Chapter 1

Real Analysis and Topology

Lectured by Someone
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Season Year

Introduction

The following are complementary reading for the course.

- G. Grimmett and D. J. A. Welsh, Probability: An Introduction, 1986
- J. K. Blitzstein and J. Hwang, Introduction to Probability, 2019
- D. F. Anderson et al, Introduction to Probability, 2018
- S. M. Ross, Introduction to Pro ability Models, 2014
- G. Grimmett and D. Stirzaker, Probability and Random Processes, 2001
- G. Grimmett and D. Stirzaker, One Thousand Exercises in Probability, 2009

Notation. Unbracketed superscripts are used to label the components of vectors, with unbracketed subscripts labellin different vectors.

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Lecture 1
Monday
30/10/2023

1 Euclidean spaces

Definition 1.0.1 ($\mathbb{R}^n$)**.** The set $\mathbb{R}^n = \{(x^1, x^2, \ldots, x^n) : x^i \in \mathbb{R},\ \forall i \in [1, n]\}$ will be considered with the operations to make it a real vector space.

1.1 Euclidean norm

Definition 1.1.1 (Inner product)**.** We will have the **inner product** on $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying:

$$\langle x, y \rangle := \sum_{i=1}^{n} x^i y^i,$$

with the **Euclidean norm** given by,

$$||\cdot|| : \mathbb{R}^n \to [0, \infty) \text{ with } ||x|| = \sqrt{\langle x, x \rangle}.$$

Proposition 1.1.2 (Properties of the Euclidean norm)**.** The Euclidean norm satisfies the following properties:

(N1) for all $x \in \mathbb{R}^n$, $||x|| \geq 0$ achieving equality iff $x = 0$,

(N2) for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, $||\lambda x|| = |\lambda| \cdot ||x||$,

(N3) for all $x, y \in \mathbb{R}^n$: $||x + y|| \leq ||x|| + ||y||$,

Theorem 1.1.3 (Cauchy-Swartz innequality)**.** For all $x, y \in \mathbb{R}^n$, $|\langle x, y \rangle| \leq ||x|| \cdot ||y||$.

Theorem 1.1.4 (Reverse triangle innequality)**.** For all $x, y \in \mathbb{R}^n$, $\big|\, ||x|| - ||y|| \,\big| \leq ||x - y||$.

Proposition 1.1.5. For $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$,

$$\max_{k \in [1,n]} \left| x^k \right| \leq ||x|| \leq \sqrt{n} \max_{k \in [1,n]} \left| x^k \right|.$$

Proof. Exercise ☐

1.2 Convergence in $\mathbb{R}^n$

Definition 1.2.1 (Open ball)**.** In $\mathbb{R}^n$ we define the **open ball** around $x \in \mathbb{R}^n$ of size $r \in \mathbb{R}$ as

$$B_r(x) := \{y \in \mathbb{R}^n : ||x - y|| < r\}.$$

This will be analoguous the the notion of open intervals used throughout analysis 1.

Definition 1.2.2 (Sequence in $\mathbb{R}^n$)**.** A **sequence** in $\mathbb{R}^n$ is an ordered list $x_0, x_1, \ldots, x_i \ldots$ with $x_i \in \mathbb{R}^n$ for all $i \in \mathbb{N}$, written $(x_i)_{i=0}^{\infty}$

Definition 1.2.3 (Convergence in $\mathbb{R}^n$)**.** We say a sequence in $\mathbb{R}^n$, $(x_i)_{i=0}^{\infty}$ **converges to** $x \in \mathbb{R}^n$ iff

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that, } \forall n \geq N,\ ||x_i - x|| < \epsilon$$

and we write $x_i \to x$ as $i \to \infty$ or $\lim\limits_{i \to \infty} x_i = x$.

Lemma 1.2.4. The sequence of vectors in $\mathbb{R}^n$, $(x_i)_{i=0}^{\infty}$, converges to some $x = (x^1, x^2, \ldots, x^n) \in \mathbb{R}^n$ iff each componenet of x_i converges to the corresponding component in x:

$$\forall k \in [1, n]\ \lim_{i \to \infty} x_i^k = x^k.$$

Proof. ($\implies$) Given $\lim\limits_{i \to \infty} x_i^k = x^k$ for all $k \in [1, n]$ we have that for all $\epsilon > 0$, $\left| x_i^k - x^k \right| < \dfrac{\epsilon}{\sqrt{n}}$ for all $i \geq N_k$ for each $k \in [1, n]$ respectively. We take $N = \max\limits_{k \in [1,n]} N_k$ and now have:

$$\max_{k \in [1,n]} \left| x_i^k - x^k \right| < \frac{\epsilon}{\sqrt{n}} \implies ||x_i - x|| \leq \sqrt{n} \max_{k \in [1,n]} \left| x_i^k - x^k \right| < \epsilon.$$

