ON SPECTRA OF 2D SCHRÖDINGER OPERATORS WITH SINGULAR POTENTIALS LOCALIZED IN A NEIGHBOURHOOD OF CURVE

YURIY GOLOVATY

Abstract. The

1. Introduction

Throughout the paper, $W_2^m(\Omega)$ stands for the Sobolev space of functions defined on a set Ω. $\sigma(A)$ is the spectrum

2. Statement of Problem and Main Results

We study the family of operators

$$H_\varepsilon = -\Delta + W + V_\varepsilon \tag{2.1}$$

in $L_2(\mathbb{R}^2)$, where the potential W increases as $|x| \to +\infty$ and the perturbation V_ε has a compact support that shrinks to a curve as $\varepsilon \to 0$. We define the family of potentials V_ε as follows. Let γ be a closed smooth curve without self-intersection points. We will denote by ω_ε the ε-neighborhood of γ, i.e., the union of all open balls with the radius ε and a center on γ. Assume that the support of V_ε lies in ω_ε. To specify the explicit dependence of V_ε on the small parameter ε we introduce curvilinear coordinates in ω_ε. Let $\alpha\colon [0,m) \to \mathbb{R}^2$, $\alpha = \alpha(s)$, be the unit-speed smooth parametrization of γ with the natural parameter s, where m is the length of γ. Then the vector $\nu = (-\dot\alpha_2, \dot\alpha_1)$ is a unit normal on γ, because $\dot\alpha_1^2 + \dot\alpha_2^2 = 1$. We define the local coordinates (s,r) in ω_ε:

$$x = \alpha(s) + r\nu(s), \qquad (s,r) \in [0,m) \times (-\varepsilon, \varepsilon). \tag{2.2}$$

The coordinate r is the signed distance from a point x to γ. Suppose that the localized potential V_ε has the form

$$V_\varepsilon\big(\alpha(s) + r\nu(s)\big) = \varepsilon^{-2}\, V\left(\varepsilon^{-1} r\right) + \varepsilon^{-1}\, U\left(s, \varepsilon^{-1} r\right), \tag{2.3}$$

where V and U are smooth functions and the supports of V and $U(s,\cdot\,)$ lie in the interval $\mathcal{I} = (-1,1)$ for all s. We note that the unperturbed operator $H_0 = -\Delta + W$ is self-adjoint in $L^2(\mathbb{R}^2)$ and its spectrum is discrete. Obviously, the operator H_ε is also self-adjoint and $\operatorname{dom} H_\varepsilon = \operatorname{dom} H_0$.

The family of potentials V_ε generally diverges in the space of distributions $\mathcal{D}(\mathbb{R}^2)$. As we will show in Propositin 2, the potentials converge only if V is a zero mean function. In this case, V_ε converges as $\varepsilon \to 0$ to $a\partial_\nu\delta_\gamma + b\delta_\gamma$, where a and b are

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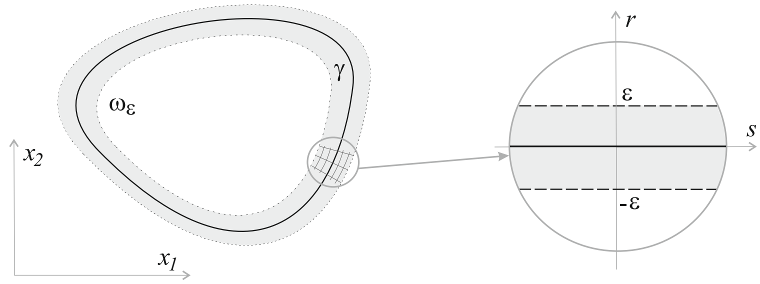

FIGURE 1. Curvilinear coordinates in the ε-neighbourhood of γ.

some functions on γ. The distribution δ_γ is the Dirac delta function supported on the curve γ, and

$$\langle a\partial_\nu\delta_\gamma + b\delta_\gamma, \phi\rangle = -\int_\gamma \partial_\nu(a\phi)\,d\gamma + \int_\gamma b\phi\,d\gamma \quad \text{for all } \phi \in C_0^\infty(\mathbb{R}^2).$$

The main task is to describe the asymptotic behaviour, as $\varepsilon \to 0$, of the spectrum of H_ε, i.e., to construct the asymptotics of eigenvalues λ^ε and eigenfunctions u_ε of the spectral problem

$$-\Delta u_\varepsilon + (W + V_\varepsilon)u_\varepsilon = \lambda^\varepsilon u_\varepsilon \quad \text{in } \mathbb{R}^2. \tag{2.4}$$

Here and afterwards λ^ε stands for an eigenvalue of the operator H_ε and u_ε stands for the corresponding eigenfunction.

We introduce some notation. The curve γ divides the plane into two domains Ω^- and Ω^+, i.e., $\mathbb{R}^2 = \Omega^- \cup \gamma \cup \Omega^+$. Suppose Ω^+ is unbounded. Let us denote by v^- and v^+ the one-side traces of v on γ.

We say that the Schrödinger operator $-\frac{d^2}{dt^2} + V$ in $L_2(\mathbb{R})$ possesses a *zero-energy resonance* if there exists a non trivial solution h of the equation $-h'' + Vh = 0$ that is bounded on the whole line. We call h the *half-bound state*. Such a solution h is unique up to a scalar factor and has nonzero limits

$$h(-\infty) = \lim_{t\to-\infty} h(t), \qquad h(+\infty) = \lim_{t\to+\infty} h(t).$$

Let $\mathcal{E} \subset (0,1)$ be an infinite set, for which zero is an accumulation point. Our main result reads as follows.

Theorem 1. *Assume that the Schrödinger operator $-\frac{d^2}{dt^2} + V$ in $L_2(\mathbb{R})$ possesses a zero-energy resonance with the half-bound state h.*

(i) Suppose that $\{\lambda^\varepsilon\}_{\varepsilon\in\mathcal{E}}$ is a sequence of eigenvalues of H_ε and $\{u_\varepsilon\}_{\varepsilon\in\mathcal{E}}$ is the corresponding sequence of eigenfunctions such that $\|u_\varepsilon\|_{L_2(\mathbb{R}^2)} = 1$. If

$$\lambda^\varepsilon \to \lambda, \qquad u_\varepsilon \to u \quad \text{in } L_2(\mathbb{R}^2) \text{ weakly} \tag{2.5}$$

as $\mathcal{E} \ni \varepsilon \to 0$, and u is a non-zero function, then λ and u are an eigenvalue and the corresponding eigenfunction of the operator $\mathcal{H} = -\Delta + W$ acting on the domain

$$\begin{aligned}\operatorname{dom}\mathcal{H} = \big\{v \in W_2^2(\Omega\setminus\gamma)\colon\ & v^+ - \theta v^- = 0,\\ & \theta\partial_\nu v^+ - \partial_\nu v^- = \big(\tfrac12(\theta^2-1)\varkappa + \mu\big)\, v^- \text{ on } \gamma\big\}.\end{aligned}$$

Here $\theta = h(+\infty)/h(-\infty)$, $\varkappa = \varkappa(s)$ is the curvature of γ, and

$$\mu(s) = \frac{1}{h^2(-\infty)} \int_{\mathcal{I}} U(s,r)h^2(r)\,dr, \quad s \in [0,m). \tag{2.6}$$

(ii) If (2.5) holds and λ is not a point of $\sigma(\mathcal{H})$, then the sequence of eigenfunctions u_ε converges to zero as $\varepsilon \to 0$ in the weak topology of $L_2(\mathbb{R}^2)$.

(iii) For each eigenvalue λ of the operator $\mathcal{H}$ and all ε small enough there exists an eigenvalue λ^ε of the operator H_ε such that $|\lambda^\varepsilon - \lambda| \leqslant c\varepsilon$ with the constant c depending only on λ.

The trivial potential $V = 0$ possesses a zero-energy resonance with a constant function as the half-bound state. Then $V_\varepsilon(x) = \varepsilon^{-1}\, U\left(s, \varepsilon^{-1} n\right)$ and $V_\varepsilon \to \mu_0 \delta_\gamma$ as $\varepsilon \to 0$ in the space of distributions, where

$$\mu_0(s) = \int_{\mathcal{I}} U(s,t)\,dt, \qquad s \in [0,m). \tag{2.7}$$

Moreover the parameter θ equals 1 and the interface conditions

$$v^+ - \theta v^- = 0, \quad \theta \partial_\nu v^+ - \partial_\nu v^- = \left(\tfrac{1}{2}(\theta^2 - 1)\varkappa + \mu\right) v^- \tag{2.8}$$

on the curve γ become

$$v^+ - v^- = 0, \quad \partial_\nu v^+ - \partial_\nu v^- = \mu_0 v^-.$$

The last conditions are exactly the same as that obtained in [1].

