Real Analysis and Topology

Lectured by Someone
Typed by Yu Coughlin
Season Year

Introduction

The following are complementary reading for the course.

- G. Grimmett and D. J. A. Welsh, Probability: An Introduction, 1986
- J. K. Blitzstein and J. Hwang, Introduction to Probability, 2019
- D. F. Anderson et al, Introduction to Probability, 2018
- S. M. Ross, Introduction to Pro ability Models, 2014
- G. Grimmett and D. Stirzaker, Probability and Random Processes, 2001
- G. Grimmett and D. Stirzaker, One Thousand Exercises in Probability, 2009

Notation. Unbracketed superscripts are used to label the components of vectors, with unbracketed subscripts labellin different vectors.

Contents

L1 1 Euclidean spaces . . . 3
1.1 Euclidean norm . . . 3
1.2 Convergence in $\mathbb{R}^n$ . . . 3
2 Continuity and limits of functions . . . 4
2.1 Open sets . . . 4
2.2 Continuity . . . 4
3 Derivative of maps of Euclidean spaces . . . 4
3.1 Total derivatives . . . 4
3.2 Directional and partial derivatives . . . 4
3.3 Higher order derivatives . . . 5
4 Inverse and implicit function theorems . . . 6
4.1 Inverse function theorem . . . 6
4.2 Implicit function theorem . . . 6
5 Metric spaces . . . 6
5.1 Introduction . . . 6
5.2 Normed vector spaces . . . 7
5.3 Open and closed sets . . . 7
5.4 Separable space . . . 8
6 Continuous maps in metric spaces . . . 8
6.1 Convergence . . . 8
6.2 Continuity of maps . . . 9
6.3 Metric homeomorphisms . . . 9
7 Topological spaces . . . 9
7.1 Topologies and their spaces . . . 9
7.2 Bases . . . 10
7.3 Closed sets . . . 10
7.4 Convergence and Hausdorff property . . . 10
7.5 Continuous maps . . . 11
7.6 Subspaces . . . 11
8 Connectedness . . . 12
8.1 Definition . . . 12
8.2 Continuous maps . . . 12
8.3 Path connected sets . . . 12
9 Compactness . . . 12
9.1 Covers . . . 12
9.2 Sequential compactness . . . 12
9.3 Continuous maps . . . 12
9.4 Arzelá-Ascoli theorem . . . 12
10 Completeness . . . 12
10.1 Banach spaces . . . 12
10.2 Fixed point theorem . . . 12

Lecture 1 Monday 30/10/2023

1 Euclidean spaces

Definition 1.0.1 ($\mathbb{R}^n$)**.** The set $\mathbb{R}^n = \{(x^1, x^2, \dots, x^n) : x^i \in \mathbb{R},\ \forall i \in [1, n]\}$ will be considered with the operations to make it a real vector space.

1.1 Euclidean norm

Definition 1.1.1 (Inner product)**.** We will have the **inner product** on $\mathbb{R}^n$ by $\langle \cdot, \cdot \rangle : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfying:

$$\langle x, y \rangle := \sum_{i=1}^{n} x^i y^i,$$

with the **Euclidean norm** given by,

$$|| \cdot || : \mathbb{R}^n \to [0, \infty) \text{ with } ||x|| = \sqrt{\langle x, x \rangle}.$$

Proposition 1.1.2 (Properties of the Euclidean norm)**.** The Euclidean norm satisfies the following properties:

(N1) for all $x \in \mathbb{R}^n$, $||x|| \geq 0$ achieving equality iff $x = 0$,

(N2) for all $x \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$, $||\lambda x|| = |\lambda| \cdot ||x||$,

(N3) for all $x, y \in \mathbb{R}^n$: $||x + y|| \leq ||x|| + ||y||$,

Theorem 1.1.3 (Cauchy-Swartz innequality)**.** For all $x, y \in \mathbb{R}^n$, $|\langle x, y \rangle| \leq ||x|| \cdot ||y||$.

Theorem 1.1.4 (Reverse triangle innequality)**.** For all $x, y \in \mathbb{R}^n$, $\big|\ ||x|| - ||y||\ \big| \leq ||x - y||$.

