

SCHRÖDINGER OPERATORS WITH A $(a\partial_\nu\delta_\gamma + b\delta_\gamma)$ -LIKE POTENTIALS

YURIY GOLOVATY

ABSTRACT. The

1. INTRODUCTION

2. STATEMENT OF PROBLEM AND MAIN RESULTS

Let us consider the family of operators

$$H_\varepsilon = -\Delta + W(x) + V_\varepsilon(x). \quad (2.1)$$

Suppose that the unperturbed operator $H_0 = -\Delta + W$ is self-adjoint in $L^2(\mathbb{R}^2)$ with a domain $\text{dom } H_0$. In addition, we suppose that $W \in L^\infty_{loc}(\mathbb{R}^2)$ and $\text{dom } H_0 \subset W_2^1(\mathbb{R}^2)$.

Let γ be a closed C^3 -curve without self-intersection points. We will denote by ω_ε the ε -neighborhood of γ , i.e., the union of all open balls with radius ε and center on γ . Suppose that potentials V_ε have compact supports that lie in ω_ε and the supports shrink to curve γ as $\varepsilon \rightarrow 0$. For this reason, $\text{dom } H_\varepsilon = \text{dom } H_0$.

To specify the dependence of V_ε on small parameter ε we introduce curvilinear coordinates in ω_ε . Let S be the circle of the same length as the length of γ . We will parameterize γ by points of the circle. Let $\alpha: S \rightarrow \mathbb{R}^2$ be the unit-speed C^3 -parametrization of γ with the natural parameter $s \in S$. Also $\nu = (-\dot{\alpha}_2, \dot{\alpha}_1)$ is the unit normal on γ , because $\dot{\alpha}_1^2 + \dot{\alpha}_2^2 = 1$. We define the local coordinates (s, r) in ω_ε by

$$x = \alpha(s) + r\nu(s), \quad (s, r) \in Q_\varepsilon = S \times (-\varepsilon, \varepsilon). \quad (2.2)$$

The coordinate r is the signed distance from a point x to γ . Therefore ω_ε is diffeomorphic to cylinder Q_ε for ε small enough. There is no loss of generality in assuming the diffeomorphism exists for $\varepsilon \in (0, 1)$.

We suppose that the localized potentials have the following structure

$$V_\varepsilon(\alpha(s) + r\nu(s)) = \varepsilon^{-2} V(\varepsilon^{-1}r) + \varepsilon^{-1} U(s, \varepsilon^{-1}r), \quad (2.3)$$

where V and U are measurable bounded functions such that

$$\text{supp } V \subset (-1, 1), \quad \text{supp } U \subset Q_1 \text{ and } \partial_s U \in L_2(\mathbb{R}^2). \quad (2.4)$$

The key assumption is that V does not depend on s .

The family of potentials V_ε generally diverges in the space of distributions $\mathcal{D}'(\mathbb{R}^2)$. As we will show in Proposition 1, the potentials converge only if V is

2000 *Mathematics Subject Classification.* Primary 34L40, 34B09; Secondary 81Q10.

a zero mean function. In this case, $V_\varepsilon \rightarrow a\partial_\nu\delta_\gamma + b\delta_\gamma$ as $\varepsilon \rightarrow 0$ for some functions a and b , where δ_γ is the Dirac delta function supported on γ and $\partial_\nu\delta_\gamma$ is the normal derivative of δ_γ at points of γ . More precisely,

$$\langle a\partial_\nu\delta_\gamma + b\delta_\gamma, \varphi \rangle = - \int_\gamma \partial_\nu(a\varphi) d\gamma + \int_\gamma b\varphi d\gamma$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$.

The main task is to construct asymptotic approximations, as $\varepsilon \rightarrow 0$, to eigenvalues and eigenfunctions of H_ε , i.e., asymptotics of eigenvalues λ^ε and eigenfunctions u_ε of spectral equation

$$-\Delta u_\varepsilon + (W + V_\varepsilon)u_\varepsilon = \lambda^\varepsilon u_\varepsilon \quad \text{in } \mathbb{R}^2. \quad (2.5)$$

We introduce some notation. The plane is divided into two domains by close curve γ . We suppose that $\mathbb{R}^2 \setminus \gamma = \Omega_{in} \cup \Omega_{out}$, where domain Ω_{out} is unbounded. Also, we say that v belongs to space $\mathcal{V} \subset L_2(\mathbb{R}^2)$ if $v|_{\Omega_{in}} \in W_2^2(\Omega_{in})$ and there exist a function w belonging to $\text{dom } H_0$ such that $v = w$ in Ω_{out} . Of course, $v|_{\Omega_{out}} \in W_{2,loc}^2(\Omega_{out})$. Let \mathcal{V}_0 be the subspaces of $L_2(\Omega_{out})$ obtained by the restriction of all elements of \mathcal{V} to Ω_{out} . We introduce two operators

$$\begin{aligned} \mathcal{D}_1 &= -\Delta + W, & \text{dom } \mathcal{D}_1 &= \{v \in \mathcal{V}_0 : v = 0 \text{ on } \gamma\}, \\ \mathcal{D}_2 &= -\Delta + W, & \text{dom } \mathcal{D}_2 &= \{v \in W_2^2(\Omega_{in}) : v = 0 \text{ on } \gamma\}. \end{aligned}$$

We also denote by $\gamma_t = \{x \in \mathbb{R}^2 : x = \alpha(s) + t\nu(s), s \in S\}$ the closed curve that is obtained from γ by flowing for “time” t along the normal vector field. Then the boundary of ω_ε consists of two curves $\gamma_{-\varepsilon}$ and γ_ε . For each $u \in \mathcal{V}$ there exist two one-side traces on γ , namely

$$u^- = \lim_{\varepsilon \rightarrow 0} u|_{\gamma_{-\varepsilon}}, \quad u^+ = \lim_{\varepsilon \rightarrow 0} u|_{\gamma_\varepsilon}. \quad (2.6)$$

We say that the Schrödinger operator $-\frac{d^2}{dt^2} + V$ in $L_2(\mathbb{R})$ possesses a *zero-energy resonance* if there exists a non trivial solution h of the equation $-h'' + Vh = 0$ that is bounded on the whole line. We call h the *half-bound state* of V . In this case, we will also simply say that potential V has a half-bound state h . Such a solution h is unique up to a scalar factor and has nonzero limits

$$h(-\infty) = \lim_{t \rightarrow -\infty} h(t), \quad h(+\infty) = \lim_{t \rightarrow +\infty} h(t)$$

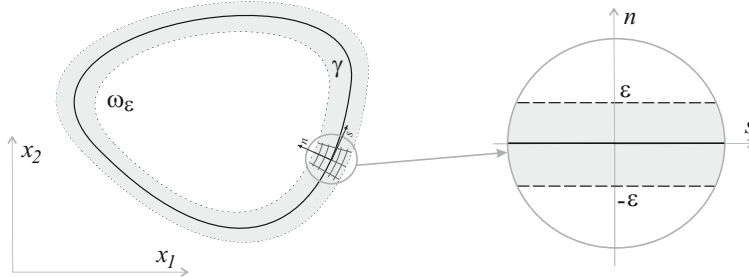


FIGURE 1. Curvilinear coordinates in the ε -neighbourhood of γ .

at both the infinities. We set

$$\theta = \frac{h(+\infty)}{h(-\infty)}. \quad (2.7)$$

Our main result reads as follows.

Theorem 1. *Let $W \in L_{loc}^\infty(\mathbb{R}^2)$ and $\text{dom } H_0 \subset W_2^1(\mathbb{R}^2)$. Assume that potentials V and U are measurable bounded functions and assumption (2.4) and (??) holds. Then the family of operators*

$$H_\varepsilon = -\Delta + W + V_\varepsilon,$$

where the perturbation V_ε is given by (2.3), converges as $\varepsilon \rightarrow 0$ in the strong resolvent sense.

If potential V possesses a zero-energy resonance with a half-bound state h , then operators H_ε converge to operator \mathcal{H} defined by $\mathcal{H}v = -\Delta v + Wv$ on functions $v \in \mathcal{V}$ obeying the interface conditions

$$u^+ - \theta u^- = 0, \quad \theta \partial_\nu u^+ - \partial_\nu u^- = \left(\frac{1}{2}(\theta^2 - 1)\varkappa + \mu\right) u^- \quad (2.8)$$

on curve γ . Here θ is given by (2.7), \varkappa is the signed curvature of γ , and

$$\mu = \frac{1}{h^2(-\infty)} \int_{-1}^1 U(\cdot, t) h^2(t) dt. \quad (2.9)$$

If potential V has no zero-energy resonance, then operators H_ε converge to the direct sum $\mathcal{D}_1 \oplus \mathcal{D}_2$ of two unperturbed operators $-\Delta + W$ in Ω_{in} and Ω_{out} respectively with the Dirichlet boundary conditions on interface γ .

