Algebra 3

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Autumn 2025

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1 Rings and Modules

1.1 Introduction

You hopefully already know the definition of a ring, so here are some examples:

- $\mathbb{Z}$, the ring of integers;
- any field like $\mathbb{F}_{p^n}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and et cetera;
- for a given ring R, the ring of polynomials $R[x]$, when we let $R = k$ a field this is an Euclidean domain admitting many parallels to $\mathbb{Z}$;
- $R[[x]]$, the ring of power series;
- $M_n(R)$ the ring of $n \times n$ matrices with entries in R, the first noncommutative example here;
- R-valued functions out of any set have a pointwise ring structure;
- For a group G we can consider the group ring $R[G]$, which we formalise as the set of finitely supported maps $f : G \to R$, addition is pointwise, and multiplication is given by

$$(fg)(x) := \sum_{uv=x} f(u)g(v)$$

  written as R-linear combinations of elements of G, this will be commutative iff G is abelian;
- we can define the quaternions as $\mathbb{H} := \mathbb{R}[Q_8]$, where Q_8 is the quaternionic group $\{\pm 1, \pm i, \pm j, \pm k\}$;
- the first Weyl algebra is roughly “the ring of polynomial valued differential operators”, defined as

$$A_1 := \mathbb{C}[x, \partial]/(\partial x - x\partial = 1)$$

  you should think of this acting of $f \in \mathbb{C}[x]$, generated by the product rule:

$$\partial x f - x\partial f = f\frac{d}{dx}xf - x\frac{d}{dx}f = f.$$

2 Matrix Lie Groups