Let us introduce two operators

$$\mathcal{D}^\pm = -\Delta + W \quad \text{in } L_2(\Omega^\pm), \qquad \operatorname{dom} \mathcal{D}^\pm = \{v \in W_2^2(\Omega^\pm)\colon\ v = 0 \text{ on } \gamma\}.$$

Theorem 2. *Suppose that the Schrödinger operator $-\frac{d^2}{dt^2} + V$ in $L_2(\mathbb{R})$ has no zero-energy resonance.*

(i) If $\lambda^\varepsilon \to \lambda$ and $u_\varepsilon \to u$ in $L_2(\mathbb{R}^2)$ weakly, as $\mathcal{E} \ni \varepsilon \to 0$, and the limit function u is different from zero, then λ is an eigenvalue of the direct sum $\mathcal{D}^- \oplus \mathcal{D}^+$ and u is the corresponding eigenfunction.

(ii) In the case that $\lambda^\varepsilon \to \lambda$ and $\lambda \notin \sigma(\mathcal{D}^- \oplus \mathcal{D}^+)$, the eigenfunctions u_ε converge to zero in $L_2(\mathbb{R}^2)$ weakly.

(iii) If $\lambda \in \sigma(\mathcal{D}^- \oplus \mathcal{D}^+)$, then for all ε small enough we can find an eigenvalue λ^ε of H_ε such that $|\lambda^\varepsilon - \lambda| \leqslant c\varepsilon$, where the constant c does not depend on ε.

The thin structure that is the singular potential V_ε can produce a series of negative eigenvalues which go to the negative infinity as $\varepsilon \to 0$ (see Fig. **??**). Although for each $\varepsilon > 0$ the number of negative eigenvalues is finite, for some potentials V_ε this number can increase infinitely as $\varepsilon \to 0$. In particular, this means that the family of operators H_ε is not uniformly bounded from below with respect to the small parameter ε.

In the case of such spectrum behavior, each real point λ can be realized as an accumulation point of the eigenvalues λ^ε of H_ε, as $\varepsilon \to 0$. In Fig. **??**, we see that the dashed line beginning at a point λ has infinitely many intersections with curves λ^ε. Theorems 1 and 2 point out the basic difference between the points of the spectra of limit operators and all other ones. This difference is that only the points of $\sigma(\mathcal{H})$ in the resonant case (or $\sigma(\mathcal{D}^- \oplus \mathcal{D}^+)$ in the non-resonant case) can be approximated by λ^ε so that the corresponding eigenfunctions u_ε have a nontrivial limit. In a particular case, the result of Theorem 2 can be improved.

Theorem 3. *Assume that the potentials V and U in (2.3) are non-negative. Then $\lambda^\varepsilon \to \lambda$ as $\varepsilon \to 0$ if and only if λ is an eigenvalue of the direct sum $\mathcal{D}^- \oplus \mathcal{D}^+$. In addition, the corresponding eigenfunctions u_ε converge in the L_2-norm to an eigenfunction u associated with the eigenvalue λ.*

3. Preliminaries

Returning to the curvilinear coordinates (s,r) given by (2.2), we see that the couple of vectors $\alpha = (\dot\alpha_1, \dot\alpha_2)$, $\nu = (-\dot\alpha_2, \dot\alpha_1)$ gives a Frenet frame for γ. The Jacobian of transformation $x = \alpha(s) + r\nu(s)$ has the form

$$J(s,r) = \begin{vmatrix} \dot\alpha_1(s) - r\ddot\alpha_2(s) & -\dot\alpha_2(s) \\ \dot\alpha_2(s) + r\ddot\alpha_1(s) & \dot\alpha_1(s) \end{vmatrix}$$
$$= \dot\alpha_1^2(s) + \dot\alpha_2^2(s) - r\big(\dot\alpha_1(s)\ddot\alpha_2(s) - \dot\alpha_2(s)\ddot\alpha_1(s)\big) = 1 - r\varkappa(s),$$

where $\varkappa = \det(\dot\alpha, \ddot\alpha)$ is the signed curvature of γ. We see that J is positive for sufficiently small r, because curvature $\varkappa$ is bounded on γ. Namely, the curvilinear coordinates (s,r) can be defined correctly on ω_ε for all $\varepsilon < \varepsilon_*$, where

$$\varepsilon_* = \min_\gamma |\varkappa|^{-1}. \tag{3.1}$$

Note also that $\varkappa$ is defined uniquely up to the re-parametrization $s \mapsto -s$. Interface conditions (2.8) contain the parameters θ, $\varkappa$ and μ which depend on the particular parametrization chosen for curve γ, i.e., the parameters change along with the change of the Frenet frame.

Proposition 1. *Operator $\mathcal{H}$ in Theorem 1 does not depend upon the choice of the Frenet frame for γ.*

Proof. Every smooth curve admits two possible orientations of the arc-length parameter and consequently two possible Frenet frames. Let us change the Frenet frame $\{\alpha, \nu\}$ to the frame $\{-\alpha, -\nu\}$ and prove that interface conditions (2.8) will remain the same. This change leads to the following transformations:

$$h(\pm\infty) \mapsto h(\mp\infty),\ \ \theta \mapsto \theta^{-1},\ \ \varkappa \mapsto -\varkappa,\ \ \mu \mapsto \theta^{-2}\mu,\ \ u_\pm \mapsto u_\mp,\ \ \partial_\nu u_\pm \mapsto -\partial_\nu u_\mp.$$

The first condition $u^+ - \theta u^- = 0$ in (2.8) transforms into $u^- - \theta^{-1}u^+ = 0$ and therefore remains unchanged. For the second condition, we obtain

$$-\theta^{-1}\partial_\nu u^- + \partial_\nu u^+ - \left(-\tfrac12(\theta^{-2} - 1)\varkappa + \theta^{-2}\mu\right) u^+ = 0.$$

Multiplying the equality by θ yields

$$\theta\partial_\nu u^+ - \partial_\nu u^- - \left(\tfrac12(\theta^2 - 1)\varkappa + \mu\right)\theta^{-1}u^+ = 0,$$

since $-\theta(\theta^{-2} - 1) = \theta^{-1}(\theta^2 - 1)$. It remains to insert u^- in place of $\theta^{-1}u^+$, in view of the first interface condition. □

The metric tensor $g = (g_{ij})$ in the orthogonal coordinates (s,r) has the form

$$g = \begin{pmatrix} J^2 & 0 \\ 0 & 1 \end{pmatrix}.$$

In fact, we have $g_{11} = |x_s|^2 = |\dot\alpha + r\dot\nu|^2 = |(1 - r\varkappa)\dot\alpha|^2 = J^2$, by the Frenet-Serret formula $\dot\nu = -\varkappa\dot\alpha$, and $g_{22} = |x_r|^2 = |\nu|^2 = 1$. In particular, the Laplace-Beltrami operator in ω_ε has the form

$$\Delta\phi = J^{-1}\left(\partial_s(J^{-1}\partial_s\phi) + \partial_r(J\partial_r\phi)\right). \tag{3.2}$$

All the results presented in Theorems 1 and 2 concern arbitrary potentials V_ε of the form (2.3) that generally diverge in the distributional sense. However, the spectra of H_ε converge to spectra of the limit operators without reference to the convergence of the potentials. The following statement shows that the convergence conditions for the potentials V_ε and the spectra of H_ε are quite different.

Proposition 2. *The family of potentials V_ε converges in the space of distributions $\mathcal{D}'(\mathbb{R}^2)$ if and only if V is a zero mean function, i.e., $\int_\mathbb{R} V\,dt = 0$. In this case,*

$$V_\varepsilon \to \beta\,\partial_\nu\delta_\gamma + (\beta\varkappa + \mu_0)\,\delta_\gamma \quad \text{in } \mathcal{D}'(\mathbb{R}^2),$$

where $\beta = -\int_\mathbb{R} rV(r)\,dr$ and μ_0 is given by (2.7).

Proof. It is evident that the sequence $\varepsilon^{-1}\,U\left(s,\varepsilon^{-1}r\right)$ converge to $\mu_0\delta_\gamma$ in $\mathcal{D}'(\mathbb{R}^2)$. Write $g_\varepsilon = \varepsilon^{-2}\,V\left(\varepsilon^{-1}r\right)$ and $n = \varepsilon^{-1}r$. Then we have

$$\begin{aligned}\int_{\mathbb{R}^2} g_\varepsilon\phi\,dx &= \int_{\omega_\varepsilon} g_\varepsilon\phi\,dx = \varepsilon^{-2}\int_{-\varepsilon}^{\varepsilon}\int_0^m V(\varepsilon^{-1}r)\phi(s,r)(1-r\varkappa(s))\,ds\,dr\\ &= \varepsilon^{-1}\int_{-1}^{1}\int_0^m V(n)\phi(s,\varepsilon n)(1-\varepsilon n\varkappa(s))\,ds\,dn = \varepsilon^{-1}\int_{-1}^{1} V(n)\,dn\int_0^m \phi(s,0)\,ds\\ &\qquad + \int_{-1}^{1} nV(n)\,dn\int_0^m \big(\partial_n\phi(s,0) - \varkappa(s)\phi(s,0)\big)\,ds + O(\varepsilon)\end{aligned}$$

as $\varepsilon\to 0$ for all $\phi\in C_0^\infty(\mathbb{R}^2)$. The sequence g_ε has a finite limit as $\varepsilon\to 0$ in $\mathcal{D}'(\mathbb{R}^2)$ iff $\int_\mathbb{R} V\,dt = 0$. In this case, we have

$$\int_{\mathbb{R}^2} g_\varepsilon\phi\,dx \to \beta\int_\gamma \left(\partial_\nu\delta_\gamma + \varkappa\delta_\gamma\right)\phi\,d\gamma,$$

which completes the proof. □

4. Formal Asymptotics of Eigenvalues

Now we will show how interface conditions (2.8) can be found by direct calculations, constructing the formal asymptotics of eigenvalues and eigenfunctions.

