SCHRÖDINGER OPERATORS WITH A $(a\partial_\nu\delta_\gamma + b\delta_\gamma)$-LIKE POTENTIALS

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Abstract. The

1. Introduction

2. Statement of Problem and Main Results

Let us consider the family of operators

$$H_\varepsilon = -\Delta + W(x) + V_\varepsilon(x). \tag{2.1}$$

Suppose that the unperturbed operator $H_0 = -\Delta + W$ is self-adjoint in $L^2(\mathbb{R}^2)$ with a domain $\operatorname{dom} H_0$. In addition, we suppose that $W \in L^\infty_{loc}(\mathbb{R}^2)$ and $\operatorname{dom} H_0 \subset W_2^1(\mathbb{R}^2)$.

Let γ be a closed C^3-curve without self-intersection points. We will denote by ω_ε the ε-neighborhood of γ, i.e., the union of all open balls with radius ε and center on γ. Suppose that potentials V_ε have compact supports that lie in ω_ε and the supports shrink to curve γ as $\varepsilon \to 0$. For this reason, $\operatorname{dom} H_\varepsilon = \operatorname{dom} H_0$.

To specify the dependence of V_ε on small parameter ε we introduce curvilinear coordinates coordinates in ω_ε. Let S be the circle of the same length as the length of γ. We will parameterize γ by points of the circle. Let $\alpha\colon\ S \to \mathbb{R}^2$ be the unit-speed C^3-parametrization of γ with the natural parameter $s \in S$. Also $\nu = (-\dot\alpha_2, \dot\alpha_1)$ is the unit normal on γ, because $\dot\alpha_1^2 + \dot\alpha_2^2 = 1$. We define the local coordinates (s, r) in ω_ε by

$$x = \alpha(s) + r\nu(s), \qquad (s, r) \in Q_\varepsilon = S \times (-\varepsilon, \varepsilon). \tag{2.2}$$

The coordinate r is the signed distance from a point x to γ. Therefore ω_ε is diffeomorphic to cylinder Q_ε for ε small enough. There is no loss of generality in assuming the diffeomorphism exists for $\varepsilon \in (0, 1)$.

We suppose that the localized potentials have the following structure

$$V_\varepsilon(\alpha(s) + r\nu(s)) = \varepsilon^{-2}\, V\left(\varepsilon^{-1} r\right) + \varepsilon^{-1}\, U\left(s, \varepsilon^{-1} r\right), \tag{2.3}$$

where V and U are measurable bounded functions such that

$$\operatorname{supp} V \subset (-1, 1), \quad \operatorname{supp} U \subset Q_1 \text{ and } \partial_s U \in L_2(\mathbb{R}^2). \tag{2.4}$$

The key assumption is that V does not depend on s.

The family of potentials V_ε generally diverges in the space of distributions $\mathcal{D}(\mathbb{R}^2)$. As we will show in Propositin 1, the potentials converge only if V is

2000 *Mathematics Subject Classification.* Primary 34L40, 34B09; Secondary 81Q10.

a zero mean function. In this case, $V_\varepsilon \to a\partial_\nu\delta_\gamma + b\delta_\gamma$ as $\varepsilon \to 0$ for some functions a and b, where δ_γ is the Dirac delta function supported on γ and $\partial_\nu\delta_\gamma$ is the normal derivative of δ_γ at points of γ. More precisely,

$$\langle a\partial_\nu\delta_\gamma + b\delta_\gamma, \varphi\rangle = -\int_\gamma \partial_\nu(a\varphi)\,d\gamma + \int_\gamma b\varphi\,d\gamma$$

for all $\varphi \in C_0^\infty(\mathbb{R}^2)$.

The main task is to construct asymptotic approximations, as $\varepsilon \to 0$, to eigenvalues and eigenfunctions of H_ε, i.e., asymptotics of eigenvalues λ^ε and eigenfunctions u_ε of spectral equation

$$-\Delta u_\varepsilon + (W + V_\varepsilon)u_\varepsilon = \lambda^\varepsilon u_\varepsilon \quad \text{in } \mathbb{R}^2. \tag{2.5}$$

We introduce some notation. The plane is divided into two domains by close curve γ. We suppose that $\mathbb{R}^2 \setminus \gamma = \Omega_{in} \cup \Omega_{out}$, where domain Ω_{out} is unbounded. Also, we say that v belongs to space $\mathcal{V} \subset L_2(\mathbb{R}^2)$ if $v|_{\Omega_-} \in W_2^2(\Omega_{in})$ and there exist a function w belonging to $\operatorname{dom} H_0$ such that $v = w$ in Ω_{out}. Of course, $v|_{\Omega_{out}} \in W^2_{2,loc}(\Omega_{out})$. Let $\mathcal{V}_0$ be the subspaces of $L_2(\Omega_{out})$ obtained by the restriction of all elements of $\mathcal{V}$ to Ω_{out}. We introduce two operators

$$\mathcal{D}_1 = -\Delta + W, \qquad \operatorname{dom}\mathcal{D}_1 = \{v \in \mathcal{V}_0 \colon\; v = 0 \text{ on } \gamma\},$$
$$\mathcal{D}_2 = -\Delta + W, \qquad \operatorname{dom}\mathcal{D}_2 = \{v \in W_2^2(\Omega_{in}) \colon\; v = 0 \text{ on } \gamma\}.$$

We also denote by $\gamma_t = \{x \in \mathbb{R}^2 \colon\; x = \alpha(s) + t\nu(s),\; s \in S\}$ the closed curve that is obtained from γ by flowing for "time" t along the normal vector field. Then the boundary of ω_ε consists of two curves $\gamma_{-\varepsilon}$ and γ_ε. For each $u \in \mathcal{V}$ there exist two one-side traces on γ, namely

$$u^- = \lim_{\varepsilon\to 0} u|_{\gamma_{-\varepsilon}}, \qquad u^+ = \lim_{\varepsilon\to 0} u|_{\gamma_\varepsilon}. \tag{2.6}$$

We say that the Schrödinger operator $-\frac{d^2}{dt^2} + V$ in $L_2(\mathbb{R})$ possesses a *zero-energy resonance* if there exists a non trivial solution h of the equation $-h'' + Vh = 0$ that is bounded on the whole line. We call h the *half-bound state* of V. In this case, we will also simply say that potential V has a half-bound state h. Such a solution h is unique up to a scalar factor and has nonzero limits

$$h(-\infty) = \lim_{t\to-\infty} h(t), \qquad h(+\infty) = \lim_{t\to+\infty} h(t)$$

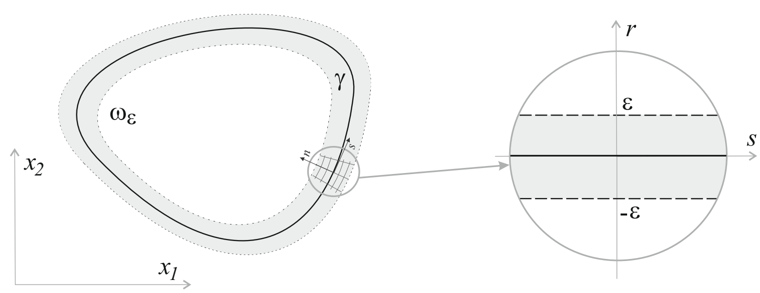

FIGURE 1. Curvilinear coordinates in the ε-neighbourhood of γ.

at both the infinities. We set

$$\theta = \frac{h(+\infty)}{h(-\infty)}. \tag{2.7}$$

Our main result reads as follows.

Theorem 1. *Let $W \in L^\infty_{loc}(\mathbb{R}^2)$ and $\operatorname{dom} H_0 \subset W_2^1(\mathbb{R}^2)$. Assume that potentials V and U are measurable bounded functions and assumption (2.4) and (**??**) holds. Then the family of operators*

$$H_\varepsilon = -\Delta + W + V_\varepsilon,$$

where the perturbation V_ε is given by (2.3), converges as $\varepsilon \to 0$ in the strong resolvent sense.

