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IMPERIAL COLLEGE LONDON

DEPARTMENT OF MATHEMATICS

SECOND-YEAR GROUP RESEARCH PROJECT

Reflection Groups

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Abstract

Type your abstract here. The abstract is a summary of the contents of the project. It should be brief but informative, and should avoid technicalities as far as possible.

Contents

1 Introduction

The introduction should attempt to set your work in the context of other work done in the field. It should demonstrate that you are aware of what you are doing, and how it relates to other work (with references). It should also provide an overview of the contents of the project. You should highlight your individual contributions and any novel result: which of the calculations, theorems, examples, proofs, conjectures, codes etc. are your own? This is how you cite a reference in the bibliography[1]. All of the commands and formatting are in ./style/header.sty

2 Polytopes

2.1 Polytopes

2.1.1 Dihedral groups

The dihedral group, D_{2n}, is the group of symmetries of a regular n-gon. Its standard presentation is given by

$$\langle r, s \mid r^n = e,\ s^2 = e,\ (rs)^2 = e \rangle$$

where r is a rotation of $2\pi/n$ and s is a reflection.

Let l_1 and l_2 be two reflection axes with an angle θ between l_1 and l_2, and s_1 and s_2 be the respective reflections. After some algebra, the composition $s_1 s_2$ turns out to be a counterclockwise rotation through 2π.

Therefore, an alternative presentation of D_{2n} is given by

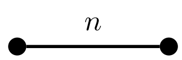
$$\langle s_1, s_2 \mid s_1^2 = e,\ s_2^2 = e,\ (s_1 s_2)^n = e \rangle$$

where s_1 and s_2 are adjacent reflections.

This shows that D_{2n} is an example of a reflection group.

Example 2.1.2. The coxeter diagram for the symmetries of a regular n-gon, also known as $I_2(n)$, looks like

$I_2(n)$



$(n \geq 4)$

Theorem 2.1.3. Let $G \curvearrowright X$ be an action of a finite group G on a finite set X. Then the number of G-orbits in X is given by:

$$\text{Number of orbits} = \frac{1}{|G|} \sum_{g \in G} |\text{Fix}(g)|$$

where $\text{Fix}(g) = \{x \in X \mid g \cdot x = x\}$.

Example 2.1.4. How many distinguishable necklaces can be made using seven different colored beads of the same size?
Let X be the 7! possible arrangements. The necklace can be turned over (a reflection) as well as rotated so we consider the dihedral group D_{14} acting on X. Using the previous theorem,

$$\text{Number of orbits} = \frac{7!}{14} = 360$$

as only the identity leaves any arrangement fixed.

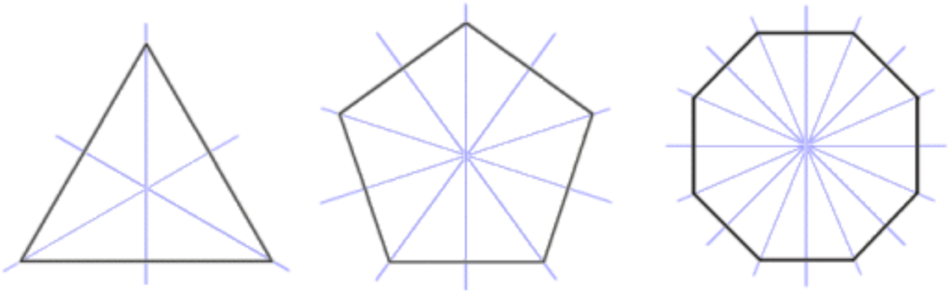


Figure 1 The lines of symmetry of a regular 3-,4- and 7-gon.

Theorem 2.1.5. Let p be a prime. Then any group G of order $2p$ is isomorphic to either the cyclic group C_{2p} or the dihedral group D_{2p}.
Proof. By Cauchy's theorem, there exists an element $a \in G$ of order p and an element $b \in G$ of order 2. Let

$$H = \langle a \rangle,$$

so H is a subgroup of G of index 2, so $H \trianglelefteq G$.
Since $H \trianglelefteq G$, conjugation by b sends H to itself. Thus, there exists $k \in \{1, 2, \dots, p-1\}$ such that

$$bab^{-1} = a^k.$$

Applying conjugation by b twice to a gives

$$a = b^2ab^{-2} = b(bab^{-1})b^{-1} = ba^kb^{-1} = (bab^{-1})^k = (a^k)^k = a^{k^2}.$$

Therefore,

$$a = a^{k^2} \implies a^{k^2-1} = e.$$

Since a has order p, this implies

$$p \mid (k^2 - 1),$$

or equivalently,

$$k^2 \equiv 1 \pmod{p}.$$

Because p is prime, this implies

$$k \equiv \pm 1 \pmod{p}.$$

- If $k \equiv 1$, then

  $$bab^{-1} = a,$$

  and b commutes with a. Hence G is abelian, and since a has order p and b has order 2, G is cyclic of order $2p$.

