The 2D solution

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{\eta}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\rho h^3}{\eta}\frac{\partial p}{\partial y}\right)=12\left\{\frac{\partial}{\partial x}\left[\frac{\rho h\left(u_A+u_B\right)}{2}\right]+\frac{\partial}{\partial y}\left[\frac{\rho h\left(v_A+v_B\right)}{2}\right]+\frac{\partial(\rho h)}{\partial t}\right\} \tag{4.3}$$

$$\frac{\partial}{\partial x}\left(\frac{\rho h^3}{\eta}\frac{\partial p}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\rho h^3}{\eta}\frac{\partial p}{\partial y}\right)=12\left\{u_{av}\frac{\partial(\rho hU)}{\partial x}+u_{av}\frac{\partial(\rho hV)}{\partial y}+\frac{\partial(\rho h)}{\partial t}\right\} \tag{4.5}$$

where:

$$u=\frac{u_A+u_B}{2} \text{ and } v=\frac{v_A+v_B}{2} \tag{4.4}$$

$U=\frac{u}{u_{av}}[-]$; $V=\frac{v}{u_{av}}[-]$ u_{av} is the average value of the entrainment motion.

Dimensionless parameters 2D:

$$\begin{cases} X=x/b \\ Y=y/a \\ \bar\rho=\rho/\rho_0 \\ \bar\eta=\eta/\eta_0 \\ H=hR_x/b^2=\frac{mm}{m^2}=[-] \\ P=p/P_h=[-] \\ \bar t=u_{av}t/R_x=\frac{\frac{m}{s}s}{m}=[-] \\ W^*=W/(E'R_xL)=\frac{N}{\frac{N}{m^2}m^2}=[-] \\ G^*=\alpha E' \end{cases} \rightarrow \begin{cases} x=bX \\ y=aY \\ \rho=\rho_0\bar\rho \\ \eta=\eta_0\bar\eta \\ h=Hb^2/R_x \\ p=P_hP \\ t=R_x\bar t/u_{av} \\ W=W^*E'R_xL \\ G^*=\alpha E' \end{cases} \rightarrow \begin{cases} dx=bdX \\ dy=adY \\ d\rho=\rho_0 d\bar\rho \\ d\eta=\eta_0 d\bar\eta \\ dh=(b^2/R_x)dH \\ dp=P_h dP \\ dt=(R_x/u_{av})d\bar t \end{cases} \tag{5.1}$$

Assuming that $\bar\rho\neq f(t)$ and considering (5.1) the Reynolds equation becomes:

$$\underbrace{\frac{\partial}{\partial X}\left(\frac{\bar\rho H^3}{\bar\eta}\frac{\partial P}{\partial X}\right)_{i,j}}_{A}+k^2\underbrace{\frac{\partial}{\partial Y}\left(\frac{\bar\rho H^3}{\bar\eta}\frac{\partial P}{\partial Y}\right)_{i,j}}_{B}=\psi\left\{\underbrace{\left(\frac{\partial(\bar\rho HU)}{\partial X}\right)_{i,j}}_{C}+k\underbrace{\left(\frac{\partial(\bar\rho HV)}{\partial Y}\right)_{i,j}}_{D}+\frac{R_x}{b}\bar\rho S^*\right\}$$

(see equation 5-33 at pg183. It's the same!!)

$$\text{where: } \psi=12\frac{u_{av}\eta_0R_x^2}{P_hb^3};\quad S^*=\frac{\partial h/\partial t}{u_{av}} \text{ and } k=\frac{b}{a} \tag{5.4}$$

Finite difference representation of Reynolds equation

- central difference method for the left hand side terms
- central difference and backwards difference method for the right hand terms.
- β is a weight factor: β=0 foreword sif. scheme. β=1 backwards dif. scheme. β=0.5 central dif. scheme.

$$\frac{\partial}{\partial X}\left(\frac{\bar\rho H^3}{\bar\eta}\frac{\partial P}{\partial X}\right)_{i,j}=$$

$$=\frac{1}{\Delta X}\left[\left(\frac{\bar\rho H^3}{\bar\eta}\frac{\partial P}{\partial X}\right)_{i+\frac12,j}-\left(\frac{\bar\rho H^3}{\bar\eta}\frac{\partial P}{\partial X}\right)_{i-\frac12,j}\right]=\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i+\frac12,j}\frac{P_{i+1,j}-P_{i,j}}{\Delta X^2}-\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i-\frac12,j}\frac{P_{i,j}-P_{i-1,j}}{\Delta X^2}=$$