Remark 1. *If potential V is identically zero, then $V_\varepsilon = \varepsilon^{-1} U(s, \varepsilon^{-1}n)$ and so obviously $V_\varepsilon \rightarrow \mu_0 \delta_\gamma$, as $\varepsilon \rightarrow 0$, in the space of distributions. Here*

$$\mu_0(s) = \int_{-1}^1 U(s, t) dt. \quad (2.10)$$

Potential $V = 0$ possesses a zero-energy resonance with constant functions as half-bound states. Hence parameter θ equals 1 and interface conditions (2.8) become $u^+ - u^- = 0$, $\partial_\nu u^+ - \partial_\nu u^- - \mu_0 u^- = 0$. These conditions are exactly the same as that obtained in [1].

3. PRELIMINARIES

Returning now to curvilinear coordinates (s, r) given by (2.2), we see that the couple of vectors $\tau = (\dot{\alpha}_1, \dot{\alpha}_2)$, $\nu = (-\dot{\alpha}_2, \dot{\alpha}_1)$ gives a Frenet frame for γ . The Jacobian of transformation $x = \alpha(s) + r\nu(s)$ has the form

$$\begin{aligned} J(s, r) &= \begin{vmatrix} \dot{\alpha}_1(s) - r\ddot{\alpha}_2(s) & -\dot{\alpha}_2(s) \\ \dot{\alpha}_2(s) + r\ddot{\alpha}_1(s) & \dot{\alpha}_1(s) \end{vmatrix} \\ &= \dot{\alpha}_1^2(s) + \dot{\alpha}_2^2(s) - r(\dot{\alpha}_1(s)\ddot{\alpha}_2(s) - \dot{\alpha}_2(s)\ddot{\alpha}_1(s)) = 1 - r\varkappa(s). \end{aligned}$$

Here $\varkappa = \det(\dot{\alpha}, \ddot{\alpha})$ is the signed curvature of γ . Note that \varkappa is a continuous function of the arc-length parameter s and the sign of $\varkappa(s)$ is defined uniquely up to the re-parametrization $s \mapsto -s$. We see that J is positive for sufficiently small n , because curvature \varkappa is bounded on γ . Namely, the curvilinear coordinates (s, r) can be defined correctly on all domains ω_ε with $\varepsilon \leq \varepsilon_*$, where $\varepsilon_* = \min_\gamma |\varkappa|^{-1}$.

However, the above we have accepted that $\varepsilon_* = 1$, since this involves no loss of generality. We also have

$$\int_{\omega_\varepsilon} f(x_1, x_2) dx_1 dx_2 = \int_{Q_\varepsilon} f(s, r)(1 - r\kappa(s)) ds dr \quad (3.1)$$

for all integrable functions f . The metric tensor $g = (g_{ij})$ of ω_ε in the orthogonal coordinates (s, r) has the form

$$g = \begin{pmatrix} J^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

In fact, we have $g_{11} = |x_s|^2 = |\dot{\alpha} + r\dot{\nu}|^2 = |(1 - r\kappa)\dot{\alpha}|^2 = J^2$, by the Frenet-Serret formula $\dot{\nu} = -\kappa\dot{\alpha}$, and $g_{22} = |x_r|^2 = |\nu|^2 = 1$. In particular, the gradient in the local coordinates becomes

$$\nabla\varphi = \frac{1}{\sqrt{g_{11}}} \partial_s \varphi \tau + \frac{1}{\sqrt{g_{22}}} \partial_r \varphi \nu = \frac{1}{J} \partial_s \varphi \tau + \partial_r \varphi \nu$$

and therefore we have

$$\nabla\varphi \cdot \nabla\psi = J^{-2} \partial_s \varphi \partial_s \psi + \partial_r \varphi \partial_r \psi. \quad (3.2)$$

The Laplace-Beltrami operator in ω_ε has also the explicit form

$$\Delta\varphi = J^{-1} (\partial_s (J^{-1} \partial_s \varphi) + \partial_r (J \partial_r \varphi)) \quad (3.3)$$

as is easy to check.

Proposition 1. *If $\int_{\mathbb{R}} V dt = 0$, then*

$$V_\varepsilon \rightarrow \beta \partial_\nu \delta_\gamma + (\beta\kappa + \mu_0) \delta_\gamma, \quad \text{as } \varepsilon \rightarrow 0,$$

in the space of distributions $\mathcal{D}'(\mathbb{R}^2)$, where $\beta = -\int_{\mathbb{R}} tV(t) dt$ and μ_0 is given by (2.10).

Proof. It is evident that potentials $\varepsilon^{-1} U(s, \varepsilon^{-1}r)$ converge to $\mu_0 \delta_\gamma$ in $\mathcal{D}(\mathbb{R}^2)$. We will prove that sequence $g_\varepsilon = \varepsilon^{-2} V(\varepsilon^{-1}r)$ converges to $\beta(\partial_\nu \delta_\gamma + \kappa \delta_\gamma)$ as $\varepsilon \rightarrow 0$, provided V is a zero-mean function. In fact, for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ we have

$$\begin{aligned} \langle g_\varepsilon, \varphi \rangle &= \int_{\omega_\varepsilon} g_\varepsilon(x) \varphi(x) dx = \frac{1}{\varepsilon^2} \int_{Q_\varepsilon} V\left(\frac{r}{\varepsilon}\right) \varphi(s, r)(1 - r\kappa(s)) ds dr \\ &= \frac{1}{\varepsilon} \int_{Q_1} V(t) \varphi(s, \varepsilon t)(1 - \varepsilon t\kappa(s)) ds dt \\ &= \frac{1}{\varepsilon} \int_{-1}^1 V(t) dt \int_S \varphi(s, 0) ds \\ &\quad + \int_{-1}^1 tV(t) dt \int_S (\partial_n \varphi(s, 0) - \kappa(s) \varphi(s, 0)) ds + O(\varepsilon), \end{aligned}$$

as $\varepsilon \rightarrow 0$. The sequence $\langle g_\varepsilon, \varphi \rangle$ has a finite limit for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ if and only if $\int_{\mathbb{R}} V dt = 0$. In this case, we have

$$\langle g_\varepsilon, \varphi \rangle \rightarrow \beta \int_\gamma (\partial_\nu \delta_\gamma + \kappa \delta_\gamma) \varphi d\gamma,$$

which completes the proof. \square

Interface conditions (2.8) contain the parameters which depend on the particular parametrization chosen for curve γ . More precisely, parameters θ , \varkappa and μ change along with the change of the Frenet frame.

Proposition 2. *Operator \mathcal{H} in Theorem 1 does not depend upon the choice of the Frenet frame for curve γ .*

Proof. Every smooth curve in the plane admits two possible orientations of arc-length parameter and consequently two possible Frenet frames. Let us change the Frenet frame $\{\tau, \nu\}$, previously introduced in Sec. 2, to the frame $\{-\tau, -\nu\}$ and prove that interface conditions (2.8) will remain the same. This change leads to the following transformations:

$$\begin{aligned} h(\pm\infty) &\mapsto h(\mp\infty), & u_\pm &\mapsto u_\mp, & \partial_\nu u_\pm &\mapsto -\partial_\nu u_\mp, \\ \theta &\mapsto \theta^{-1}, & \varkappa &\mapsto -\varkappa, & \mu &\mapsto \theta^{-2}\mu. \end{aligned}$$

The first condition $u^+ - \theta u^- = 0$ in (2.8) transforms into $u^- - \theta^{-1}u^+ = 0$ and therefore remains unchanged. As for the second condition, we have

$$-\theta^{-1}\partial_\nu u^- + \partial_\nu u^+ - \left(-\frac{1}{2}(\theta^{-2} - 1)\varkappa + \theta^{-2}\mu\right)u^+ = 0.$$

Multiplying the equality by θ yields

$$\theta\partial_\nu u^+ - \partial_\nu u^- - \left(\frac{1}{2}(\theta^2 - 1)\varkappa + \mu\right)\theta^{-1}u^+ = 0,$$

since $-\theta(\theta^{-2} - 1) = \theta^{-1}(\theta^2 - 1)$. It remains to insert u^- in place of $\theta^{-1}u^+$, in view of the first interface condition. \square

In the sequel, the normal vector field ν will be outward to domain Ω_{in} , that is to say, the local coordinate n will increase in the direction from Ω_{in} to Ω_{out} . At the end of the section, we record some technical assertion, which will be often used below. Throughout the paper, $W_2^l(\Omega)$ stands for the Sobolev space of functions defined on a set Ω .

