IMPERIAL

Imperial College London

Department of Mathematics

Second-year Group Research Project

Reflection Groups

Author:
Jialin Li (CID: 02209592)
Yu Coughlin (CID: 02386349)
Kishan Ryley (CID: 02320343)
Chengyan Hu (CID: 02389139)
Heng Wang (CID: 02395274)

Supervisor(s):
Alessio Corti

June 16, 2025

Abstract

In this paper, we first study the geometry of known finite reflection groups, then the structure and classification of finite reflection groups acting on Euclidean spaces, and give an algorithm of word reduction in Coxeter Groups. We establish a structural decomposition of Euclidean reflection groups, and code this in integers, yielding decidability of their first-order theories.

Contents

1 Introduction

In lectures, we have seen how to classify finitely generated abelian groups. A main objective of this report is to classify finite reflection groups.

Recall that a reflection is a linear operator on $\mathbb{R}^n$ which sends some non-zero vector $\mathbf{u}$ to its negative while fixing point-wise the hyperplane $H_{\mathbf{u}}$ orthogonal to $\mathbf{u}$. With the usual inner product, the reader can check that $s_{\mathbf{u}}(\mathbf{v}) = \mathbf{v} - 2\frac{\mathbf{v}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\mathbf{u}$.

We will start by looking at the lower-dimensional reflection groups and proceed to more general ones, introducing some related terminology and ideas along the way. Classification of reflection groups not only describes the symmetries of regular polytopes but also helps understand more complex algebraic structures.

We then focus on affine reflection groups. By relating them to finite reflection groups, we show some finiteness properties of them, which give us a structural theorem on Euclidean reflection groups. From this theorem, we show this structure can be interpreted in Presburger arithmetic, hence deduce the decidability of Euclidean reflection groups.

2 Dihedral groups

2.1 Dihedral groups

The dihedral group, D_{2n}, is the group of symmetries of a regular n-gon. Its standard presentation is given by

$$\langle r, s \mid r^n = s^2 = e,\ (rs)^2 = e \rangle$$

where r is a rotation of $2\pi/n$ and s is a reflection.

Let l_1 and l_2 be two reflection axes with an angle θ between l_1 and l_2, and s_1 and s_2 be the respective reflections. After some algebra, the composition $s_1 s_2$ turns out to be a counterclockwise rotation through 2θ.

Therefore, an alternative presentation of D_{2n} is given by

$$\langle s_1, s_2 \mid s_1^2 = s_2^2 = e,\ (s_1 s_2)^n = e \rangle$$

where s_1 and s_2 are adjacent reflections.

This shows that D_{2n} is an example of a finite reflection group.

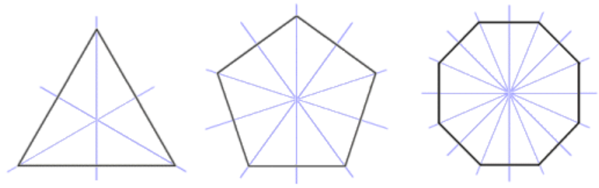
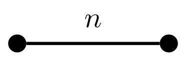


Figure 1 The lines of symmetry of a regular 3-,5- and 8-gon.

Example 2.1.1. The Coxeter diagram for the symmetries of a regular n-gon, also known as $I_2(n)$, looks like

$I_2(n)$  $(n \geq 4)$

The precise meaning of these Coxeter diagrams will be given in 4.2.

Theorem 2.1.2. Let $G \curvearrowright X$ be an action of a finite group G on a finite set X. Then the number of G-orbits in X is given by:

$$\text{Number of orbits} = \frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)|$$

where $\mathrm{Fix}(g) = \{x \in X \mid g \cdot x = x\}$.

Example 2.1.3. [1] How many distinguishable necklaces can be made using seven different colored beads of the same size?
Let X be the 7! possible arrangements. The necklace can be turned over (a reflection) as well as rotated so we consider the dihedral group D_{14} acting on X. Using the previous theorem,

$$\text{Number of orbits} = \frac{7!}{14} = 360$$

as only the identity leaves any arrangement fixed.

Theorem 2.1.4. Let p be a prime. Then any group G of order $2p$ is isomorphic to either the cyclic group C_{2p} or the dihedral group D_{2p}.

Proof. By Cauchy's theorem, there exists an element $a \in G$ of order p and an element $b \in G$ of order 2. Let

$$H = \langle a \rangle,$$

so H is a subgroup of G of index 2, so $H \trianglelefteq G$.

Since $H \trianglelefteq G$, conjugation by b sends H to itself. Thus, there exists $k \in \{1, 2, \ldots, p-1\}$ such that

$$bab^{-1} = a^k.$$

Applying conjugation by b twice to a gives

$$a = b^2ab^{-2} = b(bab^{-1})b^{-1} = ba^kb^{-1} = (bab^{-1})^k = (a^k)^k = a^{k^2}.$$

Therefore,

$$a = a^{k^2} \implies a^{k^2-1} = e.$$

Since a has order p, this implies

$$p \mid (k^2 - 1),$$

or equivalently,

$$k^2 \equiv 1 \pmod{p}.$$

Because p is prime, this implies

$$k \equiv \pm 1 \pmod{p}.$$

- If $k \equiv 1$, then

  $$bab^{-1} = a,$$

  and b commutes with a. Hence G is abelian, and since a has order p and b has order 2, G is cyclic of order $2p$.

- If $k \equiv -1$, then
$$bab^{-1} = a^{-1},$$
which is the defining relation for the dihedral group D_{2p}:
$$D_{2p} = \langle a, b \mid a^p = e, b^2 = e, bab = a^{-1} \rangle.$$

Thus, G is isomorphic to either C_{2p} or D_{2p}. □

2.2 The Infinite dihedral group

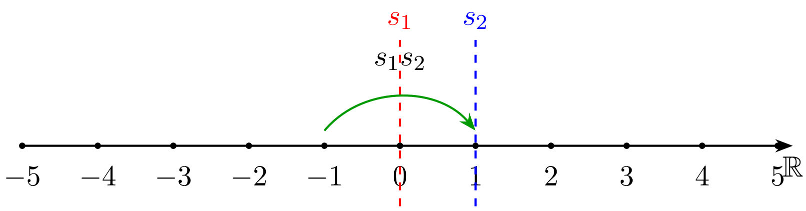
We can also consider the group generated by two reflections s_1, s_2 with order of $s_1 s_2$ infinite i.e.

$$D_\infty = \langle s_1, s_2 \mid s_1^2 = s_2^2 = 1 \rangle$$

This is the infinite dihedral group denoted by D_∞. Note every finite dihedral group is a quotient group of D_∞:

$$D_n \cong D_\infty / \langle (s_1 s_2)^n \rangle$$

D_∞ can be taken as the symmetry group of the real line, with all integers distinguished. The generators can be any two consecutive distinguished points on that line. This is an example of an affine reflection group which we will define later in 4.1, and also the simplest example.



3 Platonic solids and reflections

3.1 Platonic solids

Definition 3.1.1. A polyhedron is regular if its faces are regular polygons, all with the same number of sides, and also each vertex belongs to the same number of edges.

Theorem 3.1.2. [2] The only regular convex polyhedra are the five Platonic solids.

Proof. Before writing the proof, we introduce some notations:
V, the number of vertices;
E, the number of edges;
F, the number of faces;
n, the number of sides on a face;
r, the number of edges to which each vertex belongs.
Observe that

$$2E = nF \tag{1}$$

and

$$2E = rV \tag{2}$$

(1) comes from counting the number of pairs (e, f) where e is an edge and f is a face and e lies on f; (2) comes from counting the number of pairs (v, e) where v is a vertex and v lies on e.
Substitute into Euler's formula, we get

$$\frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E} \tag{3}$$

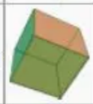
Polyhedron		Vertices	Edges	Faces
tetrahedron		4	6	4
cube / hexahedron		8	12	6
octahedron		6	12	8
dodecahedron		20	30	12
icosahedron		12	30	20

Figure 2 The five Platonic solids.

Now $n \geq 3$, as a polygon must have at least 3 sides and $r \geq 3$, since in a polyhedron a vertex must belong to at least 3 edges. By (3), we can't have both $n \geq 4$ and $r \geq 4$, since this would make the left-hand side of (3) at most $\frac{1}{2}$. It follows that either $n = 3$ or $r = 3$. If $n = 3$, then (3) becomes

$$\frac{1}{r} = \frac{1}{6} + \frac{1}{E} \tag{4}$$

The right-hand side is greater than $\frac{1}{6}$, and hence $r < 6$. Therefore, $r = 3$, 4 or 5 and $E = 6$, 12 or 30, respectively. If $r = 3$, (3) becomes

$$\frac{1}{n} = \frac{1}{6} + \frac{1}{E} \tag{5}$$

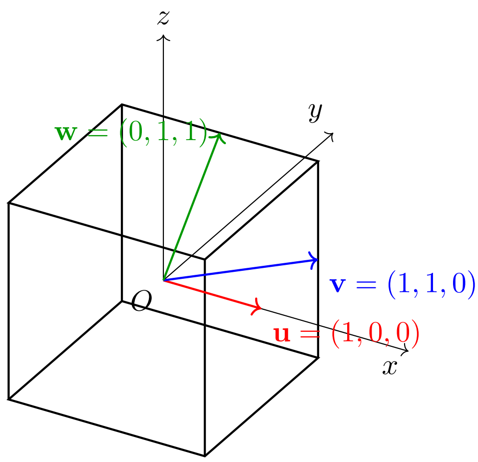
Similarly, $n = 3, 4$ or 5 and $E = 6, 12$ or 30, respectively. These parameters coincide with those in the table above. $\square$

Example 3.1.3. Let G be the group of symmetries of a dodecahedron. What is $|G|$?
Let G act on the 12 faces of the dodecahedron and fix a face. There are $|D_{10}| = 10$ symmetries which fix this face and our action is clearly transitive. By Orbit-Stabiliser theorem, $|G| = 10 \times 12 = 120$. Alternatively, this can be done by considering the fundamental domain, which is a triangle that uniquely determines the reflection. There are 120 such triangles.

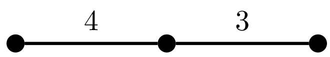
We would like to study the group of symmetries of these Platonic solids.

3.2 Cube

We start by investigating the cube.



As, seen before, $s_{\mathbf{u}}$ and $s_{\mathbf{v}}$ generates the symmetries of a square. It turns out that $\mathbf{w} = (0, 1, 1)$, orthorgonal to $\mathbf{u}$, is a clever third and sufficient choice of vector, which gives $(s_{\mathbf{u}} s_{\mathbf{w}})^2 = e$. Moreover, $\mathbf{v} \cdot \mathbf{w} = \frac{1}{2}$ so the angle between $\mathbf{v}$ and $\mathbf{w}$ is $\frac{\pi}{3}$, hence $(s_{\mathbf{v}} s_{\mathbf{w}})^3 = e$. Thus the Coxeter diagram for the symmetries of the cube, also known as BC_3, looks like



Equivalently, recall that this is equivalent to the group presentation

$$\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = e,\ (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

3.3 Dual

Before moving on to the other solids, we first introduce the concept of dual.

Definition 3.3.1. The dual of a Platonic solid is a new Platonic solid where the faces and vertices are interchanged with those of the original. This can be constructed by connecting the centers of each face of the solid, inscribing this new dual polyhedron within the original solid.

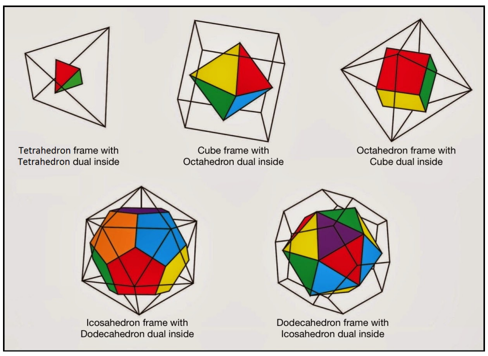
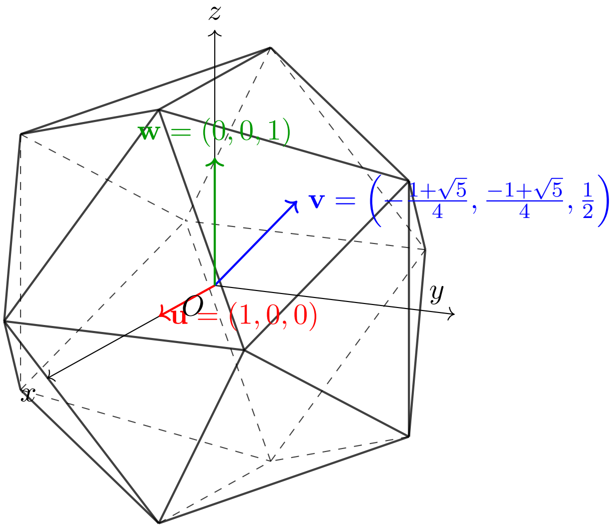


Figure 3 Duals of each Platonic solid.

