We're now going to start examining the algebraic properties of general reflection groups. To do this we will be working in V, denoting $\mathbb{R}^n$ equipped with the standard inner product, our fixed *euclidean space*.

0.1   Affine reflections

This subsection roughly follows [**?**]. When considering reflections in V, we don't want to just consider those lying in $O(V)$, we are also interested in reflections across some hyperplane not lying through the origin. To tackle this we first define some standard preliminaries:

Definition 0.1.1. A **reflection** in V is a linear transformation $s \in O(V)$ (the orthogonal group of inner product preserving linear maps) with eigenvalues $\{-1, 1\}$ with corresponding dimensions of eigenspaces:
$$\dim E_1 = n - 1 \qquad \dim E_{-1} = 1$$
for linear maps in $O(V)$ this is equivalent to fixing some hyperplane (codimension 1 subspace) and having determinant -1.

Definition 0.1.2. The **group of general affine transformations** of V is semidirect product of $GL(V)$ acting on V
$$GA(V) := V \rtimes GL(V)$$
where V acts on itself by translation. $(u, T) \in GA(V)$ acts on $v \in V$ as $(u, T) \cdot v = u + T(v)$.

Proposition 0.1.3. This is actually an action. Furthermore, the action is transitive.

Proof. Take $(u, T), (w, S) \in GA(V)$ and $v \in V$. Showing this is an action is a direct calculation:

$$(u,T)\cdot((w,S) \cdot v) = (u,T)\cdot(w + S(v)) = u+T(w)+TS(v) = (u+T(w), TS)\cdot v = ((u,T)(w,S))\cdot v$$

from the definition of the semidirect product. Suppose (u, T) and (w, S) act the same on V: as both T and S are linear the two affine transformations send 0 to u and w respectively, thus we have $u = w$; subtracting these equal translations and having equality means the two linear transformations must also be equal. $\square$

We can now consider appropriate affine versions of linear groups, the most important for us will be the **affine orthogonal group** $AO(V)$, which can be equivalently viewed as the subgroup of $GA(V)$ consisting of isometries, or as $V \rtimes O(V)$. We need stricter criteria than just the linear component of our affine transformation be a reflection to suitably capture the notion of a reflection across an affine hyperplane.

Proposition 0.1.4. For a unit normal vector α and some $k \in \mathbb{R}$, reflection across the affine hyperplane $(V, \alpha) = k$ corresponds to the affine transformation $(2k\alpha, s_\alpha)$, where s_α is the reflection along α.

Proof. Call the affine hyperplane H and choose a $v \in V$. The vector orthogonal to H that goes to v has length $k - (v, \alpha)$ so reflecting across H send v to $v+2(k-(v,\alpha))\alpha = 2k\alpha+(v-2(v,\alpha)\alpha) = (2k\alpha, s_\alpha) \cdot v$. $\square$

We call such affine transformations **affine reflections**, and we can now talk about **affine reflection groups** as those generated by affine reflection in the same way we could for orthogonal reflections.

Lemma 0.1.5. For all affine hyperplanes H and affine reflections r, the set rH is also an affine hyperplane.

Proof. Let H be $\{v \in V \mid (v, \alpha) = k\}$ for some $\alpha \in V, k \in \mathbb{R}$. As r is bijective the set rH is equal to $\{w \in V \mid (r^{-1}w, \alpha) = k\}$, by writing $r = (u, T) \in GA(V)$ this can be rewritten as $\{w \in V \mid (w, T^*(\alpha)) = k - (u, \alpha)\}$, an affine hyperplane. $\qquad\square$

Lemma 0.1.6. An affine transformations $r = (u, T)$ that fixes some affine hyperplane and is an involution must be an affine reflection.

