Schrödinger operators with a $(a\partial_\nu\delta_\gamma + b\delta_\gamma)$-like potentials

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Abstract Each chapter should be preceded by an abstract (10–15 lines long) that summarizes the content. The abstract will appear *online* at www.SpringerLink.com and be available with unrestricted access.

1 Introduction

2 Statement of Problem and Main Results

Let us consider the family of operators

$$H_\varepsilon = -\Delta + W(x) + V_\varepsilon(x). \tag{1}$$

Suppose that the unperturbed operator $H_0 = -\Delta + W$ is self-adjoint in $L^2(\mathbb{R}^2)$ with a domain $\operatorname{dom} H_0$. In addition, we suppose that $W \in L^\infty_{loc}(\mathbb{R}^2)$ and $\operatorname{dom} H_0 \subset W_2^1(\mathbb{R}^2)$.

Let γ be a closed C^3-curve without self-intersection points. We will denote by ω_ε the ε-neighborhood of γ, i.e., the union of all open balls with radius ε and center on γ. Suppose that potentials V_ε have compact supports that lie in ω_ε and the supports shrink to curve γ as $\varepsilon \to 0$. For this reason, $\operatorname{dom} H_\varepsilon = \operatorname{dom} H_0$.

To specify the dependence of V_ε on small parameter ε we introduce curvilinear coordinates coordinates in ω_ε. Let S be the circle of the same length as the length of γ. We will parameterize γ by points of the circle. Let $\alpha\colon S \to \mathbb{R}^2$ be the unit-speed C^3-parametrization of γ with the natural parameter $s \in S$. Also $\nu = (-\dot\alpha_2, \dot\alpha_1)$ is the unit normal on γ, because $\dot\alpha_1^2 + \dot\alpha_2^2 = 1$. We define

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the local coordinates (s,n) in ω_ε by

$$x = \alpha(s) + n\nu(s), \qquad (s,n) \in Q_\varepsilon = S \times (-\varepsilon, \varepsilon). \tag{2}$$

The coordinate n is the signed distance from a point x to γ. Therefore ω_ε is diffeomorphic to cylinder Q_ε for ε small enough. There is no loss of generality in assuming the diffeomorphism exists for $\varepsilon \in (0,1)$.

We suppose that the localized potentials have the following structure

$$V_\varepsilon(\alpha(s) + n\nu(s)) = \varepsilon^{-2}\, V\left(\varepsilon^{-1}n\right) + \varepsilon^{-1}\, U\left(s, \varepsilon^{-1}n\right), \tag{3}$$

where V and U are measurable bounded functions such that

$$\operatorname{supp} V \subset (-1,1), \quad \operatorname{supp} U \subset Q_1 \text{ and } \partial_s U \in L_2(\mathbb{R}^2). \tag{4}$$

The key assumption is that V does not depend on s.

The family of potentials V_ε generally diverges in the space of distributions $\mathcal{D}(\mathbb{R}^2)$. As we will show later in Propositin 1, the potentials converge only if V is a zero mean function, namely

$$V_\varepsilon(x) \to a\partial_\nu\delta_\gamma + b\delta_\gamma \quad \text{ as } \ \varepsilon \to 0,$$

where δ_γ is Dirac's delta function supported on γ, i.e., $\langle \delta_\gamma, \varphi\rangle = \int_\gamma \varphi\, d\gamma$, and a, b are some functions on γ. Therefore we shall suppose throughout that

$$\int_{\mathbb{R}} V(t)\, dt = 0. \tag{5}$$

We now introduce some notation. The plane is divided into two domains by close curve γ. We suppose that $\mathbb{R}^2 \setminus \gamma = \Omega_{in} \cup \Omega_{out}$, where domain Ω_{out} is unbounded. Let us introduce the subspace $\mathcal{V} \subset L_2(\mathbb{R}^2)$ as follows. We say that v belongs to $\mathcal{V}$ if $v|_{\Omega_-} \in W_2^2(\Omega_{in})$ and there exist a function h belonging to $\operatorname{dom} H_0$ such that $v = h$ in Ω_{out}. Of course, $v|_{\Omega_{out}} \in W^2_{2,loc}(\Omega_{out})$.

Let $\mathcal{V}_0$ be the subspaces of $L_2(\Omega_{out})$ obtained by the restriction of all elements of $\mathcal{V}$ to Ω_{out}. We introduce two operators

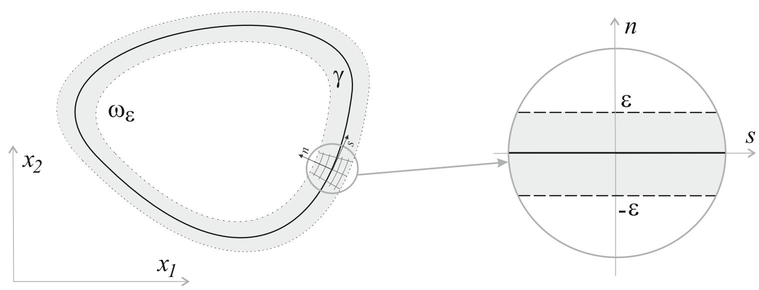

Fig. 1 Curvilinear coordinates in the ε-neighbourhood of γ.

$$\mathcal{D}_1 = -\Delta + W, \qquad \operatorname{dom}\mathcal{D}_1 = \{v \in \mathcal{V}_0\colon\ v = 0 \ \text{on}\ \gamma\},$$
$$\mathcal{D}_2 = -\Delta + W, \qquad \operatorname{dom}\mathcal{D}_2 = \{v \in W_2^2(\Omega_{in})\colon\ v = 0 \ \text{on}\ \gamma\}.$$

We also denote by $\gamma_t = \{x \in \mathbb{R}^2\colon\ x = \alpha(s) + t\nu(s),\ s \in S\}$ the closed curve that is obtained from γ by flowing for "time" t along the normal vector field. Then the boundary of ω_ε consists of two curves $\gamma_{-\varepsilon}$ and γ_ε. For each $v \in \mathcal{V}$ there exist two one-side traces on γ, namely

$$v_- = \lim_{\varepsilon\to 0} v|_{\gamma_{-\varepsilon}}, \qquad v_+ = \lim_{\varepsilon\to 0} v|_{\gamma_\varepsilon}.$$

We say that the Schrödinger operator $-\frac{d^2}{dt^2} + V$ in $L_2(\mathbb{R})$ possesses a *zero-energy resonance* if there exists a non trivial solution h of the equation $-h'' + Vh = 0$ that is bounded on the whole line. We call h the *half-bound state* of V. In this case, we will also simply say that potential V has a half-bound state h. Such a solution h is unique up to a scalar factor and has nonzero limits

$$h(-\infty) = \lim_{t\to-\infty} h(t), \qquad h(+\infty) = \lim_{t\to+\infty} h(t)$$

at both the infinities. We set

$$\theta = \frac{h(+\infty)}{h(-\infty)}. \tag{6}$$

Our main result reads as follows.

Theorem 1. *Let $W \in L^\infty_{loc}(\mathbb{R}^2)$ and $\operatorname{dom} H_0 \subset W_2^1(\mathbb{R}^2)$. Assume that potentials V and U are measurable bounded functions and assumption (4) and (5) holds. Then the family of operators*

$$H_\varepsilon = -\Delta + W + V_\varepsilon,$$

where the perturbation V_ε is given by (3), converges as $\varepsilon \to 0$ in the strong resolvent sense.

