SCHRÖDINGER OPERATORS WITH A $(a\partial_\nu\delta_\gamma + b\delta_\gamma)$-LIKE POTENTIALS

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Abstract. The

1. Introduction

Throughout the paper, $W_2^m(\Omega)$ stands for the Sobolev space of functions defined on a set Ω.

2. Statement of Problem and Main Results

Let us consider the family of operators

$$H_\varepsilon = -\Delta + W + V_\varepsilon, \tag{2.1}$$

where the potential W increases as $|x| \to +\infty$ and $W \in L^\infty_{loc}(\mathbb{R}^2)$. We define the potential V_ε as follows. Let γ be a closed smooth curve without self-intersection points. We will denote by ω_ε the ε-neighborhood of γ, i.e., the union of all open balls with radius ε and center on γ. Suppose that V_ε has a compact support that lies in ω_ε and shrinks to γ as $\varepsilon \to 0$. To specify the dependence of V_ε on small parameter ε we introduce curvilinear coordinates coordinates in ω_ε. Let S be the circle of the same length as the length of γ. We will parameterize γ by points of the circle. Let $\alpha\colon S \to \mathbb{R}^2$ be the unit-speed C^∞-parametrization of γ with the natural parameter $s \in S$. Then $\nu = (-\dot\alpha_2, \dot\alpha_1)$ is a unit normal on γ, because $\dot\alpha_1^2 + \dot\alpha_2^2 = 1$. We define the local coordinates (s, r) in ω_ε by

$$x = \alpha(s) + r\nu(s), \qquad (s,r) \in Q_\varepsilon = S \times (-\varepsilon, \varepsilon). \tag{2.2}$$

The coordinate r is the signed distance from a point x to γ. Therefore ω_ε is diffeomorphic to cylinder Q_ε for ε small enough. Suppose that the localized potential V_ε has the following structure

$$V_\varepsilon\big(\alpha(s) + r\nu(s)\big) = \varepsilon^{-2}\, V\left(\varepsilon^{-1} r\right) + \varepsilon^{-1}\, U\left(s, \varepsilon^{-1} r\right), \tag{2.3}$$

where V and U are smooth functions of compact support. Without loss of generality we can assume that the supports of V and $U(s,\cdot\,)$ lie in the interval $\mathcal{I} = (-1, 1)$ for all $s \in S$. The key assumption is that V does not depend on s. Note that the unperturbed operator $H_0 = -\Delta + W$ is self-adjoint in $L^2(\mathbb{R}^2)$ and its spectrum is discrete. Obviously, we have $\operatorname{dom} H_\varepsilon = \operatorname{dom} H_0$.

The family of potentials V_ε generally diverges in the space of distributions $\mathcal{D}(\mathbb{R}^2)$. As we will show in Propositin 2, the potentials converge only if V is a zero mean function. In this case, $V_\varepsilon \to a\partial_\nu\delta_\gamma + b\delta_\gamma$ as $\varepsilon \to 0$ for some functions a and b, where

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δ_γ is the Dirac delta function supported on γ and $\partial_\nu\delta_\gamma$ is the normal derivative of δ_γ at points of γ. More precisely,

$$\langle a\partial_\nu\delta_\gamma + b\delta_\gamma, \phi\rangle = -\int_\gamma \partial_\nu(a\phi)\,d\gamma + \int_\gamma b\phi\,d\gamma$$

for all $\phi \in C_0^\infty(\mathbb{R}^2)$.

The main task is to construct asymptotic approximations, as $\varepsilon \to 0$, to eigenvalues and eigenfunctions of H_ε, i.e., asymptotics of eigenvalues λ^ε and eigenfunctions u_ε of spectral equation

$$-\Delta u_\varepsilon + (W + V_\varepsilon)u_\varepsilon = \lambda^\varepsilon u_\varepsilon \quad \text{in} \ \ \mathbb{R}^2. \tag{2.4}$$

We introduce some notation. The plane is divided into two domains by close curve γ. We suppose that $\mathbb{R}^2 \setminus \gamma = \Omega^- \cup \Omega^+$, where domain Ω^+ is unbounded. In the sequel, the normal vector field ν on γ will be outward to domain Ω^-, that is to say, the local coordinate r will increase in the direction from Ω^- to Ω^+.

We say that v belongs to space $\mathcal{V} \subset L_2(\mathbb{R}^2)$ if $v|_{\Omega_-} \in W_2^2(\Omega^-)$ and there exist a function w belonging to $\operatorname{dom} H_0$ such that $v = w$ in Ω^+. Of course, the restriction of v to Ω^+ belongs to $W^2_{2,loc}(\Omega^+)$. Let $\mathcal{V}_0$ be the subspaces of $L_2(\Omega^+)$ obtained by the restriction of all elements of $\mathcal{V}$ to Ω^+. We introduce two operators (in the spaces?)

$$\begin{aligned}
&\mathcal{D}_1 = -\Delta + W, \qquad && \operatorname{dom}\mathcal{D}_1 = \{v \in \mathcal{V}_0\colon\ v = 0 \ \text{ on } \gamma\},\\
&\mathcal{D}_2 = -\Delta + W, \qquad && \operatorname{dom}\mathcal{D}_2 = \{v \in W_2^2(\Omega^-)\colon\ v = 0 \ \text{ on } \gamma\}.
\end{aligned}$$

We also denote by $\gamma_t = \{x \in \mathbb{R}^2\colon\ x = \alpha(s) + t\nu(s),\ s \in S\}$ the closed curve that is obtained from γ by flowing for "time" t along the normal vector field. Then the boundary of ω_ε consists of two curves $\gamma_{-\varepsilon}$ and γ_ε. For any $v \in \mathcal{V}$ there exist two one-side traces on γ, namely

$$v^- = \lim_{t\to 0-} v|_{\gamma_t}, \qquad v^+ = \lim_{t\to 0+} v|_{\gamma_t}. \tag{2.5}$$

We say that the Schrödinger operator $-\frac{d^2}{dt^2} + V$ in $L_2(\mathbb{R})$ possesses a *zero-energy resonance* if there exists a non trivial solution h of the equation $-h'' + Vh = 0$ that is bounded on the whole line. We call h the *half-bound state* of V. Such a solution h is unique up to a scalar factor and has nonzero limits

$$h(-\infty) = \lim_{t\to-\infty} h(t), \qquad h(+\infty) = \lim_{t\to+\infty} h(t)$$

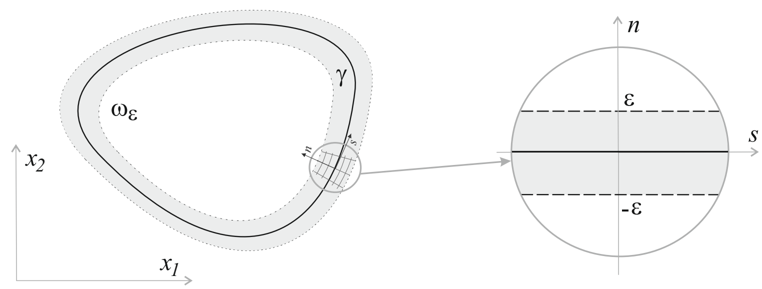

FIGURE 1. Curvilinear coordinates in the ε-neighbourhood of γ.

at both the infinities. We set
$$\theta = \frac{h(+\infty)}{h(-\infty)}. \tag{2.6}$$
We defined the operator $\mathcal{H}v = -\Delta v + Wv$, which acts on functions $v \in \mathcal{V}$ obeying the interface conditions
$$u^+ - \theta u^- = 0, \quad \theta\partial_\nu u^+ - \partial_\nu u^- = \left(\tfrac{1}{2}(\theta^2-1)\varkappa + \mu\right) u^- \tag{2.7}$$
on curve γ. Here $\varkappa$ is the signed curvature of γ and
$$\mu = \frac{1}{h^2(-\infty)} \int_{\mathcal{I}} U(\,\cdot\,, t) h^2(t)\, dt. \tag{2.8}$$
Our main result reads as follows.

Theorem 1. *Let $W \in L^\infty_{loc}(\mathbb{R}^2)$ and $\operatorname{dom} H_0 \subset W_2^1(\mathbb{R}^2)$. Assume that potentials V and U are measurable bounded functions and assumption (**??**) and (**??**) holds. Then the family of operators*
$$H_\varepsilon = -\Delta + W + V_\varepsilon,$$
where the perturbation V_ε is given by (2.3), converges as $\varepsilon \to 0$ in the strong resolvent sense.