- If $k \equiv -1$, then
$$bab^{-1} = a^{-1},$$
which is the defining relation for the dihedral group D_{2p}:
$$D_p = \langle a, b \mid a^p = e, b^2 = e, bab = a^{-1} \rangle.$$

Thus, G is isomorphic to either C_{2p} or D_{2p}.

□

2.2 The Infinite dihedral group

3 Platonic solids and reflections

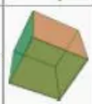
Polyhedron		Vertices	Edges	Faces
tetrahedron		4	6	4
cube / hexahedron		8	12	6
octahedron		6	12	8
dodecahedron		20	30	12
icosahedron		12	30	20

Figure 2 The five Platonic solids.

Definition 3.0.1. A polyhedron is regular if its faces are regular polygons, all with the same number of sides, and also each vertex belongs to the same number of edges.

Theorem 3.0.2. The only regular convex polyhedra are the five Platonic solids.
Proof. Before writing the proof, we introduce some notations. V, the number of vertices;
E, the number of edges;
F, the number of faces;
n, the number of sides on a face;
r, the number of edges to which each vertex belongs.
Observe that
$$2E = nF \tag{1}$$
and
$$2E = rV \tag{2}$$

(1) comes from counting the number of pairs (e,f) where e is an edge and f is a face and e lies on f; (2) comes from counting the number of pairs (v,e) where v is a vertex and v lies on e. Substitute into Euler's formula, we get

$$\frac{1}{r} + \frac{1}{n} = \frac{1}{2} + \frac{1}{E} \tag{3}$$

Now $n \geq 3$, as a polygon must have at least 3 sides and $r \geq 3$, since in a polyhedron a vertex must belong to at least 3 edges. By (3), we can't have both $n \geq 4$ and $r \geq 4$, since this would make the left-hand side of (3) at most $\frac{1}{2}$. It follows that either $n = 3$ or $r = 3$. If $n = 3$, then (3) becomes

$$\frac{1}{r} = \frac{1}{6} + \frac{1}{E} \tag{4}$$

The right-hand side is greater than $\frac{1}{6}$, and hence $r < 6$. Therefore, $r = 3$, 4 or 5 and $E = 6$, 12 or 30, respectively. If $r = 3$, (3) becomes

$$\frac{1}{n} = \frac{1}{6} + \frac{1}{E} \tag{5}$$

Similarly, n = 3,4 or 5 and E = 6,12 or 30, respectively. These parameters are coincide with those in the table above.

Example 3.0.3. Let G be the group of symmetries of a dodecahedron. What is $|G|$?
Let G act on the 12 faces of the dodecahedron and fix a face. There are $|D_{10}| = 10$ symmetries which fix this face and our action is clearly transitive. By Orbit-Stabiliser theorem, $|G| = 10 \times 12 = 120$. Alternatively, this can be done by considering the fundamental domain, which is a triangle that uniquely determines the reflection. There are 120 such triangles.

We would like to study the group of symmetries of these Platonic solids.

3.1 Tetrahedron

Each face has D_6 as a group of symmetries.
Before moving on to the other solids, we first introduce the concept of dual.

Definition 3.1.1. The dual of a Platonic solid is a new Platonic solid where the faces and vertices are interchanged with those of the original.

Remark 3.1.2. The tetrahedron is self-dual. The cube and the octahedron form a dual pair. The dodecahedron and the icosahedron form a dual pair.

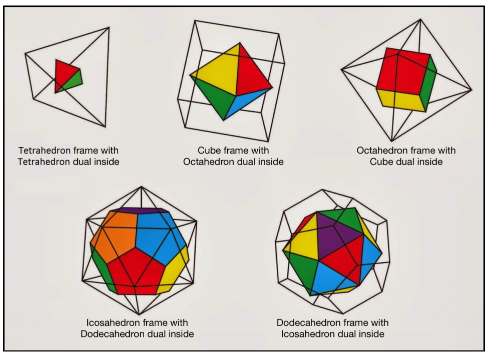


Figure 3 Duals of each Platonic solid.

4 Classification of finite reflection groups

Throughout this section, E will denote $\mathbb{R}^n$ equipped with the standard inner product, our fixed *euclidean space*.

4.1 Affine reflections

This subsection roughly follows [2]. When considering reflections in E, we don't want to just consider those lying in $O(E)$, we are also interested in reflections across some hyperplane not lying throught the origin. To tackle this we first define some standard preliminaries:

Definition 4.1.1. A **reflection** in E is a linear transformation $s \in O(E)$ (the orthogonal group of inner product preserving linear maps) with eigenvalues $\{-1, 1\}$ with corresponding dimensions of eigenspaces:

$$\dim E_1 = n - 1 \qquad \dim E_{-1} = 1$$

for linear maps in $O(E)$ this is equivalent to fixing some hyperplane (codimension 1 subspace) and having determinant -1.

