These lectures notes were taken by Yu Coughlin for 2024/2025 module Algebraic Topology by Dr Sara Veneziale at Imperial College London. As well as reading Hatcher, May, and Fomenko Fuchs' books. Some parts of these notes are added / modified very heavily compared to the lectures, some stuff from the lectures is missing, some stuff is alsmost exactly as it was in the lectures.

1 Homotopy

1.1 Introduction and Operations

Recall the definition of an abstract topological space:

> **Def 1.1** A **topological space** is a set X with a collection $\Omega \subseteq \mathcal{P}(X)$ of subsets of X called a **topology**. Ω contains both the empty set $\emptyset$ as well as X and is closed under taking arbitrary unions and finite intersections. The sets $U \in \Omega$ are called **open sets**.

To form a category we also need morphisms:

> **Def 1.2** A funcion $f : X \to Y$ between topological spaces (X, Ω_X), (Y, Ω_Y) is called **continuous** if:
>
> $$\forall U \in \Omega_Y, \quad f^{-1}(U) := \{x \in X : f(x) \in U\} \in \Omega_X$$

We can now form the first topological categories of interest in this course,

> **Def 1.3 Top** is the category of topological spaces and continuous maps. **Top**$_*$ is the category of **based** topological spaces, pairs (X, x) for X a topological space with $x \in X$, and basepoint-preserving continuous maps.

Providing the full topology everytime we want to construct a new topological space from old is too much work so we have:

> **Prop 1.1** If X is a set and $\{\Omega_i\}_{i\in I}$ is a family of topologies on X, then $\bigcap_{i\in I} \Omega_i$ is a topology on X. If (Y, Ω) is a topological space and $A \in \Omega$, then $\{U \cap A : U \in \Omega\}$ is a topology on A called the **subspace topology**.
>
> *Proof.* Easy exercise. □

Which lets us form

> **Def 1.4** Given a set X, and a collection $\mathcal{B} \subseteq \mathcal{P}(X)$, the topology **generated by** $\mathcal{B}$ is the intersection of all topologies on X containing $\mathcal{B}$.

Now we can show **Top** is (co)complete,

> **Def 1.5** Have $\{(S_i, \Omega_i)\}_{i\in I}$ in **Top** and let S be a set. For $\{f_i : S \to S_i\}_{i\in I}$ the **initial topology** $\Omega_{initial}(\{f_i\}_{i\in I})$ is the topology on S with the *minimum* collection of open subsets such that all the $f_i : (S, \Omega_{\text{initial}}) \to (S_i, \Omega_i)$ are continuous, this will be the topology generated by $\bigcup_{i\in I}\{f^{-1}(U) : U \in \Omega_i\}$. The **final topology** is defined dually as the largest topology making every $g_i : S_i \to S$ continuous, which is $\{U \subseteq S : \forall i \in I,\ g_i^{-1}(U) \in \Omega_i\}$.

> **Prop 1.2** For a small category I and a diagram $X : I \to$ **Top**, with $(S_i, \Omega_i) := X(i)$ for all i, the limit $\varprojlim X$ exists and has underlying set the limit of the underlying diagram in **Set** with the initial topology $\Omega_{\text{initial}}(\{p_i\})$ where p_i are the projection maps from the limit in **Set**.
>
> *Proof.* The required universal property is immediate from the definition of a minimal topology: for any cone (S, Ω) over $X(I)$ there is a function $f : S \to \varprojlim S_i$ making the necessary diagrams commute, where the continuity of this function is exactly the definition of the initial topology. □
>
> Dually, the colimit $\varinjlim X$ is just the colimit in **Set** with the final topology, this has a similar proof.

I'm going to declare that by requiring all of these maps be basepoint preserving the result works in **Top**$_*$ as well, without checking it.

2 Homology

2.1 Relative homology

We're building towards some sort of long exact sequence of homology:

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \cdots$$

so we need to work with some (X, A) where A is a subspace of X.

Def 2.1 For A a subspace of X we define the **relative chain complex** $C_n(X, A) := C_n(X)/C_n(A)$ where the normal boundary operator $\partial_n : C_n(X) \to C_{n-1}(X)$ sends $C_n(A)$ to $C_{n-1}(A)$ so induces a well-defined boundary operator:

$$\bar{\partial}_n : C_n(X, A) \to C_{n-1}(X, A) \qquad c + C_n(A) \mapsto \partial_n(c) + C_{n-1}(A)$$

Showing this is well-defined and $\bar{\partial}^2 = 0$ is left as an exercise. □

Now we have a chain complex we should obviously take its homology groups to get:

Def 2.2 For the same (X, A) the **relative homology groups** are:

$$H_n(X, A) := \frac{\ker(\bar{\partial}_n)}{\operatorname{im}(\bar{\partial}_{n+1})} \quad \begin{array}{l}\text{(relative cycles)}\\ \text{(relative boundaries)}\end{array}$$

Exc 2.1

This gives us a nice short exact sequence of chain complexes:

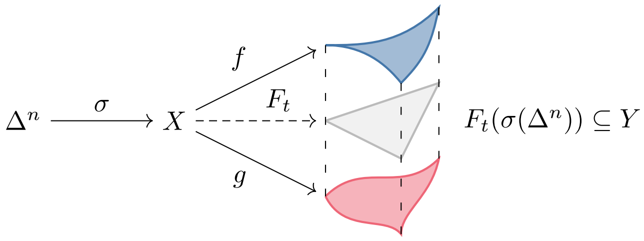
$$0 \to C_\bullet(A) \xrightarrow{i_\#} C_\bullet(X) \to C_\bullet(X, A) \to 0$$

inducing a corresponding long exact sequence in homology which is getting closer to our goal. We need to show there's an isomorphism between these relative homology groups and the reduced homology groups, which is true under mild conditions on the pair (X, A). However, this requires lots of machinery.

2.2 Homotopy invariance

Theorem (Homotopy invariance) If $f, g : X \to Y$ are homotopic maps, then $f_* = g_*$.

Proof. The homotopy F_t gives us a family of simplices in Y *continuously* interpolating between $f\sigma$ and $g\sigma$:



So the essential ingredient in this proof will be dividing $\Delta^n \times I$ into $(n+1)$-simplices.
Given a homotopy $F : X \times I \to Y$ from f to g and a singular n-simplex $\sigma : \Delta^n \to X$ we form:

$$F \circ (\sigma, \mathrm{id}) : \Delta^N \times I \xrightarrow{\sigma,\mathrm{id}} X \times I \xrightarrow{F} Y$$

and define the **prism operator** $P : C_n(X) \to C_{n+1}(X)$ by:

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma, \mathrm{id})|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]}$$

where $[v_0, \ldots, v_n] := \Delta^n \times \{0\}$ and $[w_0, \ldots, w_n] := \Delta^n \times \{1\}$. And the claim is that:

$$\partial P(\sigma) = g_\# \sigma - f_\# \sigma - P(\partial\sigma)$$

From which we see: if $\partial P(c) = g_\# c - f_\# c - P(\partial c)$ for some $c \in C_n(X)$, then if c is a cycle ($\partial c = 0$) then $\partial P(c) = g_\#(c) - f_\#(c)$, as their difference on chains is a boundary $f_* = g_*$.
We now look to prove the claim, which will be done via some ugly algebra:

$$\begin{aligned}\partial P(\sigma) &= \partial \left(\sum_{i=0}^{n} (-1)^i F \circ (\sigma, \mathrm{id})|_{[v_0,\ldots,v_i,w_i,\ldots,w_n]} \right) \\ &= \sum_{j \le i} (-1)^{i+j} F \circ (\sigma, \mathrm{id})|_{[v_0,\ldots,\hat{v}_j,\ldots,v_i,w_i,\ldots,w_n]} + \sum_{j \ge i} (-1)^{i+j+1} F \circ (\sigma, \mathrm{id})|_{[v_0,\ldots,v_i,w_i,\ldots,\hat{w}_j,\ldots,w_n]}\end{aligned}$$

all the terms with $i = j$ will cancel except for:

$$F \circ (\sigma, \mathrm{id})|_{[\hat{v}_0,w_0,\ldots,w_n]} = g_\#(\sigma) \quad \text{and} \quad F \circ (\sigma, \mathrm{id})|_{[v_0,\ldots,v_n,\hat{w}_n]} = f_\#(\sigma)$$

as otherwise we will always have both terms:

$$(-1)^{i+i} F \circ (\sigma, \mathrm{id})|_{[v_0,\ldots,v_{i-1},\hat{v}_i,w_i,\ldots,w_n]} + (-1)^{i-1+i} F \circ (\sigma, \mathrm{id})|_{[v_0,\ldots,v_{i-1},\hat{w}_{i-1},w_i,\ldots,w_n]} = 0$$

this leaves us exactly:

$$\partial P(\sigma) = g_\#(\sigma) - f_\#(\sigma) + \sum_{i<j} \ldots + \sum_{i>j} \ldots = g_\#(\sigma) - f_\#(\sigma) + P(\partial\sigma)$$

which is obvious by expanding out $\partial\sigma$ before appplying P. □

Our formulation of homotopy invariance is atually a case of a slightly more general construction.

