Chapter 1

Analysis

1 Introduction

Lecture 1 Thursday 10/01/19

The following are references.

- E Artin, Galois theory, 1994
- A Grothendieck and M Raynaud, Revêtements étales et groupe fondamental, 2002
- I N Herstein, Topics in algebra, 1975
- M Reid, Galois theory, 2014

Notation.

2 Number systems

2.1 Naturals, integers and rationals

Definition 2.1.1 (Natural numbers)**.** As in IUM, we define the **natural numbers**, $\mathbb{N}$, from the Peano axioms:

P1 0 is a natural number,

P6 if n is a natural number then $S(n)$ is a natural number where $S(n)$ is the successor of n,

P9 the principle of mathematical induction.

Clearly, there are many Peano axioms not included, these are however not particularly relevant to this course. Addition and multiplication is defined as expected and will descend to our other number systems

Definition 2.1.2 (Integers)**.** The **integers** are defined as $\mathbb{Z} := \mathbb{N} \times \mathbb{N} / \sim$ where $\sim$ is the equivalence relation given by $(a, b) \sim (c, d)$ iff $a + d = b + c$. Subtraction is defined as expected and will also descend to our other number systems.

Definition 2.1.3 (Rationals)**.** The **rationals** are defined as $\mathbb{Q} := \mathbb{Z} \times \mathbb{N}^{>0} / \sim$ where $\sim$ is the equivalence relation given by $(a, b) \sim (c, d)$ iff $ad = bc$. The equivalence class (p, q) will be written as $\frac{p}{q}$. There is an element of each equivalence class $\frac{p'}{q'}$ with $\gcd(p', q') = 1$, we say that $\frac{p'}{q'}$ is in **lowest terms**.

Theorem 2.1.4 (Axioms of the rationals)**.** With the usual operations descended from $\mathbb{N}$ and $\mathbb{Z}$, $\mathbb{Q}$ satisfies the following axioms with $a, b, c \in \mathbb{Q}$ throughout:

Q1 $a + (b + c) = (a + b) + c$ ($+$ is associative),

Q2 $\exists 0 \in \mathbb{Q}$ such that $a + 0 = a$ (0 is the additive identity of $\mathbb{Q}$),

Q3 $\forall a \in \mathbb{Q}, \exists (-a) \in \mathbb{Q}$ such that $a + (-a) = 0$ ($\mathbb{Q}$ is closed under additive inverses),

Q4 $a + b = b + a$ ($+$ is commutative), Q5 $a \times (b \times c) = (a \times b) \times c$ ($\times$ is associative),

Q6 $\exists 1 \in \mathbb{Q}$ such that $a \times 1 = a$ (1 is the multiplicative identity of $\mathbb{Q}$),

Q7 $a \times (b + c) = (a \times b) + (a \times c)$ ($\times$ is left distributive over $+$),

Q8 $(a + b) \times c = (a \times c) + (b \times c)$ ($\times$ is right distributive over $+$),

Q9 $a \times b = b \times a$ ($\times$ is commutative),

Q10 $\forall a \in \mathbb{Q}, \exists a^{-1} \in \mathbb{Q}$ such that $a \times a^{-1} = 1$ ($\mathbb{Q}$ is closed under multiplicative inverses),

Q11 for all $a \in \mathbb{Q}$ either $x < 0$, $x = 0$ or $x >=$ (Trichotomy),

Q12 for all $x, y \in \mathbb{Q}$ we have $x > 0, y > 0 \implies x + y > 0$,

Q13 for all $x \in \mathbb{Q}$ there exists a $n \in \mathbb{N}$ such that $x < n$ (Archimedean axiom).

1-4 says $(\mathbb{Q}, +)$ is an abelian group, 1-9 says $(\mathbb{Q}, +, \times)$ is a commutative ring, 1-10 says $(\mathbb{Q}, +, \times)$ is a field.

2.2 Decimal expansions

Definition 2.2.1. For $a_0 \in \mathbb{N}$ and $a_i \in [1, 9]$ for $i > 0 \in \mathbb{N}$, define the **periodic decimal**

$$a_0.a_1a_2\dots\overline{a_ia_{i+1}\dots a_j},$$

to be equal to the rational number

$$a_0 + \frac{a_1}{10} + \frac{a_2}{100} + \dots + \frac{a_i}{10^i} + \left(\frac{a_{i+1}a_{i+2}\dots a_j}{10^j}\right)\left(\frac{1}{1 - 10^{i-j}}\right).$$

Theorem 2.2.2. If $x \in \mathbb{Q}$ has 2 decimal expansions, then they will be of the form

$$x = a_0.a_1a_2\dots a_n\overline{9} = a_0.a_1a_2\dots(a_n + 1), a_n \in [0, 8].$$

Definition 2.2.3 (Real numbers)**.** The **real numbers**, $\mathbb{R}$, can be defined as:

$$\mathbb{R} := \{a_0.a_1a_2\dots : a_0 \in \mathbb{Z}, a_i \in [0, 9], \nexists N \in \mathbb{N} \text{ such that } a_i = 9 \ \forall i \geq N\}.$$

2.3 Countability

Definition 2.3.1 (Countability)**.** A set S is **countably infinite** iff there exists a bijection $f : \mathbb{N} \to S$, a set is **countable** if it is finite or countable infinite.

Theorem 2.3.2. All $S \subseteq \mathbb{N}$ are countable, $\mathbb{Z}$ and $\mathbb{Q}$ are both countable, $\mathbb{R}$ is uncountable.

3 Bounded sets

3.1 Supremums and infinums

Definition 3.1.1 (Maximum and minimum)**.** $s \in \mathbb{R}$ is the **maximum** of a set $S \subset \mathbb{R}$ iff $\forall s' \in S$, $s \geq s'$. **Minimums** are defined similarly. Maximums and minimums are unique.

Definition 3.1.2 (Bounded)**.** A non-empty set $S \subset \mathbb{R}$ is **bounded above** if there exists some $M \in \mathbb{R}$ such that $\forall s \in S$, $s \leq M$ with **bounded below** defined similarly. S is **bounded** if it is both bounded above and bounded below.

