of condensate and quasiparticles, we point out that this is distinctly different from the old two-fluid picture where some of the electrons were effectively superconducting and the rest were normal (Gorter and Casimir 1934). Here, in thermal equilibrium, all of the electronic charge is contained in the condensate, and the net charge in the quasiparticles is zero, there being equal numbers on the electron-like and the hole-like branches of the quasiparticle spectrum. This requirement need not hold in the nonequilibrium state, leading to the concept of quasiparticle charge imbalance [discussed in the review in this volume by Clarke (1985)], which has considerable bearing on several aspects of nonequilibrium superconductivity to be discussed in the present chapter.

Generally speaking, any mechanism that couples to a system so as to probe its state can also be used to perturb it from its equilibrium condition. Often, the same device is used both as a perturbation and as a probe, or in other language as both the injector and the detector. In our schematic picture of superconductivity in fig. 1, we have listed some of the major types of probes and the subsystems with which they directly interact. Supercurrents and pairtunneling couple to the condensate, while quasiparticle currents and tunneling affect the quasiparticle distribution. Electromagnetic fields couple to both electronic components, while phonon injection and external heating enter the superconductor via the phonon system.

The various types of external perturbations can also be classified in another way, according to whether their primary effect is to transfer charge or energy to the superconductor. Energy-mode perturbations (or "condensation processes") ultimately affect the magnitude of the energy gap Δ, and are similar in some respects to heating and cooling. Phonons and light are of this class, and quasiparticle tunneling has a strong energy-mode component. Charge-mode perturbations ("conversion processes") couple to electrochemical potentials, currents, and the phase of the superconducting order parameter, and produce the charge imbalance mentioned above. Examples are external currents and tunneling injection. Another equivalent set of terminology for these two modes is longitudinal and transverse (Schmid and Schön 1975a), according to
whether the induced change in the complex order parameter represented in the complex plane is parallel to (a change in the magnitude) or perpendicular to (a change in the phase) the equilibrium order parameter.

Most external perturbations can be applied in either a steady-state (DC) mode, a modulated (AC) mode, or a transient (pulsed) mode. In all of these cases, what is generally inferred is a relaxation time of the quantity being measured, usually a macroscopic variable. This is typically related to the characteristic time of the underlying microscopic process, but the macroscopic and microscopic times need not be the same. Furthermore, the DC relaxation time may even be different from the transient response time for the same physical quantity, for example the energy gap A. In addition, if spatially resolved measurements yield input on characteristic lengths as well, a more complete dynamical equation can be inferred.

One can also apply a high frequency AC field to a superconductor in order to couple to an AC collective mode or another nonequilibrium wave phenomenon. In fact, if one makes use of the AC Josephson effect, one can obtain this sort of AC coupling by applying a DC field. One can therefore use a Josephson junction (or other weak link) both to induce and to detect an AC nonequilibrium perturbation, via an analysis of the DC response. This is the basis for the measurements of the "pair-field susceptibility" (Ferrell 1969, Scalapino 1970), where a DC electromagnetic field gives rise to an effective pair field that is modulated in time and space. The resulting modulated nonequilibrium perturbation is measured using a DC supercurrent.

One of these collective modes, associated with a charge-mode perturbation (Schmid and Schön 1975b, Artemenko and Volkov 1975), has been referred to as a "charge imbalance wave". One can make use of a quantitative analogy between these waves in a generalized two-fluid

picture of superconductivity, and electromagnetic waves propagating down a transmission line (Kadin et al. 1980), to provide insight into a range of phenomena in nonequilibrium superconductivity. One particularly interesting application is to the phase-slip center (Rieger et al. 1972 and Skocpol et al. 1974a), involving the currentinduced breakdown of superconductivity in a quasi one-dimensional filament.

Another superconducting system which is rich in dynamical nonequilibrium phenomena is the microbridge, a short superconducting bridge coupling two superconducting banks essentially in equilibrium. The very large local current densities in the bridge region drive the superconducting order parameter into oscillations at the Josephson frequency, leading to a series of effects, many of which are evident in the DC $I-V$ characteristics. Because of these driven oscillations, closely-spaced microbridges or phase-slip centers also interact in a way that can lead these oscillations to become synchronized.

A set of recent reviews of the field of nonequilibrium superconductivity, in addition to those in the present volume, may be found in the proceedings of the 1980 Maratea summer school, edited by Gray (1981a). Other reviews that
complement or overlap the present treatment are those by Pals et al. (1982), Lindelof (1981), Pethick and Smith (1979, 1980), Artemenko and Volkov (1979), Tinkham (1979), Likharev (1979), Aronov and Spivak (1978), Chang and Scalapino (1978), Dmitriev and Khristenko (1978), Kaplan et al. (1976); Entin-Wohlman and Orbach (1978, 1979), and Langenberg (1974).

We continue the chapter in section 2 with a review of the theoretical framework within which we will work, with an emphasis on the quasiparticle distribution function and dynamical equations of the order parameter, working up to a discussion of charge imbalance waves. In section 3, we show how the $I-V$ characteristics of a Josephson tunnel junction can be used to determine the order parameter dynamics, and the significance of the propagating collective mode thus inferred. We follow this in section 4 with the development of a simple dynamical model for the phase-slip center, and analyze the characteristics of microbridges, both single and in combination, with an emphasis on those features that can be explained via simple semiphenomenological pictures. Finally, the chapter closes in section 5 with a brief summary and a discussion of the potential impact of nonequilibrium superconductivity on related work in the future, both basic and applied.

44.2 2. Basic theory of nonequilibrium superconductors

44.2.1 2.1. Quasiparticle distributions and self-consistency equations

For a simple BCS-superconductor with a gap A, the condensate consists of an assemblage of momentum-paired electron states ($k,-k$), with an occupation fraction v_k^2 of electron k-states (see fig. 2a) given by

$$v_k^2 = \left(1-\xi_k/\varepsilon_k\right)/2, \tag{2.1}$$

where $\xi_k = \varepsilon_k - \varepsilon_s$ is the (normal) electronic kinetic energy referred to the condensate chemical potential ε_s, and $E_k = \left[\xi_k^2 + A^2\right]^{1/2}$ is the quasiparticle excitation energy (fig. 2b). In equilibrium, ε_s coincides with the normal electronic Fermi energy ε_{F}, but this need not always be so. It is important to realize that the energy smearing in eq. (2.1) (i.e. the existence of empty states below ε_{s} and filled states above, even at $T=0$) is an essential feature of the BCS ground state, in that it allows Cooper pairs near ε_s to scatter coherently around the Fermi sphere via exchange of virtual phonons, thus lowering their energies.

A quasiparticle excitation consists of a pairing state ($k,-k$) with one and only one of the two states occupied, thereby making it unavailable for pairing. Whether this quasiparticle

represents an electron or a hole (or something in between) depends on the likelihood that this paired state would otherwise have been occupied in the condensate. For $\xi_k \ll 0$, both states would have been

Fig. 2. The effects of a shift in the condensate chemical potential $\varepsilon_{\mathrm{s }}$ on the condensate and on the quasiparticles. The dashed lines indicate equilibrium, the solid lines disequilibrium, and the dotted line in (c) local equilibrium. (a) v_k^2 gives the occupation of electrons in the condensate. (b) A shift in ε_{s} shifts the excitation spectrum, but in the absence of inelastic collisions, a given quasiparticle remains at the same value of ε_k but changes its excitation energy E_k and its effective charge q_k. (c) If the nonequilibrium quasiparticles can be characterized by a quasi-equilibrium distribution $f_0\left(\varepsilon_k - \varepsilon_n\right)$ around a "normal" chemical potential ε_n, this is actually closer to the global equilibrium distribution $f_0\left(\varepsilon_k\right)$ than to the local equilibrium distribution $f_0\left(\varepsilon_k - \varepsilon_s\right)$. The extent of the disequilibrium in the figures here is exaggerated for clarity. (From Kadin et al. 1980.)}

occupied, making the quasiparticle a hole, for $\xi_h \gg 0$ the quasiparticle is likewise clearly an electron. For $\left|\check{\zeta}_k\right| < \Delta$, the paired state may or may not be in the condensate, giving the quasiparticle a mixed character with effective charge

$$q_k = e\left[v_k^2 - \left(1 - v_k^2\right)\right] = e\xi_k/\varepsilon_k. \tag{2.2}$$

The occupation of quasiparticle states is given by the distribution function $f\left(\varepsilon_k\right)$, which in thermal equilibrium is given by the Fermi function

$$f_0\left(\varepsilon_k\right) = \left[1 + \exp\left(E_k/k_{\mathrm{B}}T\right)\right]^{-1}. \tag{2.3}$$

Note that the form of this equilibrium distribution depends implicitly through E_k on the condensate parameters Λ and ε_s. These are determined from $f\left(\varepsilon_k\right)$ using two

self-consistency equations, the first of which (the BCS gap equation) follows from energy considerations, and the second (the condensate potential equation) from charge conservation.

The BCS gap equation is usually presented in the form

$$1/V = \sum_k (1 - 2f_k)/2E_k = N(0) \int_0^{\hbar\omega_c} \mathrm{d}\varepsilon (1 - 2f(\varepsilon))/E \tag{2.4}$$

where Δ is contained implicitly in E_k. The upper limit $\hbar\omega_\mathrm{c}$ is a necessary cut-off to prevent the integral from diverging, where ω_c is of the order of the phonon Debye frequency. Here $N(0)$ is the normal electronic density of states at the Fermi level for one spin, and we have assumed that $f(\varepsilon_k)$ is isotropic in momentum space. The gap equation can be written in a more tractable form if we note that

$$\int_0^{\hbar\omega_\mathrm{c}} \frac{\mathrm{d}\varepsilon}{E} \simeq \ln(2\hbar\omega_\mathrm{c}) - \ln\Delta, \tag{2.5}$$

for $\Delta \ll \hbar\omega_\mathrm{c}$. Since $f(\varepsilon) = 0$ for the case of $\Delta(T = 0)$, we obtain

$$-\ln[A/\Delta(0)] = \int_{-\alpha}^{\alpha} \mathrm{d}\varepsilon f(\varepsilon)/\left[(\varepsilon - \varepsilon_\mathrm{s})^2 + \Delta^2\right]^{1/2}, \tag{2.6}$$

where now the integral is convergent, so that the cut-off is unnecessary.
For the condensate potential equation, we first define the total charge density in the quasiparticles as the charge imbalance (also sometimes referred to as "branch imbalance") (Tinkham and Clarke 1972)

$$Q^* = 2N(0) \int \mathrm{d}\varepsilon q(\varepsilon) f(\varepsilon) \tag{2.7}$$

In equilibrium, $Q^* = 0$, since there are equal numbers of excitations on the electron-like and the hole-like branches of the quasiparticle spectrum. This need not be true for a nonequilibrium state, but the total electronic charge must still be essentially zero. This is true because charge neutrality is a metallic property that holds over far shorter times ($\varepsilon_0/\sigma \sim 10^{-19}$ s) and far shorter distances (the Thomas-Fermi screening length ~ 1Å) than those relevant for superconductivity. The remaining charge must lie in the condensate, which can only occur if the condensate chemical potential ε_s has been shifted from its equilibrium value (Waldram 1975, Volkov and Zaitsev 1975):

$$Q^* = 2N(0)e \int \left(v^2\big|_{\varepsilon_\mathrm{s}=0} - v^2\big|_{\varepsilon_\mathrm{s}=0} \right) \mathrm{d}\varepsilon = -2N(0)e\varepsilon_\mathrm{s}. \tag{2.8}$$

Then

$$-\varepsilon_\mathrm{s} = \int_{-\infty}^{\infty} \mathrm{d}\varepsilon f(\varepsilon)(\varepsilon - \varepsilon_\mathrm{s}) / \left[(\varepsilon - \varepsilon_\mathrm{s})^2 + \Lambda^2\right]^{1/2} \tag{2.9}$$

eq. (2.9) is the condensate potential equation. Note the similar self-consistent natures of eqs. (2.6) and (2.9). Thus, the condensate is sensitive to what are in effect two moments of the quasiparticle distribution $f(\varepsilon)$. At least for small deviations from equilibrium, the gap Δ depends on the even part of $f(\varepsilon)$ about ε_s, and the potential ε_s depends on the odd part.

The key to understanding nonequilibrium superconductors lies in the realization that while the consistency equations (2.6) and (2.9) should be valid down to times of order $h/\Delta \sim 10^{11} - 10^{-12}$ s (to establish the superconducting pairing), the quasiparticle distribution $f(\varepsilon)$ can change only on the much slower time scale of the inelastic electron-phonon collision time $\tau_\mathrm{E} \sim 10^{-7} - 10^{-10}$ s (depending on the material). Therefore, although in principle Δ and ε_s

can change very quickly, in practice they are locked into instantaneous quasiparticle distributions which are changing more slowly. A similar relation also holds in the spatial domain. It is important to note, however, that this entire approach breaks down in the case of a gapless superconductor (Gor'kov and Eliashberg 1968) (for $T > T_c$, or for a high concentration of magnetic impurities), where the condensate becomes largely decoupled from $f(\varepsilon)$, and the order parameter relaxation becomes governed by intrinsic condensate times (e.g. τ_{GL}, the Ginzburg-Landau relaxation time near T_c)

In principle, the distribution $f(\varepsilon)$ has as many degrees of freedom as there are quasiparticles. In practice, a small number of simple approximate forms appear to be relevant in many cases. For example, an energy-mode perturbation can simply heat up the quasiparticles, suggesting a distribution characterized by an effective electronic temperature $T^* > T$ (Parker 1975). This does not require that the phonons also be at T^*, since it is a nonequilibrium state. This distribution has been found to apply in a number of experimental situations. An alternative distribution has been proposed for an increased number of quasiparticles, which is characterized by the equilibrium T but a systematic shift in the excitation energy E by μ^* (Owen and Scalapino 1972), but a distribution purely of this type has not been seen experimentally.

For the charge-mode distribution, the simplest nonequilibrium distribution (Pethick and Smith 1979) is one given by

$$f_0\left(\varepsilon-\varepsilon_{\mathrm{n}}\right)=\left\{1+\exp\left(\left[\left(\varepsilon-\varepsilon_{\mathrm{n}}\right)^2+\Delta^2\right]^{1/2}/k_{\mathrm{B}}T\right)\right\}^{-1} \tag{2.10}$$

This is a thermal distribution, but one which as a function of electron kinetic energy ε has its maximum at a value ε_n, the effective normal chemical potential, different from both the "global equilibrium" value $\varepsilon_{\mathrm{r}}=0$, and the "local equilibrium" (with the nonequilibrium condensate) ε_s (see fig. 2c). One can
estimate the relative sizes of ε_{n} and ε_{s} (which must have opposite signs from charge conservation) by evaluating the charge imbalance Q^* in a Taylor series expansion in $(\varepsilon_s-\varepsilon_n)$, and retaining only the lowest order term. For T close to T_{c}, one has

$$Q^*=2N(0)\int \mathrm{d}\varepsilon q\left(\varepsilon-\varepsilon_{\mathrm{s}}\right) f_0\left(\varepsilon-\varepsilon_{\mathrm{n}}\right)\approx\left(\varepsilon_{\mathrm{s}}-\varepsilon_{\mathrm{n}}\right) 2N(0)e\left(1-\pi\Delta/4k_{\mathrm{B}}T\right) \tag{2.11}$$

Comparing this to eq. (2.8), from electroneutrality we have, to lowest order in $\Lambda/k_{\mathrm{B}}T$, that

$$\varepsilon_{\mathrm{n}}\approx-\varepsilon_{\mathrm{s}}\left(\pi\Delta/4k_{\mathrm{B}}T\right). \tag{2.12}$$

Therefore, near T_c, when Δ is very small and there are many quasiparticles, a small shift in ε_{n} will require a very much larger shift in ε_{s} in the other direction in order to maintain electroneutrality.

There are a number of ways experimentally to induce a nonequilibrium value of $f(\varepsilon)$ rather directly. If a voltage is applied across a tunnel barrier between a superconductor and a normal metal, quasiparticles of one sign will tunnel into the superconductor, and quasiparticles of the other sign will be extracted. This will clearly induce a charge imbalance in the superconductor, but in addition a careful examination would show that more quasiparticles (with higher energy) are added than extracted. In general this latter effect will cause the gap to decrease (Chi and Clarke 1979, Gray 1978a). The counterelectrode of the tunnel junction can also be a superconductor, in which case it is possible to extract more quasiparticles than are injected, and to actually increase the gap in the superconductor with the smaller value of Δ, as shown experimentally by Chi and Clarke (1979).

Microwaves also act to change $f(\varepsilon)$ (Eliashberg 1970). If the energy of the microwave photon $\hbar\omega$ is less than the energy 2Δ required to break a pair, the photons cannot directly produce

additional quasiparticles, but they can redistribute the quasiparticles already present, by increasing their energies. There may be, however, a decrease in the number of quasiparticles near the gap edge, which leads from the gap equation to an increase in the magnitude of the gap. This paradoxical effect, predicted by Eliashberg (1970), gives superconductive "cooling" as a consequence of microwave heating. Gap enhancement has been observed experimentally (Kommers and Clarke 1977, Pals and Dobben 1979, 1980, Hall et al. 1980), together with a relatively larger enhancement of the critical current J_c (Klapwijk and Mooij 1976a, Latyshev and Nad' 1976). Furthermore, a slight enhancement of T_c itself has also been observed for Al films in the presence of microwaves (Klapwijk et al. 1977, Pals and Dobben 1979, Mooij et al. 1980). Note however, from the gap equation (2.6), that the enhanced gap can never be more than $A(0)$, and is usually much less because of heating effects. There is now general agreement that superconductivity is enhanced by microwaves although some anomalies remain. Dahlberg et al.
(1979) observed that critical current enhancement was not always accompanied by gap enhancement. Also Van Attekum and Ramekers (1982) and Van Attekum et al. (1983) found order parameter enhancement half that expected from critical current enhancement.

Similar critical current and gap enhancement effects has also been observed with injected monochromatic phonons rather than microwave photons (Tredwell and Jacobsen 1975, 1976, Seligson and Clarke 1983). A more complete discussion of stimulated superconductivity is carried out in the chapters of the present volume by Dmitriev et al. (1985) and Eliashberg and Ivlev (1985), and in the review by Dmitriev and Khristenko (1978).

Another type of nonequilibrium energy-mode state is one which develops a spatially inhomogeneous value of the gap, while the energy is being pumped into the system uniformly (Chang and Scalapino 1974, Baru and Sukhanov 1975, Elesin 1974, 1976, 1977a,b, Smith 1977, Eckern et al. 1979). This gap instability has been predicted for a sufficiently large input of energy from optical illumination, microwaves, and high energy tunnel injection. There has been some experimental verification of these effects (Iguchi 1978a,b, Iguchi and Langenberg 1980, Smith 1980), although it has proven difficult to unambiguously distinguish the real effect from that due to possible inhomogeneities in the sample and/or energy input (Dynes et al. 1977, Iguchi and Langenberg 1980, Gray and Willemsen 1978).

Quasiparticle tunneling is one of the standard methods for detecting a nonequilibrium distribution $f(\varepsilon)$ (Gray 1981b). Often a double tunnel junction configuration is used, where the center film is common to both junctions, and one junction (usually low resistance) is used for quasiparticle injection while the other (usually relatively high resistance) is used for detection. Since the tunnel conductance $\mathrm{d}I/\mathrm{d}V$ for voltages less than A is actually a measure of the quasiparticle density of states convoluted with a function representing thermal smearing, it is possible to determine $f(\varepsilon)$ rather directly (Smith 1980). In addition, the value of Δ is made evident by the increased current at $eV = \Delta$, enabling a check on the validity of the BCS gap equation (Gray 1981b).

Furthermore, tunnel junction probes can also be used to measure the nonequilibrium potential difference $\varepsilon_\mathrm{s} - \varepsilon_\mathrm{n}$ (Clarke 1972). Quasiparticle currents J_n are driven by a difference in the quasiparticle electrochemical potential $\mu_\mathrm{n} = \varepsilon_\mathrm{n} + e\phi$ (where ϕ is the usual electric potential), and supercurrents J_s are accelerated by a gradient in $\mu_s = \varepsilon_s + e\phi$. A normal counterelectrode will exchange charge with the quasiparticles, but not with the condensate (at least near T_c), and therefore in a potentiometric situation this normal probe will measure μ_n. On the other hand, a superconducting counterelectrode allows current flow via both the condensate and quasiparticle channels. In a stationary situation, μ_s must be constant in space, so that the superconducting probe will measure μ_s. Since μ_n will then not be constant across this junction, quasiparticle current will flow across the probe. However, this is canceled by

the counterflow of J_s, which does not need a potential gradient to drive it. The voltage

Fig. 3. Experimental configurations which yield a nonequilibrium quasiparticle potential difference. (a) Double tunnel-junction experiment of Clarke (1972). (b) Normal and superconducting microprobes near a current-carrying N/S boundary, as in the experiment of Yu and Mercereau (1972, 1975).

measured potentiometrically between the two probes (fig. 3a) will thus be

$$\Phi = (\mu_{\mathrm{n}} - \mu_{\mathrm{s}})/e = (\varepsilon_{\mathrm{n}} - \varepsilon_{\mathrm{s}})/e. \tag{2.13}$$

Furthermore, the superconducting probe can even be an extension of the superconducting film, and the normal probe can also be a normal contact via a clean normal-superconducting boundary (see fig. 3b) (Yu and Mercereau 1972, 1975).

There is also a spatial aspect to problems in nonequilibrium superconductivity. For example, for current flow at a normal-superconducting boundary (Pippard et al. 1971), the current in the normal metal couples directly (at least close to T_c) to quasiparticle current in the superconductor. Since farther into the superconductor, this same current is carried by the condensate, there will be an excess boundary resistance (Pippard et al. 1971. Hsiang and Clarke 1980) accompanied by a local build-up near the boundary of quasiparticles of one sign, i.e. a charge imbalance. More generally, quasiparticle relaxation with a time τ_{R} leads to a characteristic quasiparticle diffusion length $\Lambda = (D\tau_{\mathrm{R}})^{1/2}$ over which the disequilibrium relaxes in space for a steady-state process, where D is the quasiparticle diffusion length. For high frequencies, a propagating wave solution is possible. In both cases, some of the same types of probes (tunneling, etc.) can be used to measure this spatial variation, provided they can be made small enough - typically several μm or less. For example, small

superconducting and normal tunnel junction probes of a superconducting strip placed near a phase-slip center have been used to measure the spatial variation of the electrochemical potential μ_{s} and μ_{n} of the condensate and the quasiparticles (see fig. 4) (Dolan and Jackel 1977, Stuivinga et al. 1982, 1983).

44.2.2 2.2. Dynamical equations and relaxation times

The simplest approach to superconducting dynamics, certainly on the phenomenological level, lies with the time-dependent Ginzburg-Landau (TDGL) equation, which extends the static GL equation to handle time-dependent phenomena (Cyrot 1973). The GL free energy density functional can be written in the form

Fig. 4. Potentials near a PSC: the experiment of Dolan and Jackel (1977). (a) Experimental configuration, with arrays of superconducting and normal tunnel-junction microprobes. (b) Spatial dependence of the measured potentials. (From Dolan and Jackel 1977.)

$$f_{\mathrm{GL}} = f_{\mathrm{N}} + \alpha|\psi|^2 + (\beta/2)|\psi|^4 - \hbar^2/2m(\nabla - 2\mathrm{i}eA/\hbar)^2\psi, \tag{2.14}$$

where $\psi \sim \Delta \exp(\mathrm{i}\theta)$ is the complex superconducting order parameter. The simplest response of the condensate to changes is a diffusive one on the time scale (near T_{c}) of the GL relaxation

$$\tau_{\mathrm{GL}} = \pi\hbar/8k_{\mathrm{B}} \mid T - T_{\mathrm{c}}\mathrm{l}, \tag{2.15}$$

leading to a dynamical equation

$$\tau_{\mathrm{GL}}\partial\psi/\partial t = \pm\psi - (\beta/|\alpha|)|\psi|^2\psi + \xi^2\nabla^2\psi, \tag{2.16}$$

where $-\psi$ applies for $T > T_{\mathrm{c}}$. It can be shown that the GL coherence length ξ is actually the diffusion length for τ_{GL} [i.e. $\xi = (D\tau_{\mathrm{GL}})^{1/2}$], emphasizing the diffusive nature of the equation.

Unfortunately, the validity of eq. (2.16) is far less universal than its static equivalent, essentially because it only takes into account the condensate and does not even consider the corresponding changes in the quasiparticles. For $T > T_{\mathrm{c}}$, the quasiparticle spectrum is normal, so that the quasiparticles are essentially decoupled from the condensate and can be largely ignored. The same is effectively true for any other "gapless" superconductor. In both cases, the simple TDGL equation is expected to hold. For the other limit, for a superconductor with a gap in the excitation spectrum, the order parameter is strongly coupled to the quasiparticle distribution function. Since quasiparticle relaxation is usually slower (except extremely close

to T_c) than the corresponding intrinsic condensate relaxation times (e.g. τ_{GL}), the former provides a bottleneck in the relaxation process and must be examined in more detail.

Quasiparticle relaxation can involve either a recombination process that decreases the number of quasiparticles by two, or a scattering process that merely changes the excitation energy, but both are inelastic processes that in general require a phonon. This means that although the various relaxation processes are not all the same, most of them scale with the characteristic electron-phonon scattering time τ_{E}. This is the same time that enters into the electronic thermal conductivity in normal metals (although not the one associated with the electrical conductivity). The time τ_E is strongly temperature dependent, going as $1/T^3$ for $T \ll \theta_{\mathrm{D}}$ (Debye temperature). In fact, it is just as strongly energy-dependent as well, going as $1/E^3$ of the phonon energy (hence the subscript $\mathbf{E}$ in τ_{E}). Nevertheless, at least for small deviations from equilibrium, τ_{E} can often be considered as essentially a constant characteristic of a particular material. Most of the experimental work on nonequilibrium superconductors has been done using simple elemental materials, namely $\mathrm{Al}, \mathrm{Sn}, \mathrm{In}$ or Pb . In general, the longer τ_E the more significant the nonequilibrium effects, making A1 with $\tau_{\mathrm{E}}\left(T_{\mathrm{c}}\right) \sim 10^{-8}$ often the material of choice. The others (Sn, In, and Pb , resp.) have values $3, 1.5$, and 0.5×10^{-10} s (Kaplan et al. 1976).

Another aspect of the relaxation problem is that it has been typically approached from the two opposite ends of the temperature range, which exhibit distinctly different properties. For $T \ll T_{\mathrm{c}}$, one has $k_{\mathrm{B}} T \ll \Delta$, and for a
state not too far from equilibrium the superconducting order parameter is unlikely to be perturbed substantially from equilibrium. However, any excess quasiparticles are likely to stay around for a long time, since recombination is a two-body process, and there are few equilibrium quasiparticles with which to recombine. In particular, the recombination time τ_R is given theoretically by

$$\tau_{\mathrm{R}} \sim \tau_E \exp\left(2\Delta / k_{\mathrm{B}} T\right). \tag{2.17}$$

This exponential form has been confirmed in several experiments from steadystate measurements (Langenberg 1974).

One persistent problem with these measurements, which exists as well for higher temperatures, has been referred to in the literature as "phonon trapping". Because of finite sample size and phonon mismatch at the boundaries between the superconducting film and the insulating substrate, the nonequilibrium recombination phonon with energy 24 may "decay" into a pair of quasiparticles in the superconductor (by breaking a Cooper pair) rather than simply leaving the film. This recycling of phonons may occur, on the average, a number of times before it finally leaves the film. This problem can be somewhat minimized by using a very thin film well-bonded to a crystalline substrate with similar acoustic properties (Kaplan 1979). An alternative analysis of the same problem can be made on the macroscopic level (a "simple heating" model) by considering the relative quality of thermal links between the quasiparticle system, the phonons in the film, and the external world in equilibrium at temperature T.

For the regime with $T \sim T_{\mathrm{c}}, \Delta \ll k_{\mathrm{B}} T$, the approach is somewhat different. Here, there are plenty of equilibrium quasiparticles so that quasiparticle recombination is not a bottleneck, and scattering and recombination occur at essentially the same rate $1/\tau_E$. In this regime, the order parameter is strongly perturbed by small changes in $f(\varepsilon)$, and great attention is paid to the relaxation times associated with macroscopic variations of the order parameter, namely those of the gap Δ and the potential ε_{s}.

Because of the strong coupling of the condensate and the quasiparticles in this regime close to T_c, the time measured in an experiment depends on the nature of the experiment, even for the same physical quantity. Taking the variation of the gap as an example, as the gap changes, the states, whether occupied or not, simply ride up or down, each individual state is conserved

and remains at the same value of kinetic energy ε, provided no inelastic collisions occur (Schmid and Schön 1975a).

However, the equilibrium distribution is a function of the quasiparticle excitation energy E and hence of Δ, so that the instantaneous state of "local equilibrium" changes with Δ. Since one expects that collisions with phonons will relax the distribution toward this state of local equilibrium, rather than (directly) towards global equilibrium (with the equilibrium value of Δ), this coupled aspect of the collective relaxation requires careful consideration.

We consider first the relaxation of a longitudinal (energy) mode perturbation, in the regime close to T_c. In this case, every scattering event acts to change the energy of a quasiparticle by $\sim k_{\mathrm{B}}T$, which is more than sufficient to relax the disequilibrium. Then, the nonequilibrium distribution will relax with a time τ_{E} towards the thermal distribution corresponding to the nonequilibrium value of the gap:

$$\partial f/\partial t = \left[f_0(\varepsilon, \Delta, T) - f\right]/\tau_{\mathrm{E}} + (\dot{f})_{\mathrm{inj}}. \tag{2.18}$$

Here, f is the actual instantaneous nonequilibrium distribution, which corresponds to the gap Δ from the BCS gap equation. For example, one might have $f \approx f_0\left(\varepsilon, A, T^* > T\right)$ where $A = A_{\mathrm{eq}}\left(T^*\right)$. It is important to recognize that f is relaxing toward a non-physical distribution that it can never actually achieve, the only physical equilibrium distribution being $f_0\left(\varepsilon, A_{\mathrm{eq}}\left(T\right), T\right)$. The nonphysical distribution corresponds to "local equilibrium" with the phonons at temperature T but not equilibrium with the condensate, whereas the "global equilibrium" distribution involves equilibrium with both. For that matter, the actual distribution f is effectively in quasi-equilibrium with the condensate, although not with the thermal phonons.

Figure 5 illustrates the three relevant distributions for an initial "hot" nonequilibrium distribution f. As f relaxes toward the local equilibrium distribution $f_0(\varepsilon, \Delta, T), \Delta$ increases, causing the latter distribution to decrease and move away from f. Thus, f is relaxing toward a distribution which is continually moving away from it, so that the overall real-time relaxation time will clearly be greater than τ_{E}. Quantitatively, in evaluating the time-derivative of the gap equation (2.6), one must take into account not only the direct change in f, but also the implicit change in E :

$$-\dot{\Delta}/\Delta = \int \mathrm{d}\varepsilon \left[\dot{f}/E - \left(f/E^2\right)\left(\mathrm{d}E/\mathrm{d}\Lambda\right)\dot{\Delta}\right]. \tag{2.19}$$

Grouping terms and evaluating integrals, all to lowest order in $\Delta/\Delta_{\mathrm{eq}}$ and $\Delta/k_{\mathrm{B}}T$, one obtains an overall relaxation time

$$\tau_{\Delta} = 3.7\tau_{\mathrm{E}}\left(k_{\mathrm{B}}T/\Delta\right). \tag{2.20}$$

This is the same time that has also been obtained by other methods, and has been referred to as $\tau_{\mathrm{R}}^{(\mathrm{L})}$ (for longitudinal) (Schmid and Schön 1975a). This divergence in the transient response time near T_{c} has in fact been directly observed, by examining in real-time the characteristics of a tunnel junction irradiated by a fast laser pulse (Schuller and Gray 1976, 1977). Other investigations of superconducting relaxation using laser probes have been carried out by Chi et al. (1981) and by Gorlach and Huebner (1983). We emphasize, however, that the effective relaxation time in steady state should remain as simply τ_E, since in steady state the excitation energy E remains fixed. This is reasonable, since this should go over continuously to a simple thermal healing length for $T > T_{\mathrm{c}}$.

For the relaxation of a transverse nonequilibrium distribution (a charge imbalance), the analysis is similar, although a bit more complicated. First, the appropriate microscopic relaxation time is not simply τ_{E}, as it was for the longitudinal case. In fact, charge imbalance relaxation cannot occur at all in the normal (non-superconducting) state, when a quasiparticle

Fig. 5. Quasiparticle excitations energies (a) and distributions (b) for an energy-mode disequilibrium. The "hot" nonequilibrium (ne) distribution, the local equilibrium (ℓ e) distribution $f_{0 \left(\varepsilon, \Delta_{ne}, T\right)$, and the global equilibrium (ge) distribution $f_0\left(\varepsilon, \Delta_{\mathrm{eq}}, T\right)$ are indicated.}

is difinitely either an electron or a hole. It can occur only via scattering events that change the effective charge of quasiparticle, namely those which involve states near the gap edge with mixed electron- hole character. Therefore, for $\Delta \ll k_{\mathrm{B}} T$ (i.e. T close to T_{c}), the average inelastic scattering event can relax a charge imbalance only to the order of $\Delta / k_{\mathrm{B}} T$. More detailed calculations have shown that the effective average microscopic relaxation time for charge imbalance τ_{Q*} is

$$\tau_{\mathrm{Q}*} = \left(4 k_{\mathrm{B}} T / \pi \Delta\right) \tau_{\mathrm{E}} \tag{2.21}$$

in this regime close to T_c (Tinkham 1972, Schmid and Schön 1975a). This result has been well documented experimentally in steady-state measurements on the nonequilibrium potential ε_{s} in double-tunnel junction measurements of Clarke and co-workers (Clarke 1972, Clarke and Paterson 1974, Paterson
1978). Further confirmation has been found in steady-state measurements of the quasiparticle diffusion length near normal-superconducting boundaries and phase-slip centers (Dolan and Jackel 1977, Hsiang and Clarke 1980).

Given the above quantity $\tau_{\mathrm{Q}*}$ as the effective microscopic relaxation time for charge imbalance, one can proceed to obtain the corresponding transient relaxation time (analogously to that derived earlier for longitudinal relaxation), by writing first

$$\dot{f} = -\left[f - f_0\left(\varepsilon - \varepsilon_{\mathrm{s}}\right)\right]/\tau_{Q*}. \tag{2.22}$$

Note that $f_0\left(\varepsilon - \varepsilon_s\right)$ represents the local equilibrium distribution toward which f is relaxing, but which again is unphysical, since such an actual distribution would violate charge conservation. (Also see fig. 2c.) If the actual nonequilibrium distribution f is close to $f_0\left(\varepsilon - \varepsilon_{\mathrm{n}}\right)$, then near to T_{c}, as shown in eq. (2.12), ε_{n} is far closer to $\varepsilon_{\mathrm{F}} = 0$ than it is to ε_{s}, and for a transient process, this will enhance the relaxation towards global equilibrium. In fact, a detailed calculation shows that the transient relaxation time $\tau_{\mathrm{R}} = \tau_{\mathrm{E}}$ (Schmid and Schön 1975a, Kadin et al. 1978). This particular result has not been directly confirmed experimentally, although some consequences of it have seen.

In certain circumstances, non-phonon forms of inelastic relaxation may become relevant. For example, when the electron-phonon time is very long (e.g. in Al), the direct electron-electron scattering may become significant. Furthermore, in sufficiently disordered materials, the electron electron rate may be enhanced substantially. Another scattering time that is relevant to many of the relaxation processes in superconductors is the elastic spin-flip scattering time τ_{s} from magnetic impurities. Since these events induce pairbreaking, even a small concentration can have a large effect on gap relaxation in both the longitudinal and the transverse modes (Schmid and Schön 1975a, Lemberger and Clarke 1981a, Aspen and Goldman 1979, 1981a). Other pairbreaking interactions, including an external magnetic field and a large supercurrent, have a comparable large effect on the relaxation times (Kadin et al. 1978, Lemberger and Clarke 1981b).

Another characteristic time of the condensate (which does not involve the electron-phonon time τ_E) is the supercurrent response τ_j. Consider a superconductor, in the two-fluid sense, as a parallel combination of an inductive channel with kinetic inductance $L_h = 4\pi i^2, c^2$ (i is the magnetic penetration depth), and a quasiparticle channel with normal resistance ρ near T_c. Then, a sudden change in the current (less than the critical current I_c) will initially be carried by the (noninductive) normal channel, until the supercurrent can respond in a time

$$\tau_{\mathrm{J}} = L_{\mathrm{k}}/\rho = \tau\left(n/n_{\mathrm{s}}\right) = \begin{cases} \tau/2\left(1 - T/T_{\mathrm{c}}\right) & (\text{ clean }), \\ 2k_{\mathrm{B}}T\hbar/\pi\Delta^2 & (\text{ dirty }), \end{cases} \tag{2.23}$$

evaluated for the clean and dirty limits near T_{c}. Typically, $\tau_{\mathrm{J}} \sim 10^{-11}$ s, smaller than τ_{E}, and a factor of 5.79 smaller than τ_{GL}. The relaxation and response times of the superconductor under various conditions are summarized in table 1.

Given the varied hierarchy of relaxation times and quasiparticle distributions, the previous discussion clearly suggests that no simple universal TDGL equation is likely to be appropriate for all situations. Nevertheless, a number of attempts have been made, with varying degrees of generality (Schmid 1981). One of these (Tinkham 1981, Schmid 1978) includes the nonequilibrium part of the distribution explicitly in the form

$$\left[\alpha + \delta\alpha^* + \left(\beta/T_{\mathrm{c}}^2\right)\Delta^2 - \xi^2(0)\nabla^2\right]\Delta = 0 \tag{2.25}$$

where the quantity

$$\delta\alpha^* = \delta T^*/T_{\mathrm{c}} = \int_{-\infty}^{\infty} \delta f(\varepsilon)/E \, \mathrm{d}\varepsilon \tag{2.26}$$

is known as the control function, and can also be expressed in terms of an effective temperature change $\delta \mathrm{T}^* = \delta\alpha^* \ \mathrm{T_c}$, although the distribution does not have to be strictly thermal. The time dependence of this equation comes in given the fact that δf, the deviation of the distribution from local equilibrium, decays with a characteristic time τ_{E}, leading to an equation

$$\left[1-\psi^2+\xi^2(T)\nabla^2\right]\psi = 2\tau_\Delta/\tau_\mathrm{E}\int^t \mathrm{d}t'\partial\psi/\partial t' \exp\left(t'-t\right)/\tau_\mathrm{E} \approx 2\tau_\Delta\psi\partial\psi/\partial t \tag{2.27}$$

Table 1
Scattering and relaxation times in superconductors near $T_{c
$.}

General class	Characteristic time	Order of
Quasiparticle scattering times	Impurity elastic scattering τ	13-15
	Inelastic electron-phonon scattering τ_E	7-10
	Spin-flip scattering τ_s	$\lesssim 12$
Condensate relaxation times	Ginzburg Landau time $\tau_\mathrm{GL} \sim 0.4\hbar/k_\mathrm{B}T_\mathrm{c}\left(1-T/T_\mathrm{c}\right)$	10-11
Coupled system relaxation	Energy-mode transient relaxation $\tau_\Delta \sim 1.2\tau_\mathrm{E}\left(1-T/T_\mathrm{c}\right)^{-1/2}$	6-9
	Energy-mode steady-state relaxation	7-10
	$\sim \tau_\mathrm{E}$ Charge-mode transient relaxation	7-10
	$\sim \tau_\mathrm{E}$ Charge-mode steady-state relaxation $\tau_{Q^*} \sim 0.4\tau_\mathrm{E}\left(1-T/T_\mathrm{c}\right)^{-1/2}$	6-9

where the latter form is appropriate when f changes slowly on the scale of τ_E. Both forms implictly assume that $\tau_\mathrm{E} \gg \tau_\mathrm{GL}$, and that f is changing slowly on the scale of τ_GL.

Another generalized TDGL equation, which attempts to interpolate from a regime dominated by τ_GL to one determined by τ_E, is that of Kramer and Watts-Tobin (1978), which can be written in the form

$$\left[1-|\psi|^2+\xi^2(T)\nabla^2\right]\psi = \left[\frac{\partial}{\partial t}+\frac{2ie\phi}{h}+\frac{1}{2}\gamma^2\frac{\partial f^2}{\partial t}\right]\psi\left[\frac{u}{\left[1+\gamma^2 f^2\right]^{1/2}}\right] \tag{2.28}$$

where $\gamma = 2\tau_E\Delta/h, u = 5.79$ from TDGL theory, and $f = |\psi/\psi_\mathrm{eq}\left(T\right)|$. It appears, however, to be somewhat more restricted in its applicability than eq. (2.27) above. A further point about these two sets of equations is that both were derived primarily to deal with energy-mode nonequilibrium states. Neither contains the terms which would describe the propagating charge imbalance waves that we will discuss in section 3.

An experiment by Pals and Wolter (1979) (also Wolter et al. 1981a,b) brings out some of the issues that we have been discussing in relation to relaxation times and dynamical equations. They applied a sharp current step with $I > I_\mathrm{c}$ to a narrow superconducting Al microstrip, and measured the delay time between the application of the current step and the development of a voltage across the strip. This delay time was of order τ_1, but dependent on the current and the temperature; the larger the current the faster the development of a voltage. A simple qualitative explanation was proposed by Tinkham (1981) and subsequently made more rigorous by Geier and Schön (1982), given the condition, generally valid for Al , that $\tau_\mathrm{J} \ll \tau_\mathrm{E}$. Over a time $\sim \tau_\mathrm{J}$, the entire current step changes from normal current J_n to supercurrent J_s. This would clearly be expected to give a narrow voltage pulse, but it would be too fast to be resolved. In the presence of the supercurrent, the magnitude of the superconducting order parameter is depressed, but not totally to zero, since the quasiparticle distribution is one that is appropriate

for a larger gap. As the distribution f decays, Δ decreases along with it, until the condition is reached that the supercurrent can no longer carry the entire current. Then a voltage begins to appear, and the order parameter suddenly (i.e. over a time $\sim \tau_{\mathrm{J}}$) collapses. The more complete calculation has been made from eq. (2.27), taking the supercurrent density $j = f^2 g$, and using the full integro differential form, and it confirms the basic picture we have been discussing (Geier and Schön 1982). Subsequent experimental investigations have tended to confirm this approach in more detail (Frank et al. 1983, Frank and Tinkham 1983, Pals et al. 1982).

44.2.3 2.3. Charge imbalance dynamics and the transmission-line picture

It was believed for quite some time that the propagating mode of the superconducting order parameter, because of the electric charge, was at the
plasma frequency (Anderson 1958). A mode with an acoustic dispersion relation was, in fact, observed by Carlson and Goldman (1975, 1976), from measurements of the pair-field susceptibility (to be discussed in the next section). Subsequent analysis showed that this could occur via a counterflow of supercurrent J_{s} and "normal" quasiparticle current J_{n} (Schmid and Schön 1975b, Artemenko and Volkov 1975). What was not appreciated at the time any of this work was done was that Thouless and Tilley (1961) had much earlier discussed phonon-like modes in charged superconductors from a rather different point of view. We will examine this phenomenon from the point of view of charge imbalance, which provides a clear and consistent picture of these waves. In essence, a net inflow of J_{n} will produce a charge imbalance, which by electroneutrality must be accompanied by a net outflow of J_{s} and a shift in ε_s. This in turn acts as a restoring force to produce currents in the opposite direction. These oscillations do in fact propagate, and can be viewed as waves of charge imbalance. The waves are, however, relatively heavily damped because of charge imbalance decay (by dissipative electron-phonon collisions), damping out over a distance of order of the transverse mode quasiparticle diffusion length $\Lambda_{\mathrm{Q}}* = (D\tau_Q*)^{1/2} = 1 - 10\mu$ m.

In the regime close to T_{c}, a simple analog model for these charge imbalance waves has been developed, using signal propagation down a transmission line to represent charge imbalance waves propagating down a nonequilibrium superconductor (Kadin et al. 1980). In the relevant regime, the equations governing the transmission line are isomorphous to those in the generalized two-fluid model of the nonequilibrium superconductor (Pethick and Smith 1979). This heuristic framework will be presented below, and it will be used again in a later section for a model for the phase-slip center.

In order to deal with spatially inhomogeneous disequilibria, we use the electrochemical potentials μ_{c} for the superfluid condensate and μ_{n} for the quasiparticles, as defined earlier. Close to T_{c}, since ε_{n} is small, $\mu_{\mathrm{n}} \approx \mathrm{e}\phi$. We then use the standard GL expressions $\mu_{\mathrm{s}} = -(h/2)\mathrm{d}\theta/\mathrm{d}t$ and

$$\boldsymbol{J}_{\mathrm{s}} = n_{\mathrm{s}} e v_{\mathrm{s}} = (n_{\mathrm{s}} e/2m)\left(\hbar\nabla\theta - 2e\boldsymbol{A}\right) \tag{2.29}$$

to obtain the following equation for the time evolution of $\boldsymbol{J}_{\mathrm{s}}$:

$$\tau_{\mathrm{J}}\dot{\boldsymbol{J}}_{\mathrm{s}} = -\sigma\nabla\mu_{\mathrm{s}}/e - \sigma\dot{A} = \sigma\boldsymbol{E} + D\nabla Q^*, \tag{2.30}$$

where $E = -\nabla\varphi - \mathrm{d}A/\mathrm{d}t$ is the electric field, and $\sigma = \mathrm{n}e^2\tau/m = 2N(0)e^2D$ is the normal electronic conductivity. For the normal current density J_n, we use the usual relation

$$J_{\mathrm{n}} = \sigma E = -\sigma\nabla\mu_{\mathrm{n}}/e - \sigma\dot{A} \tag{2.31}$$

but with the understanding that this is a good approximation only near T_c, where most of the quasiparticles have a well defined charge $\pm e$. Pethick and Smith (1979) have compared the true normal current (defined by $\boldsymbol{J}_n = \boldsymbol{J} - \boldsymbol{J}_-$) with the current carried by the flow of quasiparticles (which they call $\boldsymbol{J}_n^0$), and
have demonstrated that the distinction between these two goes to zero as $\Delta/k_B T$, as do the corrections to eq. (2.31) above.

The presence of $\boldsymbol{J}_\mathbf{n}$ means that charge can leave the quasiparticle system in a given region of space not only through inelastic scattering, but also through the net inflow or outflow of quasiparticle current as determined by the term $\boldsymbol{\nabla} \cdot \boldsymbol{J}_n$. Because this acts to produce a direct change in the distribution function, it is subject to the same enhancement factor due to the condensate response as that due to charge imbalance decay or quasiparticle injection. Thus we have

$$\dot{Q}^* = (4k_B T/\pi\Delta)\left[\left(-\dot{Q}^*/\tau_{Q*}\right) - \nabla \cdot J_n\right]. \tag{2.32}$$

If we take the $\boldsymbol{\nabla} \cdot \left(\tau_J \dot{\boldsymbol{J}}_s - \boldsymbol{J}_n\right)$ from eqs. (2.30) and (2.31), and τ_J times the time derivative of eq. (2.32), and assume $\boldsymbol{\nabla} \cdot \boldsymbol{J}_{tot} = 0$ from electro-neutrality, we obtain the following partial differential equation involving only the charge imbalance Q^* :

$$D\tau_{Q*}\nabla^2 Q^* = \tau_J \tau_E \dot{Q}^* + (\tau_J + \tau_E)\dot{Q}^* + Q^* \tag{2.33}$$

This is the charge imbalance wave equation we have been seeking. In the low frequency limit $\omega \ll \tau_J^{-1}, \tau_E^{-1}$, it describes charge imbalance decay with a characteristic decay length $\Lambda_{Q*} = (D\tau_{Q*})^{1/2}$ (henceforth simply Λ will be used). This length, as pointed out earlier, has been used to characterize the penetration of an electric field into a superconductor-normal metal interface. In the high frequency limit ($\omega \gg \tau_E^{-1}, \tau_F^{-1}$), eq. (2.33) describes waves of charge imbalance, propagating with velocity $v = \Lambda/(\tau_J \tau_E)^{1/2}$. Substituting the clean and dirty limit formulas for τ_J, we have

$$v = \begin{cases} 0.53 v_F (1-t)^{1/4} & (\text{ clean }), \\ 0.61 v_F (l/\xi_0)(1-t)^{1/4} & (\text{ dirty }), \end{cases} \tag{2.34}$$

where v_F is the Fermi velocity, $l = v_F \tau$ is the mean free path, and $\xi_0 = \hbar v_F/\pi\Delta(0)$ is the BCS coherence length. These expressions are identical to those initially derived for this propagating mode within other theoretical frameworks (Schmid and Schön 1975b, Artemenko and Volkov 1975).

For the more general case where neither frequency limit is appropriate, eq. (2.33) describes damped dispersive waves of charge imbalance with the dispersion relation

$$-\Lambda^2 K^2 = (1 + i\omega\tau_J)(1 + i\omega\tau_E) \tag{2.35}$$

Thus, the two phenomena of DC electric field penetration into a superconductor and the high-frequency collective mode are both special cases of the more general phenomenon of charge imbalance dynamics in a superconductor.

These relationships can be represented in an appealing picture by reference to fig. 6. If we define the gauge-invariant nonequilibrium potential Φ

$$\Phi = \phi + \hbar\dot{\theta}/2e = -\varepsilon_s/e = Q^*/2N(0)e^2 \tag{2.36}$$

we can rewrite eqs. (2.32) and (2.33) in the form (for a one-dimensional wire with cross-sectional area A)

Fig. 6. Transmission-line equivalent of the charge imbalance wave equation. (From Kadin et al. 1980.)

$$\frac{\partial I_{\mathrm{n}}}{\partial x} + G\Phi + C\dot{\Phi} = 0, \tag{2.37}$$

$$\frac{\partial \Phi}{\partial x} + RI_{\mathrm{n}} - L\dot{I}_{\mathrm{s}} = 0, \tag{2.38}$$

where the parameters (each per unit length) are:

$$R = 1/\sigma A, \quad G = \sigma A/\Lambda^2, \quad L = \tau_{\mathrm{J}}/\sigma A, \quad C = 2N(0)e^2 A\left(\pi\Delta/4k_{\mathrm{B}}T\right), \tag{2.39}$$

This system of equations is isomorphic to the (standard) equations governing current flow on the transmission line of fig. 6. The resistive line represents the channel for the normal current I_{n} and its voltage μ_{n}/e, and the inductive line represents the condensate channel with I_s and μ_s/e (ignoring the magnetic vector potential A). The voltage between the lines is the nonequilibrium potential $\Phi = (\mu_{\mathrm{n}} - \mu_{\mathrm{c}})/e$. Equations (2.37) and (2.38) then follow directly from Kirchhoff's laws.

The circuit parameters R, L, C, and G have a simple and appealing physical interpretation. The relaxation of Φ is governed by the time constant $\tau_{\mathrm{E}} = C/G$, while the ratio L/R is the response time τ_{f} for the supercurrent when $\Phi = 0$. The resistance per unit length R is just that, while the inductance per unit length L is the usual London theory kinetic inductance of the supercurrent.

To assist in the interpretation of C and G, we introduce the quantity

$$Q_{\mathrm{n}} = (\pi\Delta/4k_{\mathrm{B}}T)\, Q^*, \tag{2.40}$$

which is a measure of the departure of the distribution function from global equilibrium, as distinguished from Q^*, which measures the departure from local equilibrium (Pethick and Smith 1979). Neglecting conversion processes.
$\mathrm{d}Q_{\mathrm{n}}/\mathrm{d}t = -\nabla \cdot \boldsymbol{J}_{\mathrm{n}}$. Thus, Q_{n} is the quantity which is analogous to the "free charge density" in the transmission line. The capacitance per unit length C is then equal to $Q_{\mathrm{n}}A\Phi$, which differs from Q^*A/Φ by the factor $\pi\Delta/4k_{\mathrm{B}}T$. This factor can be interpreted as an effective dielectric constant, representing the response of the condensate in enhancing this "free charge" in the quasiparticles. The conductance per unit length G gives the proportionality between the rate at which $Q_{\mathrm{n}}A$ changes by conversion processes ($Q^*A/\tau_{Q*} = -Q_{\mathrm{n}}A/\tau_E$) and the nonequilibrium potential Φ. Other standard parameters governing the transmission line also have direct physical significance in terms of the charge imbalance dynamics. For example, the propagation constant of the line is

$$K \equiv [(R + \mathrm{i}\omega L)(G + \mathrm{i}\omega C)]^{1/2} \tag{2.41}$$

This relates to the wave vector of the charge imbalance waves eq. (2.35) by $K = \mathrm{i}k(\omega)$, giving the low-frequency decay length $\Lambda = (RG)^{-1,2}$ and the highfrequency wave velocity $v = (LC)^{-1/2}$. The characteristic impedance of the line is

$$Z(\omega) = [(R + \mathrm{i}\omega L)/(G + \mathrm{i}\omega C)]^{1/2}, \tag{2.42}$$

which, for low frequencies, gives $Z(0) = RA$, the resistance of the line corresponding to the characteristic decay length. For many purposes, a semiinfinite length of the transmission line can be represented by this single lumped impedance $Z(\omega)$ connected between the two channels. Standard formulas for finite lengths of transmission line are also available.

The heuristic utility of this transmission line picture is illustrated for two types of measurements in fig. 7. For the spatially uniform quasiparticle injection experiment of Clarke (1972), in fig. 7a, a constant current I tunnels from a normal film through a resistive oxide barrier into a superconducting film below. This current creates the "free charge" in the normal channel at a rate $(\mathrm{d}Q_{\mathrm{n}}/\mathrm{d}t)_{\mathrm{inj}} = I/LA$, charging up the capacitors. They discharge at the same rate through the leakage conductance G. so that the current I leaves the nonequilibrium region as a pure supercurrent in the condensate channel. The induced nonequilibrium potential is therefore

$$\Phi = I/(GL) = \mathrm{I}\tau_{Q*}/\left(2N(0)e^2LA\right), \tag{2.43}$$

where L is the length of the nonequilibrium region, much larger than Λ to minimize edge effects. This result is essentially the same as that originally derived by Tinkham and Clarke (1972), for the measurement of the nonequilibrium potential measured using superconducting and normal probes in the nonequilibrium region. We re-emphasize here that if the current is suddenly turned off, the potential Φ will decay exponentially with a time $\tau_{\mathrm{E}} = C/G$ and not τ_{Q*}. Furthermore, the additional junction resistance (per unit area)

$$\mathrm{R}_{Q*} = 1/G = \tau_{Q*}/2N(0)e^2, \tag{2.44}$$

evident in fig. 7a, has been directly measured experimentally (Lemberger 1981a,b).
Another important phenomenon in which charge imbalance plays a crucial role is the excess resistance on the superconducting side of a current-carrying superconducting-normal interface (Pippard et al. 1971, Hsiang and Clarke 1980). We can represent this (again, for T near T_c) by the equivalent circuit of fig. 7b, in which the normal current connects up to the "normal" channel on the superconducting side. Over a length equal to Λ, the current leaks over to the condensate line by charge imbalance decay, and the excess resistance is $Z(0) = R\Lambda$. Furthermore, the equivalent circuit gives rather directly the exponentially decaying nonequilibrium potential near the boundary, observed by Yu and Mercereau (1972, 1975) using small normal and superconducting probes.

We reiterate that these pictures are only strictly valid very close to T_c. For lower temperatures, the fact that quasiparticles near the gap edge have both electron-like and hole-like characteristics becomes important. A fraction of the injected current of order $A/k_{\mathrm{B}}T$ enters the condensate line directly in fig. 7a, reducing the contribution to the charge imbalance. Similarly, electrons with energy $E < 1$ approaching the $\mathrm{N} - \mathrm{S}$ boundary from the normal side in fig. 7 b are reflected as holes via Andreev (1964) reflection, with superconducting pairs to carry the current in the superconductor. A microscopic analysis which calculates the fraction of current Andreev-reflected at the boundary has been used successfully to explain the observed boundary resistance over a wide range of temperatures (Hsiang and Clarke 1980). With these and other extensions, this transmission line picture should continue to provide a heuristic guide to the understanding of superconducting dynamics involving transverse disequilibria.

Fig. 7. Use of the transmission-line equivalent to model (a) uniform quasiparticle injection, and (b) current flow through a normal-superconducting boundary. (From Kadin et al. 1980.)

44.3 3. Pair-field susceptibility and Josephson tunneling

44.3.1 3.1. Introduction and experimental configuration

We have described how the use of quasiparticle tunneling provides in-sight into the relaxation processes in nonequilibrium superconductors. In this section we will review how another aspect of Josephson tunneling can be used to measure the superconducting pair-field susceptibility, which in turn allows a direct determination of the dynamical equations governing the evolution of the order parameter. Although it has been studied theoretically (Ferrell 1969, Scalapino 1970, Kadin and Goldman 1982) and experimentally (Anderson and Goldman 1970, Yoshihiro and Kajimura 1970) for more than a decade, the physical basis of the measurement of the pair-field susceptibility and its connection with more familiar aspects of the Josephson effect and superconducting order parameter dynamics have not been widely appreciated.

In magnetic systems, the most direct methods for the determination of the dynamics of the magnetization M involve the measurement of the magnetic susceptibility χ_{M}, and its dependence on the wave-vector k and frequency ω. Alternatively, the structure function $S(k\omega)$ is inferred, which is related to the susceptibility by the fluctuation- dissipation theorem. A superconducting system does not have a thermodynamic conjugate field which induces superconductivity in the same way that a laboratory magnetic field H produces a finite magnetization in a paramagnet. However, in both the superconducting and the magnetic case, within the spirit of mean field theory, one can treat the internal interaction as an effective field acting on a given region of the material. In a superconductor, this internal interaction is long range (on the order of the coherence length $\xi \sim 1000$Å) and gives rise to the well-known proximity effect in a clean contact between a superconductor and a normal metal. It has perhaps not been generally appreciated, however, that the same type of physical effect (albeit weaker) occurs across a tunneling barrier, and it is precisely this type of coupling that gives rise to the usual Josephson effects between two superconducting electrodes. Thus, even in a tunnel junction between a superconductor and a normal metal, there is a small induced order parameter on the normal

side (see fig. 8). Furthermore, there is as a consequence a small "excess" supercurrent across the junction, for finite nonzero voltages as it will turn out, which is a measure of the generalized superconducting susceptibility $\chi(k,\omega)$ or pair-field susceptibility (PFS). Similar considerations are relevant when both electrodes are superconducting, and the excess pair current for $V=0$ is also related to the susceptibility.

Fig. 8. Schematic of tunnel junction for measurement of the pair-field susceptibility for $T \sim T_{\mathrm{c} } \ll T_{\mathrm{c}}^{\prime}$. The magnetic field is in the y direction and the electric field and current flows are in the z direction. (From Carlson and Goldman 1976.)}

The geometry of the effect consists of a high-conductance tunnel junction between a strong superconductor (critical temperature $\mathrm{T}_{\mathrm{c}}^{\prime}$) and a weak superconductor ($T_{\mathrm{c}}<\mathrm{T}_{\mathrm{c}}^{\prime}$), where the measurements are made at a temperature T above, near, or slightly less than T_{c}, in order to probe the properties of the weak superconductor. In typical experiments the strong superconductor is Pb , with $T_{\mathrm{c}}^{\prime}=7.2$ K. The weak superconductor has been Sn (Anderson and Goldman 1970) ($T_{\mathrm{C}}=3.8$ K), Al in the dirty limit ($T_{\mathrm{c}} \geqslant 1.2$ K), (Anderson et al. 1972), Al with paramagnetic Er impurities (Aspen and Goldman 1981a) (T_c from 1 to 2 K , with differing concentrations), and Ag/Pb proximity sandwiches in the Cooper limit (Aspen and Goldman 1981b). A review of the experimental results has been given by Goldman (1981).

The tunnel junctions have been of the cross-film type, with a junction area of order 0.1 mm^2 defined by means of a window in an insulating (e.g. $\mathrm{Bi}_2\mathrm{O}_3$) layer. The oxide barriers have been native oxides of either Al or Pb .

For an ideal tunnel junction with $T \ll T_{\mathrm{c}}$, the quasiparticle current should be exponentially small for low voltages $V \ll \Delta$:

$$I_{\mathrm{qp}}(V) \sim (V/R_{\mathrm{NN}}) \exp\left[-\left(\Delta^{\prime}-eV\right)/k_{\mathrm{B}}T\right] \ll V/R_{\mathrm{NN}}. \tag{3.1}$$

This exponential smallness is important, because it acts as a background for a small excess

current at low, but nonzero voltages. This excess current turns
out, as we will show, to be a form of the Josephson effect which is second order in the tunnel conductance (as opposed to first order for the usual zero-voltage Josephson supercurrent). Thus, the excess current associated with the PFS cannot be detected at all in the usual high-resistance junctions, and can only be seen in low resistance junctions when the quasiparticle background is significantly reduced. In practice, the effect has been seen in junctions with normal state resistance $R < 1\Omega$, but without any pinhole shorts. The excess supercurrent can be identified as a Josephson current by its extreme sensitivity to a small magnetic field in the plane of the junction (fig. 8), one far too small to affect the electrodes themselves. Finally, the apparent violation of the AC Josephson effect (a DC supercurrent at a finite voltage) can also be understood, and does not present a problem (Kadin and Goldman 1982).

One of the basic features of the Josephson interaction, following from its dependence on the phase of the order parameter, is that it is gauge-invariant with respect to electromagnetic fields. This means, as we will show below, that a DC voltage across a Josephson junction gives rise to a temporal modulation of the interaction at the Josephson frequency $\omega_0 = 2\mathrm{eV}/\hbar$, and that similarly, a constant magnetic field in the plane of the junction leads to a spatial modulation interaction with wave vector $q_0 \sim H$. In this way, the action of a constant electromagnetic field is to produce a purely harmonic modulation of the effective pair field by which the strong superconductor affects the weak superconductor (the perturbation in the other direction is negligible). This induces a harmonic response in the weak superconductor, which then in turn leads via the Josephson effect back to a DC excess supercurrent. The final result is that

$$I_{\mathrm{ex}} \sim \left(1/R_{\mathrm{NN}}\right)^2 \chi''\left(q_0\omega_0\right), \tag{3.2}$$

where $\chi''\left(q_0\omega_0\right)$ is the imaginary part of the pair-field susceptibility. Thus the dynamics of the order parameter in the weak superconductor can be fully mapped out by examining I_{ex} as a function of V and H. Our derivation of this effect, which emphasizes phenomenological pictures rather than the most general theoretical framework (for the latter, see the review by Schön (1985) in this volume), was taken in large part from Kadin and Goldman (1982), and represents a generalization of the initial treatment of the problem by Ferrell (1969).

As we mentioned above, the weak superconducting film will be affected by the presence of a strong superconductor across the barrier. This interaction can be represented by the Josephson free energy $F_{\mathrm{J}} = -\left(\hbar I_0/2e\right)\cos\gamma$, which gives rise in the usual way to the supercurrent in the junction $I_{\mathrm{s}} = I_0 \sin\gamma$. Here γ is the gauge-invariant phase difference across the barrier, and I_0 is the critical current, which is inversely proportional to the normal-state junction resistance R_{n} and goes as $|\psi|$. If we assume that the thickness $d \ll \xi \sim 1000\text{Å}$, the interaction is uniform through the thickness and we can express it in terms of a
free energy density

$$f_J = -\left(\hbar j_0/2ed\right)\cos\gamma = -(\hbar\bar{C}/2ed)|\psi|\cos\gamma, \tag{3.3}$$

where $j_0 = \bar{C}|\psi|$ is the maximum critical current density. In the regime of interest where $|\psi|$ is small and $T \sim T_{\mathrm{c}} \ll \mathrm{T}'_{\mathrm{c}}, j_0$ is given by

$$j_0 = \left(G_{\mathrm{N}}\Delta_{\mathrm{in}}/4e\right)\ln\left(T'_{\mathrm{c}}/T\right), \tag{3.4}$$

where G_{N} is the normal conductance per unit area of the junction, and Δ_{in} is the induced BCS gap parameter related to $|\psi|$ (in the dirty limit) by

$$\Delta_{\mathrm{in}} = |\psi|\left(4k_{\mathrm{B}}Th/\pi n\tau\right)^{1/2}. \tag{3.5}$$

Consider the case where there is a constant voltage V_0 across the junction and an external magnetic field H_0 in the plane of the junction (fig. 8). Then γ can be written

$$\gamma = \theta - 2eV_0 t/\hbar - \left(2e/\hbar \int A \cdot \ \mathrm{d}z = \theta - \omega_0 t - q_0 x \right. \tag{3.6}$$

where

$$q_0 = (2e/\hbar)\,(\lambda' + d/2 + \delta)\,H_0 \tag{3.7}$$

Here the penetration depth i' of the strong superconductor is less than its thickness (which is normally the case), while $d \ll \lambda$ for the weak superconductor, and δ is the barrier thickness, usually negligible. Then

$$f_J = -\left[\psi\zeta^*(x,t) + \psi^*\zeta(x,t)\right] \tag{3.8}$$

where

$$\zeta = (\hbar\bar{C}/4ed)\exp\left[i\left(q_0 x + \omega_0 t\right)\right] \tag{3.9}$$

The quantity ζ can be viewed as an effective pair field, conjugate to the order parameter ψ, which is modulated in space and time for constant values of V and H. This will cause an induced order parameter for $T > T_c$, or an induced part of the order parameter for $T < T_c$. These two cases will be discussed separately below.

44.3.2 3.2. Pair-field susceptibility for $T > T_c$

For $T > T_c$, the simple TDGL equation is valid, but with the addition of the Josephson interaction term (eq. 3.8) to the free energy in eq. (2.14). Taking the variational derivative $\delta f/\delta\psi^*$ leads directly to

$$\tau_{\mathrm{GL}}\frac{\partial\psi}{\partial t} + \psi - \xi^2\nabla^2\psi = C_1\exp\left[\mathrm{i}\left(q_0 x + \omega_0 \mathrm{t}\right)\right] \tag{3.10}$$

where $C_1 = \hbar C/(4ed|\alpha|)$, and we have taken the other film to be at voltage V_0, and also neglect the vector potential A in $\nabla - 2\mathrm{i}eA/h$, since for small fields its effect can be largely accounted for as a shift in T_c. The quantity on the left of eq. (3.10) above is the usual TDGL equation for $T > T_c$. If the right-hand side were equal to zero, then in steady-state one would have $\psi = 0$. If the quantity on the right included thermal fluctuations, we would still have the time average of the order parameter $\langle\psi\rangle = 0$, although $\left\langle|\psi|^2\right\rangle$ would be nonzero, giving rise to excess electrical conductance of the film in the normal state above T_c (Skocpol and Tinkham 1975). In the present case, the quantity on the right is the effective pair-field which excites a particular mode of oscillation with wave vector q_0 and frequency ω_0 determined by the external electromagnetic field. The induced order parameter which results takes the form

$$\psi_{\mathrm{in}}\left(x,t\right) = C_1\exp\left[\mathrm{i}\left(q_0 x + \omega_0 t\right)\right] / \left[\mathrm{i}\omega_0\tau_{\mathrm{GL}} + \left(1 + \xi^2 q_0^2\right)\right]. \tag{3.11}$$

This can also be expressed in the language of linear response theory, where one can define the order parameter response function $\chi\left(x - x', t - t'\right)$ such that

$$\psi(x,t) = \int \mathrm{d}x' \int \mathrm{d}t'\zeta\left(x',t'\right)\chi\left(x - x', t - t'\right) \tag{3.12}$$

or in terms of Fourier transforms,

$$\psi_{\mathrm{in}}(k,\omega) = \zeta(k,\omega)\chi(k,\omega), \tag{3.13}$$

where in the present case

$$\zeta(k,\omega) = \alpha C_1 \delta\left(k - q_0\right) \delta\left(\omega - \omega_0\right) \tag{3.14}$$

$$\chi(k,\omega) = \left\{\alpha\left[i\omega\tau_{\mathrm{GL}} + \left(1 + k^2\xi^2\right)\right]\right\}^{-1} \tag{3.15}$$

so that

$$\psi_{\mathrm{in}}(x,t) = \zeta(x,t)\chi\left(q_0,\omega_0\right). \tag{3.16}$$

In other works, χ may be defined to correspond to Δ_{in} rather than to ψ_{in}, leading to the replacement of α in eq. (3.15) by $N(0)\left(T - T_c\right)T_c$. In addition, the definition of the susceptibility χ in terms of linear response theory is equivalent to the quantum-mechanical definition in terms of the commutator of operators

$$\chi\left(r - r', t - t'\right) = \begin{cases} -(i/h)\left\langle\left[\hat{\Psi}(r,t), \hat{\Psi}^{-}\left(r',t'\right)\right]\right\rangle & \mathrm{t} > t' \\ 0, & \mathrm{t} < t' \end{cases} \tag{3.17}$$

where ψ is the order parameter operator, [ ] represents the commutator, and $\rangle$ is the expectation value at finite temperature.

One can now obtain the "excess" supercurrent from the usual expression

$$I_{\mathrm{ex}} = \int_0^{\mathrm{w}}\int_0^L \mathrm{d}x\ \mathrm{d}y j_{\mathrm{J}} = (4edA/h)\,\mathrm{Im}\left(\psi_{\mathrm{in}}\zeta^*\right) = 4edA/\hbar|\zeta|^2\chi''\left(q_0\omega_0\right). \tag{3.18}$$

Fig. 9. Total I-V characteristic (points) for an $\mathrm{Al -\mathrm{Al}{2} \mathrm{O}{3}-\mathrm{Pb}$ junction at 1.96778 K . The solid line is the quasiparticle background determined from, and a least squares fit to, the data over the range $V = 150 - 200\mu$ V. The excess current is the portion of the curve above the solid lines. (From Carlson and Goldman 1976.)}

This general relationship between I_{ex} and χ is also valid below T_c, as will be shown below. For the case of $T > T_c$ here, the detailed dependences on V and H, in accordance with the simple TDGL equation,

$$I_{\mathrm{ex}} = \left(\hbar C^2 WL/4ed\alpha\right)\omega_0\tau_{\mathrm{GL}}/\left[\left(\omega_0\tau_{\mathrm{GL}}\right)^2 + \left(1 + {q_0}^2\xi^2\right)^2\right], \tag{3.19}$$

have been confirmed experimentally (Anderson and Goldman 1970, Anderson et al. 1972). Figure 9 shows a I V characteristic for an $Al_1 - Al_2O_3 - Pb$ junction, and fig. 10 presents the excess current I_{ex} obtained by subtracting off the quasiparticle background current and a small leakage current. Figure 11 shows the peak voltage V_p of the excess current voltage characteristic where in zero magnetic field $V_p = (2e/\hbar)\,(1/\tau_{GL})$.

Note that the I_{ex} is antisymmetric for the two polarities of voltage, and in fact is zero for zero voltage. This is unusual for a supercurrent, but is easily explained via a simple picture (fig. 12). Because of the $\cos\gamma$ term in the interaction energy, the ψ_{in} "wants" to point in the same direction in the complex plane as ψ' of the strong superconductor. This orientation will also yield the maximum value of the magnitude of ψ_{in} , but since the supercurrent goes as $\sin\gamma$, this corresponds to $I_{ex} = 0$. This is the case for $V = 0$. For $V > 0$, on the other hand, ψ' is rotating in the complex plane at an angular frequency $\omega_0 = 2\text{eV}/\hbar$. The induced order parameter "tries" to follow, but it has a characteristic response time τ_{GL}, so that as the frequency increases it begins to

Fig. 10. (a) Typical total current, leakage current, and quasiparticle current versus voltage, from top to bottom, respectively. (b) Excess current obtained by subtracting the leakage and quasiparticle currents from the total current in (a). In both parts the units are arbitrary.

lag behind. This gives a nonzero value of I_{ex} of the right sign. As this phase lag increases, however, the interaction energy decreases, so that the length of the vector representing ψ_{in} also decreases. Thus, as suggested in fig. 11, the projection of ψ_{in} perpendicular to ψ', which gives

the magnitude of I_{ex}, will increase to a maximum and then return to zero for high voltage.

