MATH40002 Analysis 1

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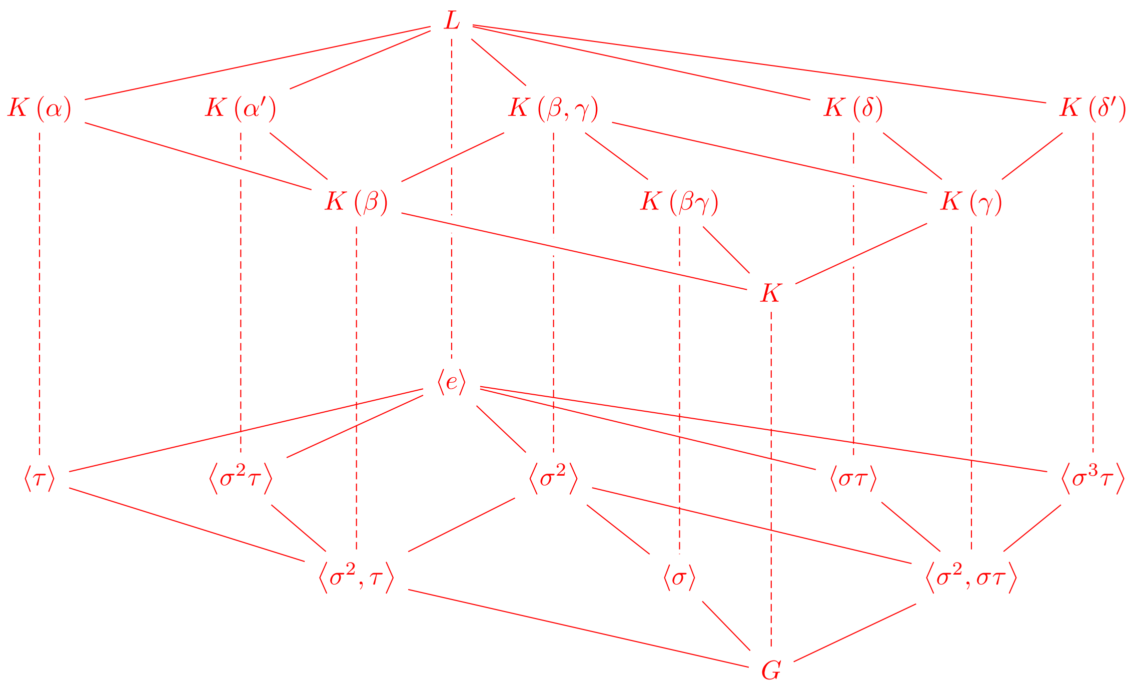

Syllabus

Number systems, decimal expansions, sup and inf, sequences, series, convergence tests, power series, continuity, closure, compactness, uniform continuity, differentiation, integration

Contents

0 Introduction

Lecture 1
Thursday
10/01/19

The following are references.

- E Artin, Galois theory, 1994
- A Grothendieck and M Raynaud, Revêtements étales et groupe fondamental, 2002
- I N Herstein, Topics in algebra, 1975
- M Reid, Galois theory, 2014

Notation.

1 Number systems

1.1 Naturals, integers and rationals

Definition 1 (Natural numbers)**.** As in IUM, we define the **natural numbers**, $\mathbb{N}$, from the Peano axioms:

P1 0 is a natural number,

P6 if n is a natural number then $S(n)$ is a natural number where $S(n)$ is the successor of n,

P9 the principle of mathematical induction.

Clearly, there are many Peano axioms not included, these are however not particularly relevant to this course. Addition and multiplication is defined as expected and will descend to our other number systems

Definition 2 (Integers)**.** The **integers** are defined as $\mathbb{Z} := \mathbb{N} \times \mathbb{N}/\sim$ where $\sim$ is the equivalence relation given by $(a, b) \sim (c, d)$ iff $a + d = b + c$. Subtraction is defined as expected and will also descend to our other number systems.

Definition 3 (Rationals)**.** The **rationals** are defined as $\mathbb{Q} := \mathbb{Z} \times \mathbb{N}^{>0}/\sim$ where $\sim$ is the equivalence relation given by $(a, b) \sim (c, d)$ iff $ad = bc$. The equivalence class (p, q) will be written as $\frac{p}{q}$. There is an element of each equivalence class $\frac{p'}{q'}$ with $\gcd(p', q') = 1$, we say that $\frac{p'}{q'}$ is in **lowest terms**.

Theorem 4 (Axioms of the rationals)**.** With the usual operations descended from $\mathbb{N}$ and $\mathbb{Z}$, $\mathbb{Q}$ satisfies the following axioms with $a, b, c \in \mathbb{Q}$ throughout:

Q1 $a + (b + c) = (a + b) + c$ ($+$ is associative),

Q2 $\exists 0 \in \mathbb{Q}$ such that $a + 0 = a$ (0 is the additive identity of $\mathbb{Q}$),

Q3 $\forall a \in \mathbb{Q}, \exists (-a) \in \mathbb{Q}$ such that $a + (-a) = 0$ ($\mathbb{Q}$ is closed under additive inverses),

Q4 $a + b = b + a$ ($+$ is commutative),

Q5 $a \times (b \times c) = (a \times b) \times c$ ($\times$ is associative),

Q6 $\exists 1 \in \mathbb{Q}$ such that $a \times 1 = a$ (1 is the multiplicative identity of $\mathbb{Q}$),

Q7 $a \times (b + c) = (a \times b) + (a \times c)$ ($\times$ is left distributive over $+$),

Q8 $(a + b) \times c = (a \times c) + (b \times c)$ ($\times$ is right distributive over $+$),

Q9 $a \times b = b \times a$ ($\times$ is commutative),

Q10 $\forall a \in \mathbb{Q}, \exists a^{-1} \in \mathbb{Q}$ such that $a \times a^{-1} = 1$ ($\mathbb{Q}$ is closed under multiplicative inverses),

Q11 for all $a \in \mathbb{Q}$ either $x < 0$, $x = 0$ or $x >=$ (Trichotomy),

Q12 for all $x, y \in \mathbb{Q}$ we have $x > 0, y > 0 \implies x + y > 0$,

Q13 for all $x \in \mathbb{Q}$ there exists a $n \in \mathbb{N}$ such that $x < n$ (Archimedean axiom).

