Functional Analysis

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Contents

1 Banach Spaces

I'll write these notes mostly over $\mathbb{R}$, everything works similarly over $\mathbb{C}$ or any complete normed field. So when I say "vector space", I mean a real one.

Definition 1.0.1. A **norm** on a vector space E is a map $\|-\| : E \to \mathbb{R}_{\geq 0}$ such that for all $x, y \in E, \lambda \in \mathbb{R}$:

$$\|x\| = 0 \iff x = 0 \qquad \|\lambda x\| = |\lambda| \|x\| \qquad \|x + y\| \leq \|x\| + \|y\|$$

called **definiteness**, **homogenaity**, and the **triangle innequality** respectively.

Common examples are $\|-\|_1$, $\|-\|_2$, and $\|-\|_\infty$ on $\mathbb{R}^n$.

Definition 1.0.2. For a compact $K \subseteq \mathbb{R}^n$ the **uniform norm** on the set of continuous functions $\mathcal{C}(K, \mathbb{R})$ is given by:

$$\|f\|_\infty := \sup_{x \in K} |f(x)| = \max_{x \in K} |f(x)|.$$

Letting K be n discrete points recovers $\|-\|_\infty$ on $\mathbb{R}^n$, how can we recover the other defintions like this?

Definition 1.0.3. A **Banach space** is a complete normed vector space.

We will show in the finite case that all normed vector spaces are complete. Note that the on the space $\mathcal{C}([0,1], \mathbb{R})$ the norm:

$$\|f\| := \int_0^1 f(x) dx$$

is not complete, many progressive continuous approximations to non-conintuous shapes are an example. The change is normally increasingly localised and extreme, the uniform norm will properly detect this.

Definition 1.0.4. Two norms, $\|-\|, \|-\|'$ on a vector space E are **equivalent** if there exists $C_1, C_2 \geq 0$ such that for all $x \in E$:

$$C_1 \|x\|' \leq \|x\| \leq C_2 \|x\|'$$

This is obviously an equivalence relation. Equivalent norms induce homeomorphic topologies by id, as:

$$B'_{r/C_2}(x) \subseteq B_r(x) \subseteq B'_{r/C_1}(x).$$

Proposition 1.0.5. All finite dimensional normed vector spaces are equivalent.

Proof. Choose any norm on a d dimensional vector space E, and let $\{e_1, \ldots, e_d\}$ be a basis for E:

$$\|x\| = \left\|\sum x_i e_i\right\| \leq \sum \|x_i e_i\| = \sum |x_i| \|e_i\| \leq \max_{1 \leq i \leq d} \|e_i\| \sum |x_i| = \max_{1 \leq i \leq d} \|e_i\| \|x\|_1.$$

Now, let K be the unit sphere for $\|-\|_1$. This is closed and bounded so, by the Heine-Borel theorem, compact. The previous step shows $x \mapsto \|x\|$ is continuous wrt $\|-\|_1$, so it will attain its infinum on K. By definiteness, this minimum, m, will be positive. And for all $x \in E$: $\|x\| = \|x\|_1 \|x / \|x\|_1\| \geq m \|x\|_1$. $\square$

From now onwards, let Ω be an open subset of $\mathbb{R}^d$ such that $\overline{\Omega}$ is compact and $\partial\Omega$ is smooth. The particulars of these restrictions isn't important.

Definition 1.0.6. The **uniform norm** for $f \in \mathcal{B}(\Omega, \mathbb{R})$ is given by $\|f\|_\infty := \sup_{x\in\Omega} |f(x)|$.

Think of $C_b^0(\Omega, \mathbb{R})$ as a subspace of $\mathcal{B}(\Omega, \mathbb{R})$.

Proposition 1.0.7. $C_b^0(\Omega, \mathbb{R})$ is closed (contains all limit points) w.r.t. the uniform norm.

Proof. Let $(f_n) \to f$, and fix some $\varepsilon > 0$. By assumption, there exists some $N \geq 0$ such that $\|f_N - f\|_\infty < \varepsilon$. As f_N is continuous, there will exist some $B_\delta(x) \subseteq \Omega$ such that for all $y \in B_\delta(x)$ we have $|f_N(x) - f_N(y)| < \varepsilon$. So we can write:

$$\begin{aligned} |f(x) - f(y)| &= |(f(x) - f_N(x)) - (f(y) - f_N(y)) + (f_N(x) - f_N(y))| \\ &\leq |(f(x) - f_N(x))| + |(f(y) - f_N(y))| + |(f_N(x) - f_N(y))| < 3\varepsilon. \end{aligned}$$

□

Proposition 1.0.8. $C_b^0(\Omega, \mathbb{R})$ is complete w.r.t. the uniform norm.

Proof. Let (f_n) be Cauchy w.r.t $\|-\|_\infty$, for any $x \in \Omega$, $(f_n(x))$ will be a Cauchy sequence in $\mathbb{R}$ so converges to a limit we will call $f(x)$. It now suffices to show $(f_n) \to f$ w.r.t. $\|-\|_\infty$ and appeal to the previous proposition. Fix $\varepsilon > 0$, by assumption there exists N such that $\forall n, m \geq N$, $\|f_n - f_m\|_\infty < \varepsilon$. Consider for all $x \in \Omega$:

$$|f_n(x) - f(x)| = \lim_{m\to\infty} |f_n(x) - f_m(x)| \leq \varepsilon.$$

As each $|f_n(x) - f(x)| \leq$ is ε, so will $\|f_n - f\|_\infty$, the supremum. □

Definition 1.0.9. Let $\mathcal{C}^k(\Omega, \mathbb{R})$ be the set of k-times bounded continuously differentiably functions, equipped with the norm:

$$\|f\|_{\mathcal{C}^k} := \sum_{|\alpha|\leq k} \|\partial^\alpha f\|_\infty \qquad \alpha \in \mathbb{N}^d$$

The simplest example is with $d = k = 1$ where $\|f\|_{\mathcal{C}^1} = \|f\|_\infty + \|f'\|_\infty$.

Proposition 1.0.10. For $\Omega \subset \mathbb{R}$, $\mathcal{C}^1(\Omega, \mathbb{R})$ with the $\|-\|_{\mathcal{C}^1}$ norm is complete.

Proof. Let (f_n) be Cauchy w.r.t. $\|-\|_{\mathcal{C}^1}$, then both (f_n) and (f_n') must be Cauchy w.r.t. $\|-\|_\infty$. So, by the previous proposition, there exists continuous, uniform limits $(f_n) \to f$, $(f_n') \to g$. Thus:

$$\|f_n - f\|_{\mathcal{C}^1} = \|f_n - f\|_\infty + \|f_n' - g\|_\infty \to 0.$$

How do we know $f' = g$ or even $f \in \mathcal{C}^1$? By FTC and the exchange of uniform limits with Riemann integrals, we have:

$$f(x) - f(y) = \lim_{n\to\infty} f_n(x) - f_n(y) = \lim_{n\to\infty} \int_x^y f_n' = \int_x^y g$$

for all $x, y \in \Omega$, and all n. □

TODO: Hölder.