The physical picture in fig. 12 indicates clearly that $I_{ex} = I_0 \sin\gamma$ is timeindependent for non-zero voltages, and is not the time-average of some oscillating current. This is true even though the coupling across the junction is time dependent. These facts would seem to contradict the AC Josephson relation, but this apparent paradox can be resolved using the concept of charge imbalance. As we pointed out earlier, the usual charge imbalance experiments involved, in effect, a tunnel injection of normal current I_n from a normal metal into a superconductor, which ultimately left the film as supercurrent I_s. This caused a build-up of "normal" charge density Q^* in the superconductor

$$Q^* = I_n \tau_{Q*}/\Omega \tag{3.20}$$

where Ω is the nonequilibrium volume. For the present case we are focussing

Fig. 11. Peak voltage $V_{\mathrm{p }}=\hbar / 2 e \tau_{\mathrm{GL}}}$ versus T for an $Al - Al_2O_3 - Pb$ junction. (From Goldman, 1981.)}

Fig. 12. Physical picture of a pair-field susceptibility measurement for $T > T_c$.
on the normal film. In the absence of superconducting order, there can be no charge imbalance since relaxation cannot occur. However, because of the interaction with the strongly superconducting electrode, the normal film has become effectively a weak "gapless" superconductor, even above T_{c}. In terms of the transmission-line picture in fig. 6, it has a superconducting channel. In effect, the excess pair current I_{ex} is a supercurrent injected into the normal film, which presumably leaves the film at the boundaries as normal current I_{n}. Therefore, this will produce a charge imbalance

$$Q^* = -I_{\mathrm{ex}}\tau_{\mathrm{R}}/\Omega \tag{3.21}$$

for some effective relaxation time τ_{R}. The negative sign is taken because the process is essentially the opposite to that described by eq. (3.20). Also, as it is a true supercurrent across the junction it should flow with no resistance, i.e. no voltage difference between the condensate electrochemical potential μ_{s} on the two sides. This is consistent only if the voltage V_0 in the quasiparticle channel across the junction is equal to the internal voltage drop Φ between the normal and superconducting channels of the weak superconductor (we are neglecting the small chemical potential shift on the strongly superconducting side). This requirement implies that the relaxation time τ_R must satisfy the relation

$$\tau_{\mathrm{R}} = 2N(0)e^2V_0Lwd/I_{\mathrm{ex}}. \tag{3.22}$$

Substituting the expression for I_{ex} in the low voltage, $q_0 = 0$, limit, one obtains

$$\tau_{\mathrm{R}} = mN(0)D/\left|\psi_{\mathrm{in}}\right|^2 = 2k_{\mathrm{B}}Th/\pi\Delta_{\mathrm{in}}^2, \tag{3.23}$$

using the theory of dirty superconductors. In fact, this relaxation time is identical to that derived by more standard methods for charge imbalance relaxation in a gapless superconductor below its T_c (Schmid and Schön, 1975a). Thus, the measurement of the pair-field susceptibility in this regime can be alternately viewed as a measurement of the steady-state charge imbalance relaxation time in a weak induced superconductor.

We emphasize that, although the results of measurements of the pair-field susceptibility must be identical to the spectrum of spontaneous thermal fluctuations (both deriving from the dynamical equations of the order parameter), the excess current I_{ex} is not a direct consequence of these fluctuations as it is determined in linear response. This may not have been widely appreciated, as the early work was developed at the same time as the effects of thermal fluctuations were being analyzed (Skocpol and Tinkham 1975). Since I_{cx} is a Josephson current, it depends on the existence of a stable phase correlation across the barrier. Such a correlation cannot result from spontaneous thermal fluctuations, which therefore do not contribute to the excess current. In fact, thermal fluctuations will act against well-defined features in the excess current, producing some broadening in the structure (Kadin and

Goldman 1982). On the other hand, in a measurement of excess electrical conductivity of a film above T_{c} only correlations within the film are important, and thermal fluctuations do contribute to these.

44.3.3 3.3. Pair-field susceptibility for $T < T_{\mathrm{c}}$

The determination of the pair-field susceptibility follows the same principles as for $T > T_c$, but it is somewhat more complicated for several reasons, both experimental and theoretical. First, there is the usual DC Josephson current for zero voltage, associated with the equilibrium value of the order parameter, which must be suppressed using a voltage and a magnetic field in order to see the (much smaller) desired effect. Shorts in the tunnel barrier are even more

deleterious here, since they can give rise to time-averaged supercurrents even for nonzero voltages. On the theoretical side, the simple TDGL equation is no longer valid except in special cases, and a sufficiently generalized dynamical equation is likely to be much more complicated. The situation is further confused by the fact that, as discussed earlier, longitudinal mode and transverse mode disequilibria tend to obey different dynamical equations, but the resulting I_{ex} yields a single susceptibility, which is a linear combination of the two modes, and makes them difficult to separate.

The most remarkable result in the experiments measuring I_{ex} below T_c was the observation of a second peak for an Al film (Carlson and Goldman 1973, 1975), which suggested the (then surprising) existence of a propagating collective mode in the superconductor, one which was later identified with the transverse collective mode (charge-imbalance waves) (Schmid and Schön 1975b, Artemenko and Volkov 1975). This peak is illustrated in fig. 13, which shows $I_{\mathrm{ex}}/V \sim S(q,\omega)$ for a range of temperatures above and below T_c. Presented in this way, the earlier peak associated with the time lag in the order parameter response is shifted to $V = 0$. The width of this peak is still a measure of this time lag, however, and the decreased width for $T < T_c$ is consistent with what we would expect for an increased time (e.g. τ_1). Furthermore, the position of the displaced peak ω_{p} is approximately proportional to q (fig. 14), in a way one would expect for resonant behavior associated with a wave of velocity $v = \omega/q$. The fact that the peak is broad, and eventually damps out entirely at low temperatures, is not surprising either, since charge imbalance waves are highly damped.

In explaining the theoretical behavior for $T < T_c$, we begin with the simple TDGL equation, even though it is valid in this regime only for gapless superconductors. In the same way as that obtained earlier, the TDGL equation including the interaction term is

$$\tau_{\mathrm{GL}}\frac{\partial\psi}{\partial t} - \left(1 - |\psi|^2/\psi_{\mathrm{eq}}^2\right)\psi - \xi^2\nabla^2\psi = C_1 \exp\left[\mathrm{i}\left(q_0 x + \omega_0 t\right)\right] \tag{3.24}$$

Fig. 13. Structure factor $I_{\mathrm{ex} } / V \sim S(g, \omega)$ in arbitrary units versus T and ω. The magnetic field is 125 Oe. Curves are shown only for the $\omega > 0$ portion and are symmetric about $\omega = 0$. $T_c(H) = 1.780$ K. Dashed lines are from Schmid and Schön (1975b). (From Carlson and Goldman 1976.)}

where the differences in sign relative to eq. (3.10) follow from the sign change of α. Since we are looking at small variations in ψ about its equilibrium value ψ_{eq} , we define

Fig. 14. (a) Position of peak in the structure factor $S(q,\omega)$ of Al plotted versus q in units of magnetic field at several temperatures. The points A-D correspond to $T = 1.771, 1.767, 1.764$ and 1.753 K , respectively. (b) Frequencies of peaks in $S(q,\omega)$ at fixed $\omega=6.32 \times 10^{-3}$. (From Carlson and Goldman 1976.)

$$\psi = \psi_{\mathrm{eq}} + \delta\psi^{\mathrm{L}} + \mathrm{i}\delta\psi^{\mathrm{T}} \tag{3.25}$$

where $\delta\psi^{\mathrm{L}}$ and $\delta\psi^{\mathrm{T}}$ are real functions that are small compared to ψ_{eq} . Substituting in and separating into real and imaginary parts, one obtains two equations

$$\left(\tau_{\mathrm{GL}}\frac{\partial}{\partial t} + 2 - \xi^2\nabla^2\right)\delta\psi^{\mathrm{L}} = C_1\cos\left(q_0 x + \omega_0 t\right), \tag{3.26}$$

$$\left(\tau_{\mathrm{GL}}\frac{\partial}{\partial t} - \xi^2\nabla^2\right)\delta\psi^{\mathrm{T}} = C_1\sin\left(q_0 x + \omega_0 t\right). \tag{3.27}$$

Clearly, the longitudinal variation $\delta\psi^L = g\psi_{\mathrm{eq}}$ exhibits somewhat different behavior than the transverse variation $\delta\psi^\top = i\theta\psi_{\mathrm{cq}}$, even without a more complicated dynamical equation. Equations (3.26) and (3.27) can be solved to yield

$$\delta\psi^L = \zeta(x,t)\chi^L\left(q_0,\omega_0\right) + \text{ c.c. }, \tag{3.28}$$

$$\delta\psi^T = \zeta(x,t)\chi^T\left(q_0,\omega_0\right) + \text{ c.c.}, \tag{3.29}$$

where

$$\chi^{\mathrm{L}}(k,\omega) = \left[2|\alpha|\left(\mathrm{i}\omega\tau_{\mathrm{GL}} + \xi^2k^2\right)\right]^{-1} \tag{3.30}$$

and

$$\chi^{\mathrm{T}}(k,\omega) = \left[2|\alpha|\left(i\omega\tau_{\mathrm{GL}} + \xi^2k^2\right)\right]^{-1}. \tag{3.31}$$

The pair field ζ is the same as in eq. (3.13) and c.c. represents the complex conjugate. Note that this is no longer simply a linear response, since there is also a "conjugate-linear" response proportional to the complex conjugate of ζ. However, as the conjugate term gives a contribution at frequency $2\omega_0$, it is only the linear response term that contributes to the time-averaged DC excess current

$$I_{\mathrm{ex}} = \frac{\hbar \bar{C}^2 w L}{4ed} \operatorname{Im}\left[\chi^{\mathrm{L}}\left(q_0, \omega_0\right) + \chi^{\mathrm{T}}\left(q_0, \omega_0\right)\right]. \tag{3.32}$$

This current is in addition to the usual Josephson supercurrent for $V = 0$, $H \simeq 0$, and $I_{\mathrm{s}} = CwL\psi_{\mathrm{eq}} \sin\gamma$. Note that the excess supercurrent is of higher order in the small coupling constant $\bar{C}$, and can only be seen when the usual supercurrent is not present or is greatly reduced in magnitude.

In eqs. (3.26) and (3.27), both $\delta\psi^{\mathrm{T}}$ and $\delta\psi^{\mathrm{L}}$ obey diffusive equations, i.e. with only a first derivative in time. This will clearly not produce the "displaced
peak" seen in fig. 13, that we have associated with propagating waves, and which requires a second derivative in time. As an example of how the present approach would deal with a generalized TDGL equation which contains the propagating mode, consider eq. (2.28) (Kramer and Watts-Tobin 1978), as modified by Skocpol (1981) (also Octavio and Skocpol 1981) to account for charge imbalance waves:

$$\frac{\tau_{\mathrm{GL}}}{\left(1+\gamma^2 f^2\right)^{1/2}}\left(\frac{\partial}{\partial t} + \frac{2\mathrm{i}e\phi}{\hbar} + \frac{2\mathrm{i}\tau_{\mathrm{E}}}{\hbar}\dot{\phi} + \frac{\gamma^2}{2}\frac{\partial f^2}{\partial t}\right)\psi = \left(1-f^2\right)\psi + \xi^2\left(\nabla - \frac{2\mathrm{i}eA}{\hbar}\right)^2\psi. \tag{3.33}$$

We will work in the experimentally accessible regime where $\gamma = 2\tau_{\mathrm{E}}\Delta(\mathrm{T})/\hbar \gg 1$. Adding in the pair field ξ and proceeding as before, one obtains the pair of equations

$$\left(\tau_A \frac{\partial}{\partial t} + 2 - \xi^2\right)\delta\psi^L = C_1 \cos\left(q_0 x + \omega_0 t\right) \tag{3.34}$$

$$\left(2\tau_{\mathrm{GL}}/\hbar\right)\left(\dot{\Phi} + \tau_{\mathrm{E}}\ddot{\Phi}\right) - \xi^2 \nabla \cdot q = \left(C_1/\psi_{\mathrm{eq}}\right)\sin\left(q_0 x + \omega_0 t\right) \tag{3.35}$$

where here $q = \nabla\theta - 2ieA/\hbar$ is the gauge-invariant phase gradient. From the first equation one obtains, as expected

$$\chi^{\mathrm{L}}(k,\omega) = \left[2|\alpha|\left(i\omega\tau_A + 2 + \xi^2 k^2\right)\right]^{-1} \tag{3.36}$$

i.e. the same as eq. (3.30) but with $\tau_\Delta = \tau_{\mathrm{GL}}\gamma$ substituted for τ_{GL}. One can solve the transverse equation using $J_{\mathrm{s}} = \psi^2$ ehq,m and applying the condition of electroneutrality in a two-fluid picture as done before eq. (2.33) to derived charge imbalance waves. The following equation in the nonequilibrium potential ϕ results:

$$\begin{aligned}&\left[\tau_{\mathrm{J}}\tau_{\mathrm{E}} + \tau_{\mathrm{E}}\frac{\partial^2}{\partial t^2} + \left(\tau_{\mathrm{J}} + \tau_{\mathrm{E}}\right)\frac{\partial}{\partial t} + 1 - \Lambda^2\nabla^2\right]\Phi \\ &\quad = \frac{\hbar\gamma}{2\tau_{\mathrm{GL}}}\frac{\hbar \mathrm{C}}{4ed|\alpha|\psi_{\mathrm{eq}}(T)}\left[\omega_0\tau_j\cos\left(q_0 x + \tau_0 t\right) + \sin\left(q_0 x + \tau_0 t\right)\right]\end{aligned} \tag{3.37}$$

From this, one obtains the transverse susceptibility

$$\chi^{\mathrm{T}}(k,\omega) = \Gamma\left(1 + \mathrm{i}\omega\tau_J\right)/2\mathrm{i}\omega\tau_{\mathrm{GL}}\left[\left(1+\mathrm{i}\omega\tau_{\mathrm{J}}\right)\left(1+\mathrm{i}\omega\tau_{\mathrm{E}}\right) + \Lambda^2 k^2\right]. \tag{3.38}$$

For high frequencies $\omega \gg \tau_{\mathrm{J}}^{-1}, \tau_{\mathrm{E}}^{-1}$, this clearly exhibits a resonance at a frequency

$$\omega = \mathrm{k}\Lambda/\left(\tau_{\mathrm{J}}\tau_{\mathrm{E}}\right)^{1/2} \tag{3.39}$$

corresponding to a wave with velocity

$$v = (A\tau_1\tau_1)^{12} = [2D, \|(T), h]^{12} \tag{3.40}$$

as expected for charge imbalance waves, and found experimentally by Carlson and Goldman (1973) in the excess current $I_{\text{ex}} \sim \chi''(k,\omega)$.

Experiments have also been carried out for $T < T_{\text{c}}$ on weakly superconducting Al films doped with paramagnetic Er ions, for varying concentrations of Er designed to give different degrees of pair-breaking via spin-flip scattering (Aspen and Goldman 1981a). These measurements of the pair-field susceptibility confirmed the theoretical expectations that a sufficient amount of pairbreaking would eliminate the propagating mode.

Another feature of the data below T_{c} is the appearance of a resonant peak at the gap voltage, largely independent of the wave vector q, which can be identified with a high-frequency resonance in the longitudinal mode contribution to $\chi''(k,\omega)$. This resonance is included in several theoretical analyses (Dinter 1977, 1978, Entin-Wohlman and Orbach, 1978, Schön and Ambegaokar 1979), and may be analogous to the "Riedel singularity" (Riedel 1964) in the Josephson response of an ideal superconducting weak link. On the other hand, an alternative interpretation based on two-particle tunneling and pair-breaking by the AC Josephson effect (Simanek and Hayward 1974) cannot be ruled out by the data alone.

Finally, an alternative way of looking at the presence of I_{ex} for $T < T_{\text{c}}$ is by noting that the AC pair field ζ acts analogously to an external AC voltage of frequency $\omega = 2\text{eV}/\hbar$ across the Josephson junction. In the latter situation the external signal at ω mixes with the internal Josephson oscillator at ω to give a DC constant voltage step, together with an oscillatory current at 2ω. The position along the constant voltage step is related to the phase difference between the external and internal oscillators. In the present case, this phase difference is fixed by the interaction that gives rise to $\text{d}\psi_{\text{in}}$, so that this additional degree of freedom is absent.

44.4 4. Phase-slip centers and microbridges

44.4.1 4.1. Phase-slip centers

It is often so that a simple geometry greatly facilitates the study of physical phenomena. A superconducting filament is one such system, which is geometrically simple, but makes for a surprisingly rich and complex system. If the cross-sectional dimensions are small compared with the coherence length $\xi(T)$ (which in practice, close to T_c, may be about a μm), the filament can be treated as effectively one-dimensional. A real version of such an idealized sample can be a thin-film microstrip or a whisker crystal, with length of order 100μ m. The current-induced breakdown of superconductivity in such a sample is still not understood in its full complexity. We will deal with the problem in terms of approximations which are amenable to physical interpretation, while including references to some of the more rigorous treatments. (See the review by Galaiko and Kopnin (1985) in the present volume.)

The resistive transition of such a sample near T_{c}, as seen in the DC $I - V$ curve, is characterized by the appearance of a series of voltage steps, each followed by a sloping plateau of approximately constant differential resistance, as shown in fig. 15. Each increment of differential resistance is about the same from one plateau to the next, at least for the first few steps. This basic picture has been observed in microstrips of Sn (Skocpol et al. 1974a), Al (Klapwijk and Mooij 1976b), In (Jillie 1976), and other elements and alloys (Boone et al. 1977, Mller and Baumann 1978), and in whisker crystals of Sn and Sn alloys (Meyer and von Minnigerode 1972, Meyer and Tidecks 1976, Tidecks and Meyer 1979). These constant increments of differential resistance suggests that the phenomena consists of the successive onset of a series of similar dissipative units arrayed along the filament. This was confirmed by Skocpol (Skocpol et al. 1974a), who fabricated a microstrip with a series of side-tabs located at intervals along the

Fig. 15. Current-voltage curve for a Sn microstrip. Note the step structure with the incremental increase in the differential resistance. The inset shows the geometry of the sample.

length of the strip, which acted as voltage probes. The
dissipative units were found to enter singly, and tended to space themselves relatively far apart. It makes sense, therefore, to speak in terms of discrete spatially localized units, although as will be discussed later, their interaction is also of interest.

Skocpol, Beasley, and Tinkham (SBT) (Skocpol et al. 1974a,b) proposed a model in which each of these units is identified as a phase-slip center (PSC). Each PSC first appears at its own local value of the critical current, as determined by the local critical temperature and cross-sectional area, and by the presence of other PSCs nearby. The differential resistance of each PSC is approximately equal to the normal-state resistance of a length $2A$, where the precise identity of Λ was unclear originally, but has since been shown to be the diffusion length for the charge imbalance relaxation time τ_{Q^*}. In the Sn sample of fig. 15, Λ was about 6μ m and the corresponding relaxation time $\sim 10^{-9}$ s.

The nonequilibrium region is not fully normal, however, since extrapolating the straight-line slope down to the current axis gives a positive value. This implies that only a portion of the current, associated with the excited quasiparticles, contributes resistance, and the remainder, a (fairly) constant fraction of the critical current, is carried as dissipationless supercurrent. This suggests that we can deal with the problem in terms of the generalized twofluid model discussed earlier, which we illustrated using the transmission-line picture.

In equilibrium, within the SBT model of the PSC, the electrochemical potentials $\mu_{\rm n}$ and μ_ς are equal far from the center of the dissipative region, but show fundamentally different behavior in the nonequilibrium region near a PSC. As indicated in fig. 16, a voltage drop across a PSC must appear as a change in both of these potentials, but the change occurs over two different length scales. A gradient in μ, produces an acceleration of the supercurrent I_s and leads to a

relaxation oscillation (or phase-slip cycle) of the order parameter within a region of order of the GL coherence length $\xi(T)$. The timeaveraged potential μ, changes abruptly within this region and is constant outside. The phase-slip process causes the time-averaged supercurrent I at the core region to be depressed below the value it has far away, so that if the total current is to remain constant, the dissipative normal current I_{n}, which is driven by gradients in μ_{11}, must make up the difference. This interconversion between normal and supercurrent must take place over the scale of Λ (typically larger than ξ), the length required for the nonequilibrium quasiparticles carrying I_{n} to come back into equilibrium with the pair condensate.

This picture of the DC effects near a PSC was confirmed quite directly in the experiment of Dolan and Jackel (1977), where a closely spaced array of superconducting and normal probes measured the spatial behavior of these time-averaged potentials. As predicted, the value of μ_n jumped abruptly at the center of the PSC, while μ_{n} relaxed exponentially back to the equilibrium value on each side of the center with a decay length which fit the form $A = (D\tau_{Q^*})^{1/2}$.

The weak divergence in $\Lambda \sim (T_{\mathrm{c}} - T)^{-1/4}$ was not apparent in many of the earlier measurements on $I - V$ curves of PSCs, in part because of slight variations in T_{c} and the difficulty of clearly separating one PSC from the next very close to the transition. Furthermore, heating-related effects farther from T_{c} have often produced distortions of the $I - V$ curves which can mask the slight divergence. Nevertheless, within the limited temperature range over which changes in differential resistance of a PSC accurately reflected changes in the decay length, Dolan and Jackel (1977) did find that this resistance corresponded to the normal resistance of a length Λ_{Q^*} and further careful work has confirmed this (Kadin et al. 1978, Stuivinga et al. 1982, 1983, Aponte and Tinkham 1983).

In the limit where $\Lambda \gg \xi$, that is being considered in fig. 16 (even if it is sometimes of marginal validity), the details of the superconducting dynamics in the core region, which produce the actual phase-slip process itself, are not crucial to the model. However, the fact that there are high-frequency Josephson oscillations in the core is important in determining the timedependent behavior of the currents and potentials in the larger region outside the core.

A simple model which incorporates the DC results of SBT, but explicitly includes AC effects, is that of Kadin et al. (1980) (KSS), which represents the problem in terms of a Josephson oscillator coupled to the transmission-line picture of propagating charge imbalance waves (fig. 17a). The oscillations in

the magnitude of the gap, which constitute the phase-slip process, are assumed to be restricted to a core region with size ξ that is small compared to the length scale A characterizing variations in the charge imbalance, and that this core region may be modeled by a discrete Josephson oscillator of zero length and zero resistance, with a current-phase relation of the form

$$I_{\mathrm{s}}(x = 0, t) = I_{\mathrm{c}} f(\Delta\theta(t)), \tag{4.1}$$

where $\Delta\theta$ is the phase difference across the core, f is some function with period 2π and amplitude unity, and I_c is the critical current of the core. The form of this function must be given ad hoc, although one would not necessarily expect the purely sinusoidal dependence appropriate for weak links small compared to ξ.I_{c} is taken to be equal to or slightly smaller than the critical current of the adjacent regions of the filament. The Josephson oscillator lies in the condensate line because it is driven by the difference in μ_s :

$$V(t) = \left[\mu_{\mathrm{s}}\left(x = 0^-, t\right) - \mu_{\mathrm{s}}\left(x = 0^+, t\right)\right]/e = \Phi\left(0^+\right) - \Phi\left(0^-\right) \tag{4.2}$$

where the condition that μ_n is continuous through the origin follows from the assumption of negligible junction resistance.

As we commented earlier, the effect of the semi-infinite line on either side of the Josephson oscillator can also be obtained by replacing the line with its characteristic impedance $Z(\omega)$, as in fig. 17b. The voltage across the junction can then be expressed as

$$V(t) = 2\int \mathrm{d}\omega \mathrm{e}^{\mathrm{i}\omega t} Z(\omega)\bar{I}_{\mathrm{n}}(0,\omega) \tag{4.3}$$

This relation assumes, of course, that there is not another PSC or a reflective discontinuity in the transmission line nearby, on the scale of Λ. In the more general case of a total current with an AC component, the AC current far from the Josephson element would be split between I_{s} and I_{n} in accordance with the ratio between R and $i\omega L$, so that a modified fig. 17b would include connections to both sides of $Z(\omega)$. One can also consider cases where the junction resistance itself is not negligible compared to $Z(\omega)$, e.g. for a small in-line tunnel junction, as in the schematic in fig. 17c.

The high-voltage DC behavior of the PSC takes a particularly simple form. At the center of the PSC most of the current $I \geqslant I_{\mathrm{c}}$ must be normal current I_{n}, which in terms of fig. 17c flows through the shunt impedance. The oscillations in $I_{,\text{ of order}}\, I_{\mathrm{c}}$ are then relatively small, so that the Josephson element is subject to an almost constant voltage bias. Therefore, the phase difference $\Delta\theta$ across the element increases nearly linearly with time, and one can express the $I-V$ relation in the form

$$\bar{V} = 2Z(0)\left(I - I_{\mathrm{s}}(\overline{0,t})\right) = 2R\Lambda\left(I - I_{\mathrm{c}}\langle f\rangle\right) \tag{4.4}$$

where

$$\langle f\rangle = \int_0^{2\pi} f(\Delta\theta)\mathrm{d}(\Delta\theta)/2\pi \tag{4.5}$$

is the average value of the periodic function f and the horizontal bars represent averages over time. In the high-voltage regime, the model predicts a constant differential resistance

$$\mathrm{d}\bar{V}/\mathrm{d}I = 2Z(0) = 2R\Lambda \equiv R_{\mathrm{eff}} \tag{4.6}$$

and an extrapolated excess current $I_{\mathrm{ex}} = \langle f\rangle I_{\mathrm{c}}$, in agreement with the SBT model. This is equivalent to a simple resistively shunted junction (RSJ) model with shunt resistance R_{eff}. Note, however, that R_{eff} is not the intrinsic junction resistance, which we have assumed to be negligible, but the resistance of the nonequilibrium region which extends to a distance Λ on either side of the core.

This leads to a characteristic voltage in the problem, $V_{\mathrm{c}} = I_{\mathrm{c}}R_{\mathrm{eff}}$, at which the fundamental Josephson frequency is

$$\omega_{\mathrm{c}} \equiv 2eI_{\mathrm{c}}R_{\mathrm{eff}}/\hbar = 4eI_{\mathrm{c}}R\Lambda/\hbar \tag{4.7}$$

For much lower voltages, the Josephson oscillations become more pulse-like and are dominated by higher harmonics of $2\mathrm{eV}/\hbar$ up to of order ω_{c}. For larger voltages, on the other hand, the oscillations become small perturbations on the DC average. One can further compare ω_{c}^{-1} to the characteristic times τ_{J} and τ_{E}. For a typical real material in a reasonable temperature range, one has $\omega_{\mathrm{c}}\tau_{\mathrm{E}} \gg 1$ and $\omega_{\mathrm{c}}\tau_{\mathrm{J}} > 1$. In the "high frequency" regime where both are very large, a particularly simple approximation becomes valid. In this case, the effective AC impedance shunting the junction in fig. 17b becomes a simple resistance

$$R_{\mathrm{ac}} = R_{\mathrm{eff}}\sqrt{\tau_{\mathrm{J}}/\tau_{\mathrm{E}}} \tag{4.8}$$

The $I-V$ curves for this situation can be expressed in terms of the solution of the resistively shunted junction (RSJ) model (see e.g. Solymar 1972) for the resistance R_{ac}, in the form

$$I = I_{\mathrm{RSJ}}\left(\bar{V}, R_{\mathrm{ac}}\right) + \bar{V}\left(1/R_{\mathrm{eff}} - 1/R_{\mathrm{ac}}\right) \tag{4.9}$$

that is, with a correction that accounts for the different de resistance. In fig. 16 we have plotted the $I - V$ relation of eq. (4.9) for various values of $\tau_{\mathrm{E}}/\tau_{\mathrm{J}}$, using the linear current-phase relation.

The most striking feature about the curves of fig. 18 is the existence of nonzero voltage solutions for currents less than the critical current I_c. so that V is a hysteretic function of I (the more customary control variable in experiments). For the special case $\tau_{\mathrm{E}} = \tau_{\mathrm{J}}$, eq. (4.9) gives exactly the RS.J curve, while if $\tau_{\mathrm{E}} < \tau_{\mathrm{J}}$, the curve lies to the right of the RSJ curve [i.e. it has a larger $I(V)$]. In the more usual case $\tau_F > \tau_j$ and this hysteresis occurs. This is true

for any current–phase relation, and since the RSJ curve always rises vertically at $I = I_{\mathrm{C}}, \tau_{\mathrm{E}} > \tau_{\mathrm{J}}$ provides the condition for hysteresis. This condition can be understood by noting that $R_{\mathrm{ac}} \ll R_{\mathrm{eff}}$, so that the voltage driving the Josephson oscillation is closer to being a constant voltage bias. The resulting $I - V$ curve then lies closer to the voltage-biased line which forms the high-voltage asymptote, and as is evident from fig. 16, it approaches this line at lower and lower voltages as τ_E/τ_J is increased.

In an actual experiment, one typically measures voltage versus current at a series of temperatures near T_c. In fig. 19, we plot theoretical curves from the KSS model in an analogous way, showing the onset and development of hysteresis. They correspond to the linear current phase relation in the highfrequency limit, and were obtained from curves as shown in fig. 16 using the temperature scaling relations ($t \equiv T/T_{\mathrm{c}}$)

$$\tau_{\mathrm{E}}/\tau_{\mathrm{J}} \sim (1-t); \quad I_{\mathrm{c}} R_{\mathrm{eff}} \sim (1-t)^{5/4} I_{\mathrm{c}} \sim (1-t)^{3/2}. \tag{4.10}$$

Within the given approximations, these curves are independent of material. The onset of hysteresis at $\tau_{\mathrm{F}/\tau_1} = 1$ would correspond to $T_{\mathrm{c}} - T = 4\mathrm{mK}$ for Sn , and 0.05 mK for Al (Kaplan et al. 1976, Chi and Clarke 1979, and Clarke and Paterson, 1974).

The development of hysteresis in fig. 19 is qualitatively very similar to that seen in many experiments on both microstrips and whiskers. Quantitative comparisons, however, are complicated by a number of factors that are difficult to fold into such a model, such as noise-induced smearing of IV curves close to T_{c} and heating effects farther from T_{c}. In particular, heating has been shown to account reasonably well for hysteresis in some cases (Skocpol et al. 1974b, Kadin et al. 1978, Likharev 1979). The intrinsic hysteresis in the KSS model is a supplementary mechanism which should become important near T_{c}, where heating-induced hysteresis disappears more rapidly. The onset of hysteresis close to T_{c} does seem to be consistent with the picture, although some rather large sample-to-sample variations remain to be explained. Furthermore, in clean tin whisker crystals, where heating effects are somewhat less severe, Meyer and Tidecks (1976) have noted a saturation effect in the hysteresis similar to that predicted by the KSS model.

Since the model consists of a driven Josephson oscillator coupled to a transmission line, it can also be used to exhibit AC effects, in particular the propagation of charge imbalance waves away from the center. Looking first at oscillations in time at the center, fig. 20 shows I_{n} and Φ for parameters appropriate to Sn at $T/T_{\mathrm{c}} = 0.99, V = I_{\mathrm{c}} R_{\mathrm{eff}}$, and $I_{\mathrm{c}} = I_{\mathrm{c}}$ (strip), using the linear current-phase relation. The oscillations repeat with the Josephson period, and

the abrupt jumps at the end of each cycle represent the discontinuous phaseslip process built into the piecewise-linear $f(\Delta\theta)$. These act as the source for the driven waves shown in fig. 21 as a function of position at a fixed time. The discontinuous nature of these jumps is again an artifact of the linear currentphase relation, but here they make the progression of the waves easily visible. Their wavelength λ is given in the high-frequency approximation as

$$\lambda = (\Lambda/\sqrt{\tau_{\mathrm{J}}\tau_{\mathrm{E}}})\,(h/2e\bar{V}), \tag{4.11}$$

which works out to 0.85Λ for the case of fig. 21 . The corresponding spatial behavior of $\mu_{\rm n}$ and $\mu_{\rm s}$ is shown in fig. 22, and clearly conforms in its time average to the earlier picture of SBT. It is also evident that the waves are decaying as they move away from the center. For the high-frequency limit in the condition $\tau_{\rm E} \gg \tau_{\rm J}$ the AC decay length $\Lambda_{\rm AC}$ is smaller than the DC decay length Λ by a factor $\sim (\tau_{\rm J}/\tau_{\rm E})^{1/2}$, which for figs. 18-20 is about 0.6 A . The difference between the two lengths is more dramatic for Al , for which $\tau_{\rm E}$ is about 80 times larger than it is for Sn . This is demonstrated in fig. 23, where $I_{\rm n}$

is plotted for Al with $T/T_{\rm c} = 0.99$ and $V = I_{\rm c}R_{\rm eff}/2$. In this case one has $\Lambda_{\rm AC} = \Lambda/8$, although in fact $\Lambda_{\rm AC}$ has in fact not decreased relative to Sn ; rather it is Λ proportional to $\tau_{\rm E}^{1,2}$ that has increased. In real terms, Λ remains on the order of a few μm, and in a later section we will consider the interaction of multiple PSCs spaced on this scale, via these driven oscillations of nonequilibrium charge imbalance.

There is also an AC voltage V_∞ across the PSC, even far away from the center, so that a PSC can act as a (weak) source of external microwaves, and conversely will show a Josephson step upon radiation with microwaves. However, this AC voltage is actually much smaller in amplitude than the value $I_c R_{\rm eff}$ that one might initially infer. In fact, it follows from the KSS model that in the regime we have been dealing with,

$$V_\infty \approx I_{\rm c}R_{\rm eff}/2\omega\,(\tau_{\rm J}\tau_{\rm E})^{1/2}\,, \tag{4.12}$$

which is not only smaller than one might expect, but in addition it has a faster roll-off with voltage (Kadin et al. 1980). This behavior is qualitatively consistent with the observation of a rather small AC Josephson step for a PSC in

Sn (Skocpol et al. 1974a) and the absence of such an effect for a PSC in Al (Klapwijk and Mooij 1976a,b).

Finally, we reiterate that the KSS picture does not represent a complete selfconsistent model of a PSC, in that the treatment of the phase-slip process in the core is not really treated as a true nonequilibrium process. In particular, Baratoff (1982a,b) has pointed out, on the basis of numerical analysis of the complete set of nonlinear equations, that the oscillations in the core may be expected to be largely decoupled from the charge imbalance waves out in the "wings". Nevertheless, the qualitative pictures in the KSS model are likely to retain some validity. For completeness, we also mention several other theoretical approaches to the self-consistent treatment of PSC: Kulik (1980), Ivlev and Kopnin (1980, 1981, 1982), and Watts-Tobin et al. (1981). The subject is by no means closed.

44.4.2 4.2. Superconducting microbridges

The Josephson effect was originally derived for and seen in tunnel junctions, but quite early on, similar phenomena were seen in a wide assortment of other weak links. In particular, the two basic types that have been studied in detail are the thin film microbridge (basically a microscopic constriction in a

superconducting film) and the point contact (a sharpened superconducting wire contacting a plate). We will concentrate on the microbridge, although in fact microbridges have evolved to become more and more like point contacts in an attempt to imitate the three-dimensional geometry of the latter. Since various extensive reviews of microbridge behavior (Lindelof, 1981, Bindslev Hansen and Lindelof 1981, Likharev 1979) have already appeared recently, the current treatment will attempt to identify several aspects of microbridge behavior that illustrate the points that we have been making in earlier sections, namely that dynamic non-equilibrium phenomena at the Josephson frequency can be reflected in the DC $I-V$ characteristics. See also the reviews by Aslamazov and Volkov (1985) and by de Lozanne and Beasley (1985) in the present volume.

In a typical tunnel junction, the connection between the two superconductors is an intervening layer of insulator, and the interaction is due to the overlap of the weak evanescent tails of the wave functions on the two sides. The current densities in the superconductor itself are rather small, and nonequilibrium effects are kept to a minimum, and are relatively easily understood. In the microbridge, on the other hand, the bridge typically consists of the same superconducting material as the banks, so that the weak interaction must be produced geometrically. In fact, in order to produce an ideal Josephson effect, all the dimensions of the bridge region must be less than the GL coherence length $\breve{\zeta}(T)$, typically less than a micron (Likharev 1979). In the voltage state, AC Josephson effects are produced by periodically driving the gap to zero by the application of a large current density. Even in the banks, the nonequilibrium effects can often not be neglected, and in the bridge region itself, these effects must necessarily be in the highly nonlinear regime. For examples of different microbridge geometries, see fig. 24.

One of the simplest of these nonequilibrium effects near the bridge, yet one of the most crucial, is Joule heating. The effective dimensionality of the bridge is of great importance here. For a one-dimensional bridge (i.e. a PSC), the hot electrons will diffuse out over quite a substantial distance (typically several microns) before the heat is finally transferred to the substrate. For two and three dimensions, on the other hand, there is a dilution process that acts to lower the temperature even without transfer of heat out of the film. Even then, a simple, general argument shows that for an ideal three-dimensional contact, the temperature T_{m} in the middle of the bridge will be enhanced above T_0 of the bath according to

$$T_{\mathrm{m}} = \left[T_0^2 + (3.2\ \mathrm{V})^2\right]^{1/2} \tag{4.13}$$

where the voltage V is given in mV and the temperatures are in K (Tinkham et al. 1977). For high-voltage (and thus high-frequency) operation, this emphasizes the necessity of trying to achieve a three-dimensional geometry, within a thin film. This has been achieved experimentally with a so-called variable-

thickness bridge (VTB) (Klapwijk and Mooij 1975, Gubankov et al. 1977, Octavio et al. 1977), where the thickness of the bridge region is much less than that of the banks. It has been in these types of microbridges that the best and most consistent Josephson performance has been found. This is fortunate, because the theoretical analyses have, for reasons of simplifying the calculations, been going in the same direction, with the "one-dimensional superconductor with ends in equilibrium" (ODSEE) (Likharev, 1979). We will discuss some of the features seen in these kinds of bridges, together with simple physical explanations of their origins.

Given the fact that these small VTBs have an almost ideal sinusoidal supercurrent-phase relation, and that the capacitance across the bridge is generally negligible, one might initially expect that the simple resistivelyshunted junction (RSJ) model might provide an acceptable first estimate of the $I - V$ characteristics of such a junction. In fact, there are several very striking deviations from this simple behavior, as suggested in fig. 25. As the current is increased above the critical current I_{c}, there is first the low-voltage "foot" (Octavio et al. 1978, Gubankov et al. 1977, 1980), followed by a fairly sharp rise in voltage at an "enhanced critical current", and several noticeable wiggles at subharmonics of the energy gap 24 . Finally, at high voltages there is a quasi-ohmic region, which however extrapolates to the current axis with a positive current.

The current understanding of the low-voltage foot is that it is a manifestation of nonequilibrium "dynamic cooling" in a short microbridge. The following physical explanation, due to Tinkham (1979, 1981), was further refined by Schmid et al. (1980) and is closely related to other calculations of Golub (1976) and Aslamazov and Larkin (1976). In this picture (see fig. 26), the bridge is so short that the quasiparticles in the bridge region are in equilibrium with the banks, provided that their energies are at least as great as the energy gap Δ_0 of the banks. In the presence of current, the gap is decreased in the bridge, and the quasiparticles with energies

$E < \Delta_0$ are essentially trapped. They can bounce back and forth within the bridge (via Andreev reflection) (Andreev 1964), but cannot leave the bridge except by an inelastic scattering event, set by the electron-phonon time τ_E. A current greater than the critical current I_{c_0} will cause the gap in the bridge to oscillate cyclically at

the Josephson frequency between Δ_0 and 0 (in the center). Now, at low frequencies $\omega\tau_{\mathrm{E}} \ll 1$, this does not present a problem; the low-energy quasiparticles have time to remain at equilibrium. In the other limit $\omega\tau_E \gg 1$, however, the low-energy quasiparticles are permanently under-represented. Their occupancies are more appropriate for higher-energy quasiparticles. Given this, and the fact that the low-energy quasiparticles are actually most important in determining the gap, the value of the gap and of the effective critical current I_{c_1} (here signifying the amplitude of the Josephson oscillations) will be enhanced above the equilibrium value I_{c_0}. For intermediate frequencies $\omega\tau_{\mathrm{E}}$ on the order of unity, there will be an intermediate degree of critical current enhancement. This enhancement will saturate at a Josephson frequency of several times $1/\tau_E$, above which the $I \cdot V$ curve takes a form similar to a RSJ with a larger value of I_{c}. The theory (Schmid et al. 1980) predicts that $I_{c_1}/I_{c_0} = 1.33$, which agrees very well with experiments (Octavio et al. 1978) after corrections for finite length have been put in.

The "excess current" suggests the presence of a time-averaged supercurrent in parallel with the normal dissipative current, although other interpretations are also possible. This excess current, which also appears in PSCs, typically has a value of $0.6 - 0.8$ of the enhanced value of I_c (Octavio et al. 1978, Gubankov et al. 1977, 1980). Deviations from a purely sinusoidal currentphase relation might be able to explain some of this, although this does not appear to be the case for these small bridges. It has also been suggested, on the basis of TDGL theory, that the excess current is really an "insufficient voltage" due to the finite response time of the gap, taken together with the rigid boundary conditions (Likharev and Yacobson 1975). Finally, a recent analysis has suggested that the main contribution to this excess current may not be due to dynamic processes at all, but rather to a static process involving transport of current across the bridge region with the help of Andreev reflection at the boundaries with the banks (Artemenko et al. 1979). This excess current will occur even in a normal superconducting contact, where clearly Josephson dynamics are not relevant. The theory predicts a value

$$I_{\mathrm{ex}} = 4\Delta/3eR_{\mathrm{N}}, \tag{4.14}$$

in the limit of small contacts of zero length, where the latter form has assumed an ideal junction $I_{\mathrm{c}}R$ product. This picture has also been extended within a unified theory to allow for a transition between a "clean" bridge contact (which shows the effect) and a "dirty" tunnel contact (which does not) in terms of the scattering at the interface (Blonder et al. 1982). Reasonable agreement with this theory has been achieved with point contacts (Divin and Nad' 1979, Blonder and Tinkham 1983).

The subharmonic gap structure is another example where the effect may or may not be related to dynamic effects at the Josephson frequency. These features, which show up most clearly in the conductance $\sigma = \mathrm{d}I/\mathrm{d}V$, are shown

for one Sn microbridge in fig. 27. They appear approximately at voltages $V = 2\Delta(T)/ne$, where $n = 1, 2, 3, \ldots$, up to as many as 12 , and with peak heights that typically tend to decrease monotonically with n. If one assumes that this relation should actually be exact, and that deviations from it are due to local heating in the bridge, then these features can be used as a thermometer for the bridge temperature, as has been done by Octavio (Octavio et al. 1977), giving good agreement with the heating model of Tinkham et al. (1977).

The origin of the subharmonic gap structure remains a subject of debate. Earlier models involved multiparticle tunneling (Schrieffer and Wilkins 1963) or self-coupling of the Josephson radiation to the microbridge (Werthamer 1966). However, the former does not explain subharmonics of index greater than two whereas the latter can account directly only for the even

subharmonics (n even). Combining the latter with single particle tunneling can give the odd subharmonics, but a quantitative fit given the similar magnitudes of both harmonics has been lacking. Lindelof (1981) has recently suggested
that rather than the usual Josephson radiation, quantized charge imbalance waves created by the microbridge could produce these features. Finally, the recent picture that used Andreev (1964) reflection to explain the excess current (Klapwijk et al. 1982, Octavio et al. 1983) has also been applied to explain subharmonic gap structure via multiple Andreev reflection of trapped quasiparticles inside the bridge. Again, this does not involve dynamics at the Josephson frequency. This picture has no difficulty with the odd and even harmonics, and furthermore can explain the features in point contacts between unlike superconductors with different values of Δ in a way that the selfcoupling models cannot. A complete quantitative test of the various models, however, requires a refinement of both the theories and the experiments on real bridges.

44.4.3 4.3. Interactions between microbridges or phase-slip centers

In attempting to understand the superconducting dynamics in a microbridge or phase-slip center, one of the more important techniques is to use a Josephson device to probe the dynamics produced by the first Josephson device (i.e. the microbridge or PSC). In fact, many of the effects discussed in previous chapters have in effect used just this technique, except that the detector was the same device as the "injector" of the nonequilibrium dynamics. There are some cases where the configuration of a strong injector and a weak probe is most illustrative. There are others in which a more symmetric configuration involving Josephson oscillators mutually influencing each other has been involved, and still others where a single junction has been used in a resonant mode as a self-detector.

Consider first the experiment of Skocpol and Jackel (1981), illustrated schematically in fig. 28a, where a superconducting tin microstrip contained a localized weak spot, and nearby, on the scale of the AC decay length for charge imbalance (about 2μ m), was located an array of small Josephson tunnel junctions, with lead counter-electrodes. For currents exceeding the critical current, a PSC was nucleated at the weak spot, and would be expected to generate charge imbalance waves which extend out to the tunnel junction probes. In particular, the condensate electrochemical potential μ_{s} should oscillate with frequency $2eV\hbar$, where V is the total voltage across the PSC. This in turn should modulate the phase difference across the junction, and should give a small constant-voltage AC Josephson step at probe voltages $V' = n\hbar\omega/2e = nV$, for integers $n = \pm 1, \ldots$, in exactly the same way as would an external microwave current. An effect of the type desired was in fact seen in the IV characteristics of the closest probe junction, although it was smaller than expected. This may have been due in part to the depressed value of the gap in the center of the PSC, also measured using the tunnel junction probes. It was not possible, however, in these measurements to obtain information on the wave nature of the inferred oscillating potentials.

The experimental configuration that we have been discussing, indicated in fig. 28b, is topologically equivalent to that in fig. 28c, used to investigate the interaction between two microbridges spaced several microns apart. These experiments, and others on arrays of microbridges, have shown that under certain circumstances the voltages across the individual bridges tend to lock together, and their Josephson oscillations to synchronize. The former effect can be seen simply by examining the DC $I - V$ characteristics, while the latter has been seen most directly by measuring coherence in the microwave radiation emitted by the bridges. This subject of the short-range interaction between two or more weak links has been reviewed by Lindelof and Bindslev Hansen (1981).

The voltage locking is illustrated by an example from Jillie et al. (1977, 1980), using In microbridges spaced less than 2μ m apart (fig. 29). Typically, the current through one of

the microbridges was held fixed, while that through the other was varied. Some results of these experiments are shown in fig. 30, and compared with a numerical simulation (Jillie et al. 1980). In addition to the static effects, which can be simply modeled in terms of the steady-state diffusion of excited quasiparticles from one bridge to the other, there are some

striking dynamic effects involving voltage locking or frequency pulling. The voltage locking, which occurs for a fairly narrow range of currents, is always accompanied by a sharp change in the differential resistance $\mathrm{d}V/\mathrm{d}I$ of the junctions. This phenomenon of voltage locking is not fundamentally different from that of a constant-voltage Josephson step under the influence of a highfrequency signal. However, since the two junctions are comparable in their oscillator strengths, they exert a similar frequency-pulling effect on eath other when their frequencies are close. As nonlinear oscillators, they also exhibit a similar (but weaker) effect when their frequencies are small integral multiples of each other (note the small "dimples" in fig. 30 b, c).

The strength of the voltage locking was characterized by the total change in the differential resistance from both bridges, and is shown for two temperatures near T_{c} in fig. 31 as a function of locking voltage. The remarkable feature of this is that the interaction strength oscillates with voltage (frequency), in a way that suggests rather strongly the presence of driven waves.

The simple model to explain the effects involved assuming that a fraction (of order $20^{\circ}{}_{\circ}$) of the quasiparticle current from each bridge also flows through the other one. This is equivalent to the schematic diagram in fig. 32. With an optimum value for this effective shunt resistance R_s that couples the two resistively-shunted oscillators, a reasonable fit to many of the features of the voltage locking could be obtained. However, this cannot account for the very striking modulation in the strength of the voltage locking interaction.

One simple modification is to introduce the idea of a delay time for the excited quasiparticles to get from one bridge to the other. If one assumes a fixed delay time, independent of frequency, together with an oscillatory excitation process, one is led directly to the phenomenon of waves in the superconductor, and back to a consideration of charge imbalance waves as a possible cause of voltage locking in the closely-spaced bridges.

This delay time can be analyzed by replacing the simple resistor coupling of fig. 32a with a delay line, which is another word for a finite length of transmission line, as used by KSS (Kadin et al. 1980) to model charge imbalance dynamics near a PSC. This is essentially equivalent to considering the region between the two bridges as a resonant cavity for the charge imbalance waves (Lindelof and Bindslev Hansen 1977).

A numerical simulation of the equations has been carried out (Blackburn 1983), but one can also make use of lumped element equivalents of the transmission line. One can then transform the problem of charge imbalance wave coupling to one with a single complex impedance shunting the two bridges. The general expression will depend on the detailed geometry and be rather complicated, but for the case where the bridges are not too close together, this shunting impedance takes the form

$$Z_{\mathrm{s}} = 3Z_0 \exp(i2kl), \tag{4.15}$$

where Z_0 again is the characteristic impedance of the line (eq. 2.42), k is its (complex) propagation vector (eq. 2.41), and $2l$ is the distance between the bridges. This form immediately gives the roll-off of the interaction with distance, and furthermore shows that the complex phase of the coupling impedance rotates both with distance and with voltage (through the frequency dependence of k). One can show that the optimum coupling is one that is inductive (i.e. pure imaginary), and that for identical bridges, the pure resistive coupling will actually yield a vanishingly small range of voltage locking. Therefore, the effective coupling strength should oscillate as a function of voltage before dying out, as in fact was observed in the experiments. Given these conditions, one can estimate the apparent wave velocity from the experiments, and it is roughly in accord with what is expected theoretically. However, although a detailed fit

has not been carried out, there are some serious detailed differences, in particular the temperature dependence of the inferred wave velocity. The theory suggests a velocity that goes as $(1 - T/T_c)^{1/4}$, whereas the data gives an exponent of order $\frac{1}{2}$. The situation is complicated by a number of factors, and remains unresolved. There have also been several other proposals to explain this voltage-locking behavior, including one (Lindelof and Bindslev Hansen 1981) that depends on DC quasiparticle injection and does not require the Josephson effect at all.

One recent experiment (Kadin et al. 1981) attempted to study the role played by charge imbalance waves near a PSC by providing a boundary to reflect the waves produced by the PSC back to the center, thereby using it as its own detector. This was done using a narrow channel of length $2L$ much less than the AC decay length, which broadened suddenly to wide banks. Assuming that a PSC was localized in the center of this channel and sending out waves in both directions, then the waves should be expected to reflect at
the ends with inversion, returning to the center with a phase lag

$$\Delta\theta = \pi + 4\pi L/\lambda. \tag{4.16}$$

This is equivalent within the transmission-line model to "shorting out" the ends at distance L from the center. One can used the lumped element equivalent of the finite length of the transmission line (fig. 32b,c) to obtain a shunting impedance

$$Z = 2Z_0 \tanh(KL), \tag{4.17}$$

which has resonances that occur when the condition $\Delta\theta = 0$ is satisfied, i.e.

$$2L = \lambda/2, 3\lambda/2, \ldots, \text{ etc. }, \tag{4.18}$$

and these would also be expected to show up as features in the differential resistance of the bridge, since varying the voltage will vary the wavelength of the charge imbalance waves. These ideas were tested on a set of narrow. In channels about 0.3μ m wide, and from 0.5μ m to 4μ m in length. These samples did indeed exhibit a set of bumps in $\mathrm{d}V/\mathrm{d}I$ in approximately the voltage range expected. Some of these features were associated with resonances in the charge imbalance waves driven by the PSC, and the velocity and temperature dependence of the waves thus inferred were in rough accord with the theoretical predictions. No other obvious sources (such as microwave resonances or subharmonic gap structure) could account for these features. On the other hand, detailed agreement with theory was lacking, suggesting at least that the model and/or the treatment of the boundary conditions were oversimplified.

There have also been attempts, both experimentally and theoretically, to examine large arrays of closely-spaced microbridges for coherent interactions. In one such experiment, a linear array of up to 2000Nb/Ta proximity-effect microbridges were fabricated with spacings that went down to less than 2μ m (Palmer and Mercereau 1974, 1977). For the smallest separations, these were found to exhibit coherent behavior of their Josephson oscillations at low frequencies. Although coupling via magnetic flux modulation in these very wide bridges (100μ m) has been proposed as a possibility, coupling via charge imbalance waves has also been considered. Within the KSS model, the finite length of transmission line of distance $2L$ between each adjacent pair of Josephson oscillators can be represented by its "tee" equivalent, as represented in fig. 32a, where $Z_1 = Z_0 \tanh(KL, 2)$ and $Z_2 = Z_0, \sinh(KL)$. For an array of such identical PSC's, driven by an external DC current, symmetry requires that the nonequilibrium potential $\Phi = 0$ halfway between the junctions. Then, no current goes through the transverse impedances Z_2, and the problem reduced essentially to the single PSC in a resonant cavity treated above, so that the array should show the same resonant features in its $I - V$ characteristics. This picture also accounts for the DC interaction between the PSCs, in that the differential resistance of

such an array of identical equally spaced PSCs in
a filament of length S takes the form

$$\mathrm{d}V/\mathrm{d}I = 2\Lambda nR \tanh(S/2n\Lambda). \tag{4.19}$$

This properly interpolates between the value of $2nAR$ for non-interacting junctions and SR for a larger number of closely spaced bridges.

Artemenko, Volkov, and Zaitsev (1978a,b) have treated the similar problem of a large array of proximity-effect bridges, using a more elaborate theoretical framework, which however is approximated by a set of equations that are virtually identical to the picture of fig. 32 before calculations are actually carried through. They have pointed out that, among other consequences, the periodic array of Josephson oscillators in the superconducting filament alters the propagation characteristics of the charge imbalance waves, leading to the development of a band structure, including forbidden gaps, in much the same way as a band structure develops for electron propagation in a periodic lattice. This also follows from fig. 32, if one replaces each Josephson oscillator by its characteristic Josephson inductance $L_j = h/2eI_c$ appropriate for small amplitudes, and then uses Kirchhoff's current and voltage laws between each of the nodes.

It is important to note that the schematic circuit of fig. 30 is essentially equivalent to the actual circuit of devices that have been artificially fabricated to permit coherent coupling of two or more Josephson oscillators, which need not be spaced very closely together, each of which can be represented as a simple RSJ. In these arrays of junctions, an external shunt (which need not be purely resistive) permits some microwave current to recirculate through the series array, and provides a phase reference onto which the individual oscillators can lock, in the same way as to an external microwave signal. Not surprisingly, since one-dimensional long-range order tends to be unstable, the long-range interaction provided by the single loop around the entire array has been found to be more effective in producing long-range phase coherence than local interaction between adjacent bridges. These are of technological as well as fundamental interest, since the total microwave power coupled from an array of n such coherently synchronized oscillators can approach n^2 times the power of an individual oscillator, with a radiation linewidth that is narrowed by a factor of n. One of the more successful configurations for obtaining this coherence involved a parallel configuration for DC biasing, but a series arrangement at microwave frequencies (Sandell et al. 1979). This ensured that the junctions were automatically locked to the same frequency, and the shunt impedance had merely to synchronize the phases of the individual oscillations. One such array of about 100 In microbridges, fabricated by electron-beam lithography, did in fact produce a measured 10,000 times the microwave power from a single junction, in agreement with the predicted scaling (Jain et al. 1981). Arrays of other kinds of junctions, including tunnel junctions, have also been examined, using various types of coupling, including coupling via a
resonant cavity or a resonance in the array itself (e.g., Davidson 1981, Bindslev Hansen et al. 1981, Bindslev Hansen 1982, Likharev et al. 1981). Finally, there has also been considerable work recently on large two-dimensional arrays of junctions (e.g., Resnick et al. 1981, Voss and Webb 1982, Abraham et al. 1982). The ultimate significance of all this work, and its possible relation to nonequilibrium superconductivity, remain to be seen.

44.5 5. Discussion and conclusions

In this chapter we have treated a number of experimental configurations which are illustrative of the richness and complexity of the phenomena in the field known as nonequilibrium superconductivity. We have reviewed the theory emphasizing physical models rather than formal mathematical development. The latter can be found in a number of the other chapters

of this volume. In our view the use of simple physical models is essential to process by which experiments to probe these phenomena are designed in a creative way. Detailed theory is of course needed for the quantitative analysis of experimental results. The chapter has been focussed strongly on the use of the AC Josephson effect to produce and to probe the dynamics of the order parameter. This approach allows for the study of high-frequency dynamics through measurement of DC current-voltage characteristics. Thus rather complicated nonequilibrium phenomena can be studied using remarkably simple tools, ammeters and voltmeters. Within this framework our detailed discussion has focussed on investigations of the pair-field susceptibility, studied using tunneling junctions, and experiments with phase-slip centers, and microbridges. Central to the discussion is the transmission-line model of dynamical charge imbalance effects, which is developed to facilitate understanding charge imbalance waves. Special attention has been paid to physical explanations of the measurement of the pair-field susceptibility which have not been widely appreciated. A number of other fundamental experiments relating to nonequilibrium superconductivity, but not necessarily involving the AC Josephson effect, are treated in the section in which the basic theory is developed.

Superconductors out of equilibrium have been excellent model systems for the study of nonequilibrium statistical mechanics, in part because the equilibrium superconducting state is so well understood. In a few instances there are remarkable parallels between measurements in the superconducting state and similar studies of other physical systems out of equilibrium. The most striking example of this is the study of the pair-field susceptibility which yields results which are similar to those obtained in neutron scattering experiments on magnetic systems used to probe the dynamics of the magnetic order parameter. Propagating charge imbalance waves in superconductors were originally discovered in a pair-field susceptibility experiment. In the absence of any
detailed theory they were identified as a propagating collective mode of the superconducting order parameter by exploiting the analogy with magnetic systems.

Recently the inelastic and elastic electron scattering times that parametrize the theory of nonequilibrium superconductivity have become further accessible to experimental study in the context of the investigation of electronic localization in thin wires (Chaudhari et al. 1980) and thin films (Gordon et al. 1983, Santhanam and Prober 1984). In the study of weak localization in thin films the internal consistency of the data and the agreement of the temperaturedependent resistance and magnetoresistance with theory is extremely quantitative. Turning the problem around, the determination of relaxation times from the analysis of localization data may stimulate detailed, first-principles microscopic calculations.

Despite the breadth of the work in nonequilibrium superconductivity, most of the quantitative experimental studies have been carried out at temperatures very close to T_c where the coupled equations for the order parameter and quasiparticle distribution function have particularly simple analytic forms and are actually differential equations. Extensions to low temperatures will probably require more extensive numerical analysis of the theory than has been previously used, as analytical results may not be obtainable. On the experimental side, future work at low temperatures will have to overcome the problem of heating. Studies of transient response over short time scales, which are possible using superconducting sampling devices e.g., Faris (1980), may be an important tool in finessing this heating problem.

It should also be noted that most experiments in the area of nonequilibrium superconductivity have been carried out in regimes rather close to equilibrium and under nearly linear conditions. Nonlinear effects, which have been hardly touched experimentally, may be a fruitful field of study when long-wavelength-high-intensity electromagnetic sources such as the Free Electron Laser become available.

Finally, the issues which are raised in nonequilibrium superconductivity are very important for the understanding of superconducting devices and possible hybrid systems containing both

superconducting and semiconducting devices. Quasiparticle injection by tunneling is central to the operation of both the superconducting transistor (Gray 1978b), the quiteron (Faris et al. 1983) and three-terminal edge junctions (Hunt and Buhrman 1983). These are threeterminal superconducting devices of a type which may be very useful in the practical application of superconducting digital electronics, i.c. for the superconducting computer. Optimization of their performance will require detailed knowledge of nonequilibrium superconductivity. In a superconducting computer, or a hybrid computer containing semiconducting active devices and narrow superconducting interconnects, there will probably be quasiparticle injection into the interconnects and current flow across superconducting
semiconducting or superconducting-normal boundaries. As the scale of integration of such systems becomes larger and the device spacing gets smaller one can expect the lengths of the superconducting interconnects to become shorter than the nonequilibrium lengths associated with the processes that we have discussed in this chapter. The consequences of this for the technology have not really been explored.

Fig. 16. (a) Time-averaged potentials and (b) currents in the SBT model of the phase-slip center. (From Kadin et al. 1980.)

Fig. 17. Equivalent circuits for a phase-slip center. (a) The ideal Josephson element is located on the inductive (condensate) line of the transmission line. (b) The equivalent of (a) as far as the currents and potentials at the center are concerned, where $Z(\omega)$ is the complex frequencydependent characteristic impedance of the transmission line. (c) A generalization of (b) to allow for cases where the intrinsic resistance $R_{3

$ of the Josephson element is not small compared to $Z(\omega)$. (From Kadin et al. 1980.)}

Fig. 18. $I-\bar{V
$ characteristics in the high-frequency limit for the linear $f(\Delta\theta)$ and various values of $\tau_{\mathrm{E}}/\tau_{\mathrm{J}}$. The curves are scaled so that all have the same critical current and high-voltage differential resistance. The values of $\tau_{\mathrm{E}}/\tau_{\mathrm{J}}$ corresponding to the curves from left to right are $1024, 256, 64, 16,\ 4, 1, \frac{1}{4}$, and $\frac{1}{16}$. The dashed line is the high-voltage asymptote. (From Kadin et al. 1980.)}

Fig. 19. I-V curves for the linear current-phase relation in the high-frequency limit for various values of $\tau_{\mathrm{E}} / \tau_{\mathrm{J}}}$, with the temperature dependence reintroduced in accordance with $\tau_{\mathrm{E}}/\tau_{\mathrm{J}} \sim (1-T/T_{\mathrm{c}})\,, \quad I_{\mathrm{c}} \sim (1-T/T_{\mathrm{c}})^{3/2}$ and $I_{\mathrm{c}} R_{\mathrm{eff}} \sim (1-T/T_{\mathrm{c}})^{5/4}$. The arrows pointing upward represent hysteretic switching at the critical current I_{c} into the dissipative state, and the down-ward-pointing arrows represent switching back into the zero-voltage state at the current I_{m}. The values of τ_E/τ_J corresponding to the curves from left to right are $\frac{1}{2}, 1, 2, 3, 4$, and 5 . (From Kadin et al. 1980.)}

Fig. 20. Time dependence within a single Josephson period of the oscillations in the center for (a) $I_{\mathrm{n }}(0, t) / I_{\mathrm{c}}$ and (b) $\Phi(0,t)/I_{\mathrm{c}}R_{\mathrm{eff}}$ using the linear $f(\Delta\theta)$ and parameters $\tau_{\mathrm{E}}/\tau_{\mathrm{J}} = 11, \omega_{\mathrm{c}}\tau_J = 2.2$, $V = I_{\mathrm{c}}R_{\mathrm{eff}}$, and $I = 1.5I_{\mathrm{c}}$, appropriate for Sn at $T/T_{\mathrm{c}} = 0.99$. Both I_{n} and Φ are discontinuous at the beginning and end of each cycle for the linear $f(\Delta\theta)$. (From Kadin et al. 1980.)}

Fig. 21. Spatial dependence (for $x > 0$) of (a) $I_{\mathrm{n} / I_{\mathrm{c}}}$ and (b) $\Phi / I_{\mathrm{c}} R_{\mathrm{eff}}$ for the same parameters as in fig. 20. The dashed curves are the DC averages. (From Kadin et al. 1980.)}

Fig. 22. Spatial dependence of the potentials $\mu_{\mathrm{n} } / e \bar{V}$ and $\mu_{\mathrm{s}}/e\bar{V}$ for the same set of parameters as in fig. 21. The behavior for $x < 0$ is included to make clear the symmetry of the solutions. The range of variation of $\mu(-\infty)/e\bar{V}$ is indicated by arrows on the left. (From Kadin et al. 1980.)}

Fig. 23. Spatial dependence of $I_{\mathrm{n} } / I_{\mathrm{c}}$ for the linear $f(\Delta\theta)$ and parameters $\tau_{\mathrm{E}} = 280\tau_{\mathrm{J}}, \omega_{\mathrm{c}}\tau_{\mathrm{J}} = 11$, $\bar{V} = 0.5I_{\mathrm{c}}R_{\mathrm{eff}}$, and $I = 1.0I_{\mathrm{c}}$, appropriate to Al at $T/T_{\mathrm{c}} = 0.99$. Note that the horizontal scale is expanded by a factor of two compared with fig. 21. (From Kadin et al. 1980.)}

Fig. 24. (a) Long microbridge; (b) two-dimensional uniform thickness microbridge; (c) variablethickness bridge. (From Octavio et al. 1977.)

Fig. 25. (a) I I'characteristics of well-cooled microbridge. A through G correspond to temperatures of $4.20, 3.68, 3.56, 3.38, 3.05, 2.52$, and 2.10 K , respectively. (b) $I - V$ characteristics of a very well-cooled microbridge. A through H correspond to $4.2, 3.73, 3.62, 3.49, 3.28, 3.18, 2.97$, and 2.10 K , respectively. (From Octavio et al. 1977.)

Fig. 26. (a) Cyclic depression of the gap. Quasiparticle levels above $\Delta_{0 $ remain in thermal equilibrium by rapid diffusive contact with the banks. Those in levels below Δ_0 (heavy shading) can be out of equilibrium. (b) Effective temperature shifts for rapid and slow modulation of Δ. (c) $I_{\mathrm{s}}(\phi)$ for $\omega\tau_E \ll 1$, showing increase in forward current. (d) $I_{\mathrm{s}}(U)$ for $\omega\tau_E \gg 1$, showing increase of effective I_{c} to $I_{\mathrm{cl}} > I_{\mathrm{c}}$. In (c) and (d), equilibrium supercurrents are shown in dashed curves for comparison. (From Tinkham 1981.)}

Fig. 27. Derivative curves showing the variation of the subharmonic gap structure with temperature. (From Octavio et al. 1977.)

Fig. 28. Detection scheme for charge imbalance waves near a PSC: (a) physical picture; (b) schematic; (c) equivalent of (b), emphasizing the similarity with the configuration for investigating coupled microbridges.

Fig. 29. SEM micrograph of one of the samples used by Jillie (1976). The light areas are a 100 nm thick In film.

Fig. 30. Experimental results (a), (b), (c) and a numerical simulation (d) for a series opposed biased pair of indium microbridges $2 \mu \mathrm{~m
$ apart showing voltage locking: (a) $I-V$ curves with I_2 held constant (91μ A) and I_1 swept; (b) the differential resistance of the two curves shown in (a); (c) the total differential resistance $\mathrm{d}V_{\mathrm{T}}/\mathrm{d}I_1 = 0$. In the locking region A ($\mathrm{d}V_{\mathrm{T}}/\mathrm{d}I_1$) is a measure of the strength of the interaction. (d) Numerical simulation using the correct parameters for the coupled bridges ($\alpha_1 - \alpha_2 = 0.21$). (From Jillie et al. 1980.)}

Fig. 31. Voltage and temperature dependence of voltage locking interaction strength, Δ, for the series opposed case. (Jillie et al. 1977.)

Fig. 32. Lumped-element equivalents for (a) a long series of closely-spaced PSCs, and (b) a single PSC in a short filament (length $2l$), using the tee section. (c) The equivalent of (b), using $Z_{1

^{\prime}(2 l)=Z_{0} \tanh (K l)$.}

45 Experimental Study of Enhanced Superconductivity by Dmitriev, Gubankov and Nad

45.1 1. Introduction

Not very long ago it was generally believed that an electromagnetic field, when applied to a superconductor, should reduce its superconducting parameters: the critical current I_c, the critical magnetic field H_c and the critical temperature T_c, that is, cause superconductivity suppression. This standpoint was based on consideration of the physical mechanisms of the interaction between an electromagnetic field and a superconductor: absorption by Cooper pairs of electromagnetic field quanta $\hbar\omega$ with energies higher than the Cooper-pair coupling energy, acceleration of superconducting electrons to supercritical velocities at which their kinetic energy is larger than the coupling energy, and electromagnetic energy absorption by normal electrons and resulting heating of the superconductor.

However, experiments in 1966-1970 revealed that an electromagnetic field with frequency about 10 GHz acting on a superconductor causes I_c, H_c and T_c to increase, that is, it enhances superconductivity (Wyatt et al. 1966a,b, Dayem and Wiegand 1967, Dmitriev et al. 1970).

A theoretical explanation of such unusual and unexpected behaviour of a superconductor in an electromagnetic field was proposed by Eliashberg (1970) who took into account the interaction of the field with quasi-particle excitations which so modifies the quasi-particle energy distribution that the superconducting pairing is enhanced.

Eliashberg's theory did not win general acceptance very easily. The reason was that it applied to a uniform superconductor, while the early experimental observations used films (bridges) and point contacts (Dmitriev et al. 1973). And only after Eliashberg and his coworkers (see the chapter by Eliashberg and Ivlev in this volume) had further developed their theory and some experiments corroborated the theoretical results the essence of the effect was understood; its principle is that under an electromagnetic field a superconductor goes to a new nonequilibrium steady state characterized by stronger superconducting properties due to quasi-particle transition from the lowenergy region near the edge of the energy gap to the higher-energy region, with the total number of quasi-particles preserved.

The papers mentioned above thus pioneered in a new branch of research: nonequilibrium superconductivity in alternating fields. It was realized that if
an electromagnetic field drives the superconductor to a new nonequilibrium state, then, the theory of the nonstationary Josephson effect should take into account the effects of nonequilibrium quasi-particles arising from the Josephson self-radiation in weakly coupled superconductors. This provided an explanation for the initial parts of the current-voltage characteristics of Josephson weak links and the peculiarities of superconductivity enhancement in superconducting contacts, which had not been physically explained by that time (Aslamazov and Larkin 1976, 1978). The concept of nonequilibrium superconductivity also became the base of work on superconductivity enhancement by sound and tunnel current. The possible superconductivity enhancement by a direct current in tunnel junctions had been suggested by Parmenter (1961). However, that paper referred to quasi-particle injection (extraction) into one of the contact films, which is radically different from enhancement by the electromagnetic field, with a constant total number of quasi-particles.

The understanding of the enhancement mechanisms has been contributed to by scientists of many nations and investigations through wide ranges of temperature and frequency on samples of various superconducting materials and various geometries: film bridges, homogeneous strips, proximity-effect bridges and varying-thickness bridges, point contacts and tunnel junctions. These investigations permitted, on one hand, to ascertain the generality of superconductivity enhancement and, on the other hand, to find the features peculiar to uniform and constricted

superconductors. There appeared a rapidly developing branch of physics dealing with nonequilibrium superconductivity, which is important both for the solution of the fundamental problems of nonequilibrium superconductivity and for its practical application. In recent years, the concept of nonequilibrium superconductivity has essentially evolved, extensively and intensively, from the initial state-of-the-art.

In this chapter, the term "nonequilibrium" will be applied to a superconductor in which the quasi-particle distribution function is different from that in thermodynamic equilibrium. The nonequilibrium state in a superconductor can be induced by:
(1) irradiating the superconductor with an electromagnetic or sound field;
(2) passing a direct current higher than the critical current, through a uniform superconductor or a superconductor containing a weak link with metallic conductivity;
(3) passing a direct tunnel current through a superconductor containing a tunnel barrier;
(4) tunnel injection (extraction) of normal excitations into the superconductor.

Each process involves complex and peculiar effects, providing both the deviation from equilibrium and the tendency to equilibrium. Such processes in
the superconductor generate a range of wonderful phenomena: enhancement and suppression of superconductivity, development of a state with a spatially inhomogeneous order parameter $\hat{\Delta}$, excess currents, collective oscillations, reduction of ultrasound absorption, to mention only some of them.

In this paper we shall only deal with nonequilibrium phenomena resulting in superconductivity enhancement and also touch upon the problem of the spatially inhomogeneous state and excess current.

45.2 2. Discovery of microwave-enhanced superconductivity

After the nonstationary Josephson effect had been predicted theoretically and observed experimentally, active search and investigation of new types of weakly coupled superconductors commenced. In particular, the IV characteristics of thin film bridges (Dayem bridges) with the radius of constriction less than 1μ m displayed current steps induced by a microwave field (Anderson and Dayem 1964).

An experimental investigation of the influence of a 10 GHz electromagnetic field on the $I-V$ curve of tin thin film bridges with the radius of constriction larger than 1μ m (the characteristic dimensions of such bridges are: length and width about 2μ m, thickness about 2000Å.$R_{4.2} \approx 0.3\Omega$) did not find the nonstationary Josephson effect, i.e. under irradiation, the $I-V$ characteristic did not develop steps. Another effect was however revealed: a radiationinduced increase in the bridge critical current (Wyatt et al. 1966a,b, Dayem and Wiegand 1967, Dmitriev et al. 1970), see fig. 1. The effect reduced as temperature went down. Investigation of the transition temperature of the bridges from the normal to the superconducting state showed that in an electromagnetic field a sample becomes superconductive at higher temperatures than in the absence of the field (fig. 2). Thus, in addition to I_c, the bridge critical temperature T_c becomes higher under the effect of an electromagnetic field.

Figure 3a represents the dependence of the bridge critical current on the radiating power for a range of values of a d.c. magnetic field perpendicular to the plane of the bridge (Dmitriev et al. 1970). One can see that this dependence is similar to $I_c(P)$ for various temperatures shown in fig. 1 . The lower curve of fig. 3a suggests that a sample in the resistive state under the influence of a magnetic field $H \gtrsim H_c$ may be converted to superconductive by applying microwave irradiation. Thus, the irradiation also causes the critical magnetic field H_c to increase, fig. 3b.

So, those early experiments ascertained that an electromagnetic field might enhance superconductivity, that is, cause an increase in the superconducting parameters of the sample: I_c, T_c

and H_c.

