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IMPERIAL COLLEGE LONDON

DEPARTMENT OF MATHEMATICS

SECOND-YEAR GROUP RESEARCH PROJECT

Reflection Groups

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Abstract

Type your abstract here. The abstract is a summary of the contents of the project. It should be brief but informative, and should avoid technicalities as far as possible.

Contents

1 Introduction

The introduction should attempt to set your work in the context of other work done in the field. It should demonstrate that you are aware of what you are doing, and how it relates to other work (with references). It should also provide an overview of the contents of the project. You should highlight your individual contributions and any novel result: which of the calculations, theorems, examples, proofs, conjectures, codes etc. are your own? This is how you cite a reference in the bibliography[1]. All of the commands and formatting are in ./style/header.sty

2 Polytopes

2.1 Polytopes

2.1.1 Dihedral groups

The dihedral group, D_{2n}, is the group of symmetries of a regular n-gon. Its standard presentation is given by

$$\langle r, s \mid r^n = e,\ s^2 = e,\ (rs)^2 = e \rangle$$

where r is a rotation of $2\pi/n$ and s is a reflection.

Let l_1 and l_2 be two reflection axes with an angle θ between l_1 and l_2, and s_1 and s_2 be the respective reflections. After some algebra, the composition $s_1 s_2$ turns out to be a counterclockwise rotation through 2π.

Therefore, an alternative presentation of D_{2n} is given by

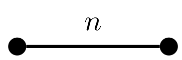
$$\langle s_1, s_2 \mid s_1^2 = e,\ s_2^2 = e,\ (s_1 s_2)^n = e \rangle$$

where s_1 and s_2 are adjacent reflections.

This shows that D_{2n} is an example of a reflection group.

Example 2.1.2. The coxeter diagram for the symmetries of a regular n-gon, also known as $I_2(n)$, looks like

$I_2(n)$



$(n \geq 4)$

Theorem 2.1.3. Let $G \curvearrowright X$ be an action of a finite group G on a finite set X. Then the number of G-orbits in X is given by:

$$\text{Number of orbits} = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|$$

where $\mathrm{Fix}(g) = \{x \in X \mid g \cdot x = x\}$.

Example 2.1.4. How many distinguishable necklaces can be made using seven different colored beads of the same size?
Let X be the 7! possible arrangements. The necklace can be turned over (a reflection) as well as rotated so we consider the dihedral group D_{14} acting on X. Using the previous theorem,

$$\text{Number of orbits} = \frac{7!}{14} = 360$$

as only the identity leaves any arrangement fixed.

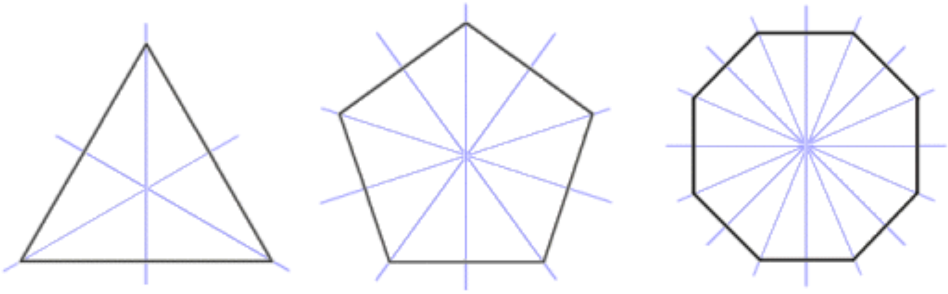


Figure 1 The lines of symmetry of a regular 3-,4- and 7-gon.

Theorem 2.1.5. Let p be a prime. Then any group G of order $2p$ is isomorphic to either the cyclic group C_{2p} or the dihedral group D_{2p}.
Proof. By Cauchy's theorem, there exists an element $a \in G$ of order p and an element $b \in G$ of order 2. Let

$$H = \langle a \rangle,$$

so H is a subgroup of G of index 2, so $H \trianglelefteq G$.
Since $H \trianglelefteq G$, conjugation by b sends H to itself. Thus, there exists $k \in \{1, 2, \ldots, p-1\}$ such that

$$bab^{-1} = a^k.$$

Applying conjugation by b twice to a gives

$$a = b^2ab^{-2} = b(bab^{-1})b^{-1} = ba^kb^{-1} = (bab^{-1})^k = (a^k)^k = a^{k^2}.$$

Therefore,

$$a = a^{k^2} \implies a^{k^2-1} = e.$$

Since a has order p, this implies

$$p \mid (k^2 - 1),$$

or equivalently,

$$k^2 \equiv 1 \pmod p.$$

Because p is prime, this implies

$$k \equiv \pm 1 \pmod p.$$

- If $k \equiv 1$, then

$$bab^{-1} = a,$$

and b commutes with a. Hence G is abelian, and since a has order p and b has order 2, G is cyclic of order $2p$.

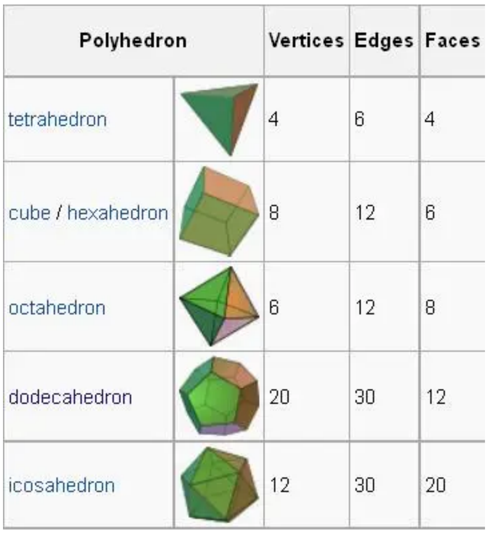
- If $k \equiv -1$, then
$$bab^{-1} = a^{-1},$$
which is the defining relation for the dihedral group D_{2p}:
$$D_p = \langle a, b \mid a^p = e, b^2 = e, bab = a^{-1} \rangle.$$

Thus, G is isomorphic to either C_{2p} or D_{2p}.

□

2.2 The Infinite dihedral group

3 Platonic solids and reflections



Polyhedron		Vertices	Edges	Faces
tetrahedron		4	6	4
cube / hexahedron		8	12	6
octahedron		6	12	8
dodecahedron		20	30	12
icosahedron		12	30	20

Figure 2 The five Platonic solids.

Definition 3.0.1. A polyhedron is regular if its faces are regular polygons, all with the same number of sides, and also each vertex belongs to the same number of edges.

Theorem 3.0.2. The only regular convex polyhedra are the five Platonic solids.
Proof. Before writing the proof, we introduce some notations. V, the number of vertices;
E, the number of edges;
F, the number of faces;
n, the number of sides on a face;
r, the number of edges to which each vertex belongs.
Observe that
$$2E = nF \tag{1}$$
and
$$2E = rV \tag{2}$$

(1) comes from counting the number of pairs (e,f) where e is an edge and f is a face and e lies on f; (2) comes from counting the number of pairs (v,e) where v is a vertex and v lies on e. Substitute into Euler's formula, we get

$$\frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E} \tag{3}$$

Now $n \geq 3$, as a polygon must have at least 3 sides and $r \geq 3$, since in a polyhedron a vertex must belong to at least 3 edges. By (3), we can't have both $n \geq 4$ and $r \geq 4$, since this would make the left-hand side of (3) at most $\frac{1}{2}$. It follows that either $n = 3$ or $r = 3$. If $n = 3$, then (3) becomes

$$\frac{1}{r} = \frac{1}{6} + \frac{1}{E} \tag{4}$$

The right-hand side is greater than $\frac{1}{6}$, and hence $r < 6$. Therefore, $r = 3$, 4 or 5 and $E = 6$, 12 or 30, respectively. If $r = 3$, (3) becomes

$$\frac{1}{n} = \frac{1}{6} + \frac{1}{E} \tag{5}$$

Similarly, n = 3,4 or 5 and E = 6,12 or 30, respectively. These parameters are coincide with those in the table above.

Example 3.0.3. Let G be the group of symmetries of a dodecahedron. What is $|G|$?
Let G act on the 12 faces of the dodecahedron and fix a face. There are $|D_{10}| = 10$ symmetries which fix this face and our action is clearly transitive. By Orbit-Stabiliser theorem, $|G| = 10 \times 12 = 120$. Alternatively, this can be done by considering the fundamental domain, which is a triangle that uniquely determines the reflection. There are 120 such triangles.

We would like to study the group of symmetries of these Platonic solids.

3.1 Tetrahedron

Each face has D_6 as a group of symmetries.
Before moving on to the other solids, we first introduce the concept of dual.

Definition 3.1.1. The dual of a Platonic solid is a new Platonic solid where the faces and vertices are interchanged with those of the original.

Remark 3.1.2. The tetrahedron is self-dual. The cube and the octahedron form a dual pair. The dodecahedron and the icosahedron form a dual pair.

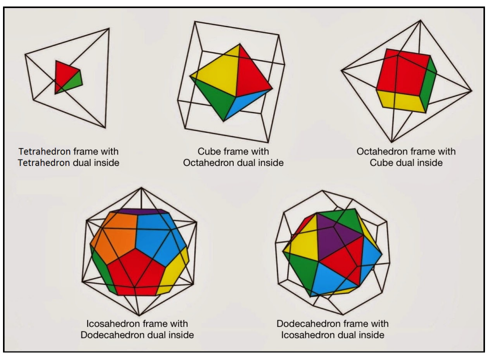


Figure 3 Duals of each Platonic solid.

