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Zad 1

Mamy zagadnienie brzegowe funkcji harmonicznej na prost $(0,a)\times(0,b)$

i i) Wiemy, że

$$\begin{cases} u_{xx} + u_{yy} = 0 \\ u(a,y) = u(x,0) = u(x,b) = 0 & x \in (0,a \\ u_x(0,y) + \lambda u(0,y) = g(y) & y \in (0,b) \end{cases}$$

Rozwiązanie z rozdzielonymi zmiennymi, czyli

$$u(x,y) = X(x)\,Y(y) \qquad ' = \frac{\partial}{\partial x} \qquad ' = \frac{\partial}{\partial y}$$

Podstawiając

$$\begin{cases} X''Y + XY'' = 0 \\ X(a)Y(y) = X(x)Y(0) = X(x)Y(b) = 0 \Rightarrow X(a) = Y(0) = Y(b) = 0 \\ X'(0)Y(y) + \lambda X(0)Y(y) = g(y) \end{cases}$$

w.p.p. $X(x)=0$ $\vee$ $Y(y)=0$ czyli $u(x,y)=0$ trywialny przypadek

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{const} = \gamma$$

więc

$$X'' + \gamma X = 0 \quad i \quad Y'' - \gamma Y = 0$$

oraz $X(a)=0$ $\quad i \quad$ $Y(0) = Y(b) = 0$

aby to było spełnione

$$Y_n(y) = \sin\left(\frac{n\pi}{b} y\right)$$

$$\cancel{\text{wtedy}}\ X_n(x) = c_1 e^{\frac{n\pi}{b}x} + c_2 e^{-\frac{n\pi}{b}x}$$

$X_n(a) = 0$ więc $c_1 + c_2 e^{-\frac{2na\pi}{b}} = 0$ czyli $c_1 = -c_2 e^{-\frac{2\pi a n}{b}}$

więc

$$X_n(x) = c_2\left(e^{-\frac{n\pi x}{b}} - e^{\frac{n\pi}{b}x} \cdot e^{-\frac{2an\pi}{b}}\right) \qquad e^{\frac{n\pi x}{b}} \cdot e^{-\frac{2an}{b}}$$

Więc otrzymujemy wzór

$$u_n = a_n \cdot \sin\left(\frac{n\pi}{b}y\right) \cdot c_2\left(e^{-\frac{n\pi x}{b}} - e^{\frac{n\pi x}{b}} \cdot e^{-\frac{2an\pi}{b}}\right) = b_n \cdot \sin\left(\frac{n\pi}{b}y\right) \cdot \left(e^{-\frac{n\pi x}{b}} - \right.$$

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Zad 1 c.d

Można uprościć u_n $e^{-\frac{n\pi \cancel{bx}}{b}} - e^{\frac{n\pi(x-2a)}{b}} = c \cdot \sinh\left(\frac{n\pi}{b}(x-a)\right)$

więc

$$u_n = c_n \cdot \sin\left(\frac{n\pi}{b} \cdot y\right) \cdot \sinh\left(\frac{n\pi}{b}(x-a)\right)$$

Z szeregów Fouriera

$$\sum_{i=1}^{\infty} u_n = u$$

Chcemy jeszcze obliczyć c_n tak aby $u_x(0,y) + \lambda y(0,y) = g(y)$

$$u_x(x,y) = \sum_{i=1}^{\infty} c_n \sin\left(\frac{n\pi}{b} y\right) \cdot \left(\frac{n\pi}{b}\right) \cdot \cosh\left(\frac{n\pi}{b}(x-a)\right)$$

więc podstawiając

$$\sum_{i=1}^{\infty} c_n \sin\left(\frac{n\pi}{b} y\right) \left(\left(\frac{n\pi}{b}\right) \cdot \cosh\left(-\frac{n\pi a}{b}\right) + \lambda \sinh\left(-\frac{n\pi a}{b}\right)\right) = g(y)$$

$$c_n \left(\left(\frac{n\pi}{b} \cdot \cosh\left(-\frac{n\pi a}{b}\right) + \lambda \sinh\left(-\frac{n\pi a}{b}\right)\right) = \frac{2}{b} \int_b^b g(s) \sin\left(\frac{n\pi s}{b}\right) ds$$

więc

$$c_n = \frac{\frac{2}{b} \int_0^b g(s) \sin\left(\frac{n\pi s}{b}\right) ds}{\left(\frac{n\pi}{b} \cdot \sinh\left(-\frac{n\pi a}{b}\right) + \lambda \sinh\left(-\frac{n\pi a}{b}\right)\right)}$$

Więc ostatecznie otrzymujemy

$$u(x,y) = \sum_{i=0}^{\infty} \left(\underbrace{\frac{\frac{2}{b} \int_0^b g(s) \sin\left(\frac{i\pi s}{b}\right) ds}{\left(\frac{i\pi}{b} \sinh\left(-\frac{i\pi a}{b}\right) + \lambda \sinh\left(-\frac{i\pi a}{b}\right)\right)}}_{= c_n}\right) \cdot \sin\left(\frac{n\pi}{b} y\right) \cdot \sinh\left(\frac{n\pi}{b}(x-a)\right)$$