Proposition 3. *Suppose that $v \in W_2^1(\Omega_{out})$ and $w \in W_2^1(\Omega_{in})$. Then*

$$\|v(\cdot, \varepsilon) - v(\cdot, 0)\|_{L_2(\gamma)} \leq c_1 \varepsilon^{1/2} \|v\|_{W_2^1(\Omega_{out})}, \quad (3.4)$$

$$\|w(\cdot, -\varepsilon) - w(\cdot, 0)\|_{L_2(\gamma)} \leq c_2 \varepsilon^{1/2} \|w\|_{W_2^1(\Omega_{in})}, \quad (3.5)$$

where the constants c_k do not depend on ε .

Proof. First we assume that v is a smooth function in Ω_{out} . Then

$$v(s, \varepsilon) - v(s, 0) = \int_0^\varepsilon \partial_t v(s, t) dt,$$

and for all $\psi \in L_2(\gamma)$ we have

$$\int_S (v(s, \varepsilon) - v(s, 0))\psi(s) ds = \int_S \int_0^\varepsilon \partial_t v(s, t)\psi(s) dt ds.$$

Therefore

$$\begin{aligned}
\left| \int_S (v(s, \varepsilon) - v(s, 0)) \psi(s) ds \right| &\leq \int_S \int_0^\varepsilon |\partial_t v(s, t)| |\psi(s)| dt ds \\
&\leq \left(\int_0^\varepsilon \int_S |\psi(s)|^2 ds dt \right)^{1/2} \left(\int_S \int_0^\varepsilon |\partial_t v|^2 dt ds \right)^{1/2} \\
&\leq c\varepsilon^{1/2} \|\psi\|_{L_2(\gamma)} \left(\int_{\Omega_{out}} |\nabla v|^2 dx \right)^{1/2} \leq c_1 \varepsilon^{1/2} \|v\|_{W_2^1(\Omega_{out})} \|\psi\|_{L_2(\gamma)}.
\end{aligned}$$

Hence (3.4) holds for all smooth functions v and then by continuity for all $v \in W_2^1(\Omega_{out})$. Similar arguments apply to the proof of (3.5). \square

4. FORMAL ASYMPTOTICS

4.1. Leading Terms. In this section we will show how interface conditions (2.8) can be found by direct calculations, constructing the formal asymptotics of eigenvalues and eigenfunctions. We look for asymptotics of λ_ε and u_ε in the form

$$\lambda^\varepsilon \approx \lambda + \varepsilon \lambda_1, \quad u_\varepsilon(x) \approx \begin{cases} u(x) + \varepsilon u_1(x) & \text{in } \mathbb{R}^2 \setminus \omega_\varepsilon, \\ v_0(s, \frac{r}{\varepsilon}) + \varepsilon v_1(s, \frac{r}{\varepsilon}) + \varepsilon^2 v_2(s, \frac{r}{\varepsilon}) & \text{in } \omega_\varepsilon. \end{cases} \quad (4.1)$$

Recall that the boundary of ω_ε consists of two curves $\gamma_{-\varepsilon}$ and γ_ε . To match two different approximations, we hereafter assume that

$$[u_\varepsilon]_{\gamma_{\pm\varepsilon}} = 0, \quad [\partial_\nu u_\varepsilon]_{\gamma_{\pm\varepsilon}} = 0, \quad (4.2)$$

where $[\cdot]_{\gamma_{\pm\varepsilon}}$ is a jump across $\gamma_{\pm\varepsilon}$. Since function u_ε solves (2.5) and domain ω_ε shrinks to γ , the leading term u must be a solution of the equation

$$-\Delta u + W u = \lambda u \quad \text{in } \mathbb{R}^2 \setminus \gamma,$$

subject to appropriate interface conditions on γ . To find these conditions, we consider equation (2.5) in the curvilinear coordinates (s, n) , where $n = r/\varepsilon$. Then in vicinity of γ the Laplacian can be written as

$$\Delta = \frac{1}{1 - \varepsilon n \kappa} \left(\varepsilon^{-2} \partial_n (1 - \varepsilon n \kappa) \partial_n + \partial_s \left(\frac{1}{1 - \varepsilon n \kappa} \partial_s \right) \right), \quad (4.3)$$

by (3.3). From this we readily deduce the asymptotic representation

$$\Delta = \varepsilon^{-2} \partial_n^2 - \varepsilon^{-1} \kappa \partial_n - n \kappa^2 \partial_n + \partial_s^2 + \varepsilon P_\varepsilon,$$

where P_ε is a partial differential operator on the second order with respect to s and the first one with respect to n whose coefficients are uniformly bounded on ε . Substituting (4.1) into (2.5) for $x \in \omega_\varepsilon$ in particular yields

$$-\partial_n^2 v_0 + V(n) v_0 = 0, \quad -\partial_n^2 v_1 + V(n) v_1 = -\kappa(s) \partial_n v_0 - U(s, n) v_0 \quad (4.4)$$

in cylinder $Q_1 = S \times (-1, 1)$. From (4.2) we see that necessarily

$$u^-(s) = v_0(s, -1), \quad u^+(s) = v_0(s, 1), \quad (4.5)$$

$$\partial_n v_0(s, -1) = 0, \quad \partial_n v_0(s, 1) = 0, \quad (4.6)$$

$$\partial_n v_1(s, -1) = \partial_\nu u^-(s), \quad \partial_n v_1(s, 1) = \partial_\nu u^+(s), \quad (4.7)$$

where u^\pm are defined by (2.6). Combining (4.6)–(4.7), we conclude that v_0 and v_1 solve boundary value problems

$$\begin{cases} -\partial_n^2 v_0 + V(n)v_0 = 0 & \text{in } Q_1, \\ \partial_n v_0(s, -1) = 0, \quad \partial_n v_0(s, 1) = 0, & s \in S; \end{cases} \quad (4.8)$$

$$\begin{cases} -\partial_n^2 v_1 + V(n)v_1 = -\kappa(s)\partial_n v_0 - U(s, n)v_0 & \text{in } Q_1, \\ \partial_n v_1(s, -1) = \partial_\nu u^-(s), \quad \partial_n v_1(s, 1) = \partial_\nu u^+(s), & s \in S \end{cases} \quad (4.9)$$

respectively. We have two boundary value problems for the “non-elliptic” partial differential operator in Q_1 . Of course, the problems can also be regarded as boundary value problems for ordinary differential equations on $(-1, 1)$, which depend on parameter $s \in S$.

Case of zero-energy resonance. Assume that operator $-\frac{d^2}{dn^2} + V$ has a zero energy resonance with half-bound state h . Since the support of V lies in interval $(-1, 1)$, the half-bound state h is constant outside this interval as a bounded solution of equation $h'' = 0$. Therefore the restriction of h to $(-1, 1)$ is a nonzero solution of the Neumann boundary value problem

$$-h'' + V(n)h = 0, \quad n \in (-1, 1), \quad h'(-1) = 0, \quad h'(1) = 0. \quad (4.10)$$

Hereafter, we fix h by additional condition $h(-1) = 1$. In view of (2.7), we have $h(1) = \theta$, since $h(\pm\infty) = h(\pm 1)$.

In this case, (4.8) admits a infinite-dimensional space of solutions

$$\mathcal{N} = \{a(s)h(n) : a \in L^2(S)\}.$$

Therefore $v_0(s, n) = a_0(s)h(n)$ for some function a_0 . From (4.5) we deduce that $u^- = a_0$ and $u^+ = \theta a_0$ and hence that

$$u^+ = \theta u^- \quad \text{on } \gamma. \quad (4.11)$$

In addition, we must set $v_0(s, n) = u^-(s)h(n)$.

