MATH40003A Linear Algebra

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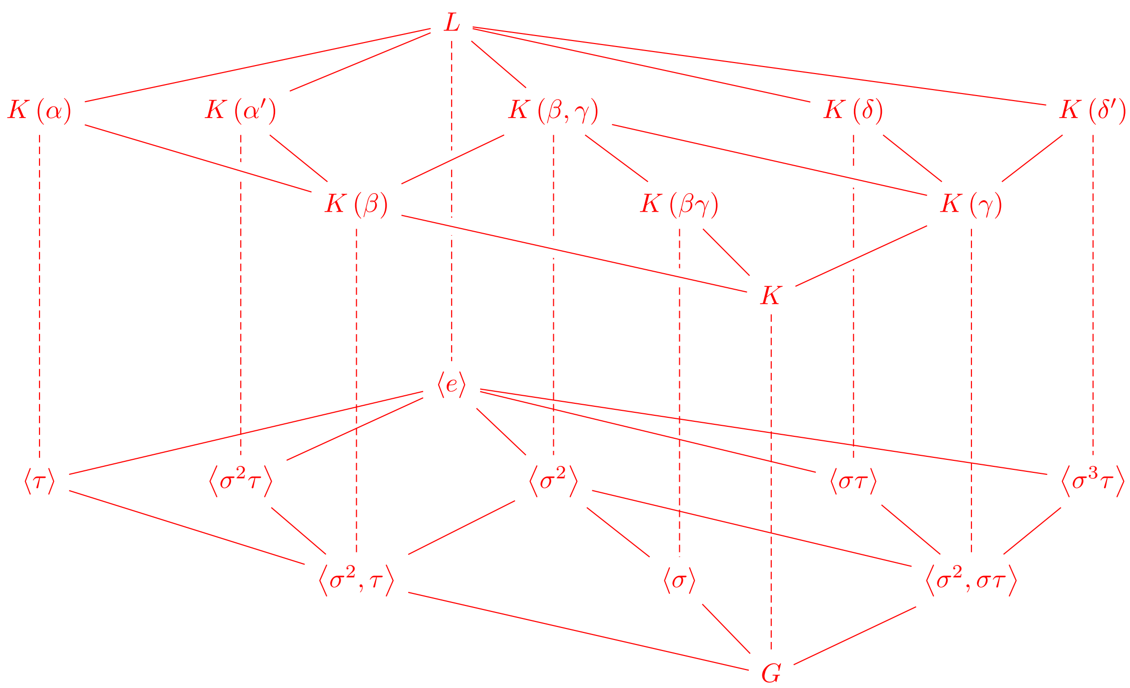

$G = \mathrm{Gal}\left(L/K\right) \cong \mathcal{D}_8$

Syllabus

Systems of linear equations, Matrices, Augmented matrices, Elementary matrices, EROs, REF & rREF, Linear maps, Fields, Vector Spaces, Subspaces, Spanning, Linear independence, Bases, Rank Nullity, Representations of bases, Determinants, Eigenvalues and eigenvectors, characteristic polynomial, diagonisability, orthogonality, Gramm-Schmidt process, symmetric matrices, spectral theorem

Contents

0 Introduction

Lecture 1 Thursday 10/01/19

The following are references.

- E Artin, Galois theory, 1994
- A Grothendieck and M Raynaud, Revêtements étales et groupe fondamental, 2002
- I N Herstein, Topics in algebra, 1975
- M Reid, Galois theory, 2014

Notation. If K is a field, or a ring, I denote the **ring of polynomials** with coefficients in K.

1 Linear Systems and matrices

1.1 Linear systems

Definition 1 (Linear system)**.** A **linear system** is a set of linear equations in the same variables.

