1 Homotopy

2 Homology

2.1 Relative homology

We're building towards some sort of long exact sequence of homology:

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \cdots$$

so we need to work with some (X, A) where A is a subspace of X.

Def 2.1 For A a subspace of X we define the **relative chain complex** $C_n(X, A) := C_n(X)/C_n(A)$ where the normal boundary operator $\partial_n : C_n(X) \to C_{n-1}(X)$ sends $C_n(A)$ to $C_{n-1}(A)$ so induces a well-defined boundary operator:

$$\bar{\partial}_n : C_n(X, A) \to C_{n-1}(X, A) \qquad c + C_n(A) \mapsto \partial_n(c) + C_{n-1}(A)$$

Showing this is well-defined and $\bar{\partial}^2 = 0$ is left as an exercise. □

Now we have a chain complex we should obviously take its homology groups to get:

Def 2.2 For the same (X, A) the **relative homology groups** are:

$$H_n(X, A) := \frac{\ker(\bar{\partial}_n)}{\mathrm{im}(\bar{\partial}_{n+1})} \quad \begin{array}{l}\text{(relative cycles)} \\ \text{(relative boundaries)}\end{array}$$

Exc 2.1

This gives us a nice short exact sequence of chain complexes:

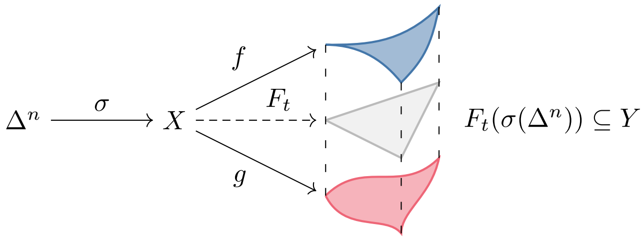
$$0 \to C_\bullet(A) \xrightarrow{i_\#} C_\bullet(X) \to C_\bullet(X, A) \to 0$$

inducing a corresponding long exact sequence in homology which is getting closer to our goal. We need to show there's an isomorphism between these relative homology groups and the reduced homology groups, which is true under mild conditions on the pair (X, A). However, this requires lots of machinery.

2.2 Homotopy invariance

Theorem (Homotopy invariance) If $f, g : X \to Y$ are homotopic maps, then $f_* = g_*$.

Proof. The homotopy F_t gives us a family of simplices in Y *continuously* interpolating between $f\sigma$ and $g\sigma$:



So the essential ingredient in this proof will be dividing $\Delta^n \times I$ into $(n+1)$-simplices.

Given a homotopy $F : X \times I \to Y$ from f to g and a singular n-simplex $\sigma : \Delta^n \to X$ we form:

$$F \circ (\sigma, \mathrm{id}) : \Delta^N \times I \xrightarrow{\sigma, \mathrm{id}} X \times I \xrightarrow{F} Y$$

and define the **prism operator** $P : C_n(X) \to C_{n+1}(X)$ by:

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}$$

where $[v_0, \ldots, v_n] := \Delta^n \times \{0\}$ and $[w_0, \ldots, w_n] := \Delta^n \times \{1\}$. And the claim is that:

$$\partial P(\sigma) = g_\# \sigma - f_\# \sigma - P(\partial \sigma)$$

From which we see: if $\partial P(c) = g_\# c - f_\# c - P(\partial c)$ for some $c \in C_n(X)$, then if c is a cycle ($\partial c = 0$) then $\partial P(c) = g_\#(c) - f_\#(c)$, as their difference on chains is a boundary $f_* = g_*$.

We now look to prove the claim, which will be done via some ugly algebra:

$$\begin{aligned}
\partial P(\sigma) &= \partial \left(\sum_{i=0}^{n} (-1)^i F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]} \right) \\
&= \sum_{j \leq i} (-1)^{i+j} F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, \hat{v}_j, \ldots, v_i, w_i, \ldots, w_n]} + \sum_{j \geq i} (-1)^{i+j+1} F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j, \ldots, w_n]}
\end{aligned}$$

all the terms with $i = j$ will cancel except for:

$$F \circ (\sigma, \mathrm{id})|_{[\hat{v}_0, w_0, \ldots, w_n]} = g_\#(\sigma) \quad \text{and} \quad F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, v_n, \hat{w}_n]} = f_\#(\sigma)$$

as otherwise we will always have both terms:

$$(-1)^{i+i} F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, v_{i-1}, \hat{v}_i, w_i, \ldots, w_n]} + (-1)^{i-1+i} F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, v_{i-1}, \hat{w}_{i-1}, w_i, \ldots, w_n]} = 0$$

this leaves us exactly:

$$\partial P(\sigma) = g_\#(\sigma) - f_\#(\sigma) + \sum_{i<j} \ldots + \sum_{i>j} \ldots = g_\#(\sigma) - f_\#(\sigma) + P(\partial \sigma)$$

which is obvious by expanding out $\partial \sigma$ before appplying P. □

Our formulation of homotopy invariance is atually a case of a slightly more general construction.

Def 2.3 Two morphisms of chain complexes $f_\bullet, g_\bullet : A_\bullet \to B_\bullet$ are called **chain homotopic** if there exists some family $P_n : A_n \to B_{n+1}$ such that $p - q = \partial P + P\partial$.

Prop 2.1 Chain homotopic maps induce the same maps on homology.

Proof. Exercise. □

So if some map $h : X \to Y$ is a homotopy equivalence, the induced map will be an isomorphism.

2.3 Excision

Theorem (Excision) Given $Z \subset A \subset X$ with $\bar{Z} \subset \mathring{A}$, the inclusion of pairs $(X/Z, A/Z) \hookrightarrow (X, A)$ induces an isomorphism of relative homology $H_n(X/Z, A/Z) \cong H_n(X, A)$.

There is an obvious equivalent formulation of the excision theorem that seems a bit more palatable: if $A, B \subset X$ such that $X = \mathring{A} \cup \mathring{B}$ then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism on relative homology.

To prove this theorem we will use the following result without proof.

Prop 2.2 Given a collection $\mathcal{U} = \{U_\alpha\}$ of subsets of X that covers X we can form,

$$C_n^{\mathcal{U}} := \{n\text{-chains in } X \text{ where each simplex is in on } U_\alpha$$

then $i : C_\bullet^{\mathcal{U}} \to (X)C_\bullet(X)$ is a **chain homotopy equivalence**, which means there exists some $s : C_\bullet(X) \to C_\bullet^{\mathcal{U}}(X)$ such that $i \circ s$ and $s \circ i$ are both chain homotopic to the identity.

the idea is that we can subdivide simplices that overlap the sheet in our open cover (by something called barycentric subdivision) until each simplex only lies in one sheet.