0.1 Polytopes

0.1.1 Dihedral groups

The dihedral group, D_{2n}, is the group of symmetries of a regular n-gon. Its standard presentation is given by

$$\langle r, s \mid r^3 = e,\ s^2 = e,\ (rs)^2 = e\rangle$$

where r is a rotation of $2\pi/n$ and s is a reflection.

Let l_1 and l_2 be two reflection axes with an angle θ between l_1 and l_2, and s_1 and s_2 be the respective reflections. After some algebra, the composition $s_1 s_2$ turns out to be a counterclockwise rotation through 2π.

Therefore, an alternative presentation of D_{2n} is given by

$$\langle s_1, s_2 \mid s_1^2 = e,\ s_2^2 = e,\ (s_1 s_2)^n = e\rangle$$

where s_1 and s_2 are adjacent reflections.

This shows that D_{2n} is an example of a reflection group.

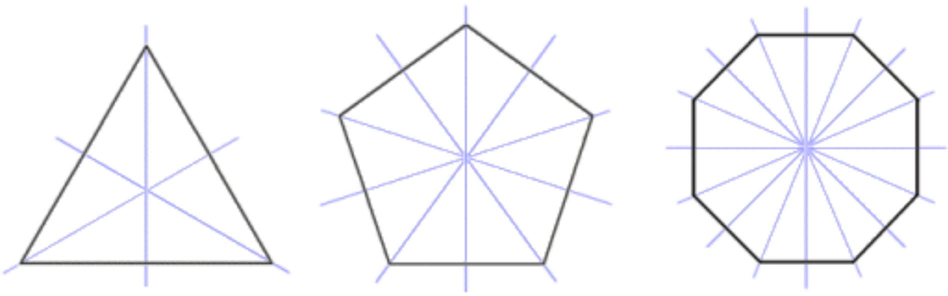
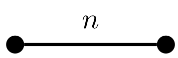


Figure 1 The lines of symmetry of a regular 3-,4- and 7-gon.

Example 0.1.2. The coxeter diagram for the symmetries of a regular n-gon, also known as $I_2(n)$, looks like

$I_2(n)$



$(n \geq 4)$

Theorem 0.1.3. Let $G \curvearrowright X$ be an action of a finite group G on a finite set X. Then the number of G-orbits in X is given by:

$$\text{Number of orbits} = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|$$

where $\mathrm{Fix}(g) = \{x \in X \mid g \cdot x = x\}$.

Example 0.1.4. How many distinguishable necklaces can be made using seven different colored beads of the same size?
Let X be the 7! possible arrangements. The necklace can be turned over (a reflection) as well as rotated so we consider the dihedral group D_{14} acting on X. Using the previous theorem,

$$\text{Number of orbits} = \frac{7!}{14} = 360$$

as only the identity leaves any arrangement fixed.

Theorem 0.1.5. Let p be a prime. Then any group G of order $2p$ is isomorphic to either the cyclic group C_{2p} or the dihedral group D_{2p}.
Proof. By Cauchy's theorem, there exists an element $a \in G$ of order p and an element $b \in G$ of order 2. Let

$$H = \langle a \rangle,$$

so H is a subgroup of G of index 2, so $H \trianglelefteq G$.
Since $H \trianglelefteq G$, conjugation by b sends H to itself. Thus, there exists $k \in \{1, 2, \ldots, p-1\}$ such that

$$bab^{-1} = a^k.$$

Applying conjugation by b twice to a gives

$$a = b^2ab^{-2} = b(bab^{-1})b^{-1} = ba^kb^{-1} = (bab^{-1})^k = (a^k)^k = a^{k^2}.$$

Therefore,

$$a = a^{k^2} \implies a^{k^2-1} = e.$$

Since a has order p, this implies

$$p \mid (k^2 - 1),$$

or equivalently,

$$k^2 \equiv 1 \pmod{p}.$$

Because p is prime, this implies

$$k \equiv \pm 1 \pmod{p}.$$

- If $k \equiv 1$, then

  $$bab^{-1} = a,$$

  and b commutes with a. Hence G is abelian, and since a has order p and b has order 2, G is cyclic of order $2p$.