Notation 2. The follow are all equivalent notation for the same linear system:

$$\begin{array}{ccccccccc} a_{11}x_1 & + & a_{12}x_2 & + & \dots & + & a_{1n}x_n & = & b_1 \\ a_{21}x_1 & + & a_{22}x_2 & + & \dots & + & a_{2n}x_n & = & b_2 \\ \vdots & & \vdots & & \ddots & & \vdots & & \vdots \\ a_{m1}x_1 & + & a_{m2}x_2 & + & \dots & + & a_{mn}x_n & = & b_m \end{array} \iff \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

$$\iff \left(\begin{array}{cccc|c} a_{11} & a_{12} & \dots & a_{1n} & b_1 \\ a_{21} & a_{22} & \dots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_m \end{array}\right).$$

1.2 Matrix algebra

Definition 3 (Matrix by elements)**.** An $m \times n$ matrix written as $A = [a_{ij}]_{m\times n}$ has the element a_{ij} in the ith row and jth column.

Definition 4 (Matrix addition)**.** If $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ then $A + B := [a_{ij} + b_{ij}]_{m\times n}$.

Definition 5 (Scalar multiplication)**.** If $A = [a_{ij}]_{m\times n}$ then $\lambda A := [\lambda a_{ij}]_{m\times n}$.

Definition 6 (Matrix multiplication)**.** If $A = [a_{ij}]_{p\times q}$ and $B = [b_{ij}]_{q\times r}$ then $AB := C = [c_{ij}]_{p\times r}$ where $c_{ij} = \sum_{k=1}^{q} a_{ik}b_{kj}$.

Theorem 7. Matrix multiplication is associative.

Remark 8. Matrix multiplication is not commutative.

1.3 EROs

Definition 9 (Elementary row operations)**.** The three **elementary row operations (EROs)** that can be performed on augmented matrixes are as follows:

1. Multiply a row by a non-zero scalar.
2. Swap two rows.
3. Add a scalar multiple of a row to another row.

Remark 10. EROs preserve the set of solutions of a linear system. Each ERO has an inverse.

Definition 11 (Equivalence of linear systems)**.** Two systems of linear equations are equivalent iff either:

1. They are both inconsistent.

2. (wlog) The augmented matrix of the first system can be transformed to the augmented matrix of the second system with just EROs.

Definition 12 (Row echelon form / Echelon form/ REF)**.** A matrix is in **row echelon form** if it satisifes the following:

1. All of the zero rows are at the bottom of the matrix,
2. The first non-zero entry in any row is 1,
3. The first non-zer entru in row is stricly to the left of the first non-zero entry in row $i+1$.

Definition 13 (Reduced row echelon form / Row reduced echelon form / rREF)**.** A matrix is im **reduced row echelon form** if it is in REF and the first non-zero entry in a row is the only non-zero entry in its column.

1.4 Matirces of note

Definition 14 (Square matirx)**.** A matrix is **square** iff it has the same number of rows and columns.

Definition 15. A square matrix ($A = [a_{ij}]_{n\times n}$) is: 1. **Upper triangular** iff $i > j \implies a_{ij} = 0$. 2. **Lower triangular** iff $i < j \implies a_{ij} = 0$. 3. **Diagonal** iff $i \neq j \implies a_{ij} = 0$.

Definition 16 (Identity matrix)**.** The **identity matrix** of size n written I_n, is the square diagonal matrix of size n with all diagonal entries equal 1.

Definition 17 (Elementary matrix)**.** An **elementary matrix** is a matrix that can be achieved by appling a single ERO to the identity matrix.

Definition 18 (Inverse)**.** For a square matrix B if there exists a matrix B^{-1} such that $\mathbf{BB}^{-1} = I = B^{-1}B$ then B^{-1} is the **inverse** of B and vice versa.

Definition 19 (Singular)**.** A matrix without an inverse is **singular**.

Theorem 20. The inverse of a matrix is unique.

Definition 21. A **transpose** of the matrix $A = [a_{ij}]_{m\times n}$ is $A^{\mathrm{T}} := [a_{ji}]_{n\times m}$.

Theorem 22. If the matrix A has an inverse then its transpose has an inverse with $(A^{\mathrm{T}})^{-1} = (A^{-1})^{\mathrm{T}}$.

Theorem 23. If a matrix $A \in M_{m\times n}$ can be reduced to I_n by a sequence of EROs then A is inevitable with A^{-1} given by applying the same sequence of EROs to I_n.