1-4 says $(\mathbb{Q}, +)$ is an abelian group, 1-9 says $(\mathbb{Q}, +, \times)$ is a commutative ring, 1-10 says $(\mathbb{Q}, +, \times)$ is a field.

1.2 Decimal expansions

Definition 5. For $a_0 \in \mathbb{N}$ and $a_i \in [1,9]$ for $i > 0 \in \mathbb{N}$, define the **periodic decimal**

$$a_0.a_1a_2\ldots\overline{a_ia_{i+1}\ldots a_j},$$

to be equal to the rational number

$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \ldots + \frac{a_i}{10^i} + \left(\frac{a_{i+1}a_{i+2}\ldots a_j}{10^j}\right)\left(\frac{1}{1-10^{i-j}}\right).$$

Theorem 6. If $x \in \mathbb{Q}$ has 2 decimal expansions, then they will be of the form

$$x = a_0.a_1a_2\ldots a_n\overline{9} = a_0.a_1a_2\ldots(a_n+1), a_n \in [0,8].$$

Definition 7 (Real numbers)**.** The **real numbers**, $\mathbb{R}$, can be defined as:

$$\mathbb{R} := \{a_0.a_1a_2\ldots : a_0 \in \mathbb{Z}, a_i \in [0,9], \nexists N \in \mathbb{N} \text{ such that } a_i = 9\ \forall i \geq N\}.$$

1.3 Countability

Definition 8 (Countability)**.** A set S is **countably infinite** iff there exists a bijection $f : \mathbb{N} \to S$, a set is **countable** if it is finite or countable infinite.

Theorem 9. All $S \subseteq \mathbb{N}$ are countable, $\mathbb{Z}$ and $\mathbb{Q}$ are both countable, $\mathbb{R}$ is uncountable.

2 Bounded sets

2.1 Supremums and infinums

Definition 10 (Maximum and minimum)**.** $s \in \mathbb{R}$ is the **maximum** of a set $S \subset \mathbb{R}$ iff $\forall s' \in S$, $s \geq s'$. **Minimums** are defined similarly. Maximums and minimums are unique.

Definition 11 (Bounded)**.** A non-empty set $S \subset \mathbb{R}$ is **bounded above** if there exists some $M \in \mathbb{R}$ such that $\forall s \in S$, $s \leq M$ with **bounded below** defined similarly. S is **bounded** if it is both bounded above and bounded below.

Theorem 12. If S is bounded then $\exists R > 0$ such that $|s| < \mathbb{R}$ for all $s \in S$.

Definition 13 (Supremum and infinum)**.** If $S \subset \mathbb{R}$ is bounded above, we say $x \in \mathbb{R}$ is the **least upper bound** or **supremum** iff x is and upper bound for S and for all $y \in \mathbb{R}$ such that y is an upper bound of S, $x \leq y$. The **infinum** is defined similarly.

2.2 Completeness

Theorem 14 (Completeness axiom)**.** For all non-empty $S \subset \mathbb{R}$, if S is bounded above then S has a supremum, and similarly for S bounded below.

2.3 Dedekind cuts

Definition 15 (Dedekind cut)**.** A non-empty set $S \subset \mathbb{Q}$ is a **Dedekind cut** if it satisfies:

1. $s \in S$ and $s > t \in \mathbb{Q} \implies t \in S$ (S is a semi-infinite interval to the left),
2. S is bounded above with no maximum.

Dedekind cuts are in the form $S_r := (-\infty, r) \cap \mathbb{Q}$.

Theorem 16 (Real numbers)**.** We can redefine the reals as the set of Dedekind cuts, $\mathbb{R} := \{S_r \subset \mathbb{Q}\}$. All operations and orderings as well as the completeness axiom are held by this new Dedekind cut definition.

Theorem 17 (Triangle innequality)**.** For all $a, b \in \mathbb{R}$ we have $|a+b| \leq |a| + |b|$.

3 Sequences

Definition 18 (Real sequence). A **real sequence** is a function $a : \mathbb{N} \to \mathbb{R}$ written (a_n). Sequences of other number systems are defined similarly.

3.1 Convergence

Definition 19 (Convergence of sequences). A real sequence (a_n) **converges** to some $a \in \mathbb{R}$ as $n \to \infty$ iff

$$\forall \epsilon > 0,\ \exists N_\epsilon \text{ such that } \forall n \geq N_\epsilon,\ |a_n - a| < \epsilon.$$

For complex series the definition is the same just with $|\cdot|$ referring to the modulus instead of the absolute value. This is written $a_n \to a$ (as $n \to \infty$).

3.2 Divergence

Definition 20 (Divergence). A sequence (a_n) **diverges** iff it doesn't converge.

Definition 21 (Divergence to infinity). A sequence (a_n) **diverges to** ∞ iff $\forall R > 0,\ \exists N \in \mathbb{N}$, such that $\forall n \geq N, a_n > R$. And similarly for a sequence diverging to $-\infty$.

3.3 Limits

Theorem 22 (Uniqueness of limits). Given a sequence (a_n) if $a_n \to a$ and $a_n \to b$, $a = b$.

Theorem 23. If a sequence (a_n) converges then (a_n) is bounded.

Theorem 24 (Algebra of limits). Given two sequences $a_n \to a$ and $b_n \to b$ the following hold:

- $a_n + b_n \to a + b$,
- $a_n b_n \to ab$ (a special case of this is $ca_n \to ca$ for a constant c),
- $\frac{a_n}{b_n} \to \frac{a}{b}$ given $b \neq 0$.

