

# IMPERIAL

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## Reflection Groups

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## **Abstract**

# Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Dihedral groups</b>	<b>2</b>
2.1	Dihedral groups . . . . .	2
2.2	The Infinite dihedral group . . . . .	4
<b>3</b>	<b>Platonic solids and reflections</b>	<b>4</b>
3.1	Cube . . . . .	5
3.2	Dual . . . . .	6
3.3	Dodecahedron and isocahedron . . . . .	6
3.4	Tetrahedron . . . . .	7
<b>4</b>	<b>Classification of finite reflection groups</b>	<b>7</b>
4.1	Affine reflections . . . . .	7
4.2	Reflection groups . . . . .	9
4.3	Coxeter presentation . . . . .	10
4.4	Classification . . . . .	13
4.5	Geometric representation . . . . .	15
<b>5</b>	<b>Uniform polytopes</b>	<b>16</b>
<b>6</b>	<b>Tits' word problem</b>	<b>16</b>
6.0	Abstract . . . . .	16
6.1	Exchange Condition . . . . .	16
6.2	M-operations . . . . .	18
6.3	Tit's Theorem . . . . .	19
<b>7</b>	<b>More on Exchange Condition</b>	<b>19</b>
<b>8</b>	<b>Euclidean Reflection Groups</b>	<b>20</b>
8.1	Structure of Euclidean Reflection Groups . . . . .	20
8.2	Some Model Theory of Euclidean Reflection Groups . . . . .	23
<b>A</b>	<b>Model-Theoretic Preliminaries</b>	<b>24</b>

# 1 Introduction

In lectures, we have seen how to classify finitely generated abelian groups. A main objective of this report is to classify finite reflection groups.

Recall that a reflection is a linear operator on  $\mathbb{R}^n$  which sends some nonzero vector  $\mathbf{u}$  to its negative while fixing pointwise the hyperplane  $H_{\mathbf{u}}$  orthogonal to  $\mathbf{u}$ . With the usual inner product, the reader can check that  $s_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - 2\frac{\mathbf{v} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}}\mathbf{u}$ .

We will start by looking at the lower-dimensional reflection groups and proceed to more general ones, introducing some related terminology and ideas along the way. Classification of reflection groups not only describes the symmetries of regular polytopes but also helps understand more complex algebraic structures.

We then focus on affine reflection groups. By relating them to finite reflection groups, we show more finiteness properties of them, which gives us a structural theorem on Euclidean reflection groups. From this theorem, we show this structure can be interpreted in Presburger arithmetic, hence deduce the decidability of Euclidean reflection groups.

## 2 Dihedral groups

### 2.1 Dihedral groups

The dihedral group,  $D_{2n}$ , is the group of symmetries of a regular  $n$ -gon. Its standard presentation is given by

$$\langle r, s \mid r^n = s^2 = e, (rs)^2 = e \rangle$$

where  $r$  is a rotation of  $2\pi/n$  and  $s$  is a reflection.

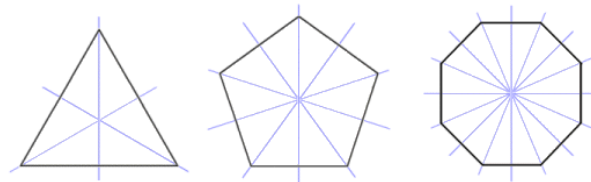
Let  $l_1$  and  $l_2$  be two reflection axes with an angle  $\theta$  between  $l_1$  and  $l_2$ , and  $s_1$  and  $s_2$  be the respective reflections. After some algebra, the composition  $s_1 s_2$  turns out to be a counterclockwise rotation through  $2\theta$ .

Therefore, an alternative presentation of  $D_{2n}$  is given by

$$\langle s_1, s_2 \mid s_1^2 = s_2^2 = e, (s_1 s_2)^n = e \rangle$$

where  $s_1$  and  $s_2$  are adjacent reflections.

This shows that  $D_{2n}$  is an example of a reflection group.



**Figure 1** The lines of symmetry of a regular 3-, 4- and 7-gon.

**Example 2.1.1.** The coxeter diagram for the symmetries of a regular  $n$ -gon, also known as  $I_2(n)$ , looks like

$$I_2(n) \quad \bullet \xrightarrow{n} \bullet \quad (n \geq 4)$$

**Theorem 2.1.2.** Let  $G \curvearrowright X$  be an action of a finite group  $G$  on a finite set  $X$ . Then the number of  $G$ -orbits in  $X$  is given by:

$$\text{Number of orbits} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

where  $\text{Fix}(g) = \{x \in X \mid g \cdot x = x\}$ .

**Example 2.1.3.** [1] How many distinguishable necklaces can be made using seven different colored beads of the same size?

Let  $X$  be the  $7!$  possible arrangements. The necklace can be turned over (a reflection) as well as rotated so we consider the dihedral group  $D_{14}$  acting on  $X$ . Using the previous theorem,

$$\text{Number of orbits} = \frac{7!}{14} = 360$$

as only the identity leaves any arrangement fixed.

**Theorem 2.1.4.** Let  $p$  be a prime. Then any group  $G$  of order  $2p$  is isomorphic to either the cyclic group  $C_{2p}$  or the dihedral group  $D_{2p}$ .

**Proof.** By Cauchy's theorem, there exists an element  $a \in G$  of order  $p$  and an element  $b \in G$  of order 2. Let

$$H = \langle a \rangle,$$

so  $H$  is a subgroup of  $G$  of index 2, so  $H \trianglelefteq G$ .

Since  $H \trianglelefteq G$ , conjugation by  $b$  sends  $H$  to itself. Thus, there exists  $k \in \{1, 2, \dots, p-1\}$  such that

$$bab^{-1} = a^k.$$

Applying conjugation by  $b$  twice to  $a$  gives

$$a = b^2 ab^{-2} = b(bab^{-1})b^{-1} = ba^k b^{-1} = (bab^{-1})^k = (a^k)^k = a^{k^2}.$$

Therefore,

$$a = a^{k^2} \implies a^{k^2-1} = e.$$

Since  $a$  has order  $p$ , this implies

$$p \mid (k^2 - 1),$$

or equivalently,

$$k^2 \equiv 1 \pmod{p}.$$

Because  $p$  is prime, this implies

$$k \equiv \pm 1 \pmod{p}.$$

- If  $k \equiv 1$ , then

$$bab^{-1} = a,$$

and  $b$  commutes with  $a$ . Hence  $G$  is abelian, and since  $a$  has order  $p$  and  $b$  has order 2,  $G$  is cyclic of order  $2p$ .

- If  $k \equiv -1$ , then

$$bab^{-1} = a^{-1},$$

which is the defining relation for the dihedral group  $D_{2p}$ :

$$D_{2p} = \langle a, b \mid a^p = e, b^2 = e, bab = a^{-1} \rangle.$$






Thus,  $G$  is isomorphic to either  $C_{2p}$  or  $D_{2p}$ .

□

## 2.2 The Infinite dihedral group

We can also consider the group generated by two reflections  $s_1, s_2$  with order of  $s_1 s_2$  infinite. This is the infinite dihedral group denoted by  $D_\infty$ . Note every finite dihedral group is a quotient group of  $D_\infty$ . The quotient map is simply sending generators to generators.  $D_\infty$  can be taken as the symmetry group of the real line, with all integers distinguished. **Add a picture?** The generators can be any two consecutive distinguished points on that line. This is an example of affine reflection groups we will define later, and also the simplest example.

## 3 Platonic solids and reflections

Polyhedron		Vertices	Edges	Faces
tetrahedron		4	6	4
cube / hexahedron		8	12	6
octahedron		6	12	8
dodecahedron		20	30	12
icosahedron		12	30	20

**Figure 2** The five Platonic solids.

**Definition 3.0.1.** A polyhedron is regular if its faces are regular polygons, all with the same number of sides, and also each vertex belongs to the same number of edges.

**Theorem 3.0.2.** [2] The only regular convex polyhedra are the five Platonic solids.

**Proof.** Before writing the proof, we introduce some notations.

$V$ , the number of vertices;

$E$ , the number of edges;

$F$ , the number of faces;  
 $n$ , the number of sides on a face;  
 $r$ , the number of edges to which each vertex belongs.  
Observe that

$$2E = nF \quad (1)$$

and

$$2E = rV \quad (2)$$

(1) comes from counting the number of pairs  $(e, f)$  where  $e$  is an edge and  $f$  is a face and  $e$  lies on  $f$ ; (2) comes from counting the number of pairs  $(v, e)$  where  $v$  is a vertex and  $v$  lies on  $e$ . Substitute into Euler's formula, we get

$$\frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E} \quad (3)$$

Now  $n \geq 3$ , as a polygon must have at least 3 sides and  $r \geq 3$ , since in a polyhedron a vertex must belong to at least 3 edges. By (3), we can't have both  $n \geq 4$  and  $r \geq 4$ , since this would make the left-hand side of (3) at most  $\frac{1}{2}$ . It follows that either  $n = 3$  or  $r = 3$ . If  $n = 3$ , then (3) becomes

$$\frac{1}{r} = \frac{1}{6} + \frac{1}{E} \quad (4)$$

The right-hand side is greater than  $\frac{1}{6}$ , and hence  $r < 6$ . Therefore,  $r = 3, 4$  or  $5$  and  $E = 6, 12$  or  $30$ , respectively. If  $r = 3$ , (3) becomes

$$\frac{1}{n} = \frac{1}{6} + \frac{1}{E} \quad (5)$$

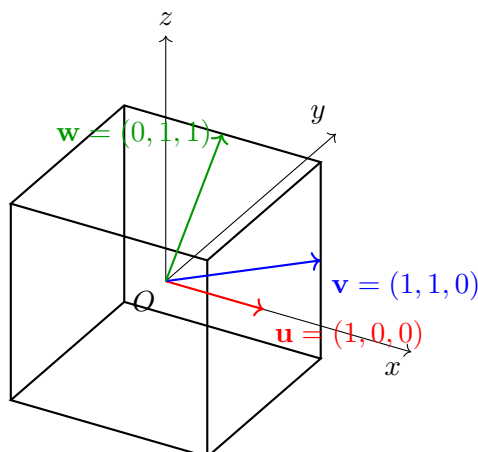
Similarly,  $n = 3, 4$  or  $5$  and  $E = 6, 12$  or  $30$ , respectively. These parameters coincide with those in the table above.  $\square$

**Example 3.0.3.** Let  $G$  be the group of symmetries of a dodecahedron. What is  $|G|$ ?  
Let  $G$  act on the 12 faces of the dodecahedron and fix a face. There are  $|D_{10}| = 10$  symmetries which fix this face and our action is clearly transitive. By Orbit-Stabiliser theorem,  $|G| = 10 \times 12 = 120$ . Alternatively, this can be done by considering the fundamental domain, which is a triangle that uniquely determines the reflection. There are 120 such triangles.

We would like to study the group of symmetries of these Platonic solids.

### 3.1 Cube

We start by investigating the cube.