If potential V possesses a zero-energy resonance with a half-bound state h, then operators H_ε converge to operator $\mathcal{H}$ defined by $\mathcal{H}v = -\Delta v + Wv$ on functions $v \in \mathcal{V}$ obeying the interface conditions

$$u_+ - \theta u_- = 0, \quad \theta\partial_\nu u_+ - \partial_\nu u_- = \left(\tfrac{1}{2}(\theta^2 - 1)\varkappa + \mu\right) u_- \tag{7}$$

on curve γ. Here θ is given by (6), $\varkappa$ is the signed curvature of γ, and

$$\mu = \frac{1}{h^2(-\infty)} \int_{-1}^{1} U(\,\cdot\,, t) h^2(t)\, dt. \tag{8}$$

If potential V has no zero-energy resonance, then operators H_ε converge to the direct sum $\mathcal{D}_1 \oplus \mathcal{D}_2$ of two unperturbed operators $-\Delta + W$ in Ω_{in} and Ω_{out} respectively with the Dirichlet boundary conditions on interface γ.

Remark 1. If potential V is identically zero, then $V_\varepsilon = \varepsilon^{-1} U\big(s, \varepsilon^{-1} n\big)$ and so obviously $V_\varepsilon \to \mu_0 \delta_\gamma$, as $\varepsilon \to 0$, in the space of distributions. Here

$$\mu_0(s) = \int_{-1}^{1} U(s,t)\, dt. \tag{9}$$

Potential $V = 0$ possesses a zero-energy resonance with constant functions as half-bound states. Hence parameter θ equals 1 and interface conditions (7) become $u_+ - u_- = 0$, $\partial_\nu u_+ - \partial_\nu u_- - \mu_0 u_- = 0$. These conditions are exactly the same as that obtained in [1].

3 Preliminaries

Returning now to curvilinear coordinates (s,n) given by (2), we see that the couple of vectors $\tau = (\dot\alpha_1, \dot\alpha_2)$, $\nu = (-\dot\alpha_2, \dot\alpha_1)$ gives a Frenet frame for γ. The Jacobian of transformation $x = \alpha(s) + n\nu(s)$ has the form

$$\begin{aligned} J(s,n) &= \begin{vmatrix} \dot\alpha_1(s) - n\ddot\alpha_2(s) & -\dot\alpha_2(s) \\ \dot\alpha_2(s) + n\ddot\alpha_1(s) & \dot\alpha_1(s) \end{vmatrix} \\ &= \dot\alpha_1^2(s) + \dot\alpha_2^2(s) - n\big(\dot\alpha_1(s)\ddot\alpha_2(s) - \dot\alpha_2(s)\ddot\alpha_1(s)\big) = 1 - n\varkappa(s). \end{aligned}$$

Here $\varkappa = \det(\dot\alpha, \ddot\alpha)$ is the signed curvature of γ. Note that $\varkappa$ is a continuous function of the arc-length parameter s and the sign of $\varkappa(s)$ is defined uniquely up to the re-parametrization $s \mapsto -s$. We see that J is positive for sufficiently small ε, because curvature $\varkappa$ is bounded on γ. Namely, the curvilinear coordinates (s,n) can be defined correctly on all domains ω_ε with $\varepsilon \leqslant \varepsilon_*$, where $\varepsilon_* = \min_\gamma |\varkappa|^{-1}$. However, the above we have accepted that $\varepsilon_* = 1$, since this involves no loss of generality. We also have

$$\int_{\omega_\varepsilon} f(x_1,x_2)\, dx_1 dx_2 = \int_{Q_\varepsilon} f(s,n)(1 - n\varkappa(s))\, ds\, dn \tag{10}$$

for all integrable functions f

Next, metric tensor $g = (g_{ij})$ of ω_ε in the orthogonal coordinates (s,n) has the form

$$g = \begin{pmatrix} J^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

In fact, we have $g_{11} = |x_s|^2 = |\dot\alpha + n\dot\nu|^2 = |(1 - n\varkappa)\dot\alpha|^2 = J^2$, by the Frenet-Serret formula $\dot\nu = -\varkappa\dot\alpha$, and $g_{22} = |x_n|^2 = |\nu|^2 = 1$. In particular, the gradient in the local coordinates becomes

$$\nabla\varphi = \frac{1}{\sqrt{g_{11}}}\,\partial_s\varphi\,\tau + \frac{1}{\sqrt{g_{22}}}\,\partial_n\varphi\,\nu = \frac{1}{J}\,\partial_s\varphi\,\tau + \partial_n\varphi\,\nu$$

and therefore we have

$$\nabla\varphi\cdot\nabla\psi = J^{-2}\partial_s\varphi\,\partial_s\psi + \partial_n\varphi\,\partial_n\psi. \tag{11}$$

The Laplace-Beltrami operator in ω_ε has also the explicit form

$$\Delta\varphi = J^{-1}\left(\partial_s(J^{-1}\partial_s\varphi) + \partial_n(J\partial_n\varphi)\right) \tag{12}$$

as is easy to check.

Proposition 1. *If $\int_{\mathbb{R}} V\,dt = 0$, then*

$$V_\varepsilon \to \beta\partial_\nu\delta_\gamma + (\beta\varkappa + \mu_0)\,\delta_\gamma, \quad \text{as } \varepsilon\to 0,$$

in the space of distributions $\mathcal{D}(\mathbb{R}^2)$, where $\beta = -\int_{\mathbb{R}} tV(t)\,dt$ and μ_0 is given by (9).

Proof. It is evident that potentials $\varepsilon^{-1}\,U\left(s,\varepsilon^{-1}n\right)$ converge to $\mu_0\delta_\gamma$ in $\mathcal{D}(\mathbb{R}^2)$. We will prove that sequence $g_\varepsilon = \varepsilon^{-2}\,V\left(\varepsilon^{-1}n\right)$ converges to $\beta\left(\partial_\nu\delta_\gamma + \varkappa\delta_\gamma\right)$ as $\varepsilon\to 0$, provided V is a zero-mean function. In fact, for all $\varphi\in C_0^\infty(\mathbb{R}^2)$ we have

$$\begin{aligned}
\langle g_\varepsilon,\varphi\rangle &= \int_{\omega_\varepsilon} g_\varepsilon(x)\varphi(x)\,dx = \frac{1}{\varepsilon^2}\int_{Q_\varepsilon} V\left(\frac{n}{\varepsilon}\right)\varphi(s,n)(1-n\varkappa(s))\,ds\,dn\\
&= \frac{1}{\varepsilon}\int_{Q_1} V(t)\varphi(s,\varepsilon t)(1-\varepsilon t\varkappa(s))\,ds\,dt\\
&= \frac{1}{\varepsilon}\int_{-1}^{1} V(t)\,dt\int_S \varphi(s,0)\,ds\\
&\qquad\qquad + \int_{-1}^{1} tV(t)\,dt\int_S\left(\partial_n\varphi(s,0) - \varkappa(s)\varphi(s,0)\right)ds + O(\varepsilon),
\end{aligned}$$

as $\varepsilon\to 0$. The sequence $\langle g_\varepsilon,\varphi\rangle$ has a finite limit for all $\varphi\in C_0^\infty(\mathbb{R}^2)$ if and only if $\int_{\mathbb{R}} V\,dt = 0$. In this case, we have

$$\langle g_\varepsilon,\varphi\rangle \to \beta\int_\gamma\left(\partial_\nu\delta_\gamma + \varkappa\delta_\gamma\right)\varphi\,d\gamma,$$

which completes the proof. □

Interface conditions (7) contain the parameters which depend on the particular parametrization chosen for curve γ. More precisely, parameters θ, $\varkappa$ and μ change along with the change of the Frenet frame.