Theorem 25. If (a_n) is a positive sequence then $a_n \to 0 \iff \frac{1}{a_n} \to +\infty$, and similarly for negative sequences.

Theorem 26 (Ratio test). If a sequence (a_n) satisfies $\left|\frac{a_{n+1}}{a_n}\right| \to L < 1$ then $a_n \to 0$.

3.4 Monotone sequences

Definition 27 (Monotonically increasing sequence). A sequence, (a_n), is **monotonically increasing** iff $\forall m, n \in \mathbb{N}$ with $n > m$ we have $a_n \geq a_m$, and similarly for monotonically decreasing and their strict equivalents.

Theorem 28 (Monotone convergence). If a sequence (a_n) is monotone increasing and bounded above then $a_n \to a := \sup\{a_i : i \in \mathbb{N}\}$ written $a_n \uparrow a$. This holds similarly for monotone decreasing sequences.

3.5 Cauchy sequences

Definition 29 (Cauchy sequence). A sequence (a_n) is a **Cauchy sequence** iff $\forall \epsilon > 0 \in \mathbb{R},\ \exists N \in \mathbb{N}$ such that $\forall n, m < N,\ |a_n - a_m| < \epsilon$.

Theorem 30 (Cauchy convergence criterion). A sequence (a_n) converges iff it is a Cauchy sequence.

3.6 Subsequences

Definition 31 (Subsequence)**.** Given a strictly monotonically increasing function $n : \mathbb{N} \to \mathbb{N}$ and a sequence (a_n), the sequence (b_n) defined by $b_i := a_{n(i)}$ is a **subsequence** of (a_n).

Theorem 32. Given a subsequence of (a_n), $(a_{n(i)})$, if $a_n \to a$ then $a_{n(i)} \to a$ as $i \to \infty$.

Theorem 33 (Bolzano-Weierstrass)**.** If a sequence (a_n) is bounded then it has a convergent subsequence.

Note 34 (Sketch of the Bolzano-Weierstrass theorem proof)**.** The proof of the Bolzano-Weierstrass theorem is an equally valuable point as the statement of the theorem itself. The idea of the proof considers the "peak points" of the sequence: if there are infinitely many peak points, then the peak points themselves form a monotonically decreasing subsequence; if there are finitely many, then the points after the final peak must have a monotonically decreasing subsequence bounded above by the final peak. By the monotone convergence theorem both of these subsequences must converge.

4 Series

Definition 35 (Infinite series)**.** An **(infinite) series** is an expression of the form $\sum_{i=1}^{\infty} a_i$ of $a_1 + a_2 + \dots$ for some sequence (a_n). The sequence **partial sums** of the series (S_n) is given by

$$S_n := \sum_{i=1}^{n} a_i = a_1 + a_2 + \dots + a_n.$$

4.1 Convergence

Definition 36 (Convergence of series)**.** The series $\sum_{i=1}^{\infty} a_i$ of (a_n) **converges** iff $S_n \to A \in \mathbb{R}$, written $\sum_{n=1}^{\infty} a_n = A$.

Theorem 37. For a sequence (a_n), $\sum_{n=1}^{\infty} a_n$ converges if $a_n \to 0$ (the converse is not true).

Theorem 38. Given a sequence non-negative sequence (a_n), the convergence of the infinite series and the boundedness of (S_n) are equivalent.

Theorem 39 (Algebra of limits for series)**.** A similar algebra of limits for series can be established from the algebra of limits for sequences acting on the partial sums of the series.

Theorem 40 (Comparison I test)**.** Given sequences $(a_n), (b_n)$ if $0 \le a_n \le b_n$ then:

- If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, $0 \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$,
- If $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ also diverges.

Theorem 41 (Comparison II test (Sandwich theorem))**.** Given sequences $(a_n), (b_n), (c_n)$ with $a_n \le b_n \le c_n$, if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} c_n$ both converge, $\sum_{n=1}^{\infty} b_n$ converges.

Theorem 42. If $\alpha > 1 \in \mathbb{R}$, $\sum_{n=1}^{\infty} \frac{1}{n^{\alpha}}$ converges.

Definition 43 (Alternating sequence)**.** A sequence (a_n) is **alternating** iff $a_{2n} \ge 0$ and $a_{2n-1} \le 0$ of vice versa for all $n \in \mathbb{N}^{>0}$.

Theorem 44. If (a_n) is alternating with $|a_n| \downarrow 0$, a_n converges and $\sum_{n=1}^{\infty} a_n$ converges.

4.2 Absolute convergence

Definition 45 (Absolute convergence)**.** Given a sequence (a_n) the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** iff $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem 46. Absolute convergence $\implies$ convergence.

Theorem 47 (Comparison III test)**.** Given sequences $(a_n), (b_n)$ with $\frac{a_n}{b_n} \to L \in \mathbb{R}$ if $\sum_{n=1}^{\infty} b_n$ is absolutely convergent then $\sum_{n=1}^{\infty} a_n$ is also absolutely convergent.

Theorem 48 (Ratio test)**.** If the sequence (a_n) is such that $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent or divergent if $r > 1$.

Theorem 49 (Root test)**.** If the sequence (a_n) is such that $|a_n|^{\frac{1}{n}} \to r < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent or divergent is $r > 1$.

Remark 50. Both the ratio test and the root test are inconclusive if $r = 1$.

4.3 Rearrangement of series

Sometimes, series are easier to deal with and have cancellations when their terms are rearranged. However, the rearrangement of terms will only preserve limits under certain conditions.

Definition 51 (Reordering)**.** Given a bijection $n : \mathbb{N} \to \mathbb{N}$ and a sequence (a_n), the sequence (b_n) with $b_i := a_{n(i)}$ is a **rearrangement** or **reordering** of (a_n).

Theorem 52. If (a_n) is a sequence satisying $a_n \to 0$, $\sum_{n:a_n \geq 0} a_n = \infty$ and $\sum_{n:a_n \leq 0} a_n = -\infty$ then $\sum_{n=1}^{\infty} a_n$ can be rearranged to converge to any $r \in \mathbb{R}$.

