Inverse Problems in Imaging
Lecture 1

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Introduction

What is an Inverse Problem ?
Generally, it means the recovery of information from observations that have been obtained by a *measurement system*

$$g = A(f)$$

" given data, find the image that produced it, assuming a model of measurements A"
In many cases, where f is an image, A is a *linear integral operator* :

$$g(y) = \int_\Omega K(x, y) f(x) \mathrm{d}x$$

K is the kernel of the integral operator.

Let's look at some examples.

Denoise : $A \equiv Identity$, $K(x, y) = \delta(x - y)$. Added noise.

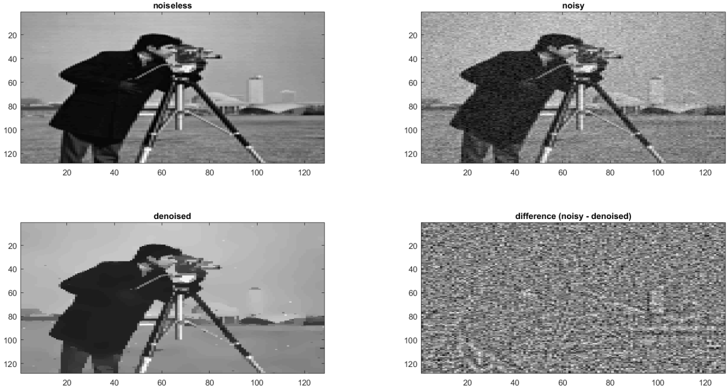


Figure: Original and noisy Camerman image.

Introduction
Example : Image Deblurring

Denoise : $A \equiv$ *convolution*, $K(x,y) = b(x-y)$. b is blurring function, assumed known.

Figure: Original and Blurred Camerman image.

"Blind deconvolution" : the function b is not known. Much harder problem!

Inpainting : $A \equiv$ *Masking*, where $K(x,y) = \delta(x-y)\mathcal{I}(x)$. $\mathcal{I}$ is an *indicator function*.

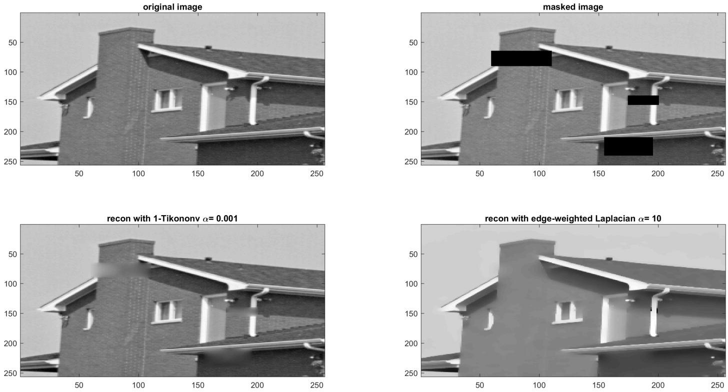
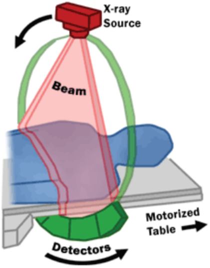


Figure: Original House image and Inpainting Results.

X-Ray CT : $A \equiv Line - projection$, $K(x,y) = \delta(s - x \cdot \hat{n})$. Where $x \cdot \hat{n} = s$ is equation of a line.



zero scattering photons $\Rightarrow$ are propagated along rays
$\boldsymbol{\ell} := x_0 + \ell \hat{n}_\perp$

$$\hat{\boldsymbol{s}} \cdot \nabla U + fU = 0 \quad \equiv \quad \mathcal{T}_f U = 0$$

whose solution

$$U = U_0 \exp\left[- \int_{\boldsymbol{\ell}} f(x_0 + \ell \hat{n}_\perp) \mathrm{d}\ell \right]$$

is the basis for the definition of the *Ray Transform*

$$g(s, \hat{n}_\perp) := -\ln\left[\frac{U}{U_0}\right] = \int_{-\infty}^{\infty} f(s\hat{n} + \ell\hat{n}_\perp)\mathrm{d}\ell \quad \equiv \quad g(s, \hat{n}_\perp) = \mathcal{R}f$$

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X-Ray CT - the Radon Transform

