0.1 Structure of Euclidean Reflection Groups

This section follows [**?**].

Let W be an affine reflection group, naturally acting on the Euclidean space V, with fixed set of affine hyperplanes $\mathcal{H}$. We fix an arbitrary chamber C as our fundamental chamber, S be the set of reflection with respect to its walls. For an n-dimensional Euclidean space V, we have the isomorphism $\mathrm{AO}(V) \cong V \rtimes \mathrm{O}(V)$. Every element $\mathrm{AO}(V)$ represents the composition of an orthogonal linear map with a transition τ_v sending x to $x+v$. Note W can be taken as a subgroup of $\mathrm{AO}(V)$.

From Section **??**, we recall some facts of affine reflection groups:

1. W is generated by S.
2. W is simply transitive on chambers.
3. $H \in \mathcal{H}$ if and only if $s_H \in W$
4. $\langle e_{s_i}, e_{s_j} \rangle = -\cos(\frac{\pi}{m_{i,j}})$, where e_{s_i}, e_{s_j} are two unit vectors normal to some walls of C, pointing into C from the walls.

Theorem 0.1.1. 1. The hyperplanes $H \in \mathcal{H}$ fall into finitely many classes under parallelism;

2. Let $\overline{W}$ be the image of W under the projection $\eta : \mathrm{AO}(V) \twoheadrightarrow \mathrm{O}(V)$, then $\overline{W}$ is finite.

Proof. (1) Let $\Phi := \{\pm e_H : H \in \mathcal{H}\}$, where e_H is the unit vector normal to H. We will show $\angle(e_1, e_2)$ can take finitely many values. Let $H_1, H_2 \in \mathcal{H}$ normal to e_1, e_2. If they are parallel, then $\angle(e_1, e_2) = \pi$. So we may assume they intersect and take $x \in H_1 \cap H_2$. Since W is transitive on chambers, and by the fact that chambers D, D' is separated by wall H' if and only if wD, wD' is separated by wH', we can choose $w \in W$ with $wx \in \overline{C}$. Then $\overline{C}$ intersects with wH_1 and wH_2, with vectors $\bar{w}e_1, \bar{w}e_2$ normal to them, where $\bar{w} = \eta(w)$. $\angle(e_1, e_2) = \angle(\bar{w}e_1, \bar{w}e_2)$ since $\bar{w}$ is an isometry. From the inner product formula we can deduce the angle between two distinct vectors cannot be smaller than $\pi/2$, so C has finitely many walls. Then there are only finitely many possibilities for angles between two vectors in Φ, hence it is finite as we are in a finite-dimensional space.

(2) Note Φ we defined is stable under action of $\overline{W}$ and reflection with respect to the hyperplanes they are perpendicular to generates $\overline{W}$. We show the natural homomorphism $\sigma : \overline{W} \to S(\Phi)$ is injective. Suppose $w \in \ker(\sigma)$, w fixes all vectors of Φ. But then $V_0 = \mathrm{Span}(\Phi)$ is also fixed by w. Write the vector space V as $V_0 \oplus V_1$ for V_1 some subspace of V. Then since every reflection s_e fixes every point of V_1 and W is generated by reflection of walls of C, so does w. We then conclude w is identity hence $\overline{W}$ is finite. $\square$

Definition 0.1.2. We say W is essential if intersection of hyperplanes H $s_H \in \overline{W}$ is a singleton.

Definition 0.1.3. A Coxeter group is irreducible if its Coxeter diagram is connected.

For a general affine reflection group, we can reduce it to essential, irreducible cases. If it is irreducible, then there are sets of hyperplanes $\mathcal{H}_i$, each hyperplane in $\mathcal{H}_i$ is orthogonal to the ones in other $\mathcal{H}_j$'s. We can decompose the space into direct sum $V_1 \oplus \cdots \oplus V_n$, where each V_i corresponds to $\mathcal{H}_i$, and the corresponding reflection group W_i is irreducible in V_i. If W_i is not essential, we can quotient the intersection of all hyperplanes out to get an essential one.

Theorem 0.1.4. Assume W is essential and irreducible in n-dimensional vector space V, then one of the following is true:

1. W is finite, C has n walls;

2. W is infinite, C has $n+1$ walls, and vectors perpendicular to any n walls form a basis. $\overline{C}$ is compact.

Proof. We first count the number of walls. Suppose the walls of C are $H_1, \dots, H_k$, with corresponding normal vectors $e_1, \dots, e_k$ pointing to C. Since W is essential, $e_1, \dots, e_k$ is a spanning set hence $k \geq n$.