Definition 4.1.2. The **group of general affine transformations** of E is semidirect product of $GL(E)$ acting on E

$$GA(E) := E \rtimes GL(E)$$

where E acts on itself by translation. $(v, T) \in GA(E)$ acts on $e \in E$ as $(v, T) \cdot e = v + T(e)$.

Proposition 4.1.3. This is actually an action. Furthermore, the action is transitive.

Proof. Take $(v, T), (u, S) \in GA(E)$ and $e \in E$. Showing this is an action is a direct calculation:

$$(v,T)\cdot((u,S)\cdot e) = (v,T)\cdot(u+S(e)) = v+T(u)+TS(e) = (v+T(u),TS)\cdot e = ((v,T)(u,S))\cdot e$$

from the definition of the semidirect product. Suppose (v, T) and (u, S) act the same on E: as both T and S are linear the two affine transformations send 0 to v and u respective, thus we have $v = u$; subtracting these equal translations and having equality means the two linear transformations must also be equal. □

We can now consider appropriate affine versions of linear groups, the most important for us will be the **affine orthogonal group** $AO(E)$, which can be equivalently viewed as the subgroup of $GA(E)$ preserving the inner product or as $E \rtimes O(E)$. We need stricter criteria than just the linear component of our affine transformation be a reflection to suitably capture the notion of a reflection across an affine hyperplane.

Proposition 4.1.4. For a unit normal vector α and some $k \in \mathbb{R}$, reflection across the affine hyperplane $(E, \alpha) = k$ corresponds to the affine transformation $(2k\alpha, s_\alpha)$, where s_α is the reflection along α.

Proof. Call the affine hyperplane H and choose a $v \in E$. The vector orthogonal to H that goes to v has length $k-(v,\alpha)$ so reflecting across H send v to $v+2(k-(v,\alpha))\alpha = 2k\alpha+(v-2(v,\alpha)\alpha) = (2k\alpha, s_\alpha)\cdot v$. □

We call such affine transformations, **affine reflections**.

Lemma 4.1.5. For all affine hyperplanes H and affine reflections r, the set rH is also an affine hyperplane.

Proof. Let H be $\{v \in V \mid (v, \alpha) = k\}$ for some $\alpha \in E, k \in \mathbb{R}$. As r is bijective the set rH is equal to $\{w \in V \mid (r^{-1}w, \alpha) = k\}$, by writing $r = (u, T) \in GA(E)$ this can be rewritten as $\{w \in V \mid (w, T^*(\alpha)) = k - (u, \alpha)\}$, an affine hyperplane. □

Lemma 4.1.6. An affine transformations $r = (u, T)$ that fixes some affine hyperplane and is an involution must be an affine reflection.

Proof. As $r^2 = \mathrm{id}$, on 0 we have $r^2(0) = u + T(u) = 0$. So as $r^2 = u + T(u) + A^2$ this means A is also an involution so we can use the primary decomposition $E = E_1 \oplus E_{-1}$ into the eigenspaces of A. Call the hyperplane r fixes H, then for any $h = v_1 + v_{-1} \in H$ (where $v_1, v_{-1} \in V_1, V_{-1}$ respectively) we have $r(v_1 + v_{-1}) = u + T(v_1 + v_{-1}) = u + v_1 - v_{-1} = v_1 + v_{-1}$ therefore $2v_{-1} = u$ so $\dim E_{-1} = 1$ and T is the reflection along $V_{-1} = \langle v_{-1} \rangle$, thus $r = \left(2 \|v_{-1}\| \hat{v}_{-1}, s_{v_{-1}}\right)$. □

Proposition 4.1.7. For all affine reflections r, s with s reflecting across the affine hyperplane H, the affine transformation rsr^{-1} is an affine reflection across rH.

Proof. First, notice $\left(rsr^{-1}\right)^2 = \mathrm{id}$ as both r and s are involutions. Also, $rsr^{-1}(rH) = rH$ as s fixes H. □

From now on, we will use the umbrella term *reflection* to refer to both linear and affine reflecitons.

4.2 Reflection groups

The plan is to show all finitely generated reflection groups are in fact Coxeter groups, which admit a nice geometric classification. This follows [1]
We are interested in groups generated by reflections in E, so throughout the next two sections fix a group $W \leq GA(E)$ which can be generated by reflections.
Consider the set $\mathcal{H}$ of all hyperplanes such that an element $w \in W$ is the reflection across. If we make a poor choice of our reflections generating W we may end up with $\mathcal{H}$ dense in E.

Example 4.2.1. Consider the following set of hyperplanes in $\mathbb{R}^2$, by previous lemma $\mathcal{H}$ is closed under the induced action from W so any such pentagon will generate a smaller pentagon, inverted in its center, this infinite descent makes $\mathcal{H}$ dense in $\mathbb{R}^2$.