In the sequel, the normal vector field ν on γ will be outward to domain Ω^-, that is to say, the local coordinate r will increase in the direction from Ω^- to Ω^+. We also denote by $\gamma_t = \{x\in\mathbb{R}^2\colon\ x = \alpha(s) + t\nu(s),\ s\in S\}$ the closed curve that is obtained from γ by flowing for "time" t along the normal vector field. Then the boundary of ω_ε consists of two curves $\gamma_{-\varepsilon}$ and γ_ε.

We look for the approximation to an eigenvalue λ_ε and the corresponding eigenfunction u_ε of (2.4) in the form

$$\lambda^\varepsilon \approx \lambda, \qquad u_\varepsilon(x) \approx \begin{cases} u(x) & \text{in } \mathbb{R}^2\setminus\omega_\varepsilon,\\ v_0\left(s,\frac{r}{\varepsilon}\right) + \varepsilon v_1\left(s,\frac{r}{\varepsilon}\right) + \varepsilon^2 v_2\left(s,\frac{r}{\varepsilon}\right) & \text{in } \omega_\varepsilon.\end{cases} \tag{4.1}$$

To match the approximations in ω_ε and $\mathbb{R}^2\setminus\omega_\varepsilon$, we hereafter assume that

$$[u_\varepsilon]_{\pm\varepsilon} = 0, \quad [\partial_r u_\varepsilon]_{\pm\varepsilon} = 0, \tag{4.2}$$

where $[w]_t$ stands for the jump of w across γ_t in the positive direction of the local coordinate r.

Since the function u_ε solves (2.4) and the domain ω_ε shrinks to γ, the function u must be a solution of the equation

$$-\Delta u + Wu = \lambda u \quad \text{in} \;\; \mathbb{R}^2 \setminus \gamma,$$

subject to appropriate interface conditions on γ. To find these conditions, we consider equation (2.4) in the curvilinear coordinates (s, n), where $n = r/\varepsilon$. Here and afterwards it will be convenient to parameterize the curve γ by the points of a circle. This will allow us not to indicate every time that the functions $v_k = v_k(s, n)$ are periodic with respect to s. Let S be the circle of the same length as the length of γ. Then ω_ε is diffeomorphic to the cylinder $Q_\varepsilon = S \times (-\varepsilon, \varepsilon)$ for ε small enough. Given a point $x \in \omega_\varepsilon$, we write

$$x = \alpha(s) + r\nu(s), \qquad (s, r) \in S \times (-\varepsilon, \varepsilon).$$

We also set $Q = S \times \mathcal{I}$. In the vicinity of γ the Laplacian can be written as

$$\Delta = \frac{1}{1 - \varepsilon n \varkappa} \left(\varepsilon^{-2} \partial_n (1 - \varepsilon n \varkappa) \partial_n + \partial_s \Big(\frac{1}{1 - \varepsilon n \varkappa} \, \partial_s \Big) \right),$$

by (3.2). From this we readily deduce the representation

$$\Delta = \varepsilon^{-2} \partial_n^2 - \varepsilon^{-1} \varkappa(s) \partial_n - n \varkappa^2(s) \partial_n + \partial_s^2 + \varepsilon P_\varepsilon, \tag{4.3}$$

where P_ε is a partial differential operator of the second order on s and the first one on n whose coefficients are uniformly bounded in Q with respect to ε.

Substituting (4.1) into (2.4) for $x \in \omega_\varepsilon$ in particular yields

$$-\partial_n^2 v_0 + V(n) v_0 = 0, \qquad -\partial_n^2 v_1 + V(n) v_1 = -\varkappa(s) \partial_n v_0 - U(s, n) v_0 \tag{4.4}$$

$$-\partial_n^2 v_2 + V(n) v_2 = -(\varkappa(s) \partial_n + U(s, n)) v_1 + (\partial_s^2 - n \varkappa^2 \partial_n - W(s, 0) + \lambda) v_0 \tag{4.5}$$

in the cylinder Q. From (4.2) we see that necessarily

$$u^-(s) = v_0(s, -1), \qquad u^+(s) = v_0(s, 1), \tag{4.6}$$

$$\partial_n v_0(s, -1) = 0, \qquad \partial_n v_0(s, 1) = 0, \tag{4.7}$$

$$\partial_n v_1(s, -1) = \partial_r u^-(s), \qquad \partial_n v_1(s, 1) = \partial_r u^+(s), \tag{4.8}$$

where $u^\pm$ are the one-side traces of u on γ. Combining (4.7)–(4.8), we conclude that v_0 and v_1 solve the boundary value problems

$$\begin{cases} -\partial_n^2 v_0 + V(n) v_0 = 0 \quad \text{in} \;\; Q, \\ \;\partial_n v_0(s, -1) = 0, \quad \partial_n v_0(s, 1) = 0, \quad s \in S; \end{cases} \tag{4.9}$$

$$\begin{cases} -\partial_n^2 v_1 + V(n) v_1 = -\varkappa(s) \partial_n v_0 - U(s, n) v_0 \quad \text{in} \;\; Q, \\ \;\partial_n v_1(s, -1) = \partial_r u^-(s), \quad \partial_n v_1(s, 1) = \partial_r u^+(s), \quad s \in S \end{cases} \tag{4.10}$$

respectively. Hence we have the boundary value problems for the "non-elliptic" partial differential operator $-\partial_n^2 + V$ in the cylinder Q. These problems can also be regarded as boundary value problems for ordinary differential equations on $\mathcal{I}$ which depend on the parameter $s \in S$.

4.1. **Case of zero-energy resonance.** Assume that operator $-\frac{d^2}{dt^2}+V$ has a zero energy resonance with half-bound state h. Since the support of V lies in the interval $\mathcal{I}$, the half-bound state h is constant outside $\mathcal{I}$ as a bounded solution of the equation $h''=0$. Therefore the restriction of h to $\mathcal{I}$ is a nonzero solution of the Neumann boundary value problem

$$-h''+V(n)h=0 \quad \text{in } \mathcal{I}, \qquad h'(-1)=0, \quad h'(1)=0. \tag{4.11}$$

Hereafter, we fix h by additional condition $h(-1)=1$. Then $h(\pm\infty)=h(\pm 1)$ and $\theta=h(1)$. In this case, (4.9) admits a infinitely many solutions $v_0(s,n)=a_0(s)h(n)$, where a_0 is an arbitrary function on S. From (4.6) we deduce that $u^-=a_0$ and $u^+=h(1)a_0=\theta a_0$ and hence that $v_0(s,n)=u^-(s)h(n)$ and

$$u^+=\theta u^- \quad \text{on } \gamma. \tag{4.12}$$

Next, problem (4.10) is in general unsolvable, since (4.9) admits nontrivial solutions. To find solvability conditions, we rewrite equation in (4.10) in the form

$$-\partial_n^2 v_1+V(n)v_1=-\big(\varkappa(s)h'(n)+U(s,n)h(n)\big)u^-(s), \tag{4.13}$$

multiply by $a(s)h(n)$, $a\in L^2(S)$, and then integrate over Q

$$\int_Q\left(-\partial_n^2 v_1+V(n)v_1\right)a(s)h(n)\,dn\,ds = -\int_Q(\varkappa(s)h'(n)+U(s,n)h(n))u^-(s)a(s)h(n)\,dn\,ds. \tag{4.14}$$

Since h is a solution of (4.11), integrating by parts twice on the left-hand side yields

$$\int_S\int_{\mathcal{I}}\left(-\partial_n^2 v_1+Vv_1\right)ah\,dn\,ds=-\int_S(\partial_n v_1 h-v_1h')\big|_{n=-1}^{n=1}a\,ds \\ -\int_S\int_{\mathcal{I}}av_1\left(-h''+Vh\right)\,dn\,ds=-\int_S\big(\theta\partial_r u^+-\partial_r u^-\big)a\,ds,$$

in view of the boundary conditions for v_1. Hence (4.14) becomes

$$\int_S\left(\theta\partial_r u^+-\partial_r u^-\right)a\,ds=\int_S u^-a\int_{\mathcal{I}}\left(\varkappa hh'+Uh^2\right)\,dn\,ds.$$

The equality $hh'=\frac12(h^2)'$ implies

$$\int_{\mathcal{I}}hh'\,dn=\tfrac12(h^2(1)-h^2(-1))=\tfrac12(\theta^2-1). \tag{4.15}$$

Therefore we obtain

$$\int_S\left(\theta\partial_r u^+-\partial_r u^-\right)a\,ds=\int_S\big(\tfrac12(\theta^2-1)\varkappa+\mu\big)u^-a\,ds$$

for all $a\in L^2(S)$, where $\mu(s)=\int_{\mathcal{I}}U(s,n)h^2(n)\,dn$. From this we deduce

$$\theta\partial_r u^+-\partial_r u^-=\big(\tfrac12(\theta^2-1)\varkappa+\mu\big)u^- \quad \text{on } \gamma,$$

which is necessary for the solvability of (4.10). In view of the Fredholm alternative, this condition is also sufficient. Moreover it is a jump condition for the normal derivative of u at the interface γ, since $\partial_\nu=\partial_r$ on γ.