Proposition 1.1.5. For $x = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$,

$$\max_{k \in [1,n]} \left| x^k \right| \leq ||x|| \leq \sqrt{n} \max_{k \in [1,n]} \left| x^k \right|.$$

Proof. Exercise □

1.2 Convergence in $\mathbb{R}^n$

Definition 1.2.1 (Open ball)**.** In $\mathbb{R}^n$ we define the **open ball** around $x \in \mathbb{R}^n$ of size $r \in \mathbb{R}$ as

$$B_r(x) := \{y \in \mathbb{R}^n : ||x - y|| < r\}.$$

This will be analoguous the the notion of open intervals used throughout analysis 1.

Definition 1.2.2 (Sequence in $\mathbb{R}^n$)**.** A **sequence** in $\mathbb{R}^n$ is an ordered list $x_0, x_1, \dots, x_i \dots$ with $x_i \in \mathbb{R}^n$ for all $i \in \mathbb{N}$, written $(x_i)_{i=0}^{\infty}$

Definition 1.2.3 (Convergence in $\mathbb{R}^n$)**.** We say a sequence in $\mathbb{R}^n$, $(x_i)_{i=0}^{\infty}$ **converges to** $x \in \mathbb{R}^n$ iff

$$\forall \epsilon > 0\ \exists N \in \mathbb{N} \text{ such that, } \forall n \geq N,\ ||x_i - x|| < \epsilon$$

and we write $x_i \to x$ as $i \to \infty$ or $\lim\limits_{i \to \infty} x_i = x$.

Lemma 1.2.4. The sequence of vectors in $\mathbb{R}^n$, $(x_i)_{i=0}^{\infty}$, converges to some $x = (x^1, x^2, \dots, x^n) \in \mathbb{R}^n$ iff each componenet of x_i converges to the corresponding component in x:

$$\forall k \in [1, n]\ \lim_{i \to \infty} x_i^k = x^k.$$

Proof. ($\implies$) Given $\lim\limits_{i \to \infty} x_i^k = x^k$ for all $k \in [1, n]$ we have that for all $\epsilon > 0$, $\left| x_i^k - x^k \right| < \frac{\epsilon}{\sqrt{n}}$ for all $i \geq N_k$ for each $k \in [1, n]$ respectively. We take $N = \max\limits_{k \in [1,n]} N_k$ and now have:

$$\max_{k \in [1,n]} \left| x_i^k - x^k \right| < \frac{\epsilon}{\sqrt{n}} \implies ||x_i - x|| \leq \sqrt{n} \max_{k \in [1,n]} \left| x_i^k - x^k \right| < \epsilon.$$

($\impliedby$) Similarly, given $\lim\limits_{i \to \infty} x_i = x \implies ||x_i - x|| < \epsilon$ for all $\epsilon > 0$:

$$\left| x_i^k - x^k \right| \leq \max_{k \in [1,n]} \left| x_i^k - x^k \right| \leq ||x_i - x|| < \epsilon,$$

therefore $\lim\limits_{i \to \infty} x_i^k = x^k$ for all $k \in [1, n]$. □

2 Continuity and limits of functions

2.1 Open sets

Definition 2.1.1 (Open set in $\mathbb{R}^n$)**.** A subset $U \subseteq \mathbb{R}^n$ is **open** in $\mathbb{R}^n$ iff:

$$\forall x \in U,\ \exists r > 0 \text{ such that } B_r(x) \subseteq U.$$

2.2 Continuity

Definition 2.2.1 (Continuity)**.** Let $A \subseteq \mathbb{R}^n$ the we have $f : A \to \mathbb{R}^m$ **continuous at** some $p \in A$ iff

$$\forall \epsilon > 0,\ \exists \delta > 0 \text{ such that } \forall x \in A \text{ with } ||x - p|| < \delta,\ ||f(x) - f(p)|| < \epsilon.$$

If f is continuous at all $p \in A$ we say f is **continuous on** A.

Theorem 2.2.2. Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^m$ with $f : A \to B$ continuous at $p \in A$. Suppose $g : B \to \mathbb{R}^l$ is continuous as $f(p)$, then $g \circ f : A \to \mathbb{R}^l$ is cotninuous at p.