If potential V possesses a zero-energy resonance with a half-bound state h, then operators H_ε converge to operator $\mathcal{H}$ defined by $\mathcal{H}v = -\Delta v + Wv$ on functions $v \in \mathcal{V}$ obeying the interface conditions

$$u^+ - \theta u^- = 0, \quad \theta\partial_\nu u^+ - \partial_\nu u^- = \left(\tfrac{1}{2}(\theta^2 - 1)\varkappa + \mu\right) u^- \tag{2.8}$$

on curve γ. Here θ is given by (2.7), $\varkappa$ is the signed curvature of γ, and

$$\mu = \frac{1}{h^2(-\infty)} \int_{-1}^{1} U(\,\cdot\,, t) h^2(t)\, dt. \tag{2.9}$$

If potential V has no zero-energy resonance, then operators H_ε converge to the direct sum $\mathcal{D}_1 \oplus \mathcal{D}_2$ of two unperturbed operators $-\Delta + W$ in Ω_{in} and Ω_{out} respectively with the Dirichlet boundary conditions on interface γ.

Remark 1. *If potential V is identically zero, then $V_\varepsilon = \varepsilon^{-1} U\left(s, \varepsilon^{-1} n\right)$ and so obviously $V_\varepsilon \to \mu_0\delta_\gamma$, as $\varepsilon \to 0$, in the space of distributions. Here*

$$\mu_0(s) = \int_{-1}^{1} U(s,t)\, dt. \tag{2.10}$$

Potential $V = 0$ possesses a zero-energy resonance with constant functions as half-bound states. Hence parameter θ equals 1 and interface conditions (2.8) become $u^+ - u^- = 0$, $\partial_\nu u^+ - \partial_\nu u^- - \mu_0 u^- = 0$. These conditions are exactly the same as that obtained in [1].

3. Preliminaries

Returning now to curvilinear coordinates (s, r) given by (2.2), we see that the couple of vectors $\tau = (\dot\alpha_1, \dot\alpha_2)$, $\nu = (-\dot\alpha_2, \dot\alpha_1)$ gives a Frenet frame for γ. The Jacobian of transformation $x = \alpha(s) + r\nu(s)$ has the form

$$J(s,r) = \begin{vmatrix} \dot\alpha_1(s) - r\ddot\alpha_2(s) & -\dot\alpha_2(s) \\ \dot\alpha_2(s) + r\ddot\alpha_1(s) & \dot\alpha_1(s) \end{vmatrix}$$
$$= \dot\alpha_1^2(s) + \dot\alpha_2^2(s) - r\big(\dot\alpha_1(s)\ddot\alpha_2(s) - \dot\alpha_2(s)\ddot\alpha_1(s)\big) = 1 - r\varkappa(s).$$

Here $\varkappa = \det(\dot\alpha, \ddot\alpha)$ is the signed curvature of γ. Note that $\varkappa$ is a continuous function of the arc-length parameter s and the sign of $\varkappa(s)$ is defined uniquely up to the re-parametrization $s \mapsto -s$. We see that J is positive for sufficiently small n, because curvature $\varkappa$ is bounded on γ. Namely, the curvilinear coordinates (s, r) can be defined correctly on all domains ω_ε with $\varepsilon \leqslant \varepsilon_*$, where $\varepsilon_* = \min_\gamma |\varkappa|^{-1}$.

However, the above we have accepted that $\varepsilon_* = 1$, since this involves no loss of generality. We also have

$$\int_{\omega_\varepsilon} f(x_1, x_2)\, dx_1 dx_2 = \int_{Q_\varepsilon} f(s, r)(1 - r\varkappa(s))\, ds\, dr \tag{3.1}$$

for all integrable functions f. The metric tensor $g = (g_{ij})$ of ω_ε in the orthogonal coordinates (s, r) has the form

$$g = \begin{pmatrix} J^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

In fact, we have $g_{11} = |x_s|^2 = |\dot\alpha + r\dot\nu|^2 = |(1 - r\varkappa)\dot\alpha|^2 = J^2$, by the Frenet-Serret formula $\dot\nu = -\varkappa\dot\alpha$, and $g_{22} = |x_r|^2 = |\nu|^2 = 1$. In particular, the gradient in the local coordinates becomes

$$\nabla\varphi = \frac{1}{\sqrt{g_{11}}}\,\partial_s\varphi\,\tau + \frac{1}{\sqrt{g_{22}}}\,\partial_r\varphi\,\nu = \frac{1}{J}\,\partial_s\varphi\,\tau + \partial_r\varphi\,\nu$$

and therefore we have

$$\nabla\varphi \cdot \nabla\psi = J^{-2}\partial_s\varphi\,\partial_s\psi + \partial_r\varphi\,\partial_r\psi. \tag{3.2}$$

The Laplace-Beltrami operator in ω_ε has also the explicit form

$$\Delta\varphi = J^{-1}\left(\partial_s(J^{-1}\partial_s\varphi) + \partial_r(J\partial_r\varphi)\right) \tag{3.3}$$

as is easy to check.

Proposition 1. *If $\int_{\mathbb{R}} V\, dt = 0$, then*

$$V_\varepsilon \to \beta\partial_\nu\delta_\gamma + (\beta\varkappa + \mu_0)\,\delta_\gamma, \quad \text{as } \varepsilon \to 0,$$

in the space of distributions $\mathcal{D}'(\mathbb{R}^2)$, where $\beta = -\int_{\mathbb{R}} tV(t)\, dt$ and μ_0 is given by (2.10).

Proof. It is evident that potentials $\varepsilon^{-1}\, U\left(s, \varepsilon^{-1} r\right)$ converge to $\mu_0\delta_\gamma$ in $\mathcal{D}(\mathbb{R}^2)$. We will prove that sequence $g_\varepsilon = \varepsilon^{-2}\, V\left(\varepsilon^{-1} r\right)$ converges to $\beta\left(\partial_\nu\delta_\gamma + \varkappa\delta_\gamma\right)$ as $\varepsilon \to 0$, provided V is a zero-mean function. In fact, for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ we have

$$\begin{aligned}
\langle g_\varepsilon, \varphi\rangle = \int_{\omega_\varepsilon} g_\varepsilon(x)\varphi(x)\, dx &= \frac{1}{\varepsilon^2}\int_{Q_\varepsilon} V\left(\frac{r}{\varepsilon}\right)\varphi(s, r)(1 - r\varkappa(s))\, ds\, dr \\
&= \frac{1}{\varepsilon}\int_{Q_1} V(t)\varphi(s, \varepsilon t)(1 - \varepsilon t\varkappa(s))\, ds\, dt \\
&= \frac{1}{\varepsilon}\int_{-1}^{1} V(t)\, dt \int_S \varphi(s, 0)\, ds \\
&\quad + \int_{-1}^{1} tV(t)\, dt \int_S \left(\partial_n\varphi(s, 0) - \varkappa(s)\varphi(s, 0)\right) ds + O(\varepsilon),
\end{aligned}$$

as $\varepsilon \to 0$. The sequence $\langle g_\varepsilon, \varphi\rangle$ has a finite limit for all $\varphi \in C_0^\infty(\mathbb{R}^2)$ if and only if $\int_{\mathbb{R}} V\, dt = 0$. In this case, we have

$$\langle g_\varepsilon, \varphi\rangle \to \beta\int_\gamma \left(\partial_\nu\delta_\gamma + \varkappa\delta_\gamma\right)\varphi\, d\gamma,$$

which completes the proof. □

Interface conditions (2.8) contain the parameters which depend on the particular parametrization chosen for curve γ. More precisely, parameters θ, $\varkappa$ and μ change along with the change of the Frenet frame.

Proposition 2. *Operator $\mathcal{H}$ in Theorem 1 does not depend upon the choice of the Frenet frame for curve γ.*

Proof. Every smooth curve in the plane admits two possible orientations of arc-length parameter and consequently two possible Frenet frames. Let us change the Frenet frame $\{\tau, \nu\}$, previously introduced in Sec. 2, to the frame $\{-\tau, -\nu\}$ and prove that interface conditions (2.8) will remain the same. This change leads to the following transformations:

$$h(\pm\infty) \mapsto h(\mp\infty), \quad u_\pm \mapsto u_\mp, \quad \partial_\nu u_\pm \mapsto -\partial_\nu u_\mp,$$
$$\theta \mapsto \theta^{-1}, \quad \varkappa \mapsto -\varkappa, \quad \mu \mapsto \theta^{-2}\mu.$$

The first condition $u^+ - \theta u^- = 0$ in (2.8) transforms into $u^- - \theta^{-1}u^+ = 0$ and therefore remains unchanged. As for the second condition, we have

$$-\theta^{-1}\partial_\nu u^- + \partial_\nu u^+ - \left(-\tfrac{1}{2}(\theta^{-2}-1)\varkappa + \theta^{-2}\mu\right)u^+ = 0.$$

Multiplying the equality by θ yields

$$\theta\partial_\nu u^+ - \partial_\nu u^- - \left(\tfrac{1}{2}(\theta^2-1)\varkappa + \mu\right)\theta^{-1}u^+ = 0,$$

since $-\theta(\theta^{-2}-1) = \theta^{-1}(\theta^2-1)$. It remains to insert u^- in place of $\theta^{-1}u^+$, in view of the first interface condition. □

In the sequel, the normal vector field ν will be outward to domain Ω_{in}, that is to say, the local coordinate n will increase in the direction from Ω_{in} to Ω_{out}. At the end of the section, we record some technical assertion, which will be often used below. Throughout the paper, $W_2^l(\Omega)$ stands for the Sobolev space of functions defined on a set Ω.