Theorem 53. If (a_n) is a sequence with absolutely convergent series, $\sum_{n:a_n \geq 0} a_n = A$ and $\sum_{n:a_n \leq 0} a_n = B$ with all arrangements of (a_n) converging to $A + B$.

4.4 Power series

Throughout this subsection $[0, \infty] := [0, \infty) \cup \{+\infty\}$.

Definition 54 (Power series)**.** For $z \in \mathbb{C}$ and a complex sequence (a_n), a **power series** is an expression in the form $\sum_{n=1}^{\infty} a_n z^n$.

Definition 55 (Radius of convergence)**.** Given the power series $\sum_{n=1}^{\infty} a_n z^n$, there exists some $R \in [0, \infty]$ such that:

- $|z| < R \implies \sum_{n=1}^{\infty} a_n z^n$ converges,
- $|z| > R \implies \sum_{n=1}^{\infty} a_n z^n$ diverges.

We cannot tell what happens when $|z| = R$ so this has to be checked separately. R is the **radius of convergence** of the power series.

Corollary 56. Given the same power series $\sum_{n=1}^{\infty} a_n z^n$, have $S := \{|z| \in \mathbb{R}^{\geq 0} : a_n z^n \to 0\}$ then

$$R := \begin{cases} \sup(S) & \text{if } S \text{ is bounded} \\ \infty & \text{otherwise} \end{cases}.$$

is the radius of convergence for the power series.

Theorem 57 (Evaluating radius of convergence from tests)**.** For the power series $\sum_{n=1}^{\infty} a_n z^n$:

- if $\left|\frac{a_{n+1}}{a_n}\right| \to a \in [0, \infty]$ then $R = \frac{1}{a}$ is the radius of convergence for the power series,
- if $|a_n|^{\frac{1}{n}} \to a \in [0, \infty]$ then $R = \frac{1}{a}$ is the radius of convergence for the power series,

Definition 58 (Cauchy product)**.** Given two series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$; their **Cauchy product** is the series $\sum_{n=0}^{\infty} \sum_{i=0}^{n} a_i b_{n-i}$.

Remark 59. If (a_n), (b_n) are the coefficients for a power series, then the Cauchy product of their series will be the coefficients of the product of the power series.

Theorem 60. If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are absolutely convergent their Cauchy product converges absolutely to $\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$.

Theorem 61. If the power series $\sum_{n=1}^{\infty} a_n z^n$, $\sum_{n=1}^{\infty} b_n z^n$ have radii of convergence R_a, R_b respectively then their Cauchy product has radius of convergence $R_c \geq \min(R_a, R_b)$.

4.5 Exponential series

Definition 62. For $z \in \mathbb{C}$, its **exponential series** is

$$E(z) := \sum_{n=}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots,$$

with $E(z)$ converging absolutely for all $z \in \mathbb{C}$.

Theorem 63 (Properties of exponential series)**.** For all $z, w \in \mathbb{C}$:
1. $E(z)E(w) = E(z + w)$, 2. $\frac{1}{E(z)} = E(-z)$, 3. $E(z) \neq 0$.

Theorem 64. For all $x \in \mathbb{Q}$, $E(x) = e^x$, with $e := E(1)$.

5 Continuity

5.1 Continuous functions

Definition 65 (Limit of real functions)**.** For a function $f : \mathbb{R} \to \mathbb{R}$ and some $a, b \in \mathbb{R}$ we have $f(x) \to b$ **as** $x \to a$ of $\lim_{x \to a} f(x) = b$ iff:

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } |x - a| < \delta \impliedby |f(x) - b| < \epsilon.$$

Definition 66 (Continuity of real functions)**.** Given the function $f : \mathbb{R} \to \mathbb{R}$

1. f is **continuous at a point** $a \in \mathbb{R}$ iff $\lim_{x \to a} f(x) = f(a)$,
2. f is **continuous (on** $\mathbb{R}$**)** iff f is continuous at all $a \in \mathbb{R}$.

Definition 67 (Discontinuity of real functions)**.** The function $f : \mathbb{R} \to \mathbb{R}$ is **discontinuous** at a point if it is not continuous at that point.

Definition 68 (Sequential continuity)**.** The function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R} \iff f(a_n) \to f(a)$ as $n \to \infty$ for all sequences (a_n) converging to a.

Remark 69. The definition for limits and continuity of complex functions is similar with $|\cdot|$ being the modulus instead of the absolute values. The same definitition also applies for functions that are continuous on certain subsets of $\mathbb{R}$ or $\mathbb{C}$.

Theorem 70. $E : \mathbb{C} \to \mathbb{C}$ given by $E(z) := \sum_{n=}^{\infty} \frac{z^n}{n!}$ is continuous on $\mathbb{C}$.

Theorem 71 (Properties of the real exponential function)**.** Given the exponential function $E : \mathbb{R} \to (0, \infty)$:

1. for all $x \in \mathbb{R}$, $E(x) > 0$,
2. $x > 0 \implies E(x) > 1$,
3. $E(x)$ is a strictly increasing function,
4. For $|x| < 1$, $|E(x) - 1| \leq \frac{|x|}{1 - |x|}$,
5. E is a continuous bijection.

Theorem 72. The inverse of $E(x) = e^x$ is the **natural logarithm** function $\ln : (0, \infty) \to \mathbb{R}$ satisfying $y = \ln x \iff x = e^y$ for all $x, y \in \mathbb{R}$.

Definition 73 (Exponentiation of positive bases)**.** For $a \in (0, \infty)$, for all $x \in \mathbb{R}$ define $a^x := E(x \ln a)$.

Definition 74 (Trigonomentric functions)**.** The **sine** and **cosine** functions are defined as:

$$\sin(\theta) := \Im[E(i\theta)], \qquad \cos(\theta) := \Re[E(i\theta)].$$

and are both continuous functions from $\mathbb{R} \to [-1, 1]$.

