

# Wave Equation Background

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## 1 Introduction

The one dimensional scalar wave equation takes the form

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad (1)$$

where  $u$  is a scalar field defined on  $x \in \Omega$  (where  $\Omega$  is some one-dimensional domain) and evolving in time for all  $t \in \tau$  (where  $\tau$  is an interval in  $\mathbb{R}$ ).  $c \in \mathbb{R}$  is a constant. In the initial-boundary formulation (the problem I solve here), we specify  $u(0, x)$  and  $u(t, \partial\Omega)$ , where  $\partial\Omega$  is the boundary of the domain.

This is the problem I will solve. For simplicity, we will assume that

$$\Omega = [0, L], \text{ where } L > 0 \in \mathbb{R} \quad (2)$$

$$\text{and } \tau = [0, T], \text{ where } T > 0 \in \mathbb{R}. \quad (3)$$

Then the initial and boundary conditions are

$$u(0, x) = f(x) \text{ where } f(x) \in L_2(\Omega), \quad (4)$$

$$u(t, 0) = \alpha, \text{ where } \alpha \in \mathbb{R} \quad (5)$$

$$\text{and } u(t, L) = \beta \text{ where } \alpha \in \mathbb{R}. \quad (6)$$

## 2 Coordinates

We will be feeding the wave equation in to a fourth-order Runge-Kutta integrator. Therefore, we would like to write the wave equation as a coupled system of first-order equations, not second order. Therefore, we let

$$s = \frac{\partial u}{\partial t} \text{ and } r = \frac{\partial u}{\partial x}. \quad (7)$$

This transforms the wave equation into a first-order equation:

$$\frac{\partial s}{\partial t} = c^2 \frac{\partial r}{\partial x}. \quad (8)$$

Additionally, if we take a derivative of  $r$  with respect to  $t$ , we get a second relationship:

$$\begin{aligned} \frac{\partial r}{\partial t} &= \frac{\partial}{\partial t} \frac{\partial}{\partial x} u = \frac{\partial}{\partial x} \frac{\partial}{\partial t} u \\ \Rightarrow \frac{\partial r}{\partial t} &= \frac{\partial s}{\partial x}. \end{aligned} \quad (9)$$

These two equations define the system we will integrate using Runge-Kutta and finite difference methods. This changes the boundary conditions somewhat. Now,

$$r(0, x) = f'(x) \quad (10)$$

$$s(t, 0) = s(t, L) = 0. \quad (11)$$

And now  $\alpha$  and  $\beta$  are constants of integration when we retrieve  $u$  by integrating  $s$ .

### 3 Finite Differences

We place the functions  $u$  and  $r$  on a discrete, one-dimensional grid with fixed spacing. Let the grid points be indexed by  $i$  ranging from 0 to  $n$ . And let the grid spacing be  $h$ .

This means the notion of a continuous derivative doesn't exist. To fix this, we use a method known as finite differences. We approximate derivatives at points on the interior by a "centered difference" formula and derivatives at points on the boundary by a "one-sided difference" formula. Let  $v$  be a scalar field on the grid. Let each element be  $v_i$ , where  $i$  is the index of the grid points. Then,

$$Dv_i = \begin{cases} \frac{v_{i+1} - v_{i-1}}{2h} & \text{if } 0 < i < n \\ \frac{v_i - v_{i-1}}{h} & \text{if } i = n \\ \frac{v_{i+1} - v_i}{h} & \text{if } i = 0 \end{cases}. \quad (12)$$

These formulae can easily be derived by using Taylor series. See, for example [1] or [2].

There is a problem, however. In the continuous case, we proved the existence of solutions using a "maximum principle" [3, 4]. And the proof of this maximum principle requires "integration by parts" to hold [4]:

$$\langle a, Db \rangle + \langle Da, b \rangle = (ab) \Big|_0^L, \quad (13)$$

for all functions  $a$  and  $b$  in  $L_2(\Omega)$ , where  $\langle a, b \rangle$  is the standard inner product on the space:

$$\langle a, b \rangle = \int_0^L ab dx. \quad (14)$$

To show stability (which we won't), we must cook up a discrete inner product so that a discrete version of integration by parts, called "summation by parts," holds.