Proposition 3. *Suppose that $v \in W_2^1(\Omega_{out})$ and $w \in W_2^1(\Omega_{in})$. Then*

$$\|v(\,\cdot\,,\varepsilon) - v(\,\cdot\,,0)\|_{L_2(\gamma)} \leqslant c_1\varepsilon^{1/2}\|v\|_{W_2^1(\Omega_{out})}, \tag{3.4}$$
$$\|w(\,\cdot\,,-\varepsilon) - w(\,\cdot\,,0)\|_{L_2(\gamma)} \leqslant c_2\varepsilon^{1/2}\|w\|_{W_2^1(\Omega_{in})}, \tag{3.5}$$

where the constants c_k do not depend on ε.

Proof. First we assume that v is a smooth function in Ω_{out}. Then

$$v(s,\varepsilon) - v(s,0) = \int_0^\varepsilon \partial_t v(s,t)\,dt,$$

and for all $\psi \in L_2(\gamma)$ we have

$$\int_S (v(s,\varepsilon) - v(s,0))\psi(s)\,ds = \int_S\int_0^\varepsilon \partial_t v(s,t)\psi(s)\,dt\,ds.$$

Therefore

$$\left|\int_S \big(v(s,\varepsilon)-v(s,0)\big)\psi(s)\,ds\right| \leqslant \int_S\int_0^\varepsilon |\partial_t v(s,t)|\,|\psi(s)|\,dt\,ds$$
$$\leqslant \left(\int_0^\varepsilon\int_S |\psi(s)|^2\,ds\,dt\right)^{1/2}\left(\int_S\int_0^\varepsilon |\partial_t v|^2\,dt\,ds\right)^{1/2}$$
$$\leqslant c\varepsilon^{1/2}\|\psi\|_{L_2(\gamma)}\left(\int_{\Omega_{out}} |\nabla v|^2\,dx\right)^{1/2} \leqslant c_1\varepsilon^{1/2}\|v\|_{W_2^1(\Omega_{out})}\|\psi\|_{L_2(\gamma)}.$$

Hence (3.4) holds for all smooth functions v and then by continuity for all $v \in W_2^1(\Omega_{out})$. Similar arguments apply to the proof of (3.5). □

4. Formal Asymptotics

4.1. **Leading Terms.** In this section we will show how interface conditions (2.8) can be found by direct calculations, constructing the formal asymptotics of eigenvalues and eigenfunctions. We look for asymptotics of λ_ε and u_ε in the form

$$\lambda^\varepsilon \approx \lambda + \varepsilon\lambda_1, \qquad u_\varepsilon(x) \approx \begin{cases} u(x) + \varepsilon u_1(x) & \text{in } \mathbb{R}^2\setminus\omega_\varepsilon,\\ v_0\left(s,\frac{r}{\varepsilon}\right) + \varepsilon v_1\left(s,\frac{r}{\varepsilon}\right) + \varepsilon^2 v_2\left(s,\frac{r}{\varepsilon}\right) & \text{in } \omega_\varepsilon. \end{cases} \tag{4.1}$$

Recall that the boundary of ω_ε consists of two curves $\gamma_{-\varepsilon}$ and γ_ε. To match two different approximations, we hereafter assume that

$$[u_\varepsilon]_{\gamma_{\pm\varepsilon}} = 0, \quad [\partial_\nu u_\varepsilon]_{\gamma_{\pm\varepsilon}} = 0, \tag{4.2}$$

where $[\,\cdot\,]_{\gamma_{\pm\varepsilon}}$ is a jump across $\gamma_{\pm\varepsilon}$. Since function u_ε solves (2.5) and domain ω_ε shrinks to γ, the leading term u must be a solution of the equation

$$-\Delta u + Wu = \lambda u \quad \text{in } \mathbb{R}^2\setminus\gamma,$$

subject to appropriate interface conditions on γ. To find these conditions, we consider equation (2.5) in the curvilinear coordinates (s,n), where $n = r/\varepsilon$. Then in vicinity of γ the Laplacian can be written as

$$\Delta = \frac{1}{1-\varepsilon n\varkappa}\left(\varepsilon^{-2}\partial_n(1-\varepsilon n\varkappa)\partial_n + \partial_s\Big(\frac{1}{1-\varepsilon n\varkappa}\,\partial_s\Big)\right), \tag{4.3}$$

by (3.3). From this we readily deduce the asymptotic representation

$$\Delta = \varepsilon^{-2}\partial_n^2 - \varepsilon^{-1}\varkappa\partial_n - n\varkappa^2\partial_n + \partial_s^2 + \varepsilon P_\varepsilon,$$

where P_ε is a partial differential operator on the second order with respect to s and the first one with respect to n whose coefficients are uniformly bounded on ε. Substituting (4.1) into (2.5) for $x \in \omega_\varepsilon$ in particular yields

$$-\partial_n^2 v_0 + V(n)v_0 = 0, \qquad -\partial_n^2 v_1 + V(n)v_1 = -\varkappa(s)\partial_n v_0 - U(s,n)v_0 \tag{4.4}$$

in cylinder $Q_1 = S\times(-1,1)$. From (4.2) we see that necessarily

$$u^-(s) = v_0(s,-1), \qquad u^+(s) = v_0(s,1), \tag{4.5}$$
$$\partial_n v_0(s,-1) = 0, \qquad \partial_n v_0(s,1) = 0, \tag{4.6}$$
$$\partial_n v_1(s,-1) = \partial_\nu u^-(s), \qquad \partial_n v_1(s,1) = \partial_\nu u^+(s), \tag{4.7}$$

where $u^\pm$ are defined by (2.6). Combining (4.6)–(4.7), we conclude that v_0 and v_1 solve boundary value problems

$$\begin{cases} -\partial_n^2 v_0 + V(n)v_0 = 0 \quad \text{in} \quad Q_1, \\ \partial_n v_0(s,-1) = 0, \quad \partial_n v_0(s,1) = 0, \quad s \in S; \end{cases} \tag{4.8}$$

$$\begin{cases} -\partial_n^2 v_1 + V(n)v_1 = -\varkappa(s)\partial_n v_0 - U(s,n)v_0 \quad \text{in} \quad Q_1, \\ \partial_n v_1(s,-1) = \partial_\nu u^-(s), \quad \partial_n v_1(s,1) = \partial_\nu u^+(s), \quad s \in S \end{cases} \tag{4.9}$$

respectively. We have two boundary value problems for the "non-elliptic" partial differential operator in Q_1. Of course, the problems can also be regarded as boundary value problems for ordinary differential equations on $(-1,1)$, which depend on parameter $s \in S$.

Case of zero-energy resonance. Assume that operator $-\frac{d^2}{dt^2}+V$ has a zero energy resonance with half-bound state h. Since the support of V lies in interval $(-1,1)$, the half-bound state h is constant outside this interval as a bounded solution of equation $h''=0$. Therefore the restriction of h to $(-1,1)$ is a nonzero solution of the Neumann boundary value problem

$$-h'' + V(n)h = 0, \quad n \in (-1,1), \qquad h'(-1) = 0, \quad h'(1) = 0. \tag{4.10}$$

Hereafter, we fix h by additional condition $h(-1)=1$. In view of (2.7), we have $h(1)=\theta$, since $h(\pm\infty) = h(\pm 1)$.

In this case, (4.8) admits a infinite-dimensional space of solutions

$$\mathcal{N} = \big\{a(s)h(n)\colon\ a \in L^2(S)\big\}.$$

Therefore $v_0(s,n) = a_0(s)h(n)$ for some function a_0. From (4.5) we deduce that $u^- = a_0$ and $u^+ = \theta a_0$ and hence that

$$u^+ = \theta u^- \quad \text{on } \gamma. \tag{4.11}$$

In addition, we must set $v_0(s,n) = u^-(s)h(n)$.

