Chapter 2 The Klein-Gordon Field

Consider the amplitude for a free particle to propagate from x_0 to x:

$$U(t) = \langle \vec{x} | e^{-iHt} | \vec{x}_0 \rangle$$

For non-relativistic case, we have

$$U(t) = \langle \vec{x} | e^{-i(\vec{p}^2/2m)t} | \vec{x}_0 \rangle = \int \frac{d^3p}{(2\pi)^3} \langle \vec{x} | e^{-i(\vec{p}^2/2m)t} | \vec{p} \rangle \langle \vec{p} | \vec{x}_0 \rangle$$

$$= \frac{1}{(2\pi)^3} \int d^3p \cdot e^{-i(\vec{p}^2/2m)t} \langle \vec{x} | \vec{p} \rangle \langle \vec{p} | \vec{x}_0 \rangle \qquad \langle \vec{p} | \vec{x} \rangle = e^{-i\vec{p}\cdot\vec{x}}$$

$$= \frac{1}{(2\pi)^3} \int d^3p \cdot e^{-i(\vec{p}^2/2m)t} e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)}$$

$$= \frac{1}{(2\pi)^3} \int d^3p \cdot e^{-i\frac{t}{2m}\vec{p}^2 + i(\vec{x}-\vec{x}_0)\cdot\vec{p}}$$

(using $\int_{-\infty}^{\infty} e^{-ax^2+bx} dx = \sqrt{\frac{\pi}{a}} e^{(b^2/4a)}$)

$$= \frac{1}{(2\pi)^3} \left(\frac{2\pi m}{it}\right)^{\frac{3}{2}} e^{im(\vec{x}-\vec{x}_0)^2/2t}$$

$$= \left(\frac{m}{2\pi i t}\right)^{\frac{3}{2}} e^{im(\vec{x}-\vec{x}_0)^2/2t}$$

$\Longrightarrow$ A particle can propagate between any two points in an arbitrary short time $\rightsquigarrow$ violation of causality.

For relativistic case: $E = \sqrt{p^2+m^2}$, similarly we have

$$U(t) = \langle \vec{x} | e^{-it\sqrt{\vec{p}^2+m^2}} | \vec{x}_0 \rangle$$

$$= \frac{1}{(2\pi)^3} \int d^3p \, e^{-it\sqrt{\vec{p}^2+m^2}} \cdot e^{i\vec{p}\cdot(\vec{x}-\vec{x}_0)}$$

$$= \frac{1}{2\pi^2 |\vec{x}-\vec{x}_0|} \int_0^{\infty} dp \cdot p \sin(p|\vec{x}-\vec{x}_0|) e^{-it\sqrt{p^2+m^2}}$$

Asymptotic analysis $\rightsquigarrow$ violation of causality.

Lagrangian Field Theory:

$$S = \int L \, dt = \int \mathcal{L}(\phi, \partial_\mu\phi) \, d^4x$$

$$0 = \delta S = \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \delta(\partial_\mu\phi) \right\}$$

$$= \int d^4x \left\{ \frac{\partial \mathcal{L}}{\partial \phi} \delta\phi - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) \delta\phi + \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)} \delta\phi\right) \right\}$$

$$\Longrightarrow \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0 \qquad \text{E-L equation}$$

If the Lagrangian contains more than one field, there is one such equation for each.

Hamiltonian Field Theory:

conjugate momentum $p \equiv \partial L / \partial \dot{q}$, $\dot{q} = \partial q / \partial t$ $\quad H = \sum p \dot{q} - L$

$$p(\vec{x}) \equiv \frac{\partial L}{\partial \dot{\phi}(\vec{x})} = \frac{\partial}{\partial \dot{\phi}(\vec{x})} \int \mathcal{L}(\phi(\vec{y}), \dot{\phi}(\vec{y}))\, d^3y$$

$$\sim \frac{\partial}{\partial \dot{\phi}(\vec{x})} \sum_y \mathcal{L}(\phi(\vec{y}), \dot{\phi}(\vec{y}))\, d^3y = \pi(\vec{x})\, d^3y$$

where $\pi(\vec{x}) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}(\vec{x})}$ is called the momentum density conjugate to $\phi(\vec{x})$.

$H = \sum_x p(\vec{x}) \dot{\phi}(x) - L$, passing to the continuum, this becomes

$H = \int d^3x \left[\pi(\vec{x}) \dot{\phi}(\vec{x}) - \mathcal{L} \right] \equiv \int d^3x\, \mathcal{H}$ → Hamiltonian density

Example: consider a theory of a single field $\phi(x)$ with

$$\mathcal{L} = \frac{1}{2} \dot{\phi}^2 - \frac{1}{2} (\nabla \phi)^2 - \frac{1}{2} m^2 \phi^2$$
$$= \frac{1}{2} (\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$$

E-L equation: $\partial_\mu (\partial^\mu \phi) + m^2 \phi = 0$ $\quad$ i.e. $(\partial^\mu \partial_\mu + m^2)\phi = 0$

$$H = \int d^3x\, \mathcal{H} = \int d^3x \left[\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right]$$

Noether's Theorem:

infinitesimal transformation: $\phi(x) \longrightarrow \phi'(x) = \phi(x) + \alpha \Delta \phi(x)$ $\quad \alpha \ll 1$

$\mathcal{L}(x) \longrightarrow \mathcal{L}(x) + \alpha \partial_\mu \mathcal{J}^\mu(x)$ $\quad$ for some $\mathcal{J}^\mu$

$$\alpha \Delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi} (\alpha \Delta \phi) + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \partial_\mu (\alpha \Delta \phi)$$
$$= \alpha\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi \right) + \alpha \left[\frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \right] \Delta \phi$$
$$= \alpha\, \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi \right) \equiv \partial_\mu \mathcal{J}^\mu$$

$\Longrightarrow \partial_\mu j^\mu(x) = 0$ $\quad$ for $\quad j^\mu(x) = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \Delta \phi - \mathcal{J}^\mu$

$\Longrightarrow$ the current $j^\mu(x)$ is conserved.

$Q \equiv \int j^0\, d^3x$ is a constant in time.

The K-G Field as Harmonic Oscillators:

$$\begin{cases} [\phi(\vec{x}), \pi(\vec{y})] = i \delta^{(3)}(\vec{x} - \vec{y}) \\ [\phi(\vec{x}), \phi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0 \end{cases}$$

Fourier trans: $\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot\vec{x}} \phi(\vec{p},t)$

$\left[\frac{\partial^2}{\partial t^2} + (|\vec{p}|^2 + m^2)\right]\phi(\vec{p},t) = 0$ the same as the equation of motion for a simple harmonic oscillator with frequency $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$

$H_{SHO} = \frac{1}{2}p^2 + \frac{1}{2}\omega^2\phi^2$

$\phi = \frac{1}{\sqrt{2\omega}}(a + a^+)$, $p = -i\sqrt{\frac{\omega}{2}}(a - a^+)$

$[\phi, p] = i \iff [a, a^+] = 1 \implies H_{SHO} = \omega(a^+a + \frac{1}{2})$

$|0\rangle$ s.t. $a|0\rangle = 0$ is an eigenstate of H_{SHO} with eigenvalue $\frac{1}{2}\omega$, the zero-point energy. Furthermore we have

$$[H_{SHO}, a^+] = \omega a^+, \quad [H_{SHO}, a] = -\omega a$$

$\implies |n\rangle \equiv (a^+)^n|0\rangle$ are eigenstates of H_{SHO} with eigenvalue $(n+\frac{1}{2})\omega$

We can find the spectrum of the K-G Hamiltonian using the same trick.

$$\begin{cases} \phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} + a^+_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}\right) \\ \pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(a_{\vec{p}} e^{i\vec{p}\cdot\vec{x}} - a^+_{\vec{p}} e^{-i\vec{p}\cdot\vec{x}}\right) \end{cases}$$, rearrange:

$$\begin{cases} \phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}}\left(a_{\vec{p}} + a^+_{-\vec{p}}\right) e^{i\vec{p}\cdot\vec{x}} \\ \pi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3} (-i)\sqrt{\frac{\omega_{\vec{p}}}{2}}\left(a_{\vec{p}} - a^+_{-\vec{p}}\right) e^{i\vec{p}\cdot\vec{x}} \end{cases}$$

$[a_{\vec{p}}, a^+_{\vec{p}'}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{p}')$, use this to verify:

$$[\phi(\vec{x}), \pi(\vec{x}')] = \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{-i}{2}\sqrt{\frac{\omega_{\vec{p}'}}{\omega_{\vec{p}}}}\left([a^+_{-\vec{p}}, a_{\vec{p}'}] - [a_{\vec{p}}, a^+_{-\vec{p}'}]\right) e^{i(\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{x}')}$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^6} \frac{-i}{2}\sqrt{\frac{\omega_{\vec{p}'}}{\omega_{\vec{p}}}}\left(-(2\pi)^3\delta^{(3)}(\vec{p}' + \vec{p}) - (2\pi)^3\delta^{(3)}(\vec{p} + \vec{p}')\right) e^{i(\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{x}')}$$

$$= \int \frac{d^3p\, d^3p'}{(2\pi)^3} i\sqrt{\frac{\omega_{\vec{p}'}}{\omega_{\vec{p}}}}\,\delta^{(3)}(\vec{p} + \vec{p}')\, e^{i(\vec{p}\cdot\vec{x} + \vec{p}'\cdot\vec{x}')}$$

$$= i\int \frac{d^3p}{(2\pi)^3} e^{i\vec{p}\cdot(\vec{x} - \vec{x}')} = i\,\delta^{(3)}(\vec{x} - \vec{x}')$$

$$\implies H = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}}\left(a^+_{\vec{p}} a_{\vec{p}} + \frac{1}{2}[a_{\vec{p}}, a^+_{\vec{p}}]\right)$$

$$H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2\right]$$

$$= \int d^3x \int \frac{d^3p\, d^3p'}{(2\pi)^6} e^{i(\vec{p}+\vec{p}')\cdot\vec{x}} \left\{ -\frac{\sqrt{\omega_{\vec{p}}\omega_{\vec{p}'}}}{4}(a_{\vec{p}} - a^+_{-\vec{p}})(a_{\vec{p}'} - a^+_{-\vec{p}'}) + \frac{-\vec{p}\cdot\vec{p}' + m^2}{4\sqrt{\omega_{\vec{p}}\omega_{\vec{p}'}}}(a_{\vec{p}} + a^+_{-\vec{p}})(a_{\vec{p}'} + a^+_{-\vec{p}'}) \right\}$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}}\left(a^+_{\vec{p}} a_{\vec{p}} + \frac{1}{2}[a_{\vec{p}}, a^+_{\vec{p}}]\right)$$

The second term is proportional to $\delta(0)$. It is the sum over all modes of the zero-point energies $\frac{1}{2}\omega_{\vec{p}}$. This is OK!