If potential V possesses a zero-energy resonance with a half-bound state h, then operators H_ε converge to operator $\mathcal{H}$.

If potential V has no zero-energy resonance, then operators H_ε converge to the direct sum $\mathcal{D}_1 \oplus \mathcal{D}_2$ of two unperturbed operators $-\Delta + W$ in Ω^- and Ω^+ respectively with the Dirichlet boundary conditions on interface γ.

Remark 1. *If potential V is identically zero, then $V_\varepsilon = \varepsilon^{-1} U\left(s, \varepsilon^{-1} n\right)$ and so obviously $V_\varepsilon \to \mu_0\delta_\gamma$, as $\varepsilon \to 0$, in the space of distributions. Here*
$$\mu_0(s) = \int_{\mathcal{I}} U(s,t)\, dt. \tag{2.9}$$
Potential $V = 0$ possesses a zero-energy resonance with constant functions as half-bound states. Hence parameter θ equals 1 and interface conditions (2.7) become $u^+ - u^- = 0$, $\partial_\nu u^+ - \partial_\nu u^- - \mu_0 u^- = 0$. These conditions are exactly the same as that obtained in [1].

Notation. $\Upsilon = \frac{1}{2}(\theta^2 - 1)\varkappa + \mu$, $Q = S \times \mathcal{I}$. We denote by Ω_ε the set $\mathbb{R}^2 \setminus \omega_\varepsilon$.

3. Preliminaries

Returning now to curvilinear coordinates (s, r) given by (2.2), we see that the couple of vectors $\tau = (\dot\alpha_1, \dot\alpha_2)$, $\nu = (-\dot\alpha_2, \dot\alpha_1)$ gives a Frenet frame for γ. The Jacobian of transformation $x = \alpha(s) + r\nu(s)$ has the form
$$J(s,r) = \begin{vmatrix} \dot\alpha_1(s) - r\ddot\alpha_2(s) & -\dot\alpha_2(s) \\ \dot\alpha_2(s) + r\ddot\alpha_1(s) & \dot\alpha_1(s) \end{vmatrix}$$
$$= \dot\alpha_1^2(s) + \dot\alpha_2^2(s) - r\big(\dot\alpha_1(s)\ddot\alpha_2(s) - \dot\alpha_2(s)\ddot\alpha_1(s)\big) = 1 - r\varkappa(s).$$
Here $\varkappa = \det(\dot\alpha, \ddot\alpha)$ is the signed curvature of γ. Note that $\varkappa$ is a C^1-function of the arc-length parameter s and the sign of $\varkappa(s)$ is defined uniquely up to the re-parametrization $s \mapsto -s$. We see that J is positive for sufficiently small n, because curvature $\varkappa$ is bounded on γ. Namely, the curvilinear coordinates (s, r) can be defined correctly on ω_ε for all $\varepsilon < \varepsilon_*$, where $\varepsilon_* = \min_\gamma |\varkappa|^{-1}$.

The metric tensor $g = (g_{ij})$ of ω_ε in the orthogonal coordinates (s, r) has the form

$$g = \begin{pmatrix} J^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

In fact, we have $g_{11} = |x_s|^2 = |\dot\alpha + r\dot\nu|^2 = |(1 - r\varkappa)\dot\alpha|^2 = J^2$, by the Frenet-Serret formula $\dot\nu = -\varkappa\dot\alpha$, and $g_{22} = |x_r|^2 = |\nu|^2 = 1$. In particular, the gradient in the local coordinates becomes

$$\nabla\phi = \frac{1}{\sqrt{g_{11}}}\,\partial_s\phi\,\tau + \frac{1}{\sqrt{g_{22}}}\,\partial_r\phi\,\nu = \frac{1}{J}\,\partial_s\phi\,\tau + \partial_r\phi\,\nu$$

and therefore we have

$$\nabla\phi\cdot\nabla\psi = J^{-2}\partial_s\phi\,\partial_s\psi + \partial_r\phi\,\partial_r\psi. \tag{3.1}$$

The Laplace-Beltrami operator in ω_ε has also the explicit form

$$\Delta\phi = J^{-1}\left(\partial_s(J^{-1}\partial_s\phi) + \partial_r(J\partial_r\phi)\right) \tag{3.2}$$

as is easy to check.

Interface conditions (2.7) contain the parameters which depend on the particular parametrization chosen for curve γ. More precisely, parameters θ, $\varkappa$ and μ change along with the change of the Frenet frame.

Proposition 1. *Operator $\mathcal{H}$ in Theorem 1 does not depend upon the choice of the Frenet frame for curve γ.*

Proof. Every smooth curve in the plane admits two possible orientations of arc-length parameter and consequently two possible Frenet frames. Let us change the Frenet frame $\{\tau, \nu\}$, previously introduced in Sec. 2, to the frame $\{-\tau, -\nu\}$ and prove that interface conditions (2.7) will remain the same. This change leads to the following transformations:

$$h(\pm\infty) \mapsto h(\mp\infty), \quad u_\pm \mapsto u_\mp, \quad \partial_\nu u_\pm \mapsto -\partial_\nu u_\mp,$$
$$\theta \mapsto \theta^{-1}, \quad \varkappa \mapsto -\varkappa, \quad \mu \mapsto \theta^{-2}\mu.$$

The first condition $u^+ - \theta u^- = 0$ in (2.7) transforms into $u^- - \theta^{-1}u^+ = 0$ and therefore remains unchanged. As for the second condition, we have

$$-\theta^{-1}\partial_\nu u^- + \partial_\nu u^+ - \left(-\tfrac{1}{2}(\theta^{-2} - 1)\varkappa + \theta^{-2}\mu\right)u^+ = 0.$$

Multiplying the equality by θ yields

$$\theta\partial_\nu u^+ - \partial_\nu u^- - \left(\tfrac{1}{2}(\theta^2 - 1)\varkappa + \mu\right)\theta^{-1}u^+ = 0,$$

since $-\theta(\theta^{-2} - 1) = \theta^{-1}(\theta^2 - 1)$. It remains to insert u^- in place of $\theta^{-1}u^+$, in view of the first interface condition. □

Proposition 2. *If $\int_{\mathbb{R}} V\,dt = 0$, then the family of potentials V_ε converges to $\beta\,\partial_\nu\delta_\gamma + (\beta\varkappa + \mu_0)\,\delta_\gamma$, as $\varepsilon \to 0$, in the space of distributions $\mathcal{D}'(\mathbb{R}^2)$, where $\beta = -\int_{\mathbb{R}} tV(t)\,dt$ and μ_0 is given by (2.9).*

Proof. It is evident that potentials $\varepsilon^{-1}U\left(s,\varepsilon^{-1}r\right)$ converge to $\mu_0\delta_\gamma$ in $\mathcal{D}'(\mathbb{R}^2)$. Write $g_\varepsilon = \varepsilon^{-2}V\left(\varepsilon^{-1}r\right)$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^2} g_\varepsilon\phi\,dx &= \varepsilon^{-2}\int_{Q_\varepsilon} V(\varepsilon^{-1}r)\phi(s,r)(1-r\varkappa(s))\,ds\,dr\\ &= \varepsilon^{-1}\int_Q V(n)\phi(s,\varepsilon n)(1-\varepsilon n\varkappa(s))\,ds\,dn = \varepsilon^{-1}\int_{\mathcal{I}} V(n)\,dn\int_S \phi(s,0)\,ds\\ &\qquad + \int_{\mathcal{I}} nV(n)\,dn\int_S \big(\partial_n\phi(s,0)-\varkappa(s)\phi(s,0)\big)\,ds + O(\varepsilon),\end{aligned}$$

as $\varepsilon\to 0$, for all $\phi\in C_0^\infty(\mathbb{R}^2)$. The last sequence has a finite limit as $\varepsilon\to 0$ for all $\phi\in C_0^\infty(\mathbb{R}^2)$ if and only if $\int_{\mathbb{R}} V\,dt = 0$. In this case, we have

$$\int_{\mathbb{R}^2} g_\varepsilon\phi\,dx \to \beta\int_\gamma \left(\partial_\nu\delta_\gamma + \varkappa\delta_\gamma\right)\phi\,d\gamma,$$

which completes the proof. □

At the end of the section, we record some technical assertion, which will be often used below.