Problem (4.9) is in general unsolvable, since $\mathcal{N} \neq \{0\}$. To find solvability conditions, we rewrite equation in (4.9) as

$$-\partial_n^2 v_1 + V(n)v_1 = -(\kappa(s)h'(n) + U(s, n)h(n))u^-(s), \quad (4.12)$$

multiply by an arbitrary element of \mathcal{N} and then integrate over Q_1

$$\begin{aligned} & \int_{Q_1} (-\partial_n^2 v_1 + V(n)v_1) a(s)h(n) dn ds \\ &= - \int_{Q_1} (\kappa(s)h'(n) + U(s, n)h(n))u^-(s)a(s)h(n) dn ds. \end{aligned} \quad (4.13)$$

Since h is a solution of (4.10), integrating by parts twice on the left-hand side yields

$$\begin{aligned} & \int_S \int_{-1}^1 (-\partial_n^2 v_1 + V(n)v_1) ah dn ds = - \int_S (\partial_n v_1 h - v_1 h') \Big|_{n=-1}^{n=1} a ds \\ &= - \int_S \int_{-1}^1 av_1 (-h'' + Vh) dn ds = - \int_S (\theta \partial_\nu u^+ - \partial_\nu u^-) a ds, \end{aligned}$$

in view of the boundary conditions for v_1 . Recall that $h(-1) = 1$ and $h(1) = \theta$. Since $hh' = \frac{1}{2}(h^2)'$, we also have $\int_{-1}^1 hh' dt = \frac{1}{2}(\theta^2 - 1)$. Therefore (4.13) becomes

$$\int_S (\theta \partial_\nu u^+ - \partial_\nu u^-) a ds = \int_S (\frac{1}{2}(\theta^2 - 1)\varkappa + \mu) u^- a ds$$

for all $a \in L^2(S)$, where $\mu(s) = \int_{-1}^1 U(s, n) h^2(n) dn$. Therefore

$$\theta \partial_\nu u^+ - \partial_\nu u^- = (\frac{1}{2}(\theta^2 - 1)\varkappa + \mu) u^- \quad \text{on } \gamma,$$

which is necessary for the solvability of (4.9). In view of the Fredholm alternative, this condition is also a sufficient one. At the same time, it is a jump condition at the interface γ for the normal derivative of u .

Therefore the leading terms λ and u of asymptotics (4.1) solve the problem

$$-\Delta u + Wu = \lambda u \quad \text{in } \mathbb{R}^2 \setminus \gamma, \quad (4.14)$$

$$u^+ - \theta u^- = 0, \quad \theta \partial_\nu u^+ - \partial_\nu u^- = (\frac{1}{2}(\theta^2 - 1)\varkappa + \mu) u^- \quad \text{on } \gamma. \quad (4.15)$$

Assume that λ is an eigenvalue of operator \mathcal{H} associated with eigenfunction u . Now we can calculate the trace u^- on γ and finally determine $v_0(s, n) = u^-(s)h(n)$.

Non-resonant case. Now suppose that problem (4.10) admits the trivial solution only, i.e., $\mathcal{N} = \{0\}$. Then $v_0 = 0$ and therefore $u^- = 0$ and $u^+ = 0$ on γ , by (4.5). We thus get

$$-\Delta u + Wu = \lambda u \quad \text{in } \mathbb{R}^2 \setminus \gamma, \quad u|_\gamma = 0.$$

Let us suppose that λ is an eigenvalue of the direct sum $\mathcal{D}_1 \oplus \mathcal{D}_2$ of two Dirichlet type operators and u is a corresponding eigenfunction. In this case, problem (4.9) has the form

$$\begin{cases} -\partial_n^2 v_1 + V(n)v_1 = 0 & \text{in } Q_1, \\ \partial_n v_1(s, -1) = \partial_\nu u^-, & \partial_n v_1(s, 1) = \partial_\nu u^+, \end{cases} \quad (4.16)$$

and admits a unique solution.

4.2. Correction terms. Upon our substituting (4.1) into equation (2.5) and conditions (4.2), we in particular discover that correction term u_1 solves the equation

$$-\Delta u_1 + Wu_1 = \lambda u_1 + \lambda_1 u \quad \text{in } \mathbb{R}^2 \setminus \gamma \quad (4.17)$$

and v_2 is a solution of the problem

$$\begin{aligned} -\partial_n^2 v_2 + V(n)v_2 &= -(\varkappa(s)\partial_n + U(s, n))v_1 \\ &\quad + (\partial_s^2 - n\varkappa^2 \partial_n - W(s, 0) + \lambda)v_0 \quad \text{in } Q_1, \end{aligned} \quad (4.18)$$

$$\partial_n v_2(\cdot, -1) = \partial_\nu u_1^- - \partial_\nu^2 u^-, \quad \partial_n v_2(\cdot, 1) = \partial_\nu u_1^+ + \partial_\nu^2 u^+, \quad s \in S. \quad (4.19)$$

In addition,

$$v_1(\cdot, -1) = u_1^- - \partial_\nu u^-, \quad v_1(\cdot, 1) = u_1^+ + \partial_\nu u^+. \quad (4.20)$$

If operator $-\frac{d^2}{dt^2} + V$ has a zero energy resonance, then u is an eigenfunction of (4.14), (4.15). Since the second condition in (4.15) holds, problem (4.9) is solvable and possesses a linear manifold of solutions

$$v_1(s, n) = a_1(s)h(n) + w(s, n), \quad a_1 \in L_2(S), \quad (4.21)$$

where w is a partial solution of (4.9) such that $w(s, -1) = 0$ for all $s \in S$. We may therefore substitute (4.21) into (4.20) and obtain

$$a_1 = u_1^- - \partial_\nu u^-, \quad \theta a_1 + w(\cdot, 1) = u_1^+ + \partial_\nu u^+,$$

from which the coupling condition

$$u_1^+ - \theta u_1^- = w(\cdot, 1) - \partial_\nu u^+ - \theta \partial_\nu u^- \quad \text{on } \gamma \quad (4.22)$$

follows. Also, we have $v_1(s, n) = (u_1^-(s) - \partial_\nu u^-(s))h(n) + w(s, n)$.

By reasoning similar to that for (4.9), we can write the solvability condition for problem (4.18), (4.19) as

$$\theta \partial_\nu u_1^+ - \partial_\nu u_1^- - \left(\frac{1}{2}(\theta^2 - 1)\varkappa + \mu\right)u_1^- = -\theta \partial_\nu^2 u^+ - \partial_\nu^2 u^- + \int_{-1}^1 f(s, n)h(n)dn, \quad (4.23)$$

where $f = (\varkappa \partial_n + U)(h \partial_\nu u_1^- - w) + (\partial_s^2 - n \varkappa^2 \partial_n - W(\cdot, 0) + \lambda)v_0$. Then (4.17), (4.22) and (4.23) taken together yield

$$-\Delta u_1 + W u_1 = \lambda u_1 + \lambda_1 u \quad \text{in } \mathbb{R}^2 \setminus \gamma, \quad (4.24)$$

$$u_1^+ - \theta u_1^- = g_0, \quad \theta \partial_\nu u_1^+ - \partial_\nu u_1^- - \left(\frac{1}{2}(\theta^2 - 1)\varkappa + \mu\right)u_1^- = g_1 \quad \text{on } \gamma, \quad (4.25)$$

where g_0 and g_1 stand for the right hand sides of (4.22) and (4.23) respectively.

Since λ is an eigenvalue of \mathcal{H} , the last problem is generally unsolvable. But there exists a unique value of λ_1 for which (4.24), (4.25) admits a solution. Let us multiply equation (4.24) by eigenfunction u , and integrate over \mathbb{R}^2 , to find

$$\lambda_1 = \|u\|_{L_2(\mathbb{R}^2)}^{-2} \int_\gamma (g_1 u^- - g_0 \partial_\nu u^+) d\gamma, \quad (4.26)$$

where we have integrated by parts in the term containing Δu_1 :

$$\begin{aligned} - \int_{\mathbb{R}^2} \Delta u_1 u dx &= - \int_{\Omega_{out}} \Delta u_1 u dx - \int_{\Omega_{in}} \Delta u_1 u dx \\ &= \int_\gamma (\partial_\nu u_1^+ u^+ - \partial_\nu u_1^- u^- - u_1^+ \partial_\nu u^+ + u_1^- \partial_\nu u^-) d\gamma - \int_{\mathbb{R}^2} u_1 \Delta u dx. \end{aligned}$$

Utilizing the coupling conditions for u and u_1 on interface γ , we have obtained

$$\begin{aligned} - \int_{\mathbb{R}^2} \Delta u_1 u dx + \int_{\mathbb{R}^2} u_1 \Delta u dx &= \int_\gamma ((\theta \partial_\nu u_1^+ - \partial_\nu u_1^-)u^- - (\theta \partial_\nu u^+ - \partial_\nu u^-)u_1^- - g_0 \partial_\nu u^+) d\gamma \\ &= \int_\gamma ((R u_1^- + g_1)u^- - R u^- u_1^- - g_0 \partial_\nu u^+) d\gamma = \int_\gamma (g_1 u^- - g_0 \partial_\nu u^+) d\gamma, \end{aligned}$$

where $R = \frac{1}{2}(\theta^2 - 1)\varkappa + \mu$.

