Real Analysis, Week 11, Spring 2024

Karen Navasardyan

AUA

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Let $A \subset \mathbb{R}$ be given and suppose that for each $n \in \mathbb{N}$ there is a function $f_n : A \to \mathbb{R}$. We'll say that $\{f_n\}$ is a **sequence of functions** on A to $\mathbb{R}$. Clearly, for each $x \in A$, such a sequence gives rise to a sequence of real numbers, namely the sequence

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Definition

Let $\{f_n\}$ be a sequence of functions on $A \subset \mathbb{R}$, let $A_0 \subset A$, and let $f : A_0 \to \mathbb{R}$. We say that the **sequence** $\{f_n\}$ **converges on** A_0 **to** f if, for each $x \in A_0$, the sequence $\{f_n(x)\}$ converges to $f(x)$.

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$$\forall \varepsilon > 0 \quad \exists n_0 = n_0(\varepsilon) \quad s.t. \quad \forall n \geq n_0 \quad |f_n(x) - f(x)| < \varepsilon$$

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If the sequence f_n is uniformly convergent on A_0 to f, then this sequence also converges pointwise on A_0 to f.

A sequence $f_n(x)$ of functions on $A \subset \mathbb{R}$ does not converge uniformly on A_0 to a function $f : A_0 \to \mathbb{R}$ if and only if for some $\varepsilon_0 > 0$ there is a subsequence f_{n_k} of f_n and a sequence x_k in A_0 such that

$$|f_{n_k}(x_k) - f(x_k)| \geq \varepsilon_0 \quad \forall k \in \mathbb{N}.$$

Example. Prove that the sequence

$$f_n(x) = \frac{nx}{1 + n^2x^2}$$

converges pointwise on $\mathbb{R}$ to $f(x) = 0$, but $f_n(x)$ does not converge on $\mathbb{R}$ uniformly.

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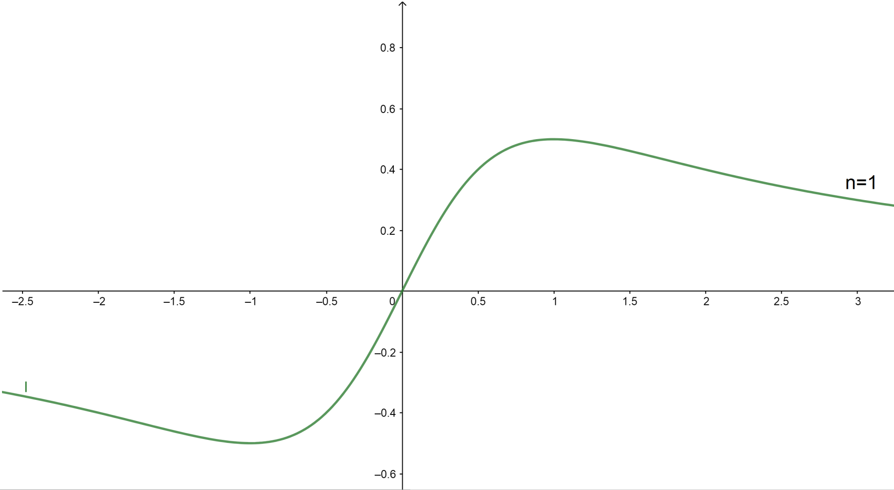

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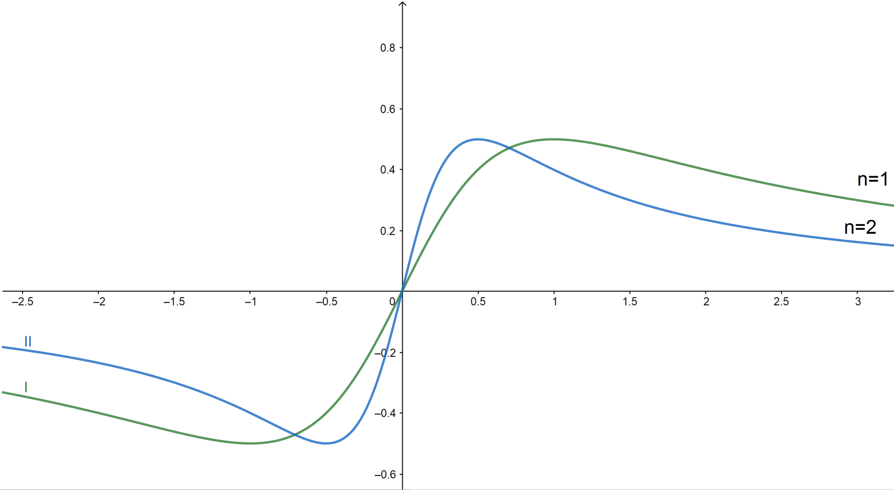