4 Classification of finite reflection groups

Throughout this section, E will denote $\mathbb{R}^n$ equipped with the standard inner product, our fixed *euclidean space*.

4.1 Affine reflections

This subsection roughly follows [2]. When considering reflections in E, we don't want to just consider those lying in $O(E)$, we are also interested in reflections across some hyperplane not lying throught the origin. To tackle this we first define some standard preliminaries:

Definition 4.1.1. A **reflection** in E is a linear transformation $s \in O(E)$ (the orthogonal group of inner product preserving linear maps) with eigenvalues $\{-1, 1\}$ with corresponding dimensions of eigenspaces:

$$\dim E_1 = n - 1 \qquad \dim E_{-1} = 1$$

for linear maps in $O(E)$ this is equivalent to fixing some hyperplane (codimension 1 subspace) and having determinant -1.

Definition 4.1.2. The **group of general affine transformations** of E is semidirect product of $GL(E)$ acting on E

$$GA(E) := E \rtimes GL(E)$$

where E acts on itself by translation. $(v, T) \in GA(E)$ acts on $e \in E$ as $(v, T) \cdot e = v + T(e)$.

Proposition 4.1.3. This is actually an action. Furthermore, the action is transitive.

Proof. Take $(v, T), (u, S) \in GA(E)$ and $e \in E$. Showing this is an action is a direct calculation:

$$(v, T) \cdot ((u, S) \cdot e) = (v, T) \cdot (u + S(e)) = v + T(u) + TS(e) = (v + T(u), TS) \cdot e = ((v, T)(u, S)) \cdot e$$

from the definition of the semidirect product. Suppose (v, T) and (u, S) act the same on E: as both T and S are linear the two affine transformations send 0 to v and u respective, thus we have $v = u$; subtracting these equal translations and having equality means the two linear transformations must also be equal. $\square$

We can now consider appropriate affine versions of linear groups, the most important for us will be the **affine orthogonal group** $AO(E)$, which can be equivalently viewed as the subgroup of $GA(E)$ preserving the inner product or as $E \rtimes O(E)$. We need stricter criteria than just the linear component of our affine transformation be a reflection to suitably capture the notion of a reflection across an affine hyperplane.

Proposition 4.1.4. For a unit normal vector α and some $k \in \mathbb{R}$, reflection across the affine hyperplane $(E, \alpha) = k$ corresponds to the affine transformation $(2k\alpha, s_\alpha)$, where s_α is the reflection along α.

Proof. Call the affine hyperplane H and choose a $v \in E$. The vector orthogonal to H that goes to v has length $k - (v, \alpha)$ so reflecting across H send v to $v + 2(k - (v, \alpha))\alpha = 2k\alpha + (v - 2(v, \alpha)\alpha) = (2k\alpha, s_\alpha) \cdot v$. $\square$

We call such affine transformations, **affine reflections**.

Lemma 4.1.5. For all affine hyperplanes H and affine reflections r, the set rH is also an affine hyperplane.

Proof. Let H be $\{v \in V \mid (v, \alpha) = k\}$ for some $\alpha \in E, k \in \mathbb{R}$. As r is bijective the set rH is equal to $\{w \in V \mid (r^{-1}w, \alpha) = k\}$, by writing $r = (u, T) \in GA(E)$ this can be rewritten as $\{w \in V \mid (w, T^*(\alpha)) = k - (u, \alpha)\}$, an affine hyperplane. $\square$

Lemma 4.1.6. An affine transformations $r = (u, T)$ that fixes some affine hyperplane and is an involution must be an affine reflection.

Proof. As $r^2 = \text{id}$, on 0 we have $r^2(0) = u + T(u) = 0$. So as $r^2 = u + T(u) + A^2$ this means A is also an involution so we can use the primary decomposition $E = E_1 \oplus E_{-1}$ into the eigenspaces of A. Call the hyperplane r fixes H, then for any $h = v_1 + v_{-1} \in H$ (where $v_1, v_{-1} \in V_1, V_{-1}$ respectively) we have $r(v_1 + v_{-1}) = u + T(v_1 + v_{-1}) = u + v_1 - v_{-1} = v_1 + v_{-1}$ therefore $2v_{-1} = u$ so $\dim E_{-1} = 1$ and T is the reflection along $V_{-1} = \langle v_{-1} \rangle$, thus $r = \big(2v_{-1}, s_{v_{-1}}\big)$. $\square$

Proposition 4.1.7. For all affine reflections r, s with s reflecting across the affine hyperplane H, the affine transformation rsr^{-1} is an affine reflection across rH.

Proof. First, notice $\big(rsr^{-1}\big)^2 = \text{id}$ as both r and s are involutions. Also, $rsr^{-1}(rH) = rH$ as s fixes H. $\square$

From now on, we will use the umbrella term *reflection* to refer to both linear and affine reflecitons.

4.2 Reflection groups

The plan is to show all finitely generated reflection groups are in fact Coxeter groups, which admit a nice geometric classification. This follows [1]

We are interested in groups generated by reflections in E, so throughout the next two sections fix a group $W \leq GA(E)$ which can be generated by reflections.

Consider the set $\mathcal{H}$ of all hyperplanes such that an element $w \in W$ is the reflection across. If we make a poor choice of our reflections generating W we may end up with $\mathcal{H}$ dense in E.

Example 4.2.1. Consider the following set of hyperplanes in $\mathbb{R}^2$, by previous lemma $\mathcal{H}$ is closed under the induced action from W so any such pentagon will generate a smaller pentagon, inverted in its center, this infinite descent makes $\mathcal{H}$ dense in $\mathbb{R}^2$.

To remedy this we will restrict the reflection groups we consider by rquiring for any compact subset $B \subset E$, the intersection $\mathcal{H} \cap B$ be finite.

Definition 4.2.2. The connected components of $E \setminus \mathcal{H}$ are called the **chambers** of W in E.

Proposition 4.2.3. The number of hyperplanes touching any single chamber is finite.

We now want to choose a chamber C_0, and consider the set of hyperplanes $\{H_1, \ldots, H_k\} \subseteq \mathcal{H}$ bounding C. We will call the corresponding reflections across these hyperplanes $\{s_1, \ldots, s_k\}$ **simple reflections**.

Theorem 4.2.4. The set of simple reflections $\{s_1, \ldots, s_k\}$ generates W.

Proof. Let W' be the subgorup of W generated by the simple reflections. Let s be one of the reflections generating W, and call the hyperplane it reflects across H. If W' acts transitively on the set of chambers then there will exists some $w \in W'$ such that $wH_i = H$ for some simple reflection s_i as H will must bound a chamber. Thus, by an earlier lemma, $ws_iw^{-1} = s$ so $s \in W'$ and $W = W'$. Now we just have to show the action of W' is transitive on chambers. Suppose it isn't, i.e. there is some chamber C such that no $w \in W'$ satisfies $wC = C_0$. Let C' be the closest chamber to C_0 in the W' orbit of C, as $C' \neq C_0$ there must be some simple hyperplane (the boundary of C_0) between them, reflecting across this must strictly decrease the distance between the two chambers contradicting the minimality of C'. Thus W' acts transitively on the set of chambers. $\square$

As a direct corollary of this proof we now know W acts transitively on the set of chambers. Before discussing this further we should examine the relations that these simple reflections satisfy.
For any two simple reflections s_i, s_j the subgroup $\langle s_i, s_j \rangle$ will be dihedral, as seen in the previous section, (note that this will be the infinite dihedral group iff the hypeplanes being reflected along are parallel), call the order of this dihedral group $2m_{ij}$. The product $s_i s_j$ will have order m_{ij} in W and so W satisfies the set of relations $(s_i s_j)^{m_{ij}} = \text{id}$ for all i, j, taking $m_{ii} = 1$. A group *presented* by these relations is called **Coxeter**.

Definition 4.2.5. A **Coxeter system** is a pair (W, S) where $S = \{s_i\}_{i \in I}$ is a generating set for W which admits the presentation:

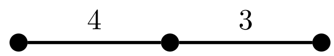
$$W = \langle S \mid (s_i s_j)^{m_{ij}} \text{ for all } i, j \in I \rangle$$

where each $m_{ij} \in \mathbb{N} \cup \{\infty\}$.

To each Coxeter system we can assign a **Coxeter diagram**: an undirected graph created by the following rules:

- Draw a node i for each $s_i \in S$;
- For each relation $(s_i s_j)^{m_{ij}}$ with $m_{ij} > 2$ draw an edge between i and j and label it with m_{ij}.

This process can be reversed to obtain a Coxeter system from any Coxeter diagram. This correspondence will associate the graph:



to the group presentation:

$$\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = e,\ (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

In the future, for the sake of readability, the 3 labels will often be excluded.
The classification of finite reflection groups goes by prooving all reflection groups are infact Coxeter groups and then classifying all the finite Coxeter groups.

4.3 Coxeter presentation

We have shown the action of W on the set of chambers is transitive, but we need a stronger notion to prove these groups are Coxeter.

Definition 4.3.1. Let G be a group acting on a set X, the action is called **simply transitive** if for all $x, y \in G$ there exists a unique $g \in G$ such that $g \cdot x = y$.

By fixing a base point $x \in X$ there is clear a 1-to-1 correspondence between X and G: associate $e \leftrightarrow x$ and for all $g \in G$, $g \leftrightarrow g \cdot x$.

Proposition 4.3.2. An action is simply transitive iff it is transitive and free.