4. Limit Spectral Problem and Asymptotics of Eigenvalues

4.1. **Formal Asymptotics.** Now we will show how interface conditions (2.7) can be found by direct calculations, constructing the formal asymptotics of eigenvalues and eigenfunctions. We look for the asymptotic approximation of λ_ε and u_ε in the form

$$\lambda^\varepsilon \approx \lambda, \qquad u_\varepsilon(x)\approx\begin{cases}u(x) & \text{in } \ \mathbb{R}^2\setminus\omega_\varepsilon,\\ v_0\left(s,\frac{r}{\varepsilon}\right)+\varepsilon v_1\left(s,\frac{r}{\varepsilon}\right)+\varepsilon^2 v_2\left(s,\frac{r}{\varepsilon}\right) & \text{in } \ \omega_\varepsilon.\end{cases} \tag{4.1}$$

Recall that the boundary of ω_ε consists of two curves $\gamma_{-\varepsilon}$ and γ_ε. To match two different approximations, we hereafter assume that

$$[u_\varepsilon]_{\pm\varepsilon}=0, \quad [\partial_r u_\varepsilon]_{\pm\varepsilon}=0, \tag{4.2}$$

where $[w]_t$ stands for the jump of w across γ_t in the positive direction of local coordinate r.

Since function u_ε solves (2.4) and domain ω_ε shrinks to γ, the function u must be a solution of the equation

$$-\Delta u + Wu = \lambda u \quad \text{in } \ \mathbb{R}^2\setminus\gamma,$$

subject to appropriate interface conditions on γ. To find these conditions, we consider equation (2.4) in the curvilinear coordinates (s,n), where $n=r/\varepsilon$. Then in vicinity of γ the Laplacian can be written as

$$\Delta = \frac{1}{1-\varepsilon n\varkappa}\left(\varepsilon^{-2}\partial_n(1-\varepsilon n\varkappa)\partial_n + \partial_s\Big(\frac{1}{1-\varepsilon n\varkappa}\,\partial_s\Big)\right),$$

by (3.2). From this we readily deduce the asymptotic representation

$$\Delta = \varepsilon^{-2}\partial_n^2 - \varepsilon^{-1}\varkappa\partial_n - n\varkappa^2\partial_n + \partial_s^2 + \varepsilon P_\varepsilon. \tag{4.3}$$

Here P_ε is a partial differential operator on the second order with respect to s and the first one with respect to n whose coefficients are uniformly bounded on ε. Substituting (4.1) into (2.4) for $x \in \omega_\varepsilon$ in particular yields

$$-\partial_n^2 v_0 + V(n)v_0 = 0, \qquad -\partial_n^2 v_1 + V(n)v_1 = -\varkappa(s)\partial_n v_0 - U(s,n)v_0 \tag{4.4}$$

in the cylinder Q. From (4.2) we see that necessarily

$$u^-(s) = v_0(s,-1), \qquad u^+(s) = v_0(s,1), \tag{4.5}$$

$$\partial_n v_0(s,-1) = 0, \qquad \partial_n v_0(s,1) = 0, \tag{4.6}$$

$$\partial_n v_1(s,-1) = \partial_r u^-(s), \qquad \partial_n v_1(s,1) = \partial_r u^+(s), \tag{4.7}$$

where $u^\pm$ are defined by (2.5). Combining (4.6)–(4.7), we conclude that v_0 and v_1 solve the boundary value problems

$$\begin{cases} -\partial_n^2 v_0 + V(n)v_0 = 0 \quad \text{in } \ Q, \\ \ \partial_n v_0(s,-1) = 0, \quad \partial_n v_0(s,1) = 0, \quad s \in S; \end{cases} \tag{4.8}$$

$$\begin{cases} -\partial_n^2 v_1 + V(n)v_1 = -\varkappa(s)\partial_n v_0 - U(s,n)v_0 \quad \text{in } \ Q, \\ \ \partial_n v_1(s,-1) = \partial_r u^-(s), \quad \partial_n v_1(s,1) = \partial_r u^+(s), \quad s \in S \end{cases} \tag{4.9}$$

respectively. We have two boundary value problems for the "non-elliptic" partial differential operator in Q. Of course, the problems can also be regarded as boundary value problems for ordinary differential equations on $\mathcal{I}$, which depend on parameter $s \in S$.

Case of zero-energy resonance. Assume that operator $-\frac{d^2}{dt^2} + V$ has a zero energy resonance with half-bound state h. Since the support of V lies in the interval $\mathcal{I}$, the half-bound state h is constant outside this interval as a bounded solution of equation $h'' = 0$. Therefore the restriction of h to $\mathcal{I}$ is a nonzero solution of the Neumann boundary value problem

$$-h'' + V(n)h = 0, \quad n \in (-1,1), \qquad h'(-1) = 0, \quad h'(1) = 0. \tag{4.10}$$

Hereafter, we fix h by additional condition $h(-1) = 1$. In view of (2.6), we have $h(1) = \theta$, since $h(\pm\infty) = h(\pm 1)$. In this case, (4.8) admits a infinitely many solutions $a(s)h(n)$, where a is an arbitrary function on S. Then $v_0(s,n) = a_0(s)h(n)$ for some a_0. From (4.5) we deduce that $u^- = a_0$ and $u^+ = \theta a_0$ and hence that $v_0(s,n) = u^-(s)h(n)$ and

$$u^+ = \theta u^- \quad \text{on } \gamma. \tag{4.11}$$

Next, problem (4.9) is in general unsolvable, since (4.8) admits nontrivial solutions. To find solvability conditions, we rewrite equation in (4.9) as

$$-\partial_n^2 v_1 + V(n)v_1 = -\big(\varkappa(s)h'(n) + U(s,n)h(n)\big)u^-(s), \tag{4.12}$$

multiply by $a(s)h(n)$ and then integrate over Q

$$\int_Q \big(-\partial_n^2 v_1 + V(n)v_1\big)\, a(s)h(n)\, dn\, ds = -\int_Q (\varkappa(s)h'(n) + U(s,n)h(n))u^-(s)a(s)h(n)\, dn\, ds. \tag{4.13}$$

Since h is a solution of (4.10), integrating by parts twice on the left-hand side yields

$$\int_S\int_{\mathcal I}\left(-\partial_n^2 v_1 + V v_1\right) ah\,dn\,ds = -\int_S (\partial_n v_1 h - v_1 h')\big|_{n=-1}^{n=1} a\,ds \\ -\int_S\int_{\mathcal I} a v_1\left(-h'' + Vh\right)\,dn\,ds = -\int_S\left(\theta\partial_r u^+ - \partial_r u^-\right)a\,ds,$$

in view of the boundary conditions for v_1. Recall that $h(-1) = 1$ and $h(1) = \theta$. Hence (4.13) becomes

$$\int_S\left(\theta\partial_r u^+ - \partial_r u^-\right)a\,ds = \int_S u^- a\int_{\mathcal I}\left(\varkappa hh' + Uh^2\right)\,dn\,ds.$$

The equality $hh' = \frac12(h^2)'$ implies

$$\int_{\mathcal I} hh'\,dn = \tfrac12(h^2(1) - h^2(-1)) = \tfrac12(\theta^2 - 1). \tag{4.14}$$

Therefore we obtain

$$\int_S\left(\theta\partial_r u^+ - \partial_r u^-\right)a\,ds = \int_S\left(\tfrac12(\theta^2-1)\varkappa + \mu\right)u^- a\,ds$$

for all $a \in L^2(S)$, where $\mu(s) = \int_{\mathcal I} U(s,n)h^2(n)\,dn$. From this we deduce

$$\theta\partial_r u^+ - \partial_r u^- = \left(\tfrac12(\theta^2-1)\varkappa + \mu\right)u^- \quad \text{on } \gamma,$$

which is necessary for the solvability of (4.9). In view of the Fredholm alternative, this condition is also sufficient. At the same time, it is a jump condition at the interface γ for the normal derivative of u.

Therefore the leading terms λ and u of asymptotics (4.1) solve the problem

$$-\Delta u + Wu = \lambda u \qquad \text{in } \mathbb{R}^2\setminus\gamma, \tag{4.15}$$

$$u^+ - \theta u^- = 0, \quad \theta\partial_\nu u^+ - \partial_\nu u^- = \left(\tfrac12(\theta^2-1)\varkappa + \mu\right)u^- \quad \text{on } \gamma. \tag{4.16}$$

This problem can be written as $(\mathcal H - \lambda)u = 0$. Assume that λ is an eigenvalue of operator $\mathcal H$ and u is an eigenfunction for this eigenvalue. Here it is not essential to us what the multiplicity of λ is, and the eigenfunction u is arbitrarily chosen. Now we can calculated the trace u^- on γ and finally determine v_0.

