Functional Analysis

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0.1 Banach Spaces

I'll write these notes mostly over $\mathbb{R}$, everything works similarly over $\mathbb{C}$ or any complete normed field. So when I say "vector space", I mean a real one.

Definition 0.1.1. A **norm** on a vector space E is a map $\|-\| : E \to \mathbb{R}_{\geq 0}$ such that for all $x, y \in E, \lambda \in \mathbb{R}$:

$$\|x\| = 0 \iff x = 0 \qquad \|\lambda x\| = |\lambda| \, \|x\| \qquad \|x + y\| \leq \|x\| + \|y\|$$

called **definiteness**, **homogenaity**, and the **triangle inequality** respectively.

Common examples are $\|-\|_1$, $\|-\|_2$, and $\|-\|_\infty$ on $\mathbb{R}^n$.

Definition 0.1.2. For a compact $K \subseteq \mathbb{R}^n$ the **uniform norm** on the set of continuous functions $\mathcal{C}(K, \mathbb{R})$ is given by:

$$\|f\|_\infty := \sup_{x \in K} |f(x)| = \max_{x \in K} |f(x)|.$$

Letting K be n discrete points recovers $\|-\|_\infty$ on $\mathbb{R}^n$, how can we recover the other defintions like this?

Definition 0.1.3. A **Banach space** is a complete normed vector space.

We will show in the finite case that all normed vector spaces are complete. Note that the on the space $\mathcal{C}([0, 1], \mathbb{R})$ the norm:

$$\|f\| := \int_0^1 f(x) dx$$

is not complete, many progressive continuous approximations to non-conintuous shapes are an example. The change is normally increasingly localised and extreme, the uniform norm will properly detect this.

Definition 0.1.4. Two norms, $\|-\|, \|-\|'$ on a vector space E are **equivalent** if there exists $C_1, C_2 \geq 0$ such that for all $x \in E$:

$$C_1 \|x\|' \leq \|x\| \leq C_2 \|x\|'$$

This is obviously an equivalence relation. Equivalent norms induce homeomorphic topologies by id, as:

$$B'_{r/C_2}(x) \subseteq B_r(x) \subseteq B'_{r/C_1}(x).$$

Proposition 0.1.5. All finite dimensional normed vector spaces are equivalent.

Proof. Choose any norm on a d dimensional vector space E, and let $\{e_1, \ldots, e_d\}$ be a basis for E:

$$\|x\| = \left\| \sum x_i e_i \right\| \leq \sum \|x_i e_i\| = \sum |x_i| \, \|e_i\| \leq \max_{1 \leq i \leq d} \|e_i\| \sum |x_i| = \max_{1 \leq i \leq d} \|e_i\| \, \|x\|_1 .$$

Now, let K be the unit sphere for $\|-\|_1$. This is closed and bounded so, by the Heine-Borel theorem, compact. The previous step shows $x \mapsto \|x\|$ is continuous wrt $\|-\|_1$, so it will attain its infinum on K. By definiteness, this minimum, m, will be positive. And for all $x \in E$: $\|x\| = \|x\|_1 \, \|x / \|x\|_1\| \geq m \|x\|_1$. $\square$

From now onwards, let Ω be an open subset of $\mathbb{R}^d$ such that $\overline{\Omega}$ is compact and $\partial\Omega$ is smooth. The particulars of these restrictions isn't important.

Definition 0.1.6. The **uniform norm** for $f \in \mathcal{B}(\Omega, \mathbb{R})$ is given by $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$.

Think of $C_b^0(\Omega, \mathbb{R})$ as a subspace of $\mathcal{B}(\Omega, \mathbb{R})$.

Proposition 0.1.7. $C_b^0(\Omega, \mathbb{R})$ is closed (contains all limit points) w.r.t. the uniform norm.

