0.1 Dihedral groups

The dihedral group, D_{2n}, is the group of symmetries of a regular n-gon. Its standard presentation is given by

$$\langle r, s \mid r^n = s^2 = e,\ (rs)^2 = e \rangle$$

where r is a rotation of $2\pi/n$ and s is a reflection.

Let l_1 and l_2 be two reflection axes with an angle θ between l_1 and l_2, and s_1 and s_2 be the respective reflections. After some algebra, the composition $s_1 s_2$ turns out to be a counterclockwise rotation through 2θ.

Therefore, an alternative presentation of D_{2n} is given by

$$\langle s_1, s_2 \mid s_1^2 = s_2^2 = e,\ (s_1 s_2)^n = e \rangle$$

where s_1 and s_2 are adjacent reflections.

This shows that D_{2n} is an example of a finite reflection group.

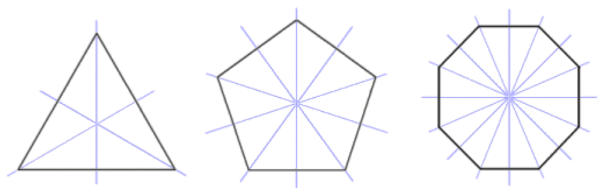
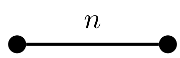


Figure 1 The lines of symmetry of a regular 3-,5- and 8-gon.

Example 0.1.1. The Coxeter diagram for the symmetries of a regular n-gon, also known as $I_2(n)$, looks like

$I_2(n)$



$(n \geq 4)$

The precise meaning of these Coxeter diagrams will be given in **??**.

Theorem 0.1.2. Let $G \curvearrowright X$ be an action of a finite group G on a finite set X. Then the number of G-orbits in X is given by:

$$\text{Number of orbits} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

where $\text{Fix}(g) = \{x \in X \mid g \cdot x = x\}$.

Example 0.1.3. **[?]** How many distinguishable necklaces can be made using seven different colored beads of the same size?
Let X be the 7! possible arrangements. The necklace can be turned over (a reflection) as well as rotated so we consider the dihedral group D_{14} acting on X. Using the previous theorem,

$$\text{Number of orbits} = \frac{7!}{14} = 360$$

as only the identity leaves any arrangement fixed.

Theorem 0.1.4. Let p be a prime. Then any group G of order $2p$ is isomorphic to either the cyclic group C_{2p} or the dihedral group D_{2p}.

Proof. By Cauchy's theorem, there exists an element $a \in G$ of order p and an element $b \in G$ of order 2. Let

$$H = \langle a \rangle,$$

so H is a subgroup of G of index 2, so $H \trianglelefteq G$.

Since $H \trianglelefteq G$, conjugation by b sends H to itself. Thus, there exists $k \in \{1, 2, \dots, p-1\}$ such that

$$bab^{-1} = a^k.$$

Applying conjugation by b twice to a gives

$$a = b^2ab^{-2} = b(bab^{-1})b^{-1} = ba^kb^{-1} = (bab^{-1})^k = (a^k)^k = a^{k^2}.$$

Therefore,

$$a = a^{k^2} \implies a^{k^2-1} = e.$$

Since a has order p, this implies

$$p \mid (k^2 - 1),$$

or equivalently,

$$k^2 \equiv 1 \pmod p.$$

Because p is prime, this implies

$$k \equiv \pm 1 \pmod p.$$

- If $k \equiv 1$, then

  $$bab^{-1} = a,$$

  and b commutes with a. Hence G is abelian, and since a has order p and b has order 2, G is cyclic of order $2p$.

- If $k \equiv -1$, then

  $$bab^{-1} = a^{-1},$$

  which is the defining relation for the dihedral group D_{2p}:

  $$D_{2p} = \langle a, b \mid a^p = e, b^2 = e, bab = a^{-1} \rangle.$$

Thus, G is isomorphic to either C_{2p} or D_{2p}. □

0.2 The Infinite dihedral group

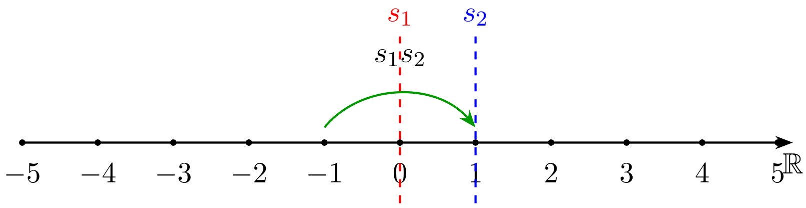
We can also consider the group generated by two reflections s_1, s_2 with order of s_1s_2 infinite i.e.