4.3. Adjustment of asymptotics.

Proposition 4 (Smoothness of asymptotics). *Let*

Proof. We set

$$v_1(s, n) = \partial_\nu u^-(s) h_1(n) - u^-(s) h_2(s, n), \quad (4.27)$$

where h_1 and h_2 solve the Cauchy problems

$$\begin{cases} -h_1'' + V(n)h_1 = 0, & t \in (-1, 1), & h_1(-1) = 0, & h_1'(-1) = 1, \\ -h_2'' + V(n)h_2 = \kappa(s)h'(n) + U(s, n)h(n), & n \in (-1, 1), \\ h_2(s, -1) = 0, & \partial_n h_2(s, -1) = 0, & s \in S \end{cases} \quad (4.28)$$

respectively. We see at once that v_1 of the form (4.27) solves equation (4.12) and satisfies boundary conditions $v_1(s, -1) = 0$ and $\partial_t v_1(s, -1) = \partial_\nu u^-(s)$. Now we show that condition $\partial_t v_1(s, 1) = \partial_\nu u^+(s)$ also holds. Recall that half-bound state h was fixed by $h(-1) = 1$. Then the Lagrange identity $(h_1 h' - h_1' h)|_{-1}^1 = 0$ implies $h_1'(1) = \theta^{-1}$. Next, multiplying the equation in (4.28) by h and integrating by parts twice yield

$$(h' h_2 - h \partial_t h_2)|_{-1}^1 = \kappa(s) \int_{-1}^1 h h' dn + \int_{-1}^1 U(s, n) h^2(n) dn.$$

Hence we have $\theta \partial_t h_2(s, 1) = -\frac{1}{2}(\theta^2 - 1)\kappa(s) - \mu(s)$, and then derive

$$\begin{aligned} \partial_t v_1(s, 1) &= \partial_\nu u^-(s) h_1'(1) - u^-(s) \partial_t h_2(s, 1) \\ &= \theta^{-1} \left(\partial_\nu u^-(s) + \left(\frac{1}{2}(\theta^2 - 1)\kappa(s) + \mu(s) \right) u^-(s) \right) = \partial_\nu u^+(s) \end{aligned}$$

in view of interface conditions (4.15). \square

5. PROOF OF THEOREM 1

We will provide a proof for the most interesting case when potential V has a zero-energy resonance. The non-resonant case, which is much easier, follows similarly. We must prove that

$$(H_\varepsilon - \zeta)^{-1} f \rightarrow (\mathcal{H} - \zeta)^{-1} f, \quad \text{as } \varepsilon \rightarrow 0, \quad (5.1)$$

for all $f \in L_2(\mathbb{R}^2)$ and some $\zeta \in \mathbb{C} \setminus \mathbb{R}$. But the resolvents of H_ε are uniformly bounded on ε , namely,

$$\|(H_\varepsilon - \zeta)^{-1}\| \leq |\operatorname{Im} \zeta|^{-1}.$$

It will thus be sufficient to prove that (5.1) holds for $f \in \mathcal{F}$, where \mathcal{F} is some dense subset of $L_2(\mathbb{R}^2)$. We suppose that $\mathcal{F} = C_0^\infty(\mathbb{R}^2 \setminus \gamma)$.

Hereafter, letters c_j denote various positive numbers independent of ε , whose values might be different in different proofs.

5.1. Approximation in $W_2^1(\mathbb{R}^2)$. Given $f \in \mathcal{F}$ and $\zeta \in \mathbb{C}$, $\operatorname{Im} \zeta \neq 0$, we must compare $u_\varepsilon = (H_\varepsilon - \zeta)^{-1} f$ and $u = (\mathcal{H} - \zeta)^{-1} f$ in $L_2(\mathbb{R})$, and show that the difference $u_\varepsilon - u$ is infinitely small in $L_2(\mathbb{R})$ -norm, as $\varepsilon \rightarrow 0$. The basic idea of the proof is to construct a suitable approximation to u_ε in $W_2^1(\mathbb{R}^2)$. The formal asymptotics (??) constructed above will be used as a starting point in the construction of this approximation. We recall that the Sobolev space $W_2^1(\mathbb{R}^2)$ contains the domains of H_0 and H_ε .

We note that $v_0 \in W_2^1(Q_1)$. This inclusion follows from the explicit form $v_0(s, t) = u^-(s)h(t)$, where u^- belongs to $W_2^{3/2}(S)$ (as a trace of $u \in W_2^2(\Omega_{in})$)

on curve γ) and $h \in W_2^2(-1, 1) \subset C^1(-1, 1)$. In view of representation formula (4.27), function v_1 does not belongs to $W_2^1(Q_1)$, since $\partial_\nu u^- \in W_2^{1/2}(S)$ in general. However the term $u^- h_2$ in (4.27) is an element of $W_2^1(Q_1)$, because $h_2 = h_2(s, t)$ possesses the additional smoothness with respect to s owing to the making more smoothness assumptions upon the curve γ and potential U . Recall that $\varkappa \in C^1(S)$ and $\partial_s U \in L_2(\mathbb{R}^2)$.

We now regularize the trace $\partial_\nu u^-$. Let $\{\beta_\varepsilon^-\}_{\varepsilon>0}$ be a sequence in $W_2^1(\gamma)$ such that $\beta_\varepsilon^- \rightarrow \partial_\nu u^-$ in $W_2^{-1/2}(\gamma)$. The sequence can be chosen in such a way that

$$\|\beta_\varepsilon^- - \partial_\nu u^-\|_{W_2^{-1/2}(\gamma)} \leq c_1 \varepsilon^{1/2} \quad \|\beta_\varepsilon^-\|_{W_2^1(\gamma)} \leq c_2 \varepsilon^{-1/2}, \quad (5.2)$$

since $|\langle \beta_\varepsilon^- - \partial_\nu u^-, \beta_\varepsilon^- \rangle| \leq \|\beta_\varepsilon^- - \partial_\nu u^-\|_{W_2^{-1/2}(\gamma)} \|\beta_\varepsilon^-\|_{W_2^1(\gamma)}$. Then the function

$$v_1^\varepsilon(s, t) = \beta_\varepsilon^-(s) h_1(t) - u^-(s) h_2(s, t) \quad (5.3)$$

belongs to $W_2^1(Q_1)$ and solves the problem

$$\begin{cases} -\partial_n^2 v_1^\varepsilon + V(t) v_1^\varepsilon = -\varkappa(s) u^-(s) h'(t) - U(s, t) h(t) & \text{in } Q_1, \\ \partial_t v_1^\varepsilon(s, -1) = \beta_\varepsilon^-(s), \quad \partial_t v_1^\varepsilon(s, 1) = \beta_\varepsilon^+(s), & s \in S, \end{cases} \quad (5.4)$$

where $\beta_\varepsilon^+ = \theta^{-1} \left(\frac{1}{2}(\theta^2 - 1) \varkappa u^- - \beta_\varepsilon^- \right)$. Thus

$$\|\beta_\varepsilon^+ - \partial_\nu u^+\|_{W_2^{-1/2}(\gamma)} \leq c_1 \varepsilon^{1/2}, \quad (5.5)$$

as $\varepsilon \rightarrow 0$, by solvability condition (??). We also have the bounds

$$\|v_1^\varepsilon\|_{L_2(Q_1)} + \|\partial_t v_1^\varepsilon\|_{L_2(Q_1)} \leq C_1, \quad \|\partial_s v_1^\varepsilon\|_{L_2(Q_1)} \leq C_2 \varepsilon^{-1/2}. \quad (5.6)$$

We introduce the function

$$y_\varepsilon(x) = \begin{cases} u(x) & \text{in } \mathbb{R}^2 \setminus \omega_\varepsilon, \\ v_0\left(s, \frac{r}{\varepsilon}\right) + \varepsilon v_1^\varepsilon\left(s, \frac{r}{\varepsilon}\right) & \text{in } \omega_\varepsilon. \end{cases} \quad (5.7)$$

It is still not smooth enough and does not belong to $W_2^1(\mathbb{R}^2)$, because it has in general jump discontinuities on curves $\gamma_{-\varepsilon}$ and γ_ε . We will show that both the jumps

$$\begin{aligned} [y_\varepsilon]_{\gamma_{-\varepsilon}} &= v_0(s, -1) - u(s, -\varepsilon) = u^-(s) - u(s, -\varepsilon), \\ [y_\varepsilon]_{\gamma_\varepsilon} &= u(s, \varepsilon) - v_0(s, 1) - \varepsilon v_1^\varepsilon(s, 1) = u(s, \varepsilon) - \theta u^-(s) - \varepsilon v_1^\varepsilon(s, 1) \\ &= u(s, \varepsilon) - u^+(s) - \varepsilon v_1^\varepsilon(s, 1) \end{aligned}$$

are small as $\varepsilon \rightarrow 0$. Recall that $v_1^\varepsilon(s, -1) = 0$.