Proof. Let $(f_n) \to f$, and fix some $\varepsilon > 0$. By assumption, there exists some $N \geq 0$ such that $\|f_N - f\|_\infty < \varepsilon$. As f_N is continuous, there will exist some $B_\delta(x) \subseteq \Omega$ such that for all $y \in B_\delta(x)$ we have $|f_N(x) - f_N(y)| < \varepsilon$. So we can write:

$$\begin{aligned} |f(x) - f(y)| &= |(f(x) - f_N(x)) - (f(y) - f_N(y)) + (f_N(x) - f_N(y))| \\ &\leq |(f(x) - f_N(x))| + |(f(y) - f_N(y))| + |(f_N(x) - f_N(y))| < 3\varepsilon. \end{aligned}$$

□

Proposition 0.1.8. $C_b^0(\Omega, \mathbb{R})$ is complete w.r.t. the uniform norm.

Proof. Let (f_n) be Cauchy w.r.t $\|-\|_\infty$, for any $x \in \Omega$, $(f_n(x))$ will be a Cauchy sequence in $\mathbb{R}$ so converges to a limit we will call $f(x)$. It now suffices to show $(f_n) \to f$ w.r.t. $\|-\|_\infty$ and appeal to the previous proposition. Fix $\varepsilon > 0$, by assumption there exists N such that $\forall n, m \geq N$, $\|f_n - f_m\|_\infty < \varepsilon$. Consider for all $x \in \Omega$:

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon.$$

As each $|f_n(x) - f(x)| \leq$ is ε, so will $\|f_n - f\|_\infty$, the supremum. □

Definition 0.1.9. Let $\mathcal{C}^k(\Omega, \mathbb{R})$ be the set of k-times bounded continuously differentiably functions, equipped with the norm:

$$\|f\|_{\mathcal{C}^k} := \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_\infty \qquad \alpha \in \mathbb{N}^d$$

The simplest example is with $d = k = 1$ where $\|f\|_{\mathcal{C}^1} = \|f\|_\infty + \|f'\|_\infty$.

Proposition 0.1.10. For $\Omega \subset \mathbb{R}$, $\mathcal{C}^1(\Omega, \mathbb{R})$ with the $\|-\|_{\mathcal{C}^1}$ norm is complete.

Proof. Let (f_n) be Cauchy w.r.t. $\|-\|_{\mathcal{C}^1}$, then both (f_n) and (f_n') must be Cauchy w.r.t. $\|-\|_\infty$. So, by the previous proposition, there exists continuous, uniform limits $(f_n) \to f$, $(f_n') \to g$. Thus:

$$\|f_n - f\|_{\mathcal{C}^1} = \|f_n - f\|_\infty + \|f_n' - g\|_\infty \to 0.$$

How do we know $f' = g$ or even $f \in \mathcal{C}^1$? By FTC and the exchange of uniform limits with Riemann integrals, we have:

$$f(x) - f(y) = \lim_{n\to\infty} f_n(x) - f_n(y) = \lim_{n\to\infty} \int_x^y f_n' = \int_x^y g$$

for all $x, y \in \Omega$, and all n. □

TODO: Hölder.

0.2 Lebesgue spaces of functions

We will keep the same fixed Ω, and continue to be interested in functions $f : \Omega \to \mathbb{R}$; and, unless stated otherwise, by:

$$\int_\Omega f(x)dx$$

I mean the integral w.r.t. the Lebesgue measure. We can generalise to functions from Ω to some arbitrary measure space, but won't in this course.

Definition 0.2.1. For some real $p \geq 1$, the **L^p-norm** on the set of measurable functions $f : \Omega \to \mathbb{R}$ is:

$$\|f\|_{L^p} := \left|\int_\Omega |f|^p \, dx\right|^{\frac{1}{p}}$$

and for $p = \infty$:

$$\|f\|_{L^\infty} := \operatorname{essup}_\Omega f = \inf\{M \in \mathbb{R},\ |f| \leq M \text{ a.e.}\}$$

Unfortunately, if $f = g$ a.e. but $f \neq g$ our norm fails to be definite. For this we consider $L^p(\Omega, \mathbb{R})$ the set of such measurable functions with finite L^p modulo equality a.e., making our norm definite.

Most of the discussion around L^p spaces has a counterpart in l^p spaces, either by adapting the proof directly, or consider the collection of sequences as a discrete measure space. Such discussions probably won't be included.