$[H, a^\dagger_{\vec{p}}] = \omega_{\vec{p}}\, a^\dagger_{\vec{p}}$, $[H, a_{\vec{p}}] = -\omega_{\vec{p}}\, a_{\vec{p}}$

$|0\rangle$ with $a_{\vec{p}}|0\rangle = 0$ for all $\vec{p}$ is the ground state or vacuum, and has $E=0$ after we drop the zero-point energy. All other eigenstates can be built by acting on $|0\rangle$ with creation operators. In general, the state $a^\dagger_{\vec{p}}\, a^\dagger_{\vec{q}} \cdots |0\rangle$ is an eigenstate of H with energy $\omega_{\vec{p}} + \omega_{\vec{q}} + \cdots$

$\vec{P} = -\int d^3x\, \pi(\vec{x}) \nabla\phi(\vec{x}) = \int \frac{d^3p}{(2\pi)^3}\, \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}}$, so $a^\dagger_{\vec{p}}$ creates momentum $\vec{p}$ and energy $\omega_{\vec{p}} = \sqrt{|\vec{p}|^2 + m^2}$. Similarly, $a^\dagger_{\vec{p}}\, a^\dagger_{\vec{q}} \cdots |0\rangle$ has momentum $\vec{p} + \vec{q} + \cdots$. It is natural to call these excitation particles.

* $a^\dagger_{\vec{p}}$, $a^\dagger_{\vec{q}}$ commute $\Longrightarrow a^\dagger_{\vec{p}}\, a^\dagger_{\vec{q}}|0\rangle$ is identical to $a^\dagger_{\vec{q}}\, a^\dagger_{\vec{p}}|0\rangle$

We conclude: K-G particles obey Bose-Einstein statistics.

* Normalization: $\langle 0|0\rangle = 1$

For $|\vec{p}\rangle \propto a^\dagger_{\vec{p}}|0\rangle$, the simplest normalization $\langle \vec{p}|\vec{q}\rangle = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$ is not Lorentz invariant.

Consider a boost in the 3-direction:

$$P_3' = \gamma(P_3 + \beta E), \quad E' = \gamma(E + \beta P_3)$$

$$\delta^{(3)}(\vec{p}-\vec{q}) = \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\frac{dp_3'}{dp_3} = \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\,\gamma\left(1+\beta\frac{dE}{dp_3}\right)$$

$$= \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\frac{\gamma}{E}(E+\beta P_3) = \delta^{(3)}(\vec{p}\,'-\vec{q}\,')\frac{E'}{E}$$

Note that $E_{\vec{p}}\,\delta^{(3)}(\vec{p}-\vec{q})$ is Lorentz invariant. Therefore we define

$|\vec{p}\rangle = \sqrt{2E_{\vec{p}}}\; a^\dagger_{\vec{p}}|0\rangle$, so that $\langle \vec{p}|\vec{q}\rangle = 2E_{\vec{p}}(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})$

On the Hilbert space, $\Lambda \longrightarrow U(\Lambda)$, a unitary operator.

$$U(\Lambda)|\vec{p}\rangle = |\Lambda\vec{p}\rangle$$

$$U(\Lambda)\, a^\dagger_{\vec{p}}\, U^{-1}(\Lambda) = \sqrt{\frac{E_{\Lambda\vec{p}}}{E_{\vec{p}}}}\; a^\dagger_{\Lambda\vec{p}}$$

Then let's switch to the Heisenberg picture.

$$\begin{cases} \phi(x) = \phi(\vec{x}, t) = e^{iHt} \phi(\vec{x}) e^{-iHt} \\ \pi(x) = \pi(\vec{x}, t) = e^{iHt} \pi(\vec{x}) e^{-iHt} \end{cases}$$

The Heisenberg equation: $i \frac{\partial}{\partial t} \hat{O} = [\hat{O}, H] \Longrightarrow$

$$i \frac{\partial}{\partial t} \phi(\vec{x}, t) = \left[\phi(\vec{x}, t), \int d^3x' \left\{ \frac{1}{2} \pi^2(\vec{x}', t) + \frac{1}{2} (\nabla \phi(\vec{x}', t))^2 + \frac{1}{2} m^2 \phi^2(\vec{x}, t) \right\} \right]$$

$$= \left[\phi(\vec{x}, t), \int d^3x' \left\{ \frac{1}{2} \pi^2(\vec{x}', t) \right\} \right]$$

$$= \int d^3x' \frac{1}{2} \left(\phi(\vec{x}, t) \pi^2(\vec{x}', t) - \pi^2(\vec{x}', t) \phi(\vec{x}, t) \right)$$

$$= \int d^3x' \frac{1}{2} \left(\phi(\vec{x}, t) \pi(\vec{x}', t) \pi(\vec{x}', t) - \pi(\vec{x}', t) \phi(\vec{x}, t) \pi(\vec{x}', t) \right.$$
$$\left. + \pi(\vec{x}', t) \phi(\vec{x}, t) \pi(\vec{x}', t) - \pi(\vec{x}', t) \pi(\vec{x}', t) \phi(\vec{x}, t) \right)$$

$$= \int d^3x' \frac{1}{2} \left([\phi(\vec{x}, t), \pi(\vec{x}', t)] \pi(\vec{x}', t) + \pi(\vec{x}', t) [\phi(\vec{x}, t), \pi(\vec{x}', t)] \right)$$

$$= \int d^3x' \frac{1}{2} \left(2i \delta^{(3)}(\vec{x} - \vec{x}') \pi(\vec{x}', t) \right)$$

$$= i \pi(\vec{x}, t)$$

$$i \frac{\partial}{\partial t} \pi(\vec{x}, t) = -i(-\nabla^2 + m^2) \phi(\vec{x}, t)$$ by similarly calculation

$$\Longrightarrow \frac{\partial^2}{\partial t^2} \phi = (\nabla^2 - m^2) \phi$$ K-G equation.

Note that $H a_{\vec{p}} = a_{\vec{p}} (H - E_{\vec{p}}) \Longrightarrow H^n a_{\vec{p}} = a_{\vec{p}} (H - E_{\vec{p}})^n$

Similarly $H a_{\vec{p}}^{+} = a_{\vec{p}}^{+} (H + E_{\vec{p}}) \Longrightarrow H^n a_{\vec{p}}^{+} = a_{\vec{p}}^{+} (H + E_{\vec{p}})^n$

$$\Longrightarrow e^{iHt} a_{\vec{p}} e^{-iHt} = a_{\vec{p}} e^{-iE_{\vec{p}} t}, \quad e^{iHt} a_{\vec{p}}^{+} e^{-iHt} = a_{\vec{p}}^{+} e^{iE_{\vec{p}} t}$$

$$e^{iHt} a_{\vec{p}} e^{-iHt} = \sum_{n=0}^{\infty} \frac{(iHt)^n}{n!} a_{\vec{p}} e^{-iHt} = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} H^n a_{\vec{p}} e^{-iHt}$$
$$= \sum_{n=0}^{\infty} a_{\vec{p}} (H - E_{\vec{p}})^n \frac{(it)^n}{n!} e^{-iHt} = a_{\vec{p}} e^{i(H - E_{\vec{p}})t} e^{-iHt} = a_{\vec{p}} e^{-iE_{\vec{p}} t}$$

$$\Longrightarrow \phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip \cdot x} + a_{\vec{p}}^{+} e^{ip \cdot x} \right) \Big|_{p^0 = E_{\vec{p}}}$$

$$\pi(\vec{x}, t) = \frac{\partial}{\partial t} \phi(\vec{x}, t)$$

Causality:

The amplitude for a particle to propagate from y to x is $\langle 0 | \phi(x) \phi(y) | 0 \rangle$, we call this quantity $D(x-y)$.

$$D(x-y) = \langle 0 | \phi(x) \phi(y) | 0 \rangle$$

$$= \langle 0| \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} (a_{\vec p} e^{-ip\cdot x} + a^\dagger_{\vec p} e^{ip\cdot x})(a_{\vec q} e^{-iq\cdot y} + a^\dagger_{\vec q} e^{iq\cdot y}) \Big|_{p^0=E_{\vec p},\ q^0=E_{\vec q}} |0\rangle$$

$$= \langle 0| \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} \Big(a_{\vec p} a_{\vec q}\, e^{-i(p\cdot x+q\cdot y)} + a_{\vec p} a^\dagger_{\vec q}\, e^{i(-px+qy)} + a^\dagger_{\vec p} a_q\, e^{i(px-qy)}$$
$$+ a^\dagger_{\vec p} a^\dagger_{\vec q}\, e^{i(px+qy)} \Big) \Big|_{p^0=E_{\vec p},\ q^0=E_{\vec q}} |0\rangle$$

$$= \langle 0| \int \frac{d^3p\, d^3q}{(2\pi)^6} \frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} \Big(a_{\vec p} a^\dagger_{\vec q}\, e^{i(-px+qy)} \Big) \Big|_{p^0=E_{\vec p},\ q^0=E_{\vec q}} |0\rangle$$

Since $\langle 0| a_{\vec p} a^\dagger_{\vec q} |0\rangle = (2\pi)^3 \delta^{(3)}(\vec p - \vec q)$, we have

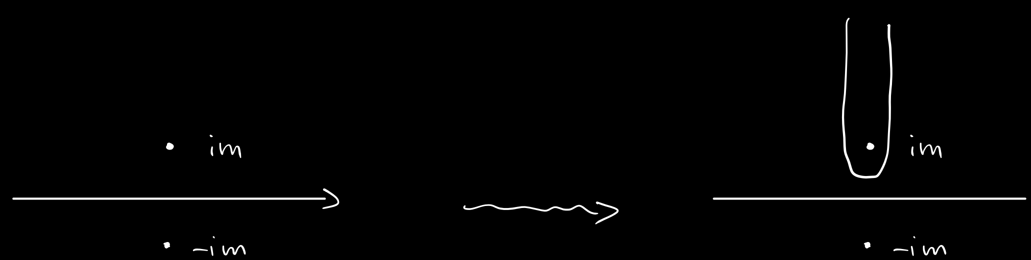
$$D(x-y) = \int \frac{d^3p\, d^3q}{(2\pi)^3} \frac{1}{2\sqrt{E_{\vec p} E_{\vec q}}} \delta^{(3)}(\vec p - \vec q)\, e^{i(\vec p\cdot \vec x - \vec q\cdot \vec y)}\, e^{-i(E_{\vec p} t - E_{\vec q} t)}$$
$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} e^{-ip(x-y)}$$

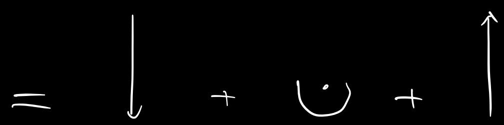
1. Consider $x^0 - y^0 = t$, $\vec x - \vec y = 0$ (time like)

