The Negative Discrete Spectrum of a Two-Dimensional Schrödinger Operator

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Abstract

In this paper we study the asymptotics of the discrete negative spectrum of a two-dimensional Schrödinger operator with a large coupling constant. In particular, some "nonstandard" formulae are obtained. © 1996 John Wiley & Sons, Inc.

1. Introduction

1.1. The Case $d \geqq 3$

For a Schrödinger operator

$$-\Delta - \alpha V(x), \qquad \alpha > 0,\ x \in \mathbb{R}^d, \tag{1.1}$$

we denote by $N(\alpha, -\gamma^2, V)$ the number of eigenvalues lying on the left of point $\lambda = -\gamma^2$, $\gamma \geqq 0$,

$$N(\alpha, V) := N(\alpha, 0, V). \tag{1.2}$$

If $d \geqq 3$ then the asymptotic formulae for $N(\alpha, -\gamma^2, V)$ when $\alpha \to \infty$ has been well studied. There are particularly complete results in the Weyl (semiclassical) case,[1] that is, under the condition

$$V \in L_{d/2}(\mathbb{R}^d), \qquad d \geqq 3. \tag{1.3}$$

The following Cwikel-Lieb-Rozenblum estimate is well known (see [11], [10], [9], and also [12] and [4]).

$$N(\alpha, -\gamma^2, V) \leqq C(d)\alpha^{d/2} \int V_+^{d/2} dx, \qquad \gamma \geqq 0,\ d \geqq 3, \tag{1.4}$$

and it is accompanied by the Weyl type asymptotics

$$\lim_{\alpha \to \infty} \alpha^{-d/2} N(\alpha, -\gamma^2, V) = (2\pi)^{-d}\omega_d \int V_+^{d/2} dx, \qquad \gamma \geqq 0,\ d \geqq 3, \tag{1.5}$$

[1] Concerning the "non-Weyl" asymptotics see [5].

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$$\omega_d := \operatorname{vol}\{x \in \mathbb{R}^d : |x| < 1\}.$$

Let us assume that $V(x) \geqq 0$. We see that

(1) Asymptotics (1.5) is valid under only one condition (1.3) which is equivalent to the finiteness of the asymptotic coefficient.

(2) Asymptotics (1.5) is valid *simultaneously* for all $\gamma \geqq 0$.

Moreover, the following property is pointed out in [11] (see also [4]):

(3) If for some $\gamma \geqq 0$

$$N(\alpha, -\gamma^2, V) = O(\alpha^{d/2}), \tag{1.6}$$

then (1.3) is fulfilled and likewise (1.4) and (1.5). Therefore the Weyl order $\alpha^{d/2}$ in the estimate (1.6) implies the Weyl asymptotic formula (1.5).

1.2. Specificity of the Case $d = 2$

The main purpose of this paper is to show that the spectral properties of the operator (1.1) which were described above are not true in the case $d = 2$. Namely let us assume again that $V \geqq 0$. Then for $d = 2$ we have

(1) The condition $V \in L_1(\mathbb{R}^2)$ is not enough to justify the Weyl asymptotics

$$\lim_{\alpha\to\infty} \alpha^{-1} N(\alpha, V) = \frac{1}{4\pi}\int V dx. \tag{1.7}$$

(2) Weyl asymptotics

$$\lim_{\alpha\to\infty} \alpha^{-1} N(\alpha, -\gamma^2, V) = \frac{1}{4\pi}\int V dx, \qquad \gamma > 0, \tag{1.8}$$

might be valid, however, at the same time $N(\alpha, V) \sim c\alpha^p$, $p > 1$. (There are examples for an arbitrary $p > 1$.)

(3) It might happen that

$$\lim_{\alpha\to\infty} \alpha^{-1} N(\alpha, V) = \frac{1}{4\pi}\int V dx + \beta, \qquad \beta > 0, \tag{1.9}$$

that is Weyl's order of asymptotics does not guarantee the Weyl formula (1.7).

Everything mentioned here follows from our main Theorem 5.1 and its applications in Sections 6.1 and 6.2.

All the phenomena described above depend on the behavior of the operator (1.1) on functions $u(x) = u(|x|)$. Weyl's asymptotics is always defined by the behavior of V on compact sets. It "competes" with the "non-Weyl" contribution defined by an auxiliary problem on semiaxis with the potential

$$\mathcal{V}(r) = \frac{1}{2\pi}\int_{-\pi}^{\pi} V(r,\theta)\, d\theta. \tag{1.10}$$

Moreover, everything is determined by the behavior of $\mathcal{V}(r)$ at infinity. The contribution of the potential (1.10) to formula (1.9) is additive. It can have the order α^p, $p > 1$, however, and then it dominates the Weyl term. For more detailed discussion see Section 6.

1.3. Contents

While studying spectral asymptotics for the Dirac operator (see [3]) the authors already met with the phenomenon where the Weyl (high energy) asymptotics "competed" with the non-Weyl (threshold with respect to energy) asymptotics. The cause of this competition was more explicit there and well observed in the momentum representation.

In the present paper we use a variational technique in coordinate representation. In Section 2 the corresponding variational problem is converted (in the spirit of the paper [1]) to a variational problem with discrete spectrum. In Section 3 we give some general preparatory material and in Section 4 we discuss properties of an auxiliary problem on semiaxis. In Section 5 the main Theorem 5.1 is proved. We give some comments in Section 6 and also prove Theorem 6.3 which deals with the complementary contribution from a local singularity. Finally, in Section 7, it is shown that similar effects when $d \geqq 3$ can be observed for the operator

$$-\Delta - (d-2)^2/4|x|^2 - \alpha V(x).$$

Our main results are formulated in terms of upper and lower limits for power type asymptotics for the counting function of an auxiliary operator on a semiaxis. We could have given more general (nonpower type) statements using the results of [15] and [16] concerning corresponding operator ideals and function ideals. Such generalizations can be obtained automatically, and we do not give them, avoiding cumbersome formulations.

1.4. The Main Notations

Let us fix notations. Assuming that $\mathfrak{x} = \{\xi_k\}$ is a complex sequence we denote by $\|\mathfrak{x}\|_p$, $0 < p \leqq \infty$, its (quasi)norm in l_p. We also need a notation for the quasinorm in the "weak l_p-class" $l_{p,\infty}$

$$\|\mathfrak{x}\|_{p,\infty}^p := \sup_{s>0} s^p \operatorname{card}\{k : |\xi_k| > s\}, \quad 0 < p < \infty.$$

Now let $\mathbb{H}$ be a Hilbert space and T be a compact operator in $\mathbb{H}$. By $s_k(T)$ we denote the sequence of its s-numbers (eigenvalues of the operator $|T| = (T^*T)^{1/2}$). Furthermore

$$n(s,T) := \operatorname{card}\{k : s_k(T) > s\},$$

$$\|T\|_p^p := \sum_k s_k^p(T), \quad 0 < p \leqq \infty, \qquad \|T\|_{p,\infty}^p := \sup_{s>0} s^p n(s,T), \quad 0 < p < \infty,$$

$$\Delta_p(T) := \limsup_{s\to 0} s^p n(s,T), \qquad \delta_p(T) := \liminf_{s\to 0} s^p n(s,T).$$

If $T = T^*$ then we put $2T_\pm = |T| \pm T$,

$$n_\pm(s,T) := n(s,T_\pm),$$

$$\Delta_p^{(\pm)}(T) := \Delta_p(T_\pm), \qquad \delta_p^{(\pm)}(T) := \delta_p(T_\pm).$$

It is clear that if $T = T^*$, then

$$n(s,T) = n_+(s,T) + n_-(s,T),$$

where n_+ is the counting function of the positive (n_- – negative) spectrum of T. By H^1 we denote the Sobolev class of the first order equipped by the standard Hilbert metric. The integral over the whole space is written without indicating the domain of integration. The unit cube in $\mathbb{R}^d$ is denoted by $\mathbb{Q}^d$; C and c shall be different constants whose values are unimportant.

2. Reduction to a Spectral Problem of a Compact Operator

2.1. A Basic Condition

Let us consider a Schrödinger operator H in $L_2(\mathbb{R}^2)$

$$H(\alpha) = -\Delta - \alpha V(x), \qquad \alpha > 0, \tag{2.1}$$

whose real-valued potential V satisfies the following condition. Denote

$$\Omega_{a,b} := \{x \in \mathbb{R}^2 : a < |x| < b\}, \qquad 0 \leqq a < b \leqq \infty, \tag{2.2}$$

$$\Omega_k := \Omega_{a,b}, \qquad a = e^{k-1},\ b = e^k,\ k \in \mathbb{Z}. \tag{2.3}$$

For $Q \in L_{\sigma,\mathrm{loc}}(\mathbb{R}^2 \setminus \{0\})$, $Q(x) \geqq 0$, $\sigma > 1$, we put

$$\eta_k(Q,\sigma) := \left(\int_{\Omega_k} |x|^{2(\sigma-1)} Q^\sigma \, dx\right)^{1/\sigma}, \qquad \sigma > 1,\ k \in \mathbb{Z}. \tag{2.4}$$

If $Q \in L_{\sigma,\mathrm{loc}}(\mathbb{R}^2)$, then we denote

$$\hat{\eta}_0(Q,\sigma) := \left(\int_{|x|<1} Q^\sigma \, dx\right)^{1/\sigma}.$$

Let us consider the sequences

$$\mathfrak{y}(Q,\sigma) := \{\eta_k(Q,\sigma)\}, \quad k \in \mathbb{Z}, \tag{2.5}$$

$$\begin{cases} \hat{\mathfrak{y}}(Q,\sigma) := \{\hat{\eta}_k(Q,\sigma)\}, & k \in \mathbb{Z}_+, \\ \hat{\eta}_k := \eta_k, & k \in \mathbb{N}. \end{cases}$$

We mostly need the sequence $\hat{\mathfrak{y}}$; the full sequence $\mathfrak{y}$ will be used only when the potential has a singularity at zero.