Suppose $k = n$. Then the intersection of all H_i's is a singleton. By shifting the planes we may assume $x = 0$. Hence all H_i's are linear and W is a finite reflection group.

Suppose $k > n$, then the list of vectors must be linearly dependent. Choose an index set I such that

$$\sum_{i \in I} \lambda_i e_i = 0$$

such that $\lambda_i \neq 0$ for all $i \in I$. We show $I = \{1, \dots, k\}$. Suppose not, then let $J = \{1, \dots, k\} \backslash I$.

Starting from $\sum_{i \in I} \lambda_i e_i = 0$, move the negative coefficients terms to the right

$$\sum_{i \in K} \lambda_i e_i = \sum_{i \in L} -\lambda_i e_i$$

where K, L is a partition of I. Then consider the inner product of left with right, since $\langle e_i, e_j \rangle$, it is non-positive. But it is also an inner product of two same vectors, then the summation is the 0 vector.

Choose $j \in J$ and take inner product

$$\sum_{i \in K} \lambda_i \langle e_i, e_j \rangle = 0$$

For $i \in K$ λ_i are positive and $\langle e_i, e_j \rangle$ are non-positive, we must have the case that $\langle e_i, e_j \rangle = 0$, which implies $s_{e_i} s_{e_j}$ has order 2. Then same thing holds for index set L. We then conclude the graph of I and J in Coxeter diagram are disconnected, contradicting irreducibility.

C now has $n+1$ walls, so C is a bounded set. In particular $\overline{C}$ is compact. Since W is simply transitive, $\bigcup_{w \in W} w\overline{C}$ is compact if W is finite, but this union covers V so W is infinite. $\square$

Definition 0.1.5. A Euclidean reflection group is an essential, infinite, irreducible, affine reflection group.

For the rest of this section, let W be a Euclidean reflection group. Let T be the kernel of the projection above restricted on W, then we have a short exact sequence

$$1 \to T \to W \to \overline{W} \to 1$$

Proposition 0.1.6. There exists point $x \in V$ such that its group of stabilizers W_x is isomorphic to $\overline{W}$. In particular, $W \cong T \rtimes W_x$.

Proof. By Theorem 0.1.1, let $\overline{\mathcal{H}}$ be the set of linear hyperplanes parallel to some affine planes of $\mathcal{H}$. $\overline{W}$ is generated by reflection about these hyperplanes. In fact W is generated by $\{s_{\overline{H_1}}, \dots, s_{\overline{H_n}}\}$, with $\overline{H_1}, \dots, \overline{H_n} \in \overline{\mathcal{H}}$. Choose affine hyperplanes $H_1, \dots, H_n$ from $\mathcal{H}$, H_i parallel to $\overline{H_i}$, and take their intersection, which is a single point x by Theorem 0.1.4. Linear parts of $s_{H_1}, \dots s_{H_n}$ generates $\overline{W}$ and since translations have no fixed points, W_x bijects to $\overline{W}$. $\square$

By Proposition 0.1.6, with possibly shifting we can assume 0 is a special point and hence $W \cong T \rtimes W_0 \cong T \rtimes \overline{W}$. We can then identify T by

$$L := \{v \in V : \tau_v \in W\}$$

as a additive group.

We then have $W \cong L \rtimes \overline{W} \leq V \rtimes \mathrm{O}(V) \cong \mathrm{AO}(V)$ We next show L is a lattice i.e. L is in the form of $\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ for $\{e_i\}_{1 \leq i \leq n}$ a basis of V

Lemma 0.1.7. Non-identity elements of L are bounded away from 0, i.e. L is a discrete subgroup of V as a topological group.

Proof. Pick $x \in C$ of the fundamental chamber. Since W is simply transitive on chambers, so is L. Let

$$U = \{v \in V : x + v \in C\}$$

U is a neighbourhood of 0 since C is open. Thus $U \cap L = \{0\}$, L is bounded away from 0. $\square$

The next lemma is from theory of topological groups:

Lemma 0.1.8. If L is a discrete subgroup of the additive group of a finite-dimensional $\mathbb{R}$-vector space, then $L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_r$ for some linearly independent vectors $e_1, \ldots, e_r$.

