The plan is to show all finitely generated reflection groups are in fact Coxeter groups, which admit a nice geometric classification. This follows[**?**]
This is a rough outline of the structure:

0.1 Reflection groups

work in n-dimensional euclidean space E. This may be spoken about in the previous section?
The origin preserving reflections of E are all contained in $O(E)$, this is also the smallest group containing all of these reflections.[citation-needed](easy induction i think?) We also want to consider reflections that don't preserve the origin, or *affine reflections* these will, similarly, all lie in the **affine orthogonal group**

$$AO(E) := E \rtimes O(E)$$

where E acts on itself by translation.
A reflection in E is an affine transformation $r \in AO(E)$ that fixes some affine hyperplane H (some translation of codimension 1 subspace).
A group $W \leq AO(E)$ is a **reflection group** if it is generated by affine reflections. If we have $W \leq O(E) \leq AO(E)$ then we recover the definition used for reflection groups of regular polytopes discussed earlier.
From now on we will fix a reflection group W and start referring to affine reflections as just reflections.
Let $\mathcal{H}$ be the set of affine hyperplanes fixed by some reflection in W.

Lemma 0.1.1. If $H \in \mathcal{H}$ is an affine hyperplane and $s \in W$ is a reflection, sH is also an affine hyperplane.

Proof. A hyperplane in any euclidean space is equivalently the locus

$$\{v \in E \mid (v, \alpha) = 0\}$$

where α is the normal vector to the hyperplane. Thus an affine hyperplane with normal α and minimal distance from the origin k will be

$$\{v \in E \mid (v, \alpha) = k\}$$

Now sH will consists of points sv such that $(v, \alpha) = k$, but as s is $AO(E)$, it preserves the inner product so $(sv, s\alpha) = (v, \alpha) = k$ so

$$sH = \{v \in E \mid (sv, s\alpha) = k\} = \{v \in E \mid (v, s^* s\alpha) = k\}$$

where s^* is the adjoint map. □

Let Φ be the set of unit normal vectors to hyperplanes $H_i \in \mathcal{H}$.

Lemma 0.1.2. For all affine hyperplanes $H \in \mathcal{H}$ with corresponding reflection $r \in W$, and reflections $s \in W$ the map srs^{-1} is a reflection over the affine hyperplane sH.

Proof. □

Proposition 0.1.3. The action of W on E stabilises Φ and $\mathcal{H}$.

Proof. Start by considering some affine hyperplane $H \in \mathcal{H}$ with corresponding reflection $r \in W$, then for all $s \in W$ the map srs^{-1} fixes sH and has determinant -1 so is a reflection, therefore $sH \in \mathcal{H}$ □

We can derive the Coxeter relations from Φ, a group satisfying such relations is called Coxeter.

Definition 0.1.4. We call a group W **Coxeter** if it admits a presentation of the form:

$$\langle r_1, \ldots, r_n \mid (r_i r_j)^{m_{ij}} \text{ for all } i, j \rangle$$

where each $m_{ij} \in \mathbb{N} \cup \{\infty\}$, and $m_{ii} = 1$ for all i. For formal reasons, we will consider the pair (W, R), where R is the set of generators in the presentation, and call this a **Coxeter system**. We call a Coxeter system finite if R is finite.

0.2 The fundamental domain

The action of W fixes Φ and $\mathcal{H}$ so acts on the connexted componenets $E \setminus \mathcal{H}$ called the set of **fundamental domains** for W.
The reflections fixing hyperplanes bounding any single fundamental domain will generate W, such reflections (hyperplanes, roots) are called **simple**.
Show the action of W on the fundamental domains is transitive

0.3 Words

The length of a word in terms of simple reflections corresponds to the number of plane between a fundamental domain and its image. This implies the action on fundamental domains is in fact **simply** transitive.

Proposition 0.3.1 (Deletion condition)**.** Given an unreduced expression $w = r_1 \cdots r_k$ there exists $1 \leq i < j \leq k$ such that $w = r_1 \cdots \hat{r_i} \cdots \hat{r_j} \cdots r_k$, where the hat means ommitance.

Proposition 0.3.2 (Exchange condition)**.** For $w = r_1 \cdots r_k$ a not necessarily reduced expression and some simple $r \in W$ with $l(wr) < l(w)$, then there exists an $1 \leq i \leq k$ s.t. $wr = r_1 \cdots \hat{r_i} \cdots r_k$.

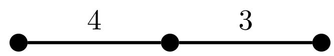
Theorem 0.3.3. Any non-trivial relation in a reflection group is a consequence of the Coxeter relations.

0.4 Classification

To any finite Coxeter system (W, R) we can associate an undirected graph called its **Coxeter diagram** by the following rules:

- Draw a node i for each $r_i \in R$;
- For each relation $(r_i r_j)^{m_{ij}}$ with $m_{ij} > 2$ draw an edge between i and j and label it with m_{ij}.

This process can be reversed to obtain a Coxeter system from any Coxeter diagram. This correspondence will associate the graph:



to the group presentation:

$$\left\langle r_1, r_2, r_3 \;\middle|\; r_1^2 = r_2^2 = r_3^2 = e,\ (r_1 r_2)^4 = (r_2 r_3)^3 = (r_1 r_3)^2 = e \right\rangle$$

For brevity the 3 labels will often be excluded.

- show the graph is well defined up isometryish

- disjoint unions of graphs correspond to products of groups
- bilinear form of a coxeter group
- if positive definite, the coxeter group is finite
- classify positive-definite forms
- all of which can be seen as reflection groups of regular polyhedra. some of which will be described in the previous section?