

# Chapter 1

# Calculus

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## Introduction

The following are suggested textbooks:

- G F Simmons, Calculus with Analytic Geometry, 1995
- J Stewart, Calculus, 2011
- S Lang, A First Course in Calculus, 1986
- S Lang, Undergraduate Analysis, 1997
- J Marsden and A Weinstein, Calculus I and Calculus II, 1985

**Note.** The actual majority of MATH40004A Calculus was a less formal and more example / application based derivation of the entirety of MATH40002 Analysis. As all of this content can be found in the corresponding document for Analysis, it isn't included in here.

Lecture 1  
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# 1 Lengths, volumes and surfaces

## 1.1 Lengths

**Theorem 1.1.1** (Arc length). The **arc length** of the curve  $y = f(x)$  along  $[a, b]$  is given by

$$\int_a^b \sqrt{1 + (f'(x))^2} \, dx$$

**Theorem 1.1.2** (Distance and velocity of parameterised curves). If a curve is parameterised by  $(x(t), y(t), z(t))$ , the **distance travelled** from time  $t_0$  to  $t$  is given by:

$$L(t) = \int_{t_0}^t \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt$$

which naturally leads to the velocity at  $t$ :

$$v(t) = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}$$

## 1.2 Volumes and volumes of revolution

**Theorem 1.2.1** (Volume). If the cross sectional area of a shape when cut by a plane at  $x = x_0$  is given by  $A(x_0)$  for all  $x_0 \in [a, b]$ , the volume of the shape is given by

$$V = \int_a^b A(x) \, dx$$

**Theorem 1.2.2** (Disk method). The **volume of revolution** of  $y = f(x)$  about the  $x$ -axis from  $x = a$  to  $x = b$  is given by,

$$V_x = \int_a^b \pi (f(x)^2) \, dx$$

**Theorem 1.2.3** (Shell method). The **volume of revolution** of  $y = f(x)$  about the  $y$ -axis from  $y = a$  to  $y = b$  is given by,

$$V_y = \int_a^b \pi (f^{-1}(x)^2) \, dx = \int_a^b 2\pi x f(x) \, dx$$

## 1.3 Surfaces

**Theorem 1.3.1.** The **surface area of revolution** of  $y = f(x)$  about the  $x$ -axis from  $x = a$  to  $x = b$  is given by,

$$S_x = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2} \, dx$$

**Theorem 1.3.2.** The **surface area of revolution** of  $y = f(x)$  about the  $y$ -axis from  $y = a$  to  $y = b$  is given by,

$$S_y = \int_a^b 2\pi x \sqrt{1 + (f'(x))^2} \, dx$$

## 1.4 Centres of mass

**Theorem 1.4.1** (1D discrete case). If we have a system of  $n$  particles each with mass  $m_k$  and position  $x_k$  we can define the **centre of mass** at  $\bar{x}$  by

$$\bar{x} = \frac{\sum_{k=1}^n m_k x_k}{\sum_{k=1}^n m_k}$$

**Theorem 1.4.2** (2D continuous case). If we have a region limited by  $f(x)$  and  $g(x)$ , give  $g(x) \leq f(x)$  for all  $x \in [a, b]$ , with uniform mass, the coordinates of the **centre of mass**,  $(\bar{x}, \bar{y})$  is

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x)) \, dx}{\int_a^b f(x) - g(x) \, dx} \quad \bar{y} = \frac{\int_a^b \frac{f(x)^2 - g(x)^2}{2} \, dx}{\int_a^b f(x) - g(x) \, dx}$$

**Theorem 1.4.3** (Pappus's theorem). If  $R$  is a region with area  $A$  lying on one side of the line  $l$ ,  $V = Ad$  is the volume obtained by rotation  $R$  about  $l$ , where  $d$  is the distance travelled by the **com** when  $R$  is rotated about  $l$ .

## 1.5 Moments of inertia

**Theorem 1.5.1.** Given a curve  $y = f(x)$  in the interval  $[a, b]$ , this is representing a wire in a given shape, and have the density per unit length of the wire at a given  $x$  be  $\rho(x)$ , the **moment of inertia** of the curve about the  $x$  and  $y$  axis respectively is given by

$$I_x = \int_a^b \rho(x) f(x)^2 \sqrt{1 + f'(x)^2} \, dx \quad I_y = \int_a^b \rho(x) x^2 \sqrt{1 + f'(x)^2} \, dx$$

## 1.6 Polar coordinates

**Definition 1.6.1** (Polar coordinates). A parameterisation of  $x, y$  is  $r, \theta$  with  $x = r \cos(\theta)$  and  $y = r \sin(\theta)$ .

**Theorem 1.6.2** (Polar arc length). The arc length of a curve,  $r = f(\theta)$  in polar coordinates between angles  $\alpha, \beta$  is given by

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta$$

**Theorem 1.6.3** (Polar area). The area of a polar curve,  $r = f(\theta)$  between angles  $\alpha, \beta$  is given by

$$A = \frac{1}{2} \int_{\alpha}^{\beta} f(\theta)^2 \, d\theta$$

# 2 Fourier series

## 2.1 Orthogonal and orthonormal function spaces

**Definition 2.1.1** (Inner product of functions). If  $f, g : [a, b] \rightarrow \mathbb{R}$  are integrable on  $[a, b]$ , the their **inner product** is defined as

$$\langle f, g \rangle := \int_a^b f(x)g(x) \, dx$$

**Definition 2.1.2** (Orthogonal and orthonormal system). If  $\mathcal{S} = \{\phi_0, \phi_1, \dots\}$  is a collection of integrable real functions on  $[a, b]$ , iff  $\langle \phi_n, \phi_m \rangle = 0$  for all  $n \neq m$  then  $\mathcal{S}$  is an **orthogonal system** on  $[a, b]$ . Furthermore,  $\mathcal{S}$  is a **orthonormal system** on  $[a, b]$  iff  $\|\phi_n\| := \langle \phi_n, \phi_n \rangle = 1$  for all  $n$ .

**Theorem 2.1.3.** The system

$$\mathcal{S} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{2\pi}}, \frac{\sin(x)}{\sqrt{2\pi}}, \frac{\cos(2x)}{\sqrt{2\pi}}, \frac{\sin(2x)}{\sqrt{2\pi}}, \dots \right\}$$

is orthonormal on all closed intervals of length  $2\pi$ .

## 2.2 Periodic functions

**Definition 2.2.1** (Periodic function). A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is **periodic** with period  $T$  iff  $f(x + T) = f(x)$  for all  $x \in \mathbb{R}$ .

**Definition 2.2.2** (Discontinuity). When periodically extending a function, if  $\lim_{x \rightarrow \xi+} f(x) \neq \lim_{x \rightarrow \xi-} f(x)$ , we

$$\text{set } f(\xi) := \frac{1}{2} \left[ \lim_{x \rightarrow \xi+} f(x) + \lim_{x \rightarrow \xi-} f(x) \right]$$

**Theorem 2.2.3** (Integral over period). If  $f(x)$  is a  $T$  periodic function, for all  $a, b \in \mathbb{R}$  we have

$$\int_{a+T}^{b+T} f(x) \, dx = \int_a^b f(x) \, dx$$

## 2.3 Trigonometric polynomials

## 2.4 Fourier series

## 3 Plane curves

## 4 Laplace transform