Proof. An action is transitive if for all $x, y \in X$ there exists some $g \in G$ such that $g \cdot x = y$, and free if there exists at most one such g; therefore these statements are equivalent. $\square$

If we know the action is trasntive a priori we only need to check the free condition on a single element. So showing our action is simply transitive ammounts to showing for all $w \in W$ such that $wC_0 = C_0$ we have $w = \text{id}$.
We will go about prooving this by finding a geometric interpretation of the length of a word $w \in W$ in terms of simple reflections.

Definition 4.3.3. For $w \in W$ define the **length** of w, $l(w)$ to be the minimal positive integer r such that $w = s_1 \cdots s_r$, a length r product of simple reflections.

And the contrasting way to compute the length: for any $w \in W$ consider the set of hyperplanes H such that C_0 and wC_0 lie on different sides of H. Call this $\mathcal{L}(w)$ and let $n(w) = |\mathcal{L}(w)|$.

Lemma 4.3.4.
1. $n(w) = n(w^{-1})$,
2. $l(w) = 1$ iff w is a simple reflection
3. $l(w) = l(w^{-1})$
4. if w can be written as $s_1 \cdots s_r$ then $\det(w) = (-1)^r$
5. $l(s_i w), l(ws_i) = l(w) \pm 1$

Proof. (1.) H separtes C_0 and $w^{-1}C_0$ iff wH separates wC_0 and C_0 — W acts isometrically. □

Lemma 4.3.5. H_i is in exactly one of $\mathcal{L}(w)$ or $\mathcal{L}(s_i w)$

Lemma 4.3.6. Choose a simple hyperplane H_i, for any hyperplane $H \neq H_i$ and $w \in W$, if $H \in \mathcal{L}(w)$ then $s_i H \in \mathcal{L}(s_i w)$.

Proof. As H separates C_0 and wC_0, $s_i H$ separates $s_i C_0$ and $s_i w C_0$, so $s_i H$ is in exactly one of $\mathcal{L}(s_i)$ or $\mathcal{L}(s_i w)$, but if $s_i H \in \mathcal{L}(s_i)$ we would have $s_i H = H_i \implies H = H_i$ a contradiction. Therefore $s_i H \in \mathcal{L}(s_i w)$. □

Corollary 4.3.7. $s_i \left(\mathcal{L}(w) \setminus \{H_i\}\right) = \mathcal{L}(s_i w) \setminus \{H_i\}$.

Proof. by applying the previous lemma to both w and $s_i w$ we get the required iff. □

Proposition 4.3.8. For all $w \in W$, we have $n(w) \leq l(w)$.

Proof. If $l(w) = 1$ then w must be some simple reflection s_i so $\mathcal{L}(w) = \{H_i\}$[citation-needed]so $n(w) = 1$. Now by induction on $l(w)$: if $l(s_i w) = l(w) + 1$ then by the previous corollary and an earlier lemma we know $n(s_i w) = n(w) \pm 1$, namely $n(s_i w) \leq n(w) + 1 = l(w) + 1 = l(s_i w)$. □

Now to show $n(w) = l(w)$ we will enumerate the hyperplanes in $\mathcal{L}(w)$.

Lemma 4.3.9. If $w = s_1 \cdots s_r$ is a reduced expression for w in terms of simple reflections, the hyperplanes:

$$H_1, s_1 H_2, s_1 s_2 H_3, \ldots s_1 \cdots s_{r-1} H_r$$

are all distinct.

Proof. Suppose, for a contraditction, that there exists some $1 \leq i < j \leq r$ such that the hyperplanes $s_1 \cdots s_{i-1} H_i$ and $s_1 \cdots s_{j-1} H_j$ are equal, by applying $s_1 \cdots s_{i-1}$ to both sides this implies $H_i = s_i \cdots s_{j-1} H_j$, by an earlier corollary this implies $s_i = (s_I \cdots s_{j-1}) s_j (s_{j-1} \cdots s_i)$ which implies $s_{i+1} \cdots s_{j-1} = s_i \cdots s_j$ contradicting the minimality of the lenght of w. □

Proposition 4.3.10. If $w = s_1 \cdots s_r$ is a reduced expression for w in terms of simple reflections:

$$\mathcal{L}(w) = \{H_1, s_1 H_2, s_1 s_2 H_3, \ldots s_1 \cdots s_{r-1} H_r\}$$

Proof. We go by induction on r. Observe the base case $\mathcal{L}(s_1) = \{H_1\}$ from an earlier lemma. Now assume the claim holds for all w with reduced length $< r$ specifically:

$$\mathcal{L}(s_1 w) = \{H_2, s_2 H_3, \ldots s_2 \cdots s_{r-1} H_r\}$$

we know from an earlier lemma H_1 is in exactly one of $\mathcal{L}(w)$ or $\mathcal{L}(s_1 w)$. If $H_1 \in \mathcal{L}(s_1 w)$ by applying s_1 we would have for some $i > 1$:

$$s_1 H_1 = H_1 = s_1 \cdots s_i H_i$$

which contradicts the previous lemma, thus $H_1 \in \mathcal{L}(w)$, combining this with the earlier corollary gives the desired result. $\square$

A direct consequence of this propsition is that if some $w \in W$ is such that $wC_0 = C_0$ then $\mathcal{L}(w) = \emptyset$ which implies $w = \text{id}$ in reduced form. We know this is sufficient to say the action of W on the set of chambers is simply transitive.
We can now prove two key combinatorial properties of the reflection group, and use these to show all relations in W are direct consequences of those in the Coxeter presentation.

Theorem 4.3.11 (Exchange Condition)**.** If $w \in W$ has reduced expression in terms of simple reflections $w = s_1 \cdots s_r$ and $l(sw) < l(w)$ for some simple reflection s, there exists some $1 \leq i \leq r$ such that $sw = ss_1 \cdots s_{i-1} s_{i+1} \cdots s_r$.

Proof. We know:

$$\mathcal{L}(w) = \{H_1, s_1 H_2, s_1 s_2 H_3, \ldots s_1 \cdots s_{r-1} H_r\}$$

by a previous proposition we know H, the hyperplane corresponding to s, must lie in $\mathcal{L}(w)$ and so for some $1 \leq i \leq r$, $H = s_1 \cdots s_{i-1} H_i$ which implies $s = s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1$ therefore $ss_1 \cdots s_{i-1} = s_1 \cdots s_i$. Substituting this back into w gives the desired result. $\square$

4.4 Classification

Definition 4.4.1. To a Coxeter system (W, S), with $n = |S|$, we can associate a real symmetric $n \times n$ matrix A, by setting $a_{ij} := -\cos(\pi/m_{ij})$. The quadratic form $Q(v) = v^\top A v$ on E is called the **associated quadratic form** of the Coxeter system.

We can realise each a_{ij} as the inner product of simple roots (α_i, α_j) corresponding the the reflection s_i, s_j. Such matrices are called **Gram matrices**.

Proposition 4.4.2. The quadratic form associated to a finite reflection group W is positive definite.

Proof. As Φ forms a basis for E there is an invertible change of basis matrix B form the standard basis to Φ, we can thus write $A = B^\top B$ and for any nonzero $v \in E$ have:

$$Q(v) = v^\top A v = v^\top B^\top B v = (Bv)^\top (Bv) = \|Bv\|^2 > 0$$

as B is a linear transformation acting on $v \neq 0$, therefore Q is positive definite. $\square$

We now want to fix a quadratic form $Q(v) = v^\top A v$ for some symmetric $A \in \mathcal{M}_n(\mathbb{R})$.

Lemma 4.4.3. The kernel of Q (the set of vectors $v \in E$ such that $Q(v) = 0$) equals the nullspace of A.

Proof. The nullspace of A obviously lies in the kernel of Q. From linear algebra 2 we know we can orthogonally diagonalise A and so we can write:

$$Q(v) = v^\top Av = v^\top P^\top DPv = (Pv)^\top D(Pv)$$

where P and D are some orthogonal and diagonal matrices respectively. Observe the right hand side is 0 iff for ever $1 \leq i \leq n$ either d_{ii} or $(Pv)_i$ are 0. This implies $D(Pv) = 0$ and thus v is in the nullspace of A. □

We will now demand all of the off-diagonal entries of A be nonpositive, note that all of the matrices for quadratic forms associated with Coxeter diagrams will satisfy this condition.

Lemma 4.4.4. The only nonzero vectors in the nullspace of A have all coordinates nonzero.

Proof. First suppose $v \neq 0$ is in the nullspace of A, then by the previous lemma we know $Q(v) = 0$. Consider the vector u given by $u_i = |v_i|$, we will have $Q(u) = u^\top Au \geq 0$, but as all off-diagonal entries of A are nonpositive this will satisfy $Q(u) \leq Q(v) = 0$. Thus u is also in the nullspace of A. So if the nullspace is nontrivial, it will always contain a vector with nonnegative coordinates.
If we suppose some, but not all, of these coordinates are zero we can consider the set of such indices I. For any $i \in I$ consider the ith term in Au:

$$(Au)_i = \sum_{j \notin I} a_{ij}|v_j| = 0$$

as all the a_{ij} are nonpositive and all the $|v_j|$ are strictly positive, we get that for all $i \in I, j \notin I$ the entries $a_{ij} = 0$ contradicting irreducibility of A. Thus the coordinates of all vectors in the nullspace of A are nonzero. □

Corollary 4.4.5. The dimension of the nullspace of A will be at most 1.

Lemma 4.4.6. The smallest eigenvalue d of A has multiplicity 1, and its coordinates will all be positive.