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} e^{-ip^0 t + i\vec p\cdot 0}$$
$$= \frac{1}{(2\pi)^3} \int d^3p \frac{1}{2\sqrt{p^2+m^2}} e^{-i\sqrt{p^2+m^2}\, t}$$
$$= \frac{1}{(2\pi)^3} \int p_r^2 \sin\theta\, dp_r\, dp_\theta\, dp_\varphi \frac{1}{2\sqrt{p_r^2+m^2}} e^{-i\sqrt{p_r^2+m^2}\, t}$$
$$= \frac{1}{2\pi^2} \int dp_r \frac{p_r^2}{2\sqrt{p_r^2+m^2}} e^{-i\sqrt{p_r^2+m^2}\, t}$$
$$= \frac{1}{4\pi^2} \int_m^\infty dE \sqrt{E^2-m^2}\, e^{-iEt}$$
$$\underset{t\to\infty}{\sim} e^{-imt}$$

2. Consider $x^0 - y^0 = 0$, $\vec x - \vec y = \vec r$

$$D(x-y) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec p}} e^{i\vec p\cdot \vec r}$$
$$= \frac{1}{(2\pi)^3} \int p_r^2\, dp_r \frac{1}{2E_{p_r}} \int dp_\varphi \int \sin p_\theta\, dp_\theta\, e^{ip_r r\cos p_\theta}$$
$$= \frac{1}{(2\pi)^2} \int_0^\infty dp \frac{p^2}{2E_p} \int_0^\pi d\theta \sin\theta\, e^{ipr\cos\theta}$$
$$= \frac{1}{(2\pi)^2} \int_0^\infty dp \frac{p^2}{2E_p} \left[\frac{-e^{ipr\cos\theta}}{ipr} \Big|_0^\pi \right]$$
$$= \frac{1}{(2\pi)^2} \int_0^\infty dp \frac{p^2}{2E_p} \frac{e^{ipr} - e^{ipr}}{ipr}$$
$$= \frac{-i}{2(2\pi)^2 r} \int_0^\infty dp \frac{p}{\sqrt{p^2+m^2}} (e^{ipr} - e^{-ipr})$$
$$= \frac{-i}{2(2\pi)^2 r} \int_{-\infty}^\infty dp \frac{p}{\sqrt{p^2+m^2}} e^{ipr}$$





$$= \frac{1}{4\pi^2 r}\int_m^\infty d\rho\, \frac{\rho}{\sqrt{\rho^2-m^2}}\, e^{-\rho r}$$

$$\underset{r\to\infty}{\sim} e^{-mr}$$

So we find that outside the light-cone, the propagation amplitude is exponentially vanishing but nonzero.

To really discuss causality, we should ask whether a measurement performed at one point can affect a measurement at another point whose separation from the first is spacelike. The simplest thing we could try to measure is the field $\phi(x)$, so we should compute the commutator $[\phi(x), \phi(y)]$; if this commutator vanishes, one measurement cannot affect the other. In fact, if the commutator vanishes for $(x-y)^2<0$, causality is preserved quite generally.

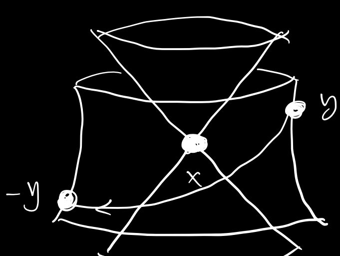
$[\phi(\vec{x}), \phi(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y}) \Longrightarrow$ the commutator vanishes for $x^0 = y^0$.

$$\begin{aligned}
&[\phi(x), \phi(y)] \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}} \int \frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{q}}}} \left[a_{\vec{p}}e^{-ip\cdot x} + a_{\vec{p}}^{\dagger}e^{ip\cdot x},\ a_{\vec{q}}e^{-iq\cdot y} + a_{\vec{q}}^{\dagger}e^{iq\cdot y}\right] \\
&= \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\left([a_{\vec{p}}, a_{\vec{q}}^{\dagger}]e^{i(-px+qy)} - [a_{\vec{q}}, a_{\vec{p}}^{\dagger}]e^{i(px-qy)}\right) \\
&= \int \frac{d^3p\,d^3q}{(2\pi)^6}\frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}(2\pi)^3\left(\delta^{(3)}(\vec{p}-\vec{q})e^{i(-px+qy)} - \delta^{(3)}(\vec{p}-\vec{q})e^{i(px-qy)}\right) \\
&= \int \frac{d^3p\,d^3q}{(2\pi)^3}\frac{1}{2\sqrt{E_{\vec{p}}E_{\vec{q}}}}\delta^{(3)}(\vec{p}-\vec{q})\left(e^{i(-px+qy)} - e^{i(px-qy)}\right) \\
&= \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\left(e^{-ip(x-y)} - e^{ip(x-y)}\right) \\
&= D(x-y) - D(y-x)
\end{aligned}$$

$(x-y) \longmapsto -(x-y)$

$(x-y)^2<0 \Longrightarrow$ we can perform a LT $\Longrightarrow [\phi(x), \phi(y)] = 0$

We conclude that no measurement in the K-G theory can affect another measurement outside the light-cone.



The K-G Propagator:

$[\phi(x), \phi(y)]$ is a c-number, we can write $[\phi(x), \phi(y)] = \langle 0|[\phi(x), \phi(y)]|0\rangle$

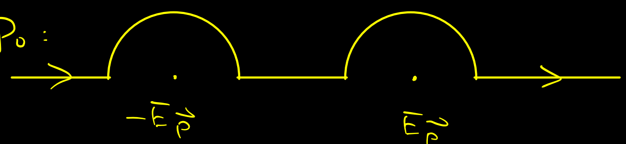
Assume $x^0 > y^0$:

$$\langle 0|[\phi(x), \phi(y)]|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \left\{ \frac{1}{2E_{\vec{p}}} e^{-ip\cdot(x-y)} \Big|_{p^0 = E_{\vec{p}}} + \frac{1}{-2E_{\vec{p}}} e^{-ip\cdot(x-y)} \Big|_{p^0 = -E_{\vec{p}}} \right\} \quad *$$

$$\underset{x^0 > y^0}{=} \int \frac{d^3p}{(2\pi)^3} \int \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-ip\cdot(x-y)}$$

$*$ P_0:



$$\int \frac{dp^0}{2\pi i} \frac{-1}{p^2 - m^2} e^{-ip\cdot(x-y)}$$

$$= \int \frac{dp^0}{2\pi i} \frac{-1}{p^{0\,2} - (\vec{p}^2 + m^2)} e^{-ip\cdot(x-y)}$$

$$= \int \frac{dp^0}{2\pi i} \frac{-1}{p^{0\,2} - E_{\vec{p}}^2} e^{-ip\cdot(x-y)}$$

Be care with the direction of path ! $\rightsquigarrow$ $= -\mathrm{Res}(-E_{\vec{p}}) - \mathrm{Res}(E_{\vec{p}})$

$= *$

For $x^0 > y^0$, we close the contour below; for $x^0 < y^0$, above $\Rightarrow 0$

$$\Longrightarrow D_R(x-y) \equiv \theta(x^0 - y^0) \langle 0|[\phi(x), \phi(y)]|0\rangle$$

$$(\partial^2 + m^2) D_R(x-y) = \left(\partial^2 \theta(x^0 - y^0)\right) \langle 0|[\phi(x), \phi(y)]|0\rangle$$
$$+ 2\left(\partial_\mu \theta(x^0 - y^0)\right)\left(\partial^\mu \langle 0|[\phi(x), \phi(y)]|0\rangle\right)$$
$$+ \theta(x^0 - y^0)(\partial^2 + m^2)\langle 0|[\phi(x), \phi(y)]|0\rangle$$

1^{st} term $= \left(\partial_\mu \delta^\mu(x^0 - y^0)\right) \langle 0|[\phi(x), \phi(y)]|0\rangle$

$= -\delta^\mu(x^0 - y^0)\, \partial_\mu \langle 0|[\phi(x), \phi(y)]|0\rangle$

$= -\delta(x^0 - y^0) \langle 0|[\pi(x), \phi(y)]|0\rangle$

2^{nd} term $= 2\,\delta(x^0 - y^0) \langle 0|[\pi(x), \phi(y)]|0\rangle$

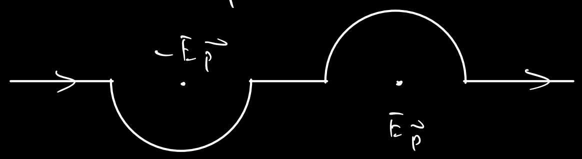
3^{rd} term $= 0$ since ϕ is K-G field

$= \delta(x^0 - y^0) \langle 0|[\pi(x), \phi(y)]|0\rangle$

$= -i\,\delta^{(4)}(x-y)$

$\Longrightarrow D_R(x-y)$ is a Green's function of the K-G operator. Since it vanishes for $x^0 < y^0$, it is the retarded Green's function.

* $\int \frac{dp^0}{2\pi i} \frac{-1}{p^2-m^2} e^{-ip\cdot(x-y)}$ can be evaluated according to four different contours, of which that we used above is only one.



is extremely useful, called Feymann prescription.

We write $D_F(x-y) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2-m^2+i\varepsilon} e^{-ip(x-y)}$

$p^2-m^2+i\varepsilon = p^{0\,2} - \vec{p}^{\,2} + m^2 + i\varepsilon = p^{0\,2} - E_{\vec{p}}^2 + i\varepsilon = 0$

$p^{0\,2} = E_{\vec{p}}^2 - i\varepsilon \sim p^0 = \pm(E_{\vec{p}} - i\varepsilon)$ are the two poles.

$$D_F(x-y) = \begin{cases} D(x-y) & \text{for } x^0 > y^0 \\ D(y-x) & \text{for } x^0 < y^0 \end{cases}$$

$$= \theta(x^0-y^0)\langle 0|\phi(x)\phi(y)|0\rangle + \theta(y^0-x^0)\langle 0|\phi(y)\phi(x)|0\rangle$$

$$\equiv \langle 0|T\phi(x)\phi(y)|0\rangle$$

time-ording symbol T, which instructs us to place the operators that follow in order with the latest to the left.

* D_F is a Green's function of the K-G operator.

The Green's function $D_F(x-y)$ is called the Feymann propagator for a Klein-Gordon particle, since it is, after all, a propagation amplitude.

Particle Creation by a Classical Source:

Consider a K-G field coupled to an external, classical source field $j(x)$. That is, consider the field equation $(\partial^2+m^2)\phi(x) = j(x)$, where $j(x)$ is some fixed, known function of space and time that is nonzero only for a finite time interval.