Proof. As $r^2 = \mathrm{id}$, on 0 we have $r^2(0) = u + T(u) = 0$. So as $r^2 = u + T(u) + T^2$ this means T is also an involution so we can use the primary decomposition $V = E_1 \oplus E_{-1}$ into the eigenspaces of T. Call the hyperplane r fixes H, then for any $h = v_1 + v_{-1} \in H$ (where $v_1, v_{-1} \in V_1, V_{-1}$ respectively) we have $r(v_1 + v_{-1}) = u + T(v_1 + v_{-1}) = u + v_1 - v_{-1} = v_1 + v_{-1}$ therefore $2v_{-1} = u$ so $\dim E_{-1} = 1$ and T is the reflection along $V_{-1} = \langle v_{-1} \rangle$, thus $r = (2v_{-1}, s_{v_{-1}})$. $\qquad\square$

Proposition 0.1.7. For all affine reflections r, s with s reflecting across the affine hyperplane H, the affine transformation rsr^{-1} is an affine reflection across rH.

Proof. First, notice $(rsr^{-1})^2 = \mathrm{id}$ as both r and s are involutions. Also, $rsr^{-1}(rH) = rH$ as s fixes H. By the previous lemma, and earlier described faithfulness of the action, this is sufficient. $\qquad\square$

From now on, we will use the umbrella term *reflection* to refer to both linear and affine reflecitons.
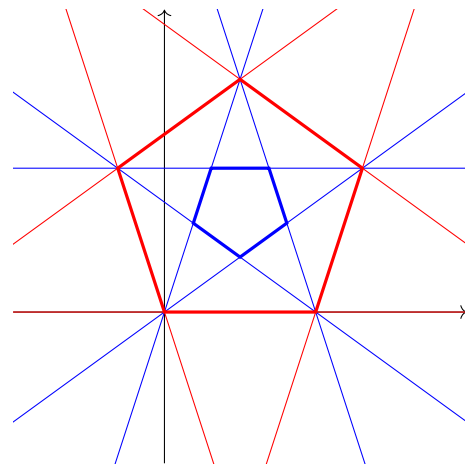
0.2 Reflection groups

We are interested in groups generated by reflections in E, so throughout the next two sections fix a group $W \leq GA(E)$ which can be generated by reflections. This roughly follows [?]

We would like to consider the set of all hyperplanes H such that there is some affine reflection $w \in W$ across H, this set will be referred to as $\mathcal{H}$. For certain choices of generators for W we may find $\mathcal{H}$ is dense in E.

Example 0.2.1. Consider the set of hyperplanes pictured in red:
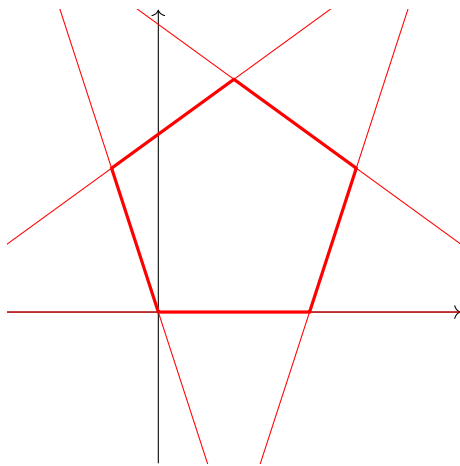
(a) Generating hyperplanes   **(b)** Resulting hyperplanes

As we have seen in the previous proposition, W acts on $\mathcal{H}$ so all the reflections of these hyperplanes, show in blue, must also lie in $\mathcal{H}$. This process can be repeated to find an arbitrarily small pentagon bounded by hyperplanes. By the presence of arbitrarily close parallel hyperplanes we can see W will contain arbitrarily small translations and thus $\mathcal{H}$ will be dense in E.

This adds technicalities and looses the discrete geometric intuition we have for reflection groups of polytopes and lattices. To remedy this we will restrict the reflection groups we consider by requiring for any compact subset $B \subset E$, the intersection $\mathcal{H} \cap B$ be finite. We can also see this immediately restricts the possible angles between hyperplanes to rational multiples of π.

The set of unitvectors $\alpha \in E$ such that there exists some $H \in \mathcal{H}$ with α normal to $\mathcal{H}$ will be written Φ. And without loss of generality we will assume Φ spans E from here onwards.