Problem (4.9) is in general unsolvable, since $\mathcal{N} \neq \{0\}$. To find solvability conditions, we rewrite equation in (4.9) as

$$-\partial_n^2 v_1 + V(n)v_1 = -\big(\varkappa(s)h'(n) + U(s,n)h(n)\big)u^-(s), \tag{4.12}$$

multiply by an arbitrary element of $\mathcal{N}$ and then integrate over Q_1

$$\int_{Q_1} \big(-\partial_n^2 v_1 + V(n)v_1\big)\, a(s)h(n)\, dn\, ds$$
$$= -\int_{Q_1} (\varkappa(s)h'(n) + U(s,n)h(n))u^-(s)a(s)h(n)\, dn\, ds. \tag{4.13}$$

Since h is a solution of (4.10), integrating by parts twice on the left-hand side yields

$$\int_S \int_{-1}^1 \big(-\partial_n^2 v_1 + V v_1\big)\, ah\, dn\, ds = -\int_S (\partial_n v_1 h - v_1 h')\big|_{n=-1}^{n=1} a\, ds$$
$$-\int_S \int_{-1}^1 a v_1\, (-h'' + Vh)\, dn\, ds = -\int_S \big(\theta\partial_\nu u^+ - \partial_\nu u^-\big) a\, ds,$$

in view of the boundary conditions for v_1. Recall that $h(-1)=1$ and $h(1)=\theta$. Since $hh'=\frac{1}{2}(h^2)'$, we also have $\int_{-1}^1 hh'\,dt=\frac{1}{2}(\theta^2-1)$. Therefore (4.13) becomes

$$\int_S \left(\theta\partial_\nu u^+ - \partial_\nu u^-\right) a\,ds = \int_S \left(\tfrac{1}{2}(\theta^2-1)\varkappa+\mu\right)u^- a\,ds$$

for all $a\in L^2(S)$, where $\mu(s)=\int_{-1}^1 U(s,n)h^2(n)\,dn$. Therefore

$$\theta\partial_\nu u^+ - \partial_\nu u^- = \left(\tfrac{1}{2}(\theta^2-1)\varkappa+\mu\right)u^- \quad \text{on } \gamma,$$

which is necessary for the solvability of (4.9). In view of the Fredholm alternative, this condition is also a sufficient one. At the same time, it is a jump condition at the interface γ for the normal derivative of u.

Therefore the leading terms λ and u of asymptotics (4.1) solve the problem

$$-\Delta u + Wu = \lambda u \qquad \text{in } \mathbb{R}^2\setminus\gamma, \tag{4.14}$$

$$u^+ - \theta u^- = 0, \quad \theta\partial_\nu u^+ - \partial_\nu u^- = \left(\tfrac{1}{2}(\theta^2-1)\varkappa+\mu\right)u^- \quad \text{on } \gamma. \tag{4.15}$$

Assume that λ is an eigenvalue of operator $\mathcal{H}$ associated with eigenfunction u. Now we can calculated the trace u^- on γ and finally determine $v_0(s,n)=u^-(s)h(n)$.

Non-resonant case. Now suppose that problem (4.10) admits the trivial solution only, i.e., $\mathcal{N}=\{0\}$. Then $v_0=0$ and therefore $u^-=0$ and $u^+=0$ on γ, by (4.5). We thus get

$$-\Delta u + Wu = \lambda u \quad \text{in } \mathbb{R}^2\setminus\gamma, \qquad u|_\gamma = 0.$$

Let us suppose that λ is an eigenvalue of the direct sum $\mathcal{D}_1\oplus\mathcal{D}_2$ of two Dirichlet type operators and u is a corresponding eigenfunction. In this case, problem (4.9) has the form

$$\begin{cases} -\partial_n^2 v_1 + V(n)v_1 = 0 \quad \text{in } Q_1, \\ \ \partial_n v_1(s,-1) = \partial_\nu u^-, \qquad \partial_n v_1(s,1) = \partial_\nu u^+, \end{cases} \tag{4.16}$$

and admits a unique solution.

4.2. **Correction terms.** Upon our substituting (4.1) into equation (2.5) and conditions (4.2), we in particular discover that correction term u_1 solves the equation

$$-\Delta u_1 + Wu_1 = \lambda u_1 + \lambda_1 u \quad \text{in } \mathbb{R}^2\setminus\gamma \tag{4.17}$$

and v_2 is a solution of the problem

$$\begin{aligned} -\partial_n^2 v_2 + V(n)v_2 = {} & -(\varkappa(s)\partial_n + U(s,n))v_1 \\ & + (\partial_s^2 - n\varkappa^2\partial_n - W(s,0)+\lambda)v_0 \quad \text{in } Q_1, \end{aligned} \tag{4.18}$$

$$\partial_n v_1(\,\cdot\,,-1) = \partial_\nu u_1^- - \partial_\nu^2 u^-, \quad \partial_n v_1(\,\cdot\,,1) = \partial_\nu u_1^+ + \partial_\nu^2 u^+, \quad s\in S. \tag{4.19}$$

In addition,

$$v_1(\,\cdot\,,-1) = u_1^- - \partial_\nu u^-, \qquad v_1(\,\cdot\,,1) = u_1^+ + \partial_\nu u^+. \tag{4.20}$$

If operator $-\frac{d^2}{dt^2}+V$ has a zero energy resonance, then u is an eigenfunction of (4.14), (4.15). Since the second condition in (4.15) holds, problem (4.9) is solvable and possesses a linear manifold of solutions

$$v_1(s,n) = a_1(s)h(n) + w(s,n), \qquad a_1\in L_2(S), \tag{4.21}$$

where w is a partial solution of (4.9) such that $w(s,-1)=0$ for all $s\in S$. We may therefore substitute (4.21) into (4.20) and obtain

$$a_1 = u_1^- - \partial_\nu u^-, \quad \theta a_1 + w(\,\cdot\,,1) = u_1^+ + \partial_\nu u^+,$$

from which the coupling condition

$$u_1^+ - \theta u_1^- = w(\,\cdot\,,1) - \partial_\nu u^+ - \theta\partial_\nu u^- \quad \text{on } \gamma \tag{4.22}$$

follows. Also, we have $v_1(s,n) = (u_1^-(s) - \partial_\nu u^-(s))h(n) + w(s,n)$.

By reasoning similar to that for (4.9), we can write the solvability condition for problem (4.18), (4.19) as

$$\theta\partial_\nu u_1^+ - \partial_\nu u_1^- - \big(\tfrac12(\theta^2-1)\varkappa+\mu\big)u_1^- = -\theta\partial_\nu^2 u^+ - \partial_\nu^2 u^- + \textstyle\int_{-1}^1 f(s,n)h(n)\,dn, \tag{4.23}$$

where $f = (\varkappa\partial_n + U)(h\partial_\nu u_1^- - w) + (\partial_s^2 - n\varkappa^2\partial_n - W(\,\cdot\,,0)+\lambda)v_0$. Then (4.17), (4.22) and(4.23) taken together yield

$$-\Delta u_1 + W u_1 = \lambda u_1 + \lambda_1 u \quad \text{in } \mathbb{R}^2\setminus\gamma, \tag{4.24}$$

$$u_1^+ - \theta u_1^- = g_0, \quad \theta\partial_\nu u_1^+ - \partial_\nu u_1^- - \big(\tfrac12(\theta^2-1)\varkappa+\mu\big)u_1^- = g_1 \quad \text{on } \gamma, \tag{4.25}$$

where g_0 and g_1 stand for the right hand sides of (4.22) and (4.23) respectively.