Proof. We prove by induction on dim$V$. Assume $L \neq 0$, by discreteness, choose $e \in L$ of minimal length δ. $L \cap \mathbb{R}e$ must contain e. Operation of L is the vector addition so $\mathbb{Z}e \subseteq L \cap \mathbb{R}e$. If $(L \cap \mathbb{R}e) \backslash \mathbb{Z}e$ is nonempty, then we would have a contradiction with minimality of length of e. So $L \cap \mathbb{R}e = \mathbb{Z}e$.

For any element $v \in L \backslash \mathbb{Z}e$, we can find $w \in \mathbb{R}e$ such that distance fo v to $\mathbb{R}e$ equals $d(v, w)$. Let $u \in \mathbb{Z}e$ be of minimal distance to v. Then $d(v, w) \geq |d(v, u) - d(w, v)| \geq |\delta - \delta/2| = \delta/2$ by triangle inequality.

Now we consider the quotient group $L/\mathbb{Z}e$ as the subgroup of additive group of the quotient space $V/\mathbb{R}e$. Since we have the canonical isomorphism $(\mathbb{R}e)^\perp \to V/\mathbb{R}e$, define a map $V \to (\mathbb{R}e)^\perp$ by $v \mapsto v^\perp$. Given any two cosets $[v], [w] \in V/\mathbb{R}e$, define $d([v], [w]) = ||v^\perp - w^\perp||$, which clearly a metric. The above is actually showing $L/\mathbb{Z}e$ is a discrete subgroup of $V/\mathbb{R}e$. The lemma follows from induction. $\square$

We are now left to show the set of linearly independent vectors given in Lemma 0.1.8 is a basis of V.

Lemma 0.1.9. In the context of Lemma 0.1.8, $r = n$.

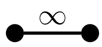
Proof. Since for any $v \in V$, there is $w \in W$ and $y \in \overline{C}$ such that $v = wy$. w is $\tau_v \overline{w}$ for $\overline{w} \in \overline{W}$ and $\tau_v \in L$. Thus v is equivalent to some $y \in \bigcup_{\overline{w} \in \overline{W}} \overline{w}C$ modulo L. It is a compact set since it is a finite union of compact ones. L is a lattice thus $r = n$. $\square$

Combining all these results, we obtain:

Theorem 0.1.10. The Euclidean reflection group W is isomorphic to $\mathbb{Z}^n \rtimes \overline{W}$, where $\overline{W}$ is a finite reflection group.

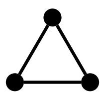
We end this section by giving some examples of Euclidean reflection groups in low dimension vector spaces.

For $\mathbb{R}^1$, there is only one finite reflection group S_2 with hyperplane 0, we can add 1 and these two points generates an affine reflection group, each element corresponds to a reflection about $k \in \mathbb{Z}$. This is called D_∞ with Coxeter diagram :



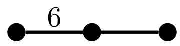
This is isomorphic to $\mathbb{Z} \rtimes S_2$.

For $\mathbb{R}^2$, for example we can tile the whole space with equilateral triangles. So let it be the chamber and we have a diagram:



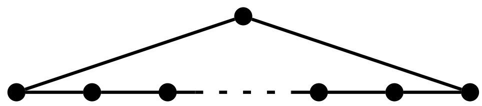
This group is isomorphic to $\mathbb{Z}^2 \rtimes D_6$.

We can tile with hexagon but themselves cannot be chambers, we can split each into 12 30-60-90 triangles. Let them be the chambers and we have a diagram $\tilde{A}_2$:



which is the group $\mathbb{Z}_2 \rtimes D_{12}$.

In fact, for many finite reflection groups $\overline{W}$, essential and irreducible, have affine analogous ones. For these groups, let V be the vector space they are acting on of dimension n. We can realise $V \rtimes \mathrm{O}(V)$ as an inner semidirect product of AO(V). Define L to be the n-rank lattice of V. Then we can define an inner semidirect product $L \rtimes \overline{W}$ which gives us the affine analogue of $\overline{W}$. An example is the finite reflection groups of type A_n, which can be realised as reflection about planes $x_i - x_j = 0$ $(i \neq j)$ in the vector space $V = \{(x_1, \dots, x_n) \in \mathbb{R}^n : \sum x_i = 0\}$ with $\mathcal{H} := \{x_i - x_j = k : k \in \mathbb{Z}, i \neq j\}$. One can check $\mathcal{H}$ in fact gives a Euclidean reflection group with the Coxeter diagram:



with n nodes in the graph.