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2 + j(x)\phi(x)$$

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right) - \frac{\partial\mathcal{L}}{\partial\phi} = \partial^2\phi + m^2\phi^2 - j(x) = 0, \text{ i.e. } (\partial^2+m^2)\phi = j$$

If $j(x)$ is turned on for only a finite time, it is easiest to solve the problem using the field equation directly. Before $j(x)$ is turned on, $\phi(x)$ has the form

$$\phi_0(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}} e^{-ip\cdot x} + a_{\vec{p}}^{\dagger} e^{ip\cdot x} \right)$$

With a source, the solution of the equation of motion can be constructed using the retarded Green's function:

$$\phi(x) = \phi_0(x) + i\int d^4y\, D_R(x-y)\, j(y)$$

$$= \phi_0(x) + i\int d^4y \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \theta(x^0-y^0) \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) j(y)$$

If we wait until all of j is in the past, $\theta = 1$ in the whole domain.

$$= \phi_0(x) + i\int d^4y \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \left(e^{-ip\cdot(x-y)} - e^{ip\cdot(x-y)}\right) j(y)$$

$$= \phi_0(x) + i\int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} \underbrace{\int d^4y \left(e^{ip\cdot y - ip\cdot x} - e^{-ip\cdot y + ip\cdot x}\right) j(y)}$$

$$= \int d^4y\, e^{ip\cdot y} j(y)\, e^{-ip\cdot x} - \int d^4y\, e^{-ip\cdot y} j(y)\, e^{ip\cdot x}$$

$$= \tilde{j}(p)\, e^{-ip\cdot x} - \int d^4y\, e^{-ip\cdot y} j(y)\, e^{ip\cdot x}$$

$$= \tilde{j}(p)\, e^{-ip\cdot x} - \tilde{j}^*(p)\, e^{ip\cdot x}$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left\{ \left(a_{\vec{p}} + \frac{i}{\sqrt{2E_{\vec{p}}}} \tilde{j}(p) \right) e^{-ip\cdot x} + \left(a_{\vec{p}}^{\dagger} - \frac{i}{\sqrt{2E_{\vec{p}}}} \tilde{j}^*(p) \right) e^{ip\cdot x} \right\}$$

$$\Longrightarrow H = \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left(a_{\vec{p}}^{\dagger} - \frac{i}{\sqrt{2E_{\vec{p}}}} \tilde{j}^*(p) \right) \left(a_{\vec{p}} + \frac{i}{\sqrt{2E_{\vec{p}}}} \tilde{j}(p) \right)$$

$$\langle 0| H |0\rangle = \langle 0| \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \frac{1}{2E_{\vec{p}}} \tilde{j}^*(p)\, \tilde{j}(p) \,|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} |\tilde{j}(p)|^2$$

Problems

2.1 Classical Electromagnetism.

(a) $\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = -\frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)$

$= -\frac{1}{4}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\nu A_\mu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu + \partial_\nu A_\mu \partial^\nu A^\mu)$

$= -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu)$

$= -\frac{1}{2}(\partial_\mu A_\nu g^{\mu a} g^{\nu b} \partial_a A_b - \partial_\mu A_\nu g^{\nu a} g^{\mu b} \partial_a A_b)$

$\frac{\partial \mathcal{L}}{\partial(\partial_\alpha A_\beta)} = -\frac{1}{2}(\delta^\alpha_\mu \delta^\beta_\nu g^{\mu a} g^{\nu b} \partial_a A_b + \partial_\mu A_\nu g^{\mu a} g^{\nu b} \delta^\alpha_a \delta^\beta_b$

$- \delta^\alpha_\mu \delta^\beta_\nu g^{\nu a} g^{\mu b} \partial_a A_b - \partial_\mu A_\nu g^{\nu a} g^{\mu b} \delta^\alpha_a \delta^\beta_b)$

$= -\frac{1}{2}(\partial^\alpha A^\beta + \partial^\alpha A^\beta - \partial^\beta A^\alpha - \partial^\beta A^\alpha)$

$= -(\partial^\alpha A^\beta - \partial^\beta A^\alpha) = -F^{\alpha\beta}$

$\frac{\partial \mathcal{L}}{\partial A_\alpha} = 0$, $\Longrightarrow$ E-L equation: $\partial_\mu F^{\mu\nu} = 0$

① $\partial_i F^{i0} = \partial_i(-F^{0i}) = \partial_i E^i = 0 \Longrightarrow \nabla \cdot E = 0$

② $\partial_\mu F^{\mu i} = 0 \Longrightarrow \partial_0 F^{0i} + \partial_j F^{ji} = 0$ i.e. $\partial_0(-E^i) - \partial_j F^{ij} = 0$

$\partial_j F^{ij} = \partial_j(-\epsilon^{ijk} B^k) = -(\vec{\nabla} \times \vec{B})^i$

$\Longrightarrow -\partial_0 E^i + (\vec{\nabla} \times \vec{B})^i = 0$ i.e. $-\frac{\partial \vec{E}}{\partial t} + \vec{\nabla} \times \vec{B} = 0$

③ $\partial^\lambda F^{\mu\nu} + \partial^\mu F^{\nu\lambda} + \partial^\nu F^{\lambda\mu} = \partial^\lambda \partial^\mu A^\nu - \partial^\lambda \partial^\nu A^\mu + \partial^\mu \partial^\nu A^\lambda - \partial^\mu \partial^\lambda A^\nu$

$+ \partial^\nu \partial^\lambda A^\mu - \partial^\nu \partial^\mu A^\lambda = 0$

Let $\lambda = 0$, $\mu = 1$, $\nu = 2$, we have

$\partial^0 F^{12} + \partial^1 F^{20} + \partial^2 F^{01} = -\frac{\partial B_z}{\partial t} - \frac{\partial E_y}{\partial x} + \frac{\partial E_x}{\partial y} = 0$

i.e. $-\left(\frac{\partial \vec{B}}{\partial t}\right)_z - (\vec{\nabla} \times \vec{E})_z = 0 \Longrightarrow \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$

④ Let $\lambda = 1$, $\mu = 2$, $\nu = 3$, we have

$\partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12} = 0 \Longrightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$, i.e. $\vec{\nabla} \cdot \vec{B} = 0$

(b) $T^{\mu\nu} = g^{\nu\sigma} T^\mu{}_\sigma = g^{\nu\sigma}\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial_\sigma A_\lambda - \mathcal{L} \delta^\mu{}_\sigma\right)$

$= \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\lambda)} \partial^\nu A_\lambda - g^{\nu\mu} \mathcal{L} = -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$

Obviously $T^{\mu\nu} \neq T^{\nu\mu}$

$\hat{T}^{\mu\nu} = T^{\mu\nu} + \partial_\lambda K^{\lambda\mu\nu} = -F^{\mu\lambda} \partial^\nu A_\lambda + \frac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} + \partial_\lambda (F^{\mu\lambda} A^\nu)$

$$= -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + F^{\mu\lambda}(\partial_\lambda A^\nu) + \underbrace{(\partial_\lambda F^{\mu\lambda})}_{= -\partial_\lambda F^{\lambda\mu} = 0} A^\nu$$

$$= -F^{\mu\lambda}\partial^\nu A_\lambda + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + F^{\mu\lambda}(\partial_\lambda A^\nu)$$

$$= F^{\mu\lambda}(\partial_\lambda A^\nu - \partial^\nu A_\lambda) + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

$$= F^{\mu\lambda}F_\lambda{}^\nu + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

$$\hat{T}^{\nu\mu} = F^{\nu\lambda}F_\lambda{}^\mu + \frac{1}{4}g^{\nu\mu}F_{\rho\sigma}F^{\rho\sigma} = F^\mu{}_\lambda F^{\lambda\nu} + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

$$= F^{\mu\lambda}F_\lambda{}^\nu + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} = \hat{T}^{\mu\nu} \quad \text{symmetric !}$$

$$\hat{T}^{00} = \frac{1}{2}(\vec{B}^2 + \vec{E}^2) \quad , \quad \hat{T}^{0i} = (\vec{E}\times\vec{B})^i$$

2.2 The complex scalar field

(a) $\pi(x) = \frac{\partial\mathcal{L}}{\partial(\dot\phi)} = \dot\phi^* \quad , \quad \hat\pi(x) = \frac{\partial\mathcal{L}}{\partial(\dot\phi^*)} = \dot\phi \quad (= \pi^*(x))$

$$[\phi(\vec{x}), \pi(\vec{y})] = [\phi^*(\vec{x}), \pi^*(\vec{y})] = i\delta^{(3)}(\vec{x}-\vec{y})$$

$$H = \int d^3x\, (\pi\dot\phi + \pi^*\dot\phi^* - \mathcal{L}) = \int d^3x\, (\pi\pi^* + \pi^*\pi - \partial_\mu\phi^*\partial^\mu\phi + m^2\phi^*\phi)$$

$$= \int d^3x\, (\pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi)$$

$$i\frac{\partial}{\partial t}\phi(\vec{x},t) = [\phi(\vec{x},t), H] = \int d^3x\, [\phi(\vec{x},t), \pi^*\pi] = i\pi^*(\vec{x},t)$$

$$i\frac{\partial}{\partial t}\pi(\vec{x},t) = [\pi(\vec{x},t), H] = \int d^3x\, [\pi(\vec{x},t), \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi]$$

$$= -i\delta^{(3)}(\vec{x}-\vec{y})\int d^3x\, (\nabla\phi^*\cdot\nabla + m^2\phi^*) \implies \dot\pi(\vec{x},t) = \nabla^2\phi^* - m^2\phi^*$$

$$\ddot\phi(\vec{x},t) = \dot\pi^*(\vec{x},t) = \nabla^2\phi - m^2\phi \qquad (\text{K-G equation})$$

(b) $$\phi(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}(a_{\vec p}e^{-ip\cdot x} + b^\dagger_{\vec p}e^{ip\cdot x})$$

$$\phi^*(x) = \int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec p}}}(b_{\vec p}e^{-ip\cdot x} + a^\dagger_{\vec p}e^{ip\cdot x})$$

$$[a_{\vec p}, a^\dagger_{\vec q}] = (2\pi)^3\delta^{(3)}(\vec p - \vec q) \quad , \quad [b_{\vec p}, b^\dagger_{\vec q}] = (2\pi)^3\delta^{(3)}(\vec p - \vec q)$$

$$H = \int d^3x\, (\pi^*\pi + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi)$$

$$= \int d^3x\, (\dot\phi\,\dot\phi^* + \nabla\phi^*\cdot\nabla\phi + m^2\phi^*\phi)$$

$$= \int d^3x \int\frac{d^3p}{(2\pi)^3}\frac{d^3q}{(2\pi)^3}\Big\{\frac{1}{2\sqrt{E_{\vec p}E_{\vec q}}}E_{\vec p}E_{\vec q}(a_{\vec p}e^{-ip\cdot x} - b^\dagger_{\vec p}e^{ip\cdot x})(-b_{\vec q}e^{-iq\cdot x} + a^\dagger_{\vec q}e^{iq\cdot x})$$

$$+ \frac{1}{2\sqrt{E_{\vec p}E_{\vec q}}}\vec p\cdot\vec q\,(a_{\vec p}e^{-ip\cdot x} - b^\dagger_{\vec p}e^{ip\cdot x})(-b_{\vec q}e^{-iq\cdot x} + a^\dagger_{\vec q}e^{iq\cdot x})$$

$$+ \frac{1}{2\sqrt{E_{\vec p}E_{\vec q}}}m^2(a_{\vec p}e^{-ip\cdot x} + b^\dagger_{\vec p}e^{ip\cdot x})(b_{\vec q}e^{-iq\cdot x} + a^\dagger_{\vec q}e^{iq\cdot x})\Big\}$$

$$= \int\frac{d^3p}{(2\pi)^3}E_{\vec p}\,(a^\dagger_{\vec p}a_{\vec p} + b^\dagger_{\vec p}b_{\vec p} + [b_{\vec p}, b^\dagger_{\vec p}])$$