CONDITION 2.1. The potential $V \in L_{\sigma,\mathrm{loc}}(\mathbb{R}^2)$ and

$$\hat{\mathfrak{y}}(|V|,\sigma) \in l_1(\mathbb{Z}_+), \qquad \sigma > 1, \tag{2.6}$$

with some $\sigma > 1$.

It is clear that (2.6) is fulfilled for any $\tilde{\sigma} \in (1,\sigma)$ if it is fulfilled for a $\sigma > 1$. Besides, it follows from (2.6) that

$$V \in L_1(\mathbb{R}^2). \tag{2.7}$$

The precise definition of the operator H under Condition 2.1 is given via its quadratic form, that is the operator in (2.1) is understood as form-sum.

2.2. Decomposition of Hilbert Spaces

Let us denote by $N(\alpha, V)$ the number of negative eigenvalues of the operator (2.1). The variational definition of $N(\alpha, V)$ is the following:

$$N(\alpha, V) = \sup \dim \mathscr{L}, \tag{2.8}$$

$$\int (|\nabla u|^2 - \alpha V |u|^2) dx < 0, \quad u \in \mathscr{L} \setminus \{0\}, \tag{2.9}$$

$$\mathscr{L} \subset H^1(\mathbb{R}^2). \tag{2.10}$$

It is essential that (2.8) holds true if we replace (2.10) by the more restrictive condition on the subspace $\mathscr{L}$

$$\mathscr{L} \subset C_0^\infty(\mathbb{R}^2). \tag{2.11}$$

Later we shall denote by r, θ the polar coordinates of $x \in \mathbb{R}^2$ and write $u(x) = u(r,\theta)$. Our standard notation is

$$u = z + g, \tag{2.12}$$

where

$$2\pi z(r) = \int_{-\pi}^{\pi} u(r,\theta) d\theta, \quad r\text{-a.e.} \tag{2.13}$$

$g = u - z$. It is clear that

$$\int_{-\pi}^{\pi} g(r,\theta)d\theta = 0, \quad r\text{-a.e.}, \tag{2.14}$$

$$\int_{-\pi}^{\pi} |\nabla_x u|^2 d\theta = \int_{-\pi}^{\pi} |\nabla_x g|^2 d\theta + 2\pi |z'(r)|^2, \quad r\text{-a.e.} \tag{2.15}$$

Let us notice that *the decomposition* (2.12), (2.13) *induces the respective orthogonal decomposition of the Hilbert spaces* $L_2(\mathbb{R}^2)$ *and* $H^1(\mathbb{R}^2)$.

Let us fix a real number $R > 0$ and, taking into account the decomposition (2.12), (2.13), we assume that functions u in (2.9) satisfy the additional condition

$$z(R) = 0, \quad R > 0. \tag{2.16}$$

Then the value (2.8) can only decrease but not more than by one. Denote this new value by $N_R(\alpha, V)$. Then

$$N_R(\alpha, V) \leqq N(\alpha, V) \leqq N_R(\alpha, V) + 1. \tag{2.17}$$

It is obvious that we only need to study the asymptotic behavior of $N_R(\alpha, V)$.

2.3. Transformation to a Compact Operator

It is well-known (see, for example, [1]) that the completion of the class $C_0^\infty(\mathbb{R}^2)$ with respect to the metric defined by the form $\int |\nabla u|^2 dx$ does not lead to a space of functions (this form degenerates on functions equal to constants). The situation changes if we can "take off" the degeneration of this form by introducing a condition of rank one. It is convenient for us to use (2.16), so that the completion mentioned above leads to a Hilbert space of functions

$$\mathcal{H}_R^1(\mathbb{R}^2) := \{u \in H^1_{\text{loc}}(\mathbb{R}^2) : \nabla u \in L_2(\mathbb{R}^2),\ z(R) = 0\}, \quad R > 0, \tag{2.18}$$

with the metric form $\int |\nabla u|^2 dx$.

Let us denote by $T = T(V, R)$ the *compact* self-adjoint operator in $\mathcal{H}_R^1(\mathbb{R}^2)$, generated by the quadratic form $\int V|u|^2 dx$. The counting function $n_+(\cdot, T)$ of its positive spectrum coincides with the counting function of the sequential maximums of the following quotient of forms

$$\frac{\int V|u|^2 dx}{\int |\nabla u|^2 dx}, \quad u \in \mathcal{H}_R^1(\mathbb{R}^2). \tag{2.19}$$

The quotient (2.19) can be restricted on functions $u \in C_0^\infty(\mathbb{R}^2)$, $z(R) = 0$. Thus

$$\begin{cases} n_+(s, T) = \sup \dim \mathcal{L}, \\ \int V|u|^2 dx / \int |\nabla u|^2 dx > s, \qquad u \in \mathcal{L} \setminus \{0\},\ s > 0, \\ \mathcal{L} \subset \{u \in C_0^\infty(\mathbb{R}^2) : z(R) = 0\}. \end{cases} \tag{2.20}$$

It is clear that (2.8), (2.9), and (2.11) (under the additional condition (2.16)) coincide with (2.20) if $\alpha s = 1$. Therefore

$$N_R(\alpha, V) = n_+(s, T(V, R)), \qquad \alpha s = 1, \ R > 0,$$

and we have obtained the following lemma.

LEMMA 2.2. *The number $N(\alpha, V)$ of negative eigenvalues of operator* (2.1) *satisfies inequalities* (2.17). *The value $N_R(\alpha, V)$ coincides with the counting function $n_+(s, (2.19))$ of the sequential maximums of the quotient* (2.19), *where $\alpha s = 1$.*

AGREEMENT 2.3. We have used the notation $n_+(s, (2.19))$ instead of the notation $n_+(s, T(V, R))$. The majority of the future arguments will be in variational terms and we shall use similar notations systematically. These notations also concern the functionals Δ_p, $\Delta_p^{(\pm)}$, δ_p, $\delta_p^{(\pm)}$ and operator norms $\|\cdot\|_p$, $\|\cdot\|_{p,\infty}$.

3. Auxiliary Statements

We collect here preliminary material which prevents us from being distracted whilst proving the main result. The formulated lemmas are rather simple but we shall prove them for the sake of completeness.

3.1. A Hardy-Type Inequality

Let us use notations (2.2) and (2.3) and introduce the Hilbert space

$$(3.1) \quad G(\Omega_{a,b}) = \Big\{ g \in H^1_{\mathrm{loc}}(\Omega_{a,b}) : \nabla g \in L_2(\Omega_{a,b}); \\ \int_{-\pi}^{\pi} g(r, \theta) d\theta = 0 \ \text{ a.e. } r \in (a, b) \Big\}, \quad 0 \leqq a < b \leqq \infty,$$

with the metric form

$$(3.2) \qquad \int_{\Omega_{a,b}} |\nabla g|^2 dx.$$

PROPOSITION 3.1. *The following inequality holds true*

$$(3.3) \qquad \int_{\Omega_{a,b}} |x|^{-2} |g|^2 dx \leqq \int_{\Omega_{a,b}} |\nabla g|^2 dx, \quad g \in G(\Omega_{a,b}).$$

Proof: It is clear that $g(r, \cdot)$ is an absolutely continuous function for almost all $r \in (a, b)$ and $g'_\theta(r, \cdot) \in L_2(-\pi, \pi)$. The decomposition of function g into the

Fourier series does not contain the free term. This fact together with Parseval identity implies

$$\int_{-\pi}^{\pi} |g(r,\theta)|^2 d\theta \leqq \int_{-\pi}^{\pi} |g'_\theta(r,\theta)|^2 d\theta .$$

It is enough now to integrate both sides of the last inequality over (a,b) with respect to the measure $r^{-1}dr$ and notice that $r^{-2}|g'_\theta|^2 \leqq |\nabla g|^2$. The proposition is proved.

It follows from (3.3) that the forms (3.2) and

$$\int_{\Omega_{a,b}} (|\nabla g|^2 + |x|^{-2}|g|^2)dx \tag{3.4}$$

define equivalent metrics in $G(\Omega_{a,b})$.

Let us consider the Hilbert space of functions

$$\mathscr{H}^1(\Omega_{a,b}) = \left\{ u \in H^1_{\text{loc}}(\Omega_{a,b}) : \int_{\Omega_{a,b}} (|\nabla u|^2 + |x|^{-2}|u|^2)dx < \infty \right\}, \tag{3.5}$$

where the metric form is defined by the integral in (3.5). It is clear that for $0 < a < b < \infty$ we obtain $\mathscr{H}^1(\Omega_{a,b}) = H^1(\Omega_{a,b})$ element-wise and their norms are equivalent. If one or both of the equalities $a = 0$ or $b = \infty$ are fulfilled, however, then this fact is already incorrect. In this case the following statement becomes informative:

PROPOSITION 3.2. *The space $G(\Omega_{a,b})$, $0 \leqq a < b \leqq \infty$, which is equipped by the metric form* (3.4), *is an (eigen)subspace in $\mathscr{H}^1(\Omega_{a,b})$.*

3.2. A Spectral Estimate

Let $Q \in L_{\sigma,\text{loc}}(\Omega_{R,\infty})$, $Q(x) \geqq 0$, $\sigma > 1$ and $R > 0$. We study the quotient of quadratic forms

$$\frac{\int_{|x|>R} Q|u|^2 dx}{\int_{|x|>R}(|\nabla u|^2 + |x|^{-2}|u|^2)dx}, \quad u \in \mathscr{H}^1(\Omega_{R,\infty}). \tag{3.6}$$