To show the L^p norm is in fact a norm, we know just have to show homogenaity, which is obvious, and the triangle inequality, which will involve some work.

Lemma 0.2.2 (Young's inequality)**.** Let $p, q, r \in \mathbb{R}$ be such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

then for all $x, y > 0$ we have

$$(xy)^r \leq \frac{r}{p} q^p + \frac{r}{q} y^q$$

Proof. On problem sheet 2. □

Proposition 0.2.3 (Hölder's inequality)**.** Given $p, q, r \in [1, \infty]$ such that

$$\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$$

if $f, g \in L^p, L^q$ respectively, then $fg \in L^r$ and $\|fg\|_{L^r} \leq \|f\|_{L^p} \|g\|_{L^q}$.

Proof. If p or $q = \infty$ this is easy, so assume $p, q < \infty$. By homogenaity we may assume

$$\|f\|_{L^p} = \|g\|_{L^q} = 1$$

and by Young's inequality we see

$$\|fg\|_{L^r}^r = \int |f|^r |g|^r \, dx \leq \int \frac{r}{p} |f|^p + \frac{r}{q} |g|^q \, dx = \frac{r}{p} \int |f|^p \, dx + \frac{r}{q} \int |g|^q \, dx = \frac{r}{p} + \frac{r}{q} = 1.$$ □

Proposition 0.2.4 (Minkowski's inequality)**.** If $f, g \in L^p$, then $\|f+g\|_{L^p} \leq \|f\|_{L^p} + \|g\|_{L^p}$.

Proof. If $p = \infty$ this is easy, so assume $p = \infty$. Pointiwse, we know:

$$|f+g|^p \leq |f+g|^{p-1} (|f| + |g|)$$

now integrate to get:

$$\|f+g\|_{L^p}^p = \int |f+g|^p \, dx \leq \int |f+g|^{p-1} (|f| + |g|) dx = \int |f+g|^{p-1} |f| \, dx + \int |f+g|^{p-1} |g| \, dx$$

applying Hölder's inequality on $\frac{p-1}{p} + \frac{1}{p} = 1$ gives this

$$\leq \left| \int |f+g|^p \, dx \right|^{1-\frac{1}{p}} (\|f\|_{L^p} + \|g\|_{L^p}) = \frac{\|f+g\|_{L^p}^p}{\|f+g\|_{L^p}} (\|f\|_{L^p} + \|g\|_{L^p})$$

multiplying out now completes the proof. □

Theorem 0.2.5. $L^p(\Omega, \mathbb{R})$ is complete w.r.t. the L^p norm.

Proof. Let (f_n) be a Cauchy sequence in L^p, as subsequences of Cauchy sequences are also Cauchy sequences, wlog we can have $\|f_{n+1} - f_n\|_{L^p} < 2^{-n}$. We want to define:

$$f(x) = f_0(x) + \sum_{n=0}^{\infty} (f_{n+1}(x) - f(x))$$

but we don't yet know if the sum is well defined, so instead consider

$$g(x) = |f_0(x)| + \sum_{n=0}^{\infty} |f_{n+1}(x) - f(x)|$$

which does exist, and call the Nth trucation g_N. By Minkowski's inequality:

$$\|g_N\|_{L^p} \le \|f\|_{L^p} + \sum_{n=0}^{N} \|f_{n+1} - f_n\|_{L^p} \le C + \sum_{n=0}^{N} 2^{-n} \le C + 2.$$

(g_N) is a monotone increasing sequence converging to g all ≥ 0, the same holds for (g_N^p) and g^p, so by the monotone convergence theorem:

$$\|g\|_{L^p}^p = \int \lim_{N\to\infty} |g_N|^p \, dx = \lim_{N\to\infty} \|g_N\|_{L^p}^p \le (C+2)^p$$

so g is finite a.e. and thus f is defined a.e.

f is certainly in L^p as $|f| \le g \in L^p$.