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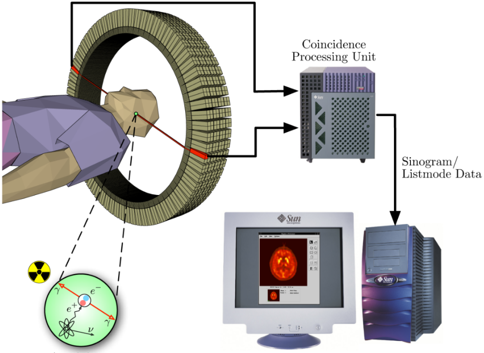
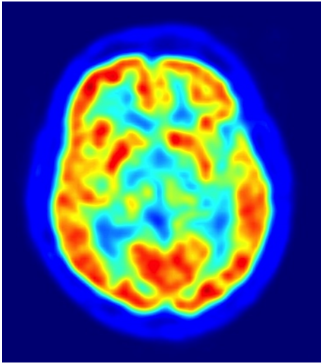
Mono-energetic CT is linear, but multispectral CT is *non linear*

$$g = \log \sum_i w_i(\lambda) \mathrm{e}^{-\mathcal{R} f_i(\lambda)}$$

PET : $A \equiv Line - projection, K(x, y) = \delta(s - x \cdot \hat{n})$. Noise is Poissonian

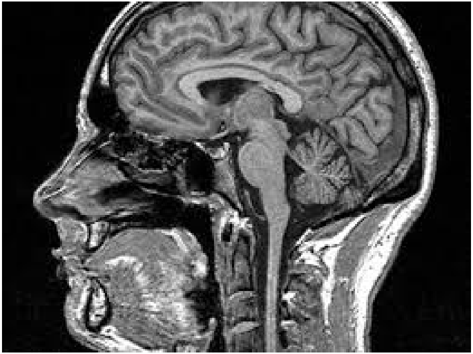




PET with attenuation is *non-linear*

$$g = \mathrm{e}^{-\mathcal{R}\mu}\mathcal{R}f$$

MRI : $A \equiv \mathit{FourierTransform}$, $K(x,y) = \mathrm{e}^{-\mathrm{i}xy}$. Noise is Rician



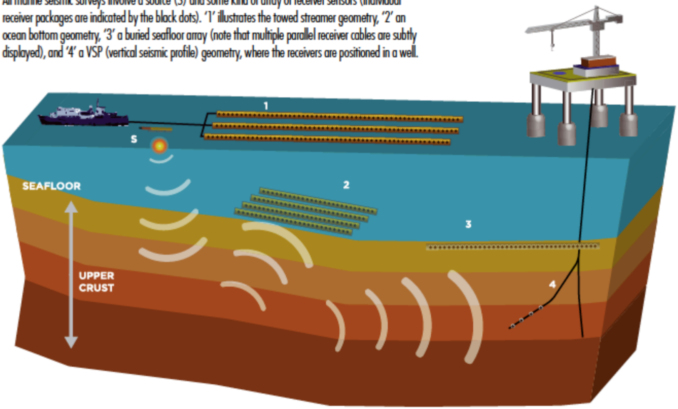
Parallel MRI is *multilinear*

$$\begin{pmatrix} g_1 \\ g_2 \\ \vdots \\ g_n \end{pmatrix} = \begin{pmatrix} \mathcal{F}[s_1 f] \\ \mathcal{F}[s_2 f] \\ \vdots \\ \mathcal{F}[s_n f] \end{pmatrix}$$

GeoPhysics : $A \equiv$ *wave equation propagation*, $K(x,y) = \delta(x - y - ct)$.

Figure 1 *(credit: Jack Caldwell)*

All marine seismic surveys involve a source (S) and some kind of array or receiver sensors (individual receiver packages are indicated by the black dots). '1' illustrates the towed streamer geometry, '2' an ocean bottom geometry, '3' a buried seafloor array (note that multiple parallel receiver cables are subtly displayed), and '4' a VSP (vertical seismic profile) geometry, where the receivers are positioned in a well.



www.ogp.org.uk/pubs/448.pdf, 20.02.13

From Literature :
Most people, if you describe a train of events to them will tell you what the result will be. There are few people, however that if you told them a result, would be able to evolve from their own inner consciousness what the steps were that led to that result. This power is what I mean when I talk of reasoning backward.
Sherlock Holmes, A Study in Scarlet, Sir Arthur Conan Doyle (1887)

Introduction
Quotations

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In 1976 Keller formulated the following very general definition of inverse problems, which is often cited in the literature:
We call two problems inverses of one another if the formulation of each involves all or part of the solution of the other. Often, for historical reasons, one of the two problems has been studied extensively for some time, while the other is newer and not so well understood. In such cases, the former problem is called the direct problem, while the latter is called the inverse problem.