1 Homotopy

2 Homology

2.1 Relative homology

We're building towards some sort of long exact sequence of homology:

$$\cdots \to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X/A) \to \cdots$$

so we need to work with some (X, A) where A is a subspace of X.

Def 2.1 For A a subspace of X we define the **relative chain complex** $C_n(X, A) := C_n(X)/C_n(A)$ where the normal boundary operator $\partial_n : C_n(X) \to C_{n-1}(X)$ sends $C_n(A)$ to $C_{n-1}(A)$ so induces a well-defined boundary operator:

$$\bar{\partial}_n : C_n(X, A) \to C_{n-1}(X, A) \qquad c + C_n(A) \mapsto \partial_n(c) + C_{n-1}(A)$$

Showing this is well-defined and $\bar{\partial}^2 = 0$ is left as an exercise. □

Now we have a chain complex we should obviously take its homology groups to get:

Def 2.2 For the same (X, A) the **relative homology groups** are:

$$H_n(X, A) := \frac{\ker(\bar{\partial}_n)}{\operatorname{im}(\bar{\partial}_{n+1})} \quad \begin{array}{l}\text{(relative cycles)}\\ \text{(relative boundaries)}\end{array}$$

Exc 2.1 TODO: relative cyclic diagram

This gives us a nice short exact sequence of chain complexes:

$$0 \to C_\bullet(A) \xrightarrow{i_\#} C_\bullet(X) \to C_\bullet(X, A) \to 0$$

inducing a corresponding long exact sequence in homology which is getting closer to our goal. We need to show there's an isomorphism between these relative homology groups and the reduced homology groups, which is true under mild conditions on the pair (X, A). However, this requires lots of machinery.

2.2 Homotopy invariance

Theorem (Homotopy invariance) If $f, g : X \to Y$ are homotopic maps, then $f_* = g_*$.

TODO: prism diagram
we want to divide this in

2.3 Excision