Definition 24. A matrix A is **orthogonal** if it has an inverse with $A^{-1} = A^{\mathrm{T}}$.

Theorem 25. An orthogonal matrix A satisfies the condition $(Ax)\cdot(Ay) = x\cdot y$, where $\cdot$ is the dot product.

2 Vector Spaces

The notion of a vector space is a structure designed to generalise that of real vectors, so before developing them we must first produce a generalisation of the real numbers.

2.1 Fields

Definition 26 (Field)**.** A **field** is a set $\mathbb{F}$ equipped with the binary operations **addition** $+ : \mathbb{F}\times\mathbb{F} \to \mathbb{F}$ and **multiplication** $\cdot : \mathbb{F}\times\mathbb{F} \to \mathbb{F}$ satisfying the follow axioms:

A1 $\forall x, y \in \mathbb{F} : x + y = y + x$ (commutativity of addition),

A2 $\forall x, y, z \in \mathbb{F} : x + (y + z) = (x + y) + z$ (associativity of addition),

A3 $\exists 0_{\mathbb{F}} \in \mathbb{F}$ such that $\forall x \in \mathbb{F} : x + 0_{\mathbb{F}} = x$, (additive identity element),

A4 $\forall x \in \mathbb{F},\ \exists(-x) \in \mathbb{F}$ such that $x + (-x) = 0_{\mathbb{F}}$, (additive inverse);

M1 $\forall x, y \in \mathbb{F} : x \cdot y = y \cdot x$ (commutativity of multiplication), M2 $\forall x, y, z \in \mathbb{F} : x \cdot (y \cdot z) = (x \cdot y) \cdot z$ (associativity of multiplication),

M3 $\exists 1_{\mathbb{F}} \in \mathbb{F}$ such that $\forall x \in \mathbb{F} : x \cdot 1_{\mathbb{F}} = x$, (multiplicative identity element),

M4 $\forall x \in \mathbb{F} \setminus \{0_{\mathbb{F}}\},\ \exists x^{-1} \in \mathbb{F}$ such that $x \cdot x^{-1} = 1_{\mathbb{F}}$, (multiplicative inverse);

D $\forall x, y, z \in \mathbb{F} : x \cdot (y + z) = x \cdot y + x \cdot z$ (distributivity of multiplication over addition).

The field $(\mathbb{F}, +, \cdot)$ is often referred to as just $\mathbb{F}$.

Example 27. The familiar sets $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ are all fields.

Theorem 28. If $p \in \mathbb{N}$ is prime with $\mathbb{F}_p = \{0, 1, \ldots, p - 1\}$ then $(\mathbb{F}_p, +_p, \cdot_p)$ is a field.

2.2 Vector spaces

Definition 29 (Vector space)**.** A **vector space** over a field $\mathbb{F}$ is a set V equipped with the binary operations **vector addition** $\oplus : V \times V \to V$ and **scalar multiplication** $\odot : \mathbb{F} \times V \to V$ satisfying the follow axioms:

A1 $\forall u, v, w \in V : u \oplus (v \oplus w) = (u \oplus v) \oplus w$ (associativity of addition),

A2 $\forall u, v \in V : u \oplus v = v \oplus u$ (commutativity of vector addition),

A3 $\exists 0_V \in V$ such that $\forall v \in V : v + 0_V = v$, (vector additive identity element),

A4 $\forall v \in V,\ \exists (-v) \in v$ such that $v + (-v) = 0_V$, (vector additive inverse),

A5 $\forall x \in \mathbb{F},\ \forall u, v \in V : x \odot (u \oplus v) = (x \odot u) \oplus (x \odot v)$ (vector distributivity 1),

A6 $\forall x, y \in \mathbb{F},\ \forall v \in V : x \cdot (x + y) \odot v = (x \odot v) \oplus (y \odot v)$ (vector distributivity 2),

A7 $\forall x, y \in \mathbb{F},\ \forall v \in V : (x \cdot y) \odot v = x \odot (y \odot v)$ (scalar multiplication associativity),

A8 $\forall v \in V : 1_F \odot v = v$, (scalar multiplication identity element).