Therefore λ and u in the approximation (4.1) must solve the problem

$$-\Delta u + Wu = \lambda u \qquad \text{in } \mathbb{R}^2 \setminus \gamma, \tag{4.16}$$

$$u^+ - \theta u^- = 0, \quad \theta \partial_\nu u^+ - \partial_\nu u^- = \big(\tfrac{1}{2}(\theta^2 - 1)\varkappa + \mu\big) u^- \quad \text{on } \gamma, \tag{4.17}$$

i.e., the spectral equation $\mathcal{H}u = \lambda u$. Assume that λ is an eigenvalue of operator $\mathcal{H}$ and u is an eigenfunction for this eigenvalue.

Now we can calculated the trace u^- on γ and finally determine $v_0(s,n) = u^-(s)h(n)$.

Since the second condition in (4.17) holds, problem (4.10) is solvable and v_1 is defined up to the term $a_1(s)h(n)$. Choose the solution v_1 so that $v_1(s,-1) = 0$ for all $s \in S$. We also suppose that v_2 solves the Cauchy problem

$$\begin{aligned} -\partial_n^2 v_2 + V(n) v_2 = {}& -(\varkappa(s)\partial_n + U(s,n))v_1 \\ & + (\partial_s^2 - n\varkappa^2 \partial_n - W(s,0) + \lambda) v_0 \quad \text{in } \ Q, \end{aligned} \tag{4.18}$$

$$v_2(s,-1) = 0, \quad \partial_n v_2(s,-1) = 0, \quad s \in S. \tag{4.19}$$

4.2. **Non-resonant case.** Now suppose that problem (4.11) admits the trivial solution only. Then $v_0 = 0$ and therefore $u^- = 0$ and $u^+ = 0$ on γ, by (4.6). We thus get

$$-\Delta u + Wu = \lambda u \quad \text{in } \ \mathbb{R}^2 \setminus \gamma, \qquad u|_\gamma = 0.$$

Let us suppose that λ is an eigenvalue of the direct sum $\mathcal{D}_1 \oplus \mathcal{D}_2$ of two Dirichlet type operators and u is a corresponding eigenfunction. In this case, problem (4.10) has the form

$$\begin{cases} -\partial_n^2 v_1 + V(n) v_1 = 0 \quad \text{in } \ Q, \\ \ \partial_n v_1(s,-1) = \partial_\nu u^-, \qquad \partial_n v_1(s,1) = \partial_\nu u^+, \end{cases} \tag{4.20}$$

and admits a unique solution. We also assume that

$$\begin{cases} -\partial_n^2 v_2 + V(n) v_2 = -\varkappa(s)\partial_n v_1 + U(s,n) v_1 \quad \text{in } \ Q, \\ \ v_2(s,-1) = 0, \quad \partial_n v_2(s,-1) = 0, \quad s \in S, \end{cases}$$

and apply the reasoning above.

5. Justification of the asymptotics

The function

$$\hat v_\varepsilon(x) = \begin{cases} u(x) & \text{in } \ \mathbb{R}^2 \setminus \omega_\varepsilon, \\ v_0\left(s, \frac{r}{\varepsilon}\right) + \varepsilon v_1\left(s, \frac{r}{\varepsilon}\right) + \varepsilon^2 v_2\left(s, \frac{r}{\varepsilon}\right) & \text{in } \ \omega_\varepsilon. \end{cases} \tag{5.1}$$

does not in general belong to $\operatorname{dom} H_\varepsilon$, because it has jump discontinuities on the boundary $\partial\omega_\varepsilon$. Let us define the function ζ plotted in Fig. 2. This function is smooth outside the origin, $\zeta(r) = 1$ for $r \in [0, \beta/2]$ and $\zeta(r) = 0$ in the set $\mathbb{R} \setminus [0, \beta)$. We here and henceforth assume that $\beta < \varepsilon_*$, where ε_* is given by (3.1). Set

$$\eta_\varepsilon = \big([\hat v_\varepsilon]_\varepsilon + [\partial_r \hat v_\varepsilon]_\varepsilon\,(r-\varepsilon)\big)\,\zeta(r-\varepsilon) + \big([\hat v_\varepsilon]_{-\varepsilon} + [\partial_r \hat v_\varepsilon]_{-\varepsilon}\,(r+\varepsilon)\big)\,\zeta(-r-\varepsilon).$$

and note that η_ε and $\partial_r\eta_\varepsilon$ have the same jumps across the boundary of ω_ε as $\hat v_\varepsilon$ and $\partial_r \hat v_\varepsilon$ respectively. In addition, η_ε is different from zero in the set $\omega_{\beta+\varepsilon} \setminus \omega_\varepsilon$ only. Therefore the function

$$v_\varepsilon(x) = \begin{cases} u(x) - \eta_\varepsilon(x) & \text{in } \ \mathbb{R}^2 \setminus \omega_\varepsilon, \\ v_0\left(s, \frac{r}{\varepsilon}\right) + \varepsilon v_1\left(s, \frac{r}{\varepsilon}\right) + \varepsilon^2 v_2\left(s, \frac{r}{\varepsilon}\right) & \text{in } \ \omega_\varepsilon \end{cases} \tag{5.2}$$

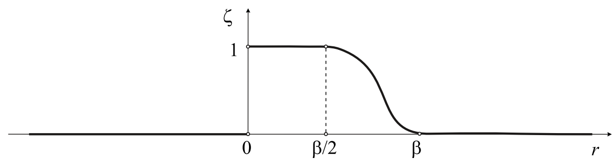

FIGURE 2. Plot of the function ζ.

belongs to the domain of $\mathcal{H}_\varepsilon$. We have not changed the asymptotics (5.1) too much, since

$$\sup_{x\in\mathbb{R}^2\setminus\omega_\varepsilon}\big(|\eta_\varepsilon(x)|+|\Delta\eta_\varepsilon(x)|\big)\leqslant c\varepsilon. \tag{5.3}$$

The last inequality is valid, because the jumps of $\hat{v}_\varepsilon$ and $\partial_r\hat{v}_\varepsilon$ across both curves $\gamma_{-\varepsilon}$ and γ_ε are infinitely small as $\varepsilon\to 0$ uniformly on $s\in S$. In fact,

$$\begin{aligned}
[\hat{v}_\varepsilon]_{-\varepsilon} &= v_0(s,-1)-u(s,-\varepsilon)=u^-(s)-u(s,-\varepsilon)=O(\varepsilon),\\
[\hat{v}_\varepsilon]_{\varepsilon} &= u(s,\varepsilon)-v_0(s,1)+O(\varepsilon)=u(s,\varepsilon)-\theta u^-(s)+O(\varepsilon)\\
&\qquad\qquad\qquad\qquad\qquad = u(s,\varepsilon)-u^+(s)+O(\varepsilon)=O(\varepsilon),\\
[\partial_r\hat{v}_\varepsilon]_{-\varepsilon} &= \varepsilon^{-1}\partial_n v_0(s,-1)+\partial_n v_1(s,-1)-\partial_r u(s,-\varepsilon)\\
&\qquad\qquad\qquad\qquad\qquad = \partial_r u^-(s)-\partial_r u(s,-\varepsilon)=O(\varepsilon),\\
[\partial_r\hat{v}_\varepsilon]_{\varepsilon} &= \partial_r u(s,\varepsilon)-\varepsilon^{-1}\partial_n v_0(s,1)-\partial_n v_1(s,1)+O(\varepsilon)\\
&\qquad\qquad\qquad\qquad\qquad = \partial_r u^+(s)-\partial_r u(s,\varepsilon)+O(\varepsilon)=O(\varepsilon),
\end{aligned}$$

as $\varepsilon\to 0$, by construction of v_k. We also have utilized in this calculation condition (4.12) and the inequality $|u(s,\pm\varepsilon)-u^\pm(s)|+|\partial_r u(s,\pm\varepsilon)-\partial_r u^\pm(s)|\leqslant c\varepsilon$.

