Chapter 1

Categories

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Season Year

Introduction

The following are complementary reading for the course.

- G. Grimmett and D. J. A. Welsh, Probability: An Introduction, 1986
- J. K. Blitzstein and J. Hwang, Introduction to Probability, 2019
- D. F. Anderson et al, Introduction to Probability, 2018
- S. M. Ross, Introduction to Pro ability Models, 2014
- G. Grimmett and D. Stirzaker, Probability and Random Processes, 2001
- G. Grimmett and D. Stirzaker, One Thousand Exercises in Probability, 2009

Contents

1 Basic definitions

1.1 Categories

Definition 1.1.1 (Category)**.** A catgeory $\mathcal{C}$ contains the following data:

1. a *collection* of objects, $\mathrm{Ob}(\mathcal{C})$,
2. for every $x, y \in \mathrm{Ob}(\mathcal{C})$ a collection of morphisms $\mathrm{Hom}_{\mathcal{C}}(x, y)$ from x to y,
3. an identity morphism $\mathrm{id}_x \in \mathrm{Hom}_{\mathcal{C}}(x, x)$ for all $x \in \mathrm{Ob}(\mathcal{C})$,
4. a composition map of morphisms, $\circ : \mathrm{Hom}_{\mathcal{C}}(y, z) \times \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$ for all $x, y, z \in \mathrm{Ob}(\mathcal{C})$.

Which satisfy the two axioms:

1. for all $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ with $x, y \in \mathrm{Ob}(\mathcal{C})$ we have $f \circ \mathrm{id}_x = f = \mathrm{id}_y \circ f$,
2. for compatible morphisms f, g, h we have $f \circ (g \circ h) = (f \circ g) \circ h$.

We will use the shorthand $x \in \mathcal{C}$ for $x \in \mathrm{Ob}\,\mathcal{C}$, $\mathrm{Hom}(x, y)$ for $\mathrm{Hom}_{\mathcal{C}}(x, y)$ when $\mathcal{C}$ is obvious and $\mathrm{End}(x)$ for $\mathrm{Hom}(x, x)$.

Note 1.1.2. Note that in our definition the term *collection* is used instead of set, this is commonplace and necessary to prevent paradoxes when constructing the category of sets.

Examples 1.1.3. The following are all categories:

1. Set with sets as objects and functions as their morphisms,
2. Grp with groups as objects and their homomorphisms as morphisms,
3. Ab, Grp restricted to abelian groups,
4. for a field k, Vect_k with k-vector spaces as objects and linear transformations as morphisms,
5. Cat with categories as objects and soon to be defined **functors** as morphisms,
6. Top, Rng, Meas, Poset, Man with their objects and morphisms all defined similarly
7. Given a category $\mathcal{C}$, $\mathcal{C}^{op}$ wich has the same opjects as $\mathcal{C}$ but $\mathrm{Hom}_{\mathcal{C}^{op}}(x, y) = \mathrm{Hom}_{\mathcal{C}}(y, x)$ for all $x, y \in \mathcal{C}$,
8. Any set X with objects as elements in X and no morphisms except the identities
9. $(\mathbb{R}, \leq)$ with objects as $\mathbb{R}$ and a morphisms from x to y iff $x \leq y$ for all $x, y \in \mathbb{R}$.

Definition 1.1.4 (Isomorphism)**.** A morphism $f \in \mathrm{Hom}(x, y)$ is an **isomorphism** iff there is a morphism $f^{-1} \in \mathrm{Hom}(y, x)$ with $f \circ f^{-1} = \mathrm{id}_y$ and $f^{-1} \circ f = \mathrm{id}_x$.

1.2 Functors

Definition 1.2.1 ((Covariant) Functor)**.** Given categories $\mathcal{C}, \mathcal{D}$ a **(covariant) functor** $F : \mathcal{C} \to \mathcal{D}$ is the following data:

1. a map $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ (also denoted F),
2. for any two objects $x, y \in \mathcal{C}$ a map $\mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{D}}(F(x), F(y))$ (also also denoted F)

satisfying the properties:

1. for all $x \in \mathcal{C}$, $F(\mathrm{id}_x) = \mathrm{id}_{F(x)}$,
2. for all x, y, z with f, g in $\mathrm{Hom}_{\mathcal{C}}(y, z), \mathrm{Hom}_{\mathcal{C}}(x, y)$, $F(f \circ g) = F(f) \circ F(g)$.

Definition 1.2.2 (Contravariant functor)**.** A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor from $\mathcal{C}^{op}$ to $\mathcal{D}$.

Definition 1.2.3 (Hom-functor)**.** The **hom-functor** for a given category $\mathcal{C}$ is $\mathrm{Hom}_{\mathcal{C}} : \mathcal{C}^{op} \times \mathcal{C} \to \mathsf{Set}$ sending a pair of elements $c, d \in \mathcal{C}$ to $\mathrm{Hom}_{\mathcal{C}}(c, d)$.