As, seen before,  $s_u$  and  $s_v$  generates the square. It turns out that  $s_u = (0, 1, 1)$ , orthorgonal to  $s_v$ , is a clever third choice of vector, which gives  $(s_u s_v)^2 = e$ . Moreover,  $u \cdot w = \frac{1}{2}$  so the angle between  $u$  and  $w$  is  $\frac{\pi}{3}$ , hence  $(s_u s_w)^3 = e$ . Thus the coxeter diagram for the symmetries of the cube, also known as  $BC_3$ , looks like



Equivalently, recall that this is equivalent to the group presentation

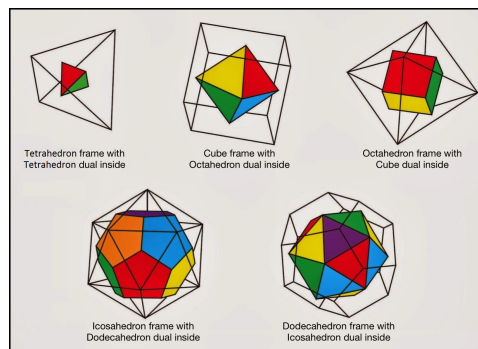
$$\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = e, (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

### 3.2 Dual

Before moving on to the other solids, we first introduce the concept of dual.

**Definition 3.2.1.** The dual of a Platonic solid is a new Platonic solid where the faces and vertices are interchanged with those of the original.

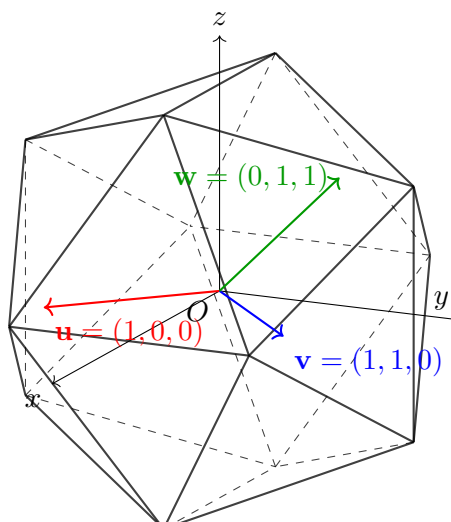
**Remark 3.2.2.** The tetrahedron is self-dual. The cube and the octahedron form a dual pair. The dodecahedron and the icosahedron form a dual pair.



**Figure 3** Duals of each Platonic solid.

**Remark 3.2.3.** A polyhedron and its dual have the same axes of symmetry, so they have the same Coxeter diagram. For example, the Coxeter diagram for the symmetries of the octahedron is again  $BC_3$  as seen before.

### 3.3 Dodecahedron and isocahedron





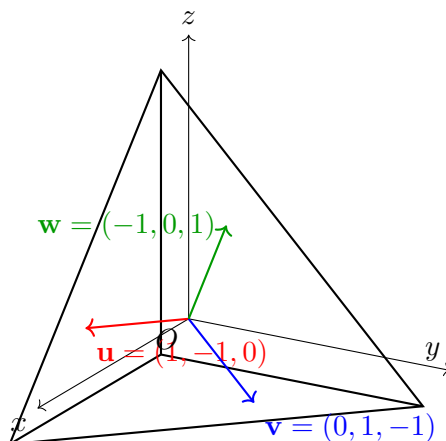
The group presentation is

$$\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = e, (s_1 s_2)^3 = (s_2 s_3)^5 = (s_1 s_3)^2 = e \rangle$$

and the Coxeter diagram, also known as  $H_3$ , is



### 3.4 Tetrahedron



The group presentation is

$$\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = e, (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

and the Coxeter diagram, also known as  $A_3$  (careful, this is NOT the alternating group  $A_3$ ), is



## 4 Classification of finite reflection groups

We're now going to start examining the algebraic properties of general reflection groups. To do this we will be working in  $V$ , denoting  $\mathbb{R}^n$  equipped with the standard inner product, our fixed *euclidean space*.

### 4.1 Affine reflections

This subsection roughly follows [3]. When considering reflections in  $V$ , we don't want to just consider those lying in  $O(E)$ , we are also interested in reflections across some hyperplane not lying through the origin. To tackle this we first define some standard preliminaries:

**Definition 4.1.1.** A **reflection** in  $E$  is a linear transformation  $s \in O(E)$  (the orthogonal group of inner product preserving linear maps) with eigenvalues  $\{-1, 1\}$  with corresponding dimensions of eigenspaces:

$$\dim E_1 = n - 1 \quad \dim E_{-1} = 1$$

for linear maps in  $O(E)$  this is equivalent to fixing some hyperplane (codimension 1 subspace) and having determinant  $-1$ .

**Definition 4.1.2.** The **group of general affine transformations** of  $E$  is semidirect product of  $GL(E)$  acting on  $E$

$$GA(E) := E \rtimes GL(E)$$

where  $E$  acts on itself by translation.  $(v, T) \in GA(E)$  acts on  $e \in E$  as  $(v, T) \cdot e = v + T(e)$ .

**Proposition 4.1.3.** This is actually an action. Furthermore, the action is transitive.

*Proof.* Take  $(v, T), (u, S) \in GA(E)$  and  $e \in E$ . Showing this is an action is a direct calculation:

$$(v, T) \cdot ((u, S) \cdot e) = (v, T) \cdot (u + S(e)) = v + T(u) + TS(e) = (v + T(u), TS) \cdot e = ((v, T)(u, S)) \cdot e$$

from the definition of the semidirect product. Suppose  $(v, T)$  and  $(u, S)$  act the same on  $E$ : as both  $T$  and  $S$  are linear the two affine transformations send 0 to  $v$  and  $u$  respectively, thus we have  $v = u$ ; subtracting these equal translations and having equality means the two linear transformations must also be equal.  $\square$

We can now consider appropriate affine versions of linear groups, the most important for us will be the **affine orthogonal group**  $AO(E)$ , which can be equivalently viewed as the subgroup of  $GA(E)$  consisting of isometries, or as  $E \rtimes O(E)$ . We need stricter criteria than just the linear component of our affine transformation be a reflection to suitably capture the notion of a reflection across an affine hyperplane.

**Proposition 4.1.4.** For a unit normal vector  $\alpha$  and some  $k \in \mathbb{R}$ , reflection across the affine hyperplane  $(E, \alpha) = k$  corresponds to the affine transformation  $(2k\alpha, s_\alpha)$ , where  $s_\alpha$  is the reflection along  $\alpha$ .

*Proof.* Call the affine hyperplane  $H$  and choose a  $v \in E$ . The vector orthogonal to  $H$  that goes to  $v$  has length  $k - (v, \alpha)$  so reflecting across  $H$  send  $v$  to  $v + 2(k - (v, \alpha))\alpha = 2k\alpha + (v - 2(v, \alpha)\alpha) = (2k\alpha, s_\alpha) \cdot v$ .  $\square$

We call such affine transformations, **affine reflections**.

**Lemma 4.1.5.** For all affine hyperplanes  $H$  and affine reflections  $r$ , the set  $rH$  is also an affine hyperplane.

*Proof.* Let  $H$  be  $\{v \in V \mid (v, \alpha) = k\}$  for some  $\alpha \in E, k \in \mathbb{R}$ . As  $r$  is bijective the set  $rH$  is equal to  $\{w \in V \mid (r^{-1}w, \alpha) = k\}$ , by writing  $r = (u, T) \in GA(E)$  this can be rewritten as  $\{w \in V \mid (w, T^*(\alpha)) = k - (u, \alpha)\}$ , an affine hyperplane.  $\square$

**Lemma 4.1.6.** An affine transformations  $r = (u, T)$  that fixes some affine hyperplane and is an involution must be an affine reflection.

*Proof.* As  $r^2 = \text{id}$ , on 0 we have  $r^2(0) = u + T(u) = 0$ . So as  $r^2 = u + T(u) + A^2$  this means  $A$  is also an involution so we can use the primary decomposition  $E = E_1 \oplus E_{-1}$  into the eigenspaces of  $A$ . Call the hyperplane  $r$  fixes  $H$ , then for any  $h = v_1 + v_{-1} \in H$  (where  $v_1, v_{-1} \in V_1, V_{-1}$  respectively) we have  $r(v_1 + v_{-1}) = u + T(v_1 + v_{-1}) = u + v_1 - v_{-1} = v_1 + v_{-1}$  therefore  $2v_{-1} = u$  so  $\dim E_{-1} = 1$  and  $T$  is the reflection along  $V_{-1} = \langle v_{-1} \rangle$ , thus  $r = (2v_{-1}, s_{v_{-1}})$ .  $\square$

**Proposition 4.1.7.** For all affine reflections  $r, s$  with  $s$  reflecting across the affine hyperplane  $H$ , the affine transformation  $rsr^{-1}$  is an affine reflection across  $rH$ .

*Proof.* First, notice  $(rsr^{-1})^2 = \text{id}$  as both  $r$  and  $s$  are involutions. Also,  $rsr^{-1}(rH) = rH$  as  $s$  fixes  $H$ .  $\square$

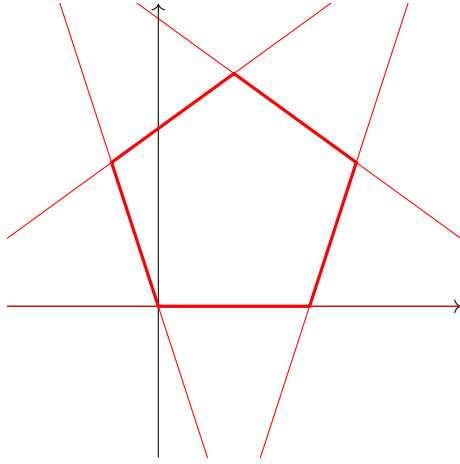
From now on, we will use the umbrella term *reflection* to refer to both linear and affine reflections.

## 4.2 Reflection groups

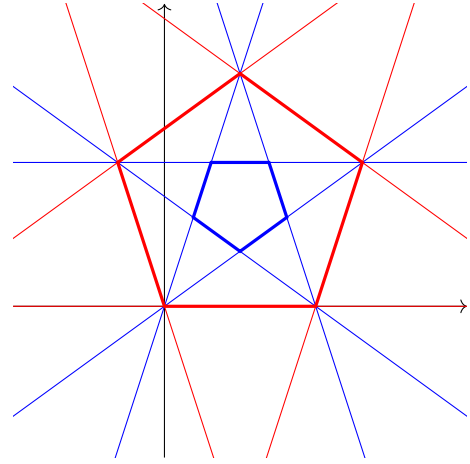
We are interested in groups generated by reflections in  $E$ , so throughout the next two sections fix a group  $W \leq GA(E)$  which can be generated by reflections. This roughly follows [4]

We would like to consider the set of all hyperplanes  $H$  such that there is some affine reflection  $w \in W$  across  $H$  this set will be referred to as  $\mathcal{H}$ . For certain choices of generators for  $W$  we may find  $\mathcal{H}$  is dense in  $E$ .

**Example 4.2.1.** Consider the set of hyperplanes pictured in red:



(a) Generating hyperplanes



(b) Resulting hyperplanes

As we have seen in the previous proposition,  $W$  acts on  $\mathcal{H}$  so all the reflections of these hyperplanes, show in blue, must also lie in  $\mathcal{H}$ . This process can be repeated to find an arbitrarily small pentagon bounded by hyperplanes. By the presence of arbitrarily close parallel hyperplanes we can see  $W$  will contain arbitrarily small translations and thus  $\mathcal{H}$  will be dense in  $E$ .