Theorem 3.1.3. If S is bounded then $\exists R > 0$ such that $|s| < \mathbb{R}$ for all $s \in S$.

Definition 3.1.4 (Supremum and infinum)**.** If $S \subset \mathbb{R}$ is bounded above, we say $x \in \mathbb{R}$ is the **least upper bound** or **supremum** iff x is and upper bound for S and for all $y \in \mathbb{R}$ such that y is an upper bound of S, $x \leq y$. The **infinum** is defined similarly.

3.2 Completeness

Theorem 3.2.1 (Completeness axiom)**.** For all non-empty $S \subset \mathbb{R}$, if S is bounded above then S has a supremum, and similarly for S bounded below.

3.3 Dedekind cuts

Definition 3.3.1 (Dedekind cut)**.** A non-empty set $S \subset \mathbb{Q}$ is a **Dedekind cut** if it satisfies:

1. $s \in S$ and $s > t \in \mathbb{Q} \implies t \in S$ (S is a semi-infinite interval to the left),
2. S is bounded above with no maximum.

Dedekind cuts are in the form $S_r := (-\infty, r) \cap \mathbb{Q}$.

Theorem 3.3.2 (Real numbers)**.** We can redefine the reals as the set of Dedekind cuts, $\mathbb{R} := \{S_r \subset \mathbb{Q}\}$. All operations and orderings as well as the completeness axiom are held by this new Dedekind cut definition.

Theorem 3.3.3 (Triangle innequality)**.** For all $a, b \in \mathbb{R}$ we have $|a + b| \leq |a| + |b|$.

4 Sequences

Definition 4.0.1 (Real sequence)**.** A **real sequence** is a function $a : \mathbb{N} \to \mathbb{R}$ written (a_n). Sequences of other number systems are defined similarly.

4.1 Convergence

Definition 4.1.1 (Convergence of sequences)**.** A real sequence (a_n) **converges** to some $a \in \mathbb{R}$ as $n \to \infty$ iff

$$\forall \epsilon > 0,\ \exists N_\epsilon \text{ such that } \forall n \geq N_\epsilon,\ |a_n - a| < \epsilon.$$

For complex series the definition is the same just with $|\cdot|$ referring to the modulus instead of the absolute value. This is written $a_n \to a$ (as $n \to \infty$).

4.2 Divergence

Definition 4.2.1 (Divergence)**.** A sequence (a_n) **diverges** iff it doesn't converge.

Definition 4.2.2 (Divergence to infinity)**.** A sequence (a_n) **diverges to** ∞ iff $\forall R > 0,\ \exists N \in \mathbb{N}$, such that $\forall n \geq N, a_n > R$. And similarly for a sequence diverging to $-\infty$.

4.3 Limits

Theorem 4.3.1 (Uniqueness of limits)**.** Given a sequence (a_n) if $a_n \to a$ and $a_n \to b$, $a = b$.

Theorem 4.3.2. If a sequence (a_n) is convergent then (a_n) is bounded.

Theorem 4.3.3 (Algebra of limits)**.** Given two sequences $a_n \to a$ and $b_n \to b$ the following hold:

- $a_n + b_n \to a + b$,
- $a_n b_n \to ab$ (a special case of this is $ca_n \to ca$ for a constant c),
- $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$ given $b \neq 0$.

Theorem 4.3.4. If (a_n) is a positive sequence then $a_n \to 0 \iff \dfrac{1}{a_n} \to +\infty$, and similarly for negative sequences.

Theorem 4.3.5 (Ratio test)**.** If a sequence (a_n) satisfies $\left|\dfrac{a_{n+1}}{a_n}\right| \to L < 1$ then $a_n \to 0$.

4.4 Monotone sequences

Definition 4.4.1 (Monotonically increasing sequence)**.** A sequence, (a_n), is **monotonically increasing** iff $\forall m, n \in \mathbb{N}$ with $n > m$ we have $a_n \geq a_m$, and similarly for monotonically decreasing and their strict equivalents.

Theorem 4.4.2 (Monotone convergence)**.** If a sequence (a_n) is monotone increasing and bounded above then $a_n \to a := \sup\{a_i : i \in \mathbb{N}\}$ written $a_n \uparrow a$. This holds similarly for monotone decreasing sequences.

4.5 Cauchy sequences

Definition 4.5.1 (Cauchy sequence)**.** A sequence (a_n) is a **Cauchy sequence** iff $\forall \epsilon > 0 \in \mathbb{R},\ \exists N \in \mathbb{N}$ such that $\forall n, m < N,\ |a_n - a_m| < \epsilon$.

Theorem 4.5.2 (Cauchy convergence criterion)**.** A sequence (a_n) converges iff it is a Cauchy sequence.

4.6 Subsequences

Definition 4.6.1 (Subsequence)**.** Given a strictly monotonically increasing function $n : \mathbb{N} \to \mathbb{N}$ and a sequence (a_n), the sequence (b_n) defined by $b_i := a_{n(i)}$ is a **subsequence** of (a_n).

Theorem 4.6.2. Given a subsequence of (a_n), $(a_{n(i)})$, if $a_n \to a$ then $a_{n(i)} \to a$ as $i \to \infty$.

Theorem 4.6.3 (Bolzano-Weierstrass)**.** If a sequence (a_n) is bounded then it has a convergent subsequence.

Note 4.6.4 (Sketch of the Bolzano-Weierstrass theorem proof)**.** The proof of the Bolzano-Weierstrass theorem is an equally valuable point as the statement of the theorem itself. The idea of the proof considers the "peak points" of the sequence: if there are infinitely many peak points, then the peak points themselves form a monotonically decreasing subsequence; if there are finitely many, then the points after the final peak must have a monotonically increasing subsequence bounded above by the final peak. By the monotone convergence theorem both of these subsequences must converge.

