# Wave Equation Background

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## 1 Introduction

The one dimensional scalar wave equation takes the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $u$ is a scalar field defined on $x \in \Omega$ (where $\Omega$ is some one-dimensional domain) and evolving in time for all $t \in \tau$ (where $\tau$ is an interval in $\mathbb{R}$). $c \in \mathbb{R}$ is a constant. In the initial-boundary formulation (the problem I solve here), we specify $u(0, x)$ and $u(t, \partial\Omega)$, where $\partial\Omega$ is the boundary of the domain.

This is the problem I will solve. For simplicity, we will assume that

$$\begin{aligned} \Omega &= [0, L], \text{ where } L > 0 \in \mathbb{R} \qquad (2)\\ \text{and } \tau &= [0, T], \text{ where } T > 0 \in \mathbb{R}. \qquad (3) \end{aligned}$$

Then the initial and boundary conditions are

$$\begin{aligned} u(0, x) &= f(x) \text{ where } f(x) \in L_2(\Omega), \qquad (4)\\ u(t, 0) &= \alpha, \text{ where } \alpha \in \mathbb{R} \qquad (5)\\ \text{and } u(t, L) &= \beta \text{ where } \alpha \in \mathbb{R}. \qquad (6) \end{aligned}$$

## 2 Coordinates

We will be feeding the wave equation in to a fourth-order Runge-Kutta integrator. Therefore, we would like to write the wave equation as a coupled system of first-order equations, not second order. Therefore, we let

$$s = \frac{\partial u}{\partial t} \text{ and } r = \frac{\partial u}{\partial x}. \tag{7}$$

This transforms the wave equation into a first-order equation:

$$\frac{\partial s}{\partial t} = c^2 \frac{\partial r}{\partial x}. \tag{8}$$

Additionally, if we take a derivative of $r$ with respect to $t$, we get a second relationship:

$$\begin{aligned} \frac{\partial r}{\partial t} &= \frac{\partial}{\partial t}\frac{\partial}{\partial x} u = \frac{\partial}{\partial x}\frac{\partial}{\partial t} u \\ \Rightarrow \frac{\partial r}{\partial t} &= \frac{\partial s}{\partial x}. \qquad (9) \end{aligned}$$

These two equations define the system we will integrate using Runge-Kutta and finite difference methods. This changes the boundary conditions somewhat. Now,

$$\begin{aligned} r(0,x) &= f'(x) & (10)\\ s(t,0) &= s(t,L) = 0. & (11) \end{aligned}$$

And now $\alpha$ and $\beta$ are constants of integration when we retrieve $u$ by integrating $s$.

## 3 Finite Differences

We place the functions $u$ and $r$ on a discrete, one-dimensional grid with fixed spacing. Let the grid points be indexed by $i$ ranging from 0 to $n$. And let the grid spacing be $h$.

This means the notion of a continuous derivative doesn't exist. To fix this, we use a method known as finite differences. We approximate derivatives at points on the interior by a "centered difference" formula and derivatives at points on the boundary by a "one-sided difference" formula. Let $v$ be a scalar field on the grid. Let each element be $v_i$, where $i$ is the index of the grid points. Then,

$$Dv_i = \begin{cases} \frac{v_{i+1}-v_{i-1}}{2h} & \text{if } 0 < i < n \\ \frac{v_i - v_{i-1}}{h} & \text{if } i = n \\ \frac{v_{i+1}-v_i}{h} & \text{if } i = 0 \end{cases} . \tag{12}$$

These formulae can easily be derived by using taylor series. See, for example [1] or [2].

There is a problem, however. In the continuous case, we proved the existence of solutions using a "maximum principle" [3, 4]. And the proof of this maximum principle requires "integration by parts" to hold [4]:

$$\langle a, Db \rangle + \langle Da, b \rangle = (ab)\Big|_0^L, \tag{13}$$

for all functions $a$ and $b$ in $L_2(\Omega)$, where $\langle a, b \rangle$ is the standard inner product on the space:

$$\langle a, b \rangle = \int_0^L ab dx. \tag{14}$$

To show stability (which we won't), we must cook up a discrete inner product so that a discrete version of integration by parts, called "summation by parts," holds.

The key turns out to be to weigh the boundary points by half in the inner product. The resulting inner product looks something like this:

$$\langle a, b \rangle = \frac{h}{2}(a_0 b_0 + a_n b_n) + h \sum_{i=1}^{n-1} a_i b_i, \tag{15}$$

where $a_i$ and $b_i$ are the components of the vectors $a$ and $b$, which exist on the discrete grid. Let's show that summation by parts does indeed hold with this inner product:

$$
\begin{aligned}
\langle a, Db\rangle + \langle Da, b\rangle &= \frac{1}{2}(a_0 Db_0 + a_n Db_n + b_0 Da_0 + b_n Da_n) + \sum_{i=1}^{n-1}(a_i Db_i + b_i Da_i) \\
&= \frac{1}{2}\left[a_0(b_1 - b_0) + b_0(a_1 - a_0) + a_n(b_n - b_{n-1}) + b_n(a_n - a_{n-1})\right] \\
&\quad + \frac{1}{2}\sum_{i=1}^{n}\left[b_i(a_{i+1} - a_{i-1}) + a_i(b_{i+1} - b_{i-1})\right] \\
&= \frac{1}{2}(2a_n b_n - 2a_0 b_0) + \frac{1}{2}(a_0 b_1 + a_1 b_0 - a_n b_{n-1} - b_n a_{n-1}) \\
&\quad + \frac{1}{2}\sum_{i=1}^{n-1}(b_i a_{i+1} - b_i a_{i-1} + a_i b_{i+1} - a_i b_{i-1}) \qquad (16)
\end{aligned}
$$

Notice that in equation (16), the $\frac{1}{2}(2a_n b_n - 2a_0 b_0)$ term is equal to $(ab)\Big|_0^L$. Furthermore, if we expand the sum from $i = 1$ to $i = n-1$, it will collapse, leaving only the initial and final terms. And these terms cancel perfectly with the $\frac{1}{2}(a_0 b_1 + a_1 b_0 - a_n b_{n-1} - b_n a_{n-1})$ terms. So, in the end, we're left with

$$\langle a, Db\rangle + \langle Da, b\rangle = (ab)\Big|_0^L. \qquad (17)$$

This means that our inner product is stable. For more information, see [5, 6, 7, 8].

# 4 Numerical Methods

When we put $r$ and $s$ on a grid, we generate two $n$-dimensional vectors that are related by the equations discussed above. To solve for the time evolution of these vectors, we can feed them into a Runge-Kutte integrator. For more information on Runge Kutte methods, see, for example, [2].

# References


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