Proposition 2. *Operator $\mathcal{H}$ in Theorem 1 does not depend upon the choice of the Frenet frame for curve γ.*

Proof. Every smooth curve in the plane admits two possible orientations of arc-length parameter and consequently two possible Frenet frames. Let us change the Frenet frame $\{\tau,\nu\}$, previously introduced in Sec. 2, to the frame $\{-\tau,-\nu\}$ and prove that interface conditions (7) will remain the same. This change leads to the following transformations:

$$h(\pm\infty)\mapsto h(\mp\infty),\quad u_\pm\mapsto u_\mp,\quad \partial_\nu u_\pm\mapsto -\partial_\nu u_\mp,$$
$$\theta\mapsto\theta^{-1},\quad \varkappa\mapsto-\varkappa,\quad \mu\mapsto\theta^{-2}\mu.$$

The first condition $u_+-\theta u_-=0$ in (7) transforms into $u_--\theta^{-1}u_+=0$ and therefore remains unchanged. As for the second condition, we have

$$-\theta^{-1}\partial_\nu u_-+\partial_\nu u_+-\left(-\tfrac12(\theta^{-2}-1)\varkappa+\theta^{-2}\mu\right)u_+=0.$$

Multiplying the equality by θ yields

$$\theta\partial_\nu u_+-\partial_\nu u_--\left(\tfrac12(\theta^2-1)\varkappa+\mu\right)\theta^{-1}u_+=0,$$

since $-\theta(\theta^{-2}-1)=\theta^{-1}(\theta^2-1)$. It remains to insert u_- in place of $\theta^{-1}u_+$, in view of the first interface condition. □

In the sequel, the normal vector field ν will be outward to domain Ω_{in}, that is to say, the local coordinate n will increase in the direction from Ω_{in} to Ω_{out}. At the end of the section, we record some technical assertion, which will be often used below. Throughout the paper, $W_2^l(\Omega)$ stands for the Sobolev space of functions defined on a set Ω.

Proposition 3. *Suppose that $v\in W_2^1(\Omega_{out})$ and $w\in W_2^1(\Omega_{in})$. Then*

$$\|v(\,\cdot\,,\varepsilon)-v(\,\cdot\,,0)\|_{L_2(\gamma)}\leqslant c_1\varepsilon^{1/2}\|v\|_{W_2^1(\Omega_{out})},\tag{13}$$
$$\|w(\,\cdot\,,-\varepsilon)-w(\,\cdot\,,0)\|_{L_2(\gamma)}\leqslant c_2\varepsilon^{1/2}\|w\|_{W_2^1(\Omega_{in})},\tag{14}$$

where the constants c_k do not depend on ε.

Proof. First we assume that v is a smooth function in Ω_{out}. Then

$$v(s,\varepsilon)-v(s,0)=\int_0^\varepsilon\partial_t v(s,t)\,dt,$$

and for all $\psi\in L_2(\gamma)$ we have

$$\int_S(v(s,\varepsilon)-v(s,0))\psi(s)\,ds=\int_S\int_0^\varepsilon\partial_t v(s,t)\psi(s)\,dt\,ds.$$

Therefore

$$
\begin{aligned}
\left|\int_S \big(v(s,\varepsilon)-v(s,0)\big)\psi(s)\,ds\right| &\leqslant \int_S\int_0^\varepsilon |\partial_t v(s,t)|\,|\psi(s)|\,dt\,ds\\
&\leqslant \left(\int_0^\varepsilon\int_S |\psi(s)|^2\,ds\,dt\right)^{1/2}\left(\int_S\int_0^\varepsilon |\partial_t v|^2\,dt\,ds\right)^{1/2}\\
&\leqslant c\varepsilon^{1/2}\|\psi\|_{L_2(\gamma)}\left(\int_{\Omega_{out}}|\nabla v|^2\,dx\right)^{1/2}\leqslant c_1\varepsilon^{1/2}\|v\|_{W_2^1(\Omega_{out})}\|\psi\|_{L_2(\gamma)}.
\end{aligned}
$$

Hence (13) holds for all smooth functions v and then by continuity for all $v\in W_2^1(\Omega_{out})$. Similar arguments apply to the proof of (14). □

4 Finding Limit Operator

It is hardly possible to guess interface conditions (7) that arise in the so-called solvable model. In this section we will show how these conditions can be found by direct calculations, constructing the formal asymptotics of function

$$
u_\varepsilon = (H_\varepsilon-\zeta)^{-1}f. \tag{15}
$$

This function is a L_2-solution of equation

$$
-\Delta u_\varepsilon + (W+V_\varepsilon-\zeta)u_\varepsilon = f \quad \text{in} \quad \mathbb{R}^2, \tag{16}
$$

for given $f\in L_2(\mathbb{R}^2)$ and $\zeta\in\mathbb{C}\setminus\mathbb{R}$. We look for asymptotics of u_ε, as $\varepsilon\to 0$, in the form

$$
u_\varepsilon(x)\sim\begin{cases} u(x)+\cdots & \text{in} \ \ \mathbb{R}^2\setminus\omega_\varepsilon,\\ v_0\left(s,\frac{n}{\varepsilon}\right)+\varepsilon v_1\left(s,\frac{n}{\varepsilon}\right)+\cdots & \text{in} \ \ \omega_\varepsilon.\end{cases} \tag{17}
$$

Recall that the boundary of ω_ε consists of curves $\gamma_{-\varepsilon}$ and γ_ε. To match two different approximations, we hereafter assume that

$$
[u_\varepsilon]_{\gamma_{\pm\varepsilon}}=0, \qquad [\partial_\nu u_\varepsilon]_{\gamma_{\pm\varepsilon}}=0, \tag{18}
$$

where $[\,\cdot\,]_{\gamma_{\pm\varepsilon}}$ is a jump across $\gamma_{\pm\varepsilon}$. Since function u_ε solves (16) and the potentials V_ε shrink to γ, the leading term u must be a solution of the equation

$$
-\Delta u + (W-\zeta)u = f \quad \text{in} \quad \mathbb{R}^2\setminus\gamma,
$$

subject to appropriate interface conditions on γ.

To find these conditions, we consider equation (16) in the curvilinear coordinates (s,t), where $t=n/\varepsilon$. Then the Laplacian can be written as

$$
\Delta=\frac{1}{1-\varepsilon t\varkappa}\left(\varepsilon^{-2}\partial_t(1-\varepsilon t\varkappa)\partial_t+\partial_s\Big(\frac{1}{1-\varepsilon t\varkappa}\,\partial_s\Big)\right), \tag{19}
$$

by (12). From this we readily deduce the asymptotic representation

$$\Delta = \varepsilon^{-2}\partial_t^2 - \varepsilon^{-1}\varkappa\partial_t - t\varkappa^2\partial_t + \partial_s^2 + \varepsilon P_\varepsilon,$$

where P_ε is a partial differential operator on the second order with respect to s and the first one with respect to t. Substituting (17) into (16) for $x \in \omega_\varepsilon$ in particular yields

$$-\partial_t^2 v_0 + V(t)v_0 = 0, \qquad -\partial_t^2 v_1 + V(t)v_1 = -\varkappa(s)\partial_t v_0 - U(s,t)v_0 \tag{20}$$

in cylinder $Q_1 = S \times (-1,1)$. From (18) we see that necessarily

$$\begin{aligned} \partial_t v_0(s,-1) &= 0, & \partial_t v_0(s,1) &= 0,\\ \partial_t v_1(s,-1) &= \partial_\nu u_-(s), & \partial_t v_1(s,1) &= \partial_\nu u_+(s),\\ u_-(s) &= v_0(s,-1), & u_+(s) &= v_0(s,1), \end{aligned} \tag{21}$$

Combining (20) and the last equalities, we conclude that v_0 and v_1 solve boundary value problems

$$\begin{cases} -\partial_t^2 v_0 + V(t)v_0 = 0 \quad \text{in } \ Q_1,\\ \partial_t v_0(s,-1) = 0, \quad \partial_t v_0(s,1) = 0, \quad s \in S; \end{cases} \tag{22}$$

$$\begin{cases} -\partial_t^2 v_1 + V(t)v_1 = -\varkappa(s)\partial_t v_0 - U(s,t)v_0 \quad \text{in } \ Q_1,\\ \partial_t v_1(s,-1) = \partial_\nu u_-(s), \quad \partial_t v_1(s,1) = \partial_\nu u_+(s), \quad s \in S \end{cases} \tag{23}$$

respectively.