Remark 3.3.2. The tetrahedron is self-dual. The cube and the octahedron form a dual pair. The dodecahedron and the icosahedron form a dual pair.

Remark 3.3.3. A polyhedron and its dual have the same planes of symmetry, so they have the same Coxeter diagram. For example, the Coxeter diagram for the symmetries of the octahedron is again BC_3 as seen before.

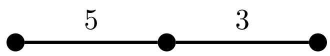
3.4 Dodecahedron and isocahedron



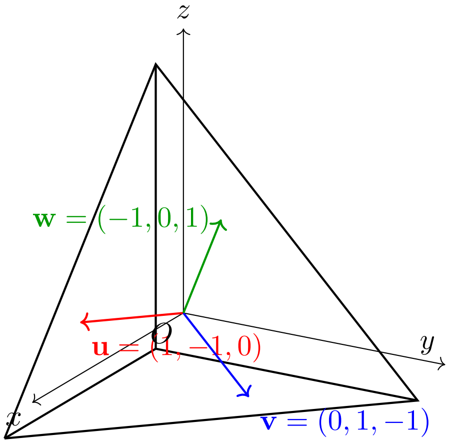
The group presentation is

$$\left\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = e,\ (s_1 s_2)^3 = (s_2 s_3)^5 = (s_1 s_3)^2 = e \right\rangle$$

and the Coxeter diagram, also known as H_3, is



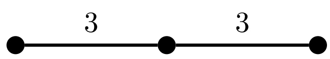
3.5 Tetrahedron



The group presentation is

$$\left\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = e,\ (s_1 s_2)^3 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \right\rangle$$

and the Coxeter diagram, also known as A_3 (careful, this is NOT the alternating group A_3), is



4 Classification of finite reflection groups

We're now going to start examining the algebraic properties of general reflection groups. To do this we will be working in V, denoting $\mathbb{R}^n$ equipped with the standard inner product, our fixed *euclidean space*.

4.1 Affine reflections

This subsection roughly follows [3]. When considering reflections in V, we don't want to just consider those lying in $O(V)$, we are also interested in reflections across some hyperplane not lying through the origin. To tackle this we first define some standard preliminaries:

Definition 4.1.1. A **reflection** in V is a linear transformation $s \in O(V)$ (the orthogonal group of inner product preserving linear maps) with eigenvalues $\{-1, 1\}$ with corresponding dimensions of eigenspaces:

$$\dim E_1 = n - 1 \qquad \dim E_{-1} = 1$$

for linear maps in $O(V)$ this is equivalent to fixing some hyperplane (codimension 1 subspace) and having determinant -1.

Definition 4.1.2. The **group of general affine transformations** of V is semidirect product of $GL(V)$ acting on V

$$GA(V) := V \rtimes GL(V)$$

where V acts on itself by translation. $(u, T) \in GA(V)$ acts on $v \in V$ as $(u, T) \cdot v = u + T(v)$.

Proposition 4.1.3. This is actually an action. Furthermore, the action is transitive.

Proof. Take $(u, T), (w, S) \in GA(V)$ and $v \in V$. Showing this is an action is a direct calculation:

$$(u,T)\cdot((w,S)\cdot v) = (u,T)\cdot(w + S(v)) = u+T(w)+TS(v) = (u+T(w), TS)\cdot v = ((u,T)(w,S))\cdot v$$

from the definition of the semidirect product. Suppose (u, T) and (w, S) act the same on V: as both T and S are linear the two affine transformations send 0 to u and w respectively, thus we have $u = w$; subtracting these equal translations and having equality means the two linear transformations must also be equal. $\square$

We can now consider appropriate affine versions of linear groups, the most important for us will be the **affine orthogonal group** $AO(V)$, which can be equivalently viewed as the subgroup of $GA(V)$ consisting of isometries, or as $V \rtimes O(V)$. We need stricter criteria than just the linear component of our affine transformation be a reflection to suitably capture the notion of a reflection across an affine hyperplane.

Proposition 4.1.4. For a unit normal vector α and some $k \in \mathbb{R}$, reflection across the affine hyperplane $(V, \alpha) = k$ corresponds to the affine transformation $(2k\alpha, s_\alpha)$, where s_α is the reflection along α.

Proof. Call the affine hyperplane H and choose a $v \in V$. The vector orthogonal to H that goes to v has length $k-(v,\alpha)$ so reflecting across H send v to $v+2(k-(v,\alpha))\alpha = 2k\alpha+(v-2(v,\alpha)\alpha) = (2k\alpha, s_\alpha)\cdot v$. $\square$

We call such affine transformations **affine reflections**, and we can now talk about **affine reflection groups** as those generated by affine reflection in the same way we could for orthogonal reflections.

Lemma 4.1.5. For all affine hyperplanes H and affine reflections r, the set rH is also an affine hyperplane.

Proof. Let H be $\{v \in V \mid (v, \alpha) = k\}$ for some $\alpha \in V, k \in \mathbb{R}$. As r is bijective the set rH is equal to $\{w \in V \mid (r^{-1}w, \alpha) = k\}$, by writing $r = (u, T) \in GA(V)$ this can be rewritten as $\{w \in V \mid (w, T^*(\alpha)) = k - (u, \alpha)\}$, an affine hyperplane. □

Lemma 4.1.6. An affine transformations $r = (u, T)$ that fixes some affine hyperplane and is an involution must be an affine reflection.

Proof. As $r^2 = \text{id}$, on 0 we have $r^2(0) = u + T(u) = 0$. So as $r^2 = u + T(u) + T^2$ this means T is also an involution so we can use the primary decomposition $V = E_1 \oplus E_{-1}$ into the eigenspaces of T. Call the hyperplane r fixes H, then for any $h = v_1 + v_{-1} \in H$ (where $v_1, v_{-1} \in V_1, V_{-1}$ respectively) we have $r(v_1 + v_{-1}) = u + T(v_1 + v_{-1}) = u + v_1 - v_{-1} = v_1 + v_{-1}$ therefore $2v_{-1} = u$ so $\dim E_{-1} = 1$ and T is the reflection along $V_{-1} = \langle v_{-1} \rangle$, thus $r = \left(2v_{-1}, s_{v_{-1}}\right)$. □

Proposition 4.1.7. For all affine reflections r, s with s reflecting across the affine hyperplane H, the affine transformation rsr^{-1} is an affine reflection across rH.

Proof. First, notice $\left(rsr^{-1}\right)^2 = \text{id}$ as both r and s are involutions. Also, $rsr^{-1}(rH) = rH$ as s fixes H. By the previous lemma, and earlier described faithfulness of the action, this is sufficient. □

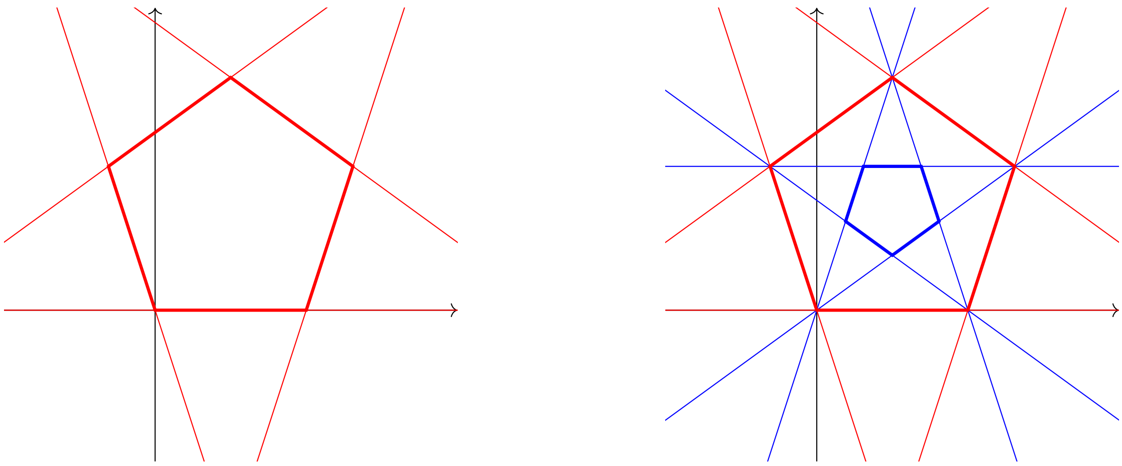
From now on, we will use the umbrella term *reflection* to refer to both linear and affine reflecitons.

4.2 Reflection groups

We are interested in groups generated by reflections in E, so throughout the next two sections fix a group $W \leq GA(E)$ which can be generated by reflections. This roughly follows [4]

We would like to consider the set of all hyperplanes H such that there is some affine reflection $w \in W$ across H, this set will be referred to as $\mathcal{H}$. For certain choices of generators for W we may find $\mathcal{H}$ is dense in E.

Example 4.2.1. Consider the set of hyperplanes pictured in red:



(a) Generating hyperplanes   **(b)** Resulting hyperplanes

As we have seen in the previous proposition, W acts on $\mathcal{H}$ so all the reflections of these hyperplanes, show in blue, must also lie in $\mathcal{H}$. This process can be repeated to find an arbitrarily small pentagon bounded by hyperplanes. By the presence of arbitrarily close parallel hyperplanes we can see W will contain arbitrarily small translations and thus $\mathcal{H}$ will be dense in E.

This adds technicalities and looses the discrete geometric intuition we have for reflection groups of polytopes and lattices. To remedy this we will restrict the reflection groups we consider by requiring for any compact subset $B \subset E$, the intersection $\mathcal{H} \cap B$ be finite. We can also see this immediately restricts the possible angles between hyperplanes to rational multiples of π.

The set of unitvectors $\alpha \in E$ such that there exists some $H \in \mathcal{H}$ with α normal to $\mathcal{H}$ will be written Φ. And without loss of generality we will assume Φ spans E from here onwards.

Definition 4.2.2. The connected components of $E \setminus \mathcal{H}$ are called the **chambers** of W in E. We will sometimes call these fundamental domain

The reflection groups we are interested in, those that appear as finite reflection groups of polytopes or lattices in whatever geometry we prefer, will have finitely many hyperplanes touching each chamber. Later on we will see this is equivalent to the group being finitely presented. A formula for the number of hyerplanes touching a chamber is given in Section A

We now want to choose a chamber C_0, and consider the set of hyperplanes $\{H_1, \dots, H_k\} \subseteq \mathcal{H}$ bounding C. We will call the corresponding reflections across these hyperplanes $\{s_1, \dots, s_k\}$ **simple reflections**.

Theorem 4.2.3. The set of simple reflections $\{s_1, \dots, s_k\}$ generates W.

Proof. Let W' be the subgorup of W generated by the simple reflections. Let s be one of the reflections generating W, and call the hyperplane it reflects across H. If W' acts transitively on the set of chambers then there will exists some $w \in W'$ such that $wH_i = H$ for some simple reflection s_i as H will must bound a chamber. Thus, by an earlier lemma, $ws_iw^{-1} = s$ so $s \in W'$ and $W = W'$. Now we just have to show the action of W' is transitive on chambers. Suppose it isn't, i.e. there is some chamber C such that no $w \in W'$ satisfies $wC = C_0$. Let C' be the closest chamber to C_0 in the W' orbit of C, as $C' \neq C_0$ there must be some simple hyperplane (the boundary of C_0) between them, reflecting across this must strictly decrease the distance between the two chambers contradicting the minimality of C'. Thus W' acts transitively on the set of chambers. $\square$

As a direct corollary of this proof we now know W acts transitively on the set of chambers. Before discussing this further we should examine the relations that these simple reflections satisfy.