LEMMA 3.3. *Let us assume that in* (3.6) $\log R =: M \in \mathbb{Z}$. *Then for the counting function of spectrum (sequential maximums) of the quotient* (3.6) *we have the estimate*

$$n(s,(3.6)) \leqq C(\sigma)s^{-1} \sum_{k \geqq M+1} \eta_k(Q,\sigma), \qquad \sigma > 1 . \tag{3.7}$$

Proof:[2] Let us consider the quotients of the type (3.6) for domains Ω_k

$$\frac{\int_{\Omega_k} Q|u|^2dx}{\int_{\Omega_k}(|\nabla u|^2 + |x|^{-2}|u|^2)dx}, \qquad u \in H^1(\Omega_k),\ k \geqq M+1. \tag{3.8.k}$$

It follows from the variational principle that

$$n(s,(3.6)) \leqq \sum_{k\geqq M+1} n(s,(3.8.k)). \tag{3.9}$$

Substituting $x = e^{k-1}y$ we reduce (3.8.k) to

$$\frac{\int_{\Omega_1} Q_k|u|^2dy}{\int_{\Omega_1}(|\nabla u|^2 + |y|^{-2}|u|^2)dy}, \quad u \in H^1(\Omega_1), \tag{3.10.k}$$

where $Q_k(y) = e^{2(k-1)}Q(e^{k-1}y)$ and the integration is carried out over the same ring Ω_1. For (3.10.k) we use the well-known estimate ([4] and [6]). Then

$$\begin{aligned} n(s,(3.8.k)) = n(s,(3.10.k)) &\leqq C(\sigma)s^{-1}\left(\int_{\Omega_1} Q_k^\sigma dy\right)^{1/\sigma} \\ &= C(\sigma)s^{-1}\left(e^{2(k-1)(\sigma-1)}\int_{\Omega_k} Q^\sigma dx\right)^{1/\sigma} \\ &\leqq C(\sigma)s^{-1}\left(\int_{\Omega_k} |x|^{2(\sigma-1)}Q^\sigma dx\right)^{1/\sigma} = C(\sigma)s^{-1}\eta_k(Q,\sigma). \end{aligned}$$

To finish the proof it only remains to use (3.9).

By analogy with the previous lemma we can establish a spectral estimate for the quotient

$$\frac{\int_{|x|<R} Q|u|^2dx}{\int_{|x|<R}(|\nabla u|^2 + |x|^{-2}|u|^2)dx}, \quad u \in \mathscr{H}^1(\Omega_{0,R}), \tag{3.11}$$

under the assumption $Q \in L_{\sigma,\mathrm{loc}}(\Omega_{0,R})$, $Q(x) \geqq 0$, $\sigma > 1$ and $R > 0$. Namely the following statement holds.

LEMMA 3.4. *Let us assume that in* (3.11) $\log R =: m \in \mathbb{Z}$. *Then for the counting function of the spectrum of the quotient* (3.11) *we have*

$$n(s,(3.11)) \leqq C(\sigma)s^{-1}\sum_{k\leqq m}\eta_k(Q,\sigma), \quad \sigma > 1. \tag{3.12}$$

[2] This argument is borrowed from [5], section 6.

The estimates (3.7) and (3.12) are of course only informative when the series on the right-hand sides converge.

4. An Auxiliary Spectral Problem on Semiaxis

4.1. Statement of the Problem

The following spectral problem on semiaxis plays the most important role in what follows below. It is convenient for us to write it in variational terms. Let us consider the quotient of quadratic forms

$$\frac{\int_A^\infty F(t)|\omega(t)|^2 dt}{\int_A^\infty |\omega'(t)|^2 dt}, \quad \omega(A) = 0, \quad A \geqq 0, \tag{4.1}$$

where function F is real and integrable on an arbitrary interval (A, A_1), $A_1 < \infty$.

PROPOSITION 4.1. *The functionals*

$$\Delta_p(4.1), \quad \delta_p(4.1), \quad \Delta_p^\pm(4.1), \quad \delta_p^\pm(4.1), \qquad p > 1/2, \tag{4.2}$$

are independent of $A \geqq 0$.

Proof: Let $A_1 > A$. Let us set for the quotient (4.1) the additional condition $\omega(A_1) = 0$. Functions $n(\cdot, (4.1))$ and $n_\pm(\cdot, (4.1))$ can only decrease but not more than by one. Then the variational problem (4.1) turns into the orthoganal sum of similar problems for the intervals (A, A_1) and (A_1, ∞). For the finite interval the respective counting function $n(s)$ can be estimated as $n(s) = O(s^{-1/2})$. This follows from the integrability of F on the interval (A, A_1) and agrees with the standard (Weyl) order of growth. Therefore if $2p > 1$, then the values (4.2) are only defined by the problem (4.1) on the semiaxis (A_1, ∞). The proof is complete.

4.2. Upper Bound

From now on let us agree to extend function F by zero on interval $(0, A)$ if $A > 0$. We denote

$$D_0 = (0,1), \quad D_k = (e^{k-1}, e^k), \qquad k \in \mathbb{N},$$

$$\zeta_0(F) = \int_{D_0} |F(t)| dt, \quad \zeta_k(F) = \int_{D_k} t|F(t)| dt, \qquad k \in \mathbb{N},$$

$$\mathfrak{z}(F) = \{\zeta_k(F)\}_{k \geqq 0}, \tag{4.3}$$

$$\nu(s, \mathfrak{z}(F)) = \operatorname{card}\{k \in \mathbb{Z}_+ : \zeta_k(F) > s\}. \tag{4.4}$$

The next statement is contained in Section 6 of the paper [5], but is not singled out. Therefore we give it here accompanied with a short proof. The details can be reconstructed from [5], where there is a discussion of some related problems for a many-dimensional case. (The same discussion concerns Proposition 4.3 which is considered below.)

PROPOSITION 4.2. *The following estimates hold*

$$\|(4.1)\|_{p,\infty} \leqq C(p)\|\mathfrak{z}(F)\|_{p,\infty}, \quad p > 1/2, \tag{4.5}$$

$$\Delta_p(4.1) \leqq C(p)^p \limsup_{s\to\infty} s^p \nu(s, \mathfrak{z}(F)), \quad p > 1/2. \tag{4.6}$$

Proof: (1) Let us assume that (4.5) is established for $A = 0$. Then for $A > 0$ the same estimate can be obtained by extending $F(t)$ and $\omega(t)$ by zero for $t \in (0, A)$. Let $A = e^{m-1}$, $m \in \mathbb{N}$. Then $\zeta_k(F) = 0$ for $k < m$, and from (4.5) we have

$$\begin{aligned}\Delta_p(4.1) \leqq \|(4.1)\|_{p,\infty}^p &\leqq C(p)^p \|\mathfrak{z}(F)\|_{p,\infty}^p \\ &= C(p)^p \sup_{s>0} s^p \operatorname{card}\{k \geqq m : \zeta_k(F) > s\}.\end{aligned}$$

Since $\Delta_p(4.1)$ is independent of A, we obtain (4.6) by taking the limit when $m \to \infty$.

(2) It now remains to prove (4.5) for $A = 0$. This can be done using an interpolation technique. For the initial estimates we use

$$\|(4.1)\| \leqq C_1 \|\mathfrak{z}(F)\|_\infty = C_1 \sup_{k\geqq 0} \zeta_k(F), \tag{4.7}$$

$$\|(4.1)\|_{1/2,\infty} \leqq C_2 \|\mathfrak{z}(F)\|_{1/2} = C_2 \left(\sum_{k\geqq 0} \zeta_k(F)^{1/2}\right)^2. \tag{4.8}$$

Applying to (4.7) and (4.8) the Lions-Peetre interpolation functor $\mathscr{K}_{\theta,\infty}$ with $\theta = 1/2p$ we obtain (4.5).

(3) Let us clarify the proof of (4.7). The Hardy inequality

$$\int_0^\infty t^{-2}|\omega(t)|^2\,dt \leqq 4\int_0^\infty |\omega'(t)|^2\,dt, \quad \omega(0) = 0,$$

allows us to estimate the quotient (4.1) (with $A = 0$) by

$$\frac{5\int_0^\infty |F(t)|\,|\omega(t)|^2\,dt}{\int_0^\infty (|\omega'(t)|^2 + t^{-2}|\omega(t)|^2)\,dt}, \quad \omega(0) = 0. \tag{4.9}$$

Let us consider the quotients

$$\frac{5\int_{D_k}|F(t)||\omega(t)|^2\,dt}{\int_{D_k}(|\omega'(t)|^2+t^{-2}|\omega(t)|^2)\,dt}, \quad k \geqq 0, \tag{4.10.k}$$

where for $k=0$ we preserve the condition $\omega(0)=0$. It can easily be seen that the quotient (4.10.0) is estimated by $5\,\zeta_0(F)$. Analogously the quotient (4.10.1) can be estimated by $c\,\zeta_1(F)$. Using dilatation we see that the estimate of the quotient (4.10.k) with $k \geqq 2$ can be reduced to the case $k=1$. As a result, the quotient (4.10.k), $k \geqq 1$, can be estimated by $c\,\zeta_k(F)$. Adding all these estimates together we obtain (4.7) with $C_1 = \max\{5, c\}$.

(4) The estimate (4.8) can also be obtained by means of quotients (4.10.k). Obviously

$$n_+(s,(4.1)) \leqq n(s,(4.9)) \leqq \sum_{k\geqq 0} n(s,(4.10.k))$$

and the same estimate holds for $n_-(s,(4.1))$. Thus

$$n(s,(4.1)) \leqq 2\sum_{k\geqq 0} n(s,(4.10.k)). \tag{4.11}$$

The first two terms in the right-hand side of (4.11) admit the standard Weyl estimate $n(s,(4.10)) \leqq Cs^{-1/2}\zeta_k(F)^{1/2}$, $k=0,1$. For the values $k \geqq 2$ the same inequality can be obtained (with the same constant C) by reducing the problem to the case $k=1$ using dilatation. The summation of these estimates shows that together with (4.11) we have

$$\sup_{s>0} s^{1/2} n(s,(4.1)) \leqq 2C\sum_{k\geqq 0}\zeta_k(F)^{1/2}. \tag{4.12}$$

The inequality (4.12) is equivalent to (4.8) with $C_2 = 4C^2$. This completes the proof.