Theorem 75 (Continuity of piecewise functions)**.** For $a, c \in \mathbb{R}$ with functions $f_1 : (-\infty, a) \to \mathbb{R}$ and $f_2 : (a, \infty) \to \mathbb{R}$, the **piecewise function** $f : \mathbb{R} \to \mathbb{R}$, defined as,

$$f(x) := \begin{cases} f_1(x) & \text{if } x < a \\ c & \text{if } x = a \\ f_2(x) & \text{if } x > a \end{cases}$$

is continuous on $\mathbb{R}$ iff both f_1 and f_2 are continuous on their respective domains and

$$\lim_{x \uparrow a} f_1(x) = \lim_{x \downarrow a} f_2(x) = c.$$

5.2 Properties of continuity

Theorem 76. For $f, g : \mathbb{R} \to \mathbb{R}$ continuous at $a \in \mathbb{R}$ the following functions are also continuous at a:

1. αf for all $\alpha \in \mathbb{R}$; 2. $f + g, f \cdot g$; 3. $\frac{f}{g}$, given $g(a) \neq 0$.

Theorem 77. The following functions (all by their well known definitions) are continuous:

1. $f(x) = x^n$, for $n \in \mathbb{N}_0$ (**monomials**);
2. $p(x) = \sum_{i=1}^{n} a_i x^i$, given (a_n) is a real sequence (**polynomials**);
3. $\frac{p(x)}{q(x)}$ at $a \in \mathbb{R}$ given p, q are polynomials with $q(a) \neq 0$ (**rational functions**);
4. $\sin(x)$, $\cos(x)$ on $\mathbb{R}$ and $\tan(x)$ whenever $\cos(x) \neq 0$, plus their reciprocals under similar conditions;
5. $f \circ g$ at $a \in \mathbb{R}$ when g is continuous at a and f is continuous at $g(a)$.

Theorem 78 (Intermediate value theorem)**.** Given $a, b \in \mathbb{R}$ with $a \leq b$, if $f : [a, b] \to \mathbb{R}$ is continuous, then for all c between $f(a)$ and $f(b)$ there exists some $x \in [a, b]$ such that $f(x) = c$.

Definition 79 (Boundedness of real functions)**.** Given some $S \subseteq \mathbb{R}$ a function $f : S \to \mathbb{R}$ is **bounded above** iff $\exists M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in \mathbb{R}$. The definitions for **bounded below** and **bounded** extend naturally from this.

Theorem 80 (Extreme value theorem)**.** Given $a, b \in \mathbb{R}$ with $a \leq b$, if $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded.

6 Properties of subsets

6.1 Open sets

Definition 81 (Open sets)**.** A set $S \subset \mathbb{R}$ is **open** iff $\forall x \in S,\ \exists \delta$ such that $(x - \delta, x + \delta) \subset S$.

Theorem 82 (Union of open sets)**.** For a collection of open sets in $\mathbb{R}$, $\{S_i\}$, given the indexing set $\mathcal{I}$ (could be countable or uncountable), $\bigcup\limits_{i \in \mathcal{I}} S_i$ is open in $\mathbb{R}$.

Theorem 83 (Finite intersections of open sets)**.** The intersection of finitely many open sets in $\mathbb{R}$ is open in $\mathbb{R}$.

6.2 Closed and compact sets

Definition 84 (Closed sets)**.** A set $S \subset \mathbb{R}$ is **closed** in $\mathbb{R}$ if all convergent subsequences of S have a limit in S.

Definition 85 (Compact sets)**.** A set $S \subset \mathbb{R}$ is **compact** in $\mathbb{R}$ if it is closed and bounded in $\mathbb{R}$.

Theorem 86. The complement of an open set is closed.

Remark 87. Not every set in $\mathbb{R}$ is either open or closed. Half-open intervals are neither open nor closed while $\mathbb{R}$ and $\emptyset$ are both open and closed.

Theorem 88. The finite union or any intersection of closed sets in $\mathbb{R}$ is closed.

Theorem 89. A set $S \subset \mathbb{R}$ is compact iff every subsequence of S has as convergent subsequence $x_{n(i)} \to x \in S$.

Theorem 90 (Extreme value theorem for compact sets)**.** If $S \subset \mathbb{R}$ is compact with $f : S \to \mathbb{R}$ continuous, there exists some $c, d \in S$ with $f(c) = \inf\limits_{x \in S} f(x)$ and $f(d) = \sup\limits_{x \in S} f(x)$.

7 Uniform continuity and convergence

7.1 Uniform continuity

Definition 91 (Uniform continuity)**.** A fuction $f : S \to \mathbb{R}$ is **uniformly continuous** iff

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x, y \in S, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Uniform continuity is a more powerful notion that continuity with f is uniformly continuous $\implies$ f is continuous.

Theorem 92. If $S \subset \mathbb{R}$ is compact and $f : S \to \mathbb{R}$ continuous then f is uniformly continuous.

7.2 Convergence of sequences of functions

Definition 93 (Pointwise convergence)**.** For some $S \subset \mathbb{R}$ with the sequence $f_1, f_2, \ldots : S \to \mathbb{R}$, f_n **converges pointwise** to some $f : S \to \mathbb{R}$ if

$$\forall x \in S,\ \forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |f(x) - f_n(x)| < \epsilon.$$

Written $\forall x \in S,\ \lim\limits_{n \to \infty} f_n(x) = f(x)$.

Definition 94 (Uniform convergence)**.** For some $S \subset \mathbb{R}$, the sequence $f_1, f_2, \ldots : S \to \mathbb{R}$ **uniformly converges** to some $f : S \to \mathbb{R}$ if

$$\forall \epsilon > 0, \ \exists N \in \mathbb{N} \text{ such that } \forall x \in S, \text{ and } \forall n > N, |f(x) - f_n(x)| < \epsilon.$$

Theorem 95. If a sequence of (uniformly) continuous functions converges uniformly to a function f then f is (uniformly) continuous.