So we obtain two boundary value problems for the "non-elliptic" partial differential operator. In fact, let ℓ be the length of S and then the length of γ. Then both problems (22) and (23) can write out as boundary value problem in rectangle $\Pi = (0,\ell) \times (-1,1)$

$$\begin{cases} -\partial_t^2 v + V(t)v = g(s,t), \qquad (s,t) \in \Pi,\\ \partial_t v(s,-1) = a_-(s), \quad \partial_t v(s,1) = a_+(s), \quad s \in S,\\ v(0,t) = v(\ell,t), \quad \partial_s v(0,t) = \partial_s v(\ell,t), \quad t \in (-1,1) \end{cases}$$

with the Neumann boundary conditions with respect to t and the periodicity conditions with respect to s. Of course, the problem can also be regarded as a boundary value problem for ordinary differential equations on $(-1,1)$, which depends on parameter $s \in (0,\ell)$. In any case, the lack of ellipticity leads to a loss of smoothness of solutions with respect to s, and this will have an considerable influence on the proof of Theorem 1.

Case of zero-energy resonance

Assume that operator $-\frac{d^2}{dt^2} + V$ has a zero energy resonance with half-bound state h. Since the support of V lies in interval $(-1,1)$, the half-bound state h is constant outside this interval as a solution of equation $h'' = 0$ which is bounded at infinity. Therefore the restriction of h to $(-1,1)$ is a nonzero solution of the Neumann boundary value problem

$$-h'' + V(t)h = 0 \quad t \in (-1,1), \qquad h'(-1) = 0, \quad h'(1) = 0. \tag{24}$$

Hereafter, we fix h by additional condition $h(-1) = 1$. In view of (6), we have $h(1) = \theta$, since $h(\pm\infty) = h(\pm 1)$.

In this case, (22) admits infinite-dimensional space of solutions

$$\mathcal{N} = \big\{a(s)h(t)\colon\ a \in L^2(S)\big\}.$$

Therefore $v_0(s,t) = a_0(s)h(t)$ for some L^2-function a_0 on S. From (21) we deduce that

$$u_- = a_0, \qquad u_+ = \theta a_0.$$

Hence $v_0(s,t) = u_-(s)h(t)$ and in particular

$$u_+ = \theta u_-. \tag{25}$$

Next, problem (23) is in general unsolvable, since $\mathcal{N} \neq \{0\}$. To find solvability conditions, we rewrite equation in (23) as

$$-\partial_t^2 v_1 + V(t)v_1 = -\big(\varkappa(s)h'(t) + U(s,t)h(t)\big)u_-(s), \tag{26}$$

multiply by an arbitrary element ψ of $\mathcal{N}$ and then integrate over Q_1:

$$\int_{Q_1} \big(-\partial_t^2 v_1 + Vv_1\big)\, ah\, dt\, ds = -\int_{Q_1} (\varkappa h' + Uh)u_-\psi\, dt\, ds. \tag{27}$$

Because $\psi = a(s)h(t)$ and h is a solution of (24), integrating by parts twice in view of the boundary conditions for v_1 yields

$$\begin{aligned}
&\int_S \left(\int_{-1}^1 \big(-\partial_t^2 v_1 + Vv_1\big)\, h\, dt\right) a\, ds \\
&\quad = -\int_S (\partial_t v_1 h - v_1 h')\big|_{-1}^1 a\, ds - \int_S \left(\int_{-1}^1 v_1 \left(-h'' + Vh\right)\, dt\right) a\, ds \\
&\quad = -\int_S \big(\theta\partial_\nu u_+ - \partial_\nu u_-\big) a\, ds.
\end{aligned}$$

Recall that $h(-1) = 1$ and $h(1) = \theta$. Returning then to (27) we see that

$$\int_S \big(\theta\partial_\nu u_+ - \partial_\nu u_-\big) a\, ds = \int_S \left(\int_{-1}^1 \left(\varkappa hh' + Uh^2\right)\, dt\right) u_- a\, ds \tag{28}$$

We also have

$$\int_{-1}^1 hh'\, dt = \tfrac{1}{2}(\theta^2 - 1),$$

since $hh' = \frac{1}{2}(h^2)'$. Therefore (28) becomes

$$\int_S \big(\theta\partial_\nu u_+ - \partial_\nu u_- - \big(\tfrac{1}{2}(\theta^2-1)\varkappa + \mu\big)u_-\big)\, a\, ds = 0,$$

where μ is given by (8). The last identity holds for all $a \in L^2(S)$ and hence the expression in the brackets vanishes on γ. We obtain the condition

$$\theta\partial_\nu u_+ - \partial_\nu u_- = \big(\tfrac{1}{2}(\theta^2-1)\varkappa + \mu\big)u_-,$$

which is necessary for the solvability of (23). In view of the Fredholm alternative, this condition is also a sufficient one. At the same time, it is a jump condition at the interface for the normal derivative of u.

Therefore the leading term of asymptotics (17) is a solution of problem

$$-\Delta u + (W-\zeta)u = f \qquad \text{in } \mathbb{R}^2 \setminus \gamma, \tag{29}$$

$$u_+ - \theta u_- = 0 \qquad \text{on } \gamma, \tag{30}$$

$$\theta\partial_\nu u_+ - \partial_\nu u_- = \big(\tfrac{1}{2}(\theta^2-1)\varkappa + \mu\big)u_- \quad \text{on } \gamma. \tag{31}$$

The problem admits a unique solution $u = (\mathcal{H}-\zeta)^{-1}f$ belonging to space $\mathcal{V}$.

Now we can calculated the trace u_- on γ and define $v_0(s,t) = u_-(s)h(t)$. Since condition (31) holds, problem (23) is solvable and possesses a linear manifold of solutions. It will be convenient for us to fix v_1 such that $v_1(s,-1) = 0$ for all $s \in S$. We set

$$v_1(s,t) = \partial_\nu u_-(s)h_1(t) - u_-(s)h_2(s,t), \tag{32}$$

where h_1 and h_2 be solutions of the Cauchy problems

$$-h_1'' + V(t)h_1 = 0, \quad t \in (-1,1), \quad h_1(-1) = 0,\ h_1'(-1) = 1;$$

$$\begin{cases} -h_2'' + V(t)h_2 = \varkappa(s)h'(t) + U(s,t)h(t), & t \in (-1,1), \\ h_2(s,-1) = 0, \quad \partial_t h_2(s,-1) = 0, & s \in S \end{cases} \tag{33}$$

respectively. We see at once that v_1 of the form (32) solves equation (26) and satisfies boundary conditions $v_1(s,-1) = 0$ and $\partial_t v_1(s,-1) = \partial_\nu u_-(s)$. Now we show that the condition $\partial_t v_1(s,1) = \partial_\nu u_+(s)$ also holds.

Recall that half-bound state h was fixed by $h(-1) = 1$. Then the Lagrange identity $(h_1h' - h_1'h)|_{-1}^{1} = 0$ implies $h_1'(1) = \theta^{-1}$. Next, multiplying the equation in (33) by h and integrating by parts twice yield

$$(h'h_2 - h\,\partial_t h_2)\big|_{-1}^{1} = \varkappa(s)\int_{-1}^{1} hh'\,dt + \int_{-1}^{1} U(s,t)h^2(t)\,dt,$$

i.e., $\theta\,\partial_t h_2(s,1) = -\frac{1}{2}(\theta^2-1)\varkappa(s) - \mu(s)$. Therefore

$$\partial_t v_1(s,1) \ = \partial_\nu u_-(s)\, h_1'(1) - u_-(s)\,\partial_t h_2(s,1)$$

$$= \theta^{-1}\left(\partial_\nu u_-(s) + \left(\tfrac{1}{2}(\theta^2-1)\varkappa(s) + \mu(s)\right)u_-(s)\right) = \partial_\nu u_+(s)$$

by (31).