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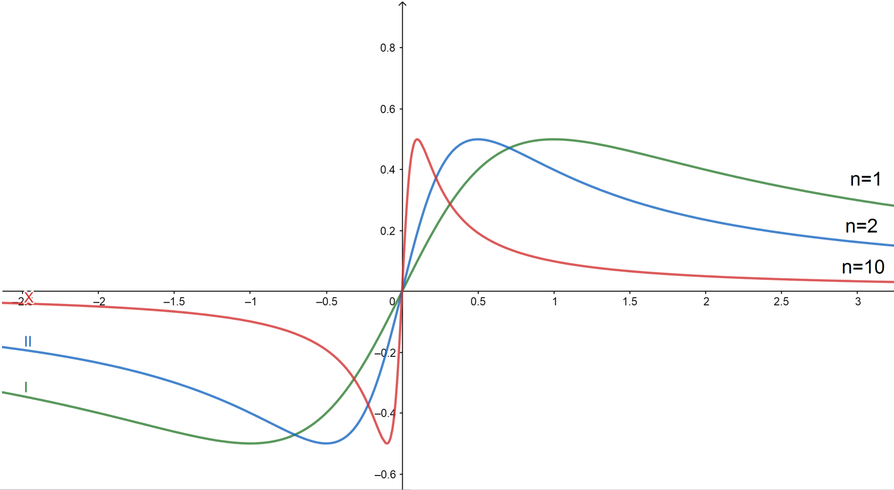

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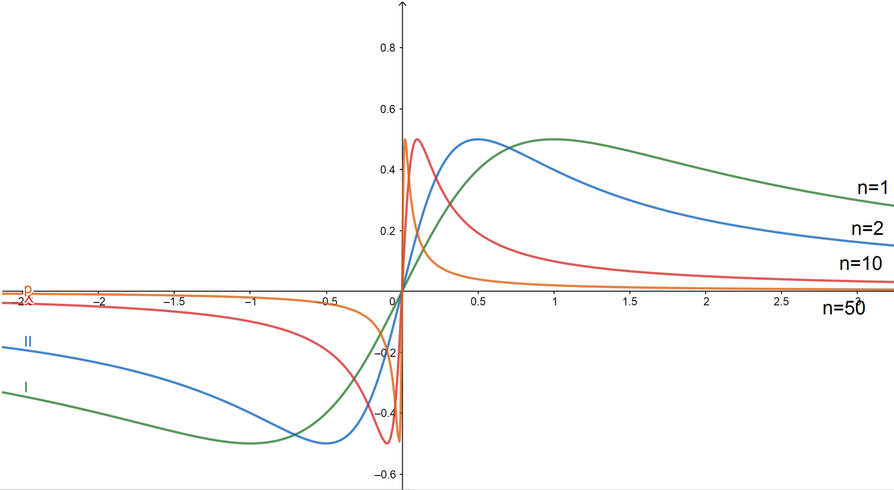

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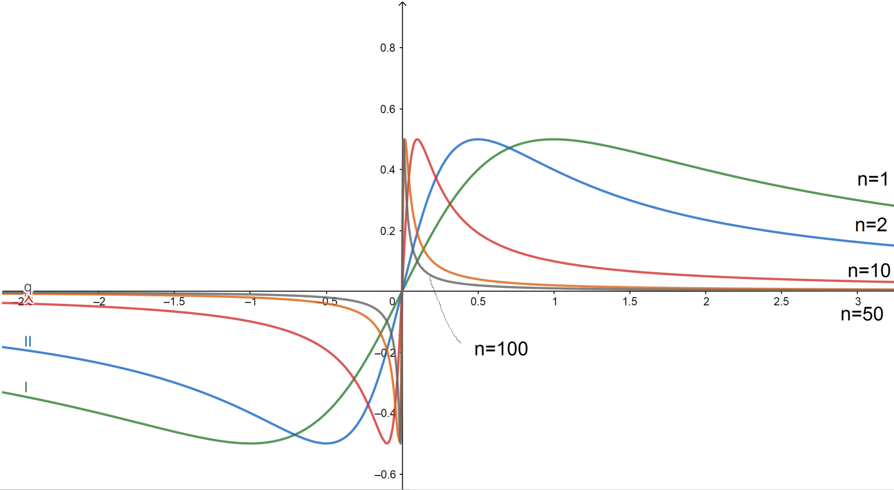

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Examples. 1) Prove that the sequence $f_n(x) = xe^{-nx}$ converges uniformly on $[0, \infty)$.
2) Study the uniformly convergence of $f_n(x) = x^n - x^{2n}$ on the interval $[0, 1]$.

Cauchy Criterion for Uniform Convergence

A sequence $f_n(x)$ of functions on $A \subset \mathbb{R}$ converges uniformly on A to a function f if and only if

$$\forall \varepsilon > 0 \quad \exists n_0 = n_0(\varepsilon) \quad s.t. \quad \forall m, n > n_0$$

$$|f_n(x) - f_m(x)| < \varepsilon \quad for\ all \quad x \in A.$$

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If the sequence of partial sums

$$S_m(x) = \sum_{n=1}^{m} f_n(x)$$

converges at any point of D to a function $f(x)$, then we say that the series converges to f pointwise on D.