Theorem 96. If, given $S \subset \mathbb{R}$, $(f_n) : S \to \mathbb{R}$ is a uniformly convergent sequence of continuous functions with $a \in S$ open in S, $\lim\limits_{n\to\infty} \lim\limits_{x\to a} f_n(x) = \lim\limits_{x\to a} \lim\limits_{n\to\infty} f_n(x)$.

7.3 Convergence of series of functions

Definition 97 (Convergence of series of functions)**.** Given $(f_n) : S \to \mathbb{R}$ defined on $S \subset \mathbb{R}$, the series $\sum\limits_{n=1}^{\infty} f_n(x)$ **converges (uniformly)** iff the sequence of partial sums $S_n(x) = \sum\limits_{n=1}^{n} f_n(x)$ converges (uniformly).

Theorem 98 (Weierstrass M-test)**.** Given continuous $(f_n) : S \to \mathbb{R}$ defined on $S \subset \mathbb{R}$,

$$\forall x \in S \text{ and } \forall i \in \mathbb{N}, \ \exists M_1, M_2, \ldots \in \mathbb{R} \text{ such that } |f_i(x)| \leq M_i \text{ and } \sum_{i=1}^{\infty} M_i \text{ converges}$$
$$\implies \sum_{n=1}^{\infty} f_i(x) \text{ converges uniformly to some continuous } g : S \to \mathbb{R}.$$

Theorem 99. If a power series $f(x) = \sum\limits_{n=1}^{\infty} f_i(x)$ has radius of convergence $R > 0$ then f is continuous on $(-R, R)$.

8 Differentiation

8.1 Differentiability

Definition 100 (Differentiability)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is **differentiable** at $a \in \mathbb{R}$, with **derivative** $f'(a) = \dfrac{d}{dx} f(x)\Big|_a$ iff

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} \text{ exists, which we set to } f'(a).$$

f is differentiable on $S \subseteq \mathbb{R}$, with derivative $\dfrac{d}{dx} f = \dfrac{df}{dx} = f' : \mathbb{R} \to \mathbb{R}$, if it is differentiable at every $x \in S$.

Examples 101. The following functions are all differentiable,

- $f(x) = x^n$, for $n \in \mathbb{N}$ on $\mathbb{R}$ with $f'(x) = nx^{n-1}$,
- $f(x) = e^x$ on $\mathbb{R}$ with $f'(x) = e^x$,
- $f(x) = \ln x$ on $\mathbb{R}^{>0}$ with $f'(x) = \frac{1}{x}$.

Theorem 102. f is differentiable $\implies$ f is continuous.

Theorems 103 (Operations on derivatives)**.** If $f, g : \mathbb{R} \to \mathbb{R}$ are both differentiable at $x = a \in \mathbb{R}$ then,

1. for all $c, d \in \mathbb{R}$, $h(x) := c \cdot f(x) + d \cdot g(x)$ is differentiable at $x = a$ with $h'(a) = c \cdot f'(a) = d \cdot g'(a)$,
2. $p(x) := f(x) \cdot g(x)$ is differentiable at $x = a$ with $p'(a) = f(a) \cdot g'(a) + f'(a) \cdot g(a)$,
3. if $f(a) \neq 0$, $q(x) := \dfrac{1}{f(a)}$ is differentiable at $x = a$ with $q'(a) = -\dfrac{f'(a)}{[f(a)]^2}$,
4. if $g(a) \neq 0$ $r(x) := \dfrac{f(x)}{g(x)}$ is differentiable at $x = a$ with $r'(a) = \dfrac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{[g(a)]^2}$.

Theorem 104 (Chain rule)**.** If $g, f : \mathbb{R} \to \mathbb{R}$ are differentiable at $x = a \in \mathbb{R}$ and $x = g(a)$ respectively then $s(x) := f \circ g(x)$ iss differentiable at $x = a$ with $s'(a) = g'(a) \cdot f' \circ g(a)$.

8.2 Local extrema and mean values

Definition 105 (Local extrema)**.** For a function $f : S \to \mathbb{R}$, f has a **local minimum** as $a \in \mathbb{R}$ iff $\exists \delta > 0$ such that $\forall y \in S$ with $|y - a| < \delta$, $f(y) \leq f(a)$, and similary for a **local maximum**.

Theorem 106. If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) and has a local maximum or minimum at $c \in (a, b)$, $f'(c) = 0$.

Theorem 107 (Rolle's theorem)**.** If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) with $f(a) = f(b)$, $\exists c \in (a, b)$ such that $f'(c) = 0$.

Theorem 108 (Mean value theorem)**.** If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b), $\exists c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Theorem 109. If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) with $f'(x) \geq 0$ for all $x \in (a, b)$ then f is monotone increasing. Similar holds for monotone/strictly increasing/decreasing or constant.

Theorem 110 (Cauchy's MVT)**.** A similar but slightly more general statement than the MVT: if $f, g : [a, b] \to \mathbb{R}$ are differentiable on (a, b), $\exists c \in (a, b)$ with $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$.

8.3 L'Hôpital's rule

Theorem 111 (L'Hôpital's rule)**.** Given $f, g : [c, d] \to \mathbb{R}$ are differentiable on (c, d) except possibly at some $a \in (c, d)$ with $g'(x) \neq 0$ on $(c, d) \setminus \{a\}$:

$$\text{if } \lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \text{ or } \infty \text{ and } \lim_{x \to a} \frac{f'(x)}{g'(x)} = L \text{ then } \lim_{x \to a} \frac{f(x)}{g(x)} = L.$$

This also applies when taking $\lim\limits_{x \to \infty}$.

Definition 112 (Higher derivatives)**.** **Higher derivatives** of $f : \mathbb{R} \to \mathbb{R}$ are defined inductively as

$$f^{(n)}(x) := \begin{cases} f(x) & \text{if } x = 0 \\ f^{(n-1)\prime}(x) & \text{otherwise} \end{cases}.$$

The existence of the nth derivative of f requires all lower order derivatives of f also exist and be differentiable.