Definition 0.2.2. The connected components of $E \setminus \mathcal{H}$ are called the **chambers** of W in E. We will sometimes call these fundamental domain

The reflection groups we are interested in, those that appear as finite reflection groups of polytopes or lattices in whatever geometry we prefer, will have finitely many hyperplanes touching each chamber. Later on we will see this is equivalent to the group being finitely presented. A formula for the number of hyerplanes touching a chamber is given in Section **??**

We now want to choose a chamber C_0, and consider the set of hyperplanes $\{H_1, \ldots, H_k\} \subseteq \mathcal{H}$ bounding C. We will call the corresponding reflections across these hyperplanes $\{s_1, \ldots, s_k\}$ **simple reflections**.

Theorem 0.2.3. The set of simple reflections $\{s_1, \ldots, s_k\}$ generates W.

Proof. Let W' be the subgorup of W generated by the simple reflections. Let s be one of the reflections generating W, and call the hyperplane it reflects across H. If W' acts transitively on the set of chambers then there will exists some $w \in W'$ such that $wH_i = H$ for some simple reflection s_i as H will must bound a chamber. Thus, by an earlier lemma, $w s_i w^{-1} = s$ so $s \in W'$ and $W = W'$. Now we just have to show the action of W' is transitive on chambers. Suppose it isn't, i.e. there is some chamber C such that no $w \in W'$ satisfies $wC = C_0$. Let C' be the closest chamber to C_0 in the W' orbit of C, as $C' \neq C_0$ there must be some simple hyperplane (the boundary of C_0) between them, reflecting across this must strictly decrease the distance between the two chambers contradicting the minimality of C'. Thus W' acts transitively on the set of chambers. $\square$

As a direct corollary of this proof we now know W acts transitively on the set of chambers. Before discussing this further we should examine the relations that these simple reflections satisfy.

For any two simple reflections s_i, s_j the subgroup $\langle s_i, s_j \rangle$ will be dihedral, as seen in the previous section, (note that this will be the infinite dihedral group iff the hypeplanes being reflected along are parallel), call the order of this dihedral group $2m_{ij}$. The product $s_i s_j$ will have order m_{ij} in W and so W satisfies the set of relations $(s_i s_j)^{m_{ij}} = \text{id}$ for all i, j, taking $m_{ii} = 1$. A group *presented* by these relations is called **Coxeter**.
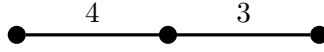
Definition 0.2.4. A **Coxeter system** is a pair (W, S) where $S = \{s_i\}_{i \in I}$ is a generating set for W which admits the presentation:

$$W = \langle S \mid (s_i s_j)^{m_{ij}} \text{ for all } i, j \in I \rangle$$

where each $m_{ij} \in \mathbb{N} \cup \{\infty\}$.

To each Coxeter system we can assign a **Coxeter diagram**: an undirected graph created by the following rules:

- Draw a node i for each $s_i \in S$;

- For each relation $(s_i s_j)^{m_{ij}}$ with $m_{ij} > 2$ draw an edge between i and j and label it with m_{ij}.

This process can be reversed to obtain a Coxeter system from any Coxeter diagram. This correspondence will associate the graph:

to the group presentation:

$$\langle s_1, s_2, s_3 \mid s_1^2 = s_2^2 = s_3^2 = e, \ (s_1 s_2)^4 = (s_2 s_3)^3 = (s_1 s_3)^2 = e \rangle$$

In the future, for the sake of readability, the 3 labels will often be excluded.

The classification of finite reflection groups goes by proving all reflection groups are in fact Coxeter groups and then classifying all the finite Coxeter groups.

0.3 Coxeter presentation

We have shown the action of W on the set of chambers is transitive, but we need a stronger notion to prove these groups are Coxeter.

Definition 0.3.1. Let G be a group acting on a set X, the action is called **simply transitive** if for all $x, y \in G$ there exists a unique $g \in G$ such that $g \cdot x = y$.

By fixing a base point $x \in X$ there is clear a 1-to-1 correspondence between X and G: associate $e \leftrightarrow x$ and for all $g \in G$, $g \leftrightarrow g \cdot x$.

Proposition 0.3.2. An action is simply transitive iff it is transitive and free.

Proof. An action is transitive if for all $x, y \in X$ there exists some $g \in G$ such that $g \cdot x = y$, and free if there exists at most one such g; therefore these statements are equivalent. $\square$

If we know the action is trasntive a priori we only need to check the free condition on a single element. So showing our action is simply transitive ammounts to showing for all $w \in W$ such that $wC_0 = C_0$ we have $w = \text{id}$.