To prove that $\lambda\in\sigma(\mathcal{H})$ is an accumulation point for some sequence $\{\lambda^\varepsilon\}_{\varepsilon>0}\subset\sigma(H_\varepsilon)$, we will apply the method of quasimodes. Let A be a self-adjoint operator in a Hilbert space L. We say a pair $(\mu,\phi)\in\mathbb{R}\times\operatorname{dom}A$ is a *quasimode* of A with the accuracy δ, if $\phi\neq 0$ and $\|(A-\mu)\phi\|_L\leqslant\delta\|\phi\|_L$.

Lemma 1 ([11, p.139])**.** *Assume (μ,ϕ) is a quasimode of A with accuracy $\delta>0$ and the spectrum of A is discrete in the interval $[\mu-\delta,\mu+\delta]$. Then there exists an eigenvalue μ_* of A such that $|\mu_*-\mu|\leqslant\delta$.*

Proof. If $\mu\in\sigma(A)$, then $\mu_*=\mu$. Otherwise the distance d_μ from μ to the spectrum of A can be computed as

$$d_\mu=\|(A-\mu)^{-1}\|^{-1}=\inf_{\psi\neq 0}\frac{\|\psi\|_L}{\|(A-\mu)^{-1}\psi\|_L},$$

where ψ is an arbitrary vector of L. Taking $\psi=(A-\mu)\phi$, we deduce

$$d_\mu\leqslant\frac{\|(A-\mu)\phi\|_L}{\|\phi\|_L}\leqslant\delta,$$

from which the assertion follows. □

Given an eigenvalue λ of $\mathcal{H}$ with eigenfunction u, we will prove that the pair (λ,v_ε) is a quasimode of H_ε with an infinitely small accuracy as $\varepsilon\to 0$, where v_ε is constructed as in (5.2) above. Write $\varrho_\varepsilon=(H_\varepsilon-\lambda)v_\varepsilon$. We see that

$$\varrho_\varepsilon=(-\Delta+W-\lambda)(u-\eta_\varepsilon)=\Delta\eta_\varepsilon-W\eta_\varepsilon+\lambda\eta_\varepsilon$$

outside ω_ε, and therefore $\sup_{x\in\mathbb{R}^2\setminus\omega_\varepsilon}|\varrho_\varepsilon(x)|\leqslant c_1\varepsilon$, because of (5.3). Note that η_ε is a function of compact support.

Recalling representation (3.2) of the Laplace operator in the local coordinates, we deduce

$$\begin{aligned}&-\Delta+W(x)+V_\varepsilon(x)=-\varepsilon^{-2}\partial_n^2+\varepsilon^{-1}\varkappa\partial_n+n\varkappa^2\partial_n-\partial_s^2-\varepsilon P_\varepsilon\\&\quad+W(s,\varepsilon n)+\varepsilon^{-2}V(n)+\varepsilon^{-1}U(s,n)=\varepsilon^{-2}\ell_0+\varepsilon^{-1}\ell_1+\ell_2+W(s,\varepsilon n)-\varepsilon P_\varepsilon,\end{aligned}$$

for $x\in\omega_\varepsilon$, where $\ell_0=-\partial_n^2+V$, $\ell_1=\varkappa\partial_n+U$ and $\ell_2=n\varkappa^2\partial_n-\partial_s^2$. Then

$$\begin{aligned}\varrho_\varepsilon&=(-\Delta+W+V_\varepsilon-\lambda)v_\varepsilon=\big(\varepsilon^{-2}\ell_0+\varepsilon^{-1}\ell_1+\ell_2+W(s,\varepsilon n)-\varepsilon P_\varepsilon-\lambda\big)\big(v_0+\varepsilon v_1+\varepsilon^2v_2\big)\\&=\varepsilon^{-2}\ell_0v_0+\varepsilon^{-1}(\ell_0v_1+\ell_1v_0)+\big(\ell_0v_2+\ell_1v_1+(\ell_2+W(s,0)-\lambda)v_0\big)\\&\quad+(W(s,\varepsilon n)-W(s,0))v_0+\varepsilon\big(\ell_1v_2+(\ell_2+W(s,\varepsilon n)-\lambda)(v_1+\varepsilon v_2)-P_\varepsilon v_\varepsilon\big).\end{aligned}\tag{5.4}$$

From our choice of v_k, we derive that the first three terms in the right-hand side vanish. Then $\sup_{x\in\omega_\varepsilon}|\varrho_\varepsilon(x)|\leqslant c_2\varepsilon$. Hence we have

$$\|(H_\varepsilon-\lambda)v_\varepsilon\|_{L_2(\mathbb{R}^2)}=\|\varrho_\varepsilon\|_{L_2(\mathbb{R}^2)}\leqslant|\omega_{2\beta}|^{1/2}\sup_{\mathbb{R}^2}|\varrho_\varepsilon|\leqslant c_3\varepsilon,$$

since $\operatorname{supp}\varrho_\varepsilon\subset\omega_{2\beta}$. On the other hand, the main contribution to the $L_2(\mathbb{R}^2)$-norm of v_ε is given by the eigenfunction u, because the norms of η_ε and v_k, $k=0,1,2$, are infinitely small as $\varepsilon\to0$. Therefore $\|v_\varepsilon\|_{L_2(\mathbb{R}^2)}\geqslant\frac12\|u\|_{L_2(\mathbb{R}^2)}$ for ε small enough. Finally, we obtain

$$\|(H_\varepsilon-\lambda)v_\varepsilon\|_{L_2(\mathbb{R}^2)}\leqslant c_3\varepsilon\leqslant2c_3\varepsilon\|u\|^{-1}_{L_2(\mathbb{R}^2)}\|v_\varepsilon\|_{L_2(\mathbb{R}^2)}\leqslant c_4\varepsilon\,\|v_\varepsilon\|_{L_2(\mathbb{R}^2)}.$$

In view of Lemma 1, there exists an eigenvalue λ^ε of H_ε such that

$$|\lambda^\varepsilon-\lambda|\leqslant c_4\varepsilon$$

for all ε small enough.

6. Proof of Main Theorem

Let $\{\lambda^\varepsilon\}_{\varepsilon>0}$ be a sequence of eigenvalues of operator H_ε and $\{u_\varepsilon\}_{\varepsilon>0}$ be the sequence of the corresponding eigenfunctions. Assume that $\|u_\varepsilon\|_{L_2(\mathbb{R}^2)}=1$.

Lemma 2. *Assume that $\lambda^\varepsilon\to\lambda$ and $u_\varepsilon\to u$ in $L_2(\mathbb{R}^2)$ weakly as $\varepsilon\to0$.*

(i) For any bounded domain $D\subset\mathbb{R}^2$ such that $\overline{D}\cap\gamma=\emptyset$ the sequence of eigenfunctions u_ε converges in $W_2^2(D)$ weakly, as $\varepsilon\to0$, to some function u. If potential W belongs to $C^2(\overline{D})$, then $u_\varepsilon\to u$ in $W_2^4(D)$ weakly.

(ii) The function u solves the equation

$$-\Delta u+Wu=\lambda u\quad\text{in }\mathbb{R}^2\setminus\gamma.\tag{6.1}$$

(iii) For any bounded domain $D\subset\mathbb{R}^2$ there exists the constant C such that

$$\|u_\varepsilon\|_{W_2^2(D\setminus\omega_\varepsilon)}\leqslant C\tag{6.2}$$

for all ε small enough.

(iv) Finally, $u_\varepsilon|_{\gamma_{-\varepsilon}}\to u^-$ and $u_\varepsilon|_{\gamma_\varepsilon}\to u^+$ in $L_2(\gamma)$ weakly as $\varepsilon\to0$.

Proof. *(i)* Recall that the support of short-range potential V_ε lies in ω_ε and chose ε so small that $D \cap \omega_\varepsilon = \emptyset$. Then for any $\phi \in C_0^\infty(D)$ we conclude from (2.4) that

$$\int_D \Delta u_\varepsilon \phi \, dx = \int_D (W - \lambda^\varepsilon) \, u_\varepsilon \phi \, dx.$$

The right-hand side has a limit as $\varepsilon \to 0$ by the assumptions, thus the left-hand side also converges for all $\phi \in C_0^\infty(D)$, i.e., $\Delta u_\varepsilon \to \Delta u$ in $L_2(D)$ weakly. From this we deduce that u_ε converges to u in $W_2^2(D)$ weakly, and hence that

$$\int_D \Delta u \phi \, dx = \int_D (W - \lambda) \, u \phi \, dx.$$

If $W \in C^2(D)$, then we can apply the Laplace operator to both sides of equality $\Delta u_\varepsilon = (W - \lambda^\varepsilon) u_\varepsilon$ and obtain

$$\int_D \Delta^2 u_\varepsilon \phi \, dx = \int_D \big((W - \lambda^\varepsilon) \, \Delta u_\varepsilon \phi + 2 \nabla W \nabla u_\varepsilon \phi + \Delta W u_\varepsilon \phi \big) \, dx.$$

The right-hand side of this identity converges as $\varepsilon \to 0$ for all $\phi \in C_0^\infty(D)$. From this we see that $\Delta^2 u_\varepsilon \to \Delta^2 u$ in $L_2(D)$ weakly, and finally that $u_\varepsilon \to u$ in $W_2^4(D)$ weakly.