Let us denote by Ω_ε the set $\mathbb{R}^2 \setminus \omega_\varepsilon$.

Lemma 1. *There exists a function $\rho_\varepsilon: \mathbb{R}^2 \rightarrow \mathbb{C}$ such that $y_\varepsilon + \rho_\varepsilon$ belongs to $W_2^1(\mathbb{R}^2)$. Moreover for the restriction of ρ_ε to Ω_ε we have the estimate*

$$\|\rho_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \leq c \varepsilon^{1/2}. \quad (5.8)$$

Proof. Let $Z_{in}: W_2^{1/2}(\gamma) \rightarrow W_2^1(\Omega_{in})$ and $Z_{out}: W_2^{1/2}(\gamma) \rightarrow W_2^1(\Omega_{out})$ be continuous extension operators such that $\text{supp } Z_{in} g \subset \Omega_{in} \cap \omega_{1/2}$ and $\text{supp } Z_{out} g \subset \Omega_{out} \cap \omega_{1/2}$ for all $g \in W_2^{1/2}(\gamma)$. The jumps $g_\varepsilon^\pm := [y_\varepsilon]_{\gamma_{\pm\varepsilon}}$ can be regarded as functions on γ . Obviously, $g_\varepsilon^\pm \in W_2^{1/2}(\gamma)$.

We set $z_\varepsilon^- = -Z_{in} g_\varepsilon^-$, $z_\varepsilon^+ = -Z_{out} g_\varepsilon^+$ and introduce function

$$\rho_\varepsilon(s, n) = \begin{cases} z_\varepsilon^+(s, n - \varepsilon) & \text{for } s \in S, n \in (\varepsilon, \varepsilon + 1/2), \\ z_\varepsilon^-(s, n + \varepsilon) & \text{for } s \in S, n \in (-\varepsilon - 1/2, -\varepsilon), \\ 0, & \text{otherwise} \end{cases} \quad (5.9)$$

in \mathbb{R}^2 for $\varepsilon < 1/2$. The function has a compact support and, in particular, it vanishes in ω_ε . Next, by construction ρ_ε has the jump discontinuities

$$[\rho_\varepsilon]_{\gamma_{\pm\varepsilon}} = z_\varepsilon^\pm(s, 0) = -[y_\varepsilon]_{\gamma_{\pm\varepsilon}}.$$

Since both the functions y_ε and ρ_ε belong to $W_2^1(\mathbb{R}^2 \setminus (\gamma_{-\varepsilon} \cup \gamma_\varepsilon))$ and

$$[y_\varepsilon + \rho_\varepsilon]_{\gamma_{\pm\varepsilon}} = 0,$$

we have $y_\varepsilon + \rho_\varepsilon \in W_2^1(\mathbb{R}^2)$. Furthermore,

$$\begin{aligned} \|\rho_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} &\leq c_1(\|z_\varepsilon^-\|_{W_2^1(\Omega_{in})} + \|z_\varepsilon^+\|_{W_2^1(\Omega_{out})}) \\ &= c_1(\|Z_{in} g_\varepsilon^-\|_{W_2^1(\Omega_{in})} + \|Z_{out} g_\varepsilon^+\|_{W_2^1(\Omega_{out})}) \\ &\leq c_2(\|g_\varepsilon^-\|_{W_2^{1/2}(\gamma)} + \|g_\varepsilon^+\|_{W_2^{1/2}(\gamma)}) \\ &\leq c_3(\|u(\cdot, -\varepsilon) - u^-\|_{W_2^{1/2}(\gamma)} + \|u(\cdot, \varepsilon) - u^+\|_{W_2^{1/2}(\gamma)}) \\ &\quad + \varepsilon\|v_1^\varepsilon(\cdot, 1)\|_{W_2^{1/2}(\gamma)} \leq c_4\varepsilon^{1/2}, \end{aligned}$$

by Proposition 3 and (5.6). In fact, the restrictions u to domains Ω_{in} and Ω_{out} belong to $W_2^2(\Omega_{in})$ and $W_2^2(\Omega_{out})$ respectively. Applying Proposition 3 to u and $\partial_s u$ yields

$$\|u(\cdot, \pm\varepsilon) - u(\cdot, \pm 0)\|_{L_2(\gamma)} + \|\partial_s u(\cdot, \pm\varepsilon) - \partial_s u(\cdot, \pm 0)\|_{L_2(\gamma)} \leq c\varepsilon^{1/2}.$$

Consequently, $\|u(\cdot, \pm\varepsilon) - u_\pm\|_{W_2^{1/2}(\gamma)} \leq \|u(\cdot, \pm\varepsilon) - u_\pm\|_{W_2^1(\gamma)} \leq c\varepsilon^{1/2}$. Finally, this follows from (5.3) that

$$\|v_1^\varepsilon(\cdot, 1)\|_{W_2^{1/2}(\gamma)} \leq c_1(\|\beta_\varepsilon^-\|_{W_2^{1/2}(\gamma)} + \|u^-\|_{W_2^{1/2}(\gamma)}) \leq c_2,$$

since $\beta_\varepsilon^- \rightarrow \partial_\nu u^-$ in $W_2^{1/2}(\gamma)$. \square

Hence, the desired approximation to u_ε in the Sobolev space $W_2^1(\mathbb{R}^2)$ has the form

$$Y_\varepsilon(x) = \begin{cases} u(x) + \rho_\varepsilon(x) & \text{in } \mathbb{R}^2 \setminus \omega_\varepsilon, \\ v_0(s, \frac{r}{\varepsilon}) + \varepsilon v_1^\varepsilon(s, \frac{r}{\varepsilon}) & \text{in } \omega_\varepsilon, \end{cases} \quad (5.10)$$

where ρ_ε is given by (5.9).

5.2. Estimate of Remainder. Let us fix $f \in C_0^\infty(\mathbb{R}^2 \setminus \gamma)$. First of all, we note that

$$\int_{\mathbb{R}^2} f \varphi dx = \int_{\Omega_\varepsilon} f \varphi dx \quad (5.11)$$

for ε small enough. We also record some other identities that will be needed below. Multiplying equation (4.14) by $\varphi \in W_2^1(\mathbb{R}^2)$ and integrating by parts over Ω_ε yield

$$\begin{aligned} \int_{\Omega_\varepsilon} (\nabla u \nabla \varphi + (W - \zeta)u\varphi) dx - \int_{\Omega_\varepsilon} f\varphi dx \\ = - \int_S (\partial_\nu u(s, \varepsilon)\varphi(s, \varepsilon) - \partial_\nu u(s, -\varepsilon)\varphi(s, -\varepsilon)) ds. \end{aligned} \quad (5.12)$$

In the same manner we can obtain from (4.8) and (5.4) that

$$\int_{Q_1} (\partial_t v_0 \partial_t \psi + V v_0 \psi) J_\varepsilon dt ds = \varepsilon \int_{Q_1} \kappa \partial_t v_0 \psi dt ds; \quad (5.13)$$

$$\begin{aligned} \int_{Q_1} (\partial_t v_1^\varepsilon \partial_t \psi + V v_1^\varepsilon \psi + U v_0 \psi) J_\varepsilon dt ds \\ = - \int_{Q_1} \kappa \partial_t v_0 \psi J_\varepsilon dt ds + \varepsilon \int_{Q_1} \kappa \partial_t v_1^\varepsilon \psi dt ds \\ + \int_S (\beta_\varepsilon^+(s)\psi(s, 1)J(s, \varepsilon) - \beta_\varepsilon^-(s)\psi(s, -1)J(s, -\varepsilon)) ds \end{aligned} \quad (5.14)$$

for all $\psi \in W_2^1(Q_1)$. For instance, let us multiply the equation in (4.8) by $\psi(s, t)J_\varepsilon(s, t)$, where $J_\varepsilon(s, t) = 1 - \varepsilon\kappa t$, and integrate over Q_1 . Then in view of boundary conditions for v_0 we deduce

$$\begin{aligned} 0 = \int_{Q_1} (-\partial_t^2 v_0 + V v_0) \psi J_\varepsilon dt ds = - \int_S (\partial_t v_0 \psi J_\varepsilon)|_{-1}^1 ds \\ + \int_{Q_1} \partial_t v_0 \partial_t (\psi J_\varepsilon) dt ds + \int_{Q_1} V v_0 \psi J_\varepsilon dt ds \\ = \int_{Q_1} (\partial_t v_0 \partial_t \psi + V v_0 \psi) J_\varepsilon dt ds - \varepsilon \int_{Q_1} \kappa \partial_t v_0 \psi dt ds, \end{aligned}$$

which establishes (5.13). Let us note here, for future use,

$$\int_{\omega_\varepsilon} g(x) dx = \varepsilon \int_{Q_1} g(s, \varepsilon t) J_\varepsilon(s, t) ds dt, \quad (5.15)$$

$$|\nabla v(x_\varepsilon)|^2 = \varepsilon^{-2} |\partial_t v(s, t)|^2 + J_\varepsilon^{-2}(s, t) |\partial_s v(s, t)|^2, \quad (5.16)$$

where $v(x_\varepsilon)$ stands for $v(s, \frac{r}{\varepsilon})$, cf. (3.1) and (3.2).