Proof. Given any $\epsilon > 0$ have $||x-p|| < \delta_f \circ \delta_g(\epsilon) \implies ||f(x)-f(p)|| < \delta_g(\epsilon) \implies ||g\circ f(x)-g\circ f(p)|| < \epsilon$. □

3 Derivative of maps of Euclidean spaces

3.1 Total derivatives

Definition 3.1.1 (Total derivate)**.** Given open $\Omega \subset \mathbb{R}^n$, the function $f : \Omega \to \mathbb{R}^m$ is **differentiable as** $p \in \Omega$ iff there is a linear linear map $\Lambda : \mathbb{R}^n \to \mathbb{R}^m$ satisfying:

$$\lim_{x \to p} \frac{||f(x) - f(p) - \Lambda(x - p)||}{||x - p||} = 0.$$

Have $Df(p) := \Lambda$ be the **total derivative** of f at p.

Remark 3.1.2. Given $f : (a, b) \to \mathbb{R}$ differentiable at $p \in (a, b)$, we have

$$\lim_{x \to p} \frac{||f(x) - f(p) - \Lambda(x - p)||}{||x - p||} = \lim_{x \to p} \frac{|f(x) - f(p) - \lambda \cdot (x - p)|}{|x - p|} = \lim_{x \to p} \left| \frac{f(x) - f(p)}{x - p} - \lambda \right| = 0$$
$$\implies \lim_{x \to p} \left| \frac{f(x) - f(p)}{x - p} \right| = \lambda, \text{ which satisfies the normal definition for a derivative.}$$

Theorem 3.1.3 (Uniqueness of total derivatve)**.** If the total derivative of a function $f : \Omega \subseteq \mathbb{R}^n \to \mathbb{R}^m$ exists, then it is unique.

Proof. □

Theorem 3.1.4 (Chain rule)**.** Let $\Omega \subset \mathbb{R}^n$, $\Omega' \subset \mathbb{R}^m$ be open and have $g : \Omega \to \Omega'$, $f : \Omega' \to \mathbb{R}^l$ differentiable at $p, g(p)$ respectively and let $h := f \circ g$, $Dh(p) = Df(g(p)) \circ Dg(p)$.

Proof. □

3.2 Directional and partial derivatives

Definition 3.2.1 (Direction derivative)**.** Suppose $\Omega \subseteq \mathbb{R}^n$ is open with $f : \Omega \to \mathbb{R}^m$ differentiable at $p \in \Omega$. For all $v \in \mathbb{R}^n$ the **directional derivative** of f at p in the direction of v is:

$$\frac{\partial f}{\partial v}(p) := \lim_{t \to 0} \frac{f(p + tv) - f(p)}{t} = Df(p)[v].$$

With the partial derivatives of f given by:

$$D_i f(p) := \frac{\partial f}{\partial e_i}(p), \text{ for all } i \in [1, n].$$

Remark 3.2.2. If the total derivative of a function exists, then so do all of its directional derivatives.

Theorem 3.2.3. If $\Omega \subset \mathbb{R}^n$ is open with $f : \Omega \to \mathbb{R}$ with all partial derivatives existing for all $x \in \Omega$. If the map $x \mapsto D_i f(x)$ is continuous at $p \in \Omega$ for all partial derivatives, then f is differentiable at p.

Proof. □

3.3 Higher order derivatives

Definition 3.3.1 (Second order partial derivatives)**.** Let $\Omega \subset \mathbb{R}^n$ be open with differentiable $f : \Omega \to \mathbb{R}$ written as $(f^1, f^2, \dots, f^n)^{\mathrm{T}}$, the ik**th second partial derivative** at p is

$$D_k D_i f^j(p) := \lim_{t \to 0} \frac{D_i f^j(p + te_k) - D_i f^j(p)}{t}.$$

This can naturally be extended to nth order partial derivatives.

Theorem 3.3.2. Given open $\Omega \subseteq \mathbb{R}^n$ and $f : \Omega \to \mathbb{R}^m$ differentiable on Ω, consider the map:

$$\begin{array}{cccll} Df & : & \Omega & \longrightarrow & \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m) \cong M_{n\times m}(\mathbb{R}) \cong \mathbb{R}^{m\times n} \\ & & p & \longmapsto & Df(p) \end{array},$$

which we can now show to be continuous or differentiable at $p \in \Omega$, when differentiable we can take $DDf(p) \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m)$. The componenents of the corresponding matrix are give by:

$$[DDf(p)[h]]_{ij} = \sum_{k=1}^{n} D_k D_i f^j(p) h^k.$$

Proof. □

Remark 3.3.3. The condition of a function being k times differentiable at a point p can is often difficult to establish, instead the continuous existence of all $k-th$ partial derivatives in a neighbourhood of p is a prefereable question which implies the former statement.