(ii) From what has already been proved, we have

$$\int_{\mathbb{R}^2} \Delta u \phi \, dx = \int_{\mathbb{R}^2} (W - \lambda) \, u \phi \, dx$$

for all test functions $\phi \in C_0^\infty(\mathbb{R}^2)$ such that $\operatorname{supp} \phi \cap \gamma = \emptyset$. Therefore u is a solution of (6.1).

(iii) Let χ_ε be the characteristic function of $D \setminus \omega_\varepsilon$. We conclude from

$$\int_{D \setminus \omega_\varepsilon} \Delta u_\varepsilon \phi \, dx = \int_{D \setminus \omega_\varepsilon} (W - \lambda^\varepsilon) \, u_\varepsilon \phi \, dx$$

that the family of functionals $\chi_\varepsilon \Delta u_\varepsilon$ in $L_2(\mathbb{R}^2)$ is pointwise bounded, since the right-hand side of the last identity converges for any $\phi \in L_2(\mathbb{R}^2)$. In view of the uniform boundedness principle, we have $\|\chi_\varepsilon \Delta u_\varepsilon\|_{L_2(\mathbb{R}^2)} \leqslant C_1$, from which (6.2) follows.

(iv) Let us introduce the function $\zeta_\varepsilon(r) = (r - \varepsilon)\zeta(r)\chi_{(\varepsilon,+\infty)}(r)$, where $\chi_{(\varepsilon,+\infty)}$ is the characteristic function of $(\varepsilon, +\infty)$ and ζ is plotted in Fig. 2. Since $\zeta_\varepsilon(\varepsilon) = 0$ and $\zeta_\varepsilon'(\varepsilon + 0) = 1$ for ε small enough, integrating by parts yields

$$\int_{\gamma_\varepsilon} u_\varepsilon a \, d\gamma = \int_{D_\varepsilon^+} (W - \lambda^\varepsilon) u_\varepsilon a \zeta_\varepsilon \, dx - \int_{D_\varepsilon^+} u_\varepsilon \Delta(a \zeta_\varepsilon) \, dx. \tag{6.3}$$

Here a is a smooth function on γ and $D_\varepsilon^+ = \omega_\beta^+ \setminus \omega_\varepsilon$. Similarly, we obtain from (6.1) the equality

$$\int_\gamma u^+ a \, d\gamma = \int_{\omega_\beta^+} (W - \lambda) u a \zeta_0 \, dx - \int_{\omega_\beta^+} u \Delta(a \zeta_0) \, dx, \tag{6.4}$$

where $\zeta_0(r) = r\zeta(r)$. Obviously,

$$\int_{D_\varepsilon^+} (W - \lambda^\varepsilon) u_\varepsilon a \zeta_\varepsilon \, dx \to \int_{\omega_\beta^+} (W - \lambda) u a \zeta_0 \, dx$$

as $\varepsilon \to 0$, because ζ_ε converges to ζ_0 uniformly on $\mathbb{R}_+$. Recalling (3.2), we can write

$$\int_{\omega_\beta^+} u_\varepsilon \Delta(a\zeta_\varepsilon)\, dx = \int_S \int_0^\beta u_\varepsilon(s,r)\zeta_\varepsilon(r)\, \partial_s \left(\frac{a'(s)}{1 - r\varkappa(s)}\right) ds\, dr$$
$$+ \int_S \int_0^\beta u_\varepsilon(s,r) a(s)\big(J(s,r)(2\zeta'(r) + (r-\varepsilon)\zeta''(r)) - \varkappa(s)(r-\varepsilon)\zeta'(r)\big)\, ds\, dr$$
$$- \int_S \int_\varepsilon^\beta u_\varepsilon(s,r) a(s)\varkappa(s)\zeta(r)\, ds\, dr, \quad (6.5)$$

provide $\varepsilon < \beta/2$. Here we used the equalities $\zeta'(r) = 0$ and $\zeta''(r) = 0$ for $r \in (0,\varepsilon)$. The right-hand side of (6.5) converges to

$$\int_S \int_0^\beta u^+(s,r)\zeta_0(r)\, \partial_s \left(\frac{a'(s)}{1 - r\varkappa(s)}\right) ds\, dr$$
$$+ \int_S \int_0^\beta u^+(s,r) a(s)\big(J(s,r)(2\zeta'(r) + r\zeta''(r)) - \varkappa(s)(\zeta(r) + r\zeta'(r))\big)\, ds\, dr,$$

which coincides with $\int_{\omega_\beta^+} u^+ \Delta(a\zeta_0)\, dx$. Therefore we conclude from (6.3) and (6.4) that $\int_{\gamma_\varepsilon} u_\varepsilon a\, d\gamma \to \int_\gamma u^+ a\, d\gamma$ for all $a \in C^\infty(\gamma)$, hence that $u_\varepsilon|_{\gamma_\varepsilon} \to u^+$ in $L_2(\gamma)$ weakly as $\varepsilon \to 0$. The proof of the weak convergence for $u_\varepsilon|_{\gamma_{-\varepsilon}}$ is similar. $\square$

6.1. **Case of zero-energy resonance.** $\Upsilon = \frac{1}{2}(\theta^2 - 1)\varkappa + \mu$.

Lemma 3. *Suppose that $\lambda^\varepsilon \to \lambda$ and $u_\varepsilon \to u$ in $L_2(\mathbb{R}^2)$ weakly as $\varepsilon \to 0$, and the one-dimensional Schrödinger operator $-\frac{d^2}{dt^2} + V$ possesses a zero-energy resonance, then λ is an eigenvalue of $\mathcal{H}$ associated with the eigenfunction u.*

The eigenvalue λ^ε and the corresponding eigenfunction u_ε satisfy the identity

$$\int_{\mathbb{R}^2} \big(\nabla u_\varepsilon \nabla\phi + (W + V_\varepsilon - \lambda^\varepsilon) u_\varepsilon \phi\big)\, dx = 0, \qquad \phi \in W_2^1(\mathbb{R}^2).$$

Let λ and u the eigenvalue and the corresponding eigenfunction of $\mathcal{H}$. Then

$$\int_{\Omega^+} \nabla u \nabla\psi\, dx + \int_{\Omega^-} \nabla u \nabla\psi\, dx + \int_{\mathbb{R}^2} (W - \lambda) u\psi\, dx + \int_\gamma \Upsilon u^- \phi^-\, d\gamma = 0$$

for all functions ψ belonging to the set $\Psi_\theta = \{f \in W_2^1(\mathbb{R}^2 \setminus \gamma) \colon f^+ = \theta f^- \text{ on } \gamma\}$.

Given $\psi \in \Psi_\theta \cap C_b^\infty(\mathbb{R}^2 \setminus \gamma)$, we construct a sequence $\{\psi_\varepsilon\}_{\varepsilon>0}$ in the space $W_2^1(\mathbb{R}^2)$ as follows. Suppose h is a half-bound state of $-\frac{d^2}{dt^2} + V$ such that $h(-1) = 1$, and functions h_1 and h_2 solve the Cauchy problems on the interval $\mathcal{I}$

$$-h_1'' + Vh_1 = 0, \quad h_1(-1) = 0, \;\; h_1'(-1) = 1; \quad (6.6)$$
$$-h_2'' + Vh_2 = \varkappa(s)h' + U(s,\cdot)h, \quad h_2(s,-1) = 0, \;\; \partial_n h_2(s,-1) = 0 \quad (6.7)$$

respectively. Let us write

$$\psi_0^\varepsilon(s,n) = \psi(s,-\varepsilon)\, h(n), \quad \psi_1^\varepsilon(s,n) = \partial_r\psi(s,-\varepsilon)\, h_1(n) - \psi(s,-\varepsilon)\, h_2(s,n).$$

and set

$$\hat\psi_\varepsilon(x) = \begin{cases} \psi(x), & \text{if } x \in \mathbb{R}^2 \setminus \omega_\varepsilon, \\ \psi_0^\varepsilon(s, \frac{r}{\varepsilon}) + \varepsilon\psi_1^\varepsilon\left(s, \frac{r}{\varepsilon}\right) & \text{if } x = (s,r) \in \omega_\varepsilon. \end{cases}$$

The function ψ_ε does not in general belong to $W_2^1(\mathbb{R}^2)$, because it has a jump discontinuity on the boundary $\partial\omega_\varepsilon$ composed of two curves $\gamma_{-\varepsilon}$ and γ_ε. Then $[\hat\psi_\varepsilon]_{-\varepsilon}=0$, by construction. Recalling the function ζ plotted in Fig. 2, we write

$$\rho_\varepsilon(x)=\begin{cases}-[\hat\psi_\varepsilon]_\varepsilon\,\zeta(r-\varepsilon), & \text{if } x=(s,r)\in\omega_{2\beta}\setminus\omega_\varepsilon,\\ 0, & \text{otherwise.}\end{cases}$$

Finally, we set $\psi_\varepsilon=\hat\psi_\varepsilon+\rho_\varepsilon$. The direct calculations show that $[\psi_\varepsilon]_\varepsilon=0$ and, therefore, ψ_ε belongs to $W_2^1(\mathbb{R}^2)$.