5 Series

Definition 5.0.1 (Infinite series)**.** An **(infinite) series** is an expression of the form $\sum_{i=1}^{\infty} a_i$ of $a_1 + a_2 + \dots$ for some sequence (a_n). The sequence **partial sums** of the series (S_n) is given by

$$S_n := \sum_{i=1}^{n} a_i = a_1 + a_2 + \dots + a_n.$$

5.1 Convergence

Definition 5.1.1 (Convergence of series)**.** The series $\sum_{i=1}^{\infty} a_i$ of (a_n) **converges** iff $S_n \to A \in \mathbb{R}$, written $\sum_{n=1}^{\infty} a_n = A$.

Theorem 5.1.2. For a sequence (a_n), $\sum_{n=1}^{\infty} a_n$ converges if $a_n \to 0$ (the converse is not true).

Theorem 5.1.3. Given a sequence non-negative sequence (a_n), the convergence of the infinite series and the boundedness of (S_n) are equivalent.

Theorem 5.1.4 (Algebra of limits for series)**.** A similar algebra of limits for series can be established from the algebra of limits for sequences acting on the partial sums of the series.

Theorem 5.1.5 (Comparison I test)**.** Given sequences $(a_n), (b_n)$ if $0 \le a_n \le b_n$ then:

- If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ converge, $0 \le \sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$,
- If $\sum_{n=1}^{\infty} a_n$ diverges, $\sum_{n=1}^{\infty} b_n$ also diverges.

Theorem 5.1.6 (Comparison II test (Sandwich theorem))**.** Given sequences $(a_n), (b_n), (c_n)$ with $a_n \le b_n \le c_n$, if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} c_n$ both converge, $\sum_{n=1}^{\infty} b_n$ converges.

Theorem 5.1.7. If $\alpha > 1 \in \mathbb{R}$, $\sum_{n=1}^{\infty} \frac{1}{n^\alpha}$ converges.

Definition 5.1.8 (Alternating sequence)**.** A sequence (a_n) is **alternating** iff $a_{2n} \ge 0$ and $a_{2n-1} \le 0$ of vice versa for all $n \in \mathbb{N}^{>0}$.

Theorem 5.1.9. If (a_n) is alternating with $|a_n| \downarrow 0$, a_n converges and $\sum_{n=1}^{\infty} a_n$ converges.

5.2 Absolute convergence

Definition 5.2.1 (Absolute convergence)**.** Given a sequence (a_n) the series $\sum_{n=1}^{\infty} a_n$ is **absolutely convergent** iff $\sum_{n=1}^{\infty} |a_n|$ converges.

Theorem 5.2.2. Absolute convergence $\implies$ convergence.

Theorem 5.2.3 (Comparison III test)**.** Given sequences $(a_n), (b_n)$ with $\frac{a_n}{b_n} \to L \in \mathbb{R}$ if $\sum_{n=1}^{\infty} b_n$ is absolutely convergent then $\sum_{n=1}^{\infty} a_n$ is also absolutely convergent.

Theorem 5.2.4 (Ratio test)**.** If the sequence (a_n) is such that $\left|\frac{a_{n+1}}{a_n}\right| \to r < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent or divergent if $r > 1$.

Theorem 5.2.5 (Root test)**.** If the sequence (a_n) is such that $|a_n|^{\frac{1}{n}} \to r < 1$ then $\sum_{n=1}^{\infty} a_n$ is absolutely convergent or divergent is $r > 1$.

Remark 5.2.6. Both the ratio test and the root test are inconclusive if $r = 1$.

5.3 Rearrangement of series

Sometimes, series are easier to deal with and have cancellations when their terms are rearranged. However, the rearrangement of terms will only preserve limits under certain conditions.

Definition 5.3.1 (Reordering)**.** Given a bijection $n : \mathbb{N} \to \mathbb{N}$ and a sequence (a_n), the sequence (b_n) with $b_i := a_{n(i)}$ is a **rearrangement** or **reordering** of (a_n).

Theorem 5.3.2. If (a_n) is a sequence satisysing $a_n \to 0$, $\sum_{n:a_n \geq 0} a_n = \infty$ and $\sum_{n:a_n \leq 0} a_n = -\infty$ then $\sum_{n=1}^{\infty} a_n$ can be rearranged to converge to any $r \in \mathbb{R}$.

Theorem 5.3.3. If (a_n) is a sequence with absolutely convergent series, $\sum_{n:a_n \geq 0} a_n = A$ and $\sum_{n:a_n \leq 0} a_n = B$ with all arrangements of (a_n) converging to $A + B$.

5.4 Power series

Throughout this subsection $[0, \infty] := [0, \infty) \cup \{+\infty\}$.

Definition 5.4.1 (Power series)**.** For $z \in \mathbb{C}$ and a complex sequence (a_n), a **power series** is an expression in the form $\sum_{n=1}^{\infty} a_n z^n$.

Definition 5.4.2 (Radius of convergence)**.** Given the power series $\sum_{n=1}^{\infty} a_n z^n$, there exists some $R \in [0, \infty]$ such that:

- $|z| < R \implies \sum_{n=1}^{\infty} a_n z^n$ converges,
- $|z| > R \implies \sum_{n=1}^{\infty} a_n z^n$ diverges.

We cannot tell what happens when $|z| = R$ so this has to be checked separately. R is the **radius of convergence** of the power series.

Corollary 5.4.3. Given the same power series $\sum_{n=1}^{\infty} a_n z^n$, have $S := \{|z| \in \mathbb{R}^{\geq 0} : a_n z^n \to 0\}$ then

$$R := \begin{cases} \sup(S) & \text{if } S \text{ is bounded} \\ \infty & \text{otherwise} \end{cases}.$$

is the radius of convergence for the power series.

Theorem 5.4.4 (Evaluating radius of convergence from tests)**.** For the power series $\sum_{n=1}^{\infty} a_n z^n$:

- if $\left|\frac{a_{n+1}}{a_n}\right| \to a \in [0, \infty]$ then $R = \frac{1}{a}$ is the radius of convergence for the power series,
- if $|a_n|^{\frac{1}{n}} \to a \in [0, \infty]$ then $R = \frac{1}{a}$ is the radius of convergence for the power series,

Definition 5.4.5 (Cauchy product)**.** Given two series $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$; their **Cauchy product** is the series $\sum_{n=0}^{\infty} \sum_{i=0}^{n} a_i b_{n-i}$.