Non-resonant case

Now suppose that problem (24) admits the trivial solution only, i.e., $\mathcal{N} = \{0\}$. Then $v_0 = 0$ and therefore $u_- = 0$ and $u_+ = 0$ on γ, by (21). We thus get

$$-\Delta u + (W - \zeta)u = f \quad \text{in } \mathbb{R}^2 \setminus \gamma, \qquad u|_\gamma = 0$$

for the leading term of asymptotics (17). The problem admits a unique solution $u \in \mathcal{V}$. Of course, $u = (\mathcal{D}_1 \oplus \mathcal{D}_2 - \zeta)^{-1} f$. In this case, problem (23) becomes

$$\begin{cases} -\partial_t^2 v_1 + V(t)v_1 = 0 \quad \text{in } Q_1, \\ \partial_t v_1(s,-1) = \partial_\nu u_-, \quad \partial_t v_1(s,1) = \partial_\nu u_+, \end{cases} \tag{34}$$

and it is also uniquely solvable.

5 Proof of Theorem 1

We will provide a proof for the most interesting case when potential V has a zero-energy resonance. The non-resonant case, which is much easier, follows similarly. We must prove that

$$(H_\varepsilon - \zeta)^{-1} f \to (\mathcal{H} - \zeta)^{-1} f, \quad \text{as } \varepsilon \to 0, \tag{35}$$

for all $f \in L_2(\mathbb{R}^2)$ and some $\zeta \in \mathbb{C} \setminus \mathbb{R}$. But the resolvents of H_ε are uniformly bounded on ε, namely,

$$\|(H_\varepsilon - \zeta)^{-1}\| \leqslant |\operatorname{Im}\zeta|^{-1}.$$

It will thus be sufficient to prove that (35) holds for $f \in \mathcal{F}$, where $\mathcal{F}$ is some dense subset of $L_2(\mathbb{R}^2)$. We suppose that $\mathcal{F} = C_0^\infty(\mathbb{R}^2 \setminus \gamma)$.

Hereafter, letters c_j denote various positive numbers independent of ε, whose values might be different in different proofs.

5.1 Approximation in $W_2^1(\mathbb{R}^2)$

Given $f \in \mathcal{F}$ and $\zeta \in \mathbb{C}$, $\operatorname{Im}\zeta \neq 0$, we must compare $u_\varepsilon = (H_\varepsilon - \zeta)^{-1} f$ and $u = (\mathcal{H} - \zeta)^{-1} f$ in $L_2(\mathbb{R})$, and show that the difference $u_\varepsilon - u$ is infinitely small in $L_2(\mathbb{R})$-norm, as $\varepsilon \to 0$. The basic idea of the proof is to construct a suitable approximation to u_ε in $W_2^1(\mathbb{R}^2)$. The formal asymptotics

(17) constructed above will be used as a starting point in the construction of this approximation. We recall that the Sobolev space $W_2^1(\mathbb{R}^2)$ contains the domains of H_0 and H_ε.

We note that $v_0 \in W_2^1(Q_1)$. This inclusion follows from the explicit form $v_0(s,t) = u_-(s)h(t)$, where u_- belongs to $W_2^{3/2}(S)$ (as a trace of $u \in W_2^2(\Omega_{in})$ on curve γ) and $h \in W_2^2(-1,1) \subset C^1(-1,1)$. In view of representation formula (32), function v_1 does not belongs to $W_2^1(Q_1)$, since $\partial_\nu u_- \in W_2^{1/2}(S)$ in general. However the term u_-h_2 in (32) is an element of $W_2^1(Q_1)$, because $h_2 = h_2(s,t)$ possesses the additional smoothness with respect to s owing to the making more smoothness assumptions upon the curve γ and potential U. Recall that $\varkappa \in C^1(S)$ and $\partial_s U \in L_2(\mathbb{R}^2)$.

We now regularize the trace $\partial_\nu u_-$. Let $\{\beta_\varepsilon^-\}_{\varepsilon>0}$ be a sequence in $W_2^1(\gamma)$ such that $\beta_\varepsilon^- \to \partial_\nu u_-$ in $W_2^{-1/2}(\gamma)$. The sequence can be chosen in such a way that

$$\|\beta_\varepsilon^- - \partial_\nu u_-\|_{W_2^{-1/2}(\gamma)} \leqslant c_1\varepsilon^{1/2} \qquad \|\beta_\varepsilon^-\|_{W_2^1(\gamma)} \leqslant c_2\varepsilon^{-1/2}, \tag{36}$$

since $|\langle\beta_\varepsilon^- - \partial_\nu u_-,\, \beta_\varepsilon^-\rangle| \leqslant \|\beta_\varepsilon^- - \partial_\nu u_-\|_{W_2^{-1/2}(\gamma)}\|\beta_\varepsilon^-\|_{W_2^1(\gamma)}$ Then the function

$$v_1^\varepsilon(s,t) = \beta_\varepsilon^-(s)h_1(t) - u_-(s)h_2(s,t) \tag{37}$$

belongs to $W_2^1(Q_1)$ and solves the problem

$$\begin{cases} -\partial_t^2 v_1^\varepsilon + V(t)v_1^\varepsilon = -\varkappa(s)u_-(s)h'(t) - U(s,t)h(t) & \text{in } \ Q_1, \\ \partial_t v_1^\varepsilon(s,-1) = \beta_\varepsilon^-(s), \quad \partial_t v_1^\varepsilon(s,1) = \beta_\varepsilon^+(s), & s \in S, \end{cases} \tag{38}$$

where $\beta_\varepsilon^+ = \theta^{-1}\left(\frac{1}{2}(\theta^2-1)\varkappa u_- - \beta_\varepsilon^-\right)$. Thus

$$\|\beta_\varepsilon^+ - \partial_\nu u_+\|_{W_2^{-1/2}(\gamma)} \leqslant c_1\varepsilon^{1/2}, \tag{39}$$

as $\varepsilon \to 0$, by solvability condition (31). We also have the bounds

$$\|v_1^\varepsilon\|_{L_2(Q_1)} + \|\partial_t v_1^\varepsilon\|_{L_2(Q_1)} \leqslant C_1, \qquad \|\partial_s v_1^\varepsilon\|_{L_2(Q_1)} \leqslant C_2\varepsilon^{-1/2}. \tag{40}$$

We introduce the function

$$y_\varepsilon(x) = \begin{cases} u(x) & \text{in } \ \mathbb{R}^2 \setminus \omega_\varepsilon, \\ v_0\left(s, \frac{n}{\varepsilon}\right) + \varepsilon v_1^\varepsilon\left(s, \frac{n}{\varepsilon}\right) & \text{in } \ \omega_\varepsilon. \end{cases} \tag{41}$$

It is still not smooth enough and does not belong to $W_2^1(\mathbb{R}^2)$, because it has in general jump discontinuities on curves $\gamma_{-\varepsilon}$ and γ_ε. We will show that both the jumps

$$[y_\varepsilon]_{\gamma_{-\varepsilon}} = v_0(s,-1) - u(s,-\varepsilon) = u_-(s) - u(s,-\varepsilon),$$

$$\begin{aligned}[y_\varepsilon]_{\gamma_\varepsilon} &= u(s,\varepsilon) - v_0(s,1) - \varepsilon v_1^\varepsilon(s,1)\\ &= u(s,\varepsilon) - \theta u_-(s) - \varepsilon v_1^\varepsilon(s,1)\\ &= u(s,\varepsilon) - u_+(s) - \varepsilon v_1^\varepsilon(s,1)\end{aligned}$$

are small as $\varepsilon \to 0$. Recall that $v_1^\varepsilon(s,-1) = 0$.

Let us denote by Ω_ε the set $\mathbb{R}^2 \setminus \omega_\varepsilon$.