Theorem 113 (Second derivative test)**.** For a second differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f'(a) = 0$ for some $a \in \mathbb{R}$,

- $f''(a) > 0 \implies f$ has a local minimum at $x = a$,
- $f''(a) < 0 \implies f$ has a local maximum at $x = a$,
- the test is inconclusive if $f''(a) = 0$.

8.4 Taylor's theorem

Definition 114 (Taylor polynomial of a function)**.** Given $f : [c, d] \to \mathbb{R}$ has an order $n \in \mathbb{N}_0$ derivative at $x = a \in (c, d)$, the **Taylor polynomial** of order n at $x = a$ is

$$P_n(x) := \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x - a)^i = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Theorem 115 (Taylor's theorem)**.** Given $f : [c, d] \to \mathbb{R}$ has an order $n + 1$, for some $n \in \mathbb{N}_0$, derivative for all $x \in (c, d)$. For $a, b \in [c, d]$ with $a \neq b$ there exists some t between a and b such that,

$$f(b) = P_n(b) + \frac{f^{(n+1)}(t)}{(n + 1)!}(b - a)^{n+1}.$$

This is a further, massive generalisation of the MVT (the case when $n = 0$).

Definition 116 (Taylor series of a function)**.** The **Taylor series**, $P(x)$, for a function $f : \mathbb{R} \to \mathbb{R}$ at $x = a$ exists if $f^{(n)}(a)$ exists for all $n \in \mathbb{N}$ and is given by

$$P(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots$$

Definition 117 (Analytic function)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is **analytic** if it equals its Taylor series.

8.5 Convexity

Definition 118 (Convexity of functions)**.** A function $f : [a, b] \to \mathbb{R}$ is **convex** iff

$$\forall c, t, d \in [a, b] \text{ with } c < t < d, f(c) + \frac{f(d) - f(c)}{d - c}(t - c) \geq f(t).$$

Theorem 119. Given the function $f : [a, b] \to \mathbb{R}$ with $f''(x)$ existing on (a, b), f is convex $\iff$ $f''(x)$ non-negative on (a, b).

8.6 Exchange of limits and derivatives

Theorem 120 (Criteria for exchange of limits and derivatives)**.** Given (f_n) is a sequence of functions with $f_n : [a, b] \to \mathbb{R}$ differentiable, if $\lim\limits_{n\to\infty} f_n(c)$ exists for some $c \in [a, b]$ and $(f_n'(x))$ converges uniformly on $[a, b]$: (f_n) converges uniformly to some differentiable f satisfying $f'(x) = \lim\limits_{n\to\infty} f_n'(x)$.

Theorem 121 (Derivatives of power series)**.** Given a power series $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$, f has a continuous derivative on $(-R, R)$ with $f'(x) = \sum\limits_{n=0}^{\infty} n a_n x^{n-1}$.

Corollary 122. Given a power series $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$, the Taylor series of f centered at $x = 0$ is $\sum\limits_{n=0}^{\infty} a_n x^n$.

8.7 Trigonometric properties

Definition 123 (π)**.** Let $S = \{y > 0 : \sin(y) = 0\}$, $\pi := \inf S$.

Definition 124 (Periodic function)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is $2L$**-periodic** iff $f(x + 2L) = f(x)$ for all $x \in \mathbb{R}$.

Theorem 125. $\sin$ and $\cos$ satisfy the following important properties: 1. $\sin(x)$ is odd, 2. $\cos(x)$ is even, 3. $\cos^2(x) + \sin^2(x) = 1$ for all $x \in \mathbb{R}$, 4. $\sin$ and $\cos$ are 2π-periodic functions.

9 Integration

9.1 Partitions

Definition 126 (Partition)**.** A **partition**, P, of the interval $[a, b] \subset \mathbb{R}$ is a finite collection of points $x_0, x_1, \ldots, x_n \in [a, b]$ such that $a = x_0 < x_1 < \ldots < x_n = b$. A partition naturally splits the domain $[a, b]$ into finitely many closed intervals.

Definition 127 (Refinement)**.** Given partitions Q, P, Q is a **refinement** of P, written $Q \prec P$, iff every point of P is also in Q.

Definition 128 (Common refinement)**.** Given paritions P, Q the **common refinement** of P and Q is the partition R containing all points in P or Q. $R \prec P$ and $R \prec Q$.

9.2 Darboux sums

Definition 129 (Darboux sums)**.** Given the bounded function $f : [a, b] \to \mathbb{R}$ and the partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, we will assign to each subintervals generated by P:

- a length, $\Delta x_i := x_{i+1} - x_i$,
- an infinum, $m_i := \inf\limits_{x_i \leq t \leq x_{i+1}} f(t)$,
- a supremum, $M_i := \sup\limits_{x_i \leq t \leq x_{i+1}} f(t)$.

Now define the **lower Darboux sum** and **upper Darboud sum** of f w.r.t. P as:

$$L(f,P) := \sum_{i=0}^{n-1} m_i \Delta x_i, \qquad U(f,P) := \sum_{i=0}^{n-1} M_i \Delta x_i \qquad \text{respectively.}$$

If $f : [a,b] \to \mathbb{R}$ is continuous then $L(f,P)$ and $U(f,p)$ exist. $L(f,P)$ is always less than or equal to $U(f,P)$.

Theorem 130 (Boundedness of refined Darboux sums)**.** If $f : [a,b] \to \mathbb{R}$ is bounded with $Q \prec P$ partitions of $[a,b]$, $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$.

Theorem 131. Given some bounded $f : [a,b] \to \mathbb{R}$, the set $\{L(f,P) : P \text{ is a partition of } [a,b]\}$ is bounded above by any upper Darboux sum on $[a,b]$ w.r.t. f.