4.3. Lower Bound

For $F \geqq 0$ the estimate (4.6) admits inversion. We shall prove somewhat more. Instead of (4.4), however, we have to consider the function

$$\nu_0(s,\mathfrak{z}(F)) = \operatorname{card}\{k \in \mathbb{N} : \zeta_k(F) > s\}.$$

It is clear that $\nu_0(s) \leqq \nu(s) \leqq \nu_0(s)+1$.

PROPOSITION 4.3. *Let us assume that in* (4.1) $F(t) \geqq 0$. *Then for some* $\gamma > 0$ *we have*

$$\nu_0(\gamma s,\mathfrak{z}(F)) \leqq 2n(s,(4.1)), \tag{4.13}$$

and consequently

$$\limsup_{s\to 0} s^p \nu(s, \mathfrak{z}(F)) \leqq 2\gamma^p \Delta_p(4.1), \quad p \geqq 1/2. \tag{4.14}$$

Proof: Let us fix an arbitrary function $f \in C_0^\infty(e^{-1/2}, e^{3/2})$, $f(t) \geqq 0$; $f(t) = 1$ for $t \in (1, e)$. Denote $u_k(t) = f(e^{1-k}t)$, $k \geqq 1$, and notice that the supports u_k and u_{k+2} do not intersect. Besides

$$\int |u_k'|^2 dt = e^{-k}\gamma, \qquad \gamma := \int |f'|^2 dt, \tag{4.15}$$

$$\int F|u_k|^2 dt \geqq \int_{D_k} F dt \geqq e^{-k}\zeta_k(F). \tag{4.16}$$

By restricting our class of function in (4.1) on the set of $u = \sum_{k\geqq 1} \beta_k u_{2k}$ and estimating (4.1) from below using (4.15) and (4.16), we obtain the quotient

$$\frac{\sum_{k\geqq 1} e^{-2k} |\beta_k|^2 \zeta_{2k}(F)}{\gamma \sum_{k\geqq 1} e^{-2k} |\beta_k|^2},$$

where both forms are sums of squares. This implies

$$\operatorname{card}\{k \in \mathbb{N} : \zeta_{2k}(F) > \gamma s\} \leqq n(s, (4.1)). \tag{4.17}$$

Similarly we have the inequality

$$\operatorname{card}\{k \in \mathbb{N} : \zeta_{2k-1}(F) > \gamma s\} \leqq n(s, (4.1)),$$

which together with (4.17) gives (4.13). The proof is complete.

Remark 4.4. In contrast to (4.6) the value $p = 1/2$ is admissible in (4.14). Furthermore it follows from (4.14) that for $F(t) \geqq 0$ the estimate $n(s, (4.1)) = O(s^{-p})$ implies $\mathfrak{z}(F) \in l_{p,\infty}$, $p \geqq 1/2$. The inequality (4.5) cannot be inverted, however, since it is impossible to estimate $\zeta_0(F)$.

4.4. Example

Let us now give an example of a function F whose $n(s, (4.1))$ has the asymptotics

$$n(s, (4.1)) \underset{s\to 0}{\sim} M(p) s^{-p}, \quad p > 1/2. \tag{4.18}$$

Let us put in (4.1)

$$F_p(t) = 2^{1/p-2} t^{-2} (\log t)^{-1/p}, \qquad t > 1,\ 2p > 1. \tag{4.19}$$

Substituting $t = \exp(2\tau)$, $\omega = (\exp\tau)v$ we transfer (4.1) to

$$\frac{\int_0^\infty \tau^{-1/p}|v|^2 d\tau}{\int_0^\infty (|v'|^2 + |v|^2) d\tau}, \quad v(0) = 0. \tag{4.20}$$

For (4.20) we write an analog of the variational inequality (2.20) where we impose the additional condition $v(\tau_0) = 0$, $\tau_0 = s^{-p}$. The problem splits and the respective quotient on the semiaxis (τ_0, ∞) does not contribute to $n(s)$ at all. Finally, we have

$$\frac{\int_0^{\tau_0} \tau^{-1/p}|v|^2 d\tau}{\int_0^{\tau_0} (|v'|^2 + |v|^2) d\tau} > s, \quad v(0) = v(\tau_0) = 0,$$

and after substituting $\tau = \tau_0 y$ we obtain

$$\frac{\int_0^1 (y^{-1/p} - 1)|v|^2 dy}{\int_0^1 |v'|^2 dy} > s^{2p}, \quad v(0) = v(1) = 0.$$

The last problem has the standard Weyl asymptotics of the order $(-1/2)$, but with respect to the parameter s^{2p}. This implies (4.18) where

$$M(p) = \frac{1}{\pi}\int_0^1 (y^{-1/p} - 1)^{1/2} dy = \frac{\Gamma(p - 1/2)}{2\sqrt{\pi}\,\Gamma(p)}. \tag{4.21}$$

Remark 4.5. When $t = 1$ the function (4.19) has a nonsummable singularity for $p \leqq 1$. Its influence, however, is diminished by the boundary condition. This fact is easier to see in the notations of (4.20). According to the corresponding result from [2], the quotient of type (4.20) (but on interval $(0, 1)$ and under condition $\omega(0) = 0$) has the Weyl asymptotics if the function standing in front of $|v|^2$ in the numerator is summable with the weight $\tau^{1-\varepsilon}$ for some $\varepsilon > 0$. In our case this is fulfilled when $2p > 1$. Therefore the singularity does not spoil the estimate $n(s) = (s^{-1/2})$ on an arbitrary finite interval, nor does it contribute to (4.18).

Remark 4.6. The asymptotic formula (4.18), with $M(p)$ defined in (4.21), obviously holds if in (4.1)

$$F(t) = F_p(t)(1 + o(1)), \quad t \to \infty.$$

5. The Main Theorem

5.1. Formulation of the Main Theorem

To formulate the main result we use the following notations. For a potential V we introduce

$$\mathcal{V}(r) := \frac{1}{2\pi}\int_{-\pi}^{\pi} V(r, \theta) d\theta, \quad r \geqq 1, \tag{5.1}$$

$$F_V(t) := e^{2t}\mathscr{V}(e^t), \quad t \geqq 0. \tag{5.2}$$

Denote by $n_+(s, V, A)$ the distribution function of spectrum of the quotient (4.1), where $F = F_V$, and let

$$\Delta_p^{(+)}(V), \quad \delta_p^{(+)}(V), \qquad p \geqq 1, \tag{5.3}$$

be the respective functionals of type (4.2) (they are independent of $A \geqq 0$). Our main result is contained in the following theorem.

THEOREM 5.1. *Let a potential V of the operator* (2.1) *satisfy Condition* 2.1 *and let $N(\alpha, V)$ be the number of negative eigenvalues of the operator* (2.1). *Then we have*

(a) *If*

$$\Delta_1(|V|) < \infty, \tag{5.4}$$

then

$$\begin{cases} \limsup\limits_{\alpha\to\infty} \alpha^{-1}N(\alpha, V) = \frac{1}{4\pi}\int V_+\,dx + \Delta_1^{(+)}(V), \\ \liminf\limits_{\alpha\to\infty} \alpha^{-1}N(\alpha, V) = \frac{1}{4\pi}\int V_+\,dx + \delta_1^{(+)}(V). \end{cases} \tag{5.5}$$

(b) *If*

$$\Delta_p(|V|) < \infty, \quad p > 1, \tag{5.6}$$

then

$$\begin{cases} \limsup\limits_{\alpha\to\infty} \alpha^{-p}N(\alpha, V) = \Delta_p^{(+)}(V), \\ \liminf\limits_{\alpha\to\infty} \alpha^{-p}N(\alpha, V) = \delta_p^{(+)}(V). \end{cases} \qquad (p > 1) \tag{5.7}$$

(c) *For the validity of the Weyl asymptotic formula*

$$\lim_{\alpha\to\infty} \alpha^{-1}N(\alpha, V) = \frac{1}{4\pi}\int V_+\,dx \tag{5.8}$$

it is sufficient to assume that

$$\Delta_1(|V|) = 0. \tag{5.9}$$

If for some $R \geqq 1$

$$V(x) \geqq 0, \quad |x| \geqq R, \tag{5.10}$$

is fulfilled, then (5.9) *is necessary for* (5.8) *to be true.*

Remark 5.2. Let (5.10) hold true and let $\mathfrak{z}(F_V)$ be a sequence (4.3) for $F = F_V$. By Propositions 4.2 and 4.3 the conditions (5.4), (5.6), and (5.9) are fulfilled if and only if we respectively have

$$\mathfrak{z}(F_V) \in l_{1,\infty}\,; \quad \mathfrak{z}(F_V) \in l_{p,\infty}\,,\ p > 1\,; \quad \lim_{s\to 0}\sup s\,\nu(s, \mathfrak{z}(F_V)) = 0\,.$$

5.2. Proof of Theorem 5.1

It follows from Lemma 2.2 that it is enough to establish the asymptotics (5.5), (5.7), and (5.8) where $N(\alpha, V)$ is changed by $n_+(s, (2.19))$, $\alpha s = 1$.