If V is a vector space over $\mathbb{F}$ we say V is an $\mathbb{F}$-vector space with $v \in V$ a **vector** and $x \in \mathbb{F}$ a **scalar**.

2.3 Subspaces

Definition 30 (Subspace)**.** A subset $W \subseteq V$ is a **subspace** of V, denoted $W \leq V$ iff:

S1 $W \neq \emptyset$,

S2 $\forall w_1, w_2 \in W : w_1 \oplus w_2 \in W$,

S3 $\forall x \in \mathbb{F},\ \forall w \in W : x \odot w \in W$.

If $W = \{0_V\}$ then W is the **trivial subspace**.

Theorem 31. Every subspace of V contains 0_V.

Theorem 32. If U and W are subspaces of V, $U \cap W$ is a subspace of V.

3 Spanning and Linear Independence

Throughout this section, assume V is an $\mathbb{F}$-vector space.

3.1 Spanning

Definition 33 (Span)**.** Given some set $\{v_1, v_2, \ldots, v_n\} \subseteq V$ define the **span** by,

$$\mathrm{Span}(\{v_1, v_2, \ldots, v_n\}) := \{u \in V : u = \sum_{i=1}^{n} \alpha_i v_i \text{ with } \alpha_i \in \mathbb{F}\}.$$

Note that the span of a subset of V is always a subspace of V.

Remark 34. If $S \subseteq V$ is infinite, $\mathrm{Span}(S)$ is the set of all **finite** linear combinations of elements of S.

Definition 35 (Spanning sets)**.** If $S \subseteq V$ and $\mathrm{Span}(S) = V$, we say S **spans** V or S is a **spanning set** for V.

3.2 Linear independence

Definition 36. The set $\{v_1, v_2, \ldots, v_n\} \subseteq V$ is **linearly independent** in V iff:

$$\sum_{i=1}^{n} \alpha_i v_i = 0_V \iff \alpha_i = 0_{\mathbb{F}} \text{ for all i} \in [1, n].$$

Theorem 37. If $S = \{v_1, v_2, \ldots, v_n\} \subseteq V$ is linearly independent in V with $v_{n+1} \in V \setminus \text{Span}(S)$ then $S \cup \{v_{n+1}\}$ is also linearly independent in V.

4 Bases

4.1 Definition

Again, assume V is an $\mathbb{F}$-vector space throughout this section.

Definition 38 (Bases)**.** A **basis** for V is linearly independent, spanning set of V. If V has a finite bases then V is said to be a **finite dimensional** vector space.

Theorem 39. Any $S \subseteq V$ is a basis for V iff every vector in V can be uniquely expressed as a linear combination of the elements of S.

Theorem 40. If V is non-trivial and S is a finite spanning set of V then S contains a basis for V.

Lemma 41 (Steinitz Echange Lemma)**.** Given some $X \subseteq V$ with $u \in \text{Span}(X)$ but $u \notin \text{Span}(X \setminus \{v\})$ for some $v \in X$, let $Y = (X \setminus \{v\}) \cup \{u\}$ then $\text{Span}(X) = \text{Span}(Y)$.

Theorem 42. Given a LI $S \subseteq V$ and spanning set $T \subseteq V$, $|S| \leq |V|$.

Corollary 43. If S and T are both bases for V, $|S| = |T|$.

4.2 Dimension

Definition 44 (Dimension of a vector space)**.** If V is finite dimensional then the **dimension** of V, $\text{dim}V$, is the size of any basis of V.

Definition 45 (Notable subspaces)**.** Let U and W both be subspaces of V, the **intersection** of U and W:

$$U \cup W := \{v \in V : v \in U \text{ and } v \in W\}$$

is a subspace of V, and the **sum** of U and W:

$$U + W := \{u + W : u \in U, w \in W\}$$

is also a subspace of V.

