Algebraic Curves

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A Commutative algebra

A.1 Noetherian modules

Definition A.1.1. Given a commutative ring R, an R-module M is **Noetherian** if all R-submodules are finitely generated.

Proposition A.1.2. *M is Noetherian iff every ascending chain of submodules stabilises.*

Proof. ($\Rightarrow$) Let $N_0 \subseteq N_1 \subseteq \cdots$ be the ascending chain, define $N = \bigcup_i N_i$ then N is finitely generated by some $\{m_1, \ldots, m_r\}$ each first appearing in $N_{i_1}, \ldots, N_{i_r}$, the chain stabilises at the maximum of these.
($\Leftarrow$) If any submodule is not finitely generated we may form the never stabilising chain $N_0 = \langle n_0 \rangle$, $N_{i+1} = N_i + \langle n_{i+1} \rangle$ where each $n_{i+1} \notin N_i$, so all submodules must be finitely generated. $\square$

Definition A.1.3. A ring R is Noetherian if it is Noetherian as an R-module.

Lemma A.1.4. All submodules of Noetherian modules are Noetherian.

Lemma A.1.5. All quotients of Noetherian modules are Noetherian.

Proposition A.1.6. If $N \leq M$ are R-modules and both N and M/N are Noetherian, so is M.

Corollary A.1.7. Any finitely generated module over a Noetherian ring is Noetherian.

A.2 Hilbert basis theorem

The leading coefficient of a polynomials $f = a_0 + a_1X + \ldots + a_nX^n$ is $a_n \in R$.

Lemma A.2.1. Given R is a Noetherian ring, if $I \trianglelefteq R[X]$, and J is the set of leading coefficients of polynomials in I, then J is an ideal of R.

Proof. If a is the leading coefficient of $f \in I$, then for any $r \in R$, ra will be the leading coefficient of $rf \in I$. If b is the leading coefficient of $g \in I$ and the degree of f and g are n and m respectively. Wlog, take $n \leq m$, then $a + b$ will be the leading coefficient of $X^{m-n}f + g \in I$ which has degree m. $\square$

As R is Noetherian, J must be finitely generated by some $\{a_1, \ldots, a_s\}$. For each a_i there must be a polynomial f_i with leading coefficient a_i, say this has degree d_i; and let d be the maximum of these.

Lemma A.2.2. For any $f \in I$, there exist $p_1, \ldots, p_i \in R[X]$ such that $\deg(f - p_1f_1 - \cdots - p_sf_s) < d$.

Proof. By induction on $\deg(f)$. For the base case, suppose $\deg(f) < d$, then set all $p_i \equiv 0$.
Now, suppose the claim holds for all polynomials of degree less than some e. Let a be the leading coefficient of f, as $a \in J$ we can write $a = r_1 a_1 + \cdots r_s a_s$ for some $r_i \in R$, now consider:

$$f - r_1 X^{e-d_1} f_1 - \cdots - r_s X^{e-d_s} f_s$$

which must not be of degree $< e$. As each $r_i X^{e-d_i} \in R[X]$ we are done. □

Theorem A.2.3 (Hilbert basis theorem)**.** If R is a Noetherian ring, then so is $R[X]$.

Proof. Let $R[X]_{\leq d}$ be the R-submodule of $R[X]$ generated by $\{1, X, \ldots, X^d\}$ this is finitely generated so Noetherian. $I \cap R[X]_{\leq d}$ is a submodule so also finitely generated, let $\{g_1, \ldots, g_k\}$ be the generating set. The previous lemma now shows I is generated by $\{g_1, \ldots, g_k, f_1, \ldots, f_s\}$. □

A.3 Nullstellensatz

A.4 Polynomial rings over UFDs are UFDs