To remedy this we will restrict the reflection groups we consider by rquiring for any compact subset $B \subset E$, the intersection $\mathcal{H} \cap B$ be finite.

Definition 4.2.2. The connected components of $E \setminus \mathcal{H}$ are called the **chambers** of W in E.

Proposition 4.2.3. The number of hyperplanes touching any single chamber is finite.

We now want to choose a chamber C_0, and consider the set of hyperplanes $\{H_1, \ldots, H_k\} \subseteq \mathcal{H}$ bounding C. We will call the corresponding reflections across these hyperplanes $\{s_1, \ldots, s_k\}$ **simple reflections**.

Theorem 4.2.4. The set of simple reflections $\{s_1, \ldots, s_k\}$ generates W.

Proof. Let W' be the subgorup of W generated by the simple reflections. Let s be one of the reflections generating W, and call the hyperplane it reflects across H. If W' acts transitively on the set of chambers then there will exists some $w \in W'$ such that $wH_i = H$ for some simple reflection s_i as H will must bound a chamber. Thus, by an earlier lemma, $ws_iw^{-1} = s$ so $s \in W'$ and $W = W'$. Now we just have to show the action of W' is transitive on chambers. Suppose it isn't, i.e. there is some chamber C such that no $w \in W'$ satisfies $wC = C_0$. Let C' be the closest chamber to C_0 in the W' orbit of C, as $C' \neq C_0$ there must be some simple hyperplane (the boundary of C_0) between them, reflecting across this must strictly decrease the distance between the two chambers contradicting the minimality of C'. Thus W' acts transitively on the set of chambers. □

As a direct corollary of this proof we now know W acts transitively on the set of chambers. Before discussing this further we should examine the relations that these simple reflections satisfy.
For any two simple reflections s_i, s_j the subgroup $\langle s_i, s_j \rangle$ will be dihedral, as seen in the previous section, (note that this will be the infinite dihedral group iff the hypeplanes being reflected along are parallel), call the order of this dihedral group $2m_{ij}$. The product $s_i s_j$ will have order m_{ij} in W and so W satisfies the set of relations $(s_i s_j)^{m_{ij}} = \text{id}$ for all i, j, taking $m_{ii} = 1$. A group *presented* by these relations is called **Coxeter**.

Definition 4.2.5. A **Coxeter system** is a pair (W, S) where $S = \{s_i\}_{i \in I}$ is a generating set for W which admits the presentation:

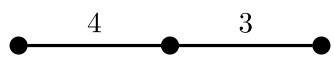
$$W = \langle S \mid (s_i s_j)^{m_{ij}} \text{ for all } i, j \in I \rangle$$

where each $m_{ij} \in \mathbb{N} \cup \{\infty\}$.

To each Coxeter system we can assign a **Coxeter diagram**: an undirected graph created by the following rules:

- Draw a node i for each $s_i \in S$;
- For each relation $(s_i s_j)^{m_{ij}}$ with $m_{ij} > 2$ draw an edge between i and j and label it with m_{ij}.

This process can be reversed to obtain a Coxeter system from any Coxeter diagram. This correspondence will associate the graph:



to the group presentation:

$$\left\langle s_1, s_2, s_3 \;\middle|\; s_1^2 = s_2^2 = s_3^2 = e,\ (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \right\rangle$$

In the future, for the sake of readability, the 3 labels will often be excluded.
The classification of finite reflection groups goes by prooving all reflection groups are infact Coxeter groups and then classifying all the finite Coxeter groups.

4.3 Coxeter presentation

The length of a word in terms of simple reflections corresponds to the number of plane between a fundamental domain and its image. This implies the action on fundamental domains is in fact **simply** transitive.

