Real Analysis
Week 9

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Convex functions

Definition

Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be **convex** on I if for any α satisfying $0 \leq \alpha \leq 1$ and any points $x_1, x_2 \in I$, we have

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2).$$

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If f is convex on I and if $x_1, x_2 \in I$, then the chord joining any two points $(x_1, f(x_1))$ and $(x_2, f(x_2))$ on the graph of f lies above the graph of f.

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Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be **concave** on I if for any α satisfying $0 \leq \alpha \leq 1$ and any points $x_1, x_2 \in I$, we have

$$f(\alpha x_1 + (1-\alpha)x_2) \geq \alpha f(x_1) + (1-\alpha) f(x_2).$$

Theorem

Let I be an open interval and let $f : I \to \mathbb{R}$ has the second order derivative on I. Then f is a convex function on I if and only if $f''(x) \geq 0$ for all $x \in I$.

Proof. $\Rightarrow$ First prove that if f has second order derivative at a point x, then

$$\lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} = f''(x). \qquad (1)$$

Now for any $x \in I$ by convexity of f we have
$f(x) = f(\frac{1}{2}(x+h) + \frac{1}{2}(x-h)) \leq \frac{1}{2}f(x+h) + \frac{1}{2}f(x-h)$,
Therefore using (1) we obtain, that $f''(x) \geq 0$.

$\Leftarrow$. Let $f''(x) \geq 0$ for all $x \in I$. By Taylor theorem for any $x_0, x \in I$ there is a point c s.t.

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(c)}{2!}(x - x_0)^2,$$

therefore

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0) \quad \forall x \in I. \tag{2}$$

Now for any $\alpha \in [0, 1]$ and $x_1, x_2 \in I$ denote
$x_0 = \alpha x_1 + (1 - \alpha)x_2$. By (2) we have
$f(x_1) \geq f(x_0) + f'(x_0)(x_1 - x_0)$
$f(x_2) \geq f(x_0) + f'(x_0)(x_2 - x_0)$. Multiplying this inequalities by α and $(1 - \alpha)$ respectively and adding them, we obtain that

$$\alpha f(x_1) + (1 - \alpha)f(x_2) \geq f(x_0) = f(\alpha x_1 + (1 - \alpha)x_2).$$

Q.E.D.

Jensen's inequality

Let $f : I \to \mathbb{R}$ be a convex function. Then for any $x_1, x_2, ..., x_n \in I$ and non-negative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$, we have

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_n f(x_n).$$

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Jensen's inequality (for concave functions)

Let $f : I \to \mathbb{R}$ be a concave function. Then for any $x_1, x_2, ..., x_n \in I$ and non-negative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$, such that $\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$, we have

$$f(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n) \geq \alpha_1 f(x_1) + \alpha_2 f(x_2) + \cdots + \alpha_n f(x_n).$$

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2. Prove that for any $a, b \in (1, \infty)$
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3. Prove that for any positive numbers $x, y, z \in [0, \pi]$
$$\sin\left(\frac{2x+y+z}{4}\right) \geq \frac{1}{2}\sin x + \frac{1}{4}\sin y + \frac{1}{4}\sin z$$

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4. Using the concavity of the function $f(x) = \ln x$ prove that for any non-negative numbers $x_1, x_2, \ldots, x_n$
$$\frac{x_1 + x_2 + \ldots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \ldots x_n}.$$

Infinite series

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If this limit does not exist or $\lim_{n\to\infty} S_n = \infty$, we say that the series is **divergent**.

Examples.

Prove that

1) If $|q| < 1$, then $\sum_{n=1}^{\infty} q^n$ is convergent and find its sum.

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2) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is divergent.

The nth Term Test

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Cauchy Criterion for Series

The series $\sum_{n=1}^{\infty} x_n$ converges if and only if

$$\forall \varepsilon > 0 \quad \exists n_0 \quad s.t. \quad \forall m > n_0 \quad \forall p \in \mathbb{N} \qquad \left| \sum_{n=m+1}^{m+p} x_n \right| < \varepsilon.$$

Theorem

Let x_n be a sequence of non-negative real numbers. Then the series $\sum_{n=1}^{\infty} x_n$ converges if and only if the sequence S_k of partial sums is bounded. In this case,

$$\sum_{n=1}^{\infty} x_n = \lim_{k\to\infty} S_k = \sup\{S_k : k \in \mathbb{N}\}.$$

Cauchy Condensation Test

Let x_n be a decreasing sequence of positive numbers. Then the series $\displaystyle\sum_{n=1}^{\infty} x_n$ is convergent if and only if $\displaystyle\sum_{n=1}^{\infty} 2^n x_{2^n}$ is convergent.

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The p–series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$.

Comparison Test

Let x_n and y_n be real sequences and suppose that for some n_0 we have

$$0 \leq x_n \leq y_n \qquad \textit{for all} \quad n \geq n_0.$$

(a) Then the convergence of $\displaystyle\sum_{n=1}^{\infty} y_n$ implies the convergence of $\displaystyle\sum_{n=1}^{\infty} x_n$.

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(a) Then the convergence of $\sum_{n=1}^{\infty} y_n$ implies the convergence of $\sum_{n=1}^{\infty} x_n$.

The divergence of $\sum_{n=1}^{\infty} x_n$ implies the divergence of $\sum_{n=1}^{\infty} y_n$.

Limit Comparison Test

Suppose that x_n and y_n are strictly positive sequences and suppose that the following limit exists $\lim\limits_{n\to\infty} \dfrac{x_n}{y_n} = r$

(a) If $r > 0$ then $\sum\limits_{n=1}^{\infty} x_n$ is convergent if and only if $\sum\limits_{n=1}^{\infty} y_n$ is convergent.

(b) If $r = 0$ and if $\sum\limits_{n=1}^{\infty} y_n$ is convergent, then $\sum\limits_{n=1}^{\infty} x_n$ is convergent.

Ratio test

Let $\sum_{n=1}^{\infty} x_n$ be a series with positive terms x_n.

1. If $\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} < 1$, then the series is convergent.

Ratio test

Let $\sum_{n=1}^{\infty} x_n$ be a series with positive terms x_n.

1. If $\limsup_{n\to\infty} \frac{x_{n+1}}{x_n} < 1$, then the series is convergent.
2. If $\liminf_{n\to\infty} \frac{x_{n+1}}{x_n} > 1$, then the series is divergent.

Root test

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Examples. Check the convergence of the following series

1. $$\sum_{n=1}^{\infty} \frac{(n!)^2(2+(-1)^n)}{(2n)!5^n}.$$

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2. $$\sum_{n=1}^{\infty} \frac{n^3(2+(-1)^n)^n}{4^n}.$$

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A series is said to be **conditionally** (or nonabsolutely) convergent if it is convergent, but it is not absolutely convergent.

Theorem

If a series $\sum_{n=1}^{\infty} x_n$ is absolutely convergent, then it is convergent.

Alternating Series Test

Let x_n be a decreasing sequence of strictly positive numbers with $\lim x_n = 0$. Then the alternating series

$$\sum_{n=1}^{\infty} (-1)^n x_n$$

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$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$ is conditionally convergent.

Dirichlet's Test

If x_n is a decreasing sequence with $\lim x_n = 0$, and if the partial sums S_n of $\sum_{n=1}^{\infty} y_n$ are bounded, then the series $\sum_{n=1}^{\infty} x_n y_n$ is convergent.

Abel's Test

If x_n is a monotonic and bounded sequence, and if the series $\sum_{n=1}^{\infty} y_n$ is convergent, then the series $\sum_{n=1}^{\infty} x_n y_n$ is also convergent.