Theorem 46. Let U and W both be subspaces of V, we have:

$$\text{dim}(U + W) = \text{dim}U + \text{dim}W - \text{dim}(U \cap W).$$

5 Matrix rank

Definition 47. Given a field $\mathbb{F}$ and a matrix $A \in M_{m\times n}(\mathbb{F})$ we have:

- the **row space** of A, $\text{RSp}(A)$, as the span of the rows of A, this is a subspace of $\mathbb{F}^n$,
- the **row rank** of A, is $\text{dim}(\text{RSp}(A))$,
- the **column space** of A, $\text{CSp}(A)$, as the span of the columns of A, this is a subspace of $\mathbb{F}^m$,
- the **column rank** of A, is $\text{dim}(\text{CSp}(A))$.

Theorem 48. For any matrix A, the row rank of A is equal to the column rank of A.

Definition 49 (Rank of a matrix)**.** The **rank** of a matrix A, $\text{rank}(A)$, is equal to the row/column rank of A.

Theorem 50. Given a field $\mathbb{F}$ and a matrix $A \in M_{n\times n}(\mathbb{F})$ with $\text{rank}(A) = n$:

- the rows of A for a basis for $\mathbb{F}^n$,
- the columns of A for a basis for $\mathbb{F}^n$,
- A is invertible.

6 Linear transformations

6.1 Definition

Definition 51 (Linear transformation)**.** Given $\mathbb{F}$-vector spaces V and W, let $T : V \to W$ be a function, T is a **linear transformation** iff the following two properties hold:

1. T **preservers vector addition**: $\forall v_1, v_2 \in V$ we have $T(v_1 +_V v_2) = T(v_1) +_W T(v_2)$,
2. T **preservers scalar multiplication**: $\forall v \in V$ and $\forall \lambda \in \mathbb{F}$ we have $\lambda T(v) = T(\lambda v)$.

Definition 52 (Identity transformation)**.** The **identity transformation** of the vector space V is the linear transformation $\text{Id}_V : V \to V$ with $\text{Id}_V(v) := v$ for all $v \in V$.

Definition 53 (Linear transformation of a matrix)**.** If $A \in M_{m\times n}(\mathbb{F})$ then we can define $T : \mathbb{F}^n \to \mathbb{F}^m$ by $T(v) := Av$, T is a linear transformation.

Theorem 54. If V and W are $\mathbb{F}$-vector space, $T(0_V) = 0_W$ and

$$v = \lambda_1 v_1 + \ldots + \lambda_n v_n \iff T(v) = \lambda_1 T(v_1) + \ldots + \lambda_n T(v_n).$$

6.2 Image and kernel

Throughout, assume $T : V \to W$ is a linear transformation and V, W are $\mathbb{F}$-vector spaces

Definition 55 (Image)**.** We define the **image** of T, denoted $\text{Im } T$, as

$$\text{Im } T := \{w \in W : \exists v \in V, T(v) = W\},$$

with $\text{Im } T$ being a subspace of W.

Definition 56 (Kernel)**.** We define the **kernel** of T, denoted $\ker T$, as

$$\ker T := \{v \in V : T(v) = 0_W\},$$

with $\ker T$ being a subspace of V.

Theorem 57. If $v_1, v_2 \in V$ then $T(v_1) = T(v_2) \iff v_1 - v_2 \in \ker T$.

Theorem 58. If $\{v_1, v_2, \ldots, v_n\}$ is a basis for V, then $\text{Im } T = \text{Span}(\{T(v_1), T(v_2), \ldots, T(v_n)\})$.

Remark 59. If T is the linear transformation for some matrix $A \in M_{m\times n}(\mathbb{F})$ then, $\ker T$ is the set of solutions for $Av = 0$, $\text{Im } T$ is the column space of A, and $\dim(\text{Im } T) = \text{rank } A$

6.3 Rank nulity

Theorem 60 (Rank Nulity Theorem)**.** If V and W are finite dimensional $\mathbb{F}$-vector spaces and $T : V \to W$ is a linear transformation, we have:

$$\dim(\text{Im } T) + \dim(\ker T) = \dim V.$$

7 Representations

7.1 Representations

7.2 Matrices of transformations

Throughout this subsection let V be an n-dimensional $\mathbb{F}$-vector space and $B = \{e_1, e_2, \ldots, e_n\}$ be a basis for V.