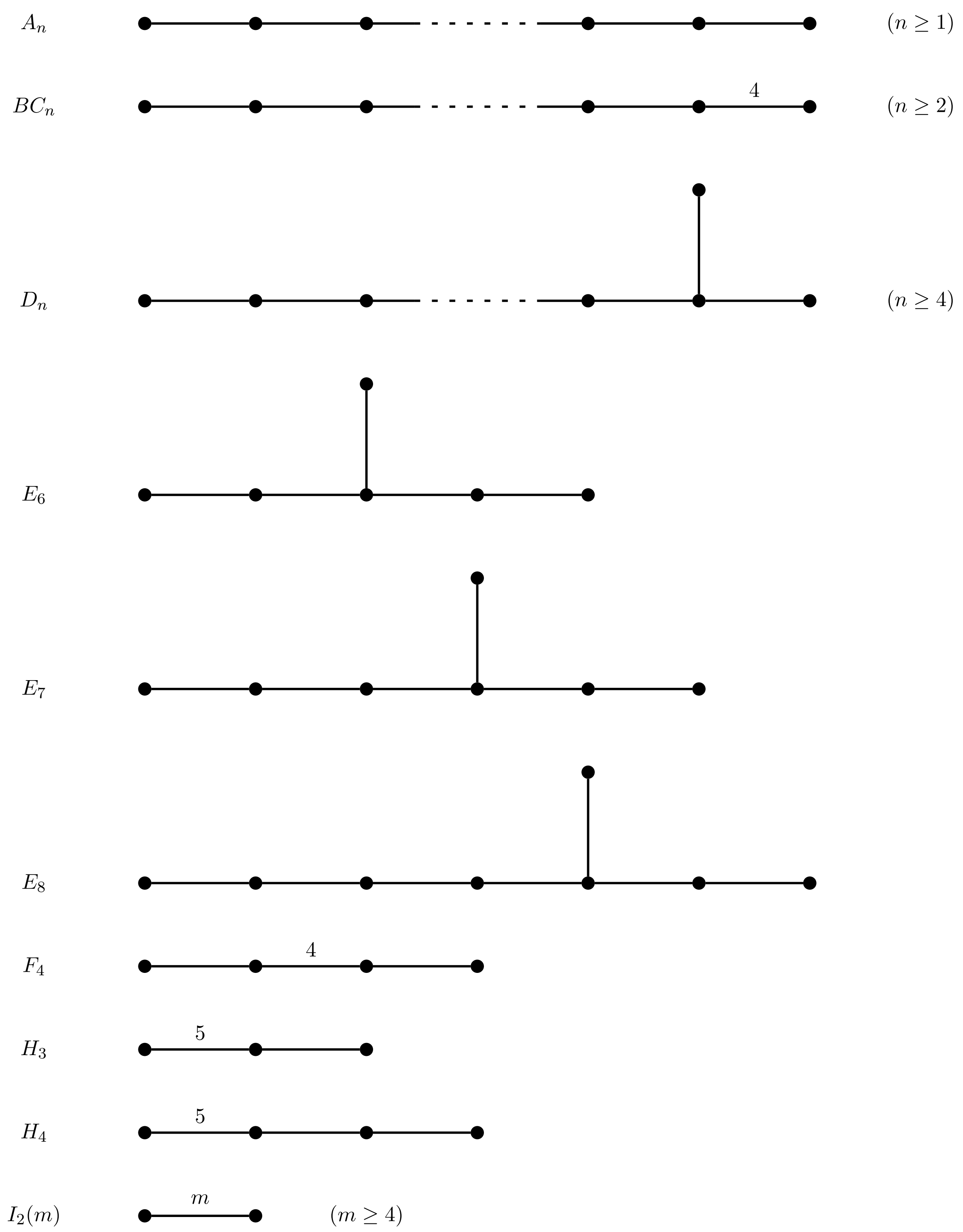
Proposition 4.3.1 (Deletion condition). Given an unreduced expression $w = r_1 \cdots r_k$ there exists $1 \le i < j \le k$ such that $w = r_1 \cdots \hat{r_i} \cdots \hat{r_j} \cdots r_k$, where the hat means ommitance.

Proposition 4.3.2 (Exchange condition). For $w = r_1 \cdots r_k$ a not necessarily reduced expression and some simple $r \in W$ with $l(wr) < l(w)$, then there exists an $1 \le i \le k$ s.t. $wr = r_1 \cdots \hat{r_i} \cdots r_k$.

Theorem 4.3.3. Any non-trivial relation in a reflection group is a consequence of the Coxeter relations.

4.4 Classification

- show the graph is well defined up isometryish
- disjoint unions of graphs correspond to products of groups
- bilinear form of a coxeter group
- if positive definite, the coxeter group is finite
- classify positive-definite forms
- all of which can be seen as reflection groups of regular polyhedra. some of which will be described in the previous section?



5 Uniform polytopes

uniform polytopes

6 Tits' word problem

words

7 Euclidean Reflection Groups

7.1 Structure of Euclidean Reflection Groups

This section follows AB8, cite in final version
Let W be an affine reflection group, naturally acting on the Euclidean space V, with fixed set of affine hyperplanes $\mathcal{H}$. We fix an arbitrary chamber C as our fundamental chamber, S be the set of reflection with respect to its walls. For an n-dimensional Euclidean space V, we have the isomorphism $\mathrm{AO}(V) \cong V \rtimes \mathrm{O}(V)$. Every element $\mathrm{AO}(V)$ represents the composition of an orthogonal linear map with a transition τ_v sending x to $x+v$. Note W can be taken as a subgroup of $\mathrm{AO}(V)$.

Theorem 7.1.1.
1. The hyperplanes $H \in \mathcal{H}$ fall into finitely many classes under parallelism;
2. Let $\overline{W}$ be the image of W under the projection $\eta : \mathrm{AO}(V) \twoheadrightarrow \mathrm{O}(V)$, then $\overline{W}$ is finite.