$$=\frac{1}{2\Delta X^2}\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i+1,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}\right]\left(P_{i+1,j}-P_{i,j}\right)-\frac{1}{2\Delta X^2}\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i-1,j}\right]\left(P_{i,j}-P_{i-1,j}\right)=$$

$$\rightarrow \boxed{A}=\frac{1}{2\Delta X^2}\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i-1,j}\right]P_{i-1,j}-\frac{1}{2\Delta X^2}\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i+1,j}+2\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i-1,j}\right]P_{i,j}+$$
$$\frac{1}{2\Delta X^2}\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i+1,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}\right]P_{i+1,j}$$

$$\frac{\partial}{\partial Y}\left(\frac{\bar\rho H^3}{\bar\eta}\frac{\partial P}{\partial Y}\right)_{i,j}=$$

$$=\frac{1}{\Delta Y}\left[\left(\frac{\bar\rho H^3}{\bar\eta}\frac{\partial P}{\partial Y}\right)_{i,j+\frac12}-\left(\frac{\bar\rho H^3}{\bar\eta}\frac{\partial P}{\partial Y}\right)_{i,j-\frac12}\right]=$$

$$=\frac{1}{2\Delta Y^2}\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j+1}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}\right]\left(P_{i,j+1}-P_{i,j}\right)-\frac{1}{2\Delta Y^2}\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j-1}\right]\left(P_{i,j}-P_{i,j-1}\right)=$$

$$\rightarrow \boxed{B}=\frac{1}{2\Delta Y^2}\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j-1}\right]P_{i,j-1}-\frac{1}{2\Delta Y^2}\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j+1}+2\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j-1}\right]P_{i,j}+$$
$$\frac{1}{2\Delta Y^2}\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j+1}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}\right]P_{i,j+1}$$

$$\boxed{C}=\left(\frac{\partial(\bar\rho HU)}{\partial X}\right)_{i,j}=(1-\beta_x)\frac{(\bar\rho HU)_{i+1,j}-(\bar\rho HU)_{i,j}}{\Delta X}+\beta_x\frac{(\bar\rho HU)_{i,j}-(\bar\rho HU)_{i-1,j}}{\Delta X}$$

$$\boxed{D}=\left(\frac{\partial(\bar\rho HV)}{\partial Y}\right)_{i,j}=(1-\beta_y)\frac{(\bar\rho HV)_{i,j+1}-(\bar\rho HV)_{i,j}}{\Delta Y}+\beta_y\frac{(\bar\rho HV)_{i,j}-(\bar\rho HV)_{i,j-1}}{\Delta Y}$$

$$\boxed{A}_{i,j}+k^2\boxed{B}_{i,j}=\psi\left\{\boxed{C}_{i,j}+k\boxed{D}_{i,j}+\frac{R_x}{b}\bar\rho S^*\right\}$$

(5.11)

$$F_i)\ F_{i,j}=\boxed{A}_{i,j}+k^2\boxed{B}_{i,j}-\psi\left\{\boxed{C}_{i,j}+k\boxed{D}_{i,j}+\frac{R_x}{b}\bar\rho S^*\right\} \tag{5.12}$$

$$F_{i,j}=\frac{1}{2\Delta X^2}\left\{\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i-1,j}\right]P_{i-1,j}-\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i+1,j}+2\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i-1,j}\right]P_{i,j}+\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i+1,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}\right]P_{i}\right.$$
$$+\frac{k^2}{2\Delta Y^2}\left\{\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j-1}\right]P_{i,j-1}-\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j+1}+2\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j-1}\right]P_{i,j}+\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j+1}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}\right]P_{i,j+1}\right\}$$
$$-\psi\frac{1}{\Delta X}\left\{(1-\beta_x)\left[(\bar\rho HU)_{i+1,j}-(\bar\rho HU)_{i,j}\right]+\beta_x\left[(\bar\rho HU)_{i,j}-(\bar\rho HU)_{i-1,j}\right]\right\}-$$
$$-k\psi\frac{1}{\Delta Y}\left\{(1-\beta_y)\left[(\bar\rho HV^*)_{i,j+1}-(\bar\rho HV^*)_{i,j}\right]+\beta_y\left[(\bar\rho HV^*)_{i,j}-(\bar\rho HV^*)_{i,j-1}\right]\right\}-\psi\frac{R_x}{b}(\bar\rho S^*)_{i,j}$$