This adds technicalities and looses the discrete geometric intuition we have for reflection groups of polytopes and lattices. To remedy this we will restrict the reflection groups we consider by requiring for any compact subset  $B \subset E$ , the intersection  $\mathcal{H} \cap B$  be finite. We can also see this immediately restricts the possible angles between hyperplanes to rational multiples of  $\pi$ .

The set of unitvectors  $\alpha \in E$  such that there exists some  $H \in \mathcal{H}$  with  $\alpha$  normal to  $H$  will be written  $\Phi$ . And without loss of generality we will assume  $\Phi$  spans  $E$  from here onwards.

**Definition 4.2.2.** The connected components of  $E \setminus \mathcal{H}$  are called the **chambers** of  $W$  in  $E$ .

The reflection groups we are interested in, those that appear as finite reflection groups of polytopes or lattices in whatever geometry we prefer, will have finitely many hyperplanes touching each chamber. Later on we will see this is equivalent to the group being finitely presented. A formula for the number of hyperplanes touching a chamber is given in Section A

We now want to choose a chamber  $C_0$ , and consider the set of hyperplanes  $\{H_1, \dots, H_k\} \subseteq \mathcal{H}$  bounding  $C_0$ . We will call the corresponding reflections across these hyperplanes  $\{s_1, \dots, s_k\}$  **simple reflections**.

**Theorem 4.2.3.** The set of simple reflections  $\{s_1, \dots, s_k\}$  generates  $W$ .

*Proof.* Let  $W'$  be the subgroup of  $W$  generated by the simple reflections. Let  $s$  be one of the reflections generating  $W$ , and call the hyperplane it reflects across  $H$ . If  $W'$  acts transitively on the set of chambers then there will exist some  $w \in W'$  such that  $wH_i = H$  for some simple

reflection  $s_i$  as  $H$  will must bound a chamber. Thus, by an earlier lemma,  $ws_iw^{-1} = s$  so  $s \in W'$  and  $W = W'$ . Now we just have to show the action of  $W'$  is transitive on chambers. Suppose it isn't, i.e. there is some chamber  $C$  such that no  $w \in W'$  satisfies  $wC = C_0$ . Let  $C'$  be the closest chamber to  $C_0$  in the  $W'$  orbit of  $C$ , as  $C' \neq C_0$  there must be some simple hyperplane (the boundary of  $C_0$ ) between them, reflecting across this must strictly decrease the distance between the two chambers contradicting the minimality of  $C'$ . Thus  $W'$  acts transitively on the set of chambers.  $\square$

As a direct corollary of this proof we now know  $W$  acts transitively on the set of chambers. Before discussing this further we should examine the relations that these simple reflections satisfy.

For any two simple reflections  $s_i, s_j$  the subgroup  $\langle s_i, s_j \rangle$  will be dihedral, as seen in the previous section, (note that this will be the infinite dihedral group iff the hyperplanes being reflected along are parallel), call the order of this dihedral group  $2m_{ij}$ . The product  $s_i s_j$  will have order  $m_{ij}$  in  $W$  and so  $W$  satisfies the set of relations  $(s_i s_j)^{m_{ij}} = \text{id}$  for all  $i, j$ , taking  $m_{ii} = 1$ . A group *presented* by these relations is called **Coxeter**.

**Definition 4.2.4.** A **Coxeter system** is a pair  $(W, S)$  where  $S = \{s_i\}_{i \in I}$  is a generating set for  $W$  which admits the presentation:

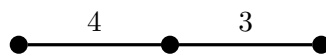
$$W = \langle S \mid (s_i s_j)^{m_{ij}} \text{ for all } i, j \in I \rangle$$

where each  $m_{ij} \in \mathbb{N} \cup \{\infty\}$ .

To each Coxeter system we can assign a **Coxeter diagram**: an undirected graph created by the following rules:

- Draw a node  $i$  for each  $s_i \in S$ ;
- For each relation  $(s_i s_j)^{m_{ij}}$  with  $m_{ij} > 2$  draw an edge between  $i$  and  $j$  and label it with  $m_{ij}$ .

This process can be reversed to obtain a Coxeter system from any Coxeter diagram. This correspondence will associate the graph:



to the group presentation:

$$\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = e, (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

In the future, for the sake of readability, the 3 labels will often be excluded.

The classification of finite reflection groups goes by proving all reflection groups are in fact Coxeter groups and then classifying all the finite Coxeter groups.

### 4.3 Coxeter presentation

We have shown the action of  $W$  on the set of chambers is transitive, but we need a stronger notion to prove these groups are Coxeter.

**Definition 4.3.1.** Let  $G$  be a group acting on a set  $X$ , the action is called **simply transitive** if for all  $x, y \in X$  there exists a unique  $g \in G$  such that  $g \cdot x = y$ .

By fixing a base point  $x \in X$  there is clear a 1-to-1 correspondence between  $X$  and  $G$ : associate  $e \leftrightarrow x$  and for all  $g \in G$ ,  $g \leftrightarrow g \cdot x$ .

**Proposition 4.3.2.** An action is simply transitive iff it is transitive and free.

*Proof.* An action is transitive if for all  $x, y \in X$  there exists some  $g \in G$  such that  $g \cdot x = y$ , and free if there exists at most one such  $g$ ; therefore these statements are equivalent.  $\square$

If we know the action is transitive a priori we only need to check the free condition on a single element. So showing our action is simply transitive amounts to showing for all  $w \in W$  such that  $wC_0 = C_0$  we have  $w = \text{id}$ .

We will go about proving this by finding a geometric interpretation of the length of a word  $w \in W$  in terms of simple reflections.

**Definition 4.3.3.** For  $w \in W$  define the **length** of  $w$ ,  $l(w)$  to be the minimal positive integer  $r$  such that  $w = s_1 \cdots s_r$ , a length  $r$  product of simple reflections.

And the contrasting way to compute the length: for any  $w \in W$  consider the set of hyperplanes  $H$  such that  $C_0$  and  $wC_0$  lie on different sides of  $H$ . Call this  $\mathcal{L}(w)$  and let  $n(w) = |\mathcal{L}(w)|$ . As the line segment between two chambers is bounded, by assumption this will only meet finitely many hyperplanes so for any  $w \in W$  we will have  $\mathcal{L}(w) \in \mathbb{N}$ .

- Lemma 4.3.4.**
1.  $n(w) = n(w^{-1})$ ,
  2.  $l(w) = 1$  iff  $w$  is a simple reflection
  3.  $l(w) = l(w^{-1})$
  4. if  $w$  can be written as  $s_1 \cdots s_r$  then  $\det(w) = (-1)^r$
  5.  $l(s_i w), l(ws_i) = l(w) \pm 1$

*Proof.* (1.)  $H$  separates  $C_0$  and  $w^{-1}C_0$  iff  $wH$  separates  $wC_0$  and  $C_0$  —  $W$  acts isometrically.  $\square$

**Lemma 4.3.5.**  $H_i$  is in exactly one of  $\mathcal{L}(w)$  or  $\mathcal{L}(s_i w)$

**Lemma 4.3.6.** Choose a simple hyperplane  $H_i$ , for any hyperplane  $H \neq H_i$  and  $w \in W$ , if  $H \in \mathcal{L}(w)$  then  $s_i H \in \mathcal{L}(s_i w)$ .

*Proof.* As  $H$  separates  $C_0$  and  $wC_0$ ,  $s_i H$  separates  $s_i C_0$  and  $s_i wC_0$ , so  $s_i H$  is in exactly one of  $\mathcal{L}(s_i)$  or  $\mathcal{L}(s_i w)$ , but if  $s_i H \in \mathcal{L}(s_i)$  we would have  $s_i H = H_i \implies H = H_i$  a contradiction. Therefore  $s_i H \in \mathcal{L}(s_i w)$ .  $\square$

**Corollary 4.3.7.**  $s_i(\mathcal{L}(w) \setminus \{H_i\}) = \mathcal{L}(s_i w) \setminus \{H_i\}$ .

*Proof.* by applying the previous lemma to both  $w$  and  $s_i w$  we get the required iff.  $\square$

**Proposition 4.3.8.** For all  $w \in W$ , we have  $n(w) \leq l(w)$ .

*Proof.* If  $l(w) = 1$  then  $w$  must be some simple reflection  $s_i$  so  $\mathcal{L}(w) = \{H_i\}$  [citation-needed] so  $n(w) = 1$ . Now by induction on  $l(w)$ : if  $l(s_i w) = l(w) + 1$  then by the previous corollary and an earlier lemma we know  $n(s_i w) = n(w) \pm 1$ , namely  $n(s_i w) \leq n(w) + 1 = l(w) + 1 = l(s_i w)$ .  $\square$

Now to show  $n(w) = l(w)$  we will enumerate the hyperplanes in  $\mathcal{L}(w)$ .

**Lemma 4.3.9.** If  $w = s_1 \cdots s_r$  is a reduced expression for  $w$  in terms of simple reflections, the hyperplanes:

$$H_1, s_1 H_2, s_1 s_2 H_3, \dots, s_1 \cdots s_{r-1} H_r$$

are all distinct.

*Proof.* Suppose, for a contradiction, that there exists some  $1 \leq i < j \leq r$  such that the hyperplanes  $s_1 \cdots s_{i-1}H_i$  and  $s_1 \cdots s_{j-1}H_j$  are equal, by applying  $s_1 \cdots s_{i-1}$  to both sides this implies  $H_i = s_i \cdots s_{j-1}H_j$ , by an earlier corollary this implies  $s_i = (s_1 \cdots s_{j-1})s_j(s_{j-1} \cdots s_i)$  which implies  $s_{i+1} \cdots s_{j-1} = s_i \cdots s_j$  contradicting the minimality of the length of  $w$ .  $\square$

**Proposition 4.3.10.** If  $w = s_1 \cdots s_r$  is a reduced expression for  $w$  in terms of simple reflections:

$$\mathcal{L}(w) = \{H_1, s_1H_2, s_1s_2H_3, \dots, s_1 \cdots s_{r-1}H_r\}$$

*Proof.* We go by induction on  $r$ . Observe the base case  $\mathcal{L}(s_1) = \{H_1\}$  from an earlier lemma. Now assume the claim holds for all  $w$  with reduced length  $< r$  specifically:

$$\mathcal{L}(s_1w) = \{H_2, s_2H_3, \dots, s_2 \cdots s_{r-1}H_r\}$$

we know from an earlier lemma  $H_1$  is in exactly one of  $\mathcal{L}(w)$  or  $\mathcal{L}(s_1w)$ . If  $H_1 \in \mathcal{L}(s_1w)$  by applying  $s_1$  we would have for some  $i > 1$ :

$$s_1H_1 = H_1 = s_1 \cdots s_iH_i$$

which contradicts the previous lemma, thus  $H_1 \in \mathcal{L}(w)$ , combining this with the earlier corollary gives the desired result.  $\square$

A direct consequence of this proposition is that if some  $w \in W$  is such that  $wC_0 = C_0$  then  $\mathcal{L}(w) = \emptyset$  which implies  $w = \text{id}$  in reduced form. We know this is sufficient to say the action of  $W$  on the set of chambers is simply transitive.

We can now prove two key combinatorial properties of the reflection group, and use these to show all relations in  $W$  are direct consequences of those in the Coxeter presentation.