Since the second condition in (4.16) holds, problem (4.9) is solvable and v_1 is defined up to the term $a_1(s)h(n)$. Choose the solution v_1 so that $v_1(s,-1) = 0$ for all $s \in S$. We also suppose that v_2 solves the Cauchy problem

$$\begin{aligned} -\partial_n^2 v_2 + V(n)v_2 = -(\varkappa(s)\partial_n + U(s,n))v_1 \\ + (\partial_s^2 - n\varkappa^2\partial_n - W(s,0) + \lambda)v_0 \quad \text{in } Q, \end{aligned} \tag{4.17}$$

$$v_2(s,-1) = 0, \quad \partial_n v_2(s,-1) = 0, \quad s \in S. \tag{4.18}$$

The function

$$\hat v_\varepsilon(x) = \begin{cases} u(x) & \text{in } \mathbb{R}^2\setminus\omega_\varepsilon, \\ v_0\left(s,\frac{r}{\varepsilon}\right) + \varepsilon v_1\left(s,\frac{r}{\varepsilon}\right) + \varepsilon^2 v_2\left(s,\frac{r}{\varepsilon}\right) & \text{in } \omega_\varepsilon. \end{cases} \tag{4.19}$$

does not in general belong to $\operatorname{dom} H_\varepsilon$, because it has jump discontinuities on the boundary $\partial\omega_\varepsilon$. We introduce the function ζ plotted in Fig. 2. This function is smooth outside the origin, $\zeta(r) = 1$ for $r \in [0,\beta/2]$ and $\zeta(r) = 0$ in the set $\mathbb{R}\setminus[0,\beta)$. We set

$$\eta_\varepsilon = \left([\hat v_\varepsilon]_\varepsilon + [\partial_r\hat v_\varepsilon]_\varepsilon\,(r-\varepsilon)\right)\zeta(r-\varepsilon) + \left([\hat v_\varepsilon]_{-\varepsilon} + [\partial_r\hat v_\varepsilon]_{-\varepsilon}\,(r+\varepsilon)\right)\zeta(-r-\varepsilon).$$

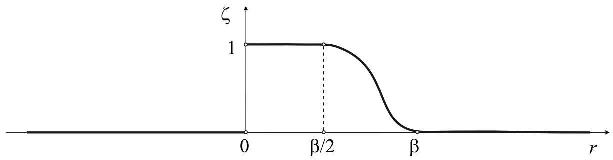

FIGURE 2. Plot of the function ζ.

and note that η_ε and $\partial_r\eta_\varepsilon$ have the same jumps across the boundary of ω_ε as $\hat v_\varepsilon$ and $\partial_r\hat v_\varepsilon$ respectively. In addition, η_ε is different from zero in the set $\omega_{\beta+\varepsilon}\setminus\omega_\varepsilon$ only. Therefore the function

$$v_\varepsilon(x)=\begin{cases}u(x)-\eta_\varepsilon(x) & \text{in } \ \mathbb{R}^2\setminus\omega_\varepsilon,\\ v_0\left(s,\frac{r}{\varepsilon}\right)+\varepsilon v_1\left(s,\frac{r}{\varepsilon}\right)+\varepsilon^2 v_2\left(s,\frac{r}{\varepsilon}\right) & \text{in } \ \omega_\varepsilon\end{cases} \tag{4.20}$$

belongs to the domain of $\mathcal{H}_\varepsilon$. We have not changed the asymptotics (4.19) too much, since

$$\sup_{x\in\mathbb{R}^2\setminus\omega_\varepsilon}\big(|\eta_\varepsilon(x)|+|\Delta\eta_\varepsilon(x)|\big)\leqslant c\varepsilon. \tag{4.21}$$

The last inequality is valid, because the jumps of $\hat v_\varepsilon$ and $\partial_r\hat v_\varepsilon$ across both curves $\gamma_{-\varepsilon}$ and γ_ε are infinitely small as $\varepsilon\to0$ uniformly on $s\in S$. In fact,

$$\begin{aligned}
[\hat v_\varepsilon]_{-\varepsilon}&=v_0(s,-1)-u(s,-\varepsilon)=u^-(s)-u(s,-\varepsilon)=O(\varepsilon),\\
[\hat v_\varepsilon]_{\varepsilon}&=u(s,\varepsilon)-v_0(s,1)+O(\varepsilon)=u(s,\varepsilon)-\theta u^-(s)+O(\varepsilon)\\
&\qquad\qquad\qquad\qquad\qquad\quad=u(s,\varepsilon)-u^+(s)+O(\varepsilon)=O(\varepsilon),\\
[\partial_r\hat v_\varepsilon]_{-\varepsilon}&=\varepsilon^{-1}\partial_n v_0(s,-1)+\partial_n v_1(s,-1)-\partial_r u(s,-\varepsilon)\\
&\qquad\qquad\qquad\qquad\qquad\quad=\partial_r u^-(s)-\partial_r u(s,-\varepsilon)=O(\varepsilon),\\
[\partial_r\hat v_\varepsilon]_{\varepsilon}&=\partial_r u(s,\varepsilon)-\varepsilon^{-1}\partial_n v_0(s,1)-\partial_n v_1(s,1)+O(\varepsilon)\\
&\qquad\qquad\qquad\qquad\qquad\quad=\partial_r u^+(s)-\partial_r u(s,\varepsilon)+O(\varepsilon)=O(\varepsilon),
\end{aligned}$$

as $\varepsilon\to0$, by construction of v_k. We also have utilized in this calculation condition (4.11) and the inequality $|u(s,\pm\varepsilon)-u^\pm(s)|+|\partial_r u(s,\pm\varepsilon)-\partial_r u^\pm(s)|\leqslant c\varepsilon$.

Non-resonant case. Now suppose that problem (4.10) admits the trivial solution only. Then $v_0=0$ and therefore $u^-=0$ and $u^+=0$ on γ, by (4.5). We thus get

$$-\Delta u+Wu=\lambda u \quad \text{in } \ \mathbb{R}^2\setminus\gamma, \qquad u|_\gamma=0.$$

Let us suppose that λ is an eigenvalue of the direct sum $\mathcal{D}_1\oplus\mathcal{D}_2$ of two Dirichlet type operators and u is a corresponding eigenfunction. In this case, problem (4.9) has the form

$$\begin{cases}-\partial_n^2 v_1+V(n)v_1=0 \quad \text{in } \ Q,\\ \ \ \partial_n v_1(s,-1)=\partial_\nu u^-, \qquad \partial_n v_1(s,1)=\partial_\nu u^+,\end{cases} \tag{4.22}$$

and admits a unique solution. We also assume that

$$\begin{cases}-\partial_n^2 v_2+V(n)v_2=-\varkappa(s)\partial_n v_1+U(s,n)v_1 \quad \text{in } \ Q,\\ \ \ v_2(s,-1)=0, \quad \partial_n v_2(s,-1)=0, \quad s\in S,\end{cases}$$

and apply the reasoning above.

4.2. **Justification of the asymptotics.** To prove that $\lambda \in \sigma(\mathcal{H})$ is an accumulation point for some sequence $\{\lambda^\varepsilon\}_{\varepsilon>0} \subset \sigma(H_\varepsilon)$, we will apply the method of quasimodes. Let A be a self-adjoint operator in a Hilbert space L. We say a pair $(\mu,\phi) \in \mathbb{R}\times \operatorname{dom} A$ is a *quasimode* of A with the accuracy δ, if $\phi \neq 0$ and $\|(A-\mu)\phi\|_L \leqslant \delta\|\phi\|_L$.

Lemma 1 ([11, p.139])**.** *Assume (μ,ϕ) is a quasimode of A with accuracy $\delta>0$ and the spectrum of A is discrete in the interval $[\mu-\delta,\mu+\delta]$. Then there exists an eigenvalue μ_* of A such that $|\mu_*-\mu| \leqslant \delta$.*

Proof. If $\mu \in \sigma(A)$, then $\mu_* = \mu$. Otherwise the distance d_μ from μ to the spectrum of A can be computed as

$$d_\mu = \|(A-\mu)^{-1}\|^{-1} = \inf_{\psi\neq 0} \frac{\|\psi\|_L}{\|(A-\mu)^{-1}\psi\|_L},$$

where ψ is an arbitrary vector of L. Taking $\psi = (A-\mu)\phi$, we deduce

$$d_\mu \leqslant \frac{\|(A-\mu)\phi\|_L}{\|\phi\|_L} \leqslant \delta,$$

from which the assertion follows. □

Given an eigenvalue λ of $\mathcal{H}$ with eigenfunction u, we will prove that the pair (λ, v_ε) is a quasimode of H_ε with an infinitely small accuracy as $\varepsilon\to 0$, where v_ε is constructed as in (4.20) above. Write $\varrho_\varepsilon = (H_\varepsilon-\lambda)v_\varepsilon$. We see that

$$\varrho_\varepsilon = (-\Delta + W - \lambda)(u-\eta_\varepsilon) = \Delta\eta_\varepsilon - W\eta_\varepsilon + \lambda\eta_\varepsilon$$

outside ω_ε, and therefore $\sup_{x\in\mathbb{R}^2\setminus\omega_\varepsilon} |\varrho_\varepsilon(x)| \leqslant c_1\varepsilon$, because of (4.21). Note that η_ε is a function of compact support.

