Chapter 1

Calculus

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Autumn 2023

Introduction

Lecture 1 Thursday 10/01/19

The following are suggested textbooks:

- G F Simmons, Calculus with Analytic Geometry, 1995
- J Stewart, Calculus, 2011
- S Lang, A First Course in Calculus, 1986
- S Lang, Undergradute Analysis, 1997
- J Marsden and A Weinstein, Calculus I and Calculus II, 1985

Note. The actual majority of MATH40004A Calculus was a less formal and more example / application based derivation of the entirety of MATH40002 Analysis. As all of this content can be found in the corresponding document for Analysis, it isn't included in here.

Contents

1 Lengths, volumes and surfaces

1.1 Lengths

Theorem 1.1.1 (Arc length)**.** The **arc length** of the curve $y = f(x)$ along $[a, b]$ is given by

$$\int_a^b \sqrt{1 + (f'(x))^2}\, \mathrm{d}x$$

Theorem 1.1.2 (Distance and velocity of parameterised curves)**.** If a curve is parameterised by $(x(t), y(t), z(t))$, the **distance travelled** from time t_0 to t is given by:

$$L(t) = \int_{t_0}^t \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}\, \mathrm{d}x$$

which naturally leads to the velocity at t:

$$v(t) = \sqrt{\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}z}{\mathrm{d}t}\right)^2}$$

1.2 Volumnes and volumes of revolution

Theorem 1.2.1 (Volume)**.** If the cross sectional area of a shape when cut by a plane at $x = x_0$ is given by $A(x_0)$ for all $x_0 \in [a, b]$, the volume of the shape is given by

$$V = \int_a^b A(x)\, \mathrm{d}x$$

Theorem 1.2.2 (Disk method)**.** The **volume of revolution** of $y = f(x)$ about the x-axis from $x = a$ to $x = b$ is given by,

$$V_x = \int_a^b \pi \left(f(x)^2\right) \mathrm{d}x$$

Theorem 1.2.3 (Shell method)**.** The **volume of revolution** of $y = f(x)$ about the y-axis from $y = a$ to $y = b$ is given by,

$$V_y = \int_a^b \pi \left(f^{-1}(x)^2\right) \mathrm{d}y = \int_a^b 2\pi x f(x)\, \mathrm{d}x$$

1.3 Surfaces

Theorem 1.3.1. The **surface area of revolution** of $y = f(x)$ about the x-axis from $x = a$ to $x = b$ is given by,

$$S_x = \int_a^b 2\pi f(x) \sqrt{1 + (f'(x))^2}\, \mathrm{d}x$$

Theorem 1.3.2. The **surface area of revolution** of $y = f(x)$ about the y-axis from $y = a$ to $y = b$ is given by,

$$S_y = \int_a^b 2\pi x \sqrt{1 + (f'(x))^2}\, \mathrm{d}x$$

1.4 Centres of mass

Theorem 1.4.1 (1D discrete case)**.** If we have a system of n particles each with mass m_k and position x_k we can define the **centre of mass** at $\bar{x}$ by

$$\bar{x} = \frac{\sum_{k=1}^{n} m_k x_k}{\sum_{k=1}^{n} m_k}$$

Theorem 1.4.2 (2D continuous case)**.** If we have a region limited by $f(x)$ and $g(x)$, give $g(x) \leq f(x)$ for all $x \in [a, b]$, with uniform mass, the coordinates of the **centre of mass**, $(\bar{x}, \bar{y})$ is

$$\bar{x} = \frac{\int_a^b x(f(x) - g(x))\,\mathrm{d}x}{\int_a^b f(x) - g(x)\,\mathrm{d}x} \qquad \bar{y} = \frac{\int_a^b f(x)^2 - g(x)^2\,\mathrm{d}x}{2\int_a^b f(x) - g(x)\,\mathrm{d}x}$$

Theorem 1.4.3 (Pappu's theorem)**.** If R is a reigon with area A lying on one side of the line l, $V = Ad$ is the volume abtained by rotation R about l, where d is the distance travelled by the **com** when R is rotated about l.

1.5 Moments of inertia

Theorem 1.5.1. Given a curve $y = f(x)$ in the interval $[a, b]$, this is representing a wire in a given shape, and have the density per unit lenght of the wire at a given x be $\rho(x)$, the **moment of inertia** of the curve about the x and y axis respectively is given by

$$I_x = \int_a^b \rho(x) f(x)^2 \sqrt{1 + f'(x)^2}\,\mathrm{d}x \qquad I_y = \int_a^b \rho(x) x^2 \sqrt{1 + f'(x)^2}\,\mathrm{d}x$$

1.6 Polar coordinates

Definition 1.6.1 (Polar coordinates)**.** A parameterisation of x, y is r, θ with $x = r\cos(\theta)$ and $y = r\sin(\theta)$.

Theorem 1.6.2 (Polar arc length)**.** The arc length of a curve, $r = f(\theta)$ in polar coordinates between angles α, β is given by

$$L = \int_\alpha^\beta \sqrt{\left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 + r^2}\,\mathrm{d}\theta$$

Theorem 1.6.3 (Polar area)**.** The area of a polar curve, $r = f(\theta)$ between angles α, β is given by

$$A = \frac{1}{2}\int_\alpha^\beta f(\theta)^2\,\mathrm{d}\theta$$

2 Fourier series

2.1 Orthogonal and orthonormal function spaces

Definition 2.1.1 (Inner product of functions)**.** If $f, g : [a, b] \to \mathbb{R}$ are integrable on $[a, b]$, the their **inner product** is defined as

$$\langle f, g \rangle := \int_a^b f(x) g(x)\,\mathrm{d}x$$

Definition 2.1.2 (Orthogonal and orthonormal system)**.** If $\mathcal{S} = \{\phi_0, \phi_1, \ldots\}$ is a collection of integrable real functions on $[a,b]$, iff $\langle \phi_n, \phi_m \rangle = 0$ for all $n \neq m$ then $\mathcal{S}$ is an **orthogonal system** on $[a,b]$. Furthermore, $\mathcal{S}$ is a **orthonormal system** on $[a,b]$ iff $||\phi_n|| := \langle \phi_n, \phi_n \rangle = 1$ for all n.