The above experimental results were obtained from samples of various

Fig. 1. Experimental dependence of critical current through Sn constriction as a function of microwave power at different temperatures. Incident power frequency is $\sim 10 \mathrm{GHz . T_{\mathrm{c}}=3.845 \mathrm{~K}$.}

superconductors: Sn , In and Al , which evidenced the general nature of the phenomenon. The bridges were however of different dimensions, including very small Josephson junctions. This created some difficulties for the physical interpretation of the quantitative results.

In order to account for the experimental data collected by that time, a phenomenological approach to the superconductivity enhancement effect was developed. It was widely discussed if the enhancement could be explained by the influence of the irradiation on fluctuations in the bridge. In the experimental temperature dependence of the bridge resistance $R(T)$ there are two steps

(Wyatt et al. 1966b), representing the transition to the superconducting state of the bridge banks at T_c^B and the bridge itself at T_c^b, such that usually $T_c^B > T_c^b$. One of the reasons could be an essentially larger influence of fluctuations on T_c^b than on T_c^B. Later, Ovchinnikov (1973) proposed a microscopic theory for this idea. The experimentally observed fact that $\delta T_c = T_c^H - T_c^b$ and the increase in T_c^b of the bridge under the effect of an electromagnetic field by δT_c suggested the idea that the field affects the bridge fluctuations, namely suppresses them and leads to a certain "order" (Hunt and Mercereau 1967, Hunt 1969).

The fluctuation model can include the additional Cooper-pair coupling energy due to fluctuation energy reduction caused by the said "order" (Dmitriev et al. 1973). Inclusion of such a contribution of the field to the energy balance equation for a bridge with a current results in a current density expression from which the superconductivity enhancement effect follows. This model was a qualitatively adequate description of the superconductivity enhancement near T_c and was based on the assumption that an electromagnetic field modulates the bridge order

Fig. 2. Experimental dependence of Sn bridge resistance as a function of microwave power at different temperatures. Incident power frequency is $\sim 10 \mathrm{GHz
. T_{\mathrm{c}}=3.84 \mathrm{~K}$.}

parameter in such a manner that its time average increases with the field. In this model the dependence of the weak-link coupling energy on the phase difference φ (Anderson 1967) had the form

$$E = E_0(1 - \cos\varphi), \quad \partial\varphi/\partial t = (2e/\hbar)V\cos\omega t$$

where V is the microwave voltage. However this model did not provide a detailed physical picture of the electromagnetic field effect on quasi-particles and therefore did not contain some experimental features of the dependences on frequency, field intensity and also temperature of the superconductivity enhancement measured later. Besides, being based on the fluctuation influence principle, the model was unable to describe the enhancement effect at temperatures sufficiently below T_{c} where fluctuations play but an insignificant role. So, there was still a need for further theoretical development.

The next step in the development of phenomenological theories was a model based on certain features of nonequilibrium superconductivity (Lindelof 1976). It is in short as follows: since $T_c^b < T_c^B$ then at a given temperature in the middle of the bridge the relative number of normal electrons is higher than that of superconducting electrons, that is, the order parameter has a minimum at the bridge centre. The electromagnetic field injects the normal electrons into the bridge banks, where they become excited and require some time to relax to equilibrium with the condensate. The law of conservation of charge requires that at the bridge centre the normal-electron deficiency should be covered by an inflow of superconducting electrons from the banks. In the average, under the electromagnetic field, the relative number of superconducting electrons in the bridge increases, whence there may be a high superconducting current. However, this model, which described the superconductivity enhancement as such, was also unable to account for its particulars at various frequencies, field intensities and temperatures.

Thus, the phenomenological models, which allowed for the presence of weak links in the superconductor (spatial inhomogeneity of the order parameter), were not able to explain the experiment in detail. Meanwhile, the microscopic theory developed for uniform superconductors (Eliashberg 1970) still wanted thorough experimental check, because most experiments by that time had been run on superconductors with the order parameter spatially inhomogeneous along the distance $\sim \xi(T)$.

A demand then arose for new advances in the investigation of superconductivity enhancement by an electromagnetic field. By experiments, uniform samples were to be studied, the similarity of the new results with those available for bridges was to be shown and the experi-

Fig. 3. Experimental dependence of the critical current of a tin bridge (a) on incident power for a range of intensities of a d.c. magnetic field perpendicular to the bridge plane and fixed temperature, (b) on d.c. magnetic field, for fixed incident power and temperature. $T_{\mathrm{c }}=3.85 \mathrm{~K}$.}

mental results were to be compared to the conclusions of Eliashberg's microscopic theory. In the theory, the objective was not only to get a more profound insight into the mechanism of superconductivity enhancement in the uniform superconductor, but also to
develop a microscopic picture of the effect allowing for the features of the inhomogeneous superconductor associated with the spatial dependence of the order parameter. The theoretical developments should then, in turn, be checked by experiments on samples adequate to the theory.

45.3 3. Microwave-enhanced superconductivity in uniform superconductors

As was mentioned above, the phenomenological explanations of superconductivity enhancement (Hunt and Mercereau 1967, Hunt 1969, Christiansen et al. 1971, Dmitriev et al. 1973) were unsuccessful. Some of the early results could not be explained, quantitative comparison of experimental data with the mostly qualitative theories was impossible and the physical nature of this strange phenomenon was still unknown.

As an alternative to the phenomenological approach, Eliashberg and colleagues (1970-1973) developed a microscopic theory of nonequilibrium states of a superconductor in an electromag-

netic field. In a series of papers they derived a superconductor dynamic theory which made allowance explicitly for inelastic processes and revealed the physics of such nonequilibrium phenomena. It was shown that because of the long energy relaxation time of excitations even in fields low compared to the critical field, essential redistribution of excitations in energy may occur with the spectrum density unchanged. For the external field frequency $\omega < 2\Delta/\hbar$, some excitations with energies $\varepsilon \sim \Delta$ move to $\varepsilon \sim \Delta + \hbar\omega$, while preserving their total number. A decrease in the occupancy of states with $\varepsilon \sim \Delta$, which play the predominant role in quasi-particle pairing and determine the gap, should lead to enlargement of the gap (see eq. (1) of the chapter by Eliashberg and Ivlev). Besides, as was shown by Chang and Scalapino (1977, 1978), since the recombination of particles with the high energy $\varepsilon \sim \Delta + \hbar\omega$ is fast, an increase of the number of high-energy particles as a result of redistribution may cause the total number of quasi-particles to decrease and thereby cause the gap to enlarge as well.

In the case of thin uniform films, one may neglect the position dependence of A. Moreover, in the neighbourhood of T_c the order parameter relaxation time τ_Δ is long (Gor'kov and Eliashberg 1969), and for the frequencies with $\omega\tau_1 \gg 1$ involved here one may neglect time oscillations of Δ as well. Thus, in theory, in a thin superconducting film under the influence of an electromagnetic field there should arise a homogeneous, stationary, nonequilibrium state with an enlarged gap. In this section we shall present results of various experimental investigations of superconductivity enhancement by an electromagnetic field, analyze them and compare with theory. These results became the experimental basis of the enhancement mechanism proposed by Eliashberg and coauthors and contributed to the complete understanding of the physical nature of the phenomenon.

Fig. 4. Dependence of critical current $\beta=I_{c (P) / I_{c}(0)$ in a Sn bridge of length 42μ m and width 1μ m on the 9.4 GHz incident power at $T_c - T = 3$mK.}

For the realization of the spatially homogeneous nonequilibrium state of a superconductor in an electromagnetic field, the energy gap should be constant and the current distribution uniform through the sample. These requirements are met by long ($L > \xi(T)$) thin ($d < \xi(T)$) narrow ($w < \lambda^2/d$) film bridges on substrates with good thermal conductivity at cryogenic temperatures. Such bridges, $\sim 100\mu$ m long, $\sim 1\mu$ m wide, were prepared by evaporation of a thin (1000Å) superconductive film through a mask represented by a thicker (about 1μ m) film (of metal or collodion) with a narrow slit (Andratskii et al. 1973, Latyshev and Nad 1975, 1976). The slit was made with a special device which provided plane-parallel cutting motion and allowed to control the cutting tool pressure and cutting angle. After evaporation the masking film was removed. Bridges produced in such a way had sharp edges and a narrow

superconducting transition (appr. 0.01 K) due to a certain inhomogeneity of T_c along the polycrystalline film of the bridge (Skocpol et al. 1974). The width uniformity was controlled by an electronic microanalyzer with accuracy to $\sim 0.1\mu$ m. Samples with large inhomogeneities were rejected. The $I-V$ characteristics were recorded by an XY-recorder with an error of $0.3^{(1)}{}_0$. The temperature maintenance and measurement accuracy was not worse than 1μ m.

The experiments (Shepard 1971, Latyshev and Nad' 1975, 1976, Klapwijk and Mooij 1976, Dmitriev and Khristenko 1978) showed that for all long
narrow bridges, with lengths from 1 to $10^3\mu$ m, irradiated by an electromagnetic field, there was a growth of the critical current I_c, and I_c as a function of the radiating power P was qualitatively similar to the dependences in fig. 4. There is a region of linear growth of I_c with P, a region of saturation, and finally a region of steeply falling I_c in a narrow power range (~ 0.1 dB). The increase of the clearly observable critical current near T_c may be even by a factor up to 50 . The saturation and fall regions are characterized by a hysteresis developing in the bridge $I-V$ characteristics (Latyshev and Nad' 1976). Figure 5 represents dependences of the critical current density j_c on the radiating power for a 1.2 mm long tin bridge at various temperatures (Latyshev and Nad' 1976). It is easily seen that in the range $T \gtrsim T_c$, i.e. from the resistance state, there occurs "induction" of the critical current (curve 1), and at $T < T_c$ it grows, reaches a maximum and there is a rapid transition to the normal state. The data presented in figs. 4 and 5 were obtained for the irradiation frequency $\omega/2\pi \sim 10$GHz. At such frequencies, though the relative growth of the critical current $\beta = I_c(P)/I_c(0)$, as is seen in fig. 5, becomes lower with temperature decrease, enhancement, i.e. $\beta > 1$, is still observed through a wide temperature range $T_c - T = 0.4$ K; for $\omega/2\pi \geqslant 30$GHz, there is enhance-

Fig. 5. Dependence of critical current density of a bridge of length 1.2 mm and width $2.4 \mu \mathrm{~m
$ on incident power at different temperatures: (1) 3.796 K ; (2) 3.778 K ; (3) 3.759 K ; (4) 3.747 K ; (5) 3.728 K . Solid curves represent calculation by eq. (1).}

ment even to the lowest temperatures studied, $T_c - T \approx 1.0$ K (Latyshev and Nad' 1974). On Al bridges, the enhancement was observed to 0.04 K (Pals and Dobben 1979). However the maximum critical current $I_c(P)$ at $T \neq 0$ never exceeded the equilibrium values of the critical current at 0 K .

From these data it is evident that in an electromagnetic field there is enhancement of the critical current not only in short but also in long uniform bridges, and not only in close proximity to T_c but far away from it as well. Two facts were inconsistent with the phenomenological explanation of superconductivity enhancement. In particular, if one explains the enhancement by fluctuation suppression in the bridge by the irradiation (Hunt 1969), then in order to account for the observed increase in I_c by 30% as we move away from T_c by 1 K , the effective temperature introduced by Hunt (1969) must exceed 2000 K , which is hardly reasonable. On the other hand, both results are consistent with the mechanism proposed by Eliashberg (1970). It also agrees with the linear rise of I_c at the initial part of $I_c(P)$, fig. 4.

However, if the effects observed are indeed superconductivity enhancement effects, not simply fluctuation suppression, then the magnitude of the bridge critical current in an electromagnetic field should exceed, apart from the experimental I_c without the field, also the theoretical maximum critical current I_c^{GL} of Ginzburg-Landau theory. For the case of uniform distribution over the bridge cross-section, I_c^{GL} was calculated with the use of the generally known formulae, which include the dependences of the penetration depth $\lambda(T)$ and the coherence length $\xi(T)$ on the electron mean free path l, i.e. the purity of the superconductive film (Latyshev and Nad 1976). These dependences for nonmagnetic impurities are specified by Gorkov's (1959) equation $\chi = \chi(\xi_0/l)$. Adding the formula for the film thickness, $d = L\rho_{300}W\,(R_{300} - R_{4.2})$ we arrive at the following relation, which does not contain the film width:

$$I_c^{GL} = \frac{c\Phi_0}{12\sqrt{3}\pi^2}\frac{\rho_{300}}{R_{300} - R_{4.2}}\frac{L\chi^{1/2}}{\lambda_p^2 - \xi_p^2}, \tag{1}$$

where Φ_0 is the flux quantum, ρ_{300} the resistivity of bulk tin samples at 300 K . R_{300} and $R_{4.2}$ the film resistances at 300 and 4.2 K respectively, and

$$\xi_p = 0.74\xi_0\,(1 - T/T_c)^{-1/2}, \quad \lambda_p = \lambda_L(0)/\sqrt{2}\,(1 - T/T_c)^{1/2}$$

with $\lambda_L(0) = 355$Å and $\xi_0 = 2300$Å for Sn . The value for χ was taken from Gor'kov's (1959) paper.

In fig. 6, for a 42μ m long and 1μ m wide bridge, the dependence calculated by eq. (1) is represented by a solid curve. The error due to inaccurate measurements of all the quantities entering eq. (1) was not larger than $7°_0$. The solid circles show the experimental points for $I_c(P)$ (the experimental error is within the circle diameter). It is seen from fig. 6 that for a narrow long uniform bridge with a uniform current distribution at $T \sim T_c$, the experimental $I_c(P)$ values are in good agreement with $I_c^{Gl.}(T)$ from theory. This means that in

such bridges, at $P = 0$, the maximum possible depairing current for the equilibrium is realized [similar results were obtained for similar bridges by Andratskii et al. (1973)]. As is seen from fig. 6, the critical current under irradiation (open circles) is much higher than the thermodynamically equilibrium critical current values and is not zero at temperatures above the equilibrium critical temperature.

Figure 6 also suggests that an increase of I_c under microwave irradiation is attended by a shift of the superconducting transition curve, $R(T)$, towards higher temperatures, which proves that the bridge critical temperature increases. Similar dependences were observed on the other bridges made of Sn and Al (Klapwijk and Mooij 1976, Latyshev and Nad' 1977, 1978, Dmitriev and Khristenko 1978, Mooij et al. 1980). When a bridge which is not irradiated ($P = 0$) is cooled, the dependence $R(T)$ first develops a small step representing the superconducting transition of the hridge banks at $T = T_c^B$. Then, at $T_c^B > T > T_c^b$, there is a plateau with a

Fig. 6. Temperature dependence of (·) the critical current of an irradiated Sn bridge of length $42 \mu \mathrm{~m
$ and width 1μ m$I_{\mathrm{c}}(0)$, (o) the critical current at the optimum power $I_{\mathrm{c}}\left(P_{\mathrm{opt}}\right)$, and the bridge resistance $R(T)$ for $P=0$ and two incident powers corresponding to attenuation 13 dB and 8.5 dB . Points ∇ and $\square$ are obtained from the slope of the bridge $I-V$ curve.}

minor resistance reduction, and finally, at $T=T_{\mathrm{c}}^{\mathrm{b}}$, the superconducting transition ($\mathrm{N} \rightarrow \mathrm{S}$) of the bridge proper occurs. Its width (at levels $0.1-0.9$ of R_{N}) is usually $\leqslant 10$mK.

If the irradiation power is relatively small, the transition, as has been mentioned, is shifted to higher temperatures and becomes essentially narrower (fig. 7 , curve for 47 dB). According to the theory (Ivlev et al. 1973) and the experimental dependences $I_c(P)$ (Latyshev and Nad' 1976), wherever in the bridge T_{c} 's are lower, the action of the irradiation leads to larger critical parameter increases than where T_c is higher. Hence, while the critical parameter grows in all parts of the bridge, it levels out over the area of the bridge.

Fig. 7. Temperature dependence of the resistance of an Al bridge of length $50 \mu \mathrm{~m
$ and width 1μ m for various incident powers (the figures show the attenuation in dB). R_{s} is the excess resistance of the $\mathrm{S}-\mathrm{N}$ boundary.}

These processes are reflected by the observed shift to higher temperatures and narrowing of the superconducting transition (Latyshev and Nad' 1977). A theoretical treatment was provided by Aslamazov (1978).

As the power increases further, the transition from the superconducting state (S) to normal (N)($\mathrm{S} \rightarrow \mathrm{N}$) takes place at higher temperatures than the $\mathrm{N} \rightarrow \mathrm{S}$ transition, i.e. temperature hysteresis appears in the $R(T)$ dependence (Klapwijk and Mooij 1976, Latyshev and Nad' 1977) and becomes larger with P. The $\mathrm{S} \rightarrow \mathrm{N}$ transition is jumplike in this case. Above a certain power, the $\mathrm{N} \rightarrow \mathrm{S}$ and $\mathrm{S} \rightarrow \mathrm{N}$ transitions cease to shift to higher temperatures and start to move back to lower temperatures. The $\mathrm{S} \rightarrow \mathrm{N}$ transition displacement is several times

larger than the width. All other conditions being equal, the magnitude of the displacement essentially depends on heat transfer: δT_{c} for samples in liquid helium is approximately twice that for samples in helium gas (Latyshev and Nad' 1977). A correlation is also seen between the dependences of I_{c} and T_{c} on the radiating power: after the maximum is attained, they both drop sharply. The maximum displacement for Sn bridges was 30 mK , though it never exceeded the difference $T_c^{\mathrm{B}} - T_c^{\mathrm{b}}$, i.e. in Sn bridges, T_c^{b} was never increased by irradiation to above $T_{\mathrm{c}}^{\mathrm{B}}$. In the fluctuation model of superconductivity enhancement the fact that T_{c} could not increase above $T_{\mathrm{c}}^{\mathrm{B}}$ was thought to be essential. On the contrary, in the microscopic theory T_c^b could be increased to above the equilibrium value of T_c^B. To clear up this question, Latyshev and Nad' (1978) performed experiments on aluminium bridges with long energy relaxation times and better heat transfer (superfluid He and better matching of the acoustic impedances of the aluminium and the quartz substrate). As is seen from fig. 7, the bridge critical temperature under the irradiation may markedly (by 13 mK) exceed $T_{\mathrm{c}}^{\mathrm{B}}$ which is free from irradiation effects, because of the low density and inhomogeneous induced microwave currents. This means that in

long bridges the banks do not essentially influence superconductivity enhancement. It is not true that irradiation merely makes $T_{\mathrm{c}}^{\mathrm{b}}$ closer to $T_{\mathrm{c}}^{\mathrm{B}}$; it causes changes in the bridge properties, and first of all (according to Eliashberg) it causes redistribution of quasi-particle energies, with the immediately resulting growth of T_{c} and I_{c}. The maximum T_{c} growth observed was in this case 73 mK (5.2% of T_{c}). Pals and Dobben (1979) observed T_{c} rise by 100 mK .

On the basis of the obtained data and the theory of Eliashberg and coauthors, the N $\rightarrow$ S and S $\rightarrow$ N transitions of a bridge in an electromagnetic field may be represented as follows. According to the theory, in an electromagnetic field, the dependence of Δ on T is not unambiguous (see the chapter by Eliashberg and Ivlev, fig. 3). By lowering the irradiated bridge temperature, one can hardly obtain the transition to the S -state at $\left(\left(T-T_{\mathrm{c}}\right)/T_{\mathrm{c}}\right)_{\max}$, because for formation of the superconducting state with a gap it is necessary that first a gap should arise (at least by fluctuations), which subsequently is enlarged by the irradiation. According to the theory, the enhancement by irradiation is realized through transfer of some of the excitations from a region with a singularity in the density of states to a higher energy region. In other words, this effect can only take place if there is a singularity in the density of states. This standpoint is supported by experimental data (figs. 6 and 7), which show that a bridge may be switched from N- to S -state by irradiation at T only slightly higher than T_{c}, where there is a possibility of fluctuational generation of S-state "germs". After such transition the bridge gets to a state with a nonequilibrium gap. If T is increased now, the bridge passes T_{c} of the N $\rightarrow$ S transition and, on reaching the maximum of $A(T,P)$ at a higher T, jumps back to the normal state. This transition is a firstorder transition. Experiment reveals indeed such a hysteresis and a steplike S $\rightarrow$ N transition.

It is seen from the temperature dependence of the critical current of an irradiated bridge as measured by the conventional method (fig. 6) that $I_{\mathrm{c}}^{\max}(T)$ drops down to zero at a certain temperature higher than T_{c0}, without irradiation (see also Klapwijk et al. 1977, Dmitriev and Khristenko 1978). By applying a special pulsed technique, allowing to avoid heating of the sample in the resistive state by microwave irradiation and providing measurement of $I_{\mathrm{c}}^{\max}(T)$ by heating of the bridge from the S-state, Pals and Dobben (1978, 1979) showed that in this environment $I_{\mathrm{c}}^{\max}(T)$ smoothly decreases to zero. A possible explanation is the following. In a bridge which is not irradiated, at $T \rightarrow T_{\mathrm{c}}$, both I_{c} and Δ smoothly tend to zero. In an irradiated bridge, at $T \rightarrow T_{\mathrm{cm}} > T_{\mathrm{c0}}$, Δ smoothly reduces to a finite value, Δ_{m}, and $I_{\mathrm{c}}^{\max}$ smoothly tends to zero (Dmitriev and Khristenko 1979). At $T_{\mathrm{cm}} > T > T_{\mathrm{co}}$, the superconducting state is metastable and as we approach T_{cm}, the magnitude of the free energy relative minimum diminishes (Schmid 1977), and the steplike transition to normal state from the state with a finite gap may be triggered by any arbitrarily small current.

Direct observations of the energy gap enhancement in a superconductor in an electromagnetic field were very important. Kommers and Clarke (1977) irradiated ($\omega/2\pi = 10$GHz) at $T \lesssim T_c$ an $\mathrm{Al} - \mathrm{Al_2O_3} - \mathrm{Al}$ tunnel junction with electrodes ranging in thickness from 800 to 3000Å and width 300μ m, on $\mathrm{BaF_2}$ substrate. Figure 8 represents the dependence of the junction differential resistance dV/dI on the voltage for a variety of incident powers. The electrode films had slightly different gaps Δ_1 and Δ_2 and at $P = 0$, in the dependence of $dV/\mathrm{d}I$ on V, the minimum **a** corresponds to $(A_1 + A_2)/e$ and the minimum **b** to $(A_1 - A_2)/e$. As is seen from fig. 8, incident power growth is attended by an appreciable displacement of the minimum a corresponding to a 80% increase in ($\Delta_1 + \Delta_2$), while the minimum **b** is displaced by no more than 3μ V. The minima **c**, **d** and **e** represent various singularities of the $I - V$ curve due to tunnelling accompanied by photon absorption or emission (Dayem and Martin 1962). There is also a hysteresis in the dependence of ($\Delta_1 + \Delta_2$) on T and a jumplike S $\rightarrow$ N transition, if temperature rises at a fixed incident power.

Fig. 8. Differential resistance $\mathrm{d V / \mathrm{d} I$ of an $\mathrm{Al} - \mathrm{Al_2O_3} - \mathrm{Al}$ tunnel junction as a function of voltage for various incident powers indicated near each curve. The irradiation frequency is 10 GHz . $T/T_c \approx 0.99$.}

Hall et al. (1980) measured I_c, T_c and Δ enhancement in a narrow Al film with four cross-shaped $Al - Al_2O_3 - Al$ tunnel junctions formed along it. Radiation of frequences 0.1 to 12 GHz was applied to the Al film through a strip line between the film and a ground plane. In the frequence range 2 to 4 GHz there was a 100% enhancement of the energy gap ($T \lesssim T_c$), increase of the critical temperature by 16 mK above the equilibrium T_{c0} and an associated increase in I_c. Thus, these direct measurements, agreeing with the above data, are unambiguous evidence that in an electromagnetic field the superconductor energy gap becomes larger.

The dependence of superconductivity enhancement on the radiation frequency was noticed in one of the first works to deal with this effect (Dayem and Wiegand 1967). It was found that at a temperature close to T_c the growing I_c of Sn bridges is only observed at frequencies above about 1 GHz . Detailed investigations of the frequency dependence of the effect were carried out by Latyshev and Nad' (1974, 1975, 1976) (see also Dmitriev and Khristenko 1978). I_c of long Sn bridges was measured as a function of the incident power for fixed frequencies at some temperature points in the range $T_c - T \lesssim 1$ K. The measurement cycles used different frequencies from 8 to 78 GHz . The maximum increase in I_c for a certain optimum power P_{opt} is convenient to characterize as

$$\beta_m = I_c\left(P_{opt}\right)/I_c(0);$$

then the enhancement condition is $\beta_m > 1$. The dependences $\beta_m - 1$ thus obtained for various frequencies typical for all samples of length $L \gg 1\mu$ m are represented in fig. 9 (Latyshev and Nad' 1974). For frequencies below 25 GHz , the I_c enhancement is maximum near T_c (where "induction" of I_c is also observed at $T > T_c$). As the temperature is lowered, the enhancement decreases and at a certain temperature disappears (within the experimental error). At above 25GHz, I_c enhancement near T_c is absent; it starts at some temperature dependent on frequency, then attains a maximum and decreases, however without vanishing even down to the lowest temperatures measured. The boundary points of the enhancement effect region of existence are given by points of intersection of the $\beta_m - 1$ curves with the $T_c - T$ axis (fig. 9). By plotting these boundary points of onset and vanishing of the I_c enhancement in a plot with the x-coordinate $\left(T_c - T\right)/T_c$ and y-coordinate $\omega/2\pi$ (fig. 10; Latyshev and Nad' 1974, 1976), one obtains two curves bounding the region of the enhancement effect: it is observed inside this enclosure and is not outside. Some of the points of fig. 10 were obtained from data of Dayem and Wiegand (1967), and Dmitriev and Khristenko (1978). They are in fairly good agreement with the results of Latyshev and Nad'.

Eliashberg's theory is that irradiation causes enlargement of the energy gap only at frequencies $\omega > \omega_c$, where the temperature-dependent critical frequency

ω_c, for $\omega < \Delta/\hbar$, is specified by

$$\omega_c^2 \ln \frac{8\Delta}{\hbar\omega_c} = 2\pi\tau_\varepsilon^{-1}\Delta/\hbar \tag{2}$$

where τ_ε is the excitation relaxation time. The dependence $\omega_c(T)/2\pi$ obtained by eq. (2) is represented by the broken curve in fig. 10 for $\tau_\varepsilon = 1.4 \times 10^{-10}$ s. The agreement of the experimental data and the theoretical curve is good enough. Note that the measurements described provide an experimental method to determine the quasi-particle relaxation time for superconductors. The value for τ_ε obtained by this method agrees fairly well with those by other methods (Clarke and Paterson 1974).

At frequencies above ω_c, as the frequency rises, the enhancement goes through a maximum and vanishes at frequencies corresponding to the upper boundary (fig. 10; Latyshev and Nad' 1974, 1976). In the vicinity of T_c the upper frequency boundary lies somewhat higher than $2A(T)$, owing to gradual rise of the competitive mechanism of pair breaking at $\omega > 2A/\hbar$. One of the results of the increasing quasi-particle number is also increasing radiation absorption

Fig. 9. Temperature dependence of the maximum enhancement of the critical current $\beta_{\mathrm{m }}=I_{\mathrm{c}}^{max}(P) / I_{\mathrm{c}}(0)$ for a variety of irradiation frequencies (in GHz).}

Fig. 10. Frequency boundaries of the region of existence of critical current enhancement. The points represent Sn bridges of various lengths from 2 to $700 \mu \mathrm{~m $ (Latyshev and Nad' 1976, Dmitriev and Khristenko 1978); ⋄ are data of Dayem and Wiegand (1967).}

leading to a higher "effective temperature" of the film (to be more accurate, it leads to larger deviation of the phonon distribution from equilibrium), and eventually to a smaller energy gap. This phenomenology is corroborated by a calculation based on the microscopic theory with allowance made for the changes in the phonon distribution function (Chang and Scalapino 1977, 1978). These authors showed that the gap growth is much smaller at frequency $\omega \gtrsim 2A/\hbar$ and the upper boundary should be a little above the dependence $2\Delta(T)$, in agreement with the experiment (fig. 10). The absence of the enhancement in Sn bridges at above 70 GHz seems to be due to the fact that at these frequencies, $\hbar\omega$ is within the order of magnitude of kT. As was shown by estimations based on a theoretical paper by Ivlev et al. (1973), the enhancement should vanish at $h\omega > kT$. Besides, the upper boundary position in this frequency range may be affected by thermal effects as well. The dependence of the enhancement on the radiation frequency was observed also by direct investigations of superconductor gap enhancement. Thus, e.g.. for an Al film, the growth of Δ was zero at frequencies $\omega/2\pi < 1$GHz, was maximum at about 4 GHz and was appreciably reduced at $\omega/2\pi > 10$GHz (Hall et al. 1980).

The theory of Eliashberg and co-workers predicted that as the power increased, the film

critical temperature would approximately double and accordingly the critical current would grow too, with saturation at high powers, $\sim (T_c/\omega)^4$. However, in an experiment (see figs. 4 and 5), at much lower powers (for a Sn bridge it was three orders of magnitude lower than predicted), the dependences $I_c(P)$ and $T_c(P)$ reach a maximum and then abruptly fall off. The maximum increment of T_c was no more than 30 mK for a Sn bridge (Latyshev and Nad' 1977) and 100 mK for an Al bridge (Pals and Dobben 1979).

To explain the disagreement between theory and experiment, it is appropriate to realize two experimental findings. First, irradiation increases the hysteresis in the $I-V$ curve, due to heat generation in the bridge; the minimum critical current I_c', realized at decreasing current, does not grow; instead it tends to zero and at $I_c' = 0$ there is a sharp decreases in I_c (Latyshev and Nad' 1976, Pals and Dobben 1979). Secondly, the I_c and T_c growth amounts, all other things being equal, depend on the heat transfer from the films: δI_c and δT_c are higher for direct contact of the bridge with liquid helium (and still higher in case of superfluid helium), than with gaseous helium (Latyshev and Nad' 1977, Klapwijk et al. 1978, Pals and Dobben 1979). These two facts evidence an essential influence of heat generation and transfer on superconductivity enhancement in bridges (see also Dmitriev et al. 1970). In the development of the microscopic theory it was supposed that heat transfer was so efficient (i.e. practically infinite), that irradiation did not drive phonons out of equilibrium. In real experimental conditions, heat transfer is certainly finite and is not likely to be high. In this case the energy given by excited quasiparticles to phonons has not enough time to flow to the thermostat, and the phonon distribution function may change markedly. This alteration may be treated very approximately as a film "effective temperature" increase. For quantitative comparison with experiment, the microwave impedance of a long bridge with $R > 10\Omega$ was measured, so that the current induced in the bridge could be determined. Figure 5 presents dependences for three helium bath temperatures (solid curves). In the calculation the only fitting parameter was the quasi-particle energy relaxation time τ_ε. The curves in fig. 5 were obtained for $\tau_\varepsilon = 1.4 \times 10^{-10}$ s, in good agreement with the value obtained from measurements of the frequency boundaries. The result of the calculation, as is seen in fig. 2, has new qualitative singularities: a maximum and a sharp decrease in the $I_c(P)$ dependence, in general agreement with experiment.

This "effective temperature" concept is of course a very rough approximation. In reality the nonequilibrium phonon distribution function may differ very essentially from the thermal one with the effective temperature. If the heat transfer is finite and there is electron phonon interaction, the phonon distribution function self-consistently tunes itself to the nonequilibrium electron distribution function. Proceeding from Eliashberg's theory and the heat transfer restriction, Chang and Scalapino (1977, 1978) calculated the nonequilibrium electron and phonon distribution functions for an irradiated superconducting film and the gap value as a function of the incident power. The resulting dependence $\Delta(P)$ has a maximum like the calculations of Latyshev and Nad' (1976) who found a maximum in $I_c(P)$. These data taken together suggest that one of the most important restrictions to the growth of the critical film parameters is due to "effective" heating, or the departure of the phonon distribution from equilibrium in a microwave field, to be more accurate.

However, even in the absence of heating there may be agents to restrict the

enhancement. One of them is superconductivity suppression because of growing microwave current amplitude under large incident power, which was theoretically considered by Aslamazov and Gavrilov (1980). Another is the possible creation of inhomogeneous states in a superconductor by irradiation ($\omega < 24/\hbar$), see Ivlev (1977) and Eckern et al. (1979). Let us consider this point in some detail. At an incident power P smaller than the power $P_{\rm opt}$ corresponding to the maximum enhancement of the critical current I_c and critical temperature T_c, there is a possibility of development of a spatially inhomogeneous state in long bridges. This is reflected by steps in $R(T)$ curves measured at small currents (fig. 11, Latyshev and Nad' 1977) and

Fig. 11. Temperature dependence of the resistance of a Sn bridge, measuring $0.15 \times 1.5 \times 47 \mu \mathrm{~m
^{3}$, for various incident powers (the figures indicate the attenuation in dB).}

steps in the $I-V$ curves at $I>I_{\mathrm{c}}$ (fig. 12, Dmitriev and Khristenko 1979). As P approaches

Fig. 12. The steplike structure of the $I-V$ curve of a narrow, long tin film ($T_{\mathrm{c }}=3.814 \mathrm{~K}$) for various electromagnetic irradiation levels (the figures show the attenuation in dB).}

P_{opt} , I_{c} and T_{c} grow as usual. At $P \sim P_{\mathrm{opt}}$, spatial inhomogeneities are levelled out, as is suggested by the narrowing of the superconducting transition curve (fig. 11, curves for attenuation 20 and 15 dB ; interpretation to be found above) and by vanishing of the steps in the $I-V$ curve (fig. 12, curves for attenuation 22 and 19.35 dB). When P_{opt} is exceeded, I_{c} and T_{c} start to decrease rapidly (I_{c} falls off to zero in a narrow power range), and the $R(T)$ and $I-V$ curves break up into a number of steps, representing the bridge transition to the resistive state. The film state is stable at horizontal sections of $R(T)$ (fig. 11). In this case, some parts of the bridge are in normal, others in superconducting states (the characteristic size of these parts is about 10μ m). If the temperature is increased from this resistive state, then for each step there are local hysteretic effects.

During further power increase at a fixed temperature (fig. 12), the bridge resistance rises by jumps of almost equal size, till the entire bridge becomes normal.

It will be noted that the steps observed at $P>P_{\mathrm{opt}}$ are associated with individual features of the bridges (viz., their inhomogeneity). For some thicker (about 3000Å) bridges, which seem more homogeneous, there were no steps in $R(T)$ curves. If steps in $R(T)$ were observed

on bridges of similar dimensions, which had equal widths, thicknesses and approximately equal lengths (varying by $1-2\mu$ m), then they were quantitatively different, i.e. were unequal in number and size. Besides, if a short narrowing ($\sim 1\mu$ m) is made in the bridge, then the number of steps increases by 1 , with all the rest, if any, unchanged. This seems to suggest that the microwave irradiation-induced steps are somehow associated with sample inhomogeneity.

A spatially inhomogeneous state may arise from instability of the homogeneous nonequilibrium superconducting state in a microwave field (Ivlev 1977, Eckern et al. 1979) with the inhomogeneities fixed at the bridge defects. This state appears and developes at powers $P > P_{\text{opt}}$, where I_{c} and T_{c} start to decrease; therefore it may prove to be among the sources of restriction to superconductivity enhancement by a microwave field.

The superconductivity enhancement effects described in this section essentially depend on the amount of inhomogeneity in the bridge. The characteristic length of the irradiation-induced inhomogeneity in Sn bridges is roughly 10μ m. It was mentioned above that superconductivity enhancement in bridges longer than $\xi(T)$ is qualitatively identical. There are differences only for bridges with $L < \xi(T)$ (in Sn bridges, $L < 1\mu$ m), whose properties largely depend on spatial gradients of the energy gap (see below). Thus, the above described experimental results corroborate and also develop Eliashberg's physical interpretation of superconductivity enhancement in uniform strips by an electromagnetic field, which is based on formation of nonequilibrium stationary states as a result of quasi-particle redistribution in energies.

45.4 4. Superconductivity enhancement by phonons and tunnel effect

As was mentioned earlier, one of the possible means to produce a nonequilibrium situation in a superconductor, apart from an electromagnetic field, is to expose it to phonon radiation (sound field). This is how an experiment was devised involving observation of superconductivity enhancement by a field of sound waves of frequency about 10 GHz (Tredwell and Jakobson 1975). The critical current was investigated as a function of the sound field intensity both for Al-Sn Josephson point contacts $[L < \xi(T),\ I_{\text{c}} \sim (1 - T/T_{\text{c}})]$ and Sn bridges, and for long bridges $\left[L > \xi(T),\ I_{\text{c}} \sim (1 - T/T_{\text{c}})^{3/2}\right]$. In both cases there was an enhancement of the critical current caused by nonthermal phonons. In short samples there was a temperature threshold of the enhancement (for $T/T_{\text{c}} > 0.996$, the critical current was suppressed), similar to the case of the electromagnetic field (see below). Calculations made by Tredwell and Jakobson based on Eliashberg's idea of a change in shape of the energy distribution function of quasi-particles under the action of a field were in fair agreement with experiment for both Josephson junctions (fig. 13a) and long bridges (fig. 13b). These results suggest that in the case of an external field the sample geometry is of minor importance, influencing only some details of the enhancement effect (see below). The main contribution comes from nonequilibrium excitations due to

changes in the interaction of superconductor quasi-particle subsystems in an external electromagnetic or sound field.

Experiments in sound waves were not restricted to small samples. Experimental investigations of sound propagation characteristics in pure single-crystalline bulk samples of gallium (Fil' et al. 1975, 1976, 1980) showed that the superconductor electron subsystem is essentially nonlinear for propagation of finite-amplitude sound oscillations. Figure 14 presents the central experimental result of the investigation; in a superconductor, the impact of powerful sound pumping on the absorption of a weak acoustic measuring signal differing in frequency from the pumping signal. One can see that in a wide temperature range the attenuation of the measuring signal, with simultaneous sound pumping of the sample, essentially decreases. This fact is not related to the amplitude-dependent internal friction; it is due to direct perturbation of the superconductor electron subsystem (Fil' et al. 1976). By varying the mutual orientation of wave

Fig. 13. (a) Relative increase in the critical current as a function of temperature for a short microbridge with $I_{\mathrm{c} } \sim\left(1-T / T_{\mathrm{c}}\right)$. Solid line shows prediction of Eliashberg's theory modified for phonon excitation. (b) Same as (a), for a long microbridge with $I_{\mathrm{c}} \sim (1-T/T_{\mathrm{c}})^{3/2}$ and two phonon power levels: (1) 0 dB , (2) -3 dB .}

vectors of two signals, it was revealed that the perturbation embraces only a comparatively narrow region of the phase space near $q \cdot v_{\mathrm{F}} = 0$ (q is the wave vector of the pumping signal and v_{F} the Fermi velocity of excitations).

Formally the nonlinear sound attenuation curves in this case (fig. 14) may be described by the BCS expression for the attenuation coefficients α_S and α_N :

$$\alpha_{\mathrm{S}}/\alpha_{\mathrm{N}} = 2(\exp \Delta(T)/kT + 1)^{-1}$$

with Δ substituted by the nonequilibrium value of $\Delta_n(T)$. Results of such a calculation are presented in the inset of fig. 14. The value of $\Delta_n(T)$ so calculated proves to have a temperature dependence

$$\Delta_{\mathrm{n}}(T)/\Delta_{\mathrm{n}}(0) = [\Delta(T)/\Delta(0)]_{\mathrm{BCS}},$$

coinciding with the BCS one to within the measurement error, but the parameter $\Delta_n(0)$ is appreciably higher than its equilibrium value.

These experimental results have not yet found an unambiguous theoretical description. An analysis of the influence of a nonequilibrium quasi-particle distribution function in a sound pumping field on the character of sound propagation in superconductors (Bulyzhenkov and Ivlev 1976) shows that this influence is only essential in the temperature range $\Delta T/T_{\mathrm{c}} \sim \omega/T_{\mathrm{c}}$, and is satisfactorily consistent with experimental data for superconducting junctions, fig. 13. For a sound frequency about 150 MHz , as used in the experiments studying sound absorption in bulk superconductors, $\omega/T_{\mathrm{c}} \lesssim 10^{-12}$, much below the temperature range in which the said effect is seen (fig. 14). Therefore this area of theoretical studies is still open to application of

Fig. 14. Experimental temperature dependence of the attenuation of the longitudinal sound in superconducting gallium. $\mathrm{O
, \triangle$ in the absence of a sound pumping field; pumping level μ_1; pumping level $\mu_2 > \mu_1$. Signal and pumping frequencies are 54.2 MHz and 150 MHz respectively. The inset shows the temperature dependence of the nonequilibrium gap as calculated from attenuation curves by the BCS formula for the sound attenuation coefficient.}

further efforts. Some recent papers describe the direct enhancing effect of sound on Cooper pairing (Bar'yakhtar and Seminozhenko 1976, 1977. Seminozhenko and Yatsenko 1980), and take into account the particular character of the nonequilibrium quasi-particle distribution in the superconductor irradiated by sound waves (Galaiko et al. 1981). However, at present neither of the mechanisms discussed in these papers has unambiguous correspondence with the experimental results. Therefore further theoretical and experimental studies are desirable, in particular direct measurements of the energy gap of a superconductor in a sound wave field.

As mentioned above, the nonequilibrium state leading to superconductivity enhancement may be set up by quasi-particle tunnelling in a superconductor tunnel junction system,

$$\mathrm{S_B - I - S_A - I - S_B}$$

(Parmenter 1961) ($\mathrm{S_B}$ and $\mathrm{S_A}$ are superconducting films, I insulators, $2A_\mathrm{B}$ and $2A_\mathrm{A}$ the energy gaps of $\mathrm{S_B}$ and $\mathrm{S_A}$ respectively, so that $\Delta_\mathrm{B} \gg \Delta_\mathrm{A}$) (fig. 15). At a temperature $T \lesssim T_\mathrm{cA}$, in the superconductor $\mathrm{S_A}$ the number of normal excitations is sufficient, while in superconductors $\mathrm{S_{B3}}$ it is negligibly small. By applying a voltage $V = 2\left(A_B - A_A\right)/e$ between two S_B contacts, one can reduce the number of normal excitations of both signs in the superconductor $\mathrm{S_A}$ through their tunnelling to the superconductors $\mathrm{S_B}$. According to BCS theory. such decrease in the quasi-particle excitation number should lead to higher A and T_c of S_A. As was found later (Kirichenko et al. 1979), quasi-particles may be extracted even in a simpler structure, $\mathrm{S_B - I - S_A}$. Moreover, quasi-particle extraction (injection) from a superconductor is not vital to superconductivity enhancement. It is sufficient to pass a direct tunnel current through a symmetric S-I-S junction. If the junction voltage is $V < 2\Delta/e$ then the tunnel current flow results in energy redistribution of quasi-particle excitations to higher energies

Fig. 15. Energy level diagrams for a system of a superconducting film of thickness L and energy gap $2 \Delta_{\mathrm{A
}}$ separated by insulating layers from superconducting contacts with energy gap 21_{B} : (a) zero bias; (b) $= 2\left(\Delta_{\mathrm{B}}-\Delta_{\mathrm{A}}\right)/e.$}

in each of the superconducting electrodes, with the total number of excitations remaining unchanged, and a superconductivity enhancement situation is realized which is similar to that considered by Eliashberg in an electromagnetic field (Peskovatskii and Seminozhenko 1976).

In the experimental environment, superconductivity enhancement resulting from tunnel current flow was observed on a symmetric $\mathrm{Al_1Al_2O_3Al}$ junction in the structure $\mathrm{AlAl_2O_3} - \mathrm{Al} - \mathrm{Al_2O_3} - \mathrm{Al}$ (Gray 1978). The first tunnel junction (generator) had low resistance, and its electrodes were in nonequilibrium. The second junction (detector) had much higher resistance and was used to measure the IV characteristics from which the change of the gap of the Al medium film was determined. The difference between the nonequilibrium and equilibrium gaps, $\delta\Delta/\Delta_0$, was up to 0.5% at $T \approx 1.2$ K (fig. 16). A similar experiment on a symmetric $\mathrm{Sn} - \mathrm{SnO}, -\mathrm{Sn} - \mathrm{SnO}, -\mathrm{Sn}$ junction yielded a maximum value of $\delta\Delta/A_0$ about $4^0{}_0$, the δA_{m} value increasing with temperature in a linear fashion (Mitsen et al. 1980), see fig. 17.

The superconductivity enhancement resulting from quasi-particle extraction from the superconductor, was experimentally studied on the structure $\mathrm{Al_B} - \mathrm{Al_2O_3} - \mathrm{Al_A} - \mathrm{Al_2O_3} - \mathrm{Al}$ (Chi and Clarke 1979). Quasi-particles were extracted from the Al_A film through the first tunnel junction to the Al_B film with a larger gap. The nonequilibrium energy gap in the $\mathrm{Al_A}$ film was measured by the second tunnel junction characteristics. The measured maximum excess of the nonequilibrium gap over equilibrium was up to 40%.

A comparison of experimental results with corollaries of theoretical works found qualitative agreement between them. However our knowledge of this

type of nonequilibrium superconductivity is still insufficient, and detailed comparison is hardly possible before new efforts bring new results.

Fig. 16. The normalized change in gap in the detector junction as a function of generator voltage. The current in the generator is also plotted to show the pair breaking structure at $V_{\mathrm{g}}=2 \Delta / e$.}

45.5 5. Superconductivity enhancement and nonequilibrium phenomena in weakly coupled superconductors

The term "weakly coupled superconductors" is the common name of superconductive structures with local suppression of the order parameter $\hat{A}$. This suppression is usually produced by placing a thin nonsuperconducting layer (an insulator or a normal metal) between two superconductors, or by making a short narrow neck between bulk superconducting electrodes (point contact or thin film bridge). If the lower order parameter area is sufficiently small, not larger than $\breve{\zeta}$, then the principal parameters of the superconductor undergo changes substantial for this scale, and therefore their spatial gradients are very high.

In this section, we shall deal with weakly coupled superconductors with metallic conductivity of superconductor-constriction-superconductor type. In such structures stationary and nonstationary Josephson effects are observed. In the case of the nonstationary Josephson effect ($\boldsymbol{V} \neq 0$), as was theoretically shown by Aslamazov and Larkin (1969), the gap in the weak link region oscillates with the Josephson frequency, $\omega = 2\mathrm{eV}/\hbar$, thus causing also oscillations of the same frequency of the superconducting and normal components of the junction current and the Josephson self-radiation [the high-frequency Josephson radiation of point contacts was first observed by Dayem and Grimes (1966), and that of thin film bridges by Gubankov et al. (1975)]. The $I-V$ curve of a junction of dimensions $L \gg l, w, L \ll \xi(T)$ (l being the electron mean free path) at $T \approx T_{\mathrm{c}}$ is, according to Aslamazov and Larkin's (1969) theory, hyperbolic,

Fig. 17. (a) Experimental dependence of the gap change on the voltage across the generator contact for several temperatures. (b) The maximum gap change as a function of temperature.

$$V = R\sqrt{I^2 - I_c^2}$$

with the critical current $I_c = cA_0^2/\sigma_N R$, where σ_N is the specific conductivity, A_0 the order parameter of bulk electrodes away from the constriction, and the constant (c was introduced by Gorkov (1959); the current across the junction is the sum of the normal (V/R) and superconducting ($I_c \sin\varphi$) currents, $\varphi = (2e/\hbar)\int V\,dt$ being the difference of the wave function phases of the superconducting electrodes. This description of weakly coupled superconductors corresponds to the so-called resistive model (McCumber 1968, Stewart 1968), which offers a satisfactory explanation to the main experimental dependences in such structures and serves as the basis for the calculation of characteristics for various Josephson devices (Likharev and Ulrich 1978. Volkov et al. 1978). However, recent theoretical and experimental studies have shown that in point contacts and small film bridges an important role is played by nonequilibrium effects. As a result, the $I-V$ curves and microwave properties of real point contacts and bridges may appreciably differ from those in the resistive model. The theoretical aspects of nonequilibrium effects in weakly coupled superconductors are treated by Aslamazov and Volkov (1985) in the present volume. Here we shall present the main experimental results for the most interesting nonequilibrium effects in weakly coupled superconductors: the superconductivity enhancement in short junctions and the excess current effect.

For the manifestation of the Josephson effect proper, the characteristic dimensions of bridge-type or point contact-type weakly coupled superconductors (L, w) should not exceed 2 or $3\xi(T)$ (Gubankov et al. 1976); otherwise, nonstationary effects are most likely to be due to motion of magnetic flux quanta (Gubankov et al. 1973). Most of the results to be presented in this section

were obtained in studies of tin bridges of variable thickness, with the following characteristic parameters.

bridge film: length $L = 0.4 - 0.6\mu$ m, width $w = 0.5 - 2\mu$ m, thickness $d = 300 - 1000$Å;

bank film: thickness $d^{\mathrm{B}} = (2-5) \times 10^3$Å;
$\xi(0) = 800 - 1000$Å, a normal-state resistance $R_{\mathrm{N}} = 0.05 - 0.5\Omega$. Variable-thickness bridges are better than point contacts for nonequilibrium effect studies, because they allow to control precisely the geometrical and other characteristic parameters of the weak coupling region.

In the discussion of characteristic parameters of these structures, it should be mentioned that the critical temperature of the bank film, $T_{\mathrm{c}}^{\mathrm{B}}$, is as a rule a few hundredths of a degree higher than the critical temperature of the bridge proper, T_c^b. This experimental fact was noted by more than one group of investigators (see above) and was ascribed to various effects: fluctuations, differences between properties of the bank and the bridge, etc. $T_{\mathrm{c}}^{\mathrm{B}}$ is usually determined by appearance of nonlinearity in the $I - V$ curve and from the dependence $R(T)$ measured at small current ($I \leqslant 1\mu$ A). In the temperature range $T_{\mathrm{c}}^{\mathrm{b}} < T < T_{\mathrm{c}}^{\mathrm{B}}$ where the structure superconductor normal-metalsuperconductor is realized, the critical current is nonzero because of the small size of the normal-metal region: its characteristic length is comparable with the coherence length of the bridge material. The temperature dependence of the critical current at $T_{\mathrm{c}}^{\mathrm{b}} < T < T_{\mathrm{c}}^{\mathrm{B}}$ is nonlinear and looks like that predicted by the theory (Aslamazov et al. 1968) for short S-N-S bridges. In case of further moderate temperature decrease ($T \lesssim T_{\mathrm{c}}^{\mathrm{b}}$), the $I_{\mathrm{c}}(T)$ dependence becomes usually linear, in agreement with the calculation of Aslamazov and Larkin (1969), and Kulik and Omelyanchuk (1978). The value of $T_{\mathrm{c}}^{\mathrm{b}}$ is found from $R(T)$ or by extrapolation of the linear part of $I_c(T)$ to the intersection with the T-axis.

The temperature evolution of the $I - V$ characteristics of one small variablethickness tin bridge is shown in fig. 18. It is representative of such structures and has been observed by several groups of authors (Octavio et al. 1978, Gubankov et al. 1977, Latyshev and Nad' 1979). It is characterized by the following features. At $T \lesssim T_{\mathrm{c}}^{\mathrm{B}}$, the bridge $I - V$ curve becomes nonlinear, there appears a superconducting part with $V = 0$. At $T \lesssim T_{\mathrm{c}}^{\mathrm{b}}$ and moderately high voltages, the $I - V$ shape becomes almost hyperbolic; as the voltage grows, there is an excess-current effect (Pankove 1966). As we move down from T_c, the situation changes: in the beginning of the IV resistive part, there appears a linear section with the differential resistance $R_{\mathrm{d}} < R$, which in the higher current region transits into a section of rapidly increasing $R_{\mathrm{d}} > R$. Near this transition one can distinguish the bump point I_{s}^0 (see the insert in fig. 18). As the temperature is lowered further [$T_{\mathrm{c}} - T > 0.5$ K], in the low V region, one can see hysteresis-shaped voltage jumps. Such temperature evolution of the $I - V$ characteristics of small bridges in the region of moderately high voltages is in agreement with Aslamazov and Larkin's (1976) calculation with allowance for

the effect of nonequilibrium electrons. The dynamic processes are in this case described by an equation similar to eq. (2.7) in the chapter by Aslamazov and Volkov.

The physical picture of the appearance of singularities in the $I - V$ curve of short bridges at sufficiently small voltages and temperatures slightly lower than T_c is as follows. As I_c is exceeded, the nonstationary Josephson effect starts in the bridge, with the gap in the constriction oscillating with Josephson frequency $\omega = 2\mathrm{eV}h$ which increases with the voltage (Aslamazov and Larkin 1969). The quasi-particles in the constriction with energy :< Δ_H receive some surplus chergy from collisions against the oscillating walls of the potential well $A(x)$, and as a result, their distribution function $f(x)$ changes. The change of $f(x)$ may be such that the time-averaged energy magnitude of the gap increases, causing growth of the bridge current, i.e. resulting in superconductivity enhancement. This mechanism is similar (with the exception of the "pumping" source) to the superconductivity enhancement in uniform bridges treated

Fig. 18. Alteration of the $I-V$ curve of a short bridge induced by temperature variation; for $T=3.641 \mathrm{~K}$, the $I-V$ curve for the bridge under a small microwave irradiation with $\omega/2\pi = 38.2$GHz is also shown. The insert shows the $I-V$ curve for the short bridge as calculated with allowance for nonequilibrium effects (Aslamazov and Larkin 1976).}

by Eliashberg (197()). At voltages $V > \tau_i^{-1}$, i.e. where the Josephson oscillation period is smaller than the energy relaxation time. there appears a linear section in the $I-V$ curve, with $R_\mathrm{d} < R$. When the current reaches the characteristic value I', there occurs a sharp increase in the time-averaged voltage. caused by the growing normal component of the current. There is only superconductivity enhancement if the constriction size is smaller than but larger than a certain characteristic length $\eta = \xi(T)\,(\delta T/T_\mathrm{c})^{1/4}$ (Aslamazov and Larkin 1976, Artemenko et al. 1979a) for which electrons with energies smaller than A_B can diffuse from the constriction. Since the value of η depends on temperature. the inequality $L > \eta$ becomes true some distance below T_c; that is, superconductivity enhancement by the "direct current" in three-dimensional contacts has a temperature threshold, which is observed experimentally (Octavio et al. 1978, Gubankov et al. 1977, Latyshev and Nad' 1979). Indeed, as is seen in fig. 18, the characteristic features of the $I-V$ curve (low R_d section in the beginning of the resistive region, "bump" region with $R_\mathrm{d} > R$ near I_c^0) appear at $\delta T = T_\mathrm{c} - T \simeq 0.02 - 0.04$ K, where L grows above η. In accordance with theoretical predictions the shape of the $I-V$ curve is near-parabolic above the bump point I'. while below. down to the linear region with low R_d. it is near-hyperbolic (Octavio 1978); for some samples the temperature dependence of $(I_\mathrm{s}^0/I_\mathrm{c})^4$ in the range $\delta T \simeq 0.08 - 0.2$ K is linear (Gubankov et al. 1977). The deviation of $(I_\mathrm{c}^0/I_\mathrm{c})^4$ from linear at small δT may be associated with the increasing error in I'. determination. and that at large δT with the appearance of heat-induced voltage jumps in the I l curve. As was found by experiments, the differential resistance of the linear $I-V$ section at $I \gtrsim I_c$ decreases with temperature in inverse proportion to $T_\mathrm{c} - T$ (Gubankov et al. 1977, Octavio et al. 1978). Figure 19 represents $R_d(T)$ in a wide temperature range for Sn bridges of different lengths from 0.5 to 0.9μ m. The dashed straight line is $A = T_\mathrm{c}\,(T_\mathrm{c} - T)^{-1}$ for $A = 0.01$. One can see that the experimen-

tal dependence $R_\mathrm{d}(T)$ goes as $(T_\mathrm{c} - T)^{-1}$. The theoretical linear part was obtained by Aslamazov and Larkin (1976) for bridges of length $\eta(T) \ll L < \xi(T)$ only for voltages $\tau_\varepsilon^{-1} < V < \tau_\varepsilon^{-1} T/\Delta$. According to this theory, $R_\mathrm{d} \sim (T_\mathrm{c} - T)^{-5/2}$. For tin bridges, L becomes larger than $\eta(T)$ at $kT/\Delta \leqslant 3$, and therefore, the range of applicability of the theoretical calculation

Fig. 19. Temperature dependence of ratio $R_{\mathrm{d} } / R_{\mathrm{N}}$ for bridges of various lengths: (•)0.5μ m; (O) 0.6μ m; (+) 0.65μ m.}

is rather narrow, making it difficult to compare theory and experiment quantitatively.

Figure 20 presents the temperature dependence of the characteristic voltage V^0 corresponding to zero second derivative (the bump point of I^0) of the $I-V$ characteristics for some of the studied samples (Gubankov et al. 1980). It is seen that V^0, depends but slightly on the tin bridge parameters and changes with temperature as $(T_{\mathrm{c}}-T)^{1/2}$. These experimental data, combined with the theory of Aslamazov and Larkin (1976), allow to estimate the energy relaxation time τ_1 for quasi-particles. For tin it is $\tau_0^{\mathrm{Sn}}=(2-6)\times 10^{-10}$ s. This range is in agreement with the value of Latyshev and Nad (1976), and Octavio et al. (1978): $\tau_\varepsilon^{\mathrm{Sn}}=(25)\times 10^{10}$ s. In a similar way, by using the $V_{\mathrm{s}}^0(T)$ dependences obtained by Hansen et al. (1978), we have the following for aluminium and lead bridges: $\tau_\varepsilon^{\mathrm{Al}}=2\times 10^{-9}$ s and $\tau_\varepsilon^{\mathrm{Pb}}=10^{-11}$ s. All these τ_ε values are several times smaller than the theoretical electron-phonon scattering time τ_0 in a normal metal at $T=T_c$. [see data presented in the paper by Gorkov and Eliashberg (1969)].

It should be noted that changes in bridge dimensions result in appreciable alteration of the $I-V$ characteristics. In very short bridges (for Sn, $L<0.5\mu$ m), in a sufficiently wide temperature range near T_{c} (about 40 mK), the $I-V$ curve was close to hyperbolic, without distinct nonequilibrium singularities at $I>I_c$, which in general agrees with the theory (Aslamazov and Larkin 1976, Artemenko et al. 1979a) which says that the energy diffusion of quasi-particles is hindered, for $L<\eta(T)$, and though the processes preserve their nonequilibrium character, the linear part at $I \gtrsim I_{\mathrm{c}}$ and the IV kink point should vanish (for details see the chapter in the present volume by Aslamazov and Volkov). As L grows towards $L\sim\xi(T)$, the $I\cdot V$ part with small differential resistance was less pronounced and was seen in a narrower temperature range located closer to T_{c}. In long bridges ($L\gg\xi(T)$) the low differential resistance part is not observed at all (Latyshev and Nad' 1979); the transition to the resistive state occurs by a jump from $V=0$ to a certain finite voltage at $I=$ const.; there is no superconductivity enhancement by "direct current". At lower temperature, the superconductivity enhancement in bridges should vanish if $L\gg\xi(T)$ (Aslamazov and Larkin 1976). This conclusion, generally speaking, conforms to experimental data, although the theory does not allow for thermal effects. In real experiments, heating of the weakly coupled area by current is usually most pronounced at low

Fig. 20. Temperature dependence of the characteristic voltage $V_{1 $ for a number of short bridges of lengths 0.5 to 1μ m, corresponding to current I_s^0. The solid curves show theoretical dependences $V_1(T)$ calculated for $\tau_\varepsilon = 2 \times 10^{-10}$ s and $\tau_\varepsilon = 6 \times 10^{-10}$ s (Gubankov et al. 1980).}

T, where L is larger than $\check{\zeta}(T)$ and
can have a certain effect upon superconductivity enhancement, suppressing the above considered singularities of the $I - V$ curve due to enhancement. An analysis of thermal effects in weakly coupled superconductors was carried out by Nad' and Polyanskii (1973) and also by Skocpol et al. (1974). They showed in particular that the hysteretic effects and voltage jumps occurring when the critical current is exceeded at relatively low temperatures, as well as the local hysteresis in the $I - V$ curve in places with gap singularities, are due to heating of the bridge film by the current.

It might be well to mention that some $I - V$ singularities similar to fig. 18 and those discussed above were observed on superconducting point contacts too (Divin and Nad' 1978). However, detailed study of these features of point contacts is rather difficult because of the impossibility of correct determination of geometric and other characteristic parameters of the weakly coupled area.

Let us now consider superconductivity enhancement in weakly coupled superconductors in an external electromagnetic field (Dmitriev et al. 1973, 1976, Dmitriev and Khristenko 1979, Gregers-Hansen and Levinsen 1971, Levinsen 1974, Klapwijk and Mooij 1976, Dayem and Wiegand 1967. Latyshev and Nad' 1974, Peskovatskii and Strizhko 1979, Octavio et al. 1978. Gubankov et al. 1977, Hansen et al. 1978, Bevza et al. 1976).

The physical causes of the superconductivity enhancement in short bridges and point contacts subjected to microwave irradiation were analyzed theoretically by Aslamazov and Larkin (1978). In contrast to long homogeneous films in which, according to Eliashberg's (1970) theory, nonequilibrium and enhancement (see the preceding section) result from direct influence of the electromagnetic field on quasi-particles, in short, spatially inhomogeneous bridges or contacts (with the characteristic dimension ~ $\check{\zeta}(T)$) these effects are associated with gap oscillations in the weakly coupled area. When there is no microwave irradiation and the bias current is higher than the critical value, these oscillations have the frequency of the Josephson selfgeneration and give rise to IV singularities similar to those considered above. Under external irradiation, when the bias current is lower than the critical value and Josephson generation does not arise, the "vibration" of the potential well forming in the constriction has the frequency of the applied irradiation and also can cause energy diffusion of electrons, shift of their distribution function, effective enhancement of the gap and superconductivity enhancement, i.e. critical

current growth.

Thus, the theory (Aslamazov and Larkin 1976, 1978) maintains that the superconductivity enhancement effects in weakly coupled superconductors depending on the bias current $I > I_c$ (i.e. the influence of the Josephson selfoscillations) and conditioned by an external electromagnetic irradiation have common nature. Experimental data corroborate this conclusion.

Experimental investigation of superconductivity enhancement by microwave irradiation in short bridges and contacts involved a wide frequency range:

Fig. 21. Temperature dependences of β values (for various microwave field frequencies) the ratio $\left(I_{\mathrm{s}}^{0}-I_{\mathrm{c}}\right) / I_{\mathrm{c}}$ of a short bridge (0.5μ m). The temperatures for which ξ and η are comparable with L are marked with broken lines.}

from 0.1 GHz to 150 GHz (Dayem and Wiegand 1967, Dmitriev et al. 1973, Latyshev and Nad' 1974, 1976, 1979, Dmitriev and Khristenko 1976, 1979. Klapwijk et al. 1977, Octavio et al. 1978, Gubankov et al. 1978). The magnitude of the effect may be characterized by the relative increase of the critical current under irradiation to the maximum value $\beta_{\mathrm{m}} = \left(I_{\mathrm{c}}^{\max} - I_{\mathrm{c}}\right) / I_{\mathrm{c}}$.

Figure 21 represents typical temperature dependences of β for various radiation frequencies and the temperature dependence of $\left(I_{\mathrm{s}}^{0} - I_{\mathrm{c}} \right) / I_{\mathrm{c}}$ (Gubankov et al. 1980). Superconductivity enhancement by irradiation is much lower at frequencies $\omega/2\pi \lesssim 10$GHz; in the range $\omega/2\pi \lesssim 1$GHz $\simeq \tau_{\varepsilon}{}^{-1}$, the effect was not seen at all. This suggests that superconductivity enhancement by irradiation is due to nonequilibrium phenomena. However, unlike the experimental results for long thin films (Latyshev and Nad' 1975, 1976, Klapwijk and Mooij 1976) and the theoretical predictions for spatially homogeneous superconductors (Eliashberg 1970), small point contacts and bridges of variable thickness did not display an increase of the critical temperature, at least at frequencies 1 to 150 GHz . Moreover, microwaveinduced superconductivity enhancement in short bridges has a temperature threshold (Jillie et al. 1975, Dmitriev et al. 1976, Dmitriev and Khristenko 1978, Gubankov et al. 1977). For example, in the bridge whose $\beta(T)$ and $\left(I_{\mathrm{s}}^{0} - I_{\mathrm{c}}\right) / I_{\mathrm{c}}$ are represented in fig. 21 , at a frequency of $\omega/2\pi \approx 10$GHz, this effect appeared and rose as the temperature was reduced, only if δT was higher than a certain threshold value: $\delta T > \delta T_{\mathrm{th}} \approx 0.02$ K.

It is obvious also from $\beta(T)$ that the maximum critical current value in a microwave field is never in excess of I_{s}^{0} and is realized at temperatures such that $L \leqslant \xi(T)$. For Sn bridges, as the radiation frequency rises, δT_{th} goes through a minimum, usually at $\omega/2\pi \simeq 10$GHz, and β has its maximum magnitude at $\omega/2\pi \simeq 40 - 50$GHz. The temperature range of superconductivity enhancement also changes in a nonmonotonic fashion: at growing frequency it widens and shifts to higher temperatures (δT_{th} decreases) and then becomes narrower, shifting downwards from T_{c} (δT_{th} increases).

The study of the frequency dependence of superconductivity enhancement in short bridges leads to the conclusion that in this case, as for long homogeneous films, there are lower and upper

frequency boundaries for occurrence of the effect. The lower boundary is determined by the Eliashberg (1970) condition (see the preceding section). The existence of the upper frequency boundary was revealed experimentally by Latyshev and Nad' (1974). At temperatures close to T_c the upper frequency boundary is determined by the energy gap magnitude A. The most probable cause may be the fact that at $\hbar\omega > 2\Delta$ the predominant process is Cooper depairing due to irradiation

Fig. 22. Differential resistance as a function of the bias voltage for a short bridge (\$0.5 \mu \mathrm{~m
\$) in the absence of irradiation ($P=0$) and under microwave irradiation ($\omega/2\pi = 9.8$GHz) corresponding to the maximum critical current enhancement.}

absorption, which leads to the usual gap diminution. However, when $\omega/2\pi \sim 100$GHz, the upper frequency boundary lies below that determined by the condition $\omega = 24/\hbar$, probably for the same reasons as in the case of long bridges.

There is a correlation between rising critical current and growing voltage V_A corresponding to the gap singularity, evidencing gap enhancement as well. Figure 22 supplies an example of the changing position of the characteristic gap singularity in $\mathrm{d}V/\mathrm{d}I$ versus $\vec{V}$ for a short bridge under the action of irradiation (Gubankov et al. 1978). One can see that V_Δ enhancement is about 2%. The small increase of V_Δ under microwave irradiation seems to be due to the fact that superconductivity enhancement by "direct current" may essentially affect the gap magnitude in short bridges even in the absence of microwave field.

As the incident power P grows, the amount $I_c(P)$ changes as follows (Dmitriev et al. 1976, Dmitriev and Khristenko 1978, Latyshev and Nad' 1979, Peskovatskii and Strizhko 1979, Gubankov et al. 1980). When the power is small, the critical current grows slowly, sometimes remains practically constant or even slightly (by some 2%) falls off. Then follows a relatively sharp increase of I_c, up to the maximum value (fig. 23). After that the critical current decreases, oscillating, approximately as $\sqrt{P}$, that is, in the same way as does the critical current of the usual Josephson junction. Thus, at a certain power, where the critical current ceases to grow, the superconductivity enhancement effect is suppressed by the nonstationary Josephson effect. This seems to be the reason that $[I_c(P)]^{\max}$ does not increase to I_s^0, as was predicted by Aslamazov and Larkin (1978).

We have considered superconductivity enhancement in short bridges for the case of $\eta < L < \xi$. The change of the $I-V$ characteristics is qualitatively different in bridges of significantly larger size (for Sn bridges, $L > 1\mu m$). In this case irradiation leads to origination of

superconducting current at $T > T_c$, and to its considerable increase (tens of times) at $T \lesssim T_c$ (Dmitriev et al. 1973, Latyshev and Nad' 1974). The I-V curve of such bridges in the absence of irradiation has neither a resistive region with small R_d for $I > I_c$, nor Josephson steps; the transition to the resistive state, both in and out of microwave field, occurs jumpwise even at comparatively high temperatures where the $I - V$ curve in the absence of irradiation has no hysteresis. Some representative dependences of I_c^{max}/I_c for bridges with various values of L are shown in fig. 24 (Latyshev and Nad' 1979). We see that with increase of the bridge length, the boundary of the enhancement shifts. Simultaneously with the shift to T_c there is growth of β, and finally at $L > L_{th} = (D/\omega)^{1/2} \approx 1\mu$ m (where $D = \frac{1}{3}c_1/l$ is the diffusion coefficient) the character of the temperature dependence of the enhancement alters: the β value is maximum at $T \gtrsim T_c$ and gently falls off as the temperature drops.

Thus, it is rather obvious that it is spatial inhomogeneity that distinguishes

Fig. 23. Critical current of a short bridge ($0.5 \mu \mathrm{~m $) as a function of the microwave power P ($\omega/2\pi = 9.8$GHz). The insert shows the initial part of $I_c(P)$ upscaled.}

Aslamazov and Larkin's mechanism of enhancement from that of Eliashberg. Indeed, it is only in a spatially inhomogeneous environment that Josephson generation can occur, and consequently superconductivity can be enhanced if a direct transport current is passed. The shift from Aslamazov and Larkin's (1976, 1978) enhancement mechanism to that of Eliashberg (1970) may be made by changing the bridge geometry, e.g., through increasing its length. As the boundary of the transition in this case, one may regard the quantity L_{th} (Latyshev and Nad' 1979): the Aslamazov-Larkin mechanism, dominates below $L = L_{th}$, the Eliashberg mechanism dominates above.

It is worth emphasizing in conclusion that for all the differences in superconductivity enhancement between spatially homogeneous and inhomogeneous systems, the fundamental source of the effect is the same: a nonequilibrium of the quasi-particle distribution function induced by external microwave radiation or by Josephson self-radiation. This should always be borne in mind when analyzing the superconductivity enhancement in various experimental environments.

Fig. 24. Temperature dependences of $I_{\mathrm{c }^{\max } / I_{\mathrm{c}}}$ for bridges of various lengths L indicated in the plot.}

45.6 6. The excess current effect

The current-voltage characteristics of short superconductive bridges and contacts have, in addition to the above peculiarities due to nonequilibrium processes, one more essential distinctive feature as opposed to the resistive model. At high voltages, the dependence $I(V)$ does not obey Ohm's law $I = V/R$; instead, it assumes an asymptotic shape shifted upwards by a certain amount I_{ex}, independent of V in a wide voltage range. This excess current was first observed experimentally by Pankove in 1966, and its presence in the $I - V$ curve is an unambiguous indication of weak coupling with metallic conductivity. Until recently however its physical nature was unclear. The range of proposed current mechanisms either based on pure phenomenology (Deaver and Pierce 1972), or applicable only to gapless superconductors (Likharev and Yakobson 1975) were not able to account for all the experimentally observed features of the effect.

One very important experimental result which contributed essentially to understanding of the excess current physics was the observation that the
excess current is directly proportional to the magnitude of the energy gap in short bridges and superconducting contacts (Divin and Nad' 1978, 1979, Zhilinski et al. 1980). It was so important in particular because the theoretical papers of Rieger et al. (1972), and Likharev and Yakobson (1975) suggested that I_{ex} should be directly proportional to the weak coupling critical current I_{c}, rather than to Δ.

Figure 25 shows the temperature dependences of the excess and critical currents of a contact with normal resistance $R_{\mathrm{N}} = 8.4\Omega$ (Divin and Nad' 1978). The measurements used stable superconducting $\mathrm{Nb}-\mathrm{Nb}$ point contacts with the principal Josephson properties depending on microshorts. I_{cx} of the contacts was measured at $V \approx 5\mathrm{mV}$, where throughout the temperature range, $V > 2\Delta/e$. The same figure shows experimental points representing the temperature

Fig. 25. Temperature dependences of (•) critical current $I_{\mathrm{c }},(\bigcirc)$ excess current I_{ex} and voltage corresponding to the gap-induced singularity $\frac{1}{2}V_{24}$ of the superconducting point contact with $R_{\mathrm{N}} = 8.4\Omega$. The solid curve shows the theoretical $I_{\mathrm{c}}(T)$ for superconducting microshorts (Kulik and Omelyanchuk 1978). The broken line is $\Delta(T)$, according to BCS theory.}

dependence of the voltage $1,2V_{2,1}$ corresponding to the gapinduced singularity of the point-contact $I - V$ curve at $V = 2\Delta/e$. As is seen, the
dependences $I_{\mathrm{c}}(T)$ and $I_{\mathrm{ex}}\,(T)$ are markedly different. $I_{\mathrm{c}}(T)$ is linear near T_{c} and somewhat off-linear at $T/T_{\mathrm{c}} < 0.6$, agreeing in general with the theoretical $I_{\mathrm{c}}(T)$ for microshorts (solid curve; Kulik and Omelyanchuk 1978). The dependence $I_{\mathrm{ex}}\,(T)$ coincides well with $\frac{1}{2}V_{21}$, and these both are in agreement with the BCS $\Delta(T)$ (broken curve). In the neighbourhood of $T_{\mathrm{c}}, I_{\mathrm{ex}} \gg I_{\mathrm{c}}$ and at $T < T_{\mathrm{c}}, I_{\mathrm{ex}} \sim I$. Weitz et al. (1978) noted that the excess current was observed in all superconducting point contacts irrespective of whether the IV curve of the irradiation contacts at $V > A/e$ had current steps, which are a characteristic indication of the nonstationary Josephson effect. In other words, the nonstationary Josephson effect cannot be the ultimate cause of the excess current.