Recalling representation (3.2) of the Laplace operator in the local coordinates, we deduce

$$\begin{aligned}
&-\Delta + W(x) + V_\varepsilon(x) = -\varepsilon^{-2}\partial_n^2 + \varepsilon^{-1}\varkappa\partial_n + n\varkappa^2\partial_n - \partial_s^2 - \varepsilon P_\varepsilon \\
&\quad + W(s,\varepsilon n) + \varepsilon^{-2}V(n) + \varepsilon^{-1}U(s,n) = \varepsilon^{-2}\ell_0 + \varepsilon^{-1}\ell_1 + \ell_2 + W(s,\varepsilon n) - \varepsilon P_\varepsilon,
\end{aligned}$$

for $x\in\omega_\varepsilon$, where $\ell_0 = -\partial_n^2 + V$, $\ell_1 = \varkappa\partial_n + U$ and $\ell_2 = n\varkappa^2\partial_n - \partial_s^2$. Then

$$\begin{aligned}
\varrho_\varepsilon &= (-\Delta+W+V_\varepsilon-\lambda)v_\varepsilon = \big(\varepsilon^{-2}\ell_0+\varepsilon^{-1}\ell_1+\ell_2+W(s,\varepsilon n)-\varepsilon P_\varepsilon-\lambda\big)\big(v_0+\varepsilon v_1+\varepsilon^2 v_2\big) \\
&= \varepsilon^{-2}\ell_0 v_0 + \varepsilon^{-1}(\ell_0 v_1 + \ell_1 v_0) + \big(\ell_0 v_2 + \ell_1 v_1 + (\ell_2 + W(s,0) - \lambda)v_0\big) \\
&\quad + (W(s,\varepsilon n) - W(s,0))v_0 + \varepsilon\big(\ell_1 v_2 + (\ell_2 + W(s,\varepsilon n) - \lambda)(v_1+\varepsilon v_2) - P_\varepsilon v_\varepsilon\big).
\end{aligned} \tag{4.23}$$

From our choice of v_k, we derive that the first three terms in the right-hand side vanish. Then $\sup_{x\in\omega_\varepsilon} |\varrho_\varepsilon(x)| \leqslant c_2\varepsilon$. Hence we have

$$\|(H_\varepsilon-\lambda)v_\varepsilon\|_{L_2(\mathbb{R}^2)} = \|\varrho_\varepsilon\|_{L_2(\mathbb{R}^2)} \leqslant |\omega_{2\beta}|^{1/2} \sup_{\mathbb{R}^2} |\varrho_\varepsilon| \leqslant c_3\varepsilon,$$

since $\operatorname{supp}\varrho_\varepsilon \subset \omega_{2\beta}$. On the other hand, the main contribution to the $L_2(\mathbb{R}^2)$-norm of v_ε is given by the eigenfunction u, because the norms of η_ε and v_k, $k=0,1,2$, are infinitely small as $\varepsilon\to 0$. Therefore $\|v_\varepsilon\|_{L_2(\mathbb{R}^2)} \geqslant \frac{1}{2}\|u\|_{L_2(\mathbb{R}^2)}$ for ε small enough. Finally, we obtain

$$\|(H_\varepsilon-\lambda)v_\varepsilon\|_{L_2(\mathbb{R}^2)} \leqslant c_3\varepsilon \leqslant 2c_3\varepsilon\|u\|^{-1}_{L_2(\mathbb{R}^2)}\,\|v_\varepsilon\|_{L_2(\mathbb{R}^2)} \leqslant c_4\varepsilon\,\|v_\varepsilon\|_{L_2(\mathbb{R}^2)}.$$

In view of Lemma 1, there exists an eigenvalue λ^ε of H_ε such that

$$|\lambda^\varepsilon - \lambda| \leqslant c_4\varepsilon$$

for all ε small enough.

5. Proof of Main Theorem

Let $\{\lambda^\varepsilon\}_{\varepsilon>0}$ be a sequence of eigenvalues of operator H_ε and $\{u_\varepsilon\}_{\varepsilon>0}$ be the sequence of the corresponding eigenfunctions. Assume that $\|u_\varepsilon\|_{L_2(\mathbb{R}^2)} = 1$.

Lemma 2. *Assume that $\lambda^\varepsilon \to \lambda$ and $u_\varepsilon \to u$ in $L_2(\mathbb{R}^2)$ weakly as $\varepsilon \to 0$. Then*

(i) $u_\varepsilon \to u$ in $W_2^4(K)$ as $\varepsilon \to 0$ for any compact set $K \subset \mathbb{R}^2$ with smooth boundary ∂K such that $K \cap \gamma = \emptyset$;

(ii) u solves the equation $-\Delta u + Wu = \lambda u$ in $\mathbb{R}^2 \setminus \gamma$;

(iii) there exists the constant C such that

$$\|u_\varepsilon\|_{W_2^2(\omega_\beta \setminus \omega_\varepsilon)} \leqslant C \tag{5.1}$$

for all $\varepsilon < \beta$;

(iv) $u_\varepsilon|_{\gamma_{-\varepsilon}} \to u^-$ and $u_\varepsilon|_{\gamma_\varepsilon} \to u^+$ in $L_2(\gamma)$ weakly as $\varepsilon \to 0$.

Proof. Let Φ_γ be the set of test functions $\phi \in C_0^\infty(\mathbb{R}^2)$ such that $\phi = 0$ in ω_β. We conclude from (2.4) that

$$\int_{\mathbb{R}^2} \Delta u_\varepsilon \phi\, dx = \int_{\mathbb{R}^2} (W - \lambda^\varepsilon)\, u_\varepsilon \phi\, dx, \quad \phi \in \Phi_\beta, \tag{5.2}$$

for all $\varepsilon < \beta$, since the support of short-range potential V_ε lies in ω_β. The right-hand side of (5.2) has a limit as $\varepsilon \to 0$ by the assumptions, thus the left-hand side also converges for all $\phi \in \Phi_\beta$, i.e., $\Delta u_\varepsilon \to \Delta u$ in $L_2(\mathbb{R}^2)$ weakly. From this we deduce that u_ε converges to u in $W_2^2(\mathbb{R}^2 \setminus \omega_\beta)$ weakly, and hence that

$$\int_{\mathbb{R}^2} \Delta u \phi\, dx = \int_{\mathbb{R}^2} (W - \lambda)\, u \phi\, dx, \quad \phi \in \Phi_\beta.$$

So u is a solution of the equation $-\Delta u + Wu = \lambda u$ in $\mathbb{R}^2 \setminus \omega_\beta$ and, therefore, in $\mathbb{R}^2 \setminus \gamma$, since β is an arbitrary positive number.

Equality (5.2) also holds for $\phi = \chi_\varepsilon \psi$, where $\psi \in L_2(\mathbb{R}^2)$ and χ_ε is the characteristic function of the domain $\omega_\beta \setminus \omega_\varepsilon$. We conclude from

$$\int_{\omega_\beta \setminus \omega_\varepsilon} \Delta u_\varepsilon \psi\, dx = \int_{\omega_\beta \setminus \omega_\varepsilon} (W - \lambda^\varepsilon)\, u_\varepsilon \psi\, dx \tag{5.3}$$

that $|(\chi_\varepsilon \Delta u_\varepsilon, \psi)_{L_2(\mathbb{R}^2)}| \leqslant c_\psi$ for any $\psi \in L_2(\mathbb{R}^2)$, since the right-hand side of (5.3) converges to $\int_{\omega_\beta} (W - \lambda)\, u\psi\, dx$ as $\varepsilon \to 0$. In view of the Banach-Steinhaus theorem, we see that

$$\|\chi_\varepsilon \Delta u_\varepsilon\|^2_{L_2(\mathbb{R}^2)} = \int_{\omega_\beta \setminus \omega_\varepsilon} |\Delta u_\varepsilon|^2\, dx \leqslant C_1,$$

from which (5.1) follows.