The lower asymptotic estimates. Let us fix $R \geqq 1$ and impose in (2.19) the additional boundary condition $u(R, \theta) = 0$. Then the variational quotient splits. For the exterior circle we, in addition, restrict the set of functions by the condition $u(x) = z(r)$ (see (2.12)). Then the respective quotients of quadratic forms are equal to

$$\frac{\int_{|x|<R} V|u|^2 dx}{\int_{|x|<R} |\nabla u|^2 dx}\,, \quad u(x)|_{|x|=R} = 0\,, \tag{5.11}$$

$$\frac{\int_R^\infty \mathscr{V}|z|^2 r dr}{\int_R^\infty |z'|^2 r dr}\,, \quad z(R) = 0\,. \tag{5.12}$$

Substituting $r = e^t$, $z(r) = \omega(t)$ we tranform the quotient (5.12) to (4.1) with $F = F_V$ and $A = \log R$. Applying the variational principle we find that

$$\begin{aligned} n_+(s, (2.19)) &\geqq n_+(s, (5.11)) + n_+(s, (5.12)) \\ &= n_+(s, (5.11)) + n_+(s, V, A)\,. \end{aligned} \tag{5.13}$$

Since $V \in L_{\sigma,\mathrm{loc}}(\mathbb{R}^2)$, $\sigma > 1$, we obtain that for the quotient (5.11) the following Weyl asymptotic formula holds (see [4] and [6])

$$\lim_{s\to 0} s\, n_+(s, (5.11)) = \frac{1}{4\pi}\int_{|x|<R} V_+\, dx\,. \tag{5.14}$$

Let the inequality (5.4) be fulfilled. We now multiply (5.13) by $s = \alpha^{-1}$ and take the upper and lower limits when $s \to 0$. After this, using (5.14), we take the limit when $R \to \infty$. If we now change $n_+(s, (2.19))$ by $N(\alpha, V)$ then we obtain that

$$\begin{cases} \lim\limits_{\alpha\to\infty} \sup \alpha^{-1} N(\alpha, V) \geqq \frac{1}{4\pi}\int V_+\, dx + \Delta_1^{(+)}(V)\,, \\ \lim\limits_{\alpha\to\infty} \inf \alpha^{-1} N(\alpha, V) \geqq \frac{1}{4\pi}\int V_+\, dx + \delta_1^{(+)}(V)\,. \end{cases} \tag{5.15}$$

Under the condition (5.6) it is enough to use the rougher inequality

$$n_+(s,(2.19)) \geqq n_+(s,V,A)$$

instead of the inequality (5.13). Finally, we have

$$\left\{\begin{array}{l}\limsup_{\alpha\to\infty} \alpha^{-p}N(\alpha,V) \geqq \Delta_p i^{(+)}(V), \\ \liminf_{\alpha\to\infty} \alpha^{-p}N(\alpha,V) \geqq \delta_p i^{(+)}(V),\end{array}\right. \qquad (p>1). \tag{5.16}$$

Formulae (5.15) and (5.16) give us the desired lower estimates.

5.3. The Upper Asymptotic Estimates

The upper asymptotic estimates are somewhat more complicated to obtain. Let us again fix $R \geqq 1$. In (2.19) we extend the set of functions u by "taking off" the condition which forces the traces of function u from inside and outside the circle $|x| = R$ to be the same. Then the problem splits again, but we now obtain from the variational principle the upper estimates. Besides, we impose an additional condition of rank one for the interior problem

$$\int_{|x|<R} u(x)dx = 2\pi \int_0^R z(r)r\,dr = 0,$$

and then cancel the condition $z(R) = 0$. Thus the counting function of the interior problem can change but not more than by one. Therefore we must consider the quotient of the quadratic forms

$$\frac{\int_{|x|<R} V|u|^2dx}{\int_{|x|<R} |\nabla u|^2dx}, \qquad \int_{|x|<R} u(x)\,dx = 0, \tag{5.17}$$

and

$$\frac{\int_{|x|>R} V|u|^2dx}{\int_{|x|>R} |\nabla u|^2dx}, \qquad z(R) = 0. \tag{5.18}$$

The variational principle and Lemma 2.2 imply the estimate

$$N(\alpha,V) \leqq n_+(s,(5.17)) + n_+(s,(5.18)) + 2, \quad \alpha s = 1. \tag{5.19}$$

For the quotient (5.17) using the assumption $V \in L_{\sigma,\mathrm{loc}}$, $\sigma > 1$, we obtain ([4] and [6]) the Weyl asymptotic formula

$$\lim_{s\to 0} s\, n_+(s,(5.17)) = \frac{1}{4\pi}\int_{|x|<R} V_+\,dx \leqq \frac{1}{4\pi}\int V_+\,dx. \tag{5.20}$$

Let us now estimate the quotient (5.18) from above. According to (2.12) and (2.15) the numerator in (5.18) can be estimated by

$$(1+\varepsilon^{-1})\int_{|x|>R}|V||g|^2dx+\int_{|x|>R}(V+\varepsilon|V|)|z|^2dx, \quad \varepsilon>0,$$

and the denominator is equal to $\int_{|x|>R}(|\nabla g|^2+|\nabla z|^2)dx$. Thus we again come to the orthogonal sum of two variational problems

$$\frac{(1+\varepsilon^{-1})\int_{|x|>R}|V||g|^2dx}{\int_{|x|>R}|\nabla g|^2dx}, \quad \int_{-\pi}^{\pi}g(r,\theta)d\theta=0, \tag{5.21}$$

and

$$\frac{\int_{|x|>R}V_\varepsilon|z|^2dx}{\int_{|x|>R}|\nabla z|^2dx}, \quad z=z(r), \quad z(R)=0, \quad V_\varepsilon:=V+\varepsilon|V|. \tag{5.22}$$

This implies

$$n_+(s,(5.18))\leqq n(s,(5.21))+n(s,(5.22)). \tag{5.23}$$

Let us now estimate the quotient (5.21) from above. We use the estimate (3.3) with $a=R$, $b=\infty$ and change the denominator in (5.21) by $\frac{1}{2}\int_{|x|>R}(|\nabla g|^2+r^{-2}|g|^2)dx$. After this we take off in (5.21) the additional condition of orthogonality (2.14) (and respectively denote the functional argument by u instead of g)

$$\frac{2(1+\varepsilon^{-1})\int_{|x|>R}|V||u|^2dx}{\int_{|x|>R}(|\nabla u|^2+|x|^{-2}|u|^2dx}. \tag{5.24}$$

We obtain the following upper estimate

$$n(s,(5.21))\leqq n(s,(5.24)). \tag{5.25}$$

Substituting $R=\exp M$, $M\in\mathbb{N}$, we apply Lemma 3.3 to the quotient (5.24) with $Q=2(1+\varepsilon^{-1})|V|$. Then (3.7) and (5.25) give

$$s\,n(s,(5.21))\leqq 2(1+\varepsilon^{-1})C(\sigma)\sum_{k\geqq M+1}\eta_k(|V|,\sigma), \tag{5.26}$$

and by (2.6) the series in the right-hand side is convergent.

5.4. The Upper Estimates (the End)

Let (5.4) be fulfilled. By comparing (5.19), (5.20), (5.23), and (5.26) we obtain (using shortened notations)

$$\begin{aligned}(5.27)\quad &\lim_{\alpha\to\infty}\left\{\begin{matrix}\sup\\ \inf\end{matrix}\right\}\alpha^{-1}N(\alpha,V)\\ &\leqq\frac{1}{4\pi}\int V_+\,dx+2(1+\varepsilon^{-1})C(\sigma)\sum_{k\geqq M+1}\eta_k(|V|,\sigma)+\left\{\begin{matrix}\Delta_1^{(+)}(V_\varepsilon)\\ \delta_1^{(+)}(V_\varepsilon)\end{matrix}\right\}.\end{aligned}$$

In the same way, under the condition (5.6), we derive

$$(5.28)\qquad \lim_{\alpha\to\infty}\left\{\begin{matrix}\sup\\ \inf\end{matrix}\right\}\alpha^{-p}N(\alpha,V)\leqq\left\{\begin{matrix}\Delta_p^{(+)}(V_\varepsilon)\\ \delta_p^{(+)}(V_e)\end{matrix}\right\},\qquad p>1.$$

The second term in the right-hand side of (5.27) *disappears* when $M\to\infty$. Let us now take the limits in (5.27) and (5.28) when $\varepsilon\to 0$. Denote by $\mathscr{D}_p^{(+)}$ either of the two functionals $\Delta_p^{(+)}$, $\delta_p^{(+)}$. Using the general properties of these functionals (see [7], chapter 11, section 6) we have the inequality

$$(5.29)\qquad |\mathscr{D}_p^{(+)}(V_\varepsilon)^{\frac{1}{p+1}}-\mathscr{D}_p^{(+)}(V)^{\frac{1}{p+1}}|\leqq\Delta_p(\varepsilon|V|)^{\frac{1}{p+1}}=\varepsilon^{\frac{p}{p+1}}\Delta_p(|V|),\quad p\geqq 1.$$

Therefore by letting $\varepsilon\to 0$ we can change V_ε to V in (5.27) and (5.28). As a result we obtain the inequalities which are inverse to (5.15) and (5.16) (respectively under conditions (5.4) and (5.6)). This gives us formulae (5.5) and (5.7).

It is enough now to check the last statement (c) of Theorem 5.1. The asymptotic formula (5.8) follows from (5.9) since we have (5.5). Let (5.10) be fulfilled. The inequality (5.13) implies that

$$n(s,V,\log R)\leqq n_+(s,(2.19)).$$

Therefore (5.8) yields $\Delta_1(V)<\infty$. Using (5.5) and (5.8) we obtain $\Delta_1(V)=0$.

Remark 5.3. Condition 2.1 in Theorem 5.1 can be weakened. Namely, the sequence of weighted L_σ-norms of V over rings in (2.6) can be changed by the sequence of Orlicz norms corresponding to the class $L\log_+L$. The estimates for $N(\alpha,V)$ which are required for this purpose, were obtained in [13]. The decomposition (2.12) was also used in [13] to estimate $N(\alpha,V)$. The contributions of the elements in this decomposition were estimated in terms independent of each other. For $V\geqq 0$ the statement of sufficiency in part (c) of Theorem 5.1 is contained in [14].