For any two simple reflections s_i, s_j the subgroup $\langle s_i, s_j \rangle$ will be dihedral, as seen in the previous section, (note that this will be the infinite dihedral group iff the hypeplanes being reflected along are parallel), call the order of this dihedral group $2m_{ij}$. The product s_is_j will have order m_{ij} in W and so W satisfies the set of relations $(s_is_j)^{m_{ij}} = \text{id}$ for all i, j, taking $m_{ii} = 1$. A group *presented* by these relations is called **Coxeter**.

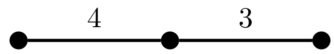
Definition 4.2.4. A **Coxeter system** is a pair (W, S) where $S = \{s_i\}_{i \in I}$ is a generating set for W which admits the presentation:

$$W = \langle S \mid (s_is_j)^{m_{ij}} \text{ for all } i, j \in I \rangle$$

where each $m_{ij} \in \mathbb{N} \cup \{\infty\}$.

To each Coxeter system we can assign a **Coxeter diagram**: an undirected graph created by the following rules:

- Draw a node i for each $s_i \in S$;
- For each relation $(s_is_j)^{m_{ij}}$ with $m_{ij} > 2$ draw an edge between i and j and label it with m_{ij}.



This process can be reversed to obtain a Coxeter system from any Coxeter diagram. This correspondence will associate the graph:

to the group presentation:

$$\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = e,\ (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

In the future, for the sake of readability, the 3 labels will often be excluded.

The classification of finite reflection groups goes by proving all reflection groups are in fact Coxeter groups and then classifying all the finite Coxeter groups.

4.3 Coxeter presentation

We have shown the action of W on the set of chambers is transitive, but we need a stronger notion to prove these groups are Coxeter.

Definition 4.3.1. Let G be a group acting on a set X, the action is called **simply transitive** if for all $x, y \in G$ there exists a unique $g \in G$ such that $g \cdot x = y$.

By fixing a base point $x \in X$ there is clear a 1-to-1 correspondence between X and G: associate $e \leftrightarrow x$ and for all $g \in G$, $g \leftrightarrow g \cdot x$.

Proposition 4.3.2. An action is simply transitive iff it is transitive and free.

Proof. An action is transitive if for all $x, y \in X$ there exists some $g \in G$ such that $g \cdot x = y$, and free if there exists at most one such g; therefore these statements are equivalent. □

If we know the action is trasntive a priori we only need to check the free condition on a single element. So showing our action is simply transitive ammounts to showing for all $w \in W$ such that $wC_0 = C_0$ we have $w = \text{id}$.

We will go about prooving this by finding a geometric interpretation of the length of a word $w \in W$ in terms of simple reflections.

Definition 4.3.3. For $w \in W$ define the **length** of w, $l(w)$ to be the minimal positive integer r such that $w = s_1 \cdots s_r$, a length r product of simple reflections.

And the contrasting way to compute the length: for any $w \in W$ consider the set of hyperplanes H such that C_0 and wC_0 lie on different sides of H. Call this $\mathcal{L}(w)$ and let $n(w) = |\mathcal{L}(w)|$. As the line segment between two chambers is bounded, by assumption this will only meet finitely many hyperplanes so for any $w \in W$ we will have $\mathcal{L}(w) \in \mathbb{N}$.

Lemma 4.3.4.

1. $n(w) = n(w^{-1})$,
2. $l(w) = 1$ iff w is a simple reflection
3. $l(w) = l(w^{-1})$
4. if w can be written as $s_1 \cdots s_r$ then $\det(w) = (-1)^r$
5. $l(s_i w), l(w s_i) = l(w) \pm 1$

Proof. (1.) H separtes C_0 and $w^{-1}C_0$ iff wH separates wC_0 and C_0 — W acts isometrically. □

Lemma 4.3.5. H_i is in exactly one of $\mathcal{L}(w)$ or $\mathcal{L}(s_i w)$

Lemma 4.3.6. Choose a simple hyperplane H_i, for any hyperplane $H \neq H_i$ and $w \in W$, if $H \in \mathcal{L}(w)$ then $s_i H \in \mathcal{L}(s_i w)$.

Proof. As H separates C_0 and wC_0, $s_i H$ separates $s_i C_0$ and $s_i w C_0$, so $s_i H$ is in exactly one of $\mathcal{L}(s_i)$ or $\mathcal{L}(s_i w)$, but if $s_i H \in \mathcal{L}(s_i)$ we would have $s_i H = H_i \implies H = H_i$ a contradiction. Therefore $s_i H \in \mathcal{L}(s_i w)$. □

Corollary 4.3.7. $s_i \left(\mathcal{L}(w) \setminus \{H_i\}\right) = \mathcal{L}(s_i w) \setminus \{H_i\}$.

Proof. by applying the previous lemma to both w and $s_i w$ we get the required iff. □

Proposition 4.3.8. For all $w \in W$, we have $n(w) \leq l(w)$.

Proof. If $l(w) = 1$ then w must be some simple reflection s_i so $\mathcal{L}(w) = \{H_i\}$[citation-needed]so $n(w) = 1$. Now by induction on $l(w)$: if $l(s_i w) = l(w) + 1$ then by the previous corollary and an earlier lemma we know $n(s_i w) = n(w) \pm 1$, namely $n(s_i w) \leq n(w) + 1 = l(w) + 1 = l(s_i w)$. □

Now to show $n(w) = l(w)$ we will enumerate the hyperplanes in $\mathcal{L}(w)$.

Lemma 4.3.9. If $w = s_1 \cdots s_r$ is a reduced expression for w in terms of simple reflections, the hyperplanes:

$$H_1, s_1 H_2, s_1 s_2 H_3, \ldots s_1 \cdots s_{r-1} H_r$$

are all distinct.

Proof. Suppose, for a contraditction, that there exists some $1 \leq i < j \leq r$ such that the hyperplanes $s_1 \cdots s_{i-1} H_i$ and $s_1 \cdots s_{j-1} H_j$ are equal, by applying $s_1 \cdots s_{i-1}$ to both sides this implies $H_i = s_i \cdots s_{j-1} H_j$, by an earlier corollary this implies $s_i = (s_I \cdots s_{j-1}) s_j (s_{j-1} \cdots s_i)$ which implies $s_{i+1} \cdots s_{j-1} = s_i \cdots s_j$ contradicting the minimality of the lenght of w. □

Proposition 4.3.10. If $w = s_1 \cdots s_r$ is a reduced expression for w in terms of simple reflections:

$$\mathcal{L}(w) = \{H_1, s_1 H_2, s_1 s_2 H_3, \ldots s_1 \cdots s_{r-1} H_r\}$$

Proof. We go by induction on r. Observe the base case $\mathcal{L}(s_1) = \{H_1\}$ from an earlier lemma. Now assume the claim holds for all w with reduced length $< r$ specifically:

$$\mathcal{L}(s_1 w) = \{H_2, s_2 H_3, \ldots s_2 \cdots s_{r-1} H_r\}$$

we know from an earlier lemma H_1 is in exactly one of $\mathcal{L}(w)$ or $\mathcal{L}(s_1 w)$. If $H_1 \in \mathcal{L}(s_1 w)$ by applying s_1 we would have for some $i > 1$:

$$s_1 H_1 = H_1 = s_1 \cdots s_i H_i$$

which contradicts the previous lemma, thus $H_1 \in \mathcal{L}(w)$, combining this with the earlier corollary gives the desired result. □

A direct consequence of this propsition is that if some $w \in W$ is such that $wC_0 = C_0$ then $\mathcal{L}(w) = \emptyset$ which implies $w = \text{id}$ in reduced form. We know this is sufficient to say the action of W on the set of chambers is simply transitive.

We can now prove two key combinatorial properties of the reflection group, and use these to show all relations in W are direct consequences of those in the Coxeter presentation.

Theorem 4.3.11 (Exchange Condition)**.** If $w \in W$ has reduced expression in terms of simple reflections $w = s_1 \cdots s_r$ and $l(sw) < l(w)$ for some simple reflection s, there exists some $1 \leq i \leq r$ such that $w = s s_1 \cdots s_{i-1} s_{i+1} \cdots s_r$.

Proof. We know:

$$\mathcal{L}(w) = \{H_1, s_1H_2, s_1s_2H_3, \dots s_1 \cdots s_{r-1}H_r\}$$

by a previous proposition we know H, the hyperplane corresponding to s, must lie in $\mathcal{L}(w)$ and so for some $1 \leq i \leq r$, $H = s_1 \cdots s_{i-1}H_i$ which implies $s = s_1 \cdots s_{i-1}s_is_{i-1} \cdots s_1$ therefore $ss_1 \cdots s_{i-1} = s_1 \cdots s_i$. Substituting this back into w gives the desired result. □

Theorem 4.3.12 (Deletion Condition)**.** If $w \in W$ has a nonreduced expression in terms of simple roots $w = s_1 \cdots s_r$ then there exists $1 \leq i, j \leq r$ such that $w = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_r$. The hats denote omission.

Proof. For this to be nonreduced there must eventually exist some i such that $l(s_i \cdots s_r) = l(s_{i+1} \cdots s_r) - 1$ so by the exchange condition $s_{i+1} \cdots s_r = s_is_{i+1} \cdots \hat{s_j} \cdots s_r$. Substituting this back into w gives the desired result. □

Informally, we can see that these weak forms of the deletion and exchange conditions are actually direct consequences of the Coxeter relations, this is geometrically motivated in subsection 4.5. The formal proof that the exchange condition (which implies the deletion condition) is a property of Coxeter groups rigourously defined as quotients of a free group is given in Section 6. We are electing to use this informal proof at this point in the report to continue the geometric nature of this section, thus the following proof that reflection groups satisfy the Coxeter presentation is the same somewhat informal argument as given in [4].

Theorem 4.3.13. Any finite reflection group satisfies the Coxeter presentation.

Proof. We want to consider the smallest nontrivial relation $w = s_1 \cdots s_r = \mathrm{id}$, in terms of simple reflections, in W which is not a consequence of the Coxeter relations.

By examining the determinants, we see $r = 2k$ must be even, thus we can write $s_1 \cdots s_{k+1} = s_{k+2} \cdots s_r$ and invoke the deletion condition which lets us rewrite $w = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_r = \mathrm{id}$ which must also not be a consequence of the Coxeter relations, this contradicts minimality. □

Note that in this proof we tacitly expect $r \geq 4$, luckily any length 2 relation is just an equality which by assumption do not occur.

4.4 Classification

Definition 4.4.1. To a Coxeter system (W, S), with $n = |S|$, we can associate a real symmetric $n \times n$ matrix A, by setting $a_{ij} := -\cos(\pi/m_{ij})$. The quadratic form $Q(v) = v^\top Av$ on E is called the **associated quadratic form** of the Coxeter system.

We can realise each a_{ij} as the inner product of simple roots (α_i, α_j) corresponding the the reflection s_i, s_j. Such matrices are called **Gram matrices**.

Proposition 4.4.2. The quadratic form associated to a finite reflection group W is positive definite.

Proof. As Φ forms a basis for E there is an invertible change of basis matrix B form the standard basis to Φ, we can thus write $A = B^\top B$ and for any nonzero $v \in E$ have:

$$Q(v) = v^\top Av = v^\top B^\top Bv = (Bv)^\top(Bv) = \|Bv\|^2 > 0$$

as B is a linear transformation acting on $v \neq 0$, therefore Q is positive definite. □

We now want to fix a quadratic form $Q(v) = v^\top Av$ for some symmetric $A \in \mathcal{M}_n(\mathbb{R})$.

Lemma 4.4.3. The kernel of Q (the set of vectors $v \in E$ such that $Q(v) = 0$) equals the nullspace of A.

Proof. The nullspace of A obviously lies in the kernel of Q. From linear algebra 2 we know we can orthogonally diagonalise A and so we can write:

$$Q(v) = v^\top Av = v^\top P^\top DPv = (Pv)^\top D(Pv)$$

where P and D are some orthogonal and diagonal matrices respectively. Observe the right hand side is 0 iff for ever $1 \leq i \leq n$ either d_{ii} or $(Pv)_i$ are 0. This implies $D(Pv) = 0$ and thus v is in the nullspace of A. □

We will now demand all of the off-diagonal entries of A be nonpositive, note that all of the matrices for quadratic forms associated with Coxeter diagrams will satisfy this condition.