We introduce the function

$$\zeta_\varepsilon(r) = (r - \varepsilon)\zeta(r)\chi_{(\varepsilon,+\infty)}(r),$$

where $\chi_{(\varepsilon,+\infty)}$ is the characteristic function of the set $(\varepsilon,+\infty)$. Since $\zeta_\varepsilon(\varepsilon)=0$ and $\zeta_\varepsilon'(\varepsilon+0)=1$ for $\varepsilon<\beta/2$, we readily deduce the equality

$$\int_{\gamma_\varepsilon} u_\varepsilon a\,d\gamma = \int_{\Omega_\varepsilon^+}(W-\lambda^\varepsilon)u_\varepsilon a\zeta_\varepsilon\,dx - \int_{\Omega_\varepsilon^+} u_\varepsilon\Delta(a\zeta_\varepsilon)\,dx, \tag{5.4}$$

where a is a smooth function on γ and $\Omega_\varepsilon^+ = \Omega^+\setminus\omega_\varepsilon$. Similarly, we have

$$\int_{\gamma} u^+ a\,d\gamma = \int_{\Omega^+}(W-\lambda)u^+ a\zeta_0\,dx - \int_{\Omega^+} u^+\Delta(a\zeta_0)\,dx, \tag{5.5}$$

where $\zeta_0(r)=r\zeta(r)$. Obviously,

$$\int_{\Omega_\varepsilon^+}(W-\lambda^\varepsilon)u_\varepsilon a\zeta_\varepsilon\,dx \to \int_{\Omega^+}(W-\lambda)u^+ a\zeta_0\,dx$$

as $\varepsilon\to0$, because ζ_ε converge to ζ_0 uniformly on $\mathbb{R}_+$. Recalling (3.2), we can write

$$\begin{aligned}\int_{\Omega_\varepsilon^+} u_\varepsilon\Delta(a\zeta_\varepsilon)\,dx = \int_S\int_0^\beta u_\varepsilon(s,r)\rho_\varepsilon(r)\,\partial_s\left(\frac{a'(s)}{1-r\varkappa(s)}\right)ds\,dr\\ + \int_S\int_0^\beta u_\varepsilon(s,r)a(s)\big(J(s,r)(2\zeta'(r)+(r-\varepsilon)\zeta''(r)) - \varkappa(s)(r-\varepsilon)\zeta'(r)\big)\,ds\,dr\\ - \int_S\int_\varepsilon^\beta u_\varepsilon(s,r)a(s)\varkappa(s)\zeta(r)\,ds\,dr,\end{aligned} \tag{5.6}$$

provide $\varepsilon<\beta/2$. Here we used the equalities $\zeta'(r)=0$ and $\zeta''(r)=0$ for $r\in(0,\varepsilon)$. The right-hand side of (5.6) converges to

$$\begin{aligned}\int_S\int_0^\beta u^+(s,r)\rho_0(r)\,\partial_s\left(\frac{a'(s)}{1-r\varkappa(s)}\right)ds\,dr\\ + \int_S\int_0^\beta u^+(s,r)a(s)\big(J(s,r)(2\zeta'(r)+r\zeta''(r)) - \varkappa(s)(\zeta(r)+r\zeta'(r))\big)\,ds\,dr,\end{aligned}$$

which coincides with $\int_{\Omega^+} u^+\Delta(a\zeta_0)\,dx$. Therefore we conclude from (5.4) and (5.5) that $\int_{\gamma_\varepsilon} u_\varepsilon a\,d\gamma \to \int_\gamma u^+ a\,d\gamma$ for all $a\in C^\infty(\gamma)$, hence that $u_\varepsilon|_{\gamma_\varepsilon}\to u^+$ in $L_2(\gamma)$ weakly as $\varepsilon\to0$. The proof of the weak convergence for $u_\varepsilon|_{\gamma_{-\varepsilon}}$ is similar. □

5.1. **Case of zero-energy resonance.**

Lemma 3. *Suppose that $\lambda^\varepsilon\to\lambda$ and $u_\varepsilon\to u$ in $L_2(\mathbb{R}^2)$ weakly as $\varepsilon\to0$, and the one-dimensional Schrödinger operator $-\frac{d^2}{dt^2}+V$ possesses a zero-energy resonance, then λ is an eigenvalue of $\mathcal{H}$ associated with the eigenfunction u.*

The eigenvalue λ^ε and the corresponding eigenfunction u_ε satisfy the identity

$$\int_{\mathbb{R}^2}\big(\nabla u_\varepsilon\nabla\phi + (W+V_\varepsilon-\lambda^\varepsilon)u_\varepsilon\phi\big)\,dx = 0, \qquad \phi\in W_2^1(\mathbb{R}^2).$$

Let λ and u the eigenvalue and the corresponding eigenfunction of $\mathcal{H}$. Then

$$\int_{\Omega^+}\nabla u\nabla\psi\,dx + \int_{\Omega^-}\nabla u\nabla\psi\,dx + \int_{\mathbb{R}^2}(W-\lambda)u\psi\,dx + \int_\gamma \Upsilon u^-\phi^-\,d\gamma = 0$$

for all functions ψ belonging to the set $\Psi_\theta=\{f\in W_2^1(\mathbb{R}^2\setminus\gamma)\colon f^+=\theta f^- \text{ on } \gamma\}$.

Given $\psi \in \Psi_\theta \cap C_b^\infty(\mathbb{R}^2 \setminus \gamma)$, we construct a sequence $\{\psi_\varepsilon\}_{\varepsilon>0}$ in the space $W_2^1(\mathbb{R}^2)$ as follows. Suppose h is a half-bound state of $-\frac{d^2}{dt^2}+V$ such that $h(-1)=1$, and functions h_1 and h_2 solve the Cauchy problems on the interval $\mathcal{I}$

$$-h_1''+Vh_1=0, \quad h_1(-1)=0, \;\; h_1'(-1)=1; \tag{5.7}$$

$$-h_2''+Vh_2=\varkappa(s)h'+U(s,\cdot)h, \quad h_2(s,-1)=0, \;\; \partial_n h_2(s,-1)=0 \tag{5.8}$$

respectively. Let us write

$$\psi_0^\varepsilon(s,n)=\psi(s,-\varepsilon)\,h(n), \quad \psi_1^\varepsilon(s,n)=\partial_r\psi(s,-\varepsilon)\,h_1(n)-\psi(s,-\varepsilon)\,h_2(s,n).$$

and set

$$\hat\psi_\varepsilon(x)=\begin{cases}\psi(x), & \text{if } x\in\mathbb{R}^2\setminus\omega_\varepsilon,\\ \psi_0^\varepsilon(s,\frac{r}{\varepsilon})+\varepsilon\psi_1^\varepsilon\left(s,\frac{r}{\varepsilon}\right) & \text{if } x=(s,r)\in\omega_\varepsilon.\end{cases}$$

The function ψ_ε does not in general belong to $W_2^1(\mathbb{R}^2)$, because it has a jump discontinuity on the boundary $\partial\omega_\varepsilon$ composed of two curves $\gamma_{-\varepsilon}$ and γ_ε. Then $[\hat\psi_\varepsilon]_{-\varepsilon}=0$, by construction. Recalling the function ζ plotted in Fig. 2, we write

$$\rho_\varepsilon(x)=\begin{cases}-[\hat\psi_\varepsilon]_\varepsilon\,\zeta(r-\varepsilon), & \text{if } x=(s,r)\in\omega_{2\beta}\setminus\omega_\varepsilon,\\ \;\,0, & \text{otherwise.}\end{cases}$$

Finally, we set $\psi_\varepsilon=\hat\psi_\varepsilon+\rho_\varepsilon$. The direct calculations show that $[\psi_\varepsilon]_\varepsilon=0$ and, therefore, ψ_ε belongs to $W_2^1(\mathbb{R}^2)$.

The first observation of the sequence $\{\psi_\varepsilon\}_{\varepsilon>0}$ is that it converges to ψ in $L_2(\mathbb{R}^2)$. In fact, we have for all $s\in S$

$$[\hat\psi_\varepsilon]_\varepsilon(s)=\psi(s,\varepsilon)-\theta\psi(s,-\varepsilon)-\varepsilon\psi_{1,\varepsilon}(s,1)=\psi(s,+0)-\theta\psi(s,-0)+O(\varepsilon)=O(\varepsilon),$$

as $\varepsilon\to0$, since $\psi(s,+0)=\theta\psi(s,-0)$. Hence $\rho_\varepsilon\to0$ in $L_2(\mathbb{R}^2)$.