We will go about prooving this by finding a geometric interpretation of the length of a word $w \in W$ in terms of simple reflections.

Definition 0.3.3. For $w \in W$ define the **length** of w, $l(w)$ to be the minimal positive integer r such that $w = s_1 \cdots s_r$, a length r product of simple reflections.

And the contrasting way to compute the length: for any $w \in W$ consider the set of hyperplanes H such that C_0 and wC_0 lie on different sides of H. Call this $\mathcal{L}(w)$ and let $n(w) = |\mathcal{L}(w)|$. As the line segment between two chambers is bounded, by assumption this will only meet finitely many hyperplanes so for any $w \in W$ we will have $\mathcal{L}(w) \in \mathbb{N}$.

Lemma 0.3.4.   1. $n(w) = n(w^{-1})$,

2. $l(w) = 1$ iff w is a simple reflection

3. $l(w) = l(w^{-1})$

4. if w can be written as $s_1 \cdots s_r$ then $\det(w) = (-1)^r$

5. $l(s_i w), l(w s_i) = l(w) \pm 1$

Proof. (1.) H separtes C_0 and $w^{-1}C_0$ iff wH separates wC_0 and C_0 — W acts isometrically. $\square$

Lemma 0.3.5. H_i is in exactly one of $\mathcal{L}(w)$ or $\mathcal{L}(s_i w)$

Lemma 0.3.6. Choose a simple hyperplane H_i, for any hyperplane $H \neq H_i$ and $w \in W$, if $H \in \mathcal{L}(w)$ then $s_i H \in \mathcal{L}(s_i w)$.

Proof. As H separates C_0 and wC_0, $s_i H$ separates $s_i C_0$ and $s_i w C_0$, so $s_i H$ is in exactly one of $\mathcal{L}(s_i)$ or $\mathcal{L}(s_i w)$, but if $s_i H \in \mathcal{L}(s_i)$ we would have $s_i H = H_i \implies H = H_i$ a contradiction. Therefore $s_i H \in \mathcal{L}(s_i w)$. $\square$

Corollary 0.3.7. $s_i\left(\mathcal{L}(w) \setminus \{H_i\}\right) = \mathcal{L}(s_i w) \setminus \{H_i\}$.

Proof. by applying the previous lemma to both w and $s_i w$ we get the required iff. $\square$

Proposition 0.3.8. For all $w \in W$, we have $n(w) \leq l(w)$.

Proof. If $l(w) = 1$ then w must be some simple reflection s_i so $\mathcal{L}(w) = \{H_i\}$[citation-needed]so $n(w) = 1$. Now by induction on $l(w)$: if $l(s_i w) = l(w) + 1$ then by the previous corollary and an earlier lemma we know $n(s_i w) = n(w) \pm 1$, namely $n(s_i w) \leq n(w) + 1 = l(w) + 1 = l(s_i w)$. $\square$

Now to show $n(w) = l(w)$ we will enumerate the hyperplanes in $\mathcal{L}(w)$.

Lemma 0.3.9. If $w = s_1 \cdots s_r$ is a reduced expression for w in terms of simple reflections, the hyperplanes:

$$H_1, s_1 H_2, s_1 s_2 H_3, \ldots s_1 \cdots s_{r-1} H_r$$

are all distinct.

Proof. Suppose, for a contraditction, that there exists some $1 \leq i < j \leq r$ such that the hyperplanes $s_1 \cdots s_{i-1} H_i$ and $s_1 \cdots s_{j-1} H_j$ are equal, by applying $s_1 \cdots s_{i-1}$ to both sides this implies $H_i = s_i \cdots s_{j-1} H_j$, by an earlier corollary this implies $s_i = (s_I \cdots s_{j-1}) s_j (s_{j-1} \cdots s_i)$ which implies $s_{i+1} \cdots s_{j-1} = s_i \cdots s_j$ contradicting the minimality of the lenght of w. $\square$

Proposition 0.3.10. If $w = s_1 \cdots s_r$ is a reduced expression for w in terms of simple reflections:

$$\mathcal{L}(w) = \{H_1, s_1 H_2, s_1 s_2 H_3, \ldots s_1 \cdots s_{r-1} H_r\}$$