($\impliedby$) Similarly, given $\lim\limits_{i \to \infty} x_i = x \implies ||x_i - x|| < \epsilon$ for all $\epsilon > 0$:

$$\left| x_i^k - x^k \right| \leq \max_{k \in [1,n]} \left| x_i^k - x^k \right| \leq ||x_i - x|| < \epsilon,$$

therefore $\lim\limits_{i \to \infty} x_i^k = x^k$ for all $k \in [1, n]$. ☐

2 Continuity and limits of functions

2.1 Open sets

Definition 2.1.1 (Open set in $\mathbb{R}^n$)**.** A subset $U \subseteq \mathbb{R}^n$ is **open** in $\mathbb{R}^n$ iff:

$$\forall x \in U,\ \exists r > 0 \text{ such that } B_r(x) \subseteq U.$$

2.2 Continuity

Definition 2.2.1 (Continuity)**.** Let $A \subseteq \mathbb{R}^n$ the we have $f : A \to \mathbb{R}^m$ **continuous at** some $p \in A$ iff

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x \in A \text{ with } ||x - p|| < \delta,\ ||f(x) - f(p)|| < \epsilon.$$

If f is continuous at all $p \in A$ we say f is **continuous on** A.

Theorem 2.2.2. Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ with $f : A \to B$ continuous at $p \in A$. Suppose $g : B \to \mathbb{R}^l$ is continuous as $f(p)$, then $g \circ f : A \to \mathbb{R}^l$ is cotninuous at p.

Proof. □

3 Derivative of maps of Euclidean spaces

3.1 Total derivatives

Definition 3.1.1 (Total derivate)**.** Given open $\Omega \subset \mathbb{R}^n$, the function $f : \Omega \to \mathbb{R}^m$ is **differentiable as** $p \in \Omega$ iff there is a linear linear map $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ satisfying:

$$\lim_{x \to p} \frac{||f(x) - f(p) - \Lambda(x - p)||}{||x - p||} = 0.$$

Have $Df(p) := \Lambda$ be the **total derivative** of f at p.

Remark 3.1.2. Given $f : (a, b) \to \mathbb{R}$ differentiable at $p \in (a, b)$, we have

$$\lim_{x \to p} \frac{||f(x) - f(p) - \Lambda(x - p)||}{||x - p||} = \lim_{x \to p} \frac{|f(x) - f(p) - \lambda \cdot (x - p)|}{|x - p|} = \lim_{x \to p} \left| \frac{f(x) - f(p)}{x - p} - \lambda \right| = 0$$
$$\implies \lim_{x \to p} \left| \frac{f(x) - f(p)}{x - p} \right| = \lambda, \text{ which satisfies the normal definition for a derivative.}$$

Theorem 3.1.3 (Uniqueness of total derivatve)**.** If the total derivative of a function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ exists, then it is unique.

Proof. □

Theorem 3.1.4 (Chain rule)**.** Let $\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^m$ be open and have $g : \Omega \to \Omega'$, $f : \Omega' \to \mathbb{R}^l$ differentiable at $p, g(p)$ respectively and let $h := f \circ g$, $Dh(p) = Df(g(p)) \circ Dg(p)$.

Proof. □

3.2 Directional and partial derivatives

Definition 3.2.1 (Direction derivative)**.** Suppose $\Omega \subseteq \mathbb{R}^n$ is open with $f : \Omega \to \mathbb{R}^m$ differentiable at $p \in \Omega$. For all $v \in \mathbb{R}^n$ the **directional derivative** of f at p in the direction of v is:

$$\frac{\partial f}{\partial v}(p) := \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t} = Df(p)[v].$$

With the partial derivatives of f given by:

$$D_i f(p) := \frac{\partial f}{\partial e_i}(p), \text{ for all } i \in [1, n].$$

Remark 3.2.2. If the total derivative of a function exists, then so do all of its directional derivatives.

Theorem 3.2.3. If $\Omega \subset \mathbb{R}^n$ is open with $f : \Omega \to \mathbb{R}$ with all partial derivatives existing for all $x \in \Omega$. If the map $x \mapsto D_i f(x)$ is continuous at $p \in \Omega$ for all partial derivatives, then f is differentiable at p.

3.3 Higher order derivatives

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