Remark 5.4.6. If (a_n), (b_n) are the coefficients for a power series, then the Cauchy product of their series will be the coefficients of the product of the power series.

Theorem 5.4.7. If $\sum_{n=1}^{\infty} a_n$, $\sum_{n=1}^{\infty} b_n$ are absolutely convergent their Cauchy product converges absolutely to $\left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$.

Theorem 5.4.8. If the power series $\sum_{n=1}^{\infty} a_n z^n$, $\sum_{n=1}^{\infty} b_n z^n$ have radii of convergence R_a, R_b respectively then their Cauchy product has radius of convergence $R_c \geq \min(R_a, R_b)$.

5.5 Exponential series

Definition 5.5.1. For $z \in \mathbb{C}$, its **exponential series** is

$$E(z) := \sum_{n=}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots,$$

with $E(z)$ converging absolutely for all $z \in \mathbb{C}$.

Theorem 5.5.2 (Properties of exponential series)**.** For all $z, w \in \mathbb{C}$:
1. $E(z)E(w) = E(z + w)$, 2. $\frac{1}{E(z)} = E(-z)$, 3. $E(z) \neq 0$.

Theorem 5.5.3. For all $x \in \mathbb{Q}$, $E(x) = e^x$, with $e := E(1)$.

6 Continuity

6.1 Continuous functions

Definition 6.1.1 (Limit of real functions)**.** For a function $f : \mathbb{R} \to \mathbb{R}$ and some $a, b \in \mathbb{R}$ we have $f(x) \to b$ **as** $x \to a$ of $\lim_{x \to a} f(x) = b$ iff:

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } |x - a| < \delta \impliedby |f(x) - b| < \epsilon.$$

Definition 6.1.2 (Continuity of real functions)**.** Given the function $f : \mathbb{R} \to \mathbb{R}$

1. f is **continuous at a point** $a \in \mathbb{R}$ iff $\lim_{x \to a} f(x) = f(a)$,
2. f is **continuous (on** $\mathbb{R}$**)** iff f is continuous at all $a \in \mathbb{R}$.

Definition 6.1.3 (Discontinuity of real functions)**.** The function $f : \mathbb{R} \to \mathbb{R}$ is **discontinuous** at a point if it is not continuous at that point.

Definition 6.1.4 (Sequential continuity)**.** The function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $a \in \mathbb{R} \iff f(a_n) \to f(a)$ as $n \to \infty$ for all sequences (a_n) converging to a.

Remark 6.1.5. The definition for limits and continuity of complex functions is similar with $|\cdot|$ being the modulus instead of the absolute values. The same definitition also applies for functions that are continuous on certain subsets of $\mathbb{R}$ or $\mathbb{C}$.

Theorem 6.1.6. $E : \mathbb{C} \to \mathbb{C}$ given by $E(z) := \sum_{n=}^{\infty} \frac{z^n}{n!}$ is continuous on $\mathbb{C}$.

Theorem 6.1.7 (Properties of the real exponential function)**.** Given the exponential function $E : \mathbb{R} \to (0, \infty)$:

1. for all $x \in \mathbb{R}$, $E(x) > 0$,
2. $x > 0 \implies E(x) > 1$,
3. $E(x)$ is a strictly increasing function,
4. For $|x| < 1$, $|E(x) - 1| \leq \frac{|x|}{1 - |x|}$,
5. E is a continuous bijection.

Theorem 6.1.8. The inverse of $E(x) = e^x$ is the **natural logarithm** function $\ln : (0, \infty) \to \mathbb{R}$ satisfying $y = \ln x \iff x = e^y$ for all $x, y \in \mathbb{R}$.

Definition 6.1.9 (Exponentiation of positive bases)**.** For $a \in (0, \infty)$, for all $x \in \mathbb{R}$ define $a^x := E(x \ln a)$.

Definition 6.1.10 (Trigonomentric functions)**.** The **sine** and **cosine** functions are defined as:

$$\sin(\theta) := \Im[E(i\theta)], \qquad \cos(\theta) := \Re[E(i\theta)].$$

and are both continuous functions from $\mathbb{R} \to [-1, 1]$.

Theorem 6.1.11 (Continuity of piecewise functions)**.** For $a, c \in \mathbb{R}$ with functions $f_1 : (-\infty, a) \to \mathbb{R}$ and $f_2 : (a, \infty) \to \mathbb{R}$, the **piecewise function** $f : \mathbb{R} \to \mathbb{R}$, defined as,

$$f(x) := \begin{cases} f_1(x) & \text{if } x < a \\ c & \text{if } x = a \\ f_2(x) & \text{if } x > a \end{cases}$$

is continuous on $\mathbb{R}$ iff both f_1 and f_2 are continuous on their respective domains and

$$\lim_{x \uparrow a} f_1(x) = \lim_{x \downarrow a} f_2(x) = c.$$

6.2 Properties of continuity

Theorem 6.2.1. For $f, g : \mathbb{R} \to \mathbb{R}$ continuous at $a \in \mathbb{R}$ the following functions are also continuous at a:

1. αf for all $\alpha \in \mathbb{R}$;
2. $f + g, f \cdot g$;
3. $\frac{f}{g}$, given $g(a) \neq 0$.

Theorem 6.2.2. The following functions (all by their well known definitions) are continuous:

1. $f(x) = x^n$, for $n \in \mathbb{N}_0$ (**monomials**);
2. $p(x) = \sum_{i=1}^{n} a_i x^i$, given (a_n) is a real sequence (**polynomials**);
3. $\frac{p(x)}{q(x)}$ at $a \in \mathbb{R}$ given p, q are polynomials with $q(a) \neq 0$ (**rational functions**);
4. $\sin(x)$, $\cos(x)$ on $\mathbb{R}$ and $\tan(x)$ whenever $\cos(x) \neq 0$, plus their reciprocals under similar conditions;
5. $f \circ g$ at $a \in \mathbb{R}$ when g is continuous at a and f is continuous at $g(a)$.