Proposition 3. *For $\psi\in\Psi_\theta\cap C_b^\infty(\mathbb{R}^2\setminus\gamma)$, we have*

$$\int_Q(\partial_n u_\varepsilon\partial_n\psi_0^\varepsilon+Vu_\varepsilon\psi_0^\varepsilon)J_\varepsilon\,dn\,ds=\varepsilon\int_Q\varkappa u_\varepsilon\partial_n\psi_0^\varepsilon\,dn\,ds, \tag{5.9}$$

$$\begin{aligned}\int_Q(\partial_n u_\varepsilon\partial_n\psi_1^\varepsilon+Vu_\varepsilon\psi_1^\varepsilon+Uu_\varepsilon\psi_0^\varepsilon)J_\varepsilon\,dn\,ds=&\\ \int_S\Big(u_\varepsilon(s,-\varepsilon)\partial_r\psi(s,-\varepsilon)\big(1+\varepsilon\varkappa(s)\big)&\\ -\theta^{-1}u_\varepsilon(s,\varepsilon)\big(\partial_r\psi(s,-\varepsilon)-\Upsilon(s)\psi(s,-\varepsilon)\big)(1-\varepsilon\varkappa(s))\Big)\,ds&\\ -\int_Q\varkappa u_\varepsilon\,\partial_n\psi_0^\varepsilon\,dn\,ds+\varepsilon\int_Q\varkappa u_\varepsilon(\varkappa\partial_n\psi_0^\varepsilon-\partial_n\psi_1^\varepsilon)\,dn\,ds.&\end{aligned} \tag{5.10}$$

where $J_\varepsilon(s,n)=1-\varepsilon n\varkappa(s)$.

Proof. The function ψ_0^ε solves the equation $-\partial_n^2 v + Vv = 0$ in Q and $h'(-1) = h'(1) = 0$. Then

$$0 = \int_Q u_\varepsilon(-\partial_n^2\psi_0^\varepsilon + V\psi_0^\varepsilon)J_\varepsilon\,dn\,ds = -\int_S \psi(s,-\varepsilon)(u_\varepsilon J_\varepsilon h')\big|_{n=-1}^{n=1}\,ds$$
$$+\int_Q(\partial_n u_\varepsilon\,\partial_n\psi_0^\varepsilon + Vu_\varepsilon\psi_0^\varepsilon)J_\varepsilon\,dn\,ds + \int_Q u_\varepsilon\,\partial_n J_\varepsilon\,\partial_n\psi_0^\varepsilon\,dn\,ds$$
$$= \int_Q(\partial_n u_\varepsilon\,\partial_n\psi_0^\varepsilon + Vu_\varepsilon\psi_0^\varepsilon)J_\varepsilon\,dn\,ds - \varepsilon\int_Q \varkappa u_\varepsilon\,\partial_n\psi_0^\varepsilon\,dn\,ds,$$

from which (5.9) follows.

Since $h(1) = \theta$, the Lagrange identity $(h_1h' - h_1'h)|_{-1}^1 = 0$ for equation (5.7) implies

$$h_1'(1) = \theta^{-1}. \tag{5.11}$$

Multiplying the equation in (5.8) by h and integrating by parts twice yield

$$(h'h_2 - h\,\partial_n h_2)\big|_{-1}^1 = \varkappa(s)\int_{\mathcal{I}} hh'\,dn + \int_{\mathcal{I}} U(s,n)h^2(n)\,dn.$$

Recalling (4.14), it follows that $\theta\,\partial_n h_2(s,1) = -\frac{1}{2}(\theta^2-1)\varkappa(s) - \mu(s)$ and finally that

$$\partial_n h_2(s,1) = -\theta^{-1}\Upsilon(s). \tag{5.12}$$

To prove (5.10), we note that ψ_1^ε is a solution of the equation

$$-\partial_n^2 v + Vv = -\varkappa\partial_n\psi_0^\varepsilon - U\psi_0^\varepsilon,$$

which follows from (5.7) and (5.8). Hence

$$\int_Q u_\varepsilon(-\partial_n^2\psi_1^\varepsilon + V\psi_1^\varepsilon + U\psi_0^\varepsilon)J_\varepsilon\,dn\,ds = -\int_Q \varkappa u_\varepsilon\,\partial_n\psi_0^\varepsilon J_\varepsilon\,dn\,ds. \tag{5.13}$$

On the other hand, integrating by parts with respect to n, we find

$$-\int_Q u_\varepsilon\,\partial_n^2\psi_1^\varepsilon J_\varepsilon\,dn\,ds = \int_Q \partial_n(u_\varepsilon J_\varepsilon)\,\partial_n\psi_1^\varepsilon\,dn\,ds - \int_S (u_\varepsilon J_\varepsilon\partial_n\psi_1^\varepsilon)\big|_{n=-1}^{n=1}\,ds$$
$$= \int_Q \partial_n u_\varepsilon\,\partial_n\psi_1^\varepsilon J_\varepsilon\,dn\,ds - \varepsilon\int_Q \varkappa u_\varepsilon\,\partial_n\psi_1^\varepsilon\,dn\,ds$$
$$-\int_S\Big(u_\varepsilon(s,\varepsilon n)J_\varepsilon(s,n)\big(\partial_r\psi(s,-\varepsilon)\,h_1'(n) - \psi(s,-\varepsilon)\,\partial_n h_2(s,n)\big)\Big)\Big|_{n=-1}^{n=1}\,ds$$
$$= \int_Q \partial_n u_\varepsilon\,\partial_n\psi_1^\varepsilon J_\varepsilon\,dn\,ds - \varepsilon\int_Q \varkappa u_\varepsilon\,\partial_n\psi_1^\varepsilon\,dn\,ds$$
$$-\int_S\Big(\theta^{-1}u_\varepsilon(s,\varepsilon)\big(\partial_r\psi(s,-\varepsilon) - \Upsilon(s)\psi(s,-\varepsilon)\big)(1-\varepsilon\varkappa(s))$$
$$- u_\varepsilon(s,-\varepsilon)\partial_r\psi(s,-\varepsilon)\big(1+\varepsilon\varkappa(s)\big)\Big)\,ds, \tag{5.14}$$

in view of initial conditions (5.7), (5.8) and equalities (5.11), (5.12). Substituting (5.14) into (5.13), we obtain (5.10). ☐

Proposition 4. *Under the assumptions of Lemma 3, for all $\psi \in \Psi_\theta \cap C_b^2(\mathbb{R}^2\setminus\gamma)$ we have*

$$\int_{\omega_\varepsilon}\big(\nabla u_\varepsilon\nabla\psi_\varepsilon + V_\varepsilon u_\varepsilon\psi_\varepsilon\big)\,dx \to \int_\gamma \Upsilon u^-\psi^-\,d\gamma,$$

as ε tends to zero.

Proof. Let us note here, for future use,

$$\int_{\omega_\varepsilon} g(x)\,dx = \varepsilon\int_Q g(s,\varepsilon n)J_\varepsilon(s,n)\,ds\,dn,$$
$$|\nabla v(x_\varepsilon)|^2 = \varepsilon^{-2}|\partial_n v(s,n)|^2 + J_\varepsilon^{-2}(s,n)\,|\partial_s v(s,n)|^2,$$

where $v(x_\varepsilon)$ stands for $v(s,\frac{r}{\varepsilon})$, cf. (3.1). Then

$$\begin{aligned}
&\int_{\omega_\varepsilon}\big(\nabla u_\varepsilon\nabla\psi_\varepsilon + V_\varepsilon u_\varepsilon\psi_\varepsilon\big)\,dx\\
&\qquad = \varepsilon^{-1}\int_Q\big(\partial_n u_\varepsilon\,\partial_n\psi_\varepsilon + \varepsilon^2 J_\varepsilon^{-2}\partial_s u_\varepsilon\,\partial_s\psi_\varepsilon + Vu_\varepsilon\psi_\varepsilon + \varepsilon U u_\varepsilon\psi_\varepsilon\big)J_\varepsilon\,dn\,ds\\
&= \varepsilon^{-1}\int_Q(\partial_n u_\varepsilon\partial_n\psi_0^\varepsilon + Vu_\varepsilon\psi_0^\varepsilon)J_\varepsilon\,dn\,ds + \int_Q(\partial_n u_\varepsilon\partial_n\psi_1^\varepsilon + Vu_\varepsilon\psi_1^\varepsilon + Uu_\varepsilon\psi_0^\varepsilon)J_\varepsilon\,dn\,ds\\
&\qquad + \varepsilon\int_Q Uu_\varepsilon\psi_1^\varepsilon J_\varepsilon\,dn\,ds + \varepsilon^2\int_Q\partial_s u_\varepsilon\,\partial_s\psi_\varepsilon J_\varepsilon^{-1}\,dn\,ds.
\end{aligned}$$