Proof. Again, from linear algebra 2 we know all the eigenvalues of a positive semidefinite matrix are nonnegative. So $A - dI$ satisfies all the requirement we placed on A for the previous lemma. As $A - dI$ is singular it will have nonempty nullspace which must be of dimension exactly 1. Thus d and $u_i = |d_i|$ are colinear so we can find an eigenvalue with all positive coordinates. □

Definition 4.4.7. Subgraph

Proposition 4.4.8. If Q is positive semidefinite and Q' comes from a proper subsystem, then Q' will be positive definite.

Proof. Suppose A' fails to be positive definite, i.e. there exists some $v \in E$ such that $Q'(v) \leq 0$, consider the vector:

$$u_i = \begin{cases} |v_i| & 1 \leq i \leq k \\ 0 & \text{otherwise} \end{cases}$$

and the value of:

$$0 \leq Q(u) = \sum_{1 \leq i,j \leq n} a_{ij}u_iu_j \leq \sum_{1 \leq i,j \leq k} a'_{ij}v_iv_j \leq 0$$

as all the $a_{ij} \leq a'_{ij} \leq 0$. This means u is in the nullspace for A and thus all the coordinates are nonzero which means $k = n$. But this then implies all $a_ij = a'_ij$. Thus the only subdiagram that fails to be positive definite is improper. □

Here is a collection of Coxeter diagrams all of which, by checking their eigenvalues on a computer, can be found to be positive definite.

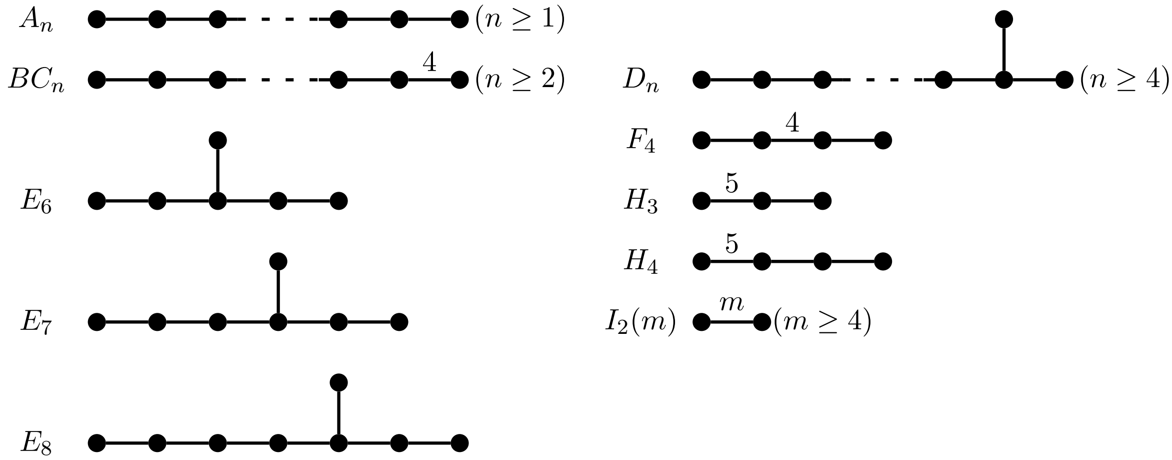


Figure 4 Positive definite Coxeter diagrams

And here is another collection who can all be founded to be only positive semidefinite:

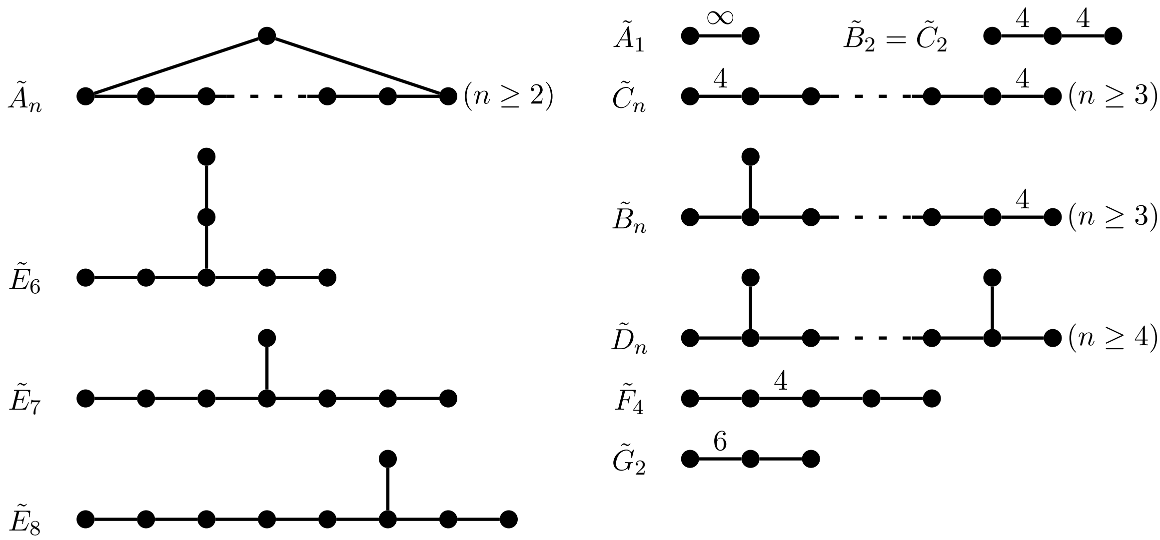


Figure 5 Positive semidefinite Coxeter diagrams

Plus two helper diagrams which are neither:

Figure 6 Helper Coxeter diagrams

Theorem 4.4.9. the diagrams of Figure 4 are all the connected diagrams with positive definite quadratic form.

Proof. The proof goes by indentifying features present in the graphs of Figure 5, observing they cannot appear as subdiagrams of those with positive definite quadratic form: e.g. no cycles as in $\tilde{A}_n$ and all edges must be finite due to $\tilde{A}_1$. Full the full proof see [1]. $\square$

Corollary 4.4.10. The following graphs categorise all finite reflection groups.

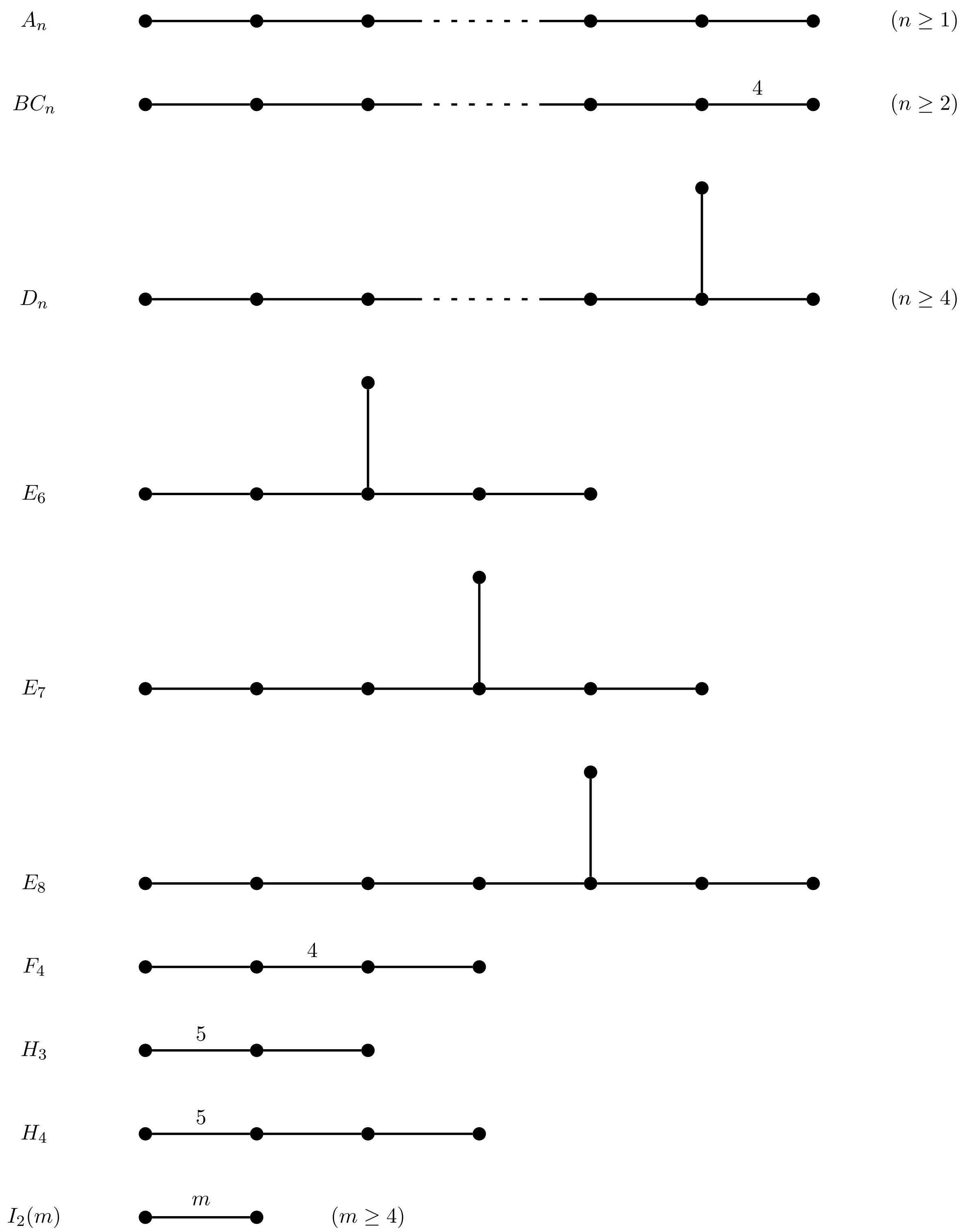
Proof. We know it suffices to classify the reflections groups corresponding to connected Coxeter diagrams, all of which must have positive definited associated quadratic form, all of which we have found. The only remaining step is to find polytopes accepting these as their finite reflection groups, lots of these examples have been seen in the previous section, and in fact every diagram of Figure 4 do occur as that of a finite reflection group. □

This solution can seem somewhat unsatisfactory as there is no immediate reason as to **why** all of the finite Coxeter groups can be realised as reflection groups.

