The plan is to show all finitely generated reflection groups are in fact Coxeter groups, which admit a nice geometric classification. This follows[**?**]
This is a rough outline of the structure:

0.1 Reflection groups

work in n-dimensional euclidean space E.
consider the group of affine transformations $GL(E) \ltimes E$, with inner product preserving subgroup O.
A reflection in E is an affine transformation $r \in O$ that fixes some affine hyperplane H (linear transformation of codimension 1 subspace).
a reflection group W is generated by reflections in the euclidean space. let $\mathcal{H}$ by the set of hyperplanes $H_i \subseteq E$ fixed by some reflection $r_i \in W$.
Choosing a unit normal vector α_i to each H_i gives a **root system** Φ.
We can derive the Coxeter relations from Φ, a group satisfying such relations is called Coxeter.

Definition 0.1.1. We call a group W **Coxeter** if it admits a presentation of the form:

$$\langle r_1, \ldots, r_n \mid (r_i r_j)^{m_{ij}} \text{ for all } i, j \rangle$$

where each $m_{ij} \in \mathbb{N} \cup \{\infty\}$, and $m_{ii} = 1$ for all i. For formal reasons, we will consider the pair (W, R), where R is the set of generators in the presentation, and call this a **Coxeter system**. We call a Coxeter system finite if R is finite.

0.2 The fundamental domain

The action of W fixes Φ and $\mathcal{H}$ so acts on the connexted componenets $E \setminus \mathcal{H}$ called the set of **fundamental domains** for W.
The reflections fixing hyperplanes bounding any single fundamental domain will generate W, such reflections (hyperplanes, roots) are called **simple**.
Show the action of W on the fundamental domains is transitive

0.3 Words

The length of a word in terms of simple reflections corresponds to the number of plane between a fundamental domain and its image. This implies the action on fundamental domains is in fact **simply** transitive.

Proposition 0.3.1 (Deletion condition)**.** Given an unreduced expression $w = r_1 \cdots r_k$ there exists $1 \leq i < j \leq k$ such that $w = r_1 \cdots \hat{r_i} \cdots \hat{r_j} \cdots r_k$, where the hat means ommitance.

Proposition 0.3.2 (Exchange condition)**.** For $w = r_1 \cdots r_k$ a not necessarily reduced expression and some simple $r \in W$ with $l(wr) < l(w)$, then there exists an $1 \leq i \leq k$ s.t. $wr = r_1 \cdots \hat{r_i} \cdots r_k$.

Theorem 0.3.3. Any non-trivial relation in a reflection group is a consequence of the Coxeter relations.

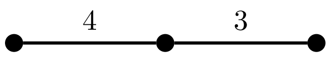
0.4 Classification

To any finite Coxeter system (W, R) we can associate an undirected graph called its **Coxeter diagram** by the following rules:

- Draw a node i for each $r_i \in R$;

- For each relation $(r_i r_j)^{m_{ij}}$ with $m_{ij} > 2$ draw an edge between i and j and label it with m_{ij}.

This process can be reversed to obtain a Coxeter system from any Coxeter diagram. This correspondence will associate the graph:

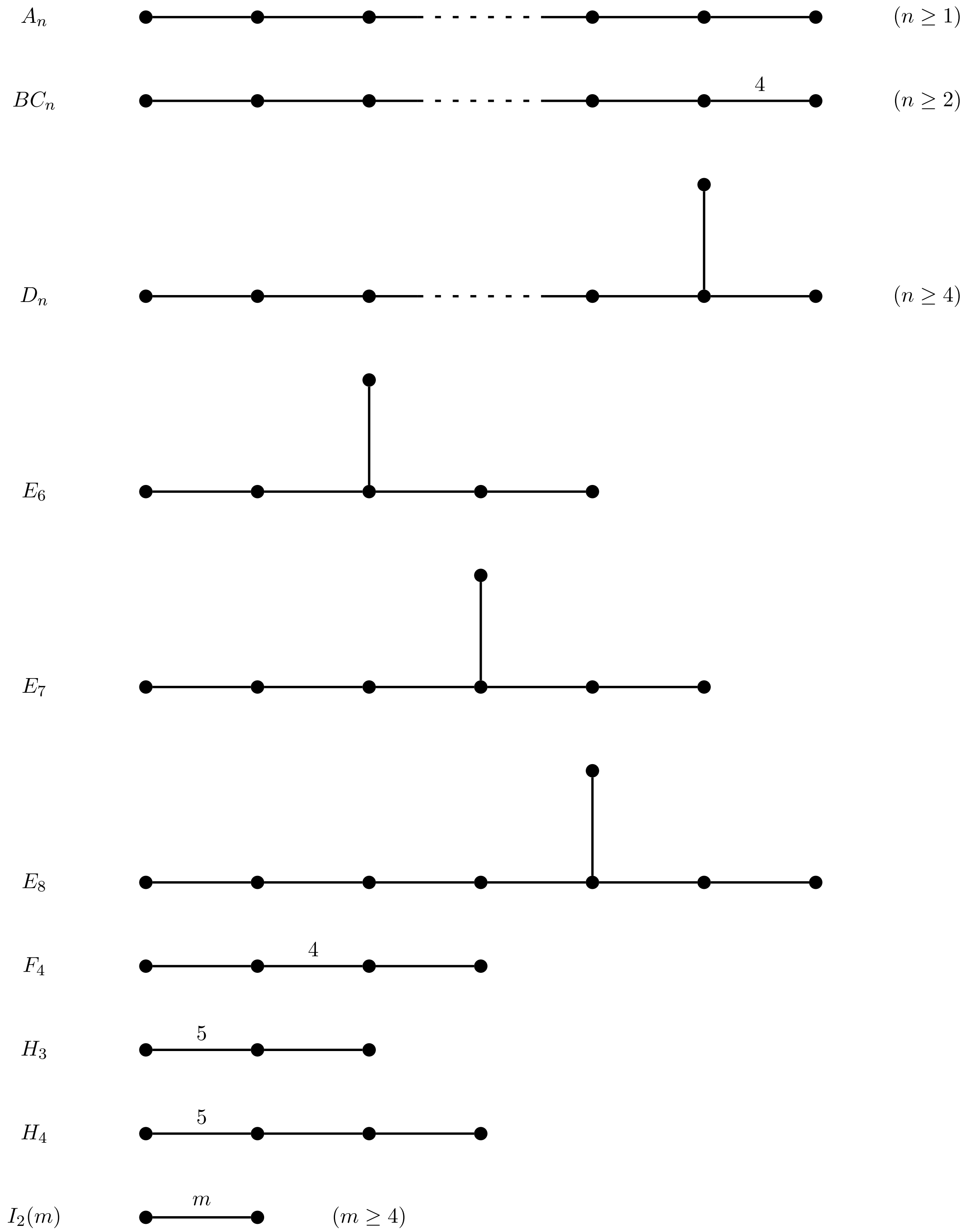


to the group presentation:

$$\left\langle r_1, r_2, r_3 \mid r_1^2 = r_2^2 = r_3^2 = e,\ (r_1 r_2)^4 = (r_2 r_3)^3 = (r_1 r_3)^2 = e \right\rangle$$

For brevity the 3 labels will often be excluded.

- show the graph is well defined up isometryish
- disjoint unions of graphs correspond to products of groups
- bilinear form of a coxeter group
- if positive define the coxeter group is finite
- classify positive-definite forms
- all of which can be seen as reflection groups of regular polyhedra. some of which will be described in the previous section?


A_n
$(n \geq 1)$
BC_n
4
$(n \geq 2)$
D_n
$(n \geq 4)$
E_6
E_7
E_8
F_4
4
H_3
5
H_4
5
$I_2(m)$
m
$(m \geq 4)$