Lemma 4.4.4. The only nonzero vectors in the nullspace of A have all coordinates nonzero.

Proof. First suppose $v \neq 0$ is in the nullspace of A, then by the previous lemma we know $Q(v) = 0$. Consider the vector u given by $u_i = |v_i|$, we will have $Q(u) = u^\top Au \geq 0$, but as all off-diagonal entries of A are nonpositive this will satisfy $Q(u) \leq Q(v) = 0$. Thus u is also in the nullspace of A. So if the nullspace is nontrivial, it will always contain a vector with nonnegative coordinates.

If we suppose some, but not all, of these coordinates are zero we can consider the set of such indices I. For any $i \in I$ consider the ith term in Au:

$$(Au)_i = \sum_{j \notin I} a_{ij}|v_j| = 0$$

as all the a_{ij} are nonpositive and all the $|v_j|$ are strictly positive, we get that for all $i \in I, j \notin I$ the entries $a_{ij} = 0$ contradicting irreducibility of A. Thus the coordinates of all vectors in the nullspace of A are nonzero. □

Corollary 4.4.5. The dimension of the nullspace of A will be at most 1.

Lemma 4.4.6. The smallest eigenvalue d of A has multiplicity 1, and its coordinates will all be positive.

Proof. Again, from linear algebra 2 we know all the eigenvalues of a positive semidefinite matrix are nonnegative. So $A - dI$ satisfies all the requirement we placed on A for the previous lemma. As $A - dI$ is singular it will have nonempty nullspace which must be of dimension exactly 1. Thus d and $u_i = |d_i|$ are colinear so we can find an eigenvalue with all positive coordinates. □

We want to consider subgraphs of the Coxeter diagram Γ associated to (W, S). For this we will introduce a Coxeter system (W', S') where $S' \subseteq S$, the Coxeter relation degrees in (W', S') are given by m'_{ij} and they will satisfy $m'_{ij} \leq m_{ij}$ for all i, j. We will call the associated Coxeter diagram, quadratic form and symmetric matrix Γ', Q', A' respectively. Assume this *subsystem* is proper i.e. $S' \subsetneq S$ or some $m'_{ij} < m_{ij}$.

Proposition 4.4.7. If Q is positive semidefinite, then Q' will be positive definite.

Proof. Suppose A' fails to be positive definite, i.e. there exists some $v \in E$ such that $Q'(v) \leq 0$, consider the vector:

$$u_i = \begin{cases} |v_i| & 1 \leq i \leq k \\ 0 & \text{otherwise} \end{cases}$$

and the value of:

$$0 \leq Q(u) = \sum_{1 \leq i,j \leq n} a_{ij}u_iu_j \leq \sum_{1 \leq i,j \leq k} a'_{ij}v_iv_j \leq 0$$

as all the $a_{ij} \leq a'_{ij} \leq 0$. This means u is in the nullspace for A and thus all the coordinates are nonzero which means $k = n$. But this then implies all $a_{ij} = a'_{ij}$. Thus the only subdiagram that fails to be positive definite is improper. □

Here is a collection of Coxeter diagrams all of which, by checking their eigenvalues on a computer, can be found to be positive definite.

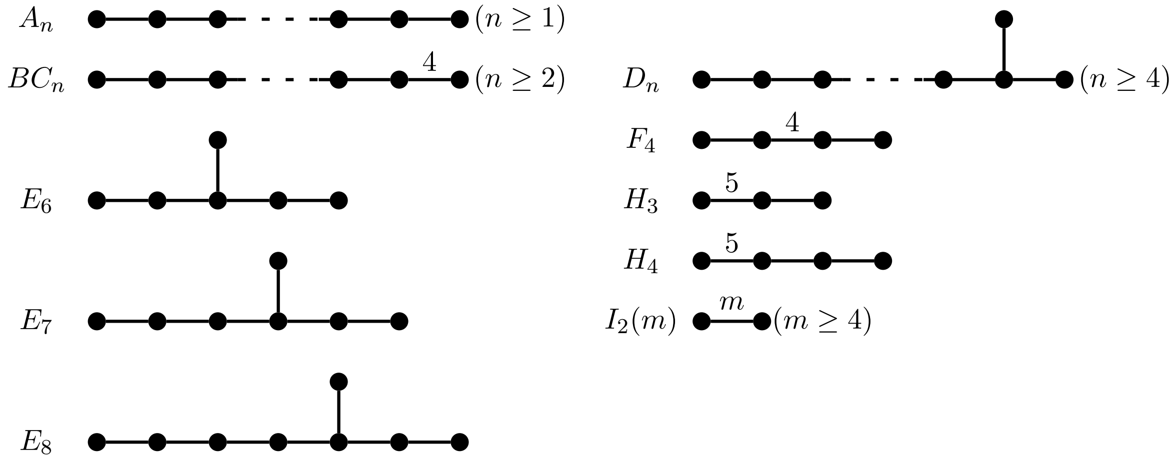


Figure 5 Positive definite Coxeter diagrams

And here is another collection who can all be founded to be only positive semidefinite:

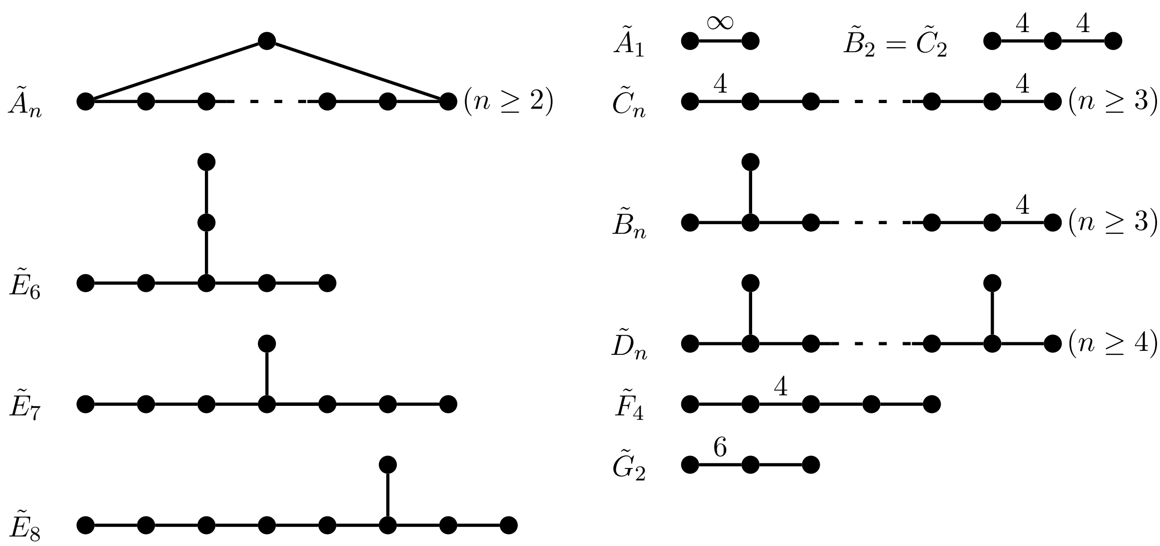


Figure 6 Positive semidefinite Coxeter diagrams

Plus two helper diagrams which are neither:

Figure 7 Helper Coxeter diagrams

Theorem 4.4.8. the diagrams of Figure 5 are all the connected diagrams with positive definite quadratic form.

Proof. The proof goes by indentifying features present in the graphs of Figure 6 and Figure 7, observing they cannot appear as subdiagrams of those with positive definite quadratic form: e.g. no cycles as in $\tilde{A}_n$ and all edges must be finite due to $\tilde{A}_1$. For the full proof see [4]. $\square$

Corollary 4.4.9. The following graphs categorise all finite reflection groups.

Proof. We know it suffices to classify the reflections groups corresponding to connected Coxeter diagrams, all of which must have positive definited associated quadratic form, all of which we have found. The only remaining step is to find polytopes admitting these Coxeter groups as their reflection groups, lots of these examples have been seen in the previous section, and in fact every diagram of Figure 5 does occur as a finite reflection group of a polytope. However, as we will see in the next section, such polytopes aren't necessarily regular. □

This solution can seem somewhat unsatisfactory as there is no immediate reason as to **why** all of the finite Coxeter groups can be realised as reflection groups.

4.5 Geometric representation

For any finite Coxeter system (W, S) consider the abstract real vector space V with basis Φ. We can define a quadratic form (and thus an inner product $\langle -, - \rangle$) on V in the same way as we did earlier. And for each generator $s \in S$ we can associate a reflection $\sigma_s : V \to V$ given natrually as $\sigma_s(v) = v - 2\langle \alpha_s, v \rangle \alpha_s$. Through some similar calculations we can find the representation this induces on W is faithful and acts orthogonally on V, the construction of the edges and vertices of a polytope demonstrated in the next section will thus show how a regular polytope can be realised from this abstract reflection group.

5 Uniform polytopes

In this section, we will only be talking about convex polytopes.

5.1 Regular polytopes and finite Coxeter groups

As we've seen already, the symmetry groups of regular polytopes are Coxeter groups. Thus we can define a map from regular polytopes to finite Coxeter groups, based on their symmetries.

Definition 5.1.1. If a polytope's symmetry group is a Coxeter, we say the polytope is *defined by* that Coxeter group. Similarly, this Coxeter group *defines* the polytope.

Naturally, we can ask: Is this map a bijection? We know that dual polyhedra have the same symmetries, so clearly the answer is no. In fact, every polytope has the same symmetry group - and hence is defined by the same Coxeter group - as its dual. This clearly shows that the map is not injective and so is not a bijection. But the question remains, is the map surjective? That is to say, does every finite Coxeter group define a regular polytope? The answer to this question is also no, but why?

For this, we make use of the fact that no two Coxeter groups can be define the same polytope. This is clear since all Coxeter groups are unique. Because of this, we can refer to the Coxeter group that defines a given polytope as the polytope's Coxeter group. With this, we only need to show that there are more Coxeter groups than regular polytopes.

Lemma 5.1.2. For $k \geq 5$, there are only 3 regular convex k-polytopes. Specifically, these are the k-simplex, the k-cube, and its dual, the k-orthoplex.[5]

Proposition 5.1.3. The map from regular polytopes to finite Coxeter groups is not surjective.

Proof. We will use the fact that a Coxeter group C_k defines a k-polytope. This k is the number of generators of C_k, or, equivalently, the number of nodes in its Coxeter diagram.

For $k = 5$, there are 3 Coxeter groups: A_5, BC_5, and D_5. From Lemma 5.1.2, we know that there are 3 regular 5-polytopes. At first glance this seems fine, but remember that duals are defined by the same Coxeter group, meaning there are only 2 unique symmetry groups of regular 5-polytopes. Hence, one of the 3 Coxeter groups does not define a regular polytope, and so the map is not surjective. □

Remark 5.1.4. With some work, you can find that A_5 defines the 5-simplex and BC_5 defines both the 5-cube and 5-orthoplex, and so D_5 does not define a regular polytope. This is actually true for any number of dimensions; A_n defines the n-simplex and BC_n defines the n-cube and n-orthoplex.

This means that $D_n, \forall n \geq 5$ as well as E_6, E_7 and E_8 do not define regular polytopes. This begs the question: Which polytopes are defined by the remaining groups? To answer this, we need to introduce a new category of polytope.

5.2 Uniform polytopes

Definition 5.2.1. A polytope is *k-face-transitive* if its symmetry group acts transitively on the k-faces on the polytope.

Remark 5.2.2. Regular k-polytopes are j-face-transitive $\forall j < k$.

Definition 5.2.3. A polytope is *uniform* if it is vertex-transitive and has uniform facets. Uniform polygons are defined as regular polygons[1].

This new category of polytopes contains the polytopes defined by D_n, E_6, E_7 and E_8, as well as many more, so let's create a way to differentiate them based on their symmetries.

5.3 Extended Coxeter diagrams

We can extend the Coxeter diagram to include more information about the specific polytope that it defines. Because of this, extended Coxeter diagrams do not represent unique Coxeter groups, but rather unique polytopes.

Definition 5.3.1. The *fundamental domain* of a uniform polytope is the[2] connected region containing exactly one point from each orbit of the points on the boundary of the polytope under its symmetry group. This can be visualised as the smallest region on the boundary of the polytope bounded by the planes of reflection of its Coxeter group.