Def 2.3 Two morphisms of chain complexes $f_\bullet, g_\bullet : A_\bullet \to B_\bullet$ are called **chain homotopic** if there exists some family $P_n : A_n \to B_{n+1}$ such that $p - q = \partial P + P\partial$.

Prop 2.1 Chain homotopic maps induce the same maps on homology.

Proof. Exercise. □

So if some map $h : X \to Y$ is a homotopy equivalence, the induced map will be an isomorphism.

2.3 Excision

> **Theorem (Excision)** Given $Z \subset A \subset X$ with $\bar{Z} \subset \mathring{A}$, the inclusion of pairs $(X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism of relative homology $H_n(X \setminus Z, A \setminus Z) \cong H_n(X, A)$.

There is an obvious equivalent formulation of the excision theorem that seems a bit more palatable: if $A, B \subset X$ such that $X = \mathring{A} \cup \mathring{B}$ then the inclusion $(B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism on relative homology.
To prove this theorem we will use the following result without proof.

> **Prop 2.2** Given a collection $\mathcal{U} = \{U_\alpha\}$ of subsets of X whose interiors cover X we can form,
>
> $$C_n^{\mathcal{U}} := \{n\text{-chains in } X \text{ where each simplex is in on } U_\alpha$$
>
> then $i : C_\bullet^{\mathcal{U}} \to C_\bullet(X)$ is a **chain homotopy equivalence**, which means there exists some $s : C_\bullet(X) \to C_\bullet^{\mathcal{U}}(X)$ such that $i \circ s$ and $s \circ i$ are both chain homotopic to the identity.

the idea is that we can subdivide simplices that overlap the sheet in our open cover (by something called barycentric subdivision) until each simplex only lies in one sheet.
Continuing with the proof skets: if we have such $A, B \subset X$ with $\mathring{A} \cup \mathring{B} = X$ then we can take $\mathcal{U} = \{A, B\}$ in the above and get $C_n^{\mathcal{U}}(X) = C_n(A) \oplus C_n(B) \subseteq C_n(X)$, this inclusion gives:

$$C_\bullet^{\mathcal{U}}(X, A) = \frac{C_\bullet^{\mathcal{U}}(X)}{C_\bullet^{\mathcal{U}}(A)} = \frac{C_\bullet^{\mathcal{U}}(X)}{C_\bullet(A)} \hookrightarrow \frac{C_\bullet(X)}{C_\bullet(A)} = C(X, A)$$

which we know is a chain homotopy equivalence. And, by using the second isomorphism theorem, we have:

$$\frac{C_n(B)}{C_n(A \cap B)} = \frac{C_n(B)}{C_n(A) \cap C_n(B)} \cong \frac{C_n(A) \oplus C_n(B)}{C_n(A)} = \frac{C_n^{\mathcal{U}}(X)}{C_n(A)} = C_n^{\mathcal{U}}(X, A)$$

the composition of which induces the isomorphism of homology:

$$H_n(B, A \cap B) \cong H_n(X, A)$$

we are finally approaching the end of our epic journey towards the long exact sequence in homology.

> **Lemma 2.1** For X a topological space and $x_0 \in X$ there is an isomorphism $H_*(X, \{x_0\}) \cong \tilde{H}_*(X)$
>
> *Proof.* Exercise, it should fall straight out of the long exact sequence of $(X, \{x_0\})$. □

Note that in this proof we didn't have to assume (X, x_0) are a good pair.

> **Prop 2.3** If (X, A) is a good pair then the quotient map $q : (X, A) \to (X/A, A/A)$ induces an isomorphism in homology: $H_n(X, A) \cong H_n(X/A, A/A) \cong \tilde{H}_n(X/A)$.

Proof. We're going to consider the diagram:

$$\begin{tikzcd}
H_n(X,A) \arrow[r, "1"] \arrow[d, "4"'] & H_n(X,V) \arrow[r, "5"] \arrow[d] & H_n(X \setminus A, V \setminus A) \arrow[d, "3"] \\
H_n(X/A, A/A) \arrow[r, "2"'] & H_n(X/A, V/A) \arrow[r, "6"'] & H_n(X/A \setminus A/A, V/A \setminus A/A)
\end{tikzcd}$$

where V is some neighbourhood that retracts to A from the fact that (X, A) are good. The strategy will be to show 4 is an isomorphism by showing all of the remaining maps are and writing 4 as their composition.
For 5 and 6 these maps are both the isomorphisms from the respective applications of excision.
For 1 we will consider the long exact sequence for the triple (X, V, A) induced by the short exact sequence of chain maps:

$$0 \to C_\bullet(V, A) \xrightarrow{\iota} C_\bullet(X, A) \xrightarrow{q} C_\bullet(X, V) \to 0$$

this long exact sequence looks like:

$$\cdots \to \cancelto{0}{H_n(V, A)} \xrightarrow{\iota_*} H_n(X, A) \xrightarrow{q*} H_n(X, V) \xrightarrow{\partial} \cancelto{0}{H_{n-1}(V, A)} \to \cdots$$