4.5 Geometric representation

For any finite Coxeter system (W, S) consider the abstract real vector space V with basis Φ. We can define a quadratic form (and thus an inner product $\langle -, - \rangle$) on V in the same way as we did earlier. And for each generator $s \in S$ we can associate a reflection $\sigma_s : V \to V$ given natrually as $\sigma_s(v) = v - 2\langle \alpha_s, v \rangle \alpha_s$. Through some similar calculations we can find the representation this induces on W is faithful and acts orthogonally on V, the construction of the edges and vertices of a polytope demonstrated in the next section will thus show how a regular polytope can be realised from this abstract reflection group.

Original finite diagram



5 Uniform polytopes

uniform polytopes

6 Tits' word problem

6.0 Abstract

Recalling the word problem for a presentation $\langle S \mid R\rangle$ of a group G, we hope to find an algorithm to reduce any word in W into a reduced word. In this paper, we totally solved this problem for Coxeter groups, and actually for all system with exchange condition, which is called Tit's theorem.
The proof is divided into several steps:
1.Every Coxeter group satisfies the exchange condition.
2.For a system with exchange condition, when do two reduced words represent the same element.
3.How to reduce a word in a system with exchange condition.
The first part of the proof is the hardest, and we focus on the most.
We will talk more on the exchange condition as the fourth part, even though part 1 is enough to prove Tit's Theorem, exchange condition show deep insight of Coxeter Groups.

6.1 Exchange Condition

Definition 1.1 A system is a tuple (W, S), where W is a group and S is a set of generators of G, such that $S = S^{-1}$ and $1 \notin S$.

Definition 1.2 For $w \in W$, the sequence s $= (s_1, ..., s_q)$,where $s_i \in S$ for all $i \in \mathbb{N}$, is called a reduced representation of w if and only if $w = s_1 s_2 ... s_q$ $(*)$ and q is the smallest among all such sequences satisfies $(*)$. We define $l(w) = q$ and call it the length of w.

Proposition 1.3 For $w, w' \in W$, we have the following facts:
(1) $l(ww') \leq l(w) + l(w')$
(2) $l(w^{-1}) = l(w)$
(3) $|l(w) - l(w')| \leq l(ww'^{-1})$
Proof: (1) and (2) are trivial. From (1), we see $l(w) \leq l(ww'^{-1}) + l(w')$ and $l(w') \leq l(w'w^{-1}) + l(w)$. From (2), we see $l(w'w^{-1}) = l(ww'^{-1})$. Putting these together gives us (3). □

Definition 1.4 If a system (W, S) satisfies the following conditions:
(1)Every element in S has order 2.
(2) Say $m(s, t)$ is the order of st, and define $I = \{(s, t) \mid s, t \in S,\ m(s, t) < \infty\}$, then $\langle S \mid (st)^{m(s,t)} = 1, (s, t) \in I\rangle$ is a presentation of W.
Then we say (W, S) is a Coxeter System and W is a Coxeter group.

Remark 1.5 A Coxeter System (W, S) satisfies the following universal property: For any group G and map $\psi : S \to G$ such that $\psi(st)^{m(st)} = e_G$,$\forall(s, t) \in I$, there exist an unique $\phi \in Hom(W, G)$ such that $\psi = \phi \circ l$. Where I is defined as in Definition 1.4, and l is the natural inclusion from S to W. Note that this is just the composition of universal properties of free groups and quotient groups.

Example 1.6 By universal property of Coxeter System, the map $s \mapsto -1$, $\forall s \in S$ extends to a homomorphism $\epsilon : W \to \{-1, 1\}$ such that $\epsilon(w) = (-1)^{l(w)}$. We call $\epsilon(w)$ the sign of w.

We want to find some invariant among representations of the same word, hence make the following definition.

Definition 1.7 For Coxeter System (W, S), say T the union of conjugacy classes of all $s \in S$. For sequence $\mathbf{s} = (s_1, ..., s_q)$, where $s_i \in S, \forall i$. Define $\Phi(\mathbf{s}) = (t_1, ..., t_q)$ such that $t_j = (s_1...s_{j-1})s_j(s_1...s_{j-1})^{-1}$. For $t \in T$, define $n(\mathbf{s}, t) := \#\{1 \leq j \leq q \mid t_j = t\}$. Finally, define $R = \{\pm 1\} \times T$.
Remark Note that we have $s_1...s_j = t_j...t_1$.

Lemma 1.8 For Coxeter System (W, S), we have the following facts:
(1) For $w \in W$ and $t \in T$, $\forall \mathbf{s} = (s_1, ..., s_q)$ such that $w = s_1...s_q$, $(-1)^{n(\mathbf{s},t)}$ has the same value. We call this value $\eta(w, t)$.
(2) For $w \in W$, consider the map $U_w : R \to R$ defined as $U_w(\epsilon, t) = (\epsilon\eta(w, t), wtw^{-1})$, then $w \mapsto U_w$ is a homomorphism from W to the permutation group of R.

Proof: For $s \in S$, we define $U_s : R \to R$ such that $(\epsilon, t) \mapsto (\epsilon(-1)^{\delta(\mathbf{s},t)}, sts^{-1})$, where $\delta_{s,t}$ is the Kronecker Symbol. For sequence $\mathbf{s} = (s_1, ..., s_q)$ in S, say $w = s_q...s_1$, $U_s = U_{s_q} \circ ... \circ U_{s_1}$.
We prove $U_{\mathbf{s}}(\epsilon, t) = (\epsilon(-1)^{n(\mathbf{s},t)}, wtw^{-1})$ by induction. $(*)$
In case $q = 0$ or 1, the proof is trivial. For $q > 1$, by induction hypothesis, say $\mathbf{s}' = (s_1, ..., s_{q-1})$,$w' = s_{q-1}...s_1$, then $U_{\mathbf{s}'} = U_{s_q}(\epsilon(-1)^{n(\mathbf{s}',t)}, w'tw^{-1})$. Hence it only remains to prove that $n(\mathbf{s}, t) = \delta_{s_q,w'tw^{-1}} + n(\mathbf{s}', t)$. Notice that $\Phi(\mathbf{s}) = (\Phi(\mathbf{s}'), w'^{-1}s_qw')$, so the proof is trivial.
We now prove the fact that $s \mapsto U_s$ can be extended to a homomorphism from W. By the universal property of Coxeter System, it saffice to prove that $\forall s, s' \in S$ such that $m(s, s') < \infty$, $(U_s \circ U_{s'})^{m(s,s')} = 1$. Define $\mathbf{s} = (s_1, ..., s_{2m(s,s')})$ such that $s_i := s$ for odd i and $s_i := s'$ for even i, so it saffice to prove $U_{\mathbf{s}} = Id$. Notice that here we have $t_j = (ss')^{j-1}s$, so $t_i \neq t_j$ for all $1 \leq i < j \leq m(s, s')$ and $t_i = t_{i+m(s,s')}$. Hence, for $t \in T$, $n(\mathbf{s}, t) = 0$ or 2. Applying the result in the previous paragraph, we see $U_{\mathbf{s}} = Id$.
Now we see $w \mapsto U_w$ is a homomorphism from W. Specifically, U_w doesn't depend on the representation chosen for w. So from $(*)$, we see $n(\mathbf{s}, t)$ is invariant among different choices of representations. This gives us (1). (2) follows easily. $\square$

Theorem 1.9 For Coxeter System (W, S), consider sequence $\mathbf{s} = (s_1, ..., s_q)$, $\Phi(\mathbf{s}) = (t_1, ..., t_q)$, $w = s_1...s_q$. Then s is an reduced representation of w if and only if $t_i \neq t_j$ for all $i \neq j$. Define $T_w := \{t \in T \mid \eta(w, t) = -1\}$, then in case s is a reduced representation of w, $T_w = \{t_1, ..., t_q\}$ and $|T_w| = l(w)$.

Proof: $\forall t \in T_w$,$n(w, t) \neq 0$. By definition of $n(w, t)$, we deduce $\forall t \in T_w$,$t \in \{t_1, ..., t_q\}$. Moreover, as $\eta(w, t)$ is independent of the choice of representation of w, T_w only depends on w. Hence $|T_w| \leq l(w)$. In case $t_i \neq t_j$ for all $i \neq j$, $n(s, t) = 0$ or 1, and hence $T_w = \{t_1, ..., t_q\}$. As $q = |T_w| \leq l(w)$, so s is a reduced representation of w.
Conversely, if there exist $i \neq j$ such that $t_i = t_j$, WLOG say $i < j$. Consider $u = s_{i+1}...s_{j-1}$, then $s_i = us_ju^{-1}$. Hence we have $s_1...s_q = s_1...s_{i-1}(us_ju^{-1})us_js_{j+1}...s_q = s_1...s_{i-1}us_js_js_{j+1}...s_q = s_1...s_{i-1}s_{i+1}...s_{j-1}s_{j+1}...s_q$. So s is not reduced. $\square$

Note that now we have a necessary and sufficient condition to check whether a word is reduced or not. We want to improve this result and get an algorithm of reducing words.