Proof. We go by induction on r. Observe the base case $\mathcal{L}(s_1) = \{H_1\}$ from an earlier lemma. Now assume the claim holds for all w with reduced length $< r$ specifically:

$$\mathcal{L}(s_1 w) = \{H_2, s_2 H_3, \ldots s_2 \cdots s_{r-1} H_r\}$$

we know from an earlier lemma H_1 is in exactly one of $\mathcal{L}(w)$ or $\mathcal{L}(s_1 w)$. If $H_1 \in \mathcal{L}(s_1 w)$ by applying s_1 we would have for some $i > 1$:

$$s_1 H_1 = H_1 = s_1 \cdots s_i H_i$$

which contradicts the previous lemma, thus $H_1 \in \mathcal{L}(w)$, combining this with the earlier corollary gives the desired result. $\square$

A direct consequence of this propsition is that if some $w \in W$ is such that $wC_0 = C_0$ then $\mathcal{L}(w) = \emptyset$ which implies $w = \mathrm{id}$ in reduced form. We know this is sufficient to say the action of W on the set of chambers is simply transitive.

We can now prove two key combinatorial properties of the reflection group, and use these to show all relations in W are direct consequences of those in the Coxeter presentation.

Theorem 0.3.11 (Exchange Condition). If $w \in W$ has reduced expression in terms of simple reflections $w = s_1 \cdots s_r$ and $l(sw) < l(w)$ for some simple reflection s, there exists some $1 \leq i \leq r$ such that $w = s s_1 \cdots s_{i-1} s_{i+1} \cdots s_r$.

Proof. We know:
$$\mathcal{L}(w) = \{H_1, s_1 H_2, s_1 s_2 H_3, \ldots s_1 \cdots s_{r-1} H_r\}$$
by a previous proposition we know H, the hyperplane corresponding to s, must lie in $\mathcal{L}(w)$ and so for some $1 \leq i \leq r$, $H = s_1 \cdots s_{i-1} H_i$ which implies $s = s_1 \cdots s_{i-1} s_i s_{i-1} \cdots s_1$ therefore $s s_1 \cdots s_{i-1} = s_1 \cdots s_i$. Substituting this back into w gives the desired result. $\qquad\square$

Theorem 0.3.12 (Deletion Condition)**.** If $w \in W$ has a nonreduced expression in terms of simple roots $w = s_1 \cdots s_r$ then there exists $1 \leq i, j \leq r$ such that $w = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_r$. The hats denote omission.

Proof. For this to be nonreduced there must eventually exist some i such that $l(s_i \cdots s_r) = l(s_{i+1} \cdots s_r) - 1$ so by the exchange condition $s_{i+1} \cdots s_r = s_i s_{i+1} \cdots \hat{s_j} \cdots s_r$. Substituting this back into w gives the desired result. $\qquad\square$

Informally, we can see that these weak forms of the deletion and exchange conditions are actually direct consequences of the Coxeter relations, this is geometrically motivated in subsection 0.5. The formal proof that the exchange condition (which implies the deletion condition) is a property of Coxeter groups rigorously defined as quotients of a free group is given in Section **??**. We are electing to use this informal proof at this point in the report to continue the geometric nature of this section, thus the following proof that reflection groups satisfy the Coxeter presentation is the same somewhat informal argument as given in [**?**].

Theorem 0.3.13. Any finite reflection group satisfies the Coxeter presentation.

Proof. We want to consider the smallest nontrivial relation $w = s_1 \cdots s_r = \text{id}$, in terms of simple reflections, in W which is not a consequence of the Coxeter relations.

By examining the determinants, we see $r = 2k$ must be even, thus we can write $s_1 \cdots s_{k+1} = s_{k+2} \cdots s_r$ and invoke the deletion condition which lets us rewrite $w = s_1 \cdots \hat{s_i} \cdots \hat{s_j} \cdots s_r = \text{id}$ which must also not be a consequence of the Coxeter relations, this contradicts minimality. $\quad\square$

Note that in this proof we tacitly expect $r \geq 4$, luckily any length 2 relation is just an equality which by assumption do not occur.