9.3 Darboux integral

Definition 132 (Darboux integrals)**.** Given a bounded function $f : [a,b] \to \mathbb{R}$, the **lower Darboux integral** and **upper Darboux integral** are:

$$\underline{\int_a^b} f(x)\,\mathrm{d}x := \sup_P L(f,P), \qquad \overline{\int_a^b} f(x)\,\mathrm{d}x := \inf_P U(f,P) \qquad \text{respectively.}$$

Definition 133 (Darboux integrability)**.** If the upper and lower Darboux integral of a bounded function $f : [a,b] \to \mathbb{R}$ are equal, f is **Darboux integrable** on $[a,b]$ with

$$\int_a^b f(x)\,\mathrm{d}x := \underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x.$$

We will now refer to Darboux integrable functions simply as **integrable**.

Theorem 134. A bounded function $f : [a,b] \to \mathbb{R}$ is integrable iff $\forall \epsilon > 0$ there exists a partition P with $U(f,P) - L(f,P) < \epsilon$. Furthermore, given a sequence of paritions (P_n) if $\lim_{n\to\infty} (U(f,P_n) - L(f,P_n)) = 0$ then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} (L(f,P_n) = \lim_{n\to\infty} (U(f,P_n).$$

Remark 135. For a bounded function $f : [a,b] \to \mathbb{R}$, f is integrable if it is, continuous, differentiable, monotone, or discontinuous at finitely many points.

9.4 Properties of integration

Theorem 136 (Monotonicity)**.** If $f, g : [a,b] \to \mathbb{R}$ are integrable with $f(x) \leq g(x)$ for all $x \in \mathbb{R}$,

$$(1) \quad \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x. \qquad (2) \quad m \cdot (b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M \cdot (b-a).$$

Theorem 137 (Boundedness)**.** If $f : [a,b] \to \mathbb{R}$ is integrable with $m \leq f(x) \leq M$ for all $x \in \mathbb{R}$,

Theorem 138 (Linearity)**.** If $f, g : [a,b] \to \mathbb{R}$ are integrable, for all $c, d \in \mathbb{R}$,

$$(3) \quad \int_a^b (cf(x) + dg(x))\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x + d\int_a^b g(x)\,\mathrm{d}x. \qquad (4) \quad \int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

Theorem 139 (Integrability on subdomains)**.** $f : [a,b] \to \mathbb{R}$ is integrable iff $\forall c \in [a,b]$, f is integrable on $[a,c]$ and $[c,b]$ with,

Theorem 140 (Composition)**.** If $f : [a,b] \to [m,M] \subset \mathbb{R}$, $g : [m,M] \to \mathbb{R}$ are integrable and continuous respectively, $h(x) := g \circ f(x)$ is integrable on $[a,b]$.

Theorem 141 (Triangle inequality). If $f : [a, b] \to \mathbb{R}$ is integrable then $|f|$ is integrable on $[a, b]$ with,

$$(6) \quad \left| \int_a^b f(x)\,\mathrm{d}x \right| \le \int_a^b |f(x)|\,\mathrm{d}x. \qquad (7) \quad \int_a^b f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x.$$

Theorem 142 (Finite point differences). If $f, g : [a, b] \to \mathbb{R}$ are integrable with $f(x) = g(x)$ except at finitely many points,

Theorem 143 (Products). If $f, g : [a, b] \to \mathbb{R}$ are integrable then $f \cdot g : [a, b] \to \mathbb{R}$ is integrable.

Theorem 144 (Maxima and minima). If $f, g : [a, b] \to \mathbb{R}$ are integrable then $\max(f, g), \min(f, g) : [a, b] \to \mathbb{R}$ are integrable.

9.5 Fundamental theorems of calculus

Theorem 145 (Fundamental theorem of calculus 1). Given continuous $f : [a, b] \to \mathbb{R}$, have $F : [a, b] \to \mathbb{R}$ with $F(x) := \displaystyle\int_a^x f(t)\,\mathrm{d}t$. F is continuous on $[a, b]$ and differentiable on (a, b). $F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem 146 (Fundamental theorem of calculus 2). Given continuous $f : [a, b] \to \mathbb{R}$ with continuous derivative on (a, b), $\displaystyle\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$.

9.6 Methods of integration

Theorem 147 (MVT). If $f : [a, b] \to \mathbb{R}$ is continuous, $\exists c \in [a, b]$ such that $\displaystyle\int_a^b f(x)\,\mathrm{d}x = f(c)(b - a)$.

Theorem 148 (Integration by parts). If $f, g : [a, b] \to \mathbb{R}$ have continuous first derivatives,

$$(2) \quad \int_a^b f(x)g'(x)\,\mathrm{d}x = \Big[f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,\mathrm{d}x. \qquad (3) \quad \int_{u(c)}^{u(d)} f(x)\,\mathrm{d}x = \int_c^d f(u(x))u'(x)\,\mathrm{d}x.$$

Theorem 149 (Integration by substitution). Given continuous $f : [a, b] \to \mathbb{R}$ if $u : [a, b] \to [c, d]$ has a continuous derivative on (c, d),

9.7 Limits and integrals

Theorem 150 (Exchanging limits and integrals). If $f_n : [a, b] \to \mathbb{R}$ is a sequence of integrable functions converging uniformly to $f : [a, b] \to \mathbb{R}$, then f is integrable with,

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x.$$

Theorem 151 (Power series integration). If the power series $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$, f is integrable on all closed subintervals of $(-R, R)$ with

$$\int_0^x f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \text{ for all } x \in (-R, R).$$

9.8 Improper integrals

Definition 152 (Improper integral). Given $f : (a, b] \to \mathbb{R}$ integrable on all $[c, b] \subset (a, b]$, the **improper integral**,

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c \downarrow a} \int_c^b f(x)\,\mathrm{d}x,$$

if the limit exists, otherwise the integral **diverges**; and similarly for other non-closed intervals or those with $\pm\infty$ as bounds.

Remark 153. When integrating over intervals with multiple undefined points, the integral is split into sums of multiple integrals each with single undefined points on their boundaries.