Chapter 3 The Dirac Field

An equation of motion is automatically Lorentz invariant if it follows from a Lagrangian that is a Lorentz scalar.

$$\Lambda: \quad x^\mu \longmapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

$$\phi(x) \longmapsto \phi'(x) = \phi(\Lambda^{-1}x)$$

We check that this transformation leaves the form of the K-G Lagrangian unchanged.

$$\partial_\mu \phi(x) \longmapsto \partial_\mu(\phi(\Lambda^{-1}x)) = (\Lambda^{-1})^\nu{}_\mu(\partial_\nu\phi)(\Lambda^{-1}x)$$

$$\begin{aligned}(\partial_\mu\phi(x))^2 \longmapsto\ & g^{\mu\nu}(\partial_\mu\phi'(x))(\partial_\nu\phi'(x)) \\ &= g^{\mu\nu}[(\Lambda^{-1})^\rho{}_\mu\partial_\rho\phi][(\Lambda^{-1})^\sigma{}_\nu\partial_\sigma\phi](\Lambda^{-1}x) \\ &= g^{\rho\sigma}(\partial_\rho\phi)(\partial_\sigma\phi)(\Lambda^{-1}x) \\ &= (\partial_\mu\phi)^2(\Lambda^{-1}x) \quad \text{transforms like a scalar}\end{aligned}$$

$$\Longrightarrow \mathcal{L}(x) \longrightarrow \mathcal{L}(\Lambda^{-1}x) \Longrightarrow S \text{ is Lorentz invariant}$$

$$\begin{aligned}(\partial^2+m^2)\phi'(x) &= [(\Lambda^{-1})^\nu{}_\mu\partial_\nu(\Lambda^{-1})^{\sigma\mu}\partial_\sigma + m^2]\phi(\Lambda^{-1}x) \\ &= (g^{\nu\sigma}\partial_\nu\partial_\sigma + m^2)\phi(\Lambda^{-1}x) = 0 \quad \text{invariant.}\end{aligned}$$

In general, any equation in which each term has the same set of uncontracted Lorentz indices will naturally be invariant under Lorentz transformations.

For simplicity, we restrict our attention to linear transformations, so that if Φ_a is an n-component multiplet, the Lorentz transformation law is given by an $n\times n$ matrix $M(\Lambda)$:

$$\Phi_a(x) \longrightarrow M_{ab}(\Lambda)\Phi_b(\Lambda^{-1}x)$$

We write it as $\Phi \longrightarrow M(\Lambda)\Phi$.

$\Lambda'' = \Lambda'\Lambda \Longrightarrow$ we require $\Phi \longrightarrow M(\Lambda')M(\Lambda)\Phi = M(\Lambda'')\Phi$

We say that M must form an n-dim representation of the Lorentz group.

$\longrightarrow$ We wonder what are the (finite-dim) matrix representations of the Lorentz Group.

For any continuous group, the transformations that lie infinitesimally close to the identity define a vector space, called the Lie algebra of the group. The basis vectors for this vector space are called the generators of the Lie algebra.

e.g. For rotation group, the generators are the angular momentum operators J^i,

$$[J^i, J^j] = i\epsilon^{ijk} J^k$$

The rotation operations are formed by exponentiating these J^i:

$$R = e^{-i\theta^i J^i} \rightsquigarrow \text{rotate by angle } |\theta| \text{ about the } \hat{\theta} \text{ axis}$$

For rotation, we have $\vec{J} = \vec{x} \times \vec{p} = \vec{x} \times (-i\vec{\nabla})$, we can write

$$J^{ij} = -i(x^i \nabla^j - x^j \nabla^i)$$

The generalization to 4-dim Lorentz transformation is now quite natural:

$$J^{\mu\nu} = i(x^\mu \partial^\nu - x^\nu \partial^\mu)$$

$$[J^{\mu\nu}, J^{\rho\sigma}] = [x^\rho\partial^\sigma - x^\sigma\partial^\rho, x^\mu\partial^\nu - x^\nu\partial^\mu]$$

$$= [x^\rho\partial^\sigma, x^\mu\partial^\nu] - [x^\sigma\partial^\rho, x^\mu\partial^\nu] - [x^\rho\partial^\sigma, x^\nu\partial^\mu] + [x^\sigma\partial^\rho, x^\nu\partial^\mu]$$

$$= x^\rho g^{\sigma\mu}\partial^\nu - x^\mu g^{\nu\rho}\partial^\sigma - x^\sigma g^{\rho\mu}\partial^\nu + x^\mu g^{\nu\sigma}\partial^\rho - x^\rho g^{\sigma\nu}\partial^\mu + x^\nu g^{\mu\rho}\partial^\sigma + x^\sigma g^{\rho\nu}\partial^\mu - x^\nu g^{\mu\sigma}\partial^\rho$$

$$= g^{\nu\rho}(x^\sigma\partial^\mu - x^\mu\partial^\sigma) - g^{\mu\rho}(x^\sigma\partial^\nu - x^\nu\partial^\sigma) - g^{\nu\sigma}(x^\rho\partial^\mu - x^\mu\partial^\rho) + g^{\mu\sigma}(x^\rho\partial^\nu - x^\nu\partial^\rho)$$

$$= i(g^{\nu\rho}J^{\mu\sigma} - g^{\mu\rho}J^{\nu\sigma} - g^{\nu\sigma}J^{\mu\rho} + g^{\mu\sigma}J^{\nu\rho})$$

Any matrices that are to represent this algebra must obey these same commutation rules.

Consider the 4×4 matrices $(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i(\delta^\mu{}_\alpha \delta^\nu{}_\beta - \delta^\mu{}_\beta \delta^\nu{}_\alpha)$

$$\mathcal{J}^{01} = \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathcal{J}^{02} = \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathcal{J}^{03} = \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$

$$\mathcal{J}^{12} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \mathcal{J}^{13} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix} \quad \mathcal{J}^{23} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \end{pmatrix}$$

$$[\mathcal{J}^{01}, \mathcal{J}^{02}] = \begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&0&0&0\\ 0&0&0&0 \end{pmatrix} - \begin{pmatrix} 0&0&0&0\\ 0&0&0&0\\ 0&1&0&0\\ 0&0&0&0 \end{pmatrix} = \begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&-1&0&0\\ 0&0&0&0 \end{pmatrix}$$

$$[\mathcal{J}^{01}, \mathcal{J}^{02}] = i\,(g^{10}\mathcal{J}^{02} - g^{00}\mathcal{J}^{12} - g^{12}\mathcal{J}^{00} + g^{02}\mathcal{J}^{10}) = \begin{pmatrix} 0&0&0&0\\ 0&0&1&0\\ 0&-1&0&0\\ 0&0&0&0 \end{pmatrix}$$

It works !

Parametrize an infinitesimal transformation as

$$V^\alpha \longrightarrow (\delta^\alpha{}_\beta - \tfrac{i}{2}\omega_{\mu\nu}(\mathcal{J}^{\mu\nu})^\alpha{}_\beta)\, V^\beta$$

where V is a 4-vector. We see this is the rotation and boost in spacetime.

Dirac trick : Suppose that we had a set of four $n\times n$ matrices γ^μ satisfying the anti-commutation relations :

$$\{\gamma^\mu, \gamma^\nu\} \equiv \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu} \times \mathbb{1}_{n\times n} \qquad (\text{Dirac Algebra})$$

Then we get an n-dim representation of the Lorentz algebra :

$$S^{\mu\nu} = \frac{i}{4}[\gamma^\mu, \gamma^\nu]$$

$$S^{\mu\nu} = \frac{i}{4}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu) = \frac{i}{4}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu - 2\gamma^\nu\gamma^\mu) = \frac{i}{2}(g^{\mu\nu} - \gamma^\nu\gamma^\mu)$$

$$[S^{\mu\nu}, \gamma^\rho] = -\frac{i}{2}[\gamma^\nu\gamma^\mu, \gamma^\rho] = -\frac{i}{2}(\gamma^\nu\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\nu\gamma^\mu)$$

$$= -\frac{i}{2}(\gamma^\nu\gamma^\mu\gamma^\rho + \gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\nu\gamma^\rho\gamma^\mu - \gamma^\rho\gamma^\nu\gamma^\mu)$$

$$= -\frac{i}{2}(\gamma^\nu\{\gamma^\mu, \gamma^\rho\} - \{\gamma^\nu, \gamma^\rho\}\gamma^\mu)$$

$$= -i(\gamma^\nu g^{\mu\rho} - g^{\nu\rho}\gamma^\mu)$$

$$[S^{\mu\nu}, S^{\rho\sigma}] = [S^{\mu\nu}, \tfrac{i}{2}(g^{\rho\sigma} - \gamma^\sigma\gamma^\rho)] = -\frac{i}{2}[S^{\mu\nu}, \gamma^\sigma\gamma^\rho]$$

$$= -\frac{i}{2}(S^{\mu\nu}\gamma^\sigma\gamma^\rho - \gamma^\sigma\gamma^\rho S^{\mu\nu}) = -\frac{i}{2}(S^{\mu\nu}\gamma^\sigma\gamma^\rho - \gamma^\sigma S^{\mu\nu}\gamma^\rho + \gamma^\sigma S^{\mu\nu}\gamma^\rho - \gamma^\sigma\gamma^\rho S^{\mu\nu})$$

$$= -\frac{i}{2}([S^{\mu\nu}, \gamma^\sigma]\gamma^\rho + \gamma^\sigma[S^{\mu\nu}, \gamma^\rho])$$

$$= -\frac{i}{2}[-i(\gamma^\nu g^{\mu\sigma} - g^{\nu\sigma}\gamma^\mu)\gamma^\rho + \gamma^\sigma(-i)(\gamma^\nu g^{\mu\rho} - g^{\nu\rho}\gamma^\mu)]$$

$$= -\frac{1}{2}(g^{\mu\sigma}\gamma^\nu\gamma^\rho - g^{\nu\sigma}\gamma^\mu\gamma^\rho + g^{\mu\rho}\gamma^\sigma\gamma^\nu - g^{\nu\rho}\gamma^\sigma\gamma^\mu)$$

$$= -\frac{1}{2}[g^{\mu\sigma}(2iS^{\rho\nu} + g^{\rho\nu}) - g^{\nu\sigma}(2iS^{\rho\mu} + g^{\rho\mu}) + g^{\mu\rho}(2iS^{\nu\sigma} + g^{\nu\sigma}) - g^{\nu\rho}$$

$$(2iS^{\mu\sigma} + g^{\mu\sigma})\;]$$

$$= i\left(g^{\nu\sigma}S^{\rho\mu} - g^{\mu\sigma}S^{\rho\nu} - g^{\mu\rho}S^{\nu\sigma} + g^{\nu\rho}S^{\mu\sigma}\right)$$

$$= i\left(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}\right)$$

This works in any dimensionality, with Lorentz or Euclidean metr.