**Theorem 4.3.11** (Exchange Condition). If  $w \in W$  has reduced expression in terms of simple reflections  $w = s_1 \cdots s_r$  and  $l(sw) < l(w)$  for some simple reflection  $s$ , there exists some  $1 \leq i \leq r$  such that  $w = ss_1 \cdots s_{i-1}s_{i+1} \cdots s_r$ .

*Proof.* We know:

$$\mathcal{L}(w) = \{H_1, s_1H_2, s_1s_2H_3, \dots, s_1 \cdots s_{r-1}H_r\}$$

by a previous proposition we know  $H$ , the hyperplane corresponding to  $s$ , must lie in  $\mathcal{L}(w)$  and so for some  $1 \leq i \leq r$ ,  $H = s_1 \cdots s_{i-1}H_i$  which implies  $s = s_1 \cdots s_{i-1}s_is_{i-1} \cdots s_1$  therefore  $ss_1 \cdots s_{i-1} = s_1 \cdots s_i$ . Substituting this back into  $w$  gives the desired result.  $\square$

**Theorem 4.3.12** (Deletion Condition). If  $w \in W$  has a nonreduced expression in terms of simple roots  $w = s_1 \cdots s_r$  then there exists  $1 \leq i, j \leq r$  such that  $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r$ . The hats denote omission.

*Proof.* For this to be nonreduced there must eventually exist some  $i$  such that  $l(s_i \cdots s_r) = l(s_{i+1} \cdots s_r) - 1$  so by the exchange condition  $s_{i+1} \cdots s_r = s_is_{i+1} \cdots \hat{s}_j \cdots s_r$ . Substituting this back into  $w$  gives the desired result.  $\square$

Informally, we can see that these weak forms of the deletion and exchange conditions are actually direct consequences of the Coxeter relations, this is geometrically motivated in subsection 4.5. The formal proof that the exchange condition (which implies the deletion condition) is a property of Coxeter groups rigorously defined as quotients of a free group is given in Section 6. We are electing to use this informal proof at this point in the report to continue the geometric nature of this section, thus the following proof that reflection groups satisfy the Coxeter presentation is the same somewhat informal argument as given in [4].

**Theorem 4.3.13.** Any finite reflection group satisfies the Coxeter presentation.

*Proof.* We want to consider the smallest nontrivial relation  $w = s_1 \cdots s_r = \text{id}$ , in terms of simple reflections, in  $W$  which is not a consequence of the Coxeter relations.

By examining the determinants, we see  $r = 2k$  must be even, thus we can write  $s_1 \cdots s_{k+1} = s_{k+2} \cdots s_r$  and invoke the deletion condition which lets us rewrite  $w = s_1 \cdots \hat{s}_i \cdots \hat{s}_j \cdots s_r = \text{id}$  which must also not be a consequence of the Coxeter relations, this contradicts minimality.  $\square$

Note that in this proof we tacitly expect  $r \geq 4$ , luckily any length 2 relation is just an equality which by assumption do not occur.

#### 4.4 Classification

**Definition 4.4.1.** To a Coxeter system  $(W, S)$ , with  $n = |S|$ , we can associate a real symmetric  $n \times n$  matrix  $A$ , by setting  $a_{ij} := -\cos(\pi/m_{ij})$ . The quadratic form  $Q(v) = v^\top A v$  on  $E$  is called the **associated quadratic form** of the Coxeter system.

We can realise each  $a_{ij}$  as the inner product of simple roots  $(\alpha_i, \alpha_j)$  corresponding to the reflection  $s_i, s_j$ . Such matrices are called **Gram matrices**.

**Proposition 4.4.2.** The quadratic form associated to a finite reflection group  $W$  is positive definite.

*Proof.* As  $\Phi$  forms a basis for  $E$  there is an invertible change of basis matrix  $B$  from the standard basis to  $\Phi$ , we can thus write  $A = B^\top B$  and for any nonzero  $v \in E$  have:

$$Q(v) = v^\top A v = v^\top B^\top B v = (Bv)^\top (Bv) = \|Bv\|^2 > 0$$

as  $B$  is a linear transformation acting on  $v \neq 0$ , therefore  $Q$  is positive definite.  $\square$

We now want to fix a quadratic form  $Q(v) = v^\top A v$  for some symmetric  $A \in \mathcal{M}_n(\mathbb{R})$ .

**Lemma 4.4.3.** The kernel of  $Q$  (the set of vectors  $v \in E$  such that  $Q(v) = 0$ ) equals the nullspace of  $A$ .

*Proof.* The nullspace of  $A$  obviously lies in the kernel of  $Q$ . From linear algebra 2 we know we can orthogonally diagonalise  $A$  and so we can write:

$$Q(v) = v^\top A v = v^\top P^\top D P v = (Pv)^\top D (Pv)$$

where  $P$  and  $D$  are some orthogonal and diagonal matrices respectively. Observe the right hand side is 0 iff for every  $1 \leq i \leq n$  either  $d_{ii}$  or  $(Pv)_i$  are 0. This implies  $D(Pv) = 0$  and thus  $v$  is in the nullspace of  $A$ .  $\square$

We will now demand all of the off-diagonal entries of  $A$  be nonpositive, note that all of the matrices for quadratic forms associated with Coxeter diagrams will satisfy this condition.

**Lemma 4.4.4.** The only nonzero vectors in the nullspace of  $A$  have all coordinates nonzero.

*Proof.* First suppose  $v \neq 0$  is in the nullspace of  $A$ , then by the previous lemma we know  $Q(v) = 0$ . Consider the vector  $u$  given by  $u_i = |v_i|$ , we will have  $Q(u) = u^\top A u \geq 0$ , but as all off-diagonal entries of  $A$  are nonpositive this will satisfy  $Q(u) \leq Q(v) = 0$ . Thus  $u$  is also in the nullspace of  $A$ . So if the nullspace is nontrivial, it will always contain a vector with nonnegative coordinates.

If we suppose some, but not all, of these coordinates are zero we can consider the set of such indices  $I$ . For any  $i \in I$  consider the  $i$ th term in  $Au$ :

$$(Au)_i = \sum_{j \notin I} a_{ij} |v_j| = 0$$

as all the  $a_{ij}$  are nonpositive and all the  $|v_j|$  are strictly positive, we get that for all  $i \in I, j \notin I$  the entries  $a_{ij} = 0$  contradicting irreducibility of  $A$ . Thus the coordinates of all vectors in the nullspace of  $A$  are nonzero.  $\square$

**Corollary 4.4.5.** The dimension of the nullspace of  $A$  will be at most 1.

**Lemma 4.4.6.** The smallest eigenvalue  $d$  of  $A$  has multiplicity 1, and its coordinates will all be positive.

*Proof.* Again, from linear algebra 2 we know all the eigenvalues of a positive semidefinite matrix are nonnegative. So  $A - dI$  satisfies all the requirement we placed on  $A$  for the previous lemma. As  $A - dI$  is singular it will have nonempty nullspace which must be of dimension exactly 1. Thus  $d$  and  $u_i = |d_i|$  are colinear so we can find an eigenvalue with all positive coordinates.  $\square$

We want to consider subgraphs of the Coxeter diagram  $\Gamma$  associated to  $(W, S)$ . For this we will introduce a Coxeter system  $(W', S')$  where  $S' \subseteq S$ , the Coxeter relation degrees in  $(W', S')$  are given by  $m'_{ij}$  and they will satisfy  $m'_{ij} \leq m_{ij}$  for all  $i, j$ . We will call the associated Coxeter diagram, quadratic form and symmetric matrix  $\Gamma', Q', A'$  respectively. Assume this *subsystem* is proper i.e.  $S' \subsetneq S$  or some  $m'_{ij} < m_{ij}$ .

**Proposition 4.4.7.** If  $Q$  is positive semidefinite, then  $Q'$  will be positive definite.

*Proof.* Suppose  $A'$  fails to be positive definite, i.e. there exists some  $v \in E$  such that  $Q'(v) \leq 0$ , consider the vector:

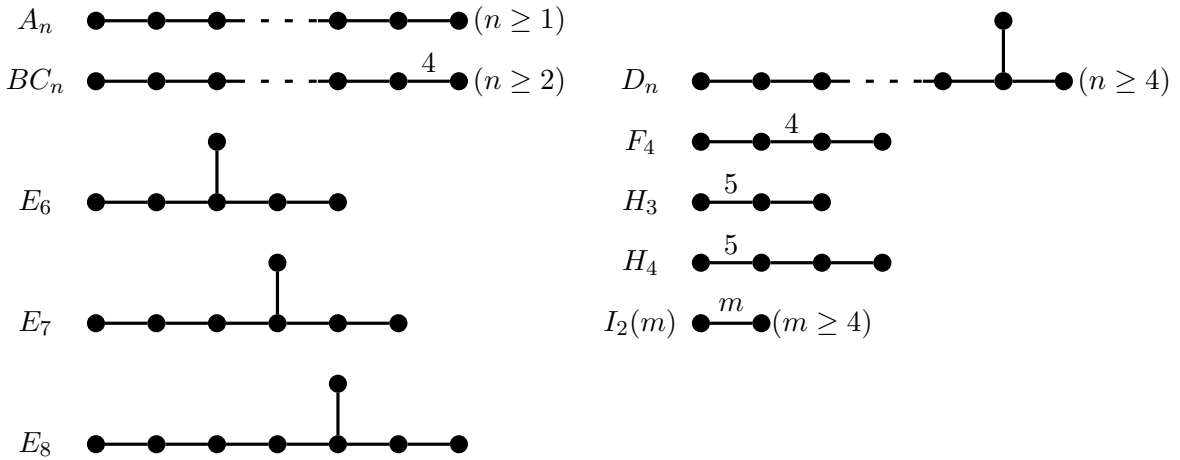
$$u_i = \begin{cases} |v_i| & 1 \leq i \leq k \\ 0 & \text{otherwise} \end{cases}$$

and the value of:

$$0 \leq Q(u) = \sum_{1 \leq i, j \leq n} a_{ij} u_i u_j \leq \sum_{1 \leq i, j \leq k} a'_{ij} v_i v_j \leq 0$$

as all the  $a_{ij} \leq a'_{ij} \leq 0$ . This means  $u$  is in the nullspace for  $A$  and thus all the coordinates are nonzero which means  $k = n$ . But this then implies all  $a_{ij} = a'_{ij}$ . Thus the only subdiagram that fails to be positive definite is improper.  $\square$

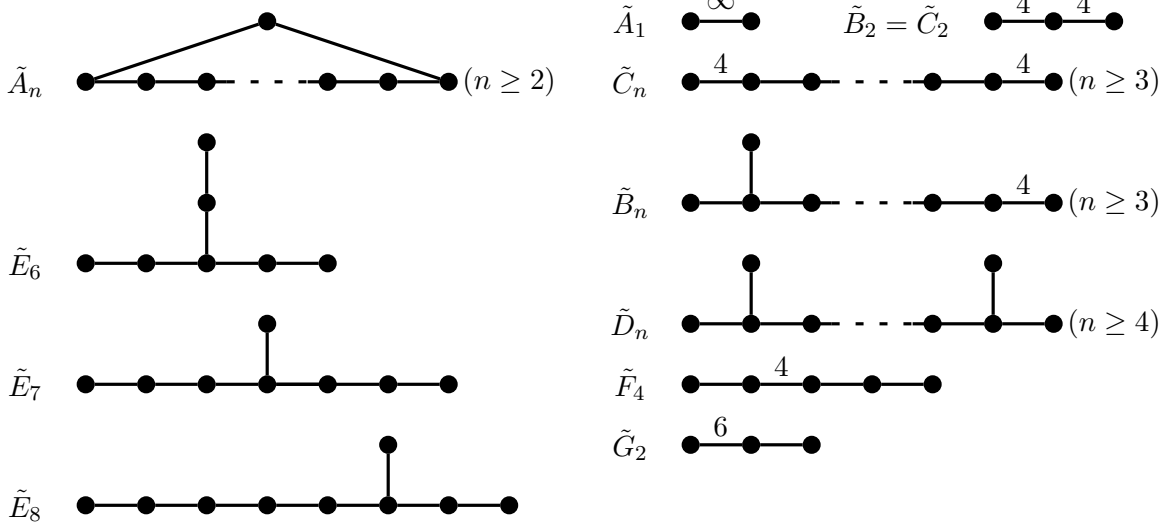
Here is a collection of Coxeter diagrams all of which, by checking their eigenvalues on a computer, can be found to be positive definite.