6. Complements and Comments to the Main Theorem

6.1. Example to the Main Theorem

Let us give an example which covers all the statements of Theorem 5.1. We consider the potential

$$(6.1)\qquad \begin{cases} V_p^\Phi(x)=2^{1/p-2}\Phi(\theta)\,r^{-2}(\log r)^{-2}(\log\log r)^{-1/p}, & r>e^2,\ 2p>1,\\ V_p^\Phi(x)=0, & r\leqq e^2,\end{cases}$$

under the condition

$$(6.2)\qquad \Phi\in L_\sigma(-\pi,\pi),\quad \sigma>1.$$

The potential (6.1) satisfies condition 2.1 with the same σ as in (6.2). We denote

$$\Phi_0 := \frac{1}{2\pi}\int_{-\pi}^{\pi} \Phi(\theta)d\theta, \qquad 2\Phi_+ := \Phi_0 + |\Phi_0|. \tag{6.3}$$

Let us construct for the potential (6.1) the respective function $F_V(t)$ using formulae (5.1) and (5.2). According to (4.19) we have $F_V(t) = \Phi_0 F_p(t)$, $t \geqq 2$, and can use the asymptotics (4.18), (4.21).

PROPOSITION 6.1. *Let $V \in L_{\sigma,\mathrm{loc}}(\mathbb{R}^2)$, $\sigma > 1$, and*

$$V(x) - V_p^{\Phi}(x) = o\,(V_p^{|\Phi|}(x)), \qquad |x| \to \infty,\ 2p > 1. \tag{6.4}$$

Then the following asymptotic formulae hold

(a) *If in* (6.4) $p = 1$, *then*

$$\lim_{\alpha\to\infty} \alpha^{-1}N(\alpha, V) = \frac{1}{4\pi}\int V_+\,dx + \frac{1}{2}\Phi_+. \tag{6.5}$$

(b) *If $p > 1$ then* (*using notation* (4.21))

$$\lim_{\alpha\to\infty} \alpha^{-p}N(\alpha, V) = \Phi_+^p M(p), \quad p > 1. \tag{6.6}$$

(c) *If $p < 1$ then asymptotics* (5.8) *is fulfilled.*

Proof: For $V = V_p^{\Phi}$ the required statement follows directly from Theorem 5.1 and the asymptotic formula (4.18), (4.21). For V satisfying (6.4), the conditions of Theorem 5.1 are also fulfilled. To be able to compute $\Delta_p^{(+)}(V) = \delta_p^{(+)}(V)$ we have to use estimates which are similar to (5.29); we have to take into account that

$$|V(x) - V_p^{\Phi}(x)| \leqq \varepsilon |V_p^{|\Phi|}(x)|, \qquad |x| > R(\varepsilon), \quad \forall \varepsilon > 0.$$

The proof is complete.

6.2. Discussion of Theorem 5.1

First of all let us make sure that the statements (1)–(3) from 1.1 are not valid if $d = 2$. Indeed:

(1) Condition 2.1 implies (2.7) and thus the coefficient in the Weyl asymptotic formula (5.8) is finite. Moreover asymptotics (5.8) is not true for $V = V_p^{\Phi}$, $\Phi_+ > 0$, $p \geqq 1$.

(2) The "lattice" condition

$$\sum_{n\in\mathbb{Z}^2} \left(\int_{Q^2+n} |V|^{\sigma} dx\right)^{1/\sigma} < \infty, \qquad \sigma > 1, \tag{6.7}$$

follows from Condition 2.1 and (6.7) implies (as was shown in [2]) that

$$\lim_{\alpha\to\infty} \alpha^{-1}N(\alpha, -\gamma^2, V) = \frac{1}{4\pi}\int V_+\,dx, \qquad \gamma > 0. \tag{6.8}$$

However, for an arbitrary $p > 1$ and $V = V_p^\Phi$, $\Phi_+ > 0$, the (non-Weyl) asymptotics[3] (6.6) of the order α^p holds.

(3) For $V = V_1^\Phi$, $\Phi_+ > 0$, the asymptotics (6.5) has the Weyl order α, but the asymptotic coefficient is different from the Weyl coefficient. (It is worth mentioning that if $\Phi_0 \leqq 0$ then the asymptotic formula (6.5) is of the Weyl type).

From the proof of Theorem 5.1 it is easy to see that the contribution to $N(\alpha, V)$ of the radial problem (non-Weyl contribution) is generated in coordinate representation by a neighborhood of infinity. On the contrary, the Weyl contribution is defined by boundary value problems on compacts. Things become different if we take the Fourier transform. It can be shown that the Weyl contribution corresponds to large momenta, but the contribution from the radial problem corresponds to small momenta (threshold effect).

When $d \geqq 3$ the Weyl asymptotic formula for $N(\alpha, V)$ also corresponds to large momenta. A possible non-Weyl asymptotics has a threshold origin. When $d \geqq 3$, however, the "addition" of these effects is impossible at least when $V \geqq 0$. If $d = 2$ then this addition becomes possible. This is described in part (a) of Theorem 5.1 and part (a) of Proposition 6.1.

6.3. About One Generalization

The results of Theorem 5.1 can be automatically carried over to elliptic operators with constant coefficients. Indeed, let us study the operator

$$\nabla_y^* h \nabla_y - \alpha\tilde{V}(y), \qquad \alpha > 0, \quad y \in \mathbb{R}^2, \tag{6.9}$$

where $h > 0$ is a real (2×2)-matrix. The substitution $y = h^{1/2}x$ transforms (6.9) to the operator (2.1) with the potential

$$V(x) = \tilde{V}(h^{1/2}x). \tag{6.10}$$

Let us notice that potentials (6.1) do not preserve their type under this transformation. It is easy to see, however, that for $\tilde{V}(y) = V_p^\Psi(y)$ the potential $V(x)$ in (6.10) satisfies condition (6.4). Generally speaking function Φ in (6.4) does not coincide with the original function Ψ.

6.4. Local Singularities

Let us consider the case where V has a point ($x = 0$) singularity, which violates the inclusion $V \in L_{\sigma,\mathrm{loc}}$ for all $\sigma > 1$. This singularity can have a strong influence

[3] Moreover there are potentials $V \geqq 0$, which satisfy Condition 2.1 for an arbitrary $\sigma > 1$, but the quotient (4.1) with $F = F_V$ generates an unbounded operator. This means that $N(\alpha, V) = \infty$ for any $\alpha > 0$. The authors are obliged to M. Solomyak for this remark.

on the asymptotic behavior of $N(\alpha, V)$ and again the functions $u(x) = u(r)$ play a main role. Instead of only one auxiliary problem of the type (4.1) there is an orthogonal sum of two such problems. The answer can be written in terms of a two-dimensional vector problem (4.1) with the diagonal matrix F. It is more convenient to use somewhat different (equivalent) notations.

Below we denote the functions (5.1) *and* (5.2) *by* $\mathcal{V}^e$ *and* F_V^e, *respectively.* Moreover, let

$$\mathcal{V}^i(r) := \frac{1}{2\pi}\int_{-\pi}^{\pi} V(r,\theta)d\theta, \quad 0 < r \leqq 1,$$
$$F_V^i(t) := e^{-2t}\mathcal{V}^i(e^{-2t}), \quad t \geqq 0.$$

By $n_+^j(s, V, A)$, $j = e, i$, we denote the distribution functions of the positive spectrum of the quotient (4.1) with $F = F_V^j$ and put

$$n_+(s, V, A) = n_+^e(s, V, A) + n_+^i(s, V, A). \tag{6.11}$$

We relate the functionals (5.3) to the distribution functions (6.11) and also the functionals $\Delta_{p,j}^{(+)}(V)$ and $\delta_{p,j}^{(+)}(V)$, $p \geqq 1$, to $n_+^j(s, V, A)$, $j = e, i$. It is clear that all these functionals are independent of A.

Instead of Condition 2.1 we use

CONDITION 6.2. The following inclusion is fulfilled for some $\sigma > 1$

$$\mathfrak{y}(|V|, \sigma) \in l_1(\mathbb{Z}) \quad (\sigma > 1), \tag{6.12}$$

where the sequence $\mathfrak{y}$ is defined in (2.4) and (2.5).

The inclusion (6.12) implies (2.7). Let us formulate an analog of Theorem 5.1 formally keeping the same notations.

THEOREM 6.3. *Let Condition* 6.2 *be fulfilled. The notations of the functionals* $\Delta_p^{(+)}(V)$, $\delta_p^{(+)}(V)$, *and* $\Delta_p(|V|)$ *are related to the distribution function* (6.11) *for* V *and* $|V|$, *respectively. Then the statements* (a) *and* (b) *and the sufficiency part of statement* (c) *of Theorem* 5.1 *are true. If* (5.10) *is supplemented by the condition*

$$V(x) \geqq 0, \quad |x| \leqq R^{-1}, \tag{6.13}$$

then the necessity part of the statement (c) is also valid.

The proof is almost the same as the proof of Theorem 5.1. Therefore we only need to give a brief explanation.

Lemma 2.2 remains valid. It is more convenient to assume that in (2.11) and (2.20) $\mathcal{L} \subset C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ (this does not change the final result). It is also helpful to take into account the fact that the set $C_0^\infty(\mathbb{R}^2 \setminus \{0\})$, restricted by the condition (2.16), is dense in $\mathcal{H}_R^1(\mathbb{R}^2)$.