For 2×2, 3×3 case, there is no four matrix that has the relation of Dirac algebra.

For 4×4 case:

All 4×4 representations of the Dirac algebra are unitarily equivalent. We therefore write one explicit realization of the Dirac algebra.

$$\gamma^0 = \begin{pmatrix} 0 & 1_{2\times 2} \\ 1_{2\times 2} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

Other repre.

$\gamma^0 = \begin{pmatrix} 1_{2\times 2} & 0 \\ 0 & 1_{2\times 2} \end{pmatrix}$ $\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$ Dirac-Pauli

Better for non-relativistic particles

$\gamma^0 = \begin{pmatrix} 0 & \sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$, $\gamma^1 = \begin{pmatrix} i\sigma^3 & 0 \\ 0 & i\sigma^3 \end{pmatrix}$ Majorana

$\gamma^2 = \begin{pmatrix} 0 & -\sigma^2 \\ \sigma^2 & 0 \end{pmatrix}$, $\gamma^3 = \begin{pmatrix} -i\sigma^1 & 0 \\ 0 & -i\sigma^1 \end{pmatrix}$

Better for Majorana fermions

This representation is called the Weyl or chiral representation.

$$\{\gamma^i, \gamma^j\} \equiv \gamma^i\gamma^j + \gamma^j\gamma^i = \begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} + \begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}$$

$$= \begin{pmatrix} -\{\sigma^i, \sigma^j\} & 0 \\ 0 & -\{\sigma^i, \sigma^j\} \end{pmatrix} = \begin{pmatrix} -2\delta^{ij} I_{2\times 2} & 0 \\ 0 & -2\delta^{ij} I_{2\times 2} \end{pmatrix} = -2\delta^{ij} I_{4\times 4}$$

$$[\gamma^i, \gamma^j] = \gamma^i\gamma^j - \gamma^j\gamma^i = \begin{pmatrix} -\sigma^i\sigma^j & 0 \\ 0 & -\sigma^i\sigma^j \end{pmatrix} - \begin{pmatrix} -\sigma^j\sigma^i & 0 \\ 0 & -\sigma^j\sigma^i \end{pmatrix}$$

$$= \begin{pmatrix} -[\sigma^i, \sigma^j] & 0 \\ 0 & -[\sigma^i, \sigma^j] \end{pmatrix} = -2i\epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

Boost: $S^{0i} = \frac{i}{4}[\gamma^0, \gamma^i] = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$

Rotation: $S^{ij} = \frac{i}{4}[\gamma^i, \gamma^j] = \frac{1}{2}\epsilon^{ijk}\begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix} \equiv \frac{1}{2}\epsilon^{ijk}\Sigma^k$

A four-component field ψ that transforms under these boosts and rotations is called a Dirac Spinor.

* The Lorentz group is noncompact $\Longrightarrow$ has no faithful, finite-dimensional representations that are unitary.

$$[\gamma^\mu, S^{\rho\sigma}] = i(\gamma^\sigma g^{\rho\mu} - g^{\sigma\mu}\gamma^\rho) = i(g^{\rho\mu}\gamma^\sigma - g^{\sigma\mu}\gamma^\rho)$$

$= i(g^{\rho\mu}\delta^\sigma{}_\nu - g^{\sigma\mu}\delta^\rho{}_\nu)\gamma^\nu$, for $\rho\sigma = 12$, $\mu\nu = 12$, we have

$$[\gamma^1, S^{12}] = i(g^{11}\delta^2{}_2 - g^{21}\delta^1{}_2)\gamma^2 = i\gamma^2$$

$(\mathcal{J}^{12})^1{}_2 = i$ correct! $\Longrightarrow [\gamma^\mu, S^{\rho\sigma}] = (\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu$

$$(1 + \tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})\gamma^\mu(1 - \tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma})$$

$$= \gamma^\mu + \tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma}\gamma^\mu - \gamma^\mu\tfrac{i}{2}\omega_{\rho\sigma}S^{\rho\sigma} + \cancel{\tfrac{1}{4}\omega_{\rho\sigma}S^{\rho\sigma}\gamma^\mu\omega_{\rho\sigma}S^{\rho\sigma}}$$

$$= \gamma^\mu - \tfrac{i}{2}\omega_{\rho\sigma}[\gamma^\mu, S^{\rho\sigma}] = \gamma^\mu - \tfrac{i}{2}\omega_{\rho\sigma}(\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu$$

$$= (1 - \tfrac{i}{2}\omega_{\rho\sigma}\mathcal{J}^{\rho\sigma})^\mu{}_\nu\gamma^\nu$$

$\Longrightarrow \Lambda_{\frac{1}{2}}^{-1}\gamma^\mu\Lambda_{\frac{1}{2}} = \Lambda^\mu{}_\nu\gamma^\nu$, where $\Lambda_{\frac{1}{2}} = e^{-\frac{i}{2}\omega_{\mu\nu}S^{\mu\nu}}$ is the spinor representation of the Lorentz transformation Λ.

Dirac Equation: $(i\gamma^\mu\partial_\mu - m)\psi(x) = 0$ Lorentz invariant

$$0 = (-i\gamma^\mu\partial_\mu - m)(i\gamma^\nu\partial_\nu - m)\psi$$

$$= (\gamma^\mu\gamma^\nu\partial_\mu\partial_\nu + m^2)\psi = (\tfrac{1}{2}\{\gamma^\mu, \gamma^\nu\}\partial_\mu\partial_\nu + m^2)\psi$$

$$= (\partial^2 + m^2)\psi \quad \text{K-G equat}$$

To write a Lagrangian for the Dirac theory, we must figure out how to multiply two Dirac spinors to form a Lorentz scalar.

Define $\bar\psi \equiv \psi^\dagger\gamma^0$. Under an infinitesimal Lorentz transformation parametrized by $\omega_{\mu\nu}$, we have $\bar\psi \longmapsto \psi^\dagger(1 + \tfrac{i}{2}\omega_{\mu\nu}(S^{\mu\nu})^\dagger)\gamma^0$

$\Longrightarrow \bar\psi \longmapsto \bar\psi\Lambda_{\frac{1}{2}}^{-1}$, then $\bar\psi\psi$ is a Lorentz scalar.

$$\mathcal{L}_{\text{Dirac}} = \bar\psi(i\gamma^\mu\partial_\mu - m)\psi$$

Weyl Spinors:

It is apparent that the Dirac representation of the Lorentz group is reducible. We can form two 2-dim repre. by considering each block separately, and writing

$$\psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$$

ψ_L: left-handed Weyl spinor $\qquad$ ψ_R: right-handed Weyl spinor

Their transformation law, under infinitesimal rotations $\vec{\theta}$ and boosts $\vec{\beta}$, are

$$\psi_L \longmapsto \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_L$$

$$\psi_R \longmapsto \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\psi_R$$

$$\psi_L^* \longmapsto \left(1 + i\vec{\theta}\cdot\frac{\vec{\sigma}^*}{2} - \vec{\beta}\cdot\frac{\vec{\sigma}^*}{2}\right)\psi_L^*$$

$$\sigma^2\psi_L^* \longmapsto \left(\sigma^2 + i\vec{\theta}\cdot\frac{-\vec{\sigma}}{2}\sigma^2 - \vec{\beta}\cdot\frac{-\vec{\sigma}}{2}\sigma^2\right)\psi_L^*$$

$$= \left(1 - i\vec{\theta}\cdot\frac{\vec{\sigma}}{2} + \vec{\beta}\cdot\frac{\vec{\sigma}}{2}\right)\sigma^2\psi_L^*$$

i.e. $\sigma^2\psi_L^*$ transforms like a right-handed Weyl spinor.

Dirac Equation:

$$(i\gamma^\mu\partial_\mu - m)\psi = \begin{pmatrix} -m & i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla}) \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla}) & -m \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$$

If $m=0$, we have $\begin{cases} i(\partial_0 + \vec{\sigma}\cdot\vec{\nabla})\psi_R = 0 \\ i(\partial_0 - \vec{\sigma}\cdot\vec{\nabla})\psi_L = 0 \end{cases}$ $\qquad$ Weyl equations

Notation: $\sigma^\mu \equiv (1, \vec{\sigma})$, $\quad \bar{\sigma}^\mu \equiv (1, -\vec{\sigma})$

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}$$

$$\begin{pmatrix} -m & i\sigma\cdot\partial \\ i\bar{\sigma}\cdot\partial & -m \end{pmatrix}\begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix} = 0 \quad . \quad i\bar{\sigma}\cdot\partial\psi_L = 0, \quad i\sigma\cdot\partial\psi_R = 0$$

Free-Particle Solutions of the Dirac Equation

Since a Dirac field ψ obeys the K-G equation, we know that it can be written as a linear combination of plane waves:

$$\psi(x) = u(p)e^{-ip\cdot x}, \quad p^2 = m^2, \quad p^0 > 0$$

Consider the rest frame, $p = p_0 = (m, \vec{0})$. The solution for general p can be found by boosting with $\Lambda_{\frac{1}{2}}$.

$$(i\gamma^\mu\partial_\mu - m)\psi(x) = (i\gamma^\mu\partial_\mu - m)(u(p)e^{-ip\cdot x})$$

$$= (i\gamma^\mu(-ip_\mu) - m)(u(p)e^{-ip\cdot x}) = 0 \implies (\gamma^\mu p_\mu - m)u(p) = 0$$

$$(\gamma^0 m - m)u(p_0) = m\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} u(p_0) = 0$$

The solutions are $u(p_0) = \sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$. We conventionally normalize ξ so that $\xi^\dagger \xi = 1$, $\sqrt{m}$ is for future convenience. Now we have the general form of $u(p)$ in the rest frame, we can obtain $u(p)$ in any other frame by boosting.