A non-example is of type $I_2(m)$ except for some specific m. There is not way to cover the whole plane using reflections using triangle with an angle $< 2\pi/12$ or of $2\pi/7$ for example.

0.2 Some Model Theory of Euclidean Reflection Groups

This section follows [**?**].

Definition 0.2.1. We say a structure $\mathcal{N}$ is interpretable in $\mathcal{M}$ if:

1. The underlying set A is $\{\overline{x} \in M^k : \mathcal{M} \models \phi(\overline{x})\}$ for some formula ϕ.

2. For every function symbols $f(\overline{x})$ of $\mathcal{N}$, there is a formula ϕ such that $\mathcal{M} \models \phi(\overline{x}, \overline{y})$ if and only if $\mathcal{N} \models f(\overline{x}) = \overline{y}$.

Definition 0.2.2. We say the theory of a group G is decidable if there is an algorithm such that for all sentences ϕ, it can decide whether ϕ is true in G.

For a definition of a sentence and an algorithm, see Appendix **??**.

Proposition 0.2.3. Euclidean reflection group G is definable in the abelian group $\mathbb{Z}$ with finitely many parameters.

This result follows from the following two lemmas:

Lemma 0.2.4. Let $G \cong \mathbb{Z}^d \rtimes_\sigma Q$ where $1 \leq d < \omega$, Q is some finite group, and $\theta : Q \to \mathrm{Aut}(\mathbb{Z}^d)$ a group homomorphism. G is interpretable in the structure $\mathcal{M} = (\mathbb{Z}^d, +, \pi_x)_{x \in Q}$ with finitely many parameters, where $+$ is the normal vector addition and $\pi_x := \sigma(x)$

Proof. We enumerate $\{\pi_x : x \in Q\}$ as $t_0, \dots, t_{n-1}$ and $t_i := (i, 0, \dots, 0)$ a d-tuple. Let $G = (\mathbb{Z}^d \times Q, \cdot)$. We represent the universe of G by a tuple (a, t_i) where $a \in M$ and $0 \leq i < n$. We need to define the group operation $(a_i, t_i) \cdot (a_j, t_j) = (a_i +_{\mathbb{Z}^d} t_i(a_j), t_i \cdot_Q t_j)$ in $\mathcal{M}$. $t_i \cdot_Q t_j$ is definable since Q in finite, we can enumerate all possible product of all elements in Q. $a_i +_{\mathbb{Z}^d} t_i(a_j)$ is definable since $+$ is a built-in function symbol of $\mathcal{M}$ and t_i is same with π_x for some x. Again this claim follows from enumerating all possibilities of t_i's. $\square$

Lemma 0.2.5. $(\mathbb{Z}^d, +, \pi_x)_{x\in Q}$ is interpretable in $(\mathbb{Z}, +, 0)$ with finitely many parameters.

Proof. Since there are finitely many π_x's, it suffices to show we can define π_x in $(\mathbb{Z}, +, 0)$ for some $x \in Q$. π_x is an automorphism of $\mathbb{Z}^d$, which is an $d \times d$ invertible matrix A over $\mathbb{Z}$. $\pi_x(\bar{b}) = A\bar{b}$ is a matrix multiplication. We only need to show we can define the element-wise multiplication part is definable. To show this, note that there are only finitely many entries in A. The multiplication of $n \times b$ where n is an entry of A and b is an arbitary integer, can be realised as

$$\underbrace{b + b + \cdots + b}_{n \text{ times}}$$

if n is positive and otherwise minus b $-n$ times. $\square$

Theorem 0.2.6. Let W be a Euclidean reflection group. Then $Th(W)$ is decidable.

Proof. W is interpretable in the structure $(\mathbb{Z}, +, 0)$ with finitely parameters by Proposition 0.2.3. But all these parameters are integers, so it is interpretable in the structure $(\mathbb{Z}, +, 0, 1)$ with no parameters. We then can translate every sentence in W back to a sentence of $(\mathbb{Z}, +, 0, 1)$. Since the Presburger arithmetic $(\mathbb{Z}, +, <, 0, 1)$ is decidable, as a definable structure in its reduct, $Th(W)$ is also decidable. $\square$