Theorem 2.1.3. The system

$$\mathcal{S} = \left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(x)}{\sqrt{2\pi}}, \frac{\sin(x)}{\sqrt{2\pi}}, \frac{\cos(2x)}{\sqrt{2\pi}}, \frac{\sin(2x)}{\sqrt{2\pi}}, \ldots \right\}$$

is orthonormal on all closed intervals of length 2π.

2.2 Periodic functions

Definition 2.2.1 (Periodic function)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is **periodic** with period T iff $f(x+T) = f(x)$ for all $x \in \mathbb{R}$.

Definition 2.2.2 (Discontinuity)**.** When periodically extending a function, if $\lim\limits_{x \to \xi+} f(x) \neq \lim\limits_{x \to \xi-} f(x)$, we set $f(\xi) := \frac{1}{2}\left[\lim\limits_{x \to \xi+} f(x) + \lim\limits_{x \to \xi-} f(x) \right]$

Theorem 2.2.3 (Integral over period)**.** If $f(x)$ is a T periodic function, for all $a, b \in \mathbb{R}$ we have

$$\int_{a+T}^{b+T} f(x)\,\mathrm{d}x = \int_a^b f(x)\,\mathrm{d}x$$

2.3 Trigonometric polynomials

Definition 2.3.1 (Trigonometric polynomial)**.** A **trigonometric polynomial** is a function in the form

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx))$$

Theorem 2.3.2. Using euler's identity we can rewrite a trigonometric polynomial

$$S_n(x) = \frac{1}{2}a_0 + \sum_{k=1}^{n} (a_k \cos(kx) + b_k \sin(kx)) \text{ as } \sum_{k=-n}^{n} \left(\gamma_k e^{ikx}\right) \text{ where } \gamma_k = \begin{cases} \frac{1}{2}a_0 & \text{if } k = 0 \\ \frac{1}{2}(a_k - ib_k) & \text{if } k \in [1, n] \\ \gamma_k^* & \text{otherwise} \end{cases}$$

2.4 Fourier series

Definition 2.4.1 (Fourier series)**.** If $f(x)$ is $2L$ periodic then its **Fourier series** is given by

$$f(x) := \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_k \cos\left(\frac{n\pi x}{L}\right) + b_k \sin\left(\frac{n\pi x}{L}\right) \right] \text{ where}$$

$$a_n := \frac{1}{\pi}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \qquad b_n := \frac{1}{\pi}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x$$

Lemma 2.4.2 (Riemann-Lebesgue)**.** If the function $f(x)$ is integrable on $[a,b]$ then

$$I_\lambda := \int_a^b g(x)\sin(\lambda x)\,\mathrm{d}x \to 0 \text{ as } \lambda \to \infty$$

Theorem 2.4.3 (Paerseval's)**.** If $f(x)$ is periodic on 2π and is represented by its Fourier series,

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f^2(x)\,\mathrm{d}x = \frac{1}{2}a_0^2 + \sum_{n=0}^{\infty}(a_n^2 + b_n^2)$$

3 Laplace transform

3.1 Definition

Definition 3.1.1 (Laplace transform)**.** The **Laplace transform** is a linear operator that when applied to a function $f(x)$ gives

$$F(p) := \mathcal{L}[f(x)] := \int_0^\infty e^{-px} f(x)\,\mathrm{d}x$$

Theorem 3.1.2 (Common Laplace tranformations)**.** These are some common functions with their Laplace transforms and the conditions for which they converge:

$f(x) = 1$	$F(p) = \frac{1}{p}$	Converges for $p > 0$
$f(x) = x$	$F(p) = \frac{1}{p^2}$	Converges for $p > 0$
$f(x) = x^n$	$F(p) = \frac{n!}{p^{n+1}}$	Converges for $p > 0$
$f(x) = e^{ax}$	$F(p) = \frac{1}{p-a}$	Converges for $p > a$
$f(x) = \sin(ax)$	$F(p) = \frac{a}{p^2+a^2}$	Converges for $p > 0$
$f(x) = \cos(ax)$	$F(p) = \frac{p}{p^2+a^2}$	Converges for $p > 0$
$f(x) = \sinh(ax)$	$F(p) = \frac{a}{p^2-a^2}$	Converges for $p > a$
$f(x) = \cosh(ax)$	$F(p) = \frac{p}{p^2-a^2}$	Converges for $p > a$

Theorem 3.1.3 (Existence of Laplace transform)**.** The Laplace transform for a function $f(x)$ exists iff there exists constants $M, c \in \mathbb{R}$ with $|f(x)| \le Me^{cx}$ ($f(x)$ is of **exponential order**).

3.2 Differentiating

Theorem 3.2.1 (Derivatives of Laplace transforms)**.** By performing DUTIS $n \in \mathbb{N}$ times we have

$$F^{(n)}(p) = \mathcal{L}[(-1)^n x^n f(x)]$$

3.3 Convolution theorem

Theorem 3.3.1 (Convolution theorem for Laplace transforms)**.** For integrable functions $f, g : \mathbb{R} \to \mathbb{R}$:

$$\mathcal{L}\left[\int_0^x f(x-t)g(t)\,\mathrm{d}t\right] = F(p)G(p)$$