This is identical with equation (5-82) for β=1

Modified Newton-Raphson method in EHL problems

Assume that $P_{i,j}$ are a set of approximate solutions to the real solutions $\bar P_{i,j}$. Therefore from equation (5.12):

$$\begin{cases}\bar F_i = f\left(\bar P_{i-1},\bar P_i,\bar P_{i+1}\right)=0\\ F_i = f\left(P_{i-1},P_i,P_{i+1}\right)\neq 0\end{cases}\text{(5.41 and 5.42)}$$

By applying Taylor's series expansion the equation (5.12) can be expressed as well:

$$\bar F_{i,j}=F_{i,j}+\frac{\partial F_{i,j}}{\partial P_{i-1,j}}\Delta P_{i-1,j}+\frac{\partial F_{i,j}}{\partial P_{i+1,j}}\Delta P_{i+1,j}+\frac{\partial F_{i,j}}{\partial P_{i,j-1}}\Delta P_{i,j-1}+\frac{\partial F_{i,j}}{\partial P_{i,j+1}}\Delta P_{i,j+1}+\frac{\partial F_{i,j}}{\partial P_{i,j}}\Delta P_{i,j}+Err=0 \qquad (5.45)$$

where: $\Delta P_{i,j}=\bar P_{i,j}-P_{i,j}$

Assuming that the truncating error is small enough to be neglected, equation (5.45) can be re-written as:

$$-F_{i,j}=\bar J_{ij,i-1j}\Delta P_{i-1,j}+\bar J_{ij,i+1j}\Delta P_{i+1,j}+\bar J_{ij,ij-1}\Delta P_{i,j-1}+\bar J_{ij,ij+1}\Delta P_{i,j+1}+\bar J_{ij,ij}\Delta P_{i,j}+Err \qquad (5.47)$$

where: $\bar J_{ij,kl}=\dfrac{\partial F_{i,j}}{\partial P_{k,l}}$

Using Gauss-Seidel iteration method:

$$\Delta P^n_{k,l}=\frac{-F_{k,l}-J_{kl,k-1l}\Delta P^n_{k-1l}-J_{kl,k+1l}\Delta P^{n-1}_{k+1l}-J_{kl,kl-1}\Delta P^n_{kl-1}-J_{kl,kl+1}\Delta P^{n-1}_{kl+1}}{J_{kl,kl}}$$ where $J_{i,k}$ are computed in the next section

(5.83)

$$\Delta P^n_{k,l}=\frac{-J[5]-J[1]\Delta P^n_{k-1l}-J[0]\Delta P^{n-1}_{k+1l}-J[3]\Delta P^n_{kl-1}-J[2]\Delta P^{n-1}_{kl+1}}{J[4]}$$

The value of the pressure for the next itereation is:

$$P^n_{i,j}=P^{n-1}_{i,j}+\Omega\Delta P^n_{i,j} \text{(5.84)}$$

Density

$$\bar\rho_{i,j}=1+\frac{0.6\cdot 10^{-9}\cdot P_h\cdot P_{i,j}}{1+1.7\cdot 10^{-9}\cdot P_h\cdot P_{i,j}}$$

Viscosity

$$\bar\eta_{i,j}=e^{[\ln\eta_0+9.67]\left[-1+\left(1+\frac{P_hP_{i,j}}{P_0}\right)^z\right]}$$

Load Balance

$$\left|\bar W-\pi\right|\le Err_W \qquad \text{(Manu) (5-104)}$$

The Jacobian

$$A)\ \bar J_{ij,i+1j}\ \bar J_{ij,i+1j}=\frac{\partial F_{i,j}}{\partial P_{i+1,j}}=$$

$$=\frac{\partial}{\partial P_{i+1,j}}\left[\begin{array}{l}\frac{1}{2\Delta X^2}\left\{\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i-1,j}\right]P_{i-1,j}-\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i+1,j}+2\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i-1,j}\right]P_{i,j}+\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i+1,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}\right]P_{i+1,j}\right\}+\\ +\frac{k^2}{2\Delta Y^2}\left\{\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j-1}\right]P_{i,j-1}-\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j+1}+2\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j-1}\right]P_{i,j}+\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j+1}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}\right]P_{i,j+1}\right\}-\\ -\psi\frac{1}{\Delta X}\left\{(1-\beta_x)\left[(\bar\rho HU)_{i+1,j}-(\bar\rho HU)_{i,j}\right]+\beta_x\left[(\bar\rho HU)_{i,j}-(\bar\rho HU)_{i-1,j}\right]\right\}-\\ -k\psi\frac{1}{\Delta Y}\left\{(1-\beta_y)\left[(\bar\rho HV^*)_{i,j+1}-(\bar\rho HV^*)_{i,j}\right]+\beta_y\left[(\bar\rho HV^*)_{i,j}-(\bar\rho HV^*)_{i,j-1}\right]\right\}-\psi\frac{R_x}{b}(\bar\rho S^*)_{i,j}\end{array}\right]$$