Since λ is an eigenvalue of $\mathcal{H}$, the last problem is generally unsolvable. But there exists a unique value of λ_1 for which (4.24), (4.25) admits a solution. Let us multiply equation (4.24) by eigenfunction u, and integrate over $\mathbb{R}^2$, to find

$$\lambda_1 = \|u\|^{-2}_{L_2(\mathbb{R}^2)}\int_\gamma \big(g_1u^- - g_0\partial_\nu u^+\big)\,d\gamma, \tag{4.26}$$

where we have integrated by parts in the term containing Δu_1:

$$\begin{aligned}-\int_{\mathbb{R}^2}\Delta u_1 u\,dx &= -\int_{\Omega_{out}}\Delta u_1 u\,dx - \int_{\Omega_{in}}\Delta u_1 u\,dx\\ &= \int_\gamma\big(\partial_\nu u_1^+u^+ - \partial_\nu u_1^-u^- - u_1^+\partial_\nu u^+ + u_1^-\partial_\nu u^-\big)\,d\gamma - \int_{\mathbb{R}^2}u_1\Delta u\,dx.\end{aligned}$$

Utilizing the coupling conditions for u and u_1 on interface γ, we have obtained

$$\begin{aligned}&-\int_{\mathbb{R}^2}\Delta u_1 u\,dx + \int_{\mathbb{R}^2}u_1\Delta u\,dx\\ &\qquad= \int_\gamma\big((\theta\partial_\nu u_1^+ - \partial_\nu u_1^-)u^- - (\theta\partial_\nu u^+ - \partial_\nu u^-)u_1^- - g_0\partial_\nu u^+\big)\,d\gamma\\ &\qquad= \int_\gamma\big((Ru_1^- + g_1)u^- - Ru^-u_1^- - g_0\partial_\nu u^+\big)\,d\gamma = \int_\gamma\big(g_1u^- - g_0\partial_\nu u^+\big)\,d\gamma,\end{aligned}$$

where $R = \frac12(\theta^2-1)\varkappa + \mu$.

4.3. **Adjustment of asymptotics.**

Proposition 4 (Smoothness of asymptotics)**.** *Let*

Proof. aa ☐

We set

$$v_1(s,n) = \partial_\nu u^-(s)h_1(n) - u^-(s)h_2(s,n), \tag{4.27}$$

where h_1 and h_2 solve the Cauchy problems

$$-h_1'' + V(n)h_1 = 0, \quad t \in (-1,1), \quad h_1(-1) = 0,\ h_1'(-1) = 1,$$

$$\begin{cases} -h_2'' + V(n)h_2 = \varkappa(s)h'(n) + U(s,n)h(n), & n \in (-1,1), \\ h_2(s,-1) = 0, \quad \partial_n h_2(s,-1) = 0, & s \in S \end{cases} \tag{4.28}$$

respectively. We see at once that v_1 of the form (4.27) solves equation (4.12) and satisfies boundary conditions $v_1(s,-1) = 0$ and $\partial_t v_1(s,-1) = \partial_\nu u^-(s)$. Now we show that condition $\partial_t v_1(s,1) = \partial_\nu u^+(s)$ also holds. Recall that half-bound state h was fixed by $h(-1) = 1$. Then the Lagrange identity $(h_1 h' - h_1' h)|_{-1}^1 = 0$ implies $h_1'(1) = \theta^{-1}$. Next, multiplying the equation in (4.28) by h and integrating by parts twice yield

$$(h'h_2 - h\,\partial_t h_2)\big|_{-1}^1 = \varkappa(s)\int_{-1}^1 hh'\,dn + \int_{-1}^1 U(s,n)h^2(n)\,dn.$$

Hence we have $\theta\,\partial_t h_2(s,1) = -\frac{1}{2}(\theta^2-1)\varkappa(s) - \mu(s)$, and then derive

$$\begin{aligned} \partial_t v_1(s,1) &= \partial_\nu u^-(s)\,h_1'(1) - u^-(s)\,\partial_t h_2(s,1) \\ &= \theta^{-1}\Big(\partial_\nu u^-(s) + \big(\tfrac{1}{2}(\theta^2-1)\varkappa(s) + \mu(s)\big)u^-(s)\Big) = \partial_\nu u^+(s) \end{aligned}$$

in view of interface conditions (4.15).

5. Proof of Theorem 1

We will provide a proof for the most interesting case when potential V has a zero-energy resonance. The non-resonant case, which is much easier, follows similarly. We must prove that

$$(H_\varepsilon - \zeta)^{-1}f \to (\mathcal{H} - \zeta)^{-1}f, \quad \text{as } \varepsilon \to 0, \tag{5.1}$$

for all $f \in L_2(\mathbb{R}^2)$ and some $\zeta \in \mathbb{C} \setminus \mathbb{R}$. But the resolvents of H_ε are uniformly bounded on ε, namely,

$$\|(H_\varepsilon - \zeta)^{-1}\| \leqslant |\operatorname{Im}\zeta|^{-1}.$$

It will thus be sufficient to prove that (5.1) holds for $f \in \mathcal{F}$, where $\mathcal{F}$ is some dense subset of $L_2(\mathbb{R}^2)$. We suppose that $\mathcal{F} = C_0^\infty(\mathbb{R}^2 \setminus \gamma)$.

Hereafter, letters c_j denote various positive numbers independent of ε, whose values might be different in different proofs.

5.1. **Approximation in $W_2^1(\mathbb{R}^2)$.** Given $f \in \mathcal{F}$ and $\zeta \in \mathbb{C}$, $\operatorname{Im}\zeta \neq 0$, we must compare $u_\varepsilon = (H_\varepsilon - \zeta)^{-1}f$ and $u = (\mathcal{H} - \zeta)^{-1}f$ in $L_2(\mathbb{R})$, and show that the difference $u_\varepsilon - u$ is infinitely small in $L_2(\mathbb{R})$-norm, as $\varepsilon \to 0$. The basic idea of the proof is to construct a suitable approximation to u_ε in $W_2^1(\mathbb{R}^2)$. The formal asymptotics (**??**) constructed above will be used as a starting point in the construction of this approximation. We recall that the Sobolev space $W_2^1(\mathbb{R}^2)$ contains the domains of H_0 and H_ε.

We note that $v_0 \in W_2^1(Q_1)$. This inclusion follows from the explicit form $v_0(s,t) = u^-(s)h(t)$, where u^- belongs to $W_2^{3/2}(S)$ (as a trace of $u \in W_2^2(\Omega_{in})$

on curve γ) and $h \in W_2^2(-1,1) \subset C^1(-1,1)$. In view of representation formula (4.27), function v_1 does not belongs to $W_2^1(Q_1)$, since $\partial_\nu u^- \in W_2^{1/2}(S)$ in general. However the term $u^- h_2$ in (4.27) is an element of $W_2^1(Q_1)$, because $h_2 = h_2(s,t)$ possesses the additional smoothness with respect to s owing to the making more smoothness assumptions upon the curve γ and potential U. Recall that $\varkappa \in C^1(S)$ and $\partial_s U \in L_2(\mathbb{R}^2)$.

We now regularize the trace $\partial_\nu u^-$. Let $\{\beta_\varepsilon^-\}_{\varepsilon>0}$ be a sequence in $W_2^1(\gamma)$ such that $\beta_\varepsilon^- \to \partial_\nu u^-$ in $W_2^{-1/2}(\gamma)$. The sequence can be chosen in such a way that

$$\|\beta_\varepsilon^- - \partial_\nu u^-\|_{W_2^{-1/2}(\gamma)} \leqslant c_1 \varepsilon^{1/2} \qquad \|\beta_\varepsilon^-\|_{W_2^1(\gamma)} \leqslant c_2 \varepsilon^{-1/2}, \tag{5.2}$$

since $|\langle \beta_\varepsilon^- - \partial_\nu u^-,\, \beta_\varepsilon^- \rangle| \leqslant \|\beta_\varepsilon^- - \partial_\nu u^-\|_{W_2^{-1/2}(\gamma)} \|\beta_\varepsilon^-\|_{W_2^1(\gamma)}$ Then the function

$$v_1^\varepsilon(s,t) = \beta_\varepsilon^-(s) h_1(t) - u^-(s) h_2(s,t) \tag{5.3}$$

belongs to $W_2^1(Q_1)$ and solves the problem

$$\begin{cases} -\partial_n^2 v_1^\varepsilon + V(t) v_1^\varepsilon = -\varkappa(s) u^-(s) h'(t) - U(s,t) h(t) \quad \text{in } \ Q_1, \\ \ \partial_t v_1^\varepsilon(s,-1) = \beta_\varepsilon^-(s), \quad \partial_t v_1^\varepsilon(s,1) = \beta_\varepsilon^+(s), \quad s \in S, \end{cases} \tag{5.4}$$

where $\beta_\varepsilon^+ = \theta^{-1}\left(\frac{1}{2}(\theta^2-1)\varkappa u^- - \beta_\varepsilon^-\right)$. Thus

$$\|\beta_\varepsilon^+ - \partial_\nu u^+\|_{W_2^{-1/2}(\gamma)} \leqslant c_1 \varepsilon^{1/2}, \tag{5.5}$$

as $\varepsilon \to 0$, by solvability condition (**??**). We also have the bounds

$$\|v_1^\varepsilon\|_{L_2(Q_1)} + \|\partial_t v_1^\varepsilon\|_{L_2(Q_1)} \leqslant C_1, \qquad \|\partial_s v_1^\varepsilon\|_{L_2(Q_1)} \leqslant C_2 \varepsilon^{-1/2}. \tag{5.6}$$