Theorem 6.2.3 (Intermediate value theorem)**.** Given $a, b \in \mathbb{R}$ with $a \leq b$, if $f : [a, b] \to \mathbb{R}$ is continuous, then for all c between $f(a)$ and $f(b)$ there exists some $x \in [a, b]$ such that $f(x) = c$.

Definition 6.2.4 (Boundedness of real functions)**.** Given some $S \subseteq \mathbb{R}$ a function $f : S \to \mathbb{R}$ is **bounded above** iff $\exists M \in \mathbb{R}$ such that $f(x) \leq M$ for all $x \in \mathbb{R}$. The definitions for **bounded below** and **bounded** extend naturally from this.

Theorem 6.2.5 (Extreme value theorem)**.** Given $a, b \in \mathbb{R}$ with $a \leq b$, if $f : [a, b] \to \mathbb{R}$ is continuous then f is bounded.

7 Properties of subsets

7.1 Open sets

Definition 7.1.1 (Open sets)**.** A set $S \subset \mathbb{R}$ is **open** iff $\forall x \in S,\ \exists \delta$ such that $(x - \delta, x + \delta) \subset S$.

Theorem 7.1.2 (Union of open sets)**.** For a collection of open sets in $\mathbb{R}$, $\{S_i\}$, given the indexing set $\mathcal{I}$ (could be countable or uncountable), $\bigcup\limits_{i \in \mathcal{I}} S_i$ is open in $\mathbb{R}$.

Theorem 7.1.3 (Finite intersections of open sets)**.** The intersection of finitely many open sets in $\mathbb{R}$ is open in $\mathbb{R}$.

7.2 Closed and compact sets

Definition 7.2.1 (Closed sets)**.** A set $S \subset \mathbb{R}$ is **closed** in $\mathbb{R}$ if all convergent subsequences of S have a limit in S.

Definition 7.2.2 (Compact sets)**.** A set $S \subset \mathbb{R}$ is **compact** in $\mathbb{R}$ if it is closed and bounded in $\mathbb{R}$.

Theorem 7.2.3. The complement of an open set is closed.

Remark 7.2.4. Not every set in $\mathbb{R}$ is either open or closed. Half-open intervals are neither open nor closed while $\mathbb{R}$ and $\emptyset$ are both open and closed.

Theorem 7.2.5. The finite union or any intersection of closed sets in $\mathbb{R}$ is closed.

Theorem 7.2.6. A set $S \subset \mathbb{R}$ is compact iff every subsequence of S has as convergent subsequence $x_{n(i)} \to x \in S$.

Theorem 7.2.7 (Extreme value theorem for compact sets)**.** If $S \subset \mathbb{R}$ is compact with $f : S \to \mathbb{R}$ continuous, there exists some $c, d \in S$ with $f(c) = \inf\limits_{x \in S} f(x)$ and $f(d) = \sup\limits_{x \in S} f(x)$.

8 Uniform continuity and convergence

8.1 Uniform continuity

Definition 8.1.1 (Uniform continuity)**.** A fuction $f : S \to \mathbb{R}$ is **uniformly continuous** iff

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x, y \in S, |x - y| < \delta \implies |f(x) - f(y)| < \epsilon.$$

Uniform continuity is a more powerful notion that continuity with f is uniformly continuous $\implies$ f is continuous.

Theorem 8.1.2. If $S \subset \mathbb{R}$ is compact and $f : S \to \mathbb{R}$ continuous then f is uniformly continuous.

8.2 Convergence of sequences of functions

Definition 8.2.1 (Pointwise convergence)**.** For some $S \subset \mathbb{R}$ with the sequence $f_1, f_2, \ldots : S \to \mathbb{R}$, f_n **converges pointwise** to some $f : S \to \mathbb{R}$ if

$$\forall x \in S,\ \forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall n \geq N,\ |f(x) - f_n(x)| < \epsilon.$$

Written $\forall x \in S,\ \lim\limits_{n \to \infty} f_n(x) = f(x)$.

Definition 8.2.2 (Uniform convergence)**.** For some $S \subset \mathbb{R}$, the sequence $f_1, f_2, \ldots : S \to \mathbb{R}$ **uniformly converges** to some $f : S \to \mathbb{R}$ if

$$\forall \epsilon > 0,\ \exists N \in \mathbb{N} \text{ such that } \forall x \in S, \text{ and } \forall n > N, |f(x) - f_n(x)| < \epsilon.$$

Theorem 8.2.3. If a sequence of (uniformly) continuous functions converges uniformly to a function f then f is (uniformly) continuous.

Theorem 8.2.4. If, given $S \subset \mathbb{R}$, $(f_n) : S \to \mathbb{R}$ is a uniformly convergent sequence of continuous functions with $a \in S$ open in S, $\lim_{n\to\infty} \lim_{x\to a} f_n(x) = \lim_{x\to a} \lim_{n\to\infty} f_n(x)$.

8.3 Convergence of series of functions

Definition 8.3.1 (Convergence of series of functions)**.** Given $(f_n) : S \to \mathbb{R}$ defined on $S \subset \mathbb{R}$, the series $\sum_{n=1}^{\infty} f_n(x)$ **converges (uniformly)** iff the sequence of partial sums $S_n(x) = \sum_{n=1}^{n} f_n(x)$ converges (uniformly).

Theorem 8.3.2 (Weierstrass M-test)**.** Given continuous $(f_n) : S \to \mathbb{R}$ defined on $S \subset \mathbb{R}$,

$$\forall x \in S \text{ and } \forall i \in \mathbb{N},\ \exists M_1, M_2, \ldots \in \mathbb{R} \text{ such that } |f_i(x)| \leq M_i \text{ and } \sum_{i=1}^{\infty} M_i \text{ converges}$$
$$\implies \sum_{n=1}^{\infty} f_i(x) \text{ converges uniformly to some continuous } g : S \to \mathbb{R}.$$

Theorem 8.3.3. If a power series $f(x) = \sum_{n=1}^{\infty} f_i(x)$ has radius of convergence $R > 0$ then f is continuous on $(-R, R)$.