Under our assumptions about potential W the function $u_\varepsilon = (H_\varepsilon - \zeta)^{-1}f$ belongs to $W_2^1(\mathbb{R}^2)$ and therefore satisfies the integral identity

$$\int_{\mathbb{R}^2} (\nabla u_\varepsilon \nabla \varphi + (W + V_\varepsilon - \zeta)u_\varepsilon \varphi) dx = \int_{\mathbb{R}^2} f\varphi dx, \quad \varphi \in W_2^1(\mathbb{R}^2). \quad (5.17)$$

To show that Y_ε is an adequate approximation to u_ε , introduce the functional

$$F_\varepsilon(\varphi) = \int_{\mathbb{R}^2} (\nabla Y_\varepsilon \nabla \varphi + (W + V_\varepsilon - \zeta)Y_\varepsilon \varphi) dx - \int_{\mathbb{R}^2} f\varphi dx, \quad (5.18)$$

defined for functions φ belonging to $W_2^1(\mathbb{R}^2)$ and prove that its norm is infinitely small as $\varepsilon \rightarrow 0$.

Lemma 2. *The functional F_ε satisfies the estimate*

$$|F_\varepsilon(\varphi)| \leq c\varepsilon^{1/2} \|\varphi\|_{W_2^1(\mathbb{R}^2)}$$

for all $\varphi \in W_2^1(\mathbb{R}^2)$.

Proof. Let us rewrite F_ε into a more detailed form

$$\begin{aligned} F_\varepsilon(\varphi) = & \int_{\omega_\varepsilon} \left(\nabla(v_0 + \varepsilon v_1^\varepsilon) \nabla \varphi + (W + V_\varepsilon - \zeta)(v_0 + \varepsilon v_1^\varepsilon) \varphi \right) dx \\ & + \int_{\Omega_\varepsilon} \left(\nabla(u + \rho_\varepsilon) \nabla \varphi + (W - \zeta)(u + \rho_\varepsilon) \varphi \right) dx - \int_{\mathbb{R}^2} f \varphi dx. \end{aligned} \quad (5.19)$$

With notation $\varphi_\varepsilon(s, t) = \varphi(s, \varepsilon t)$, we have

$$\begin{aligned} F_\varepsilon(\varphi) = & \varepsilon^{-1} \int_{Q_1} (\partial_t v_0 \partial_t \varphi_\varepsilon + V v_0 \varphi_\varepsilon) J_\varepsilon dt ds \\ & + \int_{Q_1} (\partial_t v_1^\varepsilon \partial_t \varphi_\varepsilon + V v_1^\varepsilon \varphi_\varepsilon + U v_0 \psi_\varepsilon) J_\varepsilon dt ds \\ & + \int_{\Omega_\varepsilon} (\nabla u \nabla \varphi + (W - \zeta) u \varphi) dx - \int_{\Omega_\varepsilon} f \varphi dx \\ & + \int_{\Omega_\varepsilon} (\nabla \rho_\varepsilon \nabla \varphi + (W - \zeta) \rho_\varepsilon \varphi) dx \\ & + \varepsilon \int_{Q_1} \partial_s v_0 \partial_s \varphi_\varepsilon J_\varepsilon dt ds + \varepsilon^2 \int_{Q_1} \partial_s v_1^\varepsilon \partial_s \varphi_\varepsilon J_\varepsilon dt ds \\ & + \varepsilon \int_{Q_1} (W - \zeta)(v_0 + \varepsilon v_1^\varepsilon) \varphi_\varepsilon J_\varepsilon dt ds, \end{aligned}$$

by (5.11), (5.15) and (5.16). Let us replace the first and second integrals by the right-hand sides of (5.13) and (5.14) with $\psi_\varepsilon(s, t) = \varphi(s, \varepsilon t)$ respectively, and the difference between the third and fourth ones by the right-hand side of (5.12). The other terms in the last formula are small as $\varepsilon \rightarrow 0$, because Lemma 1 and estimates (5.6) provide the bounds

$$\begin{aligned} \left| \int_{\Omega_\varepsilon} (\nabla \rho_\varepsilon \nabla \varphi + (W - \zeta) \rho_\varepsilon \varphi) dx \right| & \leq c_1 \varepsilon^{1/2} \|\varphi\|_{W_2^1(\mathbb{R}^2)}, \\ \left| \int_{Q_1} \partial_s v_0 \partial_s \varphi_\varepsilon J_\varepsilon dt ds \right| & \leq c_2 \|\varphi\|_{W_2^1(\mathbb{R}^2)}, \\ \left| \int_{Q_1} \partial_s v_1^\varepsilon \partial_s \varphi_\varepsilon J_\varepsilon dt \right| & \leq c_3 \varepsilon^{-1/2} \|\varphi\|_{W_2^1(\mathbb{R}^2)}, \\ \left| \int_{Q_1} (W - \zeta)(v_0 + \varepsilon v_1^\varepsilon) \varphi_\varepsilon J_\varepsilon dt ds \right| & \leq c_4 \|\varphi\|_{W_2^1(\mathbb{R}^2)}. \end{aligned}$$

Therefore

$$\begin{aligned} F_\varepsilon(\varphi) &= \int_{Q_1} \kappa \partial_t v_0 \varphi_\varepsilon dt ds - \int_{Q_1} \kappa \partial_t v_0 \varphi_\varepsilon J_\varepsilon dt ds + \varepsilon \int_{Q_1} \kappa \partial_t v_1^\varepsilon \varphi_\varepsilon dt ds \\ &\quad + \int_S (\beta_\varepsilon^+(s) \varphi(s, \varepsilon) J(s, \varepsilon) - \beta_\varepsilon^-(s) \varphi(s, -\varepsilon) J(s, -\varepsilon)) ds \\ &\quad - \int_S (\partial_\nu u(s, \varepsilon) \varphi(s, \varepsilon) - \partial_\nu u(s, -\varepsilon) \varphi(s, -\varepsilon)) ds + r_\varepsilon(\varphi), \end{aligned}$$

where $|r_\varepsilon(\varphi)| \leq c\varepsilon^{1/2} \|\varphi\|_{W_2^1(\mathbb{R}^2)}$. Next, F_ε in turn rearranges to become

$$\begin{aligned} F_\varepsilon(\varphi) &= \int_S (\beta_\varepsilon^+(s) - \partial_\nu u(s, \varepsilon)) \varphi(s, \varepsilon) ds \\ &\quad - \int_S (\beta_\varepsilon^-(s) - \partial_\nu u(s, -\varepsilon)) \varphi(s, -\varepsilon) ds \\ &\quad + \varepsilon \int_{Q_1} \kappa (t \partial_t v_0 + \partial_t v_1^\varepsilon) \varphi_\varepsilon dt ds \\ &\quad - \varepsilon \int_S \kappa(s) (\beta_\varepsilon^+(s) \varphi(s, \varepsilon) - \beta_\varepsilon^-(s) \varphi(s, -\varepsilon)) ds + r_\varepsilon(\varphi). \end{aligned}$$