Theorem 3.3.4 (Schwartz's theorem)**.** Suppose $\Omega \subseteq \mathbb{R}^n$ is open and $f : \Omega \to \mathbb{R}^m$ is differentiable on Ω with $D_i D_j f(p), D_j D_i f(p)$ both exist continuous only Ω; then we have

$$D_i D_j f(p) = D_j D_i f(p) \text{ for all } p \in \Omega.$$

Proof. □

Notation 3.3.5. We need the following necessary notation around an n-vector of non-negative integers, $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n) \in (\mathbb{Z}_{>0})^n$ for some $n \in \mathbb{Z}_{>0}$, to easily express Taylor's theorem in multiple dimensions:

1. $|\alpha| = \alpha_1 + \alpha_2 + \ldots + \alpha_n$,
2. $D^\alpha f = (D_1)^{\alpha_1} (D_2)^{\alpha_2} \cdots (D_n)^{\alpha_n}$,
3. for some vector $h = (h^1, h^2, \dots, h^n) \in \mathbb{R}^n$, $h^\alpha = ((h^1)^{\alpha_1}, (h^2)^{\alpha_2}, \dots, (h^n)^{\alpha_n})$,
4. $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$.

Theorem 3.3.6 (Taylor's theorem)**.** Given $p \in \mathbb{R}^n$ with $f : B_r(p) \to \mathbb{R}$, for some $r > 0$, k-times continuous differentiable on $B_r(p)$ and some $||h|| < r$; we have:

$$f(p+h) = \sum_{|\alpha| \le k-1} \frac{h^\alpha}{\alpha!} D^\alpha f(p) + R_k(p,h).$$

Where the remainder term, $R_k(p,h)$ is given by:

$$R_k(p,h) = \sum_{|\alpha| = k} \frac{h^\alpha}{\alpha!} D^\alpha f(x).$$

Proof. □

4 Inverse and implicit function theorems

4.1 Inverse function theorem

Theorem 4.1.1 (Inverse function theorem)**.** Have $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous differentiable on $\Omega \subseteq \mathbb{R}^n$ and $Df(p)$ be invertible for a $p \in \Omega$. There exists open sets $U \in \Omega$ and $V \in \mathbb{R}^n$ such that $f : U \to V$ is a bijection. Furthermore, $f^{-1} : V \to U$ is continuous differentiable on V with:

$$Df^{-1}(y) = \left[Df(f^{-1}(y))\right]^{-1}.$$

Lemma 4.1.2. Have $B_r(p) \subset \mathbb{R}^n$ with $f : B_r(p) \to \mathbb{R}^n$ contoniously differntiable. If there exists some $M \in \mathbb{R}_{>0}$ with $|D_j f^i(x)| < M$ for all $x \in B_r(p)$ then

$$||f(x) - f(y)|| \leq nM||x - y||, \text{ for all } x, y \in B_r(p).$$

Proof. □

Lemma 4.1.3. Given $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous differentiable on some $B_r(p)$ with $Df(p)$ invertible, there exists some $\delta > 0$ such that $f : B_\delta(p) \to \mathbb{R}^n$ is injective.

Proof. □

Lemma 4.1.4. $f : \mathbb{R}^n \to \mathbb{R}^n$ continuous differentiable on some $B_r(p)$ with $Df(p)$ invertible and $f : B_\delta(p) \to \mathbb{R}^n$ injective, there exists some $\kappa > 0$ with all $y \in B_\kappa(f(p))$ having a uniqe $x = B_\delta(p)$ such that $f(x) = y$.

Proof. □

Lemma 4.1.5. *Proof.* □

Proof of Theorem 4.1.1(Inverse function theorem). By Lemma 4.1.4 □

4.2 Implicit function theorem

Theorem 4.2.1 (Implicit function theorem)**.** Given $\Omega \subseteq \mathbb{R}^n$ and $\Omega' \subseteq \mathbb{R}^m$ both open with $f : \Omega \times \Omega' \to \mathbb{R}^m$ continuous differentiable on $\Omega \times \Omega'$. If there is some $p \in \Omega \times \Omega'$ with $f(p) = 0$ and $D_{n+j} f^i(p)$ invertible for $1 \leq i, j \leq m$. Then, there are open sets $A \in \Omega$ and $B \in \Omega'$ containg a and b respectively such that for all $x \in A$ ther is a unique and differentiable $g(x) \in B$ with $f(x, g(x)) = 0$.