Lemma 1. *There exists a function $\rho_\varepsilon\colon \mathbb{R}^2 \to \mathbb{C}$ such that $y_\varepsilon + \rho_\varepsilon$ belongs to $W_2^1(\mathbb{R}^2)$. Moreover for the restriction of ρ_ε to Ω_ε we have the estimate*

$$\|\rho_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \leqslant c\varepsilon^{1/2}. \tag{42}$$

Proof. Let $Z_{in}\colon W_2^{1/2}(\gamma) \to W_2^1(\Omega_{in})$ and $Z_{out}\colon W_2^{1/2}(\gamma) \to W_2^1(\Omega_{out})$ be continuous extension operators such that $\operatorname{supp} Z_{in}\, g \subset \Omega_{in} \cap \omega_{1/2}$ and $\operatorname{supp} Z_{out}\, g \subset \Omega_{out} \cap \omega_{1/2}$ for all $g \in W_2^{1/2}(\gamma)$. The jumps $g_\varepsilon^\pm := [y_\varepsilon]_{\gamma_{\pm\varepsilon}}$ can be regarded as functions on γ. Obviously, $g_\varepsilon^\pm \in W_2^{1/2}(\gamma)$.

We set $z_\varepsilon^- = -Z_{in}\, g_\varepsilon^-$, $z_\varepsilon^+ = -Z_{out}\, g_\varepsilon^+$ and introduce function

$$\rho_\varepsilon(s,n) = \begin{cases} z_\varepsilon^+(s, n-\varepsilon) & \text{for } \ s \in S,\ n \in (\varepsilon, \varepsilon + 1/2),\\ z_\varepsilon^-(s, n+\varepsilon) & \text{for } \ s \in S,\ n \in (-\varepsilon - 1/2, -\varepsilon),\\ 0, \quad \text{otherwise} \end{cases} \tag{43}$$

in $\mathbb{R}^2$ for $\varepsilon < 1/2$. The function has a compact support and, in particular, it vanishes in ω_ε. Next, by construction ρ_ε has the jump discontinuities

$$[\rho_\varepsilon]_{\gamma_{\pm\varepsilon}} = z_\varepsilon^\pm(s,0) = -[y_\varepsilon]_{\gamma_{\pm\varepsilon}}.$$

Since both the functions y_ε and ρ_ε belong to $W_2^1(\mathbb{R}^2 \setminus (\gamma_{-\varepsilon} \cup \gamma_\varepsilon))$ and

$$[y_\varepsilon + \rho_\varepsilon]_{\gamma_{\pm\varepsilon}} = 0,$$

we have $y_\varepsilon + \rho_\varepsilon \in W_2^1(\mathbb{R}^2)$. Furthermore,

$$\begin{aligned}\|\rho_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} &\leqslant c_1\big(\|z_\varepsilon^-\|_{W_2^1(\Omega_{in})} + \|z_\varepsilon^+\|_{W_2^1(\Omega_{out})}\big)\\ &= c_1\big(\|Z_{in}\, g_\varepsilon^-\|_{W_2^1(\Omega_{in})} + \|Z_{out}\, g_\varepsilon^+\|_{W_2^1(\Omega_{out})}\big)\\ &\leqslant c_2\big(\|g_\varepsilon^-\|_{W_2^{1/2}(\gamma)} + \|g_\varepsilon^+\|_{W_2^{1/2}(\gamma)}\big)\\ &\leqslant c_3\big(\|u(\,\cdot\,,-\varepsilon) - u_-\|_{W_2^{1/2}(\gamma)} + \|u(\,\cdot\,,\varepsilon) - u_+\|_{W_2^{1/2}(\gamma)}\big)\\ &\quad+\varepsilon\|v_1^\varepsilon(\,\cdot\,,1)\|_{W_2^{1/2}(\gamma)} \leqslant c_4\varepsilon^{1/2},\end{aligned}$$

by Proposition 3 and (40). In fact, the restrictions u to domains Ω_{in} and Ω_{out} belong to $W_2^2(\Omega_{in})$ and $W_2^2(\Omega_{out})$ respectively. Applying Proposition 3 to u and $\partial_s u$ yields

$$\|u(\,\cdot\,,\pm\varepsilon) - u(\,\cdot\,,\pm 0)\|_{L_2(\gamma)} + \|\partial_s u(\,\cdot\,,\pm\varepsilon) - \partial_s u(\,\cdot\,,\pm 0)\|_{L_2(\gamma)} \leqslant c\varepsilon^{1/2}.$$

Consequently, $\|u(\,\cdot\,,\pm\varepsilon)-u_\pm\|_{W_2^{1/2}(\gamma)}\leqslant\|u(\,\cdot\,,\pm\varepsilon)-u_\pm\|_{W_2^1(\gamma)}\leqslant c\varepsilon^{1/2}$. Finally, this follows from (37) that

$$\|v_1^\varepsilon(\,\cdot\,,1)\|_{W_2^{1/2}(\gamma)}\leqslant c_1(\|\beta_\varepsilon^-\|_{W_2^{1/2}(\gamma)}+\|u_-\|_{W_2^{1/2}(\gamma)})\leqslant c_2,$$

since $\beta_\varepsilon^-\to\partial_\nu u_-$ in $W_2^{1/2}(\gamma)$. $\square$

Hence, the desired approximation to u_ε in the Sobolev space $W_2^1(\mathbb{R}^2)$ has the form

$$Y_\varepsilon(x)=\begin{cases}u(x)+\rho_\varepsilon(x) & \text{in } \ \mathbb{R}^2\setminus\omega_\varepsilon,\\ v_0\left(s,\frac{n}{\varepsilon}\right)+\varepsilon v_1^\varepsilon\left(s,\frac{n}{\varepsilon}\right) & \text{in } \ \omega_\varepsilon,\end{cases}\tag{44}$$

where ρ_ε is given by (43).

5.2 Estimate of Remainder

Let us fix $f\in C_0^\infty(\mathbb{R}^2\setminus\gamma)$. First of all, we note that

$$\int_{\mathbb{R}^2}f\varphi\,dx=\int_{\Omega_\varepsilon}f\varphi\,dx\tag{45}$$

for ε small enough. We also record some other identities that will be needed below. Multiplying equation (29) by $\varphi\in W_2^1(\mathbb{R}^2)$ and integrating by parts over Ω_ε yeild

$$\begin{aligned}\int_{\Omega_\varepsilon}\big(\nabla u\nabla\varphi\ &+(W-\zeta)u\varphi\big)\,dx-\int_{\Omega_\varepsilon}f\varphi\,dx\\ &=-\int_S\big(\partial_\nu u(s,\varepsilon)\varphi(s,\varepsilon)-\partial_\nu u(s,-\varepsilon)\varphi(s,-\varepsilon)\big)\,ds.\end{aligned}\tag{46}$$

In the same manner we can obtain from (22) and (38) that

$$\int_{Q_1}\big(\partial_t v_0\,\partial_t\psi+Vv_0\psi\big)J_\varepsilon\,dtds=\varepsilon\int_{Q_1}\varkappa\,\partial_t v_0\,\psi\,dtds;\tag{47}$$

$$\begin{aligned}\int_{Q_1}&\big(\partial_t v_1^\varepsilon\,\partial_t\psi+Vv_1^\varepsilon\psi+Uv_0\psi\big)J_\varepsilon\,dt\,ds\\ &=-\int_{Q_1}\varkappa\,\partial_t v_0\psi J_\varepsilon\,dt\,ds+\varepsilon\int_{Q_1}\varkappa\,\partial_t v_1^\varepsilon\,\psi\,dt\,ds\\ &+\int_S\big(\beta_\varepsilon^+(s)\psi(s,1)J(s,\varepsilon)-\beta_\varepsilon^-(s)\psi(s,-1)J(s,-\varepsilon)\big)\,ds\end{aligned}\tag{48}$$