Definition 5.3.2. The *generating vertex* of a uniform polytope is the vertex that lies in its fundamental domain.

Definition 5.3.3. A reflection in a polytope's Coxeter group is called *inactive* if the generating vertex lies on the plane of reflection. A reflection that is not inactive is called *active*. Active reflections are denoted with rings in extended Coxeter diagrams.

We now have a way to represent different polytopes with the same Coxeter group using Coxeter diagrams. This can be used to differentiate between dual polytopes, but there are many other uniform polytopes with the same Coxeter group.

[1] This disallows polygons (and by extension polytopes) having edges of different lengths.

[2] There are multiple regions that fit this definition, but for our purposes they are all equivalent (the polytope's Coxeter group acts transitively on the domain), so we can refer to it as the fundamental domain.

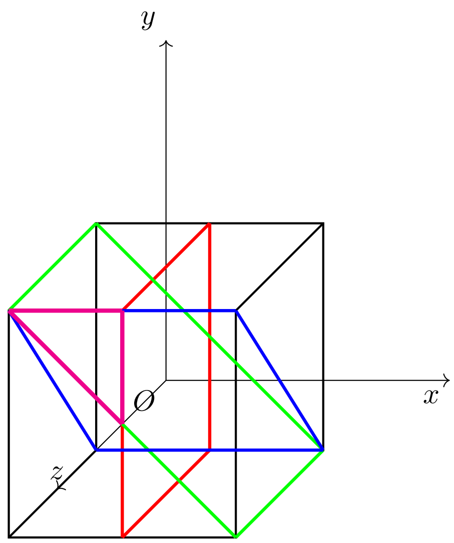


Figure 8 The fundamental domain of a cube

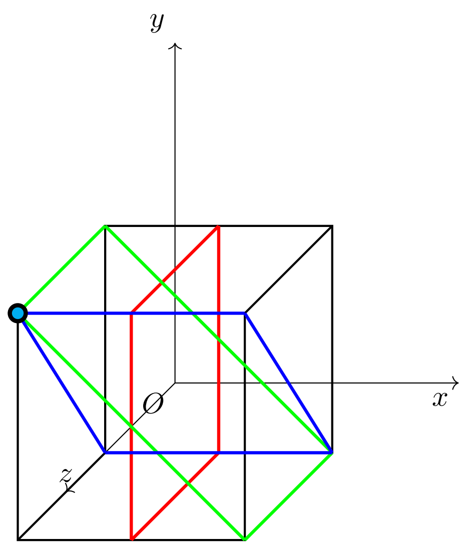


Figure 9 The generating vertex of a cube

Figure 10 The fundamental domains and generating vertices of a cube (left) and a octahedron (right)

Figure 11 An active reflection (left) and and inactive reflection (right)

Figure 12 Extended Coxeter diagrams of a cube (left) and an octahedron (right)

5.4 Rectification

Definition 5.4.1. *Rectification* is the process of creating vertices at the midpoints of the edges of a polytope and connecting vertices on adjacent edges of the polytope.

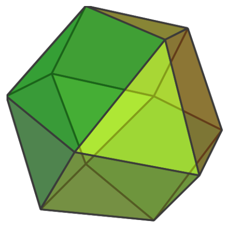


Figure 13 A rectified cube

Definition 5.4.2. A non-regular polytope is called *quasiregular* if it's vertex-transitive, edge-transitive, and has regular faces.

Proposition 5.4.3. The rectification of a regular or quasiregular polytope is uniform.

Proof. Rectification turns edges into vertices. Therefore, a rectified polytope is vertex-transitive if and only if the original polytope is edge-transitive. Additionally, its faces are uniform if and only if the original polytope is edge-transitive. □

Definition 5.4.4. *k-rectification* is the process of creating vertices at the midpoints of the k-faces of a polytope and connecting vertices on adjacent k-faces of the polytope.

Remark 5.4.5. As we've seen, the dual of a polyhedron has vertices at its faces and faces at its vertices, meaning that 2-rectification (also called birectification) produces the dual of a polyhedron. In higher dimensions, $(k-1)$-rectification produces the dual of a k-polytope.

Corollary 5.4.6. The k-rectification of a k-face-transitive polytope is uniform.

This proof is the same as for Proposition 5.4.3 but with higher-dimensional faces.

Remark 5.4.7. The rectification of a non-self-dual regular or quasiregular polytope has the same Coxeter group as that polytope. In extended Coxeter diagrams, k-rectification makes active nodes inactive, and makes nodes k away from active nodes active.

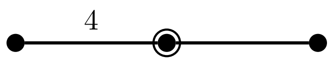


Figure 14 Extended Coxeter diagram of the rectified cube

5.5 Truncation

Definition 5.5.1. *Truncation* is the process of "cutting" the vertices of a polytope. This is done by removing the vertices of the polytope and creating a new facet in its place, bounded by the lines between equidistant points on adjacent edges. It is possible to "cut" the vertices by any amount, but *uniform truncation* truncates such that all edges are the same length[3] (and all original edges still exist).

[3]This is not always possible for general polytopes.

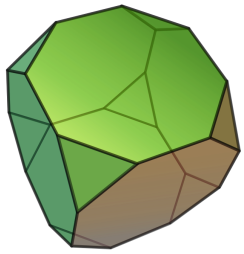


Figure 15 A uniformly truncated cube

Remark 5.5.2. Truncating a quasiregular polytope using the midpoints of the edges as the boundary points defining the new facets is the same as rectification.

Proposition 5.5.3. The uniform truncation of a uniform polytope with a uniform vertex figure is uniform.

Proof. Truncation creates facets from the vertex figure of a polytope so the vertex figure must be uniform. Uniform truncation ensures that the truncated polytope is vertex-transitive. $\square$

Definition 5.5.4. *k-truncation* is the process of "cutting" the j-faces of a polytope $\forall j < k$. It can also be seen as expanding the polytope's k-faces (moving them directly away from its center and connecting previously adjacent j-faces with $(j+1)$-faces $\forall j < k$. As before, *uniform k-truncation* creates a polytope with uniform facets.

Remark 5.5.5. 2-truncating a regular polytope is the same as rectifying its rectification[4].

Definition 5.5.6. Truncations can be combined to create a new transformation, $(k_1, k_2, ..., k_m)$-*truncation*. It "cuts" k_i-faces $\forall i \in \{1, ..., m\}$. This can be seen as expanding a polytope's k_1-faces, then expanding its k_2-faces, and so on.

Remark 5.5.7. The $(k_1, k_2, ..., k_m)$-truncation of a regular polytope has the same Coxeter group as that polytope. In extended Coxeter diagrams, $(k_1, k_2, ..., k_m)$-truncation makes nodes $(k_1, k_2, ..., k_m)$ away from active nodes active.

Figure 16 Extended Coxeter diagrams of the truncated cube (left) and $(1, 2)$-truncated cube (right)

5.6 Alternation

Definition 5.6.1. *Alternation* is the process of removing every other vertex from a polytope and connecting vertices on previously adjacent edges. It is only possible to alternate polytopes with even-sided faces.

Definition 5.6.2. The k-demicube is the alternated k-cube.

Remark 5.6.3. D_n defines the n-demicube.

[4]This can be seen more easily with extended Coxeter diagrams.

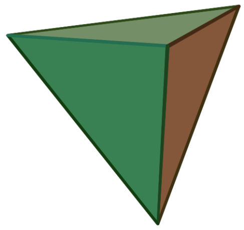


Figure 17 A 3-demicube (tetrahedron)

Definition 5.6.4. An active reflection in a polytope's Coxeter group is called *alternated* if the vertices generated by this reflection have been removed from the polytope. Alternated reflections are denoted with holes in extended Coxeter diagrams[5].

Figure 18 An alternated reflection

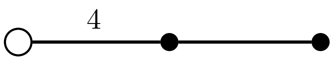


Figure 19 Extended Coxeter diagram of the 3-demicube

Theorem 5.6.5. Every finite Coxeter group defines a uniform polytope.

Proof. We will prove this by induction on the number of dimensions. In 2 dimensions, $I_2(m)$ defines the regular m-gon for $m \geq 3$, which are all uniform. In k dimensions, every finite Coxeter group that defines a k-polytope can be made by adding a new reflection to a finite Coxeter group that defines a $(k-1)$-polytope (equivalent to adding a node to a Coxeter diagram). By the induction hypothesis, this group defines a uniform $(k-1)$-polytope. Therefore, this polytope has a generating vertex in its ($(k-1)$-dimensional) fundamental domain. The k-dimensional fundamental domain (with the added reflection) contains the $(k-1)$-dimensional fundamental domain and so this vertex is the generating vertex of a (not necessarily uniform) n-polytope. We can move this vertex in the kth dimension while preserving the uniform $(k-1)$-polytope that it generates. Moving the vertex onto the new plane of reflection generates a polytope with uniform facets. Generating a polytope in this way creates a vertex-transitive polytope. $\square$

6 Tits' word problem

6.0 Abstract

This is an example of referencing: [6][7]

The word problem for a group presentation $\langle S \mid R \rangle$ involves finding an algorithm to reduce any word in the group W into a reduced word. In this paper, we solve this problem for Coxeter groups, and, more generally, for all systems satisfying the exchange condition, which is encapsulated in Tits' Theorem.

The proof of this result is divided into several steps:

- First, we show that every Coxeter group satisfies the exchange condition.

[5] These processes do not necessarily define unique polytopes, so neither do the extended Coxeter diagrams, however each extended Coxeter diagram only defines one polytope.

- Next, we examine when two reduced words represent the same element in the group.
- Lastly, we discuss how to reduce a word in a system satisfying the exchange condition.

The hardest part of the proof is the first one, and we will focus on this in more detail.

6.1 Exchange Condition

Definition 6.1.1. A system is a tuple (W, S), where W is a group and S is a set of generators for G, such that $S = S^{-1}$ and $1 \notin S$.

Definition 6.1.2. For $w \in W$, the sequence $\mathrm{s} = (s_1, \ldots, s_q)$, where $s_i \in S$ for all $i \in \mathbb{N}$, is called a **reduced representation** of w if and only if $w = s_1 s_2 \ldots s_q$ and q is the smallest such number satisfying this condition. We define $l(w) = q$, which is called the **length** of w.

Proposition 6.1.3. For $w, w' \in W$, we have the following facts:

- (1) $l(ww') \leq l(w) + l(w')$
- (2) $l(w^{-1}) = l(w)$
- (3) $|l(w) - l(w')| \leq l(ww'^{-1})$

Proof: (1) and (2) are straightforward. From (1), we see $l(w) \leq l(ww'^{-1}) + l(w')$ and $l(w') \leq l(w'w^{-1}) + l(w)$. From (2), we know $l(w'w^{-1}) = l(ww'^{-1})$. Combining these, we obtain (3). $\square$

Definition 6.1.4. If a system (W, S) satisfies the following conditions:

- (1) Every element in S has order 2.
- (2) Let $m(s, t)$ denote the order of st, and define $I = \{(s, t) \mid s, t \in S, m(s, t) < \infty\}$, then the presentation $\langle S \mid (st)^{m(s,t)} = 1, (s, t) \in I\rangle$ describes W.

Then we say that (W, S) is a Coxeter System, and W is a Coxeter group.

Remark 6.1.5. A Coxeter system (W, S) satisfies the following universal property: For any group G and map $\psi : S \to G$ such that $\psi(st)^{m(st)} = e_G$ for all $(s, t) \in I$, there exists a unique homomorphism $\phi \in \mathrm{Hom}(W, G)$ such that $\psi = \phi \circ l$, where l is the natural inclusion from S to W. This is simply the composition of the universal properties of free groups and quotient groups.

Example 6.1.6. By the universal property of Coxeter systems, the map $s \mapsto -1$ for all $s \in S$ extends to a homomorphism $\epsilon : W \to \{-1, 1\}$, such that $\epsilon(w) = (-1)^{l(w)}$. We call $\epsilon(w)$ the sign of w.