$$\bar J_{ij,i+1j}=\left[\begin{array}{l}\frac{1}{2\Delta X^2}\left\{\left[\mathrm{M}^{i,j}_{i+1,j}+\mathrm{M}^{i-1,j}_{i+1,j}\right]P_{i-1,j}-\left[\mathrm{M}^{i+1,j}_{i+1,j}+2\mathrm{M}^{i,j}_{i+1,j}+\mathrm{M}^{i-1,j}_{i+1,j}\right]P_{i,j}+\left[\mathrm{M}^{i+1,j}_{i+1,j}+\mathrm{M}^{i,j}_{i+1,j}\right]P_{i+1,j}+\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i+1,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}\right]\right\}+\\ +\frac{k^2}{2\Delta Y^2}\left\{\left[\mathrm{M}^{i,j}_{i+1,j}+\mathrm{M}^{i,j-1}_{i+1,j}\right]P_{i,j-1}-\left[\mathrm{M}^{i,j+1}_{i+1,j}+2\mathrm{M}^{i,j}_{i+1,j}+\mathrm{M}^{i,j-1}_{i+1,j}\right]P_{i,j}+\left[\mathrm{M}^{i,j+1}_{i+1,j}+\mathrm{M}^{i,j}_{i+1,j}\right]P_{i,j+1}\right\}-\\ -\psi\frac{1}{\Delta X}\left\{(1-\beta_x)\left[{}_x\mathrm{N}^{i+1,j}_{i+1,j}-{}_x\mathrm{N}^{i,j}_{i+1,j}\right]+\beta_x\left[{}_x\mathrm{N}^{i,j}_{i+1,j}-{}_x\mathrm{N}^{i-1,j}_{i+1,j}\right]\right\}-k\psi\frac{1}{\Delta Y}\left\{(1-\beta_y)\left[{}_y\mathrm{N}^{i,j+1}_{i+1,j}-{}_y\mathrm{N}^{i,j}_{i+1,j}\right]+\beta_y\left[{}_y\mathrm{N}^{i,j}_{i+1,j}-{}_y\mathrm{N}^{i,j-1}_{i+1,j}\right]\right\}-0\end{array}\right]$$

where: $\mathrm{M}^{i,j}_{k,l}=\dfrac{\partial\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}}{\partial P_{k,l}}$, ${}_x\mathrm{N}^{i,j}_{k,l}=\dfrac{\partial(\bar\rho HU)_{i,j}}{\partial P_{k,l}}$

$$B)\ \bar J_{ij,i-1j}\ \bar J_{ij,i-1j}=\frac{\partial F_{i,j}}{\partial P_{i-1,j}}$$

$$\bar J_{ij,i-1j}=\left[\begin{array}{l}\frac{1}{2\Delta X^2}\left\{\left[\mathrm{M}^{i,j}_{i-1,j}+\mathrm{M}^{i-1,j}_{i-1,j}\right]P_{i-1,j}+\left[\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i,j}+\left(\frac{\bar\rho H^3}{\bar\eta}\right)_{i-1,j}\right]-\left[\mathrm{M}^{i+1,j}_{i-1,j}+2\mathrm{M}^{i,j}_{i-1,j}+\mathrm{M}^{i-1,j}_{i-1,j}\right]P_{i,j}+\left[\mathrm{M}^{i+1,j}_{i-1,j}+\mathrm{M}^{i,j}_{i-1,j}\right]P_{i+1,j}\right\}+\\ +\frac{k^2}{2\Delta Y^2}\left\{\left[\mathrm{M}^{i,j}_{i-1,j}+\mathrm{M}^{i,j-1}_{i-1,j}\right]P_{i,j-1}-\left[\mathrm{M}^{i,j+1}_{i-1,j}+2\mathrm{M}^{i,j}_{i-1,j}+\mathrm{M}^{i,j-1}_{i-1,j}\right]P_{i,j}+\left[\mathrm{M}^{i,j+1}_{i-1,j}+\mathrm{M}^{i,j}_{i-1,j}\right]P_{i,j+1}\right\}-\\ -\psi\frac{1}{\Delta X}\left\{(1-\beta_x)\left[{}_x\mathrm{N}^{i+1,j}_{i-1,j}-{}_x\mathrm{N}^{i,j}_{i-1,j}\right]+\beta_x\left[{}_x\mathrm{N}^{i,j}_{i-1,j}-{}_x\mathrm{N}^{i-1,j}_{i-1,j}\right]\right\}-0\end{array}\right]$$