Definition 1.10 For a system (W, S), if it satisfies: $\forall w \in W, s \in S$, if $l(sw) \leq l(w)$, then for any reduced representation $\mathbf{s} = (s_1, ..., s_q)$, exist $1 \leq j \leq q$ such that $ss_1...s_{j-1} = s_1...s_j$. Then we say (W, S) satisfies the exchange condition.

Theorem 1.11 If (W, S) is a Coxeter System, then it satisfies the exchange condition.

Proof: Consider $w \in W, s \in S$ such that $l(sw) \leq l(w)$. For any reduced representation $\mathbf{s} = (s_1, ..., s_q)$ of w, consider $w' := sw$. By Example 1.6, we have $l(w') \equiv l(w) + 1 \pmod 2$. Proposition 1.3(3) gives that $|l(w) - l(w')| \leq 1$, so $l(w') = l(w) - 1$. Now pick $(s'_1, ..., s'_{l(w)-1})$ as a reduced representation of w', then $\mathbf{s}' = (s, s'_1, ..., s'_{p-1})$ is a reduced representation of w. Take $\Phi(\mathbf{s}') = (t'_1, ..., t'_p)$, then $t'_1 = s$ by definition. However Theorem 1.9 shows that $t'_i \neq t'_j$ if $i \neq j$, so $n(\mathbf{s}', s) = 1$. By Lemma 1.8, $n(\mathbf{s}', s) \equiv n(\mathbf{s}, s) \pmod 2$, so $n(\mathbf{s}, s) \neq 0$. Hence there exist j such that $s = t_j$, where t_j is an element in $\Phi(\mathbf{s})$. By definition of t_j, $ss_1...s_{j-1} = s_1...s_j$, the claim is proved. □

Theorem 1.11 is all we need to prove the Tit's theorem. But we can actually say more on exchange condition, and deduce conversely that any system such that $\forall s \in S$, s has order 2 and satisfy exchange condition is a Coxeter System. We will talk more on it in part 4.

6.2 M-operations

Definition 2.1 Consider (W, S) a system, the sequence $\mathbf{s} = (s_1, ..., s_q)$, where $s_i \in S, \forall i$ is called a word in S and $w := s_1...s_q$ is called the element in W expressed by $\mathbf{s}$. An elementary M-operation on a word in $\mathbf{s}$ is one of the following two types of operations:
(MI) Delete a subword of form (s, s) from $\mathbf{s}$.
(MII) Replace an alternating subword $(s, t, s, ...)$ by the other alternating subword $(t, s, t, ...)$, both in length $m(s, t)$.
We say a word is M-reduced if and only if its length cannot be shorten by M-operations. Clearly, any reduced word is M-reduced.

Lemma 2.2 For a system (W, S) with Exchange condition, two reduced representations express the same element in W if and only if one can be transformed to the other by MII operations.

Proof: Say $\mathbf{s} = (s_1, ..., s_q)$ and $\mathbf{r} = (r_1, ..., r_q)$ two reduced representations both express $w \in W$. We prove the lemma by induction.
In case $q = 0$,the proof is trivial. If $s_1 = r_1$, we can reduce the length of both words by 1 so the induction hypothesis can be applied directly. It suffice to prove the case $s_1 \neq r_1$. We set $m := m(s_1, r_1)$.

Claim: m is finite and there is another reduced expression $\mathbf{u}$ of w start with an alternating subword $(s_1, r_1, s_1, ...)$ of length m.

With the claim, the remaining proof is easy. Since $\mathbf{s}$ and $\mathbf{u}$ both start with s_1, by induction hypothesis, $\mathbf{s}$ can be transformed into $\mathbf{u}$ by MII operations. Now apply MII on $\mathbf{u}$ to get another word $\mathbf{u}'$ starts with $(r_1, s_1, r_1, ...)$ of length m. As $\mathbf{u}'$ and $\mathbf{r}$ both start with r_1, again by induction hypothesis, $\mathbf{u}'$ can be transformed into $\mathbf{r}$ through MII operations. The combination of all above MII operations gives the result.
The converse of the proposition is trivial. □

Proof of the Claim: Clearly, $l(r_1 w) < l(w)$ as $\mathbf{r}$ is a reduced representation of w starting with r_1. By the exchange condition there exist an index i such that $r_1 s_1 ... s_{i-1} = s_1 ... s_i$. Moreover, as $(s_1, ..., s_i)$ is a subword of $\mathbf{s}$, which is reduced, $v := (r_1, s_1, ..., s_{i-1}, s_{i+1}, ..., s_q)$ is also a reduced representation of w.
Note that $i \neq 1$. Otherwise $(s_1, s_2, ..., s_q) = (r_1, s_2, ..., s_q)$ gives arise to $s_1 = r_1$, which is a contradiction.
Define S_q to be the alternating word end in s_1 with each of its term in set $\{s_1, r_1\}$ of length q. Suppose $\mathbf{s}'$ a word start with S_{q-1} and let s' to be the element of $\{s_1, r_1\}$ that $\mathbf{s}'$ doesn't start with.
Looking at $\mathbf{s}$ and $\mathbf{r}$, we get $l(s'w) < l(w)$. Apply the exchange condition we get a reduced representation of $\mathbf{s}'$ start with s'. The removed element through exchanging process should not lie in S_{q-1} as any reduced representation with length not equal to m an element in group D_{2m} is unique. This is a fact we learned in Year 2 module Groups and Rings. If the removed term

lies in S_{q-1}, we get two different reduce representations of element in D_{2m} generated by s_1, r_1, length not equal to m after cancellation, which is a contradiction.
We hence get a reduced representation start with S_q from s$'$. Hence by induction, w has a reduced expression start with S_q for any $q \leq m(s_1, r_1)$ but we have $q \leq l(w) < \infty$. So we have m is finite.
Replace $S_{m(s_1,r_1)}$ with $(s_1, r_1, s_1, ...)$ by MII operation, we get the expecting $\mathbf{u}$. □

6.3 Tit's Theorem

Theorem 3.1 (Tit's) A word in system (W, S) with exchange condition is reduced if and only if it's M-reduced.

Proof: Suppose the word s $= (s_1, ..., s_k)$ is M-reduced. We show s is reduced by induction on k. The base case $k = 1$ is trivial.
For $k > 1$, by induction hypothesis, s$' = (s_2, ..., s_k)$ is reduced. Say w' the element expressed by s$'$. Suppose s is not reduced, then $l(w) = l(s_1\mathbf{s}') \leq k-1 < l(\mathbf{s}')$. So by exchange condition, there exist an index i such that $w' = s_1 s_2 ... s_{i-1} s_{i+1} ... s_k$. That is, w' has a reduced representation start with s_1. Apply Lemma 2.2 we see that s$'' := (s_1, s_2, ..., s_{i-1}, s_{i+1}, ..., s_k)$ can be transformed from s$'$ by MII operation. This gives a reduced representation of w start with (s_1, s_1), which is a contradiction.
The other direction of the theorem is obvious. □

Note that Tit's Theorem is far from trivial: The word in a free group can be really messy, even talk about $G = \langle x, y \mid x^n = y^2 = (xy)^2 = 1_G \rangle$ directly is not easy, see Groups and Rings module of Year 2.

Example 3.1 $G := \langle x, y \mid xyxyx = yxyxy \rangle \cong \langle x, y, a \mid xyxyx = yxyxy, a = xy \rangle \cong \langle x, a \mid aax = x^{-1}aaa \rangle \cong \langle x, a, b \mid a^2x = x^{-1}a^3, b = xa^2 \rangle \cong \langle a, b \mid a^2 = b^5 \rangle$
We can see from the example that although the relation of G is clear, what operations can be done to make reduction is hard to see. Nontrivial process is done here to transfer the group of x, y to a group of a, b and enable us to do reduction.
There are general tools like Tietze Transformation and Coset enumeration to talk about what we can do on a general group presentation, but it's out of the scope of this project.

7 More on Exchange Condition

Proposition 4.1 For system (W, S) with exchange condition such that every element in S has order 2, consider $w \in W$,$s \in S$. If s is a reduced representation of w, exactly one of the following occurs:
(1) $l(sw) = l(w)+1$,$(s, s_1, ..., s_q)$ is a reduced representation of sw. (2) $l(sw) = l(w)-1$,$\exists 1 \leq j \leq q$ such that $(s_1, ..., s_{j-1}, s_{j+1}, ..., s_q)$ is a reduced representation of sw and $(s, s_1, ..., s_{j-1}, s_{j+1}, ..., s_q)$ is a reduce representation of w.
Proof: This is a direct application of exchange condition.

8 Euclidean Reflection Groups

8.1 Structure of Euclidean Reflection Groups

This section follows AB8, cite in final version
Let W be an affine reflection group, naturally acting on the Euclidean space V, with fixed set of affine hyperplanes $\mathcal{H}$. We fix an arbitrary chamber C as our fundamental chamber, S be the

set of reflection with respect to its walls. Recall for an n-dimensional Euclidean space V, we have the isomorphism $\mathrm{AO}(V) \cong V \rtimes \mathrm{O}(V)$. Every element $\mathrm{AO}(V)$ represents the composition of an orthogonal linear map with a transition τ_v sending x to $x+v$. Note W can be taken as a subgroup of $\mathrm{AO}(V)$.

Theorem 8.1.1. 1. The hyperplanes $H \in \mathcal{H}$ fall into finitely many classes under parallelism;

2. Let $\overline{W}$ be the image of W under the projection $\eta : \mathrm{AO}(V) \twoheadrightarrow \mathrm{O}(V)$, then $\overline{W}$ is finite.