- If $k \equiv -1$, then

  $$bab^{-1} = a^{-1},$$

  which is the defining relation for the dihedral group D_{2p}:

  $$D_p = \langle a, b \mid a^p = e, b^2 = e, bab = a^{-1} \rangle.$$

Thus, G is isomorphic to either C_{2p} or D_{2p}.

$\square$

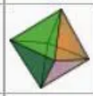
Polyhedron		Vertices	Edges	Faces
tetrahedron		4	6	4
cube / hexahedron		8	12	6
octahedron		6	12	8
dodecahedron		20	30	12
icosahedron		12	30	20

Figure 2 The five Platonic solids.

0.2 The Infinite dihedral group

1 Platonic solids and reflections

Definition 1.0.1. A polyhedron is regular if its faces are regular polygons, all with the same number of sides, and also each vertex belongs to the same number of edges.

Theorem 1.0.2. The only regular convex polyhedra are the five Platonic solids.
Proof. Before writing the proof, we introduce some notations. V, the number of vertices;
E, the number of edges;
F, the number of faces;
n, the number of sides on a face;
r, the number of edges to which each vertex belongs.
Observe that

$$2E = nF \tag{1}$$

and

$$2E = rV \tag{2}$$

(1) comes from counting the number of pairs (e,f) where e is an edge and f is a face and e lies on f; (2) comes from counting the number of pairs (v,e) where v is a vertex and v lies on e.
Substitute into Euler's formula, we get

$$\frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E} \tag{3}$$

Now $n \geq 3$, as a polygon must have at least 3 sides and $r \geq 3$, since in a polyhedron a vertex must belong to at least 3 edges. By (3), we can't have both $n \geq 4$ and $r \geq 4$, since this would make the left-hand side of (3) at most $\frac{1}{2}$. It follows that either $n = 3$ or $r = 3$. If $n = 3$, then (3) becomes

$$\frac{1}{r} = \frac{1}{6} + \frac{1}{E} \tag{4}$$

The right-hand side is greater than $\frac{1}{6}$, and hence $r < 6$. Therefore, $r = 3$, 4 or 5 and $E = 6$, 12 or 30, respectively. If $r = 3$, (3) becomes

$$\frac{1}{n} = \frac{1}{6} + \frac{1}{E} \tag{5}$$

Similarly, n = 3,4 or 5 and E = 6,12 or 30, respectively. These parameters are coincide with those in the table above.

Example 1.0.3. Let G be the group of symmetries of a dodecahedron. What is $|G|$?
Let G act on the 12 faces of the dodecahedron and fix a face. There are $|D_{10}| = 10$ symmetries which fix this face and our action is clearly transitive. By Orbit-Stabiliser theorem, $|G| = 10 \times 12 = 120$. Alternatively, this can be done by considering the fundamental domain, which is a triangle that uniquely determines the reflection. There are 120 such triangles.

We would like to study the group of symmetries of these Platonic solids.

1.1 Tetrahedron

Each face has D_6 as a group of symmetries.
Before moving on to the other solids, we first introduce the concept of dual.

Definition 1.1.1. The dual of a Platonic solid is a new Platonic solid where the faces and vertices are interchanged with those of the original.

Remark 1.1.2. The tetrahedron is self-dual. The cube and the octahedron form a dual pair. The dodecahedron and the icosahedron form a dual pair.

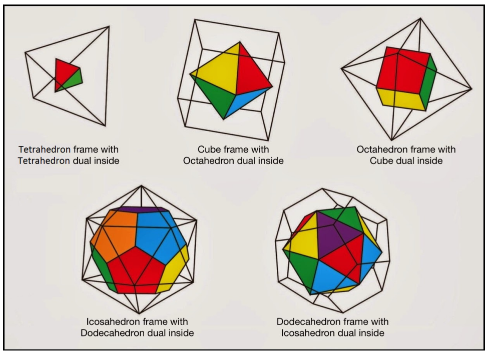


Figure 3 Duals of each Platonic solid.