If the sequence $S_n(x)$ of partial sums is uniformly convergent on D to f, we say that $\sum_{n=1}^{\infty} f_n(x)$ is **uniformly convergent** on D, or that it **converges to** f **uniformly on** D.

Cauchy Criterion

Let f_n be a sequence of functions on $D \subset \mathbb{R}$. The series $\sum\limits_{n=1}^{\infty} f_n(x)$ is uniformly convergent on D if and only if

$$\forall \varepsilon > 0 \quad \exists n_0 \quad s.t. \quad \forall m > n_0 \;\; \forall p \in \mathbb{N}$$

$$\left| \sum_{n=m+1}^{m+p} f_n(x) \right| < \varepsilon \quad for\ all \;\; x \in D.$$

Weierstrass M-Test

Let a_n be a sequence of positive real numbers such that $|f_n(x)| < a_n$ for $x \in D,\ n \in \mathbb{N}$. If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\sum_{n=1}^{\infty} f_n(x)$ is uniformly convergent on D.

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If the series of functions $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on a set A, then $f_n(x) \rightrightarrows 0$ on A.

Example. Study the uniform convergence of following series on the set A

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Interchange of limits

Theorem

Let f_n be a sequence of functions on a set $X \subset \mathbb{R}$ and suppose that f_n uniformly converges on X to a function $f : X \to \mathbb{R}$, as $n \to \infty$. If there exists the limit $\lim\limits_{x \to x_0} f_n(x)$ for each $n \in \mathbb{N}$, then

$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x).$$

Suppose $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ uniformly on X, and suppose $\lim_{x\to x_0} f_n(x) = c_n$ for each $n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} c_n$ converges and

$$\lim_{x\to x_0} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} \lim_{x\to x_0} f_n(x)$$

Theorem (Continuity)

Let f_n be a sequence of continuous functions on a set $X \subset \mathbb{R}$ and suppose that f_n uniformly converges on X to a function $f : X \to \mathbb{R}$. Then f is continuous on X.

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Suppose $\sum_{n=1}^{\infty} f_n(x)$ converges to $f(x)$ uniformly on X, and suppose f_n is continuous on X for each $n \in \mathbb{N}$. Then f is also continuous on X.

Dini's Theorem (Supplementary)

Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of continuous functions and suppose for each $x \in [a, b]$

$$f_1(x) \leq f_2(x) \leq f_3(x) \leq \cdots, \quad (or\ f_1(x) \geq f_2(x) \geq \cdots).$$

If f_n converges on $[a, b]$ to a continuous function f, then the convergence of the sequence is uniformly.

Interchange of Limit and Integral

Theorem

Let f_n be a sequence of integrable functions on $[a, b]$ and suppose that f_n converges uniformly on $[a, b]$ to f. Then f is integrable on $[a, b]$ and

$$\int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx$$

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Calculate the following limit:

$$\lim_{n\to\infty} \int_0^2 \frac{nx^2 + \ln(n+x)}{n+x^2} dx.$$

Theorem

Suppose that the functions f_n, $n \in \mathbb{N}$, are integrable on the interval $[a, b]$. If the series $\sum_{n=1}^{\infty} f_n$ converges to f uniformly on $[a, b]$, then f is integrable and

$$\int_a^b \left(\sum_{n=1}^{\infty} f_n(x) \right) dx = \int_a^b f(x)dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)dx$$

Calculate the integral

$$\int_1^2 \sum_{n=0}^{\infty} n3^{-nx} dx.$$

Interchange of Limit and Derivative

Theorem

Let (a, b) be a bounded interval and let f_n be a sequence of differentiable functions on (a, b). Suppose that there exists $x_0 \in (a, b)$ s.t. $f_n(x_0)$ converges, and that the sequence f_n' converges uniformly on (a, b). Then the sequence f_n converges uniformly on (a, b) to a differentiable function f and

$$\left(\lim_{n\to\infty} f_n(x)\right)' = f'(x) = \lim_{n\to\infty} f_n'(x).$$

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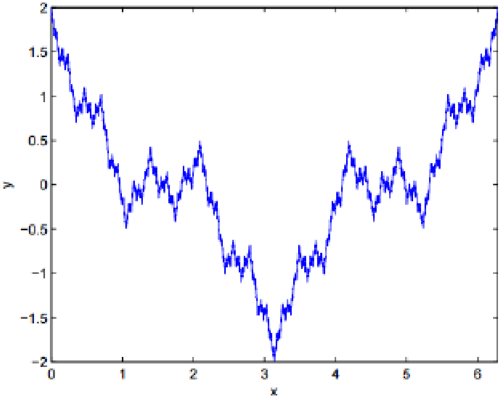

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$$\left(\sum_{n=1}^{\infty} f_n(x)\right)' = \sum_{n=1}^{\infty} f_n'(x).$$

Example.

Prove that

$$f(x) = \sum_{n=1}^{\infty} \frac{\arctan nx}{n^3 + x^2}$$

is continuous differentiable on $\mathbb{R}$.