Proof. (1) Let $\Phi := \{\pm e_H : H \in \mathcal{H}\}$, where e_H is the unit vector normal to H. We will show $\angle(e_1, e_2)$ can take finitely many values. Let $H_1, H_2 \in \mathcal{H}$ normal to e_1, e_2. If they are parallel, then $\angle(e_1, e_2) = \pi$. So we may assume they intersect and take $x \in H_1 \cap H_2$. Since W is transitive on chambers, and by the fact that chambers D, D' is separated by wall H' if and only if wD, wD' is separated by wH', we can choose $w \in W$ with $wx \in \overline{C}$. Then $\overline{C}$ intersects with wH_1 and wH_2, with vectors $\bar{w}e_1, \bar{w}e_2$ normal to them, where $\bar{w} = \eta(w)$. $\angle(e_1, e_2) = \angle(\bar{w}e_1, \bar{w}e_2)$ since $\bar{w}$ is an isometry. C has finitely many walls so there are only finitely many possibilities for angles between two vectors in Φ, hence it is finite as we are in a finite-dimensional space.
(2) Note Φ we defined is stable under action of $\overline{W}$ and reflection with respect to the hyperplanes they are perpendicular to generates $\overline{W}$. We show the natural homomorphism $\sigma : W \to S(\Phi)$ is injective. Suppose $w \in \ker(\sigma)$, w fixes all vectors of Φ. But then $V_0 = \mathrm{Span}(\Phi)$ is also fixed by w. Write the vector space V as $V_0 \oplus V_1$ for V_1 some subspace of V. Then since every reflection s_e fixes every point of V_1, so does w. We then conclude w is identity hence $\overline{W}$ is finite. □

Definition 7.1.2. We say W is essential if intersection of hyperplanes H $s_H \in \overline{W}$ is a singleton.

Definition 7.1.3. A Coxeter group is irreducible if its Coxeter diagram is connected.

For a general affine reflection group, we can reduce it to essential, irreducible cases. If it is irreducible, then there are sets of hyperplanes $\mathcal{H}_i$, each hyperplane in $\mathcal{H}_i$ is orthogonal to the ones in other $\mathcal{H}_j$'s. We can decompose the space into direct sum $V_1 \oplus \cdots \oplus V_n$, where each V_i corresponds to $\mathcal{H}_i$, and the corresponding reflection group W_i is irreducible in V_i. If W_i is not essential, we can quotient the intersection of all hyperplanes out to get an essential one.

Theorem 7.1.4. Assume W is essential and irreducible in n-dimensional vector space V, then one of the following is true:

1. W is finite, C has n walls;
2. W is infinite, C has $n+1$ walls, and vectors perpendicular to any n walls form a basis. $\overline{C}$ is compact.

Proof. □

Definition 7.1.5. A Euclidean reflection group is an infinite, irreducible, affine reflection group.

For the rest of this section, let W be a Euclidean reflection group. Let T be the kernel of the projection above restricted on W, then we have a short exact sequence

$$1 \to T \to W \to \overline{W} \to 1$$

By possibly taking quotient of the vector space V, we may assume that the intersection

$$V_0 = \bigcap_{s_H \in \overline{W}} H$$

does not have positive dimension. In this case:

Proposition 7.1.6. There exists point $x \in V$ such that its group of stabilizers W_x is isomorphic to $\overline{W}$. In particular, $W \cong T \rtimes W_x$.

Proof. By Theorem 7.1.1, let $\overline{\mathcal{H}}$ be the set of linear hyperplanes parallel to some affine planes of $\mathcal{H}$. $\overline{W}$ is generated by reflection about these hyperplanes. Choose n hyperplanes $H_1, \ldots, H_n$ from $\mathcal{H}$, each parallel to a distinct linear hyperplane of these walls and take their intersection, which is a single point x [Why? Consider this as a system of linear equations]. Linear parts of $s_{H_1}, \ldots s_{H_n}$ generates $\overline{W}$ and since translations have no fixed points, W_x bijects to $\overline{W}$. $\square$

By Proposition 7.1.6, with possibly shifting we can assume 0 is a special point and hence $W \cong T \rtimes W_0 \cong T \rtimes \overline{W}$. We can then identify T by

$$L := \{v \in V : \tau_v \in W\}$$

as an additive group.
We then have $W \cong L \rtimes \overline{W} \leq V \rtimes \mathrm{O}(V) \cong \mathrm{AO}(V)$ We next show L is a lattice i.e. L is in the form of $\mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ for $\{e_i\}_{1 \leq i \leq n}$ a basis of V

Lemma 7.1.7. Non-identity elements of L are bounded away from 0, i.e. L is a discrete subgroup of V as a topological group.

Proof. Pick $x \in C$ of the fundamental chamber. Since W is simply transitive on chambers, so is L. Let

$$U = \{v \in V : x + v \in C\}$$

U is a neighbourhood of 0 since C is open. Thus $U \cap L = \{0\}$, L is bounded away from 0. $\square$

The next lemma is from theory of topological groups:

Lemma 7.1.8. If L is a discrete subgroup of the additive group of a finite-dimensional $\mathbb{R}$-vector space, then $L = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_r$ for some linearly independent vectors $e_1, \ldots, e_r$.