0.4 Classification

Definition 0.4.1. To a Coxeter system (W, S), with $n = |S|$, we can associate a real symmetric $n \times n$ matrix A, by setting $a_{ij} := -\cos(\pi/m_{ij})$. The quadratic form $Q(v) = v^\top A v$ on E is called the **associated quadratic form** of the Coxeter system.

We can realise each a_{ij} as the inner product of simple roots (α_i, α_j) corresponding the the reflection s_i, s_j. Such matrices are called **Gram matrices**.

Proposition 0.4.2. The quadratic form associated to a finite reflection group W is positive definite.

Proof. As Φ forms a basis for E there is an invertible change of basis matrix B form the standard basis to Φ, we can thus write $A = B^\top B$ and for any nonzero $v \in E$ have:
$$Q(v) = v^\top A v = v^\top B^\top B v = (Bv)^\top (Bv) = \|Bv\|^2 > 0$$
as B is a linear transformation acting on $v \neq 0$, therefore Q is positive definite. $\qquad\square$

We now want to fix a quadratic form $Q(v) = v^\top A v$ for some symmetric $A \in \mathcal{M}_n(\mathbb{R})$.

Lemma 0.4.3. The kernel of Q (the set of vectors $v \in E$ such that $Q(v) = 0$) equals the nullspace of A.

Proof. The nullspace of A obviously lies in the kernel of Q. From linear algebra 2 we know we can orthogonally diagonalise A and so we can write:

$$Q(v) = v^\top A v = v^\top P^\top D P v = (Pv)^\top D (Pv)$$

where P and D are some orthogonal and diagonal matrices respectively. Observe the right hand side is 0 iff for ever $1 \le i \le n$ either d_{ii} or $(Pv)_i$ are 0. This implies $D(Pv) = 0$ and thus v is in the nullspace of A. $\qquad\square$

We will now demand all of the off-diagonal entries of A be nonpositive, note that all of the matrices for quadratic forms associated with Coxeter diagrams will satisfy this condition.

Lemma 0.4.4. The only nonzero vectors in the nullspace of A have all coordinates nonzero.

Proof. First suppose $v \neq 0$ is in the nullspace of A, then by the previous lemma we know $Q(v) = 0$. Consider the vector u given by $u_i = |v_i|$, we will have $Q(u) = u^\top A u \ge 0$, but as all off-diagonal entries of A are nonpositive this will satisfy $Q(u) \le Q(v) = 0$. Thus u is also in the nullspace of A. So if the nullspace is nontrivial, it will always contain a vector with nonnegative coordinates.

If we suppose some, but not all, of these coordinates are zero we can consider the set of such indices I. For any $i \in I$ consider the ith term in Au:

$$(Au)_i = \sum_{j \notin I} a_{ij} |v_j| = 0$$

as all the a_{ij} are nonpositive and all the $|v_j|$ are strictly positive, we get that for all $i \in I, j \notin I$ the entries $a_{ij} = 0$ contradicting irreducibility of A. Thus the coordinates of all vectors in the nullspace of A are nonzero. $\qquad\square$

Corollary 0.4.5. The dimension of the nullspace of A will be at most 1.

Lemma 0.4.6. The smallest eigenvalue d of A has multiplicity 1, and its coordinates will all be positive.

Proof. Again, from linear algebra 2 we know all the eigenvalues of a positive semidefinite matrix are nonnegative. So $A - dI$ satisfies all the requirement we placed on A for the previous lemma. As $A - dI$ is singular it will have nonempty nullspace which must be of dimension exactly 1. Thus d and $u_i = |d_i|$ are colinear so we can find an eigenvalue with all positive coordinates. $\quad\square$

We want to consider subgraphs of the Coxeter diagram Γ associated to (W, S). For this we will introduce a Coxeter system (W', S') where $S' \subseteq S$, the Coxeter relation degrees in (W', S') are given by m'_{ij} and they will satisfy $m'_{ij} \le m_{ij}$ for all i, j. We will call the associated Coxeter diagram, quadratic form and symmetric matrix Γ', Q', A' respectively. Assume this *subsystem* is proper i.e. $S' \subsetneq S$ or some $m'_{ij} < m_{ij}$.