Artemenko et al. (1978, 1979*b*) considered the excess current effect on the rigorous basis of the microscopic theory and showed that in a superconducting point contact or a short bridge of length

$$L < \xi(T)\,(1 - T/T_{\mathrm{c}})^{1/4} = \eta(T)$$

at $eV > A$ and any temperature, the current is determined by the following expression:

$$I = V/R + (\Delta/R)\left(\frac{1}{4}\pi^2 - 1\right)\tanh(eV/2kT) \tag{3}$$

In the voltage region $eV \geqslant \lambda$, the current across the contact is equal to the sum of the Ohmic current V/R and the excess current

$$I_{\mathrm{ex}} = (\Delta/R)\left(\frac{1}{4}\pi^2 - 1\right)\tanh(\mathrm{eV}/2kT).$$

Thus, theory as well as experiment suggested that $I_{\text{ex}} \sim \Delta$ at high V. Note that the experimental I_{ex} values represented in fig. 25 were measured at $V \approx 5\text{mV} \gtrsim 5kT$, where the value of $\tanh(eV/2kT)$ differs from 1 by less than 2%. Quantitative comparison of experimental and theoretical I_{ex} values showed that for point contacts, the theory gives I_{cx} higher by a factor of 1.5 , possibly owing to particulars of the conduction mechanism of the real point contacts (Divin and Nad' 1978, 1979). Measurements on short ($L \leq 0.5\mu$ m) bridges of variable thickness (which are physically more definite) conducted by Zhilinski et al. (1980) yielded results similar to those for the contacts; however in this case there was quantitative agreement between experimental and theoretical I_{ex} values.

The theory of Artemenko et al. (1978) was the first where the excess current was obtained as a function of voltage:

$$I_{\text{ex}} \sim \tanh(eV/2kT).$$

The experimental measurement of $I_{\text{ex}}(V)$ was carried out by Divin and Nad (1979). The $I_{\text{ex}}(V)$ values were determined by the difference between two experimental $I - V$ curves of the contact for $T \lesssim T_{\text{c}}$ $(I = V/R + I_{\text{ex}})$ and $T > T_{\text{c}}$ $(I = V/R)$. Figure 26 shows $I_{\text{ex}}(V)$ for a contact with $R = 5.9\Omega$ at two fixed

Fig. 26. Voltage dependence of the excess current I_{ex} of a superconducting point contact with $R_{\text{N}} = 5.9\Omega$ for () $T = 9.57$ K and () $T = 9.61$ K.I_{ex} is normalized to its value I_0 at $eV \gtrsim 5$kT. The solid curve is $\tanh(eV/2kT)$.}

temperatures, starting with $V \gtrsim \Delta/e$ and up to high voltages $eV > 5kT$ where $I_{\text{ex}}(V)$ is practically constant. All the I_{ex} values were normalized to the excess magnitude at $eV \gtrsim 5kT$. The solid curve represents the theoretical dependence $\tanh(eV/2kT)$. It is seen from fig. 26 that the experimental and theoretical dependences agree well. The slight difference, a little larger at lower temperatures, may be due to the increasing contribution of the Josephson current component which falls off as $1/V$ at $eV > \Delta$ (Artemenko et al. 1979b). The $I_{\text{ex}}(V)$ results for variable thickness bridges (Zhilinski et al. 1980) at temperatures not very close to T_{c} are similar to those for contacts (fig. 26). However, near T_{c} the difference between the theoretical and experimental $I_{\text{ex}}(V)$ is more pronounced for the bridge, resulting probably from the effect of self-heating on the bridge which is larger than on the contact.

The microscopic theory of Artemenko et al. (1979b) is rigorously applicable only to the weak coupling with length $L < \eta(T)$. This holds for Nb − Nb point contacts with $R \sim 10\Omega$ in

the range $T_c - T \leqslant 1$ K (Divin and Nad' 1979) and for Sn bridges of length $L \leq 0.5\mu$ m in the range $T_c - T \leq 100$mK (Zhilinski et al. 1980). As the temperature drops out of these ranges. η becomes smaller than L and accordingly, as we have mentioned, in the near-hyperbolic $I - V$ curves at $T \lesssim T_c$ there appears a linear part and a typical bump (fig. 18). However, as is shown by $I_{ex}(T)$ experiments, in bridges with $L \leqslant 0.5\mu$ m, a temperature drop reflecting an increase in L/η to 2.5 does not lead to appreciable disturbance of the proportionality between I_{ex} and A (Zhilinski et al. 1980). A decrease of I_{ex} and a pronounced departure from $I_{ex} \sim \Delta$ are found for bridges with $L \gtrsim 1\mu$ m, in qualitative agreement with excess current calculation for long bridges (Zaitsev 1980).

The excess current was also observed in the $I - V$ curves of superconductornormal metal (S-N) point contacts (Pankove 1966, Chien and Farrell 1975, Bevza et al. 1977, Gubankov and Margolin 1979, Dorozhkin 1980). As in the case of S S contacts, the main scale for the effect in S-N contacts is the energy gap magnitude Λ. For all S N contacts I_{ex} is proportional to the value of the energy gap (Gubankov and Margolin 1979, Dorozhkin 1980). The $I - V$ curve of high-resistance contacts ($R \gtrsim 10\Omega$) which seem to have a small constriction diameter and much impurity and many defects ("dirty" contacts) displays a gap-induced singularity at $V \simeq A/e$ and the contact resistance at $V \to 0$ changes nonmonotonically with temperature: $R = R_N$ at $T \simeq T_c$ and $T \to 0$; at $T \simeq 0.8T_c$ the minimum is attained, $R = 0.87R_N$ (Gubankov and Margolin 1979). In lowresistance contacts ($R < 1\Omega$) and those prepared by electrical break-down of an insulating film ("pure" contacts), resistance decreases monotonically with temperature (Chien and Farrell 1975, Dorozhkin 1980). These experimental results are in qualitative agreement with microscopic theories for "dirty" ($l \ll \max(d, \pounds)$; Artemenko et al. 1979b) and for "pure" ($l \gg \max(d, L)$; Zaitsev 1980b) contacts.

Artemenko et al. (1979b), by a microscopic approach, and Dorozhkin (1980), by common sense reasoning, interpreted the excess current as due to quasi-particles with energyes $\varepsilon < \Delta$. In short $S - S$ and $S - N$ contacts, quasiparticles with energy $\varepsilon > \Delta$ carry current in the same way as in normal $N - N$ contacts with Ohmic $I - V$ curve $I = V/R$. However, unlike $N - N$ contacts, those with one or two superconducting electrodes have one more conduction channel for quasiparticles with $\varepsilon < \Delta$. The transfer of quasi-particles from one electrode into the other, to the energy range $\varepsilon < \Delta$ under the gap, takes place, with a finite probability, by processes similar to the Andreev reflection (Andreev 1964). This surplus conduction with the energy scale Δ is responsible for the excess current in the IV characteristics of short $S - S$ and $S - N$ contacts.

45.7 7. Conclusion

The earliest experiments investigating superconductivity enhancement were one of the starting points of nonequilibrium superconductivity studies. They were followed by a series of fundamental theoretical and experimental works dealing with nonequilibrium processes and superconductivity enhancement. At present, although studies in this area are still under way, we can think of the general concept of nonequilibrium phenomena in superconductors exposed to electric and electromagnetic fields and also to sound waves.

It was found that in addition to the "depairing" studied before, these fields can also cause growth of superconductor critical parameters, i.e. enhance superconductivity. The ultimate cause of such effects lies in the changing state of the assembly of unpaired electrons (quasi-particles), leading to changes of
properties of paired (superconducting) electrons. When, e.g., an electromagnetic field or high-frequency sound act on homogeneous superconductors, quasi-particles with energy $\varepsilon \gtrsim \Delta$ near the edge of the gap absorb quanta of the irradiation and shift to a higher energy region, thus reducing the occupancy of states near the gap edge which are predominant in pairing processes.

This is equivalent to superconductor cooling and gap enhancement.

In weakly coupled superconductors with metallic conductivity, not exposed to an external high-frequency field, with a current greater than critical, quasiparticles are pumped by Josephson self-oscillations which bring a similar result: enhancement of the gap in the weak-coupling region and rise of the superconducting current ($I-V$ bump). With a current smaller than critical, applied electromagnetic irradiation causes "forced" oscillations of the gap in the bridge constriction and superconductivity enhancement. We will emphasize again that the enhancement effects in such different systems with different "pumping sources" have essentially an identical nature; that is, changing quasi-particle distribution function and the resulting enhancement of the superconductor energy gap. In very short bridges and contacts with $L < \eta(T)$, nonequilibrium processes do not seem to induce any appreciable gap enhancement, though they give rise to anomalous currents, responsible for the $I-V$ singularities and for the excess current in such systems.

However there are some problems associated with nonequilibrium processes and superconductivity enhancement which have not yet been understood exhaustively. Among these, it is appropriate to refer first to the problem of the origin of spatially inhomogeneous states in long uniform conducting channels, carrying a direct current $I > I_c$, both with and without external high-frequency fields (phase slip centres). In the case of weakly coupled superconductive systems, too little attention has yet been given to the $I-V$ singularities in the low-voltage region (linear part) and in the voltage region corresponding to the energy gap value (gap-induced singularities). The properties of very short bridges with $L < \eta(0)$ have also been studied insufficiently. Our hope is that further experimental and theoretical efforts will bring answers to these questions as well.

46 Experiments on charge imbalance in superconductors by Clarke

46.1 1. Introduction

When a disequilibrium is created between the electron-like and hole-like excitations in a superconductor, a charge imbalance, Q^*, is created. Charge imbalance is of practical interest, first, because it can be used to explain a wide variety of nonequilibrium processes that occur in superconductors, and, second, because it provides a valuable tool for measuring various electronrelaxation rates, most notably the electron–phonon scattering rate. The importance of the disequilibrium of the electron- and hole-like branches was first realized by Pippard et al. (1971) in their explanation of the excess resistance associated with the normal metal-superconductor (NS) boundary. However, the first quantitative understanding of charge imbalance was put forward by Tinkham and Clarke (1972) to explain the results of experiments in which electrons were injected via a tunnel junction into superconducting tin films (Clarke 1972). Among other results, these authors showed that the charge imbalance relaxation rate, $\tau_{Q^*}^{-1}$, due to inelastic scattering was of order $\tau_{\mathrm{E}}^{-1}\,[\Lambda(T)/k_{\mathrm{B}}T]$ near T_c, where τ_{E}^{-1} is the electron-phonon scattering rate at T_c and at the Fermi energy, and $\Delta(T)$ is the energy gap.

The first detailed paper on the theory of charge imbalance was that of Tinkham (1972): this article laid the groundwork for a great deal of the subsequent theoretical development. The next major theoretical article was that of Schmid and Schön (1975) who used a Green's function approach to distinguish two different nonequilibrium modes: a transverse mode, in which the quasiparticle populations on the two branches differ (charge imbalance). and a longitudinal mode, in which the quasiparticle population on the two branches is equal but out of equilibrium, thereby modifying the energy gap, $\Delta(T)$. Subsequently, Pethick and Smith (1979a) introduced

a two-fluid approach to relaxation and collective motion in superconductors. This approach is conceptually simple, and produces the same results as the Schmid Schön theory in the limit $\Delta(T)/k_{\mathrm{B}}T \ll 1$ in which both theories are valid.

The purpose of the present chapter is to describe experiments in which charge imbalance plays a central role for our understanding of the physical mechanisms that are involved. It is, of course, necessary to present a theoretical framework, which appears in section 2. However, I have given a minimum of theoretical discussion throughout the paper, but instead have quoted results
and listed appropriate references. Section 3 describes the early work on Sn from which it was possible to extract an accurate value for τ_{E}^{-1}. Section 4 outlines work on Al in which both elastic scattering in the presence of gap anisotropy and electron electron scattering may also relax charge imbalance. Section 5 is concerned with charge relaxation via elastic scattering in the presence of magnetic impurities or of a supercurrent. This section concludes the discussion of charge imbalance that has been generated by tunnel injection. Section 6 deals with the resistance of the NS interface in terms of charge imbalance, while section 7 describes phase-slip centers. Section 8 is concerned with charge imbalance in the presence of a temperature gradient in three different types of experiment: the voltage induced across a NS interface, the spatially decaying charge imbalance induced near the end of a superconducting film, and the charge imbalance induced by a uniform supercurrent. While all of the topics above are concerned with steady state charge imbalance, it is also possible to consider dynamic effects, such as the response to an impulse or fluctuations. Section 9 briefly describes such effects. Finally, section 10 contains some concluding remarks.

I should emphasize that this chapter is not an exhaustive account of charge imbalance phenomena, nor is the list of references by any means complete. However, I have attempted to list enough references to enable interested readers to find their way into the literature. Parts of this work have been published in an earlier review (Clarke 1981).

46.2 2. Theory of charge imbalance

In this section we outline the concepts of quasiparticle charge and charge imbalance, the generation and detection of charge imbalance by tunneling, and the relaxation of charge imbalance via electron-phonon scattering. The discussion follows the work of Tinkham and Clarke (1972). Tinkham (1972), Waldram (1975), Pethick and Smith (1979a), and Kadin et al. (1980).

46.2.1 2.1. Quasiparticle charge and charge imbalance

We begin by considering the total electronic charge, Q_{tot} , in a superconductor, the overall charge at frequencies far below the plasma frequency being maintained at zero by the equal and opposite charge of the ion cores. From the BCS theory (Bardeen et al. 1957) we have

$$Q_{\mathrm{tot}} = \frac{2}{\Omega} \sum_k \left[u_k^2 f_k + v_k^2 \left(1 - f_{-k}\right) \right] , \tag{2.1}$$

where the sum is over k states above and below k_F, and the factor of 2 arises from the sum over spin. In eq. (2.1), Ω is the volume of the superconductor,
and f_k is the occupancy of the state k, which, in thermal equilibrium, is the Fermi function. The u_k and v_k are the usual BCS factors given by

$$u_k^2 = \frac{1}{2} \left(1 + \xi_k / E_k\right) \tag{2.2}$$

and

$$v_k^2 = \frac{1}{2}\left(1 - \xi_k/E_k\right), \tag{2.3}$$

where

$$\xi_h = \frac{\hbar^2 k^2}{2m} - \mu_s \tag{2.4}$$

is the kinetic energy of the electron of mass m in the state k referred to the chemical potential of the superfluid, μ_s, and

$$E_k = \left(\Delta^2 + \xi_k^2\right)^{1/2} \tag{2.5}$$

is the quasiparticle excitation energy. The energy E_k is positive for all values of k, while ξ_k is positive for $k > k_{\mathrm{F}}$ and negative for $k < k_{\mathrm{F}}$. The quantity v_k^2 is the probability that the state k is occupied by a Cooper pair, while $u_h^2 = 1 - v_h^2$ is the probability that the state k is unoccupied by a Cooper pair.

We can regard eq. (2.1) as the sum of a superfluid contribution

$$Q_{\mathrm{s}} = \frac{2}{\Omega}\sum_k v_k^2 \tag{2.6}$$

and a quasiparticle contribution

$$Q^* = \frac{2}{\Omega}\sum_k \left(u_k^2 - v_k^2\right) f_k = \frac{2}{\Omega}\sum_k q_k f_k, \tag{2.7}$$

where we have set $f_{-k} = f_k$. In eq. (2.7) we have introduced the very important concept of the effective quasiparticle charge

$$q_k = u_k^2 - v_k^2 = \xi_k/E_k, \tag{2.8}$$

and defined the charge imbalance per unit volume, Q^*, which is the central subject of this chapter. We note that $q_k \to +1$ for $\xi_k \gg A\,(k > k_{\mathrm{F}})$, and $q_k \to -1$ for $\xi_k \ll -\Delta\,(k < k_{\mathrm{F}})$. It is evident in Eq. (2.7) that if $f_k\,(\xi_k)$ is the Fermi function, which is even in ξ_k about μ_k, the quasiparticle charge, Q^*, is zero since q_k is odd in $\check{\zeta}_h$. Thus, the total electronic charge is $(2/\Omega)\sum_h^2$ at all temperatures, as we expect.

We now turn to a nonequilibrium situation in which f_k is no longer the Fermi function. In this case, Q^* may be nonzero and μ_{s} shifted from its equilibrium value, the Fermi energy, by an amount $\delta\mu_{\mathrm{s}}$. As we shall see, Q^* and $\delta\mu_s$ are intimately related. As an example, suppose that we add excitations to the $k > k_{\mathrm{F}}$ quasiparticle branch in a particular region of the superconductor. Overall charge neutrality will ensure that pairs with an equivalent charge are removed from the region. Thus, the chemical potential of the pairs must decrease by an amount $\delta\mu_{\mathrm{s}}$, as indicated in fig. 1a. At the same time, the excitation spectrum shifts because its minimum still occurs at the pair chemical potential, as shown in fig. 1b. The effective quasiparticle charge is still $q_k = \xi_k/E_k$, but ξ_k is now referred to the shifted chemical potential, $\mu_s - \delta\mu_s$. Thus, the addition of a single electron-like excitation induces some quasiparticles near the bottom of the excitation spectrum to cross branches, producing a small increase in the charge of all the electron-like quasiparticles, and a small decrease in the charge of all the hole-like quasiparticles.

The quantities Q^* and $\delta\mu_{\mathrm{s}}$ are related as follows. By charge neutrality

Fig. 1. The effect on (a) condensate density, and (b) quasiparticle excitation spectrum, when electron-like excitations are added to the superconductor. The solid and dashed lines represent equilibrium and displacement from equilibrium, respectively.

$$Q^* + Q_\mathrm{s}|_{\delta\mu_\mathrm{s}\neq 0} = \frac{2}{\Omega}\sum_k v_k^2\Big|_{\delta\mu_\mathrm{s}=0}, \tag{2.9}$$

$$Q^* = \frac{2}{\Omega}\sum_k \left(v_k^2|_{\delta\mu_\mathrm{s}=0} - v_k^2|_{\delta\mu_\mathrm{s}\neq 0}\right). \tag{2.10}$$

Now the reduction in the number of pairs represented by the right-hand side of eq. (2.10) is just $2N(0)\delta\mu_\mathrm{c}$, where $N(0)$ is the single-spin density of states per unit volume at the Fermi energy. Thus

$$Q^* = -2N(0)\delta\mu_\mathrm{s} \tag{2.11}$$

We see that eq. (2.1) is really a self-consistent equation for μ_s : given any distribution, f_k, the constancy of Q_tot determines μ_s.

In the steady state with Q^* continuously injected at a rate $\dot{Q}_i^*$, the excess distribution relaxes at a rate $\tau_{Q^*}^{-1}$ to produce a steady state charge imbalance Q^*. Thus

$$\tau_{Q^*}^{-1} = \dot{Q}_\mathrm{i}^*/Q^*. \tag{2.12}$$

The relaxation of Q^* may occur through both inelastic and elastic processes, as we will discuss in sections 2.3 and 4 .

46.2.2 2.2. Tunneling generation and detection of charge imbalance

We follow closely the approach developed by Tink ham (1972) and Pethick and Smith (1979a) to calculate the charge imbalance generated in a superconductor (S) by quasiparticle tunnel injection from a normal metal (N), and the voltage induced by this charge imbalance across a second NIS tunnel junction at zero bias current.

When a NIS junction is biased at a voltage V, the rate at which quasiparticles are injected into the state k of the superconductor is

$$\begin{aligned}\dot{f}_k\Big|_i =& \frac{2\pi}{\hbar}|T|^2\mathcal{N}_\mathrm{n}(0)\left[u_k^2 f^0\left(E_k - eV\right) + v_k^2 f^0\left(E_k + eV\right) - f_k\right] \\ =& \frac{2\pi}{\hbar}|T|^2\mathcal{N}_\mathrm{n}(0)\left\{\frac{1}{2}\left[f^0\left(E_k + eV\right) + f^0\left(E_k - eV\right)\right] - f_k\right. \\ &\left. + \frac{1}{2}q_k\left[f^0\left(E_k - eV\right) - f^0\left(E_k + eV\right)\right]\right\}.\end{aligned} \tag{2.13}$$

Here, $|T|^2$ is the square of the tunneling matrix element, assumed to be independent of energy, and $\mathscr{N}_n(0)$ is the single-spin density of states in the volume of the metal. The Fermi functions $f^0\left(E_k \pm eV\right)$ refer to the normal metal, while f_k is the actual distribution in the superconductor, which is not, in general, the Fermi distribution.

We convert eq. (2.13) into an expression for the rate of quasiparticle charge injection by multiplying by q_k and summing over k. Noting that only even terms contribute, we find

$$\dot{Q}_{\mathrm{i}}^{*}=\frac{2\pi}{\hbar\Omega}|T|^2\mathcal{N}_{\mathrm{n}}(0)\sum_k q_k^2\left[f^0\left(E_k-eV\right)-f^0\left(E_k+eV\right)\right]. \tag{2.14}$$

We have neglected the small contribution of the even part of $q_h f_h$ since the voltages generated by the nonequilibrium distribution are very small compared with the injection voltage. It is convenient to write eq. (2.14) in the integral form:

$$\begin{aligned}\dot{Q}_{\mathrm{i}}^{*}&=\frac{4\pi}{\hbar\Omega}|T|^2\mathcal{N}_{\mathrm{n}}(0)\mathcal{N}(0)\int_0^5\frac{\xi^2}{E^2}\left[f^0(E-eV)-f^0(E+eV)\right]\mathrm{d}\zeta\\&=\frac{G_{\mathrm{NN}}}{e^2\Omega}\int_\Delta^\infty\rho^{-1}(E)\left[f^0(E-eV)-f^0(E+eV)\right]\mathrm{d}E\end{aligned} \tag{2.15}$$

where we have used $\mathrm{d}\xi=(E/\xi)\mathrm{d}E$. In eq. (2.15), $\mathcal{N}'(0)$ is the single-spin density of states in the volume of the superconductor when it is in the normal state,

$$\rho(E)=E/\left(E^2-\Delta^2\right)^{1/2}=E/|\xi|=\left|\mathrm{q}_k^{-1}\right|(E>\Delta)$$

is the normalized BCS density of states in the superconductor, and $G_{\mathrm{NN}}=4\pi e^2|T|^2\times\mathcal{N}_{\mathrm{n}}(0)\mathcal{N}(0)/\hbar$ is the conductance of the tunnel junction when the superconductor is in the normal state.

From eq. (2.13), we can obtain an expression for the net current injected into a superconductor from a normal metal. Summing over k and simplifying, we find

$$I=\frac{2\pi e}{\hbar}|T|^2\mathscr{N}_{\mathrm{n}}(0)\sum_k\left[f^0\left(E_k-eV\right)-f^0\left(E_k+eV\right)-2q_k\delta f_k\right] \tag{2.16}$$

where $\delta f_k=f_k-f^0$. We note that the E_k in f^0 are referred to the modified chemical potential. [Pethick and Smith (1979a) denote this f^0 by $f^{0(\mathrm{l.e.})}$, where "l.e." means local equilibrium; our δf_k is precisely the same as their $\delta f_k^{\mathrm{l.e}}$.] For any injection voltage of practical interest the last term in eq. (2.16) is negligible, and we can write I in the integral form:

$$\begin{aligned}I&=\frac{4\pi e}{\hbar}|T|^2\mathscr{N}_{\mathrm{n}}(0)\mathcal{N}(0)\int_0^\infty\left[f^0(E-eV)-f^0(E+eV)\right]\mathrm{d}\xi\\&=\frac{\mathrm{G}_{\mathrm{NN}}}{e}\int_\Lambda^\infty\rho(E)\left[f^0(E-eV)-f^0(E+eV)\right]\mathrm{d}E\end{aligned} \tag{2.17}$$

We will find it useful to obtain an expression for eq. (2.17) in the limit $eV\ll k_{\mathrm{B}}T$ in which we can expand the Fermi functions to give $f^0(E\pm eV)=f^0(E)\pm[\partial f^0(E)/\partial E]\,\mathrm{eV}$. This approximation leads to

$$I=G_{\mathrm{NN}}\int_\Delta^\infty 2\rho(E)\left(-\frac{\partial f^0(E)}{\partial E}\right)V\ \mathrm{d}E,\quad(eV\ll k_{\mathrm{B}}T) \tag{2.18}$$

Since the conductance, G_{NS}, is just I/V, we obtain the reduced tunneling conductance for $eV/k_{\mathrm{B}}T\to 0, g_{\mathrm{NS}}(0)=G_{\mathrm{NS}}(0)/G_{\mathrm{NN}}$:

$$g_{\mathrm{NS}}(0) \equiv \mathrm{Y}(T) = 2\int_{\Delta}^{\infty} \rho(E)\left(-\frac{\partial f^0(E)}{\partial E}\right)\mathrm{d}E \tag{2.19}$$

where $\mathrm{Y}(T)$ is the Yoshida function.

We are now in a position to calculate the voltage generated by the steady state charge imbalance as measured by a second tunneling contact to a normal metal. This voltage, V_{d}, referred to the equilibrium chemical potential of the superconductor far away from the injected region is measured with zero current flowing through the detector junction. Since eq. (2.16) is equally true for the injection and detection junctions, we can set $I = 0$, and use the low voltage expansion of the Fermi functions to obtain

$$V_{\mathrm{d}} = \frac{2\sum q_k f_k}{2e\sum\left[-\partial f^0(E)/\partial E\right]} = \frac{Q^*}{2N(0)eg_{\mathrm{NS}}(0)} \tag{2.20}$$

We have thus found a simple relation between Q^* or, equivalently, $-2N(0)\delta\mu_{\mathrm{s}}$, and the voltage measured by the detector junction. If the detector junction does not show the ideal BCS behavior, one should use the measured normalized conductance in eq. (2.20), rather than the BCS value.

To complete our description of tunneling generation and detection of charge imbalance we combine eqs. (2.12), (2.15), (2.17), and (2.20) so that we can deduce $\tau_{Q^*}^{-1}$ in terms of measurable or precisely calculable quantities. To do this, it is convenient to introduce the parameter F^* that enables us to express $\dot{Q}_{\mathrm{i}}^*$ in terms of the injection current I_{i} :

$$F^* = e\Omega\frac{\dot{Q}_i^*}{I_i} = \frac{\int_{\Delta}^{\infty}\rho^{-1}(E)\left[f^0\left(E-eV_{\mathrm{i}}\right)-f^0\left(E+eV_{\mathrm{i}}\right)\right]\mathrm{d}E}{\int_{\Delta}^{\infty}\rho(E)\left[f^0\left(E-eV_{\mathrm{i}}\right)-f^0\left(E+eV_{\mathrm{i}}\right)\right]\mathrm{d}E} \tag{2.21}$$

We have used V_{i} to denote the injection voltage. From eqs. (2.12), (2.20), and (2.21) we find

$$\tau_{Q^*}^{-1} = \frac{\dot{Q}_{\mathrm{i}}^*}{Q^*} = \frac{F^* I_{\mathrm{i}}}{2N(0)e^2\Omega g_{\mathrm{NS}}(0)V_{\mathrm{d}}} \tag{2.22}$$

The quantity F^* is readily calculable, while the quantities $I_{\mathrm{i}}, \Omega, g_{\mathrm{NS}}(0)$, and V_{d} are those measured in the experiment.

In the limit of low injection voltages, $eV_{\mathrm{i}} \ll k_{\mathrm{B}}T$, the Fermi functions in eq. (2.21) may be expanded to yield

$$F^* = Z(T)/\mathrm{Y}(T), \quad \left(eV_{\mathrm{i}} \ll k_{\mathrm{B}}T\right), \tag{2.23}$$

where

$$Z(T) \equiv 2\int_{\Delta}^{\infty}\rho^{-1}(E)\left(-\frac{\partial f^0(E)}{\partial E}\right)\mathrm{d}E \tag{2.24}$$

However, this limit is not very useful for most practical experiments with tunneling injection, where the voltage is large compared with both $\Delta(T)/e$ and $k_{\mathrm{B}}T/e$. In these limits, one obtains

$$F^* = 1 - \pi\Delta/2\left|eV_i\right|, \quad \left[eV_i \gg k_B T, \quad eV_i > \Delta(T)\right]. \tag{2.25}$$

For a typical injection voltage, $eV_{\mathrm{i}}/\Delta = 30$, we find $F^* = 0.95$.

46.2.3 2.3. Charge imbalance relaxation via electron-phonon scattering

When an electron-like excitation (for example) is injected into a superconductor, in general it not only creates a charge imbalance but also increases the energy of the quasiparticle distribution above the equilibrium value. Thus, the processes that restore thermal equilibrium involve both cooling and charge relaxation, but these are not necessarily distinct processes. For example, if the quasiparticle is injected at any energy well above both $k_{\mathrm{B}}T$ and $A(T)$, it may scatter with the emission of a phonon to a lower energy on the same quasiparticle branch. In this process, the quasiparticle has not only cooled, but also given up some of its charge because q_k has been reduced from its initial value near unity to a lower value. This kind of inelastic process is governed by the coherence factor (Tinkham 1975)

$$(uu' - vv')^2 = \frac{1}{2}\left(1 + \frac{\xi\xi' - \Delta^2}{EE'}\right) \tag{2.26}$$

A second means of phonon-mediated charge relaxation is by recombination, a process that is governed by the coherence factor (Tinkham 1975)

$$(vu' + uv')^2 = \frac{1}{2}\left(1 + \frac{\Delta^2 - \xi\xi'}{EE'}\right) \tag{2.27}$$

In addition to inelastic processes, elastic scattering from non-magnetic or magnetic impurities can also relax Q^*, provided the quasiparticle changes branches. For the case of non-magnetic impurities in an isotropic superconductor, this process is strictly forbidden by the coherence factor in eq. (2.26), as can be seen immediately by realizing that if E and E' refer to different branches, $u' = v$ and $v' = u$. On the other hand, the coherence factor does not vanish for scattering from non-magnetic impurities in the case of anisotropic superconductors, or from magnetic impurities even in the isotropic case. We defer a discussion of elastic processes until sections 4 and 5.

It is evident that the calculation of the charge imbalance relaxation rate is, in general, a very complicated undertaking. In the limit $\Delta(T) \ll k_{\mathrm{B}}T$, and in the absence of elastic processes, the situation simplifies, and one finds (Tinkham 1972, Schmid and Schön 1975, Pethick and Smith 1979a)

$$\left(F^*\tau_{Q^*}\right)^{-1} = \left(\pi\Delta/4k_{\mathrm{B}}T\right)\tau_E^{-1}, \quad [\Delta(T) \ll k_{\mathrm{B}}T]\,. \tag{2.28}$$

In eq. (2.28), ${\tau_E}^1$ is the inelastic scattering rate at T_c for a quasiparticle at the Fermi energy; the rate actually increases with the energy of the quasiparticle, but we shall neglect this dependence. For the more general case when $\Delta(T)$ is not small, one must resort to numerical calculations. Chi and Clarke (1979, 1980) solved the Boltzmann equation on a computer to find both the steady state quasiparticle distribution and the charge imbalance relaxation rate for a wide range of temperatures and several values of the injection voltage. We
briefly summarize the results of these calculations, referring to Chi and Clarke's results by "CC".

The Boltzmann equation can be written in the form

$$\dot{f}_\xi = G_{\tilde{\zeta}} - G_{\mathrm{in}\tilde{\zeta}} - G_{\mathrm{el}\xi}. \tag{2.29}$$

The first term G_ξ is the rate of injection into $\check{\zeta}_h$. while $G_{\mathrm{in}\ ;}$ and $G_{\mathrm{el}\ ;}$ are the inelastic and elastic rates at which quasiparticles scatter out of ξ_h; for the moment we set $G_{\mathrm{el:}} = 0$. In a steady state situation $\dot{f}_\xi = 0$. For tunneling from a normal metal into a superconductor the

injection rate is given by the first line of eq. (2.13) with the terms labeled with ξ rather than k :

$$G_\xi = \left[G_{\mathrm{NN}}/2N(0)e^2\Omega\right]\left\{\frac{1}{2}(1+\xi/E)\left[f^0\left(E-eV_i\right)-f^0(E)\right]\right.$$
$$\left.-\frac{1}{2}(1-\xi/E)\left[f^0(E)-f^0\left(E+eV_{\mathrm{i}}\right)\right]\right\}. \tag{2.30}$$

We have used the Fermi distribution for the distribution functions, thereby assuming that departure from equilibrium is small. The inelastic scattering rate is proportional to $x^2F(\Omega)$, the average of the product of the square of the matrix elements for the electron phonon interaction and the phonon density of states. We assume that $F(\Omega)$ is quadratic in Ω, and can be written in the form $F\left(k_{\mathrm{B}}T_{\mathrm{c}}\hbar\right)\left(\Omega k_{\mathrm{B}}T_{\mathrm{c}}\right)^2$. With this assumption, $G_{\mathrm{in}\tilde{n}}$ can be written down using the Golden Rule (Kaplan et al. 1976, Chang and Scalapino 1977, Chang 1979, CC):

$$\begin{aligned}
G_{\mathrm{in}\varsigma} =& \frac{2\pi}{\hbar}\alpha^2F\left(k_{\mathrm{B}}T_{\mathrm{c}}/\hbar\right)\\
&\times\left\{\left[1+n^0\left(E+E'\right)\right]f_\xi f_{\xi'}-n^0\left(E+E'\right)\left(1-f_\xi\right)\left(1-f_{\xi'}\right)\right\}\\
&+\frac{1}{2}\left(1+\frac{\xi\xi'}{EE'}-\frac{\Delta^2}{EE'}\right)\left(E-E'\right)^2\theta\left(E-E'\right)\\
&\times\left\{\left[1+n^0\left(E-E'\right)\right]f_\xi\left(1-f_{\xi'}\right)-n^0\left(E-E'\right)f_{\xi'}\left(1-f_\xi\right)\right\}\\
&+\frac{1}{EE'}\left(1+\frac{\xi\xi'}{EE'}-\frac{\Delta^2}{EE'}\right)\left(E-E'\right)^2\theta\left(E'-E\right)\\
&\times\left\{n^0\left(E'-E\right)f_\xi\left(1-f_{\xi'}\right)-\left[1+n^0\left(E'-E\right)\right]f_{\xi'}\left(1-f_\xi\right)\right\}\Big].
\end{aligned} \tag{2.31}$$

In eq. (2.31), $Z_1(0)$ is the electron- phonon renormalization factor, typically about 2 for most metals. () is the Heaviside function, which is zero for negative arguments and unity for positive arguments, and $n^0(\Omega)$ is the Bose Einstein distribution function for the phonons.

In solving the Boltzmann equation. it is convenient to write $f = f^0 + \delta f$, where f^0 is the Fermi function at energy $\left(\xi^2+d^2\right)^{12}$ and ξ is referred to the
shifted chemical potential. We rewrite the equation in terms of even and odd components of $\delta f_\xi, \left(\delta f_\xi+\delta f_{-\xi}\right)/2$ and $\left(\delta f_\xi-\delta f_{-\xi}\right)/2$, make initial guesses for these components and iterate the Boltzmann equation until the solution is selfconsistent. We calculate Q^* from $2N(0)\sum q_\xi\left(\delta f_\xi-\delta f_{-\xi}\right)$ and $\tau_{\mathrm{Q}^*}^{-1}$ from $\dot{Q}_{\mathrm{i}}^*/Q^*$, using eq. (2.15).

Figure 2 shows the computed value of $\left[4k_{\mathrm{B}}T\tau_{\mathrm{E}}/\pi\Delta(T)\right]\tau_{Q^*}^{-1}$ versus $A(T)k_{\mathrm{B}}T$ for three injection voltages. At T_c, the computed value is precisely unity, but as the temperature is lowered, the function first increases and then decreases. At all temperatures below $T_c, {\tau_{Q^-}}^{-1}$ increases with injection voltage, reflecting the increase in the inelastic scattering rate with increasing energy. Figure 3 shows the computed temperature dependence of $\tau_{Q^*}^{-1}$ down to $0.3T_{\mathrm{c}}$ for two values of injection voltage. The relaxation rate goes through a peak and then decreases steadily as the temperature is lowered. It should be emphasized first, that the time τ_E is itself strongly temperature dependent, and second, that in real superconductors elastic scattering in the presence of gap anisotropy is likely to increase the value of ${\tau_{Q^2}}^1$ substantially at low temperatures. Finally, we note that according to eq. (2.22) the measured voltage V_{d} is proportional to $F^*\tau_{Q^*}I_{\mathrm{i}}$. In fig. 4 we plot both ${\tau_{Q^*}}^1$ and $\left(F^*\tau_{Q^*}\right)^{-1}$ versus $eV_{\mathrm{i}}/\Delta(T)$. For $eV_{\mathrm{i}} \gtrsim 64(T)$, we observe that $F^*\tau_{Q^*}$ becomes independent of V_{i}, and that V_{d} becomes linear in I_{i}.

This concludes our theoretical survey. We now move on to discuss experimental results.

Fig. 2. $\left(4 k_{\mathrm{B} } T \tau_{\mathrm{E}} / \pi \Delta(T)\right) \tau_{Q^{*}}{-1}$ versus $\Delta(T)/k_{\mathrm{B}}T$ and T/T_{c} : computed values for $eV_{\mathrm{i}} := (\square)0.01\Delta(T)$; (O) $10\Delta(T)$; $(\triangle)10k_{\mathrm{B}}T_{\mathrm{c}}$. (Chi and Clarke 1980.)}

46.3 3. Experiments on tin

Historically, the first measurements of charge imbalance using tunneling injection were on Sn films, and it is convenient to begin with a discussion of these results because it is possible to obtain a temperature range in which the charge relaxation due to elastic scattering is negligible compared with that due to inelastic scattering (Clarke 1972, Clarke and Paterson 1974, Moody and Paterson 1979).

The configuration used in all tunneling injection experiments is shown in fig. 5. After oxidizing the Al base electrode (XX′), typically 150 nm thick, one deposits a Sn electrode (YY′), 200 to 400 nm thick. An insulating layer of varnish or (in the later experiments) SiO is used to mask off a window near the center of the injector junction. The Sn film is oxidized slightly, and a diagonal strip (ZZ′) of CuAl(3wt%Al) about 2μ m thick is evaporated to form the detector junction. To reduce the series resistance of the CuAl strip, it is covered with a Pb film; the Cu film is made sufficiently thick and dirty to eliminate Josephson tunneling between the Al and Pb films.

Typically, four samples were fabricated simultaneously on a single substrate, and the detector junctions were connected in series. The voltage across the junctions was measured with a null-balancing SQUID voltmeter (see, for example, Clarke 1976) that ensured that the detector voltage was measured at zero current. Each junction could be investigated independently by applying a current to the appropriate injector junction. The samples and the SQUID were

immersed in liquid helium, and carefully shielded against external magnetic field fluctuations. The current voltage characteristics of both the injector and detector junctions were determined: typically, the injector junctions had a resistance of about $1\Omega\,(V_{\mathrm{i}} > 2\Delta/e)$, while the detector junctions were about $10^{-5}\Omega$. The conductance of the detector junctions was often in excess of the BCS prediction, but it was found that if one used the measured values of $g_{\mathrm{NS}}(0)$ in eq. (2.22), different samples produced values of $\tau_{Q^*}^{-1}$ in good agreement with each other.

In fig. 6 we have plotted

$$g_{\mathrm{NS}}(0)\Omega V_{\mathrm{d}}/I_{\mathrm{i}} = F^{*}\tau_{Q^{*}}/2N(0)e^{2} \text{ versus } T/T_{\mathrm{c}}$$

for three Sn samples of different thicknesses, and one sample in which the Sn has been alloyed with 3wt% In (Clarke and Paterson 1974). The mean free path in the pure Sn

Fig. 3. Computed values for $(4 / \pi) \tau_{Q^{*}}^{-1}$ versus $\Delta(T)/k_{\mathrm{B}}T$ and T/T_{c} for (□)$\mathrm{eV_i} = 0.01\Delta(T)$, and (O)$\mathrm{eV_i} = 10\Delta(T)$. (Chi and Clarke 1980.)}

samples was boundary limited, while that in the alloy was estimated to be 42 nm . The fact that the data for the pure Sn samples lie on a smooth curve demonstrates' that $F^{*}\tau_{Q^{*}}$ scales with Ω. Near $T_{\mathrm{c}}, V_{\mathrm{d}}/I_{\mathrm{i}}$, which is proportional to $F^{*}\tau_{Q^{*}}$ increases rapidly with increasing temperature. The solid line in fig. 6 is a fit of the curve $g_{\mathrm{NS}}(0)\Omega V_{\mathrm{d}}/I_{\mathrm{i}} = 4.0 \times 10^{-14} k_{\mathrm{B}}T_{\mathrm{c}}/\Delta(T)\Omega\mathrm{cm}^3$ to the data at high temperatures. Using the value $N(0) = 1.39 \times 10^{22}\mathrm{eV}^{-1}\ \mathrm{cm}^{-3}$ [computed from $N(0) = 3\gamma/2\pi^2 k_{\mathrm{B}}^2$, where γ is the coefficient of the electronic heat capacity taken from Kittel (1976)] in eq. (2.22) we find $(F^{*}\tau_{Q^{*}})^{-1} = 5.6\times\ 10^9\,(\Delta/k_{\mathrm{B}}T_{\mathrm{c}})\,\mathrm{s}^{-1}$. Equating this value with the prediction of eq. (2.28), we find $\tau_{\mathrm{E}} = 1.4 \times 10^{-10}$ s.

Below $T/T_{\mathrm{c}} = 0.95, F^{*}\tau_{Q^{*}}$ is larger for the SnIn sample than for the Sn samples. This difference arises from the contribution of elastic charge relaxation in the clean samples, a topic that we consider in the following section.

46.4 4. Experiments on aluminum

46.4.1 4.1. Charge relaxation via elastic scattering from non-magnetic impurities

The coherence factor $(uu' - vv')^2$ forbids charge relaxation via elastic scattering when the energy gap is isotropic. However, in the presence of gap anisotropy a quasiparticle may be scattered between two regions of the Fermi surface at which the energy gap is different, so that the coherence factor is no longer zero. This situation was first discussed by Tinkham (1972), and subsequently by Chi and Clarke (1979) using a computer solution of the Boltzmann equation. We briefly outline the latter treatment.

The elastic scattering rate G_{els} that appears in the Boltzmann equation [eq. (2.29)] is given

Fig. 4. $\left[4 k_{\mathrm{B} } T \tau_{\mathrm{E}} / \pi \Delta(T)\right] \tau_{Q^{*}}{-1}$ and $[4k_{\mathrm{B}}T\tau_{\mathrm{E}}/\pi\Delta(T)]\,(F^{*}\tau_{Q^{*}})^{-1}$ versus $eV_{\mathrm{i}}/\Delta(T)$ for $T = 0.9T_{\mathrm{c}}$. (Chi and Clarke 1980.)}

Fig. 5. Configuration of tunneling experiment to generate and detect charge imbalance in a Sn film. A SQUID voltmeter is used to measure the voltage between the Sn and Cu films generated by the current passed between the Al and Sn films. (Clarke 1972.)

by

$$G_{\mathrm{el}\xi} = \frac{\tau_1^{-1}}{2} \left\langle \rho_{\Delta'}(E) \left[1 - \frac{\Delta\Delta'}{E^2} - \left(1 - \frac{\Delta^2}{E^2}\right)^{1/2} \left(1 - \frac{\Delta'^2}{E^2}\right)^{1/2} \right] \right\rangle_{\Delta,\Delta'} \times \left[f_\xi \left(1 - f_{\xi'}\right) - f_{\xi'} \left(1 - f_\xi\right) \right]. \tag{4.1}$$

Here, τ_1^{-1} is the elastic scattering rate of an electron when the metal is in the normal state, $\rho_{1'}(E)$ is the final density of states, one-half times the terms in the first pair of square brackets is the coherence factor, and the two terms in the second pair of square brackets are the occupation factors for elastic scattering from ξ to ξ' and the reverse process, respectively. The symbol $\rangle_{A,1'}$ indicates an angular average over the gap-anisotropy distribution. To first order, we can replace $\rho_{A'}(E)$ with $\rho_{\bar{A}}(E)$, where $\bar{A}$ is the average energy gap. To average the coherence factor we express it in terms of the mean square gap anisotropy
$\langle\delta^2\rangle = \left\langle(\Delta - \bar{\Delta})^2\right\rangle$. Neglecting terms of order $\langle\delta^2\rangle^{1/2}/\bar{\Delta}$, one can write $\left\langle(uu' - vv')^2\right\rangle = \langle\delta^2\rangle/2\left(E^2 - \bar{\Delta}^2\right)$ for $E\rangle\,\bar{\Delta}$; for $E \leqslant \bar{\Delta}$, one takes the value at $\boldsymbol{E} = \bar{\Delta} + \langle\delta^2\rangle^{1/2}/2$.

Fig. 6. $g_{\mathrm{NS }}(0) \Omega V_{\mathrm{d}} / I_{\mathrm{i}}$ versus T/T_{c} for pure Sn and SnIn . The solid line is a fit of $k_{\mathrm{B}} T_{\mathrm{c}} / \Delta(T)$ to the data for SnIn for $T/T_{\mathrm{c}} \gtrsim 0.7$. (Clarke and Paterson 1974.)}

For clean superconductors the normalized mean square gap anisotropy (Markowitz and Kadanoff 1963) $\langle a^2 \rangle = \langle \delta^2 \rangle / \bar{\Delta}^2$ is constant, independent of temperature. However, for dirty superconductors (Anderson 1959) $\langle a^2 \rangle$ is reduced from its clean limit value $\langle a^2 \rangle_0$ to a value given approximately by $\langle a^2 \rangle = \langle a^2 \rangle_0 / \left[1 + \left(\hbar / 2\tau_1 \bar{\Delta}\right)^2\right]$. Combining these expressions, we find

$$G_{\mathrm{el}\xi} = \tau_1^{-1} \frac{\langle a^2 \rangle_0}{1 + \left(\hbar / 2\tau_1 \bar{\Delta}\right)^2} \frac{\bar{\Delta}^2 E}{2\left(E^2 - \bar{\Delta}^2\right)^{3/2}} \left(f_\xi - f_{-\xi}\right) \qquad \left(E \geqslant \bar{\Delta} + \langle \delta^2 \rangle^{1/2} / 2\right) \tag{4.2}$$

where we have set $\xi' = -\xi$; this is an excellent approximation for the values of anisotropy encountered in real metals.

We can use eq. (4.2) to interpret the results obtained for Sn and SnIn in fig. 6. For the clean films we expect the mean free path, l, to be typically 300 nm , while for the SnIn film it was estimated to be 42 nm . Using $v_{\mathrm{F}} = 0.65 \times 10^6$ ms^{-1} [calculated from $v_{\mathrm{F}} = (\pi^2 k_{\mathrm{B}}^2 / e^2 \gamma)\, \sigma / l$ (Pippard 1965), where σ is the conductivity, and $\sigma / l = 9.5 \times 10^{10} \Omega^{-1}$ cm^1 (Chambers 1952)], we estimate τ_1 to be 2.2 ps and 15.5 ps for the clean Sn and SnIn films, respectively. Thus, the factor $1/\tau_1 \left[1 + \left(\hbar / 2\tau_1 \bar{\Delta}\right)^2\right]$ is about 0.85 for the Sn films and 0.19 for the SnIn film. Now below about $0.8 T_{\mathrm{c}}$, the total measured rate (including both elastic and inelastic processes) is roughly two times higher in Sn than in SnIn , leading us to conclude that the elastic contribution is

small for SnIn , but comparable with the inelastic rate in Sn . As we increase the temperature towards T_c, $\hbar/2\tau_1\bar{\Delta}$ becomes large compared with unity, and $G_{\mathrm{el}\xi}$ scales roughly as $\bar{\Delta}^4$. Thus, in the limit $T \to T_{\mathrm{c}}$, the elastic contribution becomes negligible compared with the inelastic process, and the data for the Sn and SnIn samples become indistinguishable.

Chi and Clarke (1979) measured ${\tau_{Q^*}}^1$ in Al films in which the mean free path was reduced by evaporating them in an oxygen atmosphere. The experimental configuration was similar to that shown in fig. 5, except that it was not necessary to deposit a Pb film over the Cu film. They compared their results with computer solutions of the Boltzmann equation in which eq. (4.2) was included in the limit $\left(\hbar, 2\bar{A}\tau_1\right) \gg 1$ appropriate to the experiments. Figure 7 shows $\left(F^*\tau_{Q^*}\right)^{-1}$ versus $\Delta(T)/k_{\mathrm{B}}T_{\mathrm{c}}$ for five representative samples with parameters listed in table 1. The values $v_{\mathrm{F}} = 1.36 \times 10^6$ ms^{-1} [calculated from the free electron value divided by the thermal effective electron mass (Kittel 1976)], $\rho l = 9 \times 10^{16}\Omega$ m^2 (Fickett 1971), and $N(0) = 1.74 \times 10^{28}\mathrm{eV}^{-1}$ m^{-3} (Gschneidner 1964) were used. The solid lines are the best fit to the theory using τ_{E}^{-1} and the characteristic elastic charge relaxation rate

$$\tau_{Q^*\,\mathrm{el1}}^{-1}(0) = 2\tau_1\left(k_{\mathrm{B}}T_{\mathrm{c}}\right)^2\left\langle a^2\right\rangle_0/\hbar^2$$

Fig. 7. Measured values of $\left(F^{*}\tau_{Q^{*}}\right)^{-1}$ versus Ik_BT_s for 5 samples listed in table 1 (sample numbers shown in parentheses). The solid lines are best fits to the solution of the Boltzmann equation. (Chi and Clarke 19-9.)}

as fotume parameters. For each sample. the value of ${\tau_f}^1$ was obtained from the data in the limm A(T) k_1 I. $\to$ (). Whice ? (f)) was determined from the upward curvature required to fit the data at lower temperatures. It should be realized that the actual gap anisotropy in these folms is very small. Thus. for the clearest film listed in the table: $a^2, = 2, 10)^4/a^2/0$ at $0.9I_6$.

[18] "The upward curvature was too small to enable an estimate of $\tau_{Q^*\ \mathrm{el}}^{-1}(0)$ to be made.

Table 1
Measured and calculated quantities for five Al samples

Sample	T_c(K)	1(nm)	$^{\tau}$E (ns)	$\tau_0^{-1} = e_{el}^{-1}(0)t_E$	$\langle a^2 \rangle_0$
8	1.267	78	13	0.62	0.022
11	1.306	23	8	0.(19)3	0.01^{-}
12	1.411	12.5	5.8	0.062	0.025
13	1.573	6.0	1.5	0.031	0.00^{-}e
15	2.113	1.0	1.0	(a)	(a)

The high seristantly of τ_y : {o :ers small gap ansortropes reflects the relatively slow melasic scallering rate in Al. For the cleanest films studied (not listed in table 1) the average value of τ_{f} was about 12 ns .

46.4.2 4.2. Inelastic scattering in aluminum

(his and Clarke also extracted values of τ_{ff} from their data over a wide range of mean free paths. rangeng from about $4(x)$nm to 1 nm . The transition termperatures of these films increased from about 1.2 K for the cleanest films to about 2.1 K for the dirocest film. They found that the apparent value of τ_f decreased much more rapidly than I. "which would be expected from an electron phonon meteractoron). from about 12 ns for T $\approx$ 1.2 K to about 1 ns for $I \approx 2.1$ K. This rapid increase in the inelastic scattering rate with I_c remained a puzzle for several sears. but recent usork on the enhancement of electron electron scaltering due to localizatonn effects provides a rather convincing explanatuon (cordon et al. 1983, Santhanam and Prober 1984). We briefly describe this work.

The theory of Abrahams et al. (1981) predicts that for dirty ($l < hv_{\mathrm{F}}/k_{\mathrm{B}}T$), iw ()-dimensional lin; $13k_k, T - d^2$ isamples the electron electron scattering rate is given by

$$\mathrm{re}^{-1} = \frac{e^2 R}{2\pi\hbar^2} k_{\mathrm{B}} T \ln\left(T_1/T\right), \tag{4.3}$$

where d is the film thickness. R is the resistance of a square film and T_1 is roughly 103^2 K. Cjordon et al. and Santhanam and Prober extracted values of -: from measurements of the magnetoconductance of dirty Al films. and found good agreement with eq. (4.3). Near T. Entin-Wohlman and Orbach (19'x l) showed that the charge relaxation rate due to electron electron scattering is given by

$$\left(F^* \tau_{Q^*}\right)_{ee}^{-1} = \left(\pi\Delta/4k_B T\right) \tau_{et}^{-1}, \tag{4.4}$$

so that one can immediately apply eq. (4.3) to the inelastic scattering rates measured near T, by (C. Setting $T = 1$ K in the logarithmic term in eq. (4.3) fand neglecting the temperature dependence of this term) we find $z_{:,}' \approx 1.5$, $10^8 R - T_{\mathrm{c}} \mathrm{s}^{-1}$. For the dirtiest samples studied by CC ($T_{\mathrm{c}} \approx 2$ K.$R_{-} \approx 6\Omega$), one predicts : $\approx$ 10.5 ns from eq. 14.31 . a value within a factor of 2 of the cepperimental results. Both Gordon et al. and Santhanam and Prober show that the CC data for dirty Al are quite well fitted by eq. (4.3).

For the cleanest samples studied by CC, the electron-electron scattering rate should be negligible compared with the electron-phonon scattering rate. The measured value of τ_{E}, about 12 ns , is a factor of 3 or more below the value of about 40 ns calculated by Kaplan et al. (1976) and Lawrence and Meador (1978). The magnetoconductance data of Gordon et al., however, lead to $\tau_{\mathrm{E}} \approx 12$ ns in the clean limit, a value that is very compatible with the CC

value. This discrepancy between measured and theoretical values of τ_E in Al thus appears to be well substantiated, and is worthy of further study.

We note in passing that the values of $\langle a^2 \rangle_0$ obtained by CC assume, of course, that the inelastic scattering consists of electron-phonon processes only. If, as now seems highly probable. electron-electron scattering was significant or even dominant in the dirtier samples. since the distribution in the presence of electron electron scattering differs from that in the presence of electron phonon scattering, the values of $\langle a^2 \rangle_0$ quoted for these samples will be somewhat in error.

It is also possible to determine the inelastic and elastic scattering rates as a function of energy by measuring the nonequilibrium distribution injected via a tunnel junction by means of a second tunnel junction (Kaplan et al. 1977. Chang 1977. Kirtley et al. 1978. Gray 1981). In measurements on clean Al films, Gray found $\tau_{\mathrm{f}} \approx 12$ ns, in excellent agreement with the CC result, while Kirtley et al. found $\tau_1 \approx 0.25$ ns in Al films with $T_2 \approx 2.4$ K. More surprisingly, an analysis of both experiments produced an elastic scattering time that was essentially independent of the quasiparticle energy. This result is clearly in disagreement with the prediction of eq. (4.2), and not understond in the present theoretical framework.

46.5 5. Charge relaxation in the presence of magnetic impurities or a supercurrent

46.5.1 5.1. Elastic scattering from magnetic impurities

The addition of magnetic impurities to a superconductor destroys the degeneracy between time-reversed electron states through the exchange interaction between the conduction electrons and the impurities, giving the Cooper pairs a finite lifetime. The inverse of this lifetime, the pair breaking rate, is the elastic spin-flip scattering rate for electrons in the normal metal. τ_s^{-1} (Abrikosov and Gorkov 1960). The presence of impurities can have a dramatic effect on the charge relaxation rate as was first pointed out by Schmid and Schön (SS. 1975) and later studied by Pethick and Smith (1981a,b), Entin-Wohlman and Orbach (1979), and Lemberger and Clarke (LC, 1981a). This effect occurs because the coherence factor for elastic spin-flip scattering from one quasiparticle branch to the other in an isotropic superconductor is not zero, but of the form

$$(uu' + vv')^2 = 4u^2v^2 = \Delta^2/E^2. \tag{5.1}$$

Since this factor approaches unity as $E \to A$, we expect the spin-flip scattering to have an appreciable effect on $\tau_{Q^*}^{-1}$ when $\tau_{\mathrm{S}}^{-1} \gtrsim \tau_{\mathrm{E}}^{-1}$.

Schmid and Schön found

$$(F^*\tau_{Q^*})^{-1} = \frac{\pi\Delta}{4k_{\mathrm{B}}T_{\mathrm{c}}\tau_{\mathrm{E}}}\left(1 + \frac{2\tau_{\mathrm{E}}}{\tau_{\mathrm{S}}}\right)^{1/2}, \quad \left(\tau_{Q^*}^{-1} \ll \tau_{\mathrm{E}}^{-1}\right) \tag{5.2}$$

A factor $(1 + \hbar^2\Gamma/\Delta^2\tau_{\mathrm{E}})^{1/2}$ has been omitted in eq. (5.2), where $\Gamma = (2\tau_{\mathrm{E}})^{-1} + \tau_{\mathrm{S}}^{-1}$, since it is very close to unity for all values of $\tau_{\mathrm{E}}, \tau_{\mathrm{S}}$, and Δ used in the experiments to be described later.

LC included a term

$$G_{\mathrm{sf}\xi} = \frac{1}{\tau_{\mathrm{S}}}\frac{\Delta^2}{E^2}\frac{E}{|\xi|}\left(f_\xi - f_{-\xi}\right) \tag{5.3}$$

in the Boltzmann equation to account for spin-flip scattering, and computed solutions with $G_{\mathrm{cl}j} = 0$ (since this term is negligible compared with G_{sf} for concentrations of magnetic impurities of experimental interest). Figure 8 shows $(F^*\tau_{Q^*})^{-1}$, normalized to the SS factor

$(\pi/4\tau_{\mathrm{E}})\,(1+2\tau_{\mathrm{E}}/\tau_{\mathrm{S}})^{1/2}$ versus $\Delta/k_{\mathrm{B}}T_{\mathrm{c}}$ for several values of $\tau_{\mathrm{E}}/\tau_{\mathrm{S}}$. For values of $\Delta/k_{\mathrm{B}}T_{\mathrm{c}} < 1$, the computed values of $(F^{*}\tau_{Q^{*}})^{-1}$ lie above the SS result, and, furthermore, increase monotonically compared with the SS prediction as $\tau_{\mathrm{E}}/\tau_{\mathrm{S}}$ increases.

Using a configuration similar to that in fig. 5, Lemberger and Clarke measured ${\tau_{Q^{*}}}^{1}$ in Al films doped with Er. Figure 9 shows $(F^{*}\tau_{Q^{*}})^{-1}$ versus $A/\ k_{\mathrm{B}}T_{\mathrm{c}}$ for 3 samples with the parameters listed in table 2 ; the values of $N(0)$ and ρl were the same as those used by CC. It was assumed that $\tau_{\mathrm{f}} = 12\,(1.2/T_{\mathrm{c}})^{3}$ ns; the estimated spin-flip time was taken from Craven et al. (1971), $\tau_{\mathrm{S}}^{-1} = 1.9\times10^{13} n_{\mathrm{Er}}\mathrm{S}^{1}$, where n_{Ir} is the concentration of Fratoms. The rate $(F^{*}\tau_{Q^{*}})^{-1}$ is linear in $\Lambda, k_{\mathrm{B}}T_{\mathrm{c}}$ for $\Lambda/k_{\mathrm{B}}T_{\mathrm{c}} \lesssim 0.8$, which is consistent with the SS result (eq. 5.2) provided that one assumes that this expression can be extrapolated to much higher values of $A/k_{B}T_{c}$ than one can justify theoretically. The data in this range were used to calculate the values of τ, (meas.) listed in table 2. The slope S of the linear region for all the samples studied is plotted versus the SS expression $(\pi/4\tau_{\mathrm{E}})\,(1+2\tau_{\mathrm{r}}/\tau_{\mathrm{S}})^{1\ 2}$ in fig. 10. One sees that the data generally fall on a straight line through the origin, indicating that eq. (5.2) predicts the data remarkably well for $\Delta/k_{\mathrm{B}}T_{\mathrm{c}} \lesssim 0.8$.

Figure 9 also shows curves computed from the Boltzmann equation. It is immediately apparent that the fit to the data is much worse than that of the SS prediction. The discrepency between the computed and experimental curves is further emphasized in fig. 11, where $(SF^{*}\tau_{Q^{*}})^{1}$ is plotted versus $\Lambda/k_{13}T_{\mathrm{c}}$ for one sample of each Er concentration. It is apparent that the data, normalized in this way, lie on a universal curve, in complete contrast to the computed

Fig. 8. $\left(F^{*}\tau_{Q^{*}}\right)^{-1}$ normalized to $(\pi/4\tau_{\mathrm{E}})\,(1+2\tau_{\mathrm{E}}/\tau_{\mathrm{S}})^{1/2}$ versus $\Delta/k_{\mathrm{B}}T_{\mathrm{c}}$. The curves approach the origin with unity slope. The dashed line is an extrapolation of the SS theory to low temperatures. The inset shows the region near the origin. (Lemberger and Clarke 1981a.)}

curves shown in fig. 8, in which the same quantity increases markedly with τ_E/τ_S at a given temperature.

More recently, Beyer Nielsen et al. (1982) have examined this problem using a quasiclassical Green's function approach, taking into account the broadening of the states due to the presence of the magnetic impurities. Their results lie very much closer to Lemberger and Clarke's solutions to the Boltzmann equation than to the SS results.

This marked discrepancy between the data and the solutions to the kinetic equation is extremely puzzling. It cannot be explained by invoking an additional charge relaxation mechanism in the experiments because the experimental rate $(F^*\tau_{Q^*})^1$ lies substantially below, rather than above, the computed rate. One can rule out the possibility of impurity-impurity

Fig. 9. Experimental values of $\left(F^{*}\tau_{Q^{*}}\right)^{-1}$ versus $\Delta/k_B T_c$ for samples with Er concentrations of 21 (no. 1), 81 (no. 6), and 220 (no. 7) at.ppm, with straight lines drawn through the data by eye. The other curves, which represent computer solutions to the Boltzmann equation, have the same slope as the data in the limit $\Delta/k_B T_c \to 0$. (Lemberger and Clarke 1981a.)}

Table 2
Measured and calculated parameters for AlEr films.

Sample	Er-conc (ppm)	T_c (K)	l (nm)	τ_E (ns)	τ_S (est.) (ps)	τ_S (meas.) (ps)
1	21	1.338	38	8.6	2500	1100
6	81	1.350	32	8.3	650	860
7	220	1.410	22	7.4	240	210

interactions, and of Kondo anomalies. However, it is noteworthy that the mean free path of most of the samples was sufficiently short that electron electron scattering played at least

some role in the charge relaxation. Thus, it is just possible that the discrepancy arises from the different quasiparticle

Fig. 10. S, the slope of $\left(F^{*}\tau_{Q^{*}}\right)^{-1}$ versus $\Delta/k_{\mathrm{B}}T_{\mathrm{c}}$ for small $\Delta/k_{\mathrm{B}}T_{\mathrm{c}}$, versus $(\pi/4\tau_{\mathrm{E}})\,(1+2\tau_{\mathrm{E}}/\tau_{\mathrm{S}})^{1/2}$ for all samples. The solid line through the origin has unity slope. (Lemberger and Clarke 1981a.)}

distributions that occur in the presence of electron-phonon and electronelectron scattering. Since electron-electron scattering does not reduce the energy of the quasiparticles, there will be a smaller quasiparticle population at low energies compared with that in the presence of an equal electron-phonon

scattering rate. As a result, the charge relaxation rate due to magnetic impurity scattering will be reduced, since $G_{\mathrm{sf}\xi}$ is proportional to Δ^2/E^2 (eq. 5.3). Thus, the presence of electron-electron scattering is expected to lower the overall charge relaxation rate compared with the value one would compute assuming that the measured inelastic relaxation rate was due solely to electron-phonon processes. Obviously, only a solution of the Boltzmann equation with the inclusion of an appropriate electron-electron relaxation term (Entin-Wohlman and Orbach 1981) will resolve this issue.

46.5.2 5.2. Elastic scattering in the presence of a supercurrent

In the presence of a supercurrent, the quasiparticle energies are raised by an amount $p_k \cdot v_s$, where p_h is the momentum of a quasiparticle in state k, and v_s is the superfluid velocity (Aronov 1974, Gal'perin et al. 1974). Thus, quasiparticles at the Fermi energy where v_s and p_k are in the same direction are raised by $p_{\mathrm{F}}v_{\mathrm{s}}$, while those where v_{s} and p_k are in opposite directions are lowered by $p_{\mathrm{F}}v_{\mathrm{s}}$. This current-induced anisotropy allows elastic scattering to relax charge imbalance in much the same way as gap anisotropy. Maki (1969) has shown that an appropriate electron relaxation rate for thin films $\left[d \ll (l\xi_0)^{1/2}\right]$ in the dirty limit ($l \ll \xi_0$) is

$$\frac{1}{\tau_{\mathrm{S}}} = \frac{l v_{\mathrm{F}}}{6}\left(\frac{p_{\mathrm{S}}}{\hbar}\right)^2, \tag{5.4}$$

Fig. 11. $\left(S F^{* \tau_{Q^{*}}}\right)^{-1}$ versus $A/k_{\mathrm{B}}T_{\mathrm{c}}$ for one representative sample of each impurity concentration. (Lemberger and Clarke 1981a.)}

where $\xi_0 = \hbar v_{\mathrm{F}}/\pi\Delta(0)$, and p_{S} is the momentum of a Cooper pair. We can use eq. (5.4) in eq. (5.2) or (5.3) to predict the charge relaxation rate as a function of p_{s}. For the case of magnetic impurities LC found that at a given temperature $(F^{*}\tau_{Q^{*}})^{-1}$ was proportional to $(1 + 2\tau_{\mathrm{E}}/\tau_{\mathrm{S}})^{1/2}$, as predicted by SS. Thus, we can immediately write down an expression for the detector voltage at a given temperature and injection current in the presence of a supercurrent, I_{S} :

$$V_{\mathrm{d}}\left(I_{\mathrm{S}}\right) = \frac{V_{\mathrm{d}}(0)}{\left(1 + 2\tau_{\mathrm{E}}/\tau_{\mathrm{S}}\right)^{1/2}} = \frac{V_{\mathrm{d}}(0)}{\left(1 + b^{\mathrm{SS}} I_{\mathrm{S}}^{2}\right)^{1/2}}, \tag{5.5}$$

where from eq. (5.4)

$$b^{\mathrm{SS}} = 2\tau_{\mathrm{E}}/\tau_{\mathrm{S}} I_{\mathrm{S}}^{2} = \tau_{\mathrm{E}} l v_{\mathrm{F}} p_{\mathrm{S}}^{2}/3\hbar^{2} I_{\mathrm{S}}^{2} \tag{5.6}$$

For uniform currents much less than the critical current we can use the relations (Tinkham 1975) $p_{\mathrm{S}} = 2mv_{\mathrm{S}}, j_{\mathrm{S}} = n_{\mathrm{S}} e v_{\mathrm{S}}, n_{\mathrm{S}} = mc^{2}/4\pi e^{2}\lambda^{2}$ and, in the dirty limit, $\lambda(0) \approx \lambda_{L}(0)\left(\xi_0/l\right)^{1/2}$ to find

$$b^{\mathrm{SS}} = \left(\frac{8\pi e}{\hbar c^{2}}\right)^{2} \frac{\tau_{\mathrm{E}} v_{\mathrm{F}} \xi_0^{2} \lambda_{\mathrm{L}}^{4}(0)}{3 l d^{2} w^{2}} \frac{\lambda^{4}\left(T/T_{\mathrm{c}}\right)}{\lambda^{4}(0)}. \tag{5.7}$$

Here, w is the film width, $\lambda_{\mathrm{L}}(0)$ is the London penetration depth, and $\lambda\left(T/T_{\mathrm{c}}\right)/\ \lambda(0)$ is a well-known function (Tinkham 1975).

Lemberger and Clarke (1981b) solved the Boltzmann equation in the presence of a supercurrent by replacing $1/\tau_{\mathrm{s}}$ in eq. (5.3) with eq. (5.4) to obtain

$$G_{\mathrm{S}\xi} = \frac{l v_{\mathrm{F}}}{6}\left(\frac{p_{\mathrm{S}}}{\hbar}\right)^{2} \frac{\Delta^{2}}{E|\xi|}\left(f_{\xi} - f_{-\xi}\right) \tag{5.8}$$

With this substitution, the results computed in section 5.1 can be carried over to the present case. Thus, for fixed temperature and injection voltage,

$$V_{\mathrm{d}}\left(I_{\mathrm{S}}\right) = \frac{V_{\mathrm{d}}(0)}{\left(1 + b^{\mathrm{num}}\ I_{\mathrm{S}}^{2}\right)^{1/2}} \tag{5.9}$$

Fig. 12. Typical experimental plot of $V_{\mathrm{d }}$ versus I_{s} for fixed I_{i}. The points are a fit to a function of the form $V_{\mathrm{d}}\left(I_{\mathrm{s}}\right)=V_{\mathrm{d}}(0)/\left(1+bI_{\mathrm{s}}^{2}\right)^{1/2}$, with b as the fitting parameter. (Lemberger and Clarke 1981b.)}

where $b^{\mathrm{num}}\left(T\right)$ is the numerical value produced by the computer. In general, $b^{\mathrm{num}}\left(T\right)$ will differ from $b^{\mathrm{SS}}\left(T\right)$, as is evident in section 5.1 , and we can compare both values with the measured value, $b^{\mathrm{meas}}\left(T\right)$.

Lemberger and Clarke (1981b) performed experiments using the usual geometry with a supercurrent introduced along the Al film in which the charge

imbalance is generated. To increase the uniformity of the current distribution, a Nb groundplane was sputtered onto the substrate and covered with an insulating layer before the sample was deposited. A representative plot of $V_{\mathrm{d}}\left(I_{\mathrm{S}}\right)$ versus I_{S} is shown in fig. 12. As predicted by eqs. (5.5) and (5.9), the value of $|V_{\mathrm{d}}|$ decreased quadratically with increasing $|I_{\mathrm{S}}|$ at low supercurrents, becoming linear in I_{S} at higher supercurrents. The slight asymmetry in the curves about $I_{\mathrm{S}}=0$ was due to the non-negligible value of I_{i}. After shifting the origin appropriately, curves of the form $\left(1+hI_{5}^{2}\right)^{12}$ were fitted; the quality of the fit is excellent. Experimentally determined values of h are shown in fig. 13 for two samples, together with the fitted curves h^{SS} and h^{num} . The temperature dependence of b^{SS} is in excellent agreement with the data, while b^{num} is in substantial disagreement. Furthermore, the ratio $\left\langle b^{\mathrm{SS}}\left(T\right),b^{\mathrm{meal}}\left(T\right)\right\rangle$ varies between 1/3 and 1/2, an agreement which, given the uncertainties in the values of $v_{\mathrm{F}},l,\lambda_{\mathrm{L}}(0)$, and τ_{E}, is considered quite acceptable.

We therefore draw the same conclusions as for the case of magnetic impurities: The Schmid-Schön theory fits the measured data very accurately, even at values of $\Delta/k_{\mathrm{B}}T_{\mathrm{c}}$ much larger than one could reasonably expect, while the computed solution to the Boltzmann equation does not fit the data. However, it is again just possible that this discrepancy could be resolved by the inclusion of electron electron scattering in the Boltzmann equation.

Fig. 13. Data points are measured values of b, solid and dashed lines are fits of b^{ss} and b^{num} to the data. The values of T_{c}, l, and τ_{E} are 1.370 K, 25 nm and 5.3 ns for sample 1 and 1.514 K, 11 nm, and 2.7 ns for sample 2 . The average values of $b^{\mathrm{SS}}/b^{\mathrm{meas}}$ over the temperature range studied are 0.36 and 0.51 for samples 1 and 2 ; the value $i_L(0) = 160$ nm (Meservey and Schwartz 1969) was used. (Lemberger and Clarke 1981b.)}

46.6 6. Resistance of the normal metal-superconductor interface

We now begin a discussion of a series of situations other than tunnel injection in which charge imbalance plays a role. The first of these concerns the electrical resistance of the normal metal-superconductor (NS) interface, a subject about which there exists a substantial literature. The first important experimental work was that of Pippard et al. (1971) who measured the electrical resistance of superconductor-normal-metal-superconductor (SNS) sandwiches in which the normal metal was too thick and/or too dirty to sustain a Josephson supercurrent. They observed that, near the transition temperature of the superconductor, T_c, the resistance increased rapidly with increasing temperature. Yu and Mercereau (1972) showed that the excess resistance was associated with a potential that decayed exponentially into the superconductor. Subsequently, Harding et al. (1974) studied the resistance of SNS sandwiches in which the mean free path of the superconductor was shortened by alloying, and found an additional boundary resistance at low temperatures as well as a greatly enhanced rise in resistance near T_c. More recently, Hsiang and Clarke (1980) made measurements on a series of samples in the clean limit, and used their results to obtain values of τ_E in Pb, Sn, and In.