Now fix an $\varepsilon > 0$ and consider

$$|f - f_n| = \left| f - f_0 - \sum_{k=0}^{\infty} |f_{k+1} - f_k| \right| \le |f| + |f_0| + \sum \cdots < |f| + |f_0| + g$$

so

$$|f - f_n|^p \le 3^p(|f|^p + |f_0|^0 + |g|^p)$$

the RHS is finite and doesn't depend on n so by the domianted convergence theorem $(f - f_n) \to 0$ a.e. thus $\|f - f_n\|_{L^p} \to 0$. □

Proposition 0.2.6. Continuous functions are dense in L^p.

Idea. We can reduce the situation to only deal with characterisitc functions of measurable sets $\mathbb{1}_A$, by the regularity of the Lebesgue measure we can always find a closed F and open U such that $\mu(U \setminus F) < \varepsilon$ and $F \subset A \subset U$. Now we can observe:

$$f(t) = \frac{\operatorname{dist}(t, \Omega \setminus U)}{\operatorname{dist}(t, \Omega \setminus U) + \operatorname{dist}(t, F)}$$

has suppose on U, and is 1 on F, and thus $\|f - \mathbb{1}_A\|_{L^p} < \varepsilon$. □

TODO: i have no idea what the fuck this is saying, i should learn what it means.

0.3 Convolutions and mollifiers

Definition 0.3.1. If $f \in L^1(\mathbb{R}^d)$ and $\phi \in \mathcal{C}_c^0(\mathbb{R}^d)$, then $f * \phi : \mathbb{R}^d \to \mathbb{R}$ is given by

$$(f * \phi)(x) = \int_{\mathbb{R}^d} f(x-y)\phi(y)dy$$

Some obvious first properties of the convolution are that $\operatorname{Supp}(f * \phi) \subset \operatorname{Supp}(f) + \operatorname{Supp}(\phi)$, and $f * \phi = \phi * f$ if both are in $\mathcal{C}_c^0$. To go further we need this following result we will not prove.

Proposition 0.3.2 (Generalised Minkowski inequality)**.** For *suitable* F,

$$\left\| \int F(x,y)dy \right\|_{L^p(x)} \le \int \|F(x,y)\|_{L^p(x)} \, dy$$

Instead of interchanging L^p-norms with sums, we interchange L^p-norms with integrals.

Lemma 0.3.3. If $f \in L^p$ and $\phi \in \mathcal{C}_c^0$

$$\|\phi * f\|_{L^p} \le \|\phi\|_{L^1} \|f\|_{L^p}$$

Proof. Applying the generalised Minkowsky inequality diretly to the definition of the convolution

$$\left\| \int f(x-y)\phi(y)dy \right\|_{L^p(x)} \le \int \|f(x-y)\phi(y)\|_{L^p(x)} \, dy = \int |\phi(y)| \, \|f(x-y)\|_{L^p(x)} \, dy$$

TODO: there is something missing here, idfk what □

Proposition 0.3.4. If $f \in L^p$ and $\phi \in \mathcal{C}_c^k$, then $f*\phi \in \mathcal{C}^k$ and for all $\alpha \in \mathbb{N}^d$, we have $\partial^\alpha(f*\phi) = f*\partial^\alpha\phi$.

Proof. Once again, only consider the case $k = d = 1$, so $\partial^\alpha = \partial_x$. Furthermore, it suffices to deal with $f \in L^1$ as convolution is a local process and L^p functions are locally L^1.

$$\frac{(f*\phi)(x+h) - (f*\phi)(x)}{h} = \int f(y)\frac{\phi(x+h-y) - \phi(x-y)}{h}dy$$

and the assosiation of y to the RHS of the integrand, for fixed x, is uniformly bounded and converges pointwise to $\phi'(x-y)$, the claim now follows from the dominated convergence theorem. □

The **mollifiers** φ_n are defined as $\varphi_n(x) = n^d\varphi(nx)$ wherer $\varphi \in \mathcal{C}_c^\infty$ and $\int \varphi = 1$. These looks like bumb functions with increasing height and decreasing width as $n \to \infty$.

Theorem 0.3.5. Given a sequence of mollifiers (φ_n)

1. If $f \in \mathcal{C}_c^0$, then $f * \varphi_n \to f$ uniformly.
2. If $f \in L^p$, then $f * \varphi_n \to f$ in L^p.

Proof. □