The first observation of the sequence $\{\psi_\varepsilon\}_{\varepsilon>0}$ is that it converges to ψ in $L_2(\mathbb{R}^2)$. In fact, we have for all $s\in S$

$$[\hat\psi_\varepsilon]_\varepsilon(s)=\psi(s,\varepsilon)-\theta\psi(s,-\varepsilon)-\varepsilon\psi_{1,\varepsilon}(s,1)=\psi(s,+0)-\theta\psi(s,-0)+O(\varepsilon)=O(\varepsilon),$$

as $\varepsilon\to0$, since $\psi(s,+0)=\theta\psi(s,-0)$. Hence $\rho_\varepsilon\to0$ in $L_2(\mathbb{R}^2)$.

Proposition 3. *For $\psi\in\Psi_\theta\cap C_b^\infty(\mathbb{R}^2\setminus\gamma)$, we have*

$$\int_Q(\partial_n u_\varepsilon\partial_n\psi_0^\varepsilon+Vu_\varepsilon\psi_0^\varepsilon)J_\varepsilon\,dn\,ds=\varepsilon\int_Q\varkappa u_\varepsilon\partial_n\psi_0^\varepsilon\,dn\,ds,\tag{6.8}$$

$$\begin{aligned}\int_Q(\partial_n u_\varepsilon\partial_n\psi_1^\varepsilon+Vu_\varepsilon\psi_1^\varepsilon+Uu_\varepsilon\psi_0^\varepsilon)J_\varepsilon\,dn\,ds=&\\ \int_S\Big(u_\varepsilon(s,-\varepsilon)\partial_r\psi(s,-\varepsilon)\big(1+\varepsilon\varkappa(s)\big)&\\ -\theta^{-1}u_\varepsilon(s,\varepsilon)\big(\partial_r\psi(s,-\varepsilon)-\Upsilon(s)\psi(s,-\varepsilon)\big)(1-\varepsilon\varkappa(s))\Big)\,ds&\\ -\int_Q\varkappa u_\varepsilon\,\partial_n\psi_0^\varepsilon\,dn\,ds+\varepsilon\int_Q\varkappa u_\varepsilon(\varkappa\partial_n\psi_0^\varepsilon-\partial_n\psi_1^\varepsilon)\,dn\,ds.&\end{aligned}\tag{6.9}$$

where $J_\varepsilon(s,n)=1-\varepsilon n\varkappa(s)$.

Proof. The function ψ_0^ε solves the equation $-\partial_n^2v+Vv=0$ in Q and $h'(-1)=h'(1)=0$. Then

$$\begin{aligned}0=\int_Q u_\varepsilon(-\partial_n^2\psi_0^\varepsilon+V\psi_0^\varepsilon)J_\varepsilon\,dn\,ds&=-\int_S\psi(s,-\varepsilon)(u_\varepsilon J_\varepsilon h')\big|_{n=-1}^{n=1}\,ds\\ &+\int_Q(\partial_n u_\varepsilon\,\partial_n\psi_0^\varepsilon+Vu_\varepsilon\psi_0^\varepsilon)J_\varepsilon\,dn\,ds+\int_Q u_\varepsilon\,\partial_n J_\varepsilon\,\partial_n\psi_0^\varepsilon\,dn\,ds\\ &=\int_Q(\partial_n u_\varepsilon\,\partial_n\psi_0^\varepsilon+Vu_\varepsilon\psi_0^\varepsilon)J_\varepsilon\,dn\,ds-\varepsilon\int_Q\varkappa u_\varepsilon\,\partial_n\psi_0^\varepsilon\,dn\,ds,\end{aligned}$$

from which (6.8) follows.

Since $h(1)=\theta$, the Lagrange identity $(h_1h'-h_1'h)|_{-1}^1=0$ for equation (6.6) implies

$$h_1'(1)=\theta^{-1}.\tag{6.10}$$

Multiplying the equation in (6.7) by h and integrating by parts twice yield

$$(h'h_2-h\,\partial_nh_2)\big|_{-1}^1=\varkappa(s)\int_{\mathcal I}hh'\,dn+\int_{\mathcal I}U(s,n)h^2(n)\,dn.$$

Recalling (4.15), it follows that $\theta\,\partial_nh_2(s,1)=-\frac12(\theta^2-1)\varkappa(s)-\mu(s)$ and finally that

$$\partial_nh_2(s,1)=-\theta^{-1}\Upsilon(s).\tag{6.11}$$

To prove (6.9), we note that ψ_1^ε is a solution of the equation

$$-\partial_n^2 v + Vv = -\varkappa\partial_n\psi_0^\varepsilon - U\psi_0^\varepsilon,$$

which follows from (6.6) and (6.7). Hence

$$\int_Q u_\varepsilon(-\partial_n^2\psi_1^\varepsilon + V\psi_1^\varepsilon + U\psi_0^\varepsilon)J_\varepsilon\,dn\,ds = -\int_Q \varkappa u_\varepsilon\,\partial_n\psi_0^\varepsilon J_\varepsilon\,dn\,ds. \tag{6.12}$$

On the other hand, integrating by parts with respect to n, we find

$$\begin{aligned}
-\int_Q u_\varepsilon\,\partial_n^2\psi_1^\varepsilon J_\varepsilon\,dn\,ds &= \int_Q \partial_n(u_\varepsilon J_\varepsilon)\,\partial_n\psi_1^\varepsilon\,dn\,ds - \int_S (u_\varepsilon J_\varepsilon\partial_n\psi_1^\varepsilon)\big|_{n=-1}^{n=1}\,ds\\
&= \int_Q \partial_n u_\varepsilon\,\partial_n\psi_1^\varepsilon J_\varepsilon\,dn\,ds - \varepsilon\int_Q \varkappa u_\varepsilon\,\partial_n\psi_1^\varepsilon\,dn\,ds\\
&\quad - \int_S \Big(u_\varepsilon(s,\varepsilon n)J_\varepsilon(s,n)\big(\partial_r\psi(s,-\varepsilon)\,h_1'(n) - \psi(s,-\varepsilon)\,\partial_n h_2(s,n)\big)\Big)\Big|_{n=-1}^{n=1}\,ds\\
&= \int_Q \partial_n u_\varepsilon\,\partial_n\psi_1^\varepsilon J_\varepsilon\,dn\,ds - \varepsilon\int_Q \varkappa u_\varepsilon\,\partial_n\psi_1^\varepsilon\,dn\,ds\\
&\quad - \int_S \Big(\theta^{-1}u_\varepsilon(s,\varepsilon)\big(\partial_r\psi(s,-\varepsilon) - \Upsilon(s)\psi(s,-\varepsilon)\big)(1-\varepsilon\varkappa(s))\\
&\qquad - u_\varepsilon(s,-\varepsilon)\partial_r\psi(s,-\varepsilon)\big(1+\varepsilon\varkappa(s)\big)\Big)\,ds,
\end{aligned} \tag{6.13}$$

in view of initial conditions (6.6), (6.7) and equalities (6.10), (6.11). Substituting (6.13) into (6.12), we obtain (6.9). □

Proposition 4. *Under the assumptions of Lemma 3, for all $\psi \in \Psi_\theta \cap C_b^2(\mathbb{R}^2\setminus\gamma)$ we have*

$$\int_{\omega_\varepsilon}\big(\nabla u_\varepsilon\nabla\psi_\varepsilon + V_\varepsilon u_\varepsilon\psi_\varepsilon\big)\,dx \to \int_\gamma \Upsilon u^-\psi^-\,d\gamma,$$

as ε tends to zero.