Proof. We prove by induction on dim$V$. Assume $L \neq 0$, by discreteness, choose $e \in L$ of minimal length δ. $L \cap \mathbb{R}e$ must contain e. Operation of L is the vector addition so $\mathbb{Z}e \subseteq L \cap \mathbb{R}e$. If $(L \cap \mathbb{R}e) \backslash \mathbb{Z}e$ is nonempty, then we would have a contradiction with minimality of length of e. So $L \cap \mathbb{R}e = \mathbb{Z}e$.
For any element $v \in L \backslash \mathbb{Z}e$, we can find $w \in \mathbb{R}e$ such that distance fo v to $\mathbb{R}e$ equals $d(v, w)$. Let $u \in \mathbb{Z}e$ be of minimal distance to v. Then $d(v, w) \geq |d(v, u) - d(w, v)| \geq |\delta - \delta/2| = \delta/2$ by triangle inequality.
Now we consider the quotient group $L/\mathbb{Z}e$ as the subgroup of additive group of the quotient space $V/\mathbb{R}e$. Since we have the canonical isomorphism $(\mathbb{R}e)^\perp \to V/\mathbb{R}e$, define a map $V \to (\mathbb{R}e)^\perp$ by $v \mapsto v^\perp$. Given any two cosets $[v], [w] \in V/\mathbb{R}e$, define $d([v], [w]) = ||v^\perp - w^\perp||$, which clearly a metric. The above is actually showing $L/\mathbb{Z}e$ is a discrete subgroup of $V/\mathbb{R}e$. The lemma follows from induction. $\square$

We are now left to show the set of linearly independent vectors given in Lemma 7.1.8 is a basis of V.

Lemma 7.1.9. In the context of Lemma 7.1.8, r=n.

Proof. Since $\overline{W}$ is generated by n reflections $\square$

7.2 Some Model Theory of Euclidean Reflection Groups

This section follows MPS22, cite in final version

Definition 7.2.1. We say a structure $\mathcal{N}$ is interpretable in $\mathcal{M}$ if:

1. The underlying set A is $\{\overline{x} \in M^k : \mathcal{M} \models \phi(\overline{x})\}$ for some formula ϕ.
2. For every function symbols $f(\overline{x})$ of $\mathcal{N}$, there is a formula ϕ such that $\mathcal{M} \models \phi(\overline{x}, \overline{y})$ if and only if $\mathcal{N} \models f(\overline{x}) = \overline{y}$.

Definition 7.2.2. We say the theory of a group G is decidable if there is an algorithm such that for all sentences ϕ, it can decide whether ϕ is true in G.

For a definition of sentence and algorithm, see Cite appendix

Proposition 7.2.3. Euclidean reflection group G is definable in the abelian group $\mathbb{Z}$ with finitely many parameters.

This result follows from the following two lemmas:

Lemma 7.2.4. Let $G \cong \mathbb{Z}^d \rtimes_\sigma Q$ where $1 \leq d < \omega$, Q is some finite group, and $\theta : Q \to \mathrm{Aut}(\mathbb{Z}^d)$ a group homomorphism. G is interpretable in the structure $\mathcal{M} = (\mathbb{Z}^d, +, \pi_x)_{x \in Q}$ with finitely many parameters, where $+$ is the normal vector addition and $\pi_x := \sigma(x)$

Proof. We enumerate $\{\pi_x : x \in Q\}$ as $t_0, \ldots, t_{n-1}$ and $t_i := (i, 0, \ldots, 0)$ a d-tuple. Let $G = (\mathbb{Z}^d \times Q, \cdot)$. We represent the universe of G by a tuple (a, t_i) where $a \in M$ and $0 \leq i < n$. We need to define the group operation $(a_i, t_i) \cdot (a_j, t_j) = (a_i +_{\mathbb{Z}^d} t_i(a_j), t_i \cdot_Q t_j)$ in $\mathcal{M}$. $t_i \cdot_Q t_j$ is definable since Q in finite, we can enumerate all possible product of all elements in Q. $a_i +_{\mathbb{Z}^d} t_i(a_j)$ is definable since $+$ is a built-in function symbol of $\mathcal{M}$ and t_i is same with π_x for some x. Again this claim follows from enumerating all possibilities of t_i's. □

Lemma 7.2.5. $(\mathbb{Z}^d, +, \pi_x)_{x \in Q}$ is interpretable in $(\mathbb{Z}, +, 0)$ with finitely many parameters.