9 Differentiation

9.1 Differentiability

Definition 9.1.1 (Differentiability)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is **differentiable** at $a \in \mathbb{R}$, with **derivative** $f'(a) = \frac{d}{dx} f(x)\Big|_a$ iff

$$\lim_{x\to a} \frac{f(x) - f(a)}{x - a} \text{ exists, which we set to } f'(a).$$

f is differentiable on $S \subseteq \mathbb{R}$, with derivative $\frac{d}{dx} f = \frac{df}{dx} = f' : \mathbb{R} \to \mathbb{R}$, if it is differentiable at every $x \in S$.

Examples 9.1.2. The following functions are all differentiable,

- $f(x) = x^n$, for $n \in \mathbb{N}$ on $\mathbb{R}$ with $f'(x) = nx^{n-1}$,
- $f(x) = e^x$ on $\mathbb{R}$ with $f'(x) = e^x$,
- $f(x) = \ln x$ on $\mathbb{R}^{>0}$ with $f'(x) = \frac{1}{x}$.

Theorem 9.1.3. f is differentiable $\implies$ f is continuous.

Theorems 9.1.4 (Operations on derivatives)**.** If $f, g : \mathbb{R} \to \mathbb{R}$ are both differentiable at $x = a \in \mathbb{R}$ then,

1. for all $c, d \in \mathbb{R}$, $h(x) := c \cdot f(x) + d \cdot g(x)$ is differentiable at $x = a$ with $h'(a) = c \cdot f'(a) = d \cdot g'(a)$,
2. $p(x) := f(x) \cdot g(x)$ is differentiable at $x = a$ with $p'(a) = f(a) \cdot g'(a) + f'(a) \cdot g(a)$,
3. if $f(a) \neq 0$, $q(x) := \dfrac{1}{f(a)}$ is differentiable at $x = a$ with $q'(a) = -\dfrac{f'(a)}{[f(a)]^2}$,
4. if $g(a) \neq 0$ $r(x) := \dfrac{f(x)}{g(x)}$ is differentiable at $x = a$ with $r'(a) = \dfrac{f'(a) \cdot g(a) - f(a) \cdot g'(a)}{[g(a)]^2}$.

Theorem 9.1.5 (Chain rule)**.** If $g, f : \mathbb{R} \to \mathbb{R}$ are differentiable at $x = a \in \mathbb{R}$ and $x = g(a)$ respectively then $s(x) := f \circ g(x)$ iss differentiable at $x = a$ with $s'(a) = g'(a) \cdot f' \circ g(a)$.

9.2 Local extrema and mean values

Definition 9.2.1 (Local extrema)**.** For a function $f : S \to \mathbb{R}$, f has a **local minimum** as $a \in \mathbb{R}$ iff $\exists \delta > 0$ such that $\forall y \in S$ with $|y - a| < \delta$, $f(y) \leq f(a)$, and similary for a **local maximum**.

Theorem 9.2.2. If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) and has a local maximum or minimum at $c \in (a, b)$, $f'(c) = 0$.

Theorem 9.2.3 (Rolle's theorem)**.** If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) with $f(a) = f(b)$, $\exists c \in (a, b)$ such that $f'(c) = 0$.

Theorem 9.2.4 (Mean value theorem)**.** If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b), $\exists c \in (a, b)$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Theorem 9.2.5. If $f : [a, b] \to \mathbb{R}$ is differentiable on (a, b) with $f'(x) \geq 0$ for all $x \in (a, b)$ then f is monotone increasing. Similar holds for monotone/strictly increasing/decreasing or constant.

Theorem 9.2.6 (Cauchy's MVT)**.** A similar but slightly more general statement than the MVT: if $f, g : [a, b] \to \mathbb{R}$ are differentiable on (a, b), $\exists c \in (a, b)$ with $(f(b) - f(a))g'(c) = (g(b) - g(a))f'(c)$.

9.3 L'Hôpital's rule

Theorem 9.3.1 (L'Hôpital's rule)**.** Given $f, g : [c, d] \to \mathbb{R}$ are differentiable on (c, d) except possibly at some $a \in (c, d)$ with $g'(x) \neq 0$ on $(c, d) \setminus \{a\}$:

$$\text{if } \lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0 \text{ or } \infty \text{ and } \lim_{x\to a} \frac{f'(x)}{g'(x)} = L \text{ then } \lim_{x\to a} \frac{f(x)}{g(x)} = L.$$

This also applies when taking $\lim\limits_{x\to\infty}$.

Definition 9.3.2 (Higher derivatives)**.** **Higher derivatives** of $f : \mathbb{R} \to \mathbb{R}$ are defined inductively as

$$f^{(n)}(x) := \begin{cases} f(x) & \text{if } x = 0 \\ f^{(n-1)\prime}(x) & \text{otherwise} \end{cases}.$$

The existence of the nth derivative of f requires all lower order derivatives of f also exist and be differentiable.

Theorem 9.3.3 (Second derivative test)**.** For a second differentiable function $f : \mathbb{R} \to \mathbb{R}$ with $f'(a) = 0$ for some $a \in \mathbb{R}$,

- $f''(a) > 0 \implies f$ has a local minimum at $x = a$,
- $f''(a) < 0 \implies f$ has a local maximum at $x = a$,
- the test is inconclusive if $f''(a) = 0$.

9.4 Taylor's theorem

Definition 9.4.1 (Taylor polynomial of a function)**.** Given $f : [c, d] \to \mathbb{R}$ has an order $n \in \mathbb{N}_0$ derivative at $x = a \in (c, d)$, the **Taylor polynomial** of order n at $x = a$ is

$$P_n(x) := \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x - a)^i = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

Theorem 9.4.2 (Taylor's theorem)**.** Given $f : [c, d] \to \mathbb{R}$ has an order $n + 1$, for some $n \in \mathbb{N}_0$, derivative for all $x \in (c, d)$. For $a, b \in [c, d]$ with $a \neq b$ there exists some t between a and b such that,

$$f(b) = P_n(b) + \frac{f^{(n+1)}(t)}{(n + 1)!}(b - a)^{n+1}.$$

This is a further, massive generalisation of the MVT (the case when $n = 0$).