Definition 61 (Reperesentation of a vector)**.** Given some $v \in V$ with $v = \lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n$ for $\lambda_i \in \mathbb{F}$, we define the v **with respect to (w.r.t.)** B as

$$[v]_B := \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_n \end{pmatrix} \in \mathbb{F}^n.$$

Remark 62. This must be well defined as all vectors have a unique representation in terms of every basis.

Definition 63 (Linear isomorphism)**.** A **linear isomorphism** is a bijective linear transformation.

Theorem 64. The linear transformation $T : V \to \mathbb{F}^n$ given by $T(v) := [v]_B$ is a linear isomorphism.

7.3 Matrices of transformations

Definition 65 (Representation of a linear transformation)**.** Given finite dimensional $\mathbb{F}$-vector spaces V and W with bases $B = \{v_1, v_2, \ldots, v_n\}$, $C = \{w_1, w_2, \ldots, w_n\}$ respectively, the **matrix of** T **w.r.t.** B **and** C denoted ${}_C[T]_B$ is $m \times n$ matrix with the ith column given by $[T(v_i)]_C$. ${}_B[T]_B$ is often shortened to $[T]_B$.

Remark 66. ${}_C[T]_B[v]_B = [T(v)]_C$, for all $v \in V$.

Theorem 67. Given a finite dimensional $\mathbb{F}$-vector space V with bases $B = \{v_1, v_2, \ldots, v_n\}$ and $C = \{w_1, w_2, \ldots, w_n\}$, if $v_i = \lambda_{1i} w_1 + \lambda_{2i} w_2 + \ldots + \lambda_{ni} w_n$ and P is the matrix given by $P = [\lambda_{ij}]_{n \times n}$, we have:

- $P = [X]_C$ where X is the unique linear transformation given by $X(w_i) = v_i$ for all $i \in [1, n]$,
- $P([v]_B) = [v]_C$ for all $v \in V$,
- $P = {}_C[\mathrm{Id}_V]_B$.

P is often also called the **change of basis matrix** from B to C.

Corollary 68. P is invertible with $(P)^{-1} = ({}_C[\mathrm{Id}_V]_B)^{-1} = {}_B[\mathrm{Id}_V]_C$.

Theorem 69. If $T : V \to V$ is a linear transformation $[T]_C = ({}_C[\mathrm{Id}_V]_B)[T]_B({}_B[\mathrm{Id}_V]_C)$.

8 Determinants

8.1 Definition

Definition 70 (Minor of matrix)**.** Given a matrix $A \in M(\mathbb{F})_n$ the ij**th-minor** of the matrix A, $A_{ij} \in M(\mathbb{F})_n$, is A with row i and column j removed.

Definition 71 (Determinant)**.** The **determinant** of the matrix A is defined recursively by

$$\det(A) := \begin{cases} a_{11} & \text{if } A \text{ is a matrix with a single row and column} \\ \displaystyle\sum_{j=1}^{n} (-1)^{j+1} a_{1j} \det(A_{1j}) & \text{otherwise.} \end{cases}.$$

The determinant is only a function on square matrices.

Theorem 72. If a matrix A is singular, $\det(A) = 0$.

Theorem 73. If A is invertible then the columns of A are LI.

8.2 Properties

Definition 74 (EROs)**.** The three ERO's on the matrix A to form A' have the following effects on the determinant:

- multiplying a row by $\lambda \neq 0$, $\det(A') = \lambda \det(A)$;
- swapping two rows, $\det(A') = -\det(A)$;
- adding a scalar multiple of one row to another, $\det(A') = \det(A)$.

Definition 75 (Other miscellaneous properties)**.** For obvious types:

- If A, B and C all only differ in the ith row with the ith row of C being the sum of the ith row of A and B, $\det(C) = \det(A) + \det(B)$,
- if a matrix A has two identical rows, $\det(A) = 0$,
- $\det(AB) = \det(A)\det(B)$,
- $\det(A^{\mathrm{T}}) = \det(A)$,
- $\det(I_n) = 1$.