Proof. □

5 Metric spaces

5.1 Introduction

Definition 5.1.1 (Metric)**.** A **metric** on some arbitrary set X is a function:

$$d : X \times X \to \mathbb{R}$$

that satisfies the following properties for all $x, y, z \in X$:

(M1) $d(x, y) \geq 0$ with $d(x, y) = 0$ iff $x = y$ (positibity),

(M2) $d(x, y) = d(y, x)$ (symmetry),

(M3) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle innequality).

Definition 5.1.2 (Metric space)**.** A **metric space** is a pair consisting of a set and a metric on said set, often denoted $M = (X, d)$. The elements of X are called **points** and for any two points of M, x, y, their **distance (with respect to** d**)** is $d(x, y)$.

Examples 5.1.3. The following are common examples of metric spaces:

1. have $X = \mathbb{R}$ and $d_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $d_1(x, y) := |x - y|$,

2. have $X = \mathbb{R}^n$ and have $d(x,y) := \sqrt{\sum_{i=1}^{n} (x^i - y^i)^2}$,

3. for an arbitary non-empty set X we have $d_{\text{disc}} : X \times X \to \mathbb{R}$ by $d_{\text{disc}}(x,y) := 0$ iff $x = y$ and 1 otherwise (discrete metric),

4. have X be the set of bounded real sequences, then we can have $d_\infty : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $d_\infty(x,y) := \sup_{k \geq 1} |x^k - y^k|$,

5. let X be the set of continuous real functions on $[a,b]$ with $d(f,g) := \int_{t=a}^{b} |f(t) - g(t)| \, \mathrm{d}t$.

Definition 5.1.4 (Induced metric)**.** Given the metric space (X,d) and some $Y \subset X$, we have $d_Y : Y \times Y \to \mathbb{R}$ with $d_Y(x,y) = d(x,y)$ for all $x, y \in Y$ as the **induced metric** on Y. (Y, d_Y) is a **metric subspace** of (X,d).

5.2 Normed vector spaces

Definition 5.2.1 (Normed vector spaces)**.** Given a real-vector space V, a function $||\cdots|| : V \to \mathbb{R}$ is a **norm** on V iff the following hold for all $u, v \in V$:

(N1) $||v|| \geq 0$ with $||v|| = 0$ iff $v = 0_V$,

(N2) for all $\lambda \in \mathbb{R}$, $||\lambda v|| = |\lambda| \cdot ||v||$,

(N3) $||u + v|| \leq ||u|| + ||v||$.

A vector space together with a norm is a **normed vector space**.

Lemma 5.2.2. If $(V, ||\cdot||)$ is a normed vector space, $d_{||\cdot||} : V \times V \to \mathbb{R}$ with $d_{||\cdot||}(u,v) = ||u - v||$ is a metric on V.

Proof. □

5.3 Open and closed sets

Definition 5.3.1 (ϵ-ball)**.** Given a point x in the metric space (X,d) and a real $\epsilon > 0$, the **ball** of radius ϵ centred at x is the set,

$$B_\epsilon(x) := \{y \in X : d(x,y) < \epsilon\},$$

which is sometimes referred to as a neighbourhood of x.

Definition 5.3.2 (Open sets)**.** Given metric space (X,d) a set $U \subseteq X$ is **open** in (X,d) iff, for all $\mathrm{u} \in U$ there exists some $\delta > 0$ such that $B_\delta(u) \subseteq U$.

Proposition 5.3.3. Have $\mathcal{X} = (X,d)$ a metric space, the follow hold true:

1. $\emptyset$ and $\mathcal{X}$ are open in $\mathcal{X}$,

2. for all $x \in \mathcal{X}$ and $\epsilon > 0$, $B_\epsilon(x)$ is open in $\mathcal{X}$,

3. the union of (up to uncountably many) open sets in $\mathcal{X}$ are open in $\mathcal{X}$,

4. the intersection of finitely many open sets in $\mathcal{X}$ is open in $\mathcal{X}$.

Proof. □

Definition 5.3.4 (Topological equivalence)**.** Two metrics d, d' on X are **topologically equivalent** iff $U \subseteq X$ is open in (X,d) iff it is also open in (X,d').