$$C)\ \bar J_{ij,ij+1}\ \bar J_{ij,ij+1}=\frac{\partial F_{i,j}}{\partial P_{i,j+1}}=$$

$$
\bar{J}_{ij,ij+1}=\left[\begin{array}{l}
\dfrac{1}{2\Delta X^2}\left\{\left[\mathrm{M}^{i,j}_{i,j+1}+\mathrm{M}^{i-1,j}_{i,j+1}\right]P_{i-1,j}-\left[\mathrm{M}^{i+1,j}_{i,j+1}+2\mathrm{M}^{i,j}_{i,j+1}+\mathrm{M}^{i-1,j}_{i,j+1}\right]P_{i,j}+\left[\mathrm{M}^{i+1,j}_{i,j+1}+\mathrm{M}^{i,j}_{i,j+1}\right]P_{i+1,j}\right\}+\\
+\dfrac{k^2}{2\Delta Y^2}\left\{\left[\mathrm{M}^{i,j}_{i,j+1}+\mathrm{M}^{i,j-1}_{i,j+1}\right]P_{i,j-1}-\left[\mathrm{M}^{i,j+1}_{i,j+1}+2\mathrm{M}^{i,j}_{i,j+1}+\mathrm{M}^{i,j-1}_{i,j+1}\right]P_{i,j}+\left[\mathrm{M}^{i,j+1}_{i,j+1}+\mathrm{M}^{i,j}_{i,j+1}\right]P_{i,j+1}+\left[\left(\dfrac{\bar{\rho}H^3}{\bar{\eta}}\right)_{i,j+1}+\left(\dfrac{\bar{\rho}H^3}{\bar{\eta}}\right)_{i,j}\right]\right\}-\\
-\psi\dfrac{1}{\Delta X}\left\{\left(1-\beta_x\right)\left[{}_x\mathrm{N}^{i+1,j}_{i,j+1}-{}_x\mathrm{N}^{i,j}_{i,j+1}\right]+\beta_x\left[{}_x\mathrm{N}^{i,j}_{i,j+1}-{}_x\mathrm{N}^{i-1,j}_{i,j+1}\right]\right\}-0
\end{array}\right]
$$

$$
D)\ \bar{J}_{ij,ij-1}\quad \bar{J}_{ij,ij-1}=\frac{\partial F_{i,j}}{\partial P_{i,j-1}}=
$$

$$
\bar{J}_{ij,ij-1}=\left[\begin{array}{l}
\dfrac{1}{2\Delta X^2}\left\{\left[\mathrm{M}^{i,j}_{i,j-1}+\mathrm{M}^{i-1,j}_{i,j-1}\right]P_{i-1,j}-\left[\mathrm{M}^{i+1,j}_{i,j-1}+2\mathrm{M}^{i,j}_{i,j-1}+\mathrm{M}^{i-1,j}_{i,j-1}\right]P_{i,j}+\left[\mathrm{M}^{i+1,j}_{i,j-1}+\mathrm{M}^{i,j}_{i,j-1}\right]P_{i+1,j}\right\}+\\
+\dfrac{k^2}{2\Delta Y^2}\left\{\left[\left(\dfrac{\bar{\rho}H^3}{\bar{\eta}}\right)_{i,j}+\left(\dfrac{\bar{\rho}H^3}{\bar{\eta}}\right)_{i,j-1}\right]+\left[\mathrm{M}^{i,j}_{i,j-1}+\mathrm{M}^{i,j-1}_{i,j-1}\right]P_{i,j-1}-\left[\mathrm{M}^{i,j+1}_{i,j-1}+2\mathrm{M}^{i,j}_{i,j-1}+\mathrm{M}^{i,j-1}_{i,j-1}\right]P_{i,j}+\left[\mathrm{M}^{i,j+1}_{i,j-1}+\mathrm{M}^{i,j}_{i,j-1}\right]P_{i,j+1}\right\}-\\
-\psi\dfrac{1}{\Delta X}\left\{\left(1-\beta_x\right)\left[{}_x\mathrm{N}^{i+1,j}_{i,j-1}-{}_x\mathrm{N}^{i,j}_{i,j-1}\right]+\beta_x\left[{}_x\mathrm{N}^{i,j}_{i,j-1}-{}_x\mathrm{N}^{i-1,j}_{i,j-1}\right]\right\}-0
\end{array}\right]
$$