Definition 76 (Cofactor)**.** The ijth cofactor of a matrix A is defined as,

$$c_{ij} := (-1)^{i+j}\det(A_{ij}).$$

The **matrix of cofactors** of A is defined as $C = [c_{ij}]_{n\times n}$ where c_{ij} is the ijth cofactor of A.

Theorem 77. For a matrix A with matrix of cofactors C, $C^{\mathrm{T}}A = \det(A)\mathrm{I}_n$.

Theorem 78 (Cramer's Rule)**.** ugh

Definition 79. The **determinant** of a linear transformation $T : V \to V$ where B is a basis for T, $\det(T) = \det([T]_B)$. This definition says, rather importantly, that the determinant of the matrix of linear transformation is independent of the basis that linear transformation is represented in.

9 Eigen-things

The prefix "eigen" comes from the German word "eigen" which can be roughly translated to mean "proper" or "characteristic".

9.1 Eigenvectors and eigenvalues

Definition 80 (Eigenvectors and eigenvalues)**.** Given the finite dimensional $\mathbb{F}$-vector space, V, and the linear transformation $T : V \to V$, we say $v \in V \setminus \{0_V\}$ is an **eigenvector** of T if it satisfies the equation $T(v) = \lambda v$ for some $\lambda \in \mathbb{F}$, we call λ the corresponding **eigenvalue**.

Definition 81 (Eigenspace)**.** The **eigenspace** of an eigenvalue of a given linear transformation $T : V \to V$ is the set of eigenvectors that correspond to said eigenvalue. The eigenspace of any eigenvalue λ of T is a subspace of V.

Remark 82. The eigenvectors, eigenvalues and eigenspaces of a matrix are defined obviously and do not depend on which basis the linear transformation is respresented in.

9.2 Characteristic polynomial

Theorem 83. If D is a square diagonal matrix, D^k is D with its entries raised to the power of k.

Definition 84 (Characteristic polynomial)**.** Given the finite dimensional $\mathbb{F}$-vector space V with basis B and the linear transformation $T : V \to V$, we define the **characteristic polynomial** of T, $\chi_T : \mathbb{F} \to \mathbb{F}$ by $\chi_T(\lambda) := \det(\lambda I_n - T_B)$.

Theorem 85. The characteristic polynomial of a linear transformation is independent of the basis it is represented in.

Remark 86. Therefore, the characteristic polynomial of a matrix can be defined as the characteristic polynomial of the linear transformation it represents.

9.3 Diagonalisation

Definition 87 (Diagonalisability). Given a finite dimensional $\mathbb{F}$-vector space V, a linear transformation $T : V \to V$ is **diagonalisable** if there exists a basis for V consisting of eigenvectors of T. Similarly the matrix $A \in M_n(\mathbb{F})$ is **diagonalisable** if there exists a basis to $\mathbb{F}^n$ as eigenvectors of A.

Theorem 88. If V is a n-dimensional vector space and $T : V \to V$ has n distince eigenvalues, T is diagonalisable.

Theorem 89. If a matrix $A \in M_n(\mathbb{F})$ is diagonalisably, let P be the matrix with columns as eigenvectors of A and D be the diagonal matrix with iith entry as the corresponding eigenvalue for the ith column of P, then $A = PDP^{-1}$.

10 Orthogonality

10.1 Inner product spaces*

Definition 90 (Inner product). Let V be an n-dimensional $\mathbb{F}$-vector space, a **inner product** on V is a bilinear map $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{F}$ satisfying the following:

- $\langle u, v \rangle = \langle v, u \rangle$ for all $u, v \in V$ (Symmetry),
- $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ and $\langle \lambda u, v \rangle = \lambda \langle u, v \rangle$ for all $u, v, w \in V$ and $\lambda \in \mathbb{F}$ (Bilinearity),
- $\langle v, v \rangle \geq 0$ for all $v \in V$ with equality when $v = 0_V$ (Positive-definite).

Bilinearity must hold in both arguments however it can be derived from a single argument with the symmetry propoerty.