We introduce the function

$$y_\varepsilon(x) = \begin{cases} u(x) & \text{in } \ \mathbb{R}^2 \setminus \omega_\varepsilon, \\ v_0\left(s, \frac{r}{\varepsilon}\right) + \varepsilon v_1^\varepsilon\left(s, \frac{r}{\varepsilon}\right) & \text{in } \ \omega_\varepsilon. \end{cases} \tag{5.7}$$

It is still not smooth enough and does not belong to $W_2^1(\mathbb{R}^2)$, because it has in general jump discontinuities on curves $\gamma_{-\varepsilon}$ and γ_ε. We will show that both the jumps

$$\begin{aligned} [y_\varepsilon]_{\gamma_{-\varepsilon}} &= v_0(s,-1) - u(s,-\varepsilon) = u^-(s) - u(s,-\varepsilon), \\ [y_\varepsilon]_{\gamma_\varepsilon} &= u(s,\varepsilon) - v_0(s,1) - \varepsilon v_1^\varepsilon(s,1) = u(s,\varepsilon) - \theta u^-(s) - \varepsilon v_1^\varepsilon(s,1) \\ &= u(s,\varepsilon) - u^+(s) - \varepsilon v_1^\varepsilon(s,1) \end{aligned}$$

are small as $\varepsilon \to 0$. Recall that $v_1^\varepsilon(s,-1) = 0$.

Let us denote by Ω_ε the set $\mathbb{R}^2 \setminus \omega_\varepsilon$.

Lemma 1. *There exists a function $\rho_\varepsilon\colon \mathbb{R}^2 \to \mathbb{C}$ such that $y_\varepsilon + \rho_\varepsilon$ belongs to $W_2^1(\mathbb{R}^2)$. Moreover for the restriction of ρ_ε to Ω_ε we have the estimate*

$$\|\rho_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} \leqslant c\varepsilon^{1/2}. \tag{5.8}$$

Proof. Let $Z_{in}\colon W_2^{1/2}(\gamma) \to W_2^1(\Omega_{in})$ and $Z_{out}\colon W_2^{1/2}(\gamma) \to W_2^1(\Omega_{out})$ be continuous extension operators such that $\operatorname{supp} Z_{in}\, g \subset \Omega_{in} \cap \omega_{1/2}$ and $\operatorname{supp} Z_{out}\, g \subset \Omega_{out} \cap \omega_{1/2}$ for all $g \in W_2^{1/2}(\gamma)$. The jumps $g_\varepsilon^\pm := [y_\varepsilon]_{\gamma_{\pm\varepsilon}}$ can be regarded as functions on γ. Obviously, $g_\varepsilon^\pm \in W_2^{1/2}(\gamma)$.

We set $z_\varepsilon^- = -Z_{in}\, g_\varepsilon^-$, $z_\varepsilon^+ = -Z_{out}\, g_\varepsilon^+$ and introduce function

$$\rho_\varepsilon(s,n) = \begin{cases} z_\varepsilon^+(s, n-\varepsilon) & \text{for } \ s\in S,\ n\in(\varepsilon, \varepsilon+1/2),\\ z_\varepsilon^-(s, n+\varepsilon) & \text{for } \ s\in S,\ n\in(-\varepsilon-1/2, -\varepsilon),\\ 0, \quad \text{otherwise} & \end{cases} \tag{5.9}$$

in $\mathbb{R}^2$ for $\varepsilon < 1/2$. The function has a compact support and, in particular, it vanishes in ω_ε. Next, by construction ρ_ε has the jump discontinuities

$$[\rho_\varepsilon]_{\gamma_{\pm\varepsilon}} = z_\varepsilon^\pm(s,0) = -[y_\varepsilon]_{\gamma_{\pm\varepsilon}}.$$

Since both the functions y_ε and ρ_ε belong to $W_2^1(\mathbb{R}^2\setminus(\gamma_{-\varepsilon}\cup\gamma_\varepsilon))$ and

$$[y_\varepsilon + \rho_\varepsilon]_{\gamma_{\pm\varepsilon}} = 0,$$

we have $y_\varepsilon + \rho_\varepsilon \in W_2^1(\mathbb{R}^2)$. Furthermore,

$$\begin{aligned} \|\rho_\varepsilon\|_{W_2^1(\Omega_\varepsilon)} &\leqslant c_1\big(\|z_\varepsilon^-\|_{W_2^1(\Omega_{in})} + \|z_\varepsilon^+\|_{W_2^1(\Omega_{out})}\big)\\ &= c_1\big(\|Z_{in}\, g_\varepsilon^-\|_{W_2^1(\Omega_{in})} + \|Z_{out}\, g_\varepsilon^+\|_{W_2^1(\Omega_{out})}\big)\\ &\leqslant c_2\big(\|g_\varepsilon^-\|_{W_2^{1/2}(\gamma)} + \|g_\varepsilon^+\|_{W_2^{1/2}(\gamma)}\big)\\ &\leqslant c_3\big(\|u(\,\cdot\,,-\varepsilon) - u^-\|_{W_2^{1/2}(\gamma)} + \|u(\,\cdot\,,\varepsilon) - u^+\|_{W_2^{1/2}(\gamma)}\big)\\ &+ \varepsilon\|v_1^\varepsilon(\,\cdot\,,1)\|_{W_2^{1/2}(\gamma)} \leqslant c_4\varepsilon^{1/2}, \end{aligned}$$

by Proposition 3 and (5.6). In fact, the restrictions u to domains Ω_{in} and Ω_{out} belong to $W_2^2(\Omega_{in})$ and $W_2^2(\Omega_{out})$ respectively. Applying Proposition 3 to u and $\partial_s u$ yields

$$\|u(\,\cdot\,,\pm\varepsilon) - u(\,\cdot\,,\pm 0)\|_{L_2(\gamma)} + \|\partial_s u(\,\cdot\,,\pm\varepsilon) - \partial_s u(\,\cdot\,,\pm 0)\|_{L_2(\gamma)} \leqslant c\varepsilon^{1/2}.$$

Consequently, $\|u(\,\cdot\,,\pm\varepsilon) - u_\pm\|_{W_2^{1/2}(\gamma)} \leqslant \|u(\,\cdot\,,\pm\varepsilon) - u_\pm\|_{W_2^1(\gamma)} \leqslant c\varepsilon^{1/2}$. Finally, this follows from (5.3) that

$$\|v_1^\varepsilon(\,\cdot\,,1)\|_{W_2^{1/2}(\gamma)} \leqslant c_1(\|\beta_\varepsilon^-\|_{W_2^{1/2}(\gamma)} + \|u^-\|_{W_2^{1/2}(\gamma)}) \leqslant c_2,$$

since $\beta_\varepsilon^- \to \partial_\nu u^-$ in $W_2^{1/2}(\gamma)$. □

Hence, the desired approximation to u_ε in the Sobolev space $W_2^1(\mathbb{R}^2)$ has the form

$$Y_\varepsilon(x) = \begin{cases} u(x) + \rho_\varepsilon(x) & \text{in } \ \mathbb{R}^2\setminus\omega_\varepsilon,\\ v_0\left(s, \frac{r}{\varepsilon}\right) + \varepsilon v_1^\varepsilon\left(s, \frac{r}{\varepsilon}\right) & \text{in } \ \omega_\varepsilon, \end{cases} \tag{5.10}$$

where ρ_ε is given by (5.9).