Proof. (1) Let $\Phi := \{\pm e_H : H \in \mathcal{H}\}$, where e_H is the unit vector normal to H. We will show $\angle(e_1, e_2)$ can take finitely many values. Let $H_1, H_2 \in \mathcal{H}$ normal to e_1, e_2. If they are parallel, then $\angle(e_1, e_2) = \pi$. So we may assume they intersect and take $x \in H_1 \cap H_2$. Since W is transitive on chambers, and by the fact that chambers D, D' is separated by wall H' if and only if wD, wD' is separated by wH', we can choose $w \in W$ with $wx \in \overline{C}$. Then $\overline{C}$ intersects with wH_1 and wH_2, with vectors $\bar{w}e_1, \bar{w}e_2$ normal to them, where $\bar{w} = \eta(w)$. $\angle(e_1, e_2) = \angle(\bar{w}e_1, \bar{w}e_2)$ since $\bar{w}$ is an isometry. C has finitely many walls so there are only finitely many possibilities for angles between two vectors in Φ, hence it is finite as we are in a finite-dimensional space.
(2) Note Φ we defined is stable under action of $\overline{W}$ and reflection with respect to the hyperplanes they are perpendicular to generates $\overline{W}$. We show the natural homomorphism $\sigma : W \to S(\Phi)$ is injective. Suppose $w \in \ker(\sigma)$, w fixes all vectors of Φ. But then $V_0 = \mathrm{Span}(\Phi)$ is also fixed by w. Write the vector space V as $V_0 \oplus V_1$ for V_1 some subspace of V. Then since every reflection s_e fixes every point of V_1, so does w. We then conclude w is identity hence $\overline{W}$ is finite. $\square$

Definition 8.1.2. We say W is essential if intersection of hyperplanes H $s_H \in \overline{W}$ is a singleton.

Definition 8.1.3. A Coxeter group is irreducible if its Coxeter diagram is connected.

For a general affine reflection group, we can reduce it to essential, irreducible cases. If it is irreducible, then there are sets of hyperplanes $\mathcal{H}_i$, each hyperplane in $\mathcal{H}_i$ is orthogonal to the ones in other $\mathcal{H}_j$'s. We can decompose the space into direct sum $V_1 \oplus \cdots \oplus V_n$, where each V_i corresponds to $\mathcal{H}_i$, and the corresponding reflection group W_i is irreducible in V_i. If W_i is not essential, we can quotient the intersection of all hyperplanes out to get an essential one.

Theorem 8.1.4. Assume W is essential and irreducible in n-dimensional vector space V, then one of the following is true:

1. W is finite, C has n walls;
2. W is infinite, C has $n+1$ walls, and vectors perpendicular to any n walls form a basis. $\overline{C}$ is compact.

Proof. We first count the number of walls. Suppose the walls of C are $H_1, \ldots, H_k$, with corresponding normal vectors $e_1, \ldots, e_k$ pointing to C. Since W is essential, $e_1, \ldots, e_k$ is a spanning set hence $k \geq n$.
Suppose $k = n$. Then the intersection of all H_i's is a singleton. By shifting the planes we may assume $x = 0$. Hence all H_i's are linear and W is a finite reflection group.
Suppose $k > n$, then the list of vectors must be linearly dependent. Choose an index set I such that

$$\sum_{i \in I} \lambda_i e_i = 0$$

such that $\lambda_i \neq 0$ for all $i \in I$. We show $I = \{1, \ldots, k\}$. Suppose not, then let $J = \{1, \ldots, k\} \backslash I$. Starting from $\sum_{i \in I} \lambda_i e_i = 0$, move the negative coefficients terms to the right

$$\sum_{i \in K} \lambda_i e_i = \sum_{i \in L} -\lambda_i e_i$$

where K, L is a partition of I. Then consider the inner product of left with right, by Reference the theorem regarding inner product, it is non-positive. But it is an inner product of two same vectors, then the summation is the 0 vector.
Choose $j \in J$ and take inner product

$$\sum_{i \in K} \lambda_i \langle e_i, e_j \rangle = 0$$

For $i \in K$ λ_i are positive and $\langle e_i, e_j \rangle$ are non-positive, we must have the case that $\langle e_i, e_j \rangle = 0$, which implies $s_{e_i} s_{e_j}$ has order 2. Then same thing holds for index set L. We then conclude the graph of I and J in Coxeter diagram are disconnected, contradicting irreducibility.
C now has $n+1$ walls, so C is a bounded set. In particular $\overline{C}$ is compact. Since W is simply transitive, $\bigcup_{w \in W} w\overline{C}$ is compact if W is finite, but this union covers V so W is infinite. □

Definition 8.1.5. A Euclidean reflection group is an essential, infinite, irreducible, affine reflection group.

For the rest of this section, let W be a Euclidean reflection group. Let T be the kernel of the projection above restricted on W, then we have a short exact sequence

$$1 \to T \to W \to \overline{W} \to 1$$

Proposition 8.1.6. There exists point $x \in V$ such that its group of stabilizers W_x is isomorphic to $\overline{W}$. In particular, $W \cong T \rtimes W_x$.

Proof. By Theorem 8.1.1, let $\overline{\mathcal{H}}$ be the set of linear hyperplanes parallel to some affine planes of $\mathcal{H}$. $\overline{W}$ is generated by reflection about these hyperplanes. Choose n hyperplanes $H_1, \ldots, H_n$ from $\mathcal{H}$, each parallel to a distinct linear hyperplane of these walls and take their intersection, which is a single point x by Theorem 8.1.4. Linear parts of $s_{H_1}, \ldots s_{H_n}$ generates $\overline{W}$ and since translations have no fixed points, W_x bijects to $\overline{W}$. □

By Proposition 8.1.6, with possibly shifting we can assume 0 is a special point and hence $W \cong T \rtimes W_0 \cong T \rtimes \overline{W}$. We can then identify T by

$$L := \{v \in V : \tau_v \in W\}$$

as a additive group.
We then have $W \cong L \rtimes \overline{W} \leq V \rtimes \mathrm{O}(V) \cong \mathrm{AO}(V)$ We next show L is a lattice i.e. L is in the form of $\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ for $\{e_i\}_{1 \leq i \leq n}$ a basis of V

Lemma 8.1.7. Non-identity elements of L are bounded away from 0, i.e. L is a discrete subgroup of V as a topological group.

Proof. Pick $x \in C$ of the fundamental chamber. Since W is simply transitive on chambers, so is L. Let

$$U = \{v \in V : x + v \in C\}$$

U is a neighbourhood of 0 since C is open. Thus $U \cap L = \{0\}$, L is bounded away from 0. □

The next lemma is from theory of topological groups:

Lemma 8.1.8. If L is a discrete subgroup of the additive group of a finite-dimensional $\mathbb{R}$-vector space, then $L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_r$ for some linearly independent vectors $e_1, \ldots, e_r$.

Proof. We prove by induction on dim$V$. Assume $L \neq 0$, by discreteness, choose $e \in L$ of minimal length δ. $L \cap \mathbb{R}e$ must contain e. Operation of L is the vector addition so $\mathbb{Z}e \subseteq L \cap \mathbb{R}e$. If $(L \cap \mathbb{R}e) \backslash \mathbb{Z}e$ is nonempty, then we would have a contradiction with minimality of length of e. So $L \cap \mathbb{R}e = \mathbb{Z}e$.
For any element $v \in L\backslash \mathbb{Z}e$, we can find $w \in \mathbb{R}e$ such that distance fo v to $\mathbb{R}e$ equals $d(v, w)$. Let $u \in \mathbb{Z}e$ be of minimal distance to v. Then $d(v, w) \geq |d(v, u) - d(w, v)| \geq |\delta - \delta/2| = \delta/2$ by triangle inequality.
Now we consider the quotient group $L/\mathbb{Z}e$ as the subgroup of additive group of the quotient space $V/\mathbb{R}e$. Since we have the canonical isomorphism $(\mathbb{R}e)^{\perp} \to V/\mathbb{R}e$, define a map $V \to (\mathbb{R}e)^{\perp}$ by $v \mapsto v^{\perp}$. Given any two cosets $[v], [w] \in V/\mathbb{R}e$, define $d([v],[w]) = ||v^{\perp} - w^{\perp}||$, which clearly a metric. The above is actually showing $L/\mathbb{Z}e$ is a discrete subgroup of $V/\mathbb{R}e$. The lemma follows from induction. $\square$

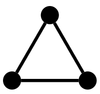
We are now left to show the set of linearly independent vectors given in Lemma 8.1.8 is a basis of V.

Lemma 8.1.9. In the context of Lemma 8.1.8, $r = n$.