Thus

$$\begin{aligned} F_\varepsilon(\varphi) &= \int_S (\beta_\varepsilon^+(s) - \partial_\nu u(s, \varepsilon)) \varphi(s, \varepsilon) ds \\ &\quad - \int_S (\beta_\varepsilon^-(s) - \partial_\nu u(s, -\varepsilon)) \varphi(s, -\varepsilon) ds + q_\varepsilon(\varphi), \end{aligned}$$

where $|q_\varepsilon(\varphi)| \leq c\varepsilon^{1/2} \|\varphi\|_{W_2^1(\mathbb{R}^2)}$. But then (5.2), (5.5) and Proposition 3 imply

$$\begin{aligned} &\left| \int_S (\beta_\varepsilon^\pm(s) - \partial_\nu u(s, \pm\varepsilon)) \varphi(s, \pm\varepsilon) ds \right| \\ &\leq \left| \int_S (\beta_\varepsilon^\pm(s) - \partial_\nu u_\pm) \varphi(s, \pm\varepsilon) ds \right| + \left| \int_S (\partial_\nu u(s, \pm\varepsilon) - \partial_\nu u_\pm) \varphi(s, \pm\varepsilon) ds \right| \\ &\leq \|\beta_\varepsilon^\pm - \partial_\nu u_\pm\|_{W_2^{-1/2}(\gamma)} \|\varphi(\cdot, \pm\varepsilon)\|_{W_2^{1/2}(\gamma)} \\ &\quad + \|\partial_\nu u(\cdot, \pm\varepsilon) - \partial_\nu u_\pm\|_{L_2(\gamma)} \|\varphi(\cdot, \pm\varepsilon)\|_{L_2(\gamma)} \leq c\varepsilon^{1/2} \|\varphi\|_{W_2^1(\mathbb{R}^2)}. \end{aligned}$$

Therefore $|F_\varepsilon(\varphi)| \leq c\varepsilon^{1/2} \|\varphi\|_{W_2^1(\mathbb{R}^2)}$ for all $\varphi \in W_2^1(\mathbb{R}^2)$, and the lemma follows. \square

5.3. The End of the Proof. From (5.17) and (5.18) we see

$$\int_{\mathbb{R}^2} \nabla(Y_\varepsilon - u_\varepsilon) \nabla \varphi dx + \int_{\mathbb{R}^2} (W + V_\varepsilon - \zeta)(Y_\varepsilon - u_\varepsilon) \varphi dx = F_\varepsilon(\varphi),$$

for all $\varphi \in W_2^1(\mathbb{R}^2)$.

If $\varphi = \overline{Y_\varepsilon - u_\varepsilon}$, then

$$\int_{\mathbb{R}^2} |\nabla(Y_\varepsilon - u_\varepsilon)|^2 dx + \int_{\mathbb{R}^2} (W + V_\varepsilon - \zeta) |Y_\varepsilon - u_\varepsilon|^2 dx = F_\varepsilon(\overline{Y_\varepsilon - u_\varepsilon}).$$

$$-\operatorname{Im} \zeta \int_{\mathbb{R}^2} |Y_\varepsilon - u_\varepsilon|^2 dx = \operatorname{Im} F_\varepsilon(\overline{Y_\varepsilon - u_\varepsilon}).$$

$$\int_{\mathbb{R}^2} |Y_\varepsilon - u_\varepsilon|^2 dx \leq |\operatorname{Im} \zeta|^{-1} |F_\varepsilon(\overline{Y_\varepsilon - u_\varepsilon})| \leq c_1 \varepsilon^{1/2} \|Y_\varepsilon - u_\varepsilon\|_{W_2^1(\mathbb{R}^2)} \leq c_2 \varepsilon^{1/4}$$

Lemma 3. *If potential V has a zero mean, then the estimate*

$$\left| \varepsilon^{-2} \int_{\omega_\varepsilon} V\left(\frac{r}{\varepsilon}\right) |\varphi|^2 dx \right| \leq c \varepsilon^{-1/2} \|\varphi\|_{W_2^1(\mathbb{R}^2)}^2$$

holds for all $\varphi, \psi \in W_2^1(\mathbb{R}^2)$.

Proof.

$$\begin{aligned} \varepsilon^{-2} \left| \int_{\omega_\varepsilon} V\left(\frac{r}{\varepsilon}\right) \varphi \psi dx \right| &= \varepsilon^{-1} \left| \int_{Q_1} V(t) \varphi(s, \varepsilon t) \psi(s, \varepsilon t) (1 - \varepsilon t \kappa(s)) dt ds \right| \\ &\leq \varepsilon^{-1} \left| \int_{Q_1} V(t) \varphi(s, \varepsilon t) \psi(s, \varepsilon t) dt ds \right| + c_1 \|\varphi\|_{W_2^1(\mathbb{R}^2)} \|\psi\|_{W_2^1(\mathbb{R}^2)} \end{aligned}$$

$$\begin{aligned} &\left| \int_{Q_1} V(t) \varphi(s, \varepsilon t) \psi(s, \varepsilon t) dt ds \right| \\ &= \left| \int_{Q_1} V(t) \left(\varphi(s, 0) + \int_0^{\varepsilon t} \partial_t \varphi(s, \tau) d\tau \right) \right. \\ &\quad \times \left. \left(\psi(s, 0) + \int_0^{\varepsilon t} \partial_t \psi(s, \tau) d\tau \right) dt ds \right| \\ &\leq \left| \int_{Q_1} V(t) \varphi(s, 0) \int_0^{\varepsilon t} \partial_t \psi(s, \tau) d\tau dt ds \right| \\ &\quad + \left| \int_{Q_1} V(t) \psi(s, 0) \int_0^{\varepsilon t} \partial_t \varphi(s, \tau) d\tau dt ds \right| \\ &\quad + \left| \int_{Q_1} V(t) \int_0^{\varepsilon t} \partial_t \varphi(s, \tau) d\tau \int_0^{\varepsilon t} \partial_t \psi(s, \tau) d\tau dt ds \right| \end{aligned}$$

$$\begin{aligned} &\left| \int_{Q_1} V(t) \varphi(s, 0) \int_0^{\varepsilon t} \partial_t \psi(s, \tau) d\tau dt ds \right| \\ &\leq c_1 \left(\int_{Q_1} |\varphi(s, 0)|^2 dt ds \right)^{1/2} \left(\int_{Q_1} \left| \int_0^{\varepsilon t} \partial_t \psi(s, \tau) d\tau \right|^2 dt ds \right)^{1/2} \\ &\leq c_1 \|\varphi\|_{W_2^1(\mathbb{R}^2)} \left(\int_{Q_1} \left| \int_0^{\varepsilon t} d\tau \right| \left| \int_{-1}^1 |\partial_t \psi|^2 d\tau \right| dt ds \right)^{1/2} \\ &\leq c_1 \varepsilon^{1/2} \|\varphi\|_{W_2^1(\mathbb{R}^2)} \|\psi\|_{W_2^1(\mathbb{R}^2)} \end{aligned}$$

□

Weinstein Excitation thresholds for nonlinear localized modes on lattices (1999)

REFERENCES

- [1] Behrndt, J., Exner, P., Holzmann, M., Lotoreichik, V. (2017). Approximation of Schrödinger operators with δ -interactions supported on hypersurfaces. *Mathematische Nachrichten*, 290(8-9), 1215-1248.
- [2] Yu. D. Golovaty, R. O. Hryniv. *On norm resolvent convergence of Schrödinger operators with δ' -like potentials*. *Journal of Physics A: Mathematical and Theoretical* **43** (2010) 155204 (14pp) (A Corrigendum: 2011 *J. Phys. A: Math. Theor.* **44** 049802)
- [3] Yu. Golovaty. *Schrödinger operators with $(\alpha\delta' + \beta\delta)$ -like potentials: norm resolvent convergence and solvable models*, *Methods of Funct. Anal. Topology* (3) **18** (2012), 243–255.
- [4] Yu. D. Golovaty and R. O. Hryniv. *Norm resolvent convergence of singularly scaled Schrödinger operators and δ' -potentials*. *Proceedings of the Royal Society of Edinburgh: Section A Mathematics* **143** (2013), 791-816.
- [5] Yu. Golovaty, *1D Schrödinger Operators with Short Range Interactions: Two-Scale Regularization of Distributional Potentials*. *Integral Equations and Operator Theory* **75**(3) (2013), 341-362.
- [6] A. V. Zolotaryuk. *Two-parametric resonant tunneling across the $\delta'(x)$ potential*. *Adv. Sci. Lett.* **1** (2008), 187-191.
- [7] A. V. Zolotaryuk. *Point interactions of the dipole type defined through a three-parametric power regularization*. *Journal of Physics A: Mathematical and Theoretical* **43** (2010), 105302.
- [8] Yu. Golovaty. *Two-parametric δ' -interactions: approximation by Schrödinger operators with localized rank-two perturbations*. *Journal of Physics A: Mathematical and Theoretical* **51**(25) (2018), 255202.
- [9] Yu. Golovaty. *Schrödinger operators with singular rank-two perturbations and point interactions*. *Integr. Equ. Oper. Theory* **90**:57 (2018).

DEPARTMENT OF MECHANICS AND MATHEMATICS, IVAN FRANKO NATIONAL UNIVERSITY OF
 LVIV, 1 UNIVERSYTETSKA STR., 79000 LVIV, UKRAINE
E-mail address: yuriy.golovaty@lnu.edu.ua