To obtain both upper and lower estimates we ought to break $\mathbb{R}^2$ into three domains: a circle $\Omega_{0,\rho}$, a ring $\Omega_{\rho,R}$, and the exterior of the circle $\Omega_{R,\infty}$. It is convenient to assume that $\rho R = 1$. For $\Omega_{0,\rho}$ and $\Omega_{R,\infty}$ all the arguments are parallel. In particular, the quotient of forms corresponding to $\Omega_{0,\rho}$ and restricted to the subset of functions $u(x) = z(r)$, after the substitution $r = e^{-t}$, $z(r) = \omega(t)$, becomes (4.1), where $F = F_V^i$, $A = \log R$. After this the estimate (3.3) must be used for both domains $\Omega_{R,\infty}$ and $\Omega_{0,\rho}$. Apart from Lemma 3.3 we ought to use (for $\Omega_{0,\rho}$) Lemma 3.4. Finally, we take into account the obvious inequalities

$$\Delta_p(|V|) \leqq \Delta_p^e(|V|) + \Delta_p^i(|V|) \leqq 2\Delta_p(|V|).$$

This is now enough to conclude that the proof of Theorem 5.1 can be converted into the proof of Theorem 6.3.

6.5. Discussion of Theorem 6.3

Near point $x = 0$ the potential

$$\text{(6.14)} \quad \begin{cases} V_{p,i}^{\Phi}(x) = 2^{1/p-2}\Phi^i(\theta)r^{-2}(\log r)^{-2}(\log\log \frac{1}{r})^{-1/p}, & r < e^{-2},\ 2p > 1, \\ V_{p,i}^{\Phi}(x) = 0, & r \geqq e^{-2} \end{cases}$$

can be considered as an analog of $V_p^{\Phi} = V_{p,e}^{\Phi}$ (see (6.1)). Proposition 6.1 has an obvious generalization in the case when the singularity is placed at $x = 0$. Let us assume that the potential V is satisfied (see (6.4)) as is the analogous condition when $|x| \to 0$. Then the statements of Proposition 6.1 remain valid, but in (6.5) one must change Φ_+ by $\Phi_+^e + \Phi_+^i$, and in (6.6) Φ_+^p must be replaced by $(\Phi_+^e)^p + (\Phi_+^i)^p$.

There is one property in which the local singularity differs from the infinite singular point. Here there is no difference in asymptotic behavior of $N(\alpha, -\gamma^2, V)$ with $\gamma > 0$ and $\gamma = 0$ (cf. [8]). In particular, let us assume that under the conditions of Theorem 6.3 the potential V has a compact support. Relatively simple variational arguments show that in this case the asymptotic formulae (5.5), (5.7), and (5.8) may be carried over to $N(\alpha, -\gamma^2, V)$ for an arbitrary $\gamma > 0$.

Another conclusion which can be made from the above statements, is the following: The condition (2.9) is also not sufficient for the Weyl asymptotics of $N(\alpha, -\gamma^2, V)$ with $\gamma > 0$.

Theorem 6.3 is not difficult to generalize in the case of several points of singularity.

7. Many-Dimensional Analog of the Main Theorem

7.1. Statement of the Problem

The natural analog of the operator (2.1) for $d \geqq 3$ (from the point of view of the behavior of the negative spectrum) is the operator

$$\text{(7.1)} \qquad H_d(\alpha) = -\Delta - \varkappa(d)|x|^{-2} - \alpha V(x), \quad \alpha > 0,$$

$$\varkappa(d) = (d-2)^2/4, \quad d \geqq 3. \tag{7.2}$$

In this paragraph we assume the following condition to be fulfilled

$$V \in L_{d/2}(\mathbb{R}^d), \quad d \geqq 3. \tag{7.3}$$

This condition has already been discussed in Section 1 (see (1.3)). The precise definition of the operator (7.1) is given via the closed in $L_2(\mathbb{R}^d)$ quadratic form

$$\int (|\nabla v|^2 - \varkappa(d)|x|^{-2}|v|^2 - \alpha V|v|^2)dx, \quad v \in H^1(\mathbb{R}^d). \tag{7.4}$$

The constant (7.2) is the best constant in the Hardy inequality

$$\varkappa(d)\int |v|^2|x|^{-2}dx \leqq \int |\nabla v|^2 dx, \quad v \in C_0^\infty(\mathbb{R}^d), \quad d \geqq 3. \tag{7.5}$$

The inequality (7.5) implies that the operator $H_d(0)$ is positive. We denote the number of negative eigenvalues of $H_d(\alpha)$ by $N_d(\alpha, V)$.

In all further notations we try to point out the similarities between (7.1) and the two-dimensional case. As before, let r, θ be the polar coordinates of $x \in \mathbb{R}^d$, but now $\theta \in \mathbb{S}^{d-1}$. Fixing a potential V we introduce the functions

$$\mathcal{V}^j(r) = \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} V(r,\theta)d\theta, \quad r_j \geqq 1, \quad j = e, i;$$

$$r_e = r, \quad r_i = r^{-1};$$

$$F_V^e(t) = e^{2t}\mathcal{V}^e(e^t), \quad F_V^i(t) = e^{-2t}\mathcal{V}^i(e^{-t}), \quad t \geqq 0.$$

As before, $n_+^j(s; V, A)$ is a counting function of the positive spectrum of the quotient (4.1) with $F = F_V^j$, $j = e, i$, and $n_+(s; V, A)$ is defined in (6.11). We shall relate the functionals

$$\Delta_{p,j}^{(+)}(V), \quad \delta_{p,j}^{(+)}(V), \qquad \Delta_p^{(+)}(V), \quad \delta_p^{(+)}(V),$$

to the distribution functions n_+^j, $j = e, i$, and n_+.

7.2. The Main Theorem in the Case $d \geqq 3$

The condition (7.3) plays the role of Condition 6.2. The above notations allow us to state a theorem which is a copy of the statement of Theorem 6.3.

THEOREM 7.1. *Let us assume that condition* (7.3) *is fulfilled. Then*

(a) *If*

$$\Delta_{d/2}(|V|) < \infty,$$

then

$$\lim_{\alpha\to\infty} \left\{ \begin{matrix} \sup \\ \inf \end{matrix} \right\} \alpha^{-d/2} N_d(\alpha, V) = \frac{\omega_d}{(2\pi)^d} \int V_+^{d/2} dx + \left\{ \begin{matrix} \Delta_{d/2}^{(+)}(V) \\ \delta_{d/2}^{(+)}(V) \end{matrix} \right\}.$$

(b) *If*

$$\Delta_p(|V|) < \infty, \quad 2p > d,$$

then

$$\lim_{\alpha\to\infty} \left\{ \begin{matrix} \sup \\ \inf \end{matrix} \right\} \alpha^{-p} N_d(\alpha, V) = \left\{ \begin{matrix} \Delta_p^{(+)}(V) \\ \delta_p^{(+)}(V) \end{matrix} \right\}, \qquad 2p > d.$$

(c) *For the validity of the Weyl asymptotic formula*

$$\lim_{\alpha\to\infty} \alpha^{-d/2} N_d(\alpha, V) = \frac{\omega_d}{(2\pi)^d} \int V_+^{d/2} dx \tag{7.6}$$

it is sufficient to assume that

$$\Delta_{d/2}(|V|) = 0. \tag{7.7}$$

If (5.10) *and* (6.13) *are satisfied for some* $R \geqq 1$ *then* (7.7) *is necessary for* (7.6) *to be true.*

Remark 7.2. Let us have in addition to (7.3)

$$\int_{|x|\leqq\varepsilon} |V|^\sigma dx < \infty, \qquad \varepsilon > 0,\ 2\sigma > d, \tag{7.8}$$

for some ε and σ. Then using Proposition 4.2 it can be easily checked that

$$\Delta_{d/2,i}(|V|) = 0.$$

This implies that in the statement of Theorem 7.1 the functional $\Delta_p^{(+)}$, $\delta_p^{(+)}$, and Δ_p can always be changed by $\Delta_{p,e}^{(+)}$, $\delta_{p,e}^{(+)}$, and $\Delta_{p,e}$, $2p \geqq d$. Thus, under the additional condition (7.8), Theorem 7.1 turns out to be an analog of Theorem 5.1.

7.3. The Beginning of the Proof of Theorem 7.1

The proof of Theorem 7.1 follows the same scheme as the proof of Theorems 5.1 and 6.3. The technical details are somewhat different, however, and we concentrate our attention mainly on these details.

First let us notice (cf. [1]) that the substitution $v = |x|^{(d-2)/2} u$ transfers the form (7.4) into

$$\int (|\nabla u|^2 - \alpha V |u|^2) q\, dx, \quad q(x) = |x|^{2-d}.$$

It is enough to make this substitution only when considering the class $C_0^\infty(\mathbb{R}^d\setminus\{0\})$ which in this case transfers into itself. We use the decomposition (2.12)

$$u = z + g, \qquad z(r) = \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} u(r,\theta)d\theta. \tag{7.9}$$

Here (cf. (2.14), (2.15))

$$\int_{\mathbb{S}^{d-1}} g(r,\theta)d\theta = 0, \tag{7.10}$$

$$\int_{\mathbb{S}^{d-1}} |\nabla_x u|^2 d\theta = \int_{\mathbb{S}^{d-1}} \left(\left|\frac{\partial g}{\partial r}\right|^2 + \frac{1}{r^2}|\nabla_\theta g|^2\right) d\theta + |\mathbb{S}^{d-1}|\,|z'(r)|^2$$

and ∇_θ is the gradient of a function defined on $\mathbb{S}^{d-1}$.

As in Section 2 (see Lemma 2.2), the study of $N_d(\alpha, V)$ for a large α is reduced to the study of the asymptotic behavior of the distribution function $n_+(s,(7.11))$, $s\alpha = 1$, of the positive spectrum of the quotient

$$\frac{\int V|u|^2 q\,dx}{\int |\nabla u|^2 q\,dx}, \qquad z(R) = 0, \quad R > 0. \tag{7.11}$$

The additional condition $z(R) = 0$ allows us to close the class $C_0^\infty(\mathbb{R}^d \setminus \{0\})$ with respect to the metric defined by the denominator in (7.11). Then the quotient (7.11) can be considered on the complete Hilbert space of functions

$$\mathscr{H}_R^1(\mathbb{R}^d) = \{u \in H^1_{\mathrm{loc}}(\mathbb{R}^d \setminus \{0\}) : \sqrt{q}\,\nabla u \in L_2(\mathbb{R}^d),\ z(R) = 0\}, \quad d \geqq 3.$$

In future we shall use notations of the type (2.2), (2.3), relating them to spherical layers in $\mathbb{R}^d$, $d \geqq 3$. Working with the quotient (7.11) we need an analog of Proposition 3.1.