Definition 91 (Norm). Given a real-vector space V with an inner product $\langle \cdot, \cdot \rangle$ the **norm** induced by the inner product of $v \in V$ is:

$$||v|| := \sqrt{\langle v, v \rangle}$$

Definition 92 (Orthogonality). Two vectors u, v in an real or complex vector space V with inner product $\langle \cdot, \cdot \rangle$ are **orthogonal** iff: $\langle u, v \rangle = 0$.

10.2 Orthonormal sets

Throughout the remainder of this section we will assume all vectors spaces are over the real or complex numbers and will use the dot product as our inner product with its induced norm.

Definition 93 (Orthogonal sets). A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in a vector space is **orthogonal** if it is pariwise orthogonal.

Definition 94 (Orthonormal sets). A set of vectors $\{v_1, v_2, \ldots, v_n\}$ in a vector space is **orthonormal** if it orthogonal and satisfies $||u_i|| = 1$ for all $i \in [1, n]$.

Theorem 95. The columns of an orthogonal matrix $P \in M_n(\mathbb{R})$ form an orthonormal set.

10.3 Gramm-Schmidt process

The Gramm-Shmidt process is a method of producing orthonormal bases.

Algorithm 96 (Gramm-Shmidt process)**.** Given a LI set $\{v_1, v_2, \ldots, v_r\} \in \mathbb{R}^n$ the **Gramm-Shmidt process** will produce the set of vectors $\{w_1, w_2, \ldots, w_r\} \in \mathbb{R}^n$ by the following:

$$\begin{aligned} w_1 &= v_1, \\ w_2 &= v_2 - \frac{w_1 \cdot v_2}{||w_1||^2} w_1, \\ w_3 &= v_3 - \left(\frac{w_1 \cdot v_3}{||w_1||^2} w_1 + \frac{w_2 \cdot v_3}{||w_2||^2} w_2 \right), \\ &\vdots \\ w_r &= v_r - \sum_{j=1}^{r-1} \frac{w_j \cdot v_r}{||w_j||^2} w_j. \end{aligned}$$

Note that each vector is the original vector v_i with its projection along all of the previous w_js subtracted and therefore $\{w_1, w_2, \ldots, w_r\}$ is orthogonal. Finally, $\{u_1, u_2, \ldots, u_r\}$, where $u_i = \frac{w_i}{||w_i||}$ for all $i \in [1, r]$, is an orthonormal set with $\text{Span}(\{u_1, u_2, \ldots, u_r\}) = \text{Span}(\{v_1, v_2, \ldots, v_r\})$.

Corollary 97. Given some vector $u \in \mathbb{R}^n$ there exists an orthogonal matrix in $M_n(\mathbb{R})$ with u as its first column.

11 Real symmetric matrices

Throughout this section, unsurprisingly, all matrices will be assumed to be real.

11.1 Introduction

Definition 98 (Self-adjoint matrices)**.** If a matrix $A \in M_n(\mathbb{R})$ is symmetric and satisfies $A(u \cdot v) = (Au) \cdot v$ for all vectors $u, v \in \mathbb{R}^n$, we say A is **self-adjoint** w.r.t. the usual scalar product.

Theorem 99. If $A \in M_n(\mathbb{R})$ is symmetric with $\lambda \in \mathbb{C}$ a root of $\chi_A(x) = 0$, $\lambda \in \mathbb{R}$.

Corollary 100. Real symmetric matrices have at least 1 real eigenvalue.

Theorem 101. If $A \in M_n(\mathbb{R})$ is symmetric with discrete eigenvalues $\lambda, \mu \in \mathbb{R}$, their corresponding eigenvectors $u, v \in \mathbb{R}^n$ satisyf $u \cdot v = 0$.

11.2 Spectral theorem

Theorem 102 (Spectral theorem)**.** If $A \in M_n(\mathbb{R})$ is symmetric, then there exists an orthonormal matrix P such that $P^{-1}AP$ is diagonal.

Corollary 103. Appropriately scaled eigenvectors of a symmetric matrix $A \in M_n(\mathbb{R})$ form an orthonormal basis for $\mathbb{R}^n$.