Proposition 0.4.7. If Q is positive semidefinite, then Q' will be positive definite.

Proof. Suppose A' fails to be positive definite, i.e. there exists some $v \in E$ such that $Q'(v) \le 0$, consider the vector:

$$u_i = \begin{cases} |v_i| & 1 \le i \le k \\ 0 & \text{otherwise} \end{cases}$$

and the value of:

$$0 \le Q(u) = \sum_{1 \le i,j \le n} a_{ij} u_i u_j \le \sum_{1 \le i,j \le k} a'_{ij} v_i v_j \le 0$$

as all the $a_{ij} \le a'_{ij} \le 0$. This means u is in the nullspace for A and thus all the coordinates are nonzero which means $k = n$. But this then implies all $a_{ij} = a'_{ij}$. Thus the only subdiagram that fails to be positive definite is improper. $\qquad\square$

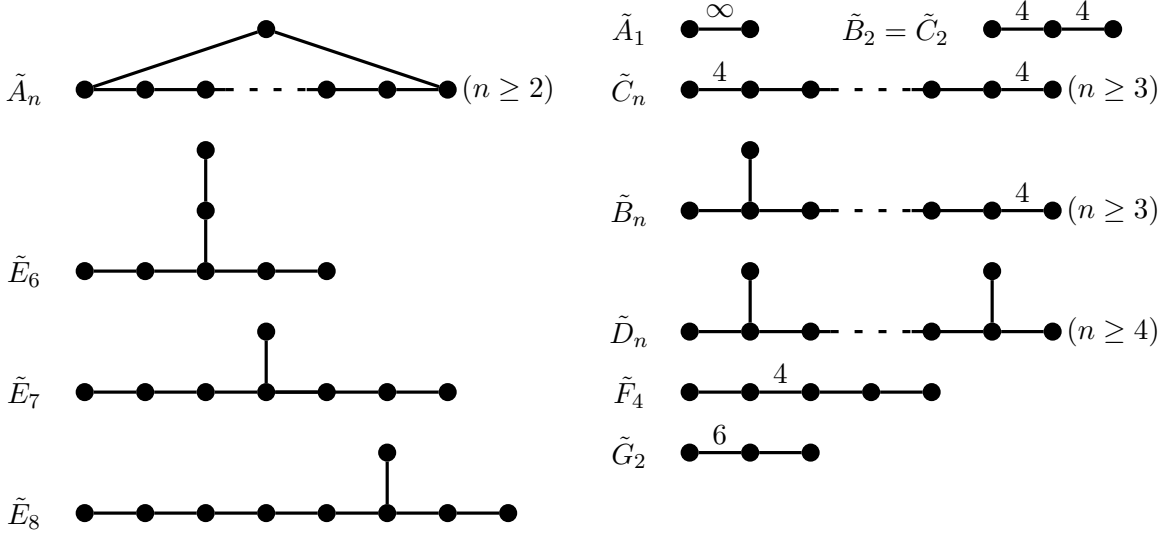Here is a collection of Coxeter diagrams all of which, by checking their eigenvalues on a computer, can be found to be positive definite.