Proof. Since there are finitely many π_x's, it suffices to show we can define π_x in $(\mathbb{Z}, +, 0)$ for some $x \in Q$. π_x is an automorphism of $\mathbb{Z}^d$, which is an $d \times d$ invertible matrix A over $\mathbb{Z}$. $\pi_x(\overline{b}) = A\overline{b}$ is simply a matrix multiplication so clearly definable over A. □

Theorem 7.2.6. Let W be a Euclidean reflection group. Then $Th(W)$ is decidable.

Proof. W is interpretable in the structure $(\mathbb{Z}, +, 0)$ with finitely parameters by Proposition 7.2.3. But all these parameters are integers, so it is interpretable in the structure $(\mathbb{Z}, +, 0, 1)$ with no parameters. We then can translate every sentence in W back to a sentence of $(\mathbb{Z}, +, 0, 1)$. Since the Presburger arithmetic $(\mathbb{Z}, +, <, 0, 1)$ is decidable, as a definable structure in its reduct, $Th(W)$ is also decidable. □

A Model Theory Preliminaries

We fix our language $\mathcal{L} = (e, \cdot)$ to be the language of groups.

Definition A.0.1. The set of $\mathcal{L}$-terms is the smallest set $\mathcal{T}$ such that

1. variables $x_i \in \mathcal{T}$ for $i = 0, 1, \ldots$;
2. if $t_1, t_2 \in \mathcal{T}$ then $(t_1 \cdot t_2) \in \mathcal{T}$

Definition A.0.2. The set of $\mathcal{L}$-formulas is the smallest set $\mathcal{F}$ such that

1. all atomic formulas $(t_1 = t_2) \in \mathcal{F}$ for $t_1, t_2 \in \mathcal{T}$;
2. $\mathcal{F}$ is closed under $\neg$ ("not"), $\vee$ ("or"), $\wedge$ ("and"), $\rightarrow$ ("implies"), $\forall$ ("for all"), $\exists$ ("there exists").

By drawing a truth table or otherwise, it suffices to define the satisfaction of $\wedge$, $\neg$, and quantifiers for $\mathcal{L}$-formulas.

Definition A.0.3. We say a formula ϕ has x as a free variable if x does not appear in any quantifiers of ϕ.

Definition A.0.4. Let ϕ be a formula with free variables $x_1, \ldots, x_n$. We inductively define a group $G \models \phi$ as follows.

1. If $\phi(\overline{x})$ is $t_1(\overline{x}) = t_2(\overline{x})$, then $G \models \phi$ if t_1 and t_2 are exactly the same term.
2. If $\phi(\overline{x})$ is $\neg\psi(\overline{x})$, then $G \models \phi(\overline{x})$ if $G \not\models \psi(\overline{x})$.
3. If $\phi(\overline{x})$ is $\psi_1(\overline{x}) \wedge \psi_2(\overline{x})$, then $G \models \phi(\overline{x})$ if $G \models \psi_1(\overline{x})$ and $G \models \psi_2(\overline{x})$.
4. If $\phi(\overline{x})$ is $\forall y \psi(\overline{x}, y)$, then $G \models \phi(\overline{x})$ if for all $a \in G$ $G \models \psi(\overline{x}, a)$.
5. If $\phi(\overline{x})$ is $\exists y \psi(\overline{x}, y)$, then $G \models \phi(\overline{x})$ if there is $a \in G$ $G \models \psi(\overline{x}, a)$.

If $G \models \phi(\overline{a})$ we say $\phi(\overline{a})$ is true in G

Definition A.0.5. We say a formula ϕ is a sentence if it does not have free variables.

To make the term algorithm clear, we recall definitions from computability theory.
We have infinitely many initial configuration of the register $R_1, R_2, \ldots$, each is a nonnegative integer.

Definition A.0.6. A register machine program is a finite sequence of instructions $I_1, \ldots, I_M$, where each I_j is one of the following:

1. Z(n): set R_n to zero;
2. S(n): increase R_n by one;
3. T(n,m): set R_n to be R_m;
4. J(n,m,s), where $1 \leq s \leq M$: if $R_n = R_m$, then go to I_s, otherwise go to the next instruction;
5. HALT

and I_m is HALT

Definition A.0.7. Suppose $A \subseteq \mathbb{N}^k$. $f : A \to \mathbb{N}$ is computable if there is a register machine program P such that:

1. If $x \in A$, then P does not halt on input x;
2. If $x \in A$, then P halts on input x with $R_1 = f(x)$.

Definition A.0.8. We say a set $A \subseteq \mathbb{N}^k$ if the function

$$\chi_A(x) = \begin{cases} 1 & \text{if } x \in A, \\ 0 & \text{if } x \notin A. \end{cases}$$

is computable.

We then can say a theory of G is decidable if the set of true sentences in G. We can make "input of ϕ" more precise using Gödel's coding.

Theorem A.0.9. The theory of $\mathbb{Z}$ in the language of $(+, 0, 1, <, \equiv_p)$ where $\equiv_p$ is the relation of "two integers are same modolu p" is complete, admits quantifier elimination hence decidable.

References

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