**Figure 5** Positive definite Coxeter diagrams

And here is another collection who can all be founded to be only positive semidefinite:





**Figure 6** Positive semidefinite Coxeter diagrams

Plus two helper diagrams which are neither:



**Figure 7** Helper Coxeter diagrams

**Theorem 4.4.8.** the diagrams of Figure 5 are all the connected diagrams with positive definite quadratic form.

*Proof.* The proof goes by indentifying features present in the graphs of Figure 6, observing they cannot appear as subdiagrams of those with positive definite quadratic form: e.g. no cycles as in  $\tilde{A}_n$  and all edges must be finite due to  $\tilde{A}_1$ . For the full proof see [4].  $\square$

**Corollary 4.4.9.** The following graphs categorise all finite reflection groups.

*Proof.* We know it suffices to classify the reflections groups corresponding to connected Coxeter diagrams, all of which must have positive defined associated quadratic form, all of which we have found. The only remaining step is to find polytopes admitting these Coxeter groups as their reflection groups, lots of these examples have been seen in the previous section, and in fact every diagram of Figure 5 does occur as a finite reflection.  $\square$

This solution can seem somewhat unsatisfactory as there is no immediate reason as to **why** all of the finite Coxeter groups can be realised as reflection groups.

## 4.5 Geometric representation

For any finite Coxeter system  $(W, S)$  consider the abstract real vector space  $V$  with basis  $\Phi$ . We can define a quadratic form (and thus an inner product  $\langle -, - \rangle$ ) on  $V$  in the same way as we did earlier. And for each generator  $s \in S$  we can associate a reflection  $\sigma_s : V \rightarrow V$  given natrually as  $\sigma_s(v) = v - 2 \langle \alpha_s, v \rangle \alpha_s$ . Through some similar calculations we can find the representation this induces on  $W$  is faithful and acts orthogonally on  $V$ , the construction of the edges and vertices of a polytope demonstrated in the next section will thus show how a regular polytope can be realised from this abstract reflection group.

## 5 Uniform polytopes

uniform polytopes

## 6 Tits' word problem

### 6.0 Abstract

Recalling the word problem for a presentation  $\langle S \mid R \rangle$  of a group  $G$ , we hope to find an algorithm to reduce any word in  $W$  into a reduced word. In this paper, we totally solved this problem for Coxeter groups, and actually for all system with exchange condition, which is called Tit's theorem.

The proof is divided into several steps:

1. Every Coxeter group satisfies the exchange condition.
2. For a system with exchange condition, when do two reduced words represent the same element.

3. How to reduce a word in a system with exchange condition.

The first part of the proof is the hardest, and we focus on the most.

We will talk more on the exchange condition as the fourth part, even though part 1 is enough to prove Tit's Theorem, exchange condition show deep insight of Coxeter Groups.

### 6.1 Exchange Condition

**Definition 1.1** A system is a tuple  $(W, S)$ , where  $W$  is a group and  $S$  is a set of generators of  $G$ , such that  $S = S^{-1}$  and  $1 \notin S$ .

**Definition 1.2** For  $w \in W$ , the sequence  $s = (s_1, \dots, s_q)$ , where  $s_i \in S$  for all  $i \in \mathbb{N}$ , is called a reduced representation of  $w$  if and only if  $w = s_1 s_2 \dots s_q (*)$  and  $q$  is the smallest among all such sequences satisfies  $(*)$ . We define  $l(w) = q$  and call it the length of  $w$ .

**Proposition 1.3** For  $w, w' \in W$ , we have the following facts:

- (1)  $l(w w') \leq l(w) + l(w')$
- (2)  $l(w^{-1}) = l(w)$
- (3)  $|l(w) - l(w')| \leq l(w w'^{-1})$

**Proof:** (1) and (2) are trivial. From (1), we see  $l(w) \leq l(w w'^{-1}) + l(w')$  and  $l(w') \leq l(w' w^{-1}) + l(w)$ . From (2), we see  $l(w' w^{-1}) = l(w w'^{-1})$ . Putting these together gives us (3).  $\square$

**Definition 1.4** If a system  $(W, S)$  satisfies the following conditions:

- (1) Every element in  $S$  has order 2.
- (2) Say  $m(s, t)$  is the order of  $st$ , and define  $I = \{(s, t) \mid s, t \in S, m(s, t) < \infty\}$ , then  $\langle S \mid (st)^{m(s, t)} = 1, (s, t) \in I \rangle$  is a presentation of  $W$ .

Then we say  $(W, S)$  is a Coxeter System and  $W$  is a Coxeter group.

**Remark 1.5** A Coxeter System  $(W, S)$  satisfies the following universal property: For any group  $G$  and map  $\psi : S \rightarrow G$  such that  $\psi(st)^{m(st)} = e_G, \forall (s, t) \in I$ , there exist an unique  $\phi \in \text{Hom}(W, G)$  such that  $\psi = \phi \circ l$ . Where  $I$  is defined as in Definition 1.4, and  $l$  is the natural inclusion from  $S$  to  $W$ . Note that this is just the composition of universal properties of free groups and quotient groups.

**Example 1.6** By universal property of Coxeter System, the map  $s \mapsto -1, \forall s \in S$  extends to a homomorphism  $\epsilon : W \rightarrow \{-1, 1\}$  such that  $\epsilon(w) = (-1)^{l(w)}$ . We call  $\epsilon(w)$  the sign of  $w$ .

We want to find some invariant among representations of the same word, hence make the following definition.

**Definition 1.7** For Coxeter System  $(W, S)$ , say  $T$  the union of conjugacy classes of all  $s \in S$ . For sequence  $\mathbf{s} = (s_1, \dots, s_q)$ , where  $s_i \in S, \forall i$ . Define  $\Phi(\mathbf{s}) = (t_1, \dots, t_q)$  such that  $t_j = (s_1 \dots s_{j-1}) s_j (s_1 \dots s_{j-1})^{-1}$ . For  $t \in T$ , define  $n(\mathbf{s}, t) := \#\{1 \leq j \leq q \mid t_j = t\}$ . Finally, define  $R = \{\pm 1\} \times T$ .

**Remark** Note that we have  $s_1 \dots s_j = t_j \dots t_1$ .

**Lemma 1.8** For Coxeter System  $(W, S)$ , we have the following facts:

(1) For  $w \in W$  and  $t \in T$ ,  $\forall \mathbf{s} = (s_1, \dots, s_q)$  such that  $w = s_1 \dots s_q$ ,  $(-1)^{n(\mathbf{s}, t)}$  has the same value. We call this value  $\eta(w, t)$ .

(2) For  $w \in W$ , consider the map  $U_w : R \rightarrow R$  defined as  $U_w(\epsilon, t) = (\epsilon \eta(w, t), wt w^{-1})$ , then  $w \mapsto U_w$  is a homomorphism from  $W$  to the permutation group of  $R$ .

**Proof:** For  $s \in S$ , we define  $U_s : R \rightarrow R$  such that  $(\epsilon, t) \mapsto (\epsilon(-1)^{\delta(\mathbf{s}, t)}, sts^{-1})$ , where  $\delta_{s,t}$  is the Kronecker Symbol. For sequence  $\mathbf{s} = (s_1, \dots, s_q)$  in  $S$ , say  $w = s_q \dots s_1$ ,  $U_s = U_{s_q} \circ \dots \circ U_{s_1}$ .

We prove  $U_s(\epsilon, t) = (\epsilon(-1)^{n(\mathbf{s}, t)}, wt w^{-1})$  by induction. (\*)

In case  $q = 0$  or  $1$ , the proof is trivial. For  $q > 1$ , by induction hypothesis, say  $\mathbf{s}' = (s_1, \dots, s_{q-1}), w' = s_{q-1} \dots s_1$ , then  $U_{\mathbf{s}'} = U_{s_q}(\epsilon(-1)^{n(\mathbf{s}', t)}, w' t w'^{-1})$ . Hence it only remains to prove that  $n(\mathbf{s}, t) = \delta_{s_q, w' t w'^{-1}} + n(\mathbf{s}', t)$ . Notice that  $\Phi(\mathbf{s}) = (\Phi(\mathbf{s}'), w'^{-1} s_q w')$ , so the proof is trivial.

We now prove the fact that  $s \mapsto U_s$  can be extended to a homomorphism from  $W$ . By the universal property of Coxeter System, it suffice to prove that  $\forall s, s' \in S$  such that  $m(s, s') < \infty$ ,  $(U_s \circ U_{s'})^{m(s, s')} = 1$ . Define  $\mathbf{s} = (s_1, \dots, s_{2m(s, s')})$  such that  $s_i := s$  for odd  $i$  and  $s_i := s'$  for even  $i$ , so it suffice to prove  $U_{\mathbf{s}} = Id$ . Notice that here we have  $t_j = (ss')^{j-1} s$ , so  $t_i \neq t_j$  for all  $1 \leq i < j \leq m(s, s')$  and  $t_i = t_{i+m(s, s')}$ . Hence, for  $t \in T$ ,  $n(\mathbf{s}, t) = 0$  or  $2$ . Applying the result in the previous paragraph, we see  $U_{\mathbf{s}} = Id$ .

Now we see  $w \mapsto U_w$  is a homomorphism from  $W$ . Specifically,  $U_w$  doesn't depend on the representation chosen for  $w$ . So from (\*), we see  $n(\mathbf{s}, t)$  is invariant among different choices of representations. This gives us (1). (2) follows easily.  $\square$

**Theorem 1.9** For Coxeter System  $(W, S)$ , consider sequence  $\mathbf{s} = (s_1, \dots, s_q)$ ,  $\Phi(\mathbf{s}) = (t_1, \dots, t_q)$ ,  $w = s_1 \dots s_q$ . Then  $\mathbf{s}$  is an reduced representation of  $w$  if and only if  $t_i \neq t_j$  for all  $i \neq j$ . Define  $T_w := \{t \in T \mid \eta(w, t) = -1\}$ , then in case  $\mathbf{s}$  is a reduced representation of  $w$ ,  $T_w = \{t_1, \dots, t_q\}$  and  $|T_w| = l(w)$ .