Now we want to find an invariant among different representations of the same word. For this, we define the following:

Definition 6.1.7. For a Coxeter system (W, S), let T denote the union of the conjugacy classes of all elements in S. For a sequence $\mathrm{s} = (s_1, \ldots, s_q)$, where $s_i \in S$ for all i, define $\Phi(\mathrm{s}) = (t_1, \ldots, t_q)$, where $t_j = (s_1 \ldots s_{j-1})s_j(s_1 \ldots s_{j-1})^{-1}$. For $t \in T$, define $n(\mathrm{s}, t) := \#\{1 \leq j \leq q \mid t_j = t\}$. Finally, define $R = \{\pm 1\} \times T$.
Remark Note that we have $s_1 \ldots s_j = t_j \ldots t_1$.

Lemma 6.1.8. For a Coxeter System (W, S), we have the following facts:

1. For $w \in W$ and $t \in T$, for every sequence $\mathbf{s} = (s_1, \ldots, s_q)$ such that $w = s_1 \ldots s_q$, the value $(-1)^{n(\mathbf{s},t)}$ is constant. We call this value $\eta(w, t)$.

2. For $w \in W$, consider the map $U_w : R \to R$ defined as

$$U_w(\epsilon, t) = (\epsilon\eta(w, t), wtw^{-1}),$$

then the map $w \mapsto U_w$ is a homomorphism from W to the permutation group of R.

Proof: For $s \in S$, define $U_s : R \to R$ by

$$U_s(\epsilon, t) = (\epsilon(-1)^{\delta(\mathbf{s},t)}, sts^{-1}),$$

where $\delta_{s,t}$ is the Kronecker delta. For a sequence $\mathbf{s} = (s_1, \ldots, s_q)$ in S, let $w = s_q \ldots s_1$, then

$$U_s = U_{s_q} \circ \cdots \circ U_{s_1}.$$

We prove that $U_{\mathbf{s}}(\epsilon, t) = (\epsilon(-1)^{n(\mathbf{s},t)}, wtw^{-1})$ by induction.

$$(*) \quad \text{In the case when } q = 0 \text{ or } 1, \text{ the proof is trivial.}$$

For $q > 1$, assume the induction hypothesis holds for $\mathbf{s}' = (s_1, \ldots, s_{q-1})$ with $w' = s_{q-1} \ldots s_1$. Then

$$U_{\mathbf{s}'} = U_{s_q}(\epsilon(-1)^{n(\mathbf{s}',t)}, w'tw'^{-1}).$$

Thus, it remains to prove that

$$n(\mathbf{s}, t) = \delta_{s_q, w'tw'^{-1}} + n(\mathbf{s}', t).$$

Notice that $\Phi(\mathbf{s}) = (\Phi(\mathbf{s}'), w'^{-1}s_q w')$, so the proof is trivial.

We now prove that the map $s \mapsto U_s$ can be extended to a homomorphism from W. By the universal property of Coxeter systems, it suffices to prove that for all $s, s' \in S$ such that $m(s, s') < \infty$, we have

$$(U_s \circ U_{s'})^{m(s,s')} = 1.$$

Define $\mathbf{s} = (s_1, \ldots, s_{2m(s,s')})$ such that $s_i := s$ for odd i and $s_i := s'$ for even i. It suffices to prove that

$$U_{\mathbf{s}} = \mathrm{Id}.$$

Notice that here we have $t_j = (ss')^{j-1}s$, so $t_i \neq t_j$ for all $1 \leq i < j \leq m(s, s')$ and $t_i = t_{i+m(s,s')}$.

Hence, for $t \in T$, $n(\mathbf{s}, t)$ is either 0 or 2. Applying the result in the previous paragraph, we see that $U_{\mathbf{s}} = \mathrm{Id}$.

Therefore, the map $w \mapsto U_w$ is a homomorphism from W. Specifically, U_w does not depend on the representation chosen for w. From $(*)$, we conclude that $n(\mathbf{s}, t)$ is invariant across different representations. This gives us statement (1). Statement (2) follows easily. □

Theorem 6.1.9. For a Coxeter System (W, S), consider a sequence $\mathbf{s} = (s_1, \ldots, s_q)$, $\Phi(\mathbf{s}) = (t_1, \ldots, t_q)$, and $w = s_1 \ldots s_q$. Then $\mathbf{s}$ is a reduced representation of w if and only if $t_i \neq t_j$ for all $i \neq j$. Define

$$T_w := \{t \in T \mid \eta(w, t) = -1\},$$

then in the case that $\mathbf{s}$ is a reduced representation of w, we have $T_w = \{t_1, \ldots, t_q\}$ and $|T_w| = l(w)$.

Proof: For all $t \in T_w$, we have $n(w, t) \neq 0$. By the definition of $n(w, t)$, we deduce that for all $t \in T_w$, $t \in \{t_1, \ldots, t_q\}$. Moreover, since $\eta(w, t)$ is independent of the choice of representation of w, the set T_w depends only on w. Hence, $|T_w| \leq l(w)$.

In the case where $t_i \neq t_j$ for all $i \neq j$, we have $n(s,t) = 0$ or 1, and thus $T_w = \{t_1, \ldots, t_q\}$. Since $q = |T_w| \leq l(w)$, it follows that s is a reduced representation of w.

Conversely, if there exist $i \neq j$ such that $t_i = t_j$, without loss of generality, assume $i < j$. Consider the subsequence $u = s_{i+1} \ldots s_{j-1}$, then we have

$$s_i = us_ju^{-1}.$$

Hence, we have

$$s_1 \ldots s_q = s_1 \ldots s_{i-1}(us_ju^{-1})us_js_{j+1} \ldots s_q = s_1 \ldots s_{i-1}us_js_js_{j+1} \ldots s_q = s_1 \ldots s_{i-1}s_{i+1} \ldots s_{j-1}s_{j+1} \ldots s_q.$$

Thus, s is not a reduced representation. □

Note that now we have a necessary and sufficient condition to check whether a word is reduced or not. We want to improve this result and develop an algorithm for reducing words.

Definition 6.1.10. For a system (W,S), if it satisfies: for all $w \in W$ and $s \in S$, if $l(sw) \leq l(w)$, then for any reduced representation $\mathrm{s} = (s_1, \ldots, s_q)$, there exists $1 \leq j \leq q$ such that $ss_1 \ldots s_{j-1} = s_1 \ldots s_j$, then we say (W,S) satisfies the *exchange condition*.

Theorem 6.1.11. If (W,S) is a Coxeter System, then it satisfies the exchange condition.

Proof: Consider $w \in W$ and $s \in S$ such that $l(sw) \leq l(w)$. For any reduced representation $\mathrm{s} = (s_1, \ldots, s_q)$ of w, consider $w' := sw$. By Example 6.1.6, we have

$$l(w') \equiv l(w) + 1 \pmod 2.$$

Proposition 6.1.3(3) gives that $|l(w) - l(w')| \leq 1$, so we conclude that $l(w') = l(w) - 1$. Now, pick $(s'_1, \ldots, s'_{l(w)-1})$ as a reduced representation of w', then $\mathrm{s}' = (s, s'_1, \ldots, s'_{p-1})$ is a reduced representation of w.

Take $\Phi(\mathrm{s}') = (t'_1, \ldots, t'_p)$, then by definition, we have $t'_1 = s$. However, Theorem 6.1.9 shows that $t'_i \neq t'_j$ if $i \neq j$, so we have

$$n(\mathbf{s}', s) = 1.$$

By Lemma 6.1.8, we know that

$$n(\mathbf{s}', s) \equiv n(\mathbf{s}, s) \pmod 2,$$

so $n(\mathbf{s}, s) \neq 0$. Hence, there exists an index j such that $s = t_j$, where t_j is an element in $\Phi(\mathrm{s})$. By the definition of t_j, we have

$$ss_1 \ldots s_{j-1} = s_1 \ldots s_j.$$

Thus, the claim is proved. □

6.2 M-operations

Definition 6.2.1. Consider a system (W,S). The sequence $\mathbf{s} = (s_1, \ldots, s_q)$, where $s_i \in S$ for all i, is called a word in S, and $w := s_1 \ldots s_q$ is called the element in W expressed by s. An elementary M-operation on a word in s is one of the following two types of operations:

- **(MI)** Delete a subword of the form (s,s) from s.
- **(MII)** Replace an alternating subword $(s,t,s,\ldots)$ by another alternating subword $(t,s,t,\ldots)$, both of length $m(s,t)$.

We say a word is M-reduced if and only if its length cannot be shortened by M-operations. Clearly, any reduced word is M-reduced.

Lemma 6.2.2. For a system (W, S) with the exchange condition, two reduced representations express the same element in W if and only if one can be transformed into the other by MII operations.

Proof: Let $\mathbf{s} = (s_1, \dots, s_q)$ and $\mathbf{r} = (r_1, \dots, r_q)$ be two reduced representations, both expressing $w \in W$. We will prove the lemma by induction.

In the case when $q = 0$, the proof is trivial. If $s_1 = r_1$, we can reduce the length of both words by 1, and the induction hypothesis can be applied directly. It suffices to prove the case where $s_1 \neq r_1$. Let $m := m(s_1, r_1)$.

Claim: m is finite, and there is another reduced expression $\mathbf{u}$ of w that starts with an alternating subword $(s_1, r_1, s_1, \dots)$ of length m.

With the claim, the remaining proof is easy. Since $\mathbf{s}$ and $\mathbf{u}$ both start with s_1, by the induction hypothesis, $\mathbf{s}$ can be transformed into $\mathbf{u}$ by MII operations. Now, apply MII on $\mathbf{u}$ to get another word $\mathbf{u}'$ that starts with $(r_1, s_1, r_1, \dots)$ of length m. Since $\mathbf{u}'$ and $\mathbf{r}$ both start with r_1, again by the induction hypothesis, $\mathbf{u}'$ can be transformed into $\mathbf{r}$ through MII operations. The combination of all the above MII operations gives the result.

The converse of the proposition is trivial. $\square$

Proof of the Claim: To prove that m is finite, consider the following. Since $\mathbf{r}$ is a reduced representation of w starting with r_1, it follows that $l(r_1 w) < l(w)$. By the exchange condition, there exists an index i such that

$$r_1 s_1 \dots s_{i-1} = s_1 \dots s_i.$$

Moreover, since $(s_1, \dots, s_i)$ is a subword of $\mathbf{s}$, which is reduced, we know that the word

$$v := (r_1, s_1, \dots, s_{i-1}, s_{i+1}, \dots, s_q)$$

is also a reduced representation of w.

Note that $i \neq 1$. Otherwise, $(s_1, s_2, \dots, s_q) = (r_1, s_2, \dots, s_q)$, which would imply $s_1 = r_1$, a contradiction.

Next, define S_q to be the alternating word ending in s_1, with each term in the set $\{s_1, r_1\}$ of length q. Suppose $\mathbf{s}'$ is a word that starts with S_{q-1}, and let s' be the element of $\{s_1, r_1\}$ that $\mathbf{s}'$ does not start with.

Looking at the relationship between $\mathbf{s}$ and $\mathbf{r}$, we see that $l(s'w) < l(w)$. Applying the exchange condition, we obtain a reduced representation of $\mathbf{s}'$ starting with s'. Importantly, the removed element in this exchange process cannot lie in S_{q-1}, since any reduced representation with a length not equal to m in a group like D_{2m} is unique. This fact is learned in the Year 2 module *Groups and Rings*. If the removed term were from S_{q-1}, we would get two different reduced representations of the same element in D_{2m} with lengths not equal to m, which would be a contradiction.

Therefore, we obtain a reduced representation starting with S_q from $\mathbf{s}'$. By induction, we conclude that w has a reduced expression starting with S_q for any $q \leq m(s_1, r_1)$, but since $q \leq l(w) < \infty$, we have that m is finite.

Finally, by replacing $S_{m(s_1, r_1)}$ with $(s_1, r_1, s_1, \dots)$ using an MII operation, we obtain the desired reduced expression $\mathbf{u}$. $\square$

6.3 Tits' Theorem

Theorem 6.3.1. (Tits') A word in a system (W, S) with the exchange condition is reduced if and only if it is M-reduced.

Proof: Suppose the word $\mathbf{s} = (s_1, \dots, s_k)$ is M-reduced. We will show that $\mathbf{s}$ is reduced by induction on k. The base case $k = 1$ is trivial.