Here $i : A \to V$ is a homotopy equivalence so $H_n(V, A) \cong 0$; thus 1, which is in fact q_*, is an isomorphism. This same argument applies to $H_n(V/A, A/A)$, so the long exact sequence for $(X/A, V/A, A/A)$ also gives isomorphisms $H_n(X/A, V/A) \cong H_n(X/A, A/A)$ which is 2. The quotient map $q : X \to X/A$ is a homeomorphism when restricted to X/A or V/A so 3 is induced by a homeomorphism of pairs, thus is an isomorphism. □

If we consider the wedge product of a collection of good pointed spaces $(X, A) = \bigvee_\alpha (X_\alpha, x_\alpha)$, which is just a quotient of the disjoint union, then we get the long isomorphism:

$$\tilde{H}_n\left(\bigvee_\alpha X_\alpha\right) \cong \tilde{H}_n(X/A) \cong H_n(X, A) =$$
$$= H_n\left(\bigsqcup_\alpha X_\alpha, \bigsqcup_\alpha \{x_\alpha\}\right) = \bigoplus_\alpha H_n(X_\alpha, x_\alpha) \cong \bigoplus_\alpha \tilde{H}_n(X_\alpha, x_\alpha) \cong \bigoplus_\alpha \tilde{H}_n(X_\alpha)$$

Theorem (Dimension invariance) If $\mathbb{R}^n \cong \mathbb{R}^m$ then $n = m$.

Proof. Exercise, we know the homology groups of the sphere. □

This next theorem is the final tool before we can prove the equivalence of simplicial and singular homology on Δ-complexes.

Theorem (Five lemma) Given the following commutative diagram of abeliang groups:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\alpha"] & B \arrow[r, "g"] \arrow[d, "\beta"] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[r, "k"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D' \arrow[r, "k'"'] & E'
\end{tikzcd}$$

if both rows are exact and $\alpha, \beta, \delta, \varepsilon$ are all isomorphisms, then γ is also an isomorphism.

Proof. We prove this with two different *four lemmas*:

$$(1)\quad \begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\alpha"'] & B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, "\gamma"] & D \arrow[d, "\delta"] \\ A' \arrow[r, "f'"'] & B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D' \end{tikzcd} \qquad (2)\quad \begin{tikzcd} B \arrow[r, "g"] \arrow[d, "\beta"'] & C \arrow[r, "h"] \arrow[d, "\gamma"'] & D \arrow[r, "k"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\ B' \arrow[r, "g'"'] & C' \arrow[r, "h'"'] & D' \arrow[r, "k'"'] & E' \end{tikzcd}$$

For (1) we assume both rows are exact; β, δ are mono; and α is epi, and show γ is mono:

- have $c \in C$ such that $\gamma(c) = 0 \in C'$ so $\delta h(c) = h'\gamma(c) = 0$ and as δ is mono, $h(c) = 0$
- so $c \in \ker h = \operatorname{im} g$, there exists $b \in B$ with $c = g(b)$ and $g'(\beta(b)) = \gamma(g(b)) = \gamma(c) = 0$
- so $\beta(b) \in \ker g' = \operatorname{im} f'$, there exists $a' \in A'$ with $f'(a') = \beta(b)$
- α epi $\implies$ there exists $a \in A$ with $a' = \alpha(a)$ and $\beta(b) = f'(\alpha(a)) = \beta(f(a))$
- β mono $\implies$ $b = f(a)$ so $c = g(b) = f(g(a)) = 0$

For (2) we assume both rows are exact; β, δ are epi; and ε is mono, and show γ is epi:

- have $c' \in C'$; as δ is epi there is a $d \in D$ with $\delta(d) = h'(c')$
- $\varepsilon(k(d)) = k'\delta(d) = k'h'(c) = 0$; as ε is mono, $k(d) = 0$
- so $d \in \ker k = \operatorname{im} h$, there exists $c \in C$ with $h(c) = d$; so $h'(\gamma(c))\delta(h(c)) = \delta(b) = c'$
- so $c' - \gamma(c) \in \ker h' = \operatorname{im} g'$, there is $b' \in B'$ with $g'(b') = c' - \gamma(c)$
- as β is epi, there is $b \in B$ with $\beta(b) = b'$; so $\gamma(g(b)) = g'(\beta(b)) = c' - \gamma(c)$
- so $c' = \gamma(g(b)) + \gamma(c) = \gamma(g(b) + c)$

therefore γ is both mono and epi, and thus an isomorphism. $\square$