$$
E)\ \bar{J}_{ij,ij}\quad \bar{J}_{ij,ij}=\frac{\partial F_{i,j}}{\partial P_{i,j}}=
$$

$$
\bar{J}_{ij,ij}=\left[\begin{array}{l}
\dfrac{1}{2\Delta X^2}\left\{\left[\mathrm{M}^{i,j}_{i,j}+\mathrm{M}^{i-1,j}_{i,j}\right]P_{i-1,j}-\left[\left(\dfrac{\bar{\rho}H^3}{\bar{\eta}}\right)_{i+1,j}+2\left(\dfrac{\bar{\rho}H^3}{\bar{\eta}}\right)_{i,j}+\left(\dfrac{\bar{\rho}H^3}{\bar{\eta}}\right)_{i-1,j}\right]-\left[\mathrm{M}^{i+1,j}_{i,j}+2\mathrm{M}^{i,j}_{i,j}+\mathrm{M}^{i-1,j}_{i,j}\right]P_{i,j}+\left[\mathrm{M}^{i+1,j}_{i,j}+\mathrm{M}^{i,j}_{i,j}\right]P_{i+1,j}\right\}+\\
+\dfrac{k^2}{2\Delta Y^2}\left\{\left[\mathrm{M}^{i,j}_{i,j}+\mathrm{M}^{i,j-1}_{i,j}\right]P_{i,j-1}-\left[\left(\dfrac{\bar{\rho}H^3}{\bar{\eta}}\right)_{i,j+1}+2\left(\dfrac{\bar{\rho}H^3}{\bar{\eta}}\right)_{i,j}+\left(\dfrac{\bar{\rho}H^3}{\bar{\eta}}\right)_{i,j-1}\right]-\left[\mathrm{M}^{i,j+1}_{i,j}+2\mathrm{M}^{i,j}_{i,j}+\mathrm{M}^{i,j-1}_{i,j}\right]P_{i,j}+\left[\mathrm{M}^{i,j+1}_{i,j}+\mathrm{M}^{i,j}_{i,j}\right]P_{i,j+1}\right\}-\\
-\psi\dfrac{1}{\Delta X}\left\{\left(1-\beta_x\right)\left[{}_x\mathrm{N}^{i+1,j}_{i,j}-{}_x\mathrm{N}^{i,j}_{i,j}\right]+\beta_x\left[{}_x\mathrm{N}^{i,j}_{i,j}-{}_x\mathrm{N}^{i-1,j}_{i,j}\right]\right\}-\psi\dfrac{R_x}{b}S^*_{i,j}Ro^{i,j}_{i,j}
\end{array}\right]
$$

Term by term:

1) $$\mathrm{M}^{i,j}_{k,l}=\frac{\partial\left(\dfrac{\bar{\rho}H^3}{\bar{\eta}}\right)_{i,j}}{\partial P_{k,l}}=\left(\frac{H^3}{\bar{\eta}}\right)_{i,j}\frac{\partial\bar{\rho}_{i,j}}{\partial P_{k,l}}+\left(\bar{\rho}H^3\right)_{i,j}\frac{\partial\left(\bar{\eta}^{-1}\right)_{i,j}}{\partial P_{k,l}}+\left(\frac{\bar{\rho}}{\bar{\eta}}\right)_{i,j}\frac{\partial\left(H^3\right)_{i,j}}{\partial P_{k,l}}=$$

$$=\left(\frac{H^3}{\bar{\eta}}\right)_{i,j}\frac{\partial\bar{\rho}_{i,j}}{\partial P_{k,l}}-\left(\frac{\bar{\rho}H^3}{\bar{\eta}^2}\right)_{i,j}\frac{\partial\bar{\eta}_{i,j}}{\partial P_{k,l}}+3\left(\frac{\bar{\rho}H^2}{\bar{\eta}}\right)_{i,j}\frac{\partial H_{i,j}}{\partial P_{k,l}}=$$

$$\boxed{\mathrm{M}^{i,j}_{k,l}=\left(\frac{H^3}{\bar{\eta}}\right)_{i,j}Ro^{i,j}_{k,l}-\left(\frac{\bar{\rho}H^3}{\bar{\eta}^2}\right)_{i,j}Et^{i,j}_{k,l}+3\left(\frac{\bar{\rho}H^2}{\bar{\eta}}\right)_{i,j}D_{m,n}}$$