for all $\psi\in W_2^1(Q_1)$. For instance, let us multiply the equation in (22) by $\psi(s,t)J_\varepsilon(s,t)$, where $J_\varepsilon(s,t)=1-\varepsilon\varkappa t$, and integrate over Q_1. Then in view of boundary conditions for v_0 we deduce

$$
\begin{aligned}
0 = \int_{Q_1} \big(-\partial_t^2 v_0 + V v_0\big)\psi J_\varepsilon \, dtds = -\int_S (\partial_t v_0 \, \psi J_\varepsilon)\big|_{-1}^{1} \, ds \\
+ \int_{Q_1} \partial_t v_0 \, \partial_t(\psi J_\varepsilon) \, dtds + \int_{Q_1} V v_0 \psi J_\varepsilon \, dtds \\
= \int_{Q_1} \big(\partial_t v_0 \, \partial_t \psi + V v_0 \psi\big) J_\varepsilon \, dtds - \varepsilon \int_{Q_1} \varkappa \, \partial_t v_0 \, \psi \, dtds,
\end{aligned}
$$

which establishes (47). Let us note here, for future use,

$$
\int_{\omega_\varepsilon} g(x) \, dx = \varepsilon \int_{Q_1} g(s, \varepsilon t) J_\varepsilon(s,t) \, ds \, dt, \tag{49}
$$

$$
|\nabla v(x_\varepsilon)|^2 = \varepsilon^{-2} |\partial_t v(s,t)|^2 + J_\varepsilon^{-2}(s,t) \, |\partial_s v(s,t)|^2, \tag{50}
$$

where $v(x_\varepsilon)$ stands for $v(s, \frac{n}{\varepsilon})$, cf. (10) and (11).

Under our assumptions about potential W the function $u_\varepsilon = (H_\varepsilon - \zeta)^{-1} f$ belongs to $W_2^1(\mathbb{R}^2)$ and therefore satisfies the integral identity

$$
\int_{\mathbb{R}^2} \big(\nabla u_\varepsilon \nabla \varphi + (W + V_\varepsilon - \zeta) u_\varepsilon \varphi\big) \, dx = \int_{\mathbb{R}^2} f \varphi \, dx, \qquad \varphi \in W_2^1(\mathbb{R}^2). \tag{51}
$$

To show that Y_ε is an adequate approximation to u_ε, introduce the functional

$$
F_\varepsilon(\varphi) = \int_{\mathbb{R}^2} \left(\nabla Y_\varepsilon \nabla \varphi + (W + V_\varepsilon - \zeta) Y_\varepsilon \varphi\right) \, dx - \int_{\mathbb{R}^2} f \varphi \, dx, \tag{52}
$$

defined for functions φ belonging to $W_2^1(\mathbb{R}^2)$ and prove that its norm is infinitely small as $\varepsilon \to 0$.

Lemma 2. *The functional F_ε satisfies the estimate*

$$
|F_\varepsilon(\varphi)| \leqslant c\varepsilon^{1/2} \|\varphi\|_{W_2^1(\mathbb{R}^2)}
$$

for all $\varphi \in W_2^1(\mathbb{R}^2)$.

Proof. Let us rewrite F_ε into a more detailed form

$$
\begin{aligned}
F_\varepsilon(\varphi) \ &= \int_{\omega_\varepsilon} \Big(\nabla(v_0 + \varepsilon v_1^\varepsilon)\nabla\varphi + (W + V_\varepsilon - \zeta)(v_0 + \varepsilon v_1^\varepsilon)\varphi\Big) \, dx \\
&+ \int_{\Omega_\varepsilon} \Big(\nabla(u + \rho_\varepsilon)\nabla\varphi + (W - \zeta)(u + \rho_\varepsilon)\varphi\Big) \, dx - \int_{\mathbb{R}^2} f\varphi \, dx.
\end{aligned}
$$

With notation $\varphi_\varepsilon(s,t) = \varphi(s, \varepsilon t)$, we have

$$
\begin{aligned}
F_\varepsilon(\varphi) \ &= \varepsilon^{-1} \int_{Q_1} \big(\partial_t v_0 \, \partial_t \varphi_\varepsilon + V v_0 \varphi_\varepsilon\big) J_\varepsilon \, dt \, ds \\
&+ \int_{Q_1} \big(\partial_t v_1^\varepsilon \, \partial_t \varphi_\varepsilon + V v_1^\varepsilon \varphi_\varepsilon + U v_0 \psi_\varepsilon\big) J_\varepsilon \, dt \, ds
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{\Omega_\varepsilon} \big(\nabla u \nabla\varphi + (W-\zeta)u\varphi\big)\,dx - \int_{\Omega_\varepsilon} f\varphi\,dx\\
&+\int_{\Omega_\varepsilon} \big(\nabla \rho_\varepsilon \nabla\varphi + (W-\zeta)\rho_\varepsilon\varphi\big)\,dx\\
&+\varepsilon\int_{Q_1} \partial_s v_0\,\partial_s\varphi_\varepsilon\,J_\varepsilon\,dt\,ds + \varepsilon^2\int_{Q_1} \partial_s v_1^\varepsilon\,\partial_s\varphi_\varepsilon\,J_\varepsilon\,dt\,ds\\
&+\varepsilon\int_{Q_1} (W-\zeta)(v_0+\varepsilon v_1^\varepsilon)\varphi_\varepsilon\,J_\varepsilon\,dt\,ds,
\end{aligned}
$$

by (45), (49) and (50). Let us replace the first and second integrals by the right-hand sides of (47) and (48) with $\psi_\varepsilon(s,t)=\varphi(s,\varepsilon t)$ respectively, and the difference between the third and fourth ones by the right-hand side of (46). The other terms in the last formula are small as $\varepsilon\to 0$, because Lemma 1 and estimates (40) provide the bounds

$$
\begin{gathered}
\Big|\textstyle\int_{\Omega_\varepsilon} \big(\nabla \rho_\varepsilon \nabla\varphi + (W-\zeta)\rho_\varepsilon\varphi\big)\,dx\Big| \leqslant c_1\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)},\\
\Big|\textstyle\int_{Q_1} \partial_s v_0\,\partial_s\varphi_\varepsilon\,J_\varepsilon\,dt\,ds\Big| \leqslant c_2\|\varphi\|_{W_2^1(\mathbb{R}^2)},\\
\Big|\textstyle\int_{Q_1} \partial_s v_1^\varepsilon\,\partial_s\varphi_\varepsilon\,J_\varepsilon\,dt\Big| \leqslant c_3\varepsilon^{-1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)},\\
\Big|\textstyle\int_{Q_1} (W-\zeta)(v_0+\varepsilon v_1^\varepsilon)\varphi_\varepsilon\,J_\varepsilon\,dt\,ds\Big| \leqslant c_4\|\varphi\|_{W_2^1(\mathbb{R}^2)}.
\end{gathered}
$$

Therefore

$$
\begin{aligned}
F_\varepsilon(\varphi) \;&= \int_{Q_1} \varkappa\,\partial_t v_0\,\varphi_\varepsilon\,dtds - \int_{Q_1} \varkappa\,\partial_t v_0\varphi_\varepsilon J_\varepsilon\,dt\,ds + \varepsilon\int_{Q_1} \varkappa\,\partial_t v_1^\varepsilon\,\varphi_\varepsilon\,dt\,ds\\
&+\int_S \big(\beta_\varepsilon^+(s)\varphi(s,\varepsilon)J(s,\varepsilon) - \beta_\varepsilon^-(s)\varphi(s,-\varepsilon)J(s,-\varepsilon)\big)\,ds\\
&-\int_S (\partial_\nu u(s,\varepsilon)\varphi(s,\varepsilon) - \partial_\nu u(s,-\varepsilon)\varphi(s,-\varepsilon))\,ds + r_\varepsilon(\varphi),
\end{aligned}
$$

where $|r_\varepsilon(\varphi)|\leqslant c\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}$. Next, F_ε in turn rearranges to become

$$
\begin{aligned}
F_\varepsilon(\varphi) = \;&\int_S \big(\beta_\varepsilon^+(s) - \partial_\nu u(s,\varepsilon)\big)\varphi(s,\varepsilon)\,ds\\
&-\int_S \big(\beta_\varepsilon^-(s) - \partial_\nu u(s,-\varepsilon)\big)\varphi(s,-\varepsilon)\,ds\\
&+\varepsilon\int_{Q_1} \varkappa\big(t\partial_t v_0 + \partial_t v_1^\varepsilon\big)\varphi_\varepsilon\,dt\,ds\\
&-\varepsilon\int_S \varkappa(s)\big(\beta_\varepsilon^+(s)\varphi(s,\varepsilon) - \beta_\varepsilon^-(s)\varphi(s,-\varepsilon)\big)\,ds + r_\varepsilon(\varphi).
\end{aligned}
$$