Proposition 7.3. *Let $g \in H^1_{\mathrm{loc}}(\Omega_{a,b})$, $0 \leqq a < b \leqq \infty$, (7.10) be satisfied and $\sqrt{q}\,\nabla g \in L_2(\Omega_{a,b})$. Then*

$$\int_{\Omega_{a,b}} |x|^{-2}|g(x)|^2 q\,dx \leqq (d-1)^{-1}\int_{\Omega_{a,b}} |\nabla g|^2 q\,dx.$$

The proof is quite similar to the proof of Proposition 3.1. We have to decompose $g(r,\cdot)$ with respect to d-dimentional spherical functions. This decomposition does not contain the free term. Furthermore, the first nonzero eigenvalue of the spherical Laplace operator $(-\Delta_\theta)$ equals $d-1$. Therefore

$$\int_{\mathbb{S}^{d-1}} |\nabla_\theta g|^2 d\theta = \int_{\mathbb{S}^{d-1}} (-\Delta_\theta g)\bar{g}\,d\theta \geqq (d-1)\int_{\mathbb{S}^{d-1}} |g|^2 d\theta. \tag{7.12}$$

It now remains to integrate (7.12) with respect to the measure $r^{-1}dr$ over the interval (a, b).

7.4. Some Spectral Estimates

Let us discuss the estimates which are going to be applied instead of Lemmas 3.3 and 3.4. Let $\Omega \subset \mathbb{R}^d$, $d \geqq 3$, be a bounded domain with a Lipschitz boundary and let $Q(x) \geqq 0$, $x \in \Omega$. For the quotient

$$\frac{\int_\Omega Q|u|^2 dx}{\int_\Omega(|\nabla u|^2 + |u|^2)dx}, \quad u \in H^1(\Omega), \tag{7.13}$$

the counting function of the spectrum has the estimate

$$n(s, (7.13)) \leqq C(\Omega)\, s^{-d/2} \int_\Omega Q^{d/2} dx, \qquad d \geqq 3. \tag{7.14}$$

(The estimate (7.14) is contained in [11]; it is explicitly written in [4]). The lower term at the denominator in (7.13) can be removed if we put an arbitrary additional condition (of rank one) which takes away the degeneracy on the constant functions. In this case the quadratic form of the denominator gives a norm in $H^1(\Omega)$ which is equivalent to the standard one and therefore the estimate of the type (7.14) remains true. Thus for the quotient

$$\frac{\int_\Omega Q(x)|u|^2 dx}{\int_\Omega |\nabla u|^2 dx}, \quad \int_\Omega u\, dx = 0, \quad u \in H^1(\Omega), \tag{7.15}$$

we have the estimate

$$n(s, (7.15)) \leqq C(\Omega)\, s^{-d/2} \int_\Omega Q^{d/2} dx, \qquad d \geqq 3. \tag{7.16}$$

Let us now consider the quotient

$$\frac{\int_{|x|>R} Q|u|^2 q\, dx}{\int_{|x|>R}(|\nabla u|^2 + |x|^{-2}|u|^2) q\, dx}, \tag{7.17}$$

on the set of all functions $u \in H^1_{\mathrm{loc}}(\Omega_{R,\infty})$ such that the finite denominator in (7.17) is finite. We shall show that

$$n(s, (7.17)) \leqq C(d)\, s^{-d/2} \int_{|x|>R} Q^{d/2} dx, \quad d \geqq 3. \tag{7.18}$$

Indeed, by using the scaling substitution we reduce (7.17) to the case $R = 1$. Then, as was shown in the proof of Lemma 3.3, we divide $\Omega_{1,\infty}$ on the layers Ω_k,

$k \geqq 1$. The estimate for every layer, by homogeneity and the scaling substitution, is reduced to the case $k = 1$. For the quotient

$$\frac{\int_{\Omega_1} Q|u|^2 q\,dx}{\int_{\Omega_1}(|\nabla u|^2 + |x|^{-2}|u|^2) q\,dx}, \tag{7.19}$$

the following estimate holds

$$n(s, (7.19)) \leqq C(d) s^{-d/2} \int_{\Omega_1} Q^{d/2} dx,$$

which obviously is a simple corollary of the estimate (7.14). Adding all the estimates which have been obtained for all Ω_k we have (7.18). Analogously it can be established that for the quotient

$$\frac{\int_{|x|<\rho} Q|u|^2 q\,dx}{\int_{|x|<\rho}(|\nabla u|^2 + |x|^{-2}|u|^2) q\,dx}, \tag{7.20}$$

the inequality

$$n(s, (7.20)) \leqq C(d)\, s^{-d/2} \int_{|x|<\rho} Q^{d/2} dx, \quad d \geqq 3, \tag{7.21}$$

is fulfilled. The inequalities (7.18) and (7.21) are the analogs of (3.7) and (3.12).

7.5. The End of the Proof of Theorem 7.1

Let us return to the quotient of forms (7.11) which were considered for $u \in \mathscr{H}^1_R(\mathbb{R}^d)$. To be able to obtain the upper and lower bounds we again divide $\mathbb{R}^d$ into three domains: $\Omega_{0,\rho}$, $\Omega_{\rho,R}$, $\Omega_{R,\infty}$ (regarding that $\rho R = 1$). In the domains $\Omega_{R,\infty}$ and $\Omega_{0,\rho}$ the final contribution to the asymptotic formula only gives the term z in the decomposition (7.9). This contribution corresponds to the positive spectrum of the quotients (4.1) where $F = F_V^e$ and $F = F_V^i$. The influence of the term g in (7.9) can be eliminated when $R \to \infty$ because of (7.18) and (7.21). Some explanations are needed in the "Weyl" zone $\Omega_{\rho,R}$. For the upper estimates we must consider the quotient

$$\frac{\int_{\Omega_{\rho,R}} V|u|^2 q\,dx}{\int_{\Omega_{\rho,R}} |\nabla u|^2 q\,dx}, \qquad \int_{\Omega_{\rho,R}} u\,dx = 0, \quad u \in H^1(\Omega_{\rho,R}). \tag{7.22}$$

The weight q is bounded from above and below on $\Omega_{\rho,R}$. Thus using (7.16) we have

$$n(s; (7.22)) \leqq C(R)\, s^{-d/2} \int_{\Omega_{\rho,R}} |V|^{d/2} dx. \tag{7.23}$$

When calculating the asymptotics of the spectrum the estimate (7.23) allows us to consider potentials V from any dense set in $L_{d/2}(\Omega_{\rho,R})$. The corresponding Weyl formula for (7.22) is well known, as for example, for a smooth V. The weight q in the asymptotic coefficient is cancelled. Finally we have

$$\lim_{s\to 0} s^{d/2} n_+(s,(7.22)) = (2\pi)^{-d}\omega_d \int_{\Omega_{\rho,R}} V_+^{d/2}\,dx. \tag{7.24}$$

The equality (7.24) replaces (5.20).

For the lower estimate we use the quotient

$$\frac{\int_{\Omega_{\rho,R}} V|u|^2 q\,dx}{\int_{\Omega_{\rho,R}} |\nabla u|^2 q\,dx}, \qquad u|_{r=\rho} = u|_{r=R} = 0, \quad u \in H^1(\Omega_{\rho,R}). \tag{7.25}$$

For this quotient there is also an estimate similar to (7.23) and, therefore, the Weyl asymptotic formula holds

$$\lim_{s\to 0} s^{d/2} n_+(s,(7.25)) = (2\pi)^{-d}\omega_d \int_{\Omega_{\rho,R}} V_+^{d/2}\,dx. \tag{7.26}$$

The equality (7.26) replaces (5.14). The rest of the proof of Theorem 7.1 is the same as in Theorems 5.1 and 6.3.

7.6. Local Singularities

As before, let the potential $V_{p,e}^{\Phi}$ be defined by (6.1) for $\Phi = \Phi^e$; the potential $V_{p,i}^{\Phi}$ be defined by (6.14). Obviously both potentials satisfy the condition (7.3). The quantity Φ_0^j, $j = e, i$, is defined (cf. (6.3)) as the mean value of the function Φ^j over the unit sphere. Nothing new is needed to be able to obtain the following statement.

PROPOSITION 7.4. *Let V satisfy condition* (7.3). *Let* (6.4) *be fulfilled for V with $V_p^{\Phi} = V_{p,e}^{\Phi}$ and*

$$V(x) - V_{p,i}^{\Phi}(x) = o\left(V_{p,i}^{|\Phi|}(x)\right), \quad |x| \to 0, \quad 2p > 1. \tag{7.27}$$

Then

(a) *If in* (6.4) *and* (7.27) $2p = d$, *then*

$$\lim_{\alpha\to\infty} \alpha^{-d/2} N_d(\alpha, V) = \frac{\omega_d}{(2\pi)^d}\int V_+^{d/2}\,dx + M\left(\frac{d}{2}\right)\left((\Phi_+^e)^{d/2} + (\Phi_+^i)^{d/2}\right).$$

(b) *If* $2p > d$, *then*

$$\lim_{\alpha\to\infty} \alpha^{-p} N_d(\alpha, V) = M(p)\left((\Phi_+^e)^p + (\Phi_+^i)^p\right).$$

(c) *If* $2p < d$, *then the asymptotic formula* (7.6) *holds.*

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