**Proof:**  $\forall t \in T_w, n(w, t) \neq 0$ . By definition of  $n(w, t)$ , we deduce  $\forall t \in T_w, t \in \{t_1, \dots, t_q\}$ . Moreover, as  $\eta(w, t)$  is independent of the choice of representation of  $w$ ,  $T_w$  only depends on  $w$ . Hence  $|T_w| \leq l(w)$ . In case  $t_i \neq t_j$  for all  $i \neq j$ ,  $n(\mathbf{s}, t) = 0$  or  $1$ , and hence  $T_w = \{t_1, \dots, t_q\}$ . As  $q = |T_w| \leq l(w)$ , so  $\mathbf{s}$  is a reduced representation of  $w$ .

Conversely, if there exist  $i \neq j$  such that  $t_i = t_j$ , WLOG say  $i < j$ . Consider  $u = s_{i+1} \dots s_{j-1}$ , then  $s_i = u s_j u^{-1}$ . Hence we have  $s_1 \dots s_q = s_1 \dots s_{i-1} (u s_j u^{-1}) u s_j s_{j+1} \dots s_q = s_1 \dots s_{i-1} u s_j s_j s_{j+1} \dots s_q = s_1 \dots s_{i-1} s_{i+1} \dots s_{j-1} s_j s_{j+1} \dots s_q$ . So  $\mathbf{s}$  is not reduced.  $\square$

Note that now we have a necessary and sufficient condition to check whether a word is reduced or not. We want to improve this result and get an algorithm of reducing words.

**Definition 1.10** For a system  $(W, S)$ , if it satisfies:  $\forall w \in W, s \in S$ , if  $l(sw) \leq l(w)$ , then for any reduced representation  $\mathbf{s} = (s_1, \dots, s_q)$ , exist  $1 \leq j \leq q$  such that  $s s_1 \dots s_{j-1} = s_1 \dots s_j$ . Then we say  $(W, S)$  satisfies the exchange condition.

**Theorem 1.11** If  $(W, S)$  is a Coxeter System, then it satisfies the exchange condition.

**Proof:** Consider  $w \in W, s \in S$  such that  $l(sw) \leq l(w)$ . For any reduced representation  $s = (s_1, \dots, s_q)$  of  $w$ , consider  $w' := sw$ . By Example 1.6, we have  $l(w') \equiv l(w) + 1 \pmod{2}$ . Proposition 1.3(3) gives that  $|l(w) - l(w')| \leq 1$ , so  $l(w') = l(w) - 1$ . Now pick  $(s'_1, \dots, s'_{l(w)-1})$  as a reduced representation of  $w'$ , then  $s' = (s, s'_1, \dots, s'_{p-1})$  is a reduced representation of  $w$ . Take  $\Phi(s') = (t'_1, \dots, t'_p)$ , then  $t'_1 = s$  by definition. However Theorem 1.9 shows that  $t'_i \neq t'_j$  if  $i \neq j$ , so  $n(s', s) = 1$ . By Lemma 1.8,  $n(s', s) \equiv n(s, s) \pmod{2}$ , so  $n(s, s) \neq 0$ . Hence there exist  $j$  such that  $s = t_j$ , where  $t_j$  is an element in  $\Phi(s)$ . By definition of  $t_j$ ,  $ss_1 \dots s_{j-1} = s_1 \dots s_j$ , the claim is proved.  $\square$

Theorem 1.11 is all we need to prove the Tits' theorem. But we can actually say more on exchange condition, and deduce conversely that any system such that  $\forall s \in S, s$  has order 2 and satisfy exchange condition is a Coxeter System. We will talk more on it in part 4.

## 6.2 M-operations

**Definition 2.1** Consider  $(W, S)$  a system, the sequence  $s = (s_1, \dots, s_q)$ , where  $s_i \in S, \forall i$  is called a word in  $S$  and  $w := s_1 \dots s_q$  is called the element in  $W$  expressed by  $s$ . An elementary M-operation on a word in  $s$  is one of the following two types of operations:

(MI) Delete a subword of form  $(s, s)$  from  $s$ .

(MII) Replace an alternating subword  $(s, t, s, \dots)$  by the other alternating subword  $(t, s, t, \dots)$ , both in length  $m(s, t)$ .

We say a word is M-reduced if and only if its length cannot be shortened by M-operations. Clearly, any reduced word is M-reduced.

**Lemma 2.2** For a system  $(W, S)$  with Exchange condition, two reduced representations express the same element in  $W$  if and only if one can be transformed to the other by MII operations.

**Proof:** Say  $s = (s_1, \dots, s_q)$  and  $r = (r_1, \dots, r_q)$  two reduced representations both express  $w \in W$ . We prove the lemma by induction.

In case  $q = 0$ , the proof is trivial. If  $s_1 = r_1$ , we can reduce the length of both words by 1 so the induction hypothesis can be applied directly. It suffice to prove the case  $s_1 \neq r_1$ . We set  $m := m(s_1, r_1)$ .

**Claim:**  $m$  is finite and there is another reduced expression  $u$  of  $w$  start with an alternating subword  $(s_1, r_1, s_1, \dots)$  of length  $m$ .

With the claim, the remaining proof is easy. Since  $s$  and  $u$  both start with  $s_1$ , by induction hypothesis,  $s$  can be transformed into  $u$  by MII operations. Now apply MII on  $u$  to get another word  $u'$  starts with  $(r_1, s_1, r_1, \dots)$  of length  $m$ . As  $u'$  and  $r$  both start with  $r_1$ , again by induction hypothesis,  $u'$  can be transformed into  $r$  through MII operations. The combination of all above MII operations gives the result.

The converse of the proposition is trivial.  $\square$

**Proof of the Claim:** Clearly,  $l(r_1 w) < l(w)$  as  $r$  is a reduced representation of  $w$  starting with  $r_1$ . By the exchange condition there exist an index  $i$  such that  $r_1 s_1 \dots s_{i-1} = s_1 \dots s_i$ . Moreover, as  $(s_1, \dots, s_i)$  is a subword of  $s$ , which is reduced,  $v := (r_1, s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_q)$  is also a reduced representation of  $w$ .

Note that  $i \neq 1$ . Otherwise  $(s_1, s_2, \dots, s_q) = (r_1, s_2, \dots, s_q)$  gives arise to  $s_1 = r_1$ , which is a contradiction.

Define  $S_q$  to be the alternating word end in  $s_1$  with each of its term in set  $\{s_1, r_1\}$  of length  $q$ . Suppose  $s'$  a word start with  $S_{q-1}$  and let  $s'$  to be the element of  $\{s_1, r_1\}$  that  $s'$  doesn't start with.

Looking at  $s$  and  $r$ , we get  $l(s'w) < l(w)$ . Apply the exchange condition we get a reduced representation of  $s'$  start with  $s'$ . The removed element through exchanging process should not lie in  $S_{q-1}$  as any reduced representation with length not equal to  $m$  an element in group  $D_{2m}$  is unique. This is a fact we learned in Year 2 module Groups and Rings. If the removed term lies in  $S_{q-1}$ , we get two different reduce representations of element in  $D_{2m}$  generated by  $s_1, r_1$ , length not equal to  $m$  after cancellation, which is a contradiction.

We hence get a reduced representation start with  $S_q$  from  $s'$ . Hence by induction,  $w$  has a reduced expression start with  $S_q$  for any  $q \leq m(s_1, r_1)$  but we have  $q \leq l(w) < \infty$ . So we have  $m$  is finite.

Replace  $S_{m(s_1, r_1)}$  with  $(s_1, r_1, s_1, \dots)$  by MII operation, we get the expecting  $u$ .  $\square$

### 6.3 Tit's Theorem

**Theorem 3.1 (Tit's)** A word in system  $(W, S)$  with exchange condition is reduced if and only if it's M-reduced.

**Proof:** Suppose the word  $s = (s_1, \dots, s_k)$  is M-reduced. We show  $s$  is reduced by induction on  $k$ . The base case  $k = 1$  is trivial.

For  $k > 1$ , by induction hypothesis,  $s' = (s_2, \dots, s_k)$  is reduced. Say  $w'$  the element expressed by  $s'$ . Suppose  $s$  is not reduced, then  $l(w) = l(s_1 s') \leq k - 1 < l(s')$ . So by exchange condition, there exist an index  $i$  such that  $w' = s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_k$ . That is,  $w'$  has a reduced representation start with  $s_1$ . Apply Lemma 2.2 we see that  $s'' := (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_k)$  can be transformed from  $s'$  by MII operation. This gives a reduced representation of  $w$  start with  $(s_1, s_1)$ , which is a contradiction.

The other direction of the theorem is obvious.  $\square$

Note that Tit's Theorem is far from trivial: The word in a free group can be really messy, even talk about  $G = \langle x, y \mid x^n = y^2 = (xy)^2 = 1_G \rangle$  directly is not easy, see Groups and Rings module of Year 2.

**Example 3.1**  $G := \langle x, y \mid xyxyx = yxyxy \rangle \cong \langle x, y, a \mid xyxyx = yxyxy, a = xy \rangle \cong \langle x, a \mid aax = x^{-1}aaa \rangle \cong \langle x, a, b \mid a^2x = x^{-1}a^3, b = xa^2 \rangle \cong \langle a, b \mid a^2 = b^5 \rangle$

We can see from the example that although the relation of  $G$  is clear, what operations can be done to make reduction is hard to see. Nontrivial process is done here to transfer the group of  $x, y$  to a group of  $a, b$  and enable us to do reduction.

There are general tools like Tietze Transformation and Coset enumeration to talk about what we can do on a general group presentation, but it's out of the scope of this project.

## 7 More on Exchange Condition

**Proposition 4.1** For system  $(W, S)$  with exchange condition such that every element in  $S$  has order 2, consider  $w \in W, s \in S$ . If  $s$  is a reduced representation of  $w$ , exactly one of the following occurs:

(1)  $l(sw) = l(w) + 1, (s, s_1, \dots, s_q)$  is a reduced representation of  $sw$ . (2)  $l(sw) = l(w) - 1, \exists 1 \leq j \leq q$  such that  $(s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_q)$  is a reduced representation of  $sw$  and  $(s, s_1, \dots, s_{j-1}, s_{j+1}, \dots, s_q)$  is a reduce representation of  $w$ .

**Proof:** This is a direct application of exchange condition.

## 8 Euclidean Reflection Groups

### 8.1 Structure of Euclidean Reflection Groups

This section follows [5].

Let  $W$  be an affine reflection group, naturally acting on the Euclidean space  $V$ , with fixed set of affine hyperplanes  $\mathcal{H}$ . We fix an arbitrary chamber  $C$  as our fundamental chamber,  $S$  be the set of reflection with respect to its walls. For an  $n$ -dimensional Euclidean space  $V$ , we have the isomorphism  $\text{AO}(V) \cong V \rtimes \text{O}(V)$ . Every element  $\text{AO}(V)$  represents the composition of an orthogonal linear map with a translation  $\tau_v$  sending  $x$  to  $x + v$ . Note  $W$  can be taken as a subgroup of  $\text{AO}(V)$ .

From Section 4, we recall some facts of affine reflection groups:

1.  $W$  is generated by  $S$ .
2.  $W$  is simply transitive on chambers.
3.  $H \in \mathcal{H}$  if and only if  $s_H \in W$
4.  $\langle e_{s_i}, e_{s_j} \rangle = -\cos(\frac{\pi}{m_{i,j}})$ , where  $e_{s_i}, e_{s_j}$  are two unit vectors normal to some walls of  $C$ , pointing into  $C$  from the walls.