Thus

$$
\begin{aligned}
F_\varepsilon(\varphi) \; &= \int_S \big(\beta_\varepsilon^+(s) - \partial_\nu u(s,\varepsilon)\big)\varphi(s,\varepsilon)\,ds \\
&\qquad\qquad - \int_S \big(\beta_\varepsilon^-(s) - \partial_\nu u(s,-\varepsilon)\big)\varphi(s,-\varepsilon)\,ds + q_\varepsilon(\varphi),
\end{aligned}
$$

where $|q_\varepsilon(\varphi)| \leqslant c\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}$. But then (36), (39) and Proposition 3 imply

$$
\begin{aligned}
&\left| \int_S \big(\beta_\varepsilon^\pm(s) - \partial_\nu u(s,\pm\varepsilon)\big)\varphi(s,\pm\varepsilon)\,ds \right| \\
&\leqslant \left| \int_S \big(\beta_\varepsilon^\pm(s) - \partial_\nu u_\pm\big)\varphi(s,\pm\varepsilon)\,ds \right| + \left| \int_S \big(\partial_\nu u(s,\pm\varepsilon) - \partial_\nu u_\pm\big)\varphi(s,\pm\varepsilon)\,ds \right| \\
&\leqslant \|\beta_\varepsilon^\pm - \partial_\nu u_\pm\|_{W_2^{-1/2}(\gamma)} \|\varphi(\,\cdot\,,\pm\varepsilon)\|_{W_2^{1/2}(\gamma)} \\
&\qquad\qquad + \|\partial_\nu u(\,\cdot\,,\pm\varepsilon) - \partial_\nu u_\pm\|_{L_2(\gamma)} \, \|\varphi(\,\cdot\,,\pm\varepsilon)\|_{L_2(\gamma)} \leqslant c\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}.
\end{aligned}
$$

Therefore $|F_\varepsilon(\varphi)| \leqslant c\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}$ for all $\varphi \in W_2^1(\mathbb{R}^2)$, and the lemma follows. □

5.3 The End of the Proof

From (51) and (52) we see

$$
\int_{\mathbb{R}^2} \nabla(Y_\varepsilon - u_\varepsilon)\nabla\varphi\,dx + \int_{\mathbb{R}^2} (W + V_\varepsilon - \zeta)(Y_\varepsilon - u_\varepsilon)\varphi\,dx = F_\varepsilon(\varphi),
$$

for all $\varphi \in W_2^1(\mathbb{R}^2)$.

If $\varphi = \overline{Y_\varepsilon - u_\varepsilon}$, then

$$
\int_{\mathbb{R}^2} |\nabla(Y_\varepsilon - u_\varepsilon)|^2\,dx + \int_{\mathbb{R}^2} (W + V_\varepsilon - \zeta)|Y_\varepsilon - u_\varepsilon|^2\,dx = F_\varepsilon(\overline{Y_\varepsilon - u_\varepsilon}).
$$

$$
-\operatorname{Im}\zeta \int_{\mathbb{R}^2} |Y_\varepsilon - u_\varepsilon|^2\,dx = \operatorname{Im} F_\varepsilon(\overline{Y_\varepsilon - u_\varepsilon}).
$$

$$
\int_{\mathbb{R}^2} |Y_\varepsilon - u_\varepsilon|^2\,dx \leqslant |\operatorname{Im}\zeta|^{-1}|F_\varepsilon(\overline{Y_\varepsilon - u_\varepsilon})| \leqslant c_1\varepsilon^{1/2}\|Y_\varepsilon - u_\varepsilon\|_{W_2^1(\mathbb{R}^2)} \leqslant c_2\varepsilon^{1/4}
$$

Lemma 3. *If potential V has a zero mean, then the estimate*

$$
\left| \varepsilon^{-2} \int_{\omega_\varepsilon} V(\tfrac{n}{\varepsilon})|\varphi|^2\,dx \right| \leqslant c\varepsilon^{-1/2}\|\varphi\|^2_{W_2^1(\mathbb{R}^2)}
$$

holds for all $\varphi, \psi \in W_2^1(\mathbb{R}^2)$.

Proof.

$$\varepsilon^{-2}\left|\int_{\omega_\varepsilon} V(\tfrac{n}{\varepsilon})\varphi\psi\,dx\right| = \varepsilon^{-1}\left|\int_{Q_1} V(t)\varphi(s,\varepsilon t)\psi(s,\varepsilon t)(1-\varepsilon t\varkappa(s))\,dt\,ds\right|$$
$$\leqslant \varepsilon^{-1}\left|\int_{Q_1} V(t)\varphi(s,\varepsilon t)\psi(s,\varepsilon t)\,dt\,ds\right| + c_1\|\varphi\|_{W_2^1(\mathbb{R}^2)}\|\psi\|_{W_2^1(\mathbb{R}^2)}$$

$$\begin{aligned}
&\left|\int_{Q_1} V(t)\varphi(s,\varepsilon t)\psi(s,\varepsilon t)\,dt\,ds\right| \\
&= \biggl|\int_{Q_1} V(t)\Bigl(\varphi(s,0)+\int_0^{\varepsilon t}\partial_t\varphi(s,\tau)\,d\tau\Bigr) \\
&\qquad\qquad\times\Bigl(\psi(s,0)+\int_0^{\varepsilon t}\partial_t\psi(s,\tau)\,d\tau\Bigr)\,dt\,ds\biggr| \\
&\leqslant \left|\int_{Q_1} V(t)\varphi(s,0)\int_0^{\varepsilon t}\partial_t\psi(s,\tau)\,d\tau\,dt\,ds\right| \\
&+\left|\int_{Q_1} V(t)\psi(s,0)\int_0^{\varepsilon t}\partial_t\varphi(s,\tau)\,d\tau\,dt\,ds\right| \\
&+\left|\int_{Q_1} V(t)\int_0^{\varepsilon t}\partial_t\varphi(s,\tau)\,d\tau\int_0^{\varepsilon t}\partial_t\psi(s,\tau)\,d\tau\,dt\,ds\right|
\end{aligned}$$

$$\begin{aligned}
&\left|\int_{Q_1} V(t)\varphi(s,0)\int_0^{\varepsilon t}\partial_t\psi(s,\tau)\,d\tau\,dt\,ds\right| \\
&\leqslant c_1\left(\int_{Q_1}|\varphi(s,0)|^2\,dt\,ds\right)^{1/2}\left(\int_{Q_1}\left|\int_0^{\varepsilon t}\partial_t\psi(s,\tau)\,d\tau\right|^2\,dt\,ds\right)^{1/2} \\
&\leqslant c_1\|\varphi\|_{W_2^1(\mathbb{R}^2)}\left(\int_{Q_1}\left|\int_0^{\varepsilon t}d\tau\right|\left|\int_{-1}^{1}|\partial_t\psi|^2\,d\tau\right|\,dt\,ds\right)^{1/2} \\
&\leqslant c_1\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}\|\psi\|_{W_2^1(\mathbb{R}^2)}
\end{aligned} \tag{53}$$

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