For $k > 1$, by the induction hypothesis, $\mathbf{s}' = (s_2, \dots, s_k)$ is reduced. Let w' be the element expressed by $\mathbf{s}'$. Suppose $\mathbf{s}$ is not reduced, then

$$l(w) = l(s_1\mathbf{s}') \leq k - 1 < l(\mathbf{s}').$$

Thus, by the exchange condition, there exists an index i such that

$$w' = s_1 s_2 \dots s_{i-1} s_{i+1} \dots s_k.$$

That is, w' has a reduced representation starting with s_1. Applying Lemma 6.2.2, we see that $\mathbf{s}'' := (s_1, s_2, \dots, s_{i-1}, s_{i+1}, \dots, s_k)$ can be transformed from $\mathbf{s}'$ by an MII operation. This gives a reduced representation of w starting with (s_1, s_1), which is a contradiction.

The other direction of the theorem is obvious. □

Note: Tits' Theorem is far from trivial. For example, when considering a group like $G = \langle x, y \mid x^n = y^2 = (xy)^2 = 1_G \rangle$, it is not immediately clear how to reduce the word of the group, as shown in the Groups and Rings module of Year 2.

Example 6.3.2. Consider the group $G := \langle x, y \mid xyxyx = yxyxy \rangle$. This can be rewritten as:

$$G \cong \langle x, y, a \mid xyxyx = yxyxy, a = xy \rangle \cong \langle x, a \mid aax = x^{-1}aaa \rangle \cong$$

$$\langle x, a, b \mid a^2x = x^{-1}a^3, b = xa^2 \rangle \cong \langle a, b \mid a^2 = b^5 \rangle.$$

In this example, we can see that while the relations defining the group G are clear, the process of simplifying or reducing these relations is far from trivial. The transition from the group generated by x and y to the one generated by a and b involves several non-trivial steps, and the operations required to perform this reduction are not immediately obvious. This illustrates the difficulty in finding a direct reduction, even when the relations are relatively simple.

There are general tools such as Tietze transformations and coset enumeration that can be used to explore what operations can be performed on a general group presentation to reduce it. However, these methods go beyond the scope of this project and are typically covered in more advanced group theory courses.

Conclusion: While Tits' Theorem provides a powerful framework for reducing words in Coxeter groups, its non-triviality is highlighted by the complexity of group presentations and the reductions that need to be applied. The example above demonstrates how the word problem in even seemingly simple groups can become quite intricate and challenging.

7 Euclidean Reflection Groups

7.1 Structure of Euclidean Reflection Groups

This section follows [8].

Let W be an affine reflection group, naturally acting on the Euclidean space V, with fixed set of affine hyperplanes $\mathcal{H}$. We fix an arbitrary chamber C as our fundamental chamber, S

be the set of reflection with respect to its walls. For an n-dimensional Euclidean space V, we have the isomorphism AO$(V) \cong V \rtimes$ O(V). Every element AO(V) represents the composition of an orthogonal linear map with a transition τ_v sending x to $x + v$. Note W can be taken as a subgroup of AO(V).

From Section **??**, we recall some facts of affine reflection groups:

1. W is generated by S.
2. W is simply transitive on chambers.
3. $H \in \mathcal{H}$ if and only if $s_H \in W$
4. $\langle e_{s_i}, e_{s_j} \rangle = -\cos(\frac{\pi}{m_{i,j}})$, where e_{s_i}, e_{s_j} are two unit vectors normal to some walls of C, pointing into C from the walls.

Theorem 7.1.1. 1. The hyperplanes $H \in \mathcal{H}$ fall into finitely many classes under parallelism;

2. Let $\overline{W}$ be the image of W under the projection $\eta : \text{AO}(V) \twoheadrightarrow \text{O}(V)$, then $\overline{W}$ is finite.

Proof. (1) Let $\Phi := \{\pm e_H : H \in \mathcal{H}\}$, where e_H is the unit vector normal to H. We will show $\angle(e_1, e_2)$ can take finitely many values. Let $H_1, H_2 \in \mathcal{H}$ normal to e_1, e_2. If they are parallel, then $\angle(e_1, e_2) = \pi$. So we may assume they intersect and take $x \in H_1 \cap H_2$. Since W is transitive on chambers, and by the fact that chambers D, D' is separated by wall H' if and only if wD, wD' is separated by wH', we can choose $w \in W$ with $wx \in \overline{C}$. Then $\overline{C}$ intersects with wH_1 and wH_2, with vectors $\bar{w}e_1, \bar{w}e_2$ normal to them, where $\bar{w} = \eta(w)$. $\angle(e_1, e_2) = \angle(\bar{w}e_1, \bar{w}e_2)$ since $\bar{w}$ is an isometry. From the inner product formula we can deduce the angle between two distinct vectors cannot be smaller than $\pi/2$, so C has finitely many walls. Then there are only finitely many possibilities for angles between two vectors in Φ, hence it is finite as we are in a finite-dimensional space.

(2) Note Φ we defined is stable under action of $\overline{W}$ and reflection with respect to the hyperplanes they are perpendicular to generates $\overline{W}$. We show the natural homomorphism $\sigma : \overline{W} \to S(\Phi)$ is injective. Suppose $w \in \ker(\sigma)$, w fixes all vectors of Φ. But then $V_0 = \text{Span}(\Phi)$ is also fixed by w. Write the vector space V as $V_0 \oplus V_1$ for V_1 some subspace of V. Then since every reflection s_e fixes every point of V_1 and W is generated by reflection of walls of C, so does w. We then conclude w is identity hence $\overline{W}$ is finite. □

Definition 7.1.2. We say W is essential if intersection of hyperplanes H $s_H \in \overline{W}$ is a singleton.

Definition 7.1.3. A Coxeter group is irreducible if its Coxeter diagram is connected.

For a general affine reflection group, we can reduce it to essential, irreducible cases. If it is irreducible, then there are sets of hyperplanes $\mathcal{H}_i$, each hyperplane in $\mathcal{H}_i$ is orthogonal to the ones in other $\mathcal{H}_j$'s. We can decompose the space into direct sum $V_1 \oplus \cdots \oplus V_n$, where each V_i corresponds to $\mathcal{H}_i$, and the corresponding reflection group W_i is irreducible in V_i. If W_i is not essential, we can quotient the intersection of all hyperplanes out to get an essential one.

Theorem 7.1.4. Assume W is essential and irreducible in n-dimensional vector space V, then one of the following is true:

1. W is finite, C has n walls;
2. W is infinite, C has $n+1$ walls, and vectors perpendicular to any n walls form a basis. $\overline{C}$ is compact.

Proof. We first count the number of walls. Suppose the walls of C are $H_1, \dots, H_k$, with corresponding normal vectors $e_1, \dots, e_k$ pointing to C. Since W is essential, $e_1, \dots, e_k$ is a spanning set hence $k \geq n$.

Suppose $k = n$. Then the intersection of all H_i's is a singleton. By shifting the planes we may assume $x = 0$. Hence all H_i's are linear and W is a finite reflection group.

Suppose $k > n$, then the list of vectors must be linearly dependent. Choose an index set I such that

$$\sum_{i \in I} \lambda_i e_i = 0$$

such that $\lambda_i \neq 0$ for all $i \in I$. We show $I = \{1, \dots, k\}$. Suppose not, then let $J = \{1, \dots, k\} \backslash I$.

Starting from $\sum_{i \in I} \lambda_i e_i = 0$, move the negative coefficients terms to the right

$$\sum_{i \in K} \lambda_i e_i = \sum_{i \in L} -\lambda_i e_i$$

where K, L is a partition of I. Then consider the inner product of left with right, since $\langle e_i, e_j \rangle$, it is non-positive. But it is also an inner product of two same vectors, then the summation is the 0 vector.

Choose $j \in J$ and take inner product

$$\sum_{i \in K} \lambda_i \langle e_i, e_j \rangle = 0$$

For $i \in K$ λ_i are positive and $\langle e_i, e_j \rangle$ are non-positive, we must have the case that $\langle e_i, e_j \rangle = 0$, which implies $s_{e_i} s_{e_j}$ has order 2. Then same thing holds for index set L. We then conclude the graph of I and J in Coxeter diagram are disconnected, contradicting irreducibility.

C now has $n + 1$ walls, so C is a bounded set. In particular $\overline{C}$ is compact. Since W is simply transitive, $\bigcup_{w \in W} w\overline{C}$ is compact if W is finite, but this union covers V so W is infinite. □

Definition 7.1.5. A Euclidean reflection group is an essential, infinite, irreducible, affine reflection group.

For the rest of this section, let W be a Euclidean reflection group. Let T be the kernel of the projection above restricted on W, then we have a short exact sequence

$$1 \to T \to W \to \overline{W} \to 1$$

Proposition 7.1.6. There exists point $x \in V$ such that its group of stabilizers W_x is isomorphic to $\overline{W}$. In particular, $W \cong T \rtimes W_x$.

Proof. By Theorem 7.1.1, let $\overline{\mathcal{H}}$ be the set of linear hyperplanes parallel to some affine planes of $\mathcal{H}$. $\overline{W}$ is generated by reflection about these hyperplanes. In fact W is generated by $\{s_{\overline{H_1}}, \dots, s_{\overline{H_n}}\}$, with $\overline{H_1}, \dots, \overline{H_n} \in \overline{\mathcal{H}}$. Choose affine hyperplanes $H_1, \dots, H_n$ from $\mathcal{H}$, H_i parallel to $\overline{H_i}$, and take their intersection, which is a single point x by Theorem 7.1.4. Linear parts of $s_{H_1}, \dots s_{H_n}$ generates $\overline{W}$ and since translations have no fixed points, W_x bijects to $\overline{W}$. □

By Proposition 7.1.6, with possibly shifting we can assume 0 is a special point and hence $W \cong T \rtimes W_0 \cong T \rtimes \overline{W}$. We can then identify T by

$$L := \{v \in V : \tau_v \in W\}$$

as a additive group.

We then have $W \cong L \rtimes \overline{W} \leq V \rtimes \mathrm{O}(V) \cong \mathrm{AO}(V)$ We next show L is a lattice i.e. L is in the form of $\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ for $\{e_i\}_{1 \leq i \leq n}$ a basis of V

Lemma 7.1.7. Non-identity elements of L are bounded away from 0, i.e. L is a discrete subgroup of V as a topological group.

Proof. Pick $x \in C$ of the fundamental chamber. Since W is simply transitive on chambers, so is L. Let
$$U = \{v \in V : x + v \in C\}$$
U is a neighbourhood of 0 since C is open. Thus $U \cap L = \{0\}$, L is bounded away from 0. $\square$

The next lemma is from theory of topological groups:

Lemma 7.1.8. If L is a discrete subgroup of the additive group of a finite-dimensional $\mathbb{R}$-vector space, then $L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_r$ for some linearly independent vectors $e_1, \ldots, e_r$.

Proof. We prove by induction on dim$V$. Assume $L \neq 0$, by discreteness, choose $e \in L$ of minimal length δ. $L \cap \mathbb{R}e$ must contain e. Operation of L is the vector addition so $\mathbb{Z}e \subseteq L \cap \mathbb{R}e$. If $(L \cap \mathbb{R}e)\backslash\mathbb{Z}e$ is nonempty, then we would have a contradiction with minimality of length of e. So $L \cap \mathbb{R}e = \mathbb{Z}e$.

For any element $v \in L\backslash\mathbb{Z}e$, we can find $w \in \mathbb{R}e$ such that distance fo v to $\mathbb{R}e$ equals $d(v, w)$. Let $u \in \mathbb{Z}e$ be of minimal distance to v. Then $d(v, w) \geq |d(v, u) - d(w, v)| \geq |\delta - \delta/2| = \delta/2$ by triangle inequality.

Now we consider the quotient group $L/\mathbb{Z}e$ as the subgroup of additive group of the quotient space $V/\mathbb{R}e$. Since we have the canonical isomorphism $(\mathbb{R}e)^\perp \to V/\mathbb{R}e$, define a map $V \to (\mathbb{R}e)^\perp$ by $v \mapsto v^\perp$. Given any two cosets $[v], [w] \in V/\mathbb{R}e$, define $d([v], [w]) = ||v^\perp - w^\perp||$, which clearly a metric. The above is actually showing $L/\mathbb{Z}e$ is a discrete subgroup of $V/\mathbb{R}e$. The lemma follows from induction. $\square$

We are now left to show the set of linearly independent vectors given in Lemma 7.1.8 is a basis of V.

Lemma 7.1.9. In the context of Lemma 7.1.8, $r = n$.