**Theorem 8.1.1.** 1. The hyperplanes  $H \in \mathcal{H}$  fall into finitely many classes under parallelism;

2. Let  $\overline{W}$  be the image of  $W$  under the projection  $\eta : \text{AO}(V) \rightarrow \text{O}(V)$ , then  $\overline{W}$  is finite.

*Proof.* (1) Let  $\Phi := \{\pm e_H : H \in \mathcal{H}\}$ , where  $e_H$  is the unit vector normal to  $H$ . We will show  $\angle(e_1, e_2)$  can take finitely many values. Let  $H_1, H_2 \in \mathcal{H}$  normal to  $e_1, e_2$ . If they are parallel, then  $\angle(e_1, e_2) = \pi$ . So we may assume they intersect and take  $x \in H_1 \cap H_2$ . Since  $W$  is transitive on chambers, and by the fact that chambers  $D, D'$  is separated by wall  $H'$  if and only if  $wD, wD'$  is separated by  $wH'$ , we can choose  $w \in W$  with  $wx \in \overline{C}$ . Then  $\overline{C}$  intersects with  $wH_1$  and  $wH_2$ , with vectors  $\bar{w}e_1, \bar{w}e_2$  normal to them, where  $\bar{w} = \eta(w)$ .  $\angle(e_1, e_2) = \angle(\bar{w}e_1, \bar{w}e_2)$  since  $\bar{w}$  is an isometry.  $C$  has finitely many walls so there are only finitely many possibilities for angles between two vectors in  $\Phi$ , hence it is finite as we are in a finite-dimensional space.

(2) Note  $\Phi$  we defined is stable under action of  $\overline{W}$  and reflection with respect to the hyperplanes they are perpendicular to generates  $\overline{W}$ . We show the natural homomorphism  $\sigma : \overline{W} \rightarrow S(\Phi)$  is injective. Suppose  $w \in \ker(\sigma)$ ,  $w$  fixes all vectors of  $\Phi$ . But then  $V_0 = \text{Span}(\Phi)$  is also fixed by  $w$ . Write the vector space  $V$  as  $V_0 \oplus V_1$  for  $V_1$  some subspace of  $V$ . Then since every reflection  $s_e$  fixes every point of  $V_1$  and  $W$  is generated by reflection of walls of  $C$ , so does  $w$ . We then conclude  $w$  is identity hence  $\overline{W}$  is finite.  $\square$

**Definition 8.1.2.** We say  $W$  is essential if intersection of hyperplanes  $H \mid s_H \in \overline{W}$  is a singleton.

**Definition 8.1.3.** A Coxeter group is irreducible if its Coxeter diagram is connected.

For a general affine reflection group, we can reduce it to essential, irreducible cases. If it is irreducible, then there are sets of hyperplanes  $\mathcal{H}_i$ , each hyperplane in  $\mathcal{H}_i$  is orthogonal to the ones in other  $\mathcal{H}_j$ 's. We can decompose the space into direct sum  $V_1 \oplus \cdots \oplus V_n$ , where each  $V_i$  corresponds to  $\mathcal{H}_i$ , and the corresponding reflection group  $W_i$  is irreducible in  $V_i$ . If  $W_i$  is not essential, we can quotient the intersection of all hyperplanes out to get an essential one.

**Theorem 8.1.4.** Assume  $W$  is essential and irreducible in  $n$ -dimensional vector space  $V$ , then one of the following is true:

1.  $W$  is finite,  $C$  has  $n$  walls;

2.  $W$  is infinite,  $C$  has  $n + 1$  walls, and vectors perpendicular to any  $n$  walls form a basis.  $\overline{C}$  is compact.

*Proof.* We first count the number of walls. Suppose the walls of  $C$  are  $H_1, \dots, H_k$ , with corresponding normal vectors  $e_1, \dots, e_k$  pointing to  $C$ . Since  $W$  is essential,  $e_1, \dots, e_k$  is a spanning set hence  $k \geq n$ .

Suppose  $k = n$ . Then the intersection of all  $H_i$ 's is a singleton. By shifting the planes we may assume  $x = 0$ . Hence all  $H_i$ 's are linear and  $W$  is a finite reflection group.

Suppose  $k > n$ , then the list of vectors must be linearly dependent. Choose an index set  $I$  such that

$$\sum_{i \in I} \lambda_i e_i = 0$$

such that  $\lambda_i \neq 0$  for all  $i \in I$ . We show  $I = \{1, \dots, k\}$ . Suppose not, then let  $J = \{1, \dots, k\} \setminus I$ .

Starting from  $\sum_{i \in I} \lambda_i e_i = 0$ , move the negative coefficients terms to the right

$$\sum_{i \in K} \lambda_i e_i = \sum_{i \in L} -\lambda_i e_i$$

where  $K, L$  is a partition of  $I$ . Then consider the inner product of left with right, since  $\langle e_i, e_j \rangle$ , it is non-positive. But it is also an inner product of two same vectors, then the summation is the 0 vector.

Choose  $j \in J$  and take inner product

$$\sum_{i \in K} \lambda_i \langle e_i, e_j \rangle = 0$$

For  $i \in K$   $\lambda_i$  are positive and  $\langle e_i, e_j \rangle$  are non-positive, we must have the case that  $\langle e_i, e_j \rangle = 0$ , which implies  $s_{e_i} s_{e_j}$  has order 2. Then same thing holds for index set  $L$ . We then conclude the graph of  $I$  and  $J$  in Coxeter diagram are disconnected, contradicting irreducibility.

$C$  now has  $n + 1$  walls, so  $C$  is a bounded set. In particular  $\overline{C}$  is compact. Since  $W$  is simply transitive,  $\bigcup_{w \in W} w\overline{C}$  is compact if  $W$  is finite, but this union covers  $V$  so  $W$  is infinite.  $\square$

**Definition 8.1.5.** A Euclidean reflection group is an essential, infinite, irreducible, affine reflection group.

For the rest of this section, let  $W$  be a Euclidean reflection group. Let  $T$  be the kernel of the projection above restricted on  $W$ , then we have a short exact sequence

$$1 \rightarrow T \rightarrow W \rightarrow \overline{W} \rightarrow 1$$

**Proposition 8.1.6.** There exists point  $x \in V$  such that its group of stabilizers  $W_x$  is isomorphic to  $\overline{W}$ . In particular,  $W \cong T \rtimes W_x$ .

*Proof.* By Theorem 8.1.1, let  $\overline{\mathcal{H}}$  be the set of linear hyperplanes parallel to some affine planes of  $\mathcal{H}$ .  $\overline{W}$  is generated by reflection about these hyperplanes. In fact  $W$  is generated by  $\{s_{\overline{H}_1}, \dots, s_{\overline{H}_n}\}$ , with  $\overline{H}_1, \dots, \overline{H}_n \in \overline{\mathcal{H}}$ . Choose affine hyperplanes  $H_1, \dots, H_n$  from  $\mathcal{H}$ ,  $H_i$  parallel to  $\overline{H}_i$ , and take their intersection, which is a single point  $x$  by Theorem 8.1.4. Linear parts of  $s_{H_1}, \dots, s_{H_n}$  generates  $\overline{W}$  and since translations have no fixed points,  $W_x$  bijects to  $\overline{W}$ .  $\square$

By Proposition 8.1.6, with possibly shifting we can assume 0 is a special point and hence  $W \cong T \rtimes W_0 \cong T \rtimes \overline{W}$ . We can then identify  $T$  by

$$L := \{v \in V : \tau_v \in W\}$$

as a additive group.

We then have  $W \cong L \rtimes \overline{W} \leq V \rtimes \text{O}(V) \cong \text{AO}(V)$  We next show  $L$  is a lattice i.e.  $L$  is in the form of  $\mathbb{Z}e_1 \oplus \dots \oplus \mathbb{Z}e_n$  for  $\{e_i\}_{1 \leq i \leq n}$  a basis of  $V$

**Lemma 8.1.7.** Non-identity elements of  $L$  are bounded away from 0, i.e.  $L$  is a discrete subgroup of  $V$  as a topological group.

*Proof.* Pick  $x \in C$  of the fundamental chamber. Since  $W$  is simply transitive on chambers, so is  $L$ . Let

$$U = \{v \in V : x + v \in C\}$$

$U$  is a neighbourhood of 0 since  $C$  is open. Thus  $U \cap L = \{0\}$ ,  $L$  is bounded away from 0.  $\square$

The next lemma is from theory of topological groups:

**Lemma 8.1.8.** If  $L$  is a discrete subgroup of the additive group of a finite-dimensional  $\mathbb{R}$ -vector space, then  $L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_r$  for some linearly independent vectors  $e_1, \dots, e_r$ .

*Proof.* We prove by induction on  $\dim V$ . Assume  $L \neq 0$ , by discreteness, choose  $e \in L$  of minimal length  $\delta$ .  $L \cap \mathbb{R}e$  must contain  $e$ . Operation of  $L$  is the vector addition so  $\mathbb{Z}e \subseteq L \cap \mathbb{R}e$ . If  $(L \cap \mathbb{R}e) \setminus \mathbb{Z}e$  is nonempty, then we would have a contradiction with minimality of length of  $e$ . So  $L \cap \mathbb{R}e = \mathbb{Z}e$ .

For any element  $v \in L \setminus \mathbb{Z}e$ , we can find  $w \in \mathbb{R}e$  such that distance from  $v$  to  $\mathbb{R}e$  equals  $d(v, w)$ . Let  $u \in \mathbb{Z}e$  be of minimal distance to  $v$ . Then  $d(v, w) \geq |d(v, u) - d(w, v)| \geq |\delta - \delta/2| = \delta/2$  by triangle inequality.

Now we consider the quotient group  $L/\mathbb{Z}e$  as the subgroup of additive group of the quotient space  $V/\mathbb{R}e$ . Since we have the canonical isomorphism  $(\mathbb{R}e)^\perp \rightarrow V/\mathbb{R}e$ , define a map  $V \rightarrow (\mathbb{R}e)^\perp$  by  $v \mapsto v^\perp$ . Given any two cosets  $[v], [w] \in V/\mathbb{R}e$ , define  $d([v], [w]) = \|v^\perp - w^\perp\|$ , which clearly a metric. The above is actually showing  $L/\mathbb{Z}e$  is a discrete subgroup of  $V/\mathbb{R}e$ . The lemma follows from induction.  $\square$

We are now left to show the set of linearly independent vectors given in Lemma 8.1.8 is a basis of  $V$ .

**Lemma 8.1.9.** In the context of Lemma 8.1.8,  $r = n$ .

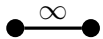
*Proof.* Since for any  $v \in V$ , there is  $w \in W$  and  $y \in \overline{C}$  such that  $v = wy$ .  $w$  is  $\tau_v \overline{w}$  for  $\overline{w} \in \overline{W}$  and  $\tau_v \in L$ . Thus  $v$  is equivalent to some  $y \in \bigcup_{\overline{w} \in \overline{W}} \overline{w}C$  modulo  $L$ . It is a compact set since it is a finite union of compact ones.  $L$  is a lattice thus  $r = n$ .  $\square$

Combining all these results, we obtain:

**Theorem 8.1.10.** The Euclidean reflection group  $W$  is isomorphic to  $\mathbb{Z}^n \rtimes \overline{W}$ , where  $\overline{W}$  is a finite reflection group.