The theory of the NS interface resistance has been widely investigated. The work of Rieger et al. (1971), who used a time-dependent Ginzburg-Landau theory, contained some essentially correct ideas, but did not produce the correct quasiparticle propagation length in the superconductor. Pippard et al. (1971) and Harding et al. (1974) used a Boltzmann equation approach that was later extended by Waldram (1975), and reviewed by Pippard (1981). The microscopic theory was developed by Schmid and Schön (1975), and has been extended by Ovchinnikov (1977, 1978), Artemenko and co-workers (Artemenko and Volkov 1977, Artemenko et al. 1978) and Krhenbhl and Watts-Tobin (1978, 1979). Hsiang and Clarke (1980) gave a simple description valid in the limit $\Delta/k_{\mathrm{B}}T \ll 1$ by adapting the tunnel injection theory that accounted quantitatively for their experimental data, and that was verified by Pethick and Smith (1981a) using their two-fluid model. Most recently, Blonder et al. (1982) have used the Bogoliubov

equations to treat the transmission and reflection of particles at the NS interface that includes a barrier of arbitrary strength.

In the clean limit $l \gg \xi(T)$ the essential picture that emerges from this work is as follows. When a charge imbalance is created in one region of a superconductor, it relaxes over a length

$$\lambda_{Q^*} = (l v_{\mathrm{F}} \tau_{Q^*}/3)^{1/2} = (D\tau_{Q^*})\,,^{1/2}, \tag{6.1}$$

where we assume $l \ll v_{\mathrm{F}}\tau_{Q^*}$. Very close to T_{c} almost all of the excitations incident from N propagate into S , so that, in the presence of an external
current, a quasiparticle current flows in the superconductor. In the usual situation where the transverse dimensions of the interface are much larger than the London penetration depth, there is no net current in the interior of the superconductor. The internal quasiparticle current is cancelled by a pair current, with a corresponding flow of supercurrent on the surface. The electric field is continuous at the interface, and the electric field, the electric potential, Q^*, and the quasiparticle current all decay exponentially into S with a characteristic length i_{Q^*} (see fig. 14). When the temperature is lowered somewhat, a substantial fraction of quasiparticles have energies $\lesssim \Delta_\infty(T)$, and are Andreev reflected (Andreev 1964) at or near the NS interface. In this process, a $k_>$ ($k_<$)quasiparticle incident from N is scattered onto the $k_<$ ($k_>$) branch, and the current carried by these two excitations continues in the superconductor as a supercurrent. Thus, there is no boundary resistance associated with these quasiparticles, and there is a discontinuity in the electric field at the interface (fig. 14). The potential is continuous at the interface, but its spatial derivative is not. In the presence of a current, the boundary scattering processes introduce disequilibrium in the quasiparticle distributions within an inelastic scattering length on either side of the interface.

Fig. 14. Variation of energy gap, $\triangle$, electric field, E, electric potential, V, normal current, j_{N}, and supercurrent, j_{S}, across an NS interface for $k_{\mathrm{B}}T/\Delta \to 0, k_{\mathrm{B}}T \sim \Delta$, and $\Delta/k_{\mathrm{B}}T \to 0$. The gap is taken to be zero in N, and the normal-state properties are assumed to be the same in the two metals.
(Hsiang and Clarke 1980.)}

As the temperature is lowered still further ($\Delta_x \gg k_{\mathrm{B}}T$), essentially all of the quasiparticles are Andreev reflected at the interface, and there is no quasiparticle current in S. Correspondingly, the electric field and potential are zero in S (fig. 14) and there is no boundary resistance.

(In fact, the Andreev scattering process occurs over a distance $\sim \xi_0$, so that the discontinuities in the electric field and the derivative of the potential extend over this region. Furthermore, there will be a small boundary resistance of order $\zeta\rho_s/A$ that is negligible in most practical situations.)

We now outline the simple model of Hsiang and Clarke (HC) that is valid in the limit $A_\infty(T)/k_\mathrm{B}T \ll 1$. We assume that the transition temperature of the normal metal is much less than T, so that we can set $A = 0$ for $x < 0$ (fig. 14). In the superconductor, Δ rises from its value at the boundary, $\Delta_0(T)$, to its full value, $A_y(T)$, over a distance of roughly the Ginzburg-Landau (1950) coherence length, $\zeta(T)$, that is always much less than i_{0^*} in the temperature range investigated experimentally. However, we note that quasiparticles with energies greater than $\Delta_\gamma(T)$ may undergo some charge relaxation in the region where Δ varies spatially. At least in the limit $\Delta \ll k_\mathrm{B}T$, this contribution to the overall relaxation rate is likely to be small, and we shall neglect it. We assume that the current densities are sufficiently low that they do not perturb Δ. We further assume that quasiparticles with energies greater than $\Delta_x(T)$ are transmitted into S with probability unity; this is the most serious approximation because some of the quasiparticles in this energy range will undergo Andreev reflection. Quasiparticles with energies $\langle A_x(T)$ are Andreev reflected at a plane taken as $x = 0$ [since $\xi(T) < i_{Q^*}$]. Finally, we assume that the quasiparticles are close to thermal equilibrium even in the vicinity of the interface; we emphasize that this is a reasonable approximation only for A, $\ll k_\mathrm{B}T$.

The charge imbalance generated by the uniform injection of a current I_i into volume Ω of a superconductor in the limit $e_\mathrm{i} \ll k_\mathrm{B}T$ is given from eqs. (2.21) to (2.23) by

$$Q^* = \frac{Z(T)}{Y(T)}\frac{I_\mathrm{i}\tau_{Q^*}}{e\Omega}, \quad (eV_\mathrm{i} \ll k_\mathrm{B}T)\,. \tag{6.2}$$

In the case of the NS interface, I_i is just the quasiparticle current injected into S , and is related to the total current, I, by

$$\mathrm{Y}(T) = I_\mathrm{i}/I. \tag{6.3}$$

Equation (6.3) follows from the realization that in a SIN tunnel junction at low voltages a fraction $[1 - \mathrm{Y}(T)]$ of the current that flows at T_c cannot flow at a temperature $T < T_c$ because there are no states available in S at energies $< .1(T)$, whereas at the NS interface, in our approximation, this same fraction $[1 - \mathrm{Y}(T)]$ of the total current is transmitted into S as a pair current.

Combining eqs. (6.1) to (6.3), and replacing the exponentially decaying Q^* with a value that is constant at the value $Q^*(0)$ for $x \leqslant \lambda_{Q^*}$ and 0 for $x > \lambda_{Q^*}$, we find

$$Q^*(0) = \frac{Z(T)I\tau_{Q^*}}{eA\lambda_{Q^*}} \tag{6.4}$$

The excess voltage, V_b, at each interface of the SNS sandwich adds to the voltage developed across the normal metal, and the total potential across the sandwich is measured with superconducting leads making metallic contact with the superconducting films. This is in contrast to the usual tunneling measurement of Q^*, where the potential is measured by a tunneling contact to a normal metal. Thus, we set $g_\mathrm{NS}(0) = 1$ in eq. (2.20), and combine the result with eq. (6.4) to obtain the boundary resistance

$$R_\mathrm{b} = \frac{V_\mathrm{b}}{I} = \frac{Z(T)\dot{\lambda}_{Q^*}\rho}{A}, \quad (\Delta \ll k_\mathrm{B}T)\,. \tag{6.5}$$

In eq. (6.5) we have set $\tau_{Q^*} = 3i_{Q^*}^2/l_\mathrm{F}$ and used the free electron model (Kittel 1976) to calculate $\rho_\mathrm{s} = 3/2e^2N(0)lv_\mathrm{F}$. As $T \to T_\mathrm{c}, Z(T) \to 1$, and the boundary resistance is just the

resistance of a length $\hat{\lambda}_{Q^*}$ of the superconductor in the normal state. As the temperature is lowered, $Z(T)$ decreases, reflecting the fact that fewer quasiparticles are able to propagate into the superconductor. At low temperatures (Clarke et al. 1979a) $Z(T) \approx (k_B T/\Delta)^{12} \exp(-\Delta/k_{\mathrm{B}}T)$, so that R_{b} vanishes exponentially as $T \to 0$, as we expect. However, we repeat our warning that the model is not expected to be quantitatively correct at intermediate temperatures.

If one measures R_{b} near T_{c} one can deduce values of τ_{Q^*}. As was discussed in sections 2.3 and 4, in general, both inelastic and elastic scattering contribute to $\tau_{Q^*}^{-1}$. We can make a crude estimate of the elastic relaxation rate in the presence of gap anisotropy for the materials used in the experiments to be described by assuming $E \sim k_{\mathrm{B}}T_{\mathrm{c}}$ in eq. (4.2). This leads us to the conclusion that electron phonon scattering should dominate at temperatures above $0.9T_{\mathrm{c}}$. Since $\Delta, k_{\mathrm{B}}T \approx 1$ at $T; T_{\mathrm{c}} = 0.9$, it appears that in the range of validity of eq. (6.5), $\Delta, k_{\mathrm{B}}T \ll 1$, the charge relaxation should be dominated by inelastic scattering, and we assume that $\tau_{Q^*} = 4k_{\mathrm{B}}T\tau_{\mathrm{E}}/\pi\Delta_\sigma(T)$.

HC measured the resistances of the SNS junctions listed in table 3. The PbBi − CuAl samples were made by evaporating the materials onto glass substrates; the CuAl was $\sim 2\mu$ m thick, while the PbBi was $\sim 20\mu$ m thick, considerably greater than i_{Q^*} over the experimental temperature range. In the remaining samples, the superconductor (up to 80μ m thick) was evaporated onto the two sides of Ir foils about 70μ m thick. The foils were cleaned by sputter etching in argon, the argon was pumped out of the system, and the superconducting material was evaporated onto each side of the foil. Two or three samples were connected in series and their resistances measured with a

Table 3
Properties of SNS junctions. ${

^{a}$}

Superconductor/ normal metal	Type	Measured τ_{E} (10^{-10} s)	Calculated $^{b}\tau_{\mathrm{E}}$ (10^{-10} s)
$Pb_{0.99}Bi_{0.01} - Cu_{0.97}Al_{0.03}$	film	0.25	0.23(Pb)
Sn − Ir	foil	2.6	2.7
$Sn_{0.99}In_{0.01}{}^{-}Ir$	foil	1.1	2.7(Sn)
In-Ir	foil	1.1	1.0

[a] Values of τ_{E} are averaged over 2 or 3 samples of each type.
[b] Kaplan et al. (1976).

SQUID voltmeter. The variation of resistance with temperature is shown in fig. 15 for one representative sample of each type. Near T_{c}, the resistance rises rapidly with increasing temperature, while at low temperatures the resistance is nearly independent of temperature. To within the experimental accuracy, the low temperature resistance was equal to the estimated resistance of the normal metal.

To compare the data with eq. (6.5), in fig. 16 we plot the measured resistance versus $Z(T)\,(k_{\mathrm{B}}T/\Delta)^{1/2}$. The solid lines are a least squares fit to the data for $T > 0.9T_{\mathrm{c}}$ ($0.96T_{\mathrm{c}}$ for Sn). The fit is good-in fact, for (a), (c) and (d) it is surprising that such good agreement extends down to temperatures as low as $0.9T_{\mathrm{c}}$ where $\Delta \sim k_{\mathrm{B}}T$. The slope of the lines in fig. 16 is $4\,(lv_{\mathrm{F}}\tau_{\mathrm{E}}/3\pi)^{1/2}\,\rho_{\mathrm{s}}/A$, and yields the average values of τ_{E} listed in table 3 . The values of τ_{E} calculated by Kaplan et al. (1976) from $\alpha^2 F$ are also listed for comparison. The agreement is generally quite good. Using the fitted values of τ_E, we have plotted the resistance predicted by eq. (6.5) in fig. 15. Except for Sn , the extrapolated lowtemperature resistance is in good agreement with the measured resistance, thus providing a good check on the consistency of our results. Particularly in the cases of PbBi and In , the fit is remarkably good even at intermediate temperatures, a result that is probably coincidental, particularly since the expression used for

τ_{Q^*} is quite inappropriate in this range. However, it may be that the increase in the elastic charge relaxation rate as the temperature is lowered tends to compensate for the decrease in the inelastic rate, thus keeping τ_{Q^*} roughly constant at temperatures below about $0.9T_c$.

As pointed out in the description of the tunneling model, the effect of Andreev reflection on the quasiparticle distribution is neglected. The detailed theory of Waldram (1975) takes into account the nonequilibrium quasiparticle distribution, but the fact that one would have to use several parameters to fit the experimental data makes a meaningful test of the theory somewhat difficult. More recently, Blonder et al. (1982) have developed a theory in which they interpose a tunneling barrier of arbitrary strength between the normal metal and the superconductor, and calculate the probabilities of reflection and

transmission as a function of this strength. They have used this model to account for the current-voltage characteristics of Cu − Nb point contacts that could be varied from tunneling to metallic contacts (Blonder and Tinkham 1983) and for the subharmonic gap structure observed in various types of weak links (Klapwijk et al. 1982, Octavio et al. 1983). The model is strictly

applicable only to the case of a hole with a radius much less than l in an insulating screen between two metals, so that the resistance is due to the constriction impeding the ballistic propagation of electrons. This leads to the assumption that the incoming electrons from the S side are described by a Fermi function, $f^0(E)$, while those coming in from the N side are described by $f^0(E - eV)$. Within this approximation, Blonder et al. compute the Andreev reflection of quasiparticles at the interface for varying strengths of the
tunneling barrier. For the case of a clean metallic contact, their result agrees with eq. (6.5) at T_c but has a somewhat different temperature dependence at lower temperatures, the exact form depending on the degree of disorder at the interface and the ratio of the Fermi velocities in the two metals. However, this picture of a small-area constriction impeding the ballistic propagation of electrons, while highly appropriate for point contacts or microbridges, is unlikely to be an appropriate description of the large area SNS sandwiches studied here.

To conclude this section on the NS interface, we note briefly that Hsiang (1980) studied the effect of a magnetic field applied parallel to the plane of SNS sandwiches in which the superconductor was $Pb_{0.98}Bi_{0.02}$, and the normal metal was Cd . The PbBi films are bulk, type-II superconductors in which the spin relaxation rate is given by (Maki 1969, de Gennes 1966, Tinkham 1975)

$$\frac{1}{\tau_s} = \frac{l v_F e H}{3c}. \tag{6.6}$$

Hsiang measured the boundary resistance as a function of applied magnetic field and temperature, and extracted values of $\tau_{Q^*}^{-1}(T, H)$. Using eq. (6.6) in eq. (5.2) he was able to estimate τ_s^{-1} as a function of H. Although there was a good deal of spread in the results, he was able to establish reasonable agreement between the measured value of τ_s^{-1} and the value predicted by eq. (6.6).

46.7 7. Phase-slip centers

A subject of major importance that has received widespread attention is the phase-slip center (PSC) in one-dimensional superconducting filaments or films. Reviews of this subject have been written by Tinkham (1979) and Skocpol (1981). Webb and Warburton (1968) first reported the appearance of discrete voltage steps in the current-voltage characteristics of Sn whiskers very close to T_c. Similar effects were observed by Meyer and von Minnigerode (1972) and Meyer (1973). However, the first understanding of the origin of the effect was put forward in the classic paper by Skocpol, Beasley and Tinkham (SBT) (1974), and we will begin with a brief outline of this work.

Figure 17 shows a typical IV characteristic obtained by SBT for the whole length of a Sn strip, 140μ m long, 4μ m wide, and 0.1μ m thick, shown inset. As the current is increased from zero, there is an initial zero-voltage region followed by a series of voltage steps to resistive regions, the dynamic resistance of which increases by approximately equal increments. As the current is reduced, there is a corresponding series of voltage steps to lower voltages, but with considerable hysteresis. The SBT explanation of these phenomena is as follows (see fig. 18).

As the current, I, is increased, eventually it exceeds the critical current, I_c, at the weakest point in the microbridge. The order parameter in this region collapses to zero and the current is forced to flow as a normal current, thereby allowing the superfluid to build up again and to resume carrying the current. This cycle repeats at a frequency 2eV/h, where V is the average voltage across the region. Each time the order parameter falls to zero, the phase difference between the two ends of the bridge slips by 2π : hence the name "phase-slip center". The width of the region over which the superfluid oscillates is roughly $2\xi(T)$. Now since the superfluid current oscillates between zero and its critical value, on the average approximatley one-half of the current in this region flows as a supercurrent, and the other half as a normal current. Since the temperature is very close to T_c, virtually all of the normal current will propagate into the superconducting regions on either side of the phase-slip region, where it will be converted into a supercurrent over a characteristic length i_{Q^*}. The time average of the pair electrochemical potential, $\bar{\mu}_{\mathrm{s}}$, is constant on either side of the phase-slip region, with a difference eV between the two sides. The time averaged quasiparticle potential, μ_{qp}, changes
smoothly with distance as indicated in fig. 18. Thus, the voltage across the PSC (at distances large compared with λ_{Q^*}) is given by

$$V = 2\left(I - \beta I_{\mathrm{c}}\right)\rho_{\mathrm{s}}\lambda_{Q^*}/A, \tag{7.1}$$

where $\beta \approx 1/2$, and we have used eq. (6.5) with $Z(T) = 1$. The dynamic resistance is

$$\partial V/\partial I = 2\rho_{\mathrm{s}}\lambda_{Q^*}/A. \tag{7.2}$$

This picture explains the essential features of fig. 17. Each step in voltage corresponds to the creation of a PSC, each of which contributes $2\rho_s\lambda_{Q^*}/A$ to the dynamic resistance. SBT were able to set outer limits on the location of each PSC by measuring the voltages across different probes on their bridge (inset, fig. 17). In the example shown in fig. 17, they found that the first four steps were produced by one PSC forming successively in each of the regions II, I, IV and III, while the fifth step was produced by a second PSC appearing in region IV. However, one very puzzling feature was the fact that the measured characteristic length for the charge imbalance relaxation was essentially independent of temperature, with a magnitude of approximately $(lv_{\mathrm{F}}\tau_{\mathrm{E}}/3)^{1/2}$. This fact prevented the SBT explanation from being a completely satisfying picture of the PSC. However, Dolan and Jackel (1977) resolved this difficulty in an elegant experiment in which they prepared a microbridge with a series of superconducting voltage leads along one side of the bridge and a series of normal voltage leads along the other. The probes were typically 2μ m apart, and formed tunnel junctions with the microbridge. In this way, they were able to measure the average superfluid and normal potentials, $V_{\mathrm{S}} = \mu_{\mathrm{s}}/e$ and $V_{\mathrm{N}} = \mu_{\mathrm{qp}}/e$, across a PSC, the position of which was defined by a small notch in the microbridge. Figure 19 shows their measured values of V_{S} and V_{N}. As expected, V_S changes abruptly, since $\xi(T)$ is less than the probe spacing, while V_N changes over a much greater length. Values of τ_{Q^*} deduced from the spatial variation of V_{N} showed the predicted $(T_{\mathrm{c}} - T)^{-1/4}$ temperature dependence, and produced the value ${\tau_{Q^*}}^{-1} = 6.3 \times 10^9\,(\Delta/k_{\mathrm{B}}T_{\mathrm{c}})\,\mathrm{s}^1$, in excellent agreement with the results of Clarke and Paterson (1974) discussed in section 3. Since this work on PSC was performed at $T \geqslant 0.98T_{\mathrm{c}}$, inelastic scattering should completely dominate the charge relaxation.

It was subsequently shown (Kadin et al. 1978) that heating effects can modify the $I - V$ characteristics so as to mask the temperature dependence of i_{Q^*}. Very near T_c, the step becomes

rounded, and, as the current is increased, the step due to the entry of the next PSC may appear before the asymptotic value of the dynamic resistance is reached. Further below T_c, local heating distorts the IV characteristic by adding curvature and creating hysteresis. Kadin et al. were able to show that when heating effects were taken into account, the temperature dependence of λ_{Q^*} was as expected. Recently, Stuivinga et al. (1983b) have presented a detailed model of heating effects in PSCs.

Kadin et al. (1978) also studied the effect of a magnetic field, H, applied parallel to the plane of tin PSCs, and found that the inferred value of τ_{Q^*} was reduced by the field. The magnetic field dependence of $\tau_{Q^*}^{-1}$ was consistent with the prediction of Schmid and Schön (eq. 5.2), with (Maki 1969, De Gennes 1966, Tinkham 1975)

$$\tau_{\mathrm{s}}^{-1} = \frac{\Delta(0,0)H^2}{hH_{\mathrm{c}}^2(0)}. \tag{7.3}$$

Here, $A(0,0)$ is the energy gap at zero temperature in zero magnetic field, and $H_c(0)$ is the parallel critical field at zero temperature. This work was the first experimental demonstration of the essential correctness of the SS theory of charge imbalance relaxation in a magnetic field.

More recently, Aponte and Tinkham (1983) made careful measurements on Sn microbridges with a series of probes, and found values of i_{Q^*} with the expected temperature dependence. Their plot of λ_{Q^*} versus $1 - T/T_{\mathrm{c}}$ is reproduced in fig. 20, and we see that the best fit to the slope is -0.28 . However, given the size of th error bars, this slope is not significantly different from $-\frac{1}{4}$. From the measured values of λ_{Q^*} they deduce

$$\tau_{Q^*} = (1.0 \pm 0.2) \times 10^{-10}\Delta(0)/\Delta(T)\mathrm{s}.$$

$\left[\tau_{Q^*}^{-1} = (5.7 \pm 1.1) \times 10^9 A(T)/k_{\mathrm{B}}T_{\mathrm{c}}\mathrm{s}^1\right.$, in excellent agreement with the results of Clarke and Paterson (1974) and Dolan and Jackel (1977).]

Klapwijk and Mooij (1976) studied PSCs in Al films, and also found that the characteristic length obtained from IV characteristics was independent of

temperature. However, recently Stuivinga et al. (1981, 1983a) measured the charge relaxation length directly using a series of probes, and found that λ_{Q^*} scaled rather accurately as $(1 - T/T_{\mathrm{c}})^{-1/4}$. They deduced a value for τ_{E} of 4 ns (using $\rho l = 4 \times 10^{-16}\Omega\ \mathrm{m}^2$). This value is about a factor of 3 smaller than that obtained by Chi and Clarke (section 4). Stuivinga et al. point out that the low value of τ_{E} rules out the possibility that elastic scattering in the presence of the supercurrent could have enhanced $\tau_{Q^*}^{-1}$ significantly, as they had suggested earlier (Stuivinga et al. 1982). Furthermore, their films were clean enough (the maximum value of $R_\square$ was about 0.4Ω) that electron-electron collisions should not have contributed in a major way to the measured inelastic relaxation rate. Thus, the very low measured value of τ_{E} remains a puzzle, although, as we shall see in section 8.2, other measurements of λ_{Q^*} in Al produce comparable values.

Very little experimental work has been carried out on the region of the core, for the obvious reason that its dimensions are so small. However, Skocpol and Jackel (1981) were able to make probes sufficiently small to measure the variation of Δ across a PSC in Sn and found that Δ was greatly suppressed in the core, as one would expect.

The interaction between neighboring PSCs or, in general, any type of weak link, is a complicated subject. Lindelof and Bindslev Hansen (1981) have given an extensive review. The effect of one PSC on the critical current of another arises from a spatially decaying charge imbalance. Such effects were first observed by Jillie et al. (1977), but the only quantitative results appear to be those of Aponte and Tinkham (1983), whose work we now briefly describe. Two neighboring PSCs were fabricated on the same strip of Sn, each PSC being nucleated at a narrowed portion of the strip so that its position was accurately known. The $I - V$ characteristics of the PSC between probes C and L

(inset, fig. 21) were measured as a function of the current flowing through the PSC between probes L and D . For values of I(LD) less than the critical current of the PSC in segment LD, there is no effect on the PSC in segment CL . For larger values of I(LD), there is a change in the apparent critical current of CL, which is increased (decreased) when the applied currents in the two segments are flowing in the same (opposite) direction. However, the magnitude of the change in I_c(CL) is not the same for the two directions of current in LD: the decrease when the currents are opposed is larger than the increase when the currents are in the same direction. This asymmetry reflects the two contributions to the interaction: a heating term, which depresses I_c(CL) irrespective of the direction of I(LD), and a term due to the diffusion of a quasiparticle current from the source PSC to the detector PSC. The diffusive quasiparticle current produced by LD will be in the same direction as the applied current I (LD), so that in the regions outside the probes L and D , there will be an exponentially decaying supercurrent in the opposite direction to I (LD) that cancels the quasiparticle current at each point. When the applied currents are in the same direction, the supercurrent induced in the detector CL by the source LD will be in the opposite direction to I(CL), thus reducing the supercurrent that flows (below the critical current of CL) and enhancing the critical current. On the other hand, when the applied currents are opposed, the critical current of CL will be depressed. By decomposing the change in I_c(CL) into symmetric and antisymmetric components, Aponte and Tinkham were able to separate out the heating and quasiparticle contributions. The normal

current in the center ($x = 0$) of the PSC, $I_N(0) = I - \beta I_c$, was obtained by dividing eq. (7.1) by eq. (7.2); β was found to be 0.6 . The quasiparticle current in a uniform strip would be $I_N(0)\exp(-x/i_{Q^*})$; however, it was necessary to correct for the diffusion of the quasiparticles into the probes. With this correction, the value of $\hat{\lambda}_{Q^*}$ yielded $\tau_{Q^*} = (1.2 \pm 0.4) \times 10^{-10}\Delta(0)/\Delta(T)$S, in excellent agreement with the value obtained from the direct measurement of i_{Q^*} described earlier. Given the fact that the heating effects were also in accord with a simple model, this picture of the effect of one PSC on the critical current of another appears to describe the experimental facts very well.

The interaction of two PSCs when both are at nonzero voltage is more complicated. Under appropriate conditions, the voltages and hence the frequencies of two nearby PSCs can become locked together (Palmer and Mercereau 1977, Jillie et al. 1977, 1980, Lindelof and Bindslev Hansen 1977, 1981). It appears most likely that the interaction arises from charge imbalance waves, a subject that we will discuss briefly in section 9 .

In concluding this section, we remark that it is rather satisfying that the simple model for the PSC proposed by Skocpol, Beasley and Tinkham gives such a good account of most of the experimental observations. One needs only to consider the time-averaged electrochemical potentials, without delving into the details of the time dependent processes, to explain all the observed phenomena with the exception of the voltage locking effect.

46.8 8. Charge imbalance induced by a temperature gradient

Although the usual thermoelectric effects observed in normal metals, namely the Seebeck and Peltier effects and the Thompson heat, vanish in superconductors, there are observable effects due to temperature gradients in superconductors. Schön (1981a) and Van Harlingen (1982) have recently reviewed the various effects that can occur. In the present section we describe three experiments in which a temperature gradient induces a charge imbalance.

46.8.1 8.1. Thermoelectric generation of charge imbalance at a NS interface

Artemenko and Volkov (1976) were the first to point out that charge imbalance could be induced by a temperature gradient across an NS interface. The quasiparticle current in a superconductor, j_{N}, contains a contribution (σ/e) ($eE - \nabla\mu_0$), where σ is the electrical conductivity of the quasiparticles, E is the electric field, and μ_0 is the quasiparticle chemical potential (which depends only on the number of electrons and the temperature). Now since the pairs in a superconductor do not accelerate in the steady state, we have $e\boldsymbol{E} - \nabla\mu_{\mathrm{c}} = 0$, so that the transport equation can be written in the form

$$\begin{aligned} j_{\mathrm{N}} &= (\sigma/e)\nabla\left(\mu_{\mathrm{s}} - \mu_0\right) L_{\mathrm{T}}(-\nabla T) \\ &= \left[\sigma/2N(0)e^2\right](-\nabla Q^*) + L_{\mathrm{T}}(-\nabla T), \end{aligned} \tag{8.1}$$

where $L_{\mathrm{T}}(T)$ is the thermoelectric coefficient, and we have made use of eq. (2.11). In the relaxation time approximation, the steady state charge imbalance is determined from the Boltzmann equation (Pethick and Smith 1979a):

$$\frac{\mathrm{d}Q^*}{\mathrm{d}t} = -\nabla\cdot j_{\mathrm{N}} - Q^*/\tau_{Q^*} = 0. \tag{8.2}$$

We now consider the one-dimensional case of a NS interface, where the normal metal occupies the region $x < 0$ and the superconductor occupies the region $x \geqslant 0$, in the presence of a temperature gradient $\mathrm{d}T/\mathrm{d}x$. In addition to the constraint $j_{\mathrm{S}}(x) + j_{\mathrm{N}}(x) = 0$ that holds throughout the superconductor, the requirement that each current be continuous across the boundary $x = 0$ imposes the condition $j_{\mathrm{S}}(0) = j_{\mathrm{N}}(0) = 0$. As a result, $\nabla\cdot j_{\mathrm{N}} \neq 0$ near the interface, and there will be a corresponding charge imbalance. Combining eqs. (8.1) and (8.2), in one dimension we find

$$j_{\mathrm{N}} - \lambda_{Q^*}^2 \frac{\mathrm{d}^2 j_{\mathrm{N}}}{\mathrm{d}x^2} = L_{\mathrm{T}}\left(-\frac{\mathrm{d}T}{\mathrm{d}x}\right) \tag{8.3}$$

and

$$Q^* - \lambda_{Q^*}^2 \frac{\mathrm{d}^2 Q^*}{\mathrm{d}x^2} = \tau_{Q^*} L_{\mathrm{T}} \frac{\mathrm{d}^2 T}{\mathrm{d}x^2} \tag{8.4}$$

For the case $\mathrm{d}^2T/\mathrm{d}x^2 = 0$, using the constraint $j_{\mathrm{N}}(0) = 0$ we can solve eqs. (8.3) and (8.4) to find

$$Q^*(x) = 2N(0)e^2\lambda_{Q^*}S\left(\frac{\mathrm{d}T}{\mathrm{d}x}\right)\exp\left(-\frac{x}{\lambda_{Q^*}}\right) \tag{8.5}$$

and

$$j_{\mathrm{N}}(x) = L_{\mathrm{T}}\left(\frac{-\mathrm{d}T}{\mathrm{d}x}\right)\left[1 - \exp\left(\frac{-x}{\lambda_{Q^*}}\right)\right] \tag{8.6}$$

where $S \equiv L_{\mathrm{T}}/\sigma$ is the thermopower in the superconducting state. Figure 22 sketches j_N, j_S and Q^* versus x. From eq. (8.5), we see that the voltage at the NS interface due to charge imbalance is

$$\frac{Q^*(0)}{2N(0)e^2} = \frac{\lambda_{Q^*}S\ \mathrm{d}T}{\mathrm{d}x}$$

These effects may be investigated experimentally by measuring the potential difference across a SNS' sandwich in the presence of a heat current (we assume

that the normal layer is thick enough to exclude Josephson tunneling). The excess voltages due to the charge imbalance at the NS interface are $-i_{Q^*}SP/\ A\kappa$ and $-\lambda'_{Q^*}S'P/A\kappa'$, where $P = A\kappa(-\mathrm{d}T/\mathrm{d}x) = A\kappa'(-\mathrm{d}T/\mathrm{d}x)'$ is the heat current, κ and κ' are the thermal conductivities of S and S$'$, and A is the cross sectional area. Thus, the total voltage drop across a SNS' sandwich is

$$V = -\left(\frac{d_{\mathrm{N}}S_{\mathrm{N}}}{A\kappa_{\mathrm{N}}} + \frac{\lambda_{Q^*}S}{A\kappa} + \frac{\lambda'_{Q^*}S'}{A\kappa'}\right)P \tag{8.7}$$

where the first term is just the contribution of the normal layer of thickness d_N and thermal conductivity κ_{N}.

Van Harlingen (1981) measured these effects in Pb − Cu − PbBi and In − Al − Sn sandwiches. The sample was mounted in a vacuum can and the voltage generated across the two superconductors (at regions far from the interfaces) in the presence of a known heat current was measured with a null-balancing SQUID voltmeter. Figure 23a shows V/P versus T for a Pb − Cu − PbBi sample. Near the transition temperature, T_c, of the Pb there is a divergence in V/P that arises from the rapid increase in λ_{Q^*} in Pb : $\lambda_{Q^*}(T) = \lambda_{Q^*}(0)(1-t)^{-1/4}$. The transition temperature of the PbBi was considerably higher than that of the Pb , so that the voltage produced by the charge imbalance in the PbBi varied relatively slowly with temperature below 7.2 K . The dip in fig. 23a is characteristic of the thermopower of Cu , and arises from the Kondo effect in the presence of magnetic impurities. The solid line is a fit to the data well below T_{c}, extrapolated to T_{c}. The difference between this line and the data near T_{c} represents the voltage, V_{s}, due to the charge imbalance in the Pb , and is plotted versus $(1-t)^{-1/4}$ in fig. 23b. We see that near $T_{\mathrm{c}}V_{\mathrm{c}}/P$ is proportional to $(1-t)^{-1/4}$, as expected. Van Harlingen estimated S from the slope of this line, $-\lambda_{Q^*}(0)S/A\kappa$, using a value for κ obtained from the measured electrical conductivity and the Wiedemann-Franz law, and the value $\lambda_{Q^*}(0) = 2.7\mu$ m

obtained from the measured electrical resistance of the SNS' junction. His value of $S = -7.8 \times 10^{-7}\mathrm{VK}^{-1}$ near T_{c} compares with the value of $-2.2\times\ 10^{-7}\mathrm{VK}^{-1}$ obtained by Christian et al. (1958) on Pb in the normal state. Given the strong dependence of S on the impurity content of the material, this discrepancy is by no means unreasonable; unfortunately, it was not possible to measure S independently in the material used in the experiment.

Van Harlingen found comparable results on In − Al − Sn sandwiches near the In transition temperature. Subsequently, Battersby and Waldram (1984) studied similar effects in Pb − Cu − Pb and Pb − Cu − PbBi sandwiches, and obtained excellent agreement between the value of the thermopower in Pb and that measured directly on a bulk sample prepared in a similar way.

46.8.2 8.2. Spatial dependence of charge imbalance induced by a temperature gradient near a superconducting boundary

Mamin et al. (1983, 1984) performed experiments designed to measure the spatial decay of the charge imbalance [eq. (8.5)] near the end of an Al film in the presence of a temperature gradient. Their sample configuration is shown in fig. 24. The Al film, deposited on a glass substrate, was 500μ m wide and typically 300 nm thick. A series of probes was fabricated on top of the Al film with a photolithographic lift-off procedure. Each probe consisted of 800 nm of Cu(3wt%Al) followed by about 5 nm of Fe and 200 nm of Pb(5wt%In). The Fe layer prevented Josephson tunneling between the Al and the PbIn. Near the end of the Al film, the probes were 2μ m wide with a 6μ m separation between centers. The voltage developed across each junction was measured with a null-balancing SQUID voltmeter relative to a wide probe far from the end of the film; the voltmeter was connected to each probe in turn by means of a mechanical

superconducting switch operated from outside the cryostat. The normal state thermopower was measured in another part of the Al film, again by means of the SQUID voltmeter. The substrate was mounted in a vacuum can, and a heater and thermometers were attached to allow a known temperature gradient to be established.

The value of $\lambda_{Q^*}(T)$ was determined by injecting current through one probe

and measuring the voltage at nearby probes. In all cases, the observed voltage was accurately exponential over as much as five decades: a representative example is shown in fig. 25. The measured value of $\lambda_{Q^*}(T)$ is plotted versus $(1 - T/T_c)$ in fig. 26: The slope is -0.22 ± 0.03. The value of $\tau_{Q^*} = 2N(0)\ e^2\lambda_{Q^*}^2/\sigma$ estimated from the measured values of λ_{Q^*} and σ and with $N(0) = 1.74 \times 10^{28}\mathrm{cV}^{-1}\ \mathrm{m}^{-1}$ (Gschneidner 1964) yields $\tau_\mathrm{E} = 1.6 \pm 0.2$ ns, assuming that only electron- phonon scattering contributes to the charge relaxation near T_c as is implied by the fact that $\lambda_{Q^*} \propto (1 - T/T_c)^{-14}$. Two other samples yielded $\tau_\mathrm{E} = 2.7 \pm 0.5$ ns and 2.3 ± 0.4 ns. The average value of τ_E, about 2 ns , is about a factor of 6 smaller than that obtained by Chi and Clarke (section 4). All three samples were sufficiently clean that charge relaxation due to electron-electron scattering should have been quite negligible. To eliminate the possibility that the Al films in the decay measurements differed in some way from those used in the CC experiments. Mamin et al. fabricated two samples in which the Al was deposited simultaneously, one in the configuration of fig. 24 and one in the configuration of fig. 5. The inferred values of τ_E were about 2 and 11 ns , respectively, in agreement with previous results. We note that the value of τ_E

obtained from the decay measurements is only about a factor of two smaller than that obtained by Stuivinga et al. (1981, 1983*a*) from measurements on phase-slip centers in Al (section 7). At the time of writing, there is no explanation for the discrepancy in the values of τ_E observed in the tunnel injection and spatial decay measurements. However, we emphasize that this discrepancy concerning τ_E does not enter into the discussion of thermoelectric effects, for which only the measured value of $\lambda_{Q^*}(T)$ is required.

We now turn to a discussion of the charge imbalance generated by a temperature gradient. Figure 27 shows $V(x)$ versus distance for a sample at four different temperatures. In contrast to the behavior observed for the current injection experiments, the data show significant downward curvature near the end of the Al film. A straight-line fit to the data points farthest from the end of the Al film yields a decay length of $24 \pm 4\mu$ m that is independent of temperature, and always longer than the temperature-dependent length measured in the current injection case. Thus, one concludes that the charge imbalance decay length measured in the thermoelectric experiment is not equal to λ_{Q^*}, presumably because the temperature gradient is non-uniform over a region near the end of the Al film where the heat flow enters the film from the substrate. In a quasi-one-dimensional model, it is easy to show that the gradient is given by

$$\frac{\mathrm{d}T}{\mathrm{d}x} = \left.\frac{\mathrm{d}T}{\mathrm{d}x}\right|_\infty \left[1 - \exp\left(-\frac{x}{\lambda_\mathrm{T}}\right)\right], \tag{8.8}$$

where $\mathrm{d}T/\ \mathrm{d}x|_\infty$ is the gradient far from the end of the film and $\lambda_\mathrm{T} = (yd/y_\mathrm{K})^{1/2}$. Here, g and d are the thermal conductivity and thickness of the Al film, and y_k is the Kapitza resistance of a unit area between the film and substrate. If we assume $d = 300$ nm, $g = 6\mathrm{Wm}^{-1}\ \mathrm{K}^{-1}$, and $y_\mathrm{K} = 5 \times 10^3\mathrm{Wm}^{-2}\ \mathrm{K}^{-1}$ at T_c (Kaplan 1979), we find $\lambda_\mathrm{T} \approx 20\mu$ m. When eq. (8.8) is inserted in eqs. (8.3) and

(8.4), one finds

$$V(x) = \frac{i_{Q^*}^2}{i_{Q^*}^2 - i_T^2}\left(i_{Q^*}\mathrm{e}^{-x/i_{Q^*} - i_T\mathrm{e}^{-x/i_T}}\right) S\,\tfrac{\mathrm{d}T}{\mathrm{d}x}\big|_x. \tag{8.9}$$

Figure 27 shows the fit of the data to eq. (8.9) using values of $i_{2,2}(T)$ from the current injection experiments, and regarding S and i_1 as adjustable parameters. The fits are generally

satisfactory. The inferred value of S near T_c was $(-2.7\pm()\,.6)/10$ " VK ^{1}, compared with a value measured in the normal state of $1-1.(15-10.(16),10^3$ VK ^{1}. Two other samples yielded values of $(-1.4\pm 0.3)\times 10^{-8}\mathrm{VK}^{-1}$ and $(-1.0\pm 0.3)\times 10^{-8}\mathrm{VK}^{-1}$ compared with normal state values of $(-()\,.94\pm(0.(0),10,1)^{-8}$ VK ^{1} and $(-1.12\pm 0.(06),$ 10 "VK 1: respectively. ()verall, the experiments provide a strong experimental verification of the theory of thermoelectric effects in superconductors. and. In particular, demonstrate that there is no discontinuity in the value of L_{T} at T_{c}.

46.8.3 8.3. Supercurrent-induced charge imbalance in the presence of a temperature gradient

A quite different kind of effect due to a temperature gradient was first pointed out by Pethick and Smith (1979b) who predicted that a charge imbalance proportionai to $\boldsymbol{I}\cdot\nabla T$ should exist in the presence of a supercurrent $\boldsymbol{I}$. Clarke ct al. (1979b) observed this effect in Sn films, and established that the measured voltage was proportional to $\boldsymbol{I}\cdot\nabla T$, as predicted. but was two to three orders of magnitude smaller than the predicted value. This discrepancy arose because the theory assumed that the elastic scattering rate was negligible compared with the inclastic rate, whereas the reverse was true in the experiment. Three further theories then appeared (Schmid and Schön 1979. Clarke and Tinkham 198(), Beyer Nielsen et al. 198() in attempts to account quantitatively for the temperature dependence and magnitude of the data. More recently. Heidel and Garland (1981) soberved similar effects in Al films, while Fjordboge et al. (1981) have given a more detailed account of their work on Sn .

The experimental configuration used by Fjordhoge et al. is shown in fig. 28. The Sn for $\mathrm{Sn}-3\mathrm{ut}''$, InI film was typically $3(x)$nm thick and 0.1 mm wide in the narrow region. After the Sn was oxidized. three Cu $\left(+3''',^{\mathrm{Al}}\right)$ disks $\sim 1\mu$ m thick were deposited. followed by three Ph strips. In a given experiment. one of the three $\mathrm{Sn}\,\mathrm{Sn}()x$ (u tunnel junctions was used to measure the quasiparticle potential relative to the pair potential with a SQUID voltmeter. The substrate was mounted in a vacuum can. with a heater at each end. and thermometers were attached to the reverse side. In figs. 29 and 3() we plot measured values of V versus I for five values of ∇T. and V versus ∇T for 10 values of I for a representative sample. The voltage is clearly proportional to $I\cdot\nabla T$. Figure 31 shows $Vy.$, (f) $I\nabla T$ versus t for the same sample: to within the experimental

resolution, all 8 samples showed the same behavior. In fig. 32, we plot $Vg_{\mathrm{NS}}\,(0)\;T/I\nabla T$ versus $(\,1-t\,)$ for the same sample; the divergence at temperatures above about $0.8T_{\mathrm{c}}$ is close to $(1-t)^{-1}$. It was found that the magnitude of the quantity $Vg_{\mathrm{NS}}(0)T(1-t)A/I\nabla T$, where A is the cross-sectional area of the films,

was very nearly the same for relatively clean Sn films in which the mean free path was limited by boundary scattering, and for SnIn films in which the mean free path was limited by impurity scattering to about 60 nm .

Figure 33 indicates the physical origin of the effect. In fig. 33a there is a thermal gradient, but no applied supercurrent. Quasiparticles moving from the left are at an effective temperature $T-\delta T$, while those moving from the right are at $T+\delta T$, where T is the local temperature. Thus, there is an imbalance in the populations of the $k>k_{\mathrm{F}}$ and $k<k_{\mathrm{F}}$ branches on the right-hand side of the Fermi surface, but an equal and opposite imbalance on the left-hand side: as a result, $Q^*=0$. If we now impose a superfluid velocity v_S (fig. 33b) the excitation energies are raised and lowered on opposite sides of the Fermi surface by an amount $p_k\cdot v_s$ (see section 5.2). The induced asymmetry ensures that the population imbalances on opposite sides of the Fermi surface no longer cancel, and the resulting charge imbalance is the origin of the observed voltage. We now briefly review the theoretical situation.

The result of Schmid and Schön (1979) can be written in the form

$$V = \frac{p_{\mathrm{F}} l}{6 e g_{\mathrm{NS}}(0)} \frac{v_{\mathrm{s}} \cdot \nabla T}{T} \frac{\Delta}{k_{\mathrm{B}} T (1-Z) \cosh^2 (\Delta / 2 k_{\mathrm{B}} T)} \ln (8 \Delta \tau_{\mathrm{E}} / \hbar) . \tag{8.10}$$

Here, p_{F} is the Fermi momentum, and Z is given by eq. (2.24). Clarke and Tinkham (1980) used a simple kinetic approach to obtain

$$V = \frac{p_{\mathrm{F}} l}{6 e g_{\mathrm{NS}}(0)} \frac{v_{\mathrm{s}} \cdot \nabla T}{T} \frac{\Delta}{k_{\mathrm{B}} T (1-Z) \cosh^2 (\Delta / 2 k_{\mathrm{B}} T)} . \tag{8.11}$$

Equation (8.10) exceeds eq. (8.11) by a factor of $\ln (8 \Delta \tau_E / \hbar)$, which varies relatively slowly with temperature over the experimental range studied, with an average value of about 6. Apart from this factor, the Schmid-Schön and Clarke-Tinkham approaches yield the same result. One can convert v_{s} into a current density by writing

$$v_{\mathrm{s}} = j_{\mathrm{s}} / n_{\mathrm{s}} e = \mu_0 j_{\mathrm{s}} \lambda^2(T) e / m$$

and using the result $i^2(T) = i_L^2(0) \left(1 + \check{\zeta}_0, l\right) (1 - t^4)^{-1}$ (Tinkham 1975) to find

$$c_{\mathrm{s}} = \frac{\mu_0 j_{\mathrm{c}} e i_1^2(0) \left(1 + \check{\zeta}_0 / l\right)}{m (1 - t^4)} . \tag{8.12}$$

Near $T_{\mathrm{c}}, 1 - Z \to \pi \Delta k_{\mathrm{B}} T, \cosh^2 (\Delta, 2 k_{\mathrm{B}} T) \to 1.$ and $1 - t^4 \to 4(1-t)$, so that the

temperature dependence of the voltage at fixed current is dominated by the temperature dependence of $v_s, (1-t)^{-1}$, in good agreement with the experimental results. The curves calculated for the two theories with $v_{\mathrm{F}} = 6.5 \times 10^5 \ \mathrm{ms}^{-1}, \lambda_{\mathrm{L}}(0) = 5 \times 10^{-8}$ m, and $\xi_0 = 2.3 \times 10^{-7}$ m are plotted in fig. 32. The curves were fitted at $T/T_{\mathrm{c}} = 0.99$ by multiplying eq. (8.11) by 2.4 , and eq. (8.10) by 0.4 . A detailed critique of these two theories can be found in the paper by Pethick and Smith (1981b).

Beyer Nielsen et al. (1980) and, subsequently, Pethick and Smith (1981a, b) have solved the Boltzmann equation in the clean limit for the case where $\tau_{\mathrm{E}}^{-1} \to 0$ and for an isotropic energy gap to obtain

$$V = \frac{p_{\mathrm{F}} l}{e g_{\mathrm{NS}}(0)} \frac{v_{\mathrm{s}} \cdot \nabla T}{T} \left[\frac{1.93 \Delta / k_{\mathrm{B}} T}{\cosh^2 (\Delta / 2 k_{\mathrm{B}} T)} + \frac{8/15}{\exp (\Delta / k_{\mathrm{B}} T) + 1} \right] . \tag{8.13}$$

Equation (8.13) is also plotted in fig. 32, and evidently has a different temperature dependence than the experimental data. In eq. (8.13), the first term in square brackets dominates the second for $T/T_{\mathrm{c}} \leqslant 0.998$, that is, throughout the experimentally realizable range. The first term arises from quasiparticles in the energy range $\Delta - p_{\mathrm{F}} v_{\mathrm{s}}$ to $\Delta + p_{\mathrm{F}} v_{\mathrm{s}}$, in which the available phase space for charge relaxation via elastic scattering is reduced. This effect is particularly pronounced for quasiparticles with energies near $\Delta - p_{\mathrm{F}} v_{\mathrm{s}}$. As a result, the charge imbalance in this "pocket" takes a relatively long time to relax compared with that at energies above $A + p_{\mathrm{F}} v_{\mathrm{s}}$, which is accounted for by
the second term in square brackets in eq. (8.13). However, it appears that, in practice, the energy levels in the pocket are sufficiently smeared out that the calculated enhancement of the relaxation time is not very significant. There are at least three mechanisms by which this smearing may occur: inelastic scattering, gap anisotropy, and impurity scattering (Schön 1981a, 1981b, Beyer Nielsen et al. 1982). When the contribution of the pocket is eliminated, the second term in square brackets becomes dominant but, unfortunately, is valid only very close to T_{c}. Beyer Nielsen et al. (1982) have used a Green's function approach to calculate the voltage in the dirty limit in the presence of electronphonon scattering, and find

$$V = \frac{2 p_{\mathrm{F}} l}{3 e g_{\mathrm{NS}}(0)} \frac{v_{\mathrm{s}} \cdot \nabla T}{T} \frac{F (\Delta / k_{\mathrm{B}} T)}{1 + \lambda} , \tag{8.14}$$

where λ is the electron-phonon interaction parameter, and F is a specified dimensionless function. Near T_c, eq. (8.14) agrees with the Schmid-Schön result, eq. (8.10). Figure 34 shows eq. (8.14) plotted versus T/T_c together with the data obtained by Fjordbge et al. (1981) for their dirtiest sample. The theoretical curve has been fitted at one temperature by scaling with a factor of 0.43 . Given that effects of anisotropy are not included in the theory, the fit is satisfactory.

As noted above, Heidel and Garland (1981) measured similar effects in Al.

The spread in their values of $V g_{\mathrm{NS}} T/I \cdot \nabla T$ from sample to sample was an order of magnitude, substantially greater than for the work on Sn . This variation perhaps reflects variations in the homogeneity of the Al films. The temperature dependence of the data fitted the SS and CT theories (eqs. 8.10 and 8.11), but not the prediction of eq. (8.13), which, of course, is valid only in the clean limit, while the samples were definitely in the dirty limit. Heidel and Garland also examined two samples with 520 and 1660 ppm Er, respectively, and found no significant difference in the magnitude of the effect compared with the samples with no magnetic impurities. This result is consistent with the idea (Clarke and Tinkham 1980) that the charge imbalance generation is a volume process in which the generation and relaxation processes cannot be independently varied. Thus, the value of Q^* is imposed by the temperature gradient, the magnitude depending only on the transport mean free path which limits the distance over which the gradient is effective in producing a nonequilibrium population. Thus, times such as τ_{E} and τ_{Q^*} play very little role in determining the magnitude of Q^*. This situation is in complete contrast to the other charge imbalance experiments described earlier, such as tunnel injection, the NS interface, phase-slip centers or generation by a temperature gradient in the absence of an applied supercurrent, where the generation occurs at a welldefined spatial homogeneity, and the generation and relaxation of Q^* are quite distinct processes. Thus, as we have seen in section 5.1, the effect of a small amount of Er in Al dramatically lowers Q^* in tunnel injection experiments.

46.9 9. Dynamical charge imbalance

All of the experiments and the associated theories described above are concerned with a steady state situation, in which the charge imbalance is maintained at a constant value by a suitable generation process. However, it is appropriate to remark briefly on dynamical charge imbalance effects where, for example, one considers the time decay of a pulse of charge imbalance. Under such circumstances, Kadin et al. (1980) show that the appropriate decay rate is τ_{E}^{-1}, rather than $\tau_{Q^*}^{-1}$. The physical reason for the difference is as follows. In a steady state measurement, the charge imbalance relaxes by means of changes in the distribution function alone, the charge, q_k, associated with a given k value remaining fixed. This leads to the well-established result $\tau_{Q^*}^{-1} = (\pi\Delta/4k_{\mathrm{B}}T)\,\tau_{\mathrm{E}}^{-1}$. On the other hand, when a pulse of charge imbalance decays with time, in addition to the relaxation of the distribution there is an accompanying change in q_h as the nonequilibrium value of μ_s decays towards its equilibrium value. Kadin et al. show that in the limit $\Delta(T)/k_{\mathrm{B}}T \ll 1$, this additional process modifies the charge imbalance relaxation rate to τ_{E}^{-1}.

Kadin et al. show that, in general, Q^* should obey a charge imbalance relaxation equation

$$D\tau_{Q^*}\nabla^2 Q^* = \tau_0\tau_E\ddot{Q}^* + (\tau_0 + \tau_E)\,\dot{Q}^* + Q^* \tag{9.1}$$

where $\tau_0 = \tau_1\,(n, n_1)$ is the response time of the supercurrent to a change in the total current (n and n, are the densities of electrons and superfluid electrons in the normal and superconducting states). In the low frequency limit, eq. (9.1) immediately yields the static

decay length ($D\tau_{Q^*})^{1/2}$ (eq. 6.1). However, in general, eq. (9.1) describes damped, dispersive charge imbalance waves with the dispersion relation

$$-\Lambda^2 k^2 = (1 + i\omega\tau_0)(1 + i\omega\tau_E), \tag{9.2}$$

where A is a characteristic length, for waves of the form $\exp[\mathrm{i}(\omega t - k \cdot r)]$.
One might expect charge imbalance waves to be generated by PSCs, so that the Josephson oscillations of two neighboring PSCs could be locked together by this mechanism, resulting in voltage locking. As mentioned in section 7, such effects have been observed, and although the evidence is perhaps not overwhelming, it seems very plausible that charge imbalance waves are responsible for this interaction.

The concept of dynamical charge imbalance also occurs in a quite different situation, namely charge imbalance fluctuations (Lemberger 1981). Whenever a charge imbalance is generated, there is an associated electrical resistance: it is, of course, this resistance that is measured directly in the measurements on the NS interface described in section 6. Lemberger (1981) has pointed out that in thermal equilibrium there must be fluctuations in Q^* related to this resistance via Nyquist's theorem (Nyquist 1928). We briefly describe this analysis.

Lemberger calculates the mean square fluctuation $\left\langle(\delta\mu_\varsigma)^2\right\rangle$ in the limit $A(T)/k_\mathrm{B}T \ll 1$, neglecting gradients in $\delta\mu_\varsigma$. Thus, the result can be applied only to situations in which $\delta\mu_s$ is spatially uniform, for example, charge imbalance injected into a superconducting film of thickness $\ll^i{}_{,Q^*}$ via a tunnel barrier of area $\geqslant i^2_{2^*}$. By combining the theory of charge imbalance with equilibrium thermodynamics, he obtains the result that in thermal equilibrium (no applied current)

$$\left\langle(\delta\mu_\mathrm{s})^2\right\rangle = 2\,(k_\mathrm{B}T)^2/\pi N(0)\Omega\Delta(T), \quad [\Delta(T) \ll k_\mathrm{B}T]. \tag{9.3}$$

For tunneling injection into a superconductor, the resistance $R_{Q^*} = V_\mathrm{d}I_\mathrm{i}$ in the limit $\Delta(T), k_\mathrm{B}T \ll 1$ is easily obtained from eq. (2.22) with $F^* = 1, y_\mathrm{ss}(0) = 1$ and $\tau_{Q^*} = 4k_\mathrm{B}T\tau_\mathrm{E}/\pi\Delta(T)$:

$$R_{Q^*} = 4k_\mathrm{B}T\tau_\mathrm{E}/2\pi N(0)e^2\Omega\Delta(T) \tag{9.4}$$

This resistance is in series with the tunneling resistance of the barrier. It was observed by Clarke and Paterson (1974) and Lemberger and Clarke (1981a) as a dip in the conductance of the detector junctions near T_c, and measured in detail by Lemberger (1981).

To apply the Niquist result, one assumes that the RC time constant of the injection junction is much less than E_t, that the resistance of the injection junction is large compared with $R_{e'^*}$. and that σ_0 is small compared with τ_E. Under these conditions, the Nyquist theorem yields

$$\left\langle(\delta\mu_\mathrm{s}e)^2\right\rangle = 4k_\mathrm{B}TR_{Q^*}B. \tag{9.5}$$

where B is the bandwidth for the fluctuations. Substituting cys. (9.3) and (9. 4) in eq. (9.5), one tinds $B = 1 + \tau_2$. Thus, the relaxation time for eharge imbalance fluetuations. as for other dynamical charge imbalance relaxation processes is τ_E

Apart from the witage locking of neighboring PSCs, there appears to be no experimental work on dynamical charge imbalance processes. This is obviously an area in which more research is needed.

46.10 10. Concluding summary

In this chapter. We have resiewed the essential deas of charge imbalance in superconductors, and discussed the various physical situations in which charge imbalance plays a role. We first considered the situation in which charge imbalance is injected via a tunnel junction at a rate

$\underset{\sim}{Y^*} = F^* l$, x^2 and then relaxes in one or more of seleral possible ways to produce a steady state voltage $I = ()^*2$ Ville'ks. (t) that is measured by a tunneling contact to a normal metal. Such measurements yeld the charge imbalance relatation rate. ... In the absence of gap anisotropt or magnetic scattering and the the limit $11T \wedge_B T \to 0$, the charge relatation is due solely to electron-phonon processes and occurs at a rate $|F^* - |^*| = 1 - 1|T|$ the V^* : meglecting electron electron scattering for the moment).

Under appropriate conditions. elastic scattering also relates charge imbalance. In the presence of nomero sap ansofropy the coherence factor mu-a be for branch crossing is no longer zero and clastic scattering produces charge relaxation at a rate that scales approximately as $A^+(T)$ near $T_{\cdot}$ This effect has been observed m both Sn and Al . The mean free path in the Al films us as reduced by sugen depenge a provess that also enhanced $l_{..}$in some cases substantally. The films with significantly enhanced transition temperatures exhboted much faster inclaster relaxation rates than could be explaned me merms of electron phonon scatterng: tecent work has explained these rates rather satustactorily an terms of electron electron scattering. The presence of magnetic mpurtase can greatly enhance - : hecause the coherence factor for branch crossing late + :!^{2} approaches units at low energies. Expermental data abtamed on Al. doped what Fr. are meacellent agreement

with the Schmid–Schön prediction $(F^* \tau_{Q^*}^{-1}) = [\pi A(T)/4k_B T_c \tau_E](1 + 2\tau_E/\tau_S)^{1/2}$ over a wherenge of balues of $a_i a_i$ for $1 h_k I_i \leqslant 0.8$. Charge relaxation can

also be induced by a supercurrent: the current induces an asymmetry across the Fermi surface that allows branch relaxation via elastic scattering. The results from experiments on dirty Al are again in excellent agreement with the Schmid-Schön result. However, computer solutions of the Boltzmann equation in the last two cases produced results that were strikingly different from the experimental data. It now seems probable that electron-electron scattering was non-negligible in all of these samples, and it is possible that the inclusion of this process in the Boltzmann equation would remove the discrepancy.

The next three sections were concerned with various experiments in which charge imbalance plays a central role. The resistance of the NS interface near T_c can be explained in terms of charge imbalance injected into the superconductor where it decays with a characteristic length λ_{Q^*}. A simple modification of the tunneling theory for the injection of Q^* yields a result that is in excellent agreement with experiments on Pb, Sn, and In , and that produces very reasonable values of τ_{E}. The voltage steps and dynamic resistance of the $I-V$ characteristics of microstrips containing phase slip centers is nicely explained in terms of the charge imbalance injected from the normal core of each PSC into the neighboring superconducting regions. The interaction of nearby PSCs at zero voltage can also be explained in terms of this static model, but the frequency locking that occurs at nonzero voltages probably involves an interaction via charge imbalance waves.

Three experiments were described in which a temperature gradient was involved. The first involved the measurement of the charge imbalance induced in a superconductor near a normal contact in the presence of a temperature gradient across the NS interface. The second was concerned with measurements of the spatial decay of the charge imbalance near the end of an Al film along which a temperature gradient was maintained. Both of these experiments involve the thermopower S of the superconductor. In each case, the agreement of theory and experiment was very satisfactory, and the measured values of S were in reasonable accord with measured normal state values. The third experiment involved the charge imbalance generated by a temperature gradient in the presence of an applied supercurrent. The agreement between the results and theoretical predictions is now in a very satisfactory state.

The last section briefly described dynamical charge imbalance effects, in particular charge imbalance waves and charge imbalance fluctuations. This is an area where there seems to be considerable scope for further experimental work.

Values of the electron-phonon scattering rate, τ_{E}^{-1}, have been extracted from experiments on tunnel injection, the NS interface, and PSCs, in which one measures the steady state value of Q^* induced by a known injection rate into a known volume, and also from measurements of i_{Q^*} in which one
measures the spatial decay of Q^* along a superconducting strip. The values of τ_E obtained for each of the metals Pb.Sn, and In are generally consistent and in good agreement with predictions based on x^2F. with an overall spread of perhaps a factor of two in each case. On the other hand, the values of τ_{F} obtained for relatively clean Al films are very puzzling. Measurements from tunneling injection and detection yield about 12ns, a value that is in good agreement with values obtained from magnetoconductance data. This value is 3 or 4 times lower than the calculated value. Furthermore, estimates of τ_F obtained from i_{Q^*} measured near PSCs or near a tunnel injection junction yield values of τ_{F} of 4 and 2 ns. respectively. Although it is now clear that electron-electron scattering can play an important or even dominant role in relatively dirty Al , it is most improbable that it plays a significant role in clean films. Thus, at present, there seems to be no explanation of these very large discrepancies in the values of τ_E.

It is noteworthy that Al was the material used in the measurements of ${\tau_{\ell^*}}^1$ in the presence of magnetic impurities or of a supercurrent. Which are in disagreement with the predictions of the Boltzmann equation. In this case it is possible, but by no means certain. that electron electron scattering may explain the discrepancy. Thus, although Al has been a widely used material in experiments on charge imbalance it is clearly a "bad actor", at least partly because the electron phonon scattering tate is so slow that other charge relaxation mechanisms are of much greater relative importance than they are in Pb, Sn, and In. Obviously, much remains to be done to resolve these issues.

Despite the problems associated with Al. however. experiments on charge imbalance are generally in remarkably good agreement with theoretical predictions, illustrating in a rather strihong way some of the fundamental subtleties of superconductivity, such as quasiparticle charge coherence factors, and a variety of pair breaking mechanisms.

Fig. 15. Circles are the measured total resistance versus $T / T_{\mathrm{c }}$ for the SNS junctions listed in table 3. The solid lines above $0.9T_{\mathrm{c}}$ [$0.96T_{\mathrm{c}}$ in (b)] are the fit to eq. (6.5), while the dashed lines show the extrapolation of the theory to lower temperatures. (Hsiang and Clarke 1980.)}

Fig. 16. Total measured resistance versus $Z(T)\left(k_{\mathrm{B
} T / \Delta\right)^{1 / 2}$ for the samples shown in fig. 15. The solid lines are a least-squares fit to the data for $T > 0.9T_{\mathrm{c}}$ [$0.96T_{\mathrm{c}}$ for (b)]. (Hsiang and Clarke 1980.)}

Fig. 17. I versus V for the whole length of the Sn bridge, $140 \mu \mathrm{~m
$ long, 4μ m wide, and 0.1μ m thick, shown inset. (Skocpol et al. 1974.)}

Fig. 18. Schematic representation of phase-slip center.

Fig. 19. Spatial variation of $V_{S
$ and V_N across a PSC in a SN microbridge at the position marked with an arrow. (Dolan and Jackel 1977.)}

Fig. 20. Temperature dependence of the quasiparticle diffusion length in Sn PSC. The best fit (solid line) for $n = 0.28$ is compared to the expected $\left(1-T / T_{c
\right)^{-1 / 4}$ divergence (dashed line). (Aponte and Tinkham 1983.)}

Fig. 21. I V characteristics of segment CL taken for different fixed values of the current through the segment LD. Plus and minus signs correspond to the cases when the currents in the two segments flow in the same or opposite directions, respectively. (Aponte and Tinkham 1983.)

Fig. 22. (a) Normal current ($j_{N
$) and supercurrent (j_S), and (b) Q^* versus distance x near the end of a superconductor in the presence of a temperature gradient $\mathrm{d}T/\mathrm{d}x$. (Mamin et al. 1984.)}

Fig. 23. (a) Plot of the measured thermoelectric voltage per unit power V/P versus T for a $\mathrm{Pb
-$ Cu – PbBi sample. The solid curve is characteristic of the normal thermopower of Cu . (b) Fit of the charge imbalance contribution V_{s}/P to the temperature dependence of $i_{Q^*}, (1-t)^{-1/4}$. (Van Harlingen 1981.)}

Fig. 24. Sample configuration for measurement of $Q^{*
$ induced by temperature gradient. Note different length scales. (Mamin et al. 1984.)}

Fig. 25. Measuring $\lambda_{Q^{*
}$ from current injection: voltage, V, normalized to injected current, I, for a series of probes. (Mamin et al. 1984.)}

Fig. 26. Temperature dependence of $\lambda_{Q^{*
}$ from current injection: λ_{Q^*} versus $(1 - T/T_c)$. (Mamin et al. 1984.)}

dT/dx = 22.6 Km⁻¹
T = 0.991 Tc
-5
-1
-0.3
dT/dx = 28.1 Km⁻¹
T = 0.978 Tc
-5
-1
-0.3
V(x) (pV)
dT/dx = 23.6 Km⁻¹
T = 0.947 Tc
-3
-1
-0.5
-0.3
dT/dx = 19.0 Km⁻¹
T = 0.913 Tc
-2

Fig. 28. Sample configuration for $\boldsymbol{I \cdot \nabla T$ experiment. (Fjordbge et al. 1981.)}

Fig. 29. V versus I for five values of ∇T. (Fjordbge et al. 1981.)

Fig. 30. V versus ∇T for ten values of I. At each value of ∇T, the voltage is defined to be zero at $I = 0$. (Fjordbge et al. 1981.)

Fig. 31. $V g_{\text {NS }}(0) / I \nabla T$ versus reduced temperature, t. (Fjordbge et al. 1981.)}

Fig. 32. $V g_{\mathrm{NS }}(0) / I \nabla T$ versus $(1-t)$. The three theoretical formulas have been fitted to the experimental data by scaling them appropriately. (Fjordboge et al. 1981.)}

Fig. 33. Schematic representation of quasiparticle excitations in presence of (a) temperature gradient and (b) temperature gradient and applied supercurrent.

Fig. 34. Comparison of theory of Beyer Nielsen et al. (1982) with data from dirty tin sample of Fjordbge et al. (1981). (Beyer Nielsen et al. 1982.)

Part XIV
Experiments: Simulations of Nonequilibrium Superconductivity

47 A cavity-QED quantum simulator of dynamical phases of a BCS superconductor by Lewis-Swan, Barberena, Cline, Young, Thompson, Maria Rey

47.1 Theory

Abstract

We propose to simulate dynamical phases of a BCS superconductor using an ensemble of cold atoms trapped in an optical cavity. Effective Cooper pairs are encoded via internal states of the atoms and attractive interactions are realized via the exchange of virtual photons between atoms coupled to a common cavity mode. Control of the interaction strength combined with a tunable dispersion relation of the effective Cooper pairs allows exploration of the full dynamical phase diagram of the BCS model, as a function of system parameters and the prepared initial state. Our proposal paves the way for the study of non-equilibrium features of quantum magnetism and superconductivity by harnessing atom-light interactions in cold atomic gases.

Introduction: The development of a generic framework to understand the properties of non-equilibrium quantum states is a long-standing challenge in modern physics. Theoretical work [Sciolla and Biroli(2011), Heyl *et al.*(2013)Heyl, Polkovnikov, and Kehrein, Heyl(2018), Žunkovič *et al.*(2018)Žunkovič, Heyl, Knap, and Silva, Jäger *et al.*(2019)Jäger, Cooper, Holland, and Morigi, Halati *et al.*(2020)Halati, Sheikhan, Ritsch, and Kollath] combined with technical advances in the control and characterization of many-body physics in cold atom experiments [Jurcevic *et al.*(2017)Jurcevic, Shen, Hauke, Maier, Brydges, Hempel, Lanyon, Heyl, Blatt, and Roos, Zhang *et al.*(2017)Zhang, Pagano, Hess, Kyprianidis, Becker, Kaplan, Gorshkov, Gong, and Monroe, Muniz *et al.*(2020)Muniz, Barberena, Lewis-Swan, Young, Cline, Rey, and Thompson, Smale *et al.*(2019)Smale, He, Olsen, Jackson, Sharum, Trotzky, Marino, Rey, and Thywissen, Tian *et al.*(2020)Tian, Yang, Qiu, Liang, Yang, Xu, and Duan, Chu *et al.*(2020)Chu, Will, Arlt, Klempt, and Rey, Baumann *et al.*(2010)Baumann, Guerlin, Brennecke, and Esslinger, Klinder *et al.*(2015)Klinder, Keßler, Wolke, Mathey, and Hemmerich, Kroeze *et al.*(2018)Kroeze, Guo, Vaidya, Keeling, and Lev] has led to new developments in this direction, such as extending the concept of phase transitions to non-equilibrium situations. Specifically, dynamical phase transitions (DPTs) [Eckstein *et al.*(2009)Eckstein, Kollar, and Werner, Schiró and Fabrizio(2010), Sciolla and Biroli(2011), Gambassi and Calabrese(2011), Smacchia *et al.*(2015)Smacchia, Knap, Demler, and Silva, Kirton *et al.*(2019)Kirton, Roses, Keeling, and Dalla Torre] have been introduced to classify distinct regimes of dynamical behaviour that arise after a sudden quench of a control parameter in a closed system. DPTs are characterized by the existence of a time-averaged order parameter that demonstrates non-analytic behaviour at the boundary between dynamical phases.

A long standing example of such dynamical phases are those predicted to emerge from quenches of Bardeen–Cooper–Schrieffer (BCS) superconductors, which has been theoretically investigated in both the condensed matter[Volkov and Kogan(1974), Barankov

et al.(2004), Yuzbashyan *et al.*(2005a)Yuzbashyan, Kuznetsov, and Altshuler, Yuzbashyan *et al.*(2005b)Yuzbashyan, Altshuler, Kuznetsov, and Enolskii, Yuzbashyan *et al.*(2006)Yuzbashyan, Tsyplyatyev, and Altshuler, Barankov and Levitov(2006), Yuzbashyan *et al.*(2015)Yuzbashyan, Dzero, Gurarie, and Foster, Collado *et al.*(2018)Collado, Lorenzana, Usaj, and Balseiro, Ojeda Collado *et al.*(2019)Ojeda Collado, Usaj, Lorenzana, and Balseiro, Ojeda Collado *et al.*(2020)Ojeda Collado, Usaj, Lorenzana, and Balseiro] and high energy communities [Pehlivan *et al.*(2011)]. However, experimental progress towards observing these phases has been limited so far to transient dynamics on rapid time-scales in terahertz pump-probe experiments [Matsunaga *et al.*(2013)Matsunaga, Hamada, Makise, Uzawa, Terai, Wang, and Shimano, Matsunaga *et al.*(2014)Matsunaga, Tsuji, Fujita, Sugioka, Makise, Uzawa, Terai, Wang, Aoki, and Shimano]. Recent proposals to enhance pairing by coupling materials to cavities and adjustable external laser driving might facilitate probing the predicted BCS phases in solid state systems [Gao *et al.*(2020)Gao, Schlawin, Buzzi, Cavalleri, and Jaksch].

Here, motivated by developments studying dynamical phase transitions in state-of-the-art quantum simulators, we present a proposal to emulate the non-equilibrium dynamics of the BCS model of superconductivity with cavity-QED [Norcia *et al.*(2018)Norcia, Lewis-Swan, Cline, Zhu, Rey, and Thompson, Muniz *et al.*(2020)Muniz, Barberena, Lewis-Swan, Young, Cline, Rey, and Thompson, Davis *et al.*(2020)Davis, Periwal, Cooper, Bentsen, Evered, Van Kirk, and Schleier-Smith, Vaidya *et al.*(2018)Vaidya, Guo, Kroeze, Ballantine, Kollár, Keeling, and Lev, Baumann *et al.*(2010)Baumann, Guerlin, Brennecke, and Esslinger, Ritsch *et al.*(2013)Ritsch, Domokos, Brennecke, and Esslinger]. Our scheme leverages the tunability and control available in this platform to map out the dynamical phase diagram over a broad range of system parameters and initial states, demonstrating the power of cavity-QED systems as quantum simulators of superconductivity and quantum magnetism [Strack and Sachdev(2011), Gopalakrishnan *et al.*(2011)Gopalakrishnan, Lev, and Goldbart, Kelly *et al.*(2020)Kelly, Rey, and Marino, Colella *et al.*(2019)Colella, Ostermann, Niedenzu, Mivehvar, and Ritsch].