5.2. **Estimate of Remainder.** Let us fix $f \in C_0^\infty(\mathbb{R}^2\setminus\gamma)$. First of all, we note that

$$\int_{\mathbb{R}^2} f\varphi\, dx = \int_{\Omega_\varepsilon} f\varphi\, dx \tag{5.11}$$

for ε small enough. We also record some other identities that will be needed below. Multiplying equation (4.14) by $\varphi \in W_2^1(\mathbb{R}^2)$ and integrating by parts over Ω_ε yeild

$$\int_{\Omega_\varepsilon} \big(\nabla u \nabla\varphi + (W-\zeta)u\varphi\big)\,dx - \int_{\Omega_\varepsilon} f\varphi\,dx \\ = -\int_S \big(\partial_\nu u(s,\varepsilon)\varphi(s,\varepsilon) - \partial_\nu u(s,-\varepsilon)\varphi(s,-\varepsilon)\big)\,ds. \quad (5.12)$$

In the same manner we can obtain from (4.8) and (5.4) that

$$\int_{Q_1} \big(\partial_t v_0\,\partial_t\psi + Vv_0\psi\big)J_\varepsilon\,dtds = \varepsilon\int_{Q_1} \varkappa\,\partial_t v_0\,\psi\,dtds; \quad (5.13)$$

$$\int_{Q_1} \big(\partial_t v_1^\varepsilon\,\partial_t\psi + Vv_1^\varepsilon\psi + Uv_0\psi\big)J_\varepsilon\,dt\,ds \\ = -\int_{Q_1} \varkappa\,\partial_t v_0\psi J_\varepsilon\,dt\,ds + \varepsilon\int_{Q_1} \varkappa\,\partial_t v_1^\varepsilon\,\psi\,dt\,ds \quad (5.14) \\ + \int_S \big(\beta_\varepsilon^+(s)\psi(s,1)J(s,\varepsilon) - \beta_\varepsilon^-(s)\psi(s,-1)J(s,-\varepsilon)\big)\;ds$$

for all $\psi \in W_2^1(Q_1)$. For instance, let us multiply the equation in (4.8) by $\psi(s,t)J_\varepsilon(s,t)$, where $J_\varepsilon(s,t) = 1-\varepsilon\varkappa t$, and integrate over Q_1. Then in view of boundary conditions for v_0 we deduce

$$0 = \int_{Q_1} \big(-\partial_t^2 v_0 + Vv_0\big)\psi J_\varepsilon\,dtds = -\int_S (\partial_t v_0\,\psi J_\varepsilon)\big|_{-1}^{1}\,ds \\ + \int_{Q_1} \partial_t v_0\,\partial_t(\psi J_\varepsilon)\,dtds + \int_{Q_1} Vv_0\psi J_\varepsilon\,dtds \\ = \int_{Q_1} \big(\partial_t v_0\,\partial_t\psi + Vv_0\psi\big)J_\varepsilon\,dtds - \varepsilon\int_{Q_1} \varkappa\,\partial_t v_0\,\psi\,dtds,$$

which establishes (5.13). Let us note here, for future use,

$$\int_{\omega_\varepsilon} g(x)\,dx = \varepsilon\int_{Q_1} g(s,\varepsilon t)J_\varepsilon(s,t)\,ds\,dt, \quad (5.15)$$

$$|\nabla v(x_\varepsilon)|^2 = \varepsilon^{-2}|\partial_t v(s,t)|^2 + J_\varepsilon^{-2}(s,t)\,|\partial_s v(s,t)|^2, \quad (5.16)$$

where $v(x_\varepsilon)$ stands for $v(s,\frac{r}{\varepsilon})$, cf. (3.1) and (3.2).

Under our assumptions about potential W the function $u_\varepsilon = (H_\varepsilon-\zeta)^{-1}f$ belongs to $W_2^1(\mathbb{R}^2)$ and therefore satisfies the integral identity

$$\int_{\mathbb{R}^2} \big(\nabla u_\varepsilon\nabla\varphi + (W+V_\varepsilon-\zeta)u_\varepsilon\varphi\big)\,dx = \int_{\mathbb{R}^2} f\varphi\,dx, \qquad \varphi \in W_2^1(\mathbb{R}^2). \quad (5.17)$$

To show that Y_ε is an adequate approximation to u_ε, introduce the functional

$$F_\varepsilon(\varphi) = \int_{\mathbb{R}^2} \left(\nabla Y_\varepsilon\nabla\varphi + (W+V_\varepsilon-\zeta)Y_\varepsilon\varphi\right)\,dx - \int_{\mathbb{R}^2} f\varphi\,dx, \quad (5.18)$$

defined for functions φ belonging to $W_2^1(\mathbb{R}^2)$ and prove that its norm is infinitely small as $\varepsilon \to 0$.

Lemma 2. *The functional F_ε satisfies the estimate*

$$|F_\varepsilon(\varphi)| \leqslant c\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}$$

for all $\varphi \in W_2^1(\mathbb{R}^2)$.

Proof. Let us rewrite F_ε into a more detailed form

$$\begin{aligned} F_\varepsilon(\varphi) = \int_{\omega_\varepsilon} \Big(\nabla(v_0 + \varepsilon v_1^\varepsilon)\nabla\varphi + (W + V_\varepsilon - \zeta)(v_0 + \varepsilon v_1^\varepsilon)\varphi\Big)\,dx \\ + \int_{\Omega_\varepsilon} \Big(\nabla(u + \rho_\varepsilon)\nabla\varphi + (W - \zeta)(u + \rho_\varepsilon)\varphi\Big)\,dx - \int_{\mathbb{R}^2} f\varphi\,dx. \end{aligned} \tag{5.19}$$

With notation $\varphi_\varepsilon(s,t) = \varphi(s,\varepsilon t)$, we have

$$\begin{aligned} F_\varepsilon(\varphi) = \varepsilon^{-1} &\int_{Q_1} \big(\partial_t v_0\,\partial_t\varphi_\varepsilon + V v_0\varphi_\varepsilon\big) J_\varepsilon\,dt\,ds \\ &+ \int_{Q_1} \big(\partial_t v_1^\varepsilon\,\partial_t\varphi_\varepsilon + V v_1^\varepsilon\varphi_\varepsilon + U v_0\psi_\varepsilon\big) J_\varepsilon\,dt\,ds \\ &+ \int_{\Omega_\varepsilon} \big(\nabla u\nabla\varphi + (W - \zeta)u\varphi\big)\,dx - \int_{\Omega_\varepsilon} f\varphi\,dx \\ &+ \int_{\Omega_\varepsilon} \big(\nabla\rho_\varepsilon\nabla\varphi + (W - \zeta)\rho_\varepsilon\varphi\big)\,dx \\ &+ \varepsilon\int_{Q_1} \partial_s v_0\,\partial_s\varphi_\varepsilon\,J_\varepsilon\,dt\,ds + \varepsilon^2\int_{Q_1} \partial_s v_1^\varepsilon\,\partial_s\varphi_\varepsilon\,J_\varepsilon\,dt\,ds \\ &+ \varepsilon\int_{Q_1} (W - \zeta)(v_0 + \varepsilon v_1^\varepsilon)\varphi_\varepsilon\,J_\varepsilon\,dt\,ds, \end{aligned}$$

by (5.11), (5.15) and (5.16). Let us replace the first and second integrals by the right-hand sides of (5.13) and (5.14) with $\psi_\varepsilon(s,t) = \varphi(s,\varepsilon t)$ respectively, and the difference between the third and fourth ones by the right-hand side of (5.12). The other terms in the last formula are small as $\varepsilon \to 0$, because Lemma 1 and estimates (5.6) provide the bounds

$$\begin{aligned} \left|\int_{\Omega_\varepsilon} \big(\nabla\rho_\varepsilon\nabla\varphi + (W - \zeta)\rho_\varepsilon\varphi\big)\,dx\right| &\leqslant c_1\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}, \\ \left|\int_{Q_1} \partial_s v_0\,\partial_s\varphi_\varepsilon\,J_\varepsilon\,dt\,ds\right| &\leqslant c_2\|\varphi\|_{W_2^1(\mathbb{R}^2)}, \\ \left|\int_{Q_1} \partial_s v_1^\varepsilon\,\partial_s\varphi_\varepsilon\,J_\varepsilon\,dt\right| &\leqslant c_3\varepsilon^{-1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}, \\ \left|\int_{Q_1} (W - \zeta)(v_0 + \varepsilon v_1^\varepsilon)\varphi_\varepsilon\,J_\varepsilon\,dt\,ds\right| &\leqslant c_4\|\varphi\|_{W_2^1(\mathbb{R}^2)}. \end{aligned}$$