Proof. Let us note here, for future use,

$$\int_{\omega_\varepsilon} g(x)\,dx = \varepsilon\int_Q g(s,\varepsilon n)J_\varepsilon(s,n)\,ds\,dn,$$
$$|\nabla v(x_\varepsilon)|^2 = \varepsilon^{-2}|\partial_n v(s,n)|^2 + J_\varepsilon^{-2}(s,n)\,|\partial_s v(s,n)|^2,$$

where $v(x_\varepsilon)$ stands for $v(s,\frac{r}{\varepsilon})$, cf. (**??**). Then

$$\begin{aligned}
&\int_{\omega_\varepsilon}\big(\nabla u_\varepsilon\nabla\psi_\varepsilon + V_\varepsilon u_\varepsilon\psi_\varepsilon\big)\,dx\\
&\qquad = \varepsilon^{-1}\int_Q\big(\partial_n u_\varepsilon\,\partial_n\psi_\varepsilon + \varepsilon^2 J_\varepsilon^{-2}\partial_s u_\varepsilon\,\partial_s\psi_\varepsilon + Vu_\varepsilon\psi_\varepsilon + \varepsilon Uu_\varepsilon\psi_\varepsilon\big)J_\varepsilon\,dn\,ds\\
&= \varepsilon^{-1}\int_Q(\partial_n u_\varepsilon\partial_n\psi_0^\varepsilon + Vu_\varepsilon\psi_0^\varepsilon)J_\varepsilon\,dn\,ds + \int_Q(\partial_n u_\varepsilon\partial_n\psi_1^\varepsilon + Vu_\varepsilon\psi_1^\varepsilon + Uu_\varepsilon\psi_0^\varepsilon)J_\varepsilon\,dn\,ds\\
&\qquad + \varepsilon\int_Q Uu_\varepsilon\psi_1^\varepsilon J_\varepsilon\,dn\,ds + \varepsilon^2\int_Q \partial_s u_\varepsilon\,\partial_s\psi_\varepsilon J_\varepsilon^{-1}\,dn\,ds.
\end{aligned}$$

In view of Proposition 3, we have

$$\int_{\omega_\varepsilon}\big(\nabla u_\varepsilon\nabla\psi_\varepsilon + V_\varepsilon u_\varepsilon\psi_\varepsilon\big)\,dx = \int_S\Big(u_\varepsilon(s,-\varepsilon)\partial_r\psi(s,-\varepsilon)\big(1+\varepsilon\varkappa(s)\big)$$

$$
\begin{aligned}
&- \theta^{-1} u_\varepsilon(s,\varepsilon)\big(\partial_r \psi(s,-\varepsilon) - \Upsilon(s)\psi(s,-\varepsilon)\big)(1-\varepsilon\varkappa(s))\Big)\, ds \\
&+ \varepsilon \int_Q u_\varepsilon\big(\varkappa^2 \partial_n \psi_0^\varepsilon - \varkappa \partial_n \psi_1^\varepsilon + U\psi_1^\varepsilon J_\varepsilon\big)\, dn\, ds + \varepsilon^2 \int_Q \partial_s u_\varepsilon\, \partial_s \psi_\varepsilon J_\varepsilon^{-1}\, dn\, ds. \qquad (6.14)
\end{aligned}
$$

For any sequence ϕ_ε bounded in the $L_2(Q)$-norm,the estimate

$$
\begin{aligned}
\left|\int_Q u_\varepsilon(s,\varepsilon n)\phi_\varepsilon(s,n)\, dn\, ds\right| &\leqslant \left(\int_Q |u_\varepsilon(s,\varepsilon n)|^2\, dn\, ds\right)^{1/2} \|\phi_\varepsilon\|_{L_2(Q)} \\
&\leqslant c\left(\varepsilon^{-1}\int_{\omega_\varepsilon} |u_\varepsilon(x)|^2\, dx\right)^{1/2} \leqslant c\varepsilon^{-1/2}\|u_\varepsilon\|_{L_2(\mathbb{R}^2)} = c\varepsilon^{-1/2}
\end{aligned}
$$

holds. Also, we have

$$
\left|\int_Q \partial_s u_\varepsilon\, \partial_s \psi_\varepsilon J_\varepsilon^{-1}\, dn\, ds\right| = \left|\int_Q u_\varepsilon\, \partial_s(J_\varepsilon^{-1}\partial_s \psi_\varepsilon)\, dn\, ds\right| \leqslant c_1 \varepsilon^{-1/2},
$$

because $\varkappa \in C^1(\gamma)$ and $\psi \in C_b^\infty(\mathbb{R}^2 \setminus \gamma)$ and, therefore, the function $\partial_s(J_\varepsilon^{-1}\partial_s\psi_\varepsilon)$ is bounded on Q uniformly on ε.

Then (6.14) implies

$$
\begin{aligned}
&\int_{\omega_\varepsilon} \big(\nabla u_\varepsilon \nabla \psi_\varepsilon + V_\varepsilon u_\varepsilon \psi_\varepsilon\big)\, dx \\
&\quad\to \int_S \Big(u(s,-0)\partial_r\psi(s,-0) - \theta^{-1}u(s,+0)\big(\partial_r\psi(s,-0) - \Upsilon(s)\psi(s,-0)\big)\Big)\, ds \\
&\qquad = \int_\gamma \Big(u^-\partial_r\psi^- - \theta^{-1}u^+\big(\partial_r\psi^- - \Upsilon\psi^-\big)\Big)\, d\gamma = \int_\gamma \Upsilon u^-\psi^-\, d\gamma,
\end{aligned}
$$

since $\theta^{-1}u^+ = u^-$. □

Proposition 5.

$$
\int_{\omega_\varepsilon} |u_\varepsilon|^2\, dx \to 0, \quad \textit{as } \varepsilon \to 0
$$

Proof.

$$
w_\varepsilon(s,r) = w_0^\varepsilon\left(s,\tfrac{r}{\varepsilon}\right) + \varepsilon w_1^\varepsilon\left(s,\tfrac{r}{\varepsilon}\right) + \varepsilon^2 w_2^\varepsilon\left(s,\tfrac{r}{\varepsilon}\right),
$$

where $w_0^\varepsilon = u_\varepsilon(s,-\varepsilon)h(n)$, and w_1^ε, w_2^ε solve the Cauchy problems

$$
\begin{aligned}
&-\partial_n^2 w_1^\varepsilon + V w_1^\varepsilon = (-\varkappa\partial_n - U)w_0^\varepsilon, \quad w_1^\varepsilon(\,\cdot\,,-1) = \partial_n w_1^\varepsilon(\,\cdot\,,-1) = 0; \\
&-\partial_n^2 w_2^\varepsilon + V w_2^\varepsilon = -(\varkappa\partial_n + U))w_1^\varepsilon + \big(\partial_s^2 - n\varkappa^2\partial_n - W(\,\cdot\,,0) + \lambda^\varepsilon\big)w_0^\varepsilon, \\
&\quad w_2^\varepsilon(\,\cdot\,,-1) = \partial_n w_2^\varepsilon(\,\cdot\,,-1) = 0
\end{aligned}
$$

respectively. All functions $w_k^\varepsilon\colon Q \to \mathbb{R}$ are bounded in $L_2(Q)$ uniformly on ε, because $\lambda^\varepsilon \to \lambda$ and $u_\varepsilon(s,-\varepsilon) \to u(s,-0)$ in $L_2(S)$ weakly and therefore $\|u_\varepsilon(\,\cdot\,,-\varepsilon)\|_{L_2(S)} \leqslant c$.

Reasoning as in (5.4) we deduce that w_ε is a solution of the equation

$$
-\Delta w_\varepsilon + (W + V_\varepsilon - \lambda^\varepsilon)w_\varepsilon = f_\varepsilon \quad \text{in } Q_\varepsilon,
$$

where $\|f_\varepsilon\|_{L_2(Q_\varepsilon)} \leqslant c_1\varepsilon$. It follows that the difference $g_\varepsilon = w_\varepsilon - u_\varepsilon$ solves the Dirichlet type boundary value problem

$$
\begin{gathered}
-\Delta g_\varepsilon + (W + V_\varepsilon - \lambda^\varepsilon)g_\varepsilon = f_\varepsilon \quad \text{in } Q_\varepsilon, \\
g_\varepsilon(s,-\varepsilon) = 0, \qquad g_\varepsilon(s,\varepsilon) = \theta u_\varepsilon(s,-\varepsilon) - u_\varepsilon(s,\varepsilon) + \varepsilon w_1^\varepsilon(s,1) + \varepsilon^2 w_2^\varepsilon(s,1).
\end{gathered}
$$

Hence

$$\|g_\varepsilon\|_{L_2(Q_\varepsilon)} \leqslant c_2(\|f_\varepsilon\|_{L_2(Q_\varepsilon)} +)$$

□

Proof of Lemma. which may be rewritten as

$$\int_{\omega_\varepsilon} \big(\nabla u_\varepsilon \nabla \psi_\varepsilon + V_\varepsilon u_\varepsilon \psi_\varepsilon\big)\, dx = -\int_{\mathbb{R}^2\setminus\omega_\varepsilon} \nabla u_\varepsilon \nabla \psi_\varepsilon\, dx - \int_{\mathbb{R}^2} (W - \lambda^\varepsilon) u_\varepsilon \psi_\varepsilon\, dx \quad (6.15)$$

The right-hand side of (6.15) has a finite limit

$$-\int_{\mathbb{R}^2} \big(\nabla u \nabla \psi + (W - \lambda) u \psi\big)\, dx$$

as $\varepsilon \to 0$. In particular, the term $\int_{\mathbb{R}^2} W u_\varepsilon \phi\, dx$ converges for any $\phi \in \operatorname{dom} H_0$ by the rapidly decay of eigenfunctions of H_ε [2, Ch.3.3]. Therefore the left-hand side of (6.15) also converges as $\varepsilon \to 0$.

□

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[12] Nazarov

[13] Perez

Department of Mechanics and Mathematics, Ivan Franko National University of Lviv, 1 Universytetska str., 79000 Lviv, Ukraine

Email address: yuriy.golovaty@lnu.edu.ua