BCS model and dynamical phases: The BCS model of superconductivity for s-wave interacting fermions is characterized by the Hamiltonian [Gurarie and Radzihovsky(2007)],

$$\hat{H} = -\chi \sum_{\mathbf{k},\mathbf{k}'} \hat{c}^\dagger_{\mathbf{k},\uparrow}\hat{c}^\dagger_{-\mathbf{k},\downarrow}\hat{c}_{\mathbf{k}',\uparrow}\hat{c}_{-\mathbf{k}',\downarrow} + \sum_{\mathbf{k},\sigma} \varepsilon_{\mathbf{k}}\hat{c}^\dagger_{\mathbf{k},\sigma}\hat{c}_{\mathbf{k},\sigma}. \tag{47.1}$$

Here, $\hat{c}^\dagger_{\mathbf{k},\sigma}$ ($\hat{c}_{\mathbf{k},\sigma}$) creates (annihilates) a fermion of momentum $\mathbf{k}$ and spin $\sigma = \uparrow, \downarrow$. The first term describes attractive s-wave interactions $\chi \geq 0$ that lead to the formation of Cooper pairs. The single-particle dispersion is $\varepsilon_{\mathbf{k}} = \mathbf{k}^2/(2m) - \mu$ with μ the chemical potential and m the particle mass. Throughout the manuscript we set $\hbar = 1$.

This "reduced" BCS model assumes that only Cooper pairs are created and destroyed with zero center-of-mass momentum and neglects pair-breaking processes, so the low-energy physics can be described using only the presence or absence of Cooper pairs at each momentum mode. The physics of the model is further simplified by introducing Anderson pseudospin-1/2 operators,

$$\hat{\sigma}^-_{\mathbf{k}} = \hat{c}_{\mathbf{k},\uparrow}\hat{c}_{-\mathbf{k},\downarrow}, \quad \hat{\sigma}^z_{\mathbf{k}} = \hat{c}^\dagger_{\mathbf{k},\uparrow}\hat{c}_{\mathbf{k},\uparrow} + \hat{c}^\dagger_{-\mathbf{k},\downarrow}\hat{c}_{-\mathbf{k},\downarrow} - 1. \tag{47.2}$$

The two eigenstates of $\hat{\sigma}^z_{\mathbf{k}}$ encode the presence/absence of a Cooper pair with momentum $\mathbf{k}$, which are created (annihilated) by $\hat{\sigma}^+_{\mathbf{k}}$ ($\hat{\sigma}^-_{\mathbf{k}}$). Equation (47.1) then becomes:

$$\hat{H} = -\chi \sum_{\mathbf{k},\mathbf{k}'} \hat{\sigma}^+_{\mathbf{k}}\hat{\sigma}^-_{\mathbf{k}'} + \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}\hat{\sigma}^z_{\mathbf{k}} = -\chi \hat{S}^+\hat{S}^- + \sum_{\mathbf{k}} \varepsilon_{\mathbf{k}}\hat{\sigma}^z_{\mathbf{k}}, \tag{47.3}$$

where $\hat{S}^\pm = \sum_{\mathbf{k}} \hat{\sigma}^\pm_{\mathbf{k}}$ are collective spin operators.

Figure 47.1: (a) BCS dynamical phases illustrated by the pairing amplitude $|\Delta(t)|$. Characteristic $t^{-1/2}$ decay of phase II is indicated by the faded line. (b) Example BCS ground-state on the Bloch sphere. The single-particle inversion $\langle\hat{\sigma}^z_{\mathbf{k}}\rangle$ correlates with the sign of the dispersion $\varepsilon_{\mathbf{k}}$ [Eq. (47.4)]. (c) BCS physics can be simulated in a cavity by encoding a spin-1/2 into a pair of internal atomic states with transition frequency ω_a, which are coupled to a single common cavity mode. The spin-1/2 atoms are divided into two ensembles (shown as blue and red) featuring mean energy splittings with opposite sign, $\pm\epsilon_0/2$.

The ground-state $|\psi\rangle_{\rm gs}$ of (47.3) within BCS theory is characterized by the expectations [Barankov and Levitov(2006)]

$$\langle\hat{\sigma}^+_{\mathbf{k}}\rangle_{\rm gs} = \frac{1}{2}\frac{\Delta_{\rm gs}}{\sqrt{\Delta^2_{\rm gs}+\varepsilon^2_{\mathbf{k}}}}, \quad \langle\hat{\sigma}^z_{\mathbf{k}}\rangle_{\rm gs} = \frac{\varepsilon_{\mathbf{k}}}{\sqrt{\Delta^2_{\rm gs}+\varepsilon^2_{\mathbf{k}}}}, \tag{47.4}$$

as shown schematically on the Bloch sphere in Fig. 47.1(b). Here, the BCS pairing gap, $\Delta_{\rm gs} \equiv \chi\langle\hat{S}^-\rangle_{\rm gs}$, is defined self-consistently.

Prior studies in superconductors and fermionic superfluids [Yuzbashyan *et al.*(2015)Yuzbashyan, Dzero, Gurarie, and Foster, 93, Barankov and Levitov(2006)] have used the BCS Hamiltonian (47.3) to describe the gap dynamics after a quench of the pairing gap from the ground-state value $\Delta_{\rm gs}$ to a final value $\Delta_{\rm f}$ [Barankov and Levitov(2006), Yuzbashyan *et al.*(2015)Yuzbashyan, Dzero, Gurarie, and Foster]. Equation (47.3) is expected to provide a valid treatment of the gap dynamics on timescales for which pair breaking processes can be neglected, provided the quench is done faster than the inverse of the quasiparticle gap.

In the mean-field (classical) limit the dynamics falls into three distinct dynamical phases according to the behaviour of the magnitude of $|\Delta(t)| = \chi|S^-(t)|$. Throughout, we adopt the notation $\mathcal{O}(t) \equiv \langle\hat{\mathcal{O}}(t)\rangle$ when making a mean-field approximation, i.e., $\langle\hat{\mathcal{O}}_1(t)\hat{\mathcal{O}}_2(t)\rangle = \langle\hat{\mathcal{O}}_1(t)\rangle\langle\hat{\mathcal{O}}_2(t)\rangle$. As $t\to\infty$ the dynamics are: Phase I) $|\Delta(t)|\to 0$, Phase II) $|\Delta(t)|\to$ const with transient oscillations that decay as $\propto t^{-1/2}$, or Phase III) $|\Delta(t)|$ features persistent oscillations. Illustrations of $|\Delta(t)|$ in each phase are shown in Fig. 47.1(a). We discuss below how these phases arise from a competition between the interactions and the distribution of single-particle splittings $\varepsilon_{\mathbf{k}}$.

BCS physics in a cavity-QED simulator: We propose to explore the phase diagram of the BCS model by emulating the Hamiltonian (47.3) in a cavity. In our proposed scheme, an ensemble of atoms is distributed in a standing wave optical lattice supported by the cavity. Each atom, which we index by the label j, encodes a spin-1/2 degree of freedom in a pair of stable internal states, $|\uparrow\rangle_j$ and $|\downarrow\rangle_j$, which map to the presence or absence of a Cooper pair, respectively. The use of the index j compared to the momentum label $\mathbf{k}$ in a real BCS superconductor will be shown to be irrelevant.

Spin-spin interactions $\propto \hat{S}^+\hat{S}^-$ are mediated by the exchange of virtual photons between atoms via a single common cavity mode (at frequency ω_c) far-detuned from the atomic resonance (at frequency ω_a) [Norcia *et al.*(2018)Norcia, Lewis-Swan, Cline, Zhu, Rey, and Thompson, Muniz *et al.*(2020)Muniz, Barberena, Lewis-Swan, Young, Cline, Rey, and Thompson, Davis *et al.*(2019)Davis, Bentsen, Homeier, Li, and Schleier-Smith, Shankar *et al.*(2019)Shankar, Salvi, Chiofalo, Poli, and Holland]. These photon-mediated interactions are analogous to the phonon-mediated interactions in a BCS superconductor. Tunable (inhomogeneous) single-particle energy shifts $\varepsilon_j\hat{\sigma}_j^z$ can be realized via external fields that generate AC Stark or Zeeman shifts of the internal atomic states.

An important ingredient for the observation of the dynamical phases I-III is the ability to prepare initial states correlated with the distribution of splittings ε_j. For example, in the BCS ground-state [Eq. (47.4)], the sign of the inversion $\langle\hat{\sigma}_{\mathbf{k}}^z\rangle$ of the Anderson pseudospins correlates with the sign of the single-particle dispersion $\varepsilon_{\mathbf{k}}$. Motivated by this case, we consider initial states where the atoms are split into a pair of ensembles where the spin configuration of the atoms in each ensemble is correlated with the sign of the ensemble's average splitting. Concretely, we consider $2N$ atoms divided into two equal ensembles and initialized as a product of coherent spin-states [Radcliffe(1971)] lying on the equatorial plane of the Bloch sphere separated by a relative azimuthal opening angle $\Delta\phi_0$: $|\psi_0\rangle = |\pi/2, \Delta\phi_0/2\rangle_+ \otimes |\pi/2, -\Delta\phi_0/2\rangle_-$ [see Fig. 47.1 and Fig. 47.2(a)] where the subscript $\pm$ denotes each ensemble. Here, $|\theta,\phi\rangle \equiv \bigotimes_j \left[\cos(\theta/2)|\downarrow\rangle_j + e^{i\phi}\sin(\theta/2)|\uparrow\rangle_j\right]$ where the product runs over $j = 1, ..., N$ or $j = N+1, ..., 2N$ atoms respectively for the $\pm$ ensembles. Lastly, following the BCS ground-state we assume a uniform distribution of splittings $\varepsilon_j \in [\pm\epsilon_0/2 - W/4, \pm\epsilon_0/2 + W/4]$ where the sign of ϵ_0 differs for each ensemble and is matched to the sign of $\pm\Delta\phi_0/2$. It is the mean $\pm\epsilon_0/2$ and characteristic width $W/2$ rather than the precise distribution of ε_j (e.g., uniform or normal) that is important to characterize the physics discussed below.

Preparation of the two ensembles and correlation with $\pm\epsilon_0$ can be achieved by spatially selective energy shifts of atoms in the cavity [Davis *et al.*(2019)Davis, Bentsen, Homeier, Li, and Schleier-Smith] [Fig. 47.1(c)] or by addressing different internal levels [Lewis-Swan *et al.*(2018)Lewis-Swan, Norcia, Cline, Thompson, and Rey, Norcia *et al.*(2018)Norcia, Lewis-Swan, Cline, Zhu, Rey, and Thompson] (see later discussion) [19].

Accessible dynamical phase diagram: In Fig. 47.2 we explore the accessible dynamical phases. Panel (b) shows the dynamical phase diagram for mean splitting $\epsilon_0/(\chi N) = 0.1$. The phase diagram is computed via a Lax analysis [Yuzbashyan *et al.*(2005b)Yuzbashyan, Altshuler, Kuznetsov, and Enolskii, Barankov and Levitov(2006), Yuzbashyan *et al.*(2015)Yuzbashyan, Dzero, Gurarie, and Foster] that is a method for integrable models, such as Eq. (47.3), to determine the frequency spectrum that rules the dynamics of the order parameter. The spectrum is extracted from the roots of $L^2(u)$, the squared norm of the Lax vector $\mathbf{L}(u)$, a polynomial defined in terms of a complex variable u that encodes the conserved quantities of the model. A spectrum with all real roots defines phase I, with one pair of complex roots phase II and with two pairs of complex roots phase III. The asymptotic behaviour of $|\Delta(t)|$ follows from the nature of the roots of $L^2(u)$, which we compute numerically [SM()].

Physically, the dynamical phases depend on the competition between single-particle dephasing generated by W and ϵ_0 and the spin-locking effect generated by the interactions with strength set by χN [Lhuillier and Laloe(1982), Gully and Mullin(1984), Johnson *et al.*(1984)Johnson, Denker, Bigelow, Lévy, Freed, and Lee, Bashkin(1986), McGuirk *et al.*(2002)McGuirk, Lewandowski, Harber, Nikuni, Williams, and Cornell, Du

[19] It should also be possible to prepare initial states that are split by their projection along the z direction, rather than the projection along x (set by $\Delta\phi_0$) that would follow even more closely the BCS ground state. However, varying the relative azimuthal opening angle gives similar physics and is more robust to typical experimental constraints [SM()]

Figure 47.2: (a) Typical initial state for opening angle $\Delta\phi_0$. The orientation of each ensemble (red and blue collective Bloch vectors) is correlated with the sign of $\pm\epsilon_0$. (b) Mean-field BCS dynamical phase-diagram as a function of $\Delta\phi_0$ andcharacteristic width W of the single-particle noise distribution, with fixed $\epsilon_0/(\chi N) = 0.1$. The phase diagram is evaluated numerically (see Ref. [SM()]) and some small structure (e.g., regions of phase II within phase III) are likely artefacts of the methods precision. (c) Time-traces of the pairing amplitude $|\Delta(t)|$ for each phase [parameters indicated by marker in (b)].

et al.(2008)Du, Luo, Clancy, and Thomas, Deutsch *et al.*(2010)Deutsch, Ramirez-Martinez, Lacroûte, Reinhard, Schneider, Fuchs, Piéchon, Laloë, Reichel, and Rosenbusch, Kleine Büning *et al.*(2011)Kleine Büning, Will, Ertmer, Rasel, Arlt, Klempt, Ramirez-Martinez, Piéchon, and Rosenbusch]. The spin-locking is induced by the existence of a many-body gap that suppresses local spin flips and favors spin alignment [Norcia *et al.*(2018)Norcia, Lewis-Swan, Cline, Zhu, Rey, and Thompson, Davis *et al.*(2020)Davis, Periwal, Cooper, Bentsen, Evered, Van Kirk, and Schleier-Smith, Rey *et al.*(2008)Rey, Jiang, Fleischhauer, Demler, and Lukin]. Such behaviour also resembles synchronization observed in arrays of coupled oscillators with dissipation [Zhu *et al.*(2015)Zhu, Schachenmayer, Xu, Herrera, Restrepo, Holland, and Rey, Y(2003)]. Other consequences associated with the many-body gap include the stabilization of localization effects in fully connected models under specific initial conditions [Santos *et al.*(2016)Santos, Borgonovi, and Celardo, Celardo *et al.*(2016)Celardo, Kaiser, and Borgonovi].

For small inhomogeneity, $W \ll \epsilon_0, \chi N$ we predict phase III dynamics independent of the opening angle $\Delta\phi_0$. Within each ensemble a gap opens between the manifold of collective states (this includes the initial fully polarized states) and those that are spatially inhomogeneous, preventing dephasing of the individual spins of each ensemble. In addition, the interplay between the homogeneous single-particle energy splitting $\pm\epsilon_0$ (that generates precession of the ensembles in opposing directions about the z-axis of the Bloch sphere), and the collective interaction (that also drives a rotation of each ensemble along a common self-generated axis set by the total transverse magnetization [Norcia *et al.*(2018)Norcia, Lewis-Swan, Cline, Zhu, Rey, and Thompson, Lewis-Swan *et al.*(2018)Lewis-Swan, Norcia, Cline, Thompson, and Rey]), leads to persistent nonlinear oscillations in the effective pairing amplitude $|\Delta(t)| = |S^+(t)|$ [see also Fig. 47.3(a)].

Phase II emerges for $2\epsilon_0 < W \lesssim \chi N$ and opening angles away from $\Delta\phi \approx \pm\pi$ [SM()]. The transition from phase III to II is driven by the ensembles no longer having a well-defined relative energy splitting correlated with their initial orientation. Thus, in contrast to phase III, spin locking of the *entire* ensemble of $2N$ atoms determines the dynamics. This means that while the pairing amplitude $|\Delta(t)|$ remains large, oscillations are transient and suppressed rather than stabilized by the interactions.

Finally, phase I emerges for $W \gtrsim \chi N$ independent of ϵ_0. Single-particle physics dominates for all initial conditions and the pairing amplitude vanishes due to rapid dephasing of the

Figure 47.3: (a) Typical trajectories of the collective Bloch vector of each ensemble (red and blue) for $W \ll \chi N, \epsilon_0$ as ϵ_0 is tuned between phases IIIa and IIIb. (b) Phase diagram characterized by amplitude $\mathcal{A} \equiv \max(|\Delta(t)|) - \min(|\Delta(t)|)$ of oscillations in $|\Delta(t)|$. The critical boundary ϵ_0^c between phases IIIa and IIIb is indicated by the red line. (c) Maximum of the total inversion difference $J_z = \left(\sum_{j\in+} \sigma_j^z - \sum_{j\in-} \sigma_j^z\right)/2$ and frequency, ω_{osc}, of oscillations of $|\Delta(t)|$ as a function of ϵ_0.

individual spins, $|\Delta(t)| \to 0$.

Beyond these three known regimes we also predict the emergence of two previously unidentified sub-phases within phase III for $W \ll \chi N, \epsilon_0$, which we label as IIIa and IIIb. These sub-phases are delineated by a critical splitting $\epsilon_0^c = \frac{\chi N}{2}[1 + \cos(\Delta\phi_0)]$ [SM()]. Phase IIIa), $\epsilon_0 < \epsilon_0^c$, is dominated by interactions and characterized by a strictly non-zero pairing amplitude, $|\Delta(t)| > 0$ that exhibits nonlinear oscillations with an approximate frequency $\omega_{osc} \propto \chi N$. Phase IIIb) is characterized by the pairing amplitude periodically vanishing, $|\Delta(t)| = 0$, and the physics is dominated by the single-particle splitting ϵ_0 such that the frequency of oscillations scales as $\omega_{osc} \propto \epsilon_0$.

In Fig. 47.3(a) we illustrate typical trajectories of the collective Bloch vector of each ensemble in sub-phases IIIa and IIIb for an initial state with $\Delta\phi_0 = 0$, which are representative of the dominant physics for $|\Delta\phi_0| \lesssim \pi$. For small $\epsilon_0 \ll \epsilon_0^c$ and $\Delta\phi_0 \ll \pi$ the Bloch vectors remain trapped close to their initial polarization due to the strong interactions, leading to $|\Delta(t)| > 0$. As ϵ_0 increases nearer to the transition, ϵ_0^c, the interactions still dominate and their interplay with the single particle term leads to a deflection of the trajectories of the Bloch vectors close to the north and south poles. Above ϵ_0^c the trajectories abruptly snap to large orbits near the equator and quickly approach the precession expected for two independent ensembles (e.g., dominated by the $\hat{\sigma}_z$ term of the Hamiltonian). Even though phase IIIb is technically absent for $\Delta\phi_0 = \pm\pi$ by our definition (as $|\Delta(0)| = 0$), we still observe rich non-trivial oscillations for $\epsilon_0 \ll \chi N$ with frequency $\omega_{osc} \propto \sqrt{\epsilon_0 \chi N}$ [SM()].

Quantitatively, the IIIa and IIIb sub-phases are delineated by abrupt changes in different observables, including the magnitude $\mathcal{A} \equiv \max(|\Delta(t)|) - \min(|\Delta(t)|)$, the frequency ω_{osc} of oscillations of $|\Delta(t)|$ [Figs. 47.3(b) and (d)] and the maximum excursion of the collective spins away from the equator of the Bloch sphere, measured by the differential inversion $J_z = \left(\sum_{j\in+} \sigma_j^z - \sum_{j\in-} \sigma_j^z\right)/2$ [Fig. 47.3(c)].

Experimental realization and robustness of proposal: In a cavity-QED experiment, the dynamical phases can be characterized by detection of intracavity light leaking out through the cavity mirrors [Norcia *et al.*(2018)Norcia, Lewis-Swan, Cline, Zhu, Rey, and Thompson]. By operating in the limit where the cavity mode is far off-resonance from the atomic transition, the virtual photons that mediate the interactions are adiabatically eliminated and slaved to the spins, such

that atomic information is imprinted onto the phase and amplitude of the cavity field via the approximate relation $a(t) \propto S^-(t) \propto \Delta(t)$ [SM(), Norcia *et al.*(2018)Norcia, Lewis-Swan, Cline, Zhu, Rey, and Thompson]. The light intensity then serves as a proxy for the BCS pairing amplitude, $|a(t)|^2 \propto |\Delta(t)|^2$, while the frequency spectrum of $a(t)$ can also be a useful diagnostic to distinguish the dynamical phases. Moreover, by continuously performing heterodyne detection of the small amount of light leaking through the cavity mirrors we are able to in principle construct time-traces of the pairing amplitude within a single experimental trial.

To demonstrate our proposal is robust to relevant decoherence and technical factors, we model an experiment where the spin-1/2 is encoded using the narrow linewidth $^1\mathrm{S}_0$-$^3\mathrm{P}_1$ optical transition of ^{87,88}Sr. Here, sub-ensembles can be prepared via spatially dependent light-shifts from the side of the cavity, or in ^{87}Sr by applying spatially dependent magnetic fields and addressing the $\pm 9/2$ nuclear spin levels of the transition. We use parameters from Ref. [Muniz *et al.*(2020)Muniz, Barberena, Lewis-Swan, Young, Cline, Rey, and Thompson] and include single-particle decoherence due to the natural linewidth of the transition $\gamma/(2\pi) = 7.5$ kHz and spatially inhomogeneous atom-light coupling arising due to the incommensurate wavelengths of the standing wave optical lattice confining the atoms and the relevant cavity mode [Muniz *et al.*(2020)Muniz, Barberena, Lewis-Swan, Young, Cline, Rey, and Thompson, Norcia *et al.*(2018)Norcia, Lewis-Swan, Cline, Zhu, Rey, and Thompson, SM()]. The latter leads to a spatial modulation of the spin-spin interactions $\chi \to \chi_{i,j}$. Our predictions should also be qualitatively relevant for other cavity-based systems that can realize an effective $\chi \hat{S}^+ \hat{S}^-$ interaction, e.g., Raman transitions [Davis *et al.*(2019)Davis, Bentsen, Homeier, Li, and Schleier-Smith, Shankar *et al.*(2019)Shankar, Salvi, Chiofalo, Poli, and Holland].

In Fig. 47.4(a) we model the transition between phases I and III as a function of the inhomogeneity strength W at fixed $\epsilon_0/(\chi N) = 0.1$ and initial state $\Delta\phi_0 = \pi$. The phases are distinguished in the frequency spectrum of the cavity field $\mathcal{F}[a](\omega)$, with phase III signaled by a pair of robust peaks in the spectrum that disappear in phase I. The peaks are consistent with the entwined but distinguishable precession of the two ensembles that leads to beating of the intensity $|a(t)|^2$ as shown in the accompanying time-trace. The oscillations in the intracavity intensity $|a(t)|^2$ are robust to the inhomogeneous interactions and the exponential decay induced by γ. The transition between the phases occurs at $W/(\chi N) \approx \pi/2$, which is consistent with the model Eq. (47.3) when the inhomogeneous atom-light coupling is taken into account by a simple rescaling to the corresponding mean value $\chi \to \overline{\chi_{i,j}} = \chi/2$ [SM()].

Similarly, the phase II-III transition can be observed by varying the initial opening angle $\Delta\phi_0$ at fixed $W/(\chi N) = \epsilon_0/(\chi N) = 0.1$. The spectrum of the intracavity field shows the signature dual peaks of phase III for $\pi/2 \lesssim \Delta\phi_0 \leq \pi$, while phase II is signaled by a single peak for $0 \leq \Delta\phi_0 \lesssim \pi/2$. The latter indicates dynamics of a single collective ensemble, with finite but non-oscillatory pairing amplitude.

Lastly, signatures of the phase IIIa-IIIb transition in the differential inversion and oscillation frequency of $|a(t)|^2$ are shown in Fig. 47.4(c). Decoherence blunts the expected cusp in the inversion, although the peak value lines up closely with the expected transition upon accounting for inhomogeneous interactions. The oscillation frequency clearly distinguishes the trivial/non-trivial regimes for $\Delta\phi_0 = \pi$. On the other hand, we find that for $\Delta\phi_0 = 0, \pi/2$ the relatively small oscillations in $|a(t)|^2$ predicted for phase IIIa are destroyed by decoherence,and instead the transition between IIIa and IIIb is marked by an abrupt vanishing of any discernible peak in the spectrum (indicated by the absence of data).

Conclusions: We have reported a proposal to observe the dynamical phases of a BCS superconductor in a cavity-QED quantum simulator. Realizing these phases via a spin degree of freedom instead of actual Cooper pairs overcomes the need to reach the ultra cold temperatures at which pairing occurs. The versatility of this platform allows us to probe the dependence of the dynamical phases on the initial state and system parameters in a controllable, isolated setting.

Figure 47.4: Dynamics of intracavity field $a(t)$. (a) Frequency spectrum of intracavity field, $\mathcal{F}[a](\omega)$, as a function of $W/(\chi N)$, and typical timetraces of $|a|^2$ in phases (i) III and (ii) I. Initial state is $\Delta\phi_0 = \pi/2$. (b) Same, but as a function of opening angle $\Delta\phi_0$ and time traces are in phases (i) II and (ii) III. Simulations are for fixed $W/(\chi N) = 0.1$. Both (a) and (b) use fixed $\epsilon_0/(\chi N) = 0.1$ and color scales are normalized. (c) Signatures of phase IIIa and phase IIIb in differential inversion J_z and oscillation frequency $\omega_{\rm osc}$ of $|a|^2$ for $W = 0$ for different $\Delta\phi_0$. Critical $\epsilon_0^{\rm c}$ for each $\Delta\phi_0$ is indicated by a vertical line. The absence of plotted results for $\omega_{\rm osc}$ below the approximate transition $\epsilon_0^{\rm c}$ for each $\Delta\phi_0$ indicates the lack of appreciable oscillations in the simulations. All relevant parameters (e.g., g, γ and κ) are taken from Refs. [Muniz *et al.*(2020)Muniz, Barberena, Lewis-Swan, Young, Cline, Rey, and Thompson, Norcia *et al.*(2018)Norcia, Lewis-Swan, Cline, Zhu, Rey, and Thompson] and results are rescaled for $N = 10^6$ (see also Ref. [SM()]).

Our predictions pave the way for future studies of more complex non-equilibrium phenomena in models of quantum magnetism and superconductivity so far not seen in real materials or high energy systems.

47.2 Supplemental Material: A cavity-QED quantum simulator of dynamical phases of a BCS superconductor

47.2.1 Lax analysis

We gain analytic insight into the dynamical phase diagram of the BCS model by employing a Lax analysis. This is a a tool that, building on the integrability of the BCS model, allows for the characterization and even solution of the asymptotic ($t \to \infty$) dynamics. In the following sections we give a brief summary of results in the classical limit, pertinent in particular for Figs. 2 and 3 of the main text. A more detailed discussion of the Lax analysis can be found in, e.g., Ref. [Yuzbashyan *et al.*(2005b)Yuzbashyan, Altshuler, Kuznetsov, and Enolskii, Yuzbashyan *et al.*(2015)Yuzbashyan, Dzero, Gurarie, and Foster] and references therein.

Lax vector and classification of dynamical phase diagram

Throughout our analysis we work with the pseudospin BCS Hamiltonian [Eq. (3) of the main text],

$$\hat{H} = \chi \hat{S}^+ \hat{S}^- + \sum_j \varepsilon_j \hat{\sigma}_j^z. \tag{47.5}$$

We will focus on the mean-field (classical) dynamics of the model generated by the approximation $\langle \hat{\mathcal{O}}_1 \hat{\mathcal{O}}_2 \rangle = \langle \hat{\mathcal{O}}_1 \rangle \langle \hat{\mathcal{O}}_2 \rangle$ and adopt the notation $\mathcal{O} \equiv \langle \hat{\mathcal{O}} \rangle$ for simplicity.

Given the form of the Hamiltonian (47.5) we also introduce the associated (mean-field) Lax vector [Yuzbashyan *et al.*(2005b)Yuzbashyan, Altshuler, Kuznetsov, and Enolskii, Barankov and Levitov(2006), Yuzbashyan *et al.*(2015)Yuzbashyan, Dzero, Gurarie, and Foster],

$$\vec{L}(u) = \frac{1}{2} \sum_j \frac{\vec{\sigma}_j(0)}{u - \varepsilon_j} - \frac{\hat{z}}{\chi}. \tag{47.6}$$

The Lax vector is explicitly defined with respect to the initial state characterized by the expectation values $\vec{\sigma}_j(0) = (\langle \hat{\sigma}_j^x(0) \rangle, \langle \hat{\sigma}_j^y(0) \rangle, \langle \hat{\sigma}_j^z(0) \rangle)$.

The mean-field dynamical phase diagram of $\hat{H}$ given the initial conditions $\vec{\sigma}_j(0)$ can be constructed by an analysis of the properties of $\vec{L}(u)$ [Yuzbashyan *et al.*(2005b)Yuzbashyan, Altshuler, Kuznetsov, and Enolskii, Barankov and Levitov(2006), Yuzbashyan *et al.*(2015)Yuzbashyan, Dzero, Gurarie, and Foster]. Specifically, the dynamical phases are defined in terms of the properties of the *complex* roots $\{u_1, u_2, ..., u_n\}$ of the equation $\vec{L}(u) \cdot \vec{L}(u) = 0$: Phase I corresponds to zero complex roots, phase II is defined by a single pair of complex roots, and phase III is accompanied by two pairs of complex roots.

To be concrete in our analysis we must specify the initial states which are fed into the definition of the Lax vector. In the main text we consider splitting the atoms inte a pair of ensembles, each of N atoms, initialized as a product of coherent spin-states [Radcliffe(1971)] lying on the equatorial plane of the Bloch sphere separated by a relative azimuthal opening angle $\Delta\phi_0$: $|\psi_0\rangle = |\pi/2, \Delta\phi_0/2\rangle_+ \otimes |\pi/2, -\Delta\phi_0/2\rangle_-$ (see Fig. 1 and Fig. 2(a) of the main text) where the subscript $\pm$ denotes each ensemble. Here, we have used $|\theta, \phi\rangle \equiv \bigotimes_j \left[\cos(\theta/2)|\downarrow\rangle_j + e^{i\phi}\sin(\theta/2)|\uparrow\rangle_j\right]$ where the product over j runs over $j = 1, ..., N$ or $j = N+1, ..., 2N$ atoms respectively for the $\pm$ ensembles. For $|\psi_0\rangle$ we then have $\vec{\sigma}_j(0) = (\cos(\Delta\phi_0/2), \pm\sin(\Delta\phi_0/2), 0)$ where the $\pm$ is correlated with the ensemble. Moreover,

we assume the single-particle energies are sampled uniformly from $\varepsilon_j \in [\pm\epsilon_0 - W/2, \pm\epsilon_n + W/2]/2$ for each ensemble. Substituting the state and single-particle dispersion into Eq. (47.6) and taking the continuum limit as $N \to \infty$ we compute the Lax vector $\vec{L}(u) = L_x(u)\hat{x} + L_y(u)\hat{y} - (1/\chi)\hat{z}$,

$$\begin{aligned} \chi L_x(u) &= \frac{2\chi N}{W}\cos\left(\frac{\Delta\phi_0}{2}\right)\left[\mathrm{ArcTanh}\left(\frac{4u}{W-2\epsilon_0}\right) + \mathrm{ArcTanh}\left(\frac{4u}{W+2\epsilon_0}\right)\right], \\ \chi L_y(u) &= \frac{\chi N}{W}\sin\left(\frac{\Delta\phi_0}{2}\right)\log\left[\frac{(4u)^2-(W-2\epsilon_0)^2}{(4u)^2-(W+2\epsilon_0)^2}\right]. \end{aligned} \tag{47.7}$$

Analytically solving for the roots of $\vec{L}(u)\cdot\vec{L}(u) = 0$ is in general not possible for arbitrary choices of system parameters $W, \epsilon_0, \chi N$ and initial state $\Delta\phi_0$. As a consequence we typically solve for the roots using a numerical algorithm, which will be detailed momentarily. However, there are three limiting cases of the parameters for which relatively simple analytic forms of the roots exist and significant insight into the phase diagram can be garnered: (i) $\Delta\phi_0 = \pi$, (ii) $W \ll \chi N, \epsilon_0$, and (iii) $\epsilon_0 \ll W, \chi N$.

For case (i), $\Delta\phi_0 = \pm\pi$, we are able to analytically solve the roots and characterize the phase diagram for any values of $W/(\chi N)$ and $\epsilon_0/(\chi N)$. A straightforward solution of $\vec{L}(u)\cdot\vec{L}(u) = 0$ yields two pairs of complex roots,

$$\begin{aligned} u_{1\pm} &= \pm\frac{1}{4}\sqrt{\frac{(W-2\epsilon_0)^2 - e^{-i\frac{W}{\chi N}}(W+2\epsilon_0)^2}{1-e^{-i\frac{W}{\chi N}}}}, \\ u_{2\pm} &= \pm\frac{1}{4}\sqrt{\frac{(W-2\epsilon_0)^2 - e^{i\frac{W}{\chi N}}(W+2\epsilon_0)^2}{1-e^{i\frac{W}{\chi N}}}}, \end{aligned} \tag{47.8}$$

which exist only for $W/(\chi N) < \pi$. If $W/(\chi N) \geq \pi$ then no complex roots exist. This allows us to diagnose the dynamical phases: For $W/(\chi N) < \pi$ the long-time dynamics are phase III, while for $W/(\chi N) \geq \pi$ the dynamics are phase I. Phase II does not exist for any choice of $W/(\chi N)$ or $\epsilon_0/(\chi N)$.

Case (ii) describes a scenario where the inhomogeneity is small compared to both the interactions and the uniform energy splitting, $W \ll \chi N, \epsilon_0$, and admits an approximate analytic solution for any $\Delta\phi_0$. To compute the roots we first expand the squared Lax vector to lowest-order in W,

$$\begin{aligned} \vec{L}(u)\cdot\vec{L}(u) &= 1 + \frac{4\chi^2N^2}{(4u^2-\epsilon_0^2)^2}\left[4u^2\cos^2\left(\frac{\Delta\phi_0}{2}\right)\right. \\ &\quad \left. + \epsilon_0^2\sin^2\left(\frac{\Delta\phi_0}{2}\right)\right] + \mathcal{O}(W^2), \end{aligned} \tag{47.9}$$

and then solve $\vec{L}(u)\cdot\vec{L}(u) = 0$ to find

$$\begin{aligned} u_{1\pm} &= \frac{1}{2}\sqrt{\epsilon_0^2 - \chi^2N^2[1+\cos(\Delta\phi_0)] \pm \frac{\chi N}{\sqrt{2}}\sqrt{\chi^2N^2[3+4\cos(\Delta\phi_0)+\cos(2\Delta\phi_0)] - 8\epsilon_0^2}}, \\ u_{2\pm} &= -\frac{1}{2}\sqrt{\epsilon_0^2 - \chi^2N^2[1+\cos(\Delta\phi_0)] \pm \frac{\chi N}{\sqrt{2}}\sqrt{\chi^2N^2[3+4\cos(\Delta\phi_0)+\cos(2\Delta\phi_0)] - 8\epsilon_0^2}}. \end{aligned} \tag{47.10}$$

The existence of these two pairs of complex roots is insensitive to the choice of $\Delta\phi_0$ and always true for $W \ll \chi N, \epsilon_0$. Thus, we predict that for small inhomogeneity the long-time dynamics is always phase III.

Lastly, case (iii) considers the case where only the inhomogeneous splitting and interactions dominate the physics, $\epsilon_0 \ll W, \chi N$. Computing the squared Lax vector to lowest-order in ϵ_0,

$$\vec{L}(u) \cdot \vec{L}(u) = 1 + 16\frac{\chi^2 N^2}{W^2}\text{ArcTanh}^2\left(\frac{4u}{W}\right)\cos^2\left(\frac{\Delta\phi_0}{2}\right), \tag{47.11}$$

yields a single pair of complex roots

$$u_\pm = \pm\frac{iW}{4}\tan\left[\frac{W}{4\chi N}\sec\left(\frac{\Delta\phi_0}{2}\right)\right], \tag{47.12}$$

only for $W/(\chi N) < 2\pi\cos(\Delta\phi_0/2)$. If $W/(\chi N) \geq 2\pi\cos(\Delta\phi_0/2)$ no complex roots exist. This allows us to diagnose: For $W/(\chi N) < 2\pi\cos(\Delta\phi_0/2)$ the dynamics is phase II, while for $W/(\chi N) \geq 2\pi\cos(\Delta\phi_0/2)$ the dynamics is phase I. We stress that these phases and the transition at $W/(\chi N) = 2\pi\cos(\Delta\phi_0/2)$ only exist in the limit $\epsilon_0 \ll W, \chi N$.

In general, we resort to a numerical search for roots of $\vec{L}(u)\cdot\vec{L}(u) = 0$. This is the procedure used to generate Fig. 2a of the main text. We use the inbuilt function *fsolve* of MATLAB 2020a to search for complex-valued roots of the squared Lax vector as a function of system parameters and initial state. An exhaustive search is performed by running many iterations of the root finding algorithm for each choice of W, ϵ_0, χN and $\Delta\phi_0$. To make this computation efficient we perform our search at fixed $\Delta\phi_0$, ϵ_0, χN and begin at small inhomogeneity $W \ll \chi N, \epsilon_0$ for which we can use the results of Eq. (47.10) as the initial guess for the algorithm. The roots are expected to change relatively smoothly as a function of W (until they vanish) and so we use the roots found for the current value of W as the subsequent starting point for the next value of W.

To validate our numerical results, we compare the roots found by the numerical search algorithm to the analytic results for cases (i)-(iii) discussed above in Fig. 47.6. For case (i), e.g., restricting to $\Delta\phi = \pi$, we observe exact quantitative agreement in terms of both the predicted roots and the presence of dynamical phases I and III (Fig. 47.6a). More generically, our numerical results for arbitrary $\Delta\phi$ (Fig. 47.6b) are consistent with the expected result of phase III for $W \ll \epsilon_0, \chi N$ [case ii)] and the boundary between phases II and I at $W/(\chi N) = 2\pi\cos(\Delta\phi_0/2)$ [case iii)]. The breakdown of the latter prediction near $\Delta\phi \approx \pm\pi$ in the numerical results is also consistent, as our assumption that $\epsilon_0 \ll W$ would be in contradiction with the predicted phase boundary.

In Fig. 47.6b (also Fig. 2b of the main text) we identify that a transition between phases III and II occurs when the inhomogeneous and uniform splittings become comparable, $W = 2\epsilon_0 \lesssim \chi N$, and $\Delta\phi \neq \pm\pi$. To support the generality of this observation we compute the phase diagram as a function of both $W/(\chi N)$ and $\epsilon_0/(\chi N)$ for fixed $\Delta\phi_0 = 0$ and present the results in Fig. 47.6c. Our data illustrates that there is always a consistent transition between phases II and III at $W = 2\epsilon_0$.

Dynamical phase diagram of alternative initial states

Throughout the main text we focused on the accessible dynamical phase diagram of the BCS model in the context for specific set of initial conditions parameterized by the state $|\psi_0\rangle = |\pi/2, \Delta\phi_0/2\rangle_+ \otimes |\pi/2, -\Delta\phi_0/2\rangle_-$ with $\Delta\phi_0 \in [-\pi, \pi]$. Here, $|\theta, \phi\rangle_\pm \equiv \bigotimes_{j\in\pm}\left[\cos(\theta/2)|\downarrow\rangle_j + e^{i\phi}\sin(\theta/2)|\uparrow\rangle_j\right]$ is a spin coherent state. This choice of state was motivated by both the fact that it shares qualitative features with the BCS ground-state and that it can be prepared relatively accurate in the experiment when taking into account all technical considerations, including inhomogeneity of the atom-light coupling (see Sec. 47.2.2 and later discussion). The former connection is of note because prior work in the literature studying the BCS dynamical phase diagram has focused on quenches from the BCS ground-state.

Figure 47.5: (a) Complex roots of the squared Lax vector, $\vec{L}(u) \cdot \vec{L}(u) = 0$, for $\Delta\phi_0 = \pi$ and $\epsilon_0/(\chi N) = 0.1$. We compare the results of the numerical search algorithm (blue markers) and the predictions of Eq. (47.8) (red lines). No complex roots exist for $W/(\chi N) \geq \pi$. (b) Numerically evaluated dynamical phase diagram as a function of $\Delta\phi_0$ and $W/(\chi N)$ at fixed $\epsilon_0/(\chi N) = 0.1$. We compare the numerical results to analytic predictions for the phase II/I boundary (red dashed line). The phase III/II transition is also indicated at $W = 2\epsilon_0$ (dashed black line). (c) Generic phase III/II transition for arbitrary splittings and fixed $\Delta\phi_0 = 0$. We indicate $W = 2\epsilon_0$ (dashed blue line) to guide the eye.

As highlighted in the main text, it is convenient to define the BCS ground-state in the pseudospin representation via the single-particle spin expectation values,

$$\langle\hat{\sigma}_j^+\rangle_{\rm gs} = \frac{1}{2}\frac{\Delta_{\rm gs}}{\sqrt{\Delta_{\rm gs}^2+\varepsilon_j^2}}, \quad \langle\hat{\sigma}_j^z\rangle_{\rm gs} = \frac{\varepsilon_j}{\sqrt{\Delta_{\rm gs}^2+\varepsilon_j^2}}, \tag{47.13}$$

where we have adopted the subscript j to mirror the conventions of the cavity-QED case for simplicity. The crucial feature of this ground-state in terms of the dynamical phase diagram is that the sign of the initial inversion $\langle\hat{\sigma}_j^z\rangle_{\rm gs}$ is correlated with the sign of the single-particle energy splitting ε_j. It is this feature that we seek to mimic with our initial state, by correlating the sign of the azimuthal angle $\pm\Delta\phi_0/2$ of each spin ensemble with the sign of the energy splitting ϵ_0. Based on our results and comparison with the literature, this type of correlation appears to be a necessary requirement for the observation of phase III.

For completeness, and mindful of the rapidly advancing technical capabilities in state-of-the-art cavity-QED experiments, we also summarize in this SM the dynamical phase diagram which might be accessed with an initial state that more closely follows the BCS ground-state: $|\psi_0'\rangle = |\frac{\pi}{2}+\frac{\Delta\theta_0}{2},0\rangle_+\otimes|\frac{\pi}{2}-\frac{\Delta\theta_0}{2},0\rangle_-$. Here, the relative opening angle $\Delta\theta_0$ between the ensembles is with respect to the elevation, which means that the signs of $\langle\hat{\sigma}_j^z\rangle$ will be correlated with the sign of the energy splitting ϵ_0 in much closer correspondence to the BCS ground-state Eq. (47.13), as can be seen from the initial expectation values $\vec{\sigma}_j(0) = (\cos(\Delta\theta_0/2), 0, \pm\sin(\Delta\theta_0/2))$. In Fig. 47.6 we present the numerically evaluated dynamical phase diagram for this state and compare to the equivalent phase diagram for $|\psi_0\rangle$. To be consistent with Fig. 2b of the main text we fix $\epsilon_0/(\chi N) = 0.1$ and probe the dependence on $W/(\chi N)$ and opening angle $\Delta\theta_0$. We limit our scan over the latter to $0 \leq \Delta\theta_0 \lessapprox \pi - \pi/8$ because our numerical algorithm performs poorly when the initial state becomes too closely aligned with the poles. We find that the dynamical phase diagram of the BCS-like state $|\psi_0'\rangle$ is very similar to the results in the main text, both qualitatively in terms of the presence of all three dynamical phases but also quantitatively in terms of the transition points as a function of $W/(\chi N)$ and $\Delta\theta_0$.

We can make the last connection quantitatively rigorous by solving for the roots of the squared Lax vector in the same limiting cases ii) and iii) from our prior analysis. First, for $W \ll \epsilon_0, \chi N$ [case ii)] we again find two pairs of complex roots (excluding when $\Delta\theta_0 = \pm\pi$).

$$\begin{aligned} u_{1\pm} &= -\frac{1}{4}\sqrt{4\epsilon_0^2 - 2\chi^2N^2 - 2\chi N\left[\chi N\cos(\Delta\theta_0) - 4\epsilon_0\sin\left(\frac{\Delta\theta_0}{2}\right)\right]} \pm \frac{i\chi N}{2}\cos\left(\frac{\Delta\theta_0}{2}\right), \\ u_{2\pm} &= \frac{1}{4}\sqrt{4\epsilon_0^2 - 2\chi^2N^2 - 2\chi N\left[\chi N\cos(\Delta\theta_0) - 4\epsilon_0\sin\left(\frac{\Delta\theta_0}{2}\right)\right]} \pm \frac{i\chi N}{2}\cos\left(\frac{\Delta\theta_0}{2}\right). \end{aligned} \tag{47.14}$$

This result indicates we should expect phase III for small enough inhomogeneity and regardless of the initial opening angle (excluding the trivial case when $\Delta\theta_0 = \pm\pi$). Secondly, for $\epsilon_0 \ll W/(\chi N)$ [case iii)] we have

$$u_\pm = \pm\frac{iW}{4}\tan\left[\frac{W}{4\chi N}\sec\left(\frac{\Delta\theta_0}{2}\right)\right], \tag{47.15}$$

for $W/(\chi N) < 2\pi\cos(\Delta\theta_0/2)$. This result demonstrates there exists a transition between phases II and I at $W/(\chi N) = 2\pi\cos(\Delta\theta_0/2)$ identically to our previous analysis of case iii) for the state $|\psi_0\rangle$.

Analytic solution of phase III from Lax analysis

The dynamics of the BCS pairing amplitude, $|\Delta(t)|$, in phase III can be solved exactly in certain cases. Specifically, by adopting an amplitude-phase parametrization of the pairing term,

Figure 47.6: Dynamical phase diagram for: (a) $|\psi_0'\rangle = |\pi/2 + \Delta\theta_0/2, 0\rangle_+ \otimes |\pi/2 - \Delta\theta_0/2, 0\rangle_-$ and (b) $|\psi_0\rangle = |\pi/2, \Delta\phi_0/2\rangle_+ \otimes |\pi/2, -\Delta\phi_0/2\rangle_-$. Parameters are identical to Fig. 2b of main text, particularly $\epsilon_0/(\chi N) = 0.1$. Some small structure, particularly the protrusion of phase III into phase II in panel (a), is likely a numerical artefact related to the breakdown of our numerical algorithm as it requires a minimum tolerance as an input to distinguish roots.

$\Delta(t) = \sqrt{R(t)}e^{i\varphi(t)}$, and substitution of this into the classical equations of motion [Yuzbashyan *et al.*(2005b)Yuzbashyan, Altshuler, Kuznetsov, and Enolskii, Barankov and Levitov(2006)] a description of the dynamics of the amplitude $R(t)$ can be reduced to the single differential equation

$$\dot{R}^2 = 4(R_+ - R)(R - R_-)(R + \tilde{R}). \tag{47.16}$$

Here, $R_\pm$ and $\tilde{R}$ are obtained from the roots of the Lax vector [Yuzbashyan *et al.*(2015)Yuzbashyan, Dzero, Gurarie, and Foster, Foster and Gurarie()]. Denoting the two pairs of complex roots as $\bar{u}_{1\pm} = \bar{u}_{1r} \pm i\bar{u}_{1i}$ and $\bar{u}_{2\pm} = \bar{u}_{2r} \pm i\bar{u}_{2i}$ we have $R_\pm = (|\bar{u}_{1i}| \pm |\bar{u}_{2i}|)^2$ and $\tilde{R} = (\bar{u}_{1r} - \bar{u}_{2r})^2$. Before proceeding further, we point out that some features of the pairing amplitude oscillations in phase III can already be predicted. Specifically, we have that the oscillation amplitude is $\mathcal{A} = \max(|\Delta(t)|) - \min(|\Delta(t)|) = \sqrt{R_+} - \sqrt{R_-}$.

For certain cases, Eq. (47.16) can be solved analytically. In particular, when $\Delta\phi_0 = \pi$ we have that $|\bar{u}_{1i}| = |\bar{u}_{2i}|$ [see Eq. (47.8)] so that $R_- = 0$. Then, a solution to Eq. (47.16) is

$$|\Delta(t)| = \sqrt{R_+}\left|\mathrm{sn}\left(t\sqrt{\tilde{R}}, -\frac{R_+}{\tilde{R}}\right)\right|, \tag{47.17}$$

where $\mathrm{sn}(u, m)$ is a Jacobi elliptic function. The nonlinear oscillations of $|\Delta(t)|$ can then be determined to have an amplitude $\mathcal{A} = \sqrt{R_+}$ and a period $T = 4K(-R_+/\tilde{R})/\sqrt{\tilde{R}}$ where $K(m)$ is the complete elliptic integral of the first kind. In the limit of $W \ll \epsilon_0 \ll \chi N$ we can simplify $R_+ = \tilde{R} \approx \epsilon_0\chi N/2$ from Eq. (47.8). This results in a pair of observations: i) the amplitude of oscillations vanishes as $\mathcal{A} = \sqrt{\epsilon_0\chi N}/\sqrt{2}$ for small splitting, and similarly ii) the frequency ω_{osc} of oscillations is approximately $\omega_{\mathrm{osc}} \sim 1/T \propto \sqrt{\epsilon_0\chi N}$. This latter result is clearly illustrated in Fig. 3d of the main text.

Figure 47.7: Amplitude $\mathcal{A} \equiv \max(|\Delta(t)|) - \min(|\Delta(t)|)$ of oscillations in $|\Delta(t)|$ obtained analytically from Eqs. (47.17) and (47.8) (solid black line) and numerical integration of mean-field equations of motion following from the Hamiltonian (47.5) for $N = 10^3$ (grey markers). Calculations are for an initial state with $\Delta\phi = \pi$, $\epsilon_0/(\chi N) = 0.1$ and varying $W/(\chi N)$. Phases I and III, equivalent to $\mathcal{A} = 0$ or $\mathcal{A} \neq 0$ in this case, are indicated by the blue and yellow backgrounds, respectively.

As further verification of both our Lax analysis and the predictions of Eq. (47.17) we compare the amplitude $\mathcal{A}$ for dynamics of the initial state with $\Delta\phi_0 = \pi$ obtained from: i) the analytic solution of the Lax roots Eq. (47.8), and ii) full numeric integration of the mean-field equations of motion following from the Hamiltonian (47.5). Results are plotted in Fig. 47.7. We observe excellent quantitative agreement, not only in terms of the boundary between phases I and III but also the predicted $\mathcal{A}$ as a function of $W/(\chi N)$. Small differences are entirely attributable to finite size effects (e.g., analytic results assume $N \to \infty$ and thus a *continuous* distribution of ε_j).

Phases IIIa and IIIb

We also report in the main text an identification of sub-phases IIIa and IIIb in the limit $W \ll \chi N, \epsilon_0$. Our classification of these sub-phases is instead related to the characterization of the real and imaginary parts of the two pairs of complex roots present for phase III.

To be concrete, we define sub-phases IIIa and IIIb by the effective order parameter R_-, which was previously introduced in the differential equation (47.16). To recap, $R_- = (|\bar{u}_{1i}| - |\bar{u}_{2i}|)^2$ where we have written the roots of the squared Lax vector in the form $\bar{u}_{1\pm} = \bar{u}_{1r} \pm i\bar{u}_{1i}$ and $\bar{u}_{2\pm} = \bar{u}_{2r} \pm i\bar{u}_{2i}$. Phase IIIa exists for $R_- = 0$ and phase IIIb for $R_- \neq 0$. Inspecting Eq. (47.16) we observe that $R_\pm$ define the maximum and minimum values of the oscillations in $|\Delta(t)|^2$, so clearly we can also interpret phase IIIa as oscillations where the pairing amplitude periodically vanishes $|\Delta| = 0$, whereas in phase IIIb the pairing amplitude is strictly greater than zero always, $|\Delta| > 0$.

Mathematically, the condition $R_- = 0$ corresponds to the case where the magnitude of the imaginary part of the Lax roots is identical for both pairs, i.e., $|\bar{u}_{1i}| = |\bar{u}_{2i}|$. In the limit of $W \ll \chi N, \epsilon_0$ we can use the results of Eq. (47.10) to identify that this condition occurs for a critical splitting

$$\epsilon_0^c = \frac{\chi N}{2}[1 + \cos(\Delta\phi_0)]. \tag{47.18}$$

47.2.2 Experimentally realistic model

In Fig. 4 of the main text we present quantitative predictions illustrative of the dynamical phases based upon state-of-the-art cavity-QED experiments. Here, we summarize the model these calculations are based upon. For the interested reader, further detail regarding the

derivation of our model can be found in Ref. [Muniz *et al.*(2020)Muniz, Barberena, Lewis-Swan, Young, Cline, Rey, and Thompson] and the associated Supplementary Information.

Emulated BCS-like Hamiltonian

We consider an ensemble of $2N$ atoms confined in a standing wave optical lattice supported by a cavity. Each atom encodes a spin-1/2 in a pair of long-lived electronic states, $|\uparrow\rangle$ and $|\downarrow\rangle$, which are separated by a narrow linewidth optical transition of frequency ω_a. A single common cavity mode couples the electronic states with single-photon Rabi frequency $2g$. In the limit that the cavity mode at frequency ω_c is far detuned from the atomic transition $|\delta_c| = |\omega_c - \omega_a| \gg g\sqrt{N}$, then the cavity field can be adiabatically eliminated and the intracavity photons serve only to mediate effective exchange interactions between the spins. Single-particle energy shifts can be generated by applying external fields to generate Stark or Zeeman shifts of the electronic states. Combining these effects with relevant sources of decoherence leads to an effective description for the atomic system in terms of a Lindblad master equation for the density matrix $\hat{\rho}_a$,

$$\frac{d\hat{\rho}_a}{dt} = -\frac{i}{\hbar}\left[\hat{H}, \hat{\rho}_a\right] + \mathcal{L}_s[\hat{\rho}_a], \tag{47.19}$$

with Hamiltonian

$$\hat{H} = \hbar \sum_{i,j} \chi_{ij} \hat{\sigma}_i^+ \hat{\sigma}_j^- + \sum_i \varepsilon_j \hat{\sigma}_i^z, \tag{47.20}$$

and decoherence due to spontaneous emission at rate γ_s described by the Lindblad jump operator,

$$\mathcal{L}_s[\hat{\rho}] = \frac{\gamma}{2} \sum_i 2\hat{\sigma}_i^- \hat{\rho} \hat{\sigma}_i^+ - \hat{\sigma}_i^+ \hat{\sigma}_i^- \hat{\rho} - \hat{\rho} \hat{\sigma}_i^+ \hat{\sigma}_i^- . \tag{47.21}$$

In the Hamiltonian (47.20) the all-to-all interactions are characterized by $\chi_{ij} = -g_i g_j/\delta_c$. The inhomogeneity is inherited from the spatial variation of the atom-light coupling, $g_j \propto g\cos(k_d j)$ for $k_d = \pi\lambda_L/\lambda_c$, due to the incommensurate wavelengths of the confining lattice, $\lambda_L = 813$ nm, and cavity mode, $\lambda_c = 689$ nm. In numerical simulations (see later discussion) we typically observe that the main consequence of inhomogeneous interactions is to effectively re-scale the interactions to their root-mean-square value, $\chi \to \chi/2$. This is also consistent with related work published in Refs. [Norcia *et al.*(2018)Norcia, Lewis-Swan, Cline, Zhu, Rey, and Thompson, Muniz *et al.*(2020)Muniz, Barberena, Lewis-Swan, Young, Cline, Rey, and Thompson].

State preparation

State preparation is realized by a combination of coherent single-particle rotations and shifts of the internal atomic levels. Specifically, we assume that all spins are initially prepared in the single-particle state $|\downarrow\rangle_j$. The cavity is suddenly injected with coherent light which is tuned to be resonant with the atomic transition [Muniz *et al.*(2020)Muniz, Barberena, Lewis-Swan, Young, Cline, Rey, and Thompson]. This process can be modelled as a single-particle term $H_{\text{rot}} = \sum_j \frac{\Omega_j}{2}\hat{\sigma}_j^y$. The inhomogeneity of the driving term $\Omega_j = \Omega_0\cos(k_d j) \propto g_j$ is again inherited from the inhomogeneous atom-light coupling g_j. In our simulations, we apply H_{rot} for a time τ_{rot} so that $\Omega_0\tau_{\text{rot}} = \pi/2$, i.e., a $\pi/2$-pulse is engineered with respect to the strongest coupled atoms, i.e., $\Omega_j = \Omega_0$. For simplicity, we assume $\Omega_0 \gg \chi N, \epsilon_0, W$ such that interactions and energy shifts can be ignored during state preparation.

After the single-particle rotation sequence, the variable opening angle $\Delta\phi_0$ of the initial state can be generated by suddenly turning on a (selective) large uniform energy splitting between the atoms of each ensemble, e.g., $\varepsilon_j = \pm\epsilon_0$, to generate precession of each ensemble about the

z-axis. This could be achieved, for example, by a spatially selective optical Stark shift of the internal levels $|\uparrow\rangle$ and $|\downarrow\rangle$.

The combination of these two coherent operations leads to initial states of the form $|\psi_0\rangle = |\psi_0\rangle_+ \otimes |\psi_0\rangle_-$ where $|\psi_0\rangle_\pm = \bigotimes_j \left[\cos(\theta_j/2)|\downarrow\rangle_j + e^{\pm i\Delta\phi/2}\sin(\theta_j/2)|\uparrow\rangle_j\right]$ with $\theta_j = (\pi/2)\cos(k_d j)$ and j runs over $1, ..., N$ or $N+1, ..., 2N$ for the respective ensembles.

Measurement of pairing amplitude via leaked light

The BCS pairing amplitude, proportional to the transverse spin coherence, can be monitored in experiment by detecting the light which leaks out through the cavity mirrors. Specifically, when the cavity field is eliminated to yield the effective spin model Eq. (47.19), at the mean-field level the intra-cavity field is related to the spins via

$$\langle \hat{a} \rangle = \frac{-2}{2\delta_c - i\kappa} \sum_j g_j \langle \hat{\sigma}_j^- \rangle. \tag{47.22}$$

We use this relation to plot the dynamics of the intracavity field in Fig. 4 of the main text.

Numerical simulation and parameters

To simulate the experimental system we solve the mean-field equations of motion associated with the master equation (47.19). Specifically, we use the software package XMDS2 [Dennis *et al.*(2013)Dennis, Hope, and Johnsson] to numerically integrate the system of equations generated by $\dot{\mathcal{O}} = \mathrm{Tr}[\hat{\mathcal{O}} \frac{d}{dt} \hat{\rho}_a]$ with the approximation $\langle \hat{\mathcal{O}}_1(t)\hat{\mathcal{O}}_2(t)\rangle = \langle \hat{\mathcal{O}}_1(t)\rangle\langle \hat{\mathcal{O}}_2(t)\rangle$.

To account for inhomogeneities in the effective spin-spin interactions and state preparation we model a system composed of two distinct ensembles ($\pm$) of atoms spatially distributed at lattice sites $j = 1, 2, ..., 2N_{\mathrm{sim}}$. Typically, we take $N_{\mathrm{sim}} = 10^2 - 10^3$ and re-scale all parameters and results to match a true atom number of $N = 10^6$. We adopt relevant parameters from Ref. [Muniz *et al.*(2020)Muniz, Barberena, Lewis-Swan, Young, Cline, Rey, and Thompson]: $g/(2\pi) = 10.9$ kHz, $\gamma_s/(2\pi) = 7.5$ kHz, $\delta_c/(2\pi) = -50$ MHz and $\kappa/(2\pi) = 153$ kHz.

48 Observing Dynamical Phases of BCS Superconductors in a Cavity QED Simulator by Young et al.

Abstract

In conventional Bardeen-Cooper-Schrieffer (BCS) superconductors [Bardeen *et al.*(1957)], electrons with opposite momenta bind into Cooper pairs due to an attractive interaction mediated by phonons in the material.

While superconductivity naturally emerges at thermal equilibrium, it can also emerge out of equilibrium when the system's parameters are abruptly changed [Yuzbashyan *et al.*(2006), Barankov and Levitov(2006), Yuzbashyan and Dzero(2006), Gurarie and Radzihovsky(2007), 36, 93, Yuzbashyan *et al.*(2015)].

The resulting out-of-equilibrium phases are predicted to occur in real materials and ultracold fermionic atoms but have not yet all been directly observed.

(in which articles there was statement that it is impossible to observe them?)

Here we realise an alternate way to generate the proposed dynamical phases using cavity quantum electrodynamics (cavity QED).

Our system encodes the presence or absence of a Cooper pair in a long-lived electronic transition in ^{88}Sr atoms coupled to an optical cavity and represents interactions between electrons as photon-mediated interactions through the cavity [Lewis-Swan *et al.*(2021), Kelly *et al.*(2022)].

(how???)

To fully explore the phase diagram, we manipulate the ratio between the single-particle dispersion and the interactions after a quench and perform real-time tracking of subsequent dynamics of the superconducting order parameter using non-destructive measurements.

We observe regimes where the order parameter decays to zero (phase I) [Barankov and Levitov(2006), Yuzbashyan and Dzero(2006)], assumes a non-equilibrium steady-state value (phase II) [Barankov and Levitov(2006), Yuzbashyan *et al.*(2006)], or exhibits persistent oscillations (phase III) [Barankov and Levitov(2006), Yuzbashyan *et al.*(2006)].

This opens up exciting prospects for quantum simulation, including the potential to engineer unconventional superconductors and to probe beyond mean-field effects like the spectral form factor [Stewart(2017), Sato and Ando(2017)], and for increasing coherence time for quantum sensing.

(which prospects? beyond mean-field effects???)

Conclusion

The demonstrated capability to emulate dynamical phases of superconductors in optical cavities opens exciting prospects for quantum simulation. For example, it will be interesting to see if our cavity simulator can engineer and probe topological superfluid phases [93, Black-Schaffer(2012), Nandkishore *et al.*(2012), Kiesel *et al.*(2012), Kiesel *et al.*(2013), Fischer *et al.*(2014), Shankar *et al.*(2022)]

(topological superfluid phases???)

and understand competing superconducting orders [Laughlin(1998), Balatsky *et al.*(2006)] in a single system

(competing superconducting orders ????),

or else enable simulation of superfluidity in phenomena relevant to high energy physics [Schäfer and Teaney(2009), Pehlivan *et al.*(2011)].

48.1 Introduction

Quantum simulation offers a path to understand a broad range of phenomena, from high-temperature superconductivity and correlated quantum magnetism in condensed matter physics [Zhou *et al.*(2021)] (???)to quarks and gluons in nuclei and matter under extreme conditions [Shuryak(2017)] (???), as well as the black hole information paradox in gravitational physics [Harlow(2016)] (???).

A fascinating and promising case is the prethermal dynamical phases [Marino *et al.*(2022)] predicted to emerge from quenches of superconductors and superfluids [Yuzbashyan *et al.*(2006), Barankov and Levitov(2006), Yuzbashyan and Dzero(2006), Gurarie and Radzihovsky(2007), 36, 93, Yuzbashyan *et al.*(2015), Volkov,Kogan (1973), Yuzbashyan *et al.*(2005), Barankov *et al.*(2004), Yuzbashyan(2008), Foster *et al.*(2014), Collado *et al.*(2023)], systems that feature Cooper pairing of electrons or neutral fermions.

(just a collection of relevant citations)

While there has been great progress in pump-probe experiments of superconductors to induce such fast quenches using THz technology, and signs of phases I and II have been observed, the intense pulses couple nonlinearly to the Cooper pairs in the superconductor and complicate a clean observation of the dynamical phases [Mansart *et al.*(2013), Matsunaga *et al.*(2013), Matsunaga *et al.*(2014)].

(I'll read it, I don't know yet, why is it so?)

For these reasons, the realisation of fermionic superfluids in ultracold atomic gases [Randeria and Taylor(2014)] has generated great excitement [Yuzbashyan *et al.*(2006), Barankov and Levitov(2006), Yuzbashyan and Dzero(2006), Gurarie and Radzihovsky(2007), 36, 93, Yuzbashyan *et al.*(2015)]; however, to date observations have been limited to spectroscopic signatures rather than the full time dynamics [Behrle *et al.*(2018)].

(spectroscopic signatures ???)

In neither system has a systematic scan of the dynamical phase diagram been performed, and in fact phase III has not been observed. (why by the way? people didn't try?)

(dynamical phase diagram ????)

Figure 48.1: **a**, The Anderson pseudospin mapping encodes the presence and absence of a Cooper pair as the up and down states of a spin-1/2 system, respectively (???). Under this mapping, the attractive interaction $\chi\hat{c}^\dagger_{\mathbf{k}}\hat{c}^\dagger_{-\mathbf{k}}\hat{c}_{-\mathbf{k}'}\hat{c}_{\mathbf{k}'}$ between electrons is equivalent to an all-to-all exchange interaction $\chi\hat{S}^+_{\mathbf{k}}\hat{S}^-_{\mathbf{k}'}$ between pseudospins.

Figure 48.2: **b**, Model parameters. The top plot shows the effective dispersion relation near the Fermi surface engineered in our system as a function of parameters δ_s and E_W, controlled using AC Stark shifts. The bottom plot visualises the ground state of a BCS superconductor using Anderson pseudospins (still haven't understood them fully...). Near the Fermi momentum, the pseudospins develop a phase-coherent superposition at a scale set by a nonzero BCS pairing gap Δ_{BCS}. This gap is self-consistently defined from the spin coherence as shown on the Bloch sphere.

Figure 48.3: **c**, Dynamical phase diagram. The three dynamical phases can be realised by varying parameters χN, δ_s, and E_W. (what is the meaning of moving through vertical or horizontal axes??) Representative dynamics of the BCS order parameter $|\Delta_{BCS}|$ for each phase are shown as insets. We explore cut H1 (dashed line) in Fig. **??** using a single ensemble of atoms and cuts V and H2 (solid lines) in Figs. 48.9 and 48.10 using two separately controlled sub-ensembles.

Figure 48.4: **d**, Cavity QED implementation of the BCS interaction. Coupling many strontium atoms to a detuned optical cavity generates infinite-range spin-exchange interactions mediated by a virtual exchange of cavity photons. This interaction also causes a field proportional to Δ_{BCS} to leak out of the cavity, providing a real-time probe of the dynamics.

(it is not clear, what were the results before?) Here, we take a step forward towards this challenge by using internal electronic states to encode effective Cooper pairs. (= do simulation)

At the heart of this implementation is the Anderson pseudospin mapping [Anderson(1958)] by which the presence or absence of Cooper pairs in a momentum mode is encoded in a pseudo spin-1/2 system. We simulate Anderson pseudospins using a long-lived electronic transition in ^{88}Sr with interactions between the spins mediated by a high finesse optical cavity. (how????)

As proposed in Refs. [Lewis-Swan *et al.*(2021), Kelly *et al.*(2022)] (read these papers!!), the scattering between Cooper pairs in condensed matter systems can be engineered in our system via the exchange of photons through the cavity (see Fig. **??d**).

(read their article, tell why it can be so??? I have no idea now)

In this way, the dynamics of a collection of interacting spin-1/2 systems maps onto the low-energy physics of a superconductor or superfluid.

We probe all three dynamical phases (phases I, II, and III) predicted to exist in BCS superconductors

(where was this prediction made???)

by utilising the high degree of control and flexibility in state initialisation, interaction control, and non-destructive measurements available when coupling long-lived atoms to an optical cavity. Behaviours intrinsic to phase I (normal phase) and phase II (finite steady-state superconductivity) have previously been observed in spin systems realized in optical cavities [Davis *et al.*(2020), Norcia *et al.*(2018a)] (read these articles!!!) and in two-level atoms interacting via collisions [Allred *et al.*(2002), Kleine *et al.*(2008), Deutsch *et al.*(2010), Smale *et al.*(2019)] (read these articles!!!).

We build on this work by clarifying the connection between these dynamical phases from the BCS model and the physics of many-body gap protection in spin systems.

(what is the second, gap protection??)

Our results also provide the first demonstration of phase III (a self-generated Floquet phase featuring persistent oscillations of the order parameter), which is predicted to dynamically emerge in superconductors via quenches from weak to strong interactions [Barankov and Levitov(2006), Yuzbashyan *et al.*(2015)]. In our system, we instead engineer this phase using flexible control of the single-particle dispersion [Lewis-Swan *et al.*(2021), Collado *et al.*(2023)]

(to understand it, I would need to read these articles), dynamically resembling the low-energy condition of a BCS superconductor.

For all experiments, we perform real-time tracking of the superconducting order parameter, enabling fast readout of the dynamics.

48.2 Experimental setup and model system

To realise dynamical phases of the BCS model, we laser cool (how is it by the way?) an ensemble of $N = 10^5 - 10^6$ ^{88}Sr atoms and trap them inside a $\lambda_{\mathrm{L}} = 813$ nm 1D optical lattice supported by a high-finesse optical cavity.

(what is an optical lattice?)

A spin-1/2 system is encoded in the electronic ground state $|\downarrow\rangle = |^1\mathrm{S}_0, m_J = 0\rangle$ and a long-lived optical excited state $|\uparrow\rangle = |^3\mathrm{P}_1, m_J = 0\rangle$.

Along this transition, we define spin operators $\hat{S}_k^- = |\downarrow\rangle\langle\uparrow|_k$ and $\hat{S}_k^z = (|\uparrow\rangle\langle\uparrow|_k - |\downarrow\rangle\langle\downarrow|_k)/2$ for single atoms with labels $k \in \{1, ..., N\}$, as well as the collective lowering operator $\hat{S}^- = \sum_k \hat{S}_k^-$ and raising operator $\hat{S}^+ = (\hat{S}^-)^\dagger$.

Assuming homogeneous atom-light coupling in the cavity and unitary dynamics, our system can be described by the Hamiltonian

$$\hat{H} = \hbar\chi\hat{S}^+\hat{S}^- + \sum_k \varepsilon_k \hat{S}_k^z. \tag{48.1}$$

The first term represents an infinite-range spin-exchange interaction described by a frequency scale χ [Norcia *et al.*(2018a)], realised using the collective coupling between the atomic ensemble and a detuned optical cavity mode.

(why????? what is the meaning of these spins?? need to study [Norcia *et al.*(2018a)] to find the derivation of this model for the gas!!)

Inhomogeneous atom-light coupling and dissipative processes (including, foremost, single-particle spontaneous decay) are present in the current implementation but do not largely change the qualitative behaviour of the targeted dynamical phases under our experimental conditions (see Methods). Previously, we have characterised this interaction [Norcia *et al.*(2018a)](read!!!) and studied collective dynamics by applying an external drive [Muniz *et al.*(2020)] (read!!!).

In this work, we go beyond the fully collective manifold by engineering a spread in single-particle energies $\varepsilon_k = \hbar\omega_k$ using applied AC Stark shifts ω_k [Baghdad *et al.*(2023), Sauerwein *et al.*(2023)]. (read!!!)

These shifts form the second term in the Hamiltonian and compete with the spin-exchange interaction.

Equation (48.1) is the so-called Richardson-Gaudin spin model [Richardson and Sherman(1964), Gaudin(1976)], which describes the low-energy physics of Bardeen-Cooper-Schrieffer (BCS) superfluids and superconductors using the Anderson pseudospin mapping [Anderson(1958)]. (read their paper!!??)

This mapping relates the presence (or absence) of a Cooper pair formed by a pair of electrons with momenta $\pm\mathbf{k}$ to a spin-up (or down) at momentum $\mathbf{k}$, as shown in Fig. **??a**.

Correspondingly, annihilating a Cooper pair maps to a spin lowering operator by the relation $\hat{S}_{\mathbf{k}}^- := \hat{c}_{\mathbf{k}}\hat{c}_{-\mathbf{k}}$, where $\hat{c}_{\pm\mathbf{k}}$ are fermionic annihilation operators.

Similarly, the spin operator $2\hat{S}_{\mathbf{k}}^z + 1 := \hat{c}_{\mathbf{k}}^\dagger\hat{c}_{\mathbf{k}} + \hat{c}_{-\mathbf{k}}^\dagger\hat{c}_{-\mathbf{k}}$ counts the number of electrons with momentum $\mathbf{k}$ or $-\mathbf{k}$. (I guess, this is explained in Anderson's paper) Our cavity system therefore

manifestly implements a BCS superconductor if one identifies the label k of an atom in the cavity with the momentum $\mathbf{k}$ of the electrons in a Cooper pair.

In this way, the first term in Eq. (48.1) is equivalent to the attractive interaction between electrons in the superconductor, and the second term can be associated with the kinetic energy or dispersion relation of the electrons. Note that the BCS model, described by Eq. (48.1), only accounts for the zero momentum collective excitations present in conventional superfluids and superconductors [Anderson(1958)].

The BCS order parameter in the Anderson mapping is defined by $\Delta_{\mathrm{BCS}} = \chi\langle\sum_{\mathbf{k}} \hat{c}_{\mathbf{k}}\hat{c}_{-\mathbf{k}}\rangle = \chi\langle\hat{S}^{-}\rangle$, as depicted in Fig. **??b**.

In equilibrium, it plays the role of the BCS pairing gap, which energetically favours many-body states where the electrons arrange in a coherent superposition between Cooper pairs and holes for states close to the Fermi energy.

Away from equilibrium, Δ_{BCS} is also predicted to characterise the three dynamical phases (I, II, and III) that arise after quenches in superconductors and superfluids [Marino *et al.*(2022)] (what is in this article?). Such dynamical phases represent distinct regimes of dynamical behaviour that arise after a sudden perturbation of a control parameter in a closed many-body system.

They are described using a time-averaged or steady-state order parameter that demonstrates non-analytic behaviour at the boundary between phases.

(a very unclear sentence. regimes use some secret steady-state order parameter? what is it??)