Therefore

$$
\begin{aligned}
F_\varepsilon(\varphi) = \int_{Q_1} \varkappa\, \partial_t v_0\, \varphi_\varepsilon\, dtds - \int_{Q_1} \varkappa\, \partial_t v_0 \varphi_\varepsilon J_\varepsilon\, dt\, ds + \varepsilon \int_{Q_1} \varkappa\, \partial_t v_1^\varepsilon\, \varphi_\varepsilon\, dt\, ds \\
+ \int_S \big(\beta_\varepsilon^+(s)\varphi(s,\varepsilon)J(s,\varepsilon) - \beta_\varepsilon^-(s)\varphi(s,-\varepsilon)J(s,-\varepsilon)\big)\, ds \\
- \int_S (\partial_\nu u(s,\varepsilon)\varphi(s,\varepsilon) - \partial_\nu u(s,-\varepsilon)\varphi(s,-\varepsilon))\, ds + r_\varepsilon(\varphi),
\end{aligned}
$$

where $|r_\varepsilon(\varphi)| \leqslant c\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}$. Next, F_ε in turn rearranges to become

$$
\begin{aligned}
F_\varepsilon(\varphi) = \int_S \big(\beta_\varepsilon^+(s) - \partial_\nu u(s,\varepsilon)\big)\varphi(s,\varepsilon)\, ds \\
- \int_S \big(\beta_\varepsilon^-(s) - \partial_\nu u(s,-\varepsilon)\big)\varphi(s,-\varepsilon)\, ds \\
+ \varepsilon \int_{Q_1} \varkappa\big(t\partial_t v_0 + \partial_t v_1^\varepsilon\big)\varphi_\varepsilon\, dt\, ds \\
- \varepsilon \int_S \varkappa(s)\big(\beta_\varepsilon^+(s)\varphi(s,\varepsilon) - \beta_\varepsilon^-(s)\varphi(s,-\varepsilon)\big)\, ds + r_\varepsilon(\varphi).
\end{aligned}
$$

Thus

$$
\begin{aligned}
F_\varepsilon(\varphi) = \int_S \big(\beta_\varepsilon^+(s) - \partial_\nu u(s,\varepsilon)\big)\varphi(s,\varepsilon)\, ds \\
- \int_S \big(\beta_\varepsilon^-(s) - \partial_\nu u(s,-\varepsilon)\big)\varphi(s,-\varepsilon)\, ds + q_\varepsilon(\varphi),
\end{aligned}
$$

where $|q_\varepsilon(\varphi)| \leqslant c\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}$. But then (5.2), (5.5) and Proposition 3 imply

$$
\begin{aligned}
&\Big|\int_S \big(\beta_\varepsilon^\pm(s) - \partial_\nu u(s,\pm\varepsilon)\big)\varphi(s,\pm\varepsilon)\, ds\Big| \\
&\leqslant \Big|\int_S \big(\beta_\varepsilon^\pm(s) - \partial_\nu u_\pm\big)\varphi(s,\pm\varepsilon)\, ds\Big| + \Big|\int_S \big(\partial_\nu u(s,\pm\varepsilon) - \partial_\nu u_\pm\big)\varphi(s,\pm\varepsilon)\, ds\Big| \\
&\leqslant \|\beta_\varepsilon^\pm - \partial_\nu u_\pm\|_{W_2^{-1/2}(\gamma)} \|\varphi(\,\cdot\,,\pm\varepsilon)\|_{W_2^{1/2}(\gamma)} \\
&+ \|\partial_\nu u(\,\cdot\,,\pm\varepsilon) - \partial_\nu u_\pm\|_{L_2(\gamma)}\, \|\varphi(\,\cdot\,,\pm\varepsilon)\|_{L_2(\gamma)} \leqslant c\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}.
\end{aligned}
$$

Therefore $|F_\varepsilon(\varphi)| \leqslant c\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}$ for all $\varphi \in W_2^1(\mathbb{R}^2)$, and the lemma follows. □

5.3. **The End of the Proof.** From (5.17) and (5.18) we see

$$
\int_{\mathbb{R}^2} \nabla(Y_\varepsilon - u_\varepsilon)\nabla\varphi\, dx + \int_{\mathbb{R}^2} (W + V_\varepsilon - \zeta)(Y_\varepsilon - u_\varepsilon)\varphi\, dx = F_\varepsilon(\varphi),
$$

for all $\varphi \in W_2^1(\mathbb{R}^2)$.

If $\varphi = \overline{Y_\varepsilon - u_\varepsilon}$, then

$$
\int_{\mathbb{R}^2} |\nabla(Y_\varepsilon - u_\varepsilon)|^2\, dx + \int_{\mathbb{R}^2} (W + V_\varepsilon - \zeta)|Y_\varepsilon - u_\varepsilon|^2\, dx = F_\varepsilon(\overline{Y_\varepsilon - u_\varepsilon}).
$$

$$-\operatorname{Im}\zeta\int_{\mathbb{R}^2}|Y_\varepsilon-u_\varepsilon|^2\,dx=\operatorname{Im}F_\varepsilon(\overline{Y_\varepsilon-u_\varepsilon}).$$

$$\int_{\mathbb{R}^2}|Y_\varepsilon-u_\varepsilon|^2\,dx\leqslant|\operatorname{Im}\zeta|^{-1}|F_\varepsilon(\overline{Y_\varepsilon-u_\varepsilon})|\leqslant c_1\varepsilon^{1/2}\|Y_\varepsilon-u_\varepsilon\|_{W_2^1(\mathbb{R}^2)}\leqslant c_2\varepsilon^{1/4}$$

Lemma 3. *If potential V has a zero mean, then the estimate*

$$\left|\varepsilon^{-2}\int_{\omega_\varepsilon}V(\tfrac{r}{\varepsilon})|\varphi|^2\,dx\right|\leqslant c\varepsilon^{-1/2}\|\varphi\|^2_{W_2^1(\mathbb{R}^2)}$$

holds for all $\varphi,\psi\in W_2^1(\mathbb{R}^2)$.

Proof.

$$\begin{aligned}\varepsilon^{-2}\left|\int_{\omega_\varepsilon}V(\tfrac{r}{\varepsilon})\varphi\psi\,dx\right|&=\varepsilon^{-1}\left|\int_{Q_1}V(t)\varphi(s,\varepsilon t)\psi(s,\varepsilon t)(1-\varepsilon t\varkappa(s))\,dt\,ds\right|\\&\leqslant\varepsilon^{-1}\left|\int_{Q_1}V(t)\varphi(s,\varepsilon t)\psi(s,\varepsilon t)\,dt\,ds\right|+c_1\|\varphi\|_{W_2^1(\mathbb{R}^2)}\|\psi\|_{W_2^1(\mathbb{R}^2)}\end{aligned}$$

$$\begin{aligned}&\left|\int_{Q_1}V(t)\varphi(s,\varepsilon t)\psi(s,\varepsilon t)\,dt\,ds\right|\\&\quad=\left|\int_{Q_1}V(t)\left(\varphi(s,0)+\int_0^{\varepsilon t}\partial_t\varphi(s,\tau)\,d\tau\right)\right.\\&\qquad\times\left.\left(\psi(s,0)+\int_0^{\varepsilon t}\partial_t\psi(s,\tau)\,d\tau\right)dt\,ds\right|\\&\quad\leqslant\left|\int_{Q_1}V(t)\varphi(s,0)\int_0^{\varepsilon t}\partial_t\psi(s,\tau)\,d\tau\,dt\,ds\right|\\&\qquad+\left|\int_{Q_1}V(t)\psi(s,0)\int_0^{\varepsilon t}\partial_t\varphi(s,\tau)\,d\tau\,dt\,ds\right|\\&\qquad\quad+\left|\int_{Q_1}V(t)\int_0^{\varepsilon t}\partial_t\varphi(s,\tau)\,d\tau\int_0^{\varepsilon t}\partial_t\psi(s,\tau)\,d\tau\,dt\,ds\right|\end{aligned}$$

$$\begin{aligned}&\left|\int_{Q_1}V(t)\varphi(s,0)\int_0^{\varepsilon t}\partial_t\psi(s,\tau)\,d\tau\,dt\,ds\right|\\&\quad\leqslant c_1\left(\int_{Q_1}|\varphi(s,0)|^2\,dt\,ds\right)^{1/2}\left(\int_{Q_1}\left|\int_0^{\varepsilon t}\partial_t\psi(s,\tau)\,d\tau\right|^2\,dt\,ds\right)^{1/2}\\&\quad\leqslant c_1\|\varphi\|_{W_2^1(\mathbb{R}^2)}\left(\int_{Q_1}\left|\int_0^{\varepsilon t}d\tau\right|\left|\int_{-1}^{1}|\partial_t\psi|^2\,d\tau\right|\,dt\,ds\right)^{1/2}\\&\quad\leqslant c_1\varepsilon^{1/2}\|\varphi\|_{W_2^1(\mathbb{R}^2)}\|\psi\|_{W_2^1(\mathbb{R}^2)}\end{aligned}$$

□

Weinstein Excitation thresholds for nonlinear localized modes on lattices (1999)

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