Definition 9.4.3 (Taylor series of a function)**.** The **Taylor series**, $P(x)$, for a function $f : \mathbb{R} \to \mathbb{R}$ at $x = a$ exists if $f^{(n)}(a)$ exists for all $n \in \mathbb{N}$ and is given by

$$P(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n = f(a) + \frac{f'(a)}{1!}(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \ldots$$

Definition 9.4.4 (Analytic function)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is **analytic** if it equals its Taylor series.

9.5 Convexity

Definition 9.5.1 (Convexity of functions)**.** A function $f : [a, b] \to \mathbb{R}$ is **convex** iff

$$\forall c, t, d \in [a, b] \text{ with } c < t < d, f(c) + \frac{f(d) - f(c)}{d - c}(t - c) \geq f(t).$$

Theorem 9.5.2. Given the function $f : [a, b] \to \mathbb{R}$ with $f''(x)$ existing on (a, b), f is convex $\iff$ $f''(x)$ non-negative on (a, b).

9.6 Exchange of limits and derivatives

Theorem 9.6.1 (Criteria for exchange of limits and derivatives)**.** Given (f_n) is a sequence of functions with $f_n : [a, b] \to \mathbb{R}$ differentiable, if $\lim\limits_{n\to\infty} f_n(c)$ exists for some $c \in [a, b]$ and $(f_n'(x))$ converges uniformly on $[a, b]$: (f_n) converges uniformly to some differentiable f satisfying $f'(x) = \lim\limits_{n\to\infty} f_n'(x)$.

Theorem 9.6.2 (Derivatives of power series)**.** Given a power series $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$, f has a continuous derivative on $(-R, R)$ with $f'(x) = \sum\limits_{n=0}^{\infty} n a_n x^{n-1}$.

Corollary 9.6.3. Given a power series $f(x) = \sum\limits_{n=0}^{\infty} a_n x^n$ with radius of convergence $R > 0$, the Taylor series of f centered at $x = 0$ is $\sum\limits_{n=0}^{\infty} a_n x^n$.

9.7 Trigonometric properties

Definition 9.7.1 (π)**.** Let $S = \{y > 0 : \sin(y) = 0\}$, $\pi := \inf S$.

Definition 9.7.2 (Periodic function)**.** A function $f : \mathbb{R} \to \mathbb{R}$ is **$2L$-periodic** iff $f(x + 2L) = f(x)$ for all $x \in \mathbb{R}$.

Theorem 9.7.3. $\sin$ and $\cos$ satisfy the following important properties: 1. $\sin(x)$ is odd, 2. $\cos(x)$ is even, 3. $\cos^2(x) + \sin^2(x) = 1$ for all $x \in \mathbb{R}$, 4. $\sin$ and $\cos$ are 2π-periodic functions.

10 Integration

10.1 Partitions

Definition 10.1.1 (Partition)**.** A **partition**, P, of the interval $[a, b] \subset \mathbb{R}$ is a finite collection of points $x_0, x_1, \ldots, x_n \in [a, b]$ such that $a = x_0 < x_1 < \ldots < x_n = b$. A partition naturally splits the domain $[a, b]$ into finitely many closed intervals.

Definition 10.1.2 (Refinement)**.** Given partitions Q, P, Q is a **refinement** of P, written $Q \prec P$, iff every point of P is also in Q.

Definition 10.1.3 (Common refinement)**.** Given paritions P, Q the **common refinement** of P and Q is the partition R containing all points in P or Q. $R \prec P$ and $R \prec Q$.

10.2 Darboux sums

Definition 10.2.1 (Darboux sums)**.** Given the bounded function $f : [a, b] \to \mathbb{R}$ and the partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$, we will assign to each subintervals generated by P:

- a length, $\Delta x_i := x_{i+1} - x_i$,
- an infinum, $m_i := \inf\limits_{x_i \leq t \leq x_{i+1}} f(t)$,
- a supremum, $M_i := \sup\limits_{x_i \leq t \leq x_{i+1}} f(t)$.

Now define the **lower Darboux sum** and **upper Darboud sum** of f w.r.t. P as:

$$L(f,P) := \sum_{i=0}^{n-1} m_i \Delta x_i, \qquad U(f,P) := \sum_{i=0}^{n-1} M_i \Delta x_i \qquad \text{respectively.}$$

If $f : [a,b] \to \mathbb{R}$ is continuous then $L(f,P)$ and $U(f,p)$ exist. $L(f,P)$ is always less than or equal to $U(f,P)$.

Theorem 10.2.2 (Boundedness of refined Darboux sums)**.** If $f : [a,b] \to \mathbb{R}$ is bounded with $Q \prec P$ partitions of $[a,b]$, $L(f,P) \leq L(f,Q) \leq U(f,Q) \leq U(f,P)$.

Theorem 10.2.3. Given some bounded $f : [a,b] \to \mathbb{R}$, the set $\{L(f,P) : P \text{ is a partition of } [a,b]\}$ is bounded above by any upper Darboux sum on $[a,b]$ w.r.t. f.

10.3 Darboux integral

Definition 10.3.1 (Darboux integrals)**.** Given a bounded function $f : [a,b] \to \mathbb{R}$, the **lower Darboux integral** and **upper Darboux integral** are:

$$\underline{\int_a^b} f(x)\,\mathrm{d}x := \sup_P L(f,P), \qquad \overline{\int_a^b} f(x)\,\mathrm{d}x := \inf_P U(f,P) \qquad \text{respectively.}$$

Definition 10.3.2 (Darboux integrability)**.** If the upper and lower Darboux integral of a bounded function $f : [a,b] \to \mathbb{R}$ are equal, f is **Darboux integrable** on $[a,b]$ with

$$\int_a^b f(x)\,\mathrm{d}x := \underline{\int_a^b} f(x)\,\mathrm{d}x = \overline{\int_a^b} f(x)\,\mathrm{d}x.$$

We will now refer to Darboux integrable functions simply as **integrable**.