The key turns out to be to weigh the boundary points by half in the inner product. The resulting inner product looks something like this:

$$\langle a, b \rangle = \frac{h}{2}(a_0b_0 + a_nb_n) + h \sum_{i=1}^{n-1} a_ib_i, \quad (15)$$

where  $a_i$  and  $b_i$  are the components of the vectors  $a$  and  $b$ , which exist on the discrete grid. Let's show that summation by parts does indeed hold with this inner product:

$$\begin{aligned}
\langle a, Db \rangle + \langle Da, b \rangle &= \frac{1}{2}(a_0Db_0 + a_nDb_n + b_0Da_0 + b_nDa_n) + \sum_{i=1}^{n-1}(a_iDb_i + b_iDa_i) \\
&= \frac{1}{2}[a_0(b_1 - b_0) + b_0(a_1 - a_0) + a_n(b_n - b_{n-1}) + b_n(a_n - a_{n-1})] \\
&\quad + \frac{1}{2} \sum_{i=1}^n [b_i(a_{i+1} - a_{i-1}) + a_i(b_{i+1} - b_{i-1})] \\
&= \frac{1}{2}(2a_nb_n - 2a_0b_0) + \frac{1}{2}(a_0b_1 + a_1b_0 - a_nb_{n-1} - b_na_{n-1}) \\
&\quad + \frac{1}{2} \sum_{i=1}^{n-1} (b_ia_{i+1} - b_ia_{i-1} + a_ib_{i+1} - a_ib_{i-1})
\end{aligned} \tag{16}$$

Notice that in equation (16), the  $\frac{1}{2}(2a_nb_n - 2a_0b_0)$  term is equal to  $(ab)\Big|_0^L$ . Furthermore, if we expand the sum from  $i = 1$  to  $i = n - 1$ , it will collapse, leaving only the initial and final terms. And these terms cancel perfectly with the  $\frac{1}{2}(a_0b_1 + a_1b_0 - a_nb_{n-1} - b_na_{n-1})$  terms. So, in the end, we're left with

$$\langle a, Db \rangle + \langle Da, b \rangle = (ab)\Big|_0^L. \tag{17}$$

This means that our inner product is stable. For more information, see [5, 6, 7, 8].

## 4 Numerical Methods

When we put  $r$  and  $s$  on a grid, we generate two  $n$ -dimensional vectors that are related by the equations discussed above. To solve for the time evolution of these vectors, we can feed them into a Runge-Kutte integrator. For more information on Runge Kutte methods, see, for example, [2].

## References

- [1] Wikipedia contributors. Finite difference. Wikipedia, The Free Encyclopedia. November 13, 2013, 20:42 UTC. Available at: [http://en.wikipedia.org/w/index.php?title=Finite\\_difference&oldid=581529514](http://en.wikipedia.org/w/index.php?title=Finite_difference&oldid=581529514). Accessed November 20, 2013.
- [2] Heath, Michael T. *Scientific Computing: An Introductory Survey*. United States: McGraw-Hill; 1997.
- [3] Wikipedia contributors. Maximum principle. Wikipedia, The Free Encyclopedia. March 23, 2013, 09:10 UTC. Available at: [http://en.wikipedia.org/w/index.php?title=Maximum\\_principle&oldid=546502732](http://en.wikipedia.org/w/index.php?title=Maximum_principle&oldid=546502732). Accessed November 20, 2013.
- [4] Bleeker, David D. and Csordas, George. *Basic Partial Differential Equations*. Boston, MA: International Press of Boston; 1996.
- [5] Diener, P, Dorband, E, Schnetter, and E, Tiglio, M. "New, efficient, and accurate high order derivatives and dissipation operators satisfying summation by parts, and applications in three-dimensional multi-block evolutions." <http://arxiv.org/abs/gr-qc/0512001>. (2005). Online resource.
- [6] Olsson, Pelle. "Summation by Parts, Projections, and Stability. I." *Mathematics of Computation*. 64.11 (1995): 1035–1065. Online resource.

- [7] Olsson, Pelle. “Supplement to Summation by Parts, Projections, and Stability. I.” *Mathematics of Computation*. 64.211 (1995): S23–S26. Online resource.
- [8] Olsson, Pelle. “Summation by Parts, Projections, and Stability II.” *Mathematics of Computation*. 64.212 (1995): 1473–1493. Online resource.