Definition 5.3.5 (Closed sets)**.** Given the metric space (X,d) with $U \subseteq X$, U is **closed** iff $X \setminus U$ is open.

Proposition 5.3.6. A set $U \subseteq X$ with (X,d) a metric spacpe is closed iff, every convergenct sequence in V has a limit in V.

Proof. □

Proposition 5.3.7. The intersection of (up to ocuntable many) closed sets in a metric space is closed; the union of finitely many sets in a metric space is closed.

Proof. □

5.4 Separable space

Definition 5.4.1 (Interior, isolated, limits and boundary points)**.** We will have (X, d) be a metric space with $V \subseteq X$ and $x \in X$:

- x is an **interor point** of V if there is some $\delta > 0$ with $B_\delta(x) \subseteq V$,
- x is an **isolated point** of V if there is some $\delta > 0$ such that $V \cap B_\delta(x) = \{x\}$,
- x is a **limit point** of V if for all $\delta > 0$, we have $(B_\delta(x) \cap V) \setminus \{x\} \neq \emptyset$,
- x is a **boundary point** of V if it is a limit point, under the previous definition, and $B_\delta(x) \setminus V \neq \emptyset$.

Remark 5.4.2. Interior and isolated points are necessarily in V, but limit points and boundary points need not be elements of V.

Definition 5.4.3 (Interior, closure and boundary)**.** Once again, we will have (X, d) a metric space with $V \subseteq X$:

- the **interior** of V is the set of all $v \in V$ with v an interior point of V, denoted V°,
- the **closure** of V is the union of V with the set of limit points of V, denoted $\overline{V}$,
- the **boundary** of V is the set of boundary points of V, denoted ∂V.

Proposition 5.4.4. $\partial V = \overline{V} \setminus V^\circ$.

Proof. □

Definition 5.4.5 (Dense set)**.** Have (X, d) a metric space, $V \subseteq X$ is **dense** in (X, d) iff $\overline{V} = X$.

Definition 5.4.6 (Separable space)**.** We say the metric space (X, d) is **separable** if there is a countable, dense set in X.

6 Continuous maps in metric spaces

6.1 Convergence

Definition 6.1.1 (Convergence in metric spaces)**.** Let $(x_n)_{n \geq 1}$ be a sequence in the metric space (X, d). We say $(x_n)_{n \geq 1}$ **converges** in (X, d) iff:

$$\exists x \in X \text{ such that, } \forall \epsilon > 0, \ \exists N \in \mathbb{Z}_{>0} \text{ with } d(x_n, x) < \epsilon \text{ for all } n \geq N.$$

And we say $(x_n)_{x \geq 1}$ converges to x in (X, d), or any other equivalent phrasing from analysis.

Definition 6.1.2 (Cauchy sequences)**.** A sequence $(x_n)_{n \geq 1}$ is **Cauchy** in (X, d) iff

$$\forall \epsilon > 0, \ \exists N \in \mathbb{Z}_{>0} \text{ such that } \forall n, m \geq N, \ d(x_n, x_m) < \epsilon.$$

Lemma 6.1.3 (Uniqueness of limits)**.** If the sequence $(x_n)_{n \geq 1}$ converges to some x in the metric space (X, d) then this limit is unique.

Proof. □

Theorem 6.1.4. Given two topologically equivalent metrics d, d' on X, the sequence $(x_n)_{n \geq 1}$ converges in (X, d) iff it also converges in (X, d').

Proof. □

6.2 Continuity of maps

Definition 6.2.1 (Continuous map)**.** Given the metric spaces $(X, d_X), (Y, d_Y)$ and $f : X \to Y$:

1. f is **continuous at** $x \in X$ iff for all $x' \in X$:
$$\forall \epsilon > 0, \exists \delta > 0 \text{ such that } d_X(x, x') < \delta \implies d_X(f(x), f(x')) < \epsilon,$$
2. f is **continuous on** $U \subseteq X$ if f is continuous at every $u \in U$,
3. f is **uniformly continuous** on $U \subseteq X$ is f is continuous on U and $\delta = \delta(\epsilon)$ does not depend on x.

Theorem 6.2.2. Let $(X, d_X), (Y, d_Y)$ be metric spaces, a function $f : X \to Y$ is continuous iff the pre-image of any open $U \subseteq Y$ is open in X.