In view of Proposition 3, we have

$$\begin{aligned}
\int_{\omega_\varepsilon}\big(\nabla u_\varepsilon\nabla\psi_\varepsilon + V_\varepsilon u_\varepsilon\psi_\varepsilon\big)\,dx &= \int_S\Big(u_\varepsilon(s,-\varepsilon)\partial_r\psi(s,-\varepsilon)\big(1+\varepsilon\varkappa(s)\big)\\
&\quad - \theta^{-1}u_\varepsilon(s,\varepsilon)\big(\partial_r\psi(s,-\varepsilon) - \Upsilon(s)\psi(s,-\varepsilon)\big)(1-\varepsilon\varkappa(s))\Big)\,ds\\
+\varepsilon\int_Q u_\varepsilon\big(\varkappa^2\partial_n\psi_0^\varepsilon - \varkappa\partial_n\psi_1^\varepsilon + U\psi_1^\varepsilon J_\varepsilon\big)\,dn\,ds &+ \varepsilon^2\int_Q\partial_s u_\varepsilon\,\partial_s\psi_\varepsilon J_\varepsilon^{-1}\,dn\,ds. \qquad (5.15)
\end{aligned}$$

For any sequence ϕ_ε bounded in the $L_2(Q)$-norm,the estimate

$$\begin{aligned}
\left|\int_Q u_\varepsilon(s,\varepsilon n)\phi_\varepsilon(s,n)\,dn\,ds\right| &\leqslant \left(\int_Q|u_\varepsilon(s,\varepsilon n)|^2\,dn\,ds\right)^{1/2}\|\phi_\varepsilon\|_{L_2(Q)}\\
&\leqslant c\left(\varepsilon^{-1}\int_{\omega_\varepsilon}|u_\varepsilon(x)|^2\,dx\right)^{1/2} \leqslant c\varepsilon^{-1/2}\|u_\varepsilon\|_{L_2(\mathbb{R}^2)} = c\varepsilon^{-1/2}
\end{aligned}$$

holds. Also, we have

$$\left|\int_Q\partial_s u_\varepsilon\,\partial_s\psi_\varepsilon J_\varepsilon^{-1}\,dn\,ds\right| = \left|\int_Q u_\varepsilon\,\partial_s(J_\varepsilon^{-1}\partial_s\psi_\varepsilon)\,dn\,ds\right| \leqslant c_1\varepsilon^{-1/2},$$

because $\varkappa\in C^1(\gamma)$ and $\psi\in C_b^\infty(\mathbb{R}^2\setminus\gamma)$ and, therefore, the function $\partial_s(J_\varepsilon^{-1}\partial_s\psi_\varepsilon)$ is bounded on Q uniformly on ε.

Then (5.15) implies

$$\begin{aligned}
&\int_{\omega_\varepsilon}\big(\nabla u_\varepsilon\nabla\psi_\varepsilon + V_\varepsilon u_\varepsilon\psi_\varepsilon\big)\,dx\\
&\quad\to\int_S\Big(u(s,-0)\partial_r\psi(s,-0) - \theta^{-1}u(s,+0)\big(\partial_r\psi(s,-0) - \Upsilon(s)\psi(s,-0)\big)\Big)\,ds\\
&\qquad\qquad = \int_\gamma\Big(u^-\partial_r\psi^- - \theta^{-1}u^+\big(\partial_r\psi^- - \Upsilon\psi^-\big)\Big)\,d\gamma = \int_\gamma\Upsilon u^-\psi^-\,d\gamma,
\end{aligned}$$

since $\theta^{-1}u^+ = u^-$. $\square$

Proposition 5.
$$\int_{\omega_\varepsilon} |u_\varepsilon|^2\,dx \to 0, \quad \text{as } \varepsilon\to 0$$

Proof.
$$w_\varepsilon(s,r) = w_0^\varepsilon\left(s,\tfrac{r}{\varepsilon}\right) + \varepsilon w_1^\varepsilon\left(s,\tfrac{r}{\varepsilon}\right) + \varepsilon^2 w_2^\varepsilon\left(s,\tfrac{r}{\varepsilon}\right),$$
where $w_0^\varepsilon = u_\varepsilon(s,-\varepsilon)h(n)$, and w_1^ε, w_2^ε solve the Cauchy problems
$$\begin{aligned}
&-\partial_n^2 w_1^\varepsilon + Vw_1^\varepsilon = (-\varkappa\partial_n - U)w_0^\varepsilon, \quad w_1^\varepsilon(\,\cdot\,,-1) = \partial_n w_1^\varepsilon(\,\cdot\,,-1) = 0;\\
&-\partial_n^2 w_2^\varepsilon + Vw_2^\varepsilon = -(\varkappa\partial_n + U))w_1^\varepsilon + \left(\partial_s^2 - n\varkappa^2\partial_n - W(\,\cdot\,,0) + \lambda^\varepsilon\right)w_0^\varepsilon,\\
&\quad w_2^\varepsilon(\,\cdot\,,-1) = \partial_n w_2^\varepsilon(\,\cdot\,,-1) = 0
\end{aligned}$$
respectively. All functions $w_k^\varepsilon\colon Q\to\mathbb{R}$ are bounded in $L_2(Q)$ uniformly on ε, because $\lambda^\varepsilon\to\lambda$ and $u_\varepsilon(s,-\varepsilon)\to u(s,-0)$ in $L_2(S)$ weakly and therefore $\|u_\varepsilon(\,\cdot\,,-\varepsilon)\|_{L_2(S)} \leqslant c$.

Reasoning as in (4.23) we deduce that w_ε is a solution of the equation
$$-\Delta w_\varepsilon + (W + V_\varepsilon - \lambda^\varepsilon)w_\varepsilon = f_\varepsilon \quad \text{in } Q_\varepsilon,$$
where $\|f_\varepsilon\|_{L_2(Q_\varepsilon)} \leqslant c_1\varepsilon$. It follows that the difference $g_\varepsilon = w_\varepsilon - u_\varepsilon$ solves the Dirichlet type boundary value problem
$$\begin{gathered}
-\Delta g_\varepsilon + (W + V_\varepsilon - \lambda^\varepsilon)g_\varepsilon = f_\varepsilon \quad \text{in } Q_\varepsilon,\\
g_\varepsilon(s,-\varepsilon) = 0, \qquad g_\varepsilon(s,\varepsilon) = \theta u_\varepsilon(s,-\varepsilon) - u_\varepsilon(s,\varepsilon) + \varepsilon w_1^\varepsilon(s,1) + \varepsilon^2 w_2^\varepsilon(s,1).
\end{gathered}$$
Hence
$$\|g_\varepsilon\|_{L_2(Q_\varepsilon)} \leqslant c_2(\|f_\varepsilon\|_{L_2(Q_\varepsilon)} +)$$
□

Proof of Lemma. which may be rewritten as
$$\int_{\omega_\varepsilon}\left(\nabla u_\varepsilon\nabla\psi_\varepsilon + V_\varepsilon u_\varepsilon\psi_\varepsilon\right)dx = -\int_{\Omega_\varepsilon}\nabla u_\varepsilon\nabla\psi_\varepsilon\,dx - \int_{\mathbb{R}^2}(W-\lambda^\varepsilon)u_\varepsilon\psi_\varepsilon\,dx \tag{5.16}$$
The right-hand side of (5.16) has a finite limit
$$-\int_{\mathbb{R}^2}\left(\nabla u\nabla\psi + (W-\lambda)u\psi\right)dx$$
as $\varepsilon\to 0$. In particular, the term $\int_{\mathbb{R}^2} Wu_\varepsilon\phi\,dx$ converges for any $\phi\in\operatorname{dom} H_0$ by the rapidly decay of eigenfunctions of H_ε [2, Ch.3.3]. Therefore the left-hand side of (5.16) also converges as $\varepsilon\to 0$.

□

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