A_n •—•—•— - - - —•—•—• $(n \geq 1)$

BC_n •—•—•— - - - —•—•—$\overset{4}{-}$• $(n \geq 2)$

D_n •—•—•— - - - —•—•—• $(n \geq 4)$

E_6

E_7

E_8

F_4 •—•—$\overset{4}{-}$•—•

H_3 •—$\overset{5}{-}$•—•

H_4 •—$\overset{5}{-}$•—•—•

$I_2(m)$ •—$\overset{m}{-}$• $(m \geq 4)$
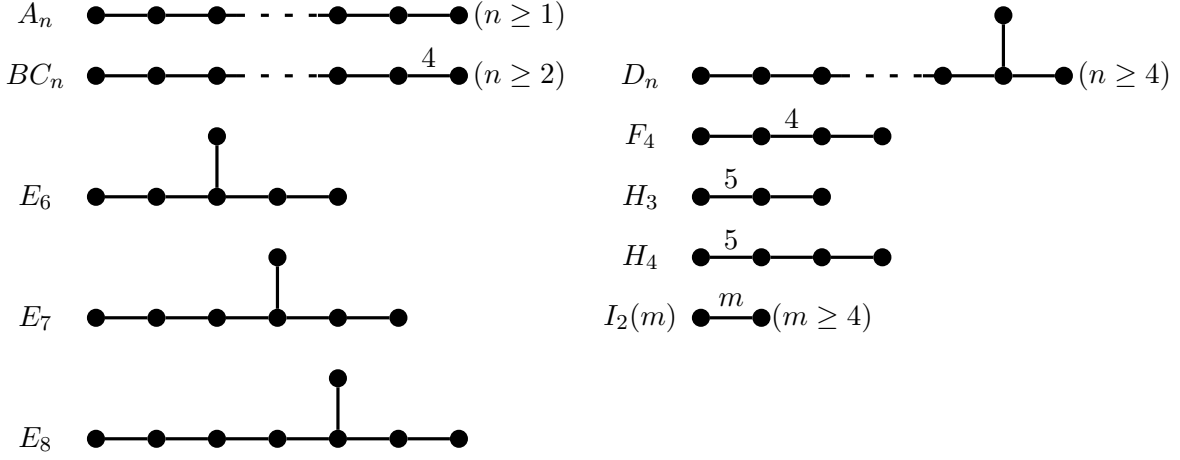
Figure 2 Positive definite Coxeter diagrams

And here is another collection who can all be founded to be only positive semidefinite:

$\tilde{A}_n$ $(n \geq 2)$

$\tilde{A}_1$ •—$\overset{\infty}{-}$•

$\tilde{B}_2 = \tilde{C}_2$ •—$\overset{4}{-}$•—$\overset{4}{-}$•

$\tilde{C}_n$ •—$\overset{4}{-}$•—•— - - - —•—•—$\overset{4}{-}$• $(n \geq 3)$

$\tilde{B}_n$ •—•—•— - - - —•—$\overset{4}{-}$• $(n \geq 3)$

$\tilde{E}_6$

$\tilde{E}_7$

$\tilde{E}_8$

$\tilde{D}_n$ $(n \geq 4)$

$\tilde{F}_4$ •—•—$\overset{4}{-}$•—•—•

$\tilde{G}_2$ •—$\overset{6}{-}$•—•

Figure 3 Positive semidefinite Coxeter diagrams

Plus two helper diagrams which are neither:

Z_4 •—•—$\overset{5}{-}$•—•

Z_5 •—•—$\overset{5}{-}$•—•—•

Figure 4 Helper Coxeter diagrams

Theorem 0.4.8. the diagrams of Figure 2 are all the connected diagrams with positive definite quadratic form.

Proof. The proof goes by indentifying features present in the graphs of Figure 3 and Figure 4, observing they cannot appear as subdiagrams of those with positive definite quadratic form: e.g. no cycles as in $\tilde{A}_n$ and all edges must be finite due to $\tilde{A}_1$. For the full proof see [**?**]. □

Corollary 0.4.9. The following graphs categorise all finite reflection groups.

8

Proof. We know it suffices to classify the reflections groups corresponding to connected Coxeter diagrams, all of which must have positive definited associated quadratic form, all of which we have found. The only remaining step is to find polytopes admitting these Coxeter groups as their reflection groups, lots of these examples have been seen in the previous section, and in fact every diagram of Figure 2 does occur as a finite reflection group of a polytope. However, as we will see in the next section, such polytopes aren't necessarily regular. □

This solution can seem somewhat unsatisfactory as there is no immediate reason as to **why** all of the finite Coxeter groups can be realised as reflection groups.

0.5 Geometric representation

For any finite Coxeter system (W, S) consider the abstract real vector space V with basis Φ. We can define a quadratic form (and thus an inner product $\langle -, - \rangle$) on V in the same way as we did earlier. And for each generator $s \in S$ we can associate a reflection $\sigma_s : V \to V$ given natrually as $\sigma_s(v) = v - 2 \langle \alpha_s, v \rangle \alpha_s$. Through some similar calculations we can find the representation this induces on W is faithful and acts orthogonally on V, the construction of the edges and vertices of a polytope demonstrated in the next section will thus show how a regular polytope can be realised from this abstract reflection group.