1.3 Natural transformations

Definition 1.3.1 (Natural transformation)**.** Given categories $\mathcal{C}, \mathcal{D}$ with functors $F, G : \mathcal{C} \to \mathcal{D}$, a **natural transformation** $\eta : F \to G$ consists of morphisms η_x for all $x \in \mathcal{C}$ such that the diagram,

$$\begin{tikzcd} F(x) \arrow[r, "F(f)"] \arrow[d, "\eta_x"] & F(y) \arrow[d, "\eta_y"] \\ G(x) \arrow[r, "G(f)"] & G(y) \end{tikzcd}$$

commutes for all $x, y \in \mathcal{C}$ and $f \in \mathrm{Hom}_{\mathcal{C}}(x, y)$.

Remark 1.3.2. By constructing the category of functors from $\mathcal{C}$ to $\mathcal{D}$, denoted $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$, morphisms are natural transformations. **Natural isomorphisms** are defined as isomorphisms in this category.

1.4 Equivalence of categories

Definition 1.4.1 (Equivalence)**.** Given categories $\mathcal{C}, \mathcal{D}$ an **equivalence of categories** is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ with natural isomorphisms $FG \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ and $\mathrm{id}_{\mathcal{C}} \xrightarrow{\sim} GF$.

Definition 1.4.2 (Adjunction)**.** An **adjuction** between categories $\mathcal{C}, \mathcal{D}$ is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that for all $x \in \mathcal{C}$ and $y \in \mathcal{D}$, there exists an $\eta_{x,y} : \mathrm{Hom}_{\mathcal{C}}(x, G(y)) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(F(x), y)$ such that the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{D}}(F(x'), y) \arrow[r, "\circ F(f)"] \arrow[d, leftrightarrow, "\eta_{x',y}"] & \mathrm{Hom}_{\mathcal{D}}(F(x), y) \arrow[r, "g\circ"] \arrow[d, leftrightarrow, "\eta_{x,y}"] & \mathrm{Hom}_{\mathcal{D}}(F(x), y') \arrow[d, leftrightarrow, "\eta_{x,y'}"] \\ \mathrm{Hom}_{\mathcal{C}}(x', G(y)) \arrow[r, "\circ f"] & \mathrm{Hom}_{\mathcal{C}}(x, G(y)) \arrow[r, "G(g)\circ"] & \mathrm{Hom}_{\mathcal{C}}(x, G(y')) \end{tikzcd}$$

commutes for all $x, x' \in \mathcal{C}$; $y, y' \in \mathcal{D}$; $f : x \to x'$ and $g : y \to y'$.

Theorem 1.4.3. If F, G form an equivalence of the categories $\mathcal{C}, \mathcal{D}$ then F, G are an adjunction.

Examples 1.4.4 (Adjunctions in group theory)**.** Consider the **forgetful functor** $F : \mathrm{Ab} \to \mathrm{Grp}$ which simply forgets the Abelian property of a group. We also have the **abeliantisation functor** $(-)^{\mathrm{ab}} : \mathrm{Grp} \to \mathrm{Ab}$ which maps $G \mapsto G^{\mathrm{ab}} := G/[G, G]$. F and $(-)^{\mathrm{ab}}$ for an adjuction between Grp and Ab.

1.5 Representable functors

Definition 1.5.1 (Yoneda functor)**.** Given some x in a category $\mathcal{C}$, there is a functor $\mathrm{Hom}_{\mathcal{C}}(-, x) : \mathcal{C}^{op} \to \mathrm{Set}$ which satisfies the required properties to have the **Yoneda functor**:

$$Y : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}).$$

Which sends an element $y \in \mathcal{C}$ to the functor from objects in $\mathcal{C}^{op}$ to the set of morphisms from these objects to y.

Lemma 1.5.2. The Yoneda functor and the hom-functor form an adjunction in Cat.

Definition 1.5.3 (Representable)**.** A functor $F \in \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ is **representable** if $F \cong Y(c)$ for some $c \in \mathcal{C}$.

Example 1.5.4. Consider the functor $F : \mathrm{Set}^{(op)} \to \mathrm{Set}$ sending a set to its powerset. F is clearly isomorphic the functor $\mathrm{Hom}(-, \{0, 1\})$ from subsets to indicator functions on X. This is the image of the Yoneda functor so F is representable.

1.6 Yoneda lemma

Theorem 1.6.1 (Yoneda lemma)**.** Given some $x \in \mathcal{C}$ and $F \in \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})$ we have

$$\mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set})}(Y(x), F) \cong F(x).$$

Remark 1.6.2. This is a generalisation of Cayley's theorem which shows that we can study a group by instead studying the permutations of its underlying set.