$Ro^{i,j}_{k,l}=\dfrac{\partial\bar{\rho}_{i,j}}{\partial P_{k,l}};\ Et^{i,j}_{k,l}=\dfrac{\partial\bar{\eta}_{i,j}}{\partial P_{k,l}};\ D_{m,n}=\dfrac{\partial H_{i,j}}{\partial P_{k,l}}$ where $m=\left|k-i+1\right|$ and $n=\left|l-j+1\right|$ (5-98,46,47)

2) $${}_x\mathrm{N}^{i,j}_{k,l}=\frac{\partial\left(\bar{\rho}HU\right)_{i,j}}{\partial P_{k,l}}=\left(HU\right)_{i,j}\frac{\partial\bar{\rho}_{i,j}}{\partial P_{k,l}}+\left(\bar{\rho}U\right)_{i,j}\frac{\partial H_{i,j}}{\partial P_{k,l}}+\left(\bar{\rho}H\right)_{i,j}\underbrace{\frac{\partial U_{i,j}}{\partial P_{k,l}}}_{assumed=0}$$

$$\boxed{{}_x\mathrm{N}^{i,j}_{k,l}=\left(HU\right)_{i,j}Ro^{i,j}_{k,l}+\left(\bar{\rho}U\right)_{i,j}D_{m,n}}$$

<u>Discussion 1 (density):</u>

$$\boxed{Ro^{i,j}_{k,l}=\frac{\partial\bar{\rho}_{i,j}}{\partial P_{k,l}}}=\frac{\partial}{\partial P_{k,l}}\left[1+\frac{0.6\cdot10^{-9}\cdot P_h\cdot P_{i,j}}{1+1.7\cdot10^{-9}\cdot P_h\cdot P_{i,j}}\right]=0.6\cdot10^{-9}\cdot P_h\frac{\partial}{\partial P_{k,l}}\left[\frac{P_{i,j}}{1+1.7\cdot10^{-9}\cdot P_h\cdot P_{i,j}}\right]=$$

$$=0.6\cdot10^{-9}\cdot P_h\left[\frac{\dfrac{\partial P_{i,j}}{\partial P_{k,l}}\left(1+1.7\cdot10^{-9}\cdot P_h\cdot P_{i,j}\right)-P_{i,j}\dfrac{\partial\left(1+1.7\cdot10^{-9}\cdot P_h\cdot P_{i,j}\right)}{\partial P_{k,l}}}{\left(1+1.7\cdot10^{-9}\cdot P_h\cdot P_{i,j}\right)^2}\right]=\frac{0.6\cdot10^{-9}\cdot P_h}{\left(1+1.7\cdot10^{-9}\cdot P_h\cdot P_{i,j}\right)^2}\frac{\partial P_{i,j}}{\partial P_{k,l}}$$

If $k\neq i$ and/or $l\neq j\rightarrow\dfrac{\partial P_{i,j}}{\partial P_{k,l}}=0\rightarrow\dfrac{\partial\bar{\rho}_{i,j}}{\partial P_{k,l}}=0$

If $k=i$ and $l=j\rightarrow\dfrac{\partial\bar{\rho}_{i,j}}{\partial P_{k,l}}=\dfrac{0.6\cdot10^{-9}\cdot P_h}{\left(1+1.7\cdot10^{-9}\cdot P_h\cdot P_{i,j}\right)^2}$

<u>Discussion 2 (viscosity):</u>

$$\boxed{Et^{i,j}_{k,l}=\frac{\partial\bar{\eta}_{i,j}}{\partial P_{k,l}}}=\frac{\partial\left[e^{\left[\ln\eta_0+9.67\right]\left[-1+\left(1+\frac{P_hP_{i,j}}{P_0}\right)^z\right]}\right]}{\partial P_{k,l}}=\frac{\partial\left[\left[\ln\eta_0+9.67\right]\left[-1+\left(1+\dfrac{P_hP_{i,j}}{P_0}\right)^z\right]\right]}{\partial P_{k,l}}\bar{\eta}_{i,j}=\left[\ln\eta_0+9.67\right]\bar{\eta}_{i,j}\frac{\partial\left[\left(1+\dfrac{P_hP_{i,j}}{P_0}\right)^z\right]}{\partial P_{k,l}}=$$

$$=\left[\ln\eta_0+9.67\right]\bar{\eta}_{i,j}z\left(1+\frac{P_hP_{i,j}}{P_0}\right)^{z-1}\frac{\partial\left[1+\dfrac{P_hP_{i,j}}{P_0}\right]}{\partial P_{k,l}}=\left[\ln\eta_0+9.67\right]\frac{zP_h\bar{\eta}_{i,j}}{P_0}\left(1+\frac{P_hP_{i,j}}{P_0}\right)^{z-1}\frac{\partial P_{i,j}}{\partial P_{k,l}}$$