In particular, the BCS model is predicted to exhibit second-order dynamical phase transitions.

(what is it??)

Figure 48.5: **a**, Tuning the single-particle dispersion. We shine an off-resonant 461 nm beam onto the atoms from outside the cavity. This generates a distribution of AC Stark shifts representing a roughly uniform density of states $\rho(\omega)$ (bottom plot).

Figure 48.6: **b**, Probing phase I and phase II. We perform a rapid $\pi/2$ pulse to prepare a highly coherent initial state, wait for 2 μs, quench to a variable $\chi N/E_W$ with $\delta_s = 0$, and then let the system evolve. The inset shows the explored parameter cut and identifies post-quench $\chi N/E_W$ values with coloured dots. The main plot shows experimental time traces of $|\Delta_{BCS}|$ (coloured curves) accompanied by numerical simulations (darker lines). Two curves are extended to demonstrate long-time coherence protection, with the $\chi N/2\pi = 0.19$ MHz trace smoothed for clarity. For $\chi N/2\pi = 1.2$ MHz, we show an ideal simulation neglecting dissipation and motional effects (dashed line), which exhibits transient Higgs oscillations. Hints of these oscillations are present in experimental data with additional damping.

Figure 48.7: **c**, Characterising the phase transition. Blue triangles show the fitted coherence time of $|\Delta_{BCS}|$ from $t = 1$ μs to 30 μs. Green circles show the time-averaged $|\Delta_{BCS}|$ between $t = 3$ μs and 8 μs, with the dark green line representing numerical simulations. In all cases, we identify a phase transition at $\chi N/2\pi = 0.2$ MHz. Error bars in all plots represent the s.e.m. of boostrap resamplings on experimental shots.

Figure 48.8: **d**, Varying initial conditions. Before $t = 0$, we shine a high-intensity 461 nm beam within 300 ns, engineering an initial phase spread $\varphi(\omega_k) \in [0, \varphi_0]$ depicted on the Bloch sphere. The phase $\varphi(\omega_k)$ applied to atom k is proportional to the post-quench frequency shift ω_k. Traces represent different φ_0 and show enhanced oscillations with increasing φ_0.

Phase I is characterised by a steady state with a vanishing order parameter $|\Delta_{\mathrm{BCS}}(t)| \to 0$ at long times. Phase II exhibits a steady state with a constant nonzero order parameter $\Delta_\infty := \lim_{t\to\infty} |\Delta_{\mathrm{BCS}}(t)| > 0$. Finally, phase III features oscillations in $|\Delta_{\mathrm{BCS}}(t)|$ that persist to long times, realising a Floquet superfluid despite not being periodically driven [Foster *et al.*(2014), 93, Yuzbashyan *et al.*(2015), 36]. The long-time behaviour of these dynamical phases admits a simpler description in terms of the Lax-reduced Hamiltonian, which is an effective Hamiltonian taking the same form of Eq. (48.1) but with rescaled parameters and a reduced number of spins [Yuzbashyan *et al.*(2015), Marino *et al.*(2022)]. Under this formulation, phases I, II, and III emerge when the Lax-reduced Hamiltonian describes effective zero-spin, one-spin, and two-spin systems respectively.

Inspired by the Lax-reduced Hamiltonian, and in order to explore all three dynamical phases, we engineer two sub-ensembles of atoms with separate control over energy shifts within each sub-ensemble. For practical convenience, we introduce experimental control in the form of an overall frequency splitting δ_{s} between two sub-ensembles and an effective frequency width E_{W} of each sub-ensemble to engineer a tunable dispersion relation ε_k as in Fig. **??b**. Phase I and phase II can also be observed using a single ensemble of atoms as shown in Fig. **??**. Both experimental setups can nonetheless be described by a common phase diagram as shown in Fig. **??c**.

We initialise all the atoms in the $|\downarrow\rangle$ state and then apply a coherent $\pi/2$ pulse through the cavity in 100 ns such that $\Omega \gg \chi N$, where Ω is the pulse Rabi frequency and χN is the characteristic interaction strength for an ensemble of N atoms. This establishes a large BCS order parameter Δ_{BCS} on a timescale faster than any other relevant dynamics, mimicking the ground state of a Hamiltonian with an infinite interaction strength χ. We then quench the system by rapidly turning on ε_k, which sets a finite ratio $\chi N/E_{\mathrm{W}}$ and a variable $\delta_{\mathrm{s}}/E_{\mathrm{W}}$, allowing us to explore the dynamical phase diagram shown in Fig. **??c**.

We measure both the pre- and post-quench dynamics of $|\Delta_{\mathrm{BCS}}|$ by monitoring light emitted by the atoms into the cavity as a function of time (see Fig. **??d**).

This light arises from a superradiance process which is suppressed when the cavity resonance is detuned from the atomic transition frequency by much more than κ, the cavity power decay linewidth [Weiner *et al.*(2012), Bohnet *et al.*(2013), Norcia *et al.*(2016)].

(what it the superradiance? what is going on in the cited articles??)

In this limit, the established cavity field adiabatically follows $\langle \hat{S}^- \rangle$, which is proportional to Δ_{BCS}. By measuring the leakage of light from the cavity in heterodyne with a local oscillator, we therefore obtain a real-time probe of Δ_{BCS}.

(how is it???)

Importantly, at the chosen detuning this probe is quasi-nondestructive, since only a small fraction of the atoms emit light over relevant timescales. In plots of $|\Delta_{\mathrm{BCS}}|$ over time, we normalise traces to the initial gap size Δ_{init} measured right after the $\pi/2$ pulse.

48.3 Phase I to phase II

We probe the phase I to phase II transition by varying the ratio $\chi N/E_{\mathrm{W}}$ between the interaction strength and the width of the single-particle energy distribution. As shown in Fig. **??a**, we shine an off-resonant 461 nm beam onto a single atomic ensemble from the side of the cavity that generates a distribution of AC Stark shifts with a spread E_{W}. Careful shaping of the 461 nm beam allows us to realise a roughly flat density of states (see Methods), resulting in a setup consistent with the $\delta_{\mathrm{s}} = 0$ line in Fig. **??c** (see Supplemental Online Material). After the initial $\pi/2$ pulse, we wait for 2 μs to let transient dynamics settle and then turn on the 461 nm beam to quench on $E_{\mathrm{W}}/2\pi = 0.83$ MHz from an initial value $E_{\mathrm{W}}^{(0)}/2\pi \ll 0.1$ MHz. The beam exhibits a rise time of roughly 50 ns, much faster than the relevant dynamics. To scan across the phase diagram in the inset of Fig. **??b**, we vary the interaction strength χN between shots by changing the atom number N.

Figure 48.9: **Phase II to phase III transition.** **a**, Engineering a bimodal energy distribution. We prepare two atomic clouds with centres separated by 3 mm and shine an off-resonant 461 nm beam centred on one cloud. This generates a density of states $\rho(\omega)$ (middle plot), equivalent to a dispersion relation $\varepsilon_k = \hbar\omega_k$ (bottom plot). **b**, Probing phase II and phase III. We prepare the same initial state as in Fig. **??b** with a $\pi/2$ pulse, quench to a finite δ_s/E_W, and then let the system evolve. The inset shows the explored parameter cut and identifies post-quench δ_s/E_W values with coloured dots. As before, coloured traces represent experimental time traces of $|\Delta_{BCS}|$, and darker lines represent numerical simulations. **c**, Ideal simulations of mean-field trajectories for the two sub-ensembles (solid and dashed curves) in phase II (magenta) and phase III (blue). The trajectories are projected onto the surface of the Bloch sphere for visual clarity. **d**, Fourier response of $|\Delta_{BCS}|^2$ for different δ_s, plotted as power spectra of the dynamics from $t = 0.5$ μs to 4 μs after subtracting slow-moving behaviour. **e**, Average oscillation amplitude between $t = 3$ μs and 8 μs. For the remaining plots, dashed lines represent ideal simulations (ignoring dissipation or motional effects), and solid dark lines correspond to full simulations. The additional dotted line represents numerical simulations rescaled by $\times 0.2$, plotted to show similar trend behaviour between experimental data and simulations. We identify a phase transition around $\delta_s/2\pi = 0.85$ MHz. **f**, Oscillation frequency of $|\Delta_{BCS}|$, measured using power spectra calculated in (**d**). We correct for systematics inferred from our data analysis and assume this correction has an uncertainty of 100%, shown by the green band. The phase transition point observed in data in panels (**e**) and (**f**) agrees well with simulations.

As shown in Fig. **??b** and **c**, we observe two distinct dynamical behaviours corresponding to phases I and II, signalled by the decay rate of $|\Delta_{BCS}|$. For experiments with sufficiently small χN, such as $\chi N/2\pi = 0.19$ MHz, $|\Delta_{BCS}|$ decays with a $1/e$ coherence time of 0.9 ± 0.1 μs. This coherence time is consistent with single-particle dephasing of $\langle \hat{S}^- \rangle$ set by the energy spread $\hbar E_W$ and is nearly constant throughout this regime. We identify the fast decay of $|\Delta_{BCS}|$ as an experimental signature of phase I. For larger interaction strengths, we observe a rapid increase in coherence time up to a maximum of 29 μs when $\chi N/2\pi = 1.2$ MHz; this constitutes an improvement of more than a factor of 30. We identify this extended coherence time regime as phase II. The residual decay of $|\Delta_{BCS}|$ in this regime can be attributed to intrinsic dissipative processes including spontaneous emission, off-resonant superradiant emission, and scattering of 461 nm light [Norcia *et al.*(2016), Norcia *et al.*(2018a)], which set a maximum predicted coherence time of 29 μs (see Methods). All experimental observations (coloured traces) are in good agreement with numerical simulations based on experimental conditions (dark lines—see Methods).

Due to the separation of timescales in the decay of $|\Delta_{\mathrm{BCS}}|$, we are able to determine the boundary between phase I and phase II in our experiment by calculating the average $|\Delta_{\mathrm{BCS}}|$ in a time window from 3 μs to 8 μs as a function of χN (see Fig. **??c**). In this analysis, phase I features a vanishing average $|\Delta_{\mathrm{BCS}}|$, while phase II sees a nonzero $|\Delta_{\mathrm{BCS}}|$ that increases with χN. The sharp rise of average $|\Delta_{\mathrm{BCS}}|$ around $\chi N/2\pi = 0.2$ MHz indicates a dynamical phase transition, which agrees with the point predicted by numerical simulations. In a spin-model picture, the BCS pairing gap corresponds to the energy gap between collective angular momentum states, which exists due to the spin-exchange interaction $\chi\hat{S}^{+}\hat{S}^{-}$ [Rey *et al.*(2008)]. Phase II corresponds to the parameter region where such interactions are sufficiently strong to protect against single-particle dephasing. As a result, the observed transition directly relates to previous experiments exploring coherence protection in other systems [Davis *et al.*(2020), Norcia *et al.*(2018a), Smale *et al.*(2019), Deutsch *et al.*(2010), Kleine *et al.*(2008), Allred *et al.*(2002)].

In BCS superconductors, the excitation of a Higgs mode is predicted to occur in phase II. This mode can be characterised by a collective damped oscillation of the order parameter $|\Delta_{\mathrm{BCS}}|$ with a characteristic frequency of $2\Delta_{\infty}$ [Yuzbashyan *et al.*(2015)]. We observe hints of Higgs oscillations by comparing the experimental trace of $|\Delta_{\mathrm{BCS}}|$ at $\chi N/2\pi = 1.2$ MHz (red curve in Fig. **??b**) with the dissipation-free simulation (dashed line in Fig. **??b**) and noticing that the first dip in the experimental trace coincides with the first cycle of Higgs oscillations (see Methods). The size of this feature can be increased experimentally by engineering an initial phase spread $\varphi(\omega_k) \in [0, \varphi_0]$ between atoms which is correlated with the post-quench frequency shifts ω_k of the atoms, as shown in Fig. **??d**. The initial state with a nonzero opening angle φ_0 shares qualitative features with the BCS ground state at finite χ up to a $\pi/2$ rotation on the Bloch sphere [Lewis-Swan *et al.*(2021)], in contrast to the initial state mimicking the BCS ground state with infinite χ in Fig. **??b**.

48.4 Phase II to phase III

We probe the phase II to phase III transition using a vertical cut through the dynamical phase diagram. To realise this, we introduce an energy splitting $\hbar\delta_{\mathrm{s}}$ between two individually addressable clouds of atoms along the cavity axis using AC Stark shifts from our 461 nm beam, as shown in Fig. 48.9**a**. In combination with a background energy spread $\hbar E_{\mathrm{W}}$ associated with lattice shifts (see Methods), this produces a bimodal density of states and a dispersion relation similar to the one proposed in Fig. **??b**. As before, we begin the experiment with a highly coherent state and with $\delta_{\mathrm{s}} = 0$. Then, we quench on a nonzero δ_{s} and let the system evolve. Between shots, we scan δ_{s} while fixing $\chi N/2\pi = 0.9$ MHz and $E_{\mathrm{W}}/2\pi \approx 0.34$ MHz to explore the vertical cut.

The resulting dynamics show a marked change in the dynamical evolution of $|\Delta_{\mathrm{BCS}}|$ over the scan as shown in Fig. 48.9**b**, which we attribute to a transition between phase II and phase III dynamics. For small δ_{s}, we either see Higgs-like oscillations which are damped after 3 μs (the trace where $\delta_{\mathrm{s}}/2\pi = 0.6$ MHz) or, for very small splittings, no oscillations resolvable above the noise floor ($\delta_{\mathrm{s}}/2\pi = 0.3$ MHz). We associate this regime with phase II since it overlaps with the previously observed phase II dynamics in parameter space. For larger δ_{s}, curves instead show large-amplitude oscillations that persist for more than 5 μs ($\delta_{\mathrm{s}}/2\pi = 1.4$ MHz). We identify the long-lived oscillations in this parameter regime as an experimental signature of phase III.

Intuitively, we can understand the difference between the two phases by identifying the two sub-ensembles of atoms with two Bloch vectors (see Fig. 48.9**c**). In phase II, a finite δ_{s} causes the Bloch vectors to precess in different directions, but the dominant scale χN locks them together to form the solid and dashed magenta orbits. In the presence of a finite E_{W}, the orbits decay, but the Bloch vectors maintain phase coherence. On the other hand, in phase III δ_{s} is large enough that the two Bloch vectors accrue an unbounded relative phase, as in the blue

orbits. The presence of interactions locks each sub-ensemble separately against a finite E_{W}, leading to persistent oscillations. This effective beating of two large spins in a macroscopic array of spin-1/2 particles is truly an interaction-driven effect since the interactions are strong enough to lock the spins within each sub-ensemble but not strong enough to lock both sub-ensembles together. In our implementation of phase III, the bimodal distribution allows us to dynamically separate the Bloch vectors of the two sub-ensembles, instead of starting with an already split distribution like in weakly interacting BCS ground states featuring a sharp Fermi edge. Despite their qualitative differences, these two situations can be dynamically connected (see Methods).

We can experimentally define a boundary between phase II and phase III using the separation of timescales observed for oscillations in $|\Delta_{\mathrm{BCS}}|$. Fig. 48.9**e** shows the average oscillation amplitude in a time window from $t = 3$ μs to 8 μs. In this analysis, we observe a sharp rise in oscillation amplitude at $\delta_{\mathrm{s}}/2\pi = 0.85$ MHz $\approx \chi N/2\pi$ as we increase δ_{s}, which we identify as a dynamical phase transition. Numerical simulations plotted in Fig. 48.9**e** agree fairly well with data in capturing trend behaviour and estimating the phase transition point. However, we see a discrepancy in the absolute size of the observed and predicted oscillation amplitudes. We attribute this to an extra dephasing mechanism (likely residual motional effects) in our system or other imperfections in the experimental sequence not captured by the theory model.

Figure 48.10: **Scan across three dynamical phases. a**, Probing phase I, II and III dynamics using time traces of $|\Delta_{\mathrm{BCS}}|$. Quenches are performed in the same manner as in Fig. 48.9**b**, except between shots we hold post-quench values of δ_{s} fixed and vary χN instead. The inset shows the explored cut through the phase diagram and identifies final $\chi N/E_{\mathrm{W}}$ values with green (phase I), blue (phase III), and red (phase II) dots. The $\chi N/2\pi = 0.2$ MHz trace is smoothed for clarity. **b**, Time average of $|\Delta_{\mathrm{BCS}}|$ in a bin from $t = 3$ μs to 8 μs vs. interaction strength. The experimental data shows signatures of a phase I to phase III transition at $\chi N/2\pi = 0.25$ MHz. **c**, Oscillation frequency of $|\Delta_{\mathrm{BCS}}|$ vs. interaction strength in a bin from $t = 0.5$ μs to 4 μs. Again, we correct for systematics inferred from our data analysis and assume this correction has an uncertainty of 100%, shown by the green band. This data identifies a phase III to phase II transition at $\chi N/2\pi = 1.0$ MHz. Experimental data and transitions in both plots are consistent with numerical simulations.

We verify the location of the phase II to phase III transition using the short-time oscillation frequency (from $t = 0.5$ μs to 4 μs) as an additional experimental signature. As can be seen in the Fourier responses in Fig. 48.9**d** and quantified in Fig. 48.9**f**, the oscillation frequency exhibits a dip vs. δ_{s} at the previously-identified phase boundary. This dip is present in roughly

the same location for experiment and theory and is expected to coincide with the phase II to phase III transition (see Supplemental Online Material).

48.5 Scan across three dynamical phases

Finally, we observe all three dynamical phases in a single cut through parameter space, as shown in Fig. 48.10**a**. We run the same experimental sequence described in Fig. 48.9, but instead scan χN between shots with $\delta_s/2\pi = 1.1$ MHz and $E_W/2\pi = 0.46$ MHz fixed. This allows us to probe phase I, phase III and then phase II by increasing atom number N.

Using order parameters established in Figs. **??** and 48.9, we determine boundaries between the three phases. As shown in Fig. 48.10**b**, the long-time average of $|\Delta_{BCS}|$ rises suddenly around $\chi N/2\pi = 0.25$ MHz in both data and simulations. This transition marks the boundary between phase I and phase III. Additionally, at $\chi N/2\pi = 1.0$ MHz we observe a dip in the short-time oscillation frequency of $|\Delta_{BCS}|$ (Fig. 48.10**c**), marking a transition between phase III and phase II. For this scan, we do not use the long-time oscillation amplitude as an order parameter due to poor signal-to-noise for smaller values of χN.

48.6 Methods

48.6.1 Experimental setup: phase I to phase II transition

To explore the phase diagram cut in Fig. **??**, we first load $10^5 - 10^6$ ^{88}Sr atoms from a magneto-optical trap into an 813 nm optical lattice supported by a high-finesse optical cavity, similar to previous experiments [Norcia *et al.*(2016), Norcia *et al.*(2018b), Norcia *et al.*(2018a), Muniz *et al.*(2020)]. The resulting atomic cloud has a temperature of roughly 15 μK, resulting in a Gaussian distribution transverse to the cavity axis with standard deviation $\sigma_y = \sigma_z = 16$ μm (coordinates defined in Extended Data Fig. 48.11**a**). Further, the cloud is extended over thousands of lattice sites, forming a distribution along the cavity axis with a standard deviation $\sigma_x = 430$ μm. We measure an axial trapping frequency of $\omega_x/2\pi = 165$ kHz, giving a Lamb-Dicke parameter of $\eta = 0.17$ for excitation with 689 nm light. At the measured temperature, $\eta^2(2\bar{n}+1) = 0.11 \ll 1$, placing the atoms in the Lamb-Dicke regime. We set a quantisation axis along $\hat{y}$ with a 2.4 G magnetic field and tune the lattice polarisation to a "magic angle" relative to this axis, such that the differential lattice shift between ground ($|^1S_0\rangle$) and excited ($|^3P_1, m_J = 0\rangle$) states vanishes [Muniz *et al.*(2020)]. Using piezoelectric actuators, we stabilise the cavity length to set the closest TEM_{00} resonance to be 51 MHz red-detuned from the atomic transition.

After loading into the lattice, we initialise the atoms with a $\hat{y}$-polarised drive through the cavity which is nominally resonant with the atomic transition. Because the drive is far off-resonance from the cavity (which has linewidth $\kappa/2\pi = 153$ kHz at 689 nm), the induced Rabi frequency is somewhat suppressed. Nonetheless, we find that roughly 5 mW of power before the cavity is sufficient to drive the atoms with a $\pi/2$ pulse in 100 ns. We allow the atoms to settle for 2 μs in order to distinguish the desired physics from transient dynamics observed after state initialisation, which we attribute to undesired excitation of sideband transitions. We then shine a 461 nm beam from the side of the cavity along the $\hat{y}$ direction, detuned from the $|^1S_0\rangle - |^1P_1\rangle$ transition by more than 10 GHz, in order to induce AC Stark shifts on the ground state. The beam has waists $(w_x, w_z) = (1030\ \mu\text{m}, 75\ \mu\text{m})$ along the $\hat{x}$ and $\hat{z}$ directions at the plane of the atoms, and its centre is displaced from the centre of the atomic cloud by $x_0 = 580$ μm along the cavity axis. From these dimensions, we calculate an atomic density of states $\rho(\omega)$ as a function of frequency shift which is roughly uniform between 0 and a maximum shift $\hbar E_W$. We estimate

that for the power and detuning used in this cut, the 461 nm beam scatters off the atoms with an average rate of $R_{\rm sc}/2\pi = 1.3$ kHz, roughly a factor of six smaller than $\gamma/2\pi = 7.5$ kHz, the spontaneous emission rate. Combined with collective emission from the atoms as described in the Readout section of the Methods, these dissipation processes set a maximum predicted coherence time in the system of 29 μs.

Figure 48.11: **Experimental configuration.** **a**, Detailed diagram of the cavity and all relevant beams. A magnetic field along $\hat{y}$ sets the quantisation axis. The 813 nm optical lattice supported by the cavity has a tunable linear polarisation. We drive a $\pi/2$ pulse with a beam polarised along $\hat{y}$ through the cavity, and during the experiment we probe the cavity resonance frequency using a second $\hat{y}$-polarised beam to measure atom number. A 461 nm beam far-detuned from the $|^1S_0\rangle - |^1P_1\rangle$ transition shines on the atoms from the side of the cavity, inducing AC Stark shifts. We probe signals transmitted through the cavity using a balanced heterodyne detector. **b**, Fluorescence image of the two atomic clouds used when scanning through phase III in Figs. 48.9 and 48.10. **c**, Frequency landscape of 689 nm beams. The atomic drive frequency $\omega_{\rm drive}$ is resonant with the atomic transition. The cavity probe frequency $\omega_{\rm cp}$ is nominally centred with the cavity resonance frequency, 51 MHz red-detuned from the atomic transition. The local oscillator used in heterodyne detection has frequency $\omega_{\rm LO}$ and is 80 MHz blue-detuned from the atomic transition.

48.6.2 Experimental setup: cuts through phase III

For the two cuts through phase III described in Figs. 48.9 and 48.10, we load the atoms in two clouds separated by 3 mm, as shown in Extended Data Fig. 48.11**b**. The left cloud has an extent described by standard deviations $(\sigma_x, \sigma_z) = (200\ \mu\text{m}, 16\ \mu\text{m})$. The right cloud has a similar extent along σ_z but is broader along the cavity axis. We tune the lattice polarisation to point along $\hat{z}$, which breaks the magic angle condition and introduces a differential trap depth between ground and excited states of 0.47 MHz for atoms experiencing peak lattice intensity. Due to their finite temperature, the atoms experience a spread in lattice intensities which leads to an inhomogeneous trap depth. We estimate the induced distribution of energy shifts by assuming the atoms occupy a 2D Gaussian distribution radially with standard deviation $\sigma_y = \sigma_z = 16\ \mu$m, compared to the lattice waist $w_y = w_z = 80\ \mu$m. This produces a peaked distribution equivalent to the narrow peak in Fig. 48.9**a**.

In these experiments, we perform a $\pi/2$ pulse as before and then immediately shine a 461 nm beam centred on the left ("bright") atomic cloud. Unlike in the previous cut, we do not wait for transient dynamics to settle after state initialisation, for the sake of simplicity. We do not see major differences between observed and expected behaviour when omitting the wait period. The beam has waists $(w_x, w_z) = (1700\ \mu\text{m}, 80\ \mu\text{m})$. We install a beam block just

before the chamber that clips the beam tail that would otherwise hit the right ("dark") atomic cloud. The 3 mm separation between clouds is sufficiently large to ensure the beam does not significantly diffract around the beam block. The beam shifts the mean energy of the bright cloud away from that of the dark cloud, introducing a tunable δ_s. While nominally, we hold E_W fixed while scanning δ_s to explore the phase II to phase III transition, in reality the finite size of the blue beam introduces an additional contribution to E_W on the bright cloud. As δ_s increases, therefore, both the size and shape of the single-particle energy distribution changes. We calculate E_W in a consistent manner by estimating the standard deviation of the bright cloud distribution and matching the result to a uniform distribution with the same standard deviation (see Supplemental Online Material). In the main text, we report the value of E_W obtained at the phase transition point for the phase II to phase III transition. As we increase the 461 nm beam power, the atoms also scatter more blue photons. At the largest applied AC Stark shift, we estimate that the bright cloud experiences a scattering rate of $R_{sc}/2\pi = 3.4$ kHz, resulting in lower coherence times for traces with large δ_s. However, this excess decoherence does not bias our measurements of oscillation amplitude and frequency at times $t \leq 8$ μs.

48.6.3 Readout

After initialisation in all experiments, the atomic ensemble establishes a small electric field inside the cavity which adiabatically follows $\langle \hat{S}^- \rangle$ [Norcia *et al.*(2018a)]. Assuming homogeneous atom-light coupling (see next section for modifications due to inhomogeneous coupling), the complex amplitude of the electric field leaking out of the cavity is given by

$$\alpha_{\text{out}}(t) = -\frac{g}{\delta_c}\sqrt{\kappa_m}\langle \hat{S}^-(t)\rangle, \tag{48.2}$$

where α_{out} has units of $\sqrt{\text{photons/s}}$. Here, $2g/2\pi = 10.6$ kHz is the single-photon Rabi frequency for an atom maximally coupled to the cavity, $\delta_c/2\pi = (\omega_c - \omega_a)/2\pi = -51$ MHz is the detuning between the cavity resonance frequency ω_c and the atomic transition frequency ω_a, and $\kappa_m/2\pi = 41$ kHz is the rate at which photons incident on the cavity mirror are transmitted. α_{out} is a form of dissipation in the system equivalent to superradiance in a detuned cavity limit. Over the region of parameter space explored in this work, we estimate that the dissipation rate never exceeds $\gamma_{SR}/2\pi = 2.3$ kHz. We measure the detuned superradiant light as it leaks out of the cavity using balanced heterodyne detection, providing us with a real-time probe of $\langle \hat{S}^- \rangle \propto \Delta_{BCS}$. In plots of $|\Delta_{BCS}|$ in the main text, we calculate the square magnitude of this quantity and average over 400-1600 shots of the experiment, taken within 2-10 minutes. We then perform background subtraction to remove vacuum noise power from the heterodyne signal. Finally, we take a signed square root of the result to return an estimate of $|\Delta_{BCS}|$ which averages to zero in the absence of a real signal. This explains why some traces dip below zero despite representing a nonnegative quantity.

Additionally, the cavity experiences a (dispersive) shift in its resonance frequency proportional to the number of atoms. We use this fact to measure atom number by sending a pulsed probe tone through the cavity and measuring the frequency shift using the transmitted light. Since this light is spectrally resolved from the light emitted by the atoms, we are able to measure both signals independently on our heterodyne detector. The different optical frequencies involved in the heterodyne beat are compared in Extended Data Fig. 48.11**c**.

Figure 48.12: **Numerical simulation of the dynamical phase diagram based on Eq. (48.3).** We identify the dynamical phases based on the long-time average (**a**) and the long-time standard deviation (**b**) of $|\Delta_{\mathrm{BCS}}(t)|$, normalised by its initial value $\Delta_{\mathrm{init}} \equiv |\Delta_{\mathrm{BCS}}(0)|$. The white solid lines mark the corresponding dynamical phase boundaries, analytically derived from Eq. (48.1), which agree with the numerical results based on Eq. (48.3). The white dashed lines mark an extra dynamical phase transition that only exists for Eq. (48.1).

48.6.4 Dynamical phase diagram

The unitary dynamics of our system is modelled by an effective atom-only Hamiltonian, given by

$$\hat{H} = \hbar\chi \sum_{jk} \zeta_j \zeta_k \hat{S}_j^+ \hat{S}_k^- + \sum_k \varepsilon_k \hat{S}_k^z, \tag{48.3}$$

where $\hat{S}_k^{+,-}, \hat{S}_k^{x,y,z}$ are the standard spin-1/2 operators on atom k. We define $\chi = -g^2\delta_c/(\delta_c^2 + \kappa^2/4)$, where g and δ_c are as defined in the previous section, and κ is the cavity linewidth. The spatial dependence of the interaction term is characterised by $\zeta_j = \cos(j\phi)$ with $\phi = \pi\lambda_L/\lambda_c$, which arises because the lattice wavelength $\lambda_L = 813$ nm is incommensurate with the cavity wavelength $\lambda_c = 689$ nm. In contrast to Eq. (48.1), Eq. (48.3) becomes non-integrable due to the inhomogeneity in the interaction term. Nevertheless, as shown in Extended Data Fig. 48.12, Eq. (48.3) leads to a similar dynamical phase diagram as Eq. (48.1) if we

1. Use a generalised superconducting order parameter $\Delta_{\mathrm{BCS}} = \chi \sum_k \zeta_k \langle \hat{S}_k^- \rangle$;
2. Interpret the $\pi/2$-pulse as a pulse along the cavity axis under the Hamiltonian $\hat{H}_{\mathrm{drive}} = \hbar\Omega \sum_k \zeta_k S_k^y$ that generates the maximum possible $|\Delta_{\mathrm{BCS}}|$, which occurs when $\Omega t = 0.586\pi$;
3. Replace the atomic number N by an effective atom number $N_{\mathrm{eff}} = N/2$, such that χN_{eff} represents the averaged interaction strength of Eq. (48.3).

We can still measure the generalised order parameter Δ_{BCS} using the field leaking out of the cavity as in the previous section, since with inhomogeneous coupling the transmitted field takes the form $\alpha_{\mathrm{out}}(t) = -\frac{g}{\delta_c}\sqrt{\kappa_m} \sum_k \zeta_k \langle \hat{S}_k^-(t) \rangle \propto \Delta_{\mathrm{BCS}}$. The dynamical phase diagram in Extended Data Fig. 48.12 is numerically calculated based on unitary evolution under Eq. (48.3), with a single-particle dispersion $\varepsilon_k/\hbar$ sampled from a uniform distribution in the frequency range $[-\delta_{\mathrm{s}}/2 - E_{\mathrm{W}}/2, -\delta_{\mathrm{s}}/2 + E_{\mathrm{W}}/2]$ and $[\delta_{\mathrm{s}}/2 - E_{\mathrm{W}}/2, \delta_{\mathrm{s}}/2 + E_{\mathrm{W}}/2]$. There χN corresponds to the averaged interaction strength of Eq. (48.3). We identify the dynamical phases based on the long-time average of $|\Delta_{\mathrm{BCS}}|$, given by

$$\mathrm{Avg}(|\Delta_{\mathrm{BCS}}|) = \lim_{T\to\infty} \frac{1}{T} \int_0^T |\Delta_{\mathrm{BCS}}(t)| dt, \tag{48.4}$$

as well as the long-time oscillation amplitude of $|\Delta_{\mathrm{BCS}}|$. Since the oscillations in $|\Delta_{\mathrm{BCS}}|$ might deviate from a sinusoidal form, for theoretical simulations it is easier to use the standard deviation as a measure of the oscillation amplitude:

$$\begin{aligned}&\mathrm{Std}(|\Delta_{\mathrm{BCS}}|)\\&=\left[\lim_{T\to\infty}\frac{1}{T}\int_0^T\Big(|\Delta_{\mathrm{BCS}}(t)|-\mathrm{Avg}(|\Delta_{\mathrm{BCS}}|)\Big)^2dt\right]^{1/2}.\end{aligned}\tag{48.5}$$

When comparing to experimental data, we measure oscillation amplitude using the Fourier spectrum because technical noise in the experiment contributes to the standard deviation of the time traces (see Fig. 48.9**d**). The dynamical phases can be characterised in theoretical simulations by

- Phase I: $\mathrm{Avg}(|\Delta_{\mathrm{BCS}}|)=0$, $\mathrm{Std}(|\Delta_{\mathrm{BCS}}|)=0$.
- Phase II: $\mathrm{Avg}(|\Delta_{\mathrm{BCS}}|)>0$, $\mathrm{Std}(|\Delta_{\mathrm{BCS}}|)=0$.
- Phase III: $\mathrm{Avg}(|\Delta_{\mathrm{BCS}}|)>0$, $\mathrm{Std}(|\Delta_{\mathrm{BCS}}|)>0$.

Figure 48.13: **Alternative approach for phase III. a**, Simulation of an alternative experimental sequence. As described by the timing sequence at the top, we simulate an experiment that prepares the initial state using a $\pi/2$ pulse, lets the system evolve under a bimodal distribution of single-particle energy (see the inset) until $|\Delta_{\mathrm{BCS}}|$ reaches its minimum value, and then quenches the system back to a continuous distribution of single-particle energies (see the inset). The theoretically predicted time trace of $|\Delta_{\mathrm{BCS}}|$ with $\chi N/E_{\mathrm{W}}=1.0$ and $\delta_{\mathrm{s,init}}/E_{\mathrm{W}}=1.6$ is shown at the bottom. The blue (grey dashed) line shows phase III dynamics under a continuous (bimodal) distribution. **b**, Long-time standard deviation of $|\Delta_{\mathrm{BCS}}(t)|$ after quenching to the continuous distribution shown in **a**. The white lines are dynamical phase boundaries for bimodal distributions (see Extended Data Fig. 48.12). Nearly all the choices of parameter for phase III using bimodal distributions can lead to phase III behaviours after quenching to the continuous distribution.

The dynamical phase boundaries (white solid lines) in Extended Data Fig. 48.12 are analytically calculated using a Lax analysis applied to Eq. (48.1), similar to the one discussed in [Lewis-Swan *et al.*(2021), Marino *et al.*(2022)], and take the following form (see Supplemental Online Material for a detailed derivation):

- Phase I to phase II:

$$\begin{aligned}&\frac{\chi N}{E_{\mathrm{W}}}=\frac{1}{\pi}\quad\text{with}\quad\frac{\delta_{\mathrm{s}}}{E_{\mathrm{W}}}\in[0,1],\\&\frac{\delta_{\mathrm{s}}}{E_{\mathrm{W}}}=1\quad\text{with}\quad\frac{\chi N}{E_{\mathrm{W}}}\in\left[\frac{1}{\pi},\frac{2}{\pi}\right].\end{aligned}\tag{48.6}$$

- Phase I to phase III:

$$\frac{\chi N}{E_{\mathrm{W}}} = \frac{2}{\pi} \quad \text{with} \quad \frac{\delta_{\mathrm{s}}}{E_{\mathrm{W}}} > 1. \tag{48.7}$$

- Phase II to phase III:

$$\frac{\delta_s}{E_{\mathrm{W}}} = \csc\left(\frac{E_{\mathrm{W}}}{\chi N}\right) \quad \text{with} \quad \frac{\chi N}{E_{\mathrm{W}}} > \frac{2}{\pi}. \tag{48.8}$$

The analytical results agree with the numerical simulations for Eq. (48.3). The only difference is that Eq. (48.1) predicts an extra dynamical phase transition marked by the white dashed line. The dynamical phase boundaries shown in Fig. **??c** are constructed by the analytical formulas above.

48.6.5 Phase III dynamics: the case of a continuous single-particle dispersion

In this manuscript, we generate phase III using a bimodal single-particle dispersion, represented with idealized assumptions by Fig. **??b** and with actual experimental conditions by Fig. 48.9**a**. Here we show that this experimentally convenient approach generates similar phase III dynamics to the one obtained in the case of a continuous dispersion but with different initial conditions.

This is done by the protocol shown in Extended Data Fig. 48.13**a**, which uses a bimodal distribution ($\delta_{\mathrm{s,init}} > E_{\mathrm{W}}$) just to generate a state with minimum $|\Delta_{\mathrm{BCS}}|$. At this point the system's dispersion is restored to be continuous by setting $\delta_{\mathrm{s,final}} = E_{\mathrm{W}}$. This approach more closely resembles the phase III quench discussed in actual BCS superconductors, where phase III is observed by quenching from a state with weak BCS paring gap $|\Delta_{\mathrm{BCS}}|$ to one with a strong pairing gap [Yuzbashyan *et al.*(2015)]. Numerical simulations based on Eq. (48.3) show that nearly all choices of parameters that lead to phase III using a bimodal distribution also lead to phase III dynamics when quenching to a continuous distribution. The only exception is a small parameter regime close to the boundary between phase III and phase II (see Extended Data Fig. 48.13**b**). Note that here we use definitions for Δ_{BCS}, the $\pi/2$ pulse, and χN which correspond to Eq. (48.3), as explained in the previous section.

48.6.6 Numerical simulations

The black dashed lines in Figs. **??**, 48.9, and 48.10 are computed from unitary evolution under Eq. (48.3) using a single-particle dispersion ε_k, sampled from the experimentally engineered distribution.

The black solid lines in the same figures are obtained by adding dissipative processes and axial motion to Eq. (48.3). The system dynamics is described by the following master equation for the density matrix $\hat{\rho}$:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \mathcal{L}(\hat{L}_c)[\hat{\rho}] + \sum_k \mathcal{L}(\hat{L}_{s,k})[\hat{\rho}] + \sum_k \mathcal{L}(\hat{L}_{\mathrm{el},k})[\hat{\rho}]. \tag{48.9}$$

The Lindblad superoperator takes the form $\mathcal{L}(\hat{L})[\hat{\rho}] = \hat{L}\hat{\rho}\hat{L}^\dagger - \frac{1}{2}(\hat{L}^\dagger\hat{L}\hat{\rho} + \hat{\rho}\hat{L}^\dagger\hat{L})$. Superradiance through the cavity is described by the jump operator

$$\hat{L}_c = \sqrt{\Gamma}\sum_k \zeta_k \hat{S}_k^-, \tag{48.10}$$

where $\Gamma = \chi\kappa/\delta_c$. Spontaneous emission from the atomic excited state is described by the jump operator

$$\hat{L}_{s,k} = \sqrt{\gamma}\hat{S}_k^- , \tag{48.11}$$

where $\gamma/2\pi = 7.5$ kHz is the spontaneous emission rate out of $^3P_1$. Single-particle decoherence is described by the jump operator

$$\hat{L}_{\mathrm{el},k} = \sqrt{2\gamma_{\mathrm{el}}}\hat{S}_k^z, \tag{48.12}$$

where γ_{el} is a fitting parameter taking into account free space scattering from the AC Stark shift beam, as well as other decoherence processes in the experiment (see Supplemental Online Material). These are the dominant dissipative processes in our system.

The axial trapping frequency of the lattice is 165 kHz and is therefore smaller than the spin-exchange interaction rate χN for most of the experiments. As a consequence, in contrast to the idealised model where atoms are assumed to be frozen, motional processes need to be accounted for, even though they are suppressed in the Lamb-Dicke regime. As shown in the Supplemental Online Material, axial motion can lead to a faster damping rate of $|\Delta_{\mathrm{BCS}}|$ oscillations. The predicted dynamical phase boundaries are nevertheless unaffected by the axial motion.

All the numerical simulations are computed using the mean-field approximation, which replaces the operators $\hat{S}_k^{x,y,z}$ by their expectation values $\langle\hat{S}_k^{x,y,z}\rangle$ in the Heisenberg equation of motion. The mean-field treatment of the BCS model is predicted to be exact in the thermodynamic limit due to the infinite-range nature of the interactions [Yuzbashyan *et al.*(2015)]. The atom number for numerical simulation is set to 5000 for the ideal conditions and 2000 for actual experimental conditions. We rescale χ to match χN with experimental values.

[H]

Figure 48.14: **Collective scaling in damped phase II oscillations.** **a**, Time dynamics of $|\Delta_{\mathrm{BCS}}|$ measured after engineering an initial phase spread over $[0, \varphi_0]$ where $\varphi_0 = 0.8\pi$ as in Fig. 2**d**, plotted in absolute frequency units (pink trace). The solid black curve represents a numerical simulation of the full system, whereas the dashed curve represents an ideal simulation neglecting dissipation and motional effects. We obtain a crude estimate of oscillation frequency in the experimental data by fitting a trough and peak to smoothed data (after subtracting slow-moving behaviour) within the first couple μs (magenta points), using these points to infer a half period of oscillation, and with uncertainties determined using a 90% amplitude threshold (pink bands). **b**, Comparing oscillation frequency estimates of experimental data (pink squares) with those of ideal simulations (black dots) for different φ_0. Theory oscillation frequencies are calculated using a Fourier transform from $t = 0$ μs to $t = 5$ μs. Error bars for experimental data are set by the minimum and maximum frequencies implied by uncertainties in the half period shown in **a**. The two frequency estimates agree within error bars. **c**, Collective scaling of oscillation frequency. For each φ_0 measured in the experiment, we plot the oscillation frequency against the long-time BCS gap Δ_∞, calculated at $t = 18$ μs for ideal simulations and at $t = 3$ μs for experimental data. The solid black line is defined by $\omega_{\mathrm{osc}} = 2\Delta_\infty$, demonstrating the expected scaling for Higgs oscillations. The dashed pink line represents a linear fit to the experimental data. The pink band shows the uncertainty in the slope assuming correlated error in ω_{osc}, such that its bounds are defined by linear fits to the data assuming maximum and minimum values for ω_{osc} as defined by the error bars.

48.6.7 Higgs-like behaviour in short-time phase II dynamics

When quenching into phase II, we observe highly damped oscillations in $|\Delta_{\mathrm{BCS}}|$, reminiscent of the Higgs oscillations predicted to arise in this regime of the BCS model. Here, we analyse traces from Fig. **??d**, in which we engineer a variable phase spread $\varphi(\omega_k) \in [0, \varphi_0]$ before quenching into phase II, to study this potential connection.

In the BCS model, Higgs oscillations can be characterised by their frequency, which should scale with the long-time BCS order parameter Δ_∞ as $\omega_{\mathrm{osc}} = 2\Delta_\infty$ [Yuzbashyan *et al.*(2015)]. We confirm this scaling in theory by measuring the oscillation frequency from $t = 0$ μs to $t = 5$ μs in idealised numerical simulations ignoring dissipation and motional effects (black dashed line in Extended Data Fig. 48.14**a**). For different values of the phase spread extent φ_0, the system reaches its steady state at a different long-time BCS gap Δ_∞. By parametrically plotting the oscillation frequency vs. $2\Delta_\infty$ as a function of φ_0 in panel **c**, we observe the expected scaling.

As discussed in the main text, oscillations in $|\Delta_{\mathrm{BCS}}|$ are consistently smaller and decay more quickly in experiment than in theory. Nonetheless, we obtain a crude estimate of the experimental oscillation frequency by measuring a half period from the first trough and peak

of $|\Delta_{\mathrm{BCS}}(t)|$, as shown in panel **a**. In panel **b**, we compare the frequency in experimental data to that of ideal simulations for different φ_0 and show that the frequencies agree within error bars. This suggests that the transient dynamics observed in $|\Delta_{\mathrm{BCS}}|$ are related to the Higgs oscillations present in theory.

Although the experimental oscillation frequency agrees with simulations, the steady-state order parameter Δ_∞ is much smaller, as can be seen in Extended Data Fig. 48.14**a**. As a result, the measured frequencies scale linearly with Δ_∞ but with a different prefactor. In panel **c**, we fit a linear relation of $\omega_{\mathrm{osc}} = (1.7^{+0.7}_{-0.4}) \times 2\Delta_\infty$ to the data, where the slope uncertainty bounds are calculated assuming errors in ω_{osc} are perfectly correlated. Most of the reduction in Δ_∞ can be captured in theory by considering dissipation and motional effects (solid black trace). We see an additional small difference in $|\Delta_{\mathrm{BCS}}|$ between full numerical simulations and experimental data, which we attribute to drifts in experimental alignments and calibration factors over time. This difference is not apparent in Fig. **??d** because we plot $|\Delta_{\mathrm{BCS}}|$ in normalised units.

48.7 Dynamical phase diagram

In this section, we perform detailed analysis of the dynamical phase diagram shown in Fig. 1**c** in the main text. We start from analytic calculation in the case of homogeneous couplings, and then generalize to the case of inhomogeneous couplings. Finally we discuss the application of our findings to experimental conditions.

48.7.1 Homogeneous model

(??? I don't understand this at all right now)

First we discuss the dynamical phases for the BCS Hamiltonian with homogeneous couplings,

$$\hat{H} = \hbar\chi\hat{S}^+\hat{S}^- + \sum_k \varepsilon_k \hat{S}^z_k. \tag{48.13}$$

We will set $\hbar = 1$. As shown in Ref. [Yuzbashyan *et al.*(2015), Lewis-Swan *et al.*(2021)], the dynamical phases can be determined using a mean-field Lax vector analysis. The Lax vector is defined as $\vec{L}(u) = L^x(u)\hat{x} + L^y(u)\hat{y} + L^z(u)\hat{z}$ with components,

$$L^x(u) = \sum_k \frac{S^x_k(0)}{u - \varepsilon_k/2}, \quad L^y(u) = \sum_k \frac{S^y_k(0)}{u - \varepsilon_k/2}, \quad L^z(u) = -\frac{1}{\chi} - \sum_i \frac{S^z_k(0)}{u - \varepsilon_k/2}, \tag{48.14}$$

where $S^{x,y,z}_k(0)$ are the expectation value of operators $\hat{S}^{x,y,z}_k$ in the initial state.

Here we consider the initial state as $S^x_k(0) = 1/2$, $S^y_k(0) = S^z_k(0) = 0$, and ε_k is chosen from a uniform distribution in the frequency range $[-\delta_{\mathrm{s}}/2 - E_{\mathrm{W}}/2, -\delta_{\mathrm{s}}/2 + E_{\mathrm{W}}/2]$ and $[\delta_{\mathrm{s}}/2 - E_{\mathrm{W}}/2, \delta_{\mathrm{s}}/2 + E_{\mathrm{W}}/2]$ (why this is the range???).

In this case, the mean-field Lax vector takes the following form: (??? there is an idea of averaging here??? not clear how MF leads to this expression.... what is N?)

$$\begin{aligned}
\chi L^x(u) &\approx \frac{\chi N}{2}\left[\frac{1}{2E_{\mathrm{W}}}\int_{-\delta_{\mathrm{s}}/2-E_{\mathrm{W}}/2}^{-\delta_{\mathrm{s}}/2+E_{\mathrm{W}}/2} \frac{dx}{u - x/2} + \frac{1}{2E_{\mathrm{W}}}\int_{\delta_{\mathrm{s}}/2-E_{\mathrm{W}}/2}^{\delta_{\mathrm{s}}/2+E_{\mathrm{W}}/2} \frac{dx}{u - x/2}\right] \\
&= \frac{\chi N}{2E_{\mathrm{W}}}\left[\ln\left(u + \frac{\delta_{\mathrm{s}}}{4} + \frac{E_{\mathrm{W}}}{4}\right) - \ln\left(u + \frac{\delta_{\mathrm{s}}}{4} - \frac{E_{\mathrm{W}}}{4}\right) + \ln\left(u - \frac{\delta_{\mathrm{s}}}{4} + \frac{E_{\mathrm{W}}}{4}\right) - \right. \\
&\qquad \left. - \ln\left(u - \frac{\delta_{\mathrm{s}}}{4} - \frac{E_{\mathrm{W}}}{4}\right)\right], \\
\chi L^y(u) &= 0, \\
\chi L^z(u) &= -1.
\end{aligned} \tag{48.15}$$

Note that $\ln z$ in the complex plane is a multivalued function. Here we take the principal value $\ln z = \ln|z| + i\mathrm{Arg}(z)$, where $\mathrm{Arg}(z)$ is the argument of z restricted in the interval $(-\pi, \pi]$. Directly combining the logarithm functions might lead to moving out of the principal branch.

One can define the dynamical phases based on the number of complex roots of equation $\vec{L}(u) \cdot \vec{L}(u) = 0$ (read Yuzbashyan): Phase I has zero complex roots, phase II has a pair of complex roots, phase III has two pairs of complex roots. Whether the complex roots have non-zero or vanishing real parts could be used for further separation of the phases. In our case, the equation $\vec{L}(u) \cdot \vec{L}(u) = 0$ takes the following form,

$$\frac{\chi N}{2E_{\mathrm{W}}}\left[\ln\left(u+\frac{\delta_{\mathrm{s}}}{4}+\frac{E_{\mathrm{W}}}{4}\right)-\ln\left(u+\frac{\delta_{\mathrm{s}}}{4}-\frac{E_{\mathrm{W}}}{4}\right)+\ln\left(u-\frac{\delta_{\mathrm{s}}}{4}+\frac{E_{\mathrm{W}}}{4}\right)-\ln\left(u-\frac{\delta_{\mathrm{s}}}{4}-\frac{E_{\mathrm{W}}}{4}\right)\right]=\pm i. \tag{48.16}$$

We find four dynamical phases based on analyzing the roots of Eq. (48.16):

- Phase I: No complex roots, which exist in the regime

$$\frac{\delta_{\mathrm{s}}}{E_{\mathrm{W}}}<1, \quad \frac{\chi N}{E_{\mathrm{W}}}<\frac{1}{\pi} \qquad \text{or} \qquad \frac{\delta_{\mathrm{s}}}{E_{\mathrm{W}}}>1, \quad \frac{\chi N}{E_{\mathrm{W}}}<\frac{2}{\pi}. \tag{48.17}$$

- Phase II: A pair of complex roots,

$$\frac{u}{E_{\mathrm{W}}}=\pm\frac{i}{4}\left[\cot\left(\frac{E_{\mathrm{W}}}{\chi N}\right)+\sqrt{\csc^2\left(\frac{E_{\mathrm{W}}}{\chi N}\right)-\frac{\delta_s^2}{E_{\mathrm{W}}^2}}\right], \tag{48.18}$$

  which exist in the regime

$$\frac{\delta_{\mathrm{s}}}{E_{\mathrm{W}}}<1, \quad \frac{\chi N}{E_{\mathrm{W}}}>\frac{1}{\pi}. \tag{48.19}$$

- Phase IIIa: Two pairs of complex roots with vanishing real parts,

$$\frac{u_1}{E_{\mathrm{W}}}=\pm\frac{i}{4}\left[\cot\left(\frac{E_{\mathrm{W}}}{\chi N}\right)+\sqrt{\csc^2\left(\frac{E_{\mathrm{W}}}{\chi N}\right)-\frac{\delta_s^2}{E_{\mathrm{W}}^2}}\right], \tag{48.20}$$

$$\frac{u_2}{E_{\mathrm{W}}}=\pm\frac{i}{4}\left[\cot\left(\frac{E_{\mathrm{W}}}{\chi N}\right)-\sqrt{\csc^2\left(\frac{E_{\mathrm{W}}}{\chi N}\right)-\frac{\delta_s^2}{E_{\mathrm{W}}^2}}\right], \tag{48.21}$$

  which exist in the regime

$$\frac{\delta_{\mathrm{s}}}{E_{\mathrm{W}}}>1, \quad \frac{\chi N}{E_{\mathrm{W}}}>\frac{2}{\pi}, \quad \frac{\delta_{\mathrm{s}}}{E_{\mathrm{W}}}<\csc\left(\frac{E_{\mathrm{W}}}{\chi N}\right). \tag{48.22}$$

  In phase IIIa, the order parameter, Δ_{BCS} oscillates around a non-zero value (non-ZOPA) as pointed out in Ref. [Collado *et al.*(2023), Lewis-Swan *et al.*(2021)].

- Phase IIIb: Two pairs of complex roots with non-zero real parts,

$$\frac{u_1}{E_{\mathrm{W}}}=\frac{1}{4}\left[\sqrt{\delta_s'^2-\csc^2\left(\frac{1}{\chi' N}\right)}\pm i\cot\left(\frac{1}{\chi' N}\right)\right], \quad \frac{u_2}{E_{\mathrm{W}}}=\frac{1}{4}\left[-\sqrt{\delta_s'^2-\csc^2\left(\frac{1}{\chi' N}\right)}\pm i\cot\left(\frac{1}{\chi' N}\right)\right], \tag{48.23}$$

  which exist in the regime

$$\frac{\delta_{\mathrm{s}}}{E_{\mathrm{W}}}>1, \quad \frac{\chi N}{E_{\mathrm{W}}}>\frac{2}{\pi}, \quad \frac{\delta_{\mathrm{s}}}{E_{\mathrm{W}}}>\csc\left(\frac{E_{\mathrm{W}}}{\chi N}\right). \tag{48.24}$$

  In phase IIIb, Δ_{BCS} oscillates with zero order parameter average (ZOPA) as explained in Ref. [Collado *et al.*(2023), Lewis-Swan *et al.*(2021)] (read these articles!).

The dynamical phases derived from the Lax analysis above are supported by numerical evidences, as shown in Fig. 48.15**a** and Fig. 48.15**b**. We numerically solve the dynamics of $\Delta_{\mathrm{BCS}} = \chi\langle\hat{S}^{-}\rangle$ under Eq. (48.13) based on mean field approximation, and then identify dynamical phases based on long-time average of $|\Delta_{\mathrm{BCS}}|$,

$$\mathrm{Avg}(|\Delta_{\mathrm{BCS}}|) = \lim_{T\to\infty}\frac{1}{T}\int_0^T |\Delta_{\mathrm{BCS}}(t)|dt, \tag{48.25}$$

and long-time oscillation amplitude of $|\Delta_{\mathrm{BCS}}|$. Since the oscillations in $|\Delta_{\mathrm{BCS}}|$ might deviates from a sinusoidal form, it is easier to use the standard deviation as a measure of the oscillation amplitude,

$$\mathrm{Std}(|\Delta_{\mathrm{BCS}}|) = \left[\lim_{T\to\infty}\frac{1}{T}\int_0^T \Big(|\Delta_{\mathrm{BCS}}(t)| - \mathrm{Avg}(|\Delta_{\mathrm{BCS}}|)\Big)^2 dt\right]^{1/2}, \tag{48.26}$$

although experimentally it's better to use the peak of Fourier spectrum to suppress the noise (see Fig. 3**d** in the main text). The dynamical phases can be characterized by

- Phase I: $\mathrm{Avg}(|\Delta_{\mathrm{BCS}}|) = 0$, $\mathrm{Std}(|\Delta_{\mathrm{BCS}}|) = 0$.
- Phase II: $\mathrm{Avg}(|\Delta_{\mathrm{BCS}}|) > 0$, $\mathrm{Std}(|\Delta_{\mathrm{BCS}}|) = 0$.
- Phase III: $\mathrm{Avg}(|\Delta_{\mathrm{BCS}}|) > 0$, $\mathrm{Std}(|\Delta_{\mathrm{BCS}}|) > 0$.

Since ε_k is chosen from a distribution with particle-hole symmetry (symmetric about 0), Δ_{BCS} becomes a real number in this case. One can further separate phase IIIa and phase IIIb by the behavior of Δ_{BCS} shown in Fig. 48.15**e** and Fig. 48.15**f**.

Figure 48.15: **Dynamical phase diagrams.** **a** and **b**, Dynamical phase diagram of the homogeneous model normalized by $\Delta_{\text{init}}/\chi N = 1/2$, where Δ_{init} is the initial value of $|\Delta_{\text{BCS}}|$. The white lines are the dynamical critical points derived from the Lax analysis. **c** and **d**, Dynamical phase diagram of the inhomogeneous model normalized by $\Delta_{\text{init}}/\chi N_{\text{eff}} = \mathcal{J}_1(\Omega\tau)$. The white lines are the same as the homogeneous model. **e**, Time evolution of Δ_{BCS} at $\delta_{\text{s}}/E_{\text{W}} = 1.1$, $\chi N/E_{\text{W}} = 1.0$ under the homogeneous model (phase IIIa). **f**, Time evolution of Δ_{BCS} at $\delta_{\text{s}}/E_{\text{W}} = 1.6$, $\chi N/E_{\text{W}} = 1.0$ under the homogeneous model (phase IIIb). **g**, Time evolution of Δ_{BCS} at $\delta_{\text{s}}/E_{\text{W}} = 1.6$, $\chi N/E_{\text{W}} = 1.0$ under the inhomogeneous model (phase III).

48.7.2 Inhomogeneous model

Here we discuss the dynamical phases for the BCS Hamiltonian with inhomogeneous coupling,

$$\hat{H} = \hbar\chi \sum_{jk} \zeta_j \zeta_k \hat{S}_j^+ \hat{S}_k^- + \sum_k \varepsilon_k \hat{S}_k^z, \tag{48.27}$$

where ζ_k is generated by random sampling of $\cos(x)$, with x chosen from a uniform distribution in the interval $[0, 2\pi)$. Similar to the homogeneous model, $\varepsilon_k/\hbar$ is still chosen from a uniform distribution in the frequency range $[-\delta_{\text{s}}/2 - E_{\text{W}}/2, -\delta_{\text{s}}/2 + E_{\text{W}}/2]$ and $[\delta_{\text{s}}/2 - E_{\text{W}}/2, \delta_{\text{s}}/2 + E_{\text{W}}/2]$. In this case, we explore the dynamical phases numerically since the Lax analysis is not applicable. As shown in Fig. 48.17**c** and Fig. 48.17**d**, one can obtain similar dynamical phases as the homogeneous model: Phase I remains the same, Phase IIIa merges into Phase II, and Phase IIIb becomes the new Phase III. The phase boundary can be roughly captured by the analytical solution of the homogeneous model. Note that χN_{eff} is the averaged interaction strength in the inhomogeneous case, where $N_{\text{eff}} = N/2$. The superconducting order parameter is defined as $\Delta_{\text{BCS}} = \chi \sum_k \zeta_k \langle \hat{S}_k^- \rangle$. The initial condition is chosen as the maximum $|\Delta_{\text{BCS}}|$ one can achieved by an external drive along the cavity axis, $\hat{H}_{\text{drive}} = \Omega \sum_k \zeta_k \hat{S}_k^y$. Assuming the initial state can be prepared by applying $\hat{H}_{\text{drive}}$ for a time τ, we have

$$\frac{\Delta_{\text{init}}}{\chi N_{\text{eff}}} \approx \frac{1}{2\pi} \int_0^{2\pi} dx \cos(x) \sin(\Omega\tau \cos(x)) = \mathcal{J}_1(\Omega\tau), \tag{48.28}$$

where $\mathcal{J}_n$ is the Bessel function of the first-kind, and the maximum of $\mathcal{J}_1(\Omega\tau)$ can be achieved at $\Omega\tau = 0.586\pi$. It is worth to mention that $\Delta_{\rm BCS}$ is a real number initially, but it becomes a complex number during the time evolution, as shown in Fig. 48.17**g**.

Figure 48.16: **Experimental control of dynamical phases.** **a** and **b**, Dynamical phase diagram for the experiment with two atomic ensembles, in terms of averaged spin-exchange interaction strength $\chi N_{\rm eff}$ and peak AC Stark shift $f_{\rm AC}$. The white lines show the predicted dynamical phase boundaries to guide the eye. The white dashed line marks a small region of phase II' due to the imbalance of $E_{\rm W}$ for the two atomic ensembles. **c**, $E_{\rm W}$ as a function of peak AC Stark shift $f_{\rm AC}$, with AC Stark shift applying to atomic cloud 1. **d**, $\delta_{\rm s}$ as a function of peak AC Stark shift $f_{\rm AC}$ (red line). The dashed line marks the place where $\delta_{\rm s} = f_{\rm AC}$.

48.7.3 Experimental control of dynamical phases

Here we elaborate on the experimental implementation of the Hamiltonian Eq. (48.27). As discussed in the previous section, we would like to approximately engineer single-particle energies $\varepsilon_k/\hbar$ sampled from a uniform distribution in the frequency range $[-\delta_{\rm s}/2 - E_{\rm W}/2, -\delta_{\rm s}/2 + E_{\rm W}/2]$ and $[\delta_{\rm s}/2 - E_{\rm W}/2, \delta_{\rm s}/2 + E_{\rm W}/2]$. The two different experimental schemes used in the main text to explore the energy distribution are summarized in the following table:

	Description	Approx. $\varepsilon_k/\hbar$
Scheme I (Fig. 2, main text)	1) Single atomic cloud 2) AC Stark shift	$[-\tilde{E}_{\rm W}/2, \tilde{E}_{\rm W}/2]$
Scheme II (Fig. 3, 4, main text)	1) Two atomic clouds 2) AC Stark shift to cloud 1	Cloud 1: $[-\delta_{\rm s}/2 - E_{\rm W}/2, -\delta_{\rm s}/2 + E_{\rm W}/2]$ Cloud 2: $[+\delta_{\rm s}/2 - E_{\rm W}/2, +\delta_{\rm s}/2 + E_{\rm W}/2]$

The first scheme is used to probe the phase I to phase II transition. We use a single atomic ensemble and apply an AC Stark shift beam with a gradient to approximately engineer $\varepsilon_k/\hbar$ from a uniform distribution $[-\tilde{E}_{\rm W}/2, \tilde{E}_{\rm W}/2]$, as discussed in the Methods. As shown in Fig. 2**a** in the main text, the distribution of atomic frequencies is not exactly uniform, so we calculate the variance of the frequency distribution experimentally. Theoretically we assign a spread $\tilde{E}_{\rm W}$ such that the uniform distibution over $[-\tilde{E}_{\rm W}/2, \tilde{E}_{\rm W}/2]$ matches the measured experimental variance. We use this scheme to probe the dynamical phase diagram at $\delta_{\rm s} = 0$ (see Fig. 1**c** in the main text).

It is worth mentioning that the uniform distribution $[-\tilde{E}_{\rm W}/2, \tilde{E}_{\rm W}/2]$ can be interpreted in two different ways: 1) $\delta_{\rm s} = 0$ and $E_{\rm W} = \tilde{E}_{\rm W}$; 2) $\delta_{\rm s} = E_{\rm W} = \tilde{E}_{\rm W}/2$. Here we prefer the first interpretation $\delta_{\rm s} = 0$ because in this scheme we only have a single control parameter

(the strength of AC Stark shift beam). Additionally, the line $\delta_s = E_W$ in the dynamical phase diagram has an implication that a small perturbation of δ_s can generate a gap in atomic frequency, which is prohibited under this mapping between experimental controls and the model parameters.

In the second scheme that probes transitions into phase III, we use two atomic ensembles and apply an AC Stark shift beam (peak AC Stark shift f_{AC}) to the first ensemble to generate a frequency splitting δ_s between the two ensembles. In contrast to the first scheme, as discussed in the Methods, here we instead use the differential lattice light shifts to engineer a frequency spread E_W for each ensemble. As shown in Fig. 3**a** in the main text, in this case we define δ_s as the mean frequency difference between the two ensembles, and E_W as the width of a uniform distribution generating the same variance.

It is worth mentioning that the Gaussian profile of the AC Stark shift beam leads to an increase in E_W for the first atomic ensemble, as well as a reduction of the expected splitting of the two ensembles $\delta_s < f_{AC}$, as shown in Fig. 48.16**c** and **d**. Using experimental parameters, we get the dynamical phase diagram as depicted in Fig. 48.16**a** and **b**. The imbalance of E_W for the two atomic ensembles can lead to a small region of phase II' marked by the white dashed line. This occurs because the spin-exchange interaction is able to lock the ensemble with smaller E_W, while the ensemble with larger E_W remains unlocked, which leads to $|\Delta_{BCS}|$ approaching a small but nonzero constant value. In the experiment, due to other dissipative processes and reduced signal-to-noise ratio for small χN, we do not observe a difference between phase I and phase II'. This is the cause of a small discrepancy between theory and experiment in Fig. 4**b** in the main text in identifying the position of the phase transition.

48.8 Short-time signatures of dynamical phases

In this section, we discuss the properties of the dynamical phases using short-time observables, since dissipative processes and noise in the experiment lead to difficulties in measuring long-time observables. In the following, we show that phase I can be characterized by the fast decay of $|\Delta_{BCS}|$, phase II can be characterized by Higgs oscillations. We further show that the phase II to phase III transition can be captured by the dip in the short-time oscillation frequency of $|\Delta_{BCS}|$. Finally, we provide an explanation of the frequency dip using an analytical solution of the two-spin BCS model.

48.8.1 Phase I: fast decay

In phase I, the single-particle energy term $\sum_k \varepsilon_k \hat{S}_k^z$ dominates over the spin-exchange interaction. To leading order, one can calculate $|\Delta_{BCS}|$ in the homogeneous model by dropping the interaction term, which gives

$$\begin{aligned}\frac{|\Delta_{BCS}|}{\chi N} &\approx \frac{1}{2N}\left|\sum_k e^{-i\varepsilon_k t/\hbar}\right| = \frac{1}{2}\left|\frac{1}{2E_W}\int_{-\delta_s/2-E_W/2}^{-\delta_s/2+E_W/2} e^{-ixt}dx + \frac{1}{2E_W}\int_{\delta_s/2-E_W/2}^{\delta_s/2+E_W/2} e^{-ixt}dx\right| \\ &= \frac{1}{2}\left|\cos\left(\frac{\delta_s}{2}\right)\right| \cdot \left|\frac{\sin(E_W t/2)}{E_W t/2}\right|.\end{aligned} \tag{48.29}$$

The decay profile of $|\Delta_{BCS}|$ is set by a sinc function with a $1/e$ coherence time t satisfying $E_W t/2\pi \approx 0.7$. For the inhomogeneous model a similar fast decay time scale of the order of $E_W t/2\pi \sim 1$ can be derived. As shown in Fig. 2**b** in the main text, we observe fast decay of $|\Delta_{BCS}|$ within 1 μs in phase I. The decay time scale for the other dynamical phases can be more than 10 times longer.

48.8.2 Phase II: Higgs oscillation

Figure 48.17: **Relation between oscillation frequency and averaged order parameter in Higgs oscillations. a**, Homogeneous model where each point is a choice of $(\chi N, \delta_{\mathrm{s}})$ in phase II (red) and phase IIIa (blue). The dashed line represents $\omega = 2\mathrm{Avg}(|\Delta_{\mathrm{BCS}}|)$. **b**, Inhomogeneous model where each point is a choice of $(\chi N_{\mathrm{eff}}, \delta_{\mathrm{s}})$ in phase II. The inset shows the points with $\delta_{\mathrm{s}} = 0$.

Higgs oscillation, generated by collective excitation of the Higgs mode in BCS superconductor, is characterized by the oscillation of $|\Delta_{\mathrm{BCS}}|$ at frequency $\omega = 2\mathrm{Avg}(|\Delta_{\mathrm{BCS}}|)$ [Yuzbashyan *et al.*(2015)]. For the homogeneous model (see Fig. 48.17**a**), we numerically confirmed this relation for all the points in phase II and phase IIIa. For the inhomogeneous model (see Fig. 48.17**b**), this relation is approximately satisfied in phase II. In experiment, we observe hints of Higgs oscillation (see Fig. 2 in the main text), which can be ideally described by the inhomogeneous model with $\delta_{\mathrm{s}} = 0$ (see the inset in Fig. 48.17**b**).

48.8.3 Transition to phase III: frequency dip

In the main text, we discuss a way to understand the phase II to phase III transition by visualising the two atomic ensembles as two large spins. For the inhomogeneous model, phase II exists in the small δ_{s} regime, where the two spins lock to each other and form a single large spin through spin-exchange interactions. In this case the many-body gap protection leads to the damped oscillations observed in phase II. Increasing δ_{s} in phase II leads to the reduction of the many-body gap, and hence to a decrease of the corresponding oscillation frequency. Phase III exists in the large δ_{s} regime, where the spin locking occurs separately in each ensemble, and the two large spin are instead precessing around each other, with a rate set by the splitting δ_{s} and the spin-exchange interaction. Increasing δ_{s} in phase III leads to a speed up of the oscillation frequency. Therefore one expects the existence of a frequency dip separating between phase II and phase III. Indeed as shown in Fig. 48.18**c** and **d**, we find good agreement between the frequency dip and the corresponding dynamical critical point. For small δ_{s}, the oscillation frequency approaches the Higgs oscillation frequency discussed in the previous subsection. For large δ_{s}, the oscillation frequency approaches δ_{s}. The reduction of oscillation frequency compared to δ_{s} indicates many-body effects in phase III. It's worth to mention that in contrast to the inhomogeneous model, the frequency dip indicates the phase IIIa to phase IIIb transition for the homogeneous model.

Figure 48.18: **Frequency dip as a signature of the phase II to phase III transition.** **a**, Mean field trajectories of the two large spin model evolving under Eq. (48.30). From left to right, Bloch spheres display trajectories with $\delta_s/(\chi N) = 0.5, 1$, and 2 respectively. **b**, Oscillation frequency of $|\Delta_{BCS}|$ in the two-spin BCS model Eq. (48.30) as a function of $\delta_s/\chi N$. The frequency dip at $\delta_s/\chi N = 1$ marks the dynamical phase transition point. **c**, Short-time frequency ω of the dynamics under inhomogeneous atom-light coupling (see Eq. (3) in the Methods). The white line marks the phase II to phase III transition, the same boundary as shown in Extended Data Fig. 2 from the Methods. **d**, Short-time frequency ω of the dynamics using experimental control parameters. The white line marks the phase II to phase III transition and represents the same boundary as in Fig. 48.16. The frequency dips match the dynamical critical points for both cases.

48.8.4 Frequency dip in the two-spin BCS model

Here we use the analytical mean field solution of the BCS Hamiltonian with two large spins ($S = N/4$ for each spin) to understand the frequency dip discussed above. In this case, the Hamiltonian simplifies to

$$\hat{H}/\hbar = \chi \hat{S}^+ \hat{S}^- + \frac{\delta_s}{2}\hat{S}_1^z - \frac{\delta_s}{2}\hat{S}_2^z, \tag{48.30}$$

where $\hat{S}^\pm = \hat{S}_1^\pm + \hat{S}_2^\pm$. The mean field equations of motion for the Hamiltonian above can then be written as

$$\begin{aligned} &\frac{d}{dt}S_1^x = 2\chi S^y S_1^z - \frac{\delta_s}{2}S_1^y, \quad \frac{d}{dt}S_1^y = -2\chi S^x S_1^z + \frac{\delta_s}{2}S_1^x, \quad \frac{d}{dt}S_1^z = -2\chi(S_2^y S_1^x - S_2^x S_1^y), \\ &\frac{d}{dt}S_2^x = 2\chi S^y S_2^z + \frac{\delta_s}{2}S_2^y, \quad \frac{d}{dt}S_2^y = -2\chi S^x S_2^z - \frac{\delta_s}{2}S_2^x, \quad \frac{d}{dt}S_2^z = -2\chi(S_1^y S_2^x - S_1^x S_2^y). \end{aligned} \tag{48.31}$$

The spin components without the hat represent the expectation value of the corresponding spin operators.