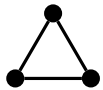
We end this section by giving some examples of Euclidean reflection groups in low dimension vector spaces.

For  $\mathbb{R}^1$ , there is only one finite reflection group  $S_2$  with hyperplane 0, we can add 1 and these two points generates an affine reflection group, each element corresponds to a reflection about  $k \in \mathbb{Z}$ . This is called  $D_\infty$  with Coxeter diagram



This is isomorphic to  $\mathbb{Z} \rtimes S_2$ .

For  $\mathbb{R}^2$ , for example we can tile the whole space with equilateral triangles. So let it be the chamber and we have a diagram



This group is isomorphic to  $\mathbb{Z}^2 \rtimes D_6$ .

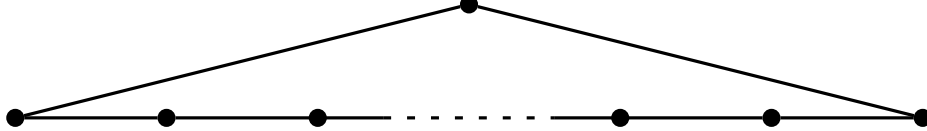
We can tile with hexagon but themselves cannot be chambers, we can split each into 12 30-60-90 triangles. Let them be the chambers and we have a diagram





which is the group  $\mathbb{Z}_2 \rtimes D_{12}$ .

In fact, for many finite reflection groups  $\overline{W}$ , essential and irreducible, have affine analogues. For these groups, let  $V$  be the vector space they are acting on of dimension  $n$ . We can realise  $V \rtimes O(V)$  as an inner semidirect product of  $AO(V)$ . Define  $L$  to be the  $n$ -rank lattice of  $V$ . Then we can define an inner semidirect product  $L \rtimes \overline{W}$  which gives us the affine analogue of  $\overline{W}$ . An example is the finite reflection groups of type  $A_n$ , which can be realised as reflection about planes  $x_i - x_j = 0$  ( $i \neq j$ ) in the vector space  $V = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum x_i = 0\}$  with  $\mathcal{H} := \{x_i - x_j = k : k \in \mathbb{Z}, i \neq j\}$ . One can check  $\mathcal{H}$  in fact gives a Euclidean reflection group with the Coxeter diagram



with  $n$  nodes in the graph.

A non-example is of type  $I_2(m)$  except for some specific  $m$ . There is not way to cover the whole plane using reflections using triangle with an angle  $< 2\pi/12$  or of  $2\pi/7$  for example.

## 8.2 Some Model Theory of Euclidean Reflection Groups

This section follows [6].

**Definition 8.2.1.** We say a structure  $\mathcal{N}$  is interpretable in  $\mathcal{M}$  if:

1. The underlying set  $A$  is  $\{\bar{x} \in M^k : \mathcal{M} \models \phi(\bar{x})\}$  for some formula  $\phi$ .
2. For every function symbols  $f(\bar{x})$  of  $\mathcal{N}$ , there is a formula  $\phi$  such that  $\mathcal{M} \models \phi(\bar{x}, \bar{y})$  if and only if  $\mathcal{N} \models f(\bar{x}) = \bar{y}$ .

**Definition 8.2.2.** We say the theory of a group  $G$  is decidable if there is an algorithm such that for all sentences  $\phi$ , it can decide whether  $\phi$  is true in  $G$ .

For a definition of a sentence and an algorithm, see Appendix A.

**Proposition 8.2.3.** Euclidean reflection group  $G$  is definable in the abelian group  $\mathbb{Z}$  with finitely many parameters.

This result follows from the following two lemmas:

**Lemma 8.2.4.** Let  $G \cong \mathbb{Z}^d \rtimes_{\sigma} Q$  where  $1 \leq d < \omega$ ,  $Q$  is some finite group, and  $\theta : Q \rightarrow \text{Aut}(\mathbb{Z}^d)$  a group homomorphism.  $G$  is interpretable in the structure  $\mathcal{M} = (\mathbb{Z}^d, +, \pi_x)_{x \in Q}$  with finitely many parameters, where  $+$  is the normal vector addition and  $\pi_x := \sigma(x)$

*Proof.* We enumerate  $\{\pi_x : x \in Q\}$  as  $t_0, \dots, t_{n-1}$  and  $t_i := (i, 0, \dots, 0)$  a  $d$ -tuple. Let  $G = (\mathbb{Z}^d \times Q, \cdot)$ . We represent the universe of  $G$  by a tuple  $(a, t_i)$  where  $a \in M$  and  $0 \leq i < n$ . We need to define the group operation  $(a_i, t_i) \cdot (a_j, t_j) = (a_i +_{\mathbb{Z}^d} t_i(a_j), t_i \cdot_Q t_j)$  in  $\mathcal{M}$ .  $t_i \cdot_Q t_j$  is definable since  $Q$  is finite, we can enumerate all possible product of all elements in  $Q$ .  $a_i +_{\mathbb{Z}^d} t_i(a_j)$  is definable since  $+$  is a built-in function symbol of  $\mathcal{M}$  and  $t_i$  is same with  $\pi_x$  for some  $x$ . Again this claim follows from enumerating all possibilities of  $t_i$ 's.  $\square$

**Lemma 8.2.5.**  $(\mathbb{Z}^d, +, \pi_x)_{x \in Q}$  is interpretable in  $(\mathbb{Z}, +, 0)$  with finitely many parameters.

*Proof.* Since there are finitely many  $\pi_x$ 's, it suffices to show we can define  $\pi_x$  in  $(\mathbb{Z}, +, 0)$  for some  $x \in Q$ .  $\pi_x$  is an automorphism of  $\mathbb{Z}^d$ , which is an  $d \times d$  invertible matrix  $A$  over  $\mathbb{Z}$ .  $\pi_x(\bar{b}) = A\bar{b}$  is a matrix multiplication. We only need to show we can define the element-wise

multiplication part is definable. To show this, note that there are only finitely many entries in  $A$ . The multiplication of  $n \times b$  where  $n$  is an entry of  $A$  and  $b$  is an arbitrary integer, can be realised as

$$\underbrace{b + b + \cdots + b}_{n \text{ times}}$$

if  $n$  is positive and otherwise minus  $b - n$  times.  $\square$

**Theorem 8.2.6.** Let  $W$  be a Euclidean reflection group. Then  $Th(W)$  is decidable.

*Proof.*  $W$  is interpretable in the structure  $(\mathbb{Z}, +, 0)$  with finitely parameters by Proposition 8.2.3. But all these parameters are integers, so it is interpretable in the structure  $(\mathbb{Z}, +, 0, 1)$  with no parameters. We then can translate every sentence in  $W$  back to a sentence of  $(\mathbb{Z}, +, 0, 1)$ . Since the Presburger arithmetic  $(\mathbb{Z}, +, <, 0, 1)$  is decidable, as a definable structure in its reduct,  $Th(W)$  is also decidable.  $\square$

## A Model-Theoretic Preliminaries

The following definitions are taken from [7].

We fix our language  $\mathcal{L} = (e, \cdot)$  to be the language of groups.

**Definition A.0.1.** The set of  $\mathcal{L}$ -terms is the smallest set  $\mathcal{T}$  such that

1. variables  $x_i \in \mathcal{T}$  for  $i = 0, 1, \dots$ ;
2. if  $t_1, t_2 \in \mathcal{T}$  then  $(t_1 \cdot t_2) \in \mathcal{T}$

**Definition A.0.2.** The set of  $\mathcal{L}$ -formulas is the smallest set  $\mathcal{F}$  such that

1. all atomic formulas  $(t_1 = t_2) \in \mathcal{F}$  for  $t_1, t_2 \in \mathcal{T}$ ;
2.  $\mathcal{F}$  is closed under  $\neg$  ("not"),  $\vee$  ("or"),  $\wedge$  ("and"),  $\rightarrow$  ("implies"),  $\forall$  ("for all"),  $\exists$  ("there exists").

By drawing a truth table or otherwise, it suffices to define the satisfaction of  $\wedge$ ,  $\neg$ , and quantifiers for  $\mathcal{L}$ -formulas.

**Definition A.0.3.** We say a formula  $\phi$  has  $x$  as a free variable if  $x$  does not appear in any quantifiers of  $\phi$ .

**Definition A.0.4.** Let  $\phi$  be a formula with free variables  $x_1, \dots, x_n$ . We inductively define a group  $G \models \phi$  as follows.

1. If  $\phi(\bar{x})$  is  $t_1(\bar{x}) = t_2(\bar{x})$ , then  $G \models \phi$  if  $t_1$  and  $t_2$  are the same element of  $G$ .
2. If  $\phi(\bar{x})$  is  $\neg\psi(\bar{x})$ , then  $G \models \phi(\bar{x})$  if  $G \not\models \psi(\bar{x})$ .
3. If  $\phi(\bar{x})$  is  $\psi_1(\bar{x}) \wedge \psi_2(\bar{x})$ , then  $G \models \phi(\bar{x})$  if  $G \models \psi_1(\bar{x})$  and  $G \models \psi_2(\bar{x})$ .
4. If  $\phi(\bar{x})$  is  $\forall y \psi(\bar{x}, y)$ , then  $G \models \phi(\bar{x})$  if for all  $a \in G$   $G \models \psi(\bar{x}, a)$ .
5. If  $\phi(\bar{x})$  is  $\exists y \psi(\bar{x}, y)$ , then  $G \models \phi(\bar{x})$  if there is  $a \in G$   $G \models \psi(\bar{x}, a)$ .

If  $G \models \phi(\bar{a})$  we say  $\phi(\bar{a})$  is true in  $G$

**Definition A.0.5.** We say a formula  $\phi$  is a sentence if it does not have free variables.

To make the term algorithm clear, we recall definitions from computability theory.

We have infinitely many initial configuration of the register  $R_1, R_2, \dots$ , each is a nonnegative integer.

**Definition A.0.6.** A register machine program is a finite sequence of instructions  $I_1, \dots, I_M$ , where each  $I_j$  is one of the following:

1.  $Z(n)$ : set  $R_n$  to zero;
2.  $S(n)$ : increase  $R_n$  by one;
3.  $T(n, m)$ : set  $R_n$  to be  $R_m$ ;
4.  $J(n, m, s)$ , where  $1 \leq s \leq M$ : if  $R_n = R_m$ , then go to  $I_s$ , otherwise go to the next instruction;
5. **HALT**

and  $I_m$  is **HALT**

**Definition A.0.7.** Suppose  $A \subseteq \mathbb{N}^k$ .  $f : A \rightarrow \mathbb{N}$  is computable if there is a register machine program  $P$  such that:

1. If  $x \in A$ , then  $P$  does not halt on input  $x$ ;
2. If  $x \in A$ , then  $P$  halts on input  $x$  with  $R_1 = f(x)$ .

**Definition A.0.8.** We say a set  $A \subseteq \mathbb{N}^k$  if the function

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

is computable.

We then can say a theory of  $G$  is decidable if the set of true sentences in  $G$ . We can make "input of  $\phi$ " more precise using Gödel's coding.

**Theorem A.0.9.** The theory of  $\mathbb{Z}$  in the language of  $(+, 0, 1, <, \equiv_p)$  where  $\equiv_p$  is the relation of "two integers are same modulo  $p$ " is complete, admits quantifier elimination hence decidable.

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