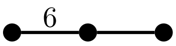
Proof. Since for any $v \in V$, there is $w \in W$ and $y \in \overline{C}$ such that $v = wy$. w is $\tau_v \overline{w}$ for $\overline{w} \in \overline{W}$ and $\tau_v \in L$. Thus v is equivalent to some $y \in \bigcup_{\overline{w} \in \overline{W}} \overline{w}C$ modulo L. It is a compact set since it is a finite union of compact ones. L is a lattice thus $r = n$. $\square$

We end this section by giving some examples of Euclidean reflection groups in low dimension vector spaces.
For $\mathbb{R}^1$, there is only one finite reflection group S_2 with hyperplane 0, we can add 1 and these two points generates an affine reflection group, each element corresponds to a reflection about $k \in \mathbb{Z}$. This is called D_∞ with Coxeter diagram

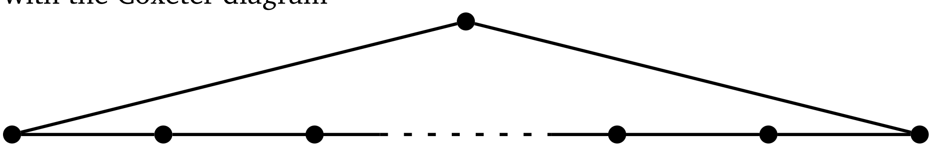
This is isomorphic to $\mathbb{Z} \rtimes S_2$.
For $\mathbb{R}^2$, for example we can tile the whole space with equilateral triangles. So let it be the chamber and we have a diagram



This group is isomorphic to $\mathbb{Z}^2 \rtimes D_6$.
We can tile with hexagon but themselves cannot be chambers, but we can split each into 12 30-60-90 triangles. Let it be the chamber and we have a diagram



which is the group $\mathbb{Z}_2 \rtimes D_{12}$.
In fact, for most finite reflection groups $\overline{W}$, essential an irreducible, have affine analogous. For these groups, let V be the vector space they are acting on of dimension n. We can realise $V \rtimes \mathrm{O}(V)$ as an inner semidirect product of $\mathrm{AO}(V)$. Define L to be the n-rank lattice of V. Then we can define an inner semidirect product $L \rtimes \overline{W}$ which gives us the affine analogue of $\overline{W}$. An example is the finite reflection groups of type A_n, which can be realised as reflection about planes $x_i - x_j = 0$ $(i \neq j)$ in the vector space $V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum x_i = 0\}$ with $\mathcal{H} := \{x_i - x_j = k : k \in \mathbb{Z}, i \neq j\}$. One can check $\mathcal{H}$ in fact gives a Euclidean reflection group with the Coxeter diagram



with n nodes in the graph.

A non-example is of type $I_2(m)$ except for some specific m. There is not way to cover the whole plane using reflections using triangle with an angle $< 2\pi/12$ or of $2\pi/7$ for example.

8.2 Some Model Theory of Euclidean Reflection Groups

This section follows MPS22, cite in final version

Definition 8.2.1. We say a structure $\mathcal{N}$ is interpretable in $\mathcal{M}$ if:

1. The underlying set A is $\{\overline{x} \in M^k : \mathcal{M} \models \phi(\overline{x})\}$ for some formula ϕ.
2. For every function symbols $f(\overline{x})$ of $\mathcal{N}$, there is a formula ϕ such that $\mathcal{M} \models \phi(\overline{x}, \overline{y})$ if and only if $\mathcal{N} \models f(\overline{x}) = \overline{y}$.

Definition 8.2.2. We say the theory of a group G is decidable if there is an algorithm such that for all sentences ϕ, it can decide whether ϕ is true in G.

For a definition of sentence and algorithm, see Cite appendix

Proposition 8.2.3. Euclidean reflection group G is definable in the abelian group $\mathbb{Z}$ with finitely many parameters.

This result follows from the following two lemmas:

Lemma 8.2.4. Let $G \cong \mathbb{Z}^d \rtimes_\sigma Q$ where $1 \leq d < \omega$, Q is some finite group, and $\theta : Q \to \mathrm{Aut}(\mathbb{Z}^d)$ a group homomorphism. G is interpretable in the structure $\mathcal{M} = (\mathbb{Z}^d, +, \pi_x)_{x \in Q}$ with finitely many parameters, where $+$ is the normal vector addition and $\pi_x := \sigma(x)$

Proof. We enumerate $\{\pi_x : x \in Q\}$ as $t_0, \ldots, t_{n-1}$ and $t_i := (i, 0, \ldots, 0)$ a d-tuple. Let $G = (\mathbb{Z}^d \times Q, \cdot)$. We represent the universe of G by a tuple (a, t_i) where $a \in M$ and $0 \leq i < n$. We need to define the group operation $(a_i, t_i) \cdot (a_j, t_j) = (a_i +_{\mathbb{Z}^d} t_i(a_j), t_i \cdot_Q t_j)$ in $\mathcal{M}$. $t_i \cdot_Q t_j$ is definable since Q in finite, we can enumerate all possible product of all elements in Q. $a_i +_{\mathbb{Z}^d} t_i(a_j)$ is definable since $+$ is a built-in function symbol of $\mathcal{M}$ and t_i is same with π_x for some x. Again this claim follows from enumerating all possibilities of t_i's. $\square$

Lemma 8.2.5. $(\mathbb{Z}^d, +, \pi_x)_{x \in Q}$ is interpretable in $(\mathbb{Z}, +, 0)$ with finitely many parameters.

Proof. Since there are finitely many π_x's, it suffices to show we can define π_x in $(\mathbb{Z}, +, 0)$ for some $x \in Q$. π_x is an automorphism of $\mathbb{Z}^d$, which is an $d \times d$ invertible matrix A over $\mathbb{Z}$. $\pi_x(\overline{b}) = A\overline{b}$ is a matrix multiplication. We only need to show we can define the element-wise multiplication part is definable. To show this, note that we all elements in A is bounded. The multiplication of $n \times b$ where n is an entry of A and b is an arbitary integer, can be realised as

$$\underbrace{b + b + \cdots + b}_{n \text{ times}}$$

if n is positive and otherwise minus b $-n$ times. $\square$

Theorem 8.2.6. Let W be a Euclidean reflection group. Then $Th(W)$ is decidable.

Proof. W is interpretable in the structure $(\mathbb{Z}, +, 0)$ with finitely parameters by Proposition 8.2.3. But all these parameters are integers, so it is interpretable in the structure $(\mathbb{Z}, +, 0, 1)$ with no parameters. We then can translate every sentence in W back to a sentence of $(\mathbb{Z}, +, 0, 1)$. Since the Presburger arithmetic $(\mathbb{Z}, +, <, 0, 1)$ is decidable, as a definable structure in its reduct, $Th(W)$ is also decidable. $\square$

A Model Theory Preliminaries

We fix our language $\mathcal{L} = (e, \cdot)$ to be the language of groups.

Definition A.0.1. The set of $\mathcal{L}$-terms is the smallest set $\mathcal{T}$ such that

1. variables $x_i \in \mathcal{T}$ for $i = 0, 1, \dots$;
2. if $t_1, t_2 \in \mathcal{T}$ then $(t_1 \cdot t_2) \in \mathcal{T}$

Definition A.0.2. The set of $\mathcal{L}$-formulas is the smallest set $\mathcal{F}$ such that

1. all atomic formulas $(t_1 = t_2) \in \mathcal{F}$ for $t_1, t_2 \in \mathcal{T}$;
2. $\mathcal{F}$ is closed under $\neg$ ("not"), $\vee$ ("or"), $\wedge$ ("and"), $\rightarrow$ ("implies"), $\forall$ ("for all"), $\exists$ ("there exists").

By drawing a truth table or otherwise, it suffices to define the satisfaction of $\wedge$, $\neg$, and quantifiers for $\mathcal{L}$-formulas.

Definition A.0.3. We say a formula ϕ has x as a free variable if x does not appear in any quantifiers of ϕ.

Definition A.0.4. Let ϕ be a formula with free variables $x_1, \dots, x_n$. We inductively define a group $G \models \phi$ as follows.

1. If $\phi(\overline{x})$ is $t_1(\overline{x}) = t_2(\overline{x})$, then $G \models \phi$ if t_1 and t_2 are exactly the same term.
2. If $\phi(\overline{x})$ is $\neg\psi(\overline{x})$, then $G \models \phi(\overline{x})$ if $G \not\models \psi(\overline{x})$.
3. If $\phi(\overline{x})$ is $\psi_1(\overline{x}) \wedge \psi_2(\overline{x})$, then $G \models \phi(\overline{x})$ if $G \models \psi_1(\overline{x})$ and $G \models \psi_2(\overline{x})$.
4. If $\phi(\overline{x})$ is $\forall y \psi(\overline{x}, y)$, then $G \models \phi(\overline{x})$ if for all $a \in G$ $G \models \psi(\overline{x}, a)$.
5. If $\phi(\overline{x})$ is $\exists y \psi(\overline{x}, y)$, then $G \models \phi(\overline{x})$ if there is $a \in G$ $G \models \psi(\overline{x}, a)$.

If $G \models \phi(\overline{a})$ we say $\phi(\overline{a})$ is true in G

Definition A.0.5. We say a formula ϕ is a sentence if it does not have free variables.

To make the term algorithm clear, we recall definitions from computability theory.
We have infinitely many initial configuration of the register $R_1, R_2, \dots$, each is a nonnegative integer.

Definition A.0.6. A register machine program is a finite sequence of instructions $I_1, \dots, I_M$, where each I_j is one of the following:

1. Z(n): set R_n to zero;
2. S(n): increase R_n by one;
3. T(n,m): set R_n to be R_m;
4. J(n,m,s), where $1 \leq s \leq M$: if $R_n = R_m$, then go to I_s, otherwise go to the next instruction;
5. HALT

and I_m is HALT

Definition A.0.7. Suppose $A \subseteq \mathbb{N}^k$. $f : A \to \mathbb{N}$ is computable if there is a register machine program P such that:

1. If $x \in A$, then P does not halt on input x;
2. If $x \in A$, then P halts on input x with $R_1 = f(x)$.

Definition A.0.8. We say a set $A \subseteq \mathbb{N}^k$ if the function

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

is computable.

We then can say a theory of G is decidable if the set of true sentences in G. We can make "input of ϕ" more precise using Gödel's coding.

Theorem A.0.9. The theory of $\mathbb{Z}$ in the language of $(+, 0, 1, <, \equiv_p)$ where $\equiv_p$ is the relation of "two integers are same modolu p" is complete, admits quantifier elimination hence decidable.

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