In the following, we assume an initial state satisfying $S_1^x = S_2^x = N/4$, $S_1^y = S_2^y = S_1^z = S_2^z = 0$. The conserved quantities of the two-spin BCS model are the total magnetisation

$$S^z = S_1^z + S_2^z = 0, \tag{48.32}$$

the total energy

$$E/\hbar = \chi S^+S^- + \frac{\delta_\mathrm{s}}{2}S_1^z - \frac{\delta_\mathrm{s}}{2}S_2^z = \chi\left(\frac{N}{2}\right)^2, \tag{48.33}$$

as well as the spin length of each of the large spins, $(S_1^x)^2 + (S_1^y)^2 + (S_1^z)^2 = (N/4)^2$, $(S_2^x)^2 + (S_2^y)^2 + (S_2^z)^2 = (N/4)^2$. Using these conserved quantities, one can derive from the mean field equations in Eq. (48.31) an equation of motion for the BCS order parameter, $\Delta_\mathrm{BCS} = \chi S^-$. To simplify the notation, we define $\Delta \equiv |\Delta_\mathrm{BCS}|/\chi N$, i.e. $\Delta^2 = S^+S^-/N^2$. From Eq. (48.32) and Eq. (48.33), we obtain

$$\frac{d}{dt}\Delta^2 = -\frac{\delta_\mathrm{s}}{\chi N^2}\frac{d}{dt}S_1^z = \frac{2\delta_\mathrm{s}}{N^2}(S_2^yS_1^x - S_2^xS_1^y), \tag{48.34}$$

which leads to

$$\begin{aligned}\frac{d^2}{dt^2}\Delta^2 &= \frac{2\delta_\mathrm{s}}{N^2}\left(S_1^x\frac{d}{dt}S_2^y + S_2^y\frac{d}{dt}S_1^x - S_1^y\frac{d}{dt}S_2^x - S_2^x\frac{d}{dt}S_1^y\right)\\ &= 4\delta_\mathrm{s}\chi\Delta^2S_1^z - \frac{2\delta_\mathrm{s}^2}{N^2}(S_1^xS_2^x + S_1^yS_2^y).\end{aligned} \tag{48.35}$$

From the above conserved quantities, we can create the equivalent expressions $\delta_\mathrm{s}S_1^z = -\chi N^2(\Delta^2 - 1/4)$, $2(S_1^xS_2^x + S_1^yS_2^y) = N^2\Delta^2 - 2\times(N/4)^2 + 2(S_1^z)^2$. Plugging these into the equation of motion gives

$$\frac{d^2}{dt^2}\Delta^2 = -6(\chi N)^2(\Delta^2)^2 + \left(2(\chi N)^2 - \delta_\mathrm{s}^2\right)\Delta^2 + \frac{\delta_\mathrm{s}^2 - (\chi N)^2}{8}. \tag{48.36}$$

The equation above can be further simplified to

$$\frac{1}{2}\left(\frac{d}{dt}\Delta\right)^2 + V(\Delta) = 0, \tag{48.37}$$

where

$$V(\Delta) = \frac{1}{2}(\chi N)^2\left(\Delta^2 - \frac{1}{4}\right)\left(\Delta^2 - \frac{1-(\delta_\mathrm{s}/\chi N)^2}{4}\right), \tag{48.38}$$

with an initial condition $\Delta = 1/2$. Eq. (48.37) can be understood as a classical particle with position Δ oscillating in the potential $V(\Delta)$. For $\delta_\mathrm{s} < \chi N$, we find Δ oscillating between $\Delta_\mathrm{max} = 1/2$ and $\Delta_\mathrm{min} = \sqrt{1-(\delta_\mathrm{s}/\chi N)^2}/2$. This is equivalent to phase II in the cases of many spins with inhomogeneous atom-light couplings, because all the oscillations damp in the large χN limit. For $\delta_\mathrm{s} > \chi N$, we find Δ oscillating between $\Delta_\mathrm{max} = 1/2$ and $\Delta_\mathrm{min} = 0$, since the definition of Δ requires $\Delta \geq 0$. This is equivalent to phase III in the cases of many spins because the phase connects to single-particle oscillations in the large δ_s limit. Therefore, a dynamical phase transition occurs at $\delta_\mathrm{s}/\chi N = 1$, which is equivalent to the phase II to phase III transition in the many-spin system.

The analytical solution of Eq. (48.37) can be written in terms of Jacobian elliptic funtions dn and cn:

$$\Delta(t) = \begin{cases} \frac{1}{2}\mathrm{dn}\left(\frac{1}{2}\chi Nt\middle|(\delta_\mathrm{s}/\chi N)^2\right) & \text{if } \delta_\mathrm{s} < \chi N \\ \frac{1}{2}\left|\mathrm{cn}\left(\frac{1}{2}\delta_\mathrm{s}t\middle|(\chi N/\delta_\mathrm{s})^2\right)\right| & \text{if } \delta_\mathrm{s} > \chi N \end{cases}. \tag{48.39}$$

The frequency of $\Delta(t)$ can be written in terms of the complete elliptic integral of the first kind $K(k^2)$:

$$\frac{\omega}{\chi N} = \begin{cases} \dfrac{\pi}{2K\Big((\delta_{\rm s}/\chi N)^2\Big)} & \text{if } \delta_{\rm s} < \chi N \\ \\ \dfrac{\delta_{\rm s}}{\chi N}\dfrac{\pi}{2K\Big((\chi N/\delta_{\rm s})^2\Big)} & \text{if } \delta_{\rm s} > \chi N \end{cases} . \tag{48.40}$$

The mean-field trajectories on the Bloch sphere are shown in Fig. 48.18**a**, and the oscillation frequency Eq. (48.40) is shown in Fig. 48.18**b**. The dynamical phase transition can also be understood from the mean field trajectories. For $\delta_{\rm s} < \chi N$, the two large spins lock to each other and oscillate near the x axis of the Bloch sphere. For $\delta_{\rm s} > \chi N$, the two large spins are unlocked and precess around the whole Bloch sphere. Near the dynamical critical point, the mean field trajectories are close to the north pole or south pole of the Bloch sphere, which leads to a slow down of the oscillations because they approach stable fixed points of the Hamiltonian.

48.9 Axial Motion

In this section, we elaborate on how to take into account axial motion present in the experimental system. Similar discussions can be found in Ref. [Muniz *et al.*(2020)]. We start with the one-dimensional Hamiltonian of our cavity QED system with two internal atomic levels ($|\uparrow\rangle$ and $|\downarrow\rangle$), given by

$$\begin{aligned}\hat{H} = &\sum_{\sigma=\{\uparrow,\downarrow\}} \int dx\ \hat{\psi}^\dagger_\sigma(x)\left[\frac{\hat{p}^2}{2M} + V_0\sin^2(k_L x)\right]\hat{\psi}_\sigma(x) + \int dx\ \hat{\psi}^\dagger_\uparrow(x)\Big[\hbar\omega_0 + U_{ac}(x)\Big]\hat{\psi}_\uparrow(x) \\ &+ \hbar g_c \int dx\ \cos(k_c x)\left[\hat{\psi}^\dagger_\uparrow(x)\hat{\psi}_\downarrow(x)\hat{a} + \hat{a}^\dagger\hat{\psi}^\dagger_\downarrow(x)\hat{\psi}_\uparrow(x)\right] + \hbar\omega_c\hat{a}^\dagger\hat{a},\end{aligned} \tag{48.41}$$

where $k_L = 2\pi/\lambda_L$ is the wavenumber of the lattice beams ($\lambda_L = 813$nm), k_c is the wavenumber of the cavity mode ($\lambda_c = 689$nm), ω_0 is the atomic transition frequency between $|\uparrow\rangle$ and $|\downarrow\rangle$ states, $U_{ac}(x)$ is the AC Stark shift applied to the atoms (including the differential light shift from the lattice beams and the transverse AC Stark shift beam), and ω_c is the frequency of cavity resonance.

Since the atoms are trapped in an optical lattice with lattice depth on the order of $10^3 E_R$, we can approximate each lattice site as an harmonic trap with axial trapping frequency $\hbar\omega_T = \sqrt{4V_0E_R}$, where $E_R = \hbar^2k_L^2/2M$ is the lattice recoil energy. We also ignore tunnelling processes between lattice sites. In this case, one can expand the atomic field operator in terms of lattice site index j and harmonic oscillator levels n:

$$\hat{\psi}_\sigma(x) = \sum_{jn} \hat{c}_{jn,\sigma}\phi_n(x - ja_L). \tag{48.42}$$

Here, $a_L = \lambda_L/2$ is the lattice spacing, and ϕ_n is the harmonic oscillator wave function for mode n, given by

$$\phi_n(x) = \frac{1}{\sqrt{2^n n!}}\left(\frac{M\omega_T}{\pi\hbar}\right)^{1/4} \mathrm{e}^{-M\omega_T x^2/2\hbar} H_n\left(\sqrt{\frac{M\omega_T}{\hbar}}x\right) \tag{48.43}$$

where $H_n(x)$ are the Hermite polynomials. Plugging this expansion into the Hamiltonian and transforming to the rotating frame of the atoms, we obtain

$$\hat{H}/\hbar = \sum_{jn\sigma} n\omega_T\hat{c}^\dagger_{jn,\sigma}\hat{c}_{jn,\sigma} + \sum_{jn}\varepsilon_{jn}\hat{c}^\dagger_{jn,\uparrow}\hat{c}_{jn,\uparrow} + g_c\sum_{jnm}\zeta^{nm}_j(\hat{c}^\dagger_{jn,\uparrow}\hat{c}_{jm,\downarrow}\hat{a} + \hat{a}^\dagger\hat{c}^\dagger_{jm,\downarrow}\hat{c}_{jn,\uparrow}) + \delta_c\hat{a}^\dagger\hat{a} \tag{48.44}$$

where $\delta_c = \omega_c - \omega_a$. For simplicity, we assume $U_{ac}(x)$ is either small or slowly varying in space and thus does not change the trap geometry. This term gives rise to an inhomogeneous transition frequency $\varepsilon_{jn} = \int dx U_{ac}(x)[\phi_n(x - ja_L)]^2/\hbar$. We calculate ζ_j^{nm} in the following way:

$$\begin{aligned}\zeta_j^{nm} &= \int dx\ \cos(k_c x)\phi_n(x - ja_L)\phi_m(x - ja_L) = \int dx\ \cos(k_c x + k_c j a_L)\phi_n(x)\phi_m(x) \\ &= \cos(j\varphi)\int dx\ \cos(k_c x)\phi_n(x)\phi_m(x) - \sin(j\varphi)\int dx\ \sin(k_c x)\phi_n(x)\phi_m(x) \\ &= \cos(j\varphi)\,\mathrm{Re}\left[(i\eta)^s e^{-\eta^2/2}\sqrt{\frac{n_<!}{n_>!}}L_{n_<}^s(\eta^2)\right] - \sin(j\varphi)\,\mathrm{Im}\left[(i\eta)^s e^{-\eta^2/2}\sqrt{\frac{n_<!}{n_>!}}L_{n_<}^s(\eta^2)\right].\end{aligned} \tag{48.45}$$

where $\varphi = \pi k_L/k_c$, $s = |n - m|$, $n_< = \min(n, m)$, $n_> = \max(n, m)$, $L_n^\alpha(x)$ are the generalised Laguerre polynomials, and $\eta = k_c\sqrt{\hbar/2M\omega_T}$ is the Lamb-Dicke parameter. In our case $\omega_T/2\pi = 165$ kHz, implying that $\eta = 0.17$. This places us in the Lamb-Dicke regime where ζ_j^{nm} is negligible for $|n - m| > 1$. It can be convenient to rewrite the Hamiltonian in terms of operators $\hat{S}^j_{n\sigma,m\sigma'} = \hat{c}^\dagger_{jn,\sigma}\hat{c}_{jm,\sigma'}$, resulting in the following form:

$$\hat{H}/\hbar = \sum_{jn\sigma} n\omega_T \hat{S}^j_{n\sigma,n\sigma} + \sum_{jn}\varepsilon_{jn}\hat{S}^j_{n\uparrow,n\uparrow} + g_c\sum_{jnm}\zeta_j^{nm}(\hat{S}^j_{n\uparrow,m\downarrow}\hat{a} + \hat{a}^\dagger\hat{S}^j_{m\downarrow,n\uparrow}) + \delta_c\hat{a}^\dagger\hat{a}. \tag{48.46}$$

In addition to the Hamiltonian dynamics, we also consider dissipation processes such as cavity loss with a rate $\kappa/2\pi = 153$ kHz, as well as spontaneous emission with a rate $\gamma/2\pi = 7.5$ kHz. The full dynamics of this open system can be described by the following Lindblad master equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \left[\hat{L}_{\mathrm{cav}}\hat{\rho}\hat{L}^\dagger_{\mathrm{cav}} - \frac{1}{2}\{\hat{L}^\dagger_{\mathrm{cav}}\hat{L}_{\mathrm{cav}}, \hat{\rho}\}\right] + \sum_{jn}\left[\hat{L}_{j,n}\hat{\rho}\hat{L}^\dagger_{j,n} - \frac{1}{2}\{\hat{L}^\dagger_{j,n}\hat{L}_{j,n}, \hat{\rho}\}\right], \tag{48.47}$$

where the jump operator for cavity loss is given by $\hat{L}_{\mathrm{cav}} = \sqrt{\kappa}\hat{a}$, and the single-particle jump operators for spontaneous emission are given by $\hat{L}_{j,n} = \sqrt{\gamma}\hat{S}^j_{n\downarrow,n\uparrow}$. Here, we assume that spontaneous emission is in the Lamb-Dicke regime.

In the experiment, δ_c is the largest frequency scale ($\delta_c \gg g_c\sqrt{N}, \kappa$), so we can adiabatically eliminate the cavity photons [Reiter and Sørensen(2012)] and obtain the following effective atom-only master equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} = -\frac{i}{\hbar}[\hat{H}_{\mathrm{eff}}, \hat{\rho}] + \left[\hat{L}_{\mathrm{col}}\hat{\rho}\hat{L}^\dagger_{\mathrm{col}} - \frac{1}{2}\{\hat{L}^\dagger_{\mathrm{col}}\hat{L}_{\mathrm{col}}, \hat{\rho}\}\right] + \sum_{jn}\left[\hat{L}_{j,n}\hat{\rho}\hat{L}^\dagger_{j,n} - \frac{1}{2}\{\hat{L}^\dagger_{j,n}\hat{L}_{j,n}, \hat{\rho}\}\right]. \tag{48.48}$$

Here, the effective Hamiltonian is given by

$$\hat{H}_{\mathrm{eff}}/\hbar = \sum_{jn\sigma} n\omega_T\hat{S}^j_{n\sigma,n\sigma} + \sum_{jn}\varepsilon_{jn}\hat{S}^j_{n\uparrow,n\uparrow} + \chi\sum_{jnm}\sum_{kpq}\zeta_j^{nm}\zeta_k^{pq}\hat{S}^j_{n\uparrow,m\downarrow}\hat{S}^k_{p\downarrow,q\uparrow}, \tag{48.49}$$

and effective collective jump operator generating superradiant decay takes the form

$$\hat{L}_{\mathrm{col}} = \sqrt{\Gamma}\sum_{jnm}\zeta_j^{nm}\hat{S}^j_{m\downarrow,n\uparrow}, \tag{48.50}$$

where $\chi = -g_c^2\delta_c/(\delta_c^2 + \kappa^2/4)$ and $\Gamma = g_c^2\kappa/(\delta_c^2 + \kappa^2/4)$. The equivalent superconducting order parameter takes the following form:

$$\Delta_{\mathrm{BCS}} = \chi\sum_{kpq}\zeta_k^{pq}\langle\hat{S}^k_{p\downarrow,q\uparrow}\rangle. \tag{48.51}$$

One can recover the inhomogeneous model discussed in the previous section by removing the axial harmonic oscillator level labels.

Similarly, the Hamiltonian for initial state preparation takes the form

$$\hat{H}_{\text{drive}}/\hbar = \sum_{jn\sigma} n\omega_T \hat{S}^j_{n\sigma,n\sigma} + \frac{1}{2}\sum_{jnm} \zeta_j^{nm}(\Omega \hat{S}^j_{n\uparrow,m\downarrow} + \Omega^* \hat{S}^j_{m\downarrow,n\uparrow}). \tag{48.52}$$

Figure 48.19: **Understanding experimental results with axial motion effects. a** Example phase II traces with $\chi N/2\pi = 1.29$MHz, $f_{\text{AC}}/2\pi = 1.1$MHz. **b** Example phase III traces with $\chi N/2\pi = 0.79$MHz, $f_{\text{AC}}/2\pi = 1.1$MHz. **c** Example phase I traces with $\chi N/2\pi = 0.15$MHz, $f_{\text{AC}}/2\pi = 1.1$MHz. The blue points are experimental data, the orange lines represent numerical simulations under ideal conditions (see Eq. (3) in the Methods), the green lines include dissipative processes on top of the ideal simulations, and the red lines consider both dissipative processes and axial motion effects.

In numerical simulations, we perform a mean-field approximation, which replaces the operators $\hat{S}^j_{p\sigma,q\sigma'}$ by their expectation values $\langle \hat{S}^j_{p\sigma,q\sigma'} \rangle$ in the Heisenberg equation of motion. We perform a random sampling of the axial harmonic oscillator mode n for each atom based on a thermal distribution of 15 μK, and we only include the modes n and $n \pm 1$ into our calculation due to the Lamb-Dicke parameter. The atom number in our simulations is set to 2000; to match χN to experimental values, we rescale χ accordingly. We also empirically take into account two additional dissipation processes to quantitatively capture the behavior of $|\Delta_{\text{BCS}}|$ at longer time scales. The first is a single-particle decoherence between electronic states, described by the jump operators $\hat{L}^{\text{el}}_{j,\sigma} = \sqrt{\gamma_{\text{el}}} \sum_n \hat{S}^j_{n\sigma,n\sigma}$ with $\gamma_{\text{el}}/2\pi < 1$kHz for Fig. 2 starting from $t = 0\mu$s, and by $\gamma_{\text{el}}/2\pi = 0.0036(f_{\text{AC}}/2\pi) + 4$kHz for Fig. 3 and Fig. 4 in the main text. The second is a single-particle decoherence between motional states, described by the jump operators $\hat{L}^{\text{mo}}_{j,n} = \sqrt{\gamma_{\text{mo}}} \sum_\sigma \hat{S}^j_{n\sigma,n\sigma}$ with $\gamma_{\text{mo}}/2\pi = 15$kHz.

Some example traces including axial motion effects are depicted in Fig. 48.19. Generally speaking, accounting for these effects allows us to more accurately predict features present in the experimentally measured evolution of $|\Delta_{\text{BCS}}|$, at the same time leaving the predicted dynamical phase boundaries unchanged. As shown in Fig. 48.19a, including axial motion effects in phase II traces allows us to capture the faster damping rate of the Higgs oscillations, as well as a slow oscillation in $|\Delta_{\text{BCS}}|$ at the axial trapping frequency. Likewise, as shown in Fig. 48.19b, including axial motion effects in phase III traces allows us to capture the faster damping rate of the oscillations in $|\Delta_{\text{BCS}}|$, although the observed damping rate is still faster than the rate predicted by theory. Finally, as shown in Fig. 48.19c, all the theory simulations of phase I dynamics are similar to the simulation under ideal conditions, indicating that axial motion does not play an significant role in this regime.

48.10 Questions about “Observing Dynamical Phases ..” by Young et al.

Why it is non-destructive?

(????)

Why the dynamical phase diagram is such?

(???? it looks very not smooth... is there an intuitive understanding of it???)

Part XV
Experiments: Shining Laser Pulses on Superconductors

49 Infrared activation of the Higgs mode by supercurrent injection in superconducting NbN by Nakamura, Iida, Murotani, Matsunaga, Terai, Shimano

Abstract

Higgs mode in superconductors, i.e. the collective amplitude mode of the order parameter does not associate with charge nor spin fluctuations, therefore it does not couple to the electromagnetic field in the linear response regime. On the contrary to this common understanding, here, we demonstrate that, if the dc supercurrent is introduced into the superconductor, the Higgs mode becomes infrared active and is directly observed in the linear optical conductivity measurement. We observed a sharp resonant peak at $\omega = 2\Delta$ in the optical conductivity spectrum of a thin-film NbN in the presence of dc supercurrent, showing a reasonable agreement with the recent theoretical prediction. The method as proven by this work opens a new pathway to study the Higgs mode in a wide variety of superconductors.

The collective modes ubiquitously exist in a variety of systems, e.g. in charge density wave (CDW) systems, ferromagnets and antiferromagnets, superfluid ^{4}He and ^{3}He, cold atomic gas systems, and superconductors, providing insights into the nature of symmetry broken ground states. In general, two types of collective modes emerge when a continuous symmtery is spontaneously broken: the phase mode and amplitude mode that correspond to the fluctuation of phase and amplitude of the order parameter, respectively (see Fig. 49.1(a)). In superconductors, the phase mode is lifted up to the high energy plasma frequency because of the screening of long range Coulomb interaction [9]. The remaining amplitude mode, recently referred to as the Higgs mode, has gained a growing interest over decades [12, 6]. Since the initial prediction made by Anderson [9], intensive theoretical studies have been devoted to elucidate the energy structure, stability, and relaxation mechanism of the Higgs mode to date. The behavior of Higgs mode has been discussed from the viewpoint of order parameter dynamics after the quantum quench [4, Barankov and Levitov(2004), 32, 19, Barankov and Levitov(2006), 38, 39, 36, 40, 93, 41, 43, 92, 17, 88, 120], which has been recently observed by terahertz pump-probe experiments in s-wave superconductors [4]. The coupling of the Higgs mode to the gauge field was initially identified in the Raman signal with the aid of strong electron-phonon coupling in $NbSe_2$, where the CDW coexists with superconductivity [12, 3, 28, 30]. Even in an s-wave superconductor without the CDW order, the nonlinear coupling between the Higgs mode and the gauge field has been elucidated in THz pump-probe response and third harmonic generation [46, 49, 55], and extensive microscopic theories have been developed to date [48, 51, 47, 61, 30]. The observability of Higgs mode in the linear response, i.e. in the optical conductivity has been addressed in two-dimensional disordered superconductors [7]. Experimentally, the finite spectral weight below the superconducting gap 2Δ observed in a disordered ultrathin NbN film sample was attributed to Higgs mode from the comparison with tunneling spectroscopy [5], whereas different origins of the spectral weight have also been suggested, i.e. disorder-induced broadning of the quasiparticle density of states [118] and the collective mode associated with the phase rather than the amplitude [117, 36].

Recently, it has been theoretically shown that under the injection of dc supercurrent, the Higgs mode linearly interacts with ac electric field polarized along the direction of the supercurrent flow. Accordingly, the Higgs mode is predicted to appear in the linear response function such as the optical conductivity spectrum $\sigma_1(\omega)$, giving rise to a polarization-dependent peak structure at the superconducting gap frequency $\omega = 2\Delta$ [57]. This effect comes from the momentum term in the action:

$$S = \int C\mathbf{Q}^2(t)|\Delta(t)|^2 dt d\mathbf{r} \tag{49.1}$$

where $\mathbf{Q}(t) = \mathbf{Q}_0 + \mathbf{Q}_\Omega(t)$ is the gauge-invariant momentum of the condensate consisting of the dc supercurrent term $\mathbf{Q}_0$ and the ac electric-field (probe-field) driven term $\mathbf{Q}_\Omega(t) = \mathrm{Re}[\mathbf{Q}_\Omega \exp(i\Omega t)]$, respectively, and $\Delta(t) = \Delta_0 + \delta\Delta(t)$ is the time-dependent superconducting order parameter. The action S includes the integral of $\delta\Delta_{2\Omega}\mathbf{Q}^2_{-\Omega}$ and $\delta\Delta_\Omega\mathbf{Q}_0\mathbf{Q}_{-\Omega}$ where $\delta\Delta_{\Omega(2\Omega)}$ denotes the Fourier component of the oscillating order parameter (Higgs mode). The first term corresponds to the quadratic coupling of the Higgs mode to the gauge field which was previously demonstrated [46]. The second term indicates linear coupling between the Higgs mode and the gauge field induced by the finite amount of condensate momentum $\mathbf{Q}_0$ parallel to the probe electric field. It is then expected that the polarization-selective excitation and detection of the Higgs mode are attainable within the linear response regime when supercurrent is injected into the system. Motivated by this theoretical prediction, here we investigated the effect of supercurrent injection on the optical conductivity in an s-wave superconductor, NbN thin film.

The optical conductivity was measured using terahertz time-domain spectroscopy (THz-TDS) technique in transmission geometry. The schematic diagram of the experimental setup is shown in Fig 49.1(b). The sample is an epitaxial NbN film of 26 nm in thickness grown on a 400-μm-thick MgO (100) substrate. The critical temperature is 14.5±0.2 K as confirmed by the transport and magnetic susceptibility profiles (see Fig. 49.1(c)). The transition width $\Delta T_c/T_c$ is only about one percent estimated from the magnetic susceptibility, indicating the high uniformity of the film. The supercurrent was injected through the Au/Ti electrodes deposited on both ends of the sample and the critical current is 3.8 A at T =5.1 K (3.7 MA/cm^2 in current density). The sample was cooled down using a ^{4}He flow cryostat in a ^{4}He atmosphere. For the measurements, the laser beam from the mode-locked Ti:sapphire oscillator (repetition rate: 80 MHz, wavelength: 800 nm, average power: 1 W, pulse width: 100 fs) was split into two with the intensity ratio of 3:1; one for the THz generation modulated by an optical chopper rotating at ≈2.3 kHz and one for the gate pulse for the electro-optic (EO) sampling. The probe THz pulse was generated by optical rectification in a ZnTe crystal, and detected by EO sampling also using a ZnTe crystal. The peak value of the probe THz electric field was below 20 V/cm which is weak enough to assure the linear response regime. Polarization of the THz pulse is determined by wire grid polarizers placed before and after the sample. The thin solid line in Fig. 49.2(a) represents the optical conductivity $\sigma_1^0(\omega)$ measured at 5 K without current injection. The superconducting gap structure is clearly identified at ≈ 5 meV. The spectrum is well fitted by the Mattis-Bardeen model with the gap energy of $2\Delta_{\mathrm{MB}} = 5.14 \pm 0.01$ meV, as shown by the dashed line in Fig. 49.2(a) [108].

To measure precisely the conductivity change $\delta\sigma_1(\omega)$ induced by the current injection with eliminating the long-term fluctuation, we repeated the THz waveform scan with and without the current injection alternatingly, and accumulated the waveform from 30–100 times. The differential spectrum $\delta\sigma_1(\omega)$ is extracted from the Fourier transform of the waveforms with and without the current, $E^{\mathrm{I}}(\omega)$ and $E^0(\omega)$, respectively, the refractive index of the substrate $n_{\mathrm{sub}}(\omega)$, and the premeasured $\sigma_1^0(\omega)$ without current injection, using the following equation commonly used for pump-probe measurements for thin film samples [99]:

$$\delta\sigma_1(\omega) = \frac{1 + n_{\mathrm{sub}}(\omega) + Z_0 d\sigma_1^0(\omega)}{Z_0 d}\left(\frac{E^0(\omega)}{E^{\mathrm{I}}(\omega)} - 1\right), \tag{49.2}$$

Figure 49.1: (a) Free energy $F(\Psi)$ with respect to the superconducting order parameter Ψ. The Higgs (amplitude) mode and the phase mode are represented by the arrows. (b) Schematic view of the terahertz transmittance experiments under the dc current. (c) Temperature dependence of the ac magnetic susceptibility χ measured at 1 Oe in 1488 Hz (open symbols) and electrical resistance R (closed symbols).

Figure 49.2: The change of the (a) real part and (b) imaginary part of the optical conductivity spectra induced by supercurrent injection taken with the THz probe electric field polarized along the direction of supercurrent. The optical conductivity spectra measured without the current are also plotted in (a) and (b) with the right axes. The dashed line depicts the Mattis-Bardeen fit and the superconducting gap estimated from the fit is marked by the vertical arrow. Shown in (c) is the change of the real part of the optical conductivity spectra taken with the THz probe electric field perpendicular to the current direction. Current (I) dependences of the peak intensity and FWHM ($= 2\sqrt{2\log 2}\sigma$) are plotted in (d). The lines are trend lines determined by the method of least squares. Details are found in the text. Small points in (d) are data of another sample (Sample #2) fabricated in very similar conditions [39].

where $Z_0 = 377\,\Omega$ is the vacuum impedance, and d is the thickness of the NbN film. At low temperature the transmission is very low below the gap energy so that the data below 4 meV are scattered [39]. Therefore in the following graphs the data above 4 meV are plotted.

The thick lines in Fig. 49.2(a) show the differential spectra of $\delta\sigma_1(\omega)$ under the currents I =0, 1.9, 2.3, 2.5, and 2.6 A at 5 K ($= 0.34T_c$) measured for the polarization parallel to the direction of the supercurrent[20]. A peak structure is clearly identified in all the data with currents between 1.9 and 2.6 A. Note that, 2.6 A is 70% of the critical current I_c at the temperature. The peak height is at most 1% with respect to the normal state conductivity σ_N=1.2×10$^4\,\Omega^{-1}$cm^{-1}. The peak position is estimated as 5.40±0.05 meV (1.30 THz), which is slightly larger than the onset energy of $\sigma_1^0(\omega)$ by amount of 0.2 meV, while it coincides with the energy of Higgs mode estimated from the time-resolved observation of the Higgs mode oscillation in previous pump-probe measurements in another NbN film with similar thickness (24 nm) [4]. Here we take into account the effect of thermal broadening [41] practically by convoluting a Gaussian distribution to the original Mattis-Bardeen spectrum function. The broadening width is estimated as 0.6 meV at 5 K and 2 meV at 13 K, which can explain the slight energy difference between the peak center and the $2\Delta_{\mathrm{MB}}$. Although the supercurrent-induced conductivity peak shows a tail in the higher energy side like the theoretically predicted one [57], here we fitted the peak assuming a Gaussian function with a constant offset for each current density: $a \cdot g(x) + b$ with $g(x) = 1/\sqrt{2\pi\sigma^2}\exp(-((x-\mu)/\sigma)^2/2)$. While the peak center μ and width σ are almost constant, the peak intensity a increases with the current as shown in Fig. 49.2(d). The solid line indicates the quadratic fit to the peak intensity and the dashed line indicates the average of the peak width determined by the method of least squares. It should be remarked that the peak width, 0.80±0.05 meV, does not vary with the temperature up to 8 K (not shown), suggesting the peak width is hardly affected by the thermal broadening effects mentioned above. Notably, this peak structure is completely absent when measured for the perpendicular polarization at the same temperature and currents as indicated in Fig. 49.2(c). These characteristics are consistent with the theoretical prediction that the spectral weight of the current-induced Higgs mode resonance should be proportional to $I^2\cos^2\theta$ where θ is the angle between the supercurrent flow and the THz electric field [57].

To establish more firm connection between the theory and experiment, we calculated the complex optical conductivity based on the theoretical work by Moor *et al.* [57] with the parameters relevant to the present experiments at T =5 K. To simulate realistic optical conductivity spectra, we introduced a broadening factor Γ for the peak by convoluting the ideal conductivity spectra with Gaussian distribution of 2Δ with FWHM ($= \Gamma$) of 0.6 meV. According to Ref. [57], the peak weight is given by a coefficient $W_Q = DQ_0^2$ with the diffusion constant $D = 4k_B/(\pi e)\,|dH_{c2}/dT|^{-1} = 9.0\times10^{-5}\,\mathrm{m^2/s}$ determined from H_{c2} measured [42]. The condensate momentum Q_0 is defined as $mv_s/\hbar$ where v_s is the condensate velocity calculated from the superfluid density n_s and the injected current density J using $J = en_sv_s$. The n_s is estimated as $5.4\times10^{26}\,\mathrm{m^{-3}}$ from the imaginary part of the optical conductivity spectrum $\sigma_2(\omega)$ using $n_s = \lim_{\omega\to0} m\sigma_2(\omega)\omega/e^2$. When the current I is 2.6 A, the W_Q is estimated as 2.3×10^{-3} meV, which is as small as $4.3\times10^{-4}\Delta$ at $T = 5$ K so that the theory in Ref. [57] is applicable.

Figures 49.3(a) and (b) show the real and imaginary parts of the calculated spectra, respectively. Compared to the Figs. 49.2(a) and (b), the calculation reasonably reproduces the characteristic peak in $\delta\sigma_1$, dispersive shape in $\delta\sigma_2$, and gradual onset of σ_1^0 at a slightly below 2Δ. The calculation also indicates that the spectral weight is transferred from the condensate at zero energy to the 2Δ peak (not shown in the figure). The transferred spectral weight is only a few percent of that of the whole condensate. The calculation shows about one order smaller signal for the perpendicular configuration (dashed lines in Figs. 49.3(a) and (b)), which is un-

[20]The oscillating structure appearing in all the differential spectra is attributed to interference effects of the THz-TDS measurements exaggerated by the differentiation. This already exists in the equilibrium spectra.

Figure 49.3: Theoretically expected changes of (a) real and (b) imaginary parts of the optical conductivity induced by supercurrent injection with respect to the normal state conductivity σ_N [57]. We set $W_Q = 0.00043\Delta$ (corresponding to 2.6 A), $T = 5\,\mathrm{K}$, the mean of $2\Delta = 5.4\,\mathrm{meV}$, and $\Gamma = 0.6\,\mathrm{meV}$. The optical conductivity without current injection is also represented as a thin line in (a).

recognizable in our experiments because of the detection limit. The calculation is in reasonable agreement with the experimental result even quantitatively, suggesting that the observed peak is attributed to that of Higgs mode.

Now we address other possible mechanisms that could also give a conductivity peak structure. Firstly, the phase mode is not plausible as it is lifted to the plasma frequency at low temperature limit due to the Anderson-Higgs mechanism. In fact, it can be lowered to the energy region near the gap, known as the Carlson-Goldman mode [43], but only at temperature close to T_c, which is not relevant to our results taken substantially below T_c. Single particle excitations caused by the current injection can also give rise to the change of optical conductivity particularly near the critical current density. They cause smearing of the density of states and shrinkage of the gap which can induce a singularity in the optical conductivity spectrum at around the gap energy [44]. Indeed in the previous experiments on impure aluminum, an absorption peak was observed at slightly lower energy than the pristine gap under the presence of strong in-plane magnetic fields ($H > 0.5H_c$) which induced effective in-plane supercurrents [45, 46]. Under such a strong field, the conductivity peak is explained within the framework of quasiparticle excitations [47]. On the other hand, in our NbN, the maximum current density corresponds to a very weak magnetic field of $0.02H_{c1} = 3 \times 10^{-5} H_{c2}$, therefore, as suggested by Moor et al. [57] the Higgs mode response dominates the observed conductivity peak.

Finally, we extended the measurement up to 14 K ($= 0.96T_c$). The real part of the conductivity change $\delta\sigma_1(\omega)$ is plotted in Fig 49.4(a). With increasing the temperature, the energy of the peak center gradually decreases in synchronization with the superconducting gap $2\Delta_{\mathrm{MB}}$ determined from the Mattis-Bardeen fit to $\sigma^0(\omega)$ as shown in Fig 49.4(b). The figure also shows the zero-temperature gap $2\Delta_0$ estimated as $4.3k_{\mathrm{B}}T_c$ [48] plotted as a horizontal line, which agrees very well with the energy of the peak center at low temperature. The peak width is almost constant at low temperature ($T \leq 10\,\mathrm{K}$), then rapidly increases at higher temperature. This temperature dependence shows a negative correlation with that of the critical current as shown in Fig. 49.4(c), presumably because both the Higgs mode and superconducting current are affected by thermally-induced quasiparticles. The polarization dependence becomes less significant at higher temperature, which is qualitatively consistent with the theoretical expectation [39]. Note that the sample temperature is precisely controlled during the measurements to negate Joule heating effects because the differential signal $\delta\sigma_1(\omega)$ is very sensitive to the sample temperature at $T > 8\,\mathrm{K}$ where the temperature coefficient of the gap has a non-zero value [39].

In summary, we have demonstrated that the Higgs mode appears in the linear optical conductivity spectrum under the supercurrent injection. A distinct peak slightly above the optical

Figure 49.4: (a) Change of the real part of the conductivity $\delta\sigma_1(\omega)$ induced by the supercurrent injection with the THz probe electric field parallel (solid lines) and perpendicular (dashed lines) to the current direction at the indicated temperatures and currents. (b) Temperature dependence of the peak center and the superconducting gap $2\Delta_{\mathrm{MB}}$ determined from the Mattis-Bardeen fit to $\sigma_1^0(\omega)$ [108]. The horizontal line shows the zero-temperature superconducting gap estimated from the T_c. (c) Temperature dependence of the peak width (square; left axis) and the critical current I_c (cross; right axis).

conductivity gap accompanied by a high energy tail is clearly observed in the film of an s-wave superconductor NbN. Based on the polarization and current density dependencies, we attribute the peak structure to the Higgs mode as recently suggested in the theoretical study [57]. The peak energy is also in agreement with the oscillation frequency of Higgs mode observed in previous time-resolved THz pump-THz probe measurements [4]. This method comprising the linear conductivity measurement and the current injection provides a new pathway to access the Higgs mode in various superconductors. Extension of the measurement scheme to p- or d-wave superconductors would be highly intriguing. The method may also be applied to study the superconductivity with competing orders in unconventional superconductors.

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50 Light-induced nonequilibrium response of the superconducting cuprate $La_{2-x}Sr_xCuO_4$ by Niwa, Yoshikawa, Tomari, Matsunaga, Song, Eisaki, Shimano

Abstract

We report the dynamics of the cuprate superconductor $La_{2-x}Sr_xCuO_4$ ($x = 0.14$) after intense photoexcitation utilizing near-infrared (800 nm) optical pump-terahertz probe spectroscopy. In the superconducting state at 5 K, we observed a redshift of the Josephson plasma resonance that sustains for hundreds of picoseconds after the photoexcitation, indicating the destruction of the c-axis superconducting coherence. We show that the metastable spectral features can be described by the photoinduced surface heating of the sample. We also demonstrate that the conventional analysis used to extract the spectra of the photoexcited surface region can give rise to artifacts in the nonequilibrium response.

50.1 Introduction

In unconventional high-temperature superconductors, it has been revealed that various electronic states emerge depending on temperature and chemical doping, such as antiferromagnetic insulator, spin density wave, stripe order, charge order, pair density wave, nematic order, and pseudogap [Kei()]. It is now commonly recognized that the elucidation of the interplay of those multiple orders as well as the unveiling of pairing glue is crucial to understand the emergence of superconductivity. In these aspects, the study of real-time dynamics of superconductivity by ultrafast spectroscopy technique has been playing important roles [Giannetti *et al.*(2016), Dal Conte *et al.*(2012)Dal Conte, Giannetti, Coslovich, Cilento, Bossini, Abebaw, Banfi, Ferrini, Eisaki, Greven, Damascelli, van der Marel, and Parmigiani, Sentef *et al.*(2013)Sentef, Kemper, Moritz, Freericks, Shen, and Devereaux]. For instance, ultrafast pump-probe experiments have shown their ability to elucidate bosonic fluctuations to which fermionic quasiparticles couple. Time-resolved observation of collective modes enables the direct access to the order parameter dynamics in nonequilibrium [4, 46, 55], and for cuprate superconductors it has been used to reveal the coupling between the superconductivity and the charge density wave order [Dakovski *et al.*(2015)Dakovski, Lee, Hawthorn, Garner, Bonn, Hardy, Liang, Hoffmann, and Turner, Hinton *et al.*(2016)Hinton, Thewalt, Alpichshev, Mahmood, Koralek, Chan, Veit, Dorow, Barišić, Kemper, Bonn, Hardy, Liang, Gedik, Greven, Lanzara, and Orenstein].

Searching for a new state of matter induced by the photoexcitation is also a fascinating subject in this respect, as it provides deeper insight into the competing orders or hidden states, with revealing new functionalities of correlated materials. Light-induced superconductivity is a highly intriguing example, where the strong photoexcitation leads to a transient emergence of superconductivity-like response above the critical temperature T_c [127, Kaiser *et al.*(2014)Kaiser, Hunt, Nicoletti, Hu, Gierz, Liu, Le Tacon, Loew, Haug, Keimer, and Cavalleri, Hu *et al.*(2014), Nicoletti *et al.*(2014)Nicoletti, Casandruc, Laplace, Khanna, Hunt, Kaiser, Dhesi, Gu, Hill, and Cavalleri, Casandruc *et al.*(2015)Casandruc, Nicoletti, Rajasekaran, Laplace, Khanna, Gu, Hill, and Cavalleri, Nicoletti *et al.*(2018)Nicoletti, Fu, Mehio, Moore, Disa, Gu, and Cavalleri]. For example in $La_{2-x}Ba_xCuO_4$, transient reappearance of Josephson plasma resonance (JPR), one of the characteristic fingerprint of superconductivity, has been observed in the c-axis terahertz (THz) response above T_c right after intense near-infrared photoexcitation [Nicoletti *et al.*(2014)Nicoletti, Casandruc, Laplace, Khanna, Hunt, Kaiser, Dhesi, Gu, Hill, and Cavalleri, Casandruc *et al.*(2015)Casandruc, Nicoletti, Rajasekaran, Laplace, Khanna, Gu, Hill, and Cavalleri]. It has been suggested that the photoexcitation suppresses the competing stripe charge order [Nicoletti *et al.*(2014)Nicoletti, Casandruc, Laplace, Khanna, Hunt, Kaiser, Dhesi, Gu, Hill, and Cavalleri, Casandruc *et al.*(2015)Casandruc, Nicoletti, Rajasekaran, Laplace, Khanna, Gu, Hill, and Cavalleri, Khanna *et al.*(2016)Khanna, Mankowsky, Petrich, Bromberger, Cavill, Möhr-Vorobeva, Nicoletti, Laplace, Gu, Hill, Först, Cavalleri, and Dhesi], although the microscopic origin of this light-induced phenomenon remains to be resolved. The photoexcited state of $La_{2-x}Ba_xCuO_4$ below T_c is even more unclear. Though a transient enhancement of JPR frequency was primarily reported [Nicoletti *et al.*(2014)Nicoletti, Casandruc, Laplace, Khanna, Hunt, Kaiser, Dhesi, Gu, Hill, and Cavalleri], recent experimental results have been suggesting an existence of photoinduced metastable phase, where a new JPR mode and significant increase of spectral weight in optical conductivity are observed [Zhang *et al.*(2018)Zhang, Wang, Shi, Lin, Zhang, Gu, Dong, and Wang, Cremin *et al.*(2019)Cremin, Zhang, Homes, Gu, Sun, Fogler, Millis, Basov, and Averitt]. Hence, the role of intense photoexcitation and its relation with the preexisting competing orders remain to be clarified.

Here, we investigated the photoexcited dynamics in the nearly optimal-doped $La_{2-x}Sr_xCuO_4$ ($x = 0.14$), one of the archetypal cuprate superconductors, by utilizing near-infrared optical-pump and THz-probe spectroscopy. In $La_{2-x}Sr_xCuO_4$, the effect of charge stripe order has been reported to be less pronounced compared to

$La_{2-x}Ba_xCuO_4$ [Vojta(2009), Wu *et al.*(2012)Wu, Buchholz, Trabant, Chang, Komarek, Heigl, Zimmermann, Cwik, Nakamura, Braden, and Schüßler-Langeheine]. In addition, previous ultrafast pump-probe studies of $La_{2-x}Sr_xCuO_4$ have presented a destruction of superconductivity after the photoexcitation [Kusar *et al.*(2005)Kusar, Demsar, Mihailovic, and Sugai, Kusar *et al.*(2008)Kusar, Kabanov, Demsar, Mertelj, Sugai, and Mihailovic, Beyer *et al.*(2011)Beyer, Städter, Beck, Schäfer, Kabanov, Logvenov, Bozovic, Koren, and Demsar]. Thus, $La_{2-x}Sr_xCuO_4$ provides an interesting platform to expose how the significance of stripe charge order affects the photoexcited state below T_c. With increasing excitation density, the JPR shifts to the low energy side which sustains for several hundreds of picoseconds, indicating a quasiequilibrium state with suppressed bulk superconducting coherence. We show that this quasiequilibrium state is well-explained by the thermalization due to the photoexcitation. At the same time, we argue the breakdown of conventional analysis used in optical pump-THz probe experiments, which is prone to produce nonexistent features in the terahertz-range spectra.

50.2 Experiments

50.2.1 Methods

We used a bulk $La_{2-x}Sr_xCuO_4$ single crystal with the doping level of $x = 0.14$ grown by floating-zone method. The mirror-polished *ac* surface of the sample (7x7x5mm) was used and masked by a metal plate with a tapered 4-mm hole. A gold mirror was also mounted on the sample holder as a reference for the reflectivity measurement by THz time-domain spectroscopy (THz-TDS). In Fig. 50.1(a) we show a schematic of near-infrared optical pump-THz probe spectroscopy. As a light source we used a Ti:Sapphire-based regenerative amplifier with the pulse energy of 4.2 mJ, repetition rate of 1 kHz, pulse duration of 100 fs, and center wavelength of 800 nm. The output of the laser was divided into three beams; each for the optical pump, the generation of the THz probe, and the gate pulse for the THz-TDS, respectively. The optical pump beam has a Gaussian profile with $1/e^2$ diameter of 9 mm, which ensures spatial uniformity of excitation density on the probed region. The THz pulse was generated by the optical rectification in a large-aperture ZnTe crystal and linearly polarized along the *c*-axis of $La_{2-x}Sr_xCuO_4$. The reflected THz pulse was detected by the electro-optic sampling in a ZnTe crystal with the gate pulse. The waveform of the probe THz pulse is shown in Fig. 50.1(b).

50.2.2 Equilibrium properties

Figures 50.1(c)-(e) show the optical properties of $La_{2-x}Sr_xCuO_4$ ($x = 0.14$) in equilibrium. In the reflectivity spectrum (Fig. 50.1(c)), a sharp plasma edge associated with the JPR is discerned below T_c. Concomitantly, the loss function spectrum (Fig. 50.1(d)) as defined by $-\Im(1/\varepsilon(\omega))$ with $\varepsilon(\omega)$ the complex dielectric function exhibits a single peak, corresponding to the longitudinal JPR mode. As temperature decreases the plasma edge shows a blueshift and approaches to 5.5 meV reflecting the development of *c*-axis superconducting coherence, which is in good agreement with the previous studies [Tamasaku *et al.*(1992)Tamasaku, Nakamura, and Uchida, Uchida *et al.*(1996)Uchida, Tamasaku, and Tajima, Basov *et al.*(1995)Basov, Mook, Dabrowski, and Timusk, Dordevic *et al.*(2003)Dordevic, Komiya, Ando, and Basov]. A notable feature is identified in the *c*-axis real-part optical conductivity $\sigma_1(\omega) = \Re[\sigma(\omega)]$ as shown in Fig. 50.1(e); the low-frequency spectral weight is gradually suppressed below T_c and exhibits gap-like behavior at the lowest temperature. This spectral feature has also been identified in previous reports on the *c*-axis optical response of $La_{2-x}Sr_xCuO_4$ [Tamasaku *et al.*(1992)Tamasaku, Nakamura, and Uchida, Basov *et al.*(1995)Basov, Mook, Dabrowski, and Timusk, Dordevic *et al.*(2003)Dordevic, Komiya, Ando, and Basov]. As the origin of this spectral peak in

Figure 50.1: (Color online) Experimental setup and optical properties in equilibrium. (a) Schematic of the optical pump-THz probe experiment and the crystal structure of $La_{2-x}Sr_xCuO_4$. Both pump and probe pulses are polarized along the c-axis direction of the crystal. (b) Time-domain waveform of the probe THz pulse generated from ZnTe. (c)-(e) Reflectivity, loss function and real-part optical conductivity of $La_{2-x}Sr_xCuO_4$ ($x = 0.14$) from 5 K to 40 K, respectively.

Figure 50.2: (Color online) (a) Pump-fluence dependence of transient reflectivity at $t_{\rm pp} = 3\,$ps. (b) Time evolution of transient reflectivity at the pump fluence of $1300\,\mu$J/cm^2.

$\sigma_1(\omega)$, two scenarios have been considered: 1) the transverse Josephson plasma mode activated by inhomogeneous distribution of inter-layer Josephson coupling constant [van der Marel and Tsvetkov(1996), Grüninger *et al.*(2000)Grüninger, van der Marel, Tsvetkov, and Erb], and 2) the in-plane inhomogeneity of superconductivity associated with the spin-density wave [Basov *et al.*(1995)Basov, Mook, Dabrowski, and Timusk, Dordevic *et al.*(2003)Dordevic, Komiya, Ando, and Basov], while its assignment remains an open issue.

50.3 Results of Photoexcited State

Now we show the result of near-infrared optical pump-THz probe spectroscopy of $La_{2-x}Sr_xCuO_4$ ($x = 0.14$) with the optical pump linearly polarized along the c-axis. We also investigated the pump polarization along the a-axis and obtained similar results. Figure 50.2(a) shows transient reflectivity measured at 3 ps after the optical pump ($t_{\rm pp} = 3\,$ps). As the pump fluence increases, the JPR plasma edge shows a gradual redshift, indicating the reduction of c-axis superconducting coherence. Figure 50.2(b) shows the time evolution of transient reflectivity at the photoexcitation density of $1300\,\mu$J/cm^2. The pump-induced redshift of JPR is most significant right after the excitation ($t_{\rm pp} = 3\,$ps), and shows slow recovery toward the equilibrium plasma frequency. The spectral features at $t_{\rm pp} = 40\,$ps and $t_{\rm pp} = 100\,$ps are almost identical, suggesting that at $t_{\rm pp} = 40\,$ps the system has reached quasiequilibrium. A hump structure in reflectivity emerges around 6–7 meV and shows the same relaxation behavior in terms of its timescale.

Here, we compare the nonequilibrium dynamics with previous studies. Ultrafast dynamics of the photoexcited $La_{2-x}Sr_xCuO_4$ has been studied by transient reflectivity measurements at 800 nm [Kusar *et al.*(2005)Kusar, Demsar, Mihailovic, and Sugai, Kusar *et al.*(2008)Kusar, Kabanov, Demsar, Mertelj, Sugai, and Mihailovic, Beyer *et al.*(2011)Beyer, Städter, Beck, Schäfer, Kabanov, Logvenov, Bozovic, Koren, and Demsar] and by THz transmittance measurements [Beyer *et al.*(2011)Beyer, Städter, Beck, Schäfer, Kabanov, Logvenov, Bozovic, Koren, and Demsar]. In optical pump-probe measurements using a thin-film $La_{2-x}Sr_xCuO_4$, the excitation density required to destroy the superconductivity at $t_{\rm pp} = 3\,$ps has been estimated to be 2.4 K/Cu [Beyer *et al.*(2011)Beyer, Städter, Beck, Schäfer, Kabanov, Logvenov, Bozovic, Koren, and Demsar], which corresponds to approximately $30\,\mu$J/cm^2 in our current experimen-

tal setup. Therefore, we expect that in our result shown in Fig. 50.2(a) the superconductivity is destroyed in all three pump fluences at $t_{\mathrm{pp}} = 3\,\mathrm{ps}$. The transient reflectivity in Fig. 50.2, however, shows only the slight redshift of JPR without complete suppression. It can be ascribed to the penetration-depth mismatch of the pump and probe pulses. The penetration depth of the THz probe is on the order of 10 µm, whereas the penetration depth at 800 nm is around 660 nm as obtained from the literature values [Uchida *et al.*(1996)Uchida, Tamasaku, and Tajima]. Therefore, in optical pump-THz probe measurements in reflection geometry, what we observe is the mixed response of small contribution from the surface photoexcited region on top of the large contribution from the equilibrium background. The consideration of the penetration-depth mismatch will be discussed in more detail in the later section.

The intense photoexcitation leads to the heating of the sample, which has been a well-known contribution in pump-probe experiments. The pump energy density required to destroy the superconductivity at $t_{\mathrm{pp}} = 100\,\mathrm{ps}$ after the photoexcitation, where the electronic system and the lattice system are considered to reach the temperature just above T_{c}, has been evaluated as 13.7 K/Cu from the saturation behavior of the pump-probe signal [Beyer *et al.*(2011)Beyer, Städter, Beck, Schäfer, Kabanov, Logvenov, Bozovic, Koren, and Demsar]. This photoexcitation intensity corresponds to approximately $150\,\mathrm{\mu J/cm^2}$, thus for the pump fluence of $410\,\mathrm{\mu J/cm^2}$ and above, we expect that photoexcited sample has been heated up above T_{c} and the superconductivity is thermally destroyed.

50.4 Analysis of the photoexcited state

50.4.1 Modeling of the quasiequilibrium state

To examine the effect of pump-induced heating in quasiequilibrium state, we consider a following model of spatial distribution of the refractive index $n^{\mathrm{H}}(\omega, z)$, assuming that all of the injected energy by photoexcitation is converted to the temperature increase in the surface region of the sample. Utilizing the result of specific heat measurement [Wen *et al.*(2004)Wen, Liu, Zhou, Xiong, Ti, Xiang, Komiya, Sun, and Ando], one can convert the pump-energy density to the amount of temperature increase at each sample depth z (See Appendix 50.7 for the details of calculation.). In Fig. 50.3(a), we show the fluence of the optical pump $F_{\mathrm{pump}}(z)$ for $1300\,\mathrm{\mu J/cm^2}$ excitation and corresponding quasiequilibrium temperature $T_{\mathrm{samp}}(z)$ obtained by converting the pump-energy density to the local temperature. Notably, we see that the photoexcitation affects the sample temperature much more deeply into the sample compared to the pump penetration depth; $F_{\mathrm{pump}}(z)$ shows an exponential decay in the length scale of the pump penetration depth, but $T_{\mathrm{samp}}(z)$ slowly recovers back to its original equilibrium temperature in 6 µm, mainly due to the small specific heat at low temperature.

From the obtained temperature distribution $T_{\mathrm{samp}}(z)$, it is now possible to construct $n^{\mathrm{H}}(\omega, z)$ by assigning equilibrium data of the refractive index at the corresponding temperature for each z, which is presented in Fig. 50.3(b). The total reponse of the sample with $n^{\mathrm{H}}(\omega, z)$ when observed by the THz pulse can be obtained as the effective refractive index $n^{\mathrm{H}}_{\mathrm{eff}}(\omega)$, by assuming the surface region of the sample as a stack of thin layers and calculating the Fresnel coefficients considering the multiple reflections (See Appendix 50.8 for the details of calculation.). In Fig. 4, we compare between the energy reflectivity measured at $t_{\mathrm{pp}} = 100\,\mathrm{ps}$ and the calculated energy reflectivity from the simulation of $n^{\mathrm{H}}_{\mathrm{eff}}(\omega)$ with various pump fluences. The simulation result describes the experimental data in quasiequilibrium state consistently. The growth of the hump structure around 6–7 meV is also apparent in the simulation. Comparing the simulated spectra with the measurement at $t_{\mathrm{pp}} = 3\,\mathrm{ps}$ in Fig. 50.2(a), the experimental result shows more pronounced redshift of JPR and the growth of hump structure. At earlier pump-probe delaytime, application of the heating simulation may not be appropriate because the sample

Figure 50.3: (Color online) Simulation of the sample heating induced by the optical excitation. (a) Pump fluence $F_{\mathrm{pump}}(z)$ at 1300 μJ/cm^2 excitation (red dotted line) and the corresponding temperature $T_{\mathrm{samp}}(z)$ (blue solid line) at the sample depth z. (b) Constructed spatial distribution of the real-part refractive index $\Re[n^{\mathrm{H}}(\omega, z)]$.

Figure 50.4: (Color online) Comparison of the experimental result with the heating simulation. (a) Pump-fluence dependence of energy reflectivity at $t_{\rm pp} = 100\,$ps and (b) corresponding energy reflectivity obtained from the heating simulation.

is in the middle of the thermalization processes to reach the quasiequilibrium state and the sample temperature cannot be clearly defined. Nevertheless, the spectral features in quasiequilibrium state in photoexcited $La_{2-x}Sr_xCuO_4$ observed at longer delaytime can be understood by considering the sample heating due to the energy injection by the optical pump.

50.4.2 Possible artifacts produced in conventional pump-probe analysis

To gain further insight toward the spectra in the photoexcited state, we must consider the penetration-depth mismatch between pump and probe in order to obtain the response functions such as the loss function and optical conductivity. In previous studies of optical pump-THz probe experiments in reflection geometry, this discrepancy was taken into account by postulating a function form for the spatial distribution of complex refractive index $n(\omega, z)$. One example is the multilayer analysis [Kaiser *et al.*(2014)Kaiser, Hunt, Nicoletti, Hu, Gierz, Liu, Le Tacon, Loew, Haug, Keimer, and Cavalleri, Zhang *et al.*(2018)Zhang, Wang, Shi, Lin, Zhang, Gu, Dong, and Wang], where one assumes that the photoinduced change of the spatial profile of the complex refractive index $n^{\rm ML}(\omega, z)$ can be written by the exponential decay with respect to the pump penetration depth $d_{\rm pump}$ i.e.,

$$n^{\rm ML}(\omega, z) = n_{\rm eq}(\omega) + (n_{\rm surf}(\omega) - n_{\rm eq}(\omega))e^{-z/d_{\rm pump}}, \tag{50.1}$$

where $n_{\rm eq}(\omega)$ and $n_{\rm surf}(\omega)$ are the equilibrium and photoexcited surface complex refractive index, respectively. One can then calculate the effective refractive index $n^{\rm ML}_{\rm eff}(\omega)$, which is performed in the same manner as the heating simulation of quasiequilibrium state in Sec. 50.4.1, and obtain the value of $n_{\rm surf}(\omega)$ such that $n^{\rm ML}_{\rm eff}(\omega)$ fits the complex refractive index obtained from experiments. Another method is the single-layer analysis [Hu *et al.*(2014), Nicoletti *et al.*(2014)Nicoletti, Casandruc, Laplace, Khanna, Hunt, Kaiser, Dhesi, Gu, Hill, and Cavalleri, Casandruc *et al.*(2015)Casandruc, Nicoletti, Rajasekaran, Laplace, Khanna, Gu, Hill, and Cavalleri, Cremin *et al.*(2019)Cremin, Zhang, Homes, Gu, Sun, Fogler, Millis, Basov, and Averitt], where one assumes that the photoexcitation creates a thin surface nonequilibrium layer

with the thickness of d_{pump} on top of the equilibrium bulk, and includes all of the photoinduced effects into the surface thin layer i.e.,

$$n^{\mathrm{SL}}(\omega, z) = \begin{cases} n_{\mathrm{surf}}(\omega) & (z \leq d_{\mathrm{pump}}), \\ n_{\mathrm{eq}}(\omega) & (\text{otherwise}). \end{cases} \tag{50.2}$$

As far as the extraction of $n_{\mathrm{surf}}(\omega)$ is concerned, both procedures are known to produce similar spectral features at the sample surface [Nicoletti *et al.*(2014)Nicoletti, Casandruc, Laplace, Khanna, Hunt, Kaiser, Dhesi, Gu, Hill, and Cavalleri, Casandruc *et al.*(2015)Casandruc, Nicoletti, Rajasekaran, Laplace, Khanna, Gu, Hill, and Cavalleri, Zhang *et al.*(2018)Zhang, Wang, Shi, Lin, Zhang, Gu, Dong, and Wang]. Here, we applied the former multilayer analysis and reconstructed the transient optical spectra at the photoexcited sample surface based on the obtained $n_{\mathrm{surf}}(\omega)$.

In Fig. 50.5(a), we present the fluence dependence of the energy reflectivity, loss function and real-part optical conductivity at $t_{\mathrm{pp}} = 100\,\mathrm{ps}$ after extracting the surface refractive index by applying the multilayer analysis to the experimental result shown in Fig. 50.4(a). The superconducting JPR shows a strong redshift even in the weak excitation regime ($41\,\mu\mathrm{J/cm^2}$), and it is completely suppressed for stronger excitation density. In the real-part conductivity, a sharp peak emerges from the edge of the conductivity gap and evolves into a prominent peak with increasing pump fluence.

However, as we demonstrated in Sec. 50.4.1 and Fig. 50.4(b), the nonequilibrium optical spectra are reproduced by taking into account the pump-induced heating effect. Supposing the effective refractive index from the heating simulation $n^{\mathrm{H}}_{\mathrm{eff}}(\omega)$ as an experimental result and applying the multilayer analysis, we can now compare the extracted photoexcited surface spectra obtained from $n_{\mathrm{surf}}(\omega)$ with the spectra at $z = 0$ of the heating simulation, which can be calculated using $n^{\mathrm{H}}(\omega, z = 0)$. We show the comparison of the extracted surface spectra and the simulated data for $1300\,\mu\mathrm{J/cm^2}$ excitation in Fig. 50.5(b). The spectra calculated from $n_{\mathrm{surf}}(\omega)$ of the simulation result also present the similar behavior of increased energy reflectivity and corresponding increase in spectral weight in the real-part optical conductivity. The effect is more exaggerated in these spectra; the reflectivity exceeds one and the loss function becomes negative. On the other hand, the actual surface spectra obtained from $n^{\mathrm{H}}(\omega, z = 0)$ of heating simulation show featureless reflectivity, loss function and optical conductivity. This comparison demonstrates a possibility that the conventional method to extract the surface response functions does not provide the actual spectra at the surface but instead resulting in artificial spectral properties.

The observation of artifacts presented in Fig. 50.5 directly points out the fragility of surface reconstruction methods. As we can clealy see in Fig. 50.3, the spatial distribution of refractive index does not follow the exponential decay. The validity of the function form of $n(\omega, z)$, both in single-layer and multilayer analysis, may break down and lead to artifacts especially under the high-intensity photoexcitation. In such a case, the acquisition of the actual form of $n(\omega, z)$ becomes crucial, but its analytical determination is a challenging task. Also, in the multilayer analysis we assume that the photoexcitation affects the spectral properties only within the pump penetration depth of the sample. However, the heating simulation suggests that the effect of photoexcitation in terms of the rise in temperature affects much deeper into the sample than the penetration depth (see *e.g.* Fig. 50.3(a)). To examine the photoexcited state more accurately in bulk samples, the pump-probe penetration depth mismatch should be resolved carefully.

Before concluding, it is worth mentioning again that the modeling of the sample heating in this study is not fully applicable in the timescale within the relaxation time where the system has not reached quasi-thermal equilibrium. Indeed, in our result at 3 ps (Fig. 50.2(a)), we observed a difference between the experimental result and the simulation. In addition,

Figure 50.5: (Color online) (a) Pump-fluence dependence of reflectivity, loss function and real-part optical conductivity at $t_{\mathrm{pp}} = 100\,\mathrm{ps}$ extracted after the multilayer analysis. (b) Reflectivity, loss function and real-part optical conductivity obtained from the multilayer analysis of $n^{\mathrm{H}}_{\mathrm{eff}}(\omega)$ for $1300\,\mu\mathrm{J/cm}^2$ excitation shown in Figs. 50.4(b) (red), and corresponding spectra calculated from $n^{\mathrm{H}}(\omega, z = 0)$ of the heating simulation (blue).

since the specific heat of phonons becomes significantly larger with increasing temperature, the effect of pump-induced temperature increase should be infinitesimal in higher equilibrium temperature, including the measurements performed above T_c, and cannot account for the emergence of JPR-like structure above T_c as observed in the stripe-phase $La_{2-x}Ba_xCuO_4$ [Nicoletti *et al.*(2014)Nicoletti, Casandruc, Laplace, Khanna, Hunt, Kaiser, Dhesi, Gu, Hill, and Cavalleri, Casandruc *et al.*(2015)Casandruc, Nicoletti, Rajasekaran, Laplace, Khanna, Gu, Hill, and Cavalleri].

50.5 Summary

We investigated the photoexcited nonequilibrium response of the high-T_c cuprate superconductor $La_{2-x}Sr_xCuO_4$ ($x = 0.14$) by utilizing near-infrared optical pump-THz probe spectroscopy. In the superconducting state, the continuous redshift of JPR was observed with increasing the photoexcitation densities, indicating the destruction of superconductivity. We demonstrated that the quasiequilibrium spectral feature appearing after photoexcitation is reproduced considering the sample surface heating induced by the pump energy. We also pointed out that conventional pump-probe analysis to extract the response function of photoexcited sample surface in optical pump-THz probe experiments can present serious artifacts in the transient spectra.

50.6 Details of the data acquision and analysis in the pump-probe measurement

The pump-induced change of the photoexcited signal at the pump-probe delay time t_{pp} was obtained by utilizing the optical chopper to measure simultaneously the time-domain signal with the optical pump $E_{wp}(t; t_{pp})$ and without pump $E_{np}(t; t_{pp})$. By performing Fourier transform we obtained complex spectra $E_{wp}(\omega; t_{pp})$ and $E_{np}(\omega; t_{pp})$. We calculated the nonequilibrium complex reflectivity $r_{pp}(\omega; t_{pp})$ by multiplying the pump-induced change to the equilibrium complex reflectivity $r_{eq}(\omega)$,

$$r_{pp}(\omega; t_{pp}) = \frac{E_{wp}(\omega; t_{pp})}{E_{np}(\omega; t_{pp})} r_{eq}(\omega). \tag{50.3}$$

Using $r_{pp}(\omega; t_{pp})$, we obtained the transient reflectivity $R_{pp}(\omega; t_{pp}) = |r_{pp}(\omega; t_{pp})|^2$, which are presented in Figs. 50.2 and 50.4(a).

The key assumption in Eq. (50.3) is that the spectral feature of $E_{np}(\omega; t_{pp})$ is identical to the spectrum measured in equilibrium state $E_{eq}(\omega)$. However, if the modulation frequency of optical chopper for the optical pump is too fast under high pump fluence, the sample presents accumulative temperature increase because the next optical pump arrives at the sample before the sample temperature recovers back to its original equilibrium temperature. In such a case, the identity between $E_{np}(\omega; t_{pp})$ and $E_{eq}(\omega)$ is violated. In order to avoid this average heating effect, we used two optical choppers with the modulation frequency of 100 Hz to limit the pump arrival to once in every 10 ms.

50.7 Details of the heating simulation

We discuss the procedure of simulation to account for the surface heating of the sample in detail. For this simulation, we consider that all of the energy injected by the intense optical excitation contributes to the sample heating in the quasiequilibrium state. The energy density of the optical pump $I(z)$ can be related with the excitation fluence by assuming an exponential

decay by,

$$\begin{aligned} I(z) &= (1-R)(-\frac{dF_{\text{pump}}(z)}{dz}) \\ &= (1-R)\frac{F_0}{d_{\text{pump}}}e^{-z/d_{\text{pump}}}, \end{aligned} \tag{50.4}$$

where F_0 is the pump fluence, $d_{\text{pump}} \approx 660\,\text{nm}$ is the penetration depth of the optical pump at 800 nm calculated from the literature values [Uchida *et al.*(1996)Uchida, Tamasaku, and Tajima], and $R \approx 0.15$ is the reflectivity at 800 nm [Uchida *et al.*(1996)Uchida, Tamasaku, and Tajima]. The absorbed energy $I(z)$ and the temperature increase can be related by utilizing specific heat, which can be described as,

$$C(T) = \gamma_0 T + \alpha T^2 + \beta T^3 + \delta T^5, \tag{50.5}$$

where $\gamma_0 T$ is the electronic specific heat, αT^2 is the specific heat arising from the d-wave superconductivity, $\beta T^3 + \delta T^5$ is the lattice specific heat including the contributions up to T^5 term, respectively. We used the following specific heat coefficients $\gamma_0 = 1.90\,\text{mJ/(mol K}^2)$, $\alpha = 0.177\,\text{mJ/(mol K}^3)$, $\beta = 0.120\,\text{mJ/(mol K}^4)$ and $\delta = 0.000\,93\,\text{mJ/(mol K}^6)$, respectively, which are taken from the fitting result of the equilibrium specific measurement of the very near doping of $La_{2-x}Sr_xCuO_4$ ($x = 0.15$) [Wen *et al.*(2004)Wen, Liu, Zhou, Xiong, Ti, Xiang, Komiya, Sun, and Ando]. The quasiequilibrium temperature $T_{\text{samp}}(z)$ can now be related to $I(z)$ through the following integral equation,

$$I(z) = \int_{T_{\text{eq}}}^{T_{\text{samp}}(z)} C(T')\,dT', \tag{50.6}$$

where T_{eq} is the equilibrium temperature. By numerically solving Eq. (50.6), we obtained $T_{\text{samp}}(z)$ for various F_0. The construction of spatial distribution of complex refractive index $n(\omega, z)$ was then carried out by assigning equilibrium complex refractive index to each z with corresponding temperature. The temperature-dependent equilibrium complex refractive index was obtained by a linear interpolation of the equilibrium result (Figs. 50.1(c)-(e)) with respect to temperature.

50.8 Calculation of the effective refractive index $n_{\text{eff}}(\omega)$

Here, we briefly discuss the method to calculate the effective complex refractive index $n_{\text{eff}}(\omega)$ from the spatially non-uniform distribution of refractive index along the z-axis $n(\omega, z)$, which has been utilized both in the multilayer analysis and the heating simulation. In order to numerically solve $n_{\text{eff}}(\omega)$, we consider the surface region of the sample as a stack of thin-film layers and calculate the reflectivity coefficients by considering infinite multiple reflections in each thin film, as schematically drawn in Fig. 50.6. In our current study, we split the sample surface of length $L = 30\,\mu\text{m}$ into $N = 1000$ layers. We denote the thickness of the each thin-film layer as $\Delta L (= L/N)$ and write $n_l(\omega) = n(\omega, l\Delta L)$ to indicate the complex refractive index of the l-th layer. When we consider the reflection coefficient of the superconducting bulk to the deepest N-th layer, we use,

$$\bar{r}_{\text{bulk}}(\omega) = \frac{n_N(\omega) - n_{\text{eq}}(\omega)}{n_N(\omega) + n_{\text{eq}}(\omega)}, \tag{50.7}$$

Figure 50.6: (Color online) A schematic of the calculation procedure of the effective complex refractive index $n_{\text{eff}}(\omega)$.

where $n_{\text{eq}}(\omega)$ is the equilibrium complex refractive index. Now, when we include the N-th layer, the reflected THz probe acquires the following reflection coefficient,

$$\begin{aligned}\bar{r}_N(\omega) &= r_{N-1,N}(\omega) \\ &\quad + t_{N-1,N}(\omega)\bar{r}_{\text{bulk}}(\omega)t_{N,N-1}(\omega)e^{i\Phi_N(\omega)} \\ &\quad \times \sum_{k=0}^{\infty}\left[r_{N,N-1}(\omega)\bar{r}_{\text{bulk}}(\omega)e^{i\Phi_N(\omega)}\right]^k \\ &= \frac{r_{N-1,N}(\omega) + \bar{r}_{\text{bulk}}(\omega)e^{i\Phi_N(\omega)}}{1 + r_{N-1,N}(\omega)\bar{r}_{\text{bulk}}(\omega)e^{i\Phi_N(\omega)}},\end{aligned} \tag{50.8}$$

where,

$$t_{i,j}(\omega) = \frac{2n_i(\omega)}{n_i(\omega) + n_j(\omega)}, \tag{50.9}$$

$$r_{i,j}(\omega) = \frac{n_i(\omega) - n_j(\omega)}{n_i(\omega) + n_j(\omega)} \tag{50.10}$$

are the Fresnel coefficients for the transmittance and reflection at the boundary of two layers with $n_i(\omega)$ and $n_j(\omega)$, and $\Phi_i(\omega) = 2n_i(\omega)\omega\Delta L/c$ gives the phase factor of the multiple reflection. Similarly, the effective Fresnel's reflection coefficient at the l-th layer $\bar{r}_l(\omega)$ can be calculated using the $\bar{r}_{l+1}(\omega)$,

$$\tilde{r}_l(\omega) = \frac{r_{l-1,l}(\omega) + \bar{r}_{l+1}(\omega)e^{i\Phi_l(\omega)}}{1 + r_{l-1,l}(\omega)\bar{r}_{l+1}(\omega)e^{i\Phi_l(\omega)}}. \tag{50.11}$$

By repeating this calculation until we reach the surface layer $l = 1$, we obtained the total effective Fresnel coefficient $\bar{r}_1(\omega)$. For the calculation of $\bar{r}_1(\omega)$, we adopt the refractive index of the sample atmosphere to be $n_{\text{air}}(\omega) = n_0(\omega) = 1$. Finally, we can calculate $n_{\text{eff}}(\omega)$ as

$$n_{\text{eff}}(\omega) = \frac{1 - \bar{r}_1(\omega)}{1 + \bar{r}_1(\omega)}. \tag{50.12}$$

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From Stability of nonequilibrium Fermi distributions with respect to Cooper pairing by Aronov, Gurevich

Comments:

2) We have learned that D. A. Kirzhnits and Yu. V. Kopaev are also studying a similar problem.
3) A similar formalism has also been used by Elesin [16] in order to solve the problem of a semiconductor in the field of a strong electromagnetic wave.
4) In this representation the factors $\mathbf{u_p}$ and $\mathbf{v_p}$ are determined by other expressions (see [11]). We take this opportunity to mention that in [11] in expression (2.18) for the collision operator the last two terms were erroneously omitted. Taking these terms into account leads to a change of the numerical coefficients in the final formulas, namely, it leads to the replacement in formula (43) of the coefficient associated with Δ/kT by $1/2 + \pi/4$ and it also leads to the addition of the term $(1 - \ln 2)/2$ in the expression for $\ln C$.
5) This implies, as one can easily see, a "negative temperature" of the superconducting excitations.
6) This state has subsequently been studied in detail experimentally. [20]
7) Equation (4.3), which we have derived, differs slightly from the corresponding equation in [12].

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[foo(a)] Note that Eq. (41.2) describes a laser pulse with definite wave vector $\mathbf{q}_\mathrm{p}$, but nonzero uncertainty in its energy $\langle E\rangle \sim \hbar/(2\tau_\mathrm{p})$. We expect that our results remain qualitatively unchanged for more realistic pulse shapes with a broader distribution of wave vectors, since $|\mathbf{q}_\mathrm{p}|$ is much smaller than the Fermi momentum.

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[foo(b)] This saturation is due to the fact that according to Pauli's principle only one quasiparticle can be excited in each mode.

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[SM()] See Supplemental Material at [URL will be inserted by publisher], which contains Refs. [Yuzbashyan *et al.*(2005b)Yuzbashyan, Altshuler, Kuznetsov, and Enolskii, Yuzbashyan *et al.*(2015)Yuzbashyan, Dzero, Gurarie, and Foster, Barankov and Levitov(2006), Radcliffe(1971), Foster and Gurarie(), Muniz *et al.*(2020)Muniz, Barberena, Lewis-Swan, Young, Cline, Rey, and Thompson, Norcia *et al.*(2018)Norcia, Lewis-Swan, Cline, Zhu, Rey, and Thompson, Dennis *et al.*(2013)Dennis, Hope, and Johnsson].

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