Proof. □

Proposition 6.2.3. If, similarly, $(X, d_X), (Y, d_Y)$ are metric spaces with $f : X \to Y$, the following are equivalent:

1. f is continuous at $x \in X$,
2. if a sequence $(x_n)_{n\geq1}$ converges to $x \in X$ then $(f(x_n))_{n\geq1}$ converges to $f(x) \in Y$.

Proof. □

6.3 Metric homeomorphisms

Definition 6.3.1 (Homeomorphism)**.** Have $(X, d_X), (Y, d_Y)$ be metric spaces, a mapping $f : X \to Y$ is a **homeomorphism** if it is a bijection with f, f^{-1} both continuous. Metric spaces with homeomorphisms between then are **homeomorphic**.

Definition 6.3.2 (Lipschitz)**.** Given metric spaces $(X, d_X), (Y, d_Y)$ and $f : X \to Y$ we say:

1. f is **Lipschitz** if there is some $M > 0$ with:
$$d_Y(f(x_1), f(x_2)) \leq M \cdot d_X(x_1, x_2) \text{ for all } x_1, x_2 \in X,$$
2. f is **bi-Lipschitz** if there is some $M_1, M_2 > 0$ with:
$$M_1 \cdot d_X(x_1, x_2) \leq d_Y(f(x_1), f(x_2)) \leq M_2 \cdot d_X(x_1, x_2) \text{ for all } x_1, x_2 \in X,$$
3. f is **isometric** if,
$$d_Y(f(x_1), f(x_2)) = d_X(x_1, x_2) \text{ for all } x_1, x_2 \in X.$$

Remark 6.3.3. An isometry between metric spaces is a bi-Lipschitz map with two unit constants.

7 Topological spaces

7.1 Topologies and their spaces

Definition 7.1.1 (Topology)**.** Given a non-emtpy set X, we say τ, a collection of subsets of X, is a **topology** on X if it satisfies the following conditions:

(T1) $\emptyset, X \subseteq \tau$,

(T2) if $X_i \in \tau$ for all i in a indexing set $\mathcal{I}$, $\bigcup_{i\in\mathcal{I}} X_i \in \tau$,

(T3) if $X_1, X_2, \ldots, X_n \in \tau$, $\bigcap_{i=1}^{m} X_i \in \tau$.

The pair (X, τ) is called a **topological space** with elements of X called **points** and elements of τ called open sets. If $x \in X$ and $x \in U \in \tau$, U is a neighbourhood of x.

Examples 7.1.2. These are some common examples of topological spaces:

1. for any set X have $\tau = \{\emptyset, X\}$, the trivial topology on X,

2. instead have τ be the collection of subsets of X, the discrete topology on X,
3. if (X, d) is a metric space, $\tau := \{U \subseteq X : U \text{ is open in } (X, d)\}$ the metric topology on X,
4. for a non-empty set X, $\tau = \{\emptyset, V, X\}$ for some non-empty $V \subset X$,
5. if $X = \{a, b\}$ and $\tau = \{\emptyset, \{a, b\}, \{b\}\}$ is the smallest toplogical space that is neither trivial nor discrete (called the Sierpinski topology).

Definition 7.1.3 (Metrisability)**.** A topological space (X, τ) is **metrisable** iff it is the topology induced by some metric.

Definition 7.1.4 (Coarser and finer topologies)**.** Given two topologies τ_1, τ_2 both on X, we say τ_1 is **coarser** than τ_2, and equivalently τ_2 is **finer** than τ_1, iff $\tau_2 \subseteq \tau_1$.

7.2 Bases

Definition 7.2.1 (Basis)**.** Given a topological space (X, τ) we call a subfamily $B \subseteq \tau$ a **basis** for τ iff every open set in τ is the union of open sets in B.

7.3 Closed sets

Definition 7.3.1 (Closed sets)**.** Given a topological space (X, τ), we say $V \subseteq X$ is **closed** iff $X \setminus V$ is open.

Proposition 7.3.2. Closed sets in any given topological space (X, τ) satisfy the followoing:

(C1) $X, \emptyset$ are closed,

(C2) if C_1, C_2 are closed, $C_1 \cup C_2$ is closed,

(C3) the (up to uncountable) intersection of closed sets is closed.