Consider a boost along the 3-direction.

Recall: $S^{0i} = -\frac{i}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & -\sigma^i \end{pmatrix}$, $\Lambda_{\frac{1}{2}} = \exp(-\frac{i}{2}\omega_{\mu\nu} S^{\mu\nu})$

* $\Lambda_{\frac{1}{2}}$ (along 3-direction) $= \exp\left[-\frac{i}{2}(\omega_{03} S^{03} + \omega_{30} S^{30})\right]$

$$= \exp\left[-i(\omega_{03} S^{03})\right] = \exp\left[-\frac{1}{2}\omega_{03}\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right] \overset{\omega_{03}=\eta}{=\!=} \exp\left[-\frac{1}{2}\eta\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right]$$

* For a 4-momentum vector:

rest frame $p^\mu = (m, \vec{0})$, boosted frame $p^\mu = (E, \vec{p})$

$$\Lambda = \exp(-\frac{i}{2}\omega_{03} \mathcal{J}^{03}) = \exp\left[\eta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]$$

$$\begin{pmatrix} E \\ p^3 \end{pmatrix} = \exp\left[\eta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} m \\ 0 \end{pmatrix} = \left[\cosh\eta\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sinh\eta\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right]\begin{pmatrix} m \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} m\cosh\eta \\ m\sinh\eta \end{pmatrix}$$

$$u(p) = \exp\left[-\frac{1}{2}\eta\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right]\sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \left[\cosh(\tfrac{1}{2}\eta)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \sinh(\tfrac{1}{2}\eta)\begin{pmatrix} \sigma^3 & 0 \\ 0 & -\sigma^3 \end{pmatrix}\right]\sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} \cosh(\frac{\eta}{2}) - \sinh(\frac{\eta}{2})\sigma^3 & 0 \\ 0 & \cosh(\frac{\eta}{2}) + \sinh(\frac{\eta}{2})\sigma^3 \end{pmatrix}\sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e^{\frac{\eta}{2}} + e^{-\frac{\eta}{2}}}{2} - \frac{e^{\frac{\eta}{2}} - e^{-\frac{\eta}{2}}}{2}\sigma^3 & 0 \\ 0 & \frac{e^{\frac{\eta}{2}} + e^{-\frac{\eta}{2}}}{2} - \frac{e^{\frac{\eta}{2}} - e^{-\frac{\eta}{2}}}{2}\sigma^3 \end{pmatrix}\sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} e^{\frac{\eta}{2}}(\frac{1-\sigma^3}{2}) + e^{-\frac{\eta}{2}}(\frac{1+\sigma^3}{2}) & 0 \\ 0 & e^{\frac{\eta}{2}}(\frac{1+\sigma^3}{2}) + e^{-\frac{\eta}{2}}(\frac{1-\sigma^3}{2}) \end{pmatrix}\sqrt{m}\begin{pmatrix} \xi \\ \xi \end{pmatrix}$$

$$= \begin{pmatrix} \sqrt{m}\left[e^{\frac{\eta}{2}}(\frac{1-\sigma^3}{2}) + e^{-\frac{\eta}{2}}(\frac{1+\sigma^3}{2})\right]\xi \\ \sqrt{m}\left[e^{\frac{\eta}{2}}(\frac{1+\sigma^3}{2}) + e^{-\frac{\eta}{2}}(\frac{1-\sigma^3}{2})\right]\xi \end{pmatrix}$$

$$= \begin{pmatrix} \left[(me^{\eta})^{\frac{1}{2}}(\frac{1-\sigma^3}{2}) + (me^{-\eta})^{\frac{1}{2}}(\frac{1+\sigma^3}{2})\right]\xi \\ \left[(me^{\eta})^{\frac{1}{2}}(\frac{1-\sigma^3}{2}) + (me^{-\eta})^{\frac{1}{2}}(\frac{1+\sigma^3}{2})\right]\xi \end{pmatrix}$$

$$\leadsto \begin{cases} \sqrt{me^{\eta}} = \sqrt{m(\cosh\eta + \sinh\eta)} \\ \sqrt{me^{-\eta}} = \sqrt{m(\cosh\eta - \sinh\eta)} \end{cases}$$

$$= \begin{pmatrix} \left[\sqrt{E+p^3}\left(\frac{1-\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1+\sigma^3}{2}\right)\right]\xi \\ \left[\sqrt{E+p^3}\left(\frac{1+\sigma^3}{2}\right) + \sqrt{E-p^3}\left(\frac{1-\sigma^3}{2}\right)\right]\xi \end{pmatrix}$$

If $\xi = \begin{pmatrix}1\\0\end{pmatrix}$, we have $u(p) = \begin{pmatrix} \sqrt{E-p^3}\begin{pmatrix}1\\0\end{pmatrix} \\ \sqrt{E+p^3}\begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix} \xrightarrow[(E \approx p^3)]{\text{large boost}} \sqrt{2E}\begin{pmatrix}\begin{pmatrix}0\\0\end{pmatrix}\\ \begin{pmatrix}1\\0\end{pmatrix}\end{pmatrix}$

If $\xi = \begin{pmatrix}0\\1\end{pmatrix}$, we have $u(p) = \begin{pmatrix} \sqrt{E+p^3}\begin{pmatrix}0\\1\end{pmatrix} \\ \sqrt{E-p^3}\begin{pmatrix}0\\1\end{pmatrix}\end{pmatrix} \xrightarrow[(E \approx p^3)]{\text{large boost}} \sqrt{2E}\begin{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix}\end{pmatrix}$

$\vec{L} = \vec{r}\times\vec{p}$

These two solutions are eigenstates of the helicity operator $\quad \vec{J} = \vec{L} + \vec{S}$

$$h \equiv \frac{1}{2}\hat{p}_i \begin{pmatrix}\sigma^i & 0\\ 0 & \sigma^i\end{pmatrix} \qquad \hat{p} := \frac{\vec{p}}{|\vec{p}|}, \quad h \equiv \hat{\vec{p}}\cdot\vec{J} = \hat{\vec{p}}\cdot\vec{S}$$

A particle with $h = +\frac{1}{2}$ is called right-handed, while one with $h = -\frac{1}{2}$ is called left-handed.

Massless limit: chirality = helicity

Recall: $\sigma^\mu = (1, \vec{\sigma})$, $\bar{\sigma}^\mu = (1, -\vec{\sigma})$, $\qquad \vec{p} = (0, 0, p^3) = (0, 0, p_z)$

$\uparrow \hat{z} \quad \uparrow \vec{p}$

$$p\cdot\sigma = p^0\sigma^0 - p^3\sigma^3 = \begin{pmatrix} E - p_z & 0 \\ 0 & E + p_z\end{pmatrix}$$

$$\sqrt{p\cdot\sigma} = \begin{pmatrix}\sqrt{E-p_z} & 0 \\ 0 & \sqrt{E+p_z}\end{pmatrix}, \quad \sqrt{p\cdot\bar{\sigma}} = \begin{pmatrix}\sqrt{E+p_z} & 0 \\ 0 & \sqrt{E-p_z}\end{pmatrix}$$

Then $u^s(p) = \begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix}$. we claim that this is the correct solution for $\forall \vec{p}$.

Pf: $(\not{p} - m)u^s(p) = \begin{pmatrix} -m & p\cdot\sigma \\ p\cdot\bar{\sigma} & -m\end{pmatrix}\begin{pmatrix}\sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix} = \begin{pmatrix} -m\sqrt{p\cdot\sigma}\,\xi^s + p\cdot\sigma\sqrt{p\cdot\bar{\sigma}}\,\xi^s \\ p\cdot\bar{\sigma}\sqrt{p\cdot\sigma}\,\xi^s - m\sqrt{p\cdot\bar{\sigma}}\,\xi^s\end{pmatrix}$

$$= \begin{pmatrix}\sqrt{p\cdot\sigma}\left(-m + \sqrt{(p\cdot\sigma)(p\cdot\bar{\sigma})}\right)\xi^s \\ \sqrt{p\cdot\bar{\sigma}}\left(\sqrt{(p\cdot\bar{\sigma})(p\cdot\sigma)} - m\right)\xi^s\end{pmatrix},$$

where $(p\cdot\sigma)(p\cdot\bar{\sigma}) = p_\mu\sigma^\mu p_\nu\bar{\sigma}^\nu = p_\mu p_\nu \sigma^\mu\bar{\sigma}^\nu$

$= p_\mu p_\nu \frac{\{\sigma^\mu, \sigma^\nu\}}{2} = g^{\mu\nu}p_\mu p_\nu = p^2 = m^2$

$\rightsquigarrow \{\sigma^i, \sigma^j\} = 2\delta^{ij}$

$\Rightarrow \{\sigma^\mu, \sigma^\nu\} = 2g^{\mu\nu}$

$= 0$

It is convenient to write the normalization condition for $u(p)$ in a Lorentz invariant way. Note that $u^\dagger u = 2E_{\vec{p}}\,\xi^\dagger\xi$. To make a Lorentzian scalar, we define $\bar{u}(p) = u^\dagger(p)\gamma^0$, then

$$\bar{u}u = 2m\,\xi^\dagger\xi \quad \text{——— Normalization condition}$$

Spin Sums

In evaluating Feynman diagrams, we will often wish to sum over the polarization states of a fermion.

$$\sum_{s=1,2} u^s(p)\bar{u}^s(p) = \sum_s \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi^s \\ \sqrt{p\cdot\bar\sigma}\,\xi^s \end{pmatrix} \left(\xi^{s\dagger}\sqrt{p\cdot\bar\sigma} \,,\ \xi^{s\dagger}\sqrt{p\cdot\sigma} \right)$$

$$= \begin{pmatrix} \sqrt{p\cdot\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\sigma}\sqrt{p\cdot\sigma} \\ \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\bar\sigma} & \sqrt{p\cdot\bar\sigma}\sqrt{p\cdot\sigma} \end{pmatrix} = \begin{pmatrix} m & p\cdot\sigma \\ p\cdot\bar\sigma & m \end{pmatrix}$$

$$\Longrightarrow \begin{cases} \sum_s u^s(p)\bar{u}^s(p) = \gamma\cdot\vec{p} + m \\ \sum_s v^s(p)\bar{v}^s(p) = \gamma\cdot\vec{p} - m \end{cases} \qquad \gamma^\mu p_\mu \equiv \not{p}$$

Dirac Matrices and Dirac Field Bilinears

Consider $\bar\Psi\Gamma\Psi$ where Γ is any 4x4 constant matrix. Can we decompose this expression into terms that have definite transformation properties under the Lorentz group? Yes, if we write Γ in the following basis

