1 Homotopy

2 Homology

2.1 Relative homology

We're building towards some sort of long exact sequence of homology:

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \cdots$$

so we need to work with some (X, A) where A is a subspace of X.

Def 2.1 For A a subspace of X we define the **relative chain complex** $C_n(X, A) := C_n(X)/C_n(A)$ where the normal boundary operator $\partial_n : C_n(X) \to C_{n-1}(X)$ sends $C_n(A)$ to $C_{n-1}(A)$ so induces a well-defined boundary operator:

$$\bar{\partial}_n : C_n(X, A) \to C_{n-1}(X, A) \qquad c + C_n(A) \mapsto \partial_n(c) + C_{n-1}(A)$$

Showing this is well-defined and $\bar{\partial}^2 = 0$ is left as an exercise. □

Now we have a chain complex we should obviously take its homology groups to get:

Def 2.2 For the same (X, A) the **relative homology groups** are:

$$H_n(X, A) := \frac{\ker(\bar{\partial}_n)}{\operatorname{im}(\bar{\partial}_{n+1})} \quad \begin{matrix} \text{(relative cycles)} \\ \text{(relative boundaries)} \end{matrix}$$

Exc 2.1

This gives us a nice short exact sequence of chain complexes:

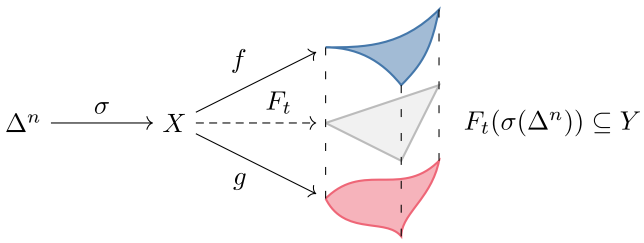
$$0 \to C_\bullet(A) \xrightarrow{i_\#} C_\bullet(X) \to C_\bullet(X, A) \to 0$$

inducing a corresponding long exact sequence in homology which is getting closer to our goal. We need to show there's an isomorphism between these relative homology groups and the reduced homology groups, which is true under mild conditions on the pair (X, A). However, this requires lots of machinery.

2.2 Homotopy invariance

Theorem (Homotopy invariance) If $f, g : X \to Y$ are homotopic maps, then $f_* = g_*$.

Proof. The homotopy F_t gives us a family of simplices in Y *continuously* interpolating between $f\sigma$ and $g\sigma$:



So the essential ingredient in this proof will be dividing $\Delta^n \times I$ into $(n+1)$-simplices.

Given a homotopy $F : X \times I \to Y$ from f to g and a singular n-simplex $\sigma : \Delta^n \to X$ we form:

$$F \circ (\sigma, \mathrm{id}) : \Delta^N \times I \xrightarrow{\sigma, \mathrm{id}} X \times I \xrightarrow{F} Y$$

and define the **prism operator** $P : C_n(X) \to C_{n+1}(X)$ by:

$$P(\sigma) = \sum_{i=0}^{n} (-1)^i F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]}$$

where $[v_0, \ldots, v_n] := \Delta^n \times \{0\}$ and $[w_0, \ldots, w_n] := \Delta^n \times \{1\}$. And the claim is that:

$$\partial P(\sigma) = g_\# \sigma - f_\# \sigma - P(\partial \sigma)$$

From which we see: if $\partial P(c) = g_\# c - f_\# c - P(\partial c)$ for some $c \in C_n(X)$, then if c is a cycle ($\partial c = 0$) then $\partial P(c) = g_\#(c) - f_\#(c)$, as their difference on chains is a boundary $f_* = g_*$.

We now look to prove the claim, which will be done via some ugly algebra:

$$\begin{aligned}
\partial P(\sigma) &= \partial \left(\sum_{i=0}^{n} (-1)^i F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, w_n]} \right) \\
&= \sum_{j \le i} (-1)^{i+j} F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, \hat{v}_j, \ldots, v_i, w_i, \ldots, w_n]} + \sum_{j \ge i} (-1)^{i+j+1} F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, v_i, w_i, \ldots, \hat{w}_j, \ldots, w_n]}
\end{aligned}$$

all the terms with $i = j$ will cancel except for:

$$F \circ (\sigma, \mathrm{id})|_{[\hat{v}_0, w_0, \ldots, w_n]} = g_\#(\sigma) \quad \text{and} \quad F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, v_n, \hat{w}_n]} = f_\#(\sigma)$$

as otherwise we will always have both terms:

$$(-1)^{i+i} F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, v_{i-1}, \hat{v}_i, w_i, \ldots, w_n]} + (-1)^{i-1+i} F \circ (\sigma, \mathrm{id})|_{[v_0, \ldots, v_{i-1}, \hat{w}_{i-1}, w_i, \ldots, w_n]} = 0$$

this leaves us exactly:

$$\partial P(\sigma) = g_\#(\sigma) - f_\#(\sigma) + \sum_{i<j} \ldots + \sum_{i>j} \ldots = g_\#(\sigma) - f_\#(\sigma) + P(\partial \sigma)$$

which is obvious by expanding out $\partial \sigma$ before appplying P. $\square$

2.3 Excision