If $k\neq i$ and/or $l\neq j\rightarrow\dfrac{\partial P_{i,j}}{\partial P_{k,l}}=0\rightarrow\dfrac{\partial\bar{\eta}_{i,j}}{\partial P_{k,l}}=0$

If $k=i$ and $l=j\rightarrow\dfrac{\partial P_{i,j}}{\partial P_{k,l}}=1\rightarrow\dfrac{\partial\bar{\eta}_{i,j}}{\partial P_{k,l}}=\left[\ln\eta_0+9.67\right]\dfrac{zP_h\bar{\eta}_{i,j}}{P_0}\left(1+\dfrac{P_hP_{i,j}}{P_0}\right)^{z-1}\quad\left(\dfrac{1}{P_0}=5.1e\text{-}9\right)$

<u>Deflection</u> (from Johnson/Manu)

$$\delta_{k,l}=\frac{2P_h}{\pi E'}\sum_{j=1}^{nx}\sum_{i=1}^{ny}P_{i,j}D_{m,n}=[m] \tag{5-45}$$

where m and n incorporate within them the effect of a pressure node (i, j) on a deflection node (k, l) and are expressed as:

$$m=\left|k-i\right|\qquad\left(m=\left|k-i+1\right|\rightarrow Manu\right) \tag{5-46}$$

$$n=\left|l-j\right|\qquad\left(n=\left|l-j+1\right|\rightarrow Manu\right) \tag{5-47}$$

and:

$$D_{m,n} = (\bar{y} - \bar{a})\ln\left[\frac{(\bar{x} - \bar{b}) + \sqrt{(\bar{y} - \bar{a})^2 + (\bar{x} - \bar{b})^2}}{(\bar{x} + \bar{b}) + \sqrt{(\bar{y} - \bar{a})^2 + (\bar{x} + \bar{b})^2}}\right] + (\bar{y} + \bar{a})\ln\left[\frac{(\bar{x} + \bar{b}) + \sqrt{(\bar{y} + \bar{a})^2 + (\bar{x} + \bar{b})^2}}{(\bar{x} - \bar{b}) + \sqrt{(\bar{y} + \bar{a})^2 + (\bar{x} - \bar{b})^2}}\right] +$$

$$+(\bar{x} + \bar{b})\ln\left[\frac{(\bar{y} + \bar{a}) + \sqrt{(\bar{y} + \bar{a})^2 + (\bar{x} + \bar{b})^2}}{(\bar{y} - \bar{a}) + \sqrt{(\bar{y} - \bar{a})^2 + (\bar{x} + \bar{b})^2}}\right] + (\bar{x} - \bar{b})\ln\left[\frac{(\bar{y} - \bar{a}) + \sqrt{(\bar{y} - \bar{a})^2 + (\bar{x} - \bar{b})^2}}{(\bar{y} + \bar{a}) + \sqrt{(\bar{y} + \bar{a})^2 + (\bar{x} - \bar{b})^2}}\right] = [m]$$

(5-48)

where:

$$\bar{b} = \frac{\Delta x}{2},\ \bar{a} = \frac{\Delta y}{2} \text{ and } \bar{x} = x_{k,l} - x_{i,j} = m\Delta x = [m],\ \bar{y} = y_{k,l} - y_{i,j} = n\Delta y = [m] \qquad (5\text{-}49)$$

$$D_{m,n} = ym\ln\left[\frac{xm + \sqrt{ym^2 + xm^2}}{xp + \sqrt{ym^2 + xp^2}}\right] + yp\ln\left[\frac{xp + \sqrt{yp^2 + xp^2}}{xm + \sqrt{yp^2 + xm^2}}\right] + xp\ln\left[\frac{yp + \sqrt{yp^2 + xp^2}}{ym + \sqrt{ym^2 + xp^2}}\right] + xm\ln\left[\frac{ym + \sqrt{ym^2 + xm^2}}{yp + \sqrt{yp^2 + xm^2}}\right]$$

where: $xm = \bar{x} - \bar{b}$; $xp = \bar{x} + \bar{b}$; $ym = \bar{y} - \bar{a}$; $yp = \bar{y} + \bar{a}$