$\mathbb{1}$	1
γ^μ	4
$\gamma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu] \equiv \gamma^{[\mu}\gamma^{\nu]} \equiv -i\sigma^{\mu\nu}$	6
$\gamma^{\mu\nu\rho} = \gamma^{[\mu}\gamma^\nu\gamma^{\rho]}$	4
$\gamma^{\mu\nu\rho\sigma} = \gamma^{[\mu}\gamma^\nu\gamma^\rho\gamma^{\sigma]}$	1
	16 total

The Lorentz-transformation properties of these matrices are easy to verify:

① $\bar\Psi\mathbb{1}\Psi$: scalar

② $\bar\Psi\gamma^\mu\Psi \xrightarrow{\Lambda} (\bar\Psi\Lambda_{\frac12}^{-1})\gamma^\mu(\Lambda_{\frac12}\Psi) = \bar\Psi\Lambda^\mu{}_\nu\gamma^\nu\Psi = \Lambda^\mu{}_\nu(\bar\Psi\gamma^\nu\Psi)$

③ $\bar\Psi\gamma^{\mu\nu}\Psi \xrightarrow{\Lambda} (\bar\Psi\Lambda_{\frac12}^{-1})\,\frac12(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)(\Lambda_{\frac12}\Psi)$

$$= \frac12\bar\Psi\Lambda_{\frac12}^{-1}\gamma^\mu\Lambda_{\frac12}\Lambda_{\frac12}^{-1}\gamma^\nu\Lambda_{\frac12}\Psi - \frac12\bar\Psi\Lambda_{\frac12}^{-1}\gamma^\nu\Lambda_{\frac12}\Lambda_{\frac12}^{-1}\gamma^\mu\Lambda_{\frac12}\Psi$$

$$= \frac12\bar\Psi\Lambda^\mu{}_\alpha\gamma^\alpha\Lambda^\nu{}_\beta\gamma^\beta\Psi - \frac12\bar\Psi\Lambda^\nu{}_\beta\gamma^\beta\Lambda^\mu{}_\alpha\gamma^\alpha\Psi$$

$$= \Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta(\bar\Psi\gamma^{\alpha\beta}\Psi)$$

④ $\bar{\psi}\gamma^{\mu\nu\rho}\psi \xrightarrow{\Lambda}$

$$(\bar{\psi}\Lambda_{\frac{1}{2}}^{-1})\frac{1}{6}(\gamma^\mu\gamma^\nu\gamma^\rho+\gamma^\rho\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\rho\gamma^\mu-\gamma^\mu\gamma^\rho\gamma^\nu-\gamma^\rho\gamma^\nu\gamma^\mu-\gamma^\nu\gamma^\mu\gamma^\rho)(\Lambda_{\frac{1}{2}}\psi)$$

$=$ Similarly, insert $\Lambda_{\frac{1}{2}}\Lambda_{\frac{1}{2}}^{-1}=\mathbb{1}$ $=\Lambda^\mu{}_\alpha\Lambda^\nu{}_\beta\Lambda^\rho{}_\sigma(\bar{\psi}\gamma^{\alpha\beta\sigma}\psi)$

⑤ Similar :)

The last two sets of matrices can be simplified by introducing

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{4!}\epsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$$

$\Longrightarrow \gamma^{\mu\nu\rho\sigma}=-i\epsilon^{\mu\nu\rho\sigma}\gamma^5$, $\gamma^{\mu\nu\rho}=i\epsilon^{\mu\nu\rho\sigma}\gamma_\sigma\gamma^5$

$$(\gamma^5)^\dagger = -i\gamma^{3\dagger}\gamma^{2\dagger}\gamma^{1\dagger}\gamma^{0\dagger} = -i\gamma^3\gamma^2\gamma^1\gamma^0 = \gamma^5$$

$$(\gamma^5)^2 = -\gamma^0\gamma^1\gamma^2\gamma^3\gamma^0\gamma^1\gamma^2\gamma^3 = \mathbb{1} \quad (\text{since } (\sigma_i)^2=\mathbb{1})$$

$$\{\gamma^5,\gamma^\mu\} = \gamma^5\gamma^\mu+\gamma^\mu\gamma^5 = i(\gamma^0\gamma^1\gamma^2\gamma^3\gamma^\mu+\gamma^\mu\gamma^0\gamma^1\gamma^2\gamma^3)=0$$

$$[\gamma^5, S^{\mu\nu}] = [\gamma^5, \tfrac{i}{4}[\gamma^\mu,\gamma^\nu]] = \tfrac{i}{4}[\gamma^5, \gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu]$$

$$=\tfrac{i}{4}([\gamma^5,\gamma^\mu\gamma^\nu]-[\gamma^5,\gamma^\nu\gamma^\mu])$$

$$=\tfrac{i}{4}(\gamma^5\gamma^\mu\gamma^\nu-\gamma^\mu\gamma^\nu\gamma^5-\gamma^5\gamma^\nu\gamma^\mu+\gamma^\nu\gamma^\mu\gamma^5)$$

$$=\tfrac{i}{4}(-\gamma^\mu\gamma^5\gamma^\nu+\gamma^\mu\gamma^5\gamma^\nu+\gamma^\nu\gamma^5\gamma^\mu-\gamma^\nu\gamma^5\gamma^\mu)=0$$

$[\gamma^5, S^{\mu\nu}]=0 \Longrightarrow$ Dirac representation is reducible (Schur's lemma)

$\gamma^5=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$, so a Dirac spinor with only left (right) handed components is an eigenstate of γ^5 with eigenvalue -1 $(+1)$.

Rewrite the table:

$\mathbb{1}$	scalar	1
γ^μ	vector	4
$\sigma^{\mu\nu}=\frac{i}{2}[\gamma^\mu,\gamma^\nu]$	tensor	6
$\gamma^\mu\gamma^5$	pseudo-vector	4
γ^5	pseudo-scalar	1

16 total

From the vector and pseudo-vector matrices we can form two currents out of Dirac field bilinears:

$$j^\mu(x) = \bar{\psi}(x)\gamma^\mu\psi(x) \qquad j^{\mu 5}(x) = \bar{\psi}(x)\gamma^\mu\gamma^5\psi(x)$$

Assume ψ satisfies the Dirac equation:

$$\partial_\mu j^\mu = (\partial_\mu\bar{\psi})\gamma^\mu\psi + \bar{\psi}\gamma^\mu(\partial_\mu\psi)$$
$$= (im\bar{\psi})\psi + \bar{\psi}(-im\psi) = 0$$

$\Longrightarrow$ j^μ is conserved if ψ satisfies the Dirac equation.

$$\partial_\mu j^{\mu 5} = (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^\mu\gamma^5(\partial_\mu\psi)$$
$$= (\partial_\mu\bar{\psi})\gamma^\mu\gamma^5\psi - \bar{\psi}\gamma^5\gamma^\mu(\partial_\mu\psi)$$
$$= (im\bar{\psi})\gamma^5\psi - \bar{\psi}\gamma^5(-im\psi) = 2im\bar{\psi}\gamma^5\psi$$

$\Longrightarrow$ $m=0$ induces $j^{\mu 5}$ conserved.

It is then useful to write

$$j^\mu_L = \bar{\psi}\gamma^\mu\underbrace{\left(\frac{1-\gamma^5}{2}\right)}_{P_L}\psi, \qquad j^\mu_R = \bar{\psi}\gamma^\mu\underbrace{\left(\frac{1+\gamma^5}{2}\right)}_{P_R}\psi$$

when $m=0$, these are the electric current densities of left-handed and right-handed.

$j^\mu(x)$, $j^{\mu 5}(x)$ are the Noether currents corresponding to the two transformations: $\psi(x) \xrightarrow{U_{1_V}} e^{i\alpha}\psi(x)$, $\psi(x) \xrightarrow{U_{1_A}} e^{i\alpha\gamma^5}\psi(x)$.

$\mathcal{L}$: $U_{1_V} \otimes U_{1_A}$ symmetry

$$\bar{\psi}_L \equiv \psi_L^\dagger\gamma^0 = (P_L\psi)^\dagger\gamma^0 = \psi^\dagger P_L^\dagger\gamma^0 = \psi^\dagger P_L\gamma^0 = \psi^\dagger\gamma^0 P_R = \bar{\psi}P_R$$

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi = \bar{\psi}i\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi$$
$$= \bar{\psi}i\gamma^\mu\partial_\mu(P_L+P_R)\psi - m\bar{\psi}(P_L+P_R)\psi \qquad (*: \gamma^\mu P_L = P_R\gamma^\mu)$$
$$= \bar{\psi}i\gamma^\mu\partial_\mu P_L^2\psi + \bar{\psi}i\gamma^\mu\partial_\mu P_R^2\psi - m\bar{\psi}P_L^2\psi - m\bar{\psi}P_R^2\psi$$
$$= \bar{\psi}P_R i\gamma^\mu\partial_\mu\psi_L + \bar{\psi}P_L i\gamma^\mu\partial_\mu\psi_R - m\bar{\psi}P_L\psi_L - m\bar{\psi}P_R\psi_R$$
$$= \bar{\psi}_L i\gamma^\mu\partial_\mu\psi_L + \bar{\psi}_R i\gamma^\mu\partial_\mu\psi_R - m\bar{\psi}_R\psi_L - m\bar{\psi}_L\psi_R$$

* if $m=0$ $\Longrightarrow$ $\mathcal{L}$ is decoupled. $\psi_L \longrightarrow \psi_L' = e^{i\alpha_L}\psi_L$

$U(1)_L \otimes U(1)_R$ $\qquad \psi_R \longrightarrow \psi_R' = e^{i\alpha_R}\psi_R$

* if $m\neq 0$, ψ_L and ψ_R are mixed by m.

Products of Dirac bilinears obey interchange relations: Fierz identities.

Core: 1° $(\sigma^\mu)_{\alpha\beta}(\sigma_\mu)_{\gamma\delta} = 2\epsilon_{\alpha\gamma}\epsilon_{\beta\delta}$, where $\sigma^\mu = (1, \vec{\sigma})$

$$(\bar{u}_{1R}\sigma^\mu u_{2R})(\bar{u}_{3R}\sigma_\mu u_{4R}) = 2\epsilon_{\alpha\gamma}\bar{u}_{1R\alpha}\bar{u}_{3R\gamma}\epsilon_{\beta\delta}u_{2R\beta}u_{4R\delta}$$

$$= -2\epsilon_{\alpha\gamma}\bar{u}_{1R\alpha}\bar{u}_{3R\gamma}\epsilon_{\delta\beta}u_{2R\beta}u_{4R\delta}$$, rearrangement

$$= -(\bar{u}_{1R}\sigma^\mu u_{4R})(\bar{u}_{3R}\sigma_\mu u_{2R})$$

2° $\epsilon_{\alpha\beta}(\sigma^\mu)_{\beta\gamma} = (\bar{\sigma}^{\mu T})_{\alpha\beta}\epsilon_{\beta\gamma}$

Quantization of the Dirac Field