Theorem 10.3.3. A bounded function $f : [a,b] \to \mathbb{R}$ is integrable iff $\forall \epsilon > 0$ there exists a partition P with $U(f,P) - L(f,P) < \epsilon$. Furthermore, given a sequence of paritions (P_n) if $\lim_{n\to\infty} (U(f,P_n) - L(f,P_n)) = 0$ then

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} (L(f,P_n) = \lim_{n\to\infty} (U(f,P_n).$$

Remark 10.3.4. For a bounded function $f : [a,b] \to \mathbb{R}$, f is integrable if it is, continuous, differentiable, monotone, or discontinuous at finitely many points.

10.4 Properties of integration

Theorem 10.4.1 (Monotonicity)**.** If $f, g : [a,b] \to \mathbb{R}$ are integrable with $f(x) \leq g(x)$ for all $x \in \mathbb{R}$,

$$(1) \quad \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x. \qquad (2) \quad m \cdot (b-a) \leq \int_a^b f(x)\,\mathrm{d}x \leq M \cdot (b-a).$$

Theorem 10.4.2 (Boundedness)**.** If $f : [a,b] \to \mathbb{R}$ is integrable with $m \leq f(x) \leq M$ for all $x \in \mathbb{R}$,

Theorem 10.4.3 (Linearity)**.** If $f, g : [a,b] \to \mathbb{R}$ are integrable, for all $c, d \in \mathbb{R}$,

$$(3) \quad \int_a^b (cf(x) + dg(x))\,\mathrm{d}x = c\int_a^b f(x)\,\mathrm{d}x + d\int_a^b g(x)\,\mathrm{d}x. \qquad (4) \quad \int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x.$$

Theorem 10.4.4 (Integrability on subdomains)**.** $f : [a,b] \to \mathbb{R}$ is integrable iff $\forall c \in [a,b]$, f is integrable on $[a,c]$ and $[c,b]$ with,

Theorem 10.4.5 (Composition)**.** If $f : [a,b] \to [m,M] \subset \mathbb{R}$, $g : [m,M] \to \mathbb{R}$ are integrable and continuous respectively, $h(x) := g \circ f(x)$ is integrable on $[a,b]$.

Theorem 10.4.6 (Triangle innequality)**.** If $f : [a, b] \to \mathbb{R}$ is integrable then $|f|$ is integrable on $[a, b]$ with,

$$(6) \quad \left| \int_a^b f(x)\,\mathrm{d}x \right| \leq \int_a^b |f(x)|\,\mathrm{d}x. \qquad (7) \quad \int_a^b f(x)\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x.$$

Theorem 10.4.7 (Finite point differences)**.** If $f, g : [a, b] \to \mathbb{R}$ are integrable with $f(x) = g(x)$ except at finitely many points,

Theorem 10.4.8 (Products)**.** If $f, g : [a, b] \to \mathbb{R}$ are integrable then $f \cdot g : [a, b] \to \mathbb{R}$ is integrable.

Theorem 10.4.9 (Maxima and minima)**.** If $f, g : [a, b] \to \mathbb{R}$ are integrable then $\max(f, g), \min(f, g) : [a, b] \to \mathbb{R}$ are integrable.

10.5 Fundamental theorems of calculus

Theorem 10.5.1 (Fundamental theorem of calculus 1)**.** Given continuous $f : [a, b] \to \mathbb{R}$, have $F : [a, b] \to \mathbb{R}$ with $F(x) := \int_a^x f(t)\,\mathrm{d}t$. F is continuous on $[a, b]$ and differentiable on (a, b). $F'(x) = f(x)$ for all $x \in [a, b]$.

Theorem 10.5.2 (Fundamental theorem of calculus 2)**.** Given continuous $f : [a, b] \to \mathbb{R}$ with continuous derivative on (a, b), $\int_a^b f'(x)\,\mathrm{d}x = f(b) - f(a)$.

10.6 Methods of integration

Theorem 10.6.1 (MVT)**.** If $f : [a, b] \to \mathbb{R}$ is continuous, $\exists c \in [a, b]$ such that $\int_a^b f(x)\,\mathrm{d}x = f(c)(b - a)$.

Theorem 10.6.2 (Integration by parts)**.** If $f, g : [a, b] \to \mathbb{R}$ have continuous first derivatives,

$$(2) \quad \int_a^b f(x)g'(x)\,\mathrm{d}x = \Big[f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\,\mathrm{d}x. \qquad (3) \quad \int_{u(c)}^{u(d)} f(x)\,\mathrm{d}x = \int_c^d f(u(x))u'(x)\,\mathrm{d}x.$$

Theorem 10.6.3 (Integration by substitution)**.** Given continuous $f : [a, b] \to \mathbb{R}$ if $u : [a, b] \to [c, d]$ has a continuous derivative on (c, d),

10.7 Limits and integrals

Theorem 10.7.1 (Exchanging limits and integrals)**.** If $f_n : [a, b] \to \mathbb{R}$ is a sequence of integrable functions converging uniformly to $f : [a, b] \to \mathbb{R}$, then f is integrable with,

$$\int_a^b f(x)\,\mathrm{d}x = \int_a^b \lim_{n\to\infty} f_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_a^b f_n(x)\,\mathrm{d}x.$$

Theorem 10.7.2 (Power series integration)**.** If the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$, f is integrable on all closed subintervals of $(-R, R)$ with

$$\int_0^x f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1} \text{ for all } x \in (-R, R).$$

10.8 Improper integrals

Definition 10.8.1 (Improper integral)**.** Given $f : (a, b] \to \mathbb{R}$ integrable on all $[c, b] \subset (a, b]$, the **improper integral**,

$$\int_a^b f(x)\,\mathrm{d}x = \lim_{c\downarrow a} \int_c^b f(x)\,\mathrm{d}x,$$

if the limit exists, otherwise the integral **diverges**; and similarly for other non-closed intervals or those with $\pm\infty$ as bounds.

Remark 10.8.2. When integrating over intervals with multiple undefined points, the integral is split into sums of multiple integrals each with single undefined points on their boundaries.