Proof. □

Definition 7.3.3 (Closure)**.** Given an open set U in the topological space (X, τ) the **closure** of U in (X_τ) is given by:

$$\overline{U} := \bigcap_{\substack{V \subseteq X \\ V \,\text{closed}, A \subseteq V}} V.$$

Definition 7.3.4 (Point of closure)**.** Given the topological space $\mathcal{X}$ with $A \subseteq \mathcal{X}$, $x \in \mathcal{X}$ is a **point of closure** of A iff every open set U with $x \in U$ has $U \cap A = \emptyset$.

Proposition 7.3.5. $\overline{A} = \{x \in X : x \text{ is a point of closure for} A\}$.

Proof. □

7.4 Convergence and Hausdorff property

Definition 7.4.1 (Convergence)**.** For a sequences $(x_n)_{n \geq 1}$ in a topological space (X, τ) we say $(x_n)_{n \geq 1}$ **converges** (in (X, τ)) to $x \in X$ iff

$$\forall T \in \tau \text{ with } x \in T,\ \exists N \in \mathbb{Z}_{>0} \text{ such that } \forall n \geq N,\ x_n \in T.$$

Definition 7.4.2 (Hausdorff)**.** A topological space (X, τ) is **Hausdorff** iff for all $x, y \in X$ with $x \neq y$ there are open sets U, V containing x, y respectively with $U \cap V = \emptyset$. With U and V **separating** x and y.

Theorem 7.4.3. Limits of convergent sequences in Hausdorff spaces are unique.

Proof. □

Definition 7.4.4 (Regular spaces)**.** A topological space (X, τ) is **regular** iff for every closed subset $C \subseteq X$ with point $p \notin C$ there are open sets $U, V \in \tau$ such that $p \in U$, $C \subseteq V$ and $U \cap V = \emptyset$.

7.5 Continuous maps

Definition 7.5.1 (Continuous map)**.** Given two topological spaces $(X, \tau_X), (Y, \tau_Y)$ the map $f : X \to Y$ is **continuous** iff $f^{-1}(U) \in \tau_X$ for all $U \in \tau_Y$.

Definition 7.5.2 (Continuity at points)**.** The map $f : X \to Y$, with $(X, \tau_X), (Y, \tau_Y)$ topological spaces, is **continuous at** $x \in X$ iff $f^{-1}(U) \in \tau_X$ for all $U \in \tau_Y$ with $f(x) \in U$.

Definition 7.5.3 (Homeomorphism)**.** A **homeomorphism** between topolgical spaces is a bijection map, f, where both f and f^{-1} are continuous. Spaces with homeomorphisms between them are **topologically equivalent**.

7.6 Subspaces

Definition 7.6.1 (Subspace)**.** If (X, τ) is a topological space and $A \subseteq X$, the **subspace topology** on A is $\tau_A = \{A \cap U : U \in \tau\}$, (A, τ_A) is a topological space called the **supspace** of (X, τ).

Proof of topological space. □

Proposition 7.6.2 (Universal property)**.** Given topological spaces $(X, \tau_X), (Y, \tau_Y)$ with $A \subseteq X$ with its supspace topology and $g : Y \to A$, g is continuous iff $i \circ g$ is continuous, where i is the inclusion map,

$$\begin{tikzcd} & X \\ Y \arrow[ur, "i \circ g"] \arrow[r, "g"'] & A \arrow[u, hook, "i"'] \end{tikzcd}.$$

Proof. □

Theorem 7.6.3. Given the topological space (X, τ) and $A \subseteq X$, the subspace topology is the only topology such that for all (Y, τ_Y), $g : Y \to A$ is continuous iff $(i \circ g)$ is continuous.

Proof. □

Lemma 7.6.4. If B is a basis for the topological space (X, τ) and $A \subseteq X$, $B_A := \{U \cap A : U \in B\}$ is a basis for τ_A.

Proof. □

Proposition 7.6.5. For a metric space (X, d) with $A \subseteq X$, the two canonical topologies on A, τ_{d_A} and τ_A are equal.

Proof. □

8 Connectedness

8.1 Definition

8.2 Continuous maps

8.3 Path connected sets

9 Compactness

9.1 Covers

9.2 Sequential compactness

9.3 Continuous maps

9.4 Arzelá-Ascoli theorem

10 Completeness

10.1 Banach spaces

10.2 Fixed point theorem