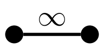
Proof. Since for any $v \in V$, there is $w \in W$ and $y \in \overline{C}$ such that $v = wy$. w is $\tau_v\overline{w}$ for $\overline{w} \in \overline{W}$ and $\tau_v \in L$. Thus v is equivalent to some $y \in \bigcup_{\overline{w}\in\overline{W}} \overline{w}C$ modulo L. It is a compact set since it is a finite union of compact ones. L is a lattice thus $r = n$. $\square$

Combining all these results, we obtain:

Theorem 7.1.10. The Euclidean reflection group W is isomorphic to $\mathbb{Z}^n \rtimes \overline{W}$, where $\overline{W}$ is a finite reflection group.

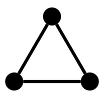
We end this section by giving some examples of Euclidean reflection groups in low dimension vector spaces.

For $\mathbb{R}^1$, there is only one finite reflection group S_2 with hyperplane 0, we can add 1 and these two points generates an affine reflection group, each element corresponds to a reflection about $k \in \mathbb{Z}$. This is called D_∞ with Coxeter diagram



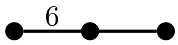
This is isomorphic to $\mathbb{Z} \rtimes S_2$.

For $\mathbb{R}^2$, for example we can tile the whole space with equilateral triangles. So let it be the chamber and we have a diagram



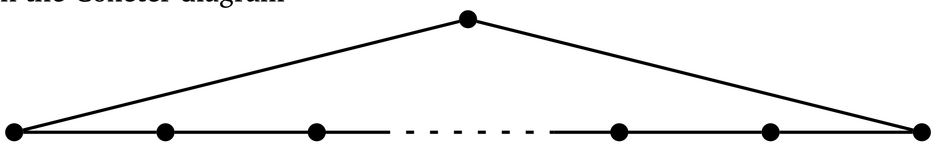
This group is isomorphic to $\mathbb{Z}^2 \rtimes D_6$.

We can tile with hexagon but themselves cannot be chambers, we can split each into 12 30-60-90 triangles. Let them be the chambers and we have a diagram



which is the group $\mathbb{Z}_2 \rtimes D_{12}$.

In fact, for many finite reflection groups $\overline{W}$, essential and irreducible, have affine analogous ones. For these groups, let V be the vector space they are acting on of dimension n. We can realise $V \rtimes \mathrm{O}(V)$ as an inner semidirect product of AO(V). Define L to be the n-rank lattice of V. Then we can define an inner semidirect product $L \rtimes \overline{W}$ which gives us the affine analogue of $\overline{W}$. An example is the finite reflection groups of type A_n, which can be realised as reflection about planes $x_i - x_j = 0$ $(i \neq j)$ in the vector space $V = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum x_i = 0\}$ with $\mathcal{H} := \{x_i - x_j = k : k \in \mathbb{Z}, i \neq j\}$. One can check $\mathcal{H}$ in fact gives a Euclidean reflection group with the Coxeter diagram


with n nodes in the graph.

A non-example is of type $I_2(m)$ except for some specific m. There is not way to cover the whole plane using reflections using triangle with an angle $< 2\pi/12$ or of $2\pi/7$ for example.

7.2 Some Model Theory of Euclidean Reflection Groups

This section follows [9].

Definition 7.2.1. We say a structure $\mathcal{N}$ is interpretable in $\mathcal{M}$ if:

1. The underlying set A is $\{\overline{x} \in M^k : \mathcal{M} \models \phi(\overline{x})\}$ for some formula ϕ.
2. For every function symbols $f(\overline{x})$ of $\mathcal{N}$, there is a formula ϕ such that $\mathcal{M} \models \phi(\overline{x}, \overline{y})$ if and only if $\mathcal{N} \models f(\overline{x}) = \overline{y}$.

Definition 7.2.2. We say the theory of a group G is decidable if there is an algorithm such that for all sentences ϕ, it can decide whether ϕ is true in G.

For a definition of a sentence and an algorithm, see Appendix **??**.

Proposition 7.2.3. Euclidean reflection group G is definable in the abelian group $\mathbb{Z}$ with finitely many parameters.

This result follows from the following two lemmas:

Lemma 7.2.4. Let $G \cong \mathbb{Z}^d \rtimes_\sigma Q$ where $1 \leq d < \omega$, Q is some finite group, and $\theta : Q \to \mathrm{Aut}(\mathbb{Z}^d)$ a group homomorphism. G is interpretable in the structure $\mathcal{M} = (\mathbb{Z}^d, +, \pi_x)_{x \in Q}$ with finitely many parameters, where $+$ is the normal vector addition and $\pi_x := \sigma(x)$

Proof. We enumerate $\{\pi_x : x \in Q\}$ as $t_0, \ldots, t_{n-1}$ and $t_i := (i, 0, \ldots, 0)$ a d-tuple. Let $G = (\mathbb{Z}^d \times Q, \cdot)$. We represent the universe of G by a tuple (a, t_i) where $a \in M$ and $0 \leq i < n$. We need to define the group operation $(a_i, t_i) \cdot (a_j, t_j) = (a_i +_{\mathbb{Z}^d} t_i(a_j), t_i \cdot_Q t_j)$ in $\mathcal{M}$. $t_i \cdot_Q t_j$ is definable since Q in finite, we can enumerate all possible product of all elements in Q. $a_i +_{\mathbb{Z}^d} t_i(a_j)$ is definable since $+$ is a built-in function symbol of $\mathcal{M}$ and t_i is same with π_x for some x. Again this claim follows from enumerating all possibilities of t_i's. $\square$

Lemma 7.2.5. $(\mathbb{Z}^d, +, \pi_x)_{x \in Q}$ is interpretable in $(\mathbb{Z}, +, 0)$ with finitely many parameters.

Proof. Since there are finitely many π_x's, it suffices to show we can define π_x in $(\mathbb{Z}, +, 0)$ for some $x \in Q$. π_x is an automorphism of $\mathbb{Z}^d$, which is an $d \times d$ invertible matrix A over $\mathbb{Z}$. $\pi_x(\overline{b}) = A\overline{b}$ is a matrix multiplication. We only need to show we can define the element-wise multiplication part is definable. To show this, note that there are only finitely many entries in A. The multiplication of $n \times b$ where n is an entry of A and b is an arbitary integer, can be realised as

$$\underbrace{b + b + \cdots + b}_{n \text{ times}}$$

if n is positive and otherwise minus b $-n$ times. $\square$

Theorem 7.2.6. Let W be a Euclidean reflection group. Then $Th(W)$ is decidable.

Proof. W is interpretable in the structure $(\mathbb{Z}, +, 0)$ with finitely parameters by Proposition 7.2.3. But all these parameters are integers, so it is interpretable in the structure $(\mathbb{Z}, +, 0, 1)$ with no parameters. We then can translate every sentence in W back to a sentence of $(\mathbb{Z}, +, 0, 1)$. Since the Presburger arithmetic $(\mathbb{Z}, +, <, 0, 1)$ is decidable, as a definable structure in its reduct, $Th(W)$ is also decidable. $\square$

A Model-Theoretic Preliminaries

The following definitions are taken from [10].

We fix our language $\mathcal{L} = (e, \cdot)$ to be the language of groups.

Definition A.0.1. The set of $\mathcal{L}$-terms is the smallest set $\mathcal{T}$ such that

1. variables $x_i \in \mathcal{T}$ for $i = 0, 1, \ldots;$
2. if $t_1, t_2 \in \mathcal{T}$ then $(t_1 \cdot t_2) \in \mathcal{T}$

Definition A.0.2. The set of $\mathcal{L}$-formulas is the smallest set $\mathcal{F}$ such that

1. all atomic formulas $(t_1 = t_2) \in \mathcal{F}$ for $t_1, t_2 \in \mathcal{T}$;
2. $\mathcal{F}$ is closed under $\neg$ ("not"), $\vee$ ("or"), $\wedge$ ("and"), $\to$ ("implies"), $\forall$ ("for all"), $\exists$ ("there exists").

By drawing a truth table or otherwise, it suffices to define the satisfaction of $\wedge$, $\neg$, and quantifiers for $\mathcal{L}$-formulas.

Definition A.0.3. We say a formula ϕ has x as a free variable if x does not appear in any quantifiers of ϕ.

Definition A.0.4. Let ϕ be a formula with free variables $x_1, \ldots, x_n$. We inductively define a group $G \models \phi$ as follows.

1. If $\phi(\overline{x})$ is $t_1(\overline{x}) = t_2(\overline{x})$, then $G \models \phi$ if t_1 and t_2 are the same element of G.
2. If $\phi(\overline{x})$ is $\neg\psi(\overline{x})$, then $G \models \phi(\overline{x})$ if $G \not\models \psi(\overline{x})$.
3. If $\phi(\overline{x})$ is $\psi_1(\overline{x}) \wedge \psi_2(\overline{x})$, then $G \models \phi(\overline{x})$ if $G \models \psi_1(\overline{x})$ and $G \models \psi_2(\overline{x})$.
4. If $\phi(\overline{x})$ is $\forall y\ \psi(\overline{x}, y)$, then $G \models \phi(\overline{x})$ if for all $a \in G$ $G \models \psi(\overline{x}, a)$.
5. If $\phi(\overline{x})$ is $\exists y\ \psi(\overline{x}, y)$, then $G \models \phi(\overline{x})$ if there is $a \in G$ $G \models \psi(\overline{x}, a)$.

If $G \models \phi(\overline{a})$ we say $\phi(\overline{a})$ is true in G

Definition A.0.5. We say a formula ϕ is a sentence if it does not have free variables.

To make the term algorithm clear, we recall definitions from computability theory.

We have infinitely many initial configuration of the register $R_1, R_2, \ldots$, each is a nonnegative integer.

Definition A.0.6. A register machine program is a finite sequence of instructions $I_1, \ldots, I_M$, where each I_j is one of the following:

1. Z(n): set R_n to zero;
2. S(n): increase R_n by one;
3. T(n,m): set R_n to be R_m;
4. J(n,m,s), where $1 \leq s \leq M$: if $R_n = R_m$, then go to I_s, otherwise go to the next instruction;
5. HALT

and I_m is HALT

Definition A.0.7. Suppose $A \subseteq \mathbb{N}^k$. $f : A \to \mathbb{N}$ is computable if there is a register machine program P such that:

1. If $x \in A$, then P does not halt on input x;
2. If $x \in A$, then P halts on input x with $R_1 = f(x)$.

Definition A.0.8. We say a set $A \subseteq \mathbb{N}^k$ if the function

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

is computable.

We then can say a theory of G is decidable if the set of true sentences in G. We can make "input of ϕ" more precise using Gödel's coding.

Theorem A.0.9. The theory of $\mathbb{Z}$ in the language of $(+, 0, 1, <, \equiv_p)$ where $\equiv_p$ is the relation of "two integers are same modulo p" is complete, admits quantifier elimination hence decidable.

References

[1] J. B. Fraleigh. *A First Course in Abstract Algebra*. Addison-Wesley, Boston, 7 edition, 1982.

[2] M. Liebeck. *A Concise Introduction to Pure Mathematics*. CRC Press, Boca Raton, FL, 4th edition, 2019.

[3] M. Berger. *Geometry I*. Universitext. Springer-Verlag, Berlin, 2009. Translated from the 1977 French original by M. Cole and S. Levy, Fourth printing of the 1987 English translation [MR0882541].

[4] J. E. Humphreys. *Reflection groups and Coxeter groups*, volume 29 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 1990.

[5] H. S. M. Coxeter. *Regular Polytopes*. Graduate Texts in Mathematics. Dover Publications, 3rd edition, 1973.

[6] N.Bourbaki. *Lie Groups and Lie Algebra,chapter 4*. Springer Cham, 2002.

[7] M. W. Davis. *The Geometry and Topology of Coxeter Groups*. Princeton University Press, 2008.

[8] P. Abramenko and K. S. Brown. *Buildings: Theory and Applications*. Graduate Texts in Mathematics. Speinger, 2008.

[9] B. Mühlherr, G. Paolini, and S. Shelah. First-order aspects of coxeter groups. *Journal of Algebra*, 595:297–346, 2022.

[10] D. Marker. *An Invitation to Mathematical Logic*. Graduate Texts in Mathematics. Springer Cham, 2024.