$$D_\infty = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \rangle$$

This is the infinite dihedral group denoted by D_∞. Note every finite dihedral group is a quotient group of D_∞:

$$D_n \cong D_\infty / \langle (s_1s_2)^n \rangle$$

D_∞ can be taken as the symmetry group of the real line, with all integers distinguished. The generators can be any two consecutive distinguished points on that line. This is an example of an affine reflection group which we will define later in **??**, and also the simplest example.



1 Platonic solids and reflections

1.1 Platonic solids

Polyhedron		Vertices	Edges	Faces
tetrahedron		4	6	4
cube / hexahedron		8	12	6
octahedron		6	12	8
dodecahedron		20	30	12
icosahedron		12	30	20

Figure 2 The five Platonic solids.

Definition 1.1.1. A polyhedron is regular if its faces are regular polygons, all with the same number of sides, and also each vertex belongs to the same number of edges.

Theorem 1.1.2. [?] The only regular convex polyhedra are the five Platonic solids.

Proof. Before writing the proof, we introduce some notations:
V, the number of vertices;
E, the number of edges;
F, the number of faces;
n, the number of sides on a face;
r, the number of edges to which each vertex belongs.
Observe that

$$2E = nF \tag{1}$$

and

$$2E = rV \tag{2}$$

(1) comes from counting the number of pairs (e, f) where e is an edge and f is a face and e lies on f; (2) comes from counting the number of pairs (v, e) where v is a vertex and v lies on e. Substitute into Euler's formula, we get

$$\frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E} \tag{3}$$

Now $n \geq 3$, as a polygon must have at least 3 sides and $r \geq 3$, since in a polyhedron a vertex must belong to at least 3 edges. By (3), we can't have both $n \geq 4$ and $r \geq 4$, since this would make the left-hand side of (3) at most $\frac{1}{2}$. It follows that either $n = 3$ or $r = 3$. If $n = 3$, then (3) becomes

$$\frac{1}{r} = \frac{1}{6} + \frac{1}{E} \tag{4}$$

The right-hand side is greater than $\frac{1}{6}$, and hence $r < 6$. Therefore, $r = 3$, 4 or 5 and $E = 6$, 12 or 30, respectively. If $r = 3$, (3) becomes

$$\frac{1}{n} = \frac{1}{6} + \frac{1}{E} \tag{5}$$

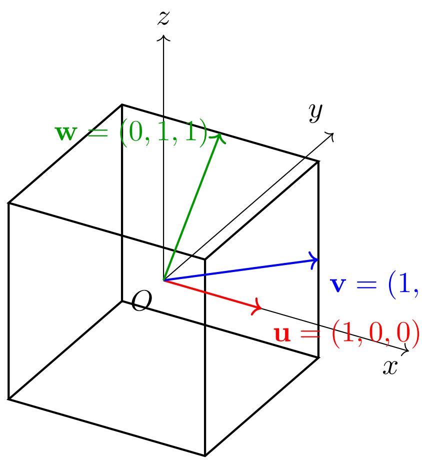
Similarly, $n = 3, 4$ or 5 and $E = 6, 12$ or 30, respectively. These parameters coincide with those in the table above. $\square$

Example 1.1.3. Let G be the group of symmetries of a dodecahedron. What is $|G|$?
Let G act on the 12 faces of the dodecahedron and fix a face. There are $|D_{10}| = 10$ symmetries which fix this face and our action is clearly transitive. By Orbit-Stabiliser theorem, $|G| = 10 \times 12 = 120$. Alternatively, this can be done by considering the fundamental domain, which is a triangle that uniquely determines the reflection. There are 120 such triangles.

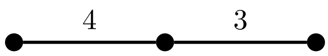
We would like to study the group of symmetries of these Platonic solids.

1.2 Cube

We start by investigating the cube.



As, seen before, $s_{\mathbf{u}}$ and $s_{\mathbf{v}}$ generates the symmetries of a square. It turns out that $\mathbf{w} = (0, 1, 1)$, orthorgonal to $\mathbf{u}$, is a clever third and sufficient choice of vector, which gives $(s_{\mathbf{u}} s_{\mathbf{w}})^2 = e$. Moreover, $\mathbf{v} \cdot \mathbf{w} = \frac{1}{2}$ so the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\frac{\pi}{3}$, hence $(s_{\mathbf{v}} s_{\mathbf{w}})^3 = e$. Thus the Coxeter diagram for the symmetries of the cube, also known as BC_3, looks like



Equivalently, recall that this is equivalent to the group presentation

$$\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = e, \ (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

1.3 Dual

Before moving on to the other solids, we first introduce the concept of dual.

Definition 1.3.1. The dual of a Platonic solid is a new Platonic solid where the faces and vertices are interchanged with those of the original. This can be constructed by connecting the centers of each face of the solid, inscribing this new dual polyhedron within the original solid.

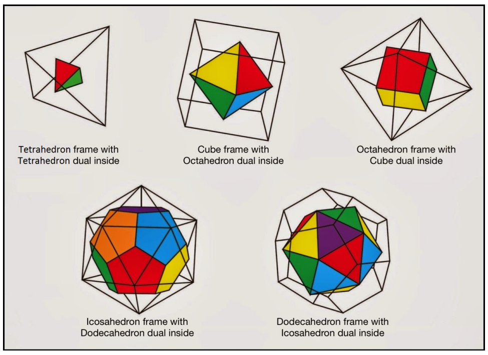
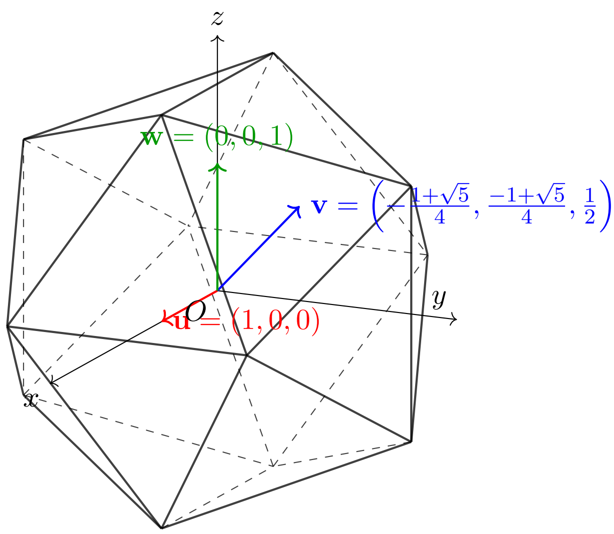


Figure 3 Duals of each Platonic solid.

Remark 1.3.2. The tetrahedron is self-dual. The cube and the octahedron form a dual pair. The dodecahedron and the icosahedron form a dual pair.

Remark 1.3.3. A polyhedron and its dual have the same planes of symmetry, so they have the same Coxeter diagram. For example, the Coxeter diagram for the symmetries of the octahedron is again BC_3 as seen before.

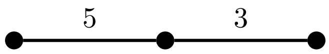
1.4 Dodecahedron and isocahedron



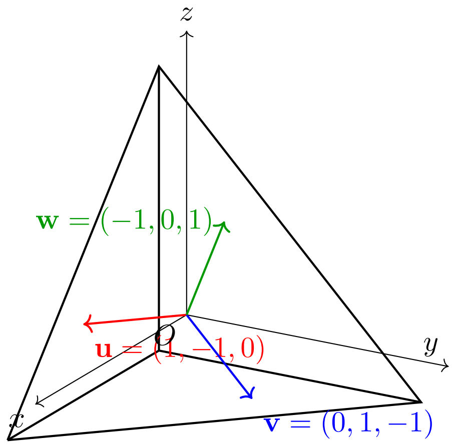
The group presentation is

$$\left\langle s_1, s_2, s_3 \;\middle|\; s_1^2 = s_2^2 = s_3^2 = e,\ (s_1 s_2)^3 = (s_2 s_3)^5 = (s_1 s_3)^2 = e \right\rangle$$

and the Coxeter diagram, also known as H_3, is



1.5 Tetrahedron



The group presentation is

$$\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = e,\ (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

